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PHENOMENOLOGICAL ASPECTS OF THE QUANTUM-MECHANICAL WORLD-VIEW

By

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SCOPE AND CONTENTS:

The contents of this paper consist of two independent but correlated topics. In Part I the history of nineteenth and twentieth century philosophy of science is traced in an effort to demonstrate the essential phenomenological aspects of the scientific methodology. Part II, on the other hand, is a technical exposition of some foundational aspects of quantum mechanics based on quantum logic. An effort is made to retain the theme that quantum mechanics is largely a phenomenological theory. As a summary, Part III constitutes an attempt to correlate the first two parts and to present tentatively some consequent reflections on the metaphysical significance of the quantum mechanical formalism.

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PHENOMENOLOGICAL ASPECTS OF THE QUANTUM-MECHANICAL WORLD-VIEW

I PHYSICS AND PHILOSOPHY

1.0 Introduction:The historical aspects of the relationship between philosophy and natural science constitute an essential chapter in the history of ideas. Out of this chapter emerge fundamental questions which as philosophers or scientists we are tempted-indeed compelled-to address. Since it is the purpose of this paper to discuss certain aspects of quantum mechanics within the context of the relationship between contemporary physics and philosophy, let us start by discussing briefly some elementary concepts which will serve to put the sequel into perspective.

To begin with, it is expedient to establish what is to be meant by philosophy. Indeed, it is imperative that we avoid labelling as *philosophical* those issues which are abstruse and speculative, while accepting as *scientific* that which is rational, systematic and logical. Indulging in this distinction can only obscure important issues by sweeping them aside as trivially inessential. Rather, let us accept as philosophical all systematic attempts to solve those problems which address themselves naturally to our intellect. Thus the mind-body problem, as well as basic epistemological issues, constitute essential aspects of philosophy within this context. Furthermore, physics, as a systematic attempt to deal with problems related to the physical world of experience, can be regarded as philosophical, at least in the sense that the philosopher is apt to embrace physics as part of his domain of enquiry.

1.1 Positivism: Adopting this point of view-or attitude-as the ground rules we may now discuss briefly the most important aspects of nineteenth and twentieth century philosophy of science. It is inevitable that any discussion of this subject will revolve very closely about *scientific positivism*, although the exact meaning of this expression may never be perfectly clarified. Largely as a result of success in mechanics, it was commonly believed, particularly by physicists, that the Newtonian world-view had provided a satisfactory, self-consistent methodology for rationalizing the physical world of experience. As it is popularly put, only the n^{th} decimal place remained to be calculated. Yet inherent in this complacency was the adoption of a particular philosophical attitude, the significance of which was well appreciated by contemporary thinkers. We find in the preface to Kirchhoff's *Mechanics* the following:

> "Mechanics is the science of motion; we define as its object the complete *description* in the simplest possible manner of such motions as

occur in nature."(1)

We find here the first expression of the popular dictum that science cannot answer the question *why* but only the question *how*, thus strongly emphasizing the descriptive rôle of a scientific theory. The distinction between *description* and *explanation* plays an important rôle in much of this paper and is connected with the issue of *ultimate reality* and whether or not physical theories tell us anything about the *essence* of this reality.

We find many of these ideas developed independently and explicitly adopted by Ernst Mach in *The Science of Mechanics*, where in addition we find the beginnings of the positivists' rally against metaphysics.

> "Regard for the true endeavor of philosophy, that of guiding into one common stream the many rills of knowledge, will not be found wanting in my work, although it takes a determined stand against the encroachments of metaphysical methods."(2)

Thus we encounter an overt awareness on the part of such men as Kirchhoff and Mach, as well as other influencial physicists such as Helmholtz, of an explicit attitude toward the meaning and significance of the mechanical methodology.

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(1) Kirchhoff, G., *Vorlesungen über mathematische Physik: Mechanik*, Leipsic, 1874.

(2) E.Mach, *The Science of Mechanics*, The Open Court Publishing Company, Chicago, 1902, preface to the first edition, p.xi.

Let us call this particular attitude *Mach-Kirchhoff positivism*.

Despite the growing tendency to call down the metaphysicians and "banish into the realm of shadows the sham ideas of the old metaphysics"(3), one encounters a growing concern for several apparently metaphysical issues such as the nature of space and time and the reality of the increasingly popular concepts of atomism. Indeed, Mach is one of the first to interpret the atomic theory as merely a model whose only rôle is to facilitate the rationalization of observed phenomena. To ascribe some objective ontological reality to the atom is, for Mach, unjustified. Similarly, absolute space and time, even causality, are branded as metaphysical if only in the sense that believing in them is tacitly a metaphysical attitude. Furthermore, this attitude is curiously related to other aspects of nineteenth century thought, particularly if we are to identify romanticism with functionalism as suggested by Mario Bunge (see Bunge [1]), thus transforming away the causal nexus in the mathematical statement of physical laws.

Turning now to the British philosophers of science, we encounter further evidence of the growing tendency toward positivism. In the work of William Kingdon Clifford, for example, can be seen a growing enthusiasm for the apparent

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(3) E. Mach, *Popular Scientific Lectures*, The Open Court Publishing Company, Chicago, 1898.

unity of science, a unity which is assumed to embrace all aspects of human knowledge and aspiration. Thus one discovers that as a social philosopher he indulges in rampant optimism for the future of man, while as a mathematician he enthusiastically assimilates the revolutionary geometries of Lobachevsky and Riemann into his philosophy of science. Indeed, his attitude toward the rôle of geometry in physics, clearly reflected by the following quote, is at once prophetic and insightful, while curiously characteristic of the contemporary style of positivism.

> "It happens that at about the beginning of this century (19^{th}) the foundations of geometry were criticized independently by two mathematicians, Lobachevsky and the immortal Gauss. And the conclusions to which these investigations led us is that, although the assumptions which were very properly made by the ancient geometers are practically exact-for such finite things as we have to deal with, and such proportions of space as we can reach-yet the truth of them for very much larger things or very much smaller things, or parts of space which are as yet beyond our reach, is a matter to be decided by experiment, when its powers are considerably increased. I want to make as clear as possible the real state of this question, because it is often supposed to be a question of words or metaphysics, whereas it is a very distinct and simple question of fact." [(4)]

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(4) W. K. Clifford, *The Common Sense of the Exact Sciences*, Alfred A. Knopf, New York, 1946, p.xxv.

Further evidence of his belief in the universality and power of scientific methodology is provided by the following statement made in an address to the British Association at Brighton in 1872.

> "Scientific thought does not mean thought about scientific subjects with long names. There are no scientific subjects. The subject of science is the human universe; that is to say, everything that is or has been, or may be related to man." (5)

Turning now to Clifford's student, the biologist-statistician Karl Pearson, we may learn more about the British style of positivism. Here we encounter an emphasis on the methodology of science, coupled with the same positivistic doctrine that all human experience constitutes the legitimate domain of science. Pearson writes in *The Grammar of Science*:

> "The unity of all science consists alone in its method, not in its material. The man who classifies facts of any kind whatever, who sees their mutual relation and describes their sequences, is applying the scientific method and is a man of science." (6)

On metaphysics, Pearson makes the following comment:

> "To say that there are certain fields - for example, metaphysics - from which science is excluded, wherein

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(5) *Ibid.*, p. xxv.

(6) K. Pearson, *The Grammar of Science*, The MacMillan Co., New York, 1911, p. 12.

> its methods have no application, is merely to say that the rules of methodological observation and the laws of logical thought do not apply to the facts, if any, which lie within such fields."[7]

Pearson's stand regarding such issues would appear to be less dogmatically anti-metaphysical. Regarding the matter of individual consciousness, for example, he suggests that it must not be taken as a fact that the other individual has an independent awareness of the universe, but rather that such a point of view be treated as an assumption, an hypothesis which simplifies one's attempt to understand the universe as he perceives it.

Returning again to the continent, we find further expression of scientific positivism in the work of Pierre Duhem, where a particularly strong phenomenalistic interpretation of theories is expressed. For Duhem, physical theory is merely a method of classifying phenomena, and in agreement with Mach, constitutes no more than an *economy of expression*. A strong Roman Catholic, he too insisted on the distinct demarcation between physics and metaphysics, thus allowing for revelation as an acceptable source of truth. In *The Aim and Structure of Physical Theory*, Duhem argues that theoretical physics is subordinate to metaphysics and that the value of a physical theory depends on the metaphysical system one adopts. In this

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(7) *Ibid.*, p. 15.

sense, he argues that theoretical constructions do not constitute explanations of underlying realities, only a reflection of phenomena. Duhem puts it this way.

> "Indeed, since antiquity there have been certain philosophers who have recognized that physical theories are by no means explanations, and that their hypotheses were not judgements about the nature of things, only premises intended to provide consequences conforming to experimental law." (8)

Finally, we should consider Henri Poincaré, whose views on the rôle of hypothesis in science have been labelled as conventionalism. His ideas embrace essentially the notion that our theoretical constructions are free creations of our mind, and as such are created so as to reflect the phenomena they are intended to correlate. He is careful, however, to avoid rampant caprice in this free creativity. In *The Foundations of Science*, he writes:

> "Here our mind can affirm, since it decrees; but let us understand that while these decrees are imposed upon our science, which, without them, would be impossible, they are not imposed upon nature." (9)

Bridgman, in *The Logic of Modern Physics*, goes even further within this context, by suggesting that the puzzling

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(8) D. Duhem, *The Aim and Structure of Physical Theory*, Princeton University Press, Princeton, New Jersey, 1954.

(9) H. Poincaré, *The Foundations of Science*, The Science Press, New York, 1929, p. 28.

correlation between mathematical constructions and natural phenomena begs a pseudo-question.

> "I am not sure that there is much meaning in this question. It is the merest truism, evident at once to unsophisticated observation, that mathematics is a human invention. Furthermore, the mathematics in which the physicist is interested was developed for the explicit purpose of *describing* the behaviour of the external world, so that it is certainly no accident that there is a correspondence between mathematics and nature." (10)

In summary, we may extract the following characteristic results regarding nineteenth century scientific positivism. Science is exhalted as a systematic methodology which promises to provide solutions to all of man's major problems. In this sense, it is seen to constitute an articulate form of *scientific humanism.* Its scope is essentially unlimited, in that it is expected to provide solutions to all our problems. Metaphysical arguments are rejected as empty and futile, even though what is meant by metaphysics is often obscure. Some writers, Duhem for example, admit metaphysical concepts through revelation, thus insisting on a clear distinction between observed truth (science) and revealed truth (religion). In any event, one finds a growing tendency to embrace a phenomenological interpretation of physical theory, thus ascribing to the

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(10) P. W. Bridgman, *The Logic of Modern Physics*, The MacMillan Co., New York, 1961, p. 60.

theorist the rôle of a free creator, whose creativity is subject to the constraints of the laws of nature. The explanatory power of our theory is questioned, even rejected, and it is held that the ultimate reality behind our observations remains mysterious and unassailed. Theoretical construction is reduced to *economy of thought*.

This growing unity in the attitude of philosophers toward science is complicated by a growing awareness of certain geometrical notions which foreshadow the relativistic and quantum revolutions to come. Witness the comments of Clifford already cited regarding the rôle of non-Euclidean geometry in the realm of the very large and the very small. Furthermore, Poincaré himself is one of the first to introduce techniques of an essentially non-quantitative nature in his solution of the stability of the n-body problem. By introducing qualitative techniques, which culminated in modern differential geometry and topology, he was able to avert the dilemma which arose upon discovery that the series expansions of Laplace diverged. The significance of topological and tensor manifold theories in the foundations of physics are only recently becoming apparent. Indeed, almost all of contemporary research in applied quantum mechanics is done using nineteenth century quantitative mathematical techniques.

Notwithstanding such minor inconsistencies, positivistic interpretations of science were widely accepted. In addition, it should be kept in mind that positivism is not restricted to the world of the natural scientist, but that *positivism* and *relativism* are key words in an analysis of any aspect of late nineteenth century thinking. Furthermore, whether contemporary science was responsible for influencing attitudes in general, or to what extent it can be understood as merely consistent with the times, is a whole new field of research with fascinating ramifications.

1.2 Logicism: Until the early part of the nineteenth century, logic was taken to mean Aristotelian logic with its traditional emphasis on the syllogism. However, as we see, nineteenth century mathematicians were to expand the power and applicability of logic considerably. In order to understand the philosophical environment in which quantum mechanics arose a century later, we must briefly review the main trends in logic during that period. It may be safely stated that early progress in logic was made possible within a climate which had finally recognized that number was not the sole object of mathematics. By this one means simply that the symbols, freely used in algebraic manipulations, need not necessarily represent numerical magnitudes. Witness the development of group theory, the elements of which may be as removed from number as geometrical operations. Early work in formal symbolic logic by George Boole can be understood as an example of this trend. Here the symbols are taken to represent classes, or equivalently, the process of selecting objects which are characterized by some *property*. Boole shows that these symbols obey a certain type of dual algebra, or in modern language, idempotent algebra (see J. Passmore [1], for a brief account).

The next stage in the development of logic, at least in the context which is relevant to this paper, is the work of Peano *et al*. In their work is found the first relatively consistent attempt to demonstrate that the foundations of arithmetic and algebra can be formulated in terms of elementary logical and set theoretic ideas, three primitive mathematical ideas (zero, number, next number) and six elementary propositions. In addition, Peano invented the logical symbolism to be adopted largely by Russell and Whitehead.

Inherent in the discussion so far is the tacit assumption that the notion of number is well understood. That such is the case was challenged by G. Frege in his work on the foundations of arithmetic. It is his contention that the subject-object antithesis with regards to number leads to difficulties not generally dealt with consistently. He argues roughly as follows. Numbers must be assigned neither objective nor subjective reality. They exist neither in space nor in the mind, but are applied freely to concepts. Here concepts must be understood not as *images in an individual mind* but rather as *objects of reason*. Frege is led in this way to an analysis of meaning in natural and symbolic languages, thus distinguishing between the sense of a sentence (its concept) and the reference of the sentence (its object). As an example, two sentences, one referring to the "morning star" and the other referring to the "evening star", convey different concepts, while retaining the same object of reference, namely Venus. The important result to be extracted here regarding Frege's work is the trend toward applying logic to the foundations of arithmetic and indeed also to natural science. Note also the trend toward what is essentially *linguistic analysis*, or in other words, an analysis of the structure of sentences intended to make reference to the external world. In the conclusions to *Foundations of Arithmetic*, Frege states:

> "I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgements and consequently *a priori*. Arithmetic thus becomes simply a development of logic, albeit a derivative one. To apply arithmetic in the physical sciences is to bring logic to bear on observed facts; calculation becomes deduction. The laws of number will not need to stand up to practical tests if they are to be applicable to the external world, for in the external world, in the whole of space and all that therein is, there are no concepts, no properties of concepts, no numbers. The laws of number, therefore, are not really applicable to external things; they are not laws of nature. They are, however, applicable to judgements holding good of things in the external world. They assert no connections between phenomena, but

connections between judgements, and among judgements are included the laws of nature." (11)

The stage is thus amply set for the work of Russell and Whitehead, which may be described for the most part as an attempt to reduce all of mathematics to logic. It is not relevant here to discuss this effort in detail. Rather, let us simply extract the growing emphasis on the rôle of logic in philosophy as a significant trend, and consider in greater detail the implications of this emphasis on Russell's later philosophical work. As already seen in the work of Frege, an emphasis on the importance of logic in the foundations of mathematics leads naturally to a study of language. This can be seen to be so when we realize that *the primitive elements of a sentential logic are sentences*, that is, natural or symbolic sentences whose function it is to make reference to some underlying concepts we wish to discuss. One is led in this way to the notion of a *proposition*. In *Our Knowledge of the External World*, Russell defines a proposition as an aggregate of symbols which can be either true or false. Now propositions may be of a strictly conceptual type, such as pure mathematical propositions. Such propositions, particularly if we are to adopt Poincaré's views regarding mathematics as free creations, may be *true or false by convention*. On the other hand, there are other

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(11) G. Frege, *Foundations of Arithmetic*, Basil Blackwell, Oxford, 1959, p. 99.

propositions which arise naturally in every day language making reference to the external world. Thus, "this object is red" is a proposition. Its truth-value is not arbitrary, however, because the redness of the object in question is an attribute to be determined by some observation process. In *Our Knowledge of the External World* Russell argues as follows. As a crude guideline, he first suggests that we distinguish between *hard* and *soft* data. He means by hard data "those which resist the solvent influence of critical reflection". Continuing, he suggests that our data are primarily the facts of sense and the laws of logic. Thus the existence of objects in the external world he correlates with our sense data of them, ascribing to their reality no more than the reality of these sensations. Thus propositions such as "there is a table before me" are taken to constitute hard data and there truth is duly taken as an hypothesis. The laws of logic obtain when we attempt to use aggregates of such propositions to deduce conclusions.

Finally, let us consider one final point of considerable importance in the next section, namely, Russell's doctrine of verifiability. To begin with, he emphasizes the psychological distinction between verifiability and truth. Now the truth-value of a proposition is independent of any empirical observation process, whereas verifiability is intrinsically *operational*. That is, it is not enough that a proposition be true in order to be verifiable, but in addition, it must be possible to discover that truth.

I conclude this section with the following quote from *Our Knowledge of the External World.*

> "Thus it is unnecessary for the enunciation of the laws of physics to assign any reality to ideal elements; it is enough to accept them as logical constructions, provided we have a way of knowing how to determine when they become actual."(12)

In summary, then, we find the following general trends in nineteenth and early twentieth century mathematics. With the realization that the elements of algebra may be very general non-numerical objects arose many new fields in pure mathematics. Of particular importance to philosophy we witness the growth of symbolic logic, and the consequent growth in the interest in linguistic analysis. This trend is seen to culminate in the work of Russell with the belief that all of mathematics can be reduced to logic, and finally, that physical theories can be constructed from propositions about the external world coupled with logical algebraic constructions on these propositions. To a certain extent, this point of view can be seen to be consistent with the contemporary positivistic attitudes described in some detail in section 1.1. Now one is led to the following question. If mathematical theories are the free creation of the theorist, as Poincaré believed, and furthermore, if mathematics is

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(12) B. Russell, *Our Knowledge of the External World*, George Allen and Unwin Ltd., London, 1969, p.17.

reducible to logic, must we conclude that the laws of logic are conventions? To a certain extent, I am prepared to answer this question in the affirmative, a point of view which I believe will be substantiated when we attempt to construct the underlying logic of quantum mechanics.

1.3 Logical Positivism: Many of the important aspects of early twentieth century philosophy of science have already appeared in the foregoing discussion of positivism and logicism. The conjunction of these two trends, *logical positivism* can thus be dealt with in a fairly straightforward manner. This section will therefore be fairly brief.

The school of philosophers, commonly known as the *Vienna Circle*, which systematically developed the following ideas was initiated by Moritz Schlick in the early 1920's. A. J. Ayer lists the following people as the principal members of the circle: Rudolf Carnap, Otto Neurath, Herbert Feigl, Friedrich Waismann, Edgar Zilsel, Philipp Frank, and Victor Kraft. Of a more mathematical or scientific bent he lists Philipp Frank, Kurt Gödel and Hans Hahn. An extensive bibliography of the logical positivists is provided in *Logical Positivism*, edited by A. J. Ayer (Ayer [1]).

Logical positivism (also called logical empiricism) embraces much of the nineteenth century positivism conjoined with an emphasis on logicism and linguistic analysis. Let us summarize their ideas by constructing a list of the major points.

(1) An emphasis on the importance of symbolic logic, influenced strongly by Russell. This manifests itself largely in an emphasis on linguistic analysis (see Carnap [1]), which gives rise to their concept of meaningfulness of a sentence, as:

(2) A sentence is said to be meaningful if its content can be verified by some empirical procedure. This leads to their:

(3) Refutation of metaphysics on the basis that sentences whose content is essentially metaphysical, or cannot be verified empirically, are meaningless according to (2).

Although many of the ideas discussed so far in this paper can be seen to have coalesced into the three points above, it is expedient to discuss to what extent it is not just a simple conjunction of otherwise unrelated ideas. The rejection of metaphysics, for example, is certainly consistent with the ideas expressed by Kirchhoff and Mach; however, one should realize, I think, that the metaphysics being rejected has been clarified to a limited extent. Whereas that which is metaphysical to Mach is obscure, the logical positivists have a least established a weak criterion for labelling metaphysical notions as those expressed by sentences which are not empirically verifiable. Indeed, logical positivism is seen to constitute a very strong form of empiricism, culminating in a fairly clear criterion for determining the meaningfulness of a statement.

Finally, the rôle of logic in this system is not arbitrary. It arises quite naturally in the foregoing discussion since the function of both natural and purely symbolic sentences, the elements of the logic, lies at the heart of logical positivism.

Bertrand Russell has criticized logical positivism on the following basis. Strictly invoking the verificationist criterion of meaning leads quite naturally to the following conclusions:

(a) That which cannot be verified or falsified is meaningless.

(b) That two propositions verified by the same occurences have the same meaning.

Russell's criticism is based on the following definition of verification. "A proposition asserting a finite number of future occurences is verified when all these occurences have taken place, and are, at some moment, perceived or remembered by some one person."(13)

If this is what we are to mean by "verified", then according to the logical positivists, statements or propositions about infinite sequences of phenomena, or about phenomena in the distant past, are branded as meaningless. Now the logical positivist would argue that it is not the

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(13) B. Russel, *Logic and Knowledge*, (edited by R. Marsh), George Allen and Unwin Ltd., London, 1968, p.375.

state of being verified which we wish to impose on propositions, but rather their potential for becoming verified (viz. verifiability), thus semantically transforming away Russell's objection. This, I would argue is operationally untenable, a point to be expanded later when we attempt to construct the concept of state in quantum mechanics.

Finally, a comment should be made regarding the methodology of the logical positivists. It is probably here where the true value of their work lies. Although it can be argued that *rigorous* philosophy is not novel to twentieth century thinkers, it is probably true that the advent of the technical devices used by the logical positivists marks an increase in the power of philosophical reasoning. Russell makes the same point in the following way:

> "I value their rigour, exactness, and attention to details, and speaking broadly, I am more hopeful of results by methods such as theirs than by any that philosophers have used in the past. What can be ascertained can be ascertained by methods such as theirs; what cannot be ascertained by such methods we must be content not to know."[14]

Wittgenstein is well remembered for putting this point more poetically as:

> "What can be said at all can be said clearly; and whereof one cannot speak, thereof must one remain

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(14) *Ibid.*, p.381.

silent."[15]

1.4 Paradigms and the Advancement of Science:Scientific optimism can be said to have reached its zenith at the end of the nineteenth century. It is interesting to note that even the realization that such ideas as non-Euclidean geometry were surely going to play increasingly important rôles did not deter Clifford *et al.* from believing in the ultimate power of the scientific methodology. Witness the quote given on page 5,-"is a matter to be decided by experiment when its powers are considerably increased"-in which the eventual realization of this requisite "increased power" is a tacit assumption. The decline of this optimism can be correlated with many aspects of historical disillusionment such as recognition of negative aspects of industrialization and the experience of two world wars. Although contemporary anti-scientific trends can be accurately labelled as sociological in origin, it should be remembered that even at its more abstract level, the methods of science are no longer judged as all-powerful. With the advent of the quantum experience, the rôle of scientific physical theory as a descriptive mechanism became increasingly apparent. Extrapolation of classical judgements became obviously suspect, as early twentieth century physicists struggled to construct new theories of the atom which would rationalize or correlate

(15) L. Wittgenstein, *Tractatus*.

the new data of experience. Thus the explanatory powers of theories were seen to be subordinated to increasingly *ad hoc* attempts to interpret new theoretical results. Epistemological and ontological questions persistently arose as the early quantum physicists continued their work. Does the electron really have some absolute ontological significance as an entity that exists in time and space? Does the measurement interaction introduce essential epistemological problems that deny once and for all a *complete* mathematical description of physical reality? In short, is quantum mechanics a complete theory in the sense discussed by Einstein (see Einstein-Podolsky-Rosen [1])?

It is issues such as these, I believe, which can be put into some perspective if discussed within the context of post-positivist philosophy of science. Let us begin, then, with a discussion of the work of Thomas Kuhn. To a certain extent this work can be understood in terms of issues quite independent from those discussed so far. The emphasis is on historical aspects of science, or to borrow a word from science itself, the dynamics of science. Thus we are concerned here not with the structure of scientific theories, but rather, the question of how science advances. Now a nineteenth century scientific positivist would have used the word "progress" rather than "advance". The distinction is academic, but I prefer the latter on the basis that it recognizes implicitly that progress towards a scientific utopia may very easily be a pipe-dream. Rather, let us say, as Kuhn would, that science advances within the context of a particular *paradigm*. By this we mean the following. In order to perform scientific activities-Kuhn would call this *normal science*-we must first adopt a set of rules which constitute the methodology of our science. In addition, we embrace certain hypotheses which, as scientists, we mutually adopt as part of our paradigm. With this equipment at our disposal we proceed to apply the methodology to the hypotheses and investigate the consequences by experimentation, say. It should be remembered, however, that experimentation constitutes an integral part of our paradigm. That is, we may be strict empiricists, and it follows that our consequent emphasis on the importance of observation constitutes part of our paradigm.

I would like to add to this notion the following point which I do not think is found explicitly in Kuhn's writings. I would say that a very important aspect of our paradigm is the *model* which we choose to invoke. By a model I take to mean any mental construction that helps us to visualize the physical problem at hand. Thus, most physicists today are apt to adopt as part of their paradigm the image of an electron as something "fuzzy", "smeared out over all of space". Regardless of the significance we attribute to this picture, it is there as part of our paradigm and as such it biases the research done. This notion of a model

I should like to emphasize at this point since later in the paper I wish to discuss the distinction between models and interpretations.

Finally, with regard to Kuhn, the rôle of revolutions in science should be discussed. The place of revolution arises naturally from the foregoing discussion. Scientific revolutions occur when our experience with the external world, our data, is no longer consistent with the paradigm we have invoked no matter how hard we try to accommodate it. In this way, the necessity of altering our paradigms is forced upon us by experience. The revolution is-for the most part-complete when our new paradigms are sufficiently formulated to justify resuming normal scientific activities to investigate their ramifications. In short, revolution is generated by crisis, an occasion we are forced to meet by changing our paradigms. In *The Structure of Scientific Revolutions* Kuhn makes the following appraisal of the rôle of crisis:

> "With the development, singly or together, of these extraordinary procedures, one other thing may occur. By concentrating scientific procedure on one area of trouble and by preparing the scientific mind to recognize scientific anomolies for what they are, crisis often proliferates new discoveries."(16)

One may fairly ask at this point the following question. To what extent do these ideas constitute a less optimistic attitude toward the powers of science, as suggested

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(16) T. Kuhn, *The Structure of Scientific Revolutions*, The University of Chicago Press, Chicago, 1971, p.88.

earlier? I think the answer is as follows. Whereas nineteenth century scientists had unlimited faith in their methodology, we have learned now to always do our research under a constantly critical eye. That is, we have learned to remain aware of the probable limitations of our paradigms. Thus we shrink from the beguiling temptation of rampant optimism, lest future experience shatter our expectations. In a sense we are heeding in a more formal way the spirit already advocated by the positivists themselves when Pearson posits as the theme of his book *The Grammar of Science*, <<*La critique est la vie de la science.*>>.[17]

1.5 Verification and Refutation: There is one final topic to be discussed which contributes to the general philosophical environment essential to the discussion of quantum mechanics which follows in Part II. This is the work of Karl Popper, which we shall now consider. We shall find in his work a contrast to both logical positivism and to the ideas of Kuhn just discussed. That is:

(a) As a post-positivist philosophical system, Popper's ideas emphasize the importance of refutation rather than verification, and

(b) As a rival to Kuhn's theory, Popper strongly deemphasizes the rôle of commitment in the advancement of science.

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(17) K. Pearson, *Op. Cit.*, quoted on the title page from Cousin.

Briefly, his ideas regarding refutation are as follow. In a spirit not unlike Poincaré he begins by asserting that scientific theories are highly conjectural. Thus a theory is born from the bold speculation of the theorist; it is not manifest nor revealed in any way. Furthermore, it is not in any way necessary in the sense that its language is arbitrary and its interpretation subject to the philosophical bias of its user. So far, so good! Nothing as yet is inconsistent with the logical positivists. Popper's ideas diverge, however, on the issue of verification. To him it is näive to place such emphasis on the rôle of verification, this criticism arising in a spirit not unlike Russell. Verification by a finite number of observations is not sufficient reason to extrapolate one's faith in the conjectured theory. Rather, one should be aggressive in his attempt to jeopardize the theory by conducting experiments which maximally test its weakest points. Now, in addition to being a fundamental revision of the verification principle, this new point of view becomes simultaneously a theory of scientific advancement. In *Conjectures and Refutations* Popper writes:

> "The way which knowledge progresses, and especially scientific knowledge, is by unjustified anticipations, by guesses, by tentative solutions, by conjectures. These conjectures are controlled by criticism; that is, by attempted refutations."[18]

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(18) K. Popper, *Conjectures and Refutations*, Harper Torchbooks, New York, 1968, p.vii.

In short, Popper's ideas constitute a formal expression of the premise that we learn by our mistakes, expanded into a systematic theory of the growth of knowledge.

It is probably clear in what sense this theory is a rival to Kuhn's paradigms. Although it is not entirely contradistinctive, it does provide the following characteristically different aspects. Whereas Kuhn's theory emphasises the rôle of crisis and adopts a certain form of scientific complacency while "normal" science is in progress, Popper is apt to reject this complacency as dangerous, holding that science advances under the continual eye of active criticism. Growth, for Popper, is a continuous process, in the sense that *commitment* impedes scientific progress. Furthermore, he would say that revolution is continual. For Kuhn, on the other hand, it is only after the transition from criticism to commitment has occured that progress can be realized. Finally, it should be noted that for Popper revolution is rational, whereas Kuhn would argue that crisis and revolution are for the most part quite irrational. That is, there is usually no rational reason for imposing the new ideas onto our paradigms, the rationale arising in retrospect.

Concluding, Imre Lakotos describes Popper's system in the following way:

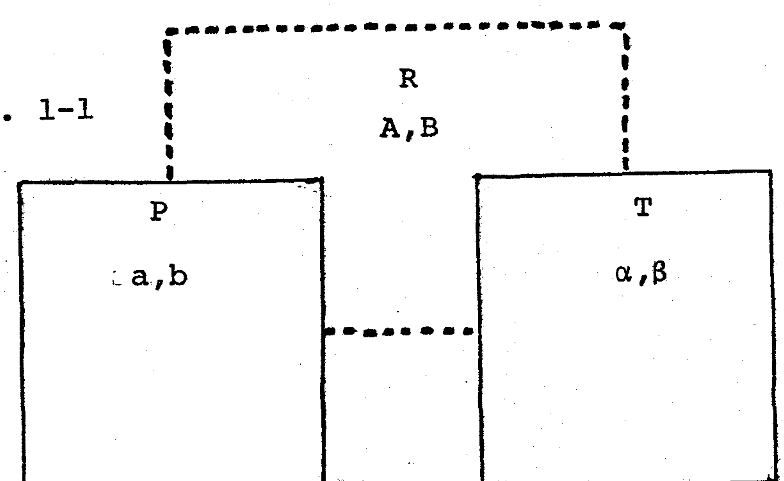
"*Belief* may be a regrettably unavoidable biological

> weakness to be kept under the control of criticism, but *commitment* is for Popper an outright crime."(19)

It is hoped that the issues brought out by the foregoing discussion serve to make it clear that in addition to structural, operational, and interpretative issues which constitute the methodology of science, there exist also certain relevant questions regarding the historical growth of science which are not completely understood.

Finally, I extract the following scheme from Popper's book *Conjectures and Refutations* which I propose to use widely in the subsequent discussions. In Fig. 1 is illustrated schematically the relationship between the world of phenomena, the world of theoretical constructions, and the world of reality which may or may not exist in some ontological sense.

Fig. 1-1



(19) I. Lakatos, A. Musgrave, *Criticism and the Growth of Knowledge*, Cambridge University Press, London, 1970, p.92.

We take P to be the set of all empirically observable phenomena. Thus $a,b\varepsilon P$ are individual phenomena which, as observers, we have detected. In a similar sense, T is to be taken as a set of symbols, and as such constitutes a theoretical representation of the observable phenomena. Taking R as the set of ontologically real objects we may make the following definitions:

Essentialism:That world-view which embraces a belief in an underlying reality. Thus the essentialist believes that phenomena reflect, in some way, elements of an ontologically meaningful physical reality.

Instrumentalism:That world-view which rejects reality as meaningless and abstruse. Thus the instrumentalist rejects ultimate explanation, accepting only phenomena as his objects of enquiry. Now one may be an instrumentalist while still retaining a belief that physical reality is in some sense meaningful, in that he contends that science cannot discover truths about this essential universe, or reflect in any way its true nature.

Popper proposes a third point of view which emphasizes the conjectural nature of a theory (see Popper [1]). I do not discuss this here, but proceed to use his scheme (we may think of it as a philosophical model) to clarify the framework in which this paper is to be understood.

Let us begin by considering the concept *electron*. We ask the following question. Is the concept "electron" to be correlated with an element of physical reality which we shall call *an electron*? This issue is discussed by Dancoff:

> "One of the things we must be most careful about is to refrain from attaching absolute reality to concepts like electrons, mass, energy, *etc.* They are useful, they are expedient, but they are also expendable."(20)

We are tempted, then, to adopt the following attitude toward things like electrons. Electrons are conceptual, theoretical constructions and as such constitute elements of our theory. If the theory which we propose (Popper would say conjecture) is successful in correlating with the world of phenomena (world of experience), then its expediency is strengthened. That is, our *electron-dependent theory* is made in some sense credible. Dirac can be seen to adopt this point of view:

> "The main object of physical science is not the provision of pictures (in the classical sense), but is the formulation of laws governing phenomena and the application of these laws to the prediction of new phenomena."(21)

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(20) Dancoff, "Does the Neutrino Really Exist?", *Bulletin of the Atomic Scientists*, 8, 139 (1952).

(21) P. A. M. Dirac, *Quantum Mechanics*, Clarendon Press, Oxford, 1958, p.10.

II QUANTUM LOGIC

2.0 Introduction:For the most part, this section may be considered during the first reading to be independent of Part I. This assertion should be taken in a technical sense since I wish to argue extensively in Part III that the following results of quantum mechanics are consistent with many of the philosophical contentions expressed in Part I. It will be found, therefore, that the contents of Part II are free from such philosophical speculations. They are written in a strictly technical style, a style not to be construed by the reader as evidence of philosophical apathy on the part of the author.

2.1 Classical Mechanics:It is common practice in many elementary texts on quantum mechanics to begin with a review of classical mechanics, an emphasis being made on the Hamiltonian formalism. By doing so, the author is then able to develop quantum mechanics by analogy, using essentially the formal arguments used originally by Heisenberg (see Heisenberg [1]). Now it is expedient to indulge in a similar approach in this paper, although we are not here interested in the Hamiltonian formalism. Indeed, since the main emphasis of this paper is on notions which are largely (but not entirely) free of dynamical aspects of quantum mechanics, the introductory review will dwell similarly on essentially, but not exclusively, kinematical matters.

It is well known that the main characteristic feature of classical mechanics is the existence of a *phase-space*. Let us begin by constructing it. First we may define a set S called the set of states. The concept of *state* is defined in such a way that the state of the system in question at time $t>0$ is uniquely determined by the appropriate physical law (the dynamics) and the state at $t=0$. To obtain more precisely what is to be meant by a state $s\varepsilon S$, we associate with each classical system an integer n called the number of *degrees of freedom*, an n-dimensional Euclidean space R^n of n-tuples and an open set M of R^n. The points of M represent the possible configurations of the system. Note that $n=3N$, where N is the number of mass-points or bodies, say, in an N-body unconstrained problem. The open sets $m\subset M$ are determined by certain logical constraints such as the impossibility of having two or more bodies at the same point in space. We see, furthermore, that the power-set of M, $\mathcal{M}=\mathcal{P}(M)$, constitutes a topology on R^n, which we may denote

$$(R^n,\mathcal{M}). \qquad (1.1)$$

We are now in a position to define the state of the system. The set S of states is the 2n-dimensional Euclidean space whose elements are the 2n-tuples $(x_1,\cdots x_n, p_1,\cdots,p_n)$, where $\underline{x}$ is the position vector and $\underline{p}$ is the momentum vector. Note that $\underline{x} \varepsilon M$. The states of the system are thus represented by points of the open set $S=M\times R^n$, a subset of R^{2n}. We may now say that the states induce a topology on R^{2n}, denoted

$$(R^{2n}, S). \tag{1.2}$$

For a detailed discussion of the importance of the topological structure mentioned above, see Mackey [1] and Abraham-Marsden [1].

We may now define classical phase-space as:

D1 Definition 1:Classical phase-space is the set of open subsets $S=M\times R^n$ of R^{2n}, otherwise known as the set of states.

Let us now consider some elementary aspects of classical dynamics. We have already said that the notion of state must be constructed in such a way that the appropriate physical law, in conjunction with the state at time t=0, leads to a knowledge of the state at time t>0. For each $s\varepsilon S$ and $t>0$ let $U_t(s)$ denote the state at time t whenever the state at t=0 was s. Now,

$$U_{t_1}(U_{t_2}(s)) \equiv U_{t_1} \circ U_{t_2}(s) = U_{t_1+t_2}(s) \tag{1.3}$$

and it follows that the set of all U_t is a *semi-group*. Indeed, since it is parametrized by the real line (ie. $t\varepsilon R^1$), our group, the *dynamical group*, is a *one-parameter semi-group*. If we now write

$$U_{-t}=U_t^{-1}, \text{ and } U_0=I, \tag{1.4}$$

then we have a *one-parameter group*.

We may now define an *orbit* as the set of all points

$U_t(s)$ for fixed s. It may turn out that the set S has sufficient structure that we may define unambiguously the *tangent vectors* at each point in the orbit. This *vector field* is called the *infinitesimal generator*. In practice, we do not usually know the dynamical group *a priori*, but must construct it from our knowledge of the infinitesimal generator. For a detailed discussion of this construction see Gudder's article in *Probabilistic Methods in Applied Mathematics* (see Bharucha-Reid [1]).

Next we consider the notion of a *dynamical-variable* or *observable*, reserving in future the term observable for the quantum-mechanical context.

D2 Definition 2: A dynamical-variable is a function defined on the phase-space S which maps states into Borel sets on the real line.

The exact meaning of this definition will be clarified later. For now, we may view dynamical-variables as quantities which can be measured, the magnitude of this measure lying in some Borel set. As an example, consider the components of the position and momentum vectors. These are dynamical variables defined as

$$X_i(\underline{x},\ \underline{p}) \equiv x_i, \text{ and} \qquad\qquad (1.5)$$
$$P_i(\underline{x},\ \underline{p}) \equiv p_i.$$

Similarly, the energy is a dynamical-variable, defined as

$$E = E(\underline{x},\ \underline{p}) \equiv \underline{p}^2/2m + V(\underline{x}) \qquad\qquad (1.6)$$

for a single particle of mass m in a conservative field represented by the potential V. Note that the rôle of the notion of mass arises quite naturally when constructing the dynamical group from the infinitesimal generator (see Gudder, *Loc. Cit.*).

A dynamical-variable is said to be an *integral* of the motion if it is differentiable and constant on orbits of the dynamical group. For example, the energy is an integral, as is well known.

2.2 The von Neumann Axioms for Quantum Mechanics: In the usual axiomatic formulation of quantum mechanics, stated originally by von Neumann (see von Neumann [1]), an emphasis is put on the existence of two primitive concepts, *state* and *observable*. In the opening remarks of a recent article by L. E. Ballentine, for example, (see Ballentine [1]) the following point of view is adopted. In any theoretical construction we may identify essentially two aspects:

(1) A mathematical formalism consisting of a set of primitive concepts (state and observable for quantum mechanics), relations between these concepts, and a dynamical law.

(2) Rules establishing a correspondence between the theoretical constructions of (1) and the world of experience.

Ballentine goes on in his paper to present the axioms of quantum mechanics in a way which invokes the density matrix formalism immediately. Let us rather proceed to posit the axioms in the following way, based on the discussion by Paul Roman in *Advanced Quantum Theory* (see Roman [1]). I quote the postulates directly.

P1 Postulate 1:All observable physical quantities correspond to Hermitian operators. The only measurable values of a physical observable are the various eigenvalues of the corresponding operator.

There would appear to be very little physical motivation to this postulate. It is not at all clear why the properties which we intuitively attribute to "observables" should appear in Hermitian operators. One can easily prove that in order for the eigenvalues to be real - an essential property for them to have if they are to represent physically measurable magnitudes -they must be Hermitian. Yet the basic premiss of the proposition-that observables are Hermitian operators-is *ad hoc* and aesthetically unacceptable, except from the purely pragmatic point of view that it works. It is true, of course, that the eigen-spectrum of arbitrary operators is not necessarily continuous, and therefore P1 contains already the essential quantum-mechanical feature of *levels*. What is lacking is a good reason for believing that these levels will correspond to those experienced in the laboratory. Of course, we may argue that such *a priori* expectation is not necessary, that only the test of experience really matters in the final analysis; however, it can now be shown that the rôle of Hermitian operators follows quite naturally from a slightly different point of view regarding the axioms, namely, that approach which treats as its primitive concepts the fundamental *propositions* which can be made about the system under observation.

P2a Postulate 2a: Any classical physical quantity must be considered to be constructed from pairs of canonically conjugate dynamical-variables. The corresponding quantum-mechanical operator is then obtained by replacing the classical canonical dynamical-variables by their corresponding quantum-mechanical operators.[22]

This postulate refers to the familiar Heisenberg approach to constructing quantum mechanics. It depends, of course, on having learned by some technique how to represent the two canonically conjugate dynamical-variables. Having done so, we construct new quantum-mechanical observables by substituting these operators in the corresponding classical function. We have seen, for example, in section 2.1 that energy is a function of the two canonically conjugate dynamical-variables $\underline{x}$ and $\underline{p}$. Thus, given that we have learned how to represent position and momentum operators quantum-mechanically,

viz.

$$x_i \rightarrow x_i \tag{2.1}$$

$$p_i \rightarrow (\hbar/i)\,\partial/\partial x_i \tag{2.2}$$

---

(22) I have substituted "dynamical-variable" where Roman uses simply "variable" to be consistent with the convention established in section 2.1 for classical observables.

then it follows immediately from P2a that the corresponding energy operator is

$$E(\underline{x},\underline{p}) = \frac{\underline{p}^2}{2m} + V(\underline{x}) \rightarrow \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_i \partial x_i} + V(\underline{x}) \quad . \tag{2.3}$$

Thus P2a tells us how to construct observables in quantum mechanics and as such is not usually considered as one of the postulates.

P2b Postulate 2b:For all pairs of the basically canonically conjugate operators the following Heisenberg commutation rules hold:

$$[q_i,q_k] = 0 \tag{2.4a}$$

$$[p_i,p_k] = 0 \tag{2.4b}$$

$$[p_i,q_k] = \frac{\hbar}{i}\,\delta_{ik} \quad . \tag{2.4c}$$

This too is not usually stated as a postulate. It is really a statement of how the two canonically conjugate operators are to be defined. That is, von Neumann has shown that operators $\underline{q}$ and $\underline{p}$ satisfying (2.4) are completely determined up to unitary equivalence. Furthermore, he has shown that any operators which commute with all the q_i and all the p_i are necessarily multiples of the identity operator and that all other operators can be constructed as functions of $\underline{q}$ and $\underline{p}$ (see von Neumann [2]).

P3a Postulate 3a:The state of a physical system is exhaustively characterized by a vector of Hilbert space upon which the operators corresponding to observables act.

In P3a we find the second primitive concept, *state*, introduced. Like P1, this point is *ad hoc*, its justification being retrospective; that is, we find that the algebra of self-adjoint operators on a Hilbert space constitutes a good theoretical representation of certain quantum-mechanical phenomena in the sense that is well known. Just as the notion of a quantum-mechanical observable can be criticized as unintuitive - in the sense that it does not immediately seem to have the properties of what we have called dynamical-variables classically - the notion of state posited here seems at first much different than a classical state. Now one may argue that we do not expect the analogies to carry over well. My point is that we can, however, arrive once again at a rationalization of the quantum-mechanical notion of state if we adopt the point of view already mentioned that the primitive elements of the theory are propositions. This point is somewhat premature, but in the sequel we shall construct the axioms out of these propositions in such a way that the rôle of states and observables is very similar to that played in classical mechanics. In this way quantum-mechanical features of the world are reflected through what turns out to be a generalized probability theory rather than through *ad hoc* postulates. Furthermore, the significance of Hilbert space in quantum mechanics is also clarified substantially through this alternative axiomatization.

P3b Postulate 3b: Actual measurement of a physical observable carries over the state-vector of the system into an eigenvector of the observed quantity, namely, into the eigenvector belonging to the observed eigenvalue of the measured observable.

This is essentially a statement of the *projection of the state vector*, and once again, it is not usually stated as a postulate in the axiomatization. It is, in fact, an independent axiom, introduced to "explain" certain aspects of *state-preparation* observed quantum-mechanically. The implications of this postulate and the relation of it to "Wigner's Friend" (23) is discussed by Gottfried in *Quantum Mechanics*, (see Gottfried [1], p. 188).

P4a Postulate 4a: If, at the instant of the measurement, the state-vector of the system is one of the eigenvectors of the measured observable, then the result of the measurement will certainly be the corresponding eigenvalue.

This follows more or less from P3b. Essentially, it guarantees that the system will not go suddenly from one eigenvector of the measured observable to another capriciously. It too is not a standard postulate. Finally, we have:

P4b Postulate 4b: In the general case, the measurement of a physical observable does not lead with certainty to a definite value. Any one of the possible eigenvalues may be obtained, but with different probabilities. The average value (expectation value) of the result of the measurement of Ω is given by the expression

---

(23) The issue of "conscious systems", discussed by Wigner, is a popular topic in many discussions on quantum mechanics and is related to the question of whether or not quantum mechanics is complete.

$$\langle\Omega\rangle = \frac{(\Phi,\Omega\Phi)}{(\Phi,\Phi)} = \frac{\langle\Phi|\Omega|\Phi\rangle}{(\Phi,\Phi)} \tag{2.5}$$

where Φ denotes the state-vector of the system just before the measurement.

This postulate is the link between theory and observation, and in the orthodox formulation is taken to characterize the *intrinsically statistical nature* of quantum mechanics. It is written in the standard notation which I do not elaborate on. Later, we shall have occasion to write the same expression in *measure-theoretic notation* which will be defined in the next section.

The foregoing discussion is intended to provide a short review of the usual axiomatic formulation of quantum mechanics in connection with some preliminary comments on the subsequent intention of the present paper. The axioms as stated are redundant and have been consequently labelled provisionally as postulates. Let us in summary extract the essential axioms as follows:

A1 <u>Axiom 1</u>:The pure states of a quantum-mechanical system are vectors in a Hilbert space $\mathcal{H}$.

A2 <u>Axiom 2</u>:The quantum-mechanical observables are represented by self-adjoint operators on $\mathcal{H}$.

A3 <u>Axiom 3</u>:The probability that an observable A has a value in a Borel set E when the system is in the state ϕ is $\langle\phi, P^A(E)\phi\rangle$ where $P^A(\ )$ is the resolution of identity for A.

I have changed the notation here somewhat prematurely, delaying its clarification until section 2.3 which we begin immediately.

2.3 Mathematical Interlude: The construction of quantum mechanics to be developed in this paper falls heavily on many mathematical subjects not normally dealt with in the standard physics curriculum. It is therefore necessary to discuss these topics briefly before proceding. The discussion will be brief since the concepts are relatively simple once basic definitions have been established.

(i) Set Theory-Borel Sets:

For our purposes no logical paradoxes will arise if we define *set* intuitively as a *collection of objects*. If every element of a set A is an element of a second set B, then we say that B *contains* A or B *is greater than* A and write $B \supset A$.

If $A \supset B$ and $B \supset A$ then we say the sets are *equal* and write $A = B$.

We define as the *union* of A and B, written $A \cup B$, the set of all objects which are either elements of A or elements of B.

We define as the *intersection* of A and B, written $A \cap B$, the set of all objects which are elements of both A and B.

We may, for our purposes, construct subsets of the sets S from properties. Thus $A = \{x:\pi(x)\}$ is a subset. If there is no $x \varepsilon S$ such that $\pi(x)$ is true, then $A = \phi$ = the empty set. We see, then, that $\phi' = S$, where the prime is taken to mean $A' = S - A$ or all those elements in S which are not in A.

We may now consider collections of subsets and speak of *classes* of sets. For example, we have already made reference in this paper to the class of *all* subsets of some set S, say, called the power-set and written $P(S)$. (In modern language we say that P is a *functor* (see MacLane-Birkhoff [1]).)

Now certain classes have interesting algebraic structure, an important example being *rings* of sets, or *Boolean rings*. We start by recalling the general definition of a ring in elementary linear algebra.

D3 <u>Definition 3</u>:A set A is called a ring if it is an additive group (w.r.t. +) and if, in addition, there exists a mapping $A \times A \to A$, $(x,y) \to x \cdot y$, satisfying

1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (3.1a)

2) $z \cdot (x+y) = z \cdot x + z \cdot y$. (3.1b)

Consider now the class R of subsets of S and take $+ \equiv \Delta$ (24)

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(24) The *difference* of two sets, A-B, is defined as $A \cap B'$. The *symmetric difference* of A and B is defined as $(A-B) \cup (B-A)$.

viz. $A+B \equiv A \Delta B.$ (3.2)

Then R forms an additive group under Δ. If we now define $\cdot \equiv \bigcap$

viz. $A \cdot B \equiv A \bigcap B$ (3.3)

then the class R forms a ring. To show this we consider the two required properties.

1) $(A \bigcap B) \bigcap C = A \bigcap (B \bigcap C) = A \bigcap B \bigcap C$ is an elementary set-theoretic result.

2) $A \bigcap (B \Delta C) = A \bigcap [(B-C) \bigcup (C-B)]$

$$= A \bigcap [(B \bigcap C') \bigcup (C \bigcap B')]$$
$$= (A \bigcap B \bigcap C') \bigcup (A \bigcap C \bigcap B')$$
$$= [B \bigcap A \bigcap (A' \bigcup C')] \bigcup [C \bigcap A \bigcap (A' \bigcup B')]$$
$$= [(B \bigcap A) \bigcap (A \bigcap C)'] \bigcup [(C \bigcap A) \bigcap (A \bigcap B)']$$
$$= (B \bigcap A) \Delta (C \bigcap A). \quad \text{Q.E.D.}$$

Note that we have in fact got a ring with identity, I, since $A \bigcap S = A$ for all A. Thus $I = S$. Furthermore, $A \bigcap A = A$; thus we have an idempotent ring.

The additive identity, it should be noted, is ϕ since $A \Delta \phi = (A - \phi) \bigcup (\phi - A) = (A \bigcap S) \bigcup \phi = A.$

It can be shown that a sufficient way of establishing that a class R is a ring is to show that for all $A, B \varepsilon R$, $A \bigcup B \varepsilon R$ and $(A-B) \varepsilon R$ (see Simmons [1], pp. 13 and 182).

We say that R is a σ-ring if it has the additional property that for every countable sequence A_i $(i \varepsilon I)$ of sets contained in $\mathcal{R}$, it follows that

$$(\bigcup_{i=1}^{\infty} A_i)\varepsilon\mathcal{R}. \tag{3.4}$$

A ring such that $S\varepsilon\mathcal{R}$ is called an *algebra* (or Boolean algebra).

We wish now to construct classes of sets which will be useful for an integration theory. First we must define the ring generated by some non-empty class of subsets of S, $\mathcal{E}$. We shall denote this ring $\mathcal{R}(\mathcal{E})$. Denote by $\{\mathcal{R}_i\}$ the family of all rings which contain $\mathcal{E}$. Then we choose to define

$$\mathcal{R}(\mathcal{E})\equiv\bigcap_i \mathcal{R}_i, \tag{3.5}$$

which is clearly still a ring. Indeed, it is the smallest ring containing the class $\mathcal{E}$. This procedure will also work for σ-rings. The Borel sets are $\mathcal{P}(\mathcal{R}(\mathcal{E}))$.[(25)]

We now let S be the real line, R^1, and seek to find an appropriate way of generating the Borel sets, written $\mathcal{B}(R^1)$. We choose as our class $\mathcal{E}$ the set of all semi-open intervals $[a,b)$, where

$$[a,b)\equiv\{x:a\leq x<b\}. \tag{3.6}$$

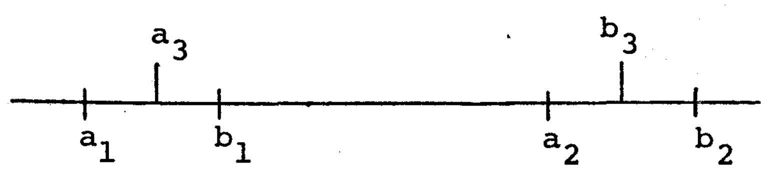
---

(25) Here $\mathcal{P}$ represents once again the power-set functor.

For our purpose, then, the Borel sets on the real line will be those sets contained in $\mathcal{P}(\mathcal{R}(E))$.

The semi-open intervals have been chosen because they form a ring under countable union. To see that this would not be the case for closed intervals, consider the finite case below. A typical element of $\mathcal{R}(E)$ would be of the form $\bigcup_{i=1}^{n} \{x: a_i \leq x \leq b_i\}$. Consider a simple case where n=2.

Fig. 2-1



Writing $A_1=[a_1,b_1]$, $A_2=[a_2,b_2]$, then $A_1 \bigcup A_2 \varepsilon \mathcal{R}(E)$. The third set shown in the figure, $A_3=[a_3,b_3]$, is also an element of $\mathcal{R}(E)$. However,

$$A_3-(A_1 \bigcup A_2)=A_3 \bigcap (A_1 \bigcup A_2)'$$
$$=A_3 \bigcap A_1' \bigcap A_2'$$
$$=(b_1,b_2) \notin \mathcal{R}(E). \qquad \text{Q.E.D.}$$

Thus the closed intervals do not generate a ring. The half-open intervals, on the other hand, do. Furthermore, they are sufficiently general to admit a meaningful integration theory. Finally, we give the standard definition of the Borel sets as:

D4 <u>Definition 4</u>:The Borel sets are those contained in the smallest σ-algebra containing the open sets.

Here we have assumed that the ring in question is an algebra, which for our purpose will generally be true. Note that by choosing to call the elements of $\mathcal{E}$ open the Borel sets are in a sense topologically induced.

(ii) Measure-Spaces:

A measure-space is a triple (S,M,μ) where S is some set, M is a σ-ring of subsets of S, and μ is a non-negative set-function. Since M is a ring, then ϕ is in M and is therefore measurable. We require that μ satisfy the following conditions in general:

1) $0 \leq \mu(A) < \infty$

2) $\mu(\phi)=0$

3) For any disjoint countable sequence of sets A_i $(i \varepsilon I)$,

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

We call property 3 *σ-additivity*. For the purpose of this paper we may take $S=R^1$, $M=\mathcal{B}(R^1)$, and

$$\mu\{[a,b)\}=b-a. \qquad (3.7)$$

We call (3.7) the *Lebesgue measure* on the real line. We may generalize this to the *Lebesgue-Stieltjes measure* by defining a real-valued, non-decreasing function $\rho(\lambda)$ and

$$\mu\{[a,b)\}=\rho(b)-\rho(a). \qquad (3.8)$$

We say that two measures μ_1 and μ_2 are *comparable* if they are defined over the same σ-ring M.

We say a measure μ_1 is *inferior* to μ_2 if all sets of μ_2-measure zero are also μ_1-measure zero, writing $\mu_1 \prec \mu_2$. Thus $\mu_1 \prec \mu_2$ or μ_1 is *absolutely continuous* with respect to μ_2 if $\mu_2(A) = 0 \Rightarrow \mu_1(A) = 0$.

(iii) Functions and Integration:

A *function* is a mapping from a *domain* $\mathcal{D}_f$ to a *codomain* D_f. Thus $x \in \mathcal{D}_f$ is mapped into $f(x) \in D_f$.

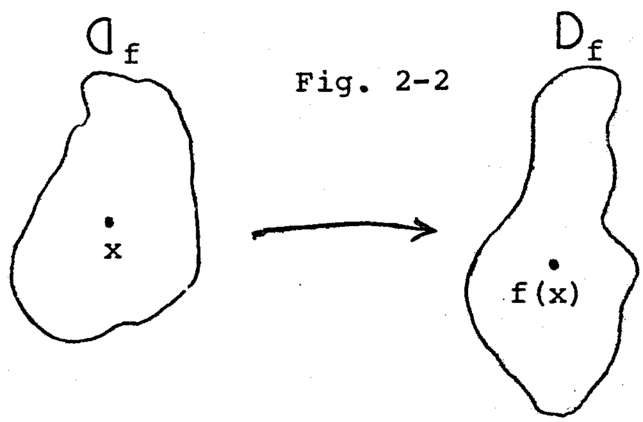


Fig. 2-2

The *graph* of a function is the set of all points of $R^1 \times R^1$ of the form $(x, f(x))$.

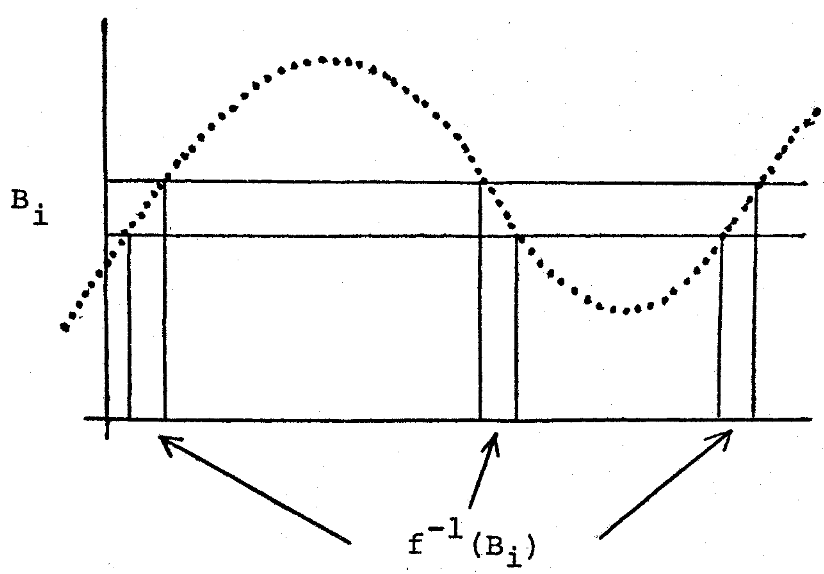


Fig. 2-3

If an inverse function exists we define the *inverse image* of a set B_i as

$$f^{-1}(B_i) = \{x : f(x) \varepsilon B_i\} \quad . \tag{3.9}$$

We now define the sequence of disjoint subsets of $\mathcal{O}_f$, $\{A_i\}$, as

$$A_i = f^{-1}(B_i) \tag{3.10}$$

where $\{B_i\}$ is a disjoint decomposition of a subset B of $\mathcal{D}_f$; also, we define *characteristic functions* as

$$\chi_{A_i}(x) = \begin{cases} 1 & \text{if} \quad x \varepsilon A_i \\ 0 & \text{if} \quad x \not\varepsilon A_i \end{cases} \quad . \tag{3.11}$$

We may now define *simple functions* as

$$f_n(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x) \quad . \tag{3.12}$$

Furthermore, we consider sequences $\{f_n(x)\}$ of such functions and define *convergence in the measure* as

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0 \tag{3.13}$$

of equivalently,

$$\lim_{n\to\infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\} = \phi \quad . \tag{3.13'}$$

We now define the *integral* of a simple function as

$$\int f_n(x) d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i) \quad , \tag{3.14}$$

and the corresponding integral of the limit function f as

$$\int f(x) d\mu = \lim_{n\to\infty} \int f_n(x) d\mu \quad . \tag{3.15}$$

If μ is the Lebesgue measure, then (3.15) defines the *Lebesgue integral*, a generalization of the Riemann integral.

(iv) Hilbert Space:

We shall be concerned largely in what follows with abstract spaces such as the Hilbert space. We begin by discussing very briefly *Banach spaces*.

D3 Definition 3: A *normed linear space* is a vector space N for which we may assign a real number $||x||$ to every element x of N such that

(1) $||x|| \geq 0$ and $||x|| = 0 \Rightarrow x = 0$

(2) $||x+y|| \leq ||x|| + ||y||$

(3) $||\alpha x|| = |\alpha| \; ||x||$, $\alpha \epsilon C$.

This space is a *metric space* if we define $d(x,y) \equiv ||x-y||$.

D4 Definition 4: A *Banach space* is a normed linear space which is *complete* under the norm.

Note that a complete metric space is one for which every *Cauchy sequence* is convergent (see Simmons [1], p. 71). We may now define *Hilbert space* as:

D5 Definition 5: A *Hilbert space H* is a complex Banach space whose norm is induced by an *inner product*, where an inner product is a function (x,y) on two elements of H having the following properties:

(1) $(\alpha x + \beta y,z) = \alpha(x,z) + \beta(y,z)$

(2) $\overline{(x,y)} = (y,x)$

(3) $(x,x) = ||x||^2$ [26] .

It should be kept in mind that Hilbert space is an abstract entity. There are, of course, many *realizations* used extensively in practice.

We shall invoke specific realizations occasionally, but for the most part the abstract structure is of most importance to us.

To conclude this section on Hilbert space we consider *bounded linear functionals* and *operators*.

A bounded linear functional $\Phi(f)$ on a Hilbert space is a function with domain H and range the complex numbers C. It must satisfy the following conditions:

(1) $\Phi(f+g) = \Phi(f) + \Phi(g)$, for all $f,g\varepsilon H$

(2) $\Phi(\lambda f) = \lambda\Phi(f)$, for all $\lambda\varepsilon C$

(3) $|\Phi(f)| < M||f||$, $(M < \infty)$.

We define as the norm $||\phi||$ of the functional the greatest lower bound of the number M in (3). We may now state and prove the following:

T1 <u>Theorem 1</u>: (Riesz's Theorem) Every bounded linear functional Φ in a Hilbert space H is of the form $\Phi(f) = (g,f)$ with g some fixed vector in H.

Proof: We let $\{\phi_n\}$ be a complete orthonormal system in H

---

(26) The notations $\bar{\alpha}$ and α^* are both taken in this paper to be the complex conjugate of the number $\alpha\varepsilon C$.

Then we may write $f = \sum_n a_n \phi_n$. Since $\Phi(f)$ is continuous, then

$$\Phi(f) = \sum_{n=1}^{\infty} a_n \Phi(\phi_n) \quad .$$

We now define

$$g \equiv \sum_{n=1}^{\infty} \Phi^*(\phi_n)\phi_n \quad .$$

Thus

$$\begin{aligned}(g,f) &= \left(\sum_{n=1}^{\infty} \Phi^*(\phi_n)\phi_n \;,\; \sum_{n=1}^{\infty} a_n\phi_n\right) \\ &= \sum_{n,m=1}^{\infty} (\Phi^*(\phi_n)\phi_n \;,\; a_m\phi_m) \\ &= \sum_{n,m=1}^{\infty} \Phi(\phi_n) a_m (\phi_n,\phi_m) \\ &= \sum_{n=1}^{\infty} \Phi(\phi_n) a_n \quad , \text{ since } (\phi_n,\phi_m) = \delta_{nm} \\ &= \Phi(f) \qquad \text{Q.E.D.}\end{aligned}$$

We may thus define a scalar product for linear functionals as

$$(\Phi_1,\Phi_2) = (g_2,g_1)$$

where $\Phi_1(f) = (g_1,f)$ and $\Phi_2(f) = (g_2,f)$.

It follows that the linear functionals are a linear manifold with scalar product, and thus constitute a Hilbert space called the *dual space* $\hat{H}$ of H.

It can be shown that the bounded linear functionals on $\hat{H}$ are again a Hilbert space which can be identified with H itself (see Jauch [1], p. 32). This duality property is what motivated Dirac's bra-ket notation.

Concluding, we may define operators A as maps of H onto itself satisfying:

(1) $A(f+g) = Af + Ag$

(2) $A(\lambda f) = \lambda Af$

(3) $||Af|| \leq M||f|| \quad , \quad 0 \leq M < \infty$.

If an $M < \infty$ does not exist, then the operator is said to be *unbounded*. The algebraic structure of operators is very involved, but for our purposes the above definition is sufficient.

(v) Spectral Theory:

We recall that the spectrum of an operator A is the set Λ of all numbers λ for which the equation $(A-\lambda I)\psi = 0$. In the finite n-dimensional case there will exist n such λ_n's. In the event $\lambda_n = \lambda_s = \ldots$, say, this case is called *degenerate* of *multiplicity* $\alpha(r)$. Now every distinct λ_r gives rise to a vector ψ_r such that $\{\psi_r\}$ is a *complete orthogonal* set if the system is non-degenerate. In the case of degeneracy, we may find $\alpha(r)$ orthogonal vectors which complete the set. We may, of course, write for the vector $f \varepsilon H$

$$f = \sum_{r=1}^{n} x_r \psi_r \quad , \tag{3.16}$$

and furthermore, we find that the operator A gives rise to the result

$$Af = f' = \sum_{r=1}^{n} x_r' \psi_r \quad , \quad x_r' = \lambda_r x_r \quad . \tag{3.17}$$

This is called the *spectral representation* of the operator A.

Now what we have effectively done is to construct a *linear isometric* image of the abstract space H, which we shall call ℓ_n^2. Consider any *orthonormal* set $\{\phi_n\}$ which spans H. Then

$$f = \sum_{r=1}^{n} a_r\phi_r \quad , \quad a_r = (f,\phi_r) \quad . \tag{3.18}$$

Thus there exists a one-to-one correspondence between $f \varepsilon H$ and $\{a_r\} \varepsilon \ell_n^2$. Furthermore, there exists a map Ω from H onto ℓ_n^2 such that

$$\{a_r\} = \Omega f \quad . \tag{3.19}$$

Now consider an operator A defined on H. We wish to establish the corresponding operator on ℓ_n^2, $\tilde{A}$, which maps Ωf into $\Omega(Af)$. Write

$$g = Af \quad , \quad \{b_r\} = \tilde{A}\{a_r\} \tag{3.20}$$

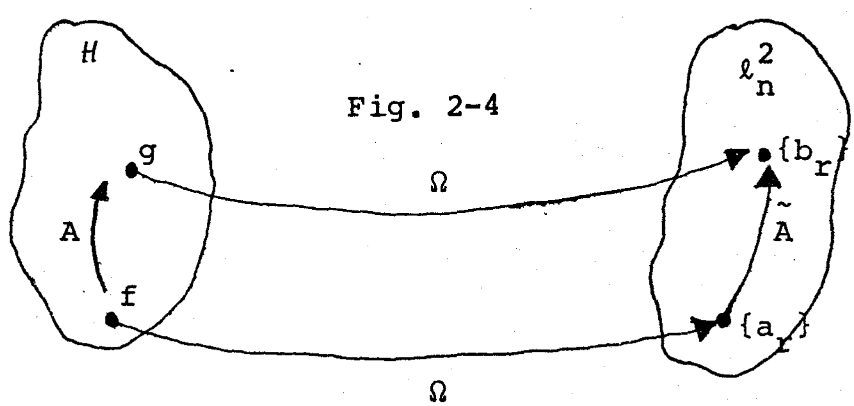


Fig. 2-4

Then

$$\Omega g = \Omega A f \quad , \quad \{b_r\} = \tilde{A}\{a_r\}$$
$$= \tilde{A}\Omega f \quad .$$

But

$$\Omega g = \{b_r\} \quad .$$

Thus

$$\Omega A f = \tilde{A}\Omega f \ .$$

Therefore,

$$\tilde{A} = \Omega A \Omega^{-1} \qquad \text{Q.E.D.}$$

In addition, ℓ_n^2 is an isometric image since:

$$f = \Sigma\ a_r \psi_r \ .$$

Thus

$$||f|| = (\Sigma_r\ a_r\psi_r \ , \ \Sigma_r\ a_r\psi_r)$$

$$= \Sigma_{rs}\ \bar{a}_r a_s (\psi_r, \psi_s)$$

$$= \Sigma_{rs}\ \bar{a}_r a_s \delta_{rs}$$

$$= \Sigma_{rs}\ |a_r|^2 \ .$$

Therefore, if we define

$$||\{a_r\}|| = \Sigma_r\ |a_r|^2 \ ,$$

then

$$\{a_r\} = \Omega f \quad \text{is isometric.} \qquad \text{Q.E.D.}$$

It should be noted that ℓ_n^2 is the realization of *H* utilized by Heisenberg in his *matrix mechanics*.

Next we consider the important notion of *projections*. Consider the complete orthonormal set of eigenvalues $\{\lambda_r\}$ already introduced. The set of vectors corresponding to a given eigenvalue λ_r spans a finite $\alpha(r)$-dimensional subspace of *H*. Call this subspace M_r. Let P_r be the projection with range M_r. Then orthogonality and completeness may be expressed by

$$P_r P_s = \delta_{rs} P_r \tag{3.21a}$$

and

$$\sum_r P_r = I \tag{3.21b}$$

and the operator A whose eigenvalues we are using may be written

$$A = \sum_r \lambda_r P_r \quad . \tag{3.21c}$$

Now we let Δ be a Borel set on the real line and define the following *projection-valued set-function* by

$$E(\Delta) = \sum_{\lambda_r \varepsilon \Delta} P_r \quad , \tag{3.22}$$

satisfying

$$\begin{aligned} E(\Delta_1) \bigcup E(\Delta_2) &= E(\Delta_1 \bigcup \Delta_2) \\ E(\Delta_1) \bigcap E(\Delta_2) &= E(\Delta_1 \bigcap \Delta_2) \\ E(0) &= 0 \\ E(R^1) &= I \quad . \end{aligned} \tag{3.23}$$

We call such a set-function a *spectral measure*.

Generalizing to the infinite-dimensional case, we write down the *spectral theorem* without proof.

T 2 Theorem 2 :To every self-adjoint operator A there corresponds a unique spectral measure $\Delta \to E(\Delta)$ defined on $\mathcal{B}(R^1)$ such that

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda \quad ,$$

where $E_\lambda = E((-\infty,\lambda])$. The integral is to be interpreted as a Lebesgue-Stieltjes integral.

(vi) Lattice Theory:

There remains one mathematical topic which will be used extensively. In this section, we consider briefly the main features of *mathematical lattices*. To begin with, what we are really interested in are *partially ordered sets*. A partially ordered set is a pair

$$\langle S, \leq \rangle \tag{3.24}$$

where S is some set and $\leq$ is an *ordering*. The ordering is said to be partial in the sense that arbitrary pairs of elements from the set S may not necessarily be ordered by $\leq$. As a prototype we shall refer throughout to the power-set of some set x, $P(x)$, ordered under set inclusion. That is, we consider

$$\langle P(x), \subseteq \rangle \quad . \tag{3.25}$$

Clearly, arbitrary subsets of x are not necessarily ordered. Note that in the general case, two elements which are ordered by $\leq$ are said to be *comparable*.

Let us now assume that the set S has a least element and a greatest element with respect to $\leq$. Then we denote these elements as θ and I respectively. Assume further that there is a relation *meet* defined on pairs of elements of S $(A \wedge B)$ and a relation *join* defined on pairs of elements of S $(A \vee B)$, and say that the partially ordered set $\langle S, \leq \rangle$ is a *lattice* if S is closed under meet and join. In our set-theoretic prototype above meet is *set-intersection* and join is *set-union*. Furthermore, $\langle P(x), \subseteq \rangle$ is trivially seen to be a lattice, if we define

$\theta = \phi =$ empty set

$I = x =$ universal set

$\wedge = \cap =$ set-intersection

$\vee = \cup =$ set-union

$\leq = \subseteq =$ containment

$' = ' =$ set complement .

The last property, set complement, makes $\langle P(x), \subseteq \rangle$ a *complemented lattice.*

2.4 <u>The Propositional Calculus of Quantum Mechanics</u>: Throughout the foregoing discussion there is one theme which I have attempted to stress, both in Part I and in Part II. Indeed, I should like to extract from this theme a point of view which will be instrumental in developing the axioms of quantum mechanics. I suggest the following premiss as a working rule to guide us, namely:

In constructing any physical theory, one has at his disposal no more than those *facts* about the physical world which manifest themselves as the result of some experiment.

We take these facts to be the empirically verifiable ones, in the sense of the logical positivists. Furthermore, I suggest that these facts can be identified as one of two types:

(1) Statements of the form

(i) The electron has spin up.

(ii) The photon is y-polarized.

(iii) The electron is in the region of space x to x+dx.

Let us call these statements the *propositions*.

(2) *Structural facts* of the form

(i) We cannot measure the position and momentum of a particle simultaneously with arbitrary accuracy.

(ii) If we prepare an electron with spin up, measure spin in some other direction, and subsequently attempt a measurement for spin up, we cannot predict the result of the measurement.

(i) <u>The Proposition System</u>:

Let us now look at the properties of our propositions. We are in a position to show:

Pr1 <u>Premiss1</u>:The set of empirically verifiable propositions form a lattice,

$$\langle L, \leq \rangle, \tag{4.1}$$

if we define

θ= ⊙ =absurd proposition

I= ▯ =trivially true proposition

$\wedge$= & =logical and

$\vee$=or =logical or

$\leq$= => =implication

'= ∿ =negation .

We now make a basic definition regarding lattices which will be important later.

D5 <u>Definition 5</u>:A lattice is said to be Boolean if its elements obey the distributive law of meet over join and join over meet.

We note that the set-theoretic prototype constructed in section 2.3(vi) is Boolean since

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\text{and} \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \ . \tag{4.2}$$

We shall show that the lattice of propositions as defined above is non-Boolean. First, however, let us consider the classic example of the two-slit experiment with an eye to discovering which of the laws of classical logic may be suspect when our reasoning leads to the well-known erroneous result. We shall conclude, incidentally, that distributivity could very well be the culprit.

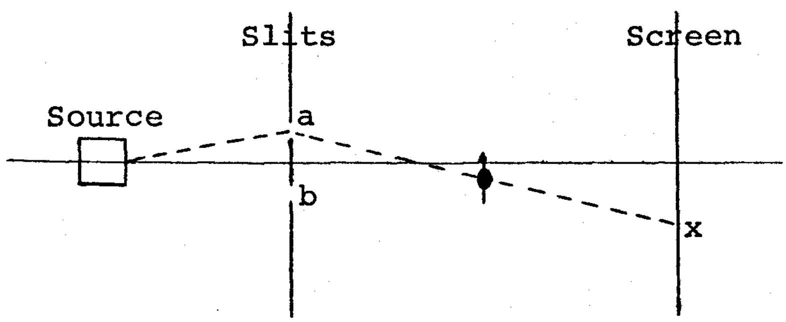


Fig. 2-5

Let us suppose, as an hypothesis, that the electron goes through just one hole. Then we may write

$$(a \vee b) \wedge \sim(a \wedge b) \quad . \tag{4.3a}$$

Thus the probability of the electron arriving at a point x on the screen is the probability of passing through a or b and arriving at x. That is

$$\Pr(x) = \Pr([a \vee b] \wedge x) \quad . \tag{4.3b}$$

By the *distributive law*, this is

$$\Pr(x) = \Pr([a \wedge x] \vee [b \wedge x]) \quad . \tag{4.3b'}$$

By the law of *total probability*, this is

$$\Pr(x) = \Pr(a \wedge x) + \Pr(b \wedge x) - \Pr(a \wedge x \wedge b \wedge x) \quad . \tag{4.3c}$$

But $\Pr(a \wedge x \wedge b \wedge x) = 0$ by (4.3a); therefore,

$$\Pr(x) = \Pr(a \wedge x) + \Pr(b \wedge x) \quad . \tag{4.3c'}$$

Now from the point of view of the single slit experiment, assuming the law of *conditional probability*,

$$\Pr(a \wedge x) = \Pr(x|a)\Pr(a) \tag{4.3d}$$

and

$$\Pr(b \wedge x) = \Pr(x|b)\Pr(b) \quad . \tag{4.3d'}$$

Assuming $\Pr(a) = \Pr(b)$, we get

$$\Pr(x) \propto \Pr(x|a) + \Pr(x|b) \quad . \tag{4.3e}$$

Thus, we are led to the conclusion that the two-slit experiment should generate an *additive* pattern, contrary to empirical evidence that it is a *superposition*. Our problem now is to decide which aspect of our argument is responsible for the contradiction. There are several possibilities:

(1) Our original hypothesis of mutually exclusive passage in (a).

(2) The use of the distributive law in passing from the conjunction

$(a \vee b) \wedge x$, (b)

to the disjunction

$(a \wedge x) \vee (b \wedge x)$. (b')

(3) The use of the law of total probability in (c).

(4) The use of the law of conditional probability in (d).

I propose that there are no physical grounds for rejecting (1), (3) or (4) above. For a complete discussion of this point, see the article by A. Fine in *Paradigms and*

Paradoxes, Colodny [1]. Having concluded that unrestricted use of distributivity leads to empirically contradictory conclusions, we may continue by formulating the following definition of *compatible* propositions.

Consider the lattice of propositions, L, and an arbitrary subset of the lattice, S. Now consider the family $\{L_i\}$ of all sublattices of L which contain S. Then define

$$\bar{L} = \bigwedge_i L_i \quad , \tag{4.4}$$

which is the smallest sublattice containing the set S. We may now establish:

D6 Definition 6: Elements of the set S are said to be pairwise compatible, written a $\leftrightarrow$ b, if $\bar{L}$ is Boolean.

Before carrying on further, let us consider a simple example of incompatible propositions. This will show in addition that the lattice of all propositions is non-Boolean. Consider as three possible propositions:

a = the photon is y-polarized

b = the photon is x-polarized

c = the photon is polarized $\pi/4$ rad. to the x-axis.

The situation is illustrated in Fig. 2-6.

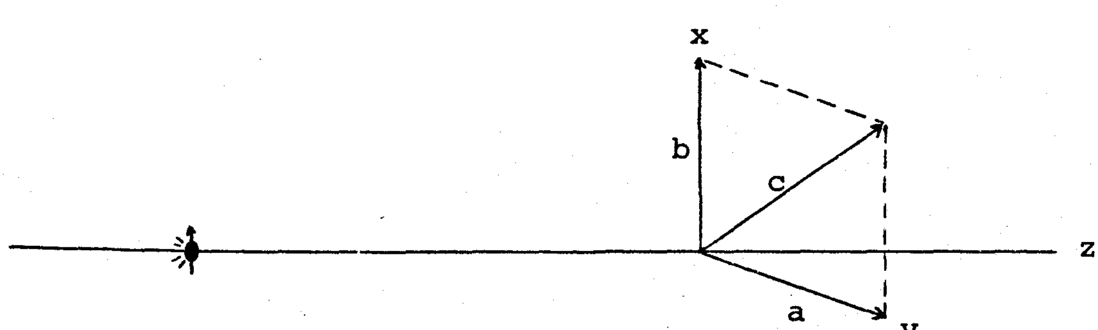


Fig. 2-6

It is easily seen that

$$c \wedge (a \vee b) = c \wedge \square = c \quad , \tag{4.5a}$$

whereas

$$(c \wedge a) \vee (c \wedge b) = \odot \vee \odot = \odot \quad . \tag{4.5b}$$

One can understand (4.5a) as reflecting a well known quantum-mechanical structural fact; namely, if we prepare photons in some arbitrary polarization and then pass them through an analyser designed to pass both x-polarized and y-polarized photons, it will pass all the photons. Similarly, (4.5b) says simply that a photon cannot be x-polarized and polarized at $\pi/4$ rad. to the x-axis simultaneously. Thus, we see that the quantum-mechanical propositions do not always obey the distributive law. The set $S = \{a, b, c\}$ considered above illustrates the consequent concept of incompatible propositions.

(ii) <u>The Lattice of Subspaces:</u>

Let us now consider some of the more familiar aspects of quantum mechanics, such as those which manifest themselves in the structure of the Hilbert space. We shall be interested primarily in the abstract Hilbert space rather than any particular realization such as a function space or ℓ^2_n.

Consider the class of all subspaces of our Hilbert space. A subspace is a space spanned by a finite number of vectors in H which is itself a linear manifold. We now assert:

Pr2 Premiss2: The class of subspaces of an abstract Hilbert space forms a complete orthocomplemented lattice, if we define

$\theta = 0$

$I = \mathcal{H}$

$M \wedge N$ = largest space contained in both M and N

$M \vee N$ = smallest space containing both M and N

$\leq \; = \; \subseteq$ = inclusion as a subspace

$M' = M^{\perp}$ = space orthogonal to M.

This lattice is non-Boolean, as demonstrated below.

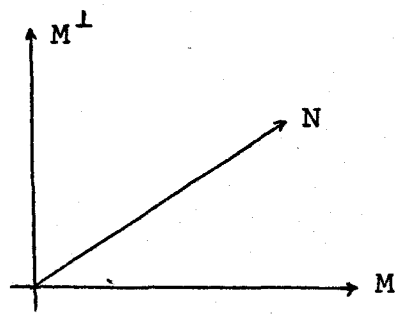


Fig. 2-7

We decompose the Hilbert space into two orthogonal subspaces, M and $M^{\perp}$, as illustrated in Fig. 2-7. Let N be any arbitrary subspace of $\mathcal{H}$ contained in neither M nor $M^{\perp}$. Then

$$N \wedge (M \vee M^{\perp}) = N \wedge \mathcal{H} = N \tag{4.6a}$$

whereas

$$(N \wedge M) \vee (N \wedge M^{\perp}) = 0 \vee 0 = 0 \quad . \tag{4.6b}$$

Thus our lattice does not obey the distributive law of meet over join. It is, therefore, non-Boolean.

Within the context of the lattice of propositions we have defined the notion of compatibility. We must now establish a similar definition for the lattice of subspaces. It would be possible, of course, to define compatibility between subspaces in a manner identical to that above. We shall, however, for physical reasons, use the following definition:

D7 Definition 7: Two subspaces, M and N of a Hilbert space $\mathcal{H}$ are said to be compatible, written $M \leftrightarrow N$, if

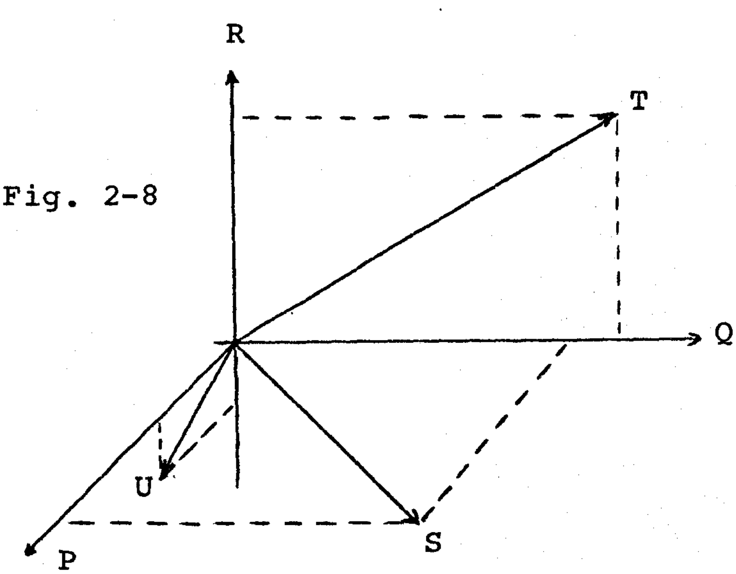
$$(M \wedge N) \vee (M \wedge N^{\perp}) = M \quad . \tag{4.7}$$

Jauch (see Jauch [1]) points out that this definition is equivalent to another formulation. Two subspaces are said to be disjoint, written $M \perp N$, if $M \subseteq N^{\perp}$. It follows that $N \subseteq M^{\perp}$. Now it turns out that two subspaces are compatible in the above sense if there exist three mutually disjoint sets M_1, N_1 and K, such that

$$M = M_1 \vee K \quad \text{and} \quad N = N_1 \vee K \quad .$$

It follows that $K = M \wedge N$, $M_1 = M \wedge K^{\perp}$ and $N_1 = N \wedge K^{\perp}$. I illustrate this point with the following example.

Fig. 2-8



We take P, Q and R as a disjoint decomposition of $\mathcal{H}$. Thus

$$\mathcal{H} = P \vee Q \vee R \quad ,$$

$$P \wedge Q = Q \wedge R = R \wedge P = 0$$

We now define

$$M = P \vee Q \quad , \quad N = Q \vee R$$

and it follows that K = Q, M_1 = R and N_1 = P. Now consider

$$\begin{aligned}(M \wedge N) \vee (M \wedge N^{\perp}) &= Q \vee (M \wedge R) \\ &= Q \vee R \\ &= M \quad ,\end{aligned}$$

which shows that $M \leftrightarrow N$ in the sense of D7. We now state:

T3 <u>Theorem 3</u>:A lattice which is compatible in the sense of D7 is also compatible in the sense of D6.

Assuming that T and S are not orthogonal in Fig. 2-8 we can illustrate these points further. First, define $L = P \vee R$. Then

$$L \wedge (T \vee S) = U \quad ,$$

whereas

$$\begin{aligned}(L \wedge T) \vee (L \wedge S) &= 0 \vee 0 \\ &= 0 \quad .\end{aligned}$$

Furthermore,

$$\begin{aligned}(T \wedge S) \vee (T \wedge S^{\perp}) &= 0 \vee 0 \\ &= 0 \neq T \quad .\end{aligned}$$

Finally, it should be pointed out that the exact restrictions to be placed on the lattice of subspaces - for physical reasons - is not well understood. In the original paper on quantum logic by Birkhoff and von Neumann (see Birkhoff and von Neumann [1]) they suggest that the following restriction be imposed. First, in any lattice, one always has,

$$x \vee (y \wedge z) \leq (x \vee y) \wedge z \quad \text{for} \quad x \leq z \ . \tag{4.8}$$

If the equality holds in (4.8) then the lattice is said to be *modular*. That the lattice be modular is the Birkhoff-von Neumann restriction. For a discussion of this point and other suggestions see Jauch (see Jauch [1], p. 83).

It can be shown that the lattice of propositions and the lattice of subspaces are *isomorphic* within the context of the above definitions of compatibility. That is, compatible propositions are mapped onto compatible subspaces. Furthermore, the essential features of quantum mechanics are contained in the lattice of propositions. By virtue of the isomorphism we may, therefore, conclude that the essential features of quantum mechanics are contained in the abstract Hilbert space, a conclusion which is well known but otherwise unsubstantiated by other than the test of experience, a sufficient but unaesthetic test.

2.5 Observables: We have now established that the abstract Hilbert space is relevant to quantum mechanics, but it remains to arrive at the familiar concepts of observables and states. In this section, we proceed to discuss observables.

We saw that in classical mechanics, dynamical-variables were defined as maps from the phase-space into Borel sets on the real line. We wish now to define quantum-mechanical observables in a similar way; that is, we wish our observables to be objects which can be measured. Our observables must establish a set of real numbers, the values of the observable. In addition, we note that each observable is associated with a proposition; namely, given any Borel set E on the real line we may make the following proposition. "The observable x has a value lying in the Borel set E." We thus define:

D8 Definition 8: An observable x is a map from the Borel sets on the real line $\mathcal{B}(R^1)$ into the lattice of propositions L, satisfying:

(1) $x(R) = \square$

(2) if $E \bigcap F = \phi$, then $x(E) \perp x(F)$

(3) $x(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} x(E_i)$, $E_i \bigcap E_j = \phi$, $i \neq j$.

The notation $a \perp b$ for two propositions $a, b \in L$ denotes *disjointness*, and is equivalent to the notion of disjoint subspaces already introduced. The above definition of observables is an example of a *σ-homomorphism*. We may thus say that an observable is a σ-homomorphism from the Borel sets on the real line into the lattice of propositions.

Two observables x and y are said to be *compatible* if $x(E) \leftrightarrow y(F)$ for every $E, F \in \mathcal{B}(R^1)$. We write $x \leftrightarrow y$.

Since there is an isomorphism between the lattice of propositions and the lattice of subspaces of the abstract Hilbert space we may identify propositions with closed subspaces. Identifying closed subspaces with their orthogonal projections, an observable may be thought of as a projection-valued measure. Finally, the spectral theorem, T2, tells us that there is a one-to-one correspondence between spectral-valued measures and self-adjoint operators. We conclude, therefore, that for computational purposes, *observables are self-adjoint operators on the abstract Hilbert space.*

2.6 States: We begin by considering the following quantum-mechanical experiment. Photons are *prepared* by passing them through a polarizer, polarized in the y direction. They are then passed through an analyser which is polarized at $\pi/4$ rad. to the y-axis. We observe that some photons will pass the analyser, while others do not; also, we are unable to predict when either event will be realized. We proceed however, to pass photons through the analyser one at a time and write a "1" in our notebook each time a photon is observed to have passed through. Similarly, we write a "0" whenever the photon does not pass. We now imagine that we could observe such events in the limit as n, the number of events, approaches infinity. We define the following *probability measure* on the proposition, a, that the photon passes:

$$p(a) = \lim_{n\to\infty} \frac{n(a)}{n} \quad , \tag{6.1}$$

where n(a) is the number of times we wrote "1" in our notebook.

Next we give the following *operational* definition of *state*.

D9 Definition 9: A state is the result of physical manipulations on the system which constitute the *preparation* of the state.

In the foregoing example, the state preparation consisted of passing the photons through a polarizer. By subsequently performing, at least conceptually, the calculation of p(a) given in (5.1), we arrive at a numerical-valued probability measure which characterizes the state of the system.

Now, if two (or more) ensembles of photons were prepared differently we could define two states characterized by $p_1(a)$ and $p_2(a)$. If these *pure states* are now physically mixed, we arrive at the familiar notion of a *mixed state*. This new state is characterized by the unique probability-measure

$$p(a) = \lambda_1 p_1(a) + \lambda_2 p_2(a) \quad , \tag{6.2a}$$

where

$$\lambda_1 + \lambda_2 = 1 \quad . \tag{6.2b}$$

In this way, we see that the pure states of the system are characterized uniquely by probability measures on the propositions, such that there is a one-to-one correspondence between propositions and pure prepared states.

We know that the lattice of propositions is isomorphic to the lattice of subspaces. We therefore conclude that *pure states are vectors in the abstract Hilbert space.*

2.7 Generalized Probability: In this final section we may summarize the foregoing results briefly by reference to the concept of a generalized probability theory, a notion introduced by Stanley Gudder (see Gudder's article in Bharucha-Reid [1]). He proceeds to construct his theory in the following way.

Consider a set L of propositions and impose upon it the following axioms:

L1 $a \leq a$ for all $a \in L$,

L2 if $a \leq b$, $b \leq c$ then $a \leq c$,

L3 if $a \leq b$, $b \leq a$ then $a = b$,

L4 There are propositions $\odot$ and $\Box$ satisfying
$\odot \leq a \leq \Box$ for all $a \in L$,

L5 $(a')' = a$ for all $a \in L$,

L6 if $a \leq b$ then $b' \leq a'$,

L7 $a \vee a' = \Box$ for all $a \in L$,

L8 if a_i is a sequence of mutually disjoint propositions then $\vee a_i$ exists,

L9 There is a full set of states M on L, where a state m is a map from L into the unit interval $[0,1] \subset R^1$ which satisfies

M1 $m(\Box) = 1$

M2 $m(\vee a_i) = \Sigma m(a_i)$.

I have not enlarged upon the notation since it may be interpreted exactly as in the previous sections. We now say that the pair (L,M) forms a *logic* and introduce compatibility between pairs of elements of L exactly as before. Furthermore, observables are introduced, once again in an identical manner to section 2.5. Clearly, all we have done so far in this section is to review the rest of Part II and adjoin the notion of a logic. In this way, however, it is evident that the resultant structure is very similar to Kolmogorov probability spaces (see Kolmogorov [1]). The difference is that our set of propositions has less structure than the σ-algebra in Kolmogorov's theory. Otherwise, probability measures are replaced by states and the random variables are replaced by observables.

III CONSPECTUS

3.0 Introduction:An effort has been made so far in this paper to sustain the theme that contemporary scientific theories are largely phenomenological in nature. In Part III we shall endeavor to draw together those threads of the theme which previously have been left deliberately hanging. In particular, two points were mentioned explicitly in Part I and then postponed, namely, the free creation of logic and the operational difficulty of defining quantum-mechanical state and the relationship of this issue to verifiability. Thus in Part III we must discuss:

(i) The extent to which quantum logic demonstrates the phenomenological aspects of quantum mechanics,

(ii) The rôle of models in quantum mechanics and the distinction between models and interpretations,

(iii) The two specific issues just listed (in light of quantum logic), and

(iv) The explanatory aspects of physical theories (explanation vs. description) and the relationship of quantum mechanics to underlying physical reality.

The attempt in the sequel to deal with the foregoing points is intended to be somewhat provisional and will tend of necessity to be largely idiosyncratic at this time.

3.1 Quantum-Mechanical Phenomena: It was seen in Part I that one's philosophical bias regarding the explanatory powers of the scientific methodology may likely designate him as an instrumentalist or an essentialist. The meaning of these two words was clarified at the end of section 1.5. Then in Part II a discussion of quantum logic was presented in an effort to demonstrate to what extent quantum mechanics can be viewed as a strongly phenomenological theory. By starting out with strict observational data (the two types of observational facts suggested in Part II) about the micro-phenomena under investigation we arrived at a mathematical formalism identical to that presented in the orthodox pedagogy. That is, starting with phenomena as our primary data, we were able to construct a mathematical theory which correlates extremely well with the wealth of observational information now available regarding the atomic domain. This assertion is to be taken in the pragmatic sense that the majority of contemporary research in atomic, solid state, and to some extent even nuclear physics, has proven highly fruitful.[27]

Does the overwhelming success of the quantum formalism

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(27) It is not being claimed here that one could expect to construct quantum mechanics historically in the manner described above. The claim is strictly retrospective; only in hindsight can we expect to recognize such a structure in our theory.

force me to believe that the elements of my theory - elements which my jargon has labelled electrons, protons, neutrons, neutrinos, *etc.* - are elements of an underlying reality which my observational data has faithfully recorded? Now it is a popular approach in quantum mechanics to say that a particle is completely and exhaustively characterized by a set of quantum-numbers. Are we therefore to conclude that the set of quantum-numbers is the only thing which is meaningful? To answer such a question in the affirmative is to adopt strict instrumentalism as your philosophical bias. I believe that in light of such issues as the foregoing discussion on quantum logic one cannot categorically reject such a point of view. However, neither is one compelled to adopt it. In this sense, I believe, one is certainly at liberty to adopt the weakened form of instrumentalism discussed in section 1.5, namely, that our theory is entirely instrumental in nature but that there does indeed exist an underlying reality which *causes* the phenomena upon which our theory is based. Let me adopt this point of view in the remaining discussion.

3.2 Models and Interpretations: The previous section has introduced a new notion to the discussion. Let me call this causal relation between the underlying reality and the observational phenomena *vertical causation*, as opposed to the traditional horizontal causal nexus one contemplates in space and time. It would be näive, of course, to believe that these two causal categories are independent, for surely if there exists an underlying reality which causes phenomena, then the causal chain we wish to understand in the observational world of experience is closely correlated with the causal link with reality alluded to above as vertical causation.

We are interested in discovering the structure of the causal connection between reality and phenomena. In particular, we wish to know to what extent the theoretical connection between phenomena enables us to draw conclusions regarding the nature of reality. Is there a one-to-one correspondence between the elements of such a reality and the theoretical constructs of our theory? If not, should we anticipate being able to construct such a theory? Clearly, a theory satisfying this condition would be complete in the sense that Einstein insisted quantum mechanics is not. Does the causal connection between elements of reality and the observational data obey a Humean "if, then always" pattern? It is questions such as these that the strict application of the scientific methodology cannot answer. The realization of syntactical relations between elements of our theory, even when/if in one-to-one correspondence with elements of reality, does not enable us to come to conclusions regarding the exact ontological nature of reality. Such lessons can only be learned when we combine our theory with semantical speculations. We may, if we choose, call such speculation metaphysical. It follows in this way that metaphysics is a meaningful domain of enquiry and contrary to the positivists we embrace it as an essential aspect of any analysis of physical theories. I propose that such semantical speculation constitutes an essential component of the *interpretation* of the physical theory.[28]

Let us now contrast this attitude toward interpretations with the notion of a model already introduced in section 1.4. As examples of models we may sight the Thomson "plum -pudding" model, the Rutherford model, and the Bohr model of the atom. Furthermore, it was suggested in section 1.4 that the image of an electron "smeared out over all space" is a model in the sense that it constitutes a psychological image which facilitates our thinking about the physical problem at hand. Indeed, I propose that a model be defined for our purposes as just such a psychological facility and that the three atomic models listed above function in exactly that way.

We now ask. To what extent do our models foster our interpretations? That is, is it sufficient to assert that the elements of our theory are in one-to-one correspondence with objects in such a model and conclude that such a correlation constitutes a satisfactory interpretation? I am claiming that the answer to this question is no. Interpretations require more than psychological satisfaction. An adequate interpretation of quantum mechanics is not realized by the mere reduction of

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(28) The notion of semantics is not to be taken here as it appears in the context of formal logic. Rather, it is to be taken in the more crude lexicographic sense of seeking meaning.

its elements to classical jargon, that is, reduction to the familiar. Familiarity does not necessarily breed intelligibility.

3.3 Selection and Correlation:We may expand the above points somewhat by discussing briefly the two issues which arose in Part I. In order to interpret any physical theory it is expedient to establish some understanding of how our theory grows. Are they, as suggested by Poincaré, freely created? We have seen in the field of geometry that Euclidean spaces are insufficient for a complete description of gravitational phenomena, at least subject to the constraint that our insistence on general covariance is justified. By invoking more generalized Riemannian geometries these difficulties can be interestingly rationalized. In light of quantum logic it is similarly realized that the unrestricted application of classical logic leads to empirical difficulties. These too can be rationalized by the introduction of a generalized non-classical logic. To conclude that the logic is freely created is, however, unjustified. That is, if we are to believe in an underlying reality whose nature is discovered by semantical speculation, then the necessity of *correlating* our theory with this reality is forced upon us. Thus certain geometrical and logical constructions may turn out to be absolutely inadequate as tools in a complete physical theory. However, those freely constructed theories which appear to be immediately adequate (they are known to correlate with observational data) are *chosen* by the theorist as physically relevant. Thus theoretical activity may very easily be a *selective* process, subject to the constraint of the necessity of correlation.

Finally, let us consider the notion of verifiability in light of quantum logic. In the process of constructing our theory we are faced continually with the problem of verifying certain propositions. Now we have seen that in quantum mechanics it is necessary to define state as a probability measure which leads to certain operational difficulties. To begin with, since state is dependent in quantum mechanics on some preparation procedure, it becomes impossible to deal with certain cosmological issues such as the state of the universe. Of greater importance is the problem of verifying certain propositions such as "the photon is y-polarized". Since a definite conclusion cannot be arrived at short of an infinite number of yes-no experiments on the ensemble, it is operationally impossible to make definitive statements about individual photons, and verifiability becomes in this sense obscured. But, of course, photons may not exist individually as elements of reality. They are, after all, part of our model, and we arrive once again at the necessity to speculate semantically in order to interpret the results.

3.4 Explanation and Description: In conclusion, let us clarify the distinction between explanation and description which has played a central rôle throughout this paper. The question we have been asking is the following. Does theoretical activity in the physical sciences constitute ultimate explanation? A strict instrumentalist would answer with a strong no, maintaining that the rôle of theory is entirely descriptive. The observational data is all there is and the sole function of a scientific theory is to describe and correlate the phenomena. Now the weakened form of instrumentalism admits a restricted explanatory rôle to scientific theory. In essence it combines a strong form of empiricism with realism in the following way. As an empirical approach to science it admits only observational data as its source of information. However, as a form of realism it contends that there does exist an underlying reality which causes the phenomena we observe. The syntactical aspects of our theory are taken to be entirely descriptive, while semantical speculation is admitted as a weakened form of explanation. What is denied is the possibility of ultimate explanation, or alternatively, ultimate interpretation. It is conceded, however, that the advent of more sophisticated theories such as quantum mechanics provides insights into the structure of reality. It is true that the world (reality, world-in-itself) is *such that quantum mechanics works very well.*

It can be fairly said that the foregoing discussion is prematurely programmatic. Notwithstanding this legitimate criticism, I present it at this time as a personal attempt to understand some of the traditional philosophical ramifications of quantum mechanics.

REFERENCES

Abraham and Marsden

[1] *Foundations of Mechanics*, W. A. Benjamin Inc., New York, 1967.

Ayer, A. J.

[1] *Logical Positivism*, The Free Press, New York, 1959.

Ballentine, L. E.

[1] "The Statistical Interpretation of Quantum Mechanics", *Rev. Mod. Phys.*, 42, 4 (1970).

Bharucha-Reid, A. T.

[1] *Probabilistic Methods in Applied Mathematics*, vol. 2, Academic Press, New York, 1970.

Birkhoff and von Neumann

[1] "The Logic of Quantum Mechanics", *Ann. Math.*, 37, 823-845 (1936).

Bridgman, P. W.

[1] *The Logic of Modern Physics*, The MacMillan Co., New York, 1961.

Carnap, R.

[1] "Science and Analysis of Language", *J.U.S.*,(1940).

Clifford, W. K.

[1] *The Common Sense of the Exact Sciences*, Alfred A. Knopf, New York, 1946.

Duhem, P.

[1] *The Aim and Structure of Physical Theory*, Princeton University Press, Princeton, 1954.

Einstein, A.

[1] "Can Quantum-Mechanical Description of Reality be Considered Complete?", *Phys. Rev.*, 47, 777 (1935).

Frege, G.

[1] *The Foundations of Arithmetic*, Basil Blackwell, Oxford, 1959.

Gottfried, K.

[1] *Quantum Mechanics*, vol. I, W. A. Benjamin Inc., New York, 1966.

Heisenberg, W.

[1] *The Physical Principles of Quantum Mechanics*, The Clarendon Press, Oxford, 1947.

Jauch, J. M.

[1] *Foundations of Quantum Mechanics*, Addison-Wesley Publishing Co., Reading, 1968.

Kolmogorov, A. N.

[1] *Foundations of the Theory of Probability*, Chelsea Publications, New York, 1950.

Lakatos, I. and Musgrave, A.

[1] *Criticism and the Growth of Knowledge*, Cambridge University Press, London, 1970.

Mach, E.

[1] *Popular Scientific Lectures*, The Open Court Publishing Co., Chicago, 1898.

[2] *The Science of Mechanics*, The Open Court Publishing Co., Chicago, 1902.

Mackey, G. W.

[1] *Mathematical Foundations of Quantum Mechanics*, W. A. Benjamin Inc., New York, 1963.

MacLane, S. and Birkhoff, G.

[1] *Algebra*, The MacMillan Co., New York, 1967.

Passmore, J.

[1] *A Hundred Years of Philosophy*, Penguin Books, 1970.

Pearson, K.

[1] *The Grammar of Science*, The MacMillan Co., Toronto, 1911.

Poincaré, H.

[1] *The Foundations of Science*, The Science Press, New York, 1929.

Popper, K.

[1] *Conjectures and Refutations*, Harper and Row Publishers, New York, 1968.

Roman, P.

[1] *Advanced Quantum Theory*, Addison-Wesley Publishing Co., Reading, 1965.

Russell, B.

[1] *Logic and Knowledge*, George Allen and Unwin Ltd., London, 1968.

[2] *Our Knowledge of the External World*, George Allen and Unwin Ltd., London, 1969.

Simmons, G.

[1] *Introduction to Topology and Modern Analysis*, McGraw-Hill Book Co., Toronto, 1963.

von Neumann, J.

[1] *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, 1955.

[2] "*Die Eindeutigkeit der Schrödingerschen Operatoren*", *Math. Ann.*, 104, 570 (1931).

Bunge, M.

[1] *Causality*, The World Publishing Co., New York, 1963.

Kirchhoff, G.

[1] *Vorlesungen über mathematische Physik: Mechanik*, Leipsic, 1874.

Kuhn, T.

[1] *The Structure of Scientific Revolution*, The University of Chicago Press, Chicago, 1942.