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SOME APPLICATIONS OF FIBRE BUNDLE TECHNIQUES IN PHYSICS

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ABSTRACT

Both the theories of differential geometry and of Lie groups and their algebras have been invaluable to the physicist. In the theory of fibre bundles and in the symplectic formulation of mechanics, these fields coalesce to provide a rich structure that enables her/him to obtain a more unified overview of the modern theories in physics. In this short work, we introduce this structure and examine its consequences in general relativity and quantum mechanics.

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INTRODUCTION

The formalism of fibre bundles has become popular in many areas of physics recently. There are many reasons for this phenomenon, and it is hoped that in this short survey some of these will become apparent. In particular, the notion of a connection in a fibre bundle is extremely useful. For the theory of general relativity, the connection, which is related to the matter via Einstein's equations, determines the geodesics of spacetime - the parameterized paths of uncharged test particles. In geometric quantization, a connection provides a homomorphism representing observables as skew hermitian operators on a Hilbert space, and in gauge theories such as electromagnetism and Yang-Mills particle fields, the connection appears as a continuous choice of gauge at each point in space time. Here, connections in general relativity and geometric quantization are dealt with briefly; for a discussion of gauge theories, see [5].

An elementary knowledge of the exterior calculus on manifolds is assumed, and an excellent introduction to this vast subject is given in the two volume series entitled "Foundation of Differential Geometry" by Kobayashi and Nomizu [1]. It is the major source for the introductory chapters.

After a somewhat detailed excursion into some of the basic definitions and results of fibre bundles and connections, a proof of Weil's theorem is presented. This result is the foundation of geometric quantization, a prescription initiated by Kostant and Souriau for quantizing classical theories in physics. Prior to the description of quantization, a brief introduction to symplectic mechanics is given, and this formalism emerges as a focus for the theories presented here. In the framework of symplectic mechanics, the correspondence between classical and quantum theories is seen more clearly, and this connection becomes rigorous through the work of Kostant and Souriau.

In the section on general relativity, the motion of a test particle is investigated with reference to symplectic theory. It is shown how the fibre bundle formalism is useful in the question of singularities, and a brief discussion of some versions of Mach's principle in relativity illustrates some unsolved problems in the theory. This last section introduces the question of uniqueness of solution to Einstein's field equations, and a related problem is discussed, with particular recourse to the holonomy group as a tool, in the last chapter. Some limitations of the method outlined there are described when the connection is not metric, as it is in Einstein's theory.

Finally, an appendix detailing some notions introduced in the section on Weil's theorem is given.

1.1 FIBRE BUNDLES

Let M be a manifold and G a Lie group.

Definition A principal fibre bundle is a manifold P, together with an action of G on P, $(u,a) \to ua \in P$, satisfying:

(1) G acts freely on P on the right: $ua = u \Longleftrightarrow a = 1$.

(2) M is the quotient space of P by the equivalence induced by G, $M = P/G$. The projection $\pi: P \to M$ is differentiable.

(3) P is locally trivial: $\forall x \in M$, $\exists U$, a neighbourhood of x, with $\pi^{-1}(U) \simeq U \times G$. By this isomorphism is meant a diffeomorphism $\psi: \pi^{-1}(U) \to U \times G$ and a map $\phi: \pi^{-1}(U) \to G$ such that $\psi(u) = (\pi(u), \phi(u))$, and $\phi(ua) = \phi(u) \cdot a$ $\forall u \in \pi^{-1}(U)$ and $a \in G$.

A principal fibre bundle P, with its base space M and structure group G will be denoted P(M,G) or sometimes more simply as P, when no confusion will arise.

For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P. As $\pi^{-1}(x) = \{ua; a \in G\}$, where u is any member of P with $\pi(u) = x$, each fibre is diffeomorphic to G.

The action of G on P induces a homomorphism σ of the Lie algebra $\hat{g}$ of G into the Lie algebra of differentiable vector fields on P. If $A \in \hat{g}$, let a_t be the one paramater subgroup of G with tangent A at 1. Then $\sigma(A)_u$ is the tangent to ua_t at $t = 0$. For $A \in \hat{g}$, $\sigma(A) = A^*$ is called the fundamental vector field corresponding to A. As G maps each fibre into itself, $(A^*)_u$ is tangent to the fibre through u. Since G G acts freely on P, A^* never vanishes if $A \neq 0$. Thus the dimension of each fibre dim $\hat{g}$ and the map that sends $A \to (A^*)_u \varepsilon T_u(P)$ for any $u \varepsilon P$ in a linear isomorphism of $\hat{g}$ onto the tangent space of the fibre through u.

For later use, we will relate our definition of a principal fibre bundle here to the definition and construction by means of an open covering. We can choose an open covering $\{U_\alpha\}$ of M such that $\pi^{-1}(U_\alpha)$ is diffeomorphic with $U_\alpha \times G$ via $u \to (\pi(u), \phi_\alpha(u))$, with $\phi_\alpha(ua) = \phi_\alpha(u)\cdot a$. On $\pi^{-1}(U_\alpha \cap U_\beta)$, $\phi_\beta(u)(\phi_\alpha(u))^{-1}$ depends only on $\pi(u)$ and not on u. We can thus define a map $\psi_{\beta\alpha}: U_\alpha \cap U_\beta \to G$ by $\psi_{\beta\alpha}(\pi(u)) = \phi_\beta(u)(\phi_\alpha(u))^{-1}$. The maps $\psi_{\beta\alpha}$ are called the transition functions for the bundle P(M,G) corresponding to the open cover $\{U_\alpha\}$. It is easy to check that

$$(*) \qquad \psi_{\gamma\alpha}(x) = \psi_{\gamma\beta}(x)\cdot\psi_{\beta\alpha}(x) \qquad \forall x \varepsilon\ U_\alpha \cap U_\beta \cap U_\gamma$$

We shall later prove a converse of all this, namely, given M with an open cover $\{U_\alpha\}$, with maps $\psi_{\beta\alpha}: U_\alpha n U_\beta \to G$, a Lie group, such that the relations (*) hold, then we can construct a differentiable principal fibre bundle P(M,G) with transition functions $\psi_{\beta\alpha}$. We will in fact only prove this in a special case in the section on Weil's theorem, but the general case is similar. For a proof, see [1].

An example of a principal fibre bundle that will be useful later is the so-called bundle of linear frames of a manifold M. Suppose the dimension of M is n. A linear frame u at $x \in M$ is an ordered basis $(X_1, \cdots, X_n)$ of the tangent space $T_x(M)$. Let $L(M)$ be the union of all linear frames u at every point of M, and π be the map that sends $u \in L(M)$ to its base point $x \in M$. The general linear group $GL(n;\mathbb{R})$ acts on $L(M)$ in the following manner. If $a = (a_{ij}) \in GL(n,\mathbb{R})$ and $u = (X_1, \cdots, X_n)$ at x, then ua by definition is the frame $(Y_1, \cdots Y_n)$ at x where $Y_i = \sum_j a_{ji} X_j$. This action is then free, and $\pi(u) = \pi(v)$ if and only if $v = ua$, for some $a \in GL(n,\mathbb{R})$. $L(M)$ is locally trivial and thus can be given a differentiable structure: let x^i, $i = 1 \cdots n$ be a local co-ordinate system in a neighbourhood U of $x \in M$. Define $f_U : U \times GL(n,\mathbb{R}) \to L(M)$ by

$$f_U(x, a_{ij}) = \left(\sum_j a_{j1} \frac{\partial}{\partial xj}, \cdots, \sum_j a_{jn} \frac{\partial}{\partial xj}\right) \text{ at } x.$$

One can verify that f_U is 1-1 and onto and $\pi(f(x,a_{ij})) = x$. This shows that $L(M)$ is locally trivial, and it can be given a smooth structure by making the f_U's diffeomorphisms. This makes $L(M)$ a principal fibre bundle: the bundle of linear frames $L(M)(M,GL(n,\mathbb{R}))$. Note that if $\pi(u) = x$, u can be viewed as an isomorphism of $\mathbb{R}^n$ onto $T_x(M)$: select a basis (e_i) $i=1 \cdots n$ for $\mathbb{R}^n$, then define $u(e_i) = X_i$, if $u = (X_1 \cdots X_n)$. If $a \in GL(n;\mathbb{R})$, the map ua is then the composition of the mappings $\mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} T_x(M)$.

Given a principal fibre bundle $P(M,G)$ and a manifold F on which G acts on the left, we construct an associated fibre bundle, with base space M and standard fibre F. On the manifold

$P\times F$, let G act on the right as follows: $a\varepsilon G$ sends $(u,\xi)\varepsilon P\times F$ to $(ua,a^{-1}\xi)\varepsilon P\times F$. Denote the quotient space of $P\times F$ under this action by $E = P\times_G F$. The projection $P\times M \to M$ that sends (u,ξ) to $\pi(u)$ induces a projection $\pi_E : E \to M$, which is onto M. Again, $\pi_E^{-1}(x)$ is called the fibre of E over x. Now each $x\varepsilon M$ has a neighbourhood U with $\pi^{-1}(U) \simeq U\times G$. Exploit this isomorphism to write the action of G on $\pi^{-1}(U)\times F$ as $(x,a,\xi) \xrightarrow{b} (x,ab,b^{-1}\xi)$ for $x\varepsilon M$, $a,b\varepsilon G$ and $\xi\varepsilon F$. Then it can be seen that $\pi_E^{-1}(U) \simeq U\times F$ by checking that $i_U : U\times F \to \pi_E^{-1}(U)$ defined by

$$i_U(x,\xi) = \overline{(f_U(x,\delta_{ij}),\xi)}$$

is one to one and onto. (Here f_U is as defined before, and the bar denotes the coset containing this element of $P\times F$). Again we make E a manifold by defining the i_U's to be diffeomorphisms. E is said to be the fibre bundle over the base M with standard fibre F and structure group G which is associated with the principal fibre bundle P. It is denoted E(M,F,G,P). Note here also that each $u\varepsilon P$ can define a mapping from F onto $\pi_E^{-1}(x)$, if $\pi(u) = x$, by letting $u\xi$, for $\xi\varepsilon F$ be the image of $(u,\xi)\varepsilon P\times F$ under the projection $P\times F \to P\times_G F = E$. These maps satisfy $(ua)\xi = u(a\xi)$ for all $u\varepsilon P$, $a\varepsilon G$ and $\xi\varepsilon F$.

Two examples of bundles of the sort introduced above are the tangent bundle T(M) and the tensor bundle $T_s^r(M)$, both of which are associated with L(M). The tangent bundle has standard fibre $\mathbb{R}^n$. The action of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ is the usual one: If $\{e_i\}$ is the standard basis for $\mathbb{R}^n$ and $v = \sum_n v_j e_j$, then $(av)_i = \sum_j a_{ij} v_j$. The fibre of $T(M)$ over $x \in M$ is diffeomorphic to $T_x(M)$ by the map $f_x: T_x(M) \to \pi_E^{-1}(x)$ defined by $f_x(V) = \overline{(u, u^{-1}(V))}$. Here $u \in \pi^{-1}(x)$ is considered as a map from $\mathbb{R}^n$ onto $T_x(M)$ as before. It is easy to show that f_x is independent of the choice of $u \in \pi^{-1}(x)$, and that it is one to one and onto. The smoothness of f_x follows by looking at local trivializations of E. Similarly, let $T^r_s(M)$ be the bundle associated with $L(M)$ with standard fibre T^r_s, where T^r_s is the tensor space of type (r,s) over $\mathbb{R}^n$. The action of $GL(n,\mathbb{R})$ on T^r_s is as follows: With $\{e_i\}$ as a the standard basis for $\mathbb{R}^n$ and $\{e^{*i}\}$ its dual basis, a tensor $A \in T^r_s$ may be written as

$$A = \sum_{i_1\cdots i_r, j_1\cdots j_s} A^{i_1\cdots i_r}{}_{j_1\cdots j_s}\, e_{i_1} \otimes\cdots\otimes e_{i_r} \otimes e^{*j_1} \otimes\cdots\otimes e^{*j_s}.$$

If $a = (a_{ij}) \in Gl(n,\mathbb{R})$ has as inverse $a^{-1} = (a^{ij})$ then the action of $Gl(a,\mathbb{R})$ is determined by

$$(aA)^{k_1\cdots k_r}{}_{\ell_1\cdots \ell_s} = \sum_{i_1\cdots i_r, \ell_1\cdots \ell_s} A^{i_1\cdots i_r}{}_{j_1\cdots j_s} \times$$

$$\times\, a_{k_1 i_1} \cdots a_{k_r i_r} \cdot a^{\ell_1 j_1} \cdots a^{\ell_s j_s}$$

A similar argument to the one used in the case of $T(M)$ shows that the fibre of $T^r_s(M)$ over x may be viewed as the tensor space of type (r,s) over the vector space $T_x(M)$. Bundles of this sort, that is, when the fibre is a vector space and G acts as a group of linear transformations, are called vector bundles.

1.2 CONNECTIONS IN FIBRE BUNDLES

Suppose $P(M,G)$ is a principal fibre bundle. For $u \in P$, let G_u be the subspace of $T_u(P)$ consisting of vectors tangent to the fibre through u.

Definition: A connection, $\Gamma = (P,Q)$, is an assignment of a subspace Q_u of $T_u(P)$ to each $u \in P$ that satisfies

(1) $T_u(P) = G_u \oplus Q_u \ \forall u \in P$ (Direct Sum)

(2) $Q_{ua} = a_* Q_u$, where $a_*: T_u(P) \to T_{ua}(P)$ is the tangent map of $u \to ua$.

(3) Q_u depends differentiably on u.

G_u and Q_u are called the vertical and horizontal subspaces respectively. By (1) each $X \in T_u(P)$ can be written uniquely as $X = vX + hX$ where $vX \in G_u$ and $hX \in Q_u$. The vector vX (respectively hX) is called the vertical (respectively horizontal) component of X. If X is a differentiable vector field, then so are vX and hX.

Γ can be characterized by a $\hat{g}$-valued 1-form ω on P called the connection form, as follows. The homomorphism σ taking $A \in \hat{g}$ to $(A^*)_u$ mentioned before is a linear isomorphism of $\hat{g}$ onto G_u, the vertical subspace. Given $X \in T_u(P)$, define $\omega(X)$ to be the unique $A \in \hat{g}$ with $(A^*)_u = vX$. Clearly $\omega(A^*) = A$, and by considering the two cases when X is vertical or horizontal, it can be verified that $a^*\omega = Ad(a^{-1})\omega$. [Here, $a^*\omega(X) = \omega(a_* X)$, and $(Ad(a^{-1})\omega)(X) = Ad(a^{-1}) \cdot \omega(X)$ where Ad denotes the adjoint representation of G in $\hat{g}$.] Conversely, given a 1-form ω satisfying these two conditions, if we set $Q_u = \{X \in T_u(P); \omega(X) = 0\}$, then this assignment will be the connection with connection form ω.

The projection $\pi: P \to M$ induces its tangent map, denoted also by π, $\pi: T_u(P) \to T_{\pi(u)}(M)$. Since $\pi(V) = 0 \Leftrightarrow V$ is vertical, π maps Q_u isomorphically onto $T_u(M)$. This allows the notion of a horizontal lift of a vector field (or vector) to be defined.

<u>Definition</u>: The horizontal lift of a vector field X on M is the unique vector field X^* on P which is horizontal and satisfies $\pi(X^*_u) = X_{\pi(u)}$.

It is easy to see that X^* is invariant by right multiplication, is differentiable if X is, and that any horizontal vector field on P which is invariant by the action of G is the lift of some vector field X on M.

Let $\tau = x_t$, $0 \leq t \leq 1$ be a piecewise differentiable curve of class C^1 on M. A horizontal lift of τ is a curve $\tau^* = u_t$, $0 \leq t \leq 1$, in P, whose tangent is everywhere horizontal, and $\pi(u_t) = x_t$. To ensure that, given u_0, a unique lift τ^* starting there exists, note firstly that the local triviality of P guarantees the existence of a curve v_t, $0 \leq t \leq 1$, with $\pi(v_t) = x_t$ and $v_0 = u_0$. The curve desired will then be of the form $u_t = v_t a_t$, where a_t is a curve in G with $a_0 = 1$. After demanding that the tangent be horizontal ($\omega(\dot{u}_t) = 0$, where $\dot{u}_t$ denotes the tangent to u_t), we only need to solve $(a_t^{-1})_* \dot{a}_t = \omega(\dot{v}_t)$, $a_0 = e$, for $0 \leq t \leq 1$. This is not difficult, and the standard theorem for first order differential equations guarantees existence and uniqueness.

Parallel displacement of a fibre along a curve in M can now be defined. Suppose $\tau = x_t$, $0 \leq t \leq 1$ is a C^1 curve in M. Let $u_0 \varepsilon P$ be an arbitrary member of the fibre above $x_0 \varepsilon M$. Then the unique lift τ^* of τ through u_0 has end point u_1 in the fibre above x_1. By varying the starting point $u_0 \varepsilon \pi^{-1}(X_0)$, we then obtain a mapping $\tau\colon \pi^{-1}(x_0) \to \pi^{-1}(x_1)$. The map is an isomorphism because ($\tau \cdot a = a \cdot \tau$) $\forall a \varepsilon G$, (horizontal curves are mapped by a to horizontal curves) and because τ^* is unique. Evidently, if a curve is C^1, the parallel displacement is independent of the specific parametrization.

Given a curve $\tau = x_t$, $0 \leq t \leq 1$, the inverse $\tau^{-1} = y_t$, $0 \leq t \leq 1$ is defined by $x_t = x_{1-t}$. Clearly the parallel displacement τ^{-1} is the inverse of τ, and if τ is a curve from x to y and μ is a curve from y to z (all in M), then the parallel displacement $\mu \cdot \tau$ is the composite of the parallel displacement of τ and μ.

We now introduce the holonomy group of a connection. Let C(x) be the loop space at a point $x \varepsilon M$, that is, the set of C^1 curves in M that begin and end at x. The set of isomorphisms of $\pi^{-1}(x)$ induced by C(x) forms a group by virtue of the preceeding remarks. This group is called the holonomy group of Γ with reference point x, and is denoted $\Phi(x)$. If we restrict attention to curves that are homotopic to $x_t=x$, we obtain a subgroup of $\Phi(x)$, called the restricted holonomy group at x which is denoted $\Phi_0(x)$.

These groups can be realized as subgroups of the structure group in the following manner. If $u\epsilon\pi^{-1}(x)$, then each $\tau\epsilon C(x)$ determines a unique $a\epsilon G$ with $\tau(v) = ua$. It can be seen that the set of elements $a\epsilon G$ that is determined by all $\tau\epsilon C(x)$ forms a subgroup of G, which we denote $\phi(u)$. It is called the holonomy group of Γ with reference point u. Similar remarks hold concerning $C_0(x)$. It is also clear that $\Phi(u) = \{a\epsilon G; u\sim ua\}$, where $u\sim v \Longleftrightarrow$ u and v can be joined by a horizontal curve. Using this fact, it can be shown that if M is connected, then the holonomy groups $\Phi(u)$, $u\epsilon P$ are all conjugate and hence isomorphic. The holonomy group is a Lie subgroup of G. For a proof of this see [1], Theorem 4.2, page 73.

Curvature Form, Structure Equations

Let P(M,G) be a principal bundle with a connection Γ with connection form ω. Let Q_u and G_u be the horizontal and vertical subspaces of $T_u(P)$ respectively. Suppose $h:T_u(P) \to Q_u$ is the horizontal projection. Define a 2-form Ω on P by

$$\Omega(X,Y) = d\omega(hX,hY)$$

The form Ω is called the curvature form of ω. Two important properties of Ω are noted:

(1) $\Omega(X,Y) = 0$ whenever either of X or Y is vertical. Any r-form ν which is zero when any of its arguments is vertical is called a tensorial r-form.

(2) $a^*\Omega = Ad(a^{-1})\Omega$.

This last property follows from the transformation rule of the connection form ω.

For the sake of completeness, the structure equation will be proven here.

Theorem (Structure Equation). If ω is a connection form and Ω its curvature form, then

$$d\omega(X,Y) = -\frac{1}{2}[\omega(X),\omega(Y)] + \Omega(X,Y) \qquad \forall X,Y\varepsilon T_u(P).$$

Proof: As both sides of the equation are bilinear and antisymmetric in X and Y, and as each vector $Z\varepsilon T_u(P)$ can be written uniquely as the sum of its horizontal and vertical components, only 3 special cases need be considered.

Case (1). X and Y are horizontal. This amounts to nothing more than the definition of Ω, as $\omega(X) = \omega(Y) = 0$.

Case (2). X and Y are vertical. We may assume $X = A_u^*$, $Y = B_u^*$ where $A,B\varepsilon\hat{g}$. Then,

$$2d\omega(A^*,B^*) = A^*(\omega(B^*))-B^*(\omega(A^*))-\omega([A^*,B^*])$$

Since $\omega(A^*) = A$, $\omega(B^*) = B$ and $[A^*,B^*] = [A,B]^*$, this reduces to $2d\omega(A^*B^*) = -\omega([A,B]^*) = -[A,B]$. But as X and Y are vertical, $\Omega(X,Y) = 0$.

Case (3). X is horizontal and Y is vertical. Extend X to a horizontal vector field on P, and let $Y = A^*$ at u, where, as before $A\varepsilon\hat{g}$ and A^* is the fundamental vector field corresponding to A. As the right-hand side of the equation vanishes, ($\omega(X) = 0$ and $\Omega(X,A^*) = 0$ as A^* is vertical) it is sufficient to show that $d\omega(X,A^*) = 0$. Again, using the normal expression for $d\omega(X,A^*)$, and noting that $\omega(A^*)$ is constant we need only prove that $\omega([X,A^*]) = 0$.

The fundamental vector field A^* is induced by (a_t) where a_t is the 1-parameter subgroup of G generated by $A\varepsilon\hat{g}$. Now we have

$$[X,A^*] = \lim_{t\to\infty} \frac{1}{t} [(a_t)_*X-X] .$$

If X is horizontal, so too is $(a_t)_*X$, and so $[X,A^*]$ is horizontal. This proves that $\omega([X,A^*]) = 0$ and we are done.

<u>Corollary</u> If both X and Y are horizontal vector fields on P, then $\omega([X,Y]) = -2\Omega(X,Y)$.

<u>Proof</u>. Use the usual expression for $d\omega$ in the structure equation just proven.

If $e_1, \cdots, e_r$ is a basis for the Algebra $\hat{g}$ and c^i_{jk} are the structure constants relative to this basis, so that

$$[e_j, e_k] = \sum_i c^i_{jk} e_i$$

write $\omega = \sum_i \omega^i e_i$ and $\Omega = \sum_i \Omega^i e_i$ (ω^i and Ω^i are now real valued forms on P). The structure equations can now be written as follows.

$$d\omega^i = -\frac{1}{2} \sum_{j,k} c^i_{jk} \omega^j \wedge \omega^k + \Omega^i \;, \qquad i = 1, \cdots, r.$$

Bianchi's identity follows easily from this expression.

Theorem (Bianchi's Identity) $\quad D\Omega = 0$.

Proof: Apply the exterior derivative to the last equation. By the definition of Ω it is sufficient to show $d\Omega(X,Y,Z) = 0$ when X, Y and Z are horizontal. We have

$$0 = dd\omega' = -\frac{1}{2} \sum_{jk} c^i_{jk} \, d\omega \wedge \omega^k + \frac{1}{2} \sum_{j,k} c^i_{jk} \, \omega^j \wedge d\omega^k + d\Omega^i.$$

As $\omega^i(X) = 0$ whenever X is horizontal, the result follows.

The statement of a theorem that relates the curvature and the holomony group will be given. It will be useful later in a uniqueness result. For a proof of the theorem, see [1] Theorem 8.1, p. 89. Let $P(u) = \{v | u \sim v\}$. As before, $u \sim v \Longleftrightarrow$ u and v can be joined by a horizontal curve.

Holonomy Theorem: Let $P(M,G)$ be a principal fibre bundle, where M is connected and paracompact. Let Γ be a connection in P, Ω the curvature form, $\Phi(u)$ the holonomy group with reference point u and $P(u)$ as above. Then the Lie algebra of $\Phi(u)$ is the subspace of $\hat{g}$ (the Lie algebra of G) spanned by all elements of the form $\Omega_v(X,Y)$, where $v \varepsilon P(u)$ and X and Y are arbitrary horizontal vectors at v.

1.3 LINEAR AND METRIC CONNECTIONS

We will now define parallel displacement in any associated bundle $E(M,F,G,P)$, when a connection is given in $P(M,G)$. In the case of the standard tensor bundles over M, it coincides with the usual notion of parallel displacement of vectors and tensors.

Given a connection Γ in $P(M,G)$, and an associated bundle $E(M,F,G,P)$, the horizontal and vertical subspaces $T_w(E)$, $w \in E$, are defined as follows: The vertical subspace F_w is the tangent space to the fibre through w. Recall the map induced by $u \in P$, $u: F \to \pi_E^{-1}(x)$, where $x = \pi(u)$, defined by $u(\xi)$ = the image of $(u,\xi) \in P \times F$ under the projection $P \times F \to E$. Choose any (u,ξ) that is mapped to w. With this ξ, consider the map $P \to E$ that takes $v \in P$ to $v(\xi)$. Q_w, the horizontal subspace, is then the image of Q_u under this map. Clearly Q_w is independent of the choice of (u,ξ), and it is not difficult to show that $T_w(E) = F_w \oplus Q_w$.

The definitions of horizontal curves and horizontal lifts are as in the case of $P(M,G)$. Again, given a curve $\tau = x_t$, $0 \leq t \leq 1$, in M and $w_0 \in E$ with $\pi_E(w_0) = x_0$, there is a unique horizontal lift $\tau^* = w_t$ starting at w_0. To see this, choose a point $(u_0,\xi) \in P \times F$ with $u_0(\xi) = w_0$, and let u_t be the lift of τ beginning at x_0 (in P). Then $w_t = u_t(\xi)$ is a lift starting at w_0. It is clearly unique.

A cross section of any associated bundle E(M,F,G,P) is a map σ: M $\to$ E such that $\pi_E \cdot \sigma$ is the identify on M. Suppose that F is a vector space (over some field K), and that G acts on F as a group of linear transformations. In this case, each fibre of E has the structure of a vector space over K, and so we can define addition of sections, and multiplication of sections by K-valued functions on M. Explicitly, if ϕ and ψ are sections of E and f is a K valued function on M, we define

$$(\phi+\psi)(m) = u(u^{-1}(\phi(m)) + u^{-1}(\psi(m)))$$

$$(f\cdot\phi)(m) = u(f(m)\cdot u^{-1}(\phi(m)))$$

where $m \in M$ and $u \in \pi^{-1}(m)$. It can be checked that the expressions on the right are independent of $u \in \pi^{-1}(m)$. We can define local sections of E in a similar manner. If $U \subset M$, we define $\Gamma(U) = \{\phi | \phi$ is a local section with domain $U\}$. In the case when E is a vector bundle $\Gamma(U)$ is a vector space. For example, sections of T(M) are just vector fields over M.

Covariant differentiation can be defined in any vector bundle E(M,F,G,P). Let $\tau_t^{t+h}: \pi_E^{-1}(x_{t+h}) \to \pi_E^{-1}(x_t)$ be the parallel displacement of fibres from x_{t+h} to x_t along a curve $\tau = x_t$, $0 \leq t \leq 1$ in M. Let ϕ be a cross section defined along τ. The covariant derivative of ϕ with respect to τ (or along τ) $\nabla_{\dot{x}_t}\phi$ is defined by

$$\nabla_{\dot{x}_t} \phi = \lim_{h \to 0} \frac{1}{h} [\tau_t^{t+h}(\phi(x_{t+h})) - \phi(x_t)].$$

Again $\dot{x}_t$ denotes the tangent to τ at x_t. Let X $T_x(M)$, and ϕ be a cross section of E defined in a neighbourhood of x. We define also the covariant derivative, $\nabla_X \phi$, of ϕ in the direction X as follows. Choose a curve $\tau = x_t - \varepsilon \leq t \leq \varepsilon$ with $X = \dot{x}_0$. Then set

$$\nabla_X \phi = \nabla_{\dot{x}_0} \phi.$$

$\nabla_X \phi$ is independent of the choice of τ. If X is a vector field and ϕ is a local cross section on $U \subset M$, then

$$\nabla : \Gamma(T(U) \times \Gamma(U) \to \Gamma(U) .$$

We now relate these definitions to the notion of the horizontal lift. Given a cross section ϕ defined on $U \subset M$, we define an F-valued function on $\pi^{-1}(U)$ as follows:

For v $\pi^{-1}(U)$, let $f(v) = v^{-1}(\phi(\pi(v)))$. (Remember, $v: F \to E$ is an isomorphism of F onto $\pi_E^{-1}(\pi(v).)$ Note that if X $T_x(M)$, and X^* is its horizontal lift to P, then we have $u(X^* f)$ $\pi_E^{-1}(x)$, if $\pi(u) = x$. We now prove the useful

<u>Lemma</u>: $\nabla_X \phi = u(X^* f)$

Proof: Choose a curve $\tau = x_t$, $a - \varepsilon \leq t \leq \varepsilon$, with $X = \dot{x}_0$. Let $\tau^* = u_t$ be the horizontal lift of τ to P beginning at u. Then $X^* = \dot{u}_0$. So,

$$X^*f = \lim_{h\to 0} \frac{1}{h}[f(u_h) - f(u)]$$

$$= \lim_{h\to 0} \frac{1}{h}[u_h^{-1}(\phi(x_h)) - u^{-1}(\phi(x))] .$$

Thus $$u(X^*f) = \lim_{h\to 0} \frac{1}{h}[u\cdot u_h^{-1}(\phi(x_h)) - \phi(x)] .$$

It is then sufficient to prove that

$$u\cdot u_h^{-1}(\phi(x_h)) = \tau_0^h(\phi(x_h)) \qquad \text{for } |h| \leq \varepsilon.$$

Set $\xi = u_h^{-1}(\phi(x_h))$. As u_t is horizontal in P, $u_t(\xi)$ is horizontal in E. Thus $\phi(x_h) = u_h(\xi)$ is obtained by parallel displacement of $u_0(\xi) = u\cdot u_h^{-1}(\phi(x_h))$ along τ from x_0 to x_h, that is,

$$\tau_o^h(\phi(x_h)) = u\ u_h^{-1}(\phi(x_h)).$$

as required.

This lemma tells us that, under the correspondence between sections and F-valued functions introduced before, if ϕ corresponds to f, then $\nabla_X\phi$ corresponds to X^*f.

Standard properties of type preserving derivations are enjoyed by ∇_X: If ϕ, ψ are cross sections and X,Y are vectors (or vector fields), then

(1) $\nabla_{X+Y}\phi = \nabla_X\phi + \nabla_Y\phi$

(2) $\nabla_X(\phi+\psi) = \nabla_X\phi + \nabla_Y\psi$

(3) $\nabla_{\lambda X}\phi = \lambda\nabla_X\phi$ if λ is a K-valued function on M

(4) $\nabla_X(\lambda\phi) = \lambda\nabla_X\phi + (X,\lambda)\cdot\phi$ if λ is a K-valued function on M.

We now consider linear and metric connections briefly.

Definitions (1) A connection in the bundle of linear frames $L(M)$ over M is called a linear connection of M.

(2) A fibre metric in a vector bundle E is an assignment, to each $x \in M$, an (a) hermitian (symmetric) inner product $g_x: \pi_E^{-1}(x) \times \pi_E^{-1}(x) \to K$. (It is hermitian if $K = \mathbb{C}$, symmetric if $K = \mathbb{R}$) such that if ϕ and ψ are differentiable cross sections of E, $g_x(\phi(x), \psi(x))$ is differentiable on M.

(3) A connection in P is called metric if there is some fibre metric in $E(M,F,G,P)$ so that parallel displacement of the fibres in E preserves g. That is, for every curve $\tau = x_t$, $0 \leq t \leq 1$ in M, the parallel displacement $\tau_1^0 = \pi_E^{-1}(x_0) \to \pi_E^{-1}(x_1)$ along τ is an isometry.

(4) The canonical form θ of $L(M)$ is the $\mathbb{R}^n$ valued 1- form on $L(M)$ defined by

$$\Theta(X) = u^{-1}(\pi_*(X)) \quad \text{for } X \in T_u(P),$$

where $\dim M = n$ and π_* is the tangent map of the projection $\pi: L(M) \to M$.

(5) Given a connection Γ in $L(M)$, the torsion form Θ is defined by

$$\Theta = D\theta$$

(Here, $D\theta(X,Y) = d\theta(hX, hY)$. $D\theta$ is the covariant derivative.)

It follows quickly from these definitions that θ tensorial 1-form, Θ is a tensorial 2-form and $a^*\theta = a^{-1} \cdot \theta$, $a^*\Theta = a^{-1} \cdot \Theta$, for $a \in GL(n, \mathbb{R})$. We will now state, without proof, the structure equations.

<u>Theorem</u> (Structure Equations).

Let ω, Θ, Ω be, respectively, the connection form, the torsion form and the curvature form of a linear connection Γ of M. Then,

(1) $$d\theta(X,Y) = -\frac{1}{2}(\omega(X)\cdot\theta(Y) - \omega(Y)\cdot\theta(X)) + \Theta(X,Y)$$

(2) $$d\omega(X,Y) = -\frac{1}{2}[\omega(X),\omega(Y)] + \Omega(X,Y)$$

$\forall X,Y \in T_u(L(M))$ and $u \in L(M)$.

The second equation has been seen already. In addition to the Bianchi identity

$$D\Omega = 0,$$

we now have

$$3D\Theta(X,Y,Z) = \mathop{\mathfrak{S}}_{X,Y,Z} \Omega(X,Y)\theta(Z)$$

where $\mathop{\mathfrak{S}}_{X,Y,Z}$ indicates a cyclic sum over X,Y and Z. This follows quickly from (1) by applying exterior differentiation to both sides. For use later, we will now introduce the more familiar curvature and torsion tensors, and restate the Bianchi identities in terms of these.

<u>Definitions</u> (1) For $X,Y, \in T_x(M)$, set

$$T(X,Y) = u(2\Theta(\bar{X},\bar{Y}))$$

where $\pi(u) = x$, $\bar{X}$ and $\bar{Y} \in T_u(L(M))$ with $\pi_*(\bar{X}) = X$, $\pi_*(\bar{Y}) = Y$.

(2) For $X,Y,Z \in T_x(M)$, set

$$R(X,Y)Z = u(2\Omega(\bar{X},\bar{Y})\cdot u^{-1}(Z))$$

where u, $\bar{X}$, $\bar{Y}$ are chosen as before.

It can be verified that the above definitions do not depend on the choice of $\bar{X}$, $\bar{Y}$ or u. T is called the torsion tensor of the connection, and R is the Riemann curvature tensor of the connection. We note here that the definition (2) may be made in a slightly more general setting, and in fact we will make use of this extended notion in the section on geometric quantization. Explicitly, for ϕ a local section of some vector bundle E(M,F,G,P) defined around $x \in M$, and $X,Y \in T_x(M)$, define the section $R(X,Y)\phi$ by

$$R(X,Y)\phi(x) = u(2\Omega(\bar{X},\bar{Y})\ u^{-1}(\phi(x))) \ .$$

Here the action of $\hat{g}$ on F is inherited from the action of G on F: it is an algebra of linear transformations. Here $u,\bar{X},\bar{Y}$ are chosen as before, and it can be shown that $R(X,Y):\Gamma(U)\to\Gamma(U)$ depends only on X,Y and not on $\bar{X},\bar{Y}$, or $u \in \pi^{-1}(x)$.

The following theorem, which follows from earlier results quickly, illustrates that these torsion and curvature tensors are the familiar ones associated with the covariant differentiation defined by the connection. We note also that the second identity holds in the more general context, Z then understood to be a section.

Theorem

Let T and R be the torsion and curvature tensors of a linear connection of M, Then $\forall X,Y,Z \in T_x(M)$,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

$$\mathfrak{S}\{R(X,Y)Z\} = \mathfrak{S}\{T(T(X,Y),Z) + \nabla_X T(Y,Z)\}$$

$$\mathfrak{S}\{(\nabla_X R)(Y,Z) + R(T(X,Y),Z)\} = 0 \ .$$

Here $\mathfrak{S}$ denotes the cyclic sum with respect to X,Y and Z. In particular, if the torsion vanishes,

$$\mathfrak{S}\{R(X,U)Z\} = 0$$

$$\mathfrak{S}\{\nabla_X R(Y,Z)\} = 0 \ .$$

WEIL'S THEOREM

A result of fundamental importance to the theory of geometric quantization is considered here. We ask the question: For an $\mathbb{R}$-valued 2-form ω on a manifold M, is there a principal fibre bundle $P(M,\pi,G)$ with connection, such that the curvature Ω satisfies $\pi^*\omega = \Omega$? In the setting of geometric quantization, an affirmative answer to this will allow us to 'quantize' certain observables.

Suppose there was $P(M,\pi,G)$ with connection Γ as above for ω. As we can always reduce a connection on $P(M,\pi,G)$ to one on the holonomy bundle, and, as the Lie algebra of the holonomy group of the connection need only be $\mathbb{R}$, it will suffice to study the case when G is abelian, with its Lie algebra, $\hat{g} = \mathbb{R}$. Then, in that situation, $G = \mathbb{R}$ or S^1. We fix now a representation of S^1: $S^1 = \{z\in\mathbb{C}/|z| = 1\}$. When $\ell n(z)$ is defined, define $\ell\tilde{n}(z) = \ell n(z)/2\pi i$, and define $\exp(r) = e^{2\pi ir}$ for $z\in S'$ and $r\in\mathbb{R}$. In the following discussion, we will consider functions $f: U \to S^1$ where $U\subset M$. We will have been able to choose U so that f(U) is simply connected, and thus there will be a continuous single valued branch of the logarithm on f(U).

Suppose that α is the connection form on $P(M,G)$. We observe that ω must be closed: $d\alpha = D\alpha = \Omega$, as G is abelian, and so $0 = d\,(d\alpha) = d\Omega = d\pi^*\omega = \pi^*d\omega$. Choose a local section s:

$U \to P$, so we have $\pi \cdot s = id_U$ and $s^*\pi^* = id_{T^*(U)}$. This implies that $d\omega|_U = s^*\pi^*d\omega = 0$. Now, choose an open contractible cover $\{U_i\}$ for M. (This will mean that $U_i \wedge U_j \wedge U_k = \emptyset$ or is contractible.) We can choose this small enough so that $\pi^{-1}(U_i) \simeq U_i \times G$. Let $\psi_j: \pi^{-1}(U_j) \to U_j \times G$ be the diffeomorphisms. Define sections $s_j: U_j \to P$ by $s_j(x) = \psi_j^{-1}(x,1))$. Let $\{c_{ij}\}$ be the transition functions for this system. We then know that $s_j = s_i c_{ij}$ on $U_i \wedge U_j$ and $c_{ij}c_{jk} = c_{ik}$ on $U_i \wedge U_j \wedge U_k$. Define 1-forms on U_i by $\alpha_i = s_i^*\alpha$. Then $d\alpha_i = \omega|_{U_i}$, and, on $U_i \wedge U_j$, $\alpha_i - \alpha_j = df_{ij}$, for some function $f_{ij}: U_i \wedge U_j \to \mathbb{R}$. We now show that we can choose f_{ij} to be $-\ell\tilde{n}c_{ij}$.

On $U_i \wedge U_j$, $s_j = s_i c_{ij}$ so

$$s_{j*} = (c_{ij})_* \cdot s_{i_*} + (L_{s_i})_* \cdot (c_{ij})_* \cdot d(\ell\tilde{n}c_{ij}).$$

Here $(c_{ij})_*$ is the tangent map of $u \to uc_{ij}$, $(Ls_i)_*$ is the tangent map of $a \to s_i(m) \cdot a$, and other symbols have their usual meanings. Hence,

$$\begin{aligned}(\alpha_i - \alpha_j)(x) &= \alpha(s_{i_*}(X)) - \alpha(s_{j*}(X)) \\ &= \alpha(s_{i_*}(X)) - \alpha((c_{ij})_* s_{i_*}(X)) \\ &\quad - \alpha((Ls_{i_*}) \cdot c_{ij_*} \cdot d(\ell\tilde{n}c_{ij})(X)) \\ &= \alpha((d\ell\tilde{n}c_{ij}(X))^*) \\ &= -d\ \ell\tilde{n}c_{ij}(X).\end{aligned}$$

Here, as usual, $(d\ell\tilde{n}c_{ij}(X))^*$ is the fundamental vector field on P corresponding to $d\ell nc_{ij}(X) \in \hat{g}$. Of course, we have used the commutativity of G here: Ad invariance means invariance, and so $a^*\alpha = \alpha$. Thus f_{ij} can be chosen to be $-\ell\tilde{n}c_{ij}$. Now, as $d(f_{ij}+f_{jk}-f_{ik}) = 0$, $f_{ij} + f_{jk} - f_{ik} = a_{ijk}$ is constant on $U_k \cap U_j \cap U_k$. But, as $c_{ij}c_{jk}c_{ik}^{-1} = 1$ there, a_{ijk} is an integer if $G = S^1$ or zero if $G = \mathbb{R}$. Employing the notions in the appendix, this means that $[\omega]$ is integral. The converse of this is the real content of Weil's theorem: If ω is a real valued closed 2-form on M, there is P(M,G) and connection with connection form α satisfying $D\alpha = \pi^*\omega$ if and only if $[\omega]$ is integral. We have shown the necessity of the condition on ω. Now we prove it is sufficient.

Suppose then, that $[\omega]$ is integral. Then we know that there is an open contractible cover $\{U_i\}$ of M, 1-forms α_i and functions f_{ij} as in (2) in the appendix. Let $c_{ij} = \exp f_{ij}$. Define $P = \dot{\bigcup}_j U_j \times S^1/\rho$ where ρ is an equivalence relation defined by $(x,g)\rho(u,h) \Longleftrightarrow x=y$ and $g = c_{ij}(X)\cdot h$ where $x \in U_i$, $y \in U_j$, and $g,h \in S^1$. ($\dot{\cup}$ means the disjoint union). Define the projection $\pi : P \to M$ by $\pi(\overline{x,h})) = x$, where the bar denotes the ρ-equivalence class. We now show that $P(M,\pi,S^1)$ is a principle fibre bundle.

The local triviality follows easily by defining diffeomorphisms

$$\psi_j : \pi^{-1}(U_j) \to U_j \times S^1 \quad \text{by}$$
$$\psi_j((\overline{x,g})) = (x,g) \ .$$

That these are good definitions is quickly checked. ψ_j is (1-1) and onto, and we can give P a differentiable structure by making the ψ_j's diffeomorphisms. That the transition functions behave is a consequence of the fact that $f_{jk}+f_{k\ell}-f_{j\ell}$

is an integer. The action of G on P is defined in the obvious manner: $(\overline{x,g}) \to (\overline{x,ga})$, and is readily seen to be free. The smoothness of π and the fact that $P/G = M$ follows from the (differentiable) local triviality.

We now define the desired connection on P. Let θ be the left invariant canonical 1-form on S^1, i.e., $\theta(A) = A$, $\forall A \in \hat{S}^1$. We know that $d\theta(X,Y) = -\frac{1}{2}[\theta(X),\theta(Y)]$, and so as S^1 is abelian, θ is closed. Define projections p_1, p_2

$$p_1 : U_j \times S^1 \to U_j$$

$$p_2 : U_j \times S^1 \to S^1 \quad \text{by}$$

$$p_1((x,g)) = x \quad \text{and} \quad p_2((x,g)) = g \ .$$

With the 1-forms α_i defined on U_i and the diffeomorphisms ψ_j, define a 1-form β_j on $\pi^{-1}(U_j)$ by

$$\beta_j = \psi_j^*(p_1^*\alpha_j + p_2^*\theta) \ .$$

By calculations similar to those done before, one can show that $\beta_j = \beta_k$ on $U_j \cap U_k$, and thus we can define a real valued 1-form α on P by $\alpha|_{U_i} = \beta_i$. Finally we check that α is a connection form, and that $D\alpha = \pi^*\omega$.

Clearly, as S^1 is abelian and θ is left invariant, α is Ad-invariant. That $\alpha(A^*) = A$ for $A \in \hat{S}^1$ follows from the defining properties of θ. Now

$$
\begin{aligned}
d\alpha|U_j &= d(\psi_j^*(p_1^*\alpha_j + p_2^*\theta)) \\
&= \psi_j^* p_1^* d\alpha_j \qquad (d\theta = 0) \\
&= \pi^* d\alpha_j \qquad (\pi = p_1 \cdot \psi_j) \\
&= \pi^* \omega|U_j
\end{aligned}
$$

and so we are finished.

SYMPLECTIC MECHANICS AND GEOMETRIC QUANTIZATION

The concept of a phase space is a very useful one in physics. Momentum phase space and velocity phase space are two common examples, and we will see later that these are, respectively, the cotangent and tangent bundles of the configuration space of the classical system. These are not the only kind however, for the phase space of an elementary relativistic particle with non zero spin is not of this sort. The salient features of a phase space are summarized in the definition of a symplectic manifold. It is here that the geometrical formulation of classical mechanics begins.

Definition: A symplectic manifold is a pair (M,ω), where M is a 2n-dimensional manifold and ω is a real 2 form on M satisfying

(1) $d\omega = 0$

(2) ω is non degenerate : $\omega(X,Y) = 0$ $\forall Y \Longleftrightarrow X = 0$.

ω is called the symplectic form of the manifold.

Observables, in the classical sense, are functions $\phi: M \to \mathbb{R}$. The set of observables on M, $\mathcal{O}$, can be given the structure of a Lie algebra in a natural way, as follows. If ω is the symplectic form, and ϕ is an observable, associate with ϕ a vector field X_ϕ on M which is determined by

$$i_{X_\phi}\omega + d\phi = 0.$$

Here, $(i_X\omega)(Y) = 2\omega(X,Y)$. The Lie bracket of $\phi,\psi \in \mathcal{O}$ is then defined by

$$\{\phi,\psi\} = 2\omega(X_\phi,X_\psi) \ .$$

This is just the classical Poisson bracket. The map $\{,\}:\mathcal{O}\times\mathcal{O}\to\mathcal{O}$ is clearly antisymmetric in its arguments; all we need show is that the Jacobi identity holds. That is,

$$\underset{\phi,\psi,\chi}{\mathfrak{S}} \{\{\phi,\psi\},\chi\} = 0$$

Note firstly from the definition that

$$\{\phi,\psi\} = X_\phi(\psi) \ .$$

We also have

$$[X_\phi,X_\psi] = X_{\{\phi,\psi\}} \qquad \text{for } \phi,\psi \in \mathcal{O}$$

This is proven in two steps. To begin, observe that

$$\begin{aligned} L_{X_\phi}\omega &= i_{X_\phi}d\omega + di_{X_\phi}\omega \\ &= 0 \qquad + d(-d\phi) \\ &= 0 \ . \end{aligned}$$

Then, if $X = X_\phi$, $Y = X_\psi$, we have

$$i_{[X,Y]}\omega = L_X i_Y \omega - i_Y L_X \omega$$

$$= i_X d i_Y \omega + d i_X i_Y \omega$$

$$= i_X d(-d\psi) + d(2\omega(Y,X))$$

$$= -d\{\phi,\psi\}$$

and so

$$[X_\phi, X_\psi] = X_{\{\phi,\psi\}}.$$

We can now prove the Jacobi identity. As ω is closed,

$$0 = d\omega(X_\phi, X_\psi, X_\chi)$$

$$= \underset{\phi,\psi,\chi}{\mathfrak{S}} [X_\phi(\omega(X_\psi,X_\chi)) - \omega([X_\psi,X_\chi],X_\phi)]$$

$$= \frac{1}{2} \underset{\phi,\psi,\chi}{\mathfrak{S}} [\{\phi,\{\psi,\chi\}\} - \{\{\psi,\chi\},\phi\}]$$

$$= \underset{\phi,\psi,\chi}{\mathfrak{S}} \{\phi,\{\psi,\chi\}\} .$$

This proves that $(\mathcal{O},\{,\})$ is indeed a Lie algebra.

As an example of a symplectic manifold, we consider the momentum phase space of a classical system, exhibit its symplectic structure and identify the Lie algebra $(\mathcal{O},\{,\})$.

Let X be the configuration space of a classical system with finitely many degrees of freedom. We assume that X is a manifold. The mechanics is formulated on $T^*(X)$, the cotangent bundle, in the following manner.

If $T(X)$ and $T(T^*X)$ denote the tangent bundles of X and $T^*(X)$ respectively, π and π^1 the projections associated with these bundles, and π_* the tangent map of π, we have the diagram

$$\begin{array}{ccc} & T(T^*(X)) & \\ \swarrow \pi' & & \pi_* \searrow \\ T^*(X) & & T(X) \end{array}$$

We define θ, the canonical one form on $T^*(X)$, naturally by

$$\theta(V) = \langle \pi^1(V)) \ , \ \pi_*(V) \rangle$$

where V is a vector in $T_m(T^*(X))$, $m \in T^*(X)$ and $\langle \, , \rangle$ is the natural pairing between $T_x(X)$ and $(T_x(M))^*$ (its dual), for $x \in X$. If $\{q^i, i=1,\ldots,n\}$ is a local co-ordinate system in $U \subset X$, we define local co-ordinates in $\{(q^i, p_j), i,j = 1,\ldots n\}$ in $T^*(U)$ where the p_j are determined by

$$\langle p_j \ , \ \frac{\partial}{\partial q^i} \rangle = \delta_{ij} \ .$$

These co-ordinates are said to be canonical. We can then write θ in these canonical co-ordinates as

$$\theta = \sum_{i=1}^{n} p_i dq^i \ .$$

The symplectic structure on $T^*(X)$ is obtained by setting $\omega = d\theta$; ω is clearly closed and can be seen to be non-degenerate by noticing that, in canonical co-ordinates,

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i .$$

In Hamiltonian formulation, the system is described by the observable $H: T^*(X) \to \mathbb{R}$, the Hamiltonian. The vector field X_H has physical significance, as the integral curves of X_H are the states of the system. In canonical co-ordinates,

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} ,$$

and so integral curves are generated by solutions of

$$\dot{q}^i = \frac{\partial H}{\partial p_i} = \{H, q^i\}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} = \{H, p_i\}$$

where $\dot{q}^i = \frac{dq^i}{ds}$, etc, and s is the parameter along the integral curve. These are the familiar Hamiltonian equations.

To identifythe Lie algebra $(\mathcal{O}, \{,\})$, observe that, in canonical co-ordinates,

$$\{f,g\} = \omega(X_g, X_f) = X_f(g) = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} .$$

This is just the classical Poisson bracket.

Geometric Quantization

Given any classical system, we can associate with it its phase space, a symplectic manifold, and a Lie algebra of observables. Suppose that (M, ω) is the phase space and $\mathfrak{A} = (\mathcal{O}, \{,\})$ is the Lie algebra. The program of quantization is to set up a Lie algebra homomorphism between $\mathfrak{A}$ and a Lie algebra of skew-hermitian operators on some Hilbert space. One way to achieve this is the geometric quantization scheme of Kostant and Souriau, which is described briefly below.

A symplectic manifold (M,ω) is said to be quantizable if $[\omega]$ is integral. The reason for this is that in that case the homomorphism mentioned above can be constructed. Let's see how. If $[\omega]$ is integral, we know by Weil's theorem that there is an S^1 principal bundle $P(M,S^1)$ with connection such that $\pi^*\omega = \Omega$, where $\pi: P \to M$ is the projection and Ω is the curvature form of the connection. Now, S^1 acts on $\mathbb{C}$ on the left naturally by multiplication: $(a,z) \to a \cdot z$, $a \in S^1$, $z \in \mathbb{C}$. We can then form the associated bundle E to P with standard fibre $\mathbb{C}$. $E(M,\mathbb{C},S^1,P)$ is then a complex vector bundle. As described before, the connection in P induces a connection and hence a covariant differentiation in E.

We now define an inner product (,) on $\Gamma(E)$ as follows. Each $u \in {}^{-1}(x)$ is an isomorphism from $\mathbb{C}$ to $\pi_E^{-1}(X)$: $u(z) = (\overline{u,z})$ as before. If $<,>$ is the usual inner product on $\mathbb{C}$, $<z_1,z_2> = z_1\bar{z}_2$, and if $x \in M$, $\psi, \phi \in \Gamma(E)$, then $<u^{-1}(\psi(x), u^{-1}(\phi(X)>$ is independent of $u \in \pi^{-1}(x)$. For if $u,v \in \pi^{-1}(x)$, $u = va$ for some a S^1, and so

$$<u^{-1}(\psi(X)),u^{-1}(\psi(X))> = <a^{-1}v^{-1}(\psi(X)),a^{-1}v^{-1}(\psi(X))>$$

$$= <v^{-1}(\psi(X)),v^{-1}(\psi(X))>$$

as S^1 is the isometry group of $<,>$.

Finally, if ϕ, $\psi \in \Gamma(E)$, define (ϕ,ψ), when it is finite, by

$$(\phi,\psi) = \int_M \langle u^{-1}\phi, u^{-1}\psi\rangle \omega^n$$

$$(\omega^n = \underbrace{\omega\wedge\cdots\wedge\omega}_{n \text{ times}})$$

The vector space of sections for which (ϕ,ϕ) is finite will be the pre-hilbert space.

We now wish to associate with each observable $\phi \in \mathcal{O}$, an operator δ_ϕ which is skew hermitian on a subspace of $\Gamma(E)$ so that this association $\phi \to \delta_\phi$ is a Lie algebra homomorphism. To this end, we define

$$\delta_\phi = \nabla_{X_\phi} - i\phi$$

and check that this is an association of the sort required.

To begin, we will calculate the curvature $R(X,Y)$. We know $R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$. As before $\nabla_X\phi = u(X^*f)$, where X^* is the horizontal lift of X to P and f is the $\mathbb{C}$-valued function on P defined by $f(u) = u^{-1}(\phi(\pi(u)))$. We shall prove here that $R(X,Y) = 2i\cdot\omega(X,Y)\cdot$, as we would expect by a previous theorem.

Now, $[\nabla_X,\nabla_Y]\phi = u([X^*,Y^*]f)$ so that $R(X,Y)\phi = u(([X^*,Y^*] - [X,Y]^*)f)$. Remembering that the horizontal component of $[X^*,Y^*]$ is $[X,Y]^*$, we have $R(X,Y)\phi = u(v[X^*,Y^*]f)$, where v is the vertical projection. Now we may choose A in the Lie algebra of S' so that $A_u^* = v[X^*, Y^*]$. Then, let a_t be a curve in S' with $(u\dot{a}_Y)_{t=0} = A_u^*$. So

$$A^* = \lim_{t\to 0} \frac{1}{t} [f(ua_t) - f(u)]$$

$$= \lim_{t\to 0} \frac{1}{t} [a_t^{-1} f(u) - f(u)]$$

$$= -Af(u).$$

The action of A on $\mathbb{C}$ here is that of a linear transformation (inherited by the action of S' on $\mathbb{C}$). Because of the simple nature of this action, we have

$$A_u^* f = -iA \cdot f(u)$$

where now A is a real number, (the Lie algebra of S' is isomorphic to the reals) the dot indicates multiplication in $\mathbb{C}$, and $i^2 = -1$. To proceed, observe that if α is the connection form, then

$$\alpha([X^*,Y^*]) = \alpha(v[X^*,Y^*]) = \alpha(A_u^*) = A \ .$$

Further, as X^* and Y^* are horizontal,

$$2\Omega(X^*,Y^*) = - \alpha([X^*,Y^*]) = -A \ .$$

This shows that $A_u^* f = 2i\Omega(X^*,Y^*)\cdot f(u)$, and so $R(X,Y)\phi =$ $u(2i\Omega(X^*,Y^*)f)$. Now we use the fact that $\pi^*\omega = \Omega$, which means

$$\begin{aligned} \Omega(X^*,Y^*) &= \pi^*\omega(X^*,Y^*) \\ &= \omega(\pi_* X^*, \ {}_*Y^*) \\ &= \omega(X,Y) \end{aligned}$$

and we have proven

$$R(X,Y)\phi = 2i\omega(X,Y)\cdot\phi \ .$$

We can now check that $\phi \to \delta_\phi$ is a Lie algebra homomorphism: If $\phi,\psi \in \mathcal{O}$, and χ is a differentiable section of E,

$$\begin{aligned} [\delta_\phi,\delta_\psi]\ \chi &= [\nabla_{X_\phi},\nabla_{X_\psi}]\chi - iX_\phi(\psi)\cdot\chi + iX_\psi(\phi)\cdot\chi \\ &= 2i\omega(X_\phi,X_\psi)\cdot\chi + \nabla_{[X_\phi,X_\psi]}\chi - 2i\{\phi,\psi\}\cdot\chi \\ &= \nabla_{[X_\phi,X_\psi]}\chi - i\{\phi,\psi\}\chi \\ &= \nabla_{X_{\{\phi,\psi\}}}\chi - i\{\phi,\psi\}\chi \\ &= \delta_{\{\phi,\psi\}}\chi \end{aligned}$$

as is desired.

To complete the association, we must show that each δ_ϕ is skew hermitian. This is done below. If $\phi \in \mathcal{O}'$, and Φ, $\Psi \in \Gamma(E)$, we need

$$(\delta_\phi \Phi, \Psi) = -(\Phi, \delta_\phi \Psi) .$$

Using the definition, and noting that ϕ is real, this is just

$$(\nabla_{X_\phi} \Phi, \Psi) + (\Phi, \nabla_{X_\phi} \Psi) = 0 .$$

Consider for a moment the fibre metric in E defined by

$$((e,f)) = \langle u^{-1}(e).u^{-1}(f)\rangle$$

where e and f are in the same fibre, and u is in the fibre (in P) above $\pi_E(e)$. We know that this definition is independent of $u \in \pi^{-1}(\pi_E(e))$. The connection in E preserves this metric: if e_t and f_t are horizontal lifts to E of a curve x_t in M, then we know that there is a horizontal curve u_t in P, and two complex numbers ξ and η such that

$$e_t = u_t(\xi) \quad \text{and} \quad f_t = u_t(\eta) .$$

Then,

$$\begin{aligned} \langle \xi, \eta \rangle &= \langle u_t^{-1}(u_t(\xi)) \;,\; u_t^{-1}(u_t(\eta))\rangle \\ &= \langle u_t^{-1}(e_t) \;,\; u_t^{-1}(f_t)\rangle \\ &= ((e_t, f_t)) . \end{aligned}$$

Thus we have

$$X_\phi((\Phi,\Psi))(x) = ((\nabla_{X_\phi}\Phi(x),\Psi(x)) + ((\Phi(x),\nabla_{X_\phi}\Psi(x))) .$$

Recalling the definition of (Φ,Ψ), we see that all we need show is

$$\int_M X_\phi((\Phi,\Psi))\omega^n = 0 .$$

Writing $f = ((\Phi,\Psi))$, we proceed. As ω^n has rank equal to $2n$, $df\wedge\omega^n = 0$. So

$$0 = i_{X_\Phi}(df\wedge\omega^n) = X_\Phi(f)\cdot\omega^n - df\wedge i_{X_\phi}\omega^n .$$

Also,

$$d(fi_{X_\phi}\omega^n) = df\wedge i_{X_\phi}\omega^n + f\cdot di_{X_\phi}\omega^n$$

$$= X_\Phi(f)\omega^n + f(L_{X_\phi}\omega^n + i_{X_\phi}d\omega^n)$$

$$= X_\phi(f)\omega^n .$$

(Remember, $L_{X_\phi}\omega = 0$.)

This shows that the integrand is exact, and so, if f has compact support, and is sufficiently differentiable, then

$$\int_M X_\phi(f)\omega^n = \int_M d(fi_{X_\phi}\omega^n) = \int_{\partial V} fi_{X_\phi}\omega^n = 0$$

where V is chosen so that it contains the support of f.

We have thus defined, for each observable $\phi\in\mathcal{O}$, an operator δ_ϕ which is skew hermitian on a subspace of $\Gamma(E)$.

(For instance this subspace may be taken as all those sections of E which are infinitely differentiable everywhere and have compact support.) The completion, $\mathcal{H}$, of this inner product space will be the Hilbert space of the quantum system. As usual, the operators' domain of definition may not be the whole of $\mathcal{H}$, but where they are defined, they will still be skew hermitian.

The association that we have made is unsatisfactory for several reasons. Perhaps the first point that strikes one is that these 'wave functions' we have constructed depend not only on x, but also on the momentum p. In this situation it is impossible at a glance to then associate (ψ,ψ) with a probability density, as is classically done. More important, however, is the fact that the representation of certain important subalgebras of observables (for instance, those connected with the Dirac problem) is not irreducible. The above problems are related, in fact, and a method known as polarization of the phase space overcomes them with some degree of success. When the polarization is carried out, the representation becomes irreducible, but it restricts the class of observables that can be quantized, and the homomorphism constructed before breaks down in general. We will not deal with this matter here. For a detailed account, see, for example, Simms and Woodhouse [2].

To conclude this section, we will briefly show how the symplectic formulation of mechanics embraces quantum theory as well.

The quantum phase space is the Hilbert space $\mathcal{H}$. An association between observables as real-valued functions on $\mathcal{H}$ and observables as skew hermitian operators on $\mathcal{H}$ is established as follows. The real valued function $\tilde{A}$ on $\mathcal{H}$ associated with a skew symmetric operator A is defined by

$$\tilde{A}(\psi) = i(A\psi,\psi) \quad , \quad i^2 = -1.$$

To continue, we need to define vector fields on $\mathcal{H}$. A curve $\psi:[0,1] \to \mathcal{H}$ is differentiable at $t \in (0,1)$ if $\psi' = \lim_{h \to 0} [\frac{\psi(t+h)-\psi(t)}{h}]$ exists. This limit is taken with respect to the Hilbert space topology. The tangent space at $\psi \in \mathcal{H}$ can thus be defined, and, as in the case of any topological vector space, (e.g. $\mathbb{R}^n$) it can be identified with $\mathcal{H}$ itself. A vector field on $\mathcal{H}$ then is simply an association to each point $\psi \in \mathcal{H}$ a vector $\phi \in \mathcal{H}$.

Any operator $A:\mathcal{H} \to \mathcal{H}$ can then be viewed as a vector field on $\mathcal{H}$, and, in the case when A and B are skew-hermitian, the vector field commutator [A,B] is identically equal to the operator commutator [A,B]. That is, $[A,B]_\psi = [A,B]\psi$. Furthermore if A is viewed as a vector field, we have $A(\tilde{B}) = i([A,B]\psi,\psi)$ $= - \widetilde{[A,B]}$.

The symplectic 2 form on $\mathcal{H}$ is defined by

$$\omega(X,Y) = \frac{i}{2}([X,Y]_\psi,\psi)$$

[remember, $[X,Y]_\psi \in \mathcal{H}$). Now, ω is antisymmetric, and is closed and real-valued if we consider it only defined on the subalgebra (of vector fields on $\mathcal{H}$) generated by skew-hermitian vector fields. This is seen quickly from previous observations. It is also almost non degenerate on this subalgebra: if $\omega(A,B) = 0\ \forall B$, then $A = \lambda i I$, $\lambda \in \mathbb{R}$, I is the identity operator. This is not too much of a problem though, for given an observable of the form $\tilde{A}$, the vector field on $\mathcal{H}$ defined by $A_\psi = A\psi$ satisfies

$$i_A \omega + d\tilde{A} = 0$$

as it ought to. We could even restrict ourselves to a subalgebra of skew-hermitian operators that does not have any centre, and this often is the case. If H is the Hamiltonian operator, integral curves of the vector field associated with H are the states of the system. These can be described very simply. The solution to $\dot{\psi}(s) = H_{\psi(s)} = H(\psi(s)), \psi(0) = \psi_0$ is just $\psi(s) = e^{sH}\cdot\psi_0$. Here the unitary operator e^{sH} is defined by its power series

$$e^{sH} = \sum_{n=0}^{\infty} \frac{(sH)^n}{n!},$$

which converges if H is continuous (as an operator). As usual, the rate of change of an observable A along $\psi(s)$ is given by $\frac{d\tilde{A}}{ds} = H(\tilde{A}) = \widetilde{[A,H]}$.

GENERAL RELATIVITY

In this section we will discuss very briefly Einstein's general theory of relativity. The free test particle will be considered through a symplectic formulation and this will reveal the close relationship between the metric and the kinetic energy. An application of the fibre bundle formalism in the study of singularities will be presented, and we will conclude the chapter with a discussion of Mach's principle and how it might appear in general relativity.

Let M be the set of all spacetime events that we wish to discuss. Assume that M has a smooth, 4 dimensional, real manifold structure. As before, let $T(M)$ and $T^*(M)$ be, respectively, the tangent and cotangent bundles of M. We will consider first how the motion of a free point particle can be described by a symplectic formulation in $T^*(M)$. If θ is the canonical 1-form on $T^*(M)$, we again set $\omega = d\theta$ as the symplectic form. We know then, from previous work, that $(T^*(M),\omega)$ is the symplectic manifold. We now need a Hamiltonian for the system.

In the special theory, $M = \mathbb{R}^4$, $T^*(M) \simeq R^4 \times R^4$, and the Lorentz metric in $T(M)$ (which is isomorphic to $\mathbb{R}^4 \times \mathbb{R}^4$ also) is given by

$$\eta_m(V,W) = V_0W_0 - V_1W_1 - V_2W_2 - V_3W_3$$

if $V = (V_0,V_1,V_2,V_3)$ and $W = (W_0,W_1,W_2,W_3)$ are vectors at m. This metric induces a metric $\eta^{\#}$ in $T^*(M)$ in the following natural manner. Define $\#_m: T_m^*(M) \to T_m(M)$ by

$$\eta_m(\#_m(p)V = \langle p,V\rangle_m \quad \forall V \in Y_m(M)$$

Here $\langle\ ,\ \rangle_m$ is the dual pairing of $T_m(M)$ and $T_m^*(M) = (T_mM)^*$. As η is non degenerate, this is a good definition, and $\#_m$ is an isomorphism. Now we define $n_m^{\#}:T_m^*(M)\times T_m^*(M) \to \mathbb{R}$ by

$$\eta_m^{\#}(p_1,p_2) = \eta_m(\#_m(p_1),\#_m(p_2))$$

for $p_1,p_2 \in T_m^*(M)$.

This is also non degenerate and has the same signature as η. The free particle Hamiltonian for the special theory is then defined by

$$H:T^*(M) \to \mathbb{R}$$

$$H(m,p) = \frac{1}{2}\,\eta_m^*(p,p).$$

In an obvious notation this is the familiar $H = \frac{p^2}{2}$.

The Hamiltonian in the general theory for the free particle is just a natural generalization of this. In the special theory, the metric η_m was independent of m. The general theory allows η_m to depend on m: let g be a metric in T(M) with signature (1,-1,-1,-1). Define, as before, a metric $g^{\#}$ in $T^*(M)$. The free particle Hamiltonian for the general theory is then

$$H(m,p) = \frac{1}{2}\, g^{\#}_{m}(p,p)$$

(or $H = \frac{1}{2} g^{ij}p_i p_j$, in an obvious notation). Just as before in the symplectic formulation, the integral curves of X_H will describe the states of the particle. In local canonical coordinates, the equations of motion, $\{H,x^i\} = \dot{x}^i$, $\{H,p_i\} = \dot{p}_i$ (notation as before) are the familiar ones:

$$p_i = g_{ij}\dot{x}^j \qquad (1)$$

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \qquad (2)$$

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $\Gamma^i_{jk} = \frac{1}{2} g^{i\ell}(g_{j\ell,k} + g_{k\ell j} - g_{jk,\ell})$ $g^{ij}g_{jk} = \delta^i_k$, and $g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}$.

Equation (2) is a geodesic, or parallel transport equation: formally, if ∇ is the covariant differentiation on T(M) associated with the unique torsion metric connection in L(M) that preserves g (that is, $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$), and we write

$$X = \sum_i \dot{x}^i \frac{\partial}{\partial x^i},$$

(2) is just

$$\nabla_X X = 0 .$$

This is just the 'free fall' equation.

Thus we have seen that, given a metric in T(M), in principle, we can determine the motion of free particles. Einstein's theory links the metric in T(M) to the matter and energy in the space time. Classically, the matter and energy has been described by a rank 2 symmetric tensor T, called the energy momentum tensor. An example of one of these will be discussed later. Now we see how Einstein related the metric to this tensor

If R is the Riemann curvature of the connection ∇ above, that is,

$$R(X,Y)Z = [\nabla_X,\nabla_Y]Z - \nabla_{[X,Y]}Z$$

then the Ricci tensor, Ric is defined by

$$\mathrm{Ric}(X,Y) = \text{trace of } Z \to R(X,Z)Y.$$

We may view $\mathrm{Ric}(X,\cdot)$ as a covector: explicitly, for $Y \in T_m(M)$, $\mathrm{Ric}(X,\cdot)(Y) = \mathrm{Ric}(X,Y)$. Recalling the isomorphism $\#_m$ defined before, we define the Ricci scalar curvature, also denoted by R, by

$$R = \text{trace of } X \to \#_m^{-1}(\mathrm{Ric}(X,\cdot)).$$

(In 'index' notation, if R^{ℓ}_{ijh} denote the components of R, then $\mathrm{Ric}_{ij} = R^{\ell}_{i\ell j}$ and $R = R_{ij}g^{ij}$.) With these definitions, Einstein's equation relating the geometry and the matter is

$$\mathrm{Ric}(X,Y) - \frac{1}{2} R\, g(X,Y) = \kappa\, T(X,Y) \qquad (3)$$

Here κ is a physical constant, the value of which can be determined by approximating (3) by a linear differential equation (in the case when the velocities of particles are small compared to the speed of light and the matter is scarce) and comparing the approximation to Newton's equation for gravitation.

Of course, one of the tests of the theory is that such a procedure described above can be carried out. There are many methods for 'deriving' this equation. The most common involve either employing an action principle or by intelligently generalizing the Newtonian gravitation equations. We will not deal with that subject here.

The fibre bundle formalism is useful not only for the setting of general relativity, but can also be exploited when dealing with important questions arising from the theory. We will outline briefly the new definition of singular points initiated by Schmidt in a 1971 paper ([6]).

In his work, Schmidt attached a boundary to any space-time on which incomplete geodesics and inextendable timelike curves of finite length and bounded acceleration terminate. The germ of his idea is to define a Riemannian metric on L(M) and to complete L(M) in the distance function associated with this Riemannian metric.

Let θ be the canonical 1-form on L(M) $(\theta_u(X) = u^{-1}(\pi_*(X))$ and ω the connection form of the unique torsion free metric connection associated to the given metric on M. θ is $\mathbb{R}^n$ valued, and ω has values in $M_n(\mathbb{R}) = \{X | X$ is an $n \times n$ matrix with real entries.$\}$ Note that $M_n(\mathbb{R})$ can be identified with $\mathbb{R}^{n^2}$: every k $1 \leq k \leq n^2$ can be written uniquely as $k = (i-1)\cdot n+j$ where $1 \leq i,j \leq n$. The identification is then

$$a_{(i-1)n+j} = a_{ij} ,$$

and it is an additive isomorphism. Now, both $\mathbb{R}^n$ and $\mathbb{R}^{n^2}$ are inner product spaces, and, with the above isomorphism, so too is $M_n(\mathbb{R})$. So, for $X,Y \in T_u(L(M))$, we can define

$$G_u(X,Y) = (\theta(X),\theta(Y))_n + (\omega(X),\omega(Y))_{n^2}$$

where $(\ ,\ )_n$ and $(\ ,\ )_{n^2}$ are, respectively, the usual inner products in $\mathbb{R}^n$ and $\mathbb{R}^{n^2}$. As it is defined, G is clearly symmetric and positive definite, and so is a genuine Reimannian metric on L(M). If M is orientable, L(M) has 2 connected components, each isometric. Let $L'(M)$ be one of these. We define the distance function on $L'(M)$ associated with G; $d(x,y) = \inf_{C \in \mathcal{C}_{x,y}} \{\int G(x,x)^{1/2} ds\}$ where $\mathcal{C}_{x,y}$ is the set of all parameterized, piecewise differential curves from x to y, X is the tangent to $C \in \mathcal{C}_{x,y}$ and s is the parameter along the curve. It is easy to show that (L(M),d) is indeed a metric space.

The completion of $(L'(M),d)$, in which $L'(M)$ is dense, will be denoted $\bar{L}(M)$, and the boundary of $L')M)$, $\bar{L}(M)-L'(M)$, will be denoted by $\dot{L}(M)$. The action of $GL_0(n,\mathbb{R})$ on $L'(M)$ is uniformly d-continuous, and can be extended in a unique fashion to $\bar{L}(M)$, where it remains so. Define $\bar{M}$ as the set of orbits of this group action on $\bar{L}(M)$, and extend π to $\bar{L}(M)$ by letting $\pi(x)$ = orbit of x. On $\bar{M}$, put the finest topology making π continuous. Schmidt then defines the b-boundary of M to be $\bar{M}$-M.

To see how the b-boundary is related to inextendable geodesics, consider the following. Suppose x_t, $0 \leq t < 1$ is a geodesic, with geodesic parameter t, such that $\lim_{t \to 1} x_t$ does not exist in M. At x_0 choose a frame $u_0 = (x_0, X_0(0), \cdots X_3(0))$ so that $g_{x_0}(X_i(0), X_j(0)) = 1$ if $i=j=0$, -1 if $i=j\neq 0$, 0 if $i\neq j$. (That is, u_0 is orthonormal at x_0.) Parallel transport this frame along x_t, and call the curve in L(M) so generated u_t. If $\dot{x}_t = \Sigma a_i(t) X_i(t)$, where $u_t = (x_t, X_0(t), \cdots, X_3(t))$, we then know that $\nabla_{\dot{x}_t} X_i(t) = 0$, $0 \leq i \leq 3$ and so $a_i(t) = \pm g(\dot{x}_t, X_i(t))$. This implies that $\frac{da_i(t)}{dt} = \pm \dot{x}_s g(\dot{x}_t, X_i(t)) = 0$ and so $a_i(t)$ is constant along x_t. But $u_t(\theta(\dot{u}_t)) = \pi_*(\dot{u}_t) = \dot{x}_t$, by the definition of θ, and we then have

$$<\theta(\dot{u}_t), \theta(\dot{u}_t)> = \sum_{i=0}^{3} a_i^2(t) = k,$$

which is a constant. Recalling that u_t is parallel, so that $\omega(\dot{u}_t) = 0$, the length of $L_0^\alpha(u_t)$ of u_t in (L(M),G) from t=0 to $t=\alpha$, $(\alpha < 1)$, is simply

$$L_0^\alpha(u_t) = \int_0^\alpha <\theta(u_t), \theta(u_t)>^{1/2} dt.$$

Now choose $\{t_n\} \subset [0,1]$ with $\lim_{n\to\infty} t_n = 1$. Then u_{t_n} is Cauchy in $(\bar{L}(M), d)$ as $d(v_{t_n}, u_{t_m}) \leq \left| \int_{t_n}^{t_m} <\theta(\dot{u}_t), \theta(\dot{u}_t)>^{1/2} dt \right| = \sqrt{k}.\ |t_m - t_n|$.

As $(\bar{L}(M), d)$ is complete, $u_{t_n} \to u \in \bar{L}(M)$ and so we define, in $\bar{M}$, $\lim_{t\to 1} x_t = \pi(u)$.

A similar argument will show that inextendable timelike curves of finite length and bounded acceleration are extendable in $\bar{M}$. It is interesting to observe that if the bundle metric d is complete, that is, if (L(M),d) is complete, then the connection of M is geodesically complete. The converse, however, is not true. There is an example due to Geroch [6] of a space time which is geodesically complete but contains inextendable timelike curves with bounded acceleration and finite length.

To conclude this chapter, we consider the relation between Einstein's general theory and Mach's principle in cosmology. Loosely stated, Mach's principle says that the matter in the universe completely determines the physics of the universe. We will attempt here to give some rigorous meaning to "the physics of the universe", to see how Mach's principle may be stated formally in the general theory, and to investigate the consequences of some of these formal versions.

To begin then, we must decide what we mean by "the physics of the universe". There are several ways we can give this meaning. If we are interested only in the parameterized paths of test particles, then it is clear from earlier work that we need only know the geodesics of spacetime. What do we need in order to determine them? We know that the energy observable, H, is sufficient, as integral curves of X_H are the states of the particle. To determine the energy, we must know the metric tensor g, as we have seen. However, any metric $\bar{g}$ related to g by a constant conformal factor will give the same energy and geodesics. Moreover, we may as well just require knowledge of the connection ∇, for this determines the geodesics. It turns out, in the generic case, that all of these are equivalent: the connection will determine the metric to within a constant conformal factor. (Of course, we must assume that ∇ is a torsion free metric connection to do this.) This equivalence is discussed in the next chapter. In general relativity, all knowledge of the matter in the universe is contained in the matter-energy tensor T, so if we agree that "the physics of the universe" is to mean knowledge of the geodesics, then Mach's principle can be restated formally as: Any two spacetimes with the same energy momentum tensor T should have the same connection ∇. This of course amounts to the question of the uniqueness of solution to Einstein's field equations.

There is another way of understanding "the physics of the universe" that points to more than just the uniqueness problem in general relativity. Free test particles are interesting, but perhaps more important is the motion of particles and extended bodies that constitute the matter in the universe. For the moment, we will consider 'motion' to mean only the unparameterized paths of these bodies. Now it is clear that if we can isolate a particle or a body, then we can isolate its contribution to T, and hence in this simple way, determine the motion of that body. (Remember, T is defined over all spacetime.) In this form, Mach's principle, which would state that any two spacetimes with the same matter tensor

T should have the same unparameterized paths for particles and extended bodies, would automatically be satisfied. This is important to realize. More significant is the consistency problem that it unveils: through the work of Einstein, Hoffmann and Infeld, it was shown, at least in some 'physically reasonable' cases, that Einstein's equations once solved for the metric g, allow the (parameterized) trajectories of the matter constituents to be determined. This means that there is another way, quite independent of the above prescription involving only T, for determining the motion of the gravitating bodies. To carry out this second procedure, we do need to solve Einstein's equations for g, however. A priori, we do not know whether these two methods will agree, and this is the consistency problem. We point out that in this second method, because we need to solve for g in Einstein's equations, there is also the question of the uniqueness of the solution. This means that for a fixed T, we may have to check several metrics for compatibility.

The solution of the consistency question will be useful in many ways. Firstly, it may tell us what 'physically reasonable' matter tensors T will look like, for these should only have compatible solutions g. Secondly, in the case when non uniqueness occurs (and it will), and there is at least one compatible solution, we can examine the non compatible metrics and perhaps extract some criteria that will enable us us to restrict our attention to a subclass of metrics that is to be considered as candidates for solutions. These criteria may even be useful in the uniqueness question.

Lastly we consider another interpretation of "the physics of the universe", one that might come closest to what we all really mean. Such an interpretation has at its core the symplectic formulation of mechanics, and is in terms of observables and the Lie algebra they form. As we have seen, this formulation of mechanics embraces almost all physics.

In any physical problem, be it classical or quantum, the fundamental elements are the observables, for these determine not only the states of the system, but all the other relevant information as well. In the chapter on geometric quantization, it was shown how we could obtain a representation of the classical Lie algebra of observables as skew hermitian operators (the observables there) on the quantum phase space. This suggests that if we can isolate the relevant subalgebra of observables on the classical level, the corresponding quantum observables that are of importance will also be determined.

Of course, any real valued function on the phase space is an observable, but not all of these are significant. What is the subalgebra that is important in a physical problem? We have seen that the energy observable, which is connected intimately to the metric, is of paramount importance for determining the parameterized paths of test particles in general relativity. There are others, naturally, that are relevant to particular problems, such as the angular momentum, or the spin, for instance. Just what the momentum is, is not entirely clear, as no co-ordinate free prescription for it exists as yet. Angular momentum too, should not need symmetries for its definition, and perhaps not even for its conservation (see [8]).

Attempts at defining angular momentum in general relativity have been made (see [9], for example) and these suggest that the holonomy group could be very useful. Each member of the holonomy Lie algebra at m (and, more generally, each member of the full Lorentz Lie algebra L_m at m; where $L_m = \{A/A:T_m(M) \rightarrow T_m(M) \text{ and } g_m(A(V),A(W)) = g_m(V,W)\ \forall V,W\in T_m(M)\}$), via the action of the holonomy group on $T_m(M)$, generates a vector field on $T_m(M)$. (In exactly the same way that the fundamental vector fields A^* on a principle fibre bundle are generated.) These vector fields can be mapped, via the tangent map of the exponential map in differential geometry, to local vector fields on M. Now each vector field on M is an observable in the following simple manner. $V\in T_x(M)$ maps $T^*(M)$ into $\mathbb{R}$ by $V(p) = p(V)$, for $p\in T^*(M)$. Thus each Lorentz Lie algebra element, and in particular each member of the holonomy Lie algebra, is associated with a locally defined observable on the phase space. (In fact the map: Lie Algebra element $\rightarrow$ locally defined vector field, is a one to one homomorphism into the Lie algebra of vector fields on M.) In the case when M is Minkowskii space, the Lie algebra of vector fields on M generated by all of these locally defined vector fields (for all $m \in M$) is finite dimensional, and its group is the Poincaré group. The observables generated are the usual angular and linear momentum. It must be pointed out in this case that the holonomy Lie algebra is trivial, and it is the Lorentz Lie algebra that generates these observables. However, in the generic case, the holonomy Lie algebra is the full Lorentz Lie algebra and this example gives us confidence in the meaning of the observables generated then. For more details, see [9].

This shows us that the metric structure, in particular the holonomy group, may enable us to determine the relevant observables. The metric structure that we needed was the connection ∇ (which was used to define the exponential map). This of course determines the holonomy group and the metric to within a constant conformal factor in the generic case. So we arrive at the situation encountered before. If Mach's principle is to state that the matter in the universe determines the relevant Lie algebra of observables, then the formal version of this is again: Any two spacetimes with the same matter tensor T should have the same connection ∇.

To summarize, if we are to attach to Mach's principle, in any of these versions, any significance at all, then the uniqueness question in the solution of Einstein's equations is a very real, physical one. It is not simply the abstract physically meaningless endeavour of mathematicians who like to keep theories tidy. Physicists who believe in Mach's principle should be just as concerned with the uniqueness problem in general relativity.

PROBLEMS OF UNIQUENESS AND THE HOLONOMY GROUP

The question of the uniqueness of the solution to Einstein's equations is a very difficult one and a vast amount of literature is devoted to it. Here we consider the related, and somewhat simpler question: what conditions on the curvature of a connection guarantee that the connection is the only one with the specified curvature?

In general, the answer is that there aren't any such conditions. For instance, for any connection with connection form ω in any principal fibre bundle P(M.G) where the structure group's Lie algebra has a non empty centre, if we choose an arbitrary closed real valued 1-form λ on M, then the 1-form $\omega' = \omega + (\pi^* \lambda)\cdot I$ is a connection form, and it has the same curvature as ω for any I, a member of the centre. If we restrict ourselves to torsion free connection in L(M), the above construction fails and the question is open again. However, when dealing with the (smaller) class of torsion free metric connections, we can solve the problem. A paper by Ihrig [11] establishes the following result: Suppose dim $M \geq 4$. Let R be the Riemann tensor of a pseudometric. Suppose further that R is broad and total at every point of M. Then this pseudometric is the unique pseudometric (to within a constant conformal factor) that has R as its curvature.

R is broad at $m \in M$ if for every vector $X \in T_m(M)$ there are vectors Y, $Z \in T_m(M)$ so that $\{R(Y,Z)X, Y, Z\}$ is a linearly independent set. R is said to be total at $m \in M$ if $\{R_m(X,Y)\cdot, | X,Y \in T_m(M)\}$, which is a set of endomorphisms of $T_m(M)$, generates a vector space V_m of endomorphisms of dimension $n(n-1)/2$. By the holonomy theorem, we know that V_m is contained in the Lie algebra of the holonomy group. The totality of R at m forces V_m to be the Lie algebra of the holonomy group, which must then be the whole group of the isometries of the metric g_m. Then we know the metric at m to within a constant, so that if R is total at all points in M, g is determined to within a conformal factor. The condition of broadness forces this factor to be constant. This result shows us that, at least in the case of torsion free metric connections, if we force the holonomy group to be as large as it possibly can be, as Ihrig's first condition on the curvature did, then we obtain some information on uniqueness. It is interesting to determine whether or not this method is of any use in the problems considered before.

Consider again the question raised before when we stipulate that the holonomy group be as large as it can. In the case of an arbitrary connected principal fibre bundle P(M,G), a well known theorem [1] displays, if dim M $\geq$ 2, a connection in P with holonomy group equal to G_0, the identity component of G. The constructed curvature does not have any analogous properties to those in Ihrig's theorem, but it is interesting nevertheless that, if we examine the construction, we see it is possible to have two distinct connections, both whose holonomy groups are G_0, and both sharing the same curvature. Hence we obtain no information on uniqueness by this simple method. However, in the case of torsion free connections in $L(M)$, whether a similar failure occurs had not been considered, to the author's knowledge. The above construction in $L(M)$ does not produce a torsion free connection there. To conclude this section, we show that, indeed, in this more special case, the same non-uniqueness occurs. To do this, we construct, for any manifold M, with $\dim M \geq 3$, a torsion free connection whose holonomy group is $GL_0(n, \mathbb{R})$. (When the manifold is not $\mathbb{R}$ orientable, the construction yields a holonomy group which is the whole of $GL(n, \mathbb{R})$.) Then, by a glance at the construction, one can see that the connection can be altered without changing either the curvature or the holonomy group.

The construction is mainly technical. Choose any $m \in M$ and (U, x^i) a local co-ordinate system around m, so that $|x^i| \leq 2n$ on U. ($n = \dim M$ here). Fix n real numbers $\alpha_1 \cdots \alpha_n$ so that

$$-\frac{1}{4} < \alpha_1 < \cdots < \alpha_n < \frac{1}{4}$$

and choose n C^∞ functions $g_1 \cdots g_n$ defined on $(0,1)$ with

$$\left.\frac{dg_j}{dt}\right|_{\alpha_k} = -\delta^j_k \qquad 1 \leq j,k \leq n.$$

Now consider the following sets:

$$U_1 = \{x \in U \,|\, | < x^3 < 2 \quad ; \quad |x^j| < \tfrac{1}{2} \text{ if } j \neq 3\}$$

and, for $k \neq 1$,

$$U_k = \{x\ U|\ |x^1-(k-2)| < \frac{1}{2}\ ;\ |x^j| < \frac{1}{2} \quad \text{for } j \neq 1.\}$$

$$F_1 = \{x \in M | \frac{5}{4} \le x^3 \le \frac{7}{4}\ ;\ |x^j| \le \frac{1}{4} \quad \text{for } j \neq 3\}$$

and, for $k \neq 1$

$$F_k = \{x \in M|\ \ |x^1-(k-2)| \le \frac{1}{4}\ ;\ |x^j| \le \frac{1}{4}\ ,\quad j \neq 1\}\ .$$

Define n C^∞ functions on U, $f_1 \ldots . f_n$ so that $f_i \equiv 1$ on F_i, $f_i \equiv 0$ outside U_i.

Consider now the torsion free connections Γ_i defined on U_i by their covariant derivatives ∇_i:

$$\nabla_{nX_n} X_n = \sum_k g_k(x^1 + (2-n)) \cdot X_k$$

$$\nabla_{nX_j} X_k = 0 \quad \text{otherwise}$$

and, for $j \neq n$

$$\nabla_{jX_j} X_j = \sum_k g_k(x^{j+1}) X_k$$

$$\nabla_{jX_i} X_h = 0 \qquad \text{otherwise}$$

(Here, $X_i = \frac{\partial}{\partial x^i}$).

We calculate the Riemann curvature, R_ℓ, associated with each Γ_ℓ:

$$R_\ell(X_i, X_j) X_h = [\nabla_{\ell X_i}, \nabla_{\ell X_j}] X_k\ , \quad \text{as}$$

$$[X_i, X_j] = 0.$$

On U_ℓ, $\ell \neq n$, $R_\ell(X_i,X_j)X_k$ is zero unless $i = k = \ell$, $j = \ell + 1$.
Then,

$$R_\ell(X_\ell,X_{\ell+1})X_\ell = - \sum_k \frac{dg_k}{dx^{\ell+1}} X_k .$$

For $\ell = n$,

$$R_n(X_n,X_1)X_n = - \sum_k \frac{dg_k}{dx^1} \cdot X_k$$

is the only non zero expression.

Define points $P_\ell^j \in U_\ell$ by

$$P_1^j = (0,\alpha_j,\tfrac{3}{2},0\cdots0)$$

$$P_n^j = ((n-2)+\alpha_j,0,0,\cdots0)$$

and, if $\ell \neq 1$ or n,

$$P_k^j = ((\ell-2),0_{jj}0_j\alpha_j,0\cdots0)$$

where α_j is in the $(\ell+1)$th place.

So, for $\ell \neq n$, at P_ℓ^j,

$$R_\ell(X_\ell,X_{\ell+1})X_\ell = X_j ,$$

and, in particular

$$R_\ell(X_\ell,X_{\ell+1})X_k = 0 \quad \text{for } k \neq \ell. \tag{1}$$

If $\ell = n$, at P_n^j

$$R_n(X_n,X_1)X_n = X_j ,$$

and, in particular

$$R_n(X_n,X_1)X_k = 0 \quad \text{for } k \neq n . \tag{2}$$

With these facts in mind, define a connection Γ on U by

$$\Gamma = \sum_{i=1}^{n} f_i \Gamma_i \qquad \text{(i.e. component-wise)} .$$

As defined, Γ is torsion free, and the curvature R of Γ satisfies, by virtue of (1) and (2)

$$R(X_\ell, X_{\ell+1})X_\ell = X_j$$

$$R(X_\ell, X_{\ell+1})X_k = 0 \qquad \text{if } k \neq j$$

$$\text{at } P_\ell^j \quad , \quad \ell \neq n.$$

and

$$R(X_\ell X_1)X_\ell = X_j$$

$$R(X_\ell X_1)X_k = 0 \qquad \text{if } k \neq j$$

$$\text{at } P_n^j .$$

Now consider the frame $u \in L(M)$ defined at $x \in U$ by $u = (x, X_1 \cdots X_n)$. It is not difficult to see that there is a path in U, passing through all the points P_i^j, $1 \leq i, j \leq n$, along which u is parallel. By the holonomy theorem, we see that, at P_i^j, the element $E_i^j \in \widehat{G\ell}(n, \mathbb{R})$ is generated (E_i^j is the only matrix whose only non zero entry is a 1 at the intersection of the ith row and the jth column.) This proves that the holonomy group of Γ is $GL_0(h, \mathbb{R})$, and if M is not $\mathbb{R}$ orientable, it is $GL(n, \mathbb{R})$. Now extend the connection defined on L(U) to L(M), keeping it torsion free. The resulting connection will satisfy the requirements of our proposition, and we are done.

APPENDIX

Let K be a ring. Here $\check{H}^2(M,K)$ and $H^2(M,K)$ will denote, respectively, the second Čech cohomology module and the second singular cohomology module of M over K. Also, $H^2_D(M,\mathbb{R})$ will denote the second de Rham cohomology module of M over $\mathbb{R}$). The isomorphisms (in the case of paracompact M) $H^2_D(M,\mathbb{R}) \overset{f}{\approx} H^2(M,\mathbb{R}) \overset{g}{\approx} \check{H}^2(M,\mathbb{R})$ are well known. The inclusion $i: \mathbb{Z} \to \mathbb{R}$ induces the homomorphisms $i^* : H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})$ and $i_1^* : \check{H}^2(M,\mathbb{Z}) \to \check{H}^2(m,\mathbb{R})$.

Definition: (1) $[\omega] \in H^2_D(M,\mathbb{R})$ is integral if $f([\omega])$ lies in $i^*(H^2(M,\mathbb{Z}))$.

There are several characterizations of this property. We shall discuss three here. The first is the definition above. the others are:

(2) $[\omega] \in H^2_D(M,\mathbb{R})$ is integral if there is a locally finite contractible open covering $\{U_i\}$ of M, with 1-forms α_i defined on U_i, smooth functions f_{ij} defined on U in U_j so that $\omega|_{U_i} = d\alpha_i$, $\alpha_i - \alpha_j = df_{ij}$ and $f_{ij} + f_{jk} - f_{ik} = a_{ijk}$ is an integer when it is defined. (Note that the f_{ij}, a_{ijk} depend only on $[\omega]$).

(3) $[\omega] \in H^2_D(M,\mathbb{R})$ is integral if, for all 2-cycles S, $\int_S \omega \quad \mathbb{Z}$

The implication 3 ==> 1 is a consequence of Rham's theorem (the isomorphism f above), 1 ==> 2 can be proven by an argument along these lines: given $K \in [K] \in H^2(M, \mathbb{Z})$, with $f([\omega]) = [k]$, we know that $K(S) = \int_S \omega \in \mathbb{Z}$ for S a 2-cycle. We can then choose $k' \in [k]$ with $k'(S) = k(S)$ for any 2-cycle S, and also with $k'(C) \in \mathbb{Z}$ for any 2-chain C. Choose a contractible, locally finite open cover $\{U_i\}$ of M, and choose $x_i \in U_i$. Let $\{h_i\}$ be a (smooth) partition of unity subordinate to $\{U_i\}$. Define $a_{ijk} = k'((X_i, X_j, X_k))$ whenever $U_i \cap U_j \cap U_k \neq 0$. Then define $f_{ij} = \sum_k a_{ijk} h_k$, $\alpha_i = \sum_j df_{ij} h_j$. It can then be checked that these functions and 1-forms fulfill the conditions in (2).

Fin ally, 2 ==> 3 follows by an application of Stokes theorem.

We present these alternative descriptions here because definition 3 is most often quoted in literature, but the characterization 2 is essential in the proof of Weil's theorem.

BIBLIOGRAPHY

1. S. Kobayashi and K. Nomizu: Foundations of Differential Geometry, Volumes I and II. (Interscience Publishers, 1969).

2. D. Simms and N. Woodhouse: Lectures on Geometric Quantization (Lecture Notes in Physice #53, Springer-Verlag, 1976).

3. B. Kostant: Lecture Notes in Mathematics #170 (Springer, Berlin, 1970).

4. D. K. Sen: Fields and/or Particles (Ryerson press, Toronto, 1968).

5. W. Dreschler: Fibre Bundle Techniques in Gauge Theories (Lecture Notes in Physics #67, Springer Verlag, 1977).

6. B. Schmidt: A New Definition of Singular Points in General Relativity. (General Relativity and Gravitation, Vol. 1, #3 (1971) pp 269-280).

7. E. Ihrig: An Exact Determination of the Gravitational Potentials g_{ij} in terms of the Gravitational Fields R^{ℓ}_{ijk} (Journal of Mathematical Physics, Vol. 16, #1 (1975)).

8. R. Woodside: Momenta Without Symmetries (To be published).

9. E. Ihrig: Lie Algebra as a Unifying Concept in Modern Physics (Canadian Math. Bull.)

10. G. Rosensteel and E. Ihrig: Geometric Quantization of the CM(3) Model (To be published).

11. E. Ihrig: The Uniqueness of g_{ij} in terms of R^{ℓ}_{ijk} (International Journal of Theoretical Physics, Vol. 14, #1 (1975)).