RAMSEY NUMBERS

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PAVOL HELL

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AUTHOR: Pavol Hell

SUPERVISOR: Dr. G. O. Sabidussi

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SCOPE AND CONTENTS: This thesis gives new finite and asymptotic estimates of the Ramsey numbers using certain number-theoretical considerations; it also contains a brief historical survey on determination of Ramsey numbers and related number-theoretical problems.

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TABLE OF CONTENTS

RAMSEY NUMBERS

INTRODUCTION

The history of Ramsey's theorem and the Ramsey numbers may be considered to have 3 starting points - 3 pioneer works:

(1) Ramsey's paper [17] in which the Ramsey theorem is for the first time stated and proved as an auxiliary combinatorial theorem for certain logical considerations (see § 1).

(2) Greenwood-Gleason in [11] noted that one of the problems on a Putnam competition has deep combinatorial roots, phrased the general problem of determining the Ramsey numbers and found the first numbers using finite field theory (see § 2).

(3) Schur's work [18] originating from number theory gave rise to a series of number-theoretical results which provide some estimates for Ramsey numbers, as was observed by Moser and Abbott, [1].

In this thesis § 1 and 2 are devoted to a survey of the first two approaches and then in § 3 and 4 the last approach is followed. A new number-theoretical problem is introduced (r-problem) which in its special case (s-problem) is shown to be equivalent with cyclic Ramsey's problem. Using some estimates of the numbers r and s, Ramsey numbers are estimated.

Except Schur's theorems 6 and 10, all theorems and proofs of § 3, 4 are original.

§1. RAMSEY'S THEOREM

In December 1928 F. P. Ramsey formulated the following theorem and used it for finding a procedure to determine the consistency of the logical formulae:

Theorem 1 (Ramsey's theorem)
Given any r, n and μ (integers), we can find an m_o such that if $m \geq m_o$ and the r-combinations (subsets with exactly r elements) of any Γ_m (set with m elements) are divided in any manner into μ mutually exclusive classes $C_i, i = 1,2,.. \mu$, then Γ_m must contain a subset Δ_n (subset with n elements) such that all the r-combinations of members of Δ_n belong to the same C_i.

This theorem became very important in graph theory, especially for the case r = 2 (we are interested mostly in this case). For this case we may reformulate the theorem in graph-theoretic language. We call a complete graph with n vertices an n-clique and denote it by K_n.

Theorem 2 Given any integers $k \geq 2$, $\ell \geq 2$, there exists an m_o, such that if $m \geq m_o$ then each graph with m vertices has either a k-clique or a set of ℓ independent vertices.

Proof One of the simplest proofs of Ramsey theorem when stated in this form is by Erdős and Szekeres [8]:

Obviously if $\kappa = 2$, then each graph with at least ℓ vertices has either an edge (i.e., a 2-clique) or does not have any edges and hence has a set of ℓ independent vertices.

For fixed κ_o and arbitrary ℓ let $m_o(\ell)$ be as desired in the theorem. When $m \geq \ell(m_o(\ell) + 1)$, then each graph G with m vertices has either a set of ℓ independent vertices, or the maximal number of independent vertices is $N < \ell$. Thus from a set S of N independent vertices there goes an edge to each of the remaining vertices. Then there is a vertex $v \in S$ from which at least $\frac{m-N}{N}$ edges are going out of S. Since $\frac{m-N}{N} > \frac{m-\ell}{\ell} \geq m_o(\ell)$, there is a κ_o-clique in the neighbourhood of v. Thus there is a $(\kappa_o + 1)$-clique in G.

The following is a natural generalization of this theorem:

<u>Theorem 3</u> Let $\kappa_1, \kappa_2, \ldots, \kappa_p$ be integers, $\kappa_i \geq 2$. Then there exists a maximal integer $R = R(\kappa_1, \kappa_2, \ldots, \kappa_p)$ such that K_R can be divided into p mutually edge-disjoint subgraphs $G_1, G_2, \ldots, G_p$ in such a way that G_i does not contain a κ_i-clique for $i = 1, 2, \ldots, p$.

Theorem 2 is a special case of theorem 3 for $p = 2$.

Many proofs are known, using various inductive inequalities.

Greenwood and Gleason in [11] used

$R(\kappa_1, \kappa_2, \ldots, \kappa_p) \leq R(\kappa_1 - 1, \kappa_2, \ldots, \kappa_p) + R(\kappa_1, \kappa_2 - 1, \ldots, \kappa_p) + \ldots + R(\kappa_1, \ldots, \kappa_p - 1) + p - 1$;

a stronger form of this inequality is available however - proved by

Kalbfleisch in [12]:

<u>Lemma 1</u> $R(\kappa_1,..,\kappa_p) \le R(\kappa_1-1,\kappa_2,..,\kappa_p) +..+ R(\kappa_1,\kappa_2,..,\kappa_p-1) + 1$ for $\kappa_i \ge 3$.

We shall prove theorem 3 and lemma 1 together with theorem 1 in a general form of Ramsey's theorem:

<u>Theorem 4</u> Let $\kappa_1,\kappa_2,\dots,\kappa_p$ and r be natural numbers, $\kappa_i \ge r$. Then there exists a maximal natural number $S = S(r;\kappa_1,\dots,\kappa_p)$ such that all the r-combinations (subsets with r elements) of a set with S elements can be divided into p mutually disjoint classes $T_1,T_2,\dots,T_p$ in such a way that T_i does not contain all the r-combinations of any subset with κ_i elements.

<u>Proof</u> (According to Erdős and Szekeres [8]):

Clearly $S(1;\kappa_1,\kappa_2,\dots,\kappa_p) = \kappa_1 + \kappa_2 +\dots+ \kappa_p-p$,

$S(i;i,\kappa_2,\dots,\kappa_p) = S(i;\kappa_2,\dots,\kappa_p)$ and $S(i;\kappa) = \kappa - 1$.

Now the proof is ready by an induction if we prove the following lemma:

<u>Lemma 2</u> $S(i;\kappa_1,\kappa_2,\dots,\kappa_p) \le S\big(i-1;S(i;\kappa_1-1,\kappa_2,\dots,\kappa_p) + 1, S(i;\kappa_1,\kappa_2-1,\dots,\kappa_p) + 1,\dots,S(i;\kappa_1,\dots,\kappa_p-1) + 1\big) + 1.$

Since $S(2;\kappa_1,\dots,\kappa_p) = R(\kappa_1,\dots,\kappa_p)$, lemma 2 implies lemma 1.

<u>Proof</u> Take a set with

$$S\big(i-1;\ S(i;\kappa_1-1,\dots,\kappa_p) + 1,\dots,S(i;\kappa_1,\dots,\kappa_p-1) + 1\big) + 2$$

elements, denote by A one fixed element.

Let the i-combinations be divided into mutually disjoint sets

$T_1, T_2, .., T_p$.

The i-combinations containing the element A may be considered as (i-1)-combinations (by removal of A).

Define a partition of all (i-1)-combinations of $S\big(i-1; S(i;\kappa_1-1,..,\kappa_p)+1,..,S(i;\kappa_1,..,\kappa_p-1)+1\big)+1$ elements into sets $T'_1,..,T'_p$ by $(A_1,A_2,..,A_{i-1}) \epsilon T'_\ell$ if and only if $(A_1,A_2,..,A_{i-1},A) \epsilon T_\ell$.

Then a certain T'_j contains all the (i-1)-combinations of some $S(i;\kappa_1,\kappa_2,..,\kappa_j-1,..,\kappa_p)+1$ elements and hence there is either a set of k_ℓ elements with all the i-combinations in T_ℓ for $\ell \neq j$, or a set of (κ_j-1) elements with all the i-combinations in T_j and thus together with A a set of κ_j elements with all the i-combinations in T_j.

§2. THE RAMSEY NUMBERS

Let us note some properties of the Ramsey numbers (similar ones hold for $S(r;\kappa_1,\ldots,\kappa_p)$) :

(R1) $R(\kappa_1,\ldots,\kappa_p)$ is invariant under a permutation of the κ_i's

(R2) $R(2,\kappa_2,\ldots,\kappa_p) = R(\kappa_2,\ldots,\kappa_p)$

(R3) $R(\kappa) = \kappa-1$

(R4) For each $n \le R(\kappa_1,\kappa_2,\ldots,\kappa_p)$ there is a partition of K_n into p edge-disjoint subgraphs $G_1,G_2,\ldots,G_p$ such that no G_i contains a κ_i-clique.

The precise determination of the Ramsey numbers is a difficult combinatorial problem. The first estimates of these numbers were obtained from the proofs of Ramsey's theorem. For example, Szekeres [8] noticed in 1935 that if

$$R(\kappa,\ell) < R(\kappa-1,\ell) + R(\kappa,\ell-1)$$

then $R(\kappa,\ell)$ has to be smaller than the binomial coefficient, i.e.,

(1.) $$R(\kappa,\ell) < \binom{\kappa+\ell-2}{\kappa-1}$$

In [6] (1952) it was proved that

(2.) $$R_p(\kappa) < p^{(\kappa-2).p\,+\,1}$$

where we write $R_p(\kappa)$ instead of $R(\underbrace{\kappa,\kappa,\ldots,\kappa}_{\text{p times}})$.

In 1947, Erdős proved a lower bound in [2]:

(3.) $$R(\kappa,\kappa) \ge 2^{\frac{\kappa}{2}}$$

The formulation of the problem - to find $R(\kappa_1,\kappa_2,\ldots,\kappa_p)$ -

was introduced by R. E. Greenwood and A. M. Gleason in 1955. They gave also the first exact values:

$R(3,3) = 5$, $R(3,4) = 8$, $R(3,5) = 13$, $R(4,4) = 17$ and $R(3,3,3) = 16$.

They also obtained

(4.) $\quad R_p(3) \leq [p!\, e]$

which is better than the value p^{p+1} of (2.) and in a weaker form will be proved in §4, using different methods.

Since then only two new Ramsey numbers have been evaluated:

$R(3,6) = 17$ and $R(3,7) = 22$ see [10], [15]. From [10], [13], [14], [19] we obtain the following table of present estimates of $R(k, \ell)$:

$k \backslash \ell$	3	4	5	6	7	8	9	10	11	12	13	14
3	5	8	13	17	22	26/29	35/36	38/43	42/53	48/62	57/72	62/83
4	8	17	24/28	33/44								
5	13	24/28	37/56	50/95								
6	17	33/44	50/95	101/181								

(lower estimate/upper estimate)

New general estimates include

(5.) $\quad R(3, \ell) \leq C.\, \ell^2 . \dfrac{\log \log \ell}{\log \ell}$

$(k \geq 3)$ $R(k, \ell) \leq C.\, \ell^{k-1} . \dfrac{\log \log \ell}{\log \ell}$ for suitable constant C,

proved in [10], and

(6.) $\quad R(3, \ell) > \dfrac{C.\, \ell^2}{\log \ell^2}$ $\qquad$ (see [4], [5], [7]).

We shall illustrate the non-constructive lower estimates of Ramsey numbers by proving (3.).

Theorem 5 $\quad R(k,k) \geq 2^{\frac{k}{2}}$

Proof by [2]: Take $N \leq 2^{\frac{k}{2}}$ vertices. There are $\binom{N}{k}$ k-cliques in K_N and if we take one fixed k-clique there are still $2^{\binom{N}{2}-\binom{k}{2}}$

subgraphs of K_N containing this κ-clique. Hence the number of all the subgraphs containing a κ-clique is less than

$$\binom{N}{\kappa}.2^{\binom{N}{2}-\binom{\kappa}{2}} < \frac{N^\kappa}{\kappa!}.2^{\binom{N}{2}-\binom{\kappa}{2}} < \frac{2^{\binom{N}{2}}}{2}$$

(since $\frac{N^\kappa}{\kappa!} < \frac{2^{\binom{\kappa}{2}}}{2}$ for $N \leq 2^{\frac{\kappa}{2}}$, $\kappa \geq 3$),

i.e., less than half of the number of all subgraphs of K_N. Thus there exists a subgraph without κ-clique such that its complement is also without κ-clique.

§3. NUMBER - THEORETICAL PROBLEMS

While investigating the solvability of the congruence $x^m + y^m = z^m \pmod{p}$, I. Schur proved the following theorem, [18]:

Theorem 6 (Schur's theorem)

Let $N > m!\,e$. In any partition of the numbers $1,2,..,N$ into m disjoint sets there is a set which contains two numbers together with their difference.

Def. A. A set S of natural numbers is called sum-free if $a\in S, b\in S$, $a>b$ implies $a-b \notin S$

Def. B. A set S of natural numbers is called sum-free if $a\in S, b\in S$ implies $a + b \notin S$

Obviously definitions A and B are equivalent; the term sum-free was introduced in [1]. Schur posed the following problem: What is the greatest natural number N_m such that one can divide the set $\{1,2,..,N_m\}$ into m sum-free sets?

From his theorem this number exists and $N_m \le [m!e]$.

Proof of Schur's theorem: Let $1,2,..,N$ be partitioned into sum-free sets $Z_1, Z_2,..,Z_m$ and let Z_1 be the largest set, $Z_1 = \{x_1, x_2,..,x_{n_1}\}$, say, where $x_1 < x_2 < .. < x_{n_1}$. Thus $N \le n_1 m$. Let Z_2 contain the greatest number of the (n_1-1) differences $x_2-x_1,..,x_{n_1}-x_1$ (these differences do not lie in Z_1), say $x_{k_1}-x_1, x_{k_2}-x_1,..,x_{k_{n_2}}-x_1$. Thus $n_1-1 \le n_2(m-1)$.

Continuing this procedure we obtain a chain $n_1 > n_2 > \dots > n_{p_o} = 1$

fulfilling $n_p - 1 \le n_{p+1}(m-p)$, i.e., $\frac{n_p}{(m-p)!} \le \frac{n_{p+1}}{(m-p-1)!} + \frac{1}{(m-p)!}$ for

$p = 1,2,\dots,p_o-1$.

Thus $\frac{n_1}{(m-1)!} \le \frac{1}{(m-1)!} + \frac{1}{(m-2)!} + \dots \frac{1}{(m-p_o)!} < e$ and $N \le n_1 . m < m!e$

Š. Znám in [20], [21] generalized Schur's problem as follows:

A set S of natural numbers is <u>κ-thin</u>, $(\kappa \ge 3)$ if $a_1, a_2, \dots, a_{\kappa-1} \in S$ $\Rightarrow a_1 + a_2 + \dots + a_{\kappa-1} \notin S$

Thus 3-thin sets are the same as sum-free sets. From [16] it follows that the following number exists:

> $f(\kappa, p)$ is the maximal natural number such that one can partition $\{1,2,\dots,f(\kappa,p)\}$ into p disjoint κ-thin sets.

Thus $f(3,p) = N_p$

Schur also proved a lower estimate for N_m, viz., $N_{m+1} \ge 3.\ N_m + 1$, see §4. This gives us $N_m \ge \frac{3^m-1}{2}$.

In [20] this was generalized to $f(\kappa,p+1) \ge \kappa . f(\kappa,p) + (\kappa-2)$ and thus

$$(7.)\quad f(\kappa,p) \ge \frac{\kappa-2}{\kappa-1}(\kappa^p-1)$$

Already for $\kappa = 3$, this estimate was improved in [1]. It was proved there, that

$$(8.)\quad f(3,p) > 89^{\frac{p}{4}-C.\log p}$$ for some constant C and sufficiently large p.

Znám's problem is a generalization of Schur's problem based on definition B of a sum-free set. If we look at definition A it seems to be natural to generalize Schur's problem in another way:

A set S of natural numbers is said to be <u>κ-poor</u> ($\kappa \geq 2$) if in every $(\kappa-1)$-tuple $a_1, a_2, \ldots, a_{\kappa-1} \in S$ with $a_1 < a_2 < \ldots < a_{\kappa-1}$, there is a couple a_i, a_j $(i > j)$ with $(a_i - a_j) \notin S$.

<u>Note 1</u>. 3-poor $\equiv$ sum-free $\equiv$ 3-thin.

<u>Note 2</u>. Only the empty set is 2-poor, and if we extend Znám's definition of a κ-thin set to $\kappa=2$, then obviously 2-poor $\equiv$ empty $\equiv$ 2-thin.

<u>Note 3</u>. While in κ-thin sets we have to check all the $(\kappa-1)$-tuples, in κ-poor sets we consider only $(\kappa-1)$-tuples of different elements. Moreover the following is true:

<u>Proposition 1</u> Every κ-thin set is κ-poor

<u>Proof</u> We have already noted this for $\kappa = 2,3$. Let $\kappa \geq 4$ and suppose S is not κ-poor, i.e., the elements $x_1 < x_2 < \ldots < x_{\kappa-1}$ of S have all their differences in S.

Then $x_1 + (x_2 - x_1) + (x_3 - x_2) + \ldots + (x_{\kappa-1} - x_{\kappa-2}) = x_{\kappa-1}$ is a sum of $\kappa-1$ elements of S belonging to S. Thus S is not κ-thin.

<u>Note 4</u>. Every κ-poor set is obviously $(\kappa+1)$-poor, while there are κ-thin sets which are not $(\kappa+1)$-thin - for example, $\{1, \kappa\}$ is such a set.

Let us state now the following problem:

<u>r-Problem</u> What is the maximal integer $r = r(\kappa_1, \kappa_2, \ldots, \kappa_p)$ ($\kappa_i \geq 2$ integers)

such that the set $\{1,2,\ldots,r\}$ can be divided into p pairwise disjoint sets $A_1, A_2, \ldots, A_p$ in such a way that each A_i is κ_i-poor? The existence of such an $r(\kappa_1,\kappa_2,\ldots,\kappa_p)$ is guaranteed by the following theorem:

__Theorem 7__ $r(\kappa_1,\kappa_2,\ldots,\kappa_p) \leq R(\kappa_1,\kappa_2,\ldots,\kappa_p) - 1$

Proof If the set $\{1,2,\ldots,r\}$ is partitioned into disjoint sets $A_1,A_2,\ldots,A_p$ and each A_i is κ_i-poor, then the complete graph with the vertices $0,1,2,\ldots,r$ can be divided into p edge-disjoint subgraphs $G_1,G_2,\ldots,G_p$ by definning (i,j) to be an edge in G_ℓ if and only if $|i-j| \in A_\ell$. Suppose the vertices $i_1 < i_2 \ldots < i_{\kappa_\ell}$ form a κ_ℓ-clique in G_ℓ. Then $(i_2-i_1) < (i_3-i_1) < \ldots < (i_{\kappa_\ell}-i_1)$ are elements of A_ℓ and so are all their differences, contrary to our assumption that A_ℓ is κ_ℓ-poor.

The idea of this proof is very similar to that of [1], [20], where $f(\kappa,p) \leq R_p(\kappa) - 1$ is proved. Since

$$(9.) \quad f(\kappa,p) \leq r_p(\kappa)$$

by proposition 1 (we denote $r(\underbrace{\kappa,\kappa,\ldots,\kappa}_{\text{p-times}})$ by $r_p(\kappa)$), our theorem 7 implies theorems of the type $f(\kappa,p) \leq R_p(\kappa) - 1$ from [1] or [20]. Theorem 7 or the theorems of the type $f(\kappa,p) \leq R_p(\kappa) - 1$ from [1], [20] and [21] also allow us to bound the Ramsey numbers from below.

__Properties of the numbers r__

(r1) $\quad r(\kappa_1,\ldots,\kappa_p)$ is invariant under permutation of the κ_i's (r2) $r(2,\kappa_2,\ldots\kappa_p) = r(\kappa_2,\ldots,\kappa_p)$

(r3) $r(\kappa) = \kappa - 2$

(r4) For each $n < r(\kappa_1,\kappa_2,\ldots,\kappa_p)$ there is a partition of the set $\{1,2,\ldots,n\}$ into p disjoint sets $A_1,A_2,\ldots,A_p$ such that each A_i is κ_i-poor.

Theorem 8 Let M be a κ-poor set of natural numbers.

Define $M + d = \{m+d \mid m \in M\}$ and $M_1 = \max M$, $M_2 = \min M$.

If $d > 2M_1 - M_2$, then $N = M \cup (M+d) \cup (M+2d) \cup \ldots \cup (M+nd)$ is also κ-poor for an arbitrary natural number n.

Proof: Suppose the distinct natural numbers $b_1, b_2, \ldots, b_{\kappa-1} \in N$ fulfill $|b_i - b_j| \in N$ for all $i \neq j$.

The sets $M, M+d, \ldots, M+n.d$ are pairwise disjoint in view of the following inequality:

$$\max(M+t.d) = M_1 + t.d < 2M_1 + t.d < M_2 + (t+1).d = \min(M+(t+1).d),$$

$$t = 0,1,\ldots,n-1.$$

We now define $C_j = b_j - t.d.$ for $b_j \in M + t.d$ $(t=0,1,\ldots,n)$.

Since

$$\max(M+(t-1).d) \leq M_1 + (t-1).d + (M_1 - M_2) < t.d < \min(M + t.d),$$

we have $t.d \notin N$ for $t = 1,2,\ldots,n$, and hence $b_i - b_j \neq t.d$ whenever $i \neq j$. Thus the C_i are again pairwise different. Moreover, $C_i \in M$, $i = 1,\ldots,\kappa-1$.

Take $C_i < C_j$ arbitrary. Then there are two cases:

(a) $C_i = b_i - t.d$ & $C_j = b_j - t.d$, i.e., $b_i, b_j \in (M+t.d)$ and the difference

$C_j - C_i = b_j - b_i \in N$. Since $b_j - b_i \le M_1 + td - M_2 - td < 2M_1 - M_2 < d$, it follows that $C_j - C_i \in M$.

(b) $C_i = b_i - t.d$ & $C_j = b_j - s.d$, i.e., $b_i \in (M+t.d)$ & $b_j \in (M+s.d)$ for $s \neq t$. Here necessarily $t<s$, since if $t>s$, then $b_i > b_j$ (since $M+t.d$ and $M+s.d$ are disjoint) and $b_i - t.d < b_j - s.d$; thus
$\max(M+(t-s-1)d) = M_1 + (t-s-1)d < M_2 - M_1 + (t-s)d \le b_i - b_j < (t-s)d < \min(M+(t-s)d)$,
i.e., $b_i - b_j \notin N$, contrary to our assumption.

Now the difference $C_j - C_i = b_j - b_i - (s-t).d$, where $b_j - b_i \in M + (s-t).d$, since

$(s-t).d < b_j - b_i \le M_1 - M_2 + (s-t).d < (s-t+1).d$ and we assumed $b_j - b_i \in N$.

Thus $C_j - C_i \in M$, and we have shown that if N is not κ-poor, then neither is M.

<u>Corollary</u> If the set M can be partitioned into p disjoint sets $A_1, A_2, \ldots, A_p$ where each A_i is κ_i-poor, and if $d > 2M_1 - M_2$, then the set $N = M \cup (M+d) \cup \ldots \cup (M+n.d)$ can also be partitioned into p disjoint sets $B_1, B_2, \ldots, B_p$ such that each B_i is κ_i-poor.

<u>Proof</u> Put $B_i = A_i \cup (A_i + d) \cup \ldots \cup (A_i + n.d)$ and use theorem 8;

$d > 2M_1 - M_2 > 2 \max A_i - \min A_i$ for all $i = 1, 2, \ldots, p$.

Put $f_t(m) = r_m(t)$ and define $g_t(n)$ to be the smallest number of t-poor sets into which $\{1, 2, \ldots, n\}$ can be partitioned.

<u>Theorem 9</u> $f_t(\kappa.m + g_t(\kappa.f_t(m))) \ge (2.f_t(m) + 1)^{\kappa} - 1$

<u>Proof</u> Let $X = 2.f_t(m)+1$ and write the numbers $1,2,\ldots,X^\kappa-1$ in base X.

Define $A = \{a_o+a_1X+\ldots+a_{\kappa-1}X^{\kappa-1} \mid a_i \le f_t(m),\ i = 0,\ldots,\kappa-1\}$ and $B_j=\{a_o+a_1X+\ldots+a_{\kappa-1}X^{\kappa-1} \mid a_i \le f_t(m)$ for $i<j$ and $a_j>f_t(m)\}$ $j = 0,1,\ldots,\kappa-1$.

Partition the set $\{1,2,\ldots,\kappa.f_t(m)\}$ into $g=g_t(\kappa.f_t(m))$ disjoint t-poor sets $M_1,M_2,\ldots,M_g$ and the set $\{1,2,\ldots,f_t(m)\}$ into m t-poor disjoint sets $N_1,N_2,\ldots,N_m$.

Then $A = \bigcup_{\ell=1}^{g} A_\ell$ where $A_\ell = \{ \sum_{i=o}^{\kappa-1} a_iX^i \in A \mid \sum_{i=o}^{\kappa-1} a_i \in M_\ell\}$ is clearly a partition into disjoint sets. Each A_ℓ is t-poor, since if $z_1<z_2<\ldots<z_{t-1}$, elements of A_ℓ, have all their differences in A_ℓ

and $z_i = \sum_{j=o}^{\kappa-1} a_j^iX^j$, then $a_j^1 \le a_j^2 \le \ldots \le a_j^{t-1}$ for all $j=0,1,\ldots,\kappa-1$

and hence $\sum_{j=o}^{\kappa-1} a_j^1 < \sum_{j=o}^{\kappa-1} a_j^2 <\ldots< \sum_{j=o}^{\kappa-1} a_j^{t-1}$ are elements of M_ℓ with

all their differences in M_ℓ, contrary to the assumption that M_ℓ is t-poor. Furthermore $B_j= \bigcup_{\ell=1}^{m} B_j^\ell$, where $B_j^\ell = \{ \sum_{i=o}^{\kappa-1} a_iX^i \in B_j \mid$ there is a (uniquely determined) $b = \bar{a}_j \in N_\ell$, such that $a_j \equiv -\bar{a}_j \pmod X\}$, is a partition into disjoint sets. If $z_1<z_2<\ldots<z_{t-1}$, elements of B_j^ℓ, have all their differences in B_j^ℓ and $z_s = \sum_{i=o}^{\kappa-1} a_i^sX^i$, then $a_i^1 \le a_i^2 \le \ldots \le a_i^{t-1}$ for all $i<j$ and $f_t(m)<a_j^{t-1} <a_j^{t-2} <\ldots<a_j^1$; the difference $a_j^r - a_j^s = C_{rs} \in N_\ell$, $(r<s)$, for a suitable $C_{rs} \le f_t(m)$. Thus we have $\bar{a}_j^1 < \bar{a}_j^2 <\ldots<\bar{a}_j^{t-1}$, elements of N_ℓ, such that their differences $\bar{a}_j^s-\bar{a}_j^r \equiv C_{rs} \pmod X$. Since $\bar{a}_j^s$, $\bar{a}_j^r$ and C_{rs} are elements of N_ℓ, and therefore are smaller than or equal to $f_t(m)$, we have $\bar{a}_j^s - \bar{a}_j^r = C_{rs}$. So, all the sets B_j^ℓ are t-poor.

Theorem 9 and its proof are generalizations of [1], where this theorem is proved for t=3.

We shall make use of this theorem later.

§4. SYMMETRY AND CYCLICITY

Let us look at Schur's proof of the theorem:

Theorem 10 $N_{m+1} \geq 3.\ N_m + 1$

Proof Suppose $\{1,2,\ldots,N_m\}$ is paritioned into sum-free and pairwise disjoint sets $A_1, A_2, \ldots A_m$.

Then, putting $B_j = \{3a \mid a \in A_j\} \cup \{3a-1 \mid a \in A_j\}$ for $j=1,2,\ldots,m$

and $B_{m+1} = \{3a-2 \mid a = 1,2,\ldots,N_m+1\}$,

we obtain a partition $\{1,2,\ldots,3\ N_m+1\} = \bigcup_{j=1}^{m+1} B_j$. The sets B_j are obviously disjoint and sum-free (e.g., if $3a \in B_j$ and $3b-1 \in B_j$, then $3(a+b)-1 \in B_j$ implies $a+b \in A_j$ contrary to the sum-freeness of A_j).

This construction has an interesting property: If the sets A_j fulfill the condition

$$a \in A_j \Rightarrow N_m+1-a \in A_j$$

then the sets B_j fulfill the condition

$$a \in B_j \Rightarrow (3N_m+1)+1-a \in B_j$$

Proof B_{m+1} fulfills the condition, since $3N_m+2-(3a-2) = 3(N_m+2-a)-2$.

For $j = 1,2,\ldots,m$ $\quad 3N_m+2-3a = 3\ (N_m+1-a)-1$

and $\quad 3N_m+2-(3a-1) = 3(N_m+1-a)$

A set $M \subset \{1,2,\ldots,n\}$ fulfilling $a \in M \Rightarrow (n+1)-a \in M$ is called n-symmetric.

A partition $\{1,2,..,n\} = \bigcup_{\kappa=1}^{p} A_\kappa$ is called a <u>symmetric partition</u> if all the sets A_i are n-symmetric.

Consider all $n \leq r(\kappa_1,..,\kappa_p)$ such that there is a symmetric partition of $\{1,2,...,n\}$ into disjoint sets $A_1,A_2,...,A_p$ where A_i is κ_i-poor. Then there is a greatest such n and we denote it by $s(\kappa_1,\kappa_2,..,\kappa_p)$. Clearly

$$s(\kappa_1,\kappa_2,...,\kappa_p) \leq r(\kappa_1,\kappa_2,...,\kappa_p)$$

Since Schur's construction preserves symmetry (as we proved) we can start with $N_1 = 1$ (which division is trivially symmetric) and obtain the lower bound

(10.) $\quad s_p(3) \geq \frac{3^p - 1}{2}$

From Znám's construction in [20] one can easily see, that it also preserves symmetry and

(11.) $\quad s_p(\kappa) \geq \frac{\kappa-2}{\kappa-1}(\kappa^p - 1)$

An example of a non-symmetric division is that of Baumert, published in [1] for $r_4(3) = 44$

A_1: 1 3 5 15 17 19 26 28 40 42 44

A_2: 2 7 8 18 21 24 27 33 37 38 43

A_3: 4 6 13 20 22 23 25 30 32 39 41

A_4: 9 10 11 12 14 16 29 31 34 35 36

Note, that only 2 pairs 12 and 33, and 15 and 30 are destroying the symmetry.

It will be proved later that there is no symmetrical division of $\{1,2,...,44\}$ into sum-free sets, thus the same example also provides a case where $s(\kappa_1,\dots,\kappa_p) < r(\kappa_1,\dots,\kappa_p)$.

The construction in the proof of theorem 9 does not preserve symmetry ($1 \in A$ while $X^{\kappa}-1 \in B_1$).

Most of the known Ramsey numbers have been reached by very regular paritions of the complete graph; Kalbfleisch calls them "regular colorings", see [12], [14]. Actually, these partitions are strongly connected with the symmetrical partitions of natural numbers. A graph G with n vertices is called n-cyclic if there is an $(n-1)$-symmetric set S such that if we identify the vertices of G with the natural numbers modulo n, then (r,s) is an edge in G if and only if $|r-s| \in S$.

A cyclic graph (defined also by Graever and Yackel in [10] is then completely described by the couple (n,M) where n is a natural number and M is an $(n-1)$-symmetric set of natural numbers.

A cyclic partition of K_n is a partition into n-cyclic subgraphs.

The cyclic Ramsey number $C(\kappa_1,\kappa_2,\dots,\kappa_p)$ is the greatest number n such that K_n has a cyclic partition into edge-disjoint subgraphs $G_1,G_2,\dots,G_p$ such that no G_i contains a κ_i-clique.

Remark $C(\kappa_1,\kappa_2,\dots,\kappa_p)$ is identical with Kalbfleisch's $L(\kappa_1,\kappa_2,\dots,\kappa_p)$ (see [12], [14]).

The connection of symmetry and cyclicity may be seen from the following theorem:

<u>Theorem 11</u> $C(\kappa_1,\kappa_2,..,\kappa_p)-1=s(\kappa_1,\kappa_2,..,\kappa_p)\le r(\kappa_1,\kappa_2,..,\kappa_p)\le R(\kappa_1,\kappa_2,..,\kappa_p)-1$

<u>Proof</u> We only have to prove the first equality.

To prove $s(\kappa_1,\kappa_2,..,\kappa_p) \le C(\kappa_1,\kappa_2,..,\kappa_p)-1$ we can repeat the proof of theorem 7 and note that if $\{1,2,..,s\} = \bigcup_{j=1}^{p} A_j$ is a symmetrical partition, then the subgraphs $G_1,G_2,...,G_p$ are $(s + 1)$-cyclic.

Also $C(\kappa_1,\kappa_2,...,\kappa_p)-1 \le s(\kappa_1,\kappa_2,...,\kappa_p)$, since having a cyclic partition of the complete graph on the vertices $0,1,2,...,C-1$ into edge-disjoint subgraphs $G_1,G_2,...,G_p$ where no G_i contains a κ_i-clique, we may define the partition $\{1,2,...,C-1\} = \bigcup_{i=1}^{p} A_i$ by putting $n\epsilon A_i$ if and only if $(0,n)$ is an edge in G_i. This is a symmetrical partition into disjoint sets and each A_i is κ_i-poor, for if $n_1,n_2,...,n_{\kappa_i-1}$ are distinct numbers from A_i such that all their differences are in A_i, then G_i contains the κ_i-clique on the vertices $0,n_1,n_2,...,n_{\kappa_i-1}$.

<u>Properties of s-numbers and cyclic Ramsey numbers</u>

(s1) $s(\kappa_1,\kappa_2,..,\kappa_p)$ is invariant under permutation of the κ_i's

(s2) $s(2,\kappa_2,..,\kappa_p) = s(\kappa_2,..,\kappa_p)$

(s3) $s(\kappa) = \kappa-2$

There does not hold any property similar to (r4). A counter-example follows from the next proposition:

<u>Proposition 2</u> There is no $(3\kappa + 2)$-symmetric sum-free set.

<u>Proof</u> $(\kappa+1) + (\kappa+1) = (3\kappa+2) + 1 - (\kappa+1)$

Thus $s_p(3) \nmid 3.\kappa + 2.$

From proposition 2 it follows also that $s_4(3)=N_4$ is either 40 or 42 or 43.

From theorem 11:

(C1) $C(\kappa_1,\ldots,\kappa_p)$ is invariant under permutation of the κ_i's

(C2) $C(2,\kappa_2,\ldots,\kappa_p) = C(\kappa_2,\ldots,\kappa_p)$

(C3) $C(\kappa) = \kappa - 1$

and no property of type (R4) holds.

From theorems 6 and 11, $C_p(3) \le [m!e] + 1$, which is a weaker form of (4.). Kalbfleisch in [12] and [14] determined some values of $C(\kappa_1,\ldots,\kappa_p) = s(\kappa_1,\ldots,\kappa_p) + 1$:

k \ ℓ	3	4	5	6	7	8	9	10	11	12	13	14
3	5	8	13	16	21	26	35	38	45	48	≥57	≥62
4	8	17	24	33								
5	13	24	37/56	50/95								
6	16	33	50/95	101/181								

Table of $C(\kappa, \ell)$

Moreover, $C(3,3,3) = 14$ since $s(3,3,3) \le r(3,3,3) = 13$, and Kalbfleisch [14] gives $C(3,3,4) = 29$, $C(4,4,4) \ge 79$

$C_4(3) \ge 41$, $C_5(3) \ge 101$, $C_6(3) \ge 277$.

We know already that $C_4(3) = 41$, 43 or 44; other such results will be improved in Corollaries 1,2.

Theorem 12 $s(\kappa_1,\kappa_2,\ldots,\kappa_p) \ge (\kappa_1-1)s(\kappa_2,\kappa_3,\ldots,\kappa_p) + (\kappa_1-2).[s(\kappa_2,\ldots,\kappa_p)+1]$, $p>1$

Proof Put $s = s(\kappa_2,\kappa_3,\ldots,\kappa_p)$

Let $\{1,2,\ldots,s\} = \bigcup_{i=2}^{p} A_i$ be a symmetrical partition, A_i κ_i-poor, pairwise disjoint.

Then let $d = 2.s + 1$ and define the sets $B_1, B_2, \ldots, B_p$ by

$$B_i = \bigcup_{n=o}^{\kappa_1-2} (A_i + n.d) \quad \text{for} \quad i = 2,3,\ldots,p$$

and $$B_1 = \bigcup_{n=o}^{\kappa_1-3} (A + n.d), \quad \text{where } A = \{s + 1, s + 2, \ldots, 2s + 1\}.$$

Since $A \cup \bigcup_{i=2}^{p} A_i = \{1,2,\ldots,2s + 1\}$ we obtain a partition of the

set $\{1,2,\ldots,(\kappa_1-1).s + (\kappa_1-2).(s + 1)\}$ into p disjoint sets $B_1, B_2, \ldots, B_p$.

Take $b \in B_i$ $(i = 2,3,\ldots,p)$, then $b = a + n.d$ for $a \in A_i, 0 \leq n \leq \kappa_1-2$ and

$(\kappa_1-1).s+(\kappa_1-2)(s+1)+1-(a+n.d)=(\kappa_1-2-n).s+(\kappa_1-2-n)(s+1)+(s+1)-a=$

$=(\kappa_1-2-n).d + (s + 1) - a \in B_i$,

since A_i is s-symmetric. Similarly $b \in B_1$, $b=a+n.d$, $a \in A$, $0 \leq n \leq \kappa_1-3$, implies

$(\kappa_1-1).s+(\kappa_1-2)(s+1)+1-(a+n.d)=(\kappa_p-3-n).d+3s+2-a \quad \in B_1$, since A is $(3s + 1)$-symmetric.

Thus we have a symmetric partition.

The sets B_i are κ_i-poor $(i = 2,3,\ldots,p)$ by the corollary of theorem 8, since $d > 2.s-1$.

Also the set B_1 is κ_1-poor, since for every (κ_1-1)-tuple of distinct elements of B_1 there is a sequence $A + n_o.d$ containing two of them, say a and b (there are only (κ_1-2) such sequences) and $|a-b| \leq s$. Thus $|a-b| \notin B_1$.

<u>Remark</u> Obviously if we do not care about symmetry we may repeat the above proof and obtain

(12.) $r(\kappa_1,\ldots,\kappa_p) \geq (\kappa_1-1).r(\kappa_2,\ldots,\kappa_p) + (\kappa_1-2).[r(\kappa_2,\ldots,\kappa_p) + 1]$

<u>Theorem 13</u> $s(\kappa_1,\kappa_2,\ldots,\kappa_p) \geq \sum_{i=1}^{p} (\kappa_i-2) \, . \prod_{j<i} (2\kappa_j-3)$

<u>Proof</u>
$$s(\kappa_1\kappa_2,\ldots,\kappa_p)\geq(\kappa_1-1).s(\kappa_2,\ldots,\kappa_p)+(\kappa_1-2).[s(\kappa_2,\ldots,\kappa_p)+1] =$$
$$=(2\kappa_1-3).s(\kappa_2,\ldots,\kappa_p)+(\kappa_1-2)\geq$$
$$\geq(2\kappa_1-3)(2\kappa_2-3).s(\kappa_3,\ldots,\kappa_p)+(2\kappa_1-3)(\kappa_2-2)+(\kappa_1-2)\geq\ldots\geq$$
$$\geq(2\kappa_1-3)(2\kappa_2-3)\ldots(2\kappa_{p-1}-3)(\kappa_p-2) +$$
$$+(2\kappa_1-3)(2\kappa_2-3)\ldots(2\kappa_{p-2}-3)(\kappa_{p-1}-2)+\ldots+(2\kappa_1-3)(\kappa_2-2)+(\kappa_1-2).$$

In the last step we have used property (s3).

Considering property (s1) one may try to improve the above theorem by ordering the κ_i's in such a way that the expression on the right would be maximal. However, this does not give any improvement since the following is true.

<u>Theorem 14</u> $\sum_{i=1}^{p} (\kappa_i-2).\prod_{j<i} (2\kappa_j-3)$ is invariant under permutation of the κ_i's.

<u>Proof</u> Since (κ_1-2) is trivially invariant we can continue by induction on p.

If the expression is invariant for p-1, then it is clear from
$$\sum_{i=1}^{p} (\kappa_i-2). \prod_{j<i} (2\kappa_j-3)= \sum_{i=1}^{p-1} (\kappa_i-2). \prod_{j<i} (2\kappa_j-3)+(\kappa_p-2). \prod_{j<p} (2\kappa_j-3)$$

that it suffices to consider the transpositions $\kappa_p \leftrightarrow \kappa_t$.

Since $1 + 2.\sum_{t<i<p}(\kappa_i-2).\prod_{t<j<i}(2\kappa_j-3) = 1 + 2(\kappa_{t+1}-2) +$

$+ 2(\kappa_{t+2}-2)(2\kappa_{t+1}-3) +\ldots+ 2(\kappa_{p-1}-2).\prod_{t<j<p-1}(2\kappa_j-3) =$

$=(2\kappa_{t+1}-3) +[(2\kappa_{t+2}-3)(2\kappa_{t+1}-3)-(2\kappa_{t+1}-3)]+\ldots+[\prod_{t<j<p}(2\kappa_j-3) -$

$- \prod_{t<j<p-1}(2\kappa_j-3)] = \prod_{t<j<p}(2\kappa_j-3)$, the difference of our expression

before and after the transposition $\kappa_p \leftrightarrow \kappa_t$ is equal to

$$(\kappa_t-2).\prod_{j<t}(2\kappa_j-3)+(2\kappa_t-3).\sum_{i=t+1}^{p-1}(\kappa_i-2).\prod_{j<t}.\prod_{t<j<i}+(\kappa_p-2)(2\kappa_t-3).\prod_{j<t}.\prod_{t<j<p} -$$

$$-(\kappa_p-2).\prod_{j<t}+(2\kappa_p-3).\sum_{i=t+1}^{p-1}(\kappa_i-2).\prod_{j<t}.\prod_{t<j<i}+(\kappa_t-2)(2\kappa_p-3).\prod_{j<t}.\prod_{t<j<p} =$$

$$=(\kappa_t-\kappa_p).\prod_{j<t}(2\kappa_j-3).[1+2.\sum_{t<i<p}(\kappa_i-2).\prod_{t<j<i}(2\kappa_j-3)-\prod_{t<j<p}(2\kappa_j-3)] = 0$$

Corollary 1 of theorem 13: $R_p(\kappa)-1 \geq r_p(\kappa) \geq C_p(\kappa)-1=s_p(\kappa) \geq \frac{1}{2}((2\kappa-3)^p-1)$

Corollary 2 $R(\kappa,\ell)-1 \geq r(\kappa,\ell) \geq C(\kappa,\ell)-1=s(\kappa,\ell) \geq 2\kappa\ell-3(\kappa+\ell) + 4$

Corollary 1 is an improvement of Znám's result $R_p(\kappa)-1 \geq \frac{\kappa-2}{\kappa-1}(\kappa^p-1)$ (see [7]).

We can obtain some better estimates using theorem 12 on some known values of r and s (or their lower estimates):

So using Baumert's result $r_4(3) = 44$ and the remark after theorem 12 we get

<u>Estimate 1</u>: $R_p(3) \geq r_p(3) + 1 \geq 3^{p-4}.44 + \frac{3^{p-4}-1}{2} + 1$ for $p \geq 4$

Using $s(4,4)=16$ from [11], $s(5,5) \geq 36$ from [14] and $s(6,6) \geq 100$ from [14]:

<u>Estimate 2</u>: $C_p(4)=s_p(4)+1 \geq 5^{p-2}.16 + \frac{5^{p-2}-1}{2} + 1$, $p \geq 2$

<u>Estimate 3</u>: $C_p(5)=s_p(5)+1 \geq 7^{p-2}.36 + \frac{7^{p-2}-1}{2} + 1$, $p \geq 2$

<u>Estimate 4</u>: $C_p(6)=s_p(6)+1 \geq 9^{p-2}.100 + \frac{9^{p-2}-1}{2} + 1$, $p \geq 2$.

Estimates 1-4 improve the following presently known results:

$R_4(3) \geq 41$ (see [19], [26]) to $R_4(3) \geq 45$

$C_5(3) \geq 101$ and $C_6(3) \geq 277$ from [26] to $C_5(3) \geq 122$ and

$C_6(3) \geq 365$, ($R_5(3) \geq 136$ and $R_6(3) \geq 401$),

$C_3(4) \geq 79$ (see [26]) to $C_3(4) \geq 83$,

$R_3(5) \geq 199$ (see [24]) to $C_3(5) \geq 256$

Moreover they give $C_4(4) \geq 413$, $C_4(5) \geq 1789$, $C_3(6) \geq 905$, etc.

Finally let us improve the asympotical estimates. From [20] (estimate (7.) here), Corollary 1 and estimates 1-4 it follows that $R_p(k) > (2k-3)^p.\mathrm{Const}$ (from [20] only $R_p(k) > k^p.\mathrm{Const}$). Thus our function $g_t(n)$ defined in §3 fulfills

(13.) $g_t(n) < \log n$

and $f_t(k.m + \log(k.f_t(m))) > (2.f_t(m) + 1)^k - 1$

by theorem 9.

For $t = 3$ and $m = 4$ this gives $r_\kappa(3) > 89^{\frac{\kappa}{4}-C.\log\kappa}$, stated in [1] (see (8.) above), and also for $t = 4,5,6$ and $m = 2$ the following estimates:

<u>Estimate 5</u> $R_\kappa(4) > r_\kappa(4) > 33^{\frac{\kappa}{2}-C.\log\kappa}$ for a constant C and sufficiently large κ.

<u>Estimate 6</u> $R_\kappa(5) > r_\kappa(5) > 73^{\frac{\kappa}{2}-C.\log\kappa}$ for a constant C and sufficiently large κ.

<u>Estimate 7</u> $R_\kappa(6) > r_\kappa(6) > 201^{\frac{\kappa}{2}-C.\log\kappa}$ for a constant C and sufficiently large κ.

These are obviously better than $C.5^\kappa$, $C.7^\kappa$, $C.9^\kappa$, respectively.

Estimates 1-7 can easily be improved if we find exact values, or good estimates of $r_\kappa(n)$ for small κ.

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