TRIPLES -- A UNIVERSAL ALGEBRAIC APPROACH

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by

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Preface

It has become clear in the past five or six years that the categories of algebras arising from triples are a natural generalization of the equational classes of Birkhoff-type universal algebra. From their discovery about ten years ago (by Godement), triples have been studied mainly with a view to their homological applications. In fact, as far as I know, the only paper to appear up to this time which concerns itself with the detailed universal algebra of triple algebras is the doctoral dissertation of E. Manes.

This thesis is an attempt to give a reasonably complete introduction to the study of triple algebras, together with an analysis of algebras which arise from triples in the category of sets. The material in Chapters 0, 1, 2 represents a collection of results from various papers appearing in the literature. The material in §3.1 and §3.2 can be found in a more general setting in Manes' dissertation, but the approach used here (through congruences as in classical universal algebra) is, as far as I know, original. The tripleableness theorem in §3.2 has been around for some time but the only proofs appearing in the literature are somewhat incomplete, and are readable only for avid category-theorists. Therefore I have included a complete, relatively clear proof most of which was learned from J. Beck, and the rest of which follows from the material on congruences in §3.1. Finally a short appendix has been included to give an idea of the notion of rank of a triple and its importance with respect to classical universal algebra. For reasons of space and time it was not feasible to make a detailed study of this within the text of the thesis.

Finally I should like to emphasize that in Chapter 3, the techniques of classical universal algebra are employed wherever possible. In certain places (for example in the construction of colimits) more elegant purely category theoretical arguments are ignored, but as pointed out above, this paper is meant to be in the spirit of universal algebra rather than pure category theory.

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Chapter 0 BASIC THEORY OF ADJOINT FUNCTORS

§0.1 Preliminary Comments

0.1.1 The reader is expected to have a grasp of the calculus of functors and natural transformations ("Godements cinq règles", etc.). Beyond this only elementary category theory is required, for example the basic theory of types of morphisms, limits and colimits. In particular none of the theory of additive categories will be needed. The above knowledge is in probably its most accessible form in Mitchell [10], and we will refer the reader to this book for any unexplained concepts.

0.1.2 We will now lay down some notational conventions with the inevitable warning that such rules are made to be broken.

(1) Upper case script letters for categories: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{K}$ etc.

(2) The letters K, L, X, Y, Z for objects of categories.

(3) Lower case Roman letters for morphisms: f, g, h etc.

(4) Upper case Roman letters for functors (except for K, L, X, Y, Z): F, G, T etc.

(5) Lower case Greek letters for natural transformations: η, μ etc. If we have $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ functors, then by saying $F \xrightarrow{\phi} G$ is a natural transformation, we mean ϕ is a family of maps $(\phi X)_{X \in \mathcal{A}}$, $FX \xrightarrow{\phi X} GX$ satisfying the usual requirement.

(6) For the value of the functor F at the object X (resp. morphism f) we write FX (resp. Ff) rather than F(X). Similarly we indicate composition of arrows by concatenation except where this is confusing, in which case we use the composition

"circle" e.g. we write fh but $\phi T o \mu$ rather than $\phi T \mu$.

(7) The set of all morphisms $X \xrightarrow{f} Y$ in $\mathcal{A}$ is denoted $\mathcal{A}(X,Y)$.

(8) We use the symbols $\prod$ and $\coprod$ for product and coproduct, respectively.

§0.2 Adjoint Functors

0.2.1 The notion of an adjoint situation is fundamental to the study of triples. Indeed the latter may be regarded as the investigation of the algebraic content of adjoint pairs. For this reason we will give a cursory review of the relevant concepts.

Given a pair of functors $\mathcal{A} \underset{F}{\overset{U}{\leftrightarrows}} \mathcal{B}$ we say that F is left adjoint to U (U is right adjoint to F) iff there is a natural equivalence $(F-,-) \xrightarrow{\varkappa} (-,U-)$. We note that the two functors involved here have domain $\mathcal{B} \times \mathcal{A}$ and range $\mathcal{S}$ (category of sets) and that they are contravariant in the first variable and covariant in the second. We write $\varkappa : F \dashv U \ (\mathcal{A},\mathcal{B})$ to describe this situation (following Eilenberg-Moore [3]).

An entirely equivalent formulation of adjointness arises from considering (in the above situation) pairs of natural transformation (η,ε) where $id_{\mathcal{B}} \xrightarrow{\eta} UF$ and $FU \xrightarrow{\varepsilon} id_{\mathcal{A}}$, satisfying $\varepsilon F \circ F\eta = id_F$, $U\varepsilon \circ \eta U = id_U$. Such pairs are in one-one correspondence with natural equivalences described above. Specifically, given $\varkappa$ define (η,ε) by $\eta X = \varkappa(X,FX)(id_{FX})$ and $\varepsilon L = \varkappa(UL,L)^{-1}\ (id_{UL})$. On the other hand, given (η,ε) define $(FX,L) \xrightarrow{\varkappa(X,L)} (X,UL)$ by $\varkappa(X,F)\ (f) = Uf \circ \eta X$. One then checks that the inverse to $\varkappa(X,L)$ is given by $\varkappa(X,L)^{-1}\ (g) = \varepsilon L \circ Fg$. It is then an easy exercise in the calculus of natural transformations to check that the above passages effect a bijection of the proper type. We write $(F, U, \eta, \varepsilon, \mathcal{A}, \mathcal{B})$ or $F \dashv U\ (\eta, \varepsilon)$ when making reference to an adjointness given in this second way. η and ε are called the front and back adjunctions, respectively, of the adjointness. We observe that this second formulation lends itself more readily to computations than the first, and our subsequent considerations will generally take this line of approach.

0.2.2 The notion of conjugate transformations will also be needed. Given adjoint situations $\varkappa: F \dashv U\ (\mathcal{A}, \mathcal{B})$ and $\varkappa': F' \dashv U'\ (\mathcal{A}, \mathcal{B})$ and natural transformations $F' \xrightarrow{\phi} F$, $U \xrightarrow{\psi} U'$ we say ϕ is conjugate to ψ, written $\phi \dashv \psi$, if the following diagram commutes:

$$\begin{tikzcd}
(F-,-) \arrow[r, "\varkappa"] \arrow[d, "{(\phi,-)}"'] & (-,U-) \arrow[d, "{(-,\psi)}"] \\
(F'-,-) \arrow[r, "\varkappa'"] & (-,U'-)
\end{tikzcd}$$

This means, of course that for any $X \in \mathcal{B}$ and $L \in \mathcal{A}$ and for any $FX \xrightarrow{f} L$ in $\mathcal{A}$, $\psi L \circ \varkappa(X,L)(f) = \varkappa'(X,L)(f \circ \phi X)$

If we have $\varkappa \sim (\eta, \varepsilon)$ and $\varkappa' \sim (\eta', \varepsilon')$ it is straightforward to verify that $\phi \dashv \psi$ iff the following two diagrams commute:

$$\begin{tikzcd}
\mathrm{id} \arrow[r, "\eta"] \arrow[d, "\eta'"'] & UF \arrow[d, "\psi F"] \\
U'F' \arrow[r, "U'\phi"'] & U'F
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F'U \arrow[r, "\phi U"] \arrow[d, "F'\psi"'] & FU \arrow[d, "\varepsilon"] \\
F'U' \arrow[r, "\varepsilon'"'] & \mathrm{id}
\end{tikzcd}$$

Every transformation $F' \xrightarrow{\phi} F$ or $U \xrightarrow{\psi} U'$ has a unique conjugate.

Given ϕ, put $\psi = U \xrightarrow{\eta' U} U'F'U \xrightarrow{U'\phi U} U'FU \xrightarrow{U'\varepsilon} U'$

given ψ, put $\phi = F' \xrightarrow{F'\eta} F'UF \xrightarrow{F'\psi F} F'U'F \xrightarrow{\varepsilon' F} F$.

For example, suppose we are given ϕ and wish to construct ψ. For such a ψ we must have $\psi = U'\varepsilon' \circ \eta' U' \circ \psi = U'\varepsilon' \circ U'F'\psi \circ \eta' U = U'\varepsilon \circ U'\phi U \circ \eta' U$ (since $\phi \dashv \psi$). This shows uniqueness of ψ, given ϕ. Next we show $\phi \dashv \psi$ with ψ given by the above formula. $\psi F \circ \eta = U'\varepsilon F \circ U'\phi UF \circ \eta' UF \circ \eta = U'\varepsilon F \circ U'\phi UF \circ U'F'\eta \circ \eta' = U'\varepsilon F \circ U'F\eta \circ U'\phi \circ \eta = U'\phi \circ \eta$. Furthermore, $\varepsilon' \circ F'\psi = \varepsilon' \circ F'U'\varepsilon \circ F'U'\phi U \circ F'\eta' U = \varepsilon \circ \varepsilon' FU \circ F'U\phi U \circ F'\eta' U = \varepsilon \circ \phi U \circ \varepsilon' F'U \circ F'\eta' U = \varepsilon \circ \phi U$. The proof that ϕ exists and is uniquely determined by ψ is entirely analogous.

The reader can easily check that the above correspondences are functorial. That is, if $F'' \xrightarrow{\phi'} F' \xrightarrow{\phi} F$ and $\phi \dashv \psi$, $\phi' \dashv \psi'$ then $\phi\phi' \dashv \psi'\psi$. Moreover the formulas for computing conjugates show $id_F \dashv id_U$. From this it follows that for $\phi \dashv \psi$, ϕ is an equivalence iff ψ is an equivalence. For example if ψ is given and has inverse ψ^{-1} then the conjugate of ψ is $\phi = \varepsilon' F \circ F'\psi F \circ F'\eta$ and the conjugate of ψ^{-1} is $\overline{\phi} = \varepsilon F' \circ F\psi^{-1} F' \circ F\eta'$. Since functorial correspondences preserve isomorphisms we must have $\overline{\phi} = \phi^{-1}$.

0.2.3 Another fact about adjoint situations is the fundamental existence theorem of P. Freyd. This theorem (usually called the "Adjoint Functor Theorem") gives a partial answer to the question of when a given functor has a left adjoint. We recall a basic property of adjoint functors if $F \dashv U$ then F preserves all existing colimits and U preserves all existing limits.

<u>Definition</u>: Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor. Then a <u>set</u> of objects $(Ai)_{i \in I}$ of is said to be a solution set for $K \in \mathcal{K}$ relative to U if for each $A \in \mathcal{A}$ and map $K \xrightarrow{f} UA$ there is an index $i_0 \in I$ and maps $K \xrightarrow{j} UA_{i_0}$ and $A_{i_0} \xrightarrow{\alpha} A$ with $f = U\alpha \circ j$.

<u>Theorem</u> (P. Freyd): Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor with $\mathcal{A}$ complete and locally small. Then U has a left adjoint iff it preserves limits and every object of $\mathcal{K}$ has a solution set relative to U.

Proof: See Mitchell [10], pages 124-126.

0.2.4 Finally we say a subcategory $\mathcal{K}' \subseteq \mathcal{K}$ is a reflective subcategory iff the inclusion functor $\mathcal{K}' \xrightarrow{J} \mathcal{K}$ has a left adjoint R, say $R \dashv J$ (ρ, ε). In this situation ε is pointwise epi and $\mathcal{K}'$ is a full subcategory iff ε is natural equivalence. (In a general adjoint situation $F \dashv U$ (η, ε) U is faithful iff ε is pointwise epi and U is full and faithful iff ε is an equivalence).

Equivalently (modulo a suitably powerful axiom of choice) $\mathcal{K}'$ is a reflective subcategory of $\mathcal{K}$ iff for each $X \in \mathcal{K}$ we have an object $RX \in \mathcal{K}'$ and a "reflection" map $X \xrightarrow{\rho X} RX$ such that for any $X \xrightarrow{f} Y$ with $Y \in \mathcal{K}'$, there is a unique $RX \xrightarrow{\check{f}} Y$ in $\mathcal{K}'$ with $f = \check{f} \circ \rho X$.

If a diagram in $\mathcal{K}'$ has a colimit in $\mathcal{K}$, then following this colimit by its reflection in $\mathcal{K}'$ gives the colimit of the diagram in $\mathcal{K}'$.

Chapter I TRIPLES

§1.1 Triples and their Algebras

1.1.1 A triple in a category $\mathcal{K}$ is a 3-list $\mathbf{T} = (T,\eta,\mu)$ where $\mathcal{K} \xrightarrow{T} \mathcal{K}$ is a functor and η,μ are natural transformations $id \xrightarrow{\eta} T$, $TT \xrightarrow{\mu} T$ satisfying $\mu o T\eta = \mu o \eta T = id_T$ and $\mu o T\mu = \mu o \mu T$. That is, the following diagrams commute:

$$\begin{tikzcd} T \arrow[r, shift left, "\eta T"] \arrow[r, shift right, "T\eta"'] \arrow[dr, equal] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd}$$

η,μ are called the unit and multiplication respectively of the triple T and the above axioms are referred to as the unitary and associative axioms. Cotriples are defined dually.

The history of and motivation for the notion of triple can be found in Beck [1], or in the introduction of [11].

1.1.2 Of principal interest to us is the category $\mathcal{K}^{\mathbf{T}}$ of algebras arising from a triple $\mathbf{T}$ in $\mathcal{K}$. The objects of $\mathcal{K}^{\mathbf{T}}$ are pairs (X,ϑ) where $X \in \mathcal{K}$ and ϑ: $TX \longrightarrow X$ is a map in $\mathcal{K}$ satisfying the following unitary and associative laws:

$$\begin{tikzcd} X \arrow[r, "\eta^X"] \arrow[dr, equal] & TX \arrow[d, "\vartheta"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} TTX \arrow[r, "\mu^X"] \arrow[d, "T\vartheta"'] & TX \arrow[d, "\vartheta"] \\ TX \arrow[r, "\vartheta"'] & X \end{tikzcd}$$

ϑ is called the operation of the algebra (X,ϑ).

For objects (X,ϑ) and (Y,ξ) in $\mathcal{K}^{\mathbb{T}}$ the hom set $\mathcal{K}^{\mathbb{T}}((X,\vartheta),(Y,\xi))$ consists of all maps $f \in \mathcal{K}(X,Y)$ satisfying the following homomorphism rule:

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \vartheta\downarrow & & \downarrow\xi \\ X & \xrightarrow{f} & Y \end{array} \qquad \text{commutes.}$$

In general, when considering such a map f as a morphism in $\mathcal{K}^{\mathbb{T}}$ (a "T-homomorphism") we will write it as [f] for emphasis. It is easy to verify that the composition law [g][f] = [gf] makes $\mathcal{K}^{\mathbb{T}}$ into a category, which we call the category of T-algebras. The following proposition more or less justifies the terminology "algebra".

1.1.3 Proposition: In the above described situation the following facts obtain:

(1) For each $X \in \mathcal{K}$, $(TX,\mu X)$ is a T-algebra. Furthermore this assignment together with $X \xrightarrow{f} Y \rightsquigarrow (TX,\mu X) \xrightarrow{[Tf]} (TY,\mu Y)$ defines a functor $F^{\mathbb{T}}: \mathcal{K} \longrightarrow \mathcal{K}^{\mathbb{T}}$.

(2) For each $X \in \mathcal{K}$, $F^{\mathbb{T}}X$ is the T-algebra freely generated by X in the following sense:

(a) if we have a monomorphism $(Y,\xi) \xrightarrow{[f]} (TX,\mu X)$ and a factorization

$$\begin{array}{ccc} Y & \xrightarrow{f} & TX \\ g\uparrow & \nearrow_{\eta X} & \\ X & & \end{array}$$

then f (and hence [f]) is an isomorphism. That is, if we think of $X \xrightarrow{\eta X} TX$ as the embedding of generators then this fact may be interpreted as "X generates $F^{\mathbb{T}}X$".

(b) For any $\mathbf{T}$-algebra (Y,ξ) and any map $X \xrightarrow{f} Y$ there is a unique $\mathbf{T}$-homomorphism $(TX,\mu X) \xrightarrow{[\tilde{f}]} (Y,\xi)$ with $\tilde{f}\eta X = f$. This is the familiar extension property of free algebras in the classical universal algebra sense.

(3) We have a built-in functor $\mathcal{K}^{\mathbf{T}} \xrightarrow{U^{\mathbf{T}}} \mathcal{K}$ defined by $U^{\mathbf{T}}(X,\vartheta) = X$ and $U^{\mathbf{T}}[f] = f$. Moreover $F^{\mathbf{T}} \dashv U^{\mathbf{T}}$ via $(\eta_{\mathbf{T}},\varepsilon_{\mathbf{T}})$ where $\eta_{\mathbf{T}} = \eta$ and $\varepsilon_{\mathbf{T}}(X,\vartheta) = [\vartheta]\colon (TX,\mu X) \longrightarrow (X,\vartheta)$.

Proof: (1) That $(TX,\mu X)$ is a $\mathbf{T}$-algebra is expressed by the original triple axioms. That Tf is a homomorphism is simply the naturality of μ. That $F^{\mathbf{T}}$ is functorial is a consequence of the functorialness of T.

(2) (a) We will show f is split epi. In the presence of the fact that f is mono, this ensures f is an isomorphism. By consulting the axiom which is required for a map to be a homomorphism, one can immediately see $U^{\mathbf{T}}$ reflects isomorphisms, so $[f]$ will be an isomorphism. In our situation f has a right inverse, namely $\xi \circ Tg$, for $f \circ \xi \circ Tg = \mu X \circ Tf \circ Tg = \mu X \circ T\eta X = id_{TX}$.

(b) We claim $\tilde{f} = \xi \circ Tf$ is the unique extension. $\tilde{f}$ is a homomorphism since $\tilde{f} \circ \mu X = \xi \circ Tf \circ \mu X = \xi \circ \mu Y \circ TTf = \xi \circ T\xi \circ TTf = \xi \circ T\tilde{f}$. Also $\tilde{f} \circ \eta X = \xi \circ Tf \circ \eta X = \xi \circ \eta Y \circ f = f$. $\tilde{f}$ is the only possible extension since given $(TX,\mu X) \xrightarrow{[g]} (Y,\xi)$ with $g \circ \eta X = f$, we must have $g = g \circ \mu X \circ T\eta X = \xi \circ Tg \circ T\eta X = \xi \circ Tf = \tilde{f}$.

(3) That ϑ is a $\mathbf{T}$-homomorphism from $(TX,\mu X)$ to (X,ϑ) is a consequence of the associative law for algebra structures. For each $X \in \mathcal{K}$, $(\varepsilon_{\mathbf{T}}F^{\mathbf{T}} \circ F^{\mathbf{T}}\eta_{\mathbf{T}})X = \varepsilon_{\mathbf{T}}(TX,\mu X) \circ [T\eta X] = [\mu X] \circ [T\eta X] = [id_{TX}] = id_{F^{\mathbf{T}}X}$. For each $(X,\vartheta) \in \mathcal{K}^{\mathbf{T}}$, $(U^{\mathbf{T}}\varepsilon_{\mathbf{T}} \circ \eta_{\mathbf{T}}U^{\mathbf{T}})(X,\vartheta) = \vartheta \circ \eta X = id_X = id_{U^{\mathbf{T}}(X,\vartheta)}$. This verifies the adjointness assertion.

§1.2 Triples as Invariants of Adjointness

1.2.1 Consider an adjointness $(F,U,\eta,\varepsilon,\mathcal{A},\mathcal{K})$. Then this adjointness generates a triple in $\mathcal{K}$, namely $\mathbf{T} = (T,\eta,\mu)$ where $T = UF$, η is the front adjunction of the adjointness, and $\mu = U\,\varepsilon\,F$. That this is indeed a triple is a consequence of the two properties of the front and back adjunction as given in 0.2.1.

In the above situation we have a canonical functor $\mathcal{A} \xrightarrow{\Phi} \mathcal{K}^{\mathbf{T}}$ given by $\Phi X = (UX, U\varepsilon X)$, $\Phi f = [Uf]$. $U\varepsilon X$ is a $\mathbf{T}$-operation since $U\varepsilon X \circ \eta UX = (U\varepsilon \bullet \eta U)X = id_{UX}$ and $U\varepsilon X \circ T(U\varepsilon X) = U\varepsilon X \circ UFU\varepsilon X = U(\varepsilon \circ FU\varepsilon)X = U(\varepsilon \circ \varepsilon FU)X = U\varepsilon X \circ U\varepsilon FUX = U\varepsilon X \circ \mu UX$. Furthermore if $X \xrightarrow{f} Y$ in $\mathcal{A}$ then $(UX, U\varepsilon X) \xrightarrow{[Uf]} (UY, U\varepsilon Y)$ since $Uf \circ U\varepsilon X = U(f \circ \varepsilon X) = U(\varepsilon Y \circ FUf) = U\varepsilon Y \circ UFUf = U\varepsilon Y \circ TUf$.

It is also clear that we have $\Phi \circ F = F^{\mathbf{T}}$ and $U^{\mathbf{T}} \circ \Phi = U$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "\Phi"] \arrow[dr, shift left, "U"] & & \mathcal{K}^{\mathbf{T}} \arrow[dl, shift left, "U^{\mathbf{T}}"] \\
& \mathcal{K} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^{\mathbf{T}}"] &
\end{tikzcd}
$$

Since we will find categories $\mathcal{K}^{\mathbf{T}}$ particularly appealing objects, it is natural to inquire about the properties of the functor Φ above. Before we can do this conveniently we require some additional machinery.

1.2.2 <u>Definitions</u>: For a category $\mathcal{K}$ we define a new category $AD(\mathcal{K})$ as follows: The objects of $AD(\mathcal{K})$ will be adjoint situations $(F, U, \eta, \varepsilon, \mathcal{A}, \mathcal{K})$. The morphisms between two such will be the functors C commuting with the right adjoints in the following way:

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "C"] \arrow[dr, "U"'] & & \mathcal{A}' \arrow[dl, "U'"] \\
& \mathcal{K} &
\end{tikzcd}
$$

It is clear that functorial composition makes AD($\mathcal{K}$) into a (large) category.

We also define a category Trip ($\mathcal{K}$) where objects are triples in $\mathcal{K}$.

For $\mathbf{T}_1 = (T_1, \eta_1, \mu_1)$ and $\mathbf{T}_2 = (T_2, \eta_2, \mu_2)$ a morphism $\mathbf{T}_1 \xrightarrow{\phi} \mathbf{T}_2$ will be a natural transformation $T_1 \xrightarrow{\phi} T_2$ satisfying the following laws:

unitary law:
$$\begin{tikzcd} T_1 \arrow[rr, "\phi"] & & T_2 \\ & id \arrow[ul, "\eta_1"] \arrow[ur, "\eta_2"'] & \end{tikzcd}$$

multiplicative law:
$$\begin{tikzcd} T_1^2 \arrow[r, "\phi\phi"] \arrow[d, "\mu_1"'] & T_1^2 \arrow[d, "\mu_1"] \\ T_1 \arrow[r, "\phi"'] & T_2 \end{tikzcd}$$
commutes, where
$$\phi\phi = T_2\phi \circ \phi T_1 = \phi T_2 \circ T_1\phi$$

Composing two such as natural transformations again gives a triple map (quite trivial with the observation that for $T_1 \xrightarrow{\phi} T_2 \xrightarrow{\psi} T_3$, one has $(\psi \circ \phi)(\psi \circ \phi) = \psi\psi \circ \phi\phi)$. This is, indeed, the composition we use for Trip ($\mathcal{K}$).

1.2.3 We have seen (1.2.1) how objects of AD($\mathcal{K}$) give rise to objects of Trip ($\mathcal{K}$). We will now extend this to a contravariant functor AD($\mathcal{K}$) $\xrightarrow{R}$ Trip ($\mathcal{K}$), the "structure" functor of functorial semantics (Lawvere [5], Linton [6], [7]).

We accomplish this in the following way: Suppose we have $(F_i, U_i, \eta_i, \varepsilon_i, \mathcal{A}_i, \mathcal{K})$ $i = 1,2$ objects of AD($\mathcal{K}$). We will refer to them by their right adjoints U_i. Now suppose we have an AD($\mathcal{K}$) morphism $U_1 \xrightarrow{C} U_2$. Say U_i generates $\mathbf{T}_i = (T_i, \eta_i, \mu_i)$. Define a system of maps $U_2F_2K \xrightarrow{\gamma K} U_1F_1K$, for each $K \in \mathcal{K}$, by $\gamma K = U_2\varepsilon_2 CF_1 K \circ U_2F_2\eta_1 K$ (we have

$$U_2F_2K \xrightarrow{U_2F_2\eta_1 K} (U_2F_2U_1F_1K = U_2F_2U_2CF_1K) \xrightarrow{U_2\varepsilon_2 CF_1 K} U_1F_1K)$$

Proposition: The system of maps (γK) defined above constitutes a triple map $\mathbf{T}_2 \xrightarrow{\gamma} \mathbf{T}_1$. If we put $R(C) = \gamma$ then R becomes a contravariant functor.

Proof: (a) Naturality of γ: This is immediate since $\gamma = U_2\varepsilon_2 CF_1 \circ U_2F_2\eta_1$ the composition of two natural transformations.

(b) Unitary law: $\gamma \circ \eta_2 = U_2\varepsilon_2 CF_1 \circ U_2F_2\eta_1 \circ \eta_2$

$= U_2\varepsilon_2 CF_1 \circ \eta_2 U_1F_1 \circ \eta_1 = (U_2\varepsilon_2 \circ \eta_2 U_2)CF_1 \circ \eta_1 = (id_{U_2})CF_1 \circ \eta_1 = \eta_1$

(c) Multiplicative law: Consult the diagram on the following page.

(d) Functorialness of R: If we have a third adjointness U_3 and a map $U_2 \xrightarrow{D} U_3$ with corresponding $\delta = U_3\varepsilon_3 DF_2 \circ U_3F_3\eta_2$ we must show $U_2\varepsilon_2 CF_1 \circ U_2F_2\eta_1 \circ U_3\varepsilon_3 DF_2 \circ U_3F_3\eta_2 = U_3\varepsilon_3 DCF_1 \circ U_3F_3\eta_1$.

This is a straightforward but complex naturality calculation:

$$U_2\varepsilon_2 CF_1 \circ U_2F_2\eta_1 \circ U_3\varepsilon_3 DF_2 \circ U_3F_3\eta_2 = U_2\varepsilon_2 CF_1 \circ U_3\varepsilon_3 DF_2U_1F_1 \circ U_3F_3U_3DF_2\eta_1 \circ U_3F_3\eta_2$$

$$= U_3\varepsilon_3 DCF_1 \circ U_3F_3U_3D\varepsilon_2 CF_1 \circ U_3F_3\eta_2U_1F_1 \circ U_3F_3\eta_1$$

$$= U_3\varepsilon_3 DCF_1 \circ U_3F_3(U_2\varepsilon_2 \circ \eta_2U_2)CF_1 \circ U_3F_3\eta_1 = U_3\varepsilon_3 DCF_1 \circ U_3F_3\eta_1$$

1.2.4 The structure functor R above roughly speaking assigns to an adjointness its algebraic structure. We will now define a semantics functor $\mathrm{Trip}(\mathcal{K}) \xrightarrow{M} \mathrm{AD}(\mathcal{K})$ which interprets a triple as its category of algebras.

Diagram for Proposition 1.2.3

$$
\begin{tikzcd}[column sep=large, row sep=huge]
U_2F_2U_2F_2 \arrow[r, "U_2F_2U_1F_1\eta_1"] \arrow[rr, bend left=30, "T_2\gamma"] \arrow[d, "U_2\varepsilon_2F_2"'] \arrow[dr, phantom, "\mathrm{I}"] & U_2F_2U_2F_2U_1F_1 \arrow[r, "U_2F_2U_2\varepsilon_2CF_1"] \arrow[dr, "U_2\varepsilon_2F_2U_1F_1"] & U_2F_2U_1F_1 \arrow[r, "U_2F_2\eta_1U_1F_1"] \arrow[rr, bend left=30, "\gamma T_1"] \arrow[drr, "U_2\varepsilon_2CF_1"] \arrow[d, phantom, "\mathrm{II}"] & U_2F_2U_1F_1U_1F_1 \arrow[r, "U_2\varepsilon_2CF_1U_1F_1"] \arrow[l, bend left=40, "U_2F_2U_1\varepsilon_1F_1", "\mathrm{IV}"'] \arrow[dr, phantom, "\mathrm{III}"] & U_1F_1U_1F_1 \arrow[d, "U_1\varepsilon_1F_1\ (= U_2C\varepsilon_1F_1)"] \\
U_2F_2 \arrow[rr, "U_2F_2\eta_1"'] & & U_2F_2U_1F_1 \arrow[rr, "U_2\varepsilon_2CF_1"'] & & U_1F_1
\end{tikzcd}
$$

I commutes by naturality of ε_2

II commutes by naturality of ε_2

III commutes since $\varepsilon_2C \circ F_2U_1\varepsilon_1 = \varepsilon_2C \circ F_2U_2C\varepsilon_1 = C\varepsilon_1 \circ \varepsilon_2CF_1U_1$ (by naturality)

Since in IV we have $U_2F_2U_1\varepsilon_1F_1 \circ U_2F_2\eta_1U_1F_1 = \mathrm{id}$, the large "triangle" on the right commutes, hence the outer diagram commutes

Specifically, for a triple $\mathbf{T}$ put $M\mathbf{T} = (F^{\mathbf{T}}, U^{\mathbf{T}}, \eta_{\mathbf{T}}, \varepsilon_{\mathbf{T}}, \mathcal{K}^{\mathbf{T}}, \mathcal{K})$. For a triple map $\mathbf{T}_1 \xrightarrow{\phi} \mathbf{T}_2$ in Trip $(\mathcal{K})$ put $\mathcal{K}^{\mathbf{T}_2} \xrightarrow{M\phi} \mathcal{K}^{\mathbf{T}_1}$ by $M\phi\ (X,\vartheta) = (X, \vartheta \circ \phi X)$, $M\phi\ [f] = [f]$.

Proposition A: M is a contravariant functor.

Proof: If $(X,\vartheta) \in \mathcal{K}^{\mathbf{T}_2}$ then $\vartheta \circ \phi X$ is a $\mathbf{T}_1$-structure on X, for

$$\vartheta \circ \phi X \circ \eta_1 = \vartheta \circ \eta_2 = id_X$$

$$\vartheta \circ \phi X \circ T_1(\vartheta \circ \phi X) = \vartheta \circ T_2\vartheta \circ T_2\phi X \circ \phi T_1 X = \vartheta \circ \mu_2 X \circ T_2 \phi X \circ \phi T_1 X$$

$$= \vartheta \circ \phi X \circ \mu_1 X.$$

Also if $(X,\vartheta) \xrightarrow{[f]} (Y,\xi)$ in $\mathcal{K}^{\mathbf{T}_2}$ then f is a homomorphism from $M\phi(X,\vartheta)$ to $M\phi(Y,\xi)$, since $f \circ \vartheta \circ \phi X = \xi \circ T_2 f \circ \phi X = \xi \circ \phi Y \circ T_1 f$. Thus we have $M\phi$ is a morphism in AD$(\mathcal{K})$. It is quite clear that M is functorial (contravariantly of course).

$M\phi$ is called semantical interpretation of ϕ. Because of the "Hom" nature of $M\phi$ on structures we write sometimes $M\phi = \mathcal{K}^{\phi}$.

Before exploring the connection between M and R we need to extend our concept of adjointness to contravariant functors. Suppose $\mathcal{A} \underset{M}{\overset{R}{\rightleftarrows}} \mathcal{B}$ are contravariant functors. Then we say $R \dashv M\ (\phi, \varepsilon)$ for $id_{\mathcal{A}} \xrightarrow{\phi} MR$, $id_{\mathcal{B}} \xrightarrow{\varepsilon} RM$ iff $\phi M \circ M\varepsilon = id_M$, $\varepsilon R \circ R\phi = id_R$. We call ϕ the front adjunction and ε the back adjunction.

The motivation for the above definition is simply that according to this we have $R \dashv M\ (\eta,\varepsilon)$ $\quad R^* \dashv M_*\ (\eta,\varepsilon^*)$ (covariant functors) using the notation of Mitchell [10] page 50. That is, if R and M are considered as having codomain and domain $\mathcal{B}^{op}$ respectively, R is left adjoint to M as in §0.2. It is clear that the remarks of §0.2 can (and will) be interpreted anew in the situation of contravariant adjoint functors.

Proposition B: With respect to the situation $AD(\mathcal{K}) \underset{M}{\overset{R}{\rightleftarrows}} Trip(\mathcal{K})$ the following facts obtain:

(1) $RM = id_{Trip(\mathcal{K})}$

(2) The maps Φ defined in 1.2.1 constitute a natural transformation $id_{AD(\mathcal{K})} \xrightarrow{\tilde{\Phi}} MR$

(3) $\tilde{\Phi}$ and $id_{Trip(\mathcal{K})} \xrightarrow{id} RM$ are front and back adjunctions for an adjointness of contravariant functors $R \longmapsto M$.

Proof: (1) Take a triple $\mathbf{T} = (T,\eta,\mu)$ in $\mathcal{K}$. Then $RM\mathbf{T}$ is the triple generated by $(F^{\mathbf{T}},U^{\mathbf{T}},\eta_{\mathbf{T}},\varepsilon_{\mathbf{T}},\mathcal{K}^{\mathbf{T}},\mathcal{K})$. But this is $(U^{\mathbf{T}}F^{\mathbf{T}},\eta^{\mathbf{T}},U^{\mathbf{T}}\varepsilon_{\mathbf{T}}F^{\mathbf{T}})$.

We have $U^{\mathbf{T}}F^{\mathbf{T}} = T$

$$\eta_{\mathbf{T}} = \eta$$

$$U^{\mathbf{T}}\varepsilon_{\mathbf{T}}F^{T}(K) = U^{\mathbf{T}}\varepsilon_{\mathbf{T}}(TK,\mu K) = U^{\mathbf{T}}[\mu K] = \mu K$$

all of these facts follow from Proposition 1.1.3.

Given a map $\mathbf{T}_1 \xrightarrow{\phi} \mathbf{T}_2$ in $Trip(\mathcal{K})$, put $\gamma = RM\phi$. Then from 0.4.3,

$$\begin{aligned}
\gamma K &= (UT_1\varepsilon_{\mathbf{T}_1}M\phi F^{T_2} \circ U^{T_1}F^{T_1}\eta_{\mathbf{T}_2})K \\
&= \mu_2 K \circ \phi T_2 K \circ T_1\eta_{\mathbf{T}_2}K \\
&= \mu_2 K \circ T_2\eta_{\mathbf{T}_2}K \circ \phi K \\
&= \phi K
\end{aligned}$$

This shows RM is the identity on morphisms as well as objects.

(2) Let us have $(F_i,U_i,\eta_i,\varepsilon_i,\mathcal{A}_i,\mathcal{K})$ $i = 1,2$ objects of $AD(\mathcal{K})$ and $\mathcal{A}_1 \xrightarrow{C} \mathcal{A}_2$ a map between them in $AD(\mathcal{K})$. To show $\tilde{\Phi}$ is natural we must establish the commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "\Phi_1"] \arrow[d, "C"'] & \mathcal{K}^{\mathbb{T}_1} \arrow[d, "MR(C)"] \\ \mathcal{A}_2 \arrow[r, "\Phi_2"] & \mathcal{K}^{\mathbb{T}_2} \end{tikzcd}$$

where $\mathbb{T}_i = (T_i, \eta_i, \mu_i) = R(F_i, U_i, \eta_i, \varepsilon_i, \mathcal{A}_i, \mathcal{K})$

Let $X \in \mathcal{A}_1$. Then $\Phi_2 C\, X = (U_2 C\, X,\ U_2\varepsilon_2 CX) = (U_1 X, U_2\varepsilon_2 CX)$. On the other hand $MR(C) \circ \Phi_1 X = (U_1 X,\ U_1\varepsilon_1 X \circ R(C)U_1 X)$. Now these two algebras are equal since

$$\begin{aligned} U_1\varepsilon_1 X \circ R(C)X &= U_1\varepsilon_1 X \circ U_2\varepsilon_2 CF_1U_1X \circ U_2F_2\eta_1U_1X \\ &= (U_2\varepsilon_2 C \circ U_2F_2U_1\varepsilon_1 \circ U_2F_2\eta_1U_1)X \\ &= U_2\varepsilon_2 CX \end{aligned}$$

For a morphism f in $\mathcal{A}_1$, $MR(C) \circ \Phi_1\, f = MR(C)[U_1 f] = [U_1 f]$

$$\Phi_2 \circ C\, f \quad = [U_2 Cf] = [U_1 f].$$

(3) Since the back adjunction is the identity the laws expressing the adjointness take the simple form

$$\tilde{\Phi}M = id_M\,,\ R\tilde{\Phi} = id_R\,.$$

To establish the first take any triple $\mathbb{T}$ in $\mathcal{K}$. Then

$$\tilde{\Phi}M\mathbb{T}: M\mathbb{T} \longrightarrow MRM\mathbb{T} = M\mathbb{T}$$

i.e.

$$\begin{tikzcd} \mathcal{K}^{\mathbb{T}} \arrow[rr, "\Phi"] \arrow[dr, "U^{\mathbb{T}}"'] & & \mathcal{K}^{\mathbb{T}} \arrow[dl, "U^{\mathbb{T}}"] \\ & \mathcal{K} & \end{tikzcd} \qquad \Phi = \tilde{\Phi}M\mathbb{T}$$

$$\Phi(X,\vartheta) = (U^{\mathbf{T}}(X,\vartheta),\ U^{\mathbf{T}}\varepsilon_{\mathbf{T}}(X,\vartheta))$$

$$= (X,\vartheta)$$

$$\Phi[f] = [U^{\mathbf{T}}[f]] = [f].$$

Hence $\Phi = id_{\mathcal{K}^{\mathbf{T}}}$ qed.

To establish the second identity, take $(F,U,\eta,\varepsilon,\mathcal{A},\mathcal{K})$ in $AD(\mathcal{K})$, generating a triple $\mathbf{T} = (T,\eta,\mu)$.

Then we have

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\Phi"] \arrow[dr, "U"'] & & \mathcal{K}^{\mathbf{T}} \arrow[dl, "U^{\mathbf{T}}"] \\ & \mathcal{K} & \end{tikzcd}$$

and $\mathbf{T} \xrightarrow{R(\Phi)} \mathbf{T}$

Putting $R(\phi) = \gamma$ we have, for each $K \in \mathcal{K}$, $TK \xrightarrow{\gamma K} TK$ defined by

$$\begin{aligned}\gamma K &= U^{\mathbf{T}}\varepsilon_{\mathbf{T}}\Phi FK \circ U^{\mathbf{T}}F^{\mathbf{T}}\eta K \\ &= U^{\mathbf{T}}\varepsilon_{\mathbf{T}}(UFK, U\varepsilon FK) \circ T\eta K \\ &= U\varepsilon FK \circ T\eta K \\ &= \mu K \circ T\eta K \\ &= id_{TK}.\end{aligned}$$

This completes the proof of the proposition. ∎

<u>Corollary</u>: The following facts hold:

(1) M is a dual isomorphism onto the full subcategory generated by the images of objects of $Trip(\mathcal{K})$ under M.

(2) Suppose $(F,U,\eta,\varepsilon,\mathcal{A},\mathcal{K}) \in AD(\mathcal{K})$, generating $\mathbf{T} = (T,\eta,\mu)$. Let $\mathbf{H}$ be any triple and $\mathcal{A} \xrightarrow{C} \mathcal{K}^{\mathbf{H}}$ any map in $AD(\mathcal{K})$. Then there is a unique functor D, a map in $AD(\mathcal{K})$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Phi"] \arrow[dr, "C"'] & \mathcal{K}^{T} \arrow[d, "D"] \\ & \mathcal{K}^{H} \end{tikzcd}$$

Namely take D = MR(C).

Proof: (1) M is 1-1 on objects since it has a left inverse. M is full and faithful since the back ajunction is an isomorphism.

(2) This merely expresses the fact that Alg($\mathcal{K}$) is a reflective subcategory of AD($\mathcal{K}$) with reflections Φ (because of the adjointness and the fact that M may be viewed as the inclusion of Alg($\mathcal{K}$)).

Two remarks are in order. First, the semantics functor R is full. Proposition B above shows that R is full when restricted to Alg($\mathcal{K}$) but the more general assertion seems to require a highly technical proof, which will be omitted here. The second remark is that what we have done here is by no means the whole story on structure-semantics. The study of these concepts was initiated by Lawvere [5] and generalized by Linton [7]. Their approach was through algebraic "theories" (see Linton [6]) which are equivalent to triples but whose theory of structure-semantics lends itself more readily to generalization. We will ignore this notion of algebraic "theories" since we find the machinery of triples more efficient than (if not quite as intuitive as) that of "theories". For a full (but sketchy) account of the relationship between "theories" and triples see

Linton [7]. For the somewhat messy details of the crux of this relationship see Davis [2].

§1.3 Tripleability

1.3.1 The functors Φ discussed in the above sections are called (for now obvious reasons) the semantical comparison functors.

Given a functor $\mathcal{A} \xrightarrow{U} \mathcal{K}$ with a left adjoint and semantical comparison functor Φ we say that U is

(1) of descent type if Φ is full and faithful,

(2) tripleable or of effective descent type (E.D.T.) if Φ is an equivalence

(3) precisely tripleable if Φ is an isomorphism.

The fact that U enjoys any of the above properties is independent of the choice of left adjoint and ajunctions. Moreover if $U' \cong U$ (naturally equiv.) then U' has (1) or (2) iff U has (1) or (2) respectively. This is an immediate corollary of the following proposition. (Observe that if U is precisely tripleable then U' is E.D.T. but not necessarily precisely tripleable).

Proposition: Let $(F,U,\eta,\varepsilon,\mathcal{A},\mathcal{K})$ and $(F',U',\eta',\varepsilon',\mathcal{A},\mathcal{K})$ be objects of $AD(\mathcal{K})$ with $F' \xrightarrow{\phi} F$, $U \xrightarrow{\psi} U'$ conjugate natural equivalences. If we denote the triples generated by the adjointnesses by $\mathbf{T}$, $\mathbf{T}'$ respectively, with semantical comparison functors Φ, Φ' then $UF \xrightarrow{\psi\phi^{-1}} U'F'$ is a triple isomorphism and the following diagram commutes up to natural equivalence.

$$\begin{tikzcd}
 & \mathcal{K}^{T} \\
\mathcal{A} \arrow[ur, "\Phi"] \arrow[dr, "\Phi'"'] & \\
 & \mathcal{K}^{T'} \arrow[uu, "M(\psi\phi^{-1})"']
\end{tikzcd}$$

If $U = U'$ and $\psi = id_U$ then the natural equivalence is the identity transformation and the diagram commutes absolutely.

Proof: $\psi\phi^{-1}$ satisfies the unitary law: $\psi\phi^{-1} \circ \eta = U'\phi^{-1} \circ \psi F \circ \eta$ $= U'\phi^{-1} \circ U'\phi \circ \eta'$ (since $\phi \dashv \psi$) $= \eta'$.

Multiplicative law: consult the diagram on page 20.

Commutativity of the diagram: On objects $M(\psi\phi^{-1}) \circ \Phi' X =$ $(U'X, U'\varepsilon'X \circ \psi\phi^{-1}U'X)$. Define a natural equivalence $\Phi \xrightarrow{\alpha} M(\psi\phi^{-1}) \circ \Phi'$ by $(UX, U\varepsilon X) \xrightarrow{\alpha X} (U'X, U'\varepsilon X \circ \psi\phi^{-1}U'X)$, $\alpha X = [\psi X]$. First we must check that each αX is a homomorphism:

$$\begin{aligned}
U'\varepsilon'X \circ \psi\phi^{-1}U'X \circ T(\alpha X) &= (U'\varepsilon' \circ \psi\phi^{-1}U' \circ UF\psi)X \\
&= (U'\varepsilon' \circ \psi F'U' \circ U\phi^{-1}U' \circ UF\psi)X \\
&= (\psi \circ U\varepsilon' \circ U\phi^{-1}U' \circ UF\psi)X \\
&= (\psi \circ U(\varepsilon' \circ F'\psi \circ \phi^{-1}U))X \\
&= (\psi \circ U(\varepsilon \circ \phi U \circ \phi^{-1}U))X \\
&= \psi X \circ U\varepsilon X
\end{aligned}$$

also each αX is an isomorphism (it being clear that the functors U^{T} preserve and reflect isomorphisms).

Thus to complete the proof we need only check that α is natural: Suppose we have $X \xrightarrow{f} Y$ in $\mathcal{A}$. Then we must establish commutativity of

Illustration that $\psi\phi^{-1}$ is multiplicative

$$
\begin{tikzcd}
UFUF \arrow[rr, "UFU\phi^{-1}"] \arrow[rrrr, bend left, "UF\psi\phi^{-1}"] \arrow[ddd, "U\varepsilon F"'] \arrow[ddrr, phantom, "\mathrm{VI}"] & & UFUF' \arrow[rr, "UF\psi F'"] \arrow[drr, "U\phi^{-1}UF'"] \arrow[ddrr, equal] \arrow[dr, phantom, "\mathrm{V}"] & & UFU'F' \arrow[rr, "U\phi^{-1}U'F'"] \arrow[rrrr, bend left, "\psi\phi^{-1}U'F'"] \arrow[dr, phantom, "\mathrm{I}"] & & UF'U'F' \arrow[rr, "\psi F'U'F'"] \arrow[dr, phantom, "\mathrm{II}"] & & U'F'U'F' \arrow[ddd, "U'\varepsilon' F'"] \\
 & & & & \bullet \arrow[urr, "UF'\psi F'"] \arrow[rrr, "\psi F'UF'"] \arrow[d, "U\phi UF'"] \arrow[drrr, phantom, "\mathrm{IV}"] & & & \bullet \arrow[ur, "U'F'\psi F'"] \arrow[d, "U'\phi UF'"] \arrow[dr, phantom, "\mathrm{III}"] & \\
 & & & & \bullet \arrow[rrr, "\psi FUF'"] \arrow[d, "U\varepsilon F'"] \arrow[drrr, phantom, "\mathrm{VII}"] & & & \bullet \arrow[dr, "U'\varepsilon F'"] & \\
UF \arrow[rrrr, "U\phi^{-1}"'] & & & & UF' \arrow[rrrr, "\psi F'"'] & & & & U'F'
\end{tikzcd}
$$

The breakdown of the diagram should be attacked in the order in which the squares are numbered.

All squares but III commute by naturality: III commutes by one of the laws expressing $\phi \dashv \psi$ (see 0.2.2).

$$\begin{tikzcd}
(UX, U\varepsilon X) \arrow[r, "\alpha X"] \arrow[d, "{[Uf]}"'] & (U'X, U'\varepsilon'X \circ \psi\phi^{-1}U'X) \arrow[d, "{[U'f]}"] \\
(UY, U\varepsilon Y) \arrow[r, "\alpha Y"'] & (U'Y, U'\varepsilon'Y \circ \psi\phi^{-1}U'Y)
\end{tikzcd}$$

But this is just naturality of ψ.

(Note that for $U = U'$ and $\psi = id_U$ we have $\alpha X = id_{(UX, U\varepsilon X)}$; the fact that $U'\varepsilon'X \circ \psi\phi^{-1}U'X = U\varepsilon X$ is a consequence of the proof that αX is a homomorphism.) ∎

1.3.2 Before proceeding further with general concepts we will give examples of tripleable functors.

<u>Example 1</u>: The "canonical" example is that of the underlying set functor of a primitive class of universal algebras of some type (with possibly infinitary operations -- see Slominsky [12]). Such functors are in fact precisely tripleable. This will follow immediately from a tripleableness theorem we will prove later. However, assuming an elementary knowledge of this kind of algebra, as in Slominsky [12], we can indicate how the inverse of the semantical comparison functor is constructed: Namely let (X, ϑ) be an element of S where $\mathbf{T}$ is generated by the standard adjointness of an equational class of type $\Delta = (a_\xi)_{\xi<\alpha}$. Let F_ξ be the corresponding operation on FX, and define operations f_ξ on X by $f_\xi((x_\eta)_{\eta<\alpha_\xi}) = \vartheta(F_\xi((x_\eta)_{\eta<\alpha_\xi}))$. This gives an algebra $(X, (f_\xi)_{\xi<\alpha})$ of the proper type which one then shows, using the properties of ϑ, is a homomorphic image of $(FX, (F_\xi)_{\xi<\alpha})$ and hence is in the original primitive class. Finally one proves $\Phi(X, (f_\xi)_{\xi<\alpha}) = (X, \vartheta)$. The extension of this construction to an inverse for Φ is then straightforward to carry out.

<u>Example 2</u>: The following are several functors $\mathcal{A} \xrightarrow{U} \mathcal{S}$ which are tripleable, but which do <u>not</u> fall into the class of examples given above.

(a) $\mathcal{A}$ is the category of compact T_2 topological spaces, U the usual underlying set functor.

(b) $\mathcal{A}$ is the category of complete atomic boolean algebras and complete boolean homomorphisms, U is the usual underlying set functor.

(c) $\mathcal{A}$ the category of compact left (or right) unitary R modules with R a discrete ring with unit, and morphisms are continuous module homomorphisms. U takes such a module to the underlying set of its abelian group.

(d) $\mathcal{A}$ the category of compact universal algebras from some specified equational class (e.g. compact groups, lattices, etc.) and U the obvious underlying set functor. Note: I know of no result which states that such functors are never of the type considered in Example 1, but it seems to be true in many non-trivial cases (i.e. when $\mathcal{A}$ contains algebras with more than 1 element). I know of no counterexample to such a conjecture.

Remark: We will indicate in the appendix how one can determine whether or not a tripleable functor falls into the class of examples given by Example 1 above. (a) of Example 2 will be established later. Proofs of (b) and (c) are indicated in the appendix. (d) is true in a more general setting, namely that of "compact $\mathbb{T}$-algebras" for triples $\mathbb{T}$ in sets. To give an adequate discussion of this (which we will not attempt) one requires the notion of tensor product of triples and related machinery. We refer the reader to Manes [9] for details.

Example 3: $\mathcal{A} \xrightarrow{U} \mathcal{T}$ is tripleable where $\mathcal{A}$ is the category whose objects are all pairs $(A, \mathfrak{D})$ where A is an algebra of some primitive class $\mathcal{V}$ and $\mathfrak{D}$ is a topology on the underlying set of A which is compatible with the algebra operations. Morphisms in $\mathcal{A}$ are continuous algebra homomorphisms. U is the obvious functor to $\mathcal{T}$, the category of topological spaces and continuous maps. $\mathcal{A}$ is called the category of topological $\mathcal{V}$-algebras. One observes that the free topological $\mathcal{V}$-algebra on a space $(X, \mathfrak{D})$ is obtained by forming the free $\mathcal{V}$-algebra FX and endowing this with the finest topology on FX which is compatible with the operations of FX and whose restriction to X is coarser than $\mathfrak{D}$. Once this is noted, one can mimic the proof sketched for Example 1, inserting the word "continuous" in the appropriate places.

Example 4: Inclusion of full reflective subcategories are tripleable.
Proof: Suppose we have $\mathcal{K}' \xrightarrow{J} \mathcal{K}$ such an inclusion with left adjoint (which we know exists) R. By comments made in 0.2.4 we can see there is no loss of generality in assuming the back adjunction is the identity. Let ρ be the front adjunction:

$$\begin{tikzcd}
\mathcal{K}' \arrow[rr, "\Phi"] \arrow[dr, shift left, "J"] & & \mathcal{K}^{\mathbf{T}} \arrow[dl, shift left, "U^{\mathbf{T}}"] \\
& \mathcal{K} \arrow[ul, shift left, "R"] \arrow[ur, shift left, "F^{\mathbf{T}}"] &
\end{tikzcd}$$

For $X \in \mathcal{K}'$ we have $\Phi X = (X, id_X)$.
For $f\colon X \longrightarrow Y$ in $\mathcal{K}'$, $\Phi f = f$.
It is clear Φ is full, faithful and 1-1 on objects.
Now take $(Y, \vartheta) \in \mathcal{K}^{\mathbf{T}}$. Then $RY \xrightarrow{\vartheta} Y$ in $\mathcal{K}$.

Then $\vartheta \circ \rho Y = \mathrm{id}_Y$ and we have
$$\begin{tikzcd} Y \arrow[r, "\rho Y"] \arrow[dr, "\rho Y"'] & RY \arrow[d, "\rho Y \circ \vartheta"] \\ & RY \end{tikzcd}$$
commutes.

By uniqueness of reflection-induced maps, we must have
$\rho Y \circ \vartheta = \mathrm{id}_{RY} \Longrightarrow \rho Y = \vartheta^{-1}$. Then we have the commutative diagram

$$\begin{tikzcd} RY \arrow[r, "R(\rho Y)=\mathrm{id}_{RY}"] \arrow[d, "\vartheta"'] & RY \arrow[d, "\mathrm{id}_{RY}"] \\ Y \arrow[r, "\rho Y"'] & RY \end{tikzcd}$$

, which shows $(Y,\vartheta) \xrightarrow{[\rho Y]} (RY, \mathrm{id}_{RY})$ is an isomorphism, i.e. (Y,ϑ) is isomorphic in $\mathcal{K}^{\mathbf{T}}$ to ΦRY.

This shows Φ is representative and hence an equivalence. ∎

Evidently a replete (closed under isomorphic images) full reflective subcategory inclusion is precisely tripleable.

<u>Remark</u>: Recalling 1.2.4, Proposition B, we note the appealing fact that for any category $\mathcal{K}$, the dual of Trip $(\mathcal{K})$ is itself tripleable over AD$(\mathcal{K})$.

This set of examples is intended only to give an idea of what tripleableness might entail. In general it is safe to say that most functors which are in some sense forgetting an algebraic type of structure (loosely speaking) are tripleable. A notable counterexample to this rough generalization is the underlying set functor for complete Boolean algebras. This functor does not even have a left adjoint (Gaifman [4]) and hence has no chance to be tripleable. To express the algebraicity of this category one needs to generalize even further to the notion of equationally definable class as exposed in Linton [6].

We will close off this chapter by mentioning a counterexample to two obvious conjectures. Namely suppose one has functors $\mathcal{A} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{B} \underset{F'}{\overset{U'}{\rightleftarrows}} \mathcal{C}$ with $F \dashv U$, $F' \dashv U'$. Then it is well-known that one has $FF' \dashv U'U$. Now three statements can be made:

(1) In general U and U' tripleable $\not\Rightarrow$ $U'U$ tripleable.

(2) $U'U$ tripleable $\not\Rightarrow$ U' tripleable.

(3) If U' is faithful, $U'U$ tripleable $\Longrightarrow$ U tripleable.

Re (1): $\mathcal{A}$ = Torsion free abelian groups

$\mathcal{B}$ = Abelian groups

$\mathcal{C}$ = Sets

U = obvious inclusion

U' = obvious underlying set functor

U is tripleable since it is the inclusion of a reflective subcategory.

U' is tripleable since it is the underlying set functor of an equational class of universal algebras. However $U'U$ is not tripleable. We will not give a proof here, but we remark that it can be shown that if an additive category is tripleable over sets via some functor it must be abelian. But the category of torsion free groups is additive and not abelian.

Re (2): $\mathcal{A}$ = compact T_2 spaces

$\mathcal{B}$ = all topological spaces

$\mathcal{C}$ = sets

U = inclusion

U' = usual forgetful functor.

U is tripleable since it is the inclusion of a reflective subcategory. U'U is tripleable as we have remarked above, and will prove later. However U' is not tripleable since the triple it generates (with left adjoint forming the discrete space on a set) is the identity triple (all components are the relevant identities; such a triple exists in any category) and the semantical comparison functor is essentially U', evidently not an equivalence of categories.

Re (3): This will be a simple corollary of Beck's tripleableness theorem, given in 2.2.3.

Chapter 2 BASIC CONSTRUCTION PRINCIPLES IN $\mathcal{K}^{\mathbf{T}}$

§2.1 Limits in $\mathcal{K}^{\mathbf{T}}$

2.1.1 Limits and certain kinds of colimits can be computed in $\mathcal{K}^{\mathbf{T}}$ from knowledge of construction of the same kind of thing in $\mathcal{K}$. We will now elucidate this, working throughout this chapter with a fixed but otherwise arbitrary category $\mathcal{K}$ and triple $\mathbf{T}$ in $\mathcal{K}$.

Definition: Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor and $\Delta \xrightarrow{D} \mathcal{A}$ a diagram (i.e. Δ a small category, D a functor). Suppose limit UD exists. Then we say U creates the limit of D if for any model $(K \xrightarrow{\pi_\delta} UD\delta \mid \delta \in \Delta)$ of limit UD, with projections π_δ, there is a family $(A \xrightarrow{\psi_\delta} D\delta \mid \delta \in \Delta)$ with the following two properties:

(1) $U\psi_\delta = \pi_\delta$ all $\delta \in \Delta$ (entailing $UA = K$) and the family $(\psi\delta \mid \delta \in \Delta)$ is the only one in $\mathcal{A}$ with this property.

(2) The family $(\psi\delta \mid \delta \in \Delta)$ is model for limit D.

In the obvious way we also define "U creates the colimit of D". We say U creates isomorphisms if U creates limits of diagrams over the one point scheme . i.e. for any $A \in \mathcal{A}$ and isomorphism $X \xrightarrow{f} UA$ in $\mathcal{K}$ there is a map $A' \xrightarrow{g} A$ in $\mathcal{A}$ such that $Ug = f$, g is unique with respect to this property and also g is an isomorphism.

A weaker notion than creates is also useful. Namely in the above situation we say that U effectively constructs the limit of D if for any model $(K \xrightarrow{\pi\delta} UD\delta \mid \delta \in \Delta)$ of limit UD there is a family $(A \xrightarrow{\pi_\delta} D\delta \mid \delta \in \Delta)$ with the following two properties:

(1) The family $(U\psi\delta \mid \delta \in \Delta)$ is a limit for UD.

(2) The family $(\psi\delta \mid \delta \in \Delta)$ is a limit for D.

Similarly "U effectively constructs colimits" may be formulated. It is quite clear that this is a weaker notion than that of creation of constructions.

Examples:

(1) The underlying set functor for topological spaces effectively constructs limits and colimits but does not create them (the uniqueness condition fails, for example there are in general many topologies on the cartesian product of the underlying sets of a family of spaces which make the projections continuous).

(2) The underlying set functor for compact T_2 spaces creates limits (e.g. there is only one compact T_2 topology on the cartesian product of the underlying sets of a family of compact T_2 spaces making the projections continuous).

Proposition: $\mathcal{K}^{\mathbf{T}} \xrightarrow{U^{\mathbf{T}}} \mathcal{K}$ creates limits of arbitrary diagrams.

Proof: First of all we treat the case of an empty diagram. It is actually quite obvious that $U^{\mathbf{T}}$ creates a singleton in $\mathcal{K}^{\mathbf{T}}$ from $1 \in \mathcal{K}$. For there is exactly one map $T1 \xrightarrow{\sigma} 1$ which must be an algebra map making $(1,\sigma)$ into a singleton in $\mathcal{K}^{\mathbf{T}}$, simply because every square with 1 in its terminus must commute.

Let $\Delta \xrightarrow{D} \mathcal{K}^{\mathbf{T}}$ be a non-empty diagram. Put $D(\delta) = (X_\delta, \vartheta_\delta)$. Then suppose $(X \xrightarrow{\pi_\delta} X_\delta \mid \delta \in \Delta)$ is a limit for $U^{\mathbf{T}}D$. Let $\alpha\colon \delta \longrightarrow \delta'$ be any map in Δ say $D\alpha = [f_\alpha]$. Then we have

$$\begin{tikzcd} TX \arrow[r, "T\pi_\delta"] \arrow[dr, "T\pi_{\delta'}"'] & TX_\delta \arrow[r, "\vartheta_\delta"] \arrow[d, "Tf_\alpha"] & X_\delta \arrow[d, "f_\alpha"] \\ & TX_{\delta'} \arrow[r, "\vartheta_{\delta'}"'] & X_{\delta'} \end{tikzcd}$$

Since this diagram commutes, $(TX \xrightarrow{\vartheta_\delta \circ T\pi_\delta} X_\delta \mid \delta \in \Delta)$ is a compatible family for $U^T D$, hence there is a unique $\vartheta : TX \longrightarrow X$ with $\pi_\delta \circ \vartheta = \vartheta_\delta \circ T\pi_\delta$ all δ. Hence to complete the proof we must show ϑ is an algebra structure and that $(X, \vartheta) \xrightarrow{[\pi_\delta]} (X_\delta, \vartheta_\delta)$ is a limit for D. (The defining equation for ϑ say that the maps π_δ will automatically be homomorphisms).

We simply make use of the fact that the projections π_δ are "jointly monomorphic" in $\mathcal{K}$ i.e. $\pi_\delta f = \pi_\delta g$ all $\delta \implies f = g$.

$$\pi_\delta \circ \vartheta \circ \mu X = \vartheta_\delta \circ T\pi_\delta \circ \mu X = \vartheta_\delta \circ \mu X_\delta \circ TT\pi_\delta = \vartheta_\delta \circ T\vartheta_\delta \circ TT\pi_\delta$$

$$= \vartheta_\delta \circ T(\pi_\delta \circ \vartheta) = \pi_\delta \circ \vartheta \circ T\vartheta \text{ for all } \delta \in \Delta \implies \vartheta \circ \mu X = \vartheta \circ T\vartheta$$

Also $\pi_\delta \circ \vartheta \circ \eta X = \vartheta_\delta \circ T\pi_\delta \circ \eta X = \vartheta_\delta \circ \eta X_\delta \circ \pi_\delta = \pi_\delta$ for all δ

$$\implies \vartheta \circ \eta X = id_X.$$

Hence ϑ is an algebra structure.

Finally suppose $(Y, \xi) \xrightarrow{[\psi_\delta]} (X_\delta, \vartheta_\delta)$ is a compatible family. Then there is a unique $Y \xrightarrow{f} X$ in $\mathcal{K}$ with $\pi_\delta f = \psi_\delta$ all $\delta \in \Delta$. Claim f is a homomorphism. But $\pi_\delta \circ f \circ \xi = \psi_\delta \circ \xi = \vartheta_\delta \circ T\psi_\delta = \vartheta_\delta \circ T\pi_\delta \circ Tf$ $= \pi_\delta \circ \vartheta \circ Tf$. Hence $f \circ \xi = \vartheta \circ Tf$ and f is a homomorphism ∎

§2.2 Contractible Coequalizers

2.2.1 We now define a concept (originally introduced by Jon Beck) which is fundamental to the study of triples.

Definition: Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be maps in a category $\mathcal{K}$. Then the pair (f,g) is said to be contractible, with contraction d if $d\colon Y \longrightarrow X$ and $fd = id_Y$, $gdf = gdg$. For $\mathcal{A} \xrightarrow{U} \mathcal{K}$ and $A \underset{k}{\overset{h}{\rightrightarrows}} B$, (h,k) is said to be U-contractible if (Uh, Uk) is contractible in $\mathcal{K}$.

It is not simply contractible pairs which concern us, but coequalizers of such pairs.

Proposition: For $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$ in a category $\mathcal{K}$, the following are equivalent:

(1) (f,g) is contractible and $h = \operatorname{coequ}(f,g)$.

(2) $hf = hg$ and $\exists$ $Z \xrightarrow{d_0} Y \xrightarrow{d_1} X$ with $hd_0 = id_Z$, $fd_1 = id_Y$, $gd_1 = d_0h$.

Proof: (1) $\Longrightarrow$ (2) Suppose (f,g) is contractible with contraction d_1. Then since $gd_1f = gd_1g$ $\exists$ unique $d_0\colon Z \longrightarrow Y$ with $d_0h = gd_1$ (since h is coequ(f,g))

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[dr, "gd_1"'] & Z \arrow[d, dashed, "d_0"] \\ & & Y \end{tikzcd}$$

Now $hd_0h = dgd_1 = hfd_1 = h \Longrightarrow hd_0 = id_Z$ (since h is epi).

(2) $\Longrightarrow$ (1) Claim d_1 is a contraction for (f,g). $fd_1 = id_Y$ by hypothesis and $gd_1g = d_0hg = d_0hf = gd_1f$. Claim $h = \operatorname{coequ}(f,g)$. Suppose we have $Y \xrightarrow{\ell} W$ with $\ell f = \ell g$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[dr, "\ell"'] & Z \arrow[d, "\ell d_0"] \\ & & W \end{tikzcd}$$

Then $\ell d_0 h = \ell g d_1 = \ell f d_1 = \ell$.

Moreover, ℓd_0 is unique with this property since $\alpha h = \ell \Rightarrow \alpha h d_0 = \ell d_0$ $\Rightarrow \alpha = \ell d_0$. ∎

2.2.2 This last characterization shows that any functor preserves coequalizers of contractible pairs. This is important because of the following result.

<u>Proposition</u>: Let $\Delta \xrightarrow{D} \mathcal{K}^{\mathbf{T}}$ be a diagram such that colimit $U^{\mathbf{T}}D$ exists and such that T and T^2 preserve this colimit. Then $U^{\mathbf{T}}$ creates the colimit of D.

Proof: First of all suppose Δ is empty. Then the hypothesis is that $\mathcal{K}$ has a cosingleton $\emptyset$ and $T\emptyset$, $T^2\emptyset$ are also cosingletons. Then there is exactly one map $T\emptyset \xrightarrow{\gamma} \emptyset$ and

$$\begin{tikzcd} \emptyset \arrow[r, "\eta\emptyset"] \arrow[dr, equal] & T\emptyset \arrow[d, "\gamma"] \\ & \emptyset \end{tikzcd} \qquad \begin{tikzcd} T^2\emptyset \arrow[r, "\mu\emptyset"] \arrow[d, "T\gamma"'] & T\emptyset \arrow[d, "\gamma"] \\ T\emptyset \arrow[r, "\gamma"'] & \emptyset \end{tikzcd}$$

commute since all objects in the diagram are cosingletons. Then $(\emptyset,\gamma) \in \mathcal{K}^{\mathbf{T}}$ and is a cosingleton in $\mathcal{K}^{\mathbf{T}}$ because $\forall$ (X,ϑ) the unique map $\emptyset \xrightarrow{f} X$ in $\mathcal{K}$ is forced to be a homomorphism since $T\emptyset$ is a cosingleton. Next suppose $\Delta \neq \emptyset$ and put $D\delta = (X_\delta, \vartheta_\delta)$ for $\delta \in \Delta$. Let $(X_\delta \xrightarrow{u_\delta} X)_{\delta \in \Delta}$ be a model for colimit $U^{\mathbf{T}}D$.

We require $TX \xrightarrow{\vartheta} X$ with $\vartheta . Tu_\delta = u_\delta . \vartheta_\delta$, all $\delta \in \Delta$. Since $(Tu_\delta)_{\delta \in \Delta}$ are the injections of a colimit, ϑ is uniquely determined if it exists. But it does exist in view of the fact that $(u_\delta . \vartheta_\delta)$ is clearly a compatible family for the diagram $TU^{\mathbf{T}}D$:

$$\begin{tikzcd}
TX_\delta \arrow[rr, "Tu_\delta"] \arrow[dr, "Tf_\alpha"] \arrow[dd, "\vartheta_\delta"'] & & TX \arrow[dd, dashed, "\vartheta"] \\
& TX_{\delta'} \arrow[ur, "Tu_{\delta}'"'] \arrow[dd, "\vartheta_{\delta'}"] & \\
X_\delta \arrow[rr, "u_\delta"] \arrow[dr, "f_\alpha"'] & & X \\
& X_{\delta'} \arrow[ur, "u_{\delta'}"'] &
\end{tikzcd}$$

That ϑ is a $\mathbf{T}$-operation follows from the following two diagrams:

$$\begin{tikzcd}
X_\delta \arrow[r, "\eta X_\delta"] \arrow[d, "u_\delta"'] & TX_\delta \arrow[l, dashed, bend left, "\vartheta_\delta"] \arrow[d, "Tu_\delta"] \\
X \arrow[r, "\eta X"'] & TX \arrow[l, dashed, bend right, "\vartheta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2X_\delta \arrow[r, "T^2u_\delta"] \arrow[d, shift right, "\mu X_\delta"'] \arrow[d, shift left, dashed, "T\vartheta_\delta"] & T^2X \arrow[d, shift left, "\mu X"] \arrow[d, shift right, dashed, "T\vartheta"'] \\
TX_\delta \arrow[r, "Tu_\delta"] \arrow[d, "\vartheta_\delta"'] & TX \arrow[d, "\vartheta"] \\
X_\delta \arrow[r, "u_\delta"] & X
\end{tikzcd}$$

$$\begin{aligned}
\vartheta \circ \eta X \circ u_\delta &= \vartheta \circ Tu_\delta \circ \eta X_\delta \\
&= u_\delta \circ \vartheta_\delta \circ \eta X_\delta \\
&= u_\delta \quad \text{all } \delta \in \Delta \\
\Longrightarrow \vartheta \circ \eta X &= id_X
\end{aligned}$$

$$\begin{aligned}
\vartheta \circ T\vartheta \circ T^2u_\delta &= \vartheta \circ T(\vartheta \circ Tu_\delta) \\
&= \vartheta \circ Tu_\delta \circ T\vartheta_\delta \\
&= u_\delta \circ \vartheta_\delta \circ T\vartheta_\delta \\
&= u_\delta \circ \vartheta_\delta \circ \mu X_\delta \\
&= \vartheta \circ Tu_\delta \circ \mu X_\delta \\
&= \vartheta \circ \mu X \circ T^2u_\delta \quad \text{all } \delta \in \Delta \\
\Longrightarrow \vartheta \circ T\vartheta &= \vartheta \circ \mu X
\end{aligned}$$

Now we have a family $((X_\delta, \vartheta_\delta) \xrightarrow{[u_\delta]} (X, \vartheta))_{\delta \in \Delta}$ in $\mathcal{K}^{\mathbf{T}}$; all that remains is to show that if $((X_\delta, \vartheta_\delta) \xrightarrow{[v_\delta]} (Y, \xi))$ is a compatible family for D, the map $X \xrightarrow{f} Y$ induced in $\mathcal{K}$ is actually a homomorphism: Consider the following:

$$\begin{tikzcd}
TX_\delta \arrow[r, "Tu_\delta"] \arrow[rr, bend left, "T\nu_\delta"] \arrow[d, "\vartheta_\delta"'] & TX \arrow[r, "Tf"] \arrow[d, "\vartheta"'] \arrow[dr, phantom, "?"] & TY \arrow[d, "\xi"] \\
X_\delta \arrow[r, "u_\delta"'] \arrow[rr, bend right, "\nu_\delta"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

we have $\xi \circ Tf \circ Tu_\delta = \xi \circ T\nu_\delta = \nu_\delta \circ \vartheta_\delta = f \circ u_\delta \circ \vartheta_\delta = f \circ \vartheta \circ Tu_\delta$ since this holds for all $\delta \in \Delta$, we have $\xi Tf = f\vartheta$. ∎

<u>Corollary 1</u>: $U^{\mathbf{T}}$ creates coequalizers of $U^{\mathbf{T}}$-contractible pairs.

<u>Corollary 2</u>: In $\mathcal{K}^{\mathbf{T}}$ consider $(T^2X, \mu TX) \overset{[\mu X]}{\underset{[T\vartheta]}{\rightrightarrows}} (TX, \mu X) \xrightarrow{[\vartheta]} (X, \vartheta)$

Then $[\vartheta]$ is the coequalizer created by $U^{\mathbf{T}}$ from the contractible coequalizer diagram

$$\begin{tikzcd}
T^2X \arrow[r, shift left, "\mu X"] \arrow[r, shift right, "T\vartheta"'] & TX \arrow[r, "\vartheta"] \arrow[l, bend left=60, "\eta TX"] & X \arrow[l, bend left=60, "\eta X"]
\end{tikzcd}$$

in $\mathcal{K}$.

Proof: (1) The hypothesis says that for we have a diagram D on the scheme $\cdot \rightrightarrows \cdot$ and that $U^{\mathbf{T}}D$ is contractible and has a coequalizer. By the characterization of Proposition A, <u>any</u> functor (in particular T and T^2) will preserve this coequalizer and hence the proposition above applies.

(2) We have only to check that the described diagram in $\mathcal{K}$ is in fact a contractible coequalizer. But $\mu X \circ \eta TX = id_{TX}$ and $\vartheta \circ \eta X = id_X$. Further, $T\vartheta \circ \eta TX = \eta X \circ \vartheta$ by naturality of η. ∎

2.2.3 The motivation for the study of contractible coequalizers is embodied in Corollary 2 above. For suppose we have an adjointness $(F,U,\eta,\varepsilon,\mathcal{A},\mathcal{K})$ with generated triple $\mathbf{T}$ and semantical comparison functor Φ. Further suppose we are attempting to construct an inverse to Φ. Take an arbitrary $(X,\vartheta)\in\mathcal{K}^{\mathbf{T}}$ and consider the following:

In $\mathcal{A}$ — In $\mathcal{K}^{\mathbf{T}}$

$$\begin{tikzcd} FUFX \arrow[r, shift left, "\varepsilon FX"] \arrow[r, shift right, "F\vartheta"'] & FX \arrow[r, dashed, "\Phi^{-1}[\vartheta]"] & \Phi^{-1}(X,\vartheta) \arrow[r, Rightarrow, "\Phi"] & (UFUFX, U\varepsilon FUX) \arrow[r, shift left, "{[U\varepsilon X]}"] \arrow[r, shift right, "{[UF\vartheta]}"'] & (UFX, U\varepsilon X) \arrow[r, "{[\vartheta]}"] & (X,\vartheta) \end{tikzcd}$$

U — $U^{\mathbf{T}}$

In $\mathcal{K}$

$$\begin{tikzcd} UFUFX \arrow[r, shift left, "U\varepsilon FX"] \arrow[r, shift right, "UF\vartheta"'] & UFX \arrow[r, "\vartheta"] \arrow[l, bend left=60, "\eta UFX"] & X \arrow[l, bend left=60, "\eta X"] \end{tikzcd}$$

Note: $(\varepsilon FX, F\vartheta)$ is U-contractible.

If Φ^{-1} exists we will have $\Phi^{-1}[\vartheta] = \text{coequ}(\varepsilon FX, F\vartheta)$. Hence it makes sense to insist that U create coequalizers in this kind of situation. We will now state two theorems regarding tripleability (due essentially to Jon Beck). Since we will not require them in this paper, no proof will be given, although the crux of the method of proof has been outlined immediately above.

Theorem: Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor having a left adjoint. The following are equivalent.

(1) U is precisely tripleable.

(2) U is tripleable and creates isomorphisms.

(3) U creates coequalizers of U-contractible pairs.

Theorem: Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor having a left adjoint. The following are equivalent.

(1) U is tripleable.

(2) U reflects isomorphisms and effectively constructs coequalizers of U-contractible pairs.

(3) $\mathcal{A}$ has and U preserves and reflects coequalizers of those pairs (f,g) for which (Uf,Ug) is contractible and has a coequalizer.

For detailed proofs see Manes [9] or Davis [2]. For a breakdown of the effect of various operating parts of the conditions see Linton [8].

§2.3 Subalgebras and Epimorphisms in $\mathcal{K}^{T}$

2.3.1 If $(X,\vartheta) \in \mathcal{K}^{T}$, a subalgebra of (X,ϑ) is any monomorphism $(X_1,\vartheta_1) \xrightarrow{[i]} (X,\vartheta)$. Observe that [i] is a monomorphism iff i is a monomorphism in $\mathcal{K}$ [U^T reflects monos since it is faithful, and preserves them since, being a right adjoint, it preserves limits. Recall that in any category, μ is mono $\Longleftrightarrow$ $\begin{array}{ccc} \cdot & \xrightarrow{id} & \cdot \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \mu} \\ \cdot & \xrightarrow{\mu} & \cdot \end{array}$ is a pullback].

Proposition A: Let $(X,\vartheta) \in \mathcal{K}^{T}$ and $X_1 \xrightarrow{i} X$ a monomorphism in $\mathcal{K}$. Then X_1 has a T-structure making i a homomorphism (i.e. X_1 "is" a subalgebra) iff we have $TX_1 \xrightarrow{\vartheta_1} X_1$ with $i\vartheta_1 = \vartheta Ti$. Moreover ϑ_1 is uniquely determined.

Proof: Clearly ϑ_1 is uniquely determined since i is a monomorphism. Suppose we do have $i\vartheta_1 = \vartheta Ti$. Then $i\vartheta_1\eta X_1 = \vartheta Ti\eta X_1 = \vartheta\eta X i = i$

$$\Longrightarrow \vartheta_1\eta X_1 = id_{X_1}. \text{ Similarly } i\vartheta_1\mu X_1 = \vartheta Ti\mu X_1 = \vartheta\mu X T^2 i$$

$$= \vartheta T\vartheta T^2 i = \vartheta TiT\vartheta_1 = i\vartheta_1 T\vartheta_1 \Longrightarrow \vartheta_1\mu X_1 = \vartheta_1 T\vartheta_1 .$$

The above theorem does not hold in general for epimorphisms but it does hold for split epimorphisms (those having a right inverse, i.e., retractions).

<u>Proposition B</u>: Let $(X,\vartheta) \in \mathcal{K}^{\mathbf{T}}$ and $X \xrightarrow{p} Y$ a retraction in $\mathcal{K}$. Then there is a $\mathbf{T}$-structure on Y making p into a homomorphism iff $\exists$ $\xi: TY \longrightarrow Y$ with $\xi Tp = p\vartheta$. ξ is uniquely determined by this.

Proof: We make use only of the fact that Tp and T^2p are again split epi. Then since Tp is epi ξ is uniquely determined. If we do have $\xi\, Tp = p\vartheta$, then $\xi\eta Yp = \xi Tp\eta X = p\vartheta\eta X = p \Rightarrow \eta Y = id_Y$ since p epi. Also $\xi T\xi T^2p = \xi TpT\vartheta = p\vartheta T\vartheta = p\vartheta\mu X = \xi Tp\mu X = \xi\mu YT^2p \Rightarrow \xi T\xi = \xi\mu Y$ since T^2p is epi. ∎

If $\mathcal{K}$ is locally small and has intersections, $\mathcal{K}^{\mathbf{T}}$ will have intersections by 2.2.1. Then for $(X,\vartheta) \in \mathcal{K}^{\mathbf{T}}$ and $X_1 \xrightarrow{i} X$ a monomorphism in $\mathcal{K}$, one can define the subalgebra generated by X_1 to be $\bigcap\{(A,\xi) \in \mathcal{K}^{\mathbf{T}}$: (A,ξ) is a subalgebra of (X,ϑ) and i factors through the injection of A$\}$. With reasonable hypotheses on $\mathcal{K}$, one can recapture many classical theorems about subalgebra generation. Some of these we will consider where $\mathcal{K} = \mathcal{S}$ (the category of sets) in the next chapter. For a fairly general analysis (where $\mathcal{K}$ is a "regular" category and $\mathbf{T}$ is a "regular" triple) consult Manes [9].

Chapter 3 TRIPLES IN THE CATEGORY OF SETS

§3.1 Constructions in S^T.

3.1.1 In this paragraph we will be concerned with construction of limits and colimits in categories $\mathbf{S}^{\mathbf{T}}$ where $\mathbf{S}$ is of course the category of sets and mappings. Construction of limits is totally settled by §2.1. To deal with colimits we will introduce the notion of a congruence and proceed in a manner entirely analogous to that of classical universal algebra. We will work with a fixed triple $\mathbf{T}$ in $\mathbf{S}$.

3.1.2 Regarding limits in $\mathbf{S}^{\mathbf{T}}$ we recall that $U^{\mathbf{T}}$ creates them, which implies first of all that $\mathbf{S}^{\mathbf{T}}$ is complete, and that we have canonical models for limits, since $\mathbf{S}$ has. In particular this means that for any two maps $(X,\vartheta) \underset{[S]}{\overset{[f]}{\rightrightarrows}} (Y,\xi)$ in $\mathbf{S}^{\mathbf{T}}$, $\{x \in X \mid f(x) = g(x)\}$ has a unique algebra structure making its inclusion into X an algebra homomorphism, and this subalgebra of (X,ϑ) is in fact the equalizer in $\mathbf{S}^{\mathbf{T}}$ of $([f],[g])$. In the same way, given a family $(X_i, \vartheta_i)_{i \in I}$ of algebras in $\mathbf{S}^{\mathbf{T}}$ there is a unique $\mathbf{T}$-structure on $\prod_{i \in I} X_i$, the cartesian product in $\mathbf{S}$, making the projections into homomorphisms. With this structure $\prod_{i \in I} X_i$ becomes a model for $\prod_{i \in I} (X_i, \vartheta_i)$ in $S^{\mathbf{T}}$. From the fact that $U^{\mathbf{T}}$ creates intersections we see that for $(X,\vartheta) \in S^{\mathbf{T}}$ and $A \subseteq X$ the subalgebra generated by A exists and its underlying set can be taken to be $\bigcap\{X' \subseteq X : X'$ has a structure making it a subalgebra of (X,ϑ) and $A \subseteq X'\}$. This is of course familiar in the context of universal algebra. A less familiar example is the case of compact T_2 spaces (which will later be shown to be of the form $S^{\mathbf{T}} \xrightarrow{U^{\mathbf{T}}} S$). Here subalgebra generation consists of forming the topological closure. Later we will give an explicit construction for subalgebra generation.

3.1.3 In classical universal algebra congruences are usually defined to be equivalence relations satisfying certain substitution laws with respect to the operations. It is well known that in this setting congruences may also be described as equivalence relations which are "subalgebras of the product" in the obvious sense. This latter approach works admirably well in the study of categories $S^{\mathbf{T}}$.

<u>Definition</u>: Let $(X,\vartheta) \in S^{\mathbf{T}}$. A congruence relation on (X,ϑ) is an equivalence relation $R \subseteq X \times X$ such that R is the underlying set of a subalgebra of $(X,\vartheta) \times (X,\vartheta)$ (in the sense that the inclusion $R \xrightarrow{r} X \times X$ is to be a homomorphism).

Since the intersection of subalgebras (resp. equivalence relations) is again a subalgebra (resp. equivalence relation) intersections of congruences are congruences, i.e. the congruences on an algebra form a closure system and hence a complete lattice.

We will now establish the fundamental classical results about congruences.

<u>Proposition A</u>: Let $R \subseteq X \times X$ be a congruence on $(X,\vartheta) \in S^{\mathbf{T}}$. Then there is a unique T-structure $\vartheta/_R$ on $X/_R$ such that the canonical map $X \xrightarrow{\nu_R} X/_R$ is a homomorphism. In fact $[\nu_R]$ is the coequalizer in $S^{\mathbf{T}}$ of the $U^{\mathbf{T}}$-contractible pair $(R, \vartheta') \underset{[\pi_2 r]}{\overset{[\pi_1 r]}{\rightrightarrows}} (X,\vartheta)$. ($\pi_1, \pi_2$ are the projections of $(X,\vartheta) \times (X,\vartheta)$ and r is the inclusion $R \longrightarrow X \times X$).

Proof: Since R is a congruence (in particular a subalgebra of the product) we have a diagram $(R, \vartheta') \xrightarrow{[r]} (X,\vartheta) \times (X,\vartheta) \underset{[\pi_2]}{\overset{[\pi_1]}{\rightrightarrows}} (X,\vartheta)$ in $S^{\mathbf{T}}$. By Corollary 1 of 2.2.2 we need only exhibit d_0, d_1 such that

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1 r"] \arrow[r, shift right, "\pi_2 r"'] & X \arrow[r, "\pi_2 r"] \arrow[l, bend left=60, "d_1"] & X/_R \arrow[l, bend left=60, "d_0"] \end{tikzcd}$$ is a contractible coequalizer diagram.

Take d any section of ν_R, i.e. any map with $\nu_R d_0 = id_{X/_R}$. Define d_1 by $d_1(x) = (x, d_0\nu_R(x))$. Then we clearly have $\nu_R d_0 = id_{X/_R}$, $\pi_1 r d_1 = id_X$, $\pi_2 r d_1 = d_0 \nu_R$. By Proposition A of 2.2.2 we do in fact have a contractible coequalizer diagram. ∎

<u>Lemma A</u>: Let $$\begin{tikzcd} (X,\vartheta) \arrow[r, "{[f]}"] \arrow[dr, "{[g]}"'] & (Y,\xi) \\ & (Z,\zeta) \end{tikzcd}$$ be a diagram in S with f onto. Suppose we have a factorization $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & Z \end{tikzcd}$$ in S. Then h is a homomorphism.

Proof: $h\xi Tf = hf\vartheta = g\vartheta = \zeta Tg = \zeta ThTf \Rightarrow h\xi = \zeta Th$ since f onto $\Rightarrow$ f split epi $\Rightarrow$ Tf split epi $\Rightarrow$ Tf onto. ∎

<u>Proposition B</u>: Let $(X,\vartheta) \xrightarrow{[f]} (Y,\xi)$ in $S^{\mathbf{T}}$. Let R be the kernel relation of f, i.e. $R = \{(x,x') \subseteq X \times X \mid f(x) = f(x')\}$. Let $X \xrightarrow{\nu_R} X/_R \xrightarrow{i} Y$ be the canonical splitting of f in S. Then R is a congruence, i is a monomorphism in $S^{\mathbf{T}}$, $f(X) \subseteq Y$ admits a subalgebra structure ξ' so that $(X/_R, \vartheta/_R) \xrightarrow{[i]} (f(X), \xi')$ is an isomorphism, and $(X,\vartheta) \xrightarrow{[\nu_R]} (X/_R, \vartheta/_R) \xrightarrow{[i]^R} (Y,\xi)$ is the categorical image factorization of [f] in $S^{\mathbf{T}}$.

Proof: The inclusion of R in X x X is the equalizer in S of $X \times X \underset{f\pi_2}{\overset{f\pi_1}{\rightrightarrows}} Y$. Since $U^{\mathbf{T}}$ creates this equalizer (see 1.1.2), R admits a subalgebra structure and hence is a congruence. By the previous lemma i is a homomorphism since ν_R is onto. Now $X/_R \xrightarrow{i} f(X)$ is 1-1 onto $\Longrightarrow$ $TX/_R \xrightarrow{Ti} Tf(X)$ is 1-1 onto and we can transport the structure $\vartheta/_R$ to $f(X)$ by $\xi' = i\,\vartheta/_R Ti^{-1}$. Then denoting the inclusion $f(X) \xrightarrow{j} Y$ we have $j\xi' Ti = ji\,\vartheta/_R Ti^{-1}\,Ti = i\,\vartheta/_R = \xi Ti = \xi TjTi \Rightarrow j\xi' = \xi Tj \Longrightarrow (f(X),\xi')$ is a subalgebra of (Y,ξ). Clearly $(X/_R, \vartheta/_R) \xrightarrow{[i]} (f(X),\xi')$ is an isomorphism. Now finally if $(X,\vartheta) \xrightarrow{[h]} (Z,\zeta) \xrightarrow{[k]} (Y,\xi)$ is a factorization of $[f]$ through a monomorphism $[k]$, at the level of sets we have $X/_R \xrightarrow{\ell} Z$ with $k\ell = i$ and $\ell\nu_R = h$. By the lemma again ℓ is a homomorphism and this shows we have an image factorization of $[f]$. ∎

<u>Corollary</u> (Explicit subalgebra generation): Let $(X,\vartheta) \in S^{\mathbb{T}}$ and $X' \subseteq X$. Then the underlying set of the subalgebra generated by X' is the set theoretic image of $TX' \xrightarrow{Ti} TX \xrightarrow{\vartheta} X$ where $X' \xrightarrow{i} X$ is the inclusion map.

Proof: We have a homomorphism $F^{\mathbb{T}}X' \xrightarrow{[\vartheta Ti]} (X,\vartheta)$ (which is the free extension of the inclusion $X' \xrightarrow{i} X$). By the proposition the set theoretical image Y of this map carries a subalgebra structure, say ξ. First of all $X' \subseteq Y$: Consider

$$\begin{tikzcd} TX' \arrow[r, "Ti"] & TX \arrow[d, shift left, "\vartheta"] \\ X' \arrow[u, "\eta X'"] \arrow[r, "i"'] & X \arrow[u, shift left, "\eta X"] \end{tikzcd}$$

Then $\forall\ x \in X'$ we have $x = i(x) = \vartheta\,\eta Xi(x) = \vartheta Ti\eta X'(x) \in Y$.

Next let (Z,σ) be any subalgebra of (X,ϑ) with $X' \overset{k}{\subseteq} Z \overset{j}{\subseteq} X$ with inclusion maps k,j as shown. Claim $Y \subseteq Z$.

Consider

$$\begin{tikzcd}
TX' \arrow[r, "Tk"] & T2 \arrow[r, "Tj"] \arrow[d, "\sigma"] & TX \arrow[d] \\
X' \arrow[r, hook, "k"] & Z \arrow[r, "j"] & X
\end{tikzcd}$$

Take any $y \in Y$. Then $$\begin{aligned} y &= \vartheta\, Ti(x') \text{ some } x' \in TX' \\ &= \vartheta\, TjTk(x') \\ &= j\, \sigma\, Tk(x') \in j(Z) = Z. \quad \blacksquare \end{aligned}$$

Example: In the category of groups the algebra structure associated with a group G is the extension of the identity $F(|G|) \longrightarrow G$ where $|G|$ is the underlying set of G, i.e. ϑ Takes strings $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ $\varepsilon_i = \pm 1$ and "multiplies" them in G. Then the above corollary gives the basic fact that for $X' \subseteq |G|$, the subgroup of G generated by X' is obtained by "multiplying" all strings $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ of elements of X' (i.e. all elements of TX' are multiplied by the action of ϑ restricted to TX').

Next we establish the familiar "isomorphism theorems" of universal algebra. First we need a lemma.

Lemma B: Let $(X, \vartheta) \xrightarrow{[f]} (Y, \xi)$ be a map in $S^{\mathbf{T}}$ and suppose $Y' \subseteq Y$ carries a subalgebra structure ξ'. Then $f^{-1}(Y')$ (the set-theoretic inverse image) carries a structure making it a subalgebra of (X, ϑ).

Proof: It is sufficient by Proposition A of 2.3.1 to show we have a factorization

$$\begin{tikzcd}
Tf^{-1}(Y') \arrow[r, "Ti"] \arrow[d, dashed] & TX \arrow[d, "\vartheta"] \\
f^{-1}(Y') \arrow[r, "i"] & X
\end{tikzcd}$$

where i is the inclusion.

Equivalently $\mathrm{Im}(\vartheta o Ti) \subseteq f^{-1}(Y')$

or $\mathrm{Im}(f \circ \vartheta \circ Ti) \subseteq Y'$.

Let j be the inclusion of Y' in Y. We have the following diagram in which all squares commute:

$$\begin{tikzcd}
Tf^{-1}(Y') \arrow[rrr, "T\bar{f}"] \arrow[dr, "Ti"] & & & TY' \arrow[dl, "Tj"'] \arrow[ddd, dash, "\xi'"] \\
& TX \arrow[r, "Tf"] \arrow[d] & TY \arrow[d, "\xi"] & \\
& X \arrow[r, "f"] & Y & \\
f^{-1}(Y') \arrow[ur, "i"] \arrow[rrr, "\bar{f}"] & & & Y' \arrow[ul, "j"']
\end{tikzcd}$$

$\bar{f}$ is the restriction of f to $f^{-1}(Y')$.

Then $f\vartheta Ti = \xi TfTi = \xi TjT\bar{f} = j\ \xi'\ T\bar{f}$.

Hence $\mathrm{Im}(f\vartheta Ti) = \mathrm{Im}(j\xi'T\bar{f}) \subseteq \mathrm{Im}j = Y'$. ∎

<u>Proposition C</u> (First Isomorphism Theorem): Let $(X,\vartheta) \in S^{T}$ and suppose $X' \subseteq X$ carries a subalgebra structure ϑ'. Then for any congruence R on (X,ϑ), the restriction of R to X', say R', is a congruence on the subalgebra (X',ϑ'). Then we have a monomorphism $i': X'/_{R'} \longrightarrow X/_{R}$ by $i'[x]_{R'} = [x]_{R}$. i is a homomorphism and the following diagram commutes

$$\begin{tikzcd}
(X',\vartheta') \arrow[r, "{[i]}"] \arrow[d, "{[\nu_{R'}]}"'] & (X,\vartheta) \arrow[d, "{[\nu_R]}"] \\
(X'/_{R'},\ \vartheta'/_{R'}) \arrow[r, "{[i']}"] & (X/_R,\ \vartheta/_R)
\end{tikzcd}$$

(i is the inclusion $X' \subseteq X$)

Proof: R' is clearly an equivalence relation. Consider the map $(X',\vartheta') \times (X',\vartheta') \xrightarrow{[i] \times [i]} (X,\vartheta) \times (X,\vartheta)$. R' is the inverse image of R under this map and since R is a subalgebra of $(X,\vartheta) \times (X,\vartheta)$, by Lemma B, R' is a subalgebra of $(X',\vartheta') \times (X',\vartheta')$, therefore a congruence. The commutativity of the diagram is by definition of i', and the latter is a homomorphism by Lemma A. ◼

<u>Proposition D</u> (Second isomorphism theorem): Let $(X,\vartheta) \in S^{T}$ and R a congruence on (X,ϑ). Then the sublattice of congruences on (X,ϑ) which contain R is isomorphic to the lattice of congruences of the quotient algebra $(X/_R, \vartheta/_R)$. Namely to a congruence ψ on (X,ϑ), $\psi \supseteq R$, we associate a congruence $\psi/_R$ on $(X/_R, \vartheta/_R)$ defined by $[x]_R \ \psi/_R \ [y]_R$ iff $x \ \psi \ y$. Then we have the following commutative diagram:

$$\begin{tikzcd}
(X,\vartheta) \arrow[r, "{[\nu_R]}"] \arrow[d, "{[\nu_\psi]}"'] & (X/_R,\ \vartheta/_R) \arrow[d, "{[\nu_{\psi/R}]}"] \\
(X/_\psi,\ \vartheta/_\psi) \arrow[r, "{[j]}"'] & (X/_R,\ \vartheta/_R)/_{\psi/R}
\end{tikzcd}$$

with [j] an isomorphism defined by $j([x]_\psi) = [[x]_R]_{\psi/R}$

Proof: Consider the map $(X,\vartheta) \times (X,\vartheta) \xrightarrow{[\nu_R] \times [\nu_R]} (X/_R, \vartheta/_R) \times (X/_R, \vartheta/_R)$. It is a basic set theoretical fact that we can establish an order preserving bijection of the set of equivalence relations on X which contain R, and the set of equivalence relations on $X/_R$ by assigning, to $\psi \subseteq X \times X$, $(\nu_R \times \nu_R)(\psi) = \psi/_R$ and to $\Sigma \subseteq X/_R \times X/_R$, $(\nu_R \times \nu_R)^{-1}(\Sigma)$. Proposition B and Lemma B ensure that this correspondence effects an order preserving bijection of the congruences on (X,ϑ) which contain R and the congruences on $(X/_R, \vartheta/_R)$. The commutativity of the diagram is by definition of j; j is 1-1, onto by basic set theoretical considerations; j is a homomorphism by Lemma A. ∎

3.1.4 We can now construct coequalizers and coproducts (and hence all colimits).

First of all suppose we have $(X,\vartheta) \underset{[g]}{\overset{[f]}{\rightrightarrows}} (Y,\xi)$ in $S^{\mathbf{T}}$. Put $\mathcal{C} = \{Q \text{ congruences on } (Y,\xi) \mid Q \supseteq \{(f(x),g(x)) \mid x \in X\}\}$. Observe $\mathcal{C} \neq \emptyset$ since the congruence $Y \times Y \in \mathcal{C}$. Put $R = \bigcap \mathcal{C}$. Then we claim $(Y,\xi) \xrightarrow{[\nu_R]} (Y/_R, \xi/_R)$ is coeq([f],[g]). But if $(Y,\xi) \xrightarrow{[h]} (Z,\zeta)$ with $[h][f] = [h][g]$ then clearly the kernel relation of h is in $\mathcal{C}$ which means $R \subseteq \mathrm{Ker}(h)$ and hence we have a unique factorization

$$\begin{tikzcd} Y \arrow[r, "\nu_R"] \arrow[dr, "h"'] & Y/_R \arrow[d, "k"] \\ & Z \end{tikzcd}$$

at the level of sets.

Since ν_R is onto h is actually a homomorphism, showing that $[\nu_R]$ has the required universality property. Since $R \supseteq \{(f(x),g(x)) \mid x \in X\}$, clearly $\nu_R f = \nu_R g$ and thus $[\nu_R] = \mathrm{coequ}([f],[g])$.

Now to form the coproduct of a family $(X_i, \vartheta_i)_{i \in I}$ in $S^{\mathbf{T}}$ we recall the construction familiar from universal algebra. Namely if $(A_i)_{i \in I}$ is a family of algebras of an equational class $\mathcal{A}$, one forms the disjoint union of the underlying sets $\dot{U}A_i$, then the free algebra in $\mathcal{A}$ on this set, i.e. $F(\dot{U}A_i;\mathcal{A})$. Then one factors out the smallest congruence R with the property that $A_i \longrightarrow F(UA_i;\mathcal{A}) \xrightarrow{\nu_R} F(UA_i;\mathcal{A})/_R$ is a homomorphism for all $i \in I$.

We will follow this procedure in constructing coproducts in $S^{\mathbf{T}}$. Namely denote the disjoint sum of the X_i by $\amalg X_i$ with injections u_i. Then we have the embeddings of the underlying set of the given algebras by $X_i \xrightarrow{\eta X_i} U^{\mathbf{T}}F^{\mathbf{T}}X_i \xrightarrow{U^{\mathbf{T}}F^{\mathbf{T}}u_i} U^{\mathbf{T}}F^{\mathbf{T}}\amalg X_i$. Let $\mathfrak{C}$ be the set of all congruences Q on $F^{\mathbf{T}}\amalg X_i$ with the property that $\nu_R \circ U^{\mathbf{T}}F^{\mathbf{T}}u_i \circ \eta X_i$ is a homomorphism from (X_i, ϑ_i) to $(T\amalg X_{i/Q}, \mu\amalg X_{i/Q})$ for all $i \in I$. $\mathfrak{C}$ is not empty since $T\amalg X_i \times T\amalg X_i$ (the congruence identifying everything) is in it. Put $R = \bigcap \mathfrak{C}$. Claim first that $R \in \mathfrak{C}$.

If not, for some $i \in I$ the following diagram does <u>not</u> commute:

$$\begin{tikzcd}
TX_i \arrow[r, "T\eta X_i"] \arrow[d, "\vartheta_i"'] & T^2X_i \arrow[r, "T^2u_i"] & T^2\amalg X_i \arrow[r, "T\nu_R"] \arrow[d, dashed, "\mu\amalg X_i"] & T(T\amalg X_{i/R}) \arrow[d, "\mu\amalg X_{i/R}"] \\
X_i \arrow[r, "\eta X_i"] & TX_i \arrow[r, "Tu_i"] & T\amalg X_i \arrow[r, "\nu_R"] & T\amalg X_{i/R}
\end{tikzcd}$$

$\Longrightarrow$ for some $x \in TX_i$, $\nu_R \circ Tu_i \circ \eta X_i \circ \vartheta_i(x) \neq \mu\amalg X_{i/R} \circ T\nu_R \circ T^2u_i \circ T\eta X_i\ (x)$

$$= \nu_R \circ \mu\amalg X_i \circ T^2u_i \circ T\eta X_i\ (x)$$

$\Longrightarrow$ for some $Q \in \mathfrak{C}$, $\nu_Q \circ Tu_i \circ \eta X_i \circ \vartheta_i(x) \neq \nu_Q \circ \mu\amalg X_i \circ T^2u_i \circ T\eta X_i(x)$

$$= \mu\amalg X_{i/Q} \circ T\nu_Q \circ T^2u_i \circ T\eta X_i(x)$$

But this contradicts the fact that $\nu_Q \circ Tu_i \circ \eta X_i$ is a homomorphism.

Thus we have homomorphism $(X_i, \vartheta_i) \xrightarrow{[\nu_i]} F^T\amalg X_{i/R}$, $\nu_i = \nu_R \circ Tu_i \circ \eta X_i$. Claim the $[\nu_i]$ form a coproduct scheme in S^T. Suppose we have maps $(X_i, \vartheta_i) \xrightarrow{[\phi_i]} (Y, \xi)$. Then at the level of sets we have a unique $X_i \xrightarrow{\phi} Y$ with $\phi u_i = \phi$. Then we have maps $F^T\amalg X_i \xrightarrow{[T\phi]} F^TY \xrightarrow{[\xi]} (Y, \xi)$ in S^T. Let the kernel relation of $[\xi]\bullet[T\phi]$ be Q. Consider the following diagram:

$$\begin{tikzcd}
X_i \arrow[r, "\eta X_i"] \arrow[ddrr, "\phi_i"'] & TX_i \arrow[r, "Tu_i"] \arrow[dr, "T\phi_i"'] & T\amalg X_i \arrow[r, "\nu_R"] \arrow[d, "T\phi"] \arrow[dr, "\nu_Q"] & T\amalg X_{i/R} \arrow[d, "\psi"] \\
 & & TY \arrow[d, "\xi"] & T\amalg X_{i/Q} \arrow[dl, "i"] \\
 & & Y &
\end{tikzcd}$$

We claim that $Q \in \mathcal{C}$. Observing that i is a monomorphism, $i\,\nu_Q Tu_i\eta X_i \vartheta_i = \xi T\tilde{\phi} Tu_i\eta X_i \vartheta_i = \xi T\phi_i\eta X_i\vartheta_i = \xi\eta Y\phi_i\vartheta_i = \phi_i\vartheta_i$ and $i\mu\amalg X_{i/Q} T\nu_Q T^2u_i T\eta X_i = \xi TiT\nu_Q T^2u_i T\eta X_i = \xi T\xi T^2\tilde{\phi}\, T^2u_i T\eta X_i = \xi T\xi T^2\phi_i T\eta_i = \xi T\phi_i = \phi_i\vartheta_i$.

Hence $\nu_Q Tu_i\eta X_i\vartheta_i = \mu\amalg X_{i/Q} T\nu_Q T^2u_i T\eta X_i \Rightarrow \nu_Q Tu_i\eta X_i$ is a homomorphism $\Rightarrow Q \in \mathcal{C} \Rightarrow R \subseteq Q$. Hence we have a factorization $\nu_Q = \psi\nu_R$. Since ν_R is onto ψ is a homomorphism and $[i\psi]$ has the property that $[i\psi] \circ [\nu_i] = \phi_i$ all $i \in I$. Moreover $[i\psi]$ is unique for having this property. Suppose we had $[\omega] : (T\amalg X_{i/R}, \mu\amalg X_{i/R}) \longrightarrow (Y, \xi)$ with this property. Then $\phi_i = \omega\nu_R Tu_i\eta X_i = \omega\nu_R\eta\amalg X_i u_i$ all $i \in I$; similarly $i\psi\nu_R Tu_i\eta X_i = i\psi\nu_R\eta\amalg X_i u_i = \phi_i$ all $i \in I$. Since u_i are injections of a coproduct in S, $\omega\nu_R\eta\amalg X_i = i\psi\nu_R\eta\amalg X_i$. But then we have $F^T\amalg X_i \overset{[\omega\nu_R]}{\underset{[i\psi\nu_R]}{\rightrightarrows}} (Y\ \xi)$ homomorphisms agreeing on the generators (see Proposition 1.3.2) and hence $\omega\nu_R = i\psi\nu_R$. Since ν_R is onto, $\omega = i\psi$. ∎

§3.2 A Tripleableness Theorem

3.2.1 We are now in a position to prove a very useful tripleableness theorem for the category S. This theorem was originally proved in a restricted form by Lawvere [5], then generalized by Linton [6]. The Lawvere-Linton theorems were proved in the context of algebraic theories but as we have noted before theories are in a sense equivalent to triples and theorems can be translated.

Theorem: Let $\mathcal{A} \xrightarrow{U} S$ be a functor having a left adjoint. Then U is tripleable iff the following three conditions hold:

L1: $\mathcal{A}$ has coequalizers and kernel pairs.

L2: A map $A \xrightarrow{f} B$ in $\mathcal{A}$ is a coequalizer $\Leftrightarrow$ Uf is onto.

L3: For a pair of maps $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{A}$, (Uf,Ug) a kernel pair $\Rightarrow$ (f,g) a kernel pair.

Remark: In the notation of the theorem statement, the pair of maps (f,g) is a kernel pair iff $\exists\, B \xrightarrow{h} C$ with

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ B \arrow[r, "h"'] & C \end{tikzcd}$$

a pullback. (f,g) is called the kernel pair of h.

One can easily establish the following for categories with kernel pairs and coequalizers:

(1) If a pair of maps is a kernel pair, it is the kernel pair of its coequalizer.

(2) If a map is a coequalizer, it is the coequalizer of its kernel pair.

(3) $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, maps in S, are a kernel pair $\Leftrightarrow$ the map $X \longrightarrow Y \times Y$ by $x \rightsquigarrow (f(x),g(x))$ is 1-1 and has an equivalence relation as image.

If one interprets kernel pairs as being embedding of kernel relations of maps, i.e. "congruences", then L3 is simply saying that congruences are "equivalence relations which are subalgebras of the product", loosely speaking. Following the proof of the theorem we will give an example where L1, L2 hold but L3 fails, which will elucidate this somewhat.

Proof of the theorem: ($\Longrightarrow$) Supposing U is tripleable, the semantical comparison functor is an equivalence of categories and it is clearly sufficient to show L1, L2, L3 hold for the functors $S^{\mathbf{T}} \xrightarrow{U^{\mathbf{T}}} S$.

L1 : Categories $S^{\mathbf{T}}$ do have coequalizers: a construction was given in 3.1.4. Kernel pairs follow from 3.1.2.

L2 : Suppose $(X,\vartheta) \xrightarrow{[f]} (Y,\xi)$ is a coequalizer, say of $(Z,\omega) \overset{[h]}{\underset{[k]}{\rightrightarrows}} (X,\vartheta)$. Then by 1.1.4 we can construct a coequalizer of these maps as a quotient $(X,\vartheta) \xrightarrow{[\nu_R]} (X/_R, \vartheta/_R)$. By the universality properties we have an isomorphism [q] with $[q][\nu_R] = [f]$. But then q is 1-1, onto and ν_R is onto $\Longrightarrow$ f is onto. On the other hand if [f] is onto we have

$$\begin{tikzcd}
(X,\vartheta) \arrow[rr, "{[f]}"] \arrow[dr, "{[\nu_R]}"'] & & (Y,\xi) \arrow[dl, "{[i]}"] \\
& (X/_R, \vartheta/_R) &
\end{tikzcd}$$

with R = Ker f and [i] an isomorphism.

But $[\nu_R]$ is coequ($[\pi_1 r]$, $[\pi_2 r]$) as in Proposition A of 3.1.3. $\Longrightarrow$ [f] is also coequalizer of this pair.

L3 : Suppose we have $(X,\vartheta) \overset{[f]}{\underset{[g]}{\rightrightarrows}} (Y,\xi)$ in $S^{\mathbf{T}}$ with (f,g) a kernel pair in S. That means the image of $X \xrightarrow{\gamma} Y \times Y$ by $x \rightsquigarrow (f(x),g(x))$ is an equivalence relation, say R. But in $S^{\mathbf{T}}$ we have the "same" map $(X,\vartheta) \xrightarrow{[\gamma]} (Y,\xi) \times (Y,\xi)$.

Its set theoretical image is R and by Proposition B of 1.1.3, R must then carry a subalgebra structure and hence is a congruence. Thus we have

$$\begin{tikzcd} (X,\vartheta) \arrow[r, "{[f]}"] \arrow[d, "{[g]}"'] & (Y,\xi) \arrow[d, "{[\nu_R]}"] \\ (Y,\xi) \arrow[r, "{[\nu_R]}"'] & (Y/_R,\xi/_R) \end{tikzcd}$$

commutes in $S^{\mathbf{T}}$.

Now when we apply $U^{\mathbf{T}}$ to this diagram it is easy to see we get a pullback. Since $U^{\mathbf{T}}$ creates limits, in particular it reflects them, showing ([f],[g]) is the kernel pair of $[\nu_R]$.

($\Longleftarrow$)
(The following proof is one given by J. Beck in an invited lecture at McMaster University in the spring of 1969. It has its roots in theorems by Linton, Lawvere and Freyd). This proof will make use of the fact, proved above, that $U^{\mathbf{T}}$ has properties L1, L2, L3, and also of the fact that $\Phi F = F^{\mathbf{T}}$.

1. Φ is faithful: Now U is faithful since the back adjunction ε is pointwise epi by L2 ($U\varepsilon$ is split epi with section ηU, hence ε must be a coequalizer and therefore epi). Since $U^{\mathbf{T}}\Phi = U$, Φ must also be faithful.

2. Φ is full: First note that Φ is full on Hom sets of the form $\mathcal{A}(FX,A)$ for we have the following bijective correspondences:

$(FX,A) = (X,UA) = (X,U^{T}\Phi A) = (F^{T}X,\Phi A) = (\Phi FX,\Phi A)$ by

$$f \rightsquigarrow Uf.\eta X \rightsquigarrow Uf.\eta X \rightsquigarrow \underset{\displaystyle Uf}{\underset{\shortparallel}{U\varepsilon A \circ T(Uf.\eta X)}} \rightsquigarrow Uf = \Phi f$$

Now we look at an arbitrary Hom set $\mathcal{A}(A,B)$. Take any $\Phi A \xrightarrow{[f]} \Phi B$ and put $A' \overset{k_1}{\underset{k_2}{\rightrightarrows}} FUA$ = kernel pair of εA. Then εA is a coequalizer

$\Rightarrow$ $\varepsilon A = \mathrm{coequ}(k_1,k_2)$. Apply Φ to this obtaining

$$\begin{tikzcd} \Phi A' \arrow[r, shift left, "\Phi k_1"] \arrow[r, shift right, "\Phi k_2"'] & \Phi FUA \arrow[r, "\Phi\varepsilon A"] \arrow[dr, dashed, "\Phi h"'] & \Phi A \arrow[d, "{[f]}"] \\ & & \Phi B \end{tikzcd}$$

Since Φ is bijective on (FUA,B) we have a unique $h\colon FUA \longrightarrow B$ with $\Phi h = [f]\Phi\varepsilon A$. Then Φ faithful $\Rightarrow hk_1 = hk_2 \Rightarrow \exists\,!\ A \xrightarrow{y} B$ with $y\varepsilon A = h$. But then $\Phi y \Phi\varepsilon A = \Phi h = [f]\Phi\varepsilon A \Rightarrow \Phi y = f$ since $\Phi\varepsilon A$ is onto (actually a coequalizer since $U^{\mathbf{T}}\Phi\varepsilon A = U\varepsilon A$ which is onto). Thus Φ is full.

3. Φ is representative: This is the most technical part of the proof and will be split into several steps for readability. The diagram on the following page is what the argument centres around and the proof will consist of constructing the diagram and then collapsing it (by proving certain maps are isomorphisms) in an intelligent way.

Take any $(X,\vartheta) \in S^{\mathbf{T}}$ we must find $A \in \mathcal{A}$ and an isomorphism $(X,\vartheta) \xrightarrow{[\zeta]} \Phi A$.

(1) Construct $[p_1]$, $[p_2]$ to be the kernel pair of $[\vartheta]$ and $[t_1]$, $[t_2]$ the kernel pair of $[\sigma]$. Since Φ is full and faithful there are unique f_i, g_i with $\Phi f_i = [p_i\sigma]$ and $\Phi g_i = [t_i\tau]$.

(2) Put $q = \mathrm{coequ}(f_1,f_2)$ and $q' = \mathrm{coequ}(g_1,g_2)$. Put (k_1,k_2) = kernel pair of q. Then since $qf_1 = qf_2$ $\exists\,!$ $FR \xrightarrow{r} K$ with $k_ir = f_i$. Now since $U^{\mathbf{T}}\Phi = U$ and since $U,U^{\mathbf{T}}$ satisfy L2, L3 we have the following observations: $(\Phi k_1,\Phi k_2)$ = ker pair Φq and $\Phi q = \mathrm{coequ}(\Phi k_1,\Phi k_2)$.

(3) $[\zeta]$ is uniquely determined by $[\zeta]\,[\vartheta] = \Phi q$ since $[\vartheta] = \mathrm{coequ}([p_1\sigma],[p_2\sigma])$ (because $[\sigma]$ is epi) and $\Phi q[p_1\sigma] = \Phi q\Phi f_1 = \Phi q\Phi f_2 = \Phi q[p_2\sigma]$. Similarly

Diagram to Show Φ is Representative

$$
\begin{tikzcd}
(TW,\mu W) \arrow[r,"{[\tau]}"] \arrow[ddd, equal] & (W,\tau) \arrow[r, shift left, "{[t_1]}"] \arrow[r, shift right, "{[t_2]}"'] & (TR,\mu R) \arrow[rr,"{[\sigma]}"] \arrow[ddd, equal] & & (R,\sigma) \arrow[dl, "{[\xi]}"'] \arrow[dd, "{[\psi]}"] \arrow[r, shift left, "{[p_1]}"] \arrow[r, shift right, "{[p_2]}"'] & (TX,\mu X) \arrow[r,"{[\vartheta]}"] \arrow[ddd, equal] & (X,\vartheta) \arrow[ddd, "{[\zeta]}"] \\
& & & \Phi Q \arrow[dr, "\Phi e"] & & & \\
& & & & \Phi K \arrow[dr, shift left, "\Phi k_1"] \arrow[dr, shift right, "\Phi k_2"'] & & \\
\Phi FW \arrow[rr, shift left, "\Phi g_1"] \arrow[rr, shift right, "\Phi g_2"'] & & \Phi FR \arrow[uur, "\Phi q'"] \arrow[urr, "\Phi r_1"] \arrow[rrr, shift left, "\Phi f_1"] \arrow[rrr, shift right, "\Phi f_2"'] & & & \Phi FX \arrow[r, "\Phi q"'] & \Phi\overline{X}
\end{tikzcd}
$$

we have a unique ξ with $[\xi][\sigma] = \Phi q'$. We have a unique $[\psi]$ with $\Phi k_i[\psi] = [p_i]$ since $\Phi q[p_1][\sigma] = \Phi q[p_2][\sigma] \Rightarrow \Phi q[p_1] = \Phi q[p_2]$ and $(\Phi k_1, \Phi k_2)$ is the kernel pair of Φq. Observe that $[\psi][\sigma] = \Phi r$ since $\Phi k_i[\psi]\,[\sigma] = [p_i][\sigma] = \Phi f_i = \Phi k_i \Phi r$. Finally e arises, uniquely determined by $eq' = r$ since $\Phi r \Phi g_1 = [\psi][\sigma][t_1][\tau] = [\psi][\sigma][t_2][\tau] = \Phi r \Phi g_2$ $\Rightarrow rg_1 = rg_2$, but $q' = \text{coequ}(g_1, g_2)$. It follows that $\Phi e \circ [\xi] = [\psi]$ since they agree when preceeded by the epimorphism $[\sigma]$.

(4) We now proceed to collapse the diagram. First $[\psi]$ is a monomorphism. If for example $[\psi][\alpha] = [\psi][\beta]$ then $\Phi k_i[\psi][\alpha] = \Phi k_i[\psi][\beta] \Rightarrow [p_i][\alpha] = [p_i][\beta] \Rightarrow [\alpha] = [\beta]$ since $([p_1],[p_2])$ is a kernel pair.

(5) $[\xi]$ is an isomorphism. ξ is onto since $\xi\sigma = Uq'$ and the latter is onto. ξ is mono, hence 1-1, since $[\psi]$ is $(\Phi e[\xi] = [\psi])$. Hence ξ is iso in S $\Rightarrow [\xi]$ iso in $S^{\mathbf{T}}$.

(6) e is an isomorphism. Since $[\xi]$ is iso and $[\psi]$ is mono, Φe is mono. But Φ, being faithful reflects monos $\Rightarrow$ e mono. Also because $[\xi]$ is iso and $([p_1],[p_2])$ is a kernel pair of $[\vartheta]$, $(\Phi k_1 \Phi e,\ \Phi k_2 \Phi e)$ is a kernel pair of $[\vartheta]$. But this means e is split epi since $(k_1 e,\ k_2 e)$ is a kernel pair of q'', say $\Rightarrow q''k_1 = q''k_2$ (e is mono) $\Rightarrow \exists\,!\ \bar{e}: K \longrightarrow Q$ with $k_i e\bar{e} = k_i$. But (k_1, k_2) kernel pair $\Rightarrow e\bar{e} = id$. Then e being split epi and mono is iso.

(7) $[\zeta]$ is an isomorphism: This is now trivial since $[\psi]$ is iso because $[e]$, $[\xi]$ are which means the kernel pairs $(\Phi k_1, \Phi k_2)$ and $([p_1],[p_2])$ are isomorphic, which in turn means their coequalizers must be isomorphic by the induced map $[\zeta]$. ∎

<u>Corollary</u> (to the arguments above): If U satisfies L1 and L2, U is of descent type (i.e. Φ is full and faithful). ∎

Remark: Linton [8] has made a detailed study of this theorem, extracting the categorical essence of the above arguments, and has shown that if one restricts the coequalizers required in L1 (to those of certain special U-contractible pairs) and if one replaces "onto" in L2 by "epi" then the theorem holds as stated for S any category with coequalizers and kernel pairs, in which every epi splits (i.e. is a retraction).

Often when working in S we are interested in the case in which functors $\mathcal{A} \xrightarrow{U} S$ are precisely tripleable. The following general lemma is of use in this respect.

Lemma: let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor with a left adjoint. The following are equivalent:

(1) U is precisely tripleable.

(2) U is tripleable and creates isomorphisms.

Proof:

(1) $\Rightarrow$ (2). Clearly U must be tripleable and since $U^{\mathbf{T}}\Phi = U$, if Φ is an isomorphism U creates isomorphisms since $U^{\mathbf{T}}$ does.

(2) $\Rightarrow$ (1). Φ is 1-1 on objects. For if $\Phi A = \Phi B$, since Φ is full and faithful $\exists\,!$ iso $A \xrightarrow{f} B$ with $\Phi f = id_{\Phi A}$. Then $Uf = id_{UB}$ but $B \xrightarrow{id} B$ is the unique isomorphism created from id_{UB} in $\mathcal{A} \Rightarrow B = A$ and $f = id_A$.

To show Φ is onto, take any $(X, \vartheta) \in \mathcal{K}^{\mathbf{T}}$ and isomorphism $\Phi A \xrightarrow{[\zeta]} (X, \vartheta)$. Then, in $\mathcal{K}$, $UA \xrightarrow{\zeta} X$ is iso $\Rightarrow$ there is exactly one map $A \xrightarrow{f} A'$ in $\mathcal{A}$ with $Uf = \zeta$ and f is an isomorphism. $\Rightarrow$ $\Phi A \xrightarrow{\Phi f} \Phi A'$ is iso in $\mathcal{K}^{\mathbf{T}}$, but $\Phi f = [\zeta]$ and since $U^{\mathbf{T}}$ creates isomorphisms we must have $\Phi A' = (X, \vartheta)$. ∎

3.2.2 Examples.

Compact T_2 Spaces: Let $\mathcal{A}$ denote the category of compact T_2 spaces and continuous maps and U the usual underlying set functor, then U is precisely tripleable. First of all U has a left adjoint, namely $S \xrightarrow{F} \mathcal{A}$ given by FX = Stone Čech compactification βX of X made discrete and Ff = usual extension of $X \xrightarrow{f} Y \xrightarrow{\eta Y} \beta Y$ to continuous $\beta X \xrightarrow{Ff} \beta Y$. It is clear U creates isomorphisms. $\mathcal{A}$ inherits completeness and cocompleteness from $\mathcal{T}$op, of which it is a reflective subcategory (limits are computed as in $\mathcal{T}$op, colimits are computed in $\mathcal{T}$op, then followed by the Stone-Čech compactification). Hence L1 is satisfied. L2 holds since the epimorphisms in $\mathcal{A}$ are precisely the onto maps (by a simple "pasting over closed subspaces" argument). Hence coequalizers are onto. Onto maps are coequalizers since they are quotient maps and the arguments of Proposition A of 3.1.3 can be used almost verbatim. Regarding L3, Suppose $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{A}$ with Uf,Ug a kernel pair. Then the map $X \longrightarrow Y \times Y$ by $x \rightsquigarrow (f(x),g(x))$ <u>in $\mathcal{A}$</u> is an injection onto an equivalence relation R, which then must be closed in Y x Y. This means $Y/_R$ is compact T_2. Then

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\nu_R"] \\ Y \arrow[r, "\nu_R"'] & Y/_R \end{tikzcd}$$

is a pullback in $\mathcal{A}$. For suppose we have $Z \underset{k}{\overset{h}{\rightrightarrows}} Y$ with $\nu_R k = \nu_R k$. Then in S the above diagram is a pullback and we have a unique $Z \xrightarrow{\alpha} X$ with $f\alpha = h$, $g\alpha = k$. But then α is automatically continuous since $f\alpha$ is really (up to isomorphisms in $\mathcal{A}$) $Z \xrightarrow{\alpha} X \subsetneq Y \times Y \xrightarrow{\pi_1} Y$ and similarly $g\alpha = \pi_2\alpha$. But $\pi_1\alpha = h$ and $\pi_2\alpha = k$ are continuous, hence α must be (since the product of topological spaces is an initial structure).

<u>Stone Spaces</u>: Let SS be the category of Stone spaces (compact and 0-dimensional). Then SS is a full reflective subcategory of $\mathcal{A}$. The reflection consists

of factoring by the component relation (i.e. identifying exactly those elements which are in the same component). Let $SS \xrightarrow{U} S$ be the usual underlying set functor. Then this still has the Stone-Čech compactification functor described above as left adjoint, — the Stone-Čech compactification of a discrete space is always 0-dimensional. Thus the triple generated by this adjointness is again the β-triple and U here has no chance of being tripleable. We will show explicitly that L3 fails (one can easily see that L1, L2 hold and that as a result U is of descent type).

For let X be any non-trivial (i.e. card $X \geq 2$) connected space and $|X|$ the underlying set of X made discrete. Now consider the map $\beta(|X|) \xrightarrow{\varepsilon^X} X$ the back adjunction (extension of the identity $|X| \longrightarrow X$). Let this have kernel relation R. Then R as a subspace of $\beta(|X|) \times \beta(|X|)$ is 0-dimensional and we have $R \underset{g}{\overset{f}{\rightrightarrows}} \beta(|X|)$, the projections. Now (Uf, Ug) is a kernel pair in S but the claim is that (f,g) is not a kernel pair in SS. For if (f,g) is a kernel pair it is the kernel pair of its coequalizer. Coeq(f,g) in SS is obtained by taking the component reflection of coeq(f,g) in $\mathcal{A}$. But coeq(f,g) in $\mathcal{A}$ is $\beta(|X|) \xrightarrow{\varepsilon^X} X$ and the reflection of X in SS is 1, the one point space in SS, since X is connected. Thus if (f,g) is a kernel pair in SS, the following is a pullback:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "g"'] & \beta(|X|) \arrow[d] \\ \beta(|X|) \arrow[r] & 1 \end{tikzcd}$$

But this is not the case. For example define $1 \underset{k}{\overset{h}{\rightrightarrows}} \beta(|X|)$ so that $h(o) = x_1 \in |X| \subseteq \beta(|X|)$ and $k(o) = x_2 \in |X| \subseteq \beta(|X|)$ where $x_1 \neq x_2$.

Then the pair $(x,x_2) \notin R$ so (h,k) cannot factor through (f,g). ∎

Note that in this example Φ is, up to categorical isomorphism, the inclusion of SS $\subseteq \mathcal{A}$.

Universal Algebra: Equational classes of universal algebras (of possibly infinitary type), equipped with their usual underlying set functors are precisely tripleable. L1, L2 present no problems, and as pointed out before, L3 simply says that "the equivalence relations which can be factored out of algebras are precisely those which are subalgebras of the product". With this in mind L3 can be established in the same way as was done for compact T_2 spaces above.

§3.3 The Triple-theoretic Birkhoff Theorem

3.3.1 The classical theorem of Birkhoff in universal algebra says that the class of algebras of a given type which satisfy a specified set of equations is closed under HSP (homomorphic images, subalgebras and products) and that conversely if a class of algebras of a given type is closed under HSP, it consists of precisely those algebras satisfying some set of equations. We can recapture this theorem in the context of triples provided we interpret "equations" as "triples" and "adding equations" as "exhibiting a new triple and a pointwise onto triple map from the original triple". The precise formulation of the theorem will perhaps clarify this suggested interpretation.

3.3.2. We require the following lemma, whose proof consists solely of tedious naturality calculations and will be omitted.

<u>Lemma A</u>: Suppose we have adjoint situations $(F_1, U_1, \eta_1, \varepsilon_1, \mathcal{A}, \mathcal{K})$ and $(F_2, U_2, \eta_2, \varepsilon_2, \mathcal{B}, \mathcal{A})$. Then we can "compose" these adjointness to obtain $(F_2F_1, U_1U_2, U_1\eta_2F_1 \circ \eta_1, \varepsilon_2 \circ F_2\varepsilon_1U_2, \mathcal{B}, \mathcal{K})$. Denoting the triples generated by $\mathbf{T}$, $\mathbf{S}$, $\mathbf{H}$ respectively we have the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{A}^{\mathbf{S}} \arrow[drr, "U"] \arrow[ddr, "U^{\mathbf{S}}"'] & & \\
& \mathcal{B} \arrow[ul, "\Phi_{\mathbf{S}}"'] \arrow[r, "\Phi_{\mathbf{H}}"'] \arrow[d, "U_2"'] & \mathcal{K}^{\mathbf{H}} \arrow[d, "\mathcal{K}^{\phi}"'] \arrow[ddl, "U^{\mathbf{H}}"] \\
& \mathcal{A} \arrow[r, "\Phi_{\mathbf{T}}"] \arrow[d, "U_1"'] & \mathcal{K}^{\mathbf{T}} \arrow[dl, "U^{\mathbf{T}}"'] \\
& \mathcal{K} &
\end{tikzcd}
$$

where $\phi = U_1\eta_2F_1$, a triple map $\mathbf{T} \longrightarrow \mathbf{H}$, and U is defined by $U(A, \vartheta) = (U_1A, U_1\vartheta \circ U_1U_2F_2\varepsilon_1A)$, $U[f] = [U_1f]$. ∎

Note: in a certain sense U_1 is acting on triples $\mathbf{S}$ in $\mathcal{A}$ "pulling them down" into triples $\mathbf{H}$ in $\mathcal{K}$. This actually extends to a functor (for fixed U_1, F_1 etc.) $\mathrm{Trip}(\mathcal{A}) \longrightarrow \mathrm{Trip}(\mathcal{K})$, or more generally a contravariant functor $\mathrm{AD}(\mathcal{A}) \longrightarrow \mathrm{AD}(\mathcal{K})$. Certain amusing results can be proved about these functors but we shall forego this as it has no bearing on the task at hand.

<u>Lemma B</u>: For a triple $\mathbf{T} = (T, \eta, \mu)$ in S and a full subcategory $\mathcal{B} \subsetneq S^{\mathbf{T}}$, the following are equivalent:

(1) $\mathcal{B}$ is closed under (all models of) products and subobjects.

(2) $\mathcal{B}$ is replete (i.e. closed under isomorphisms) and reflective with pointwise onto reflections.

Proof:

(1) $\Longrightarrow$ (2). $\mathcal{B}$ is replete simply because it is closed under all models of unary products, also $\mathcal{B}$ will be closed under products and equalizers, hence will be complete since $S^{\mathbf{T}}$ is, and the assumption says that the inclusion functor $\mathcal{B} \xrightarrow{J} S^{\mathbf{T}}$ preserves limits. By the theorem of 0.2.4 we need only verify the solution set condition to establish the existence of a left adjoint for J. Given any (X,ϑ), let $\mathcal{C} = \{(X/_R, \vartheta/_R) \mid R$ a congruence on (X,ϑ) and $(X/_R, \vartheta/_R) \in \mathcal{B}\}$. $\mathcal{C}$ is then a set and we claim it is a solution set for (X,ϑ), relative to J. For any $(Y,\xi) \in \mathcal{B}$ and $(X,\vartheta) \xrightarrow{[f]} (Y,\xi)$ in $S^{\mathbf{T}}$ we have that $f(X)$ carries a structure ξ' making it a subalgebra of (Y,ξ). Denoting the kernel relation of $[f]$ by R, we have $(X/_R, \vartheta/_R) \cong (f(X),\xi')$ and hence $\mathcal{C} \ni (X/_R, \vartheta/_R)$. But $[f]$ factors through $[\nu_R]$ as in Proposition B of 3.1.3. We have

$$\begin{tikzcd}
(X,\vartheta) \arrow[r] \arrow[dr, "{[f]}"'] & J(X/_R, \vartheta/_R) \arrow[d, "{J[i]}"] \\
& J(Y,\xi)
\end{tikzcd}$$

showing $\mathcal{C}$ is a solution set, and J has a left adjoint R. We have reflection maps $(X,\vartheta) \xrightarrow{[\rho]} R(X,\vartheta)$. There must be onto, since the image of $[\rho]$ will also have the reflector property and hence the inclusion of the image of $[\rho]$, by uniqueness of reflection induced maps, must be an isomorphism.

(2) $\Longrightarrow$ (1). $\mathcal{B}$, being replete and reflective, is closed under all limits. We need only check that $\mathcal{B}$ is closed under subobjects. Take $(X,\theta) \in \mathcal{B}$ and $(X',\theta') \xrightarrow{[i]} (X,\theta)$ a subalgebra. Then we have

$$\begin{tikzcd}
(X',\theta') \arrow[r, "{[\rho]}"] \arrow[dr, "{[i]}"'] & R(X',\theta') \arrow[d, "{[j]}"] \\
& (X,\theta)
\end{tikzcd}$$

with [j] induced by the reflection [ρ]. Since [i] is 1-1, [ρ] must be 1-1. But [ρ] by hypothesis is onto $\Longrightarrow$ [ρ] is an isomorphism. Since $\mathcal{B}$ is replete, $(X',\theta') \in \mathcal{B}$. ∎

Before presenting the main theorem we give some convenient terminology. If $\mathbf{T}$ is a triple in S and $\mathbf{T} \xrightarrow{\phi} \mathbf{T}'$ a map in Trip(S) then we say $\mathbf{T}'$ is a quotient triple of $\mathbf{T}$ with quotient map ϕ if ϕ is pointwise onto. A subcategory $\mathcal{B} \subseteq S^{\mathbf{T}}$ is HSP if it is closed under products, subobjects and homomorphic images.

<u>Theorem</u>: For a triple $\mathbf{T}$ in S, there is essentially a 1-1 correspondence between full HSP subcategories of $S^{\mathbf{T}}$ and quotient triples $\mathbf{T} \xrightarrow{\phi} \mathbf{T}'$. Specifically, if $\mathcal{B} \subseteq S^{\mathbf{T}}$ is a full HSP subcategory then the restriction of $U^{\mathbf{T}}$ to $\mathcal{B}$ is precisely tripleable and the resulting triple is a quotient of $\mathbf{T}$. Conversely, given a quotient $\mathbf{T} \xrightarrow{\phi} \mathbf{T}'$, the semantical interpretation $S^{\mathbf{T}'} \longrightarrow S^{\mathbf{T}}$ is an embedding onto a full HSP subcategory. Moreover, if we start with $\mathcal{B}$, pass to the quotient triple and then take the semantical interpretation we have a commutative diagram:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "J"] \arrow[dr, "\Phi"'] & S^{\mathbf{T}} \arrow[d, "S^{\phi}"] \\ & S^{\mathbf{T}'} \end{tikzcd}$$

with Φ an isomorphism.

On the other hand starting with a quotient $\mathbf{T} \xrightarrow{\phi} \mathbf{T}'$ and applying the constructions above, we will arrive back at a diagram

$$\begin{tikzcd} T \arrow[r, "\phi"] \arrow[dr, "\psi"'] & T' \arrow[d, "\alpha"] \\ & T'' \end{tikzcd}$$

with α an isomorphism.

(the reason for the somewhat unappealing intrusion of α is that when generating a triple from the restriction of $U^{\mathbf{T}}$ to the image of $S^{\mathbf{T}'} \longrightarrow S^{\mathbf{T}}$, we must make a choice of left adjoint, which is of course only determined up to natural equivalence).

Proof: Suppose $\mathcal{B} \subseteq S^{\mathbf{T}}$ is closed under HSP. Then by Lemma B above, is a reflective subcategory of $S^{\mathbf{T}}$, with reflections $[\rho(X,\mathcal{S})]$ which are pointwise onto. Denoting the inclusion of $\mathcal{B}$ by J, then J has a left adjoint R and hence $U^{\mathbf{T}}J$ has left adjoint $RF^{\mathbf{T}}$. This adjointness generates a triple $\mathbf{T}'$ and by Lemma A above we have a triple map $\mathbf{T} \xrightarrow{U^{\mathbf{T}}\rho F^{\mathbf{T}}} \mathbf{T}'$ which is pointwise onto since ρ is. It is clear that $U^{\mathbf{T}}J$ creates isomorphisms so to show that $U^{\mathbf{T}}J$ is tripleable we need only verify conditions L1, L2, L3 of the theorem of 1.2.1. Now $\mathcal{B}$ has coequalizers and kernel pairs since it is a reflective subcategory of $S^{\mathbf{T}}$ which has them. Coequalizers of maps in $\mathcal{B}$ are obtained by constructing the coequalizer in $S^{\mathbf{T}}$, which is onto, and following this by the reflection which is onto. Hence coequalizers in $\mathcal{B}$ are onto. On the other hand an onto map is the coequalizer of its kernel pair in $S^{\mathbf{T}}$, but this kernel pair by construction arises from a subobject of a product, hence is in $\mathcal{B}$ and thus an onto map of $\mathcal{B}$ is a coequalizer in $\mathcal{B}$. Finally if a pair of maps $([f],[g])$ in $\mathcal{B}$ is such that (f,g) is a kernel pair, then $([f],[g])$ is the kernel pair of its coequalizer in $S^{\mathbf{T}}$, but this coequalizer, being onto, is already in $\mathcal{B}$. Thus $([f],[g])$ is a kernel pair in $\mathcal{B}$. From this we conclude that $U^{\mathbf{T}}J$ is precisely tripleable.

Now suppose $\mathbf{T} \xrightarrow{\phi} \mathbf{T}'$ is a quotient in Trip(S). Then $S^{\mathbf{T}'} \xrightarrow{S^{\phi}} S^{\mathbf{T}}$ is an embedding of a full subcategory. For $S^{\phi}(X,\vartheta) = S^{\phi}(Y,\xi) \Rightarrow X = Y$ and $\phi X = \xi\phi Y \Rightarrow \vartheta = \xi$ since ϕX is onto. Hence S^{ϕ} is 1-1 on objects. Also S^{ϕ} is faithful. To show S^{ϕ} is full, take $S^{\phi}(X,\vartheta) \xrightarrow{[f]} S^{\phi}(Y,\xi)$ $\Rightarrow f\vartheta\phi X = \xi\phi YTf = \xi T'f\phi X \Rightarrow f\vartheta = \xi T'f$ (since ϕX is onto) $\Rightarrow$ f is a homomorphism from (X,ϑ) to (Y,ξ) in $S^{\mathbf{T}'}$. Let $(X_i, \vartheta_i\phi X_i)_{i \in I}$ be a family of objects from the image of S^{ϕ}. Then form the product $\prod_{i \in I}(X_i, \vartheta_i)$ in $S^{\mathbf{T}'}$. This product is $(\Pi X_i, \vartheta)$ where $\pi_i\, \vartheta = \vartheta_i T'\pi_i$ for each i. Then $S^{\phi}\Pi(X_i, \vartheta_i) = (\Pi X_i, \vartheta.\phi\Pi X_i)$ and $\pi_i\vartheta\phi\Pi X_i =$ $\vartheta_i T'\, \pi_i\phi\Pi X_i = \vartheta_i\phi X_i T\, \pi_i$. Hence the projections are homomorphisms from $S^{\phi}\Pi(X_i, \vartheta_i)$ to $S^{\phi}(X_i, \vartheta_i)$. But this uniquely determines $S^{\phi}\Pi(X_i, \vartheta_i)$ as the product in $S^{\mathbf{T}}$ of the $S^{\phi}(X_i, \vartheta_i)$. This shows the image of S^{ϕ} is closed under products. Now let $(X', \vartheta') \xrightarrow{[i]} (X, \vartheta\phi X)$ be a subalgebra of $S^{\phi}(X,\vartheta)$. We claim ϑ' factors through $\phi X'$. We have:

$$\begin{tikzcd} TX' \arrow[rr, "Ti"] \arrow[dr, "\phi X'"] \arrow[dd, "\vartheta'"'] & & TX \arrow[d, "\phi X"] \\ & T'X' \arrow[r, "T'i"] & T'X \arrow[d, "\vartheta"] \\ X' \arrow[rr, "i"] & & X \end{tikzcd}$$

Take $x_1, x_2 \in TX'$ and suppose $\phi X'(x_1) = \phi X'(x_2)$. Then $\vartheta'(x_1) = \vartheta'(x_2)$ because $\phi X'(x_1) = \phi X'(x_2) \Rightarrow \vartheta T'i\phi X'(x_1) = \vartheta T'i\phi X'(x_2) \Rightarrow \vartheta\phi XTi(x_1) = \vartheta\phi XTi(x_2) \Rightarrow i\vartheta'(x_1) = i\vartheta'(x_2) \Rightarrow \vartheta'(x_1) = \vartheta'(x_2)$. Hence ϑ' does factor through $\phi X'$, say $\vartheta' = \vartheta''\phi X'$. But then $i\vartheta''\phi X' = i\vartheta' = \vartheta T'i\phi X' \Rightarrow i\vartheta'' = \vartheta T'i$, which is sufficient for (X', ϑ'') to be in $S^{T'}$. But then $(X', \vartheta') = S^{\phi}(X', \vartheta'')$ and the image of S^{ϕ} is closed under subobjects. Suppose $(X, \vartheta\phi X) \in$ image S^{ϕ} and $(X, \vartheta\phi X) \xrightarrow{[f]} (Y, \xi)$ an onto map in S^{T}. We have

$$\begin{tikzcd} TX \arrow[rr, "Tf"] \arrow[d, "\phi X"'] & & TY \arrow[dl, "\phi Y"'] \arrow[dd, "\xi"] \\ T'X \arrow[r, "T'f"] \arrow[d, "\vartheta"'] & T'Y & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

Claim ξ factors through ϕY. Since f is onto we can choose a section $Y \xrightarrow{d} X$ with $fd = id_Y$. Then Td, $T'd$ are sections for Tf, $T'f$ respectively. Take $y_1, y_2 \in TY$ with $\phi Y(y_1) = \phi Y(y_2)$. Then $\xi(y_1) = \xi TfTd(y_1) = f\vartheta\phi XTd(y_1) = f\vartheta T'd\phi Y(y_1) = f\vartheta T'd\phi Y(y_2) = f\vartheta\phi XTd(y_2) = \xi TfTd(y_2) = \xi(y_2)$.

Hence ξ factors through ϕY, say $\xi = \xi'\phi Y$ and $\xi' T' f \phi X = f \vartheta \phi X \Rightarrow \xi' T' f = f\vartheta$. Since f is onto this is sufficient to guarantee that $(Y, \xi') \in S^{T'}$. Clearly $(X, \vartheta\phi X) \xrightarrow{[f]} (Y, \xi)$ is the image under S^{ϕ} of $(X, \vartheta) \xrightarrow{[f]} (Y, \xi')$ showing that the image of S^{ϕ} is closed under homomorphic images.

Now suppose we start with $\mathcal{B}$, generate a quotient $\mathbf{T} \xrightarrow{\phi} \mathbf{T}'$ and then consider the semantical interpretation of ϕ.

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "J"] \arrow[dr, "\Phi"'] & & S^{\mathbf{T}} \\ & S^{\mathbf{T}'} \arrow[ur, "S\phi"'] & \end{tikzcd}$$

As above, say $R \dashv J$ with adjunctions (ρ, σ) (recall σ will be a natural equivalence). Then the triple $\mathbf{T}'$ explicitly is $(U^{\mathbf{T}}JRF^{\mathbf{T}}, U^{\mathbf{T}}\rho F^{\mathbf{T}} \circ \eta, U^{\mathbf{T}}J(\sigma \circ R\varepsilon_{\mathbf{T}}J)RF^{\mathbf{T}})$. The adjunctions of $RF^{\mathbf{T}} \dashv U^{\mathbf{T}}J$ are $(U^{\mathbf{T}}\rho F^{\mathbf{T}} \circ \eta, \sigma \circ R\varepsilon_{\mathbf{T}}J)$. Then
$$\begin{aligned} S\,\Phi(X, \vartheta) &= S^{\phi}(X, U^{\mathbf{T}}J(\sigma \circ R\varepsilon_{\mathbf{T}}J)(X, \vartheta)) \\ &= (X, U^{\mathbf{T}}J(\sigma \circ R\varepsilon_{\mathbf{T}}J)(X, \vartheta) \circ U^{\mathbf{T}}\rho F^{\mathbf{T}}X) \end{aligned}$$

But
$$\begin{aligned} U^{\mathbf{T}}J(\sigma \circ R\varepsilon_{\mathbf{T}}J)(X, \vartheta) \circ U^{\mathbf{T}}\rho F^{\mathbf{T}}X &= U^{\mathbf{T}}J(\sigma(X, \vartheta) \circ R[\vartheta]) \circ U^{\mathbf{T}}\rho F^{\mathbf{T}}X \\ &= U^{\mathbf{T}}(J\sigma(X, \vartheta) \circ JR[\vartheta] \circ \rho F^{\mathbf{T}}X) \\ &= U^{\mathbf{T}}(J\sigma(X, \vartheta) \circ \rho(X, \vartheta) \circ [\vartheta]) \\ &= U^{\mathbf{T}}((J\sigma \circ \rho J)(X, \vartheta) \circ [\vartheta]) \\ &= \vartheta . \end{aligned}$$

Hence $S^{\phi}\Phi = J$ on objects. Clearly this is also the case for morphisms.

Now finally suppose we start with a quotient $\mathbf{T} \xrightarrow{\phi} \mathbf{T}'$ then take the semantical interpretation $S^{\mathbf{T}'} \xrightarrow{S\phi} S^{\mathbf{T}}$ and construct the quotient $\mathbf{T} \xrightarrow{\psi} \mathbf{T}''$ arising from the image $\mathcal{B}$ of S^{ϕ}. Then we have a commutative diagram:

$$\begin{tikzcd} S^{T'} \arrow[r, "K"] \arrow[dr, "S^{\phi}"'] & \mathcal{B} \arrow[r, "\Phi"] \arrow[d, "J"] & S^{T''} \arrow[dl, "S^{\psi}"] \\ & S^{T} & \end{tikzcd}$$

$S^{\phi} = JK$, the factorization through the image. Now since ΦK is an isomorphism, when we apply the structure functor to the above we get

$$\begin{tikzcd} T' \arrow[rr, "\alpha"] \arrow[dr, "\phi"'] & & T'' \arrow[dl, "\psi"] \\ & T & \end{tikzcd}$$

with α an isomorphism. $\square$

3.3.3 <u>Examples</u>.

(1) If one knew that for any given type, the category of all universal algebras of that type were tripleable, this proposition would show that all primitive classes are already tripleable (since they are HSP subcategories of categories of all algebras of their type).

(2) By analogy with universal algebra we call a category $S^{\mathbf{T}}$ equationally complete if it has no non-trivial HSP subcategories (i.e. no HSP proper subcategories containing algebras with more than 1 point). It is easy to check that the category of compact T_2 spaces is equationally complete. For if an HSP subcategory contains a non-trivial compact space, it contains the two element discrete spaces, hence all closed subspaces of all products of two element spaces. But this is already all Stone spaces, and every compact T_2 space is the homomorphic image of a Stone space (for example look at the back adjunction of the adjointness).

Appendix

Categories tripleable over sets are more general than primitive classes of universal algebras, even if infinitary operations are admitted.

Definition: Let $\mathbf{T} = (T,\eta,\mu)$ be a triple in sets. Then the rank of $\mathbf{T}$ is the least regular cardinal α with the following property: For any set X and $x \in TX$ there is a subset $X' \overset{i}{\hookrightarrow} X$ with $\mathrm{card}(X') < \alpha$ and $x \in$ image (Ti). If such a cardinal does not exist, we say $\mathbf{T}$ has no rank.

The underlying set functor of a primitive class of algebras of a type with dimension α (as in Slominski [12]) is tripleable, as we have seen, and one can prove this triple has rank α. Conversely every triple of rank α is isomorphic to a triple obtained in this way. These results, unpublished, appear to be due to Linton. The mathematical tools required to prove them can be found in Manes [9] and Linton [6], [7].

As examples, the triple describing compact T_2 spaces has no rank. The triple describing the category of complete atomic Boolean algebras (the dual of the category of sets) has no rank. The triple describing the category of B_*-algebras (the dual of the category of compact T_2 spaces), with underlying set functor "taking the unit ball", is tripleable with rank $\aleph_1$. The latter two examples are consequences of the following theorem which is for the most part a corollary of the tripleableness theorem of §3.2.

Theorem: Let $\mathcal{A}$ be a category with cokernel pairs and equalizers in which every monomorphism is an equalizer. Suppose $\mathcal{A}$ has an injective cogenerator q together with all powers of q. Then $\mathcal{A}^{op}$ is tripleable over sets via the functor $\mathcal{A}^{op}(q,-)$ iff for all $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{A}$, $(\ (f,q),\quad (g,q))$ a kernel pair $\Rightarrow$ (f,g) a cokernel pair. The triple arising is isomorphic to $\mathbf{T} = (T,\eta,\mu)$ where:

$$TX = \mathcal{A}(q^X,q),\quad Tf = \mathcal{A}(q^f,q)$$

$$\eta X:\quad X \longrightarrow \mathcal{A}(q^X,q) \text{ by } x \rightsquigarrow \pi_x$$

$$\mu X:\quad \mathcal{A}(q^{\mathcal{A}(q^X,q)},q) \longrightarrow \mathcal{A}(q^X,q) \quad \text{by } \mu X = \mathcal{A}(jX,q)$$

where $\qquad jX:\ q^X \longrightarrow q^{\mathcal{A}(q^X,q)}$ by $\pi_\alpha jX = \alpha.$

(All triples $\mathbf{T}$ in sets have the above form for a suitable cogenerator $q \in (S^{\mathbf{T}})^{op}$, which in the general case need not be injective. One can take q the dual object of $F^{\mathbf{T}}(1)$.)

The rank of the above triple is the smallest regular cardinal α with the following property:

For any set X and map $q^X \xrightarrow{f} q$ there is a subset $X' \xhookrightarrow{i} X$ with card(X') $< \alpha$ such that f factors through the projection $q^X \xrightarrow{q^i} q^{X'}$.

The example of complete atomic Boolean algebras is obtained by taking $\mathcal{A}$ = sets and q = 2 (a two-element set). The example of B_*-algebras is obtained by taking $\mathcal{A}$ = compact T_2 spaces and q = the unit ball of the complex plane. One can also show, using this theorem, that for a ring R with unit, the category of compact unitary right R-modules (the dual of the category of all unitary left R-modules, by Pontrjagin duality) is tripleable, with usual underlying set functor. In this case one takes $\mathcal{A}$ = category of unitary left R-modules and q = Ab(R,C) where

C is the multiplicative group of complex numbers of norm 1, and Ab(R,C) is the group of abelian group homomorphisms from R (as an abelian group) to C, with its natural left R-module structure. One can show that for non-trivial rings R, such triples never have a rank (c.f. 1.3.2, example 2(d)).

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