ON CYCLIC STEINER QUADRUPLE SYSTEMS

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

(August 1971)

MASTER OF SCIENCE (1971)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario.

TITLE: On cyclic Steiner quadruple systems

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SUPERVISOR: Dr. A. Rosa

NUMBER OF PAGES: iii, 32

SCOPE AND CONTENTS:

This thesis is a contribution to the theory of Steiner quadruple systems. S-cyclic Steiner quadruple systems are defined and then as a main result it is shown that there exists exactly one S-cyclic Steiner quadruple system of order 20.

ACKNOWLEDGEMENTS

The author wishes to express his deep gratitude to Dr. A. Rosa for his valuable advice and guidance in the preparation of this thesis.

The author is also grateful to the McMaster University for providing financial assistance.

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Chapter I

Introduction

In this thesis we are concerned with the existence of a particular kind of Steiner quadruple systems. The origins of the problem go back as far as 1852 when J. Steiner [12] posed a problem equivalent to the construction of systems $S(k, k+1, v)$; on the other hand, E. H. Moore [7] seems to be the first to define in 1896 the systems $S(t, k, v)$ (see Def. 3 below) although the notion of a Steiner system can be traced back to be found stated implicitly by Kirkman in 1847 [6].

Let us start with necessary definitions:

Definition 1: A tactical configuration $C[k, \ell, \lambda, v]$ is a collection of k-subsets of a v-set such that each ℓ-subset of the v-set is contained in exactly λ sets of the tactical configuration.

Here v, k, ℓ, λ are positive integers, and $\ell \leq k \leq v$.

A necessary condition for the existence of a tactical configuration $C[k, \ell, \lambda, v]$ is known to be

$$(1) \qquad \lambda \binom{v-h}{\ell-h} \Big/ \binom{k-h}{\ell-h} = \text{integer}, \quad h = 0, 1, \ldots, \ell-1$$

(see, e.g. [5]). The problem of determining the values of k, ℓ and λ for which this condition is also sufficient, is not yet solved completely.

Another combinatorial configurations, balanced incomplete block designs (briefly BIBD's) became widely known primarily because of their use in statistics.

Definition 2: A BIBD is an arrangement of v distinct objects into b blocks such that each block contains exactly k distinct objects, each object occurs in exactly r different blocks, and every pair of distinct objects occurs together in exactly λ blocks.

Here it is required $3 \leqq k \leqq v$ in order to eliminate some trivial designs. Counting in two different ways the total number of occurences of elements, and the number of occurences of all pairs containing a fixed element, respectively, we get the well-known necessary conditions for the existence of a BIBD (see, e.g., [3])

(2) $\quad v.r = b.k$

(3) $\quad \lambda(v-1) = r(k-1)$

Comparing definitions 1 and 2, we see that BIBD's are tactical configurations with $\ell = 2$. On the other hand, if we take tactical configurations with $\lambda = 1$ we get the kind of configurations we are going to be primarily interested in:

Definition 3: A Steiner system $S(t, k, v)$ is a collection of k-subsets of a v-set such that each t-subset of the v-set is a subset of exactly one k-subset of the system.

A Steiner system on a v-set is said to be of order v. Here, of course, $2 \leqq t < k \leqq v$. Steiner systems $S(2, 3, v)$ are called

Steiner triple systems, Steiner systems $S(3, 4, v)$ are called Steiner quadruple systems, generally Steiner systems $S(k-1, k, v)$ are called Steiner k-tuple systems. Obviously, every Steiner system with $t = 2$ is a BIBD with $\lambda = 1$, and conversely; moreover, every Steiner system with $t > 2$ is a BIBD with $\lambda > 1$ (since every tactical configuration $C[k, \ell, \lambda, v]$ is also a tactical configuration $C[k, s, \lambda, v]$ for every $2 \leq s < \ell$) but the converse is not true in general (e.g., from the 4 non-isomorphic BIBD (8, 14, 7, 4, 3), only one is a Steiner quadruple system of order 8; see [8]).

Steiner triple systems are the "smallest" non-trivial Steiner systems (and, of course, smallest non-trivial BIBD's, too). It was proved by Kirkman [6] in 1847, and subsequently by many other authors, that the conditions (1) which in this case read

(4) $\quad v \equiv 1 \text{ or } 3 \pmod 6$

are not only necessary but also sufficient for the existence of Steiner triple systems.

Similarly, it was proved by Hanani [4], although more than a century later, that the necessary conditions for the existence of Steiner quadruple systems

(5) $\quad v \equiv 2 \text{ or } 4 \pmod 6$

are also sufficient; the proof by recursive method is rather involved (see our remarks in Chapter 2). Hanani in one of his more recent papers [5] proved that condition (1) is sufficient also for $k = 4, \ell = 3$ and every λ. Actually, he proved the following theorem:

The necessary and sufficient condition for the existence of a tactical configuration $C[4, 3, \lambda, v]$ is

$$(6)\quad \begin{cases} \lambda v \equiv 0 \pmod 2 \\ \lambda(v-1)(v-2) \equiv 0 \pmod 3 \\ \lambda v(v-1)(v-2) \equiv 0 \pmod 8 \end{cases}$$

In the next we shall be interested mostly in cyclic systems.

Definition 4: A Steiner system $S(t, k, v)$ is said to be cyclic if its automorphism group contains the cyclic group of order v.

Here by an automorphism group we understand the permutation group acting on elements of the Steiner system as letters; therefore, if S is cyclic there exists an automorphism (permutation) consisting of a single cycle of length v.

The necessary and sufficient condition for the existence of cyclic Steiner triple systems is known to be

$$(7)\quad v \equiv 1 \text{ or } 3 \pmod 6,\ v \neq 9$$

(the proof was given by Peltesohn [9]; see also Rosa [11]). However, the question about necessary and sufficient conditions for the existence of cyclic Steiner quadruple systems appears far from being settled; only few results about the existence or non-existence of cyclic Steiner quadruple systems are available (see Guregová-Rosa [2], Fitting [1]). Our thesis is intended to be a contribution in this direction.

In Chapter II, some general, mostly known results about

Steiner quadruple systems are discussed. For instance, the construction is shown how to obtain a Steiner quadruple system of order 2n from a one of order n, with an illustrating example.

Chapter III is devoted solely to cyclic Steiner quadruple systems and contains the main result we have obtained. So called S-cyclic Steiner quadruple systems are defined there, and with the aid of a CDC 6400 computer it is shown that there is exactly one S-cyclic Steiner quadruple system of order 20. It is perhaps worth remarking even at this stage that the only known cyclic Steiner quadruple systems (i.e. the ones of orders 10, 20, 26, 34, 50) are all S-cyclic.

Chapter II

Steiner quadruple systems

According to Definition 3, a Steiner quadruple system of order v is a collection of 4-subsets (called quadruples) of a v-set such that each 3-subset (or triple) of the v-set is a subset of exactly one quadruple. Hanani [4] proved the following fundamental theorem concerning the existence of Steiner quadruple systems:

Theorem: The necessary and sufficient condition for the existence of a Steiner quadruple system of order v is

(00) $v \equiv 2$ or $4 \pmod 6$.

The proof of necessity is quite easy: The number of quadruples in a $S(3, 4, v)$ equals $\binom{v}{3}/\binom{4}{3} = \frac{v(v-1)(v-2)}{24}$ which must obviously be an integer. On the other hand, the number of quadruples containing a fixed element and containing a fixed pair of elements, respectively, equals $\binom{v-1}{2}/\binom{3}{2} = \frac{(v-1)(v-2)}{6}$ and $\frac{v-2}{2}$, respectively, which again must be integers. The necessity of (00) follows.

The proof of sufficiency, as it is given in [4], is complicated enough: it uses recursive constructions which necessitate separate dealing with the following cases (which cover evidently all the possibilities):

(1) $v \equiv 4$ or $8 \pmod{12}$

(2) $v \equiv 4$ or $10 \pmod{18}$

(3) $v \equiv 34 \pmod{36}$

(4) $v \equiv 26 \pmod{36}$

(5) $v \equiv 2$ or $10 \pmod{24}$

(6) $v \equiv 14$ or $38 \pmod{72}$

In addition, for the cases $v = 14$ and $v = 38$ separate direct constructions were required.

We shall not reproduce here the whole recursive proof. Instead, we shall discuss a construction for obtaining an $S(3, 4, 2v)$ from an $S(3, 4, v)$. This construction is included in Hanani's proof and covers essentially the case (1); however, it was known for a long time (see, e.g., [1]). In addition, we shall discuss another, new construction for obtaining an $S(3, 4, 2v)$ from an $S(3, 4, v)$ described in [14]. Since we shall be concerned with SQS of order 20 and since it is well-known that the Steiner quadruple system of order 10 is unique, it is clear why we consider these constructions relevant.

Let S' and S'' be two SQS of the same order v constructed on two disjoint sets X' and X'', and let $\mathcal{B}'$ and $\mathcal{B}''$ be their respective sets of quadruples. Let f be any bijection from X' to X'' and consider the set $X = X' \cup X''$, $|X| = 2v$. Let $\mathcal{B}_0$ denote the collection of all 4-subsets of X of the form $\{x', y', f(x'), f(y')\}$ where $x', y' \in X'$, $x' \neq y'$.

<u>Construction 1</u>: [1, 7, 14]. For every quadruple

$$B' = \{x', y', z', t'\} \in \mathcal{B}', \text{ let}$$

$$\mathcal{B}_{B'} = \{\{x', y', f(z'), f(t')\}, \{f(x'), f(y'), z', t'\},$$
$$\{x', f(y'), f(z'), t'\}, \{f(x'), y', z', f(t')\},$$
$$\{x', f(y'), z', f(t')\}, \{f(x'), y', f(z'), t'\}\}.$$

Then the set of quadruples

$$\mathcal{B} = \mathcal{B}' \cup \mathcal{B}'' \cup \mathcal{B}_0 \cup (\bigcup_{B' \varepsilon \mathcal{B}'} \mathcal{B}_{B'})$$

is easily checked to form a Steiner quadruple system of order $2v$ on the set X containing two disjoint subsystems of order v, namely the ones on X' and on X''.

Construction 2: [14]. For every quadruple $B' = \{x', y', z', t'\} \varepsilon \mathcal{B}'$ let

$$\mathcal{B}^*_{B'} = \{\{f(x'), y', z', t'\}, \{x', f(y'), z', t'\},$$
$$\{x', y', f(z'), t'\}, \{x', y', z', f(t')\}\},$$

and for every quadruple $B'' = \{x'', y'', z'', t''\} \varepsilon \mathcal{B}''$, let

$$\mathcal{B}^*_{B''} = \{\{f^{-1}(x''), y'', z'', t''\}, \{x'', f^{-1}(y''), z'', t''\},$$
$$\{x'', y'', f^{-1}(z''), t''\}, \{x'', y'', z'', f^{-1}(t'')\}\}.$$

Then the set of quadruples

$$\mathcal{B}^* = \mathcal{B}_0 \cup (\bigcup_{B' \varepsilon \mathcal{B}'} \mathcal{B}^*_{B'}) \cup (\bigcup_{B'' \varepsilon \mathcal{B}''} \mathcal{B}^*_{B''})$$

is easily checked to form a Steiner quadruple system of order $2v$ on the set X.

<u>Example 1:</u> Take the unique SQS of order 10 in the form

$$\{i,\ i+1,\ i+3,\ i+4\},\ \{i,\ i+1,\ i+2,\ i+6\},\ \{i,\ i+2,\ i+4,\ i+7\},$$

$$i = 0,\ 1,\ \ldots,\ 9,$$

where integers are taken modulo 10 (see [2]), and apply to it Construction 1, and also Construction 2 to obtain two SQS of order 20; we denote those systems by $S_F(20)$ and $S_D(20)$, respectively. Let our 20 elements be $0, 1, \ldots, 9, \bar{0}, \bar{1}, \ldots, \bar{9}$.

The system $S_F(20)$, i.e. the SQS obtained by Construction 1, will consist of the following quadruples:

$$\left.\begin{array}{l}
\{i,\ i+1,\ i+3,\ i+4\},\ \{i,\ i+1,\ i+2,\ i+6\},\ \{i,\ i+2,\ i+4,\ i+7\}, \\
\{\bar{i},\ \overline{i+1},\ \overline{i+3},\ \overline{i+4}\},\ \{\bar{i},\ \overline{i+1},\ \overline{i+2},\ \overline{i+6}\},\ \{\bar{i},\ \overline{i+2},\ \overline{i+4},\ \overline{i+7}\}, \\
\{i,\ i+1,\ \overline{i+3},\ \overline{i+4}\},\ \{i,\ \overline{i+1},\ i+3,\ \overline{i+4}\},\ \{\bar{i},\ i+1,\ i+3,\ \overline{i+4}\}, \\
\{i,\ \overline{i+1},\ \overline{i+3},\ i+4\},\ \{\bar{i},\ i+1,\ \overline{i+3},\ i+4\},\ \{\bar{i},\ \overline{i+1},\ i+3,\ i+4\}, \\
\{i,\ i+1,\ \overline{i+2},\ \overline{i+6}\},\ \{i,\ \overline{i+1},\ i+2,\ \overline{i+6}\},\ \{\bar{i},\ i+1,\ i+2,\ \overline{i+6}\}, \\
\{i,\ \overline{i+1},\ \overline{i+2},\ i+6\},\ \{\bar{i},\ i+1,\ \overline{i+2},\ i+6\},\ \{\bar{i},\ \overline{i+1},\ i+2,\ i+6\}, \\
\{i,\ i+2,\ \overline{i+4},\ \overline{i+7}\},\ \{i,\ \overline{i+2},\ i+4,\ \overline{i+7}\},\ \{\bar{i},\ i+2,\ i+4,\ \overline{i+7}\}, \\
\{i,\ \overline{i+2},\ \overline{i+4},\ i+7\},\ \{\bar{i},\ i+2,\ \overline{i+4},\ i+7\},\ \{\bar{i},\ \overline{i+2},\ i+4,\ i+7\},
\end{array}\right\} i=0,1,\ldots,9$$

$\{i,\ j,\ \bar{i},\ \bar{j}\}$ $i, j = 0, 1, \ldots, 9$, $i \neq j$; the numbers in quadruples are reduced modulo 10.

The system $S_D(20)$, i.e. the SQS obtained by Construction 2, will consist of the following quadruples:

$$\{\overline{i}, i+1, i+3, i+4\}, \{i, \overline{i+1}, i+3, i+4\}, \{i, i+1, \overline{i+3}, i+4\},$$

$$\{i, i+1, i+3, \overline{i+4}\}, \{\overline{i}, i+1, i+2, i+6\}, \{i, \overline{i+1}, i+2, i+6\},$$

$$\{i, i+1, \overline{i+2}, i+6\}, \{i, i+1, i+2, \overline{i+6}\}, \{\overline{i}, i+2, i+4, i+7\},$$

$$\{i, \overline{i+2}, i+4, i+7\}, \{i, i+2, \overline{i+4}, i+7\}, \{i, i+2, i+4, \overline{i+7}\},$$

$$\{\overline{i}, \overline{i+1}, \overline{i+3}, i+4\}, \{\overline{i}, \overline{i+1}, i+3, \overline{i+4}\}, \{\overline{i}, i+1, \overline{i+3}, \overline{i+4}\},$$

$$\{i, \overline{i+1}, \overline{i+3}, \overline{i+4}\}, \{\overline{i}, \overline{i+1}, \overline{i+2}, i+6\}, \{\overline{i}, \overline{i+1}, i+2, \overline{i+6}\},$$

$$\{\overline{i}, i+1, \overline{i+2}, \overline{i+6}\}, \{i, \overline{i+1}, \overline{i+2}, \overline{i+6}\}, \{\overline{i}, \overline{i+2}, \overline{i+4}, i+7\},$$

$$\{\overline{i}, \overline{i+2}, i+4, \overline{i+7}\}, \{\overline{i}, i+2, \overline{i+4}, \overline{i+7}\}, \{i, \overline{i+2}, \overline{i+4}, \overline{i+7}\},$$

$\{i, j, \overline{i}, \overline{j}\}$, $i, j = 0, 1, \ldots, 9$, $i \neq j$; the numbers in quadruples are reduced modulo 10.

It is shown in [14] that any two SQS of order $2v$, one obtained by Construction 1 and the other by Construction 2, are necessarily non-isomorphic. Consequently, $S_F(20)$ and $S_D(20)$ are non-isomorphic.

Denote by $Q(v)$ the number of non-isomorphic SQS of order v. The following values or bounds are known for the function $Q(v)$ (cf. [14]):

v	2	4	8	10	14	16	20	22	26	28	32
$Q(v)$	1	1	1	1	4	≥ 8	≥ 2	≥ 2	≥ 1	≥ 20	≥ 16

Let us state for the sake of completeness that, apart from the described constructions, there is a well-known "product rule" for obtaining an $S(3, 4, mn)$ from an $S(3, 4, m)$ and an $S(3, 4, n)$. Also Rokowska [10] has described several constructions which in some cases (notably $v = 22$) have led to quadruple systems non-isomorphic to those obtained by Hanani in [4]. However, these constructions altogether leave us far from what have been attained for Steiner triple systems (comprehensive results on embedding, extension etc.).

Although the question of the existence of Steiner quadruple systems was settled by Hanani in [4] it seems desirable to have a direct or, at least a simpler proof than the one given by Hanani. It is thought that one possible way of trying to get such a proof would be attempting to construct cyclic SQS, although from what we already know about cyclic Steiner quadruple systems, it can be taken for granted that the problem is of substantially higher level of complexity than the one about constructing cyclic Steiner triple systems.

Chapter III

Cyclic Steiner quadruple systems

According to the Definition 4, a Steiner quadruple system $S(3, 4, v)$ is cyclic if its automorphism group contains a permutation consisting of a single cycle of length v. From now on we assume that the elements of SQS are the numbers $0, 1, 2, \ldots, v-1$, and since we shall be concerned only with the question of the existence of cyclic SQS, we may assume without loss of generality the cyclic automorphism to be the permutation

$$C = (0\ 1\ 2\ \ldots\ v-1).$$

Under these assumptions we can say that a Steiner quadruple system S is cyclic if it satisfies the condition

$$(*) \qquad (i, j, k, m) \in S \Longrightarrow (i+1, j+1, k+1, m+1) \in S,$$

where the numbers are taken modulo v.

To an arbitrary quadruple (i_1, i_2, i_3, i_4), $i_1 < i_2 < i_3 < i_4$, of a cyclic SQS of order v on elements $0, 1, 2, \ldots, v-1$ we may uniquely assign an associated quadruple (briefly a-quadruple) $Q = [a_1, a_2, a_3, a_4]$ where

$$(0) \qquad a_k = \min(|i_{k+1} - i_k|, v - |i_{k+1} - i_k|), \ k = 1, 2, 3, 4, \ i_5 = i_1.$$

The set of all different quadruples (two quadruples are regarded as different if they differ in at least one element) with the same a-quadruple Q is said to be a cyclic set with the a-quadruple Q and is denoted by $M(Q)$; the cyclic set $M(Q)$ is actually an orbit of quadruples (with a-quadruple Q) under the action of the cyclic group C_v generated by C. The number of elements of the cyclic set $M(Q)$ (= the length of the orbit of quadruples) is said to be the period of the a-quadruple Q and is denoted by $P(Q)$. Obviously $P(Q)$ must be a divisor of v. Two a-quadruples are regarded as equal if they differ only by a cyclic permutation of their elements; otherwise they are different.

It follows trivially from (*) that a cyclic SQS either contains simultaneously all the $P(Q)$ quadruples or contains no quadruple from the cyclic set $M(Q)$.

Let us turn now to graph-theoretical interpretation. The word graph here will mean an undirected graph without loops, with multiple edges allowed. Denote by the symbol $\langle n, k \rangle$ the graph with n vertices in which any two distinct vertices are joined by exactly k edges. By an s-vertex-clique we shall mean a subgraph with s vertices in which any two distinct vertices are joined by precisely one edge (i.e. it is the graph $\langle s, 1 \rangle$).

By a decompostion of a graph G into subgraphs $G_1, \ldots, G_r$ we always mean an edge-disjoint decomposition, i.e. every edge of G belongs to exactly one of the subgraphs $G_1, \ldots, G_r$ and the union of all subgraphs $G_1, \ldots, G_r$ is G. A cyclic decomposition of a graph G with v vertices into subgraphs $G_1, \ldots, G_r$ is a decomposition of G such that for every i there is an index j ($i, j \in \{1, \ldots, r\}$) such that $\bar{C}(G_i) = G_j$ where $\bar{C}$ is a permutation of vertices of G consisting of a single cycle of length v.

It is easy to see that the problem of finding an SQS of order v (a cyclic SQS of order v, respectively) is equivalent to the problem of finding a decomposition (a cyclic decomposition, respectively) $K = \{K_1, \ldots, K_r\}$ of the graph $< v, \frac{v-2}{2} >$ into 4-vertex-cliques such that each triangle with the vertices from $< v, \frac{v-2}{2} >$ occurs in exactly one clique of the decomposition K (here $r = v(v-1).(v-2)/24$). Here the elements of the SQS correspond obviously to the vertices of the graph $< v, \frac{v-2}{2} >$, the quadruples of the SQS correspond to the 4-vertex-cliques K_i.

In the next we shall use both combinatorial as well as graph-theoretical interpretation and terminology. Consider an arbitrary 4-vertex-clique K. The lengths of the edges of the clique K which lie on its "circumference" correspond to the elements of the corresponding a-quadruple (see Fig. 1). From the numbers $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_4$, $a_4 + a_1$, the two least numbers give the lengths of the "diagonals" of the clique. Denote these numbers by b_1, b_2. Consequently,

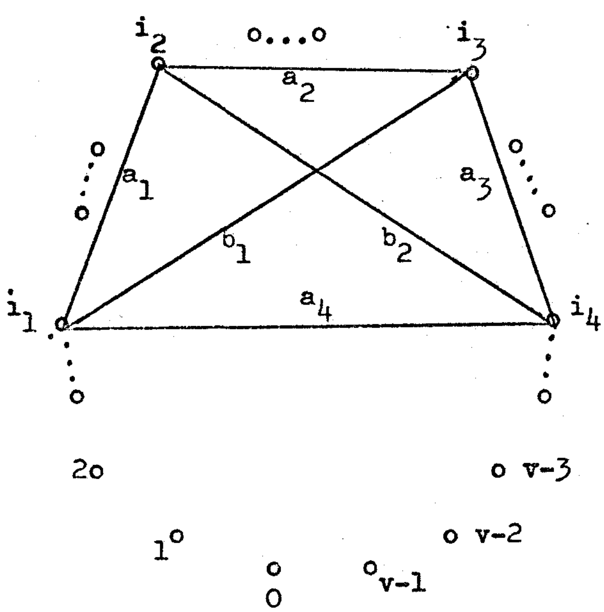

Figure 1

the six numbers $a_1, a_2, a_3, a_4, b_1, b_2$ give us the lengths of the edges of the given clique K. Denote now by $\alpha_s(K)$, $s = 1, 2, \ldots, \frac{1}{2}v$, the number of occurrences of an edge of the length s in the clique K. Obviously we must have

$$\sum_{s=1}^{\frac{1}{2}v} \alpha_s(K) = 6.$$

Further, each clique K contains 4 triangles: without loss of generality we may assume them to be the triangles with the length of edges (in the shown order)

(**) $\quad a_1a_2b_1,\ a_2a_3b_2,\ a_3a_4b_1,\ a_4a_1b_2$

Defining now $\alpha_s(Q)$ analogously to $\alpha_s(K)$, we can formulate some necessary conditions for the existence of a cyclic SQS of order v:

<u>Proposition 1</u>: For any cyclic Steiner quadruple system S of order v,

$$\sum_{Q} \alpha_s(Q)P(Q)/v = \frac{v-2}{2}, \quad s = 1, 2, \ldots, \frac{v}{2} - 1;$$

$$\sum_{Q} \alpha_s(Q)\,\frac{P(Q)}{v} = \frac{v-2}{4}, \quad s = \frac{1}{2}v$$

where the sum is extended over all different a-quadruples Q which correspond to the cyclic sets occuring in S.

<u>Proof</u>: Follows easily from graph-theoretical interpretation.

<u>Proposition 2:</u> Let S be a cyclic SQS containing the cyclic set $M(Q)$ and let $m_1 m_2 m_3$ be one of the triples (**) corresponding to the a-quadruple Q. Let $Q_1, Q_2, \ldots, Q_t$ be all the remaining a-quadruples to which the same triple $m_1 m_2 m_3$ corresponds (together with some other three triples). Then

(i) If two of the numbers m_1, m_2, m_3 are equal then S does not contain any cyclic set $M(Q_i)$, $i = 1, 2, \ldots, t$.

(ii) If all three numbers m_1, m_2, m_3 are mutually different then S contains at most one cyclic set $M(Q_i)$, $i \in \{1, 2, \ldots, t\}$.

<u>Proof:</u> Assume first that two of the numbers m_1, m_2, m_3 are equal; let, without loss of generality, say, $m_1 = m_2$. Let, say, i_1, i_2, i_3 be the elements of S whose cyclic differences defined by (0) are given by the numbers m_1, m_2, m_3. If S contains a cyclic set $M(Q_i)$ for any $i \in \{1, 2, \ldots, t\}$ then the triple of elements i_1, i_2, i_3 must belong to two different quadruples of S which is a contradiction with the definition of a Steiner quadruple system. Assume now that all three numbers m_1, m_2, m_3 are mutually different. Let, say, i_1, i_2, i_3 be the elements of S whose cyclic differences defined by (0) are given by the numbers m_1, m_2, m_3 (in this order), and let i_4 be such that the cyclic differences of the elements i_1, i_2, i_4 are the numbers m_1, m_3, m_2 (in this order). It is clear now that if S contains more than one cyclic set $M(Q_i)$ then either the triple i_1, i_2, i_3 or the triple i_1, i_2, i_4 must belong to two different quadruples of S which is again a contradiction.

<u>Proposition 3</u>: Let the four triples $T_1 = m_1m_2m_3$, $T_2 = m_1m_2m_3$, T_3, T_4 correspond to an a-quadruple Q and let $P(Q) = v$. Then no cyclic SQS of order v contains the cyclic set $M(Q)$.

<u>Proof</u>: Again, if any cyclic SQS would contain such a cyclic set $M(Q)$, there would exist a triple of elements occuring in two different quadruples (which would this time belong to the same cyclic set) - a contradiction.

Notice that the condition $P(Q) = v$ cannot be omitted here; without it the Proposition 3 would be false.

<u>Proposition 4</u>: Let the four triples

$$T_1' = m_1m_2m_3, \quad T_2' = n_1n_2n_3, \quad T_3', T_4'$$

and

$$T_1'' = m_1m_2m_3, \; T_2'' = n_1n_3n_2, \; T_3'', \; T_4''$$

correspond to a-quadruples Q' and Q'', respectively. Then no Steiner quadruple system of order v contains simultaneously the cyclic sets $M(Q')$ and $M(Q'')$.

<u>Proof</u>: One proceeds similarly as in the proof of (ii) in Proposition 2.

The necessary conditions contained in Proposition 1 - 4 were used by Guregová and Rosa [2] to investigate the existence of cyclic SQS up to the order 16. The Table 1 gives a survey of the known results on the existence of cyclic Steiner Quadruple systems.

Table 1

v	Existence of a cyclic Solution	Solution	Reference
4	yes	trivial (1,2,3,4)	
8	no		[2]
10	yes	(i,i+1,i+3,i+4), (i,i+1,1+2,i+6), (i,i+2,i+4,i+7), where i=0,1, ..., 9, addition taken modulo 10	[2]
14	no		[2]
16	no		[2]
20	?		
22	?		
26	yes		[1],[2]
28	?		
32	?		
34	yes		[1],[15]
38,40, 44,46	?		
50	yes		[15]

The first order for which the existence of a cyclic SQS is undecided is $v = 20$. We shall fill in this gap by showing the existence of a cyclic quadruple system of order 20. In fact, we shall prove a little more than that but first we have to introduce two definitions:

Definition 3.1: An a-quadruple $[a_1, a_2, a_3, a_4]$ is called symmetric if at least one of the following two conditions is satisfied:

(i) the length of the diagonals are equal (i.e. $b_1 = b_2$)

(ii) either $a_1 = a_2$ and $a_3 = a_4$ or $a_1 = a_4$ and $a_2 = a_3$.

Definition 3.2: A cyclic SQS of order v is called S-cyclic if all the a-quadruples corresponding to the cyclic sets occuring in it are symmetric.

Now we are able to formulate our

Main result: There exists exactly one S-cyclic Steiner quadruple system of order 20.

The procedure we use to derive the main result is essentially the one used by Guregová-Rosa in [2]; however, due to the comparatively large order $v = 20$ and, consequently, large number of a-quadruples, we restricted ourselves to symmetric quadruples, i.e. we were looking only for S-cyclic quadruple systems of order 20. This restriction may prove not to be so severe as it may appear at first since all the known cyclic SQS (see Table 1) are in fact S-cyclic; in other words there is not a single cyclic Steiner quadruple system known (of any order) which is not S-cyclic.

We start with the list of all possible a-quadruples corresponding to all possible quadruples on the elements 0, 1, 2, ..., 19. This list is given in Table 2 and the total number of a-quadruples in it is 145. The a-quadruples (17), (86), (125) and (142) have period 10 and the a-quadruple (145) has period 5; all the other a-quadruples have period 20.

Table 2

No.	Triangles				Associated Quadruples
1	1 1 2	1 1 2	1 2 3	1 3 2	[1 1 1 3]
2	1 1 2	1 2 3	1 3 4	2 2 4	[1 1 2 4]
3	1 2 3	1 3 2	1 3 4	1 4 3	[1 2 1 4]
4	1 1 2	1 3 4	1 4 5	2 3 5	[1 1 3 5]
5	1 3 4	1 4 3	1 4 5	1 5 4	[1 3 1 5]
6	1 1 2	1 4 5	1 5 6	2 4 6	[1 1 4 6]
7	1 4 5	1 5 4	1 5 6	1 6 5	[1 4 1 6]
8	1 1 2	1 5 6	1 6 7	2 5 7	[1 1 5 7]
9	1 5 6	1 6 5	1 6 7	1 7 6	[1 5 1 7]
10	1 1 2	1 6 7	1 7 8	2 6 8	[1 1 6 8]
11	1 6 7	1 7 6	1 7 8	1 8 7	[1 6 1 8]
12	1 1 2	1 7 8	1 8 9	2 7 9	[1 1 7 9]
13	1 7 8	1 8 7	1 8 9	1 9 8	[1 7 1 9]
14	1 1 2	1 8 9	1 9 10	2 8 10	[1 1 8 10]
15	1 8 9	1 9 8	1 9 10	1 10 9	[1 8 1 10]
16	1 1 2	1 9 10	1 10 9	2 9 9	[1 1 9 9]
17	1 9 10	1 9 10	1 10 9	1 10 9	[1 9 1 9]
18	1 2 3	1 4 5	2 2 4	2 5 3	[1 2 2 5]
19	1 2 3	1 3 2	2 3 5	2 5 3	[1 2 5 2]
20	1 2 3	1 5 6	2 3 5	3 3 6	[1 2 3 6]

No.	Triangles				Associated Quadruples
21	1 2 3	1 4 3	2 6 4	3 3 6	[1 2 6 3]
22	1 3 4	1 5 6	2 5 3	2 6 4	[1 3 2 6]
23	1 2 3	1 6 7	2 4 6	3 4 7	[1 2 4 7]
24	1 2 3	1 5 4	2 7 5	3 7 4	[1 2 7 4]
25	1 4 5	1 6 7	2 6 4	2 7 5	[1 4 2 7]
26	1 2 3	1 7 8	2 5 7	3 5 8	[1 2 5 8]
27	1 2 3	1 6 5	2 8 6	3 8 5	[1 2 8 5]
28	1 5 6	1 7 8	2 7 5	2 8 6	[1 5 2 8]
29	1 2 3	1 8 9	2 6 8	3 6 9	[1 2 6 9]
30	1 2 3	1 7 6	2 9 7	3 9 6	[1 2 9 6]
31	1 6 7	1 8 9	2 8 6	2 9 7	[1 6 2 9]
32	1 2 3	1 9 10	2 7 9	3 7 10	[1 2 7 10]
33	1 2 3	1 8 7	2 10 8	3 10 7	[1 2 10 7]
34	1 7 8	1 9 10	2 9 7	2 10 8	[1 7 2 10]
35	1 2 3	1 10 9	2 8 10	3 8 9	[1 2 8 9]
36	1 2 3	1 9 8	2 9 9	3 9 8	[1 2 9 8]
37	1 8 9	1 10 9	2 9 9	2 10 8	[1 8 2 9]
38	1 3 4	1 6 7	3 3 6	3 7 4	[1 3 3 7]
39	1 3 4	1 4 3	3 4 7	3 7 4	[1 3 7 3]
40	1 3 4	1 7 8	3 4 7	4 4 8	[1 3 4 8]
41	1 3 4	1 5 4	3 8 5	4 4 8	[1 3 8 4]
42	1 4 5	1 7 8	3 7 4	3 8 5	[1 4 3 8]
43	1 3 4	1 8 9	3 5 8	4 5 9	[1 3 5 9]
44	1 3 4	1 6 5	3 9 6	4 9 5	[1 3 9 5]
45	1 5 6	1 8 9	3 8 5	3 9 6	[1 5 3 9]

No.		Triangles			Associated Quadruples
46	1 3 4	1 9 10	3 6 9	4 6 10	[1 3 6 10]
47	1 3 4	1 7 6	3 10 7	4 10 6	[1 3 10 6]
48	1 6 7	1 9 10	3 9 6	3 10 7	[1 6 3 10]
49	1 3 4	1 10 9	3 7 10	4 7 9	[1 3 7 9]
50	1 3 4	1 8 7	3 9 8	4 9 7	[1 3 9 7]
51	1 7 8	1 10 9	3 9 8	3 10 7	[1 7 3 9]
52	1 3 4	1 9 8	3 8 9	4 8 8	[1 3 8 8]
53	1 8 9	1 9 8	3 8 9	3 9 8	[1 8 3 8]
54	1 4 5	1 8 9	4 4 8	4 9 5	[1 4 4 9]
55	1 4 5	1 5 4	4 5 9	4 9 5	[1 4 9 4]
56	1 4 5	1 9 10	4 5 9	5 5 10	[1 4 5 10]
57	1 4 5	1 6 5	4 10 6	5 5 10	[1 4 10 5]
58	1 5 6	1 9 10	4 9 5	4 10 6	[1 5 4 10]
59	1 4 5	1 10 9	4 6 10	5 6 9	[1 4 6 9]
60	1 4 5	1 7 6	4 9 7	5 9 6	[1 4 9 6]
61	1 6 7	1 10 9	4 9 7	4 10 6	[1 6 4 9]
62	1 4 5	1 9 8	4 7 9	5 7 8	[1 4 7 8]
63	1 4 5	1 8 7	4 8 8	5 8 7	[1 4 8 7]
64	1 7 8	1 9 8	4 8 8	4 9 7	[1 7 4 8]
65	1 5 6	1 10 9	5 5 10	5 9 6	[1 5 5 9]
66	1 5 6	1 6 5	5 6 9	5 9 6	[1 5 9 5]
67	1 5 6	1 9 8	5 6 9	6 6 8	[1 5 6 8]
68	1 5 6	1 7 6	5 8 7	6 6 8	[1 5 8 6]
69	1 6 7	1 9 8	5 8 7	5 9 6	[1 6 5 8]
70	1 5 6	1 8 7	5 7 8	6 7 7	[1 5 7 7]

No.	Triangles				Associated Quadruples
71	1 7 8	1 8 7	5 7 8	5 8 7	[1 7 5 7]
72	1 6 7	1 8 7	6 7 7	6 6 8	[1 6 6 7]
73	1 6 7	1 7 6	6 7 7	6 7 7	[1 6 7 6]
74	2 2 4	2 2 4	2 4 6	2 6 4	[2 2 2 6]
75	2 2 4	2 3 5	2 5 7	3 7 4	[2 2 3 7]
76	2 3 5	2 5 3	2 5 7	2 7 5	[2 3 2 7]
77	2 2 4	2 4 6	2 6 8	4 4 8	[2 2 4 8]
78	2 4 6	2 6 4	2 6 8	2 8 6	[2 4 2 8]
79	2 2 4	2 5 7	2 7 9	4 5 9	[2 2 5 9]
80	2 5 7	2 7 5	2 7 9	2 9 7	[2 5 2 9]
81	2 2 4	2 6 8	2 8 10	4 6 10	[2 2 6 10]
82	2 6 8	2 8 6	2 8 10	2 10 8	[2 6 2 10]
83	2 2 4	2 7 9	2 9 9	4 7 9	[2 2 7 9]
84	2 7 9	2 9 7	2 9 9	2 9 9	[2 7 2 9]
85	2 2 4	2 8 10	2 10 8	4 8 8	[2 2 8 8]
86	2 8 10	2 8 10	2 10 8	2 10 8	[2 8 2 8]
87	2 3 5	2 6 8	3 3 6	3 8 5	[2 3 3 8]
88	2 3 5	2 5 3	3 5 8	3 8 5	[2 3 8 3]
89	2 3 5	2 7 9	3 4 7	4 9 5	[2 3 4 9]
90	2 3 5	2 6 4	3 9 6	4 5 9	[2 3 9 4]
91	2 4 6	2 7 9	3 7 4	3 9 6	[2 4 3 9]
92	2 3 5	2 8 10	3 5 8	5 5 10	[2 3 5 10]
93	2 3 5	2 7 5	3 10 7	5 5 10	[2 3 10 5]
94	2 5 7	2 8 10	3 8 5	3 10 7	[2 5 3 10]
95	2 3 5	2 9 9	3 6 9	5 6 9	[2 3 6 9]

No.	Triangles				Associated Quadruples
96	2 3 5	2 8 6	3 9 8	5 9 6	[2 3 9 6]
97	2 6 8	2 9 9	3 9 6	3 9 8	[2 6 3 9]
98	2 3 5	2 10 8	3 7 10	5 7 8	[2 3 7 8]
99	2 3 5	2 9 7	3 8 9	5 8 7	[2 3 8 7]
100	2 7 9	2 10 8	3 8 9	3 10 7	[2 7 3 8]
101	2 4 6	2 8 10	4 4 8	4 10 6	[2 4 4 10]
102	2 4 6	2 6 4	4 6 10	4 10 6	[2 4 10 4]
103	2 4 6	2 9 9	4 5 9	5 9 6	[2 4 5 9]
104	2 4 6	2 7 5	4 9 7	5 6 9	[2 4 9 5]
105	2 5 7	2 9 9	4 9 5	4 9 7	[2 5 4 9]
106	2 4 6	2 10 8	4 6 10	6 6 8	[2 4 6 8]
107	2 4 6	2 8 6	4 8 8	6 6 8	[2 4 8 6]
108	2 6 8	2 10 8	4 8 8	4 10 6	[2 6 4 8]
109	2 4 6	2 9 7	4 7 9	6 7 7	[2 4 7 7]
110	2 7 9	2 9 7	4 7 9	4 9 7	[2 7 4 7]
111	2 5 7	2 10 8	5 5 10	5 8 7	[2 5 5 8]
112	2 5 7	2 7 5	5 7 8	5 8 7	[2 5 8 5]
113	2 5 7	2 9 7	5 6 9	6 7 7	[2 5 6 7]
114	2 5 7	2 8 6	5 7 8	6 7 7	[2 5 7 6]
115	2 6 8	2 9 7	5 7 8	5 9 6	[2 6 5 7]
116	2 6 8	2 8 6	6 6 8	6 6 8	[2 6 6 6]
117	3 3 6	3 3 6	3 6 9	3 9 6	[3 3 3 9]
118	3 3 6	3 4 7	3 7 10	4 10 6	[3 3 4 10]
119	3 4 7	3 7 4	3 7 10	3 10 7	[3 4 3 10]
120	3 3 6	3 5 8	3 8 9	5 9 6	[3 3 5 9]

No.	Triangles				Associated Quadruples
121	3 5 8	3 8 5	3 8 9	3 9 8	[3 5 3 9]
122	3 3 6	3 6 9	3 9 8	6 6 8	[3 3 6 8]
123	3 6 9	3 9 6	3 8 9	3 9 8	[3 6 3 8]
124	3 3 6	3 7 10	3 10 7	6 7 7	[3 3 7 7]
125	3 7 10	3 7 10	3 10 7	3 10 7	[3 7 3 7]
126	3 4 7	3 8 9	4 4 8	4 9 7	[3 4 4 9]
127	3 4 7	3 7 4	4 7 9	4 9 7	[3 4 9 4]
128	3 4 7	3 9 8	4 5 9	5 8 7	[3 4 5 8]
129	3 4 7	3 8 5	4 8 8	5 7 8	[3 4 8 5]
130	3 5 8	3 9 8	4 8 8	4 9 5	[3 5 4 8]
131	3 4 7	3 10 7	4 6 10	6 7 7	[3 4 6 7]
132	3 4 7	3 9 6	4 7 9	6 7 7	[3 4 7 6]
133	3 6 9	3 10 7	4 7 9	4 10 6	[3 6 4 7]
134	3 5 8	3 10 7	5 7 8	5 5 10	[3 5 5 7]
135	3 5 8	3 8 5	5 7 8	5 8 7	[3 5 7 5]
136	3 5 8	3 9 6	5 6 9	6 6 8	[3 5 6 6]
137	3 6 9	3 9 6	5 6 9	5 9 6	[3 6 5 6]
138	4 4 8	4 4 8	4 8 8	4 8 8	[4 4 4 8]
139	4 4 8	4 5 9	4 9 7	5 7 8	[4 4 5 7]
140	4 5 9	4 9 5	4 7 9	4 9 7	[4 5 4 7]
141	4 4 8	4 6 10	4 10 6	6 6 8	[4 4 6 6]
142	4 6 10	4 6 10	4 10 6	4 10 6	[4 6 4 6]
143	4 5 9	4 10 6	5 5 10	5 6 9	[4 5 5 6]
144	4 5 9	4 9 5	5 6 9	5 9 6	[4 5 6 5]
145	5 5 10	5 5 10	5 5 10	5 5 10	[5 5 5 5]

Since a Steiner quadruple system of order 20 contains 285 quadruples, three cases can occur:

(i) the cyclic SQS consists of 12 cyclic sets with a-quadruples of period 20, 4 cyclic sets with a-quadruples of period 10 and one cyclic set with a-quadruple of period 5;

(ii) the cyclic SQS consists of 13 cyclic sets with a-quadruples of period 20, 2 cyclic sets with a-quadruples of period 10 and one cyclic set with a-quadruples of period 5;

(iii) the cyclic SQS consists of 14 cyclic sets with a-quadruples of period 20 and one cyclic set with a-quadruple of period 5.

In all three cases the cyclic SQS must contain a cyclic set with a-quadruple of period 5, and since there is only one such cyclic set available (namely the one with the a-quadruple (145) from Table 2), the cyclic sets with the following a-quadruples cannot occur by Proposition 2 in any cyclic SQS of order 20 (and therefore these a-quadruples may be removed from the list in Table 2):

(56), (57), (65), (92), (93), (111), (134), (143).

According to Proposition 3, the following a-quadruples may be also removed from the list: (1), (73), (74), (84), (116), (117), (138).

Unfortunately this leaves us still with too large a number of a-quadruples, therefore at this point we select from them only the symmetric ones. Thus from now on we operate with the reduced list of a-quadruples containing the following symmetric a-quadruples (with numbers from Table 2):

(#) (3), (5), (7), (9), (11), (13), (15), (16), (19), (39), (53), (55), (66), (71), (76), (78), (80), (82), (85), (88), (102), (110), (112), (119), (121), (123), (124), (127), (135), (137), (140), (141), (144).

For the a-quadruples from (#), using the Proposition 2 and 4 a so-called system of prohibitions $Z = Z(Q)$ was formed, i.e., for any given a-quadruple Q the a-quadruples $Q_1, Q_2, \ldots, Q_t$ are found such that the cyclic sets with any of the a-quadruple Q_i, $i \in \{1, 2, \ldots, t\}$ cannot occur in a cyclic SQS of order 20 simultaneously with the cyclic set with the a-quadruple Q.

For the further computations we used the CDC 6400 computer. The following three cases have to be dealt with separately:

Case 1. We have to find all possible combinations of 12 a-quadruples from ($\neq$) satisfying the system of prohibitions Z and giving (by Proposition 1) in the sum of numbers $\alpha_s(Q)$ the vector S_1 (see Table 3).

Case 2. We have to find all possible combinations of 13 a-quadruples from ($\neq$) satisfying the system of prohibitions Z and giving (by Proposition 1) in the sum of numbers $\alpha_s(Q)$ one of the vectors S_2 to S_7 (see Table 3).

Case 3. We have to find all possible combinations of 14 a-quadruples satisfying the system of prohibitions Z such that the sum of the numbers $\alpha_s(Q)$ will give the vector S_8 (see Table 3).

Table 3

s	1	2	3	4	5	6	7	8	9	10
s_1	8	8	8	8	8	8	8	8	8	0
s_2	8	9	8	9	8	9	8	9	8	2
s_3	8	9	9	8	8	8	9	9	9	2
s_4	9	8	8	9	8	9	8	8	9	2
s_5	9	8	9	8	8	8	9	8	9	2
s_6	9	9	8	8	8	8	8	9	9	2
s_7	8	8	9	9	8	9	9	8	8	2
s_8	9	9	9	9	8	9	9	9	9	4

The computer was required to find all such combinations in all of the three above cases. Let us remark that if such a combination of a-quadruples is found it does not produce automatically a cyclic Steiner quadruple system; in other words, even if all necessary conditions given by Propositions 1-4 are satisfied they are in general not sufficient for the existence of a cyclic SQS (cf. [2]); however they are sufficient if we deal only with symmetric a-quadruples. In our case, the procedure used was the usual back-track procedure (see, e.g., [13]); the corresponding program for case 3 in FORTRAN 4 is given in Appendix 1.

The running time (for all three cases together) was 846.093 seconds. For cases 1 and 2, there was no output, and for the case 3 there exists exactly one combination of 14 a-quadruples from (#) satisfying all the above requirements. In accordance with what we said in the previous paragraph, we obtained the following S-cyclic Steiner quadruple system of order 20 (denote it by S_C (20)):

(i,i+1,i+3,i+4), (i,i+1,i+2,i+11), (i,i+1,i+5,i+16), (i,i+2,i+6,i+8), (i,i+2,i+4,i+12), (i,i+3,i+6,i+13), (i,i+3,i+9,i+14), (i,i+1,i+6,i+7), (i,i+1,i+9,i+12), (i,i+1,i+8,i+13), (i,i+2,i+7,i+9), (i,i+2,i+5,i+17), (i,i+3,i+7,i+16), (i,i+4,i+8,i+14), (i,i+5,i+10,i+15), where

i = 0, 1, 2, ..., 19 and the numbers in quadruples are taken mod 20.

As it was mentioned in Chapter 2, there are two non-isomorphic Steiner quadruple systems of order 20 known so far [14]: $S_F(20)$ and $S_D(20)$. It is conjectured that $S_C(20)$ is not isomorphic to any of those two systems but we were not able yet to establish this with certainty. On the other hand, the cyclic system $S_C(20)$ contains 2 subsystems of order 10 (one on the even numbers and the other on the odd ones) similarly as $S_F(20)$ does.

It is hoped that further investigation of the structure of the cyclic SQS of order 20 will prove useful for the task of finding a simpler (direct) proof of the existence of Steiner quadruple systems of every admissible order v.

APPENDIX 1

Computer Program for Case 3

```
        Dimension JA(33, 10), JB(33, 3), IA(14), IS(10)
        Read (5, 24) ((JA(I, J), J = 1, 10), I = 1, 33)
24      Format (70I1)
        Read (5, 25) ((JB(I, J), J = 1, 3), I = 1, 33)
25      Format (3I2)
        Read (5, 28) (IS(I), I = 1, 10)
28      Format (10I1)
        M = 0
        I = 1
        K = 1
410     IA(I) = K
        I = I + 1
        IF(I.EQ.15) Go To 401
403     K = K + 1
        IF(K.EQ.34) Go To 402
        KK = I - 1
        DO 10KL = 1, KK
        DO 10 J = 1, 3
        KB = IA(KL)
        IF(K.EQ.JB(KB,J)) Go To 403
10      Continue
        Go To 410
```

```
401     DO 11 II = 1, 10
        ISUM = 0
        DO 12 JJ = 1, 14
        IAJ = IA(JJ)
12      ISUM = ISUM + JA(IAJ, II)
        IF (ISUM. NE. IS(II)) Go To 404
11      Continue
        M = M + 1
        Write (6, 26) (IA(L), L = 1, 14)
26      Format (14I6)
404     I = 14
        Go to 403
402     I = I - 1
        IF (I.EQ.1) Go To 409
        K = IA(I)
        Go To 403
409     IF (M.NE.0) Go To 408
        Write (6, 27)
27      Format (1H1, *NO Combination Available*)
408     Stop
        End
```

References

[1] Fitting, F., Zyklische Lösungen des Steiner'schen Problems, Nieuw Arch. Wisk. (2) 11 (1915).

[2] Guregová, M. and Rosa, A., Using the computer to investigate cyclic Steiner quadruple systems, Mat. Čas. 18 (1968) 229-239.

[3] Hall, M. Jr., Combinatorial theory, Blaisdell, Walthem, Mass. 1967.

[4] Hanani, H., On quadruple systems, Canadian J. Math. 12 (1960), 145-157.

[5] Hanani, H., On some tactical configurations, Canadian J. Math. 15(1963), 702-722.

[6] Kirkman, Rev. Thomas., On a problem in combinations, Camb. and Dublin Math. J. 2 (1847), 191-204.

[7] Moore, E. H., Tactical memoranda, American J. Math. 18 (1896), 264-303.

[8] Mullin, R. C., Stanton, R. G., Uniqueness theorems in balanced incomplete block designs, J. Combinatorial Theory 7 (1969), 37-48.

[9] Peltesohn, R., Eine Lösung der beiden Heffterschen Differenzenprobleme, Compositio Math. 6 (1939), 251-257.

[10] Rokowska, B., Some new constructions of 4-tuple systems, Colloquium Mathematicum, Vol. XVII, 1967, FASC. I, 111-121.

[11] Rosa, A., Poznámka o cyklických Steinerových systémoch trojíc, Mat.-fyz. Čas. 16(1966), 285-290.

[12] Steiner, J., Combinatorische Aufgabe, J. Reine Angew. Math 45, (1853), 181-182.

[13] Walker, R. J., An enumerative technique for a class of combinatorial problems, Chapter 7 in R. Bellman and M. Hall, Jr., (editors), Combinatorial Analysis, Proc. Sympos. Appl. Math., 10, 91-94, American Mathematical Society, Providence, Rhode Island, 1960.

[14] Doyen, J., Vandensavel, M., Non isomorphic Steiner quadruple systems, (to appear).

[15] Rosa, A., Unpublished result.