MATROIDS ON COMPLETE BOOLEAN ALGEBRAS

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SCOPE AND CONTENTS: The approach to a theory of non-finitary matroids, as outlined by the author in [20], is here extended to the case in which the relevant closure operators are defined on arbitrary complete Boolean algebras, rather than on the power sets of sets. As a preliminary to this study, the theory of derivatives of operators on complete Boolean algebras is developed and the notion, having interest in its own right, of an analytic closure operator is introduced. The class of B-matroidal closure operators is singled out for especial attention and it is proved that this class is closed under Whitney duality. Also investigated is the class of those closure operators which are both matroidal and topological.

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INTRODUCTION

Ever since the early days, matroid theory has had two unreconciled aspects. Van der Waerden in Moderne Algebra [42] abstracted the common features of linear and algebraic dependence, giving axioms for what would now be called a finitary matroid, not necessarily finite. In his fundamental paper [43], Whitney, considering only the finite case, introduced the concept of matroid duality (along with many other of the basic ideas of the subject) - this was by abstracting from the duality of planar graphs. Until recently, work on matroids has followed either the former path - infinite (finitary) matroids allowed but no duality, or the latter - finite matroids only and hence duality (in potential at least). The truth of the matter is that there does not exist a completely adequate theory of duality for the class of finitary matroids and it was this fact which provided the initial motivation for considering the possibility of dropping the requirement that matroidal closure operators be finitary, the hope being that thereby a class of closure operators would be obtained which both admitted duality and contained the class of finitary matroids as a subclass.

The question then arose as to which axioms to take. In the case of finitary matroids one has a closure operator f on a set E satisfying the exchange condition:

p in $f(S \cup \{q\})$, p not in $f(S)$ implies q in $f(S \cup \{p\})$, together with the finitary condition:

p in $f(S)$ implies p in $f(T)$ for some finite subset T of S. The simple omission of the finitary condition, so that only exchange is retained, gives too wide a class of closure operators: every T_1 topological closure operator is included for example; and, more to the point, no satisfactory duality theory is obtained. Now the finitary and exchange conditions together have as a consequence the following minimality condition:

$X \subseteq f(S \cup Y)$ implies that there exists a minimal subset Z of Y such that $X \subseteq f(S \cup Z)$

(see results (84) and (35) below). It therefore seemed reasonable to take as axioms the exchange condition together with this minimality condition and on this basis a satisfactory duality was obtained which behaved for the closure operators considered exactly as does Whitney's original duality for finite matroids, the latter indeed being a special case of the former. It was found subsequently that the closure operators satisfying the exchange and minimality conditions could be described in a somewhat more simple way - they are precisely the B-matroidal closure operators as defined in [20]. (The equivalence of these two approaches to B-matroidal closure operators is established by (35), in the more general Boolean setting of this thesis.)

At the same time, this work did not really get to the essence of matroid duality. Now what one wants in mathematics, one postulates; it was found that the most convenient way to postulate matroid duality was to use the "derived set" operator or <u>derivative</u> d corresponding to a given closure operator f on the set E, where $d(X) = \{p;\ p \in f(X \setminus \{p\})\}$ for all $X \subseteq E$. Then $f(X) = X \cup d(X)$ for all $X \subseteq E$ and duality is obtained by defining $d^*(X) = E \setminus d(E \setminus X)$ for all $X \subseteq E$ and requiring that g, where $g(X) = X \cup d^*(X)$ for all $X \subseteq E$, be a closure operator on E also. The closure operators f for which this happens are then, in a very natural sense, the most general for which a precise duality holds such that one has closure operators both sides of the duality, and they are the closure operators which are termed <u>matroidal</u> in [20].

It turns out that the class of all matroidal closure operators is considerably more extensive than the class of B-matroidal clsoure operators; in particular, fascinatingly, the former class has a non-trivial intersection with the class of topological closure operators (a preliminary study of this intersection is made in Chapter 4 and in the first part of Appendix 3). In the course of a literature search, it was found that a concept of duality, essentially equivalent to that expressed above in terms of derivatives, had been set out by Sierpinski in 1945 [38];

however, he was not concerned with closure operators but only with preclosures = expansive functions (drop the idempotency requirement). The definition of matroids and B-matroids (on sets), together with the theorem that the dual of a B-matroid is again a B-matroid (so that B-matroids are matroids), was written up in [20] (in which a third, intermediate, and less interesting class of matroids, the so-called C-matroids, was also discussed).

By this time, it had become apparent that frequently one was dealing, not with the elements of the set E so much, but rather with its subsets - that is, one was doing "atomless mathematics" wherein, instead of working with a set E, one works with a Boolean algebra A, often taken to be complete, treating it as if it was the power set of E. For example: many (all?) of the things done in straight Boolean algebra are analogues of purely set-theoretical things (a good instance is Theorem 22.6 in Sikorski's book [39]); measure theory and probability, as in the books of Carathéodory [7] and Kappos [25]; Büchi's paper [6] and in particular his Boolean version of relations ("die Paarung von Gefügen"); and general topology done the Boolean way as in the books of Nöbeling [32] and Rasiowa-Sikorski [33] (see Chapter III). Büchi's eloquent plea for atomless mathematics (see [6]; quoted by Linton [30]) adds to the desirability of developing an atomless matroid theory (based on that for sets, as sketched above); such a theory is the subject matter of the present thesis.

The first thing to check is the notion of derivative for a closure operator on a Boolean algebra and here the formulae of Hammer ([17], p. 32) and Nöbeling ([32], p. 66) provide an immediate solution - in the case of a complete algebra. Simple examples show that a satisfactory notion of derivative does not exist for all closure operators on an incomplete algebra and for this reason it was decided to base everything at this stage on a complete Boolean algebra (= CBA) (and after all, power sets are CBA's, and analysts work with the reals in preference to the rationals).

In Chapter 1, various properties of operators on CBA's and their derivatives are obtained (noteworthy being the curious identity for derivatives of closure operators implicit in (6)). The concept of an analytic closure operator, defined in the second section of the chapter, is basic to the subsequent theory and is of independent interest in that all closure operators on atomic CBA's are analytic - but not all closure operators on arbitrary CBA's. Thus, given some condition which is known to hold for closure operators on atomic CBA's but which fails to hold for closure operators on CBA's in general, we may still hope to extend it to all analytic closure operators. The discussion of separators in Section 4 of Chapter 1, whilst elementary, is more thorough than is customary even in the atomic case.

Certain exchange conditions, known to hold for matroidal closure operators in the finitary case, are studied in Chapter 2 and some of their consequences are given. Although the concept of a matroidal closure operator is only defined in the next chapter, most of Chapter 2 is closely related to this concept and many of the results of Chapter 2 are later applied to matroidal closure operators. In (24) it is shown that the dual of the condition given in (6) is equivalent to the exchange condition (super-$E_{\frac{1}{2}}$) most characteristic of matroidal closure operators.

General matroidal closure operators are defined in Chapter 3 - though the main results of the chapter concern the concept of a B-matroidal closure operator. In particular, it is shown that the duality theorem for B-matroidal closure operators together with their characterization by the exchange and minimality conditions (mentioned above for the atomic case), extend very satisfactorily to the general CBA situation ((34) and (35)).

The discussion of topological closure operators on CBA's given in Chapter 4 perforce deals with a number of generalities not especially relevant to those topological closure operators which are also matroidal. The main results which do concern the latter type of closure operator specifically are: (49), in which two very simple characterizations of T_1 matroidal topological closure operators are given; (56), which, together with the subsequent discussion, describes completely the closure operators which are both topological and B-matroidal; and the results in the fourth section of the chapter, where a study (albeit rudimentary) is made of the interesting closure operators, necessarily both matroidal and topological, whose derivatives are Boolean endomorphisms of the given CBA - we call these <u>Hewitt-Katětov-closure operators</u> after Hewitt and Katětov who, independently, discovered them (see [19] and [26]).

To illustrate how derivatives etc. come out in a particular case, a fairly thorough analysis of the closure operators associated with a section of a CBA is carried out in Appendix 1, and some specific examples are mentioned.

Appendix 2 contains a discussion of various low-grade separation axioms (between T_0 and T_1) for closure operators on a CBA; except for the results on the "passage to the T_0 case" listed in (73), most of this is of marginal relevance to matroidal closure operators (with the possible exception of the abortive (79)) but it was included on account of its cautionary value: it is seen how a number of conditions, which are nice in the atomic case, split up into a morass of hair-splitting distinctions on general CBA's.

In the first part of Appendix 3, some examples of T_1 matroidal topological closure operators on atomic CBA's are described and an indication ((83)) is given of the sort of "pathologies" which arise when the T_2 axiom holds. The second part of the same appendix contains some simple results concerning B-matroidal closure operators on atomic CBA's.

Appendix 4 is a reprinting of [21], in which are characterized those graphs whose polygons and two-way infinite arcs give rise to matroids (in the same way as, in Whitney's original paper, the polygons of a finite graph give rise to a finite matroid on the set of edges of the graph). This provides a good illustration of the various notions involved in this thesis, such as derivatives, bases, duality, (B-)matroidal closure operators etc.

To summarize, it appears that the basic theory of B-matroidal closure operators on a CBA is in a fairly satisfactory state - as one would expect, this theory being closely analogous to that for the finite case - but that not too much yet is known about more general matroidal closure operators, though a beginning has been made with those which in addition are topological.

PRELIMINARIES

The basic definitions and results which we shall need from lattice theory and Boolean algebra may be found in the books of Birkhoff [4], Halmos [14], and Sikorski [39]; for general topology, see Kelley [27]. Of our notation and terminology, we mention here the following (that not mentioned hereis completely standard):

Let L be a lattice. For each a,b in L with $a \leq b$, write $[a,b] = \{x;\ a \leq x \leq b\}$, $(a,b) = \{x;\ a < x < b\}$ and similarly for $[a,b)$ and $(a,b]$, $a \precsim b$ iff $(a,b) = \phi$, $a \prec b$ (b <u>covers</u> a) iff $a \precsim b$, $a \neq b$, and $M(a,b)$ (<u>a and b form a modular pair</u>) iff $(x \vee a) \wedge b \leq x \vee (a \wedge b)$ for all $x \leq b$. L is a <u>\-lattice</u> iff for each a,b in L there exists a smallest c $(= a\backslash b)$ in L such that $a \leq b \vee c$; L is a <u>/-lattice</u> iff it satisfies the dual condition; and L is a <u>(\,/)-lattice</u> iff it is both a \-lattice and a /-lattice.

Suppose that L has a smallest element 0 - then L is <u>disjunctive</u> iff for each a,b in L with $a < b$ there exists c in L such that $0 \neq c \leq b$ and $a \wedge c = 0$; and L is <u>left-complemented</u> (Wilcox [44]) iff for each a,b in L there exists c in L such that $c \leq b$, $a \vee c = a \vee b$, $M(c,a)$, and $a \wedge c = 0$.

A <u>section</u> of L is a subset S of L such that $a < b$, b in S implies a in S; dually for a <u>cosection</u> of L.

Suppose that L is complete and that $X \subseteq L$ - then the <u>$\bigvee$-closure</u>,

$J(X)$, of X(in L) is $\{\bigvee Y;\ Y \subseteq X\}$, X is <u>$\bigvee$-closed</u> iff $J(X) = X$, and X is <u>$\bigvee$-dense</u> iff $J(X) = L$; dually for the <u>$\bigwedge$-closure</u>, $M(X)$, of X etc.. Under the same circumstances, $L(X)$ denotes the smallest complete sublattice of L containing X and, for L a complete Boolean algebra, $C(X)$ denotes the smallest complete subalgebra of L containing X.

CHAPTER 1

OPERATORS ON COMPLETE BOOLEAN ALGEBRAS AND THEIR DERIVATIVES

1. Operators and Derivatives in General

An operator on a CBA (= complete Boolean algebra) A is an order-preserving map $f: A \to A$. We shall denote the set of all operators on A by $\mathcal{O}(A)$. $\mathcal{O}(A)$ is a monoid under composition, fg being defined by $(fg)(x) = f(g(x))$, with the identity map $\Gamma = \Gamma_A$ on A as identity element. Further, $\mathcal{O}(A)$ is a complete sublattice of the complete lattice A^A, where in both cases the order relation and complete lattice operations are the pointwise ones. We recall that for a complete lattice L to be a $(\backslash,/)$-lattice it is necessary and sufficient that L be infinitely distributive - that is, for all x and $\{y_i\}$ in L, $x \wedge (\bigvee_i y_i) = \bigvee_i (x \wedge y_i)$ and $x \vee (\bigwedge_i y_i) = \bigwedge_i (x \vee y_i)$. Now any Boolean algebra is a $(\backslash,/)$-lattice, with $x\backslash y = x \wedge y'$ and $x/y = x \vee y'$, where ' denotes complementation. From the resulting infinite distributivity of our CBA A, it follows that A^A, and in turn $\mathcal{O}(A)$, are also infinitely distributive. The fact that $\mathcal{O}(A)$ is thus a $(\backslash,/)$-lattice enables us to make the following definitions. If f is an operator on a CBA A, we call

$f \setminus \upharpoonright$ the *derivative* of f and $f / \upharpoonright$ the *coderivative* of f, where $f \setminus \upharpoonright$ and $f / \upharpoonright$ are to be taken in $\mathcal{O}(A)$.

Remark on duality. As with "derivative" and "coderivative" above, we shall refer to the order-theoretic dual of an order-theoretic notion "P" by writing "co-P". As a rule, we shall not state both each order-theoretical result and the dual result. Since a CBA A possesses a natural antiautomorphism, namely complementation, we can define an associated antiautomorphism * of the lattice $\mathcal{O}(A)$ by putting $f^*(x) = f(x')'$ (notice that * is an *automorphism* of the monoid $\mathcal{O}(A)$). Given an order-theoretic property P of operators on A, an operator f on A will then be co-P iff the "dual" operator f^* is P. If we wished, we could in this way refer only for instance to derivatives and never to coderivatives; however we prefer to use * rarely and to give explicit recognition to the order-theoretic duality present.

To begin the study of derivatives and coderivatives, we remark that several of their properties follow immediately from the results on /-lattices (= Brouwerian lattices; see Birkhoff [4], pp. 45-47, 125, 128-131, 216-229, 280-281), together with the corresponding dual results $\setminus$-lattices. We shall not list these properties but mention the following: the mappings $f \to f \setminus \upharpoonright$ and $d \to \upharpoonright \vee d$ establish a bijection between the interval $[\upharpoonright, 1]$ of $\mathcal{O}(A)$ and the set of derivatives (of operators) on A; an operator is a derivative iff it is its own derivative; and if d is the derivative of an operator f then the equality $\upharpoonleft \vee d = \upharpoonleft \vee f$ shows that $d(0) = f(0)$.

As mentioned already, the $\bigvee$ and $\bigwedge$ operations in $\mathcal{O}(A)$ are given by the pointwise formulae; we now give an explicit formula for the $\setminus$ operation in $\mathcal{O}(A)$ (and, on dualizing, obtain a formula for / of course). Since in the case of $f \setminus \upharpoonleft$ this formula was used by Hammer ([17], p. 32) as the definition of the derivative of f, we shall refer to it as <u>Hammer's formula</u>.

(1) Let f and g be operators on a CBA A. Then for all x in A

$$(f \setminus g)(x) = \bigvee\{f(u)\setminus g(u);\ \ u \leq x\}.$$

<u>Proof</u>. Let $h(x)$ denote the right-hand side of this equation - then clearly h is an operator on A. We have to show that $f \leq g \vee h$ and that if $f \leq g \vee k$ for some operator k on A then $h \leq k$. Certainly $f \leq g \vee h$ since $f(x) \leq g(x) \vee (f(x)\setminus g(x)) \leq$ $\leq g(x) \vee h(x)$ for each x in A. Suppose that k is an operator on A such that $f \leq g \vee k$. Then for each x in A and all $s \leq x$ we have $f(s) \leq g(s) \vee k(s)$ and hence $f(s)\setminus g(s) \leq k(s) \leq$ $\leq k(x)$, from which it follows that $h(x) \leq k(x)$ as we wished. Q.E.D.

Nöbeling ([32], p. 66) has used quite a different formula for his definiton of derivative. The following result establishes the equivalence of <u>Nöbeling's formula</u> with the present definition.

(2) Let f be an operator on a CBA A and let d be the derivative of f. Then for all x in A

$$d(x) = \bigvee\{u;\ u \leq f(x\setminus u)\}.$$

Proof. Let $e(x)$ denote the right-hand side of this equation. Then $e(x) \leq d(x)$ since if $u \leq f(x \setminus u)$ then clearly $u \leq d(x)$. To obtain the opposite inequality let $s \leq x$ and put $f(s) \setminus s = u$. Then $s \leq x \setminus u$ and therefore $u \leq f(s) \leq f(x \setminus u)$. It follows that $f(s) \setminus s \leq e(x)$ for each $s \leq x$ and hence, by Hammer's formula, that $d(x) \leq e(x)$. Q.E.D.

As a corollary we have:

(3) Let S be a $\bigvee$-dense subset of a CBA A and let f be an operator on A with derivative d. Then for all x,y in A

$$y \wedge d(x) = \bigvee\{u \text{ in } S;\ u \leq y \wedge f(x \setminus u)\}$$

and

$$y \vee d(x) = \bigvee\{u \text{ in } S;\ u \leq y \vee f(x \setminus u)\}.$$

Proof. The relevant fact here is that $\{u;\ u \leq f(x \setminus u)\}$ is a section of A. Thus in $y \wedge d(x) = \bigvee\{y \wedge u;\ u \leq f(x \setminus u)\}$, which we obtain from Nöbeling's formula by infinite distributivity, each $y \wedge u$ is the join of elements v in S satisfying the inequality $v \leq y \wedge f(x \setminus v)$. The first equation of (3) follows and the second may be verified in a similar manner. Q.E.D.

These sharpened forms of Nöbeling's formula apply in particular to an atomic CBA A, when we can take S to be the set of atoms of A.

For atomic CBA's, the notion of derivativity turns out to be self-dual (contrary to the entry in the 5^{th} row and 6^{th} column of Hammer's table in [18], p. 59):

(4) Let d be an operator on an atomic CBA A. Then the following conditions are equivalent

(i) d is a derivative

(ii) d is a coderivative

(iii) $p \leq d(x)$ implies $p \leq d(x \setminus p)$ for all atoms p of A,

(iv) $x \prec y$ implies $y \leq x \vee d(x)$ or $y \wedge d(y) \leq x$ for all x,y in A.

<u>Proof</u>. The pattern of proof is as follows. We first show, for any CBA A, that (iii) and (iv) are equivalent and that each is implied by (i). Then in the atomic case we show that (iii) implies (i), so that in this case (i), (iii), and (iv) are equivalent - however, since (iv) is a self-dual condition and the dual of (i) is (ii), it must be the case that all four conditions are equivalent.

<u>For any CBA A, (iii) and (iv) are equivalent.</u> Assume that (iii) holds and that $x \prec y$ - then $y = x \vee p$ for some atom $p \nleq x$. Suppose that $y \wedge d(y) \nleq x$. Then $p \leq d(y)$ and by (iii) $p \leq d(y \setminus p) = d(x)$. Thus $y = x \vee p \leq x \vee d(x)$. Now assume that (iv) holds and that p is an atom and x an element of A such that $p \leq d(x)$. If $p \nleq x$ then $x = x \setminus p$ and $p \leq d(x \setminus p)$. If $p \leq x$ then $x \setminus p \prec x$ and by (iv) we have $x \leq (x \setminus p) \vee d(x \setminus p)$ or $x \wedge d(x) \leq x \setminus p$. The latter alternative cannot hold since $p \leq x \wedge d(x)$. Hence $x \leq (x \setminus p) \vee d(x \setminus p)$ from which it follows that $p \leq d(x \setminus p)$.

<u>For any CBA A, (i) implies (iii).</u> Assume that (i) holds and that p is an atom and x an element of A such that $p \le d(x)$. Then by the first equation of (3), with $S = A$, $y = p$, and $f = d$, we must have $p \le d(x \setminus p)$.

<u>For an atomic CBA A, (iii) implies (i).</u> Assume that (iii) holds and denote the derivative of d by e. Then from the first equation of (3), with S = the set of atoms of A, $y = 1$, and $f = d$, we have $e(x) = \bigvee\{p$ an atom of $A;\ p \le d(x \setminus p)\}$. By (iii), the right-hand side of this equals $\bigvee\{p$ an atom of $A;$ $p \le d(x)\}$ which in turn equals $d(x)$. Q.E.D.

We shall show later that atomicity is a necessary condition for the self-duality of derivativity, in fact proving somewhat more than this (see (8)). In any case, an operator which is both a derivative and a coderivative will be said to be a <u>biderivative</u>.

(5) Let d be an operator on a CBA A. Then the following conditions are equivalent

(i) d is a derivative

(ii) $s \wedge d(s) = \bigvee\{u;\ u \le s \wedge d(s \setminus u)\}$ for all s in A,

(iii) $s \setminus d(s) = \bigwedge\{v;\ v \le s \le v \vee d(v)\}$ for all s in A,

(iv) $(s \wedge d(s))' = \bigwedge\{w;\ w \vee s = w \vee d(s \wedge w) = 1\}$ for all s in A.

<u>Proof.</u> We first obtain the equivalence, for each element s of

A, of the three equations in (5). This will be done if we verify that $s \wedge x' = y$, $s \wedge y' = x$, and $x' = z$ where x, y, and z are the right-hand sides of these equations (in the order in which they appear). Now $s \wedge x' = y$ and $s \wedge y' = x$ iff $x \wedge y = 0$ and $x \vee y = s$, and the truth of these two latter relations follows by a routine computation from the following fact: if u and v are elements of A such that $u \wedge v = 0$ and $u \vee v = s$ then $u \leq s \wedge d(s \setminus u)$ iff $v \leq s \leq v \vee d(v)$. The relation $x' = z$ follows in a similar way from the fact that if $u' = w$ then $u \leq s \wedge d(s \setminus u)$ iff $w \vee s = w \vee d(s \wedge w) = 1$. It remains to be proved that (i) and (ii) are equivalent. If (i) holds then (ii) follows as a particular case of the first equation of (3). Suppose that (ii) holds and put $m = \bigvee\{u;\ u \leq d(s \setminus u)\}$. Since clearly $d(s) \geq m$, (i) will follow by Nöbeling's formula if we show that $n = d(s) \setminus m = 0$. Now $s \wedge n = 0$ on account of (ii) and therefore $n \leq d(s) = d(s \setminus n)$. Hence $n \leq m$, so that $n = 0$ as required. Q.E.D.

2. Closure Operators

An operator f on a CBA A is a <u>closure</u> operator iff $\upharpoonleft \leq f = f^2$. A coclosure operator is called an <u>interior</u> operator; f is a closure operator iff f^* is an interior operator (this is the usual association of interior operators with closure operators familiar from topology). A handy fact concerning a closure operator f is that $f(x) \leq f(y)$ implies $f(x \vee z) \leq f(y \vee z)$ (and likewise with ='s in place of ≤ 's).

Let f be a closure operator with derivative d on a CBA A and let x be an element of A. Then x is <u>f-closed</u> if $f(x) = x$ (equivalently, if $d(x) \leq x$); <u>f-open</u> if x' is f-closed; <u>f-clopen</u> if both f-closed and f-open; <u>f-dense</u> if $f(x) = 1$ (equivalently, if $x \vee d(x) = 1$); <u>f-pithy</u> if $x \leq d(x)$; <u>f-discrete</u> if $x \wedge d(x) = 0$; <u>f-perfect</u> if both f-closed and f-pithy; and <u>f-basic</u> if both f-dense and f-discrete. (Here and elsewhere we could have formulated our definitions, and many of the easier consequent results, in a slightly different and indeed more general way, but we have chosen the present course, with the closure operator f in first place, since it is ultimately closure operators that are our main concern.) Amongst the many properties enjoyed by the above concepts, we mention the following, which are on the whole well-known - and which in any event are straightforward (when, as here, there is no ambiguity, we have omitted the prefix f).

(a) The set of closed elements coincides with the range $f(A)$ of f. $f(A)$, being a $\bigwedge$-closed subset of A, is itself a complete lattice: the meet in $f(A)$ of a subset X of $f(A)$ coincides with $\bigwedge X$ and the join in $f(A)$ of X, which we denote by $\overline{\bigvee} X$, is given by $\overline{\bigvee} X = f(\bigvee X)$ ($\bigwedge X$ and $\bigvee X$ denote the meet and join respectively of X in A always). We call $f(A)$ the lattice <u>of</u> f. The mappings $f \mapsto f(A)$ and $L \mapsto (x \mapsto \bigwedge\{a;\ x \le a \text{ in } L\})$ establish a bijection between the set of all closure operators f on A and the set of all $\bigwedge$-closed subsets L of A. If x is closed then so is $d(x)$.

(b) x is pithy iff, whenever $0 \neq y \le x$, we have $u \le f(x \setminus u)$ for some u such that $0 \neq u \le y$ (this is an easy consequence of (3)). The set of pithy elements is a $\bigvee$-closed subset of A. If x is pithy then so is the closure $d(x)$ of x (so that $d(x)$ is in fact perfect) - expressed differently, $x \le d(x)$ implies $d(x) = d^2(x)$.

(c) x is discrete iff for no u such that $0 \neq u \le x$ do we have $u \le f(x \setminus u)$ (this is also an easy consequence of (3)). Further conditions equivalent to the discreteness of x are: $f(u) \wedge x = u$ for all $u \le x$; x is of the form $y \setminus d(y)$. The set of discrete elements is a section of A, the join of which we denote by $j(f)$. x is discrete and closed iff $d(x) = 0$ and iff each subelement of x is closed. x is basic iff it is minimal dense. The set of basic elements is an antichain in A

as may be seen from the fact that $y \wedge d(y) \leq x \leq y \leq x \vee d(x)$ implies $x = y$ (we have $x = x \vee (y \wedge d(y)) \geq x \vee (y \wedge d(x)) = y \wedge (x \vee d(x)) = y$).

(d) Suppose that $f(0) = 0$. Then we call the largest element of A, every subelement of which is a join of closed elements, the $\underline{T_1\text{-part}}$ of f and denote it by $t(f)$ (it is easy to see that such a largest element exists). If $t(f) = 1$ - equivalently, if $f(A)$ is $\bigvee$-dense in A - we say that f is T_1 (this is Nöbeling's definition; see [32], p. 77). In any case we have the relation $j(f) \wedge t(f) = \bigvee d^{-1}(0)$.

We next prove a curious result which turns out to be useful later on.

(6) Let f be a closure operator with derivative d on a CBA A. Then $d(x) \leq y$ implies $d(x \vee d(x \wedge y)) \leq y$ for all x,y in A.

Proof. Suppose that $d(x) \leq y$ but that $d(x \vee d(x \wedge y)) \nleq y$. Then $y' \wedge d(x \vee d(x \wedge y)) \neq 0$ and by the first equation of (3) there exists an element s of A such that $0 \neq s \leq y' \wedge d((x \vee d(x \wedge y))\backslash s)$ From $s \leq y'$ it follows that $s \wedge y = 0$ and also that $d(x \wedge y) \leq d(x\backslash s)$. Using this latter inequality and the idempotency of $1 \vee d$ we obtain

$s \leq d((x\backslash s) \vee (d(x \wedge y)\backslash s)) \leq d((x\backslash s) \vee d(x \wedge y)) \leq d((x\backslash s) \vee d(x\backslash s)) \leq (x\backslash s) \vee d(x\backslash s)$. Thus $s \leq d(x\backslash s) \leq d(x) \leq y$ so that $s = s \wedge y = 0$, a contradiction. Q.E.D.

<u>Corollary</u>. If f and d are as in (6) and $d(x) \leq y$ then there is an element $z \geq x$ such that $y \wedge z$ is closed and $d(z) \leq y$.

<u>Proof</u>. If we put $z = x \vee d(x \wedge y)$ then $z \geq x$ and $d(z) \leq y$ by (6); also $y \wedge z = (x \wedge y) \vee d(x \wedge y) = f(x \wedge y)$ and this is closed. Q.E.D.

Let us call the condition: $d(x) \leq y$ implies $d(x \vee d(x \wedge y)) \leq y$ of (6) <u>condition (I)</u>. (I) may be expressed as an identity by writing $d(x) \vee z$ in place of y and taking the meet of both sides of the second inequality with $d(x) \vee z$. If d is any operator whatsoever satisfying (I) then d satisfies the identity $x \vee d(x) \vee d(x \vee d(x)) = x \vee d(x)$ expressing the idempotency $(\upharpoonright \vee d)^2 = \upharpoonright \vee d$ and also the identity $d(x \vee d(x \wedge d(x))) = d(x)$ - that is, $d(\upharpoonright \vee d(\upharpoonright \wedge d)) = d$. (The first of these may be obtained by putting $y = x \vee d(x)$ in (I) and the second by putting $y = d(x)$ in (I).) However, small finite examples show that neither of these two identities implies the other (and that the second does not imply the first even when d is a derivative). The question as to which identities are satisfied by the derivatives of closure operators has apparently not been investigated - nor, for that matter, the (doubtful) existence of non-trivial (finite) identities satisfied by all derivatives, or by no operators other than derivatives.

Another result we shall need concerning derivatives of closure operators is as follows.

(7) Let f be a closure operator with derivative d on a CBA A. If x,y,z are elements of A such that $f(x) \leq f(y)$ and $y \wedge z = 0$ then $z \wedge d(x \vee z) \leq z \wedge d(y \vee z)$.

Proof. By virtue of the first equation of (3) (with $x \vee z$ in place of x and z in place of y) it is sufficient to show that $u \leq z \wedge f((x \vee z) \setminus u)$ implies $u \leq z \wedge d(y \vee z)$ for each u in A. But this is the case since if $u \leq z$ then $y \wedge u = 0$ and hence $f((x \vee z) \setminus u) \leq f(x \vee (z \setminus u)) \leq f(y \vee (z \setminus u)) = f((y \vee z) \setminus u)$, the second inequality here being a consequence of the fact that f is a closure operator. Q.E.D

We now come to a basic definition. Let f be a closure operator on a CBA A. Then we say that f is analytic iff the derivative of f is a coderivative.

(8) Every closure operator on a CBA A is analytic iff A is atomic.

Proof. It follows from (4) that if A is atomic then every closure operator on A is analytic. To obtain the converse let f be the closure operator on A such that $f(0) = 0$ and $f(x) = 1$ for $x \neq 0$, let d be the derivative of f, and let e be the coderivative of d. Then Hammer's formula gives

$d(x) = \bigvee\{f(u)\backslash u;\ u \leq x\} = \bigvee\{u';\ 0 \neq u \leq x\} = 0$ for $x = 0$, p' for $x =$ an atom p, and $= 1$ otherwise. By the dual of Hammer's formula we have $e(0) = \bigwedge\{d(v)/v;\ v \geq 0\}$ and after a little calculation obtain $e(0) = (\bigvee P)'$, where P denotes the set of atoms of A. If f is analytic then $e(0) = d(0) = 0$ so that $\bigvee P = 1$ and A is atomic. Q.E.D.

This result clearly has as a consequence the necessity of atomicity for the self-duality of derivativity. The following criteria for analyticity result immediately from the dual of (5).

(9) Let f be a closure operator with derivative d on a CBA A. Then the following conditions are equivalent

(i) f is analytic,

(ii) $f(s) = \bigwedge\{u;\ u \geq s \vee d(s/u)\}$ for all s in A,

(iii) $s/f(s) = \bigvee\{v;\ v \wedge d(v) \leq s \leq v\}$ for all s in A,

(iv) $f(s)' = \bigvee\{w;\ w \wedge s = w \wedge d(s \vee w) = 0\}$ for all s in A.

If f is a closure operator with derivative d on a CBA A, let us say that f is <u>analytic at an element</u> s of A iff the equations in (ii), (iii), and (iv) here hold for that value of s. (Note that these three equations are equivalent, the equivalence of the three dual equations having been obtained in the course of proving (5).) We see that f is analytic iff it is analytic at each s in A, that f is analytic at 0 iff $j(f) = f(0)'$, and that f is analytic at a closed element s of A iff the following equivalent equations hold:

$$s = \bigwedge\{u;\ u \geq s \vee d(s/u)\}$$

$$1 = \bigvee\{v;\ v \wedge d(v) \leq s \leq v\}$$

$$s' = \bigvee\{w;\ w \wedge s = w \wedge d(s \vee w) = 0\}.$$

(10) Let f be a closure operator on a CBA A and let s be an element of A. Then f is analytic at s iff f is analytic at f(s).

Proof. Using the equation of (9) (iv) we have that f is analytic at s iff $f(s)' = \bigvee\{w;\ w \wedge s = w \wedge d(s \vee w) = 0\}$ and that f is analytic at f(s) iff $f(s)' = \bigvee\{w;\ w \wedge f(s) = w \wedge d(f(s) \vee w) = 0\}$. The result will follow if we show the sets involved in these two joins to be the same. Now clearly $w \wedge f(s) = w \wedge d(f(s) \vee w) = 0$ implies $w \wedge s = w \wedge d(s \vee w) = 0$. Conversely if $w \wedge s = w \wedge d(s \vee w) = 0$ then $w \wedge f(s) = (w \wedge s) \vee (w \wedge d(s)) \leq (w \wedge s) \vee (w \wedge d(s \vee w)) = 0$ and by (7) $w \wedge d(f(s) \vee w) \leq w \wedge d(s \vee w) = 0$. Q.E.D.

As an immediate consequence of (10) we have:

Corollary. A closure operator on a CBA A is analytic iff it is analytic at each closed element of A.

3. Reductions

Let s be an element of a CBA A and write A^s for the interval $[0,s]$ of A. We define the reduction mapping $f \mapsto f^s$ of $\mathcal{O}(A)$ to $\mathcal{O}(A^s)$ by putting $f^s(x) = f(x) \wedge s$ and call f^s the reduction of f to A^s (the term "reduction" is Tutte's ([41], p.7)). Dually we write A_s for $[s,1]$ and f_s for the coreduction of f to A_s.

(11) Let s be an element of a CBA A. Then the reduction mapping of $\mathcal{O}(A)$ to $\mathcal{O}(A^s)$

(i) preserves the operations of $\bigvee, \bigwedge, \setminus$, and coderivation,

(ii) preserves the properties of being the identity, a derivative, a closure operator, and an analytic closure operator, and also the duals of these properties,

(iii) fails in general to preserve the operations of composition, /, and *, though the inequalities $(fg)^s \geq f^s g^s$, $(f/g)^s \leq f^s/g^s$, and $(f^*)^s \leq (f^s)^*$ always hold.

Proof. It is obvious that reduction preserves Γ and a straightforward computation shows that it preserves $\bigvee$ and $\bigwedge$.

Let f and g be operators on A. Then (firstly) for each x in A we have $(f\setminus g)^s(x) = (f\setminus g)(x) \wedge s = \bigvee\{f(u)\setminus g(u);\ u \leq x\} \wedge s = \bigvee\{f^s(u)\setminus g^s(u);\ u \leq x\}$. Since $\bigvee$ and $\setminus$ in A^s are $\bigvee$ and $\setminus$ in A restricted to A^s, this last quantity evidently equals $(f^s\setminus g^s)(x)$.

Thus $(f\backslash g)^s = f^s\backslash g^s$, that is, reduction preserves $\backslash$. The fact that reduction thereby necessarily preserves the operation of derivation shows that it also preserves the property of being a derivative - for if $f \backslash \upharpoonleft = f$ then $f^s \backslash \upharpoonleft = (f\backslash \upharpoonleft)^s = f^s$.

Before showing that reduction preserves coderivation, let us obtain the inequality in (iii) which involves $/$. We know that $f \ge g \wedge (f/g)$ and since reduction preserves $\wedge$ we have $f^s \ge g^s \wedge (f/g)^s$, from which it follows that $(f/g)^s \le f^s/g^s$ as required. To see that this inequality can be strict sometimes, let A be the 4-element CBA and take s to be one of the atoms of A, f to be $\upharpoonleft_A$, and g to be the other automorphism of A. Then $f/g = \upharpoonleft_A$ so that $(f/g)^s = \upharpoonleft$, whereas $f^s/g^s = \upharpoonleft/0 = 1 > \upharpoonleft$ (here $\upharpoonleft$, 0, and 1 are the obvious operators on A^s).

This inequality for $/$ has as a particular case the inequality $\le$ in the equation $(f/\upharpoonleft)^s = f^s/\upharpoonleft$ which expresses the fact that reduction preserves coderivation. We obtain the reverse inequality as follows.

Let x be an element of A^s. Then $(f/\upharpoonleft)^s(x) = (f/\upharpoonleft)(x) \wedge s = \bigwedge\{(f(v)/v) \wedge s;\ v \ge x\}$ --- (α). On the other hand, $(f^s/\upharpoonleft)(x) = \bigwedge\{f^s(w)/w;\ w \ge x\}$, where here everything takes place in A^s. Now the $\bigwedge$, being non-empty, coincides with the $\bigwedge$ in A; however y/z in A^s (where y and z are elements of A^s) is the same as $(y/z) \wedge s$ taken in A; furthermore $w \ge x$ in A^s means

$s \geq w \geq x$ in A. Thus with everything taking place in A we find that $(f^s/\upharpoonright)(x) = \bigwedge\{(f^s(w)/w) \wedge s;\ s \geq w \geq x\} = \bigwedge\{(f(w)/w) \wedge s;\ s \geq w \geq x\}$ --- (β). The desired inequality will be obtained if each $(f(v)/v) \wedge s$ with $v \geq x$ (as in (α)) is $\geq$ some $(f(w)/w) \wedge s$ with $s \geq w \geq x$ (as in (β)) and on taking $w = v \wedge s$ we see that this is indeed the case. The fact that reduction preserves coderivation implies, as for the derivative case, that it also preserves the property of being a coderivative.

The inequality in (iii) for composition is easily verified: if x is in A^s then $(fg)^s(x) = (fg)(x) \wedge s \geq f(g(x) \wedge s) \wedge s = (f^s g^s)(x)$. To see that the inequality can be strict, use the same example as given above but with both f and g equal to the non-identical automorphism of A.

Now let f be a closure operator on A. Then from $f \geq \upharpoonright$ follows $f^s \geq \upharpoonright$ and thence $(f^s)^2 \geq f^s$. However $f^s = (f^2)^s \geq (f^s)^2$ from what has just been proved. Hence $(f^s)^2 = f^s$ and f^s is a closure operator on A^s. Let f be an interior operator on A - then from $f \leq \upharpoonright$ follows $f^s \leq \upharpoonright$.
Also for each x in A^s we have $f(x) \leq x \leq s$ and hence $(f^s)^2(x) = f(f(x) \wedge s) \wedge s = f^2(x) \wedge s = f(x) \wedge s = f^s(x)$. Thus $(f^s)^2 = f^s$ and f^s is an interior operator on A^s.

It is an immediate consequence of what has been proved already that reduction preserves the property of being an analytic closure operator and also the dual property.

We come finally to the inequality involving $*$: for x in A^s, $(f^*)^s(x) = f(x')' \wedge s$ and $(f^s)^*(x)$ reduces to $f(s \wedge x')' \wedge s$. For the example to show that strictness may occur, let A be the 4-element CBA with s one of its atoms again and define $f(1) = s$, $f(x) = 0$ otherwise. Q.E.D.

A number of the facts listed in (11), for instance that reduction preserves the property of being a closure operator, are traditional - as are many of the following remarks, up to but not including (12). In this remarks, f is a closure operator with derivative d on a CBA A and s,x are elements of A. Then f^s is a closure operator on A^s and f_s is a closure operator on A_s. Let us say that x is (f-)closed <u>in</u> s if x is in A^s and is f^s-closed, and that x is (f-)closed <u>over</u> s if x is in A_s and is f_s-closed; similarly for pithyness, discreteness, etc. (notice that we allow ourselves to omit the prefix f, but not f^s or f_s). We shall sometimes also say that x is a <u>base of</u> s when x is basic in s. The following are evident:

x is closed in s if $f(x) \wedge s = x$, (equivalently, if $d(x) \wedge s \leq x \leq s$)	x is closed over s if x is closed and $\leq s$,
x is dense in s if $x \leq s \leq f(x)$, (equivalently, if $x \leq s \leq x \vee d(x)$)	x is dense over s if x is dense and $\geq s$,
x is pithy in s if x is pithy and $\leq s$,	x is pithy over s if $s \leq x \leq d(x) \vee s$,
x is discrete in s if x is discrete and $\leq s$.	x is discrete over s if $x \wedge d(x) \leq s \leq x$.

The lattice $f^s(A^s)$ of f^s is $f(A) \wedge s$; in particular if s is closed then $f^s(A^s)$ is the interval $[f(0),s]$ of $f(A)$. The lattice of f_s is $f(A) \overline{\vee} f(s)$ = the interval $[f(s), 1]$ of $f(A)$.

Now let s and t be elements of A such that $s \leq t$ and write A_s^t for the interval $[s,t]$ of A. Then $(f_s)^t =$ $= (f^t)_s = f_s^t$ say, the <u>bireduction</u> of f to A_s^t (this goes through for an arbitrary operator f on A). We shall not dwell on the properties of bireduction except to mention that if s and t are closed then the lattice of f_s^t is just the interval $[s,t]$ of the lattice of f.

(12) A closure operator f on a CBA A is analytic at an element s of A iff f_s is analytic at s.

The proof of this is trivial.

(13) A closure operator f on a CBA A is analytic iff $\bigvee\{v;\ v$ is discrete over $s\} = 1$ for each closed element s of A.

(Note that in a notation introduced earlier we can write $j(f_s)$ for $\bigvee\{v;\ v$ is discrete over $s\}$.)

This result is just a rewording of the corollary to (10). Nevertheless it might be of interest insofar as it leads to a sensible definition of analyticity of closure operators on arbitrary complete lattices.

4. Separators

Let B and C be CBA's. We define the <u>cartesian product</u> mapping $(g,h) \mapsto g\times h$ of $\mathcal{O}(B)\times\mathcal{O}(C)$ to $\mathcal{O}(B\times C)$ by putting $(g\times h)(y,z) = (g(y), h(z))$ and call $g\times h$ the <u>cartesian product</u> of g and h. This construction is as well-behaved as it is possible to imagine. Thus for example $(g\times h)\backslash\!\upharpoonleft = (g\backslash\!\upharpoonleft)\times(h\backslash\!\upharpoonleft)$ and if g and h are (analytic) closure operators then so is $g\times h$ and the lattice of $g\times h$ is just the product of the lattices of g and h.

It is desirable to know when a given operator f on a CBA A can be decomposed (up to isomorphism) as a cartesian product $f = g\times h$ for a suitable decomposition $A \cong B\times C$ of A. The following definition is appropriate here: an element s of A is an <u>f-separator</u> iff $f(x) = [f(x \wedge s) \wedge s] \vee [f(x \wedge s') \wedge s']$ for all x in A (the term "separator" is Tutte's ([41], p.8)).

(14) (a) Let g and h be operators on CBA's B and C respectively. Then the elements $(1,0)$ and $(0,1)$ of $B\times C$ are $(g\times h)$-separators.

(b) Let f be an operator on a CBA A and let s be an f-separator. Then associated with the decomposition $A \overset{\alpha}{\cong} A^s \times A^{s'}$ of A (given by the mappings $x \overset{\alpha}{\mapsto} (x \wedge s, x \wedge s')$ and $(y,z) \overset{\alpha^{-1}}{\mapsto} y \vee z)$ we have the decomposition $f \cong f^s \times f^{s'}$ of f (more precisely, we have $f = \alpha^{-1}(f^s \times f^{s'})\alpha)$.

The verification of this is trivial.

(15) Let f be an operator on a CBA A. Then an element s of A is an f-separator iff $f(x \wedge s) \wedge s = f(x) \wedge s$ and $f(x \vee s) \vee s = f(x) \vee s$ for all x in A.

Proof. Let s be an f-separator and let x be in A. Then $f(x) \wedge s = ([f(x \wedge s) \wedge s] \vee [f(x \wedge s') \wedge s']) \wedge s = f(x \wedge s) \wedge s$ and $f(x) \vee s \leq f(x \vee s) \vee s = [f((x \vee s) \wedge s) \wedge s] \vee [f((x \vee s) \wedge s') \wedge s'] \vee s = f(x \wedge s') \vee s \leq f(x) \vee s$, so that $f(x) \vee s = f(x \vee s) \vee s$. Conversely, suppose that s satisfies these conditions and put $u = [f(x \wedge s) \wedge s] \vee [f(x \wedge s') \wedge s']$, where x is an arbitrary element of A. Then $u \wedge s = f(x \wedge s) \wedge s = f(x) \wedge s$ and $u \vee s = f(x \wedge s') \vee s = f((x \wedge s') \vee s) \vee s = f(x \vee s) \vee s = f(x) \vee s$. On account of the distributivity of A, it follows that $u = f(x)$. Q.E.D.

This result shows that the notion of an f-separator is self-dual. (Incidentally, the condition given in (15) leads most naturally to the alternative decomposition $A \cong A^s \times A_s$, $f \cong f^s \times f_s$.)

(16) Let f be an operator on a CBA A. Then the set of all f-separators is a Boolean subalgebra of A.

<u>Proof</u>. It is immediate from the definition that 0 is an f-separator and that the complement of an f-separator is an f-separator. Thus it enough to show that, given f-separators s and t, $s \wedge t$ is also an f-separator. Using the fact that s and t satisfy the conditions of (15) we obtain

$f(x \wedge (s \wedge t)) \wedge (s \wedge t) = f(x) \wedge (s \wedge t)$ and

$f(x \vee (s \wedge t)) \vee (s \wedge t) = [f(x \vee (s \wedge t)) \vee s] \wedge [f(x \vee (s \wedge t)) \vee t] = [f(x \vee (s \wedge t) \vee s) \vee s] \wedge [f(x \vee (s \wedge t) \vee t) \vee t] = [f(x \vee s) \vee s] \wedge [f(x \vee t) \vee t] = [f(x) \vee s] \wedge [f(x) \vee t] = f(x) \vee (s \wedge t)$.

Thus $s \wedge t$ is an f-separator by (15). Q.E.D.

(17) Let f be an operator on a CBA A and let s be an f-separator. Then s is also an $(f \setminus \upharpoonright)$-separator and a $(\upharpoonright \vee f)$-separator.

<u>Proof</u>. Write d for $f \setminus \upharpoonright$ and g for $\upharpoonright \vee f$ and let x be an arbitrary element of A. Then using (3) we have

$$d(x \wedge s) \wedge s = \bigvee\{u;\ u \le s \wedge f((x \wedge s)\setminus u)\}$$
$$= \bigvee\{u;\ u \le s \wedge f((x \setminus u) \wedge s)\}$$
$$= \bigvee\{u;\ u \le s \wedge f(x \setminus u)\} \quad \text{(since } s \text{ is an f-separator)}$$
$$= d(x) \wedge s$$

and $d(x \vee s) \vee s = \bigvee\{u;\ u \le s \vee f((x \vee s) \setminus u)\}$
$$\le \bigvee\{u;\ u \le s \vee f((x\setminus u) \vee s)\} \quad \text{(since } (x \vee s)\setminus u \le (x\setminus u) \vee s)$$
$$= \bigvee\{u;\ u \le s \vee f(x\setminus u)\} \quad \text{(since } s \text{ is an f-separator)}$$
$$= d(x) \vee s \le d(x \vee s) \vee s,$$

so that $d(x \vee s) \vee s = d(x) \vee s$. Thus s is a d-separator.

For g we have $g(x \wedge s) \wedge s = ((x \wedge s) \vee f(x \wedge s)) \wedge s = (x \wedge s) \vee (f(x \wedge s) \wedge s) = (x \wedge s) \vee (f(x) \wedge s) = (x \vee f(x)) \wedge s = g(x) \wedge s$ and $g(x \vee s) \vee s = (x \vee s \vee f(x \vee s)) \vee s = x \vee s \vee f(x) = g(x) \vee s$. Hence s is a g-separator. Q.E.D.

We next obtain a further condition for an element to be a separator.

Lemma. Let f be an operator on a CBA A and let s be an element of A. Then the following conditions are equivalent.

(i) $f(s) \wedge f(s') = f(0)$ and $f(x) = f(x \wedge s) \vee f(x \wedge s')$ for all x in A,

(ii) $f(x \wedge s) = f(x) \wedge f(s)$ and $f(x \vee s) = f(x) \vee f(s)$ for all x in A,

(iii) $f(s) \vee f(s') = f(1)$ and $f(x \wedge s) = f(x) \wedge f(s)$, $f(x \wedge s') =$
$= f(x) \wedge f(s')$ for all x in A.

Proof. Suppose that (i) holds. Then $f(x) \wedge f(s) =$

$[f(x \wedge s) \wedge f(s)] \vee [f(x \wedge s') \wedge f(s)] = f(x \wedge s) \vee f(0) = f(x \wedge s)$, where $f(x \wedge s') \wedge f(s) = f(0)$ on account of the inequalities $f(0) \leq f(x \wedge s') \wedge f(s) \leq f(s') \wedge f(s) = f(0)$. Also $f(x) \vee f(s) \leq$ $\leq f(x \vee s) = f((x \vee s) \wedge s) \vee f((x \vee s) \wedge s') = f(s) \vee f(x \wedge s') \leq$ $\leq f(s) \vee f(x)$ so that $f(x) \vee f(s) = f(x \vee s)$. Thus (i) implies (ii).

Now suppose that (ii) holds. Then $x = s'$ in $f(x \vee s) =$ $= f(x) \vee f(s)$ gives $f(s) \vee f(s') = f(1)$ and to obtain (iii) we only have to show that $f(x \wedge s') = f(x) \wedge f(s')$ for all x. Now we have $f(x \wedge s') \wedge f(s) = f((x \wedge s') \wedge s) = f(0)$, and $(f(x) \wedge f(s')) \wedge f(s) = f(x) \wedge f(s \wedge s') = f(0)$, and we also have $f(x \wedge s') \vee f(s) = f((x \wedge s') \vee s) = f(x \vee s)$ and $(f(x) \wedge f(s')) \vee$ $\vee f(s) = (f(x) \vee f(s)) \wedge (f(s') \vee f(s)) = f(x \vee s) \wedge f(1) = f(x \vee s)$. It follows by the distributivity of A that $f(x \wedge s') = f(x) \wedge f(s')$ as required.

Finally suppose that (iii) holds. Then $x = s'$ in $f(x \wedge s) = f(x) \wedge f(s)$ gives $f(s) \wedge f(s') = f(0)$; also $f(x \wedge s) \vee f(x \wedge s') = (f(x) \wedge f(s)) \vee (f(x) \wedge f(s')) =$ $f(x) \wedge [f(s) \vee f(s')] = f(x) \wedge f(1) = f(x)$. Q.E.D.

Note that condition (ii) here is self-dual - thus the duals of conditions (i) and (iii) are also equivalent to the conditions of this lemma. Let us call an element s satisfying these conditions an <u>f-respector.</u>

(18) Let f be an operator on a CBA A and let s be an element of A. Then the following conditions are equivalent.

(i) s is an f-separator,

(ii) s is an f-respector and $f(s) \leq s \vee f(0)$, $f(s') \leq$ $\leq s' \vee f(0)$,

(iii) s is an f-respector and $s \wedge f(1) \leq f(s) \leq s \vee f(0)$.

Proof. Suppose that (i) holds. Then $x = 0$ in $f(x \vee s) \vee s =$ $= f(x) \vee s$ gives $f(s) \vee s = f(0) \vee s$ so that $f(s) \leq s \vee f(0)$, and since s' is also an f-separator we obtain $f(s') \leq s' \vee f(0)$ similarly. The following inequalities show that s satisfies the conditions in (i) of the above lemma and hence that s is an f-respector: $f(0) \leq f(s) \wedge f(s') \leq (s \vee f(0)) \wedge (s' \vee f(0)) =$ $f(0)$ and for all x in A $f(x \wedge s) \vee f(x \wedge s') \leq f(x) =$ $[f(x \wedge s) \wedge s] \vee [f(x \wedge s') \wedge s'] \leq f(x \wedge s) \wedge f(x \wedge s')$. Thus (i) implies (ii). Now if (ii) holds then $s \wedge f(1) = s \wedge (f(s) \wedge f(s')) \leq$ $\leq s \wedge (f(s) \vee s' \vee f(0)) = s \wedge f(s) \leq f(s)$ and hence (iii) holds. To prove that (iii) implies (i), suppose that (iii) holds and let x be any element of A. Then $f(x) \wedge s \geq f(x \wedge s) \wedge s = f(x) \wedge$ $\wedge$ $f(s) \wedge s \geq f(x) \wedge s \wedge f(1) \wedge s = f(x) \wedge s$ so that $f(x \wedge s) \wedge s =$ $f(x) \wedge s$ - and dually $f(x \vee s) \vee s = f(x) \vee s$. Hence s is an f-separator by (15). Q.E.D.

We now discuss what happens when f is a closure operator.

(19) Let f be a closure operator on a CBA A. Then

(a) $f(0)$ and all its subelements are f-separators, and

(b) an element s of A is an f-separator iff it is of the form $c\backslash u$, where c is a closed f-separator and $u \leq f(0)$.

Proof. To show (a), suppose $s \leq f(0)$. Then $f(s) = f(0)$ and we see that $f(x \wedge s) = f(0) = f(x) \wedge f(s)$, $f(x \vee s) = f(x) = f(x) \vee f(s)$, and $s \wedge f(1) \leq f(s) \leq s \vee f(0)$ so that s satisfies the conditions of (18) (iii) and is therefore an f-separator.

To prove (b), first let s be an f-separator and put $f(s) = c$ and $u = c\backslash s$, so that $s = c\backslash u$. Then c is an f-separator, as follows from the fact that it satisfies the conditions of (18) (iii): we have $f(x) \wedge f(c) \geq f(x \wedge c) \geq f(x \wedge s) = f(x) \wedge f(s) = f(x) \wedge f(c)$ $f(x \vee c) = f(x \vee s) = f(x) \vee f(s) = f(x) \vee f(c)$, and $c \wedge f(1) = = f(c) = c \vee f(0)$ (each equals c). Also, since s is an f-separator, we can write $c = f(s) = [f(s \wedge s) \wedge s] \vee [f(s \wedge s') \wedge s'] = s \vee f(0)$, so that $u \leq f(0)$. To obtain the converse implication in (b), let c be a closed f-separator and $u \leq f(0)$. It is then easy to check that $s = c\backslash u$ satisfies the conditions of (18) (iii) and is thus an f-separator. Q.E.D.

The fact that $f(0)$ is an f-separator when f is a closure operator shows that with only a slight loss in generality we may study closure operators f with $f(0) = 0$. Indeed, we do this whenever $f(0) \neq 0$ proves to be at all inconvenient or untidy.

(20) Let f be a closure operator with derivative d on a CBA A and suppose that $f(0) = 0$. Then for each element s of A the following conditions are equivalent

(i) s is an f-separator,

(ii) s is an f-respector,

(iii) s is a d-separator

(iv) s is clopen and in the centre of $f(A)$.

Proof. If s is an f-respector then $f(s) \wedge s' \leq f(s) \wedge f(s') =$ $= f(0)$ so that $f(s) \leq s \vee f(0)$, and similarly $f(s') \leq s' \vee f(0)$. It follows by (18) that conditions (i) and (ii) of the present result are equivalent. Moreover, the same holds for conditions (i) and (iii) by (17). To show that (i) implies (iv), let s be an f-separator. Then s' is also an f-separator by (16) and s, s' are both in $f(A)$ on account of (19)(b) and the fact that $f(0) = 0$. Now (and this is actually a general fact concerning lattices with 0 and 1) it is not difficult to see that for the elements s, s' of the lattice $f(A)$ to be complementary central elements of $f(A)$

it is necessary and sufficient that $(a \wedge s) \bar{\vee} (a \wedge s') = a =$ $= (a \bar{\vee} s') \wedge (a \bar{\vee} s')$ for all a in $f(A)$. The fact that s and s' do indeed satisfy these conditions is a consequence of the second identity in (i) of the above lemma which, together with its dual, gives $a = f(s) = f(a \wedge s) \vee f(a \wedge s') = (a \wedge s) \vee (a \wedge s') \leq$ $\leq (a \wedge s) \bar{\vee} (a \wedge s') \leq a$ and $a = f(a) = f(a \vee s) \wedge f(a \vee s') =$ $= (a \bar{\vee} s) \wedge (a \bar{\vee} s')$. The proof of (20) will now be completed by our showing that (iv) implies (ii). To do this, we suppose that s satisfies (iv) and deduce that it satisfies the duals of the conditions in (i) of the lemma above. It is obvious that $f(s) \vee f(s') = f(1)$ (this holds for every s in A). Now s' as well as s is in $f(A)$ since s is clopen, and also $f(x)$ is in $f(A)$ for each x in A. Because s is in the center of $f(A)$, we thus have $f(x) = f(x) \bar{\vee} (s \wedge s') = (f(x) \bar{\vee} s) \wedge (f(x) \bar{\vee} s') =$ $= f(x \vee s) \wedge f(x \vee s')$ as required. Q.E.D.

CHAPTER 2

EXCHANGE AXIOMS

1. $E_{\frac{1}{2}}$ and E_1

Let f be a closure operator on a CBA A and let x be an element of A. We say that:

x is $\underline{f\text{-}E_{\frac{1}{2}}}$ iff, for all s and y in A such that $y \le f(s \vee x)$ and $y \nleq f(s)$, we have $u \le f(s \vee y \vee (x\backslash u))$ for some u, $0 \neq u \le x$; that

x is $\underline{f\text{-}E_1}$ iff, for all s and y in A such that $y \le f(s \vee x)$ and $y \nleq f(s)$, we have $u \le f(s \vee y)$ for some u, $0 \neq u \le x$; that

f is $\underline{E_i}$ iff the set of $f\text{-}E_i$ elements is $\bigvee$-dense in A; and that

f is $\underline{\text{super-}E_i}$ iff every element of A is $f\text{-}E_i$ (where in the last two definitions $i = \frac{1}{2}, 1$).

Evidently f is $E_{\frac{1}{2}}$ if it is E_1, f is super-$E_{\frac{1}{2}}$ if it is super-E_1, and f is E_i if it is super-E_i, $i = \frac{1}{2}, 1$.

(21) Let f be a closure operator on an atomic CBA A. Then the following conditions are equivalent

(i) f is $E_{\frac{1}{2}}$,

(ii) f is E_1,

(iii) $p \leq f(s \vee q)$, $p \nleq f(s)$ implies $q \leq f(s \vee p)$ for all elements s and atoms p,q of A.

<u>Proof</u>. For any CBA it is the case that (ii) implies (i) and, since each atom q of A is in every $\bigvee$-dense subset of A, that (i) implies (iii) also. Suppose that A is atomic. Then (iii) asserts that every atom q of A is f-E_1 (if $y \leq f(s \vee q)$ and $y \nleq f(s)$ then $p \nleq f(s)$ for some atom $p \leq y$ so that by (iii) $q \leq f(s \vee p) \leq f(s \vee y)$) and this implies (ii). Q.E.D.

Note that, although an <u>atom</u> is f-$E_{\frac{1}{2}}$ iff it is f-E_1, this equivalence is not true in general for all elements, even in the atomic case. To see this, it is convenient to use some terms still to be defined: there exists a matroidal (and therefore super-$E_{\frac{1}{2}}$) closure operator f on the 8-element CBA such that $f(0) = 0$ and yet which is not a quantifier (and therefore not super-E_1).

(22) Let f be an $E_{\frac{1}{2}}$ analytic closure operator on a CBA A and let x and s be elements of A. If x is maximal discrete in s then x is a base of s.

<u>Proof</u>. We have to prove that $s \leq f(x)$. Suppose that this is

not the case. Then $s \wedge f(x)' \neq 0$ and since in the equation $f(x)' = \bigvee\{w;\ w \wedge x = w \wedge d(x \vee w) = 0\}$ of (9)(iv) the set on the right-hand side is a section of A, the fact that f is $E_{1\frac{1}{2}}$ implies that there exists an f-$E_{1\frac{1}{2}}$ element w of A such that $0 \neq w \leq s \wedge f(x)'$ and $w \wedge d(x \vee w) = 0$, where d is the derivative of f. Now $x < x \vee w \leq s$ so that by the maximality of x, $x \vee w$ cannot be discrete. Since $w \wedge d(x \vee w) = 0$, this means that $x \wedge d(x \vee w) \neq 0$. From (3), there hence exists an element u of A such that $0 \neq u \leq x \wedge f((x\backslash u) \vee w)$. Now $u \nleq f(x\backslash u)$ by the discreteness of x and therefore, since w is f-$E_{1\frac{1}{2}}$, there exists v such that $0 \neq v \leq w$, $v \leq f((x\backslash u) \vee (w\backslash v) \vee u))$. But then $0 \neq v \leq d(x \vee (w\backslash v))$ which is contrary to the equation $w \wedge d(x \vee w) = 0$. Q.E.D.

(23) The lattice of an $E_{1\frac{1}{2}}$ analytic closure operator f on a CBA A is relatively disjunctive.

Proof. By saying that a lattice L is relatively disjunctive we mean that each interval [a,b] of L is disjunctive - equivalently that whenever $a \leq b < c$ in L there is an element d of L such that $a < d \leq c$ and $b \wedge d = a$. So let a,b, and c be elements of f(A) such that $a \leq b < c$. Using the equation of (9)(iv) as in the proof of (22), we can obtain an f-$E_{1\frac{1}{2}}$ element w of A such that $0 \neq w \leq c$, $w \wedge b = w \wedge d(b \vee w) = 0$.

Let $d = f(a \vee w)$. Then $a < d \leq c$ and $e = b \wedge d \geq a$. Suppose that the inequality here is strict. Then $e \leq f(a \vee w)$ and $e \nleq f(a)$ so that since w is f-$E_{\frac{1}{2}}$ we have $u \leq f(a \vee e \vee (w\backslash u))$ for some u, $0 \neq u \leq w$. But then $0 \neq u \leq f(b \vee (w\backslash u))$ which is contrary to the equation $w \wedge d(b \vee w) = 0$. Hence $b \wedge d = a$ and d satisfies the required conditions. Q.E.D.

It can be shown by a similar but somewhat easier argument that this result remains valid when the hypothesis '$E_{\frac{1}{2}}$+ analytic' is replaced by 'E_1'.

(24) Let f be a closure operator with derivative d on a CBA A. Then the following conditions are equivalent

(i) f is super-$E_{\frac{1}{2}}$,

(ii) $y \leq d(x)$ implies $y \leq d(x \wedge d(x \vee y))$ for all x,y in A,

(iii) $y \leq d(x)$ implies that there exists $z \leq x$ such that $y \vee z$ is pithy and $y \leq d(z)$, for all x,y in A,

(iv) same as (ii) but under the additional condition $x \wedge y = 0$,

(v) same as (iii) but under the additional condition $x \wedge y = 0$.

Proof. We obtain the implications (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii), (iv)$\Longrightarrow$(i), and (v)$\Longrightarrow$(i). Since (ii)$\Longrightarrow$(iv) and (iii)$\Longrightarrow$(v) are immediate, this will give the theorem.

<u>(i) $\Longrightarrow$ (ii)</u>. Let $y \leq d(x)$, put $z = x \wedge d(x \vee y)$, and suppose that $y \not\leq d(z)$. Then $0 \neq y \wedge d(z)' = y \wedge d(z)' \wedge d(x)$ and by (3) there is an element u of A such that $0 \neq u \leq y \wedge d(z)' \wedge f(x \wedge u')$. From this it follows that $u \leq f((z \wedge u') \vee (x \wedge z' \wedge u'))$, $u \not\leq f(z \wedge u')$. Therefore, if (i) holds we have $v \leq f((z \wedge u') \vee u \vee (x \wedge z' \wedge u' \wedge v'))$ for some v such that $0 \neq v \leq x \wedge z' \wedge u'$. Now $(z \wedge u') \vee u \vee$ $\vee (x \wedge z' \wedge u' \wedge v') = z \vee u \wedge (x \wedge v') \leq (x \vee y) \wedge v'$ so that we obtain $v \leq d(x \vee y)$ which with $v \leq x$ gives $v \leq z$, an impossibility. Thus (i) implies (ii).

<u>(ii) $\Longrightarrow$ (iii)</u>, We first note, dually to a remark made in the discussion following (6), that (ii) implies the idempotency of $\upharpoonleft \wedge d$ (take $y = x \wedge d(x)$). Now if $y \leq d(x)$ and (ii) holds then $z = x \wedge d(x \vee y)$ satisfies the conditions of (iii); for certainly $z \leq x$ and $y \leq d(z)$ - and $y \vee z$ is pithy since it equals $(x \vee y) \wedge d(x \vee y)$ which is pithy on account of the idempotency of $\upharpoonleft \wedge d$.

<u>(iv) $\Longrightarrow$ (i) and (v) $\Longrightarrow$ (i)</u>. In the hypothesis of the defining condition for an element x of A to be f-$E_{\frac{1}{2}}$, it is easy to see that we can without loss of generality take s and y such that $y \leq f(s \vee x)$, $y \wedge f(s) = x \wedge s = x \wedge y = 0$, $y \neq 0$. Let us suppose we are given elements x,s, and y satisfying these conditions - then $y \leq d(s \vee x)$ and $(s \vee x) \wedge y = 0$.

Hence if (iv) holds we have $y \leq d((s \vee x) \wedge d(s \vee x \vee y))$ so that, since $y \nleq d(s)$, it must be the case that $x \wedge d(s \vee x \vee y) \neq 0$. Alternatively if (v) holds then for some $z \leq s \vee x$ we have $y \leq d(z)$ and $z \leq d(z \vee y)$, where the fact that $y \nleq d(s)$ this time implies that $x \wedge z \neq 0$ and a fortiori that $x \wedge d(s \vee x \wedge y) \neq 0$ again. Hence under either assumption $x \wedge d(s \vee x \vee y) \neq 0$ and by (3) there exists an element u of A such that $0 \neq u \leq x \wedge f(s \vee (x\backslash u) \vee y)$ as is required for (i) to hold. Q.E.D.

(25) Let f be a super $E_{\frac{1}{2}}$ closure operator on a CBA A. If $x \leq f(y)$ and y is discrete then there exists a smallest element $z \leq y$ such that $x \leq f(z)$.

<u>Proof</u>. It is sufficient to consider the case $x \wedge y = 0$ since if $x \wedge y \neq 0$ and z_0 is the smallest element $\leq y$ such that $x\backslash y \leq f(z_0)$ then it is easily seen that $z = z_0 \vee (x \wedge y)$ is the smallest element $\leq y$ such that $x \leq f(z)$. Suppose therefore that $x \wedge y = 0$. Then $x \leq d(y)$ and by (24) there exists $z \leq y$ such that $x \vee z$ is pithy and $x \leq d(z)$ (the proof of (24) shows that $z = y \wedge d(x \vee y)$ is such an element z.) Let z_1 be any element $\leq y$ such that $x \leq f(z_1)$. If $z \nleq z_1$ then since $x \vee z$ is pithy we have $u \leq f(x \vee (z\backslash u))$ for some u, $0 \neq u \leq z\backslash z_1$. But $f(x \vee (z\backslash u)) \leq f(z_1 \vee (z\backslash u)) \leq f(y\backslash u)$, and

$0 \neq u \leq f(y\backslash u)$ is contrary to the discreteness of y. Thus $z \leq z_1$. Q.E.D.

(26) Suppose we have the following situation:

a super-$E_{\frac{1}{2}}$ closure operator f on a CBA A, elements a,b of $f(A)$, and bases x,y of a,b respectively. Let $M(a,b)$ denote the statement that a and b form a modular pair in the lattice $f(A)$. Then the following conditions are equivalent.

(i) $x \vee y$ is a base of $a \vee b$,

(ii) $M(a,b)$ and $x \wedge y$ is a base of $a \wedge b$.

(Note that (i) holds iff $x \vee y$ is discrete, and iff $x \vee y$ is a base of $a \mathbin{\bar{\vee}} b$).

Proof. Assume that (i) holds. We show first that $x \wedge y$ is a base of $a \wedge b$. Suppose not - then since $x \wedge y$ is discrete it must be the case that $a \wedge b \nleq f(x \wedge y)$. Now $a \wedge b \leq f(x) =$ $= f((x \wedge y) \vee (x \wedge y'))$ and, since f is super-$E_{\frac{1}{2}}$, there exists u such that $0 \neq u \leq x \wedge y'$, $u \leq f((a \wedge b) \vee (x \wedge y) \vee (x \wedge y' \wedge u'))$. But then $0 \neq u \leq f(y \vee (x \wedge y) \vee (x \wedge y' \wedge u')) =$ $f((x \vee y) \wedge u')$ which is contrary to the discreteness of $x \vee y$. To obtain $M(a,b)$ we have to show that $(c \mathbin{\bar{\vee}} a) \wedge b \leq c \mathbin{\bar{\vee}} (a \wedge b)$ - that is, $f(c \vee a) \wedge b \leq f(c \vee (a \wedge b))$ - for all closed $c \leq b$. Take an element $c \leq b$. Then $f(c \vee a) \wedge b \leq f(c \vee a) =$

$f(c \vee (a \wedge b) \vee (x \wedge y'))$ since $a = f(x) = f((x \wedge y) \vee (x \wedge y')) = f((a \wedge b) \vee (x \wedge y'))$. Hence if $f(c \vee a) \wedge b \nleq f(c \vee (a \wedge b))$ then because f is super-$E_{\frac{1}{2}}$ there exists u such that $0 \neq u \leq x \wedge y'$, $u \leq f(c \vee (a \wedge b) \vee (f(c \vee a) \wedge b) \vee (x \wedge y' \wedge u'))$. Now $c \vee (a \wedge b) \vee (f(c \vee a) \wedge b) \leq b = f(y)$ and, since $y \wedge u = 0$, $y \vee (x \wedge y' \wedge u') = (x \vee y) \wedge u'$. Thus $0 \neq u \leq f((x \vee y) \wedge u')$ which is again contrary to the discreteness of $x \vee y$. Therefore (i) implies (ii).

Now assume that (ii) holds and suppose that $x \vee y$ is not discrete. Then either

a) there exists u such that $0 \neq u \leq x \wedge y$, $u \leq f((x \vee y) \wedge u') = f((x \wedge u') \vee (y \wedge x' \wedge u'))$, or

b) there exists v such that $0 \neq v \leq y \wedge x'$, $v \leq f(x \vee (y \wedge v'))$, or

c) the same as b) but with x and y interchanged.

If a) holds then, since $u \leq f(x \wedge u')$ (x is discrete), there exists v such that $0 \neq v \leq y \wedge x' \wedge u'$, $v \leq f((x \wedge u') \vee u \vee (y \wedge x' \wedge u' \wedge v')) = f(x \vee (y \wedge v'))$. It follows that we may without loss of generality assume that case b) occurs. From $y \wedge v' \leq b$ and $M(a,b)$ we obtain $f((y \wedge v') \vee a) \wedge b \leq f((y \wedge v') \vee (a \wedge b))$. This implies that $0 \neq v \leq f((y \wedge v') \vee (a \wedge b)) = f((y \wedge v') \vee (x \wedge y)) = f(y \wedge v')$ which however is contrary to the discreteness of y. Q.E.D.

Super-E_1 closure operators f satisfying the additional condition $f(0) = 0$ will be studied in Chapter 4; under this condition they turn out to be precisely the same as quantifiers.

2. $\underline{S_{\frac{1}{2}}}$ and $\underline{S_1}$

We now introduce some restricted forms of the above exchange axioms. In doing this, it will be convenient to consider only closure operators f with $f(0) = 0$. So let f be such a closure operator on a CBA A and let x be an element of A. Then we say that:

x is $\underline{f\text{-}S_{\frac{1}{2}}}$ iff, for all y in A such that $0 \neq y \leq f(x)$, we have $u \leq f(y \vee (x\backslash u))$ for some u, $0 \neq u \leq x$; that

x is $\underline{f\text{-}S_1}$ iff, for all y in A such that $0 \neq y \leq f(x)$, we have $u \leq f(y)$ for some u, $0 \neq u \leq x$ (i.e. we have x meets $f(y)$); that

f is $\underline{S_i}$ iff the set of $f\text{-}S_i$ elements is $\bigvee$-dense in A; and that

f is $\underline{\text{super-}S_i}$ iff every element of A is $f\text{-}S_i$ (where in the last two definitions $i = \frac{1}{2}, 1$).

It is clear that these S-conditions satisfy the same implications between themselves as do the E-conditions, and that, given $f(0) = 0$, each E-condition implies the corresponding S-condition.

(27) Let f be a closure operator on an atomic CBA A such that $f(0) = 0$. Then the following conditions are equivalent

(i) f is $S_{1/2}$,

(ii) f is S_1,

(iii) $p \leq f(q)$ implies $q \leq f(p)$ for all atoms p,q of A.

The proof of this is similar to that of (21).

(28) The lattice of an $S_{1/2}$ analytic closure operator f on a CBA A is disjunctive.

The proof of this is similar to that of (23). Here again the result remains valid when the hypothesis '$S_{1/2}$ + analytic' is replaced by 'S_1'.

<u>Lemma</u>. Let f be a closure operator on a CBA A such that $f(0) = 0$ and let x be an element of A.

(a) x is f-S_1 iff $x \wedge a = 0$ implies $f(x) \wedge a = 0$ for all closed a.

(b) If $f(x \wedge a) = f(x) \wedge a$ for all closed a then x is f-S_1, and the converse holds if f is S_1.

Proof. (a) Let x and a be such that x is f-S_1, a is closed, and $x \wedge a = 0$. Putting $y = f(x) \wedge a$ we have $f(y) \leq a$ so that $x \wedge f(y) = 0$ and hence $y = 0$ since x is f-S_1 and $y \leq f(x)$. Conversely if x is not f-S_1 and $0 \neq f(x)$, $x \wedge f(y) = 0$ then $a = f(y)$ is closed and disjoint from x but not from $f(x)$.

(b) The first statement of (b) is an immediate consequence of (a). To complete the proof of (b) we have to show that $f(x \wedge a) = f(x) \wedge a$ for all closed a, under the supposition that f is S_1 and x is f-S_1. Make this supposition and let a be closed. Then certainly $f(x \wedge a) \leq f(x) \wedge a$. If the inequality here is strict then since f is S_1 there exists an f-S_1 element x_1 such that $0 \neq x_1 \leq f(x) \wedge a \wedge f(x \wedge a)'$. Since x is f-S_1, we have $y = x \wedge f(x_1) \neq 0$ and, since x_1 is f-S_1, we have $x_1 \wedge f(y) \neq 0$. But $y \leq x \wedge a$ since $x_1 \leq a$ and hence $f(y) \leq f(x \wedge a)$, so that x_1 is both disjoint and not disjoint from $f(x \wedge a)$, a contradiction. Q.E.D.

(29) Let f be a closure operator on a CBA A such that $f(0) = 0$. Then the following conditons are equivalent.

(i) f is S_1,

(ii) $\{x;\ x \wedge a = 0$ implies $f(x) \wedge a = 0$ for all closed $a\}$ is $\bigvee$-dense in A,

(iii) $\{x;\ f(x \wedge a) = f(x) \wedge a$ for all closed $a\}$ is $\bigvee$-dense in A,

(iv) for all closed a and all elements $y \ngeq a$, there exists a closed b with $0 \neq b \leq a$ such that the only closed element $\leq b \wedge y$ is 0.

(v) the $\bigvee$-closure $J(f(A))$ of $f(A)$ in A is a Boolean subalgebra (necessarily complete) of A.

<u>Proof</u>. The equivalence of (i), (ii), and (iii) is an immediate consequence of the preceding lemma. To show that (i) and (iv) are equivalent, suppose first that (i) holds and let a be closed and $y \ngeq a$. Then there exists a non-zero f-S_1 element $x \leq a\backslash y$. Putting $b = f(x)$ we see that b is closed and $0 \neq b \leq a$; also if c is closed and $c \leq b \wedge y$ then $x \wedge c = 0$, so that $c = b \wedge c = f(x) \wedge c = 0$. Thus (i) implies (iv). Suppose conversely that (iv) holds and let z be any non-zero element of A. Put $a = f(z)$ and $y = a\backslash z$ - then $y \ngeq a$ and there exists a closed element b as described in (iv). Put $x = z \wedge b$. Then $0 \neq x \leq z$ (if $x = 0$ then b is non-zero, closed, and $\leq b \wedge y$, contrary to (iv)) and if c is closed and $x \wedge c = 0$ then $f(x) \wedge c$

is closed and $\leq b \wedge x' = b \wedge y$, so that $f(x) \wedge c = 0$. Thus every non-zero element z contains a non-zero f-S_1 element x and we have shown that (iv) implies (i).

In (v) the $\bigvee$-closure $J(f(A))$ of $f(A)$ in A, being simply the set of joins in A of all subsets of $f(A)$, is obviously a $\bigvee$-closed subset of A. Therefore in order to show that $J(f(A))$ is a (necessarily complete) subalgebra of A it is sufficient to show that it is closed under complementation. So let $a = \bigvee a_i$ be an arbitrary member of $J(f(A))$, where the a_i's are in $f(A)$, and suppose that f is S_1. Then a' is a join $\bigvee x_j$ of f-S_1 elements x_j where, for each a_i and x_j, $a_i \wedge x_j = 0$ and hence $a_i \wedge f(x_j) = 0$. It follows that $a' = \bigvee f(x_j)$ so that a' is in $J(f(A))$ and we have proved that (i) implies (v). Suppose conversely that (v) holds - we argue that (iv) must then hold also. For let $y \not\geq a$ where a is closed and let z denote the join of all closed elements $\leq y$. Then z is in $J(f(A))$ and hence by (v) so also is $a \wedge z'$. Therefore since $a \wedge z' \neq 0$ there is a non-zero closed element $b \leq a \wedge z'$ and this element b clearly fulfils the requirements of (iv). Q.E.D.

Using condition (v) of (29) we see that every T_1 closure operator is a fortiori S_1 (indeed S_1 is itself a sort of separation axiom, as also is $S_{\frac{1}{2}}$, and we shall study it in this light in Appendix 2).

(30) If f is an S_1 closure operator on a CBA A then the f-separators are precisely the central elements of $f(A)$ (c.f. (20).).

<u>Proof</u>. By (20) this will be proved if we show that each central element s of $f(A)$ is clopen. Let t be the unique complement of s in $f(A)$. Then $s' \wedge t'$ is in $J(f(A))$ by (29). But if a is in $f(A)$ and $a \leq s' \wedge t'$ then $a = a \wedge (s \mathbin{\bar{\vee}} t) = (a \wedge s) \mathbin{\bar{\vee}} (a \wedge t) = 0$. It follows that $s' \wedge t' = 0$ so that $s \vee t = 1$, $t = s'$, and s is clopen. Q.E.D.

We shall use the following result in Chapter 4:

(31) If f is an S_1 closure operator on a CBA A and $k = \bigvee d^{-1}(0)$ then k' is pithy.

<u>Proof</u>. k is clearly in $J(f(A))$ and hence, by (29)(v), so also is k'. Since the join of pithy elements is pithy, the result will therefore follow if we can show that every closed element contained in k' is pithy. Let a be closed, $a \leq k'$. Then $d(a)$ is also closed and hence $a \backslash d(a)$ is in $J(f(A))$. Now $a \backslash d(a)$ is discrete - therefore the closed elements whose join is $a \backslash d(a)$ are all discrete and, being closed, are thus all in $d^{-1}(0)$. It follows that $a \backslash d(a) \leq k$ and hence that $a \backslash d(a) = 0$. Thus $a \leq d(a)$ and a is pithy as required. Q.E.D.

Super-S_1 closure operators are the same as quantifiers, so they will be studied in Chapter 4. Super-$S_{\frac{1}{2}}$ closure operators do not appear to be of much interest for our purposes (a super-$S_{\frac{1}{2}}$ closure operator which is not $E_{\frac{1}{2}}$ is obtained by taking the 8-element CBA with closed elements 0,1, the atoms, and one further element).

CHAPTER 3

MATROIDS AND B-MATROIDS

1. Matroids

A closure operator f on a CBA A will be said to be a matroidal closure operator iff it is analytic and has a derivative d such that $\upharpoonright \wedge d$ is idempotent. The derivative of a matroidal closure operator will be called a matroidal derivative. Whitney duality for matroids is none other than the self-duality of the following evident condition for matroidal derivativity: an operator d on a CBA A is matroidal derivative iff d is a biderivative and $\upharpoonright \vee d$, $\upharpoonright \wedge d$ are both idempotent.

(32) Let f be a closure operator with derivative d on a CBA A. Then the following conditions are equivalent

- (i) f is matroidal,
- (ii) f is analytic and super-$E_{\frac{1}{2}}$,
- (iii) f is analytic and $y \leq d(x)$ implies that there exists $z \leq x$ such that $y \vee z$ is pithy, for all x,y in A,
- (iv) same as (iii) but with the second part of (iii) under the additional condition $x \wedge y = 0$.

Before proving this, we first obtain the following

Lemma. (a) For all operators d on a CBA A, $\lceil \wedge d$ is idempotent iff

(π_1) $y \leq d(x)$ implies that there exists $z \leq x$ such that $y \vee z \leq d(y \vee z)$, for all x,y in A.

(b) For all derivatives d on a CBA A, $\lceil \wedge d$ is idempotent iff

(π_2) $y \leq d(x)$ and $x \wedge y = 0$ implies that there exists $z \leq x$ such that $y \vee z \leq d(y \vee z)$, for all x,y in A.

Proof of lemma. We first note that (π_1) is equivalent to:

(π_1') $y \leq d(x)$ implies that there exists w such that

$$y \leq w \leq x \vee y, \quad w \leq d(w),$$

and that (π_2) is equivalent to the analogous condition (π_2').

(a) Clearly $\lceil \wedge d$ is idempotent iff $u \wedge d(u) \leq d(u \wedge d(u))$ for all u in A. It follows that if $\lceil \wedge d$ is idempotent and $y \leq d(x)$ then $w = (x \vee y) \wedge d(x \vee y)$ satisfies (π_1'). Suppose conversely that (π_1') holds and let u be an arbitrary element of A. Since $u \wedge d(u) \leq d(u)$ there exists w such that $u \wedge d(u) \leq w \leq u$, $w \leq d(w)$. But then $w \leq d(w) \leq d(u)$ so that $w \leq u \wedge d(u)$ and hence $w = u \wedge d(u)$. Thus $u \wedge d(u) \leq d(u \wedge d(u))$ as required.

(b) On account of (a) it will be sufficient to prove that (π_2') implies (π_1') (given that d is a derivative). So suppose that (π_2') holds and let $y \leq d(x)$. Then $y = \bigvee\{u;\ u \leq y \wedge f(x\backslash u)\}$ by (3). For each u in this join we have $u \leq d(x\backslash u)$, $(x\backslash u) \wedge u = 0$ and hence by (π_2') there exists v such that $u \leq v \leq (x\backslash u) \vee u = x \vee u$, $v \leq d(v)$. It is then easily verified that

$w = \bigvee\{v;\ u \leq v \leq x \vee u$ and $v \leq d(v)$ for some u such that $u \leq y \wedge f(x\backslash u)\}$ is as described in (π_1'). Q.E.D.

Proof of (32). (i) implies (ii) by the dual of (6) together with the fact that (24)(ii) implies (24)(i); the fact that (24)(i) implies (24)(iii) shows that (ii) implies (iii); (iii) trivially implies (iv); and (iv) implies (i) by virtue of part (b) of the lemma just proved. Q.E.D.

Since matroidal closure operators are super-$E_{\frac{1}{2}}$, many of the results of the previous chapter apply to them (note for example the four very similar conditions, involving the hypothesis $y \leq d(x)$, on a closure operator f which arise from (24)(iii), (v) and (32)(iii), (iv) and which, when coupled with analyticity, are each equivalent to the matroidality of f). It would be pleasant to be able to report that matroidal closure operators are E_1, but whether this is actually so - and whether indeed $E_{\frac{1}{2}}$ implies E_1 in general - is at present unresolved (apart from certain cases - see (21) and (44)(b)).

In view of (32), it seems doubtful whether matroidal derivatives satisfy any identities not derivable (for arbitrary operators) from condition (I) and its dual (see the discussion following (6)). Amongst the identities which are so derivable we have those expressing the idempotency of $\upharpoonright \vee d$ and $\upharpoonright \wedge d$, and also $d(\upharpoonright \vee d(\upharpoonright \wedge d)) = d$ and its dual. The joint idempotency of $\upharpoonright \vee d$ and $\upharpoonright \wedge d$ gives rise to a further pair of identities which might be worth mentioning, namely $d^2(\upharpoonright \vee d) = d(\upharpoonright \vee d)$ and its dual. (This may be seen as follows. We have $\upharpoonright \vee d \geq d(\upharpoonright \vee d)$ from the idempotency of $\upharpoonright \vee d$ and hence $d(\upharpoonright \vee d) \geq d^2(\upharpoonright \vee d)$. The left-hand side of this inequality is $(\upharpoonright \wedge d)(\upharpoonright \vee d)$ by the idempotency of $\upharpoonright \vee d$ again and the right-hand side is $\geq (\upharpoonright \wedge d)^2(\upharpoonright \vee d) = (\upharpoonright \wedge d)(\upharpoonright \vee d)$ by the idempotency of $\upharpoonright \wedge d$. The result follows.) A simple fact which follows from the idempotency of $\upharpoonright \wedge d$ is that we not only have:

> x pithy implies $d(x)$ perfect, but also the dual result:
> x closed implies $d(x)$ perfect - expressed differently,
> $d(x) \leq x$ implies $d(x) = d^2(x)$.

(33) Bireductions and cartesian products of matroidal closure operators and matroidal derivatives are again such.

For bireductions, this is an easy consequence of the definitions together with (11) and its dual; for cartesian products it is trivial.

Let us say that a lattice L is a <u>matroid lattice</u> iff L is isomorphic to the lattice of some matroidal closure operator on a CBA. Then from (33) we have

<u>Corollary.</u> Intervals and products of matroid lattices are again such.

2. <u>B-Matroids</u>

A closure operator f on a CBA A will be said to be a <u>B-matroidal closure operator</u> iff, for all s in A, each discrete subelement of s is contained in a base of s. The derivative of a B-matroidal closure operator will be called a <u>B-matroidal derivative.</u> Whitney duality holds for B-matroids, since, as is clear from the characterization given below, the notion of a B-matroidal derivative is self-dual. This self-duality shows that a B-matroidal derivative is indeed a matroidal derivative and hence that a B-matroidal closure operator is a matroidal closure operator.

(34) An operator d on a CBA A is a B-matroidal derivative iff d satisfies the following interpolation condition

IC: for all elements s,t,x,z of A such that $x \wedge d(x) \leq s \leq x \leq z \leq t \leq z \vee d(z)$, there exists an element y of A such that $y \wedge d(y) \leq s$, $x \leq y \leq z$, and $t \leq y \vee d(y)$.

<u>Proof</u>. Let d be a B-matroidal derivative and let s,t,x,z satisfy the hypothesis of IC. Let u be a base of s (0, being a discrete subelement of s, is contained in some base of s) - we claim that $u \vee (x\backslash s)$ is a base of x. Certainly u is contained in some base v of x and we clearly have $v \leq u \vee (x\backslash s)$; the reverse inequality follows from the fact that

$$\begin{aligned}
x\backslash s &= x \wedge s' \wedge (v \vee d(v)) \\
&= (x \wedge s' \wedge v) \vee (x \wedge s' \wedge d(v)) \\
&\leq v \vee (x \wedge d(x) \wedge s') \\
&\leq v \vee (s \wedge s') = v.
\end{aligned}$$

Now there exists a base $u \vee w$ of z such that $w \wedge u = 0$ and $x\backslash s \leq w$. Put $y = s \vee w$ - then $x \leq y \leq z$. Also $y \wedge d(y) =$ $= (s \vee w) \wedge d(s \vee w) \leq s \vee (w \wedge d(s \vee w)) \leq s \vee (w \wedge d(u \vee w))$ by (7) and this equals s since $u \vee w$ is discrete. Finally $y \vee d(y) \geq t$ on account of the idempotency of $\upharpoonleft \vee d$ and the relations $y \vee d(y) \geq z$, $z \vee d(z) \geq t$. Thus d satisfies IC.

Now suppose that d is given to satisfy IC. To show that d is a B-matroidal derivative we have to prove the following three facts, wherein we have put $\upharpoonleft \vee d = f$:

(i) d is a derivative (and is therefore necessarily the derivative of f);

(ii) f is idempotent (and thus a closure operator); and

(iii) f is a B-matroidal closure operator.

To do this we first prove the following lemma.

Lemma. Let d be an operator on a CBA A.

(a) If d satisfies IC in the restricted case $x = s$, $z = t$ then d is a biderivative.

(b) If d satisfies IC in the restricted case $s = 0$ then $\upharpoonleft \vee d$ is idempotent.

Proof of (a). We verify that $t \backslash d(t) = \bigwedge\{y;\ y \leq t \leq y \vee d(y)\}$ for all t in A; by (5) this will show that d is a derivative. The self-duality of the hypothesis of (a) will then imply that d is a biderivative. Now to obtain the above equation for a given t in A it is easily seen to be sufficient to show that if $t \backslash d(t) \leq s < t$ then there exists y such that $s \leq y < t \leq y \vee d(y)$. So let $t \backslash d(t) \leq s < t$ and let y be such that $y \wedge d(y) \leq s \leq y \leq t \leq y \vee d(y)$. We want $y \neq t$ - and this is the case since if $y = t$ then besides $s \geq t \wedge d(t)'$ we also have $s \geq t \wedge d(t)$, so that $s \geq t$, a contradiction.

Proof of (b). Assume that d satisfies the restricted case $s = 0$ of IC. We first show that if $x \leq z \leq x \vee d(x)$ and $x \wedge d(x) = 0$ then $(\upharpoonleft \vee d)(z) = (\upharpoonleft \vee d)(x)$. Suppose that x and z satisfy $x \leq z \leq x \vee d(x)$, $x \wedge d(x) = 0$. Then we can apply IC with $s = 0$ and $t = z \vee d(z)$ and we obtain y such that $x \leq y \leq z$, $y \wedge d(y) = 0$, $y \vee d(y) = z \vee d(z)$. Now, as remarked earlier, $y \wedge d(y) \leq x \leq y \leq x \vee d(x)$ implies $x = y$ for any operator d on A. Thus $x = y$

in the present case and hence $z \vee d(z) = x \vee d(x)$. We can now show that $\upharpoonright \vee d$ is idempotent. Let z be any element of A and apply IC with $s = x = 0$, $t = z$. This gives an element x such that $x \leq z \leq x \vee d(x)$, $x \wedge d(x) = 0$. From what has just been shown, this implies that $(\upharpoonright \vee d)(z) = (\upharpoonright \vee d)(x)$ and thence, with z replaced by $(\upharpoonright \vee d)(z)$, that $(\upharpoonright \vee d)^2(z) = (\upharpoonright \vee d)(x)$. Thus $(\upharpoonright \vee d)^2(z) = (\upharpoonright \vee d)(z)$ as required. Q.E.D.

Returning to the proof of (34), we see that this lemma gives us the first two of the three facts which have to be proved in order to show that d is a B-matroidal derivative. The third fact, stating that f is B-matroidal closure operator, follows directly from the first two facts together with the fact that d satisfies IC (here, as in (b) of the lemma, the case $s = 0$ only is required). Q.E.D.

(Note. It is possible to prove (34) using the arguments of [20] - see results (10), (11), and (12) in [20]; the proof of (34) given here is obviously better however.)

The following two examples show that IC cannot be replaced in (34) by the restricted forms of IC occuring in the above two lemmas. Let A be the finite CBA with four atoms p_1, p_2, p_3, p_4 and define $d(\bigvee_i p_i) = \bigvee_i p_{i+1}$, where p_5 is p_1. Then d satisfies IC with $x = s$, $z = t$ but $\upharpoonright \vee d$ is not idempotent. Next let A be the finite CBA with two atoms p_1 and p_2 and put $d(1) = 1$, $d(x) = p_1$

otherwise. Then d satisfies IC with $s = 0$ yet d is not a derivative.

(35) Let f be a closure operator on a CBA A. Then f is B-matroidal iff it is $E_{\frac{1}{2}}$ and satisfies the following minimality condition

MC: for all elements s,x,y of A such that $x \leq f(s \vee y)$ there exists a minimal element z of A such that $x \leq f(s \vee z)$, $z \leq y$.

<u>Proof</u>. Suppose that f is a B-matroidal closure operator. Then f is $E_{\frac{1}{2}}$ since by (32) it is super-$E_{\frac{1}{2}}$. To show that f satisfies MC let $x \leq f(s \vee y)$. Let u and v be bases of s and $s \vee y$ respectively with $u \leq v$ - such exist on account of f being B-matroidal - and denote by w the smallest subelement of v such that $x \leq f(w)$ - this exists by (25). Put $z = w \backslash u$. Then $z \leq y$ and $x \leq f(s \vee z)$, the latter since $w \leq s \vee z$. Also, if $z_1 < z$ then $f(s \vee z_1) = f(u \vee s_1)$ and since $w \nleq u \vee z_1 \leq v$ we cannot have $x \leq f(s \vee z_1)$. Hence z is minimal such that $x \leq f(s \vee z)$, $z \leq y$.

Now suppose that f is $E_{\frac{1}{2}}$ and satisfies MC. We show first that f is super-$E_{\frac{1}{2}}$. Let $y \leq f(s \vee x)$, $y \nleq f(s)$ and let z be minimal such that $y \leq f(s \vee z)$, $z \leq x$. Then $z \neq 0$ and, since f is $E_{\frac{1}{2}}$, z contains a non-zero f-$E_{\frac{1}{2}}$ element, z_1 say.

By the minimality of z, $y \leq f(s \vee (z\backslash z_1))$, whereas $y \leq f(s \vee z) = f(s \vee (z\backslash z_1) \vee z_1)$. The fact that z_1 is f-$E_{1/2}$ thus implies the existence of an element u such that $0 \neq u \leq z_1$ and $u \leq f(s \vee (z\backslash z_1) \vee y \vee (z_1\backslash u)) = f(s \vee y \vee (z\backslash u))$. It follows that $0 \neq u \leq x$ and $u \leq f(s \vee y \vee (x\backslash u))$ as required for f to be super-$E_{1/2}$. To show that f is B-matroidal, let x be a discrete subelement of an element s of A. By MC there exists a minimal element y such that $y \leq s$ and $f(s) = f(x \vee y)$ - note that then $x \wedge y = 0$. We claim that $x \vee y$ is a base of s, equivalently, that $x \vee y$ is discrete. For suppose not - then there exists either a) a non-zero element $u \leq x$ such that $u \leq f((x\backslash u) \vee y)$ or b) a non-zero element $v \leq y$ such that $v \leq f(x \vee (y\backslash v))$. If a) holds then, from $u \nleq f(x\backslash u)$ (x is discrete) and the fact that f is super-$E_{1/2}$, we obtain a non-zero $v \leq y$ such that $v \leq f((x\backslash u) \vee u \vee (y\backslash v)) = f(x \vee (y\backslash v))$ so that b) holds in any case. However, b) is clearly incompatible with the minimality of y. Thus $x \vee y$ is a base of s and this shows that f is B-matroidal. Q.E.D.

(36) Bireductions and cartesian products of B-matroidal closure operators and B-matroidal derivatives are again such.

This is straightforward (c.f. (32); here we use (33) in addition).

Let us say that a lattice L is a <u>B-matroid lattice</u> iff

L is isomorphic to the lattice of some B-matroidal closure operator on a CBA. Then from (36) we have

> Corollary. Intervals and products of B-matroid lattices are again such.

(37) A B-matroid lattice is left-complemented.

Proof. Let f be a B-matroidal closure operator on a CBA A and let a and b be in $f(A)$. Take a base x of a, extend it to a base $x \vee y$ of $a \vee b$, where $x \wedge y = 0$, and put $c = f(y)$. Then certainly $c \leq b$ (since $y \leq b$) and $a \mathbin{\bar{\vee}} c = a \mathbin{\bar{\vee}} b$. Now x is a base of a, y is a base of c, and $x \vee y$ is a base of $a \vee c$. It follows by (26) that $M(c,a)$ and that $x \wedge y$ is a base of $a \wedge c$ - so that $a \wedge c = f(x \wedge y) = f(0)$, the smallest element of $f(A)$. Q.E.D.

> Corollary. A B-matroid lattice is:
>
> (a) relatively complemented;
>
> (b) semimodular in the sense of MacLane [31]; and
>
> (c) semimodular in the sense that the relation $M(a,b)$ is symmetric.

These are well-known and in any case easy consequences of left-complementedness for arbitrary lattices with 0.

(38) Let f be a B-matroidal closure operator on a CBA A and suppose that f is S_1. Then the set of all f-separators is a complete Boolean subalgebra of A.

Proof. By (30), part (a) of the preceding corollary, and Janowitz's result [24] that the centre of a complete relatively complemented lattice L is a complete sublattice of L, the set S of all f-separators is a complete sublattice of $f(A)$. In particular, S is a $\bigwedge$-closed subset of $f(A)$ and this, together with the fact that $f(A)$ is a $\bigwedge$-closed subset of A, implies that S is a $\bigwedge$-closed subset of A. Since S is a subalgebra of A by (16), the result follows. Q.E.D.

The symmetry of $M(a,b)$ in a B-matroid lattice, obtained in (37), can also be seen directly from the following improvement of (26).

(39) Let f be a B-matroidal closure operator on a CBA A and let a and b be elements of A. Then the following conditions are equivalent

(i) $M(a,b)$,

(ii) there exist bases x and y of a and b respectively such that $x \vee y$ is a base of $a \vee b$,

(iii) for all bases x and y of a and b respectively for which $x \wedge y$ is a base of $a \wedge b$, $x \vee y$ is a base of $a \vee b$.

<u>Proof</u>. From (26) it follows immediately that (ii) implies (i) and that (i) implies (iii). Since f is B-matroidal, there exist bases x and y respectively of any given pair a and b of elements of A which in addition are such that $x \wedge y$ is a base of $a \wedge b$. To see this, take a base z of $a \wedge b$ and extend it to bases x and y of a and b respectively. Then $z \leq x \wedge y \leq a \wedge b$ and, since z and $x \wedge y$ are respectively dense and discrete in $a \wedge b$, we must have $z = x \wedge y$, and $x \wedge y$ is a base of $a \wedge b$. On account of this fact, (26) also shows that (iii) implies (ii). Q.E.D.

Some results specific to matroidal closure operators on <u>atomic</u> CBA's will be discussed in Appendix 3.

CHAPTER 4

TOPOLOGICAL MATROIDS

1. Topological Closure Operators

A $\vee$-operator on a CBA A is an operator f on A such that $f(\bigvee X) = \bigvee f(X)$ for all finite subsets X of A; equivalently, such that $f(0) = 0$ and $f(x \vee y) = f(x) \vee f(y)$ for all x,y in A.

(40) (a) The join of $\vee$-operators is again a $\vee$-operator,

(b) If f is a $\vee$-operator and g is any operator then $f \backslash g$ is a $\vee$-operator.

(c) Reductions and cartesian products of $\vee$-operators are again such. If f is a $\vee$-operator then so is the coreduction f_s of f, provided that $f(s) \leq s$.

Proof. (a) and (c) are easily verified. To obtain (b), let X be any finite subset of A. Then $(f\backslash g)(\bigvee X) = \bigvee\{f(u)\backslash g(u);\ u \leq \bigvee X\}$ by (1). Now for $u \leq \bigvee X$ we have $f(u) = f(u \wedge (\bigvee X)) = \bigvee f(u \wedge X)$ so that $f(u)\backslash g(u) = (\bigvee_{x \in X} f(u \wedge x))\backslash g(u) = \bigvee_{x \in X}(f(u \wedge x)\backslash g(u)) \leq \bigvee_{x \in X}(f(u \wedge x)\backslash g(u \wedge x)) \leq \bigvee_{x \in X}(f\backslash g)(x)$. Therefore $(f\backslash g)(\bigvee X) \leq \bigvee(f\backslash g)(X)$ and since the reverse inequality is trivial we have proved (b). Q.E.D.

As an immediate consequence of parts (a) and (b) of this result we have:

Corollary. An operator $f \geq \upharpoonleft$ is a $\vee$-operator iff its derivative is a $\vee$-operator.

Note that this corollary applies in particular to any closure operator f. A closure operator which is also a $\vee$-operator will be called a *topological closure operator* and the derivative of such an operator will be called a *topological derivative*. We see from (40) that reductions and cartesian products of topological closure operators are again such and that if f is a topological closure operator then so is f_s when s is f-closed.

(41) A closure operator f on a CBA A is a topological closure operator iff $f(A)$ is a $\vee$-closed subset of A.

Proof. This result is well-known but we include a proof for the sake of completeness: If f is topological and X is a finite subset of $f(A)$ then $f(\bigvee X) = \bigvee f(X) = \bigvee X$ so that $\bigvee X$ is in $f(A)$, that is, $f(A)$ is $\vee$-closed. If $f(A)$ is given to be $\vee$-closed and X is a finite subset of A then $\bigvee f(X)$ is in $f(A)$ - hence from $\bigvee X \leq \bigvee f(X)$ we obtain $f(\bigvee X) \leq \bigvee f(X)$ so that $f(\bigvee X) = \bigvee f(X)$. Q.E.D.

(42) A topological closure operator is analytic iff it is analytic at 0.

Proof. Recall that a closure operator f with $f(0) = 0$ is analytic at 0 iff $j(f) = 1$. Let f be a topological closure operator which is analytic at 0. To show that f is analytic it is enough, by the corollary to (10), to show that it is analytic at each closed element s, equivalently, that $s' = \bigvee\{w;\ w \wedge s = w \wedge d(s \vee w) = 0\}$, where d is the derivative of f. Now in this join $d(s \vee w) = d(s) \vee d(w)$ so that, since $d(s) \leq s$, we simply have the join of the elements w which are discrete and $\leq s'$. Since the join of all discrete elements is 1 and the set of discrete elements is a section, the join considered does equal s' as required. Q.E.D.

(43) Let f be a topological closure operator on a CBA A and let s be an element of A. Then the following conditions are equivalent

(i) s is an f-separator,

(ii) s is clopen,

(iii) s is in the centre of $f(A)$. (c.f.(20) and (30).)

Proof. Suppose first that s is clopen. Then $f(s) \wedge f(s') = s \wedge s' = 0 = f(0)$ and $f(x) = f((x \wedge s) \vee (x \wedge s')) =$

$f(x \wedge s) \vee f(x \wedge s')$ for all x in A. It follows by the lemma preceding (17) that s is an f-respector. Now suppose that s is central in $f(A)$, with t as its complement in $f(A)$. Then, since $f(A)$ is a sublattice of A by (41), t must be the complement s' of s in A, and thus s is clopen. The combination of these two facts with (20) leads directly to the result. Q.E.D.

The various exchange conditions introduced in Chapter 2 share in the general simplification occasioned by topologicality:

(44) (a) For topological closure operators the properties E_i and S_i are equivalent and so also are the properties super-E_i and super-S_i, where $i = \frac{1}{2}, 1$.

(b) For analytic topological closure operators the properties $E_{\frac{1}{2}}$, E_1, $S_{\frac{1}{2}}$, and S_1 are all equivalent.

Proof. The proof of this rests on the following:

Lemma. Let f be a topological closure operator on a CBA A and let x be an element of A. Then

(c) x is f-E_i iff x is f-S_i, where $i = \frac{1}{2}, 1$ and

(d) for x discrete, x is f-$S_{\frac{1}{2}}$ iff x is f-S_1.

Proof of lemma. (c) follows from the definitions and the fact that (for f topological) $y \leq f(s \vee x)$ and $y \nleq f(s)$ implies

$z \leq f(x)$ and $z \nleq f(0)$, where $z = y \wedge f(x)$. (d) follows from the corollary to (40) and the fact that, for any closure operator f with $f(0) = 0$, an element x is

$$f\text{-}S_{\frac{1}{2}} \text{ iff } x \wedge y = x \wedge d(x \vee y) = 0 \text{ implies } y \wedge d(x) = 0$$

and

$$f\text{-}S_1 \text{ iff } x \wedge y = x \wedge d(y) = 0 \text{ implies } y \wedge d(x) = 0.$$

(These equivalences follow easily from the definitions.) Q.E.D.

(44) can now be proved: (44)(a) follows immediately from part (c) of the lemma; (44)(b) follows from part (d) of the lemma by virtue of the fact that, for a closure operator f which is analytic at 0 and has $f(0) = 0$, the f-discrete elements are $\bigvee$-dense. Q.E.D.

The remaining results of this section are essentially all familiar from the atomic case and are only given here since they will be used in later sections.

(45) Let f be a topological closure operator on a CBA A and let x and y be elements of A. Then

(a) x open and y pithy implies $x \wedge y$ pithy, and

(b) x discrete and y pithy implies $(x \wedge y)'$ dense.

Proof. For (a) we have $x \wedge y \leq x \wedge d(y) =$

$x \wedge [d(x \wedge y) \vee d(x' \wedge y)] \leq x \wedge [d(x \wedge y) \dot{\vee} x'] \leq d(x \wedge y)$ (where $d(x' \wedge y) \leq d(x') \leq x'$ since x' is closed). The inequalities $x \wedge y \leq x \wedge d(y) = x \wedge [d(x \wedge y) \vee d(x' \wedge y)] \leq d(x' \wedge y) \leq d((x \wedge y)')$ (where $x \wedge d(x \wedge y) = 0$ since x is discrete) give (b). Q.E.D.

A closure operator f on a CBA A will be said to be (i) pithy iff 1 is f-pithy and (ii) perfectly disconnected (Semadeni [36], pp. 33/34) iff, for all x,y in A, $x \wedge y = 0$ implies $d(x) \wedge d(y) = 0$, where d is the derivative of f.

(46) Let f be a topological closure operator with derivative d on a CBA A. Then

(a) the following conditions are equivalent

(i) $d^* \leq d$,

(ii) f is pithy

(iii) every open element is pithy;

(b) the following conditions are also equivalent

(i) $d \leq d^*$,

(ii) f is perfectly disconnected,

(iii) $d(x \wedge y) = d(x) \wedge d(y)$ for all x,y in A; and

(c) if f is perfectly disconnected then every pithy element is open.

Proof. (a) We prove that (i) and (iii) are equivalent whether f is topological or not. First note that, for any element x of A, the element $f^*(x) = x \wedge d^*(x)$ (= the interior of x) is always open and that x is open iff $x = f^*(x)$, that is, iff $x \leq d^*(x)$. It is clear from this latter remark that (i) implies (iii). On the other hand, if (iii) holds and x is any element of A then the elements $x \wedge d^*(x)$ and $x' \wedge d^*(x')$, being open, are pithy - from which it follows that $x \wedge d^*(x) \leq d(x)$ and $x' \wedge d(x)' \leq d(x')$. From the second inequality we obtain $d(x')' \leq x \vee d(x)$ and thence $x' \wedge d^*(x) \leq x' \wedge (x \vee d(x)) \leq d(x)$ which combined with the first inequality yields $d^*(x) \leq d(x)$. Thus (iii) implies (i). Now suppose that f is topological. Then 1 is open since $f(0) = 0$ and hence (iii) implies (ii). If (ii) holds then for each element x of A we have $d(x) \vee d(x') = d(1) = 1$ and hence $d^*(x) \leq d(x)$ so that (ii) implies (i). This gives (a).

(b) Here it is the case that (i) and (ii) are equivalent whether f is topological or not, since (as is immediate from the definition) f is perfectly disconnected iff $d(x) \wedge d(x') = 0$ for all x in A. Suppose that f is topological and perfectly disconnected. Then for all x,y in A we have $d(x) \wedge d(y) =$ $[d(x \wedge y) \vee d(x \wedge y')] \wedge [d(x \wedge y) \vee d(x' \wedge y)] = d(x \wedge y)$ since $x \wedge y$, $x \wedge y'$, and $x' \wedge y$ are pairwise disjoint. Thus (ii) and (iii) are equivalent (it is obvious that (iii) implies (ii) whenever $f(0) = d(0) = 0$) and we have (b).

(c) If x is pithy then $x \leq d(x) \leq d^*(x)$ by (b)(i) and hence x is open. Q.E.D.

The converse of (c) in (46) is not true - see the discussion following (64) below.

A closure operator f on a CBA A will be said to be <u>extremally disconnected</u> iff, for all x in A, x open implies $f(x)$ open. If we say that an element x of A is <u>f-regular closed</u> iff it is of the form $f(y)$ for some open element y (equivalently, iff $ff^*(x) = x$) and that x is <u>f-regular open</u> iff x' is f-regular closed (equivalently, iff x is of the form $f(z)$ for some closed element z, and iff $f^*f(x) = x$) then f will be extremally disconnected iff any two of the classes of clopen elements, regular closed elements, and regular open elements are equal. Since for f topological the class of regular closed elements is known to be a CBA under the order inherited from A, it follows that if f is extremally disconnected and topological then the class of clopen elements is a CBA under this order (though this CBA will not in general be a complete sub-algebra of A - c.f. (38)). A perfectly disconnected closure operator f is always extremally disconnected: if x is open then $d(f(x)') \leq d(x') \leq x' \wedge d(x)' = f(x)'$ ($d(x') \leq x'$ since x' is closed and $d(x') \leq d(x)'$ since f is perfectly disconnected), so that $f(x)'$ is closed and $f(x)$ is open. It is well-known that the converse implication here fails to hold even in the topological case.

(47) Let f be a topological closure operator with derivative d on a CBA A. Then the following hold:

(a) if f is T_1 then $d^2 \leq d$; and

(b) if $d^2 \leq d$ and f is analytic and S_1 then f is T_1

Proof. We first prove the following result: For any topological closure operator f, the set of elements x for which $d(x)$ is closed (that is, for which $d^2(x) \leq d(x)$) is $\bigvee$-closed. Let S denote this set; we show that $s \wedge (d^2(x)\backslash d(x)) = 0$ for each x in A and s in S such that $s \leq x$. This gives the result since always $d^2(x)\backslash d(x) \leq (x \vee d(x))\backslash d(x) \leq x$ so that if x is in $J(S)$ then it will follow that $d^2(x)\backslash d(x) = 0$, and x is in S. Suppose therefore that $s \leq x$ where x is in A and s is in S. Put $a = f(x\backslash s)$ - then a is closed and $x \leq a \vee s$. Hence $d(x) \leq d(x \wedge a) \vee d(x \wedge s) \leq a \vee d(s)$ so that $d(x)\backslash a \leq d(s)$ and therefore $d^2(x) \leq d(d(x) \wedge a) \vee d(d(x)\backslash a) \leq a \vee d^2(s) \leq a \vee d(s) =$ $(x\backslash s) \vee d(x\backslash s) \vee d(s) = (x\backslash s) \vee d(x)$. This implies that $s \wedge (d^2(x)\backslash d(x)) \leq s \wedge ((x\backslash s) \vee d(x)) \wedge d(x)'$ and since this latter term is 0, we have $s \wedge (d^2(x)\backslash d(x)) = 0$ as claimed.

(a) follows immediately from this result since clearly $f(A) \subseteq S$ and by definition, f is T_1 iff $f(A)$ is $\bigvee$-dense in A.

To obtain (b), we show that if f is any closure operator with $f(0) = 0$ which is analytic at 0, S_1, and such that the set S defined above is $\bigvee$-dense in A then f is T_1. Since

f is S_1 and analytic at 0 the elements which are discrete and f-S_1 are $\bigvee$-dense in A. Hence if we show that each such element is the join of closed elements we will have shown that f is T_1. So let x be discrete and f-S_1 and let s be an element of S such that $s \leq x$. Then $d(s)$ is closed (since s is in S) and disjoint from x (since x is discrete) and therefore $d(s)$ is disjoint from $f(x)$ (since x is f-S_1) - but then $d(s) = 0$, and s is closed. Since x is the join of such elements s, x is the join of closed elements as required. Q.E.D.

Part (a) of this result is the extension to the general CBA case of Exercise D(c) in Chapter 1 of Kelley's General Topology [27]. The fact that the set S as defined here is $\bigvee$-closed is possibly new even for the atomic case. It has as an immediate consequence the result of Yang referred to in the same exercise of Kelley, namely that if d is a topological derivative on an atomic CBA then $d(x)$ is closed for each element x of A iff $d(p)$ is closed for each atom p of A. In Appendix 2 on separation axioms we shall make some further remarks on the various conditions considered in (47), along with some related conditions.

(48) Let f be a topological closure operator on a CBA A with derivative d such that $d^2 \leq d$. If x and y are elements of A such that x is dense in y and y is pithy then x is pithy.

<u>Proof</u>. From $y \leq x \vee d(x)$ we obtain $d(y) \leq d(x) \vee d^2(x) = d(x)$ and hence $x \leq y \leq d(y) \leq d(x)$. Q.E.D.

We have paid a certain amount of attention to the identities satisfied by the derivatives of closure operators of various types; in the topological case condition (I) gives us no more than we already have from idempotency. Precisely: for any $\vee$-operator d on a CBA A, d satisfies condition (I) iff $\upharpoonright \vee d$ is idempotent. (To see this let d be any $\vee$-operator. Then in the first place it is clear that $\upharpoonright \vee d$ is idempotent iff $d^2 \leq \upharpoonright \vee d$. Suppose that $d(x) \leq y$ - then $d(x \vee d(x \wedge y)) = d(x) \vee d^2(x \wedge y) \leq d(x) \vee (x \wedge y) \vee d(x \wedge y) \leq y$ as required for conditon (I). The converse implication from condition (I) to the idempotency of $\upharpoonright \vee d$ has previously been shown to hold for all d in $\mathfrak{U}(A)$.)

2. T_1 Matroidal Topological Closure Operators

Let f be a matroidal topological closure operator on a CBA A. Then f is S_1 by (32) and (44)(b), and the analysis of Appendix 2 enables us to resolve f into an analytic quantifier acting on A and a T_1 matroidal topological closure operator acting on the CBA $J(f(A))$ (see (29)(v)). We may therefore divide the study of matroidal topological closure operators into that of analytic quantifiers and of T_1 matroidal topological closure operators. The former will be considered in the next section and the latter, briefly, here.

(49) Let f be an analytic topological closure operator on a CBA A. Then the following conditions are equivalent.

(i) f is T_1 and matroidal,

(ii) every discrete element is closed,

(iii) the discrete elements form an ideal of A.

Proof. Let d denote the derivative of f. To show that (i) and (ii) are equivalent, suppose first that (i) holds and let x be discrete. Then from $d(x) \leq d(x)$ and the fact that f is super-$E_{\frac{1}{2}}$ we obtain $d(x) = d(x \wedge d(x \vee d(x))) = d(x \wedge (d(x) \vee d^2(x))) =$ $= d(x \wedge d(x)) = 0$ (where $d(x) \vee d^2(x) = d(x)$ by (47)(a) and $d(x \wedge d(x)) = 0$ since x is discrete) - thus x is closed. Now suppose that (ii) is given to hold. Then, since f is analytic, the discrete elements and therefore also the closed elements are $\bigvee$-dense so that f is T_1. To show that f is matroidal we have to show that $\upharpoonright \wedge d$ is idempotent. Let x be any element of A - then $x \backslash d(x)$ is discrete and hence closed, so that $d(x \backslash d(x)) = 0$. This implies that $d(x) = d((x \wedge d(x)) \vee (x \backslash d(x))) = d(x \wedge d(x))$ and a fortiori that $x \wedge d(x) \leq d(x \wedge d(x))$ as required for the idempotency of $\upharpoonright \wedge d$.

The equivalence of (ii) and (iii) may be obtained as follows. If (ii) holds then the set of discrete elements $= d^{-1}(0)$ and this is an ideal since d is a $\vee$-operator. On the other hand, if (ii) fails and x is discrete but not closed, let s be a non-zero discrete subelement of $d(x)$ (such exists since $d(x) \neq 0$ and f is analytic). Then x and s are discrete yet $x \vee s$ is not discrete, and (iii) fails also. Q.E.D.

(50) Let f be a T_1 matroidal topological closure operator with derivative d on a CBA A. Then:

(a) $d^2 = d$;

(b) every scattered element is discrete; and

(c) x open implies $x \wedge d(x)$ open, for all x in A.

<u>Proof</u>. (a) In the course of proving (49) we showed that $d(x) = d(x \wedge d(x))$ for all x in A - i.e. that $d = d(\upharpoonright \wedge d)$; the fact that $d^2 = d$ then follows from the inequality $d^2 \leq d = d(\upharpoonright \wedge d) \leq d^2$.

(b) An element x is scattered iff x contains no non-zero pithy element. Now $x \wedge d(x)$ is pithy so that if x is scattered we must have $x \wedge d(x) = 0$, and x is discrete.

(c) Let x be an element of A. Then $x \backslash d(x)$ is discrete and therefore closed. Thus if x is open, so also is $x \wedge d(x) = x \wedge (x \backslash d(x))'$. Q.E.D.

We see from (50)(a) and earlier results that, for a matroidal topological closure operator with derivative d, the three conditions:

$$f \text{ is } T_1;\quad d^2 \leq d;\text{ and }\quad d^2 = d,$$

are equivalent. In connection with (b) in (50) we should remark that discrete elements are always scattered - so that, relative to any T_1 matroidal topological closure operator, the properties: discrete; discrete and closed; and scattered, are equivalent.

A few further necessary and sufficient conditions for the matroidality of a T_1 topological closure operator, together with some further simple consequences of such matroidality, could be given here but none of these is particularly attractive. In Appendix 3 (see (83)) we obtain two results on topological matroids for the T_2 atomic case which give some idea of the pathologies such topological spaces must possess. If matroidal topological spaces (atomicity understood) turned out to have some application to, say, analysis, this would be very helpful in providing a direction to their further investigation. However, the existence of even T_3

matroidal topological spaces (non-discrete) is an open question at present. In view of the apparent difficulty in producing a $T_{3\frac{1}{2}}$ pithy topological space with just <u>one</u> point which is not a limit point of any discrete set (see Fine and Gillman [12], 2.6) it would seem to be not at all easy to construct a non-discrete $T_{3\frac{1}{2}}$ topological space which is matroidal and in which by (49) <u>no</u> point is a limit point of a discrete set.

3. Supertopological Closure Operators and Quantifiers

A <u>$\bigvee$-operator</u> on a CBA A is an operator f on A such that $f(\bigvee X) = \bigvee f(X)$ for all subsets X of A.

(51) (a) The join of $\bigvee$-operators is again a $\bigvee$-operator

(b) If f is a $\bigvee$-operator and g is any operator then $f\backslash g$ is a $\bigvee$-operator.

(c) Reductions and cartesian products of $\bigvee$-operators are again $\bigvee$-operators. If f is a $\bigvee$-operator then so is the coreduction f_s of f, provided that $f(s) \leq s$.

<u>Corollary</u>. An operator $f \geq \upharpoonleft$ is a $\bigvee$-operator iff its derivative is a $\bigvee$-operator.

The proofs of these results are similar to those of (40) and its corollary.

(52) Let f be a $\bigvee$-operator on a CBA A and let S be a $\bigvee$-dense subset of A. Then

(a) $f(x) = \bigvee\{f(s);\ s \text{ in } S,\ s \leq x\}$ for all x in A; and

(b) $(f\backslash g)(x) = \bigvee\{f(s)\backslash g(s);\ s \text{ in } S,\ s \leq x\}$ for all x in A and operators g on A.

<u>Proof.</u> (a) is an immediate consequence of the definitions. For (b), Hammer's formula gives $(f\backslash g)(x) = \bigvee\{f(u)\backslash g(u);\ u \leq x\} \geq \bigvee\{f(s)\backslash g(s); s \text{ in } S,\ s \leq x\}$ and the reverse inequality follows from the fact that, for $u \leq x$,

$$\begin{aligned} f(u)\backslash g(u) &= \bigvee\{f(s);\ s \text{ in } S,\ s \leq u\} \backslash g(u) \\ &= \bigvee\{f(s)\backslash g(u);\ s \text{ in } S,\ s \leq u\} \\ &\leq \bigvee\{f(s)\backslash g(s);\ s \text{ in } S,\ s \leq u\} \\ &\leq \bigvee\{f(s)\backslash g(s);\ s \text{ in } S,\ s \leq x\}. \quad \text{Q.E.D.} \end{aligned}$$

(It is not difficult to show that the truth of (a) in (52) for all $\bigvee$-dense subsets S of A is also a sufficient condition for f to be a $\bigvee$-operator, and similarly for (b) - indeed if $f \geq \upharpoonright$ it is enough to consider only $g = \upharpoonright$ in (b).)

A closure operator which is also a $\bigvee$-operator will be called a <u>supertopological closure operator</u> and the derivative of such an operator will be called a <u>supertopological derivative</u>.

(53) A closure operator f on a CBA A is supertopological iff f(A) is a $\bigvee$-closed subset of A.

The proof of this is similar to that of (41).

(54) If f is a supertopological closure operator on a CBA A then the join of any directed set of discrete elements of A is again discrete.

Proof. Let D be a directed set of discrete elements and put $\bigvee D = s$. Then from the fact that the derivative d of f is a $\bigvee$-operator (by the corollary to (51)), we obtain $s \wedge d(s) = \bigvee\{x \wedge d(y);\ x,y \text{ in } D\}$. Now for each x,y in D there exists z in D such that $x,y \leq z$ – hence $x \wedge d(y) \leq z \wedge d(z) = 0$. It follows that $s \wedge d(s) = 0$. Q.E.D.

We now consider quantifiers.

(55) Let f be a closure operator on a CBA A such that $f(0) = 0$. Then the following conditions are equivalent

(i) every closed element is open (equivalently, every open element is closed),

(ii) f(A) is a subalgebra (necessarily complete) of A,

(iii) f is topological and f(A) is complemented,

(iv) every closed element is an f-separator,

(v) f is topological and $f^*f = f$,

(vi) f is topological and $f^*f \geq \Gamma$,

(vii) f is super-E_1,

(viii) f is super-S_1,

(ix) $x \wedge a = 0$ implies $f(x) \wedge a = 0$ for all x and all closed a,

(x) $f(x \wedge a) = f(x) \wedge a$ for all x and all closed a,

(xi) f is supertopological and S_1.

(Most of this result is well-known: see Banaschewski [2], Bergmann [3], Davis [9], Halmos [15], Rubin [34], and Wright [46]. For the sake of completeness we prove the equivalence of all the conditions listed.)

Proof. The equivalence of the two conditions mentioned in (i) is obvious, as is the fact that (ii) implies (i). Since $f(A)$ is a $\bigwedge$-closed subset of A for any closure operator f, it is also obvious that (i) implies (ii) and that if $f(A)$ is a subalgebra of A then it is a complete subalgebra of A.

It follows from (41) that (ii) implies (iii). The fact that (iii) implies (i) may be seen as follows, where we are supposing that (iii) is given to hold. Since f is topological, $f(A)$ is a sublattice of A containing 0 and 1. Therefore, for each closed element a, the complement of a in $f(A)$ coincides with its complement in A - hence a' is also closed as is required for (i).

(i) implies (iv) by virtue of (43) and the already proved fact that if (i) holds then f is topological; and (iv) implies (i) by virtue of (20).

To see that (i) implies (v), let x be any element of A - then $f(x)$, being closed, is also open and hence $f^*f(x) = f(x)$ as desired. Clearly (v) implies (vi), and (vi) implies (i) since if a is closed then (vi) gives $a \leq f^*f(a) = f^*(a)$, and this implies that a is open.

Of the next four conditions, (vii) and (viii) are equivalent by (44)(a) and (viii), (ix), and (x) are equivalent by the lemma preceding (29) (just as for the first three conditions in (29)); also (ix) may be seen to be equivalent to (i) as follows. Suppose that (i) holds and let $x \wedge a = 0$ where a is closed - then $x \leq a'$ and hence $f(x) \leq a'$ since a' is closed also - thus $f(x) \wedge a = 0$. Suppose that (ix) holds and let a be closed - then from $a \wedge a' = 0$ we obtain $f(a') \wedge a = 0$ and this implies that a is open.

To conclude the proof of (55), we remark that (ii) and (xi) are equivalent on account of the fact that if f is any closure operator on a CBA A then by (53):

f is supertopological iff $f(A)$ is a complete sublattice of A; and by (29), given that $f(0) = 0$,

f is S_1 iff $J(f(A))$ is a complete subalgebra of A. Q.E.D.

A closure operator satisfying the conditions of (55), including $f(0) = 0$, is called a quantifier.

(56) Let f be a closure operator on a CBA A. Then f is B-matroidal and topological iff it is an analytic quantifier.

Proof. Suppose first that f is B-matroidal and topological. Then f is certainly analytic. To show that f is a quantifier, let a be closed and let x be a base of a'. Now f is super-$S_{\frac{1}{2}}$ by (32) and therefore x is f-$S_{\frac{1}{2}}$; but then x is f-S_1 by part (d) of the lemma to (44). Hence from $x \wedge a = 0$ we conclude $f(x) \wedge a = 0$, that is, $f(a') \wedge a = 0$, whence a is open. Now suppose that f is an analytic quantifier. Then f is surely topological. To show that f is B-matroidal, let x be a discrete subelement of an element s of A. Then by (54) and Zorn's Lemma, x is contained in a maximal discrete subelement y of s and y must be a base of s by (22). Q.E.D.

It is perhaps worth mentioning here one or two alternative ways in which one may view the situation of a quantifier f on a CBA A together with an f-base s. Given such a situation, let w be the operator on A defined by $w(x) = f(x \wedge s)$ for all x

in A. We then claim that w is a complete idempotent endomorphism of A with range $w(A) = f(A)$ (so that, in the terminology of Halmos [15], w is a complete "constant" in the monadic algebra (A,f) - or, as we prefer, that w is a complete <u>f-witness</u>, to borrow the other term used by Halmos in the same context) and also that the restriction u of f (or, equally well, of w) to A^s is a complete morphism of A^s to A with range $f(A)$ such that $A^s \xrightarrow{u} A \xrightarrow{\hat{s}} A^s = 1_{A^s}$ and $A \xrightarrow{\hat{s}} A^s \xrightarrow{u} A = w$, where $\hat{s}: x \longmapsto x \wedge s$.

In the first place, w is clearly a $\bigvee$-operator, so that to show it is a complete endomorphism it is enough to verify that it preserves complements. Let x be in A - then $x \wedge s \wedge f(x' \wedge s) = 0$ by the discreteness of s and hence $f(x \wedge s) \wedge f(x' \wedge s) = 0$, that is, $w(x) \wedge w(x') = 0$. Therefore since $w(x) \vee w(x') = w(1) = f(s) = 1$ we have $w(x') = w(x)'$ as required. Now let a be in $f(A)$ - then $w(a) = f(a \wedge s) =$ $= a \wedge f(s) = a$ from which it follows, in view of the obvious inclusion $w(A) \subseteq f(A)$, that $w(A) = f(A)$ and that w is idempotent. The statements concerning u are now easily seen to be true (the fact that $u(x) \wedge s = f(x) \wedge s = x$ for all x in A^s follows directly from the discreteness of s).

Now let us suppose we are given an arbitrary complete idempotent endomorphism w of a CBA A. Then w is of course a complete f-witness, where f is the quantifier on A such that

$f(A) = w(A)$. We want to show that there exists an f-base s such that $w(x) = f(x \wedge s)$ for all x in A. If s is any element which satisfies this equation for all x in A then s must be the smallest element x with $w(x) = 1$: for clearly $w(s) = 1$ and if $w(x) = 1$ then $f(x' \wedge s) = w(x') = 0$ so that $x' \wedge s = 0$ and $x \geq s$. This shows also that such an element s is minimal subject to $f(s) = 1$ and is therefore an f-base. So let us take $s = \bigwedge w^{-1}(1)$ = the smallest element x with $w(x) = 1$. Then for each x in A we have $w(x) = w(x \wedge s) \leq f(x \wedge s)$ (if w is any f-witness then $f^* \leq w \leq f$, as follows on taking w's in the inequality $f^*(x) \leq x \leq f(x)$). Also $w(w(x) \vee x') = w^2(x) \vee w(x') = w(x) \vee w(x') = 1$ so that $w(x) \vee x' \geq s$, that is, $w(x) \geq x \wedge s$ and hence $w(x) \geq f(x \wedge s)$. Thus $w(x) = f(x \wedge s)$ for all x in A and, as already remarked, s must be an f-base.

Finally, suppose we are given an element s of a CBA A together with a complete morphism u of A^s to A such that $A^s \xrightarrow{u} A \xrightarrow{\hat{s}} A^s = \upharpoonright_A s$ ($\hat{s}$ as above). Let f be the quantifier on A with $f(A) = u(A^s)$. Then u is the restriction of f to A^s - equivalently: for each x in A^s, $u(x)$ is the smallest element of $u(A^s)$ which contains x, this being the case since if $x \leq u(y)$ then $x \leq u(y) \wedge s = y$ and hence $u(x) \leq u(y)$. Also, s is an f-base since $f(x) \wedge s = u(x) \wedge s = x$ for all x in A^s (so that s is discrete) and $f(s) = u(s) = 1$ (so that s is dense).

It is obvious that the complete f-witness constructed from s as above coincides with the complete idempotent endomorphism of A given by $A \xrightarrow{\hat{s}} A^s \xrightarrow{u} A$.

Let (A,f) be a monadic algebra and let x be an element of A. Then an f-witness w is said to be a witness <u>for</u> x iff $f(x) = w(x)$. If A and w are complete and s is the f-base corresponding to w as above then this says that $f(x) = f(x \wedge s)$, equivalently, that s contains a base (necessarily equal to $x \wedge s$) of x. Now the monadic algebra (A,f) is said to be <u>rich</u> iff there is an f-witness for each element of A. Analogously we may say that a complete monadic algebra (A,f) is <u>completely rich</u> iff there is a complete f-witness for each element of A. By what was just stated, this is the same as saying that for each element x of A there is a base of 1 which contains a base of x. It follows that (A,f) is completely rich iff f is analytic (if f is completely rich then a fortiori every element of A has a base so that the discrete elements are $\bigvee$-dense, and f is analytic; conversely if f is analytic then by (56) it is B-matroidal and hence every element has a base which extends to a base of 1). This fact leads to a representation (closely analogous to the one obtained by Halmos [15], [16]) for analytic quantifiers in terms of analytic quantifiers of a particular type, which we now describe.

Let C be any CBA and I any index set and take ∇ to be the quantifier on the CBA C^I with $\nabla(C^I)$ consisting of the constant functions from I to C; we call ∇ the <u>diagonal</u> quantifier on C^I. For x in C^I we have: $(\nabla(x))(i) = \bigvee_j x(j)$ for all i; $((\nabla\backslash\upharpoonright)(x))(i) = \bigvee_{i\neq j} x(j)$ for all i (this follows from Hammer's formula); x is ∇-discrete iff $x(i) \wedge x(j) = 0$ for all i,j such that $i \neq j$; x is ∇-dense iff $\bigvee_i x(i) = 1$; and x is a ∇-base iff the $x(i)$'s constitute a partition of 1 with possibly zero parts. Also ∇ is analytic since if x is any non-zero element of C^I, say $x(k) \neq 0$, define y in C^I to have the value $x(k)$ at k and to be zero elsewhere - then y is a non-zero discrete subelement of x.

Now let f be any analytic quantifier on a CBA A and let W denote the set of complete f-witnesses. Define the evaluation mapping $e:A \longrightarrow f(A)^W$ by $e(x)(w) = w(x)$ for all x in A and w in W. Then:

(a) e is a complete morphism of CBA's (this is clear);

(b) e is 1-1 (if $e(x) = 0$ then, taking w in W such that $w(x) = f(x)$, we obtain $f(x) = 0$ and hence $x = 0$); and

(c) e is a morphism of monadic algebras from the algebra (A,f) to the algebra $(f(A)^W, \nabla)$ (where ∇ is the diagonal quantifier on $f(A)^W$)

[that is, $e(f(x)) = \nabla(e(x))$ for all x in A (for each w in W we have $(e(f(x))(w) = w(f(x)) = f(x)$ and $(\nabla(e(x)))(w) = \bigvee_w (e(x))(w) = \bigvee_w w(x) = f(x)$, the last equality holding since each $w(x) \leq f(x)$ and some $w(x) = f(x))$].

Thus (A,f) is represented as a complete subalgebra of the complete monadic algebra $(f(A)^W, \nabla)$. (We note that the analyticity of f is necessary in b) and in c) since they each imply, for $x \neq 0$ in A, that $w(x) \neq 0$ for some w in W - equivalently, that $f(x \wedge s) \neq 0$ for some f-base s, which in turn implies that x contains some discrete element $\neq 0$).

4. Hewitt-Katětov Closure Operators

An HK (= Hewitt-Katětov) closure operator on a CBA A is a closure operator on A whose derivative is an endomorphism of A (where by an endomorphism is meant a Boolean endomorphism, not necessarily complete). An HK derivative is the derivative of an HK closure operator. It is clear that HK closure operators and HK derivatives are topological. The reason for the title "Hewitt-Katětov" is that, for the 'maximal' spaces of Hewitt [19] and the 'maximal pithy' spaces of Katětov [26], the corresponding closure operators are HK. Since $d^* = d$ for any endomorphism d of a CBA, the following result applies in particular to any HK closure operator:

(57) Let f be a closure operator on a CBA A and suppose that the derivative d of f satisfies $d^* = d$. Then f is matroidal and extremally disconnected. Also $d(0) = 0$, $d(1) = 1$, and, for each x in A,

x is pithy iff x' is closed (i.e. iff x is open)

x is discrete iff x' is dense,

x is perfect iff x' is clopen (i.e. iff x is clopen), and

x is basic iff x' is basic.

<u>Proof</u>. It is clear that f is matroidal and that the four equivalences concerning an element x of A hold. From the equivalence of openness and pithyness, together with the fact that for any closure operator the closure of a pithy element is pithy, it follows that f is extremally disconnected. From $d(0) = d^*(0) = d(1)'$ and $d(0) \leq d(1)$ we see that $d(0) = 0$ and $d(1) = 1$. Q.E.D.

It does not follow from the hypotheses of (57) that f is perfectly disconnected - a counterexample is provided by geometry #(1,8,28,38c,1) in [5].

(58) Let f be a topological closure operator with derivative d on a CBA A. Then the following conditions are equivalent

(i) f is an HK closure operator,

(ii) $d^* = d$,

(iii) f is pithy and perfectly disconnected,

(iv) d is a $\wedge$-operator.

(The definition of a $\wedge$-operator is dual to that for a $\vee$-operator.)

Proof. It is immediate that (i) implies (ii) and that (ii) implies (iv); the equivalence of (ii) and (iii) is a consequence of (46); and (iv) implies (i) since an operator is an endomorphism iff it is simultaneously a $\vee$-operator and a $\wedge$-operator. Q.E.D.

(59) (a) Let f be an HK closure operator on a CBA A and let s be an element of A. Then f^s is an HK closure operator iff s is open and f_s is an HK closure operator iff s is closed.

(b) Cartesian products of HK closure operators are again such.

Proof. f^s, which is certainly a topological closure operator (see the remarks following (4)), will be an HK closure operator iff its derivative d^s (where d is the derivative of f) is a $\wedge$-operator. Now d^s preserves all non-empty finite meets since d does so. Hence the condition for f^s to be an HK closure operator is that $d^s(s) = s$; equivalently, that s is open. A similar argument gives the result for f_s. (b) is trivial. Q.E.D.

(60) Let f be an HK closure operator with derivative d on a CBA A. Then $d^3 = d$.

Proof. Since d is a $\vee$-operator, the identity $d^2(\upharpoonleft \vee d) = d(\upharpoonleft \vee d)$

(see the discussion preceding (33)) reduces to $d^3 \vee d = d \vee d^2$; taking $*$'s and using $d* = d$ gives $d^3 \wedge d = d \wedge d^2$. These two equations together imply that $d^3 = d$, $\mathcal{O}(A)$ being a distributive lattice. Q.E.D.

We now show how the resolution of an S_1 closure operator into a quantifier and a T_1 closure operator (to be discussed in general in Appendix 2) may be given in a simple alternative form for an HK closure operator: every HK closure operator is isomorphic to an HK quantifier $\times$ a T_1 HK closure operator ((62)), where predictably, an <u>HK quantifier</u> is an HK closure operator which is also a quantifier. We first prove

(61) Let f be a closure operator with derivative d on a CBA A. Then the following conditions are equivalent

- (i) f is an HK quantifier,
- (ii) f is a supertopological HK closure operator,
- (iii) f is an HK closure operator and d is 1-1,
- (iv) f is a topological closure operator and $d^2 = 1$,
- (v) f is a topological closure operator with a complementary pair of bases,
- (vi) f is a quantifier such that, for each x in A x is pithy iff x is open,
- (vii) to within isomorphism, A is a square and $f(A)$ is its diagonal.

(Note that in (iii), the condition 'd is 1-1' is equivalent to the equation $d^{-1}(0) = 0$.)

<u>Proof.</u> <u>(i)$\Longleftrightarrow$(ii)</u>. In view of the fact that every HK closure operator, being matroidal and topological, is S_1, (see the beginning of the second section of this chapter), the equivalence of (i) and (ii) is an immediate consequence of (55).

<u>(i)$\Longleftrightarrow$(iv)</u>. Suppose that (i) holds. Writing the equation $f*f = f$ (see (55)(v)) in terms of d, we obtain $d \vee d^2 = d \vee \upharpoonleft$; taking *'s and using $d* = d$ gives $d \wedge d^2 = d \wedge \upharpoonleft$ also. Since $\mathcal{O}(A)$ is a distributive lattice, these two equations yield $d^2 = \upharpoonleft$, and (iv) holds. Suppose conversely that (iv) is given to hold. Then from $1 \geq d(1) \geq d^2(1)$ we obtain $d(1) = 1$, and f is pithy. For all x,y in A, $d(d(x) \wedge d(y)) \leq d^2(x) \wedge d^2(y) = x \wedge y$ and hence $x \wedge y = 0$ implies $d(x) \wedge d(y) = 0$ - that is, f is perfectly disconnected. Thus f is an HK closure operator by (59). $d^2 = \upharpoonleft$ gives $d \vee d^2 = d \vee \upharpoonleft$ which in terms of f is $f*f = f$, so that f is a quantifier by (55).

<u>(iii)$\Longleftrightarrow$(iv)</u>. By what has just been proved, (iv) implies (iii). Suppose that (iii) holds and let x be any element of A. Then $d(d^2(x) + x) = d^3(x) + d(x) = d(x) + d(x) = 0$, where $d^3(x) = d(x)$ by (60). Hence $d^2(x) + x = 0$, that is, $d^2(x) = x$.

<u>(i)$\Longleftrightarrow$(v)</u>. If (i) holds then f is an analytic quantifier and by (56) there exists an f-base, s say. By (57), s' is also

a base and hence (v) holds. Suppose conversely that (v) is given to hold and that s and s' are complementary bases. Then $d(s) = s'$ and $d(s') = s$ so that $d(1) = 1$ and f is pithy. To show that f is perfectly disconnected, it is sufficient to show that $d(x) \wedge d(y) = 0$ for all disjoint subelements x and y of s - for if x and y are arbitrary disjoint elements of A then $d(x) \wedge d(y) = (d(x \wedge s) \vee d(x \wedge s')) \wedge (d(y \wedge s) \vee d(y \wedge s'))$ and this will vanish since in its expansion: $d(x \wedge s) \wedge d(y \wedge s) = 0$ by what we are about to prove, and $d(x \wedge s') \wedge d(y \wedge s') = 0$ by a similar argument (with s' in place of s); also $d(x \wedge s) \wedge d(y \wedge s') \leq d(s) \wedge d(s') = s' \wedge s = 0$, and likewise for $d(x \wedge s') \wedge d(y \wedge s)$. So let $x \wedge y = 0$, $x \vee y \leq s$ and put $z = d(x) \wedge d(y) = f(x) \wedge f(y)$. Then z is a closed subelement of s' and hence $d(z) = 0$ since s' is discrete. Thus $s = d(s') = d(z) \vee d(s'\backslash z) = d(s'\backslash z) \leq f(s'\backslash z)$, from which it follows that $s'\backslash z$ is dense. By the discreteness of s', this implies that $z = 0$ as required. We may now apply (58) to deduce that f is an HK closure operator. To show that f is an HK quantifier it is enough to show that $d^{-1}(0) = 0$ since we have already proved that (iii) implies (i). Suppose that $d(z) = 0$ - then $s'\backslash z$ is dense as above and hence $z \wedge s' = 0$, s' being discrete; similarly $z \wedge s = 0$, and therefore $z = 0$.

(v)$\Leftrightarrow$(vi). It is immediate from (46) that (i) implies (vi); thus (v) implies (vi). Suppose that (vi) holds. We show first that f is analytic - by (41) it is sufficient to show that each non-zero element x of A contains a non-zero discrete element. If every subelement of x is closed then $d(x) = 0$ (see Chapter 1, Section 2, c)) and hence x is itself discrete. If some subelement y of x is not closed then, since closedness, openness, and pithyness are the same under the present circumstances, y is not pithy either and $y \backslash d(y)$ is a non-zero discrete subelement of x. From the analyticity of f it follows by (56) that f is B-matroidal and therefore there exists an f-base, s say. Since $d(1) = 1$ (1 is closed and thus pithy) we have $s = s \wedge d(1) = s \wedge (d(s) \vee d(s')) = s \wedge d(s')$ so that $s \leq d(s')$. This implies that s' is dense and hence there exists an f-base $t \leq s'$. Now $s \vee t$, being the join of two disjoint dense elements, is pithy and thus closed. Therefore $s \vee t = 1$ and s and t are complementary bases: we have verified that (v) holds.

<u>(i)$\Longleftrightarrow$(vii)</u>. Suppose that (i) holds and let s and t be complementary bases (we have (v)). Then $d(s) = t$, $d(t) = s$, and, in view of the fact that $d^2 = \upharpoonright$ (we have (iv)), it is clear that A^s and A^t are isomorphic under d (suitably restricted). Writing $A^s \cong A^t \cong B$ say, we obtain $A \cong A^s \times A^t \cong B^2$ and it is easily seen that under this isomorphism $f(A)$ corresponds to the diagonal of B^2. The converse - namely that if B is a CBA and ∇ is the diagonal quantifier on B^2 (see the penultimate paragraph of the preceding section) then ∇ is an HK quantifier - is straightforward. Q.E.D.

(62) Let f be an HK closure operator on a CBA A. Write $t = t(f)$, $s = t'$, $g = f^s$, $B = A^s$, $h = f^t$, and $C = A^t$. Then g is an HK quantifier on B, h is a T_1 HK closure operator on C, and s and t are complementary separators - so that $f = g \times h$.

<u>Proof</u>. Since f is analytic, $j(f) = 1$ and therefore $t = t(f) \wedge j(f) = \bigvee d^{-1}(0)$ (see Chapter 1, Section 2, d)), where d is the derivative of f. Since (as previously remarked), f is S_1, (31) implies that s is pithy and hence s is open by (57) (also $t = s'$ is closed). Thus g is an HK closure operator by (59)(a). To show that g is an HK quantifier it is sufficient by (61) to show that $e^{-1}(0) = 0$, where $e = d^s$ is the derivative of g. So let $x \leq s$ be such that $e(x) = d(x) \wedge s = 0$. Then for every subelement y of x we have $y = f(y)\backslash t$, and this is in $J(f(A))$ since f is S_1 (use (29)(v)). It follows that $x \leq t$, and hence $x = 0$ as desired. The fact that g is an HK quantifier implies by (61) that there exist complementary bases x and y of s. We then have $f(s) \wedge t = (x \vee d(x)) \wedge t = d(x) \wedge t \leq d(x)$, and $f(s) \wedge t \leq d(y)$ similarly. However, $d(x) \wedge d(y) = 0$ since f is perfectly disconnected. It follows that $f(s) \wedge t = 0$ and therefore that s is closed. s is thus clopen and by (43) s is an f-separator. Moreover, the fact that t is open implies by (59)(a) that h is an HK closure operator

and it is obvious from the definition of t that h is T_1. Q.E.D.

(63) Let d be an operator on a CBA A. Then

(a) the following conditions are equivalent

(i) d is the derivative of an HK quantifier

(ii) d is an automorphic derivative of period 2

(iii) d is an automorphism of period 2 such that $\{x;\ d(x) \neq x\}$ is $\bigvee$-dense,

(iv) d is a complete HK derivative (complete as an endomorphism of A, that is); and

(b) the following conditions are also equivalent

(i) d is the derivative of a T_1 HK closure operator,

(ii) d is an idempotent endomorphic derivative,

(iii) d is an idempotent endomorphism such that $\{x;\ d(x) = 0\}$ is $\bigvee$-dense.

Proof. We first note the following fact:

A $\vee$-operator d on a CBA A is a coderivative iff $\{x;\ x \wedge d(x) = 0\}$ is $\bigvee$-dense.

The proof of this is similar to that of (42) (which follows immediately from it), except that one relies directly on the dual of condition (iv) in (5).

(a) The equivalence of (i) and (ii) follows from that of (i) and (iv) in (61) since $d^2 = \upharpoonright$ implies $d^2 \leq \upharpoonright \vee d$. The equivalence of (i) and (iv) follows from that of (i) and (ii) in (61) in view of the corollary to (51). It is clear that (ii) implies (iii) and for the converse we have that if s is non-zero and $x \le s$ is such that $d(x) \ne x$ then $y = x \backslash d(x)$ satisfies $0 \ne y \le s$, $y \wedge d(y) = 0$ - for if $x \le d(x)$ then $d(x) \le d^2(x) = x$ and hence $d(x) = x$, a contradiction. Thus (given (iii)) the above-stated fact applies to show that d is a coderivative so that, since $d^* = d$, d is a derivative and we have (ii).

(b) The equivalence of (i) and (ii) follows from the remark immediately subsequent to the proof of (50) since $d^2 = d$ implies $d^2 \le \restriction \vee d$; (i) implies (iii) for any T_1 analytic topological closure operator; and by the above-stated fact, (iii) implies that d is a coderivative, and hence (iii) implies (ii). Q.E.D.

We will use condition (iii) in part (b) of this result as one way of constructing examples of T_1 HK closure operators in Appendix 3. In the remainder of this chapter we give two results which are reformulations and extensions to the CBA setting of results due to Hewitt [19] and Katětov [26]. It does not appear that Hewitt's paper proves quite what is stated below and we do not refer to any specific theorem therein; as far as Katětov's paper is concerned however, our results are virtually the same as his Theorem 2 (modulo the equivalence of (i) and (iii) in (58)).

It would have been possible to similarly reformulate for CBA's various of their other results - for instance, the properties of Hewitt's MI-spaces (note especially his Theorem 33 from which it follows by our (49) that MI-spaces are matroidal); nevertheless it was felt that this would be a somewhat barren exercise and that an adequate impression of what to expect along such lines has been given.

(64) Let f be a T_1 topological closure operator on a CBA A. Then the following conditions are equivalent

(i) f is an HK closure operator,

(ii) for each x in A, x is pithy iff x is open.

<u>Proof</u>. (i) implies (ii) by (57). Suppose that (ii) holds. Then 1, being open, is pithy so that f is pithy and we will have obtained (i) if we show that f is perfectly disconnected. To this end we first remark that if $x \leq y \leq f(x)$ and x is open then so is y since the like implication for pithyness always holds; this shows in particular that f is extremally disconnected. If f is not perfectly disconnected then it is not difficult to see with the aid of Nöbeling's formula that there exist disjoint elements x,y and z of A with $x \vee y \vee z = 1$, $z \leq f(x) \wedge f(y)$, and $z \neq 0$. Put $u = f^*(f(x) \wedge f(y))$ - then u is clopen since f is extremally disconnected. Moreover $z \leq u$, as may be seen by

the following argument: Let $t = z \wedge f(f^*(x))$. Then $f^*(x) \leq f^*(x) \vee t \leq f(f^*(x))$ and hence $f^*(x) \vee t$ is open. But $f^*(x) \vee t$ is disjoint from y so that, since $t \leq z \leq f(y)$, we must have $t = 0$. Thus $z \leq (f(f^*(x)))' = f^*(f(x')) = f^*(f(y))$ (note that $f(x') = f(y \vee z) = f(y)$). Similarly $z \leq f^*(f(x))$ and thus $z \leq f^*(f(x)) \wedge f^*(f(y)) = f^*(f(x) \wedge f(y)) = u$ as claimed. Now from $u \leq f(x)$ and the fact that u' is closed it follows that $x \wedge u$ is dense in u (we have $u \leq f(x) = f(x \wedge u) \vee f(x \wedge u') \leq f(x \wedge u) \vee u'$, so that $u \leq f(x \wedge u)$). Since u is pithy (it is open) and f is T_1, (47)(a) and (48) now show that $x \wedge u$ is pithy also - hence $x \wedge u$ is open. In view of the fact that $x \wedge u \wedge y = 0$, this implies that $x \wedge u \wedge f(y) = 0$; therefore $x \wedge u = 0$ since $u \leq f(y)$. Because u is open we obtain $f(x) \wedge u = 0$ which, since $z \leq f(x) \wedge u$, gives $z = 0$, a contradiction. Q.E.D.

The requirement that f be T_1 in (64) cannot be dropped, not even in the atomic case (though in this case anyway, as follows from Hewitt's Theorem 9 [19] and our (65), it is sufficient to suppose that f is T_0). To obtain a counterexample, take any ultraspace (Fröhlich [13]) for which the corresponding ultrafilter is non-principal and replace each isolated point by a pair of points, each in the closure of the other. Then in the resulting topological space the pithy sets are the same as the open sets, yet the associated closure operator is not HK (by taking just one point out of each of the pairs, we obtain a set of points which, together with its complement, has the single non-isolated point of the original space as a limit point - thus our space is not perfectly disconnected). There exist simpler counterexamples - on the 8-element CBA for instance - but the one given is S_1 and shows therefore that we cannot replace T_1 by S_1 in (64). The question still remains as to whether a T_1 topological closure operator, such that every pithy element is open, is necessarily perfectly disconnected (c.f. (46)(c)). The answer is "Not necessarily", a counterexample being furnished by the Stone space X of any infinite superatomic Boolean algebra A (Day [10] defines superatomicity; his Theorem 1 shows that X is clairsemé (= scattered), so that the only pithy set is the empty set which is certainly open; and X is not perfectly disconnected, for if it were it would be extremely disconnected and A would be complete - but an infinite atomic CBA has atomless homomorphs and is thus not superatomic. (Actually, the Stone space of no infinite Boolean algebra is perfectly disconnected.)).

(65) Let f be any topological closure operator on a CBA A. Then the following conditions are equivalent

(i) for each x in A, x is pithy iff x is open,

(ii) amongst the topological closure operators on A, f is minimal pithy.

Proof. We first remark on the following immediate consequence of (45)(a): for any pithy topological closure operator each open element is pithy. In particular, if g is a pithy topological closure operator on A and $g \leq f$ then for each element x of A we have

(iii) x is g-open implies x is g-pithy

(iv) x is f-open implies x is g-open, and

(v) x is g-pithy implies x is f-pithy.

((iii) is because g is pithy: (iv) and (v) are because $g \leq f$). We deduce that if f satisfies (i) then for each x in A

(vi) x is g-open iff x is f-open.

Hence $g = f$ and we have shown that (i) implies (ii).

Now suppose that (ii) holds. Then from the pithyness of f it follows by our initial remark that every f-open element is f-pithy. We therefore have to show that every f-pithy element is f-open. Suppose that s is f-pithy but not f-open. We define a new topological closure operator g on A by taking as the g-open elements the elements of the form $x \vee (y \wedge s)$ where x and y are arbitrary f-open elements. We note that $g^s = f^s$ and that $g^{s'} = f^{s'}$, as may be readily verified by looking at the open elements corresponding to these closure operators. Now it is clear that $g < f$, the inequality being strict since s is g-open but not f-open. On account of (ii) it follows that g is not pithy.

Hence there exists a non-zero, g-open, and g-discrete element t (for example, we may take $e(1)'$ for t, where e is the derivative of g). The fact that t is g-discrete and that $g^s = f^s$ shows that $t \wedge s$ is f-discrete - and similarly $t \wedge s'$ is f-discrete. Write $t = x \vee (y \wedge s)$ where x and y are f-open. Then $t \wedge s = (x \vee y) \wedge s$ which is f-pithy by (46)(a). $t \wedge s$ is thus both f-discrete and f-pithy; hence $t \wedge s = 0$. It follows that t itself is both f-discrete ($t = t \wedge s'$) and f-pithy ($t = x \vee (y \wedge s) = x$ since $t \wedge s = 0$, and x is f-open - hence f-pithy by our initial remark); therefore $t = 0$, a contradiction. Q.E.D.

APPENDIX 1

SECTION CLOSURE OPERATORS

Let S be a section of a CBA A. Define f in $\mathcal{O}(A)$ by $f(x) = x$ for x in S, $f(x) = 1$ for x not in S. Then f is a closure operator on A and we call it the <u>section closure operator</u> associated with S. Notice that $f(A) = S \cup \{1\}$. In order to analyze f, we first introduce some notions which concern the 'geometry' of S.

(66) Let S be a section of a CBA A. Then for each x in A there is a smallest $y \leq x$ such that $[y,x) \subseteq S$.

<u>Proof.</u> Put $\Delta(x) = \{y;\ y \leq x,\ [y,x) \subseteq S\}$ and $y_0 = \bigwedge \Delta(x)$; we wish to show that y_0 is in $\Delta(x)$. Let u be in $[y_0,x)$ - then $u = y_0 \vee u = \bigwedge\{y \vee u;\ y \text{ in } \Delta(x)\}$ and, since $u < x$, there must exist some y in $\Delta(x)$ for which $y \vee u < x$; but then $y \vee u$ is in S and hence u is in S. It follows that y_0 is in $\Delta(x)$. Q.E.D.

We denote the smallest $y \leq x$ such that $[y,x) \subseteq S$ by $\delta(x)$. Note that $\delta(x) = 0$ for x in S and that, for x not in S, $\delta(x) = x$ except when x 'touches' S.

We set $\delta_1(x) = \bigwedge\{y;\ \delta(x) \le y \preceq x\}$ and $\delta_2(x) = \bigvee\{y;\ \delta(x) \preceq y \le x\}$ – then $x = \delta_1(x) \vee \delta_2(x)$ and $\delta(x) = \delta_1(x) \wedge \delta_2(x)$.

Dually, considering the cosection $A\backslash S$ in place of the section S, we define $\varepsilon(x)$ to be the largest $z \ge x$ such that $(x,z] \subseteq A\backslash S$ and put $\varepsilon_1(x) = \bigvee\{z;\ x \preceq z \le \varepsilon(x)\}$, $\varepsilon_2(x) = \bigwedge\{z;\ x \le z \preceq \varepsilon(x)\}$, so that $x = \varepsilon_1(x) \wedge \varepsilon_2(x)$ and $\varepsilon(x) = \varepsilon_1(x) \vee \varepsilon_2(x)$.

(67) Let S be a section of a CBA A and let f be the associated closure operator. Let d denote the derivative of f and e the coderivative of d. Then

(a) $x \wedge d(x) = \delta(x)$ for all x in A,

(b) $d(x) = \delta(x) \vee x' = e(x)$ for x not in S,

(c) $d(x) = 0$, $e(x) = \varepsilon_2(x) \wedge x'$, and $x \wedge e(x) = \varepsilon_2(x)$ for x in S.

<u>Proof</u>. Hammer's formula gives $x \wedge d(x) = \bigvee\{x\backslash u;\ u \le x,\ u \notin S\} = z$ say. Let u be in $[z,x)$ and suppose that u is not in S. Then $x\backslash u \le z$ and hence $x \le z \vee u = u$ which is not the case. It follows that z is in the set $\Delta(x)$ introduced in the proof of (66). Let y be in $\Delta(x)$ and suppose that $y \not\ge z$ – then there must exist u not in S such that $u \le x$ and $x\backslash u \not\le y$ – – i.e. such that $y \vee u < x$. Hence $y \vee u$ is in S, contrary to the fact that u is not in S. This shows that $z = \delta(x)$,

the smallest element of $\Delta(x)$, and we have (a). It follows that $d(x) = (x \wedge d(x)) \vee (x' \wedge f(x)) = \delta(x) \vee (x' \wedge f(x))$ which equals 0 for x in S and $\delta(x) \vee x'$ for x not in S. $e(x)$ remains to be determined (note that $x \wedge e(x) = \varepsilon_2(x)$ follows from $e(x) = \varepsilon_2(x) \wedge x'$ since $\varepsilon_2(x) \geq x$). We first remark on the following consequence of what we have already proved, namely that

$$\bigvee\{u';\ u \leq x,\ u \notin S\} = \delta(x) \vee x' \quad \text{for x not in S}$$

(both sides equal $d(x)$, the left on account of Hammer's formula and the right by what has already been proved) and hence dually that

$$\bigwedge\{v';\ v \geq x,\ v \in S\} = \varepsilon(x) \wedge x' \quad \text{for x in S.}$$

By the dual of Hammer's formula, $e(x) = \bigwedge\{d(v)/v;\ v \geq x\}$. Take x not in S - then, since $\delta(v) \geq v'$ for v not in S, we have $e(x) = \bigwedge\{d(v);\ v \geq x\} = d(x)$. Take x in S. Then

$$\begin{aligned} e(x) &= \bigwedge\{d(v)/v;\ v \geq x,\ v \in S\} \wedge \bigwedge\{d(v)/v;\ v > x,\ v \notin S\} \\ &= \bigwedge\{v';\ v \geq x,\ v \in S\} \wedge \bigwedge\{d(v);\ v > x,\ v \notin S\} \\ &= \varepsilon(x) \wedge x' \wedge \bigwedge\{d(v);\ v > x,\ v \notin S\} \ \ldots\ldots\ (\alpha). \end{aligned}$$

We claim that

$$e(x) = \varepsilon(x) \wedge x' \wedge \bigwedge\{d(y);\ y \succ x \vee \delta(y),\ y \notin S\} \ \ldots\ldots\ (\beta).$$

By (α), we certainly have $\geq$ in (β). The reverse inequality will be obtained if we show that, for every v not in S such that $v > x$, we either have

$$\text{(i)} \quad d(v) \geq \varepsilon(x) \wedge x',$$

or

(ii) $d(v) \geq d(y)$ for some y not in S such that $y \succ x \vee \delta(y)$.

Let $v > x$, v not in S - then $v \geq x \vee \delta(v)$ ($v \geq \delta(v)$ always holds). If $v = x \vee \delta(v)$ then $d(v) = \delta(v) \vee v' = \delta(v) \vee (x' \wedge \delta(v)') = \delta(v) \vee x' \geq x' \geq \varepsilon(x) \wedge x'$ which is possibility (i). If $v \succ x \vee \delta(v)$ then possibility (ii) holds with $y = v$. Therefore we may suppose that $v > x \vee \delta(v)$, v not $\succ$ $x \vee \delta(v)$. We can write $v = v_1 \vee v_2$ where $v_1, v_2 < v$ and $v_1 \wedge v_2 = x \vee \delta(v)$. Under such circumstances v_1 and v_2 are in S and hence

$$d(v) \geq v' = v_1' \wedge v_2' \geq \bigwedge\{v'; v \geq x, v \in S\} = \varepsilon(x) \wedge x'$$

which is possibility (i) again. Thus (β) holds.

We can show now that $e(x) = \varepsilon_2(x) \wedge x'$ $(=\bigwedge\{z \wedge x';$ $x \leq z \precsim \varepsilon(x)\}$ by the definition of $\varepsilon_2(x)$).

Let $y \succ x \vee \delta(y)$, y not in S, and put $z = \varepsilon(x) \wedge (x \vee d(y))$. Then

$x \leq z \precsim \varepsilon(x)$ $(x \vee \delta(y) \prec y \Longrightarrow x \vee \delta(y) \vee y' \precsim 1 \Longrightarrow \varepsilon(x) \wedge (x \vee \delta(y) \vee y')$ $\precsim \varepsilon(x)$; $\delta(y) \vee y' = d(y)$); also $z \wedge x' \leq d(y)$. In view of (β) it follows that $e(x) \geq \varepsilon_2(x) \wedge x'$ since in any case $\varepsilon(x) \wedge x' \geq \varepsilon_2(x) \wedge x'$. Now let $x \leq z \precsim \varepsilon(x)$. If $z = \varepsilon(x)$ then $z \wedge x' = \varepsilon(x) \wedge x' \geq e(x)$. If $z \prec \varepsilon(x)$ put $y = \varepsilon(x) \wedge (x \vee z')$. Then $y \succ x$ and hence $\delta(y) \leq x$ since x is in S; thus $y \succ x \vee \delta(y)$. Also, the fact that y is in $(x, \varepsilon(x)]$ implies

that y is not in S. Therefore from (β) we have
$e(x) \leq \varepsilon(x) \wedge x' \wedge d(y) = \varepsilon(x) \wedge x' \wedge (\delta(y) \vee y') \leq \varepsilon(x) \wedge x' \wedge (x \vee \varepsilon(x)' \vee (x' \wedge z)) = \varepsilon(x) \wedge x' \wedge z \leq z \wedge x'$; hence
$e(x) \leq \varepsilon_2(x) \wedge x'$. Q.E.D.

(68) Let S be a section of a CBA A and let $f, d, \delta, \varepsilon, \varepsilon_2$ be as above. Then f is analytic iff $\varepsilon_2(x) = x$ for all x in S, and $\upharpoonleft \wedge d = \delta$ is idempotent iff $\delta(x) = 0$ or x for all x in $A \backslash S$.

Proof. The first equivalence is an immediate consequence of (67), and the second follows from the fact that if $\delta(x) \neq x$ then $\delta(x)$ is in S and hence $\delta^2(x) = 0$. Q.E.D.

This last result enables us to describe completely the matroidal section closure operators on an atomic CBA A. Let us say that elements x, y of A are on the same level iff $x \backslash y$ and $y \backslash x$ are both joins of a finite number of atoms, the same number of atoms being required for each.

(69) Let S be a section of an atomic CBA A and let f be the associated closure operator. Then the following conditions are equivalent

(i) f is matroidal,

(ii) f is E_1,

(iii) if x is in S and y is on the same level as x then y is in S.

Proof. We know from (32) and (21) that (i) implies (ii). Suppose that (ii) holds; we verify that (iii) holds by induction on the number n of atoms contained in $x \backslash y$. The case $n = 0$ is trivial. Suppose that (iii) holds whenever $n = k$, and let $n = k+1$. Then there exists an element z and atoms p,q of A such that $x \backslash (z \vee p)$ and $(z \vee p) \backslash x$ each contain k atoms and $y = z \vee q$. Then $z \vee p$ is in S (supposing that x is in S). If y is not in S then $f(y) = 1$ and we have $p \leq f(z \vee q)$, $p \nleq f(z)$ (the latter since z is in S and is therefore closed). By (21)(iii), this gives $q \leq f(z \vee p)$, contrary to the fact that $z \vee p$ is closed. Hence y is in S and we have shown that (ii) implies (iii). To see that (iii) implies (i) we note that $\delta(x) = \bigwedge\{y;\ y \precsim x,\ y \in S\}$ for all x in A (as is clear from the definition of δ and the fact that A is atomic). It follows that if (iii) holds then, for x not in S, $\delta(x) = 0$ if x covers some element of S, and $\delta(x) = x$ otherwise. Thus $\restriction \wedge d$ is idempotent by (68) and, since f is analytic by (8), f is indeed matroidal. Q.E.D.

The conditions under which f is topological-matroidal, or B-matroidal, will be apparent from (69) and the results to be obtained in the remainder of this appendix, in which we apply the analysis of (67) and (68) to the cases: S an ideal; S determined by a cross-cut. We might remark here that, as a consequence of (69), the section closure operators introduced by Dlab in [11] are matroidal.

(70) Let S be a section of a CBA A and let f be the associated closure operator. Then f is topological iff either S is an ideal or $S = A\setminus\{1\}$.

Proof. By (41) f is topological iff $f(A)$ is $\vee$-closed. Here $f(A) = S \cup \{1\}$ and this is clearly $\vee$-closed when S is an ideal or $A\setminus\{1\}$. Suppose conversely that $S \cup \{1\}$ is given to be $\vee$-closed, so that for all x,y in S either $x \vee y$ is in S or $x \vee y = 1$. Then if S is not an ideal there exist x,y in S with $x \vee y = 1$. But this implies that every element z of A is the join of the elements $z \wedge x$, $z \wedge y$ of S and therefore either z is in S or $z = 1$; hence $S = A\setminus\{1\}$ or A. Q.E.D.

Note that $S = A\setminus\{1\}$ and $S = A$ give the same f, namely $f = \upharpoonleft$. Thus to obtain a topological section closure operator on A we can always take the section to be an ideal and there is a bijection between topological section closure operators on A and ideals of A.

(71) Let S be an ideal of a CBA A and let $f, d, \delta, \varepsilon, \varepsilon_2$ be as above. Let P be the set of atoms of A. Then

(a) For x not in S, $\delta(x) = x$ unless x covers an element y of S, in which case $\delta(x) = y$; for x in S, $\varepsilon(x) = x \vee (\bigvee S)'$ and $\varepsilon_2(x) = x \vee (\bigvee(S \cup P))'$.

(b) f is analytic iff $\bigvee(S \cup P) = 1$; $\upharpoonright \wedge d = \delta$ is idempotent iff $S = \{0\}$ or $S \supsetneqq P$; f is matroidal iff $\bigvee S = 1$ or $S = \{0\}$ and A is atomic.

Proof. (a) If x is not in S and $\delta(x) \neq x$ then $\delta(x)$ is in S and if x does not cover $\delta(x)$ there exist y_1, y_2 in $[\delta(x), x)$ and hence in S such that $x = y_1 \vee y_2$ - but then x is in S, a contradiction. If x covers an element y of S then certainly $\delta(x) \leq y$ and from what we have just seen we must have $\delta(x) = y$. Note that in this case $x \backslash y$ is in $P \backslash S$ (if $x \backslash y$ is in S then so is $x = (x \backslash y) \vee y$). To obtain the formula for ε, suppose first that $x < u \leq x \vee (\bigvee S)'$ - then u is not in S for otherwise $u \leq \bigvee S$ which with $u \leq x \vee (\bigvee S)'$ implies that $u \leq x$, a contradiction. This proves that $\varepsilon(x) \geq x \vee (\bigvee S)'$. To prove the reverse inequality, observe that for an arbitrary element s of S, $(\varepsilon(x) \wedge s) \vee x$ is in $S \cap [x, \varepsilon(x)]$ and is thus equal to x. That is, $\varepsilon(x) \wedge s \leq x$ for all s in S and thus $\varepsilon(x) \wedge (\bigvee S) \leq x$ so that $\varepsilon(x) \leq x \vee (\bigvee S)'$ as required. The formula for ε_2 is an easy consequence of that for ε.

(b) From (68) we know that f is analytic iff $\varepsilon_2(x) = x$ for all x in S and, in view of the formula for $\varepsilon_2(x)$ stated in (a), this is equivalent to the assertion that $\bigvee(S \cup P) = 1$. Alternatively, we can rely on (42) which by (67)(a) asserts that f is analytic iff $\bigvee\{x;\ \delta(x) = 0\} = 1$ - but, as we see from part (a) of the present result, $\delta(x) = 0$ iff x is in $S \cup P$. Now by (68) again, $\upharpoonright \wedge d = \delta$ is idempotent iff $\delta(x) = 0$ or x for all x not in S. This fails precisely when there exists an x in $A\backslash S$ covering a non-zero element y of S, and this occurs iff $S \neq \{0\}$ and there is an atom not in S (if we have x and y as described then $S \neq \{0\}$ and there is an atom not in S (if we have x and y as described then $S \neq \{0\}$ and $x\backslash y$ is in $P\backslash S$ - conversely if y is in $S\backslash\{0\}$ and p is in $P\backslash S$ then $x = y \vee p$ is in $A\backslash S$ and covers y). Since f is matroidal iff it is analytic and $\upharpoonright \wedge d$ is idempotent, the final statement of the result follows immediately from what has been shown already. Q.E.D.

A particular case in which this result applies to give a matroidal closure operator f is when S is a non-principal maximal ideal of A; the closure operators arising in this way may be characterized as the connected HK closure operators (a closure operator f is <u>connected</u> iff the only f-separators are 0 and 1), and as the pithy topological door closure operators (a closure operator is <u>door</u> iff every element is either open or closed).

A cross-cut of a CBA A (or of any ordered set for that matter) is a maximal antichain C in A which satisfies the following interpolation property:

If $x \leq a$, $x \leq z$, and $c \leq z$, where a and c are in C, then there exists b in C such that $x \leq b \leq z$.

(72) Let C be a cross-cut of a CBA A and let $S = \{x;\ x < \text{some element of } C\}$, $T = \{z;\ z < \text{some element of } C\}$. Let $f, d, \delta, \varepsilon, \varepsilon_2$ correspond to the section S as before. Then

(a) $\{S, C, T\}$ is a partition of A,

(b) If x is covered by an element of C then A_x is atomic with $A_x \cap C$ as its set of atoms.

(c) $\delta(x) = 0$ for x in C and $\delta(x) = x$ for x in T; $\varepsilon(x) = 1$ and $\varepsilon_2(x) = x$ for those x in S which are covered by some element of C and $\varepsilon(x) = \varepsilon_2(x) = x$ for the remaining x in S; $d(x) = 0$ for x in S, $d(x) = x'$ for x in C, and $d(x) = 1$ for x in T.

(d) f is B-matroidal; the set of bases of an element s of A is $\{s\}$ for s in S and $A^s \cap C$ for s in $A \backslash S$.

Proof. (a) S, C, T are pairwise disjoint since C is an antichain, and $S \cup C \cup T = A$ since every element of A is comparable with some element of C, C being a maximal antichain.

(b) This will be proved if we show that for each $u > x$ there exists c in $A_x \cap C$ with $u \geq c$. Suppose that $x \prec a$

in C and let $u > x$. Suppose if possible that u is in S. Then by the interpolation property there exists b in C for which $u < b \leq u \vee a$. Now from $x \prec a$ it follows that $u = u \vee x \preceq u \vee a$. Hence we must have $b = u \vee a$ which, since C is an antichain, can only happen if $u \leq a = b$. But then $x < u \leq a$ which with $x \prec a$ implies $u = a$, contradicting the supposition that u is in S. Thus either u is itself in C or, if u is in T, there exists by the interpolation property some c in C with $x < c \leq u$, as desired.

(c) Let x be in C - then $[0,x) \subseteq S$ and hence $\delta(x) = 0$. Let x be in T and suppose that $\delta(x) \neq x$ - then $\delta(x)$ is in S and by the interpolation property $[\delta(x),x) \cap C \neq \phi$, contrary to $[\delta(x),x) \subseteq S$. Let x be in S, x covered by some element of C - then $\varepsilon(x) = 1$ and $\varepsilon_2(x) = x$ by part (b) of the present result. Let x be in S, x not covered by any element of C, and suppose that $\varepsilon(x) \neq x$. Then $\varepsilon(x)$ is not in S and by the interpolation property there exists c in $(x,\varepsilon(x)] \cap C$. Since x is not covered by c, there exists y in (x,c) - but then y is in $(x,\varepsilon(x)] \cap S$, contrary to the inclusion $(x,\varepsilon(x)] \subseteq A\backslash S$. $d(x)$ may be obtained from $\delta(x)$ using (67)(b) and (c).

(d) The discrete elements are precisely those x for which $\delta(x) = 0$, that is, they are the elements of $S \cup C$. The assertion in (d) which concerns bases is now obvious and it follows by the interpolation property that f satisfies the defining condition for B-matroidal closure operators. Q.E.D.

It is easy to see that the converse of (72)(d) is true in the sense that if f is a B-matroidal section closure operator on a CBA A then there exists a cross-cut C of A which gives rise to f in the way described (take C to be the set of f-bases).

The simplest examples of cross-cuts on CBA's are as follows. Let A be an atomic CBA, say $A = \mathcal{P}(E)$, and let k be a non-negative integer. Then the subset C of A which consists of all subsets of E having exactly k elements is a cross-cut of A, and so also is $C^* = \{c'; \ c \text{ in } C\}$. (It is clear in general that if C is a cross-cut of a CBA A then so is C^* as just defined.) For E finite, it is easy to see (use (72)(b) say) that there are no other types of cross-cut on A; it is an unsolved problem ([22], pp. 343-344) whether all cross-cuts on A are given in this way when E is infinite.

Other examples of cross-cuts on CBA's may be obtained by considering a measure algebra (A,μ) ([14], p. 67) with A atomless and letting $C_t = \{x; \ \mu(x) = t\}$ where $0 \leq t \leq 1$. It is obvious that these are not the only cross-cuts on A: just shift μ a bit, say by expressing $1 = a \vee b$ where $\mu(a) = \mu(b) = \frac{1}{2}$ and defining a new measure algebra (A,μ') with $\mu'(a) = \frac{1}{3}$, $\mu'(b) = \frac{2}{3}$ (so that $\mu'(x) = \frac{2}{3}\,\mu(x \wedge a) + \frac{4}{3}\mu(x \wedge b)$). Of course it is still conceivable that every cross-cut on A can be obtained as a C_t for some t and some μ such that (A,μ) is a measure algebra; this is an open question. It should also be remarked that if A has atoms then C_t is not necessarily a cross-cut. Indeed it is not difficult to show that if we take any measure algebra (A,μ) for which A is the power set of a countably infinite set then C_t is a cross-cut only when $t = 0$ or 1.

APPENDIX 2

SEPARATION AXIOMS

We suppose throughout this appendix, except where explicitly stated otherwise, that f is a closure operator on a CBA A such that $f(0) = 0$, and we write d for the derivative of f. The remarks which now follow will be useful later on.

For f to be analytic at 0, which condition itself may be regarded as a sort of separation axiom, the cardinality of $f(A)$ cannot be too small: we have $|f(A)| \geq |A^x|$ for all discrete x (if x is discrete then $f(u) \wedge x = u$ for all u in A^x). In particular, if $|A^x| = |A|$ for all non-zero x in A (such a CBA is said to be weakly homogeneous by Sikorski [39], p. 107) and $|A| > |f(A)|$, then there can be no non-zero discrete elements at all, and this is the same as saying that $d = f$ (if there are no non-zero discrete elements then, for each x in A, the element $x \backslash d(x)$, which is invariably discrete, equals 0 so that $x \leq d(x)$ and hence $f(x) = d(x)$; the converse is equally easy); f is thus as non-analytic as it can be.

We will also need the following lemma (in which it is not necessary that $f(0) = 0$).

<u>Lemma.</u> Let f be any closure operator on a CBA A. Then f is analytic iff, for all closed a and all x such that $0 \neq x \leq a'$, there exists y such that $0 \neq y \leq x$ and $f(a \vee z) < f(a \vee y)$ for all $z < y$.

<u>Proof.</u> By the corollary to (10), f is analytic iff it is analytic at each closed element a of A. Now for a closed, f is analytic at a iff $a' = \bigvee\{w;\ w \wedge a = w \wedge d(a \vee w) = 0\}$, that is, iff for all x such that $0 \neq x \leq a'$ there exists y such that $0 \neq y \leq x$, $y \wedge d(a \vee y) = 0$. Thus the result will be obtained if we show that, for all a and y with $a \wedge y = 0$, we have $y \wedge d(a \vee y) = 0$ iff $f(a \vee z) < f(a \vee y)$ for $z < y$. Suppose that $y \wedge d(a \vee y) = 0$ and let $z < y$. Then $f(a \vee z) = a \vee z \vee d(a \vee z) \leq a \vee z \vee d(a \vee y) < a \vee y \vee d(a \vee y) = f(a \vee y)$. Suppose conversely that $y \wedge d(a \vee y) \neq 0$ - then by (3) there exists u such that $0 \neq u \leq y \wedge f((a \vee y)\backslash u)$. Let $z = y\backslash u$. Then $z < y$ and $(a \vee y)\backslash u = a \vee z$ so that $u \leq f(a \vee z)$ and hence $f(a \vee y) = f(a \vee z \vee u) = f(a \vee z)$. Q.E.D.

<u>T_0 and T_1</u>

The following definition has the advantage of permitting the customary "passage to the T_0 case" to be carried through most naturally:

$$f \text{ is } T_0 \text{ iff } C(f(A)) = A.$$

(Note. The notations $J(X)$, $M(X)$, $L(X)$, and $C(X)$, where $X \subseteq A$, are as introduced in the Preliminaries and X'' denotes $\{x';\ x \text{ in } X\}$.)

Suppose that f is not necessarily T_0 - then we can associate with f a T_0 closure operator g on $C(f(A))$ by defining $g(x) = f(x)$ for all x in $C(f(A))$. The following result gives (in a somewhat more general situation) various properties of this T_0-ization process.

(73) Let $\bar{A}$ be a complete subalgebra of A such that $f(A) \subseteq \bar{A}$, let g be the closure operator on $\bar{A}$ defined by $g(x) = f(x)$ for all x in $\bar{A}$, and let $^-$ be the quantifier on A with $\bar{A}$ as its range. Then

(a) $f(x) = g(\bar{x})$ for all x in A.

(b) For all x in A, x is f-discrete iff it is $^-$ discrete and $\bar{x}$ is g-discrete.

(c) f is analytic iff both g and $^-$ are analytic.

(d) f is matroidal iff both g and $^-$ are matroidal.

(e) f is B-matroidal iff both g and $^-$ are B-matroidal.

(f) f is topological iff g is topological.

Proof. (a) We have $f(x) \leq f(\bar{x}) \leq f(f(x) = f(x)$; hence $f(x) = f(\bar{x}) = g(\bar{x})$.

(b) Suppose that x is f-discrete. Then x is $^{-}$ discrete since $^{-} \leq f$. To show that $\bar{x}$ is g-discrete, let u be in A, $u \leq \bar{x}$. Then $g(u) \wedge \bar{x} = f(u) \wedge \bar{x} = \overline{f(u) \wedge x} = \bar{u} = u$ as desired (the second equality here holds because $^{-}$ is a quantifier with respect to which $f(u)$ is closed). Now suppose that x is $^{-}$ discrete and $\bar{x}$ is g-discrete. Let u be in A, $u \leq x$. Then $f(u) \wedge x = g(\bar{u}) \wedge \bar{x} \wedge x = \bar{u} \wedge x = u$. Thus x is f-discrete.

(c) Suppose that f is analytic. Then the f-discrete elements are $\bigvee$-dense and hence by (b) so also are the $^{-}$ discrete elements. By (42) this implies that $^{-}$ is analytic. To see that g is analytic, we apply the lemma above. Take a in $f(A)$ and x in $\bar{A}$ such that $0 \neq x \leq a'$; since f is analytic there exists y in A such that $0 \neq y \leq x$ and $f(a \vee z) < f(a \vee y)$ for $z < y$. Then $\bar{y}$ is in $\bar{A}$ and $0 \neq \bar{y} \leq \bar{x} = x$. Take z in $\bar{A}$, $z < \bar{y}$ and put $w = y \wedge z$. We must have $w < y$ for if $w = y$ then $y \leq z$ and hence $\bar{y} \leq \bar{z} = z$, a contradiction.
Also $\bar{w} = \overline{y \wedge z} = \bar{y} \wedge z = z$, where the middle equality holds since $^{-}$ is a quantifier, and therefore $f(w) = f(z)$. It follows that $g(a \vee z) = f(a \vee z) = f(a \vee w) < f(a \vee y) = g(\overline{a \vee y}) = g(a \vee \bar{y})$ as required (here $\overline{a \vee y} = a \vee \bar{y}$ since $^{-}$ is topological and a is $^{-}$ closed).

Now suppose that g and $^{-}$ are analytic. Take a in $f(A)$ and x in A such that $0 \neq x \leq a'$; we wish to obtain y_1 in A such that $0 \neq y_1 \leq x$ and $f(a \vee z_1) < f(a \vee y_1)$ for $z_1 < y_1$,

z_1 in A. Since the $^-$ discrete elements are $\bigvee$-dense, we may clearly suppose that x is $^-$ discrete. Now $0 \neq x \leq \overline{a'} = a'$ and hence there exists y in $\bar{A}$ such that $0 \neq y \leq \bar{x}$ and $f(a \vee z) < f(a \vee y)$ for all z in $\bar{A}$ such that $z < y$. Put $y_1 = x \wedge y$. Then $\bar{y}_1 = \bar{x} \wedge y = y$ so that $0 \neq y_1 \leq x$. Let z_1 in A satisfy $z_1 < y_1$. Then $\bar{z}_1 = \bar{y}_1 = y$, the inequality being strict since x, and therefore also y_1, is $^-$ discrete. Put $z = \bar{z}_1 \vee (y\backslash\bar{y}_1)$ - then z is in $\bar{A}$ and $z < y$ (it is easy to see that $y\backslash z = \bar{y}_1\backslash\bar{z}_1$). Hence $f(a \vee z) < f(a \vee y)$, that is, $f(a \vee \bar{z}_1 \vee (y\backslash\bar{y}_1)) < f(a \vee \bar{y}_1 \vee (y\backslash\bar{y}_1))$. It follows that $f(a \vee \bar{z}_1) < f(a \vee \bar{y}_1)$ and this is the same as $f(a \vee z_1) < f(a \vee y_1)$.

(d) Suppose that f is matroidal - then f is analytic and hence by (c) so are g and $^-$; $^-$ is in fact B-matroidal by (56). By (32), g will be matroidal if it is super-$E_{\frac{1}{2}}$. Let x,s,y be elements of $\bar{A}$ such that $y \leq f(s \vee x)$ and $y \nleq f(s)$; we want to show that there exists v in $\bar{A}$ such that $0 \neq v \leq x$ and $v \leq f(s \vee y \vee (x\backslash v))$. Replacing x by a $^-$ base of x does not affect $f(s \vee x)$ and we can without loss of generality suppose that x is $^-$ discrete. Since f is super-$E_{\frac{1}{2}}$, there exists u in A such that $0 \neq u \leq x$, $u \leq f(s \vee y \vee (x\backslash u))$ and the fact that x is $^-$ discrete implies that $\bar{u} \wedge x = u$ and hence that $x\backslash\bar{u} = x\backslash u$. Thus we have $\bar{u}$ in $\bar{A}$, $0 \neq \bar{u} \leq x$, and $\bar{u} \leq f(s \vee y \vee (x\backslash\bar{u}))$ as required.

Suppose that g and $^-$ are matroidal - to show that f

is matroidal it is sufficient by (c) and (32) again to show that f is super-$E_{\frac{1}{2}}$. Let x,s,y be elements of A such that $y \leq f(s \vee x)$, $y \nleq f(s)$. Then $\bar{y} \leq f(\bar{s} \vee \bar{x})$, $\bar{y} \nleq f(\bar{s})$ and hence there exists v in $\bar{A}$ such that $0 \neq v \leq \bar{x}$, $v \leq f(\bar{s} \vee \bar{y} \vee (\bar{x}\backslash v))$. Put $u = v \wedge x$ - then $0 \neq u \leq x$ by the usual argument depending on the fact that $\bar{\ }$ is a quantifier; also $\overline{x\backslash u} \vee v = \overline{(x\backslash u) \vee v} = \overline{x \vee v} \geq \bar{x}$ so that $\bar{x}\backslash v \leq \overline{x\backslash u}$ and hence $v \leq f(\bar{s} \vee \bar{y} \vee \overline{x\backslash u}) = f(s \vee y \vee (x\backslash u))$.

(e) Suppose that f is B-matroidal. Then $\bar{\ }$ is B-matroidal as in (d). To show that g is B-matroidal, let s and x be in $\bar{A}$ with x a g-discrete subelement of s. Let y be a $\bar{\ }$ base of x. Then by (b) y is an f-discrete subelement of s and therefore, since f is B-matroidal, there exists an f-base z of s such that $y \leq z$. We claim that $\bar{z}$ is a g-base of s such that $x \leq \bar{z}$: $\bar{z}$ is g-discrete by (b); $\bar{z}$ is g-dense in s since $s \leq f(z) = g(\bar{z})$; $\bar{z} \leq s$ since $z \leq s$ and $\bar{s} = s$; and $x \leq \bar{z}$ since $x = \bar{y}$ and $y \leq z$.

Now suppose that g and $\bar{\ }$ are B-matroidal. Let s and x be in A with x an f-discrete subelement of s. By (b), x is a $\bar{\ }$ discrete subelement of s and hence there exists a $\bar{\ }$ base y of s with $x \leq y$; also by (b), $\bar{x}$ is a g-discrete subelement of $\bar{s}$ and hence there exists a g-base z of $\bar{s}$ with $\bar{x} \leq z$. We claim that $y \wedge z$ is an f-base of s such that $x \leq y \wedge z$: $y \wedge z$

is f-discrete by (b) since it is $\bar{\ }$ discrete $(y \wedge z \leq y)$ and $\overline{y \wedge z} = \bar{y} \wedge z$ is g-discrete $(\bar{y} \wedge z \leq z)$; $y \wedge z$ is f-dense in s since $f(y \wedge z) = f(\overline{y \wedge z}) = f(\bar{y} \wedge z) = f(\bar{s} \wedge z) = f(z) = g(z) \geq s$; and clearly $x \leq y \wedge z$.

(f) This is immediate from (41). Q.E.D.

There are a number of equivalent conditions which are reminiscent of the classical T_0 axiom and which in the atomic case are indeed equivalent to it, and to T_0 as defined here. We say that f is R_0 iff for all disjoint $x, y \neq 0$ in A there exists a in $f(A)$ such that either $x \wedge a \neq 0$ and $y \wedge a' \neq 0$, or $x \wedge a' \neq 0$ and $y \wedge a \neq 0$.

(74) The following conditions are equivalent

(i) f is R_0,

(ii) for all disjoint $x, y \neq 0$ in A there exist $x_1, y_1 \neq 0$ in A such that $x_1 \leq x$, $y_1 \leq y$, and $f(x_1) \neq f(y_1)$,

(iii) for all disjoint $x, y \neq 0$ in A there exist $x_1, y_1 \neq 0$ in A such that $x_1 \leq x$, $y_1 \leq y$, and either $f(x_1) \wedge y_1 = 0$ or $x_1 \wedge f(y_1) = 0$,

(iv) for all x in A such that x is neither 0 nor an atom there exists a in $f(A)$ such that $x \wedge a \neq 0$ and $x \wedge a' \neq 0$,

(v) for all x in A such that x is neither 0 nor an atom there exists y in A such that $0 \neq y \leq x$ and $f(y) < f(x)$.

Proof. (i) $\Longrightarrow$ (iii). If $x \wedge a \neq 0$, $y \wedge a' = 0$ then $x_1 = x \wedge a$, $y_1 = y \wedge a'$ satisfy (iii) with $f(x_1) \wedge y_1 = 0$.

(iii) $\Longrightarrow$ (ii). Trivial.

(ii) $\Longrightarrow$ (i). Suppose that $f(x_1) \not\geq f(y_1)$ - then $f(x_1) \not\geq y_1$ and $a = f(x_1)$ satisfies $x \wedge a \neq 0$, $y \wedge a' \neq 0$.

(i) $\Longrightarrow$ (iv). Write $x = y \vee z$, where y and z are non-zero and disjoint, and apply R_0.

(iv) $\Longrightarrow$ (v). Take $y = x \wedge a$.

(v) $\Longrightarrow$ (iv). Take $a = f(y)$.

(iv) $\Longrightarrow$ (i). Apply (iv) with $x \vee y$ in place of x. Q.E.D.

It is particularly easy to see from condition (ii) of this result that R_0 is equivalent, when A is atomic, to the classical T_0 axiom stating that distinct points have distinct closures.

(75) (a) T_0 implies R_0, and the converse holds when A is atomic.

(b) If f is analytic at 0 and A is atomless then f is R_0

Proof. (a) For each x in A, write $A(x) = A_x \cup A^{x'} = \{y;\ x \wedge y = 0 \text{ or } x \wedge y' = 0\}$ - then this is clearly a complete subalgebra of A. Now (74)(iv) states that $f(A) \nsubseteq A(x)$ for each x in A not 0 or an atom. But if $f(A) \subseteq A(x)$ for some x, and f is T_0, then we must have $A(x) = A$, and it is clear that this only happens when x is 0 or an atom. By (74), this proves that T_0 implies R_0. Now suppose that f is R_0 and let p be an atom of A. Then, applying the defining condition for R_0 with $x = p$, we see that p is the meet of all the elements of $f(A) \cup f(A)'$ which contain it. Thus every atom of A is in $C(f(A))$ and if A is atomic this implies that $C(f(A)) = A$.

(b) Suppose that f is analytic at 0 and that A is atomless - we verify that condition (iv) of (74) holds. Let x be non-zero - then, since f is analytic at 0, x contains a non-zero discrete element y. Let z be such that $0 < z < y$. Then $f(z) \wedge y = z$ since y is discrete, and $a = f(z)$ meets x but does not contain x, as required. Q.E.D.

We give an example to show that R_0 may not be replaced by T_0 in (75)(b) (so that T_0 and R_0 are not in general equivalent). Let C be an atomless CBA and let ∇ be the diagonal quantifier on C^2 as defined in the discussion following (56). Then as shown in that discussion ∇ is analytic, and of course C^2 is atomless; but ∇ is not T_0.

We have defined T_0 to mean that $C(f(A)) = A$; the other T_i axioms to be considered here may be regarded, more or less, as stating how nicely $f(A)$ completely generates A. Thus for example T_1 states that $J(f(A)) = A$ already and is the strongest axiom we shall study. Examples of T_i axioms which are equivalent to T_0 in the atomic case on account of the equation $MJ(X) = JM(X)$ (which holds for all subsets X of an atomic CBA by virtue of the complete distributivity of the latter) are:

$MJ(f(A) \cup f(A)') = A$; $MJ(\{a\backslash b;\ a,b \text{ in } f(A)\}) = A$ etc. .

It is worth noting that these axioms, and especially T_1, impose a cardinality condition on $f(A)$ vis-a-vis A. Thus if f is T_1 then clearly $|A| \leq 2^{|f(A)|}$. Whereas no such condition holds in the case of T_0: Solovay [40] has shown that there exist arbitrarily large CBA's A which are completely generated by a countable subset X and hence by $M(X)$ where $|M(X)| \leq 2^{\aleph_0}$ - but $M(X)$ is of the form $f(A)$ for some closure operator f on A. A similar remark applies to the T_i axiom $L(f(A)) = A$ which therefore is not equivalent to T_1 in general - but which does reduce to T_1 in the atomic case, when $MJ = JM$. Between $L(f(A)) = A$ and T_1 there is a multitude of T_i axioms, for example $MJ(f(A)) = A$ and T_0 + "$f(A)' \subseteq J(f(A))$"; we shall mention this last inclusion again (see (79)).

If we look for conditions which resemble the classical T_1 axiom and yet which are appropriate to the non-atomic case (as R_0 resembles the classical T_0 axiom), we find the following obvious result:

(76) f is T_1 iff for all disjoint $x,y \neq 0$ in A there exists a in $f(A)$ such that $x \wedge a \neq 0$ and $y \wedge a = 0$.

We also find these plausible conditions:

f is R_{1a} iff for all disjoint $x,y \neq 0$ in A there exists a in $f(A)$ such that $x \wedge a \neq 0$ and $y \wedge a' \neq 0$; and

f is R_{1b} iff for all disjoint $x,y \neq 0$ in A there exist $x_1,y_1 \neq 0$ in A such that $x_1 \leq x$, $y_1 \leq y$, and $f(x_1) \wedge f(y_1) = 0$.

(77) The following conditions are equivalent

(i) f is R_{1a},

(ii) for all disjoint $x,y \neq 0$ in A there exist $x_1,y_1 \neq 0$ in A such that $x_1 \leq x$, $y_1 \leq y$, and $f(x_1) \wedge y_1 = 0$,

(iii) for all disjoint $x,y \neq 0$ in A there exist $x_1,y_1 \neq 0$ in A such that $x_1 \leq x$, $y_1 \leq y$, and $f(x_1) \wedge y_1 = x_1 \wedge f(y_1) = 0$.

Proof. (i)$\Longrightarrow$(ii). Take $x_1 = x \wedge a$, $y_1 = y \wedge a'$.

(ii)$\Longrightarrow$(i). Take $a = f(x_1)$.

(ii)$\Longrightarrow$(iii). Let x_1,y_1 be as in (ii) - then, applying (ii) to the pair y_1,x_1, we obtain $x_2,y_2 \neq 0$ such that $x_2 \leq x_1$, $y_2 \leq y_1$, and $x_2 \wedge f(y_2) = 0$; but $f(x_2) \wedge y_2 = 0$ also, being contained in $f(x_1) \wedge y_1$.

(iii). trivially implies (ii). Q.E.D.

It is clear that R_{1a} and R_{1b} coincide with the classical

T_1 axiom in the atomic case - indeed that

(78) T_1 implies R_{1b}, R_{1b} implies R_{1a}, and both converses hold when A is atomic; R_{1a} implies R_0.

Let f be an analytic quantifier $\neq \upharpoonleft$ on an atomless CBA - then, as we have already seen, f is R_0 and, since the equations $f(x_1) \wedge y_1 = 0$, $x_1 \wedge f(y_1) = 0$, $f(x_1) \wedge f(y_1) = 0$ are now equivalent, f is actually R_{1b}. It follows that R_{1a} and R_{1b} do not in general imply T_0, still less T_1.

Given some T_i axiom, we define the corresponding "S_i" axiom by saying that f is S_i iff the T_0-ization of f (as described at the beginning of this appendix) is T_i; then f will be T_i iff it is S_i and T_0. As is clear from (29), S_1 and T_1 provide an example of such a pair of conditions S_i, T_i. (For A atomic, S_1 has been discussed by Csaszar [8].)

Let A be atomic - then $S_{1/2}$ coincides with S_1 (and hence the T_i axiom $T_{1/2} = S_{1/2} + T_0$ coincides with T_1). Whether $S_{1/2} = S_1$ (or $T_{1/2} = T_1$) in general is unresolved; the only theorems in this direction are (44)(b) and the following very partial result:

(79) If f is analytic and $S_{1/2}$ then $f(A)' \subseteq J(f(A))$.

<u>Proof.</u> Let a be closed - then $a' = \bigvee\{w;\ w \wedge a = w \wedge d(a \vee w) = 0\}$ since f is analytic at a (see just before (10)), and since f

is $S_{\frac{1}{2}}$ we need only take the join over those w's which are f-$S_{\frac{1}{2}}$ (the set of w's in the original join is a section). For the resulting w's we must have $f(w) \wedge a = 0$: if not, $y = f(w) \wedge a$ satisfies $0 \neq y \leq f(w)$ and then there exists u such that $0 \neq u \leq w$, $u \leq d(y \vee (w \backslash u)) \leq d(a \vee w)$, a contradiction.
It follows that a' is the join of the closed elements $f(w)$ and we have the result. Q.E.D.

It seems reasonable to conjecture that if f is analytic and $S_{\frac{1}{2}}$ then it is S_1.

$\underline{T_D \text{ and } T_H}$

These are two instances of T_i axioms which do not reduce to T_0 or T_1 when A is atomic. The T_D axiom was introduced, for the atomic topological case, by Aull and Thron [1] and it is the following adaption to the general case of one of their equivalent conditions which we find it most convenient to take as the definition:

$$f \text{ is } T_D \text{ iff } J(\{a \backslash b;\ a,b \text{ in } f(A)\}) = A.$$

Now it is a well-known and in any case trivial fact that, for any closure operator f on a CBA A, an element x of A is of the form $a \backslash b$ for some f-closed elements a and b iff $f(x) \backslash x = d(x) \backslash x$ is closed. Thus

$$f \text{ is } T_D \text{ iff } J(\{x;\ d(x) \backslash x \text{ is closed}\}) = A.$$

Clearly T_1 implies T_D and T_D implies T_0; the examples given by Aull and Thron show that neither of these implications can be reversed even when A is atomic and f is topological (though it is easily seen that, as Aull and Thron remark, T_D and T_0 coincide when A is finite and f is topological). T_D also implies the sort of cardinality condition encountered earlier: if f is T_D then $|A| \leq 2^{|f(A)|^2}$.

Besides T_D, we wish to consider also the condition derived most directly from Aull and Thron's definition of T_D. We say that

$$f \text{ is (AT) iff } J(\{x;\ d(x) \text{ is closed}\}) = A.$$

To see that (AT) and T_D are not equivalent in general, observe that the complete embedding of a CBA B into another CBA A constructed by Kripke [29] (B does not have to be complete in Kripke's embedding but we are taking it to be complete) is such that A is homogeneous (by the argument on p. 106 of Sikorski's book [39]) and of cardinality greater than that of B. Thus if f is the quantifier on A with $f(A) = B$ then, by the remarks made at the beginning of this appendix, the derivative d of f is just f and every $d(x)$ is closed, yet f is not T_0 and thus not T_D.

We summarize various facts concerning T_D and (AT) in:

(80) (a) f is (AT) if $J(\{x;\ d(d(x)\backslash x) \leq d(x)\}) = A$;

T_D implies (AT).

(b) The following conditions are equivalent

(i) f is analytic at 0 and T_D,

(ii) f is analytic at 0 and (AT),

(iii) $J(\{x;\ d^2(x) \le d(x)\backslash x\}) = A$.

(c) $d^2 \le d$ implies that f is (AT), and the converse holds when f is topological.

Proof. (a) Since $\{x;\ d(x) \text{ is closed}\} = \{x;\ d^2(x) \le d(x)\} \subseteq \{x;\ d(d(x)\backslash x) \le d(x)\}$, (AT) certainly implies that J of the latter set equals A. Suppose conversely that this is given to hold; we will then have obtained (AT) if we show that every non-zero x, for which $d(d(x)\backslash x) \le d(x)$, contains a non-zero y for which $d^2(y) \le d(y)$. Take such an x. If $x\backslash d(x) = 0$ – that is, if $x \le d(x)$ – then $d^2(x) = d(x)$ and we can use x as our y. If $x\backslash d(x) \ne 0$, let y be such that $0 \ne y \le x\backslash d(x)$, $d(d(y)\backslash y) \le d(y)$. Then since $x\backslash d(x)$ is discrete so is y also and hence $d(y)\backslash y = d(y)$ – thus $d^2(y) \le d(y)$ and we again have a suitable y. This proves the first part of (a) and on account of it and the obvious inclusion $\{x;\ d(x)\backslash x \text{ is closed}\} \subseteq \{x;\ d(d(x)\backslash x) \le d(x)\}$, T_D implies (AT).

(b) It is obvious that the set of all discrete elements has the same intersection with the fout sets $\{x;\ d(x)\backslash x \text{ is closed}\}$, $\{x;\ d(x) \text{ is closed}\}$, $\{x;\ d(d(x)\backslash x) \le d(x)\}$, and $\{x;\ d^2(x) \le d(x)\backslash x\}$. It follows that if f is analytic at 0 and any one of these four sets is $\bigvee$-dense, then so are the others. This shows that (i) and (ii) are equivalent and imply (iii). Since the fourth of the four sets just mentioned is always contained in the other three, it remains for us to prove that if (iii) holds then f is analytic at 0. This however follows immediately from the observation that if $d^2(x) \leq d(x)\backslash x$ and x is non-zero then the discrete subelement $x\backslash d(x)$ of x is also non-zero (otherwise $x \leq d(x)$, so that $d^2(x) = d(x)$ and hence $x \leq d(x) \leq d(x)\backslash x$, which implies $x = 0$).

(c) It is trivial that $d^2 \leq d$ implies (AT). That the converse holds when f is topological is a consequence of the fact that in such a case the set $\{x;\ d(x)$ is closed$\}$ is $\bigvee$-closed (this was proved in the course of proving (47)(a)). Q.E.D.

Although (AT) is not a T_i axiom (the example preceding (80) showed that (AT) does not imply T_0), the condition $T_{D'} = (AT) + T_0$ is of course a T_i axiom in the sense that it implies T_0; and it is implied by T_D, being equivalent to it when f is analytic. The following example shows that $T_{D'}$ is not equivalent to T_D in general, even for topological closure operators. By the result of Solovay [40] already referred to, there exists a CBA A which is completely generated by a countable set X and which satisfies $|A| > 2^{2^{\aleph_0}}$. Let f be the closure operator on A with $f(A) = MJ(X)$. Then $|f(A)| \leq 2^{2^{\aleph_0}}$ so that $|A| \geq 2^{|f(A)|}$ and hence f is not T_D. Now A (as constructed by Solovay) is homogeneous and since $|A| > |f(A)|$ we have $d = f$ as explained previously. Thus $d^2 = d$ and f is (AT); indeed, since $C(f(A)) = A$, f is $T_{D'}$. Also f is topological by (41) since $f(A)$, being of the form $M(Y)$ for a $\vee$-closed subset Y of A, is itself $\vee$-closed (let $\bigwedge Y_1$, $\bigwedge Y_2$ be in $M(Y)$ where $Y_1, Y_2 \subseteq Y$ - then $(\bigwedge Y_1) \vee (\bigwedge Y_2) = \bigwedge\{y_1 \vee y_2;\ y_1 \text{ in } Y_1,\ y_2 \text{ in } Y_2\}$).

It is very easy to see that (AT) does not imply $d^2 \le d$ without the assumption that f is topological: take an atomic CBA A having 4 atoms, let S consist of 0 and the atoms of A, and let f be the closure operator on A associated with the section S as in Appendix 1; then f is T_1 yet it is not true that $d^2 \le d$ (nor that the set $\{x;\ d(x) \text{ is closed}\}$ is $\bigvee$-closed incidentally).

The last separation axiom we shall consider is the fourth in the quartet of which the first three are R_0, T_1 (as expressed in (76)), and R_{1a}. We say that

f is T_H iff for all disjoint $x, y \neq 0$ in A there exists a in $f(A)$ such that either $x \wedge a \neq 0$ and $y \wedge a = 0$, or $x \wedge a = 0$ and $y \wedge a \neq 0$.

(81) T_1 implies T_H and T_H implies T_0.

Proof. It is immediate from (76) that T_1 implies T_H. To show that T_H implies T_0, suppose that f is T_H but not T_0 and

and let $\overline{}$ be the quantifier on A such that $\overline{A} = C(f(A))$. Then there exists x in A but not in $\overline{A}$ and we have $\overline{x} > x$, $\overline{x'} > x'$. Then by T_H there is a closed element a meeting exactly one of $\overline{x} \wedge x'$ and $\overline{x'} \wedge x$; suppose that $\overline{x} \wedge x' \wedge a = 0$, $\overline{x'} \wedge x \wedge a \neq 0$. From the former it follows that $x \wedge a = \overline{x} \wedge a$, and thus that $x \wedge a$ is in $\overline{A}$. But by the usual quantifier identity, this implies that $\overline{x'} \wedge x \wedge a = \overline{x' \wedge x \wedge a} = 0$, a contradiction. Q.E.D.

Let A be atomic and let $\leq_f$ be the quasiorder on the set P of atoms of A defined by $p \leq_f q$ iff $p \leq f(q)$. It happens that a number of the weak separation axioms which have been considered can be described in terms of $(P, \leq_f)$. For example, T_0 states that $(P, \leq_f)$ is an order relation and T_1 that it is an antichain. Of the axioms introduced by Aull and Thron in [1], T_F, T_{YS}, and T_Y state that f is T_0 and $(P, \leq_f)$ contains respectively no  , no  or  , and no  or  ; T_{FF} states that f is T_0 and $(P, \leq_f)$ is of one of the three forms:

antichain; antichain + ; antichain +  .

On the other hand, it is not difficult to see from some of the examples they give ([1], p. 34) that their axioms T_D, T_{DD}, and T_{UD} are not able to be described in terms of $(P, \leq_f)$ only. The following result shows that T_H can be so described.

(82) Let A be atomic. Then f is T_H iff it is T_0 and $(P,\leq_f)$ satisfies the descending chain condition (DCC).

<u>Proof</u>. Suppose that $(P,\leq_f)$ does not satisfy the DCC and let $p_0 >_f p_1 >_f p_2$ etc. - then $x = \bigvee\{p_n;\ n \text{ even}\}$, $y = \bigvee\{p_n;\ n \text{ odd}\}$ violate the defining condition for T_H. Suppose conversely that f is T_0 but not T_H and let x,y be a pair of elements for which the defining condition for T_H fails. Let p_0 be an atom $\leq x$ - then $f(p_0)$ meets y. Let p_1 be an atom $\leq y \wedge f(p_0)$ - then $f(p_1)$ meets x. Let p_2 be an atom $\leq x \wedge f(p_1)$, etc.. We see that $(P,\leq_f)$ does not satisfy the DCC. Since by (81) T_H always implies T_0, the theorem is proved. Q.E.D.

It is clear from this result that T_H lies strictly between T_F and T_0 in the table of implications on p.34 of [1] (and that T_H coincides with T_0 when A is finite). Examples 4.1 and 4.4 of [1] show that T_H does not imply T_D and that T_D does not imply T_H respectively.

APPENDIX 3

SOME REMARKS ON TOPOLOGICAL MATROIDS AND B-MATROIDS IN THE ATOMIC CASE

For our atomic CBA we may take the power set $\mathcal{P}(E)$ of a set E and we speak of an operator on $\mathcal{P}(E)$ as being on E. If f is a closure operator on E, the pair (E,f) will sometimes be called a space and we say that a space (E,f) is topological or matroidal, etc., according as f possesses the property in question.

Topological Matroids

As remarked in Chapter 4, the analysis of Appendix 2 on separation axioms enables us to confine our attention to those matroidal topological closure operators which are T_1 and to analytic quantifiers; the latter can be dismissed in the atomic case - for then every quantifier is analytic and a quantifier on E simply corresponds to a partition of E. We have already encountered the simplest examples of T_1 matroidal topological closure operators: it follows from (70) and (71)(b) that a section closure operator f will be such iff f is the closure operator associated with an ideal S of $\mathcal{P}(E)$ such that $\bigcup S = E$, equivalently, such that S contains the ideal of finite subsets of E. Of course none of the resulting topological spaces is T_2, except in the case $S = \mathcal{P}(E)$ which gives the discrete topology on E. Nevertheless, T_2 matroidal topological spaces do exist in plenty, as we shall see.

To the best of my knowledge, the only instances occuring in the literature of T_1 matroidal topological spaces, other than those derived from ideals of $\mathcal{P}(E)$ as just described, are the HK spaces found by Hewitt [19] and Katětov [26]. They rely on the facts stated in (64) and (65), constructing a T_1 minimal pithy topological closure operator on E by a direct application of Zorn's Lemma, it being evident that if $\{f_i\}$ is a chain of pithy topological closure operators on E and, for each i, G_i denotes the set of f_i-open sets, then $G = \bigcup_i G_i$ is a basis of open sets for a pithy topological closure operator f on E such that $f \leq$ each f_i, f being pithy since there are no singletons in G (there are none in any of the G_i's). It is true that Hewitt also discusses the (presumably) wider class of MI-spaces - namely those pithy T_0 topological spaces in which every dense set is open (and which, by his Theorem 33 and our (49), are matroidal); but the only examples which he gives of such spaces are the HK spaces obtained by the procedure just outlined.

Another way in which T_1 HK closure operators may be arrived at is as follows. Let C be any atomless CBA and let C be represented as a subalgebra of some $\mathcal{P}(E)$ (say by means of Stone's Representation Theorem). Since C is an injective Boolean algebra it is a Boolean retract of $\mathcal{P}(E)$ and there exists an idempotent endomorphism d of $\mathcal{P}(E)$ with range C. We claim that $\{X;\ d(X) = 0\}$ is $\bigvee$-dense in $\mathcal{P}(E)$, from which it follows by (63)(b) that d is the derivative of a T_1 HK closure operator on E. Since $\mathcal{P}(E)$ is atomic, what we have to show is that $d(\{p\}) = 0$ for every p in E. Let p be in E – then for each A in C which contains p we have $d(\{p\}) \leq d(A) = A$; since $d(\{p\})$ is in C it follows that $d(\{p\}) \leq$ the meet in C of $\{A;\ p \in A \in C\} = M_p$ say. The fact that C is atomless implies that $M_p = 0$ – for if $M_p \neq 0$ then there exists N in C such that $0 \neq N < M_p$ and, replacing N by $M_p \setminus N$ if necessary, we can take N to contain p – but then $M_p \leq N$ by the definition of M_p, a contradiction. Thus $d(\{p\}) = 0$ for every p in E and our claim is substantiated.

Using either Hewitt's and Katětov's approach, or the one just described, it is easy to produce T_2 HK closure operators. In the former approach, take a minimal pithy topological closure operator $f \leq$ a given T_2 pithy topological closure operator – then f is automatically T_2. For the latter approach, note first that the clopen algebra of any T_1 HK closure operator f with derivative d on a CBA A is just $d(A)$ ($\dot{x}$ is f-clopen iff it is f-perfect by (57), and $d(x) = x$ is in $d(A)$ since $d^2 = d$). Therefore in the latter approach the clopen algebra of the HK closure operator f obtained is just C and to ensure that f is T_2 it is enough that the sets in C distinguish the points of E (as will be the case, for instance, when Stone's Theorem is used to represent C as a subalgebra of $\mathcal{P}(E)$).

We have already remarked on our inability to produce any non-discrete T_3 matroidal topological spaces – still less do we know whether there exist any T_3 HK spaces. If X was such a space then the clopen sets in X would form a basis of open sets for X, X being extremally disconnected by (57), and hence X would in fact be $T_{3\frac{1}{2}}$.

The two methods mentioned above for producing T_1 matroidal topological spaces (Hewitt's and Katětov's, and the one using the injectivity of CBA's) produce a special kind thereof (HK) but they are purely existential. A more constructive method, whereby from a given T_1 pithy topological space X one obtains a specific T_1 pithy matroidal topological space Y, is as follows. On the same underlying set as X take a new topological space X_1 having as a basis of open sets the sets of the form $U \setminus N$, where U is open in X and N is nowhere-dense in X (note that since the union of two nowhere-dense sets is again nowhere-dense, the family of sets of the stated form is closed under finite intersections). X_1, being finer than X, is still T_1 and X_1 is still pithy since none of the basic open sets $U \backslash N$ for X_1 is a singleton (a non-empty set of the form $U \backslash N$ is never nowhere-dense - otherwise $U = (U \backslash N) \cup (U \cap N)$ would itself be nowhere-dense which is impossible; and singletons in a T_1 pithy space are always nowhere-dense). Now iterate this process transfinitely many times, taking the union of the families of open sets already constructed to obtain a basis for the new open sets at limit ordinals, and desist upon reaching constancy - at $X_\alpha = Y$ say. Then, besides being T_1 and pithy, Y also has the property that every set of the form $U \backslash N$, U open and N nowhere-dense in Y, is itself open in Y - equivalently, every nowhere-dense set is closed. Now the discrete sets in a T_1 pithy topological space are nowhere-dense (if U is an open subset of the closure of some set D then $U \cap D$ is dense in U, and U is pithy - hence $U \cap D$ is pithy by (48); if D is discrete, this implies that $U \cap D = \phi$ so that $U = \phi$ and D is nowhere-dense). Thus every discrete set in our space Y is closed, and Y is matroidal by (49). And if we take X to be T_2, then Y will surely be T_2 also.

(We can modify this construction slightly by enlarging the topology on X only enough to make every discrete set closed, rather than every nowhere-dense set: in place of the sets $U \backslash N$ for the basic open sets of the new topology at each stage, use only the sets $U\setminus D$, where D is a finite union of discrete sets. There are presumably many further variations.)

(83) Let X be a T_2 matroidal topological space. Then

(a) The only countably compact subsets of X are the finite subsets of X.

(b) No non-isolated point of X has a totally ordered basis of neighbourhoods.

Proof. (a) It is well-known that every infinite T_2 topological space has an infinite discrete subset. Thus if Y is an infinite subset of X then Y contains an infinite discrete set D. D is closed by (49), and Y cannot be countably compact since an infinite discrete subset of a countably compact set is never closed.

(b) This will be proved if we show that:

If p is a non-isolated point in a T_2 space X and p has a totally ordered basis of neighbourhoods then p is a limit point of some discrete subset D of X.

Without loss of generality we can take $\{M_\alpha;\ \alpha < \lambda\}$ to be a base of neighbourhoods of p where the M_α's are open, λ is an initial ordinal, and $\alpha < \beta < \lambda$ implies $M_\beta \subset M_\alpha$. Put $\Lambda = \{\alpha; \alpha < \lambda\}$. We prove that there exist functions $f:\Lambda \to \Lambda$, $q:\Lambda \to X$, and $N:\Lambda \to \mathcal{P}(X)$ such that, for all $\alpha < \beta < \lambda$,

(i) N_α is an open set containing q_α and disjoint from $M_{f(\alpha)}$, and

(ii) $M_{f(\beta)} \cup N_\beta \subseteq M_{f(\alpha)}$.

Suppose that such functions have been defined on $\{\alpha;\alpha < \lambda\}$ for some $\gamma < \lambda$. Then $\{f(\alpha); \alpha < \gamma\}$ is not cofinal in Λ and $\delta = \bigvee\{f(\alpha); \alpha < \gamma\}$ is in Λ . Choose q_γ in $M_\delta \backslash \{p\}$ and let N_γ and $M_{f(\gamma)}$ be disjoint open neighbourhoods, both contained in M_δ, of q_γ and p respectively. The only thing to check about our prolongation of f to $\{\alpha;\alpha \leq \gamma\}$ is that $M_{f(\gamma)} \cup N_\gamma \subseteq M_{f(\alpha)}$ for $\alpha < \gamma$ and this happens since the left-hand side is contained in M_δ which in turn is contained in $M_{f(\alpha)}$. Let $D = \{q_\alpha ;\alpha < \lambda\}$. Since $N_\alpha \cap N_\beta \subseteq N_\alpha \cap M_{f(\alpha)} = \phi$ whenever $\alpha < \beta < \gamma$, D is certainly discrete. Also $\alpha < \beta < \gamma$ implies $M_{f(\beta)} \subset M_{f(\alpha)}$ and this implies $f(\alpha) < f(\beta)$; hence $\alpha \leq f(\alpha)$ for all $\alpha < \lambda$. Thus for $\alpha < \lambda$ we have $q_\alpha \in N_{\alpha+1} \subseteq M_{f(\alpha)} \subseteq M_\alpha$, from which it follows that p is a limit point of D. Q.E.D.

Part (a) of this result extends a result of Kirch [28] stating that MI-spaces satisfy (a). From part (b) we have

<u>Corollary</u>. No metrizable topological space is matroidal, unless it is discrete.

B-Matroids

(Note. Of the results mentioned here, (84) and (85) have already appeared in [20] and (86) in [23].)

(84) Let f be a closure operator on a set E and suppose that f is E_1 and satisfies: p in $f(X)$ implies p in $f(Y)$ for some finite subset Y of X (for all p in E and $X \subseteq E$). Then f is B-matroidal.

Proof. Apply Zorn's Lemma and (22) to verify the defining condition for a B-matroidal closure operator. Q.E.D.

(85) Let f be a B-matroidal closure operator on a set E. Then the coatoms of the lattice of f are $\bigwedge$-dense in this lattice.

Proof. The result will be obtained if we show, for each closed set A and each p not in A, that there exists a maximal proper closed set H which contains A but not p. Let X be a base of A. Since f is E_1, it is easily seen that $X \cup \{p\}$ is discrete and can therefore be extended to a base $X \cup \{p\} \cup Y$ of E, where $(X \cup \{p\}) \cap Y = \phi$. Let $H = f(X \cup Y)$. Then H is a closed set containing A but not p. Also $f(H \cup \{p\}) = E$ and this, together with the fact that f is E_1, shows easily that H is a maximal proper closed set. Q.E.D.

(86) Let f be a B-matroidal closure operator on a set E. Then, under the assumption of the Generalized Continuum Hypothesis (GCH), all f-bases have the same cardinality.

Proof. We prove the following result, from which the theorem follows in view of the easily verified fact that (for f B-matroidal) the set $\mathcal{B}$ of f-bases satisfies the hypotheses (i) and (ii) of this result:

Let $\mathcal{B}$ be a set of subsets of a set E satisfying

(i) no one member of $\mathcal{B}$ is properly contained in another, and

(ii) if B_1 and B_2 are in $\mathcal{B}$ and A, C are subsets of E such that $A \subseteq B_1$, $B_2 \subseteq C$, and $A \subseteq C$ then there exists B in $\mathcal{B}$ such that $A \subseteq B \subseteq C$.

Then if the GCH is true the members of $\mathcal{B}$ all have the same cardinality.

Proof. Let B_1 and B_2 be in $\mathcal{B}$. If B_1 is infinite then, using Sierpinski's construction [37] and the GCH (see also Wolk [45]), we obtain a chain $\mathcal{C}$ of subsets of B_1 such that $|\mathcal{C}| = 2^{|B_1|}$. For each C in $\mathcal{C}$, (ii) shows that there exists a subset D of B_2 such that $C \cup D$ is in $\mathcal{B}$ and $C \cap D = \phi$. If we select exactly one D for each C then by (i) the resulting D's will be distinct and B_2 must have at least $2^{|B_1|}$ subsets. From

$2^{|B_1|} \leq 2^{|B_2|}$ and the GCH we obtain $|B_1| \leq |B_2|$. If B_1 is finite then a similar (and in this case familiar) argument leads to the same conclusion: take $|\mathcal{C}| = |B_1| + 1$ and choose the D's to form a chain themselves, as is clearly possible when $\mathcal{C}$ is finite. Likewise, we may show that $|B_2| \leq |B_1|$. Q.E.D.

To conclude this appendix, we specialize (38) to the atomic case.

(87) Let f be a B-matroidal closure operator on a set E. Then E is partitioned by the f-separators E_i for which the only f^{E_i}-separators are ϕ and E_i.

Proof. We can split off $f(\phi)$ by (19)(a); furthermore, by the same result, the singletons of the elements of $f(\phi)$ are all f-separators and clearly satisfy the stated condition. It follows that we may without loss of generality take $f(\phi) = \phi$. Then f is S_1 by (27), and by (38) the set of f-separators is a complete subalgebra of $\mathcal{P}(E)$ and, as such, is itself an atomic CBA. The result follows on taking the E_i's to be the atomic f-separators (it is easy to check that if E_i is any f-separator then the f^{E_i}-separators are precisely the f-separators contained in E_i). Q.E.D.

Let us say that an operator f on a CBA A is connected iff the only f-separators are 0 and 1. Then, defining the cartesian product of an arbitrary family of operators exactly as we did for two operators in Chapter 1, and observing that the appropriate analogue of (14) still holds, we have

Corollary. A B-matroidal closure operator on a set is isomorphic to the cartesian product of connected B-matroidal closure operators on sets.

(87) and its corollary generalize to B-matroids on sets the decomposition into connected components first introduced by Whitney [43] for finite matroids and then extended to finitary matroids (as described in (84)) by Sasaki and Fujiwara [35].

APPENDIX 4

INFINITE GRAPHS AND MATROIDS

1. Introduction

Bean [1] has considered the problem of extending finite matroid theory to the nonfinitary case, and in particular the extent to which this theory may be carried over to infinite graphs when as circuits we take, not only the polygons, but also the two-way infinite arcs of the graph considered. Here we characterize those graphs whose polygons and two-way infinite arcs give rise to matroids in the sense of [3]. It will be helpful if we first describe the relevant definitions and results of [3], to which we refer for the proofs of these results.

2. Spaces and Matroids

The notion of a *space* on a set E used here is nearly the same as that of a Fréchet (V)-space, as described for example in Chapter I of Sierpinski's "General Topology" [6]. For our purposes, a space on E is most conveniently specified by means of the *derived set* operator ∂, which maps the set of subsets of E into itself and which satisfies the conditions: $\partial A \subseteq \partial B$ whenever $A \subseteq B \subseteq E$; $x \in \partial(A\backslash x)$ whenever $x \in \partial A$ and $A \subseteq E$. This concept of a space is in fact slightly more general than that of a Fréchet (V)-space, the latter being obtained when one further insists that $\partial\varnothing = \varnothing$. The class of spaces is very wide, containing for instance all topological spaces. Indeed, any set E, on which a closure operator $A \mapsto \bar{A}$ is defined, becomes a space if we put $\partial A = \{x \in E;\ x \in \overline{A\backslash x}\}$. The spaces which arise in this way are characterized by the condition $\partial(A \cup \partial A) \subseteq A \cup \partial A$ for all $A \subseteq E$, and we call a space satisfying this condition a *transitive* space. (The closure operator associated with a transitive space may be recovered from ∂ by means of the equation $\bar{A} = A \cup \partial A$.)

The examples of spaces most relevant here are obtained as follows. Let $E(G)$ denote the edge-set of a graph G and, for $A \subseteq E(G)$, let ∂A be the set of x in $E(G)$ for which there exists a polygon of G with edge-set P such that $x \in P \subseteq A \cup x$. Then $(E(G), \partial)$ is a transitive space (indeed a matroid, according to the definition below) in which the minimal dependent sets are the edge-sets of the polygons of G. Our task is to consider what happens when we admit two-way infinite arcs, as well as polygons, in the definition of ∂. We still obtain a space, but we shall see [(7) below] that this space will only be transitive when a certain type of subgraph is excluded from G.

This last circumstance suggests that there is some point in studying spaces which are not necessarily transitive, a suggestion which is reinforced by the fact that the duality originally defined by Whitney [8] for the class of finite matroids actually extends to the class of all spaces (but not to the class of transitive spaces). This fact was discovered by Sierpinski [5] (see also [6, p. 15]), who was apparently unaware of Whitney's work, or at least of its relation to his own. (Incidentally, in order to stay within the class of Fréchet (V)-spaces, Sierpinski had to impose the (slight) restriction of applying duality only to those Fréchet (V)-spaces which were "dense-in-themselves.")

The *dual* of a space (E, ∂) is defined to be the space (E, ∂^*) with $\partial^* A = E \backslash \partial(E \backslash A)$ for all $A \subseteq E$. (One readily verifies that (E, ∂^*) is a space. Note that $\partial^{**} = \partial$.) If P is any property of spaces, we say that a space (E, ∂) is *dually P* if and only if (E, ∂^*) is P. It may be verified that a space (E, ∂) is dually transitive if and only if $A \cap \partial A \subseteq \partial(A \cap \partial A)$ for all $A \subseteq E$. We define a *matroid* to be a space (E, ∂) which is both transitive and dually transitive. Certainly for the case of E finite, this notion of matroid is coextensive with the usual one.

A pleasant feature of spaces is that the relations established by Tutte [7], between the reductions and contractions of a finite matroid and its dual, carry over to spaces in general. If (E, ∂) is a space and $S \subseteq E$, define the *subspace* of (E, ∂) on S to be the space $(S, \partial \cdot S)$ with $(\partial \cdot S)A = S \cap \partial A$ for all $A \subseteq S$. Write $\partial \times S$ for $(\partial^* \cdot S)^*$ and refer to a space of the form $(T, (\partial \cdot S) \times T)$, where $T \subseteq S \subseteq E$, as a *minor* of (E, ∂). Then the results 3.331 through 3.36 of [7] remain valid for arbitrary spaces.

A subset A of a space (E, ∂) is said to be *dense* (or *spanning*) if $A \cup \partial A = E$; *discrete* (or *independent*) if $A \cap \partial A = \varnothing$; and a *base* if both dense and discrete. Then A is dense in (E, ∂) if and only if $E \backslash A$ is discrete in (E, ∂^*); and A is a base of (E, ∂) if and only if $E \backslash A$ is a base of (E, ∂^*).

We say that a space (E, ∂) is *finitely transitive* if $x \in \partial(A \cup y)$ and $y \in \partial A$ implies $x \in \partial A$ for all $A \subseteq E$ and distinct $x, y \in E \backslash A$; and we say that (E, ∂) is *exchange* if $x \in \partial(A \cup y)$ implies $x \in \partial A$ or $y \in \partial(A \cup x)$ for all $A \subseteq E$ and distinct x, $y \in E\backslash A$. A transitive space is finitely transitive, and a space is finitely transitive if and only if it is dually exchange (thus a dually transitive space is exchange). A space (E, ∂) is said to be B_1 if each discrete set in it is contained in a base of it; (E, ∂) is a *B-matroid* if it is transitive and each of its subspaces is B_1. The class of B-matroids, like the class of matroids, is closed under the taking of duals—from which it follows that B-matroids are indeed matroids. The classes of transitive spaces, matroids, finitely transitive spaces, exchange spaces, and B-matroids are each closed under the taking of minors.

Besides the two above-mentioned works of Sierpinski, we might also mention, in connection with general spaces, Hammer's papers on extended topology (see [2]) and the paper [4] of Schmidt, who discusses a number of the concepts arising in matroid theory.

3. Graph Terminology

Graphs are undirected, possibly with loops and "multiple" edges. A monovalent vertex of a graph G is called an *end* of G. Let G be a connected nonnull graph with e ends and its remaining vertices divalent. Then G is a *polygon* if it is finite and $e = 0$; G is a *two-way infinite arc* if it is infinite and $e = 0$; G is a *ray* if $e = 1$; and G is a *finite arc* if $e = 2$. A graph is *rayed*, or *rayless*, according as it contains or does not contain a ray. Let H, K, and L be subgraphs of some graph. Then H and K are *connected via* L if H and K have a vertex in common or if there is a finite arc in L with one end in H and the other in K. If a graph H is obtained from a graph G by replacing the edges of G with finite arcs then H is a *subdivision* of G. We will not distinguish between subgraphs having no isolated vertices and their corresponding edge-sets.

4. The Space $(E(G), \partial)$ Associated with a Graph G

Let G be any graph. For each $A \subseteq E(G)$, define ∂A to be the set of x in $E(G)$ such that $x \in P \subseteq A \cup x$ for some polygon or two-way infinite arc P in G. Then $(E(G), \partial)$ is a space, which we call the space *associated with* G. Note that a subset A of $E(G)$ is discrete in $(E(G), \partial)$ if and only if A contains no polygons or two-way infinite arcs.

(1) *The space $(E(H), \partial)$ associated with a subgraph H of a graph G is the same as the subspace on $E(H)$ of the space $(E(G), \partial)$ associated with G.*

This is an immediate consequence of the definitions.

(2) *Let G be a graph. Then every minor of $(E(G), \partial)$ is dually transitive (therefore also exchange) and finitely transitive.*

PROOF: It is sufficient to consider only the space $(E(G), \partial)$ itself since the properties in question are preserved under the taking of minors.

To show that $(E(G), \partial)$ is dually transitive, suppose that $x \in A \cap \partial A$; we wish to deduce that $x \in \partial(A \cap \partial A)$. Since $x \in \partial A$, there is a polygon or two-way infinite arc P such that $x \in P \subseteq A \cup x$, where $A \cup x = A$ since $x \in A$. Then clearly $P \subseteq \partial P \subseteq \partial A$ and hence $x \in P \subseteq A \cap \partial A$, so that $x \in \partial(A \cap \partial A)$ as required.

To show that $(E(G), \partial)$ is finitely transitive, we first remark that, as shown in Corollary (2-16) of [1], the polygons and two-way infinite arcs of any graph satisfy Whitney's postulate (C_2) for circuits, namely that if P_1 and P_2 are circuits and $y \in P_1 \cap P_2$, $x \in P_1 \backslash P_2$, then there is a circuit $P_3 \subseteq P_1 \cup P_2$ such that $x \in P_3$, $y \notin P_3$. Now suppose that $x \in \partial(A \cup y)$ and $y \in \partial A$, where x and y are distinct and not in A. We wish to conclude that $x \in \partial A$. Let P_1 and P_2 be polygons or two-way infinite arcs such that $x \in P_1 \subseteq A \cup y \cup x$ and $y \in P_2 \subseteq A \cup y$. Then $x \notin P_2$; also we may clearly suppose that $y \in P_1$. By (C_2), there exists P_3 such that $P_3 \subseteq P_1 \cup P_2$, $x \in P_3$, and $y \notin P_3$. It follows that $x \in P_3 \subseteq A \cup x$, and we have $x \in \partial A$ as required.

The next result extends Lemma (2.62) of [1] and is the main fact needed in the characterization of those graphs whose associated space is matroidal.

(3) *Let G be a graph and let $A \subseteq D \subseteq E(G)$ be such that, for each x in $E(G) \backslash D$, there is a polygon or two-way infinite arc P, having only finitely many vertices in common with each rayless component of A, for which $x \in P \subseteq D \cup x$. Then there exists a set B which is minimal with respect to the requirement that it be dense in $(E(G), \partial)$ and such that $A \subseteq B \subseteq D$.*

PROOF: Let $\mathscr{M}$ be maximal such that

(i) $\mathscr{M}$ is a set of rays in D containing a ray from each rayed component of A,

(ii) the only polygons in $A \cup (\bigcup \mathscr{M})$ are those already in A, and

(iii) no two rays of $\mathscr{M}$ are connected via A.

Next let B be maximal such that

(iv) $A \cup (\bigcup \mathscr{M}) \subseteq B \subseteq D$,

(v) the only polygons in B are those already in A, and

(vi) no two rays of $\mathscr{M}$ are connected via B.

[Any set $\mathscr{M}$ of rays in A, exactly one from each rayed component of A, will satisfy (i), (ii), and (iii); and if $\mathscr{M}$ satisfies (i), (ii), and (iii) then $B = A \cup (\bigcup \mathscr{M})$ will satisfy (iv), (v), and (vi). Zorn's lemma then guarantees the existence of a maximal $\mathscr{M}$ and of a maximal corresponding B.]

We have

(a) *If vertices u and v of G are connected via D then either they are connected via B or each is connected via B to a ray of $\mathcal{M}$.*

(b) *If a ray R in D has only finitely many vertices in common with each rayless component of A then R is connected via A to a ray of $\mathcal{M}$.*

PROOF OF (a): Suppose that (a) fails for some pair of vertices of G. Then there is a minimal finite arc C in D whose ends u and v fail to satisfy (a). We cannot have $C \subseteq B$; let x in $C \backslash B$ have ends u' and v', where u, u', v', v occur in that order in C. By the maximality of B, either $B \cup x$ fails to satisfy (v), or $B \cup x$ fails to satisfy (vi) [$B \cup x$ necessarily satisfies (iv)]. In the former case, u' and v' are connected via B; and in the latter, each is connected via B to a ray of $\mathcal{M}$. Also, u and u' satisfy (a) by the minimality of C, and so likewise do v' and v. It follows that u and v satisfy (a) after all, contrary to what was supposed. Hence (a) holds.

PROOF OF (b): Suppose that R is a ray for which (b) fails. Then R has only finitely many vertices in common with each component of A (indeed, none at all with the rayed components of A) and we may modify R according to the following description (in which the ordering of the vertices of R implicitly referred to is the natural one, beginning at the end). Let u_1 be the first vertex of R in A, let A_1 be the component of A containing u_1, and let v_1 be the last vertex of R in A_1. Replace the finite arc in R with ends u_1 and v_1 by a finite arc in A_1 with the same ends. Now do the same for the ray in R whose end is the vertex of R immediately following v_1, and continue in this way all along R (the process terminates if a ray in R is reached having no vertices in A). The ray R' which results in this way from R is like R in that it is in D and is not connected via A to any ray in $\mathcal{M}$—and in addition it is such that $A \cup (\bigcup \mathcal{M}) \cup R'$ contains no polygons not already in A. This however is contrary to the maximality of $\mathcal{M}$. Thus (b) holds.

We can now show that if A and D satisfy the hypotheses of (3) then B is dense in $(E(G), \partial)$. For let x be in $E(G) \backslash B$ and let u and v be the ends of x. We distinguish three cases. In the first case, u and v are connected via B. If this is so then there is a polygon Q such that $x \in Q \subseteq B \cup x$, and we have $x \in \partial B$. In the second case, u and v are not connected via B but they are connected via D. If this happens then by (a) each of u and v is connected via B to a ray of $\mathcal{M}$, so that each is the end of a ray in B (the two rays have no vertices in common since u and v are not connected via B). Hence there is a two-way infinite arc Q such that $x \in Q \subseteq B \cup x$, and we again have $x \in \partial B$. In the last case, u and v are not connected via D. Then in particular x is not in D and there must exist a two-way infinite arc P having only finitely many vertices in common with each rayless component of A such that $x \in P \subseteq D \cup x$. From (b) we see that the ray in $P\backslash x$ with end u is connected via A to a ray of $\mathcal{M}$. Since A and $P\backslash x$ are subsets of D, u is connected via D to a ray of $\mathcal{M}$. By virtue of (a), this implies that u is in fact connected via B to a ray of $\mathcal{M}$ and u is therefore the end of a ray in B. Applying the same reasoning to v, we obtain the situation already arrived at in the second case. Thus we have $x \in \partial B$ whenever $x \in E(G)\backslash B$, that is, B is dense in $(E(G), \partial)$.

It remains for us to prove that B is a *minimal* set dense in $(E(G), \partial)$ and such that $A \subseteq B \subseteq D$. This will be done if we show that

(c) *The only polygons and two-way infinite arcs in B are those already in A.*

Condition (v) states that (c) is true as far as polygons are concerned. Suppose that P is a two-way infinite arc in B but not in A. By (v), each ray in B satisfies the hypothesis of (b) above and is therefore connected via A to a ray of $\mathcal{M}$. Together with (vi), this implies that there exists a (unique) ray R of $\mathcal{M}$ to which each ray in P is connected via A. Let x be in $P\backslash A$, let P_1 and P_2 be the two rays into which $P\backslash x$ falls, and let B_1 and B_2 be the two components of $B\backslash x$ containing P_1 and P_2, respectively [B_1 and B_2 are distinct by (v)]. Since P_1 and P_2 are both connected via A to R, x must be in R. It follows that $P \subseteq A \cup R$. Now one of B_1 and B_2, say B_2, contains a ray in R. Then $P_1\backslash R$ is a ray in A and by (i), the component A_1 of A which contains $P_1\backslash R$ must also contain a ray S of $\mathcal{M}$. But then S is connected via A to R, yet, being contained in B_1, S is distinct from R—and this is contrary to (iii). This completes the proof of (3).

It is convenient to have available the following dualized form of (3).

(4) *Let G be a graph and let $X \subseteq S \subseteq E(G)$. Then X is contained in a base Y of the subspace $(S, \partial^* \cdot S)$ of $(E(G), \partial^*)$ provided that the following condition holds: for each x in X there exists a polygon or two-way infinite arc P, having only finitely many vertices in common with each rayless component of $E(G)\backslash S$, such that $X \cap P = x$.*

Proof: Assume that this condition does hold. Then $A = E(G)\backslash S$ and $D = E(G)\backslash X$ satisfy the hypotheses of (3). Hence there exists a minimal set B such that $A \subseteq B \subseteq D$ and B is dense in $(E(G), \partial)$. Then $Y = E(G)\backslash B$ is a maximal set such that $X \subseteq Y \subseteq S$ and Y is discrete in $(E(G), \partial^*)$. It follows that Y is a maximal discrete set in the subspace $(S, \partial^* \cdot S)$ of $(E(G), \partial^*)$. Since $(E(G), \partial)$ is finitely transitive by (2), $(E(G), \partial^*)$, and hence also $(S, \partial^* \cdot S)$, is exchange. Now a maximal discrete set in an exchange space is actually a base of that space. Thus Y is a base of $(S, \partial^* \cdot S)$.

(5) *Let G be a graph. Then*

(i) *every minor of* $(E(G), \partial)$ *is* B_1, *and*
(ii) *every subspace of* $(E(G), \partial)$ *is dually* B_1.

(Note that since B_1 is the property that every discrete set is contained in a base, dually B_1 is the property that every dense set contains a base.)

PROOF OF (i): By (1) we need only show that every minor of $(E(G), \partial)$ of the form $(S, \partial \times S)$ is B_1—equivalently, that every subspace $(S, \partial^* \cdot S)$ of $(E(G), \partial^*)$ is dually B_1. We make use of the following results.

(d) *A base of a dense subspace of a transitive space is a base of the whole space.*

(e) *If every subspace of a transitive space has a base then every subspace of the space is dually* B_1.

[Result (d) is a direct consequence of transitivity. The conclusion of (e) states that if D is a dense set in a subspace $(S, \partial \cdot S)$ of the given transitive space then D contains a base of $(S, \partial \cdot S)$. By the hypothesis (e), the subspace $(D, \partial \cdot D)$ has a base, B say. Also $(S, \partial \cdot S)$, being a subspace of a transitive space, is itself transitive. Hence B is a base of $(S, \partial \cdot S)$ by (d).]

Returning to the proof of (i), we see from (e) and the transitivity of $(E(G), \partial^*)$ [$(E(G), \partial)$ is dually transitive from (2)] that (i) will be obtained if we show that every subspace $(S, \partial^* \cdot S)$ of $(E(G), \partial^*)$ has a base Y. This last fact however follows immediately from (4) on taking $X = \varnothing$.

PROOF OF (ii): By (1) we need only show that $(E(G), \partial)$ is dually B_1—equivalently, that in $(E(G), \partial)$ every dense set D contains a base B. Let D be dense in $(E(G), \partial)$ and put $A = \varnothing$: then the hypotheses of (3) are fulfilled. Hence there exists a minimal set B such that $B \subseteq D$ and B is dense in $(E(G), \partial)$. Thus B is a minimal dense set in $(E(G), \partial)$. Now a minimal dense set in a finitely transitive space is in fact a base of that space. Hence B is a base of $(E(G), \partial)$, in view of (2).

Condition (i) of (5) shows that if $(E(G), \partial)$ is transitive then it is a B-matroid. On looking at (4), we see that what prevents the transitive space $(E(G), \partial^*)$ from being a B-matroid is the presence in G of some two-way infinite arc having infinitely many vertices in common with some rayless and connected subgraph of G. We now analyze this situation further.

5. THE BEAN GRAPH

This is the graph shown in Fig. 1 of [1] and in Fig. 1 here.

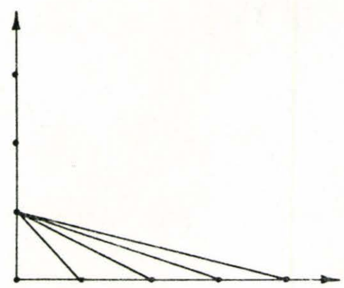

FIG. 1

(6) *If a graph G contains a two-way infinite arc P having infinitely many vertices in common with a rayless and connected subgraph C of G then G has a subdivision of the Bean graph as subgraph.*

PROOF: Replacing C if necessary by a maximal tree contained in C, we may without loss of generality take C to be a tree. We may further suppose that C is the union of finite arcs having their ends in P, for the union of the arcs in C of this type itself satisfies the hypotheses on C. Now since C is rayless, infinite, and connected, it must have an infinite-valent vertex, v say. For each edge x in C which is incident with v, let C_x be a minimal finite arc in C containing x and having v as one end and a vertex of P as the other end. Then it is easy to see that $P \cup \left(\bigcup_x C_x\right)$ contains a subdivision of the Bean graph.

The main theorem of this paper can now be proved.

(7) *Let G be a graph. Then the following conditions are equivalent:*

(i) $(E(G), \partial)$ *is a matroid;*
(ii) $(E(G), \partial)$ *is a B-matroid;*
(iii) *G has no subdivision of the Bean graph as subgraph.*

PROOF: The equivalence of (i) and (ii) is a consequence of condition (i) of (5) and the fact that any B-matroid is a matroid. To see that (i) implies (iii), suppose that G contains a subdivision H of the Bean graph and let A be the set of edges in H which are drawn in full in Fig. 2. Then the edge marked x is not in $A \cup \partial A$ but is in $\partial(A \cup \partial A)$. Thus $(E(G), \partial)$ is not transitive and is therefore not a matroid. Finally suppose that (iii) holds. We then claim that every subspace $(S, \partial^* \cdot S)$ of $(E(G), \partial^*)$ is B_1. For let X be discrete in $(S, \partial^* \cdot S)$: then X is discrete in $(E(G), \partial^*)$, and therefore $E(G) \backslash X$ is dense in $(E(G), \partial)$. Hence for each x in X there exists a polygon or two-way infinite arc P such that $x \in P \subseteq (E(G) \backslash X) \cup x$, or, what is the same thing, such that $X \cap P = x$.

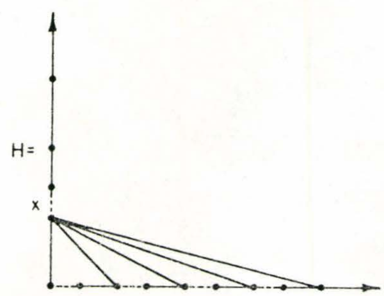

FIG. 2. The broken lines represent single edges and the full lines represent finite arcs.

From (iii) and (6), it is also the case that P has only finitely many vertices in common with each rayless component of $E(G)\backslash S$. Hence the condition of (4) holds, and X is contained in some base Y of $(S, \partial^* \cdot S)$, as required for our claim. Since $(E(G), \partial^*)$ is in any case transitive, we have shown that $(E(G), \partial^*)$ is a B-matroid. The fact that the class of B-matroids is closed under the taking of duals now gives (ii).

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