ULTRASONIC ATTENUATION IN METALS

THEORY OF ULTRASONIC ATTENUATION IN METALS DUE TO INTERACTIONS WITH CONDUCTION ELECTRONS

By

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ABSTRACT

Working within the framework of the linearized Boltzmann equation for the conduction electrons the existing theoretical treatments of ultrasonic attenuation in metals are extended to include realistic descriptions of the electronic structure and electron-lattice interaction. A variational solution of the Boltzmann equation which allows the inclusion of phonon drag effects is derived. An anisotropic scattering time solution is also presented. Both of these solutions are applied to calculation of the attenuation coefficient in pure metals and dilute alloys.

The theory of the effects of electron-electron collisions on the ultrasonic attenuation in metals is also examined.

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CHAPTER I

INTRODUCTION

Ultrasonic vibrations in solids are observed to show a simple exponential attenuation over a large range of frequencies and applied stresses (1,2). At least ten distinct physical mechanisms that contribute to this attenuation have been postulated for different types of solids under various conditions (3). In large single crystals of reasonably pure non-magnetic metals, however, it is believed that just two of these mechanisms are important. Since the attenuation in such crystals is reduced by annealing and increased by neutron irradiation, part of the attenuation is attributed to the interaction of the ultrasonic vibrations with dislocations. There is a vast literature dealing with experimental and theoretical investigations of this contribution to the attenuation (4). The other important cause of attenuation in metals is the direct interaction of ultrasonic vibrations with conduction electrons. That this effect is important is suggested by experiments which show that the attenuation in the superconducting state of such metals as lead, tin and aluminum is dramatically lower than that in the normal state of the same metals at the same temperature. In most metals this direct interaction with the conduction electrons is important for temperatures below about 20 K (8). Previous theoretical treatments of this contribution to the attenuation have used very simplified models for many of the details of the physics of the metals considered (9,10,11). The prime purpose of the present thesis is to extend the theory of ultrasonic attenuation in metals due to conduction electrons to include realistic descriptions of the electronic structure and the electron-lattice interaction. Some proposals for numerical calculations of the attenuation in real metals will also be made.

Chapter II of this thesis is a brief review of the theory of the electron-phonon interaction in the orthogonalized plane wave pseudopotential formalism. The theory of scattering of electrons by substitutional impurities is also examined.

Chapter III is a discussion of the current state of the theory of ultrasonic attenuation due to conduction electrons. In section 3.1 the Boltzmann equation for the electron distribution function in a metal which is disturbed by an impressed acoustic wave is derived. Sections 3.2 and 3.3 are examinations of two approximate methods which have been used to solve the Boltzmann equation and obtain results for the attenuation. In section 3.2 the isotropic scattering time approximation (9,10) is used to define the range of frequencies which will be of interest in this work ("low frequency regime") and to derive certain approximate results that will be needed in Chapter IV. Section 3.3 is a brief review of the approximation proposed by Bhatia and Moore (11) to solve the Boltzmann equation. This approximation will serve as the basis of the more sophistocated treatment to be presented in Chapter V.

In Chapter IV a variational theorem for the Boltzmann equation is derived and then used to investigate several aspects of ultrasonic attenuation in metals. Section 4.1 is the derivation of the variational solution to the Boltzmann equation. This variational solution is a distinct improvement over the existing theory because it allows the inclusion of realistic treatments of the electronic structure and electron-lattice scattering processes. Section 4.2 presents the application of the variational formalism to the zero temperature attenuation in dilute alloys. The variational method is applied to the attenuation in ideal crystals in section 4.3. Consideration is given to both the temperature and volume dependence of the attenuation. Section 4.4 is a treatment of the effects of phonon drag on the ultrasonic attenuation.

In Chapter V an "anisotropic scattering time" solution to the Boltzmann equation is proposed. Sections 5.1 and 5.2 examine this solution in ideal crystals and dilute alloys respectively.

Chapter VI presents a theory of the effects of electron-electron collisions on the ultrasonic attenuation.

Throughout this thesis SI units are employed.

CHAPTER II

ELECTRON-LATTICE INTERACTION

In the orthogonalized plane wave pseudopotential formalism (12) one distinguishes between core electrons, which are describable by wavefunctions localised on the atomic sites, and conduction electrons whose wavefunctions are written as linear combinations of plane waves. In principle the conduction electron wavefunctions can be determined by solving a Schrodinger equation with a realistic electron-ion potential. The wavefunction determined in this way must also be orthogonal to the core states. It turns out that the effect of this orthogonalization can be accounted for by adding a repulsive term to the electron-ion potential used in the Schrodinger equation. When this term is added the new total "potential" is called the pseudopotential. The orthogonalization term tends to nearly cancel the attractive electron-ion potential and the resulting pseudopotential is weak enough that the electron-ion interaction can be treated in first order perturbation theory. It is also weak enough that the conduction electron (pseudo)wavefunctions can be adequately described by a relatively few plane waves.

One can attempt to calculate pseudopotentials for various ions from first principles but a much simpler way of obtaining a useful pseudopotential for a particular metal is to take a parametrized model form and fit the parameters to some measurable property of the metal concerned. An example that is of some interest in the present work is the open core Ashcroft model pseudopotential which has the real space form

$$W^{\circ}(\underline{r}) = -\frac{Ze^2}{4\pi\varepsilon_0 r} \qquad r > R_c$$

$$W^{\circ}(\underline{r}) = 0 \qquad r < R_c \qquad (2.1)$$

where Z is the valence of the ion, e is the elementary charge, ε_0 is the permittivity of free space and R_c is a parameter to be fitted. It has the matrix element

$$\langle \underline{k}+q | W^{\circ} | \underline{k} \rangle = \frac{Ze^2}{\Omega_0 \varepsilon_0 q^2} \cos(qR_c) \qquad (2.2)$$

where Ω_0 is the atomic volume. Inside the metal, of course, the conduction electrons screen $W(\underline{q})$ and the effective form factor is

$$W(q) = \frac{W^{\circ}(q)}{\varepsilon^*(q)} \qquad (2.3)$$

where ε^* is the dielectric function. For a metal with a spherical fermi surface one can write (13)

$$\varepsilon^*(q) = 1 + [1 - f(q)] \frac{Ze^2}{\Omega_0 \varepsilon_0 q^2} \left(\frac{2}{3}E_F\right)^{-1} F(q) \qquad (2.4)$$

where

$$F(q) = \frac{1}{2} + \frac{4k_F^2 - q^2}{8k_F q} \ln \left| \frac{2k_F + q}{2k_F - q} \right| \tag{2.5}$$

and E_F and k_F are the fermi energy and wavevector. The factor $f(q)$ is a function which accounts for exchange and correlation effects. Theoretical formulae for $f(q)$ are available at different levels of sophistocation. The simplest form for ε^* is the Hartree form which simply sets $f(q)=0$. A more realistic treatment by Singwi et al. (14) gives

$$f(q) = A\left(1 - e^{-B(q/k_F)^2}\right) \tag{2.6}$$

where A and B are weakly dependent on the density of conduction electrons.

The pseudopotential formalism can be used to calculate the transition rates of electrons scattered by phonons or by substitutional impurities. The derivations are straightforward and only the results will be presented here.

First consider electron-phonon scattering under the assumption that the electron wavefunctions are describable by m orthogonalized plane waves, i.e.

$$\psi_{\underline{k}}(\underline{r}) = \sum_{k_n}^{m} a_{\underline{k}_n}(\underline{k})\, e^{i\underline{k}_n\cdot\underline{r}}\, e^{i\underline{k}\cdot\underline{r}} \tag{2.7}$$

where the $\underline{k}_n$ are reciprocal lattice vectors and the $a_{\underline{k}_n}$ are expansion coefficients. Also assume that the phonon states, labelled by wavevector $\underline{q}$ and branch index λ, are populated as they would be in thermal equilibrium, i.e.

$$n_{q\lambda} = n^{o}_{q\lambda} = \left[e^{\beta\hbar\omega_\lambda(q)} - 1 \right]^{-1} \tag{2.8}$$

where

$$\beta = (k_B T)^{-1}$$

and $\omega_\lambda(q)$ is the angular frequency of the phonon $(q\lambda)$. Then the intrinsic transition rate for an electron in the state $\underline{k}$ to be scattered into an empty state $\underline{k}'$ is (15)

$$Q_{\underline{k}\underline{k}'} = \frac{2\pi}{\hbar} \sum_{\lambda} |g_{\underline{k}'\underline{k}\lambda}|^2 n^{o}_{q\lambda} \left[\delta(E_{\underline{k}'} - E_{k} - \hbar\omega_\lambda(q)) \right.$$

$$\left. + e^{\beta\hbar\omega_\lambda(q)} \delta(E_{\underline{k}'} - E_{\underline{k}} + \hbar\omega_\lambda(q)) \right] \tag{2.9}$$

where

$$q = \underline{k}' - \underline{k}$$

and

$$g_{\underline{k}'\underline{k}\lambda} = -i \left(\frac{\hbar}{2MN_i\,\omega_\lambda(q)} \right)^{1/2}$$

$$\times\ \underline{\varepsilon}(q\lambda) \cdot \left\{ \sum_{\underline{k}_n}^{m} \sum_{\underline{k}_n'}^{m} a^{*}_{\underline{k}_n'}(\underline{k}')\, a_{\underline{k}_n}(\underline{k}) \left(\underline{k}' + \underline{k}_n' - \underline{k} - \underline{k}_n \right) \right.$$

$$\left. \times\ \langle \underline{k}' + \underline{k}_n' | W | \underline{k} + \underline{k}_n \rangle \right\} \tag{2.10}$$

In (2.10) $\underline{\varepsilon}(q\lambda)$ is the normalized polarisation vector of the mode $(q\lambda)$, M is the ionic mass and there are a total of N_i ions in the crystal.

The transition rates for scattering by substitutional impurities can also be calculated using pseudopotentials and first order perturbation theory. The $Q_{\underline{kk}'}$ applicable to scattering by a single impurity is simply

$$\frac{2\pi}{\hbar} \left| \langle \underline{k} | V_{scatt} | \underline{k}' \rangle \right|^2 \delta (E_{\underline{k}} - E_{\underline{k}'}) \qquad (2.11)$$

where V_{scatt} is the change in the total pseudopotential of the crystal when the impurity is added. It is easy to show that for one-OPW electron states $^{(16)}$

$$\langle \underline{k} | V_{scatt} | \underline{k}+q \rangle = \frac{1}{N_t} \left\{ (S(q)-1)\, W_H(q) + W_I(q) \right\}$$

where N_t is the total number of electrons in the crystal and $W_H(\underline{q})$ and $W_I(\underline{q})$ are the host and impurity pseudopotential form factors. $S(\underline{q})$ is the structure factor and it accounts for the effects of distortion of the lattice about the impurity. $S(\underline{q})$ is defined as

$$S(q) = \sum_{\underline{l}} \exp\left(-i q \cdot (\underline{l} + \underline{u}^{l}) \right)$$

where $\underline{l}$ is the equilibrium position of an ion in the perfect lattice and $\underline{u}^{l}$ is the displacement of the ion originally at $\underline{l}$ due to the presence of the defect.

CHAPTER III

THE BOLTZMANN EQUATION AND ITS ELEMENTARY SOLUTIONS

3.1 The Boltzmann Equation

In this section the basic physics of the interaction of sound waves with conduction electrons will be examined. The Boltzmann equation for the electron distribution and an expression for the attenuation in terms of this distribution will be derived. In sections 3.2 and 3.3 simple attempts at approximate solutions of the Boltzmann equation will be reviewed with an emphasis on results that will be of value in formulating the more realistic solutions to be presented in Chapters IV and V.

The first complete theory of ultrasonic attenuation due to conduction electrons was presented by Pippard (9). He pointed out that when a longitudinal sound wave passes through a metal the electron and ion charge densities may not follow each other exactly. Thus there will be a non-zero space charge density which will move with the sound wave and produce a periodic electrostatic field within the metal. A transverse sound wave causes no buildup of electric charge but the electron and ion current densities associated with the wave may not exactly cancel and the net current that results can produce an electric field by induction. These electric fields accelerate the conduction electrons and this acceleration is opposed by various scattering mechanisms (e.g. scattering by phonons, impurities and other electrons). When the electrons are scattered they may transfer energy from the sound wave to random thermal excitations of the lattice. In Pippard's original paper he treated these effects by following the motion of individual electrons. Steinberg (10) recast Pippard's work in terms of a Boltzmann equation for the conduction electrons. In both of these papers the conduction electrons were treated in the free electron approximation. A derivation of the required Boltzmann equation for a metal with a non-spherical fermi surface is presented below.

Define a distribution function $f_{\underline{k}}(\underline{r},t)$ to be the number of electrons with wavevector $\underline{k}$ and some particular spin at the point $\underline{r}$ at time t. If there is no disturbance of the metal $f_{\underline{k}}(\underline{r},t)$ will be given by the familiar Fermi-Dirac law

$$f_{\underline{k}}(\underline{r},t) = f^{0}_{\underline{k}}(\underline{r},t) = \left[e^{\beta(E_{\underline{k}}-\mu)} + 1 \right]^{-1} \qquad (3.1.1)$$

where $E_{\underline{k}}$ is the energy of an electron in the state $\underline{k}$ and μ is the chemical potential. The distribution function must, of course, satisfy the Boltzmann equation

$$\frac{\partial f_{\underline{k}}}{\partial t} + \underline{v}_{\underline{k}} \cdot \underline{\nabla}_{\underline{r}} f_{\underline{k}} - \left(\underline{\nabla}_{\underline{r}} E_F + e\underline{E} \right) \cdot \underline{\nabla}_{\underline{k}} f_{\underline{k}} = \left.\frac{\partial f_{\underline{k}}}{\partial t}\right]_{scatt} \qquad (3.1.$$

where $\underline{v}_{\underline{k}}$ is the velocity of an electron in the state $\underline{k}$, $\underline{E}$ is the electric field, E_F is the local fermi energy (which may vary because of fluctuations in the electron density), the charge on the electron is $-e$ and $\left.\frac{\partial f_{\underline{k}}}{\partial t}\right]_{scatt}$ is the rate of change of $f_{\underline{k}}$ due to scattering by the lattice and by other electrons. It is convenient to linearize the

Boltzmann equation by assuming that $f_{\underline{k}}$ is different from $f^{\circ}_{\underline{k}}$ only for $\underline{k}$ near the fermi surface. This allows one to write

$$f_{\underline{k}} = f^{\circ}_{\underline{k}} - \phi_{\underline{k}} \frac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}} \tag{3.1.3}$$

Then

$$\nabla_{\underline{k}} f^{\circ}_{\underline{k}} = \frac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}} \frac{\partial E_{\underline{k}}}{\partial \underline{k}} = \hbar \frac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}} \underline{V}_{\underline{k}} \tag{3.1.4}$$

If

$$\underline{\nabla}_{\underline{k}} f_{\underline{k}} >> \underline{\nabla}_{\underline{k}} (f_{\underline{k}} - f^{\circ}_{\underline{k}})$$

then equation (3.1.2) becomes

$$\left[\frac{\partial \phi_{\underline{k}}}{\partial t} + \underline{V}_{\underline{k}} \cdot \underline{\nabla}_{\underline{r}} \phi_{\underline{k}} - \underline{\nabla}_{\underline{r}} E_F \cdot \underline{V}_{\underline{k}} - e\underline{E} \cdot \underline{V}_{\underline{k}} \right] \frac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}} = - \frac{\partial f_{\underline{k}}}{\partial t} \Big]_{scatt} \tag{3.1.5}$$

When a long sinusoidal acoustic wave train moves through a metal one expects the ion velocity, the electric field and the deviation function $\phi_{\underline{k}}$ to all vary as

$$e^{i(\omega_s t - q_s \cdot \underline{r})} \tag{3.1.6}$$

where $\underline{q}_s$ and ω_s are the wavevector and angular frequency of the sound wave. Thus (3.1.5) can be written as

$$\left[(i\omega_s - i\, q_s \cdot \underline{V}_{\underline{k}}) \phi_{\underline{k}} + i\, q_s \cdot \underline{V}_{\underline{k}} \frac{\partial E_F}{\partial N} n - e\underline{E} \cdot \underline{V}_{\underline{k}} \right] \frac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}} = \frac{-\partial f_{\underline{k}}}{\partial t} \Big]_{scatt} \tag{3.1.7}$$

where the number of electrons per unit volume, N, is equal to the number per unit volume in the undisturbed metal, N_c, plus n. As shown in reference (10) all effects due to temperature gradients can be ignored for present purposes.

The current density, $\underline{J}$, at any point is equal to the sum of the current density carried by the ions and that carried by the electrons. That is

$$\underline{J} = N_o e \underline{u} - \frac{e}{4\pi^3\hbar} \int_{FS} \frac{\phi_k \underline{V}_k}{|\underline{V}_k|} d\underline{S}_k \qquad (3.1.8)$$

where $\underline{u}$ is the velocity of the ions and $dS_{\underline{k}}$ is an element of area on the fermi surface. The current density can be related to the electric field by Maxwell's equations. For transverse sound waves the relevant equations are

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} = \mu_o \underline{J} \qquad (3.1.9)$$

$$\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 \qquad (3.1.10)$$

where $\underline{B}$ is the magnetic field and μ_o is the permeability of free space. Taking the time derivative of (3.1.9) and the curl of (3.1.10) produces

$$\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{B}) = \frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} + \mu_o \frac{\partial \underline{J}}{\partial t} \qquad (3.1.11)$$

and

$$\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{B}) = -\underline{\nabla} \times \underline{\nabla} \times \underline{E} \tag{3.1.12}$$

Equating the right hand sides of (3.1.11) and (3.1.12) gives

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E} + \frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} = -\mu_0 \frac{\partial \underline{J}}{\partial t} \tag{3.1.13}$$

For an electric field and a current density that vary like plane sinusoidal waves (3.1.13) becomes

$$q_s^2 \underline{E} - \frac{\omega_s^2}{c^2} \underline{E} = -i \omega_s \mu_0 \underline{J} \tag{3.1.14}$$

Since the speed of sound is much less than c the second term on the left hand side of (3.1.14) is negligible and the electric field is given by

$$\underline{E} = \frac{-i \omega_s \mu_0 \underline{J}}{q_s^2} \tag{3.1.15}$$

For longitudinal waves

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E} = 0$$

and the only significant electric field is an electrostatic field. If there is a net charge density ρ_e then

$$\underline{\nabla} \cdot \underline{E} = -i q_s \underline{E} = \frac{\rho_e}{\varepsilon_0} \tag{3.1.16}$$

By continuity, however,

$$\underline{J} = \rho_c \underline{v}_s \tag{3.1.17}$$

where $\underline{v}_s$ is the sound velocity. Putting (3.1.16) and (3.1.17) together yields the expression relating $\underline{E}$ and $\underline{J}$ for longitudinal waves

$$\underline{E} = \frac{i \underline{J}}{\omega_s \varepsilon_0} \tag{3.1.18}$$

Equations (3.1.7), (3.1.8), (3.1.15) and (3.1.18) provide a prescription for calculating the distribution function of the conduction electrons in the presence of an acoustic wave. All that remains is to find an expression for the scattering term that allows (3.1.7) to be solved. If only scattering by the lattice is considered then the scattering term has the well known form

$$\frac{\partial f_{\underline{k}}}{\partial t}\Big]_{scatt} = \frac{\Omega}{8\pi^3} \int \left[f_{\underline{k}'}(1-f_{\underline{k}}) - f_{\underline{k}}(1-f_{\underline{k}'}) \right] Q_{\underline{k}\,\underline{k}'}\, d\underline{k}' \tag{3.1.19}$$

where $Q_{\underline{k}\,\underline{k}'}$ is the transition rate of electrons from a filled state $\underline{k}$ to an empty state $\underline{k}'$. The relation

$$\frac{\partial f^0_{\underline{k}}}{\partial E_{\underline{k}}} = -\beta f^0_{\underline{k}} (1 - f^0_{\underline{k}}) \tag{3.1.20}$$

allows (3.1.19) to be written as

$$\frac{\partial f_{\underline{k}}}{\partial t}\Big]_{scatt} = \frac{\beta \Omega}{8\pi^3} \int (\phi_{\underline{k}'} - \phi_{\underline{k}})\, P_{\underline{k}\,\underline{k}'}\, d\underline{k}' \tag{3.1.21}$$

where

$$P_{\underline{k}\underline{k}'} = f^{\circ}_{\underline{k}} (1 - f^{\circ}_{\underline{k}'}) \, Q_{\underline{k}\underline{k}'} \qquad (3.1.22)$$

$Q_{\underline{k}\underline{k}'}$ is, of course, the sum of transition rates due to phonons and other lattice imperfections (such as impurities or dislocations). In an undisturbed metal these transition probabilities can be calculated using the formulae of Chapter II. When considering the scattering of electrons by a lattice that is disturbed by an impressed ultrasonic wave, however, it is physically reasonable to suppose that the scattering will cause the electron distribution at each point to relax towards the distribution that would be in local equilibrium with the moving and dilated lattice. Indeed since the experimentally obtainable wavelengths of ultrasonic waves are much larger than either the wavelengths of most thermal phonons or the dimensions of most other lattice imperfections, the transition rates of Chapter II can be used as long as the equilibrium electron distribution which occurs implicitly in equation (3.1.21) is taken to be the distribution that is in local equilibrium with the disturbed lattice. In other words the collisions must be viewed in local frames of reference moving with the ions. This is referred to in the literature as the "collision drag assumption". In the case of impurity scattering it is immediately apparent that this assumption is reasonable. The formal justification (given in references (17) and (18)) when considering scattering by thermal phonons is a much more subtle problem since it involves constructing wavefunctions that localise the conduction electrons to regions with dimensions much smaller than ultrasonic wavelengths. All previous theoretical treatments of ultrasonic attenuation have made the collision drag assumption (9,10,11,19). The practical consequence of assuming collision drag is the modification of (3.1.21) to read

$$\frac{\partial f_{\underline{k}}}{\partial t}\Big]_{scatt} = -\frac{\beta\Omega}{8\pi^3}\int\left[\phi_{\underline{k}} - \tilde{\phi}_{\underline{k}} - (\phi_{\underline{k}'} - \tilde{\phi}_{\underline{k}'})\right] P_{\underline{k}\underline{k}'}\, d\underline{k}' \qquad (3.1.23)$$

where the $P_{\underline{kk}'}$ are those that would be calculated for an undisturbed metal and $\tilde{\phi}_{\underline{k}}$ is the deviation function that brings the distribution defined by (3.1.4) into equilibrium with the moving lattice. For transverse waves

$$f^0_{\underline{k}} - f^0_{\underline{k}}(\underline{u}) = \frac{\partial f^0_{\underline{k}}}{\partial E_{\underline{k}}}\,\frac{\partial E_{\underline{k}}}{\partial \underline{V}_{\underline{k}}}\cdot\underline{u}$$

$$= \frac{\partial f^0_{\underline{k}}}{\partial E_{\underline{k}}}\,\frac{\partial E_{\underline{k}}}{\partial \underline{k}}\cdot\frac{\partial \underline{k}}{\partial \underline{V}_{\underline{k}}}\cdot\underline{u} \qquad (3.1.24)$$

And thus

$$\phi_{\underline{k}} = \underline{V}_{\underline{k}}\cdot\underline{\underline{M}}_{\underline{k}}\cdot\underline{u} \qquad (3.1.25)$$

where $\underline{\underline{M}}$ is the effective mass tensor whose elements are

$$M_{ij} = \hbar\,\frac{\partial k_i}{\partial (V_{\underline{k}})_j} \qquad (3.1.26)$$

For electrons which are describable by a scalar effective mass, m^*, equation (3.1.25) reduces to

$$\tilde{\phi}_{\underline{k}} = m^* \underline{v}_{\underline{k}} \cdot \underline{u} \tag{3.1.27}$$

For longitudinal waves a term must be added to (3.1.25) to account for changes in the local fermi energy due to dilation and compression of the lattice, i.e.

$$\tilde{\phi}_{\underline{k}} = \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{k} \cdot \underline{u} + \frac{\partial E_F}{\partial N} n_0 \tag{3.1.28}$$

where n_0 is the number of electrons per unit volume in excess of N_0 if the electrons follow the ions exactly. The equation of continuity gives

$$-e n \underline{v}_s = \underline{J}_e \tag{3.1.29}$$

where J_e is the electron current density. If the electrons followed the ions exactly the electron current would be the negative of the ion current. That is

$$\underline{J}_e = -N_0 e \underline{u} \tag{3.1.30}$$

so

$$n_0 = \frac{N_0 u}{v_s} \tag{3.1.31}$$

It is easy to verify that for a metal with a spherical fermi surface

$$n_0 \frac{\partial E_F}{\partial N} = \frac{N_0 u}{V_s} \frac{\partial E_F}{\partial N} = \frac{m^* v_F^2 u}{3 V_s} \qquad (3.1.32)$$

It is now necessary to find an expression relating the electron distribution function to the attenuation. If the amplitude of an acoustic wave propogating along the z-axis is proportional to

$$e^{-\alpha z}$$

then α is called the amplitude attenuation coefficient and has units of nepers/unit length. This is the definition of α that shall be used throughout this thesis. Care should be taken in comparing the algebraic expressions for α in this thesis with those elsewhere in the literature because there are about half a dozen different definitions of α. Since the energy density, W, of an acoustic wave is proportional to the square of its amplitude the equation for energy dissipation can be written

$$\frac{dW}{dz} = -2\alpha W$$

or

$$\alpha = -\frac{1}{2W}\frac{dW}{dz} = \frac{-1}{2WV_s}\frac{dW}{dt} \qquad (3.1.33)$$

where $-\frac{dW}{dt}$ is the rate at which energy is transferred from the impressed sound wave to thermal excitations of the medium through which it travels. In the situation of present interest the energy lost by the sound wave is that which is given up to the lattice by the conduction electrons during scattering processes, i.e.

$$\frac{dW}{dt} = \frac{2}{8\pi^3}\left\langle \int E_{\underline{k}}(\underline{u})\, \frac{\partial f_{k}}{\partial t}\Big]_{scatt}\, d\underline{k} \right\rangle \tag{3.1.34}$$

where $E_{\underline{k}}(\underline{u})$ is the energy of an electron in the state $\underline{k}$ when viewed in a frame of reference moving with the ions and where the angular brackets indicate that an average is to be taken over one complete cycle of the sound wave. Equation (3.1.34) is intuitively reasonable and a formal justification is given in references (17) and (18). Now to first order in $\underline{u}$

$$E_{\underline{k}}(\underline{u}) \simeq -\frac{\partial E_{k}}{\partial \underline{v}_{k}}\cdot \underline{u} + E_{\underline{k}}(0)$$

$$= -\underline{v}_{\underline{k}}\cdot \underline{\underline{M}}_{\underline{k}}\cdot \underline{u} + E_{\underline{k}}(0) \tag{3.1.35}$$

Remembering that the energy density in a sound wave is

$$1/2\, \rho\, |u|^2$$

where ρ is the density of the medium, α can be written as

$$\alpha = \frac{1}{2v_s\,\rho\,|u|^2}\,\frac{2}{8\pi^3}\,\mathrm{Re}\left(\int \underline{v}_{\underline{k}}\cdot \underline{\underline{M}}_{k}\cdot \underline{u}^{*}\,\frac{\partial f_{k}}{\partial t}\Big]_{scatt}\, d\underline{k}\right) \tag{3.1.36}$$

Re denotes that the real part is to be taken and complex conjugation is indicated by the asterisk.

3.2 Isotropic Scattering Time Approximation

The simplest phenominological approach to a solution of (3.1.7) involves assuming the existence of a relaxation time τ such that

$$\frac{\partial f_{\underline{k}}}{\partial t}\Big]_{scatt} = -\frac{\partial f^{0}_{\underline{k}}}{\partial E_{\underline{k}}}\frac{\partial \phi_{\underline{k}}}{\partial t} = \frac{\partial f^{0}_{\underline{k}}}{\partial E_{\underline{k}}}\frac{\phi_{\underline{k}} - \tilde{\phi}_{\underline{k}}}{\tau} \qquad (3.2.1)$$

Such an assumption is familiar in the theory of electrical conductivity (20,21) and has been widely used in treatments of the ultrasonic attenuation in metals (9,10,19). Clearly the values of τ that apply to the two problems must be of about the same order of magnitude, although they need not be identical (11). Thus for a particular metal at a particular temperature

$$\tau \sim \frac{m\sigma}{Ne^{2}} \qquad (3.2.2)$$

where σ is the electrical conductivity. With the assumption (3.2.1) the linearized Boltzmann equation becomes

$$i(q_{s}\cdot \underline{v}_{\underline{k}} - \omega_{s})\phi_{\underline{k}} - i q_{s}\cdot \underline{v}_{\underline{k}}\frac{\partial E_{F}}{\partial N} n + e\underline{E}\cdot \underline{v}_{\underline{k}} = \frac{\phi_{\underline{k}} - \tilde{\phi}_{\underline{k}}}{\tau} \qquad (3.2.3)$$

Since both n and $\underline{E}$ depend on integrals of $\phi_{\underline{k}}$ over the fermi surface it is convenient to attempt a solution of this equation by iteration. In particular a first estimate for $\phi_{\underline{k}}$ can be made by substituting $\phi_{\underline{k}} = \tilde{\phi}_{\underline{k}}$ into the left hand side of (3.2.3). For transverse waves this gives

$$\phi'_{\underline{k}} = \tilde{\phi}_{\underline{k}} + i(q_{s}\cdot \underline{v}_{\underline{k}} - \omega_{s})\tau\tilde{\phi}_{\underline{k}} \qquad (3.2.4)$$

Substituting $\phi^1_{\underline{k}}$ into the left hand side produces

$$\phi^2_{\underline{k}} = \tilde{\phi}_{\underline{k}} + i(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\tau\,\tilde{\phi}_{\underline{k}} - (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)^2\tau^2\,\tilde{\phi}_{\underline{k}}$$

$$+ \frac{e^2\mu_0}{q_s^2}\,\omega_s^2\tau\;\frac{\underline{v}_{\underline{k}} \cdot \hat{\underline{u}}}{4\pi^3}\int_{FS} \frac{(\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{k} \cdot \underline{u})(\underline{v}_{\underline{k}} \cdot \hat{\underline{u}})}{\hbar\,|\underline{v}_{\underline{k}}|}\, dS_{\underline{k}} \qquad (3.2.5)$$

It is easy to see that the last term and hence the electric field will be negligible if

$$q_s v_F \tau << 1$$

and

$$e^2\mu_0 v_s^2 k_F^2 v_F \tau << \hbar (q_s v_F \tau)^2 \qquad (3.2.6)$$

where v_F is the fermi velocity. It is also clear that this iterative procedure will converge if the inequalities of (3.2.6) are satisfied. In practice (3.2.6) is generally satisfied by sound waves with frequencies of the order of $10\text{-}10^5$ Hz. This is the frequency range of prime interest in this thesis and will be referred to as the "low frequency regime".

It is convenient to define another deviation function by

$$\psi_{\underline{k}} = \psi^R_{\underline{k}} + i\,\psi^I_{\underline{k}} = \frac{\phi_{\underline{k}} - \tilde{\phi}_{\underline{k}}}{u} \qquad (3.2.7)$$

From equation (3.2.5) it is clear that in the low frequency regime

$$|\psi_k^R| << |\psi_{\underline{k}}^R| << \left|\frac{\tilde{\Phi}_k}{u}\right| \tag{3.2.8}$$

This inequality will play an important part in the derivation of the variational solution of the Boltzmann equation to be presented in the next chapter. An algebraic expression for the attenuation in the low frequency regime for a metal with a spherical fermi surface can be obtained by substituting $\phi_{\underline{k}}^2$ into equation (3.1.36)

$$\alpha = \frac{1}{2 v_s \rho |u|^2} \frac{2}{8\pi^3} \text{Re} \left\{ \int (m^* \underline{v}_{\underline{k}} \cdot \underline{u}^*) \frac{(m^* \underline{v}_k \cdot \underline{u})}{\tau} \right.$$

$$\left. \times \left(\omega_s - q_s \cdot \underline{v}_{\underline{k}}\right)^2 \tau^2 \left(\frac{-\partial f_{\underline{k}}^0}{\partial E_{\underline{k}}}\right) d\underline{k} \right\} \tag{3.2.9}$$

The familiar relation

$$\int d\underline{k} \longrightarrow \frac{1}{\hbar v_F} \int dS_k$$

and the fact that

$$\omega_s << q_s v_F$$

can be used to reduce (3.2.9) to

$$\frac{1}{8\pi^3} \frac{m^{*2} v_F^3 K_F^2 q_s^2 \tau}{\hbar v_s \rho} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \cos^2\theta \sin^2\theta \cos^2\phi$$

$$= \frac{m^{*2} v_F^3 k_F^2 q_s^2 \tau}{30 \pi^2 \hbar v_s \rho} \qquad (3.2.10)$$

For free electrons

$$m^* v_F = m v_F = \hbar k_F$$

and

$$3\pi^2 N_0 = k_F^3$$

so the expression for α can be simplified to

$$\alpha = \frac{N_0 m v_F^2 \omega_s^2 \tau}{10 \, v_s^3 \rho} \qquad (3.2.11)$$

This result was first obtained by Pippard $^{(9)}$.

In the case of longitudinal waves the same iterative procedure can be used. The only difference is the existence of a non-zero fluctuation in the density of electrons. The first two iterations produce

$$\phi^1_{\underline{k}} = \tilde{\phi}_{\underline{k}} + i(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\tau \tilde{\phi}_{\underline{k}} - i q \cdot \underline{v}_{\underline{k}} \tau n_0 \frac{\partial E_F}{\partial N}$$

$$= \tilde{\phi}_{\underline{k}} + i(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\tau \, \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \underline{u} - i\omega_s \tau n_0 \frac{\partial E_F}{\partial N} \qquad (3.2.12)$$

and

$$\phi^2_{\underline{k}} = \tilde{\phi}_{\underline{k}} + i(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\tau \, \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \underline{u} - i\omega_s \tau n_0 \frac{\partial E_F}{\partial N}$$

$$- (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)^2 \tau^2 \, \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \underline{u}$$

$$+ \; \omega_s \tau^2 (q_s \cdot \underline{v}_{\underline{k}} - \omega_s) \, n_0 \frac{\partial E_F}{\partial N}$$

$$- \frac{e^2}{\varepsilon_0} \tau \, \underline{v}_{\underline{k}} \cdot \hat{\underline{u}} \quad \frac{1}{4\pi^3} \int \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \underline{u} \quad \frac{\underline{v}_{\underline{k}} \cdot \hat{\underline{u}}}{\hbar |\underline{v}_{\underline{k}}|} \, dS_{\underline{k}} \qquad (3.2.13)$$

If

$$q_s v_F \tau << 1$$

and

$$e^2 v_s^2 k_F^2 v_F \tau << \varepsilon_0 \hbar (q_s v_F \tau)^2 \qquad (3.2.14)$$

then it is easy to verify that the last term, and hence the electric field, is negligible and that

$$|\psi_{\underline{k}}^R| \; << \; |\psi_{\underline{k}}^I| << \left| \frac{\tilde{\phi}_{\underline{k}}}{u} \right| \qquad (3.2.15)$$

and that

$$\left| \left(\frac{n - n_0}{u} \right)^R \right| \quad << \quad \left| \left(\frac{n}{u} \right)^I \right| \quad << \quad \left| \frac{n_0}{u} \right| \qquad (3.2.16)$$

and finally that

$$\left| \left(\frac{n}{u} \right)^I \right| \; = \; \frac{1}{4\pi^3 \hbar} \left| \int_{FS} \psi_{\underline{k}}^I \frac{dS_{\underline{k}}}{|\underline{v}_{\underline{k}}|} \right| << \frac{k_F^2}{4\pi^3 \hbar v_F} \, |\psi_{\underline{k}}^I| \qquad (3.1.17)$$

These inequalities will be used in the next chapter.

Once again it is straightforward to use equation (3.2.13) to obtain an expression for the attenuation of longitudinal sound waves in the low frequency regime in a metal with a spherical fermi surface.

$$\alpha = \frac{1}{2\rho V_s |u|^2} \frac{2}{8\pi^3} \mathrm{Re} \left[\int_{FS} \frac{m^* \underline{V}_k \cdot \underline{u}^*}{\hbar |\underline{V}_k| \tau} \right.$$

$$\times \left((q_s \cdot \underline{V}_k - \omega_s)^2 \tau^2 \, m^* \underline{V}_k \cdot \underline{u} + \omega_s (q_s \cdot \underline{V}_k - \omega_s) \tau^2 \frac{m^* V_F^2 u}{3 V_s} \right) dS_k \tag{3.2.18}$$

Remembering that

$$\omega_s << q_s V_F$$

allows α to be written as

$$\alpha = \frac{m^{*2} V_F^3 K_F^2 q_s^2 \tau}{8\pi^3 \rho V_s \hbar} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \left(\cos^4\theta - 1/3 \cos^2\theta \right)$$

$$= \frac{2 m^{*2} V_F^3 K_F^2 q_s^2 \tau}{45 \pi^2 \hbar V_s \rho} \tag{3.2.19}$$

For free electrons this is

$$\alpha = \frac{2 N_0 m V_F^2 \omega^2 \tau}{15 V_s^3 \rho} \tag{3.2.20}$$

which is the standard result (9,10).

It is interesting to note that the linear dependence of α on τ in (3.2.10) and (3.2.19) has a simple physical interpretation (22,23). When an ultrasonic wave with a frequency in the low frequency regime travels through a metal it may be considered as propogating through the conduction electron gas with the lattice entering the problem only as a cause of electron scattering. The acoustic attenuation in a fluid is proportional to the fluid's viscosity and elementary kinetic theory shows that the viscosity of a dilute gas is proportional to the mean scattering time τ (24). Such a treatment of the attenuation by this simple-minded analogy between the electron gas and a classical fluid breaks down at higher frequencies, however. When the electron mean free path becomes comparable to the wavelength of the ultrasonic wave ($q_s v_F \tau \sim 1$) a small number of electrons which have velocities almost perpendicular to q_s ($\underline{q}_s \cdot v_{\underline{k}} = \omega_s$) travel in a way that always keeps them at the same phase of the wave. These electrons are accelerated by the electric field and absorb energy from the sound wave in a resonant fashion. As the frequency of the ultrasonic wave increases these few electrons account for a larger share of the attenuation until in the limit

$$q_s v_F \tau \gg 1$$

the attenuation coefficient becomes independent of τ.

If τ in the expressions for the attenuation (3.2.10) and (3.2.19) is taken to be the relaxation time that governs the electrical conductivity then clearly, for free electrons,

$$\alpha/\sigma = \frac{m^2 v_F^2 \omega^2}{10\, v_s^3 \rho e^2} \quad \text{(transverse)}$$

$$= \frac{2 m^2 v_F^2 \omega^2}{15\, v_s^3 \rho e^2} \quad \text{(longitudinal)} \qquad (3.2.21)$$

The prediction of (3.2.21) that α/σ should be virtually independent of temperature has received experimental verification in tin, aluminum and potassium (25,26,27,28) but the measured values of α/σ tend to be about 1.5 times larger than those of (3.2.21). In the following chapters it will be shown that there is no compelling theoretical reason to believe that α must be exactly proportional to σ and it would be of interest to perform numerical calculations of α/σ using the more realistic theory to be presented below.

3.3 The Bhatia and Moore Analysis

The isotropic scattering time that appears in the Boltzmann equation for the electrical conductivity in the standard treatments has an expression in terms of an integral over the transition rates $P_{\underline{kk}'}$ which is exact if $P_{\underline{kk}'}$ depends only on the magnitude of $(\underline{k}-\underline{k}')$ (20). No such expression can be found for the τ that appears in equation (3.2.3) even for the case of isotropic scattering. In order to avoid this problem Bhatia and Moore (11) solved the Boltzmann equation for a metal with a spherical fermi surface by assuming that the scattering term could be written

$$\left.\frac{\partial f_{\underline{k}}}{\partial t}\right]_{scatt} = \frac{\partial f^0_{\underline{k}}}{\partial E_{\underline{k}}} \sum_{LM} \frac{(c_{LM}-\tilde{c}_{LM})}{\tau_L} Y_{LM}(\hat{\underline{k}}) \qquad (3.3.1)$$

where

$$\phi_{\underline{k}} = \sum_{LM} c_{LM} Y_{LM}(\hat{\underline{k}}) \qquad (3.3.2)$$

$$\tilde{\phi}_{\underline{k}} = \sum_{LM} \tilde{c}_{LM} Y_{LM}(\hat{\underline{k}}) \qquad (3.3.3)$$

and the Y_{LM} are spherical harmonics. It is convenient to define the Y_{LM} as in reference (11), i.e.

$$Y_{LM}(\theta,\phi) = P_{LM}(\cos\theta)\cos(M\phi) \qquad 0 \leq M \leq L$$

$$Y_{LM}(\theta,\phi) = P_{L|M|}(\cos\theta)\sin(M\phi) \qquad -L \leq M < 0$$

where the P_{LM} are associated Legendre polynomials ($P_{10} = \cos\theta$,

$P_{20} = (3/2)\cos^2\theta - 1/2$ etc.). This definition means that

$$-\frac{\partial f_{\underline{k}}}{\partial t}\Big]_{scatt} = \sum_{LM} \frac{(c_{LM}-\tilde{c}_{LM})}{\tau_L} Y_{LM}(\hat{\underline{k}}) = \sum_{LM} (c_{LM}-\tilde{c}_{LM}) \frac{\Omega\beta}{8\pi^3\hbar v_F}$$

$$\times \int_{FS} \left(Y_{LM}(\hat{\underline{k}}) - Y_{LM}(\hat{\underline{k}}')\right) P_{\underline{k}\underline{k}'}\, dS_{\underline{k}'} \qquad (3.3.4)$$

Now if $P_{\underline{kk}'}$ depends only on the magnitude of $(\underline{k}-\underline{k}')$ then it can be shown that (11)

$$\int_{FS} \left[Y_{LM}(\hat{\underline{k}}) - Y_{LM}(\hat{\underline{k}}')\right] P_{\underline{k}\underline{k}'}\, dS_{\underline{k}'}$$

$$= \int_{FS} \left[Y_{LM}(\hat{\underline{k}}) - P_{L0}(\hat{\underline{k}}\cdot\hat{\underline{k}}')\, Y_{LM}(\hat{\underline{k}}')\right] P_{\underline{k}\underline{k}'}\, dS_{\underline{k}'} \qquad (3.3.5)$$

so

$$-\frac{\partial f^{0}_{\underline{k}}}{\partial \epsilon_{\underline{k}}} \frac{1}{\tau_L} = \frac{\Omega\beta}{8\pi^3\hbar v_F} \int_{FS} \left(1 - P_{L0}(\hat{\underline{k}}\cdot\hat{\underline{k}}')\right) P_{\underline{k}\underline{k}'}\, dS_{\underline{k}'} \qquad (3.3.6)$$

τ_1 is just the usual relaxation time used in the theory of electrical conductivity.

For transverse waves in the low frequency regime the electric field can be neglected and

$$\omega_s << q_s v_F$$

so the Boltzmann equation can be written

$$\sum_{LM} \frac{(c_{LM} - \tilde{c}_{LM})}{\tau_L} Y_{LM}(\theta, \phi) = i q_s v_F \cos\theta \sum_{LM} c_{LM} Y_{LM}(\theta, \phi) \qquad (3.3.7)$$

Clearly the only non-zero $\tilde{c}_{LM}$ for a wave with q_s along the z-axis and $\underline{u}$ along the x-axis is $\tilde{c}_{11}$ and

$$\tilde{c}_{11} = m^* v_F u \qquad (3.3.8)$$

Using the identity (11)

$$\cos\theta \, Y_{L1}(\theta, \phi) = \frac{(L+1)}{2L+1} Y_{L-1,1}(\theta, \phi) + \frac{L}{2L+1} Y_{L+1,1}(\theta, \phi) \qquad (3.3.9)$$

equation (3.3.7) can be written as an infinite set of coupled linear equations, i.e.

$$\frac{c_{11} - m^* v_F u}{\tau_1} = i q_s v_F \frac{3}{5} c_{21}$$

$$\frac{c_{21}}{\tau_2} = i q_s v_F \frac{1}{3} c_{11} + i q_s v_F \frac{4}{7} c_{31}$$

$$\frac{c_{31}}{\tau_3} = i q_s v_F \frac{2}{5} c_{21} + i q_s v_F \frac{5}{9} c_{41} \qquad (3.3.10)$$

etc. Clearly if $q_s v_F \tau_L \ll 1$ the coefficients c_{L1} must rapidly decrease with increasing L. If one sets all the $c_{L1}=0$ for L greater than two the remaining two equations can be easily solved to yield

$$c_{11} - \tilde{c}_{11} = 1/5\, m^* v_F\, u\, q_s^2 v_F^2 \tau_1 \tau_2$$

$$c_{21} = 1/3\, i\, q_s v_F m^* v_F\, u\, \tau_2 \qquad (3.3.11)$$

Now

$$\alpha = \frac{1}{\rho v_s} \frac{k_F^2 m^* v_F}{8\pi^3 \hbar v_F} \int_{4\pi} \frac{(c_{11} - \tilde{c}_{11})}{\tau_1} \cos^2\phi \sin^3\theta \, d\theta\, d\phi$$

$$= \frac{m^{*2} v_F^3 k_F^2 q_s^2 \tau_2}{30\pi^2 \hbar v_s \rho} \qquad (3.3.12)$$

which is just (3.2.10) with τ replaced by τ_2.

The same procedure can be used for longitudinal waves and the result is simply

$$\alpha_{long} = \frac{4}{3} \alpha_{trans} \times \frac{V_s^{trans}}{V_s^{long}} \qquad (3.3.13)$$

which is just (3.2.19) with τ_2 replacing τ.

It is apparent that by truncating the equations at a higher value of L one can obtain solutions that are valid for higher frequencies. These solutions take the form of products of factors of the form $(q_s v_F \tau_L)$. The first few terms of these solutions for both transverse and longitudinal waves are given in reference (11).

This brief discussion of the Bhatia and Moore analysis has been included here because

(i) for the case of isotropic scattering it gives exact results in the low frequency regime which can be compared to those obtained by the variational procedure described in the next chapter

and (ii) it will serve as the basis for the anisotropic scattering time approximation to be presented in Chapter V.

CHAPTER IV

THE VARIATIONAL SOLUTION OF THE BOLTZMANN EQUATION

4.1 The Variational Principle

One of the most powerful techniques for the study of the electrical conductivity of metals is the use of the variational principle for the Boltzmann equation(21,29). The similarity of the results for α and σ in the elementary solutions presented in the preceding chapter suggest that there may be an analogous variational principle applicable to the Boltzmann equation for ultrasonic attenuation, at least in the low frequency regime. To apply the variational principle it is necessary to force the Boltzmann equation into the form of a linear inhomogeneous integral equation with a positive definite kernel and to relate the kernel to the quantity of interest, in this case the attenuation coefficient. This is done below, first for transverse wave and then for longitudinal waves.

The Boltzmann equation for transverse waves in the low frequency regime can be written

$$\frac{\beta\Omega}{8\pi^3}\int(\psi_{\underline{k}} - \psi_{\underline{k}'})\, P_{\underline{k}\underline{k}'}\, d\underline{k}'$$

$$= -i(q_s\cdot\underline{v}_{\underline{k}} - \omega_s)\left[\psi_{\underline{k}} + \underline{v}_{\underline{k}}\cdot\underline{\underline{M}}_{\underline{k}}\cdot\hat{\underline{u}}\right]\frac{\partial f^0_{\underline{k}}}{\partial E_{\underline{k}}} \qquad (4.1.1)$$

where the electric field has been neglected and the usual Boltzmann equation has been divided by u. Under the familiar "Bloch assumption", i.e. that the phonon distribution is unaffected by electron-phonon scattering (see section 4.4), the transition rates $P_{\underline{k}\underline{k}'}$ will be real. The real part of equation (4.1.1) is thus

$$-\frac{1}{u}\frac{\partial f_{\underline{k}}^{0}}{\partial t}\Big]^{R}_{scatt} = \frac{\beta\Omega}{8\pi^{3}}\int \left(\psi_{\underline{k}}^{R} - \psi_{\underline{k}'}^{R}\right) P_{\underline{k}\underline{k}'}\, d\underline{k}'$$

$$= \left(q_{s}\cdot \underline{v}_{\underline{k}} - \omega_{s}\right)\psi_{\underline{k}}^{I}\,\frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}} \qquad (4.1.2)$$

where the superscript ^{R} denotes the real part. α is given by

$$\alpha = \frac{1}{\rho v_{s}|u|^{2}}\,\mathrm{Re}\left[\frac{1}{8\pi^{3}}\int \frac{\partial f_{\underline{k}}}{\partial t}\Big]_{scatt}\left(\underline{v}_{\underline{k}}\cdot \underline{\underline{M}}_{\underline{k}}\cdot \underline{u}^{*}\right) d\underline{k}\right]$$

$$= \frac{-1}{8\pi^{3}\rho v_{s}}\int \left(q_{s}\cdot \underline{v}_{\underline{k}} - \omega_{s}\right)\underline{v}_{\underline{k}}\cdot \underline{\underline{M}}_{\underline{k}}\cdot \hat{\underline{u}}\;\psi_{\underline{k}}^{I}\,\frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}}\, d\underline{k} \qquad (4.1.3)$$

The imaginary part of (4.1.1) is

$$\frac{\beta\Omega}{4\pi^{3}}\int \left(\psi_{\underline{k}}^{I} - \psi_{\underline{k}'}^{I}\right) P_{\underline{k}\underline{k}'}\, d\underline{k}' = \left(q_{s}\cdot \underline{v}_{\underline{k}} - \omega_{s}\right)\psi_{\underline{k}}^{R}\,\frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}}$$

$$= \left(q_{s}\cdot \underline{v}_{\underline{k}} - \omega_{s}\right)\underline{v}_{\underline{k}}\cdot \underline{\underline{M}}_{\underline{k}}\cdot \hat{\underline{u}}\;\frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}} \qquad (4.1.4)$$

From (3.2.8) $\psi_{\underline{k}}^{R}$ is insignificant and (4.1.4) can be written as

$$P\,\psi_{\underline{k}}^{I} = X_{\underline{k}} \tag{4.1.5}$$

where

$$X_{\underline{k}} = -\left(\underline{q}_s \cdot \underline{v}_{\underline{k}} - \omega_s\right) \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \underline{\hat{u}} \; \frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}} \tag{4.1.6}$$

and

$$P\,\psi_{\underline{k}} = \frac{\beta\,\Omega}{8\pi^3} \int \left(\psi_{\underline{k}} - \psi_{\underline{k}'}\right) P_{\underline{k}\underline{k}'}\, d\underline{k}' \tag{4.1.7}$$

If one defines

$$\langle \psi\, \xi \rangle = \frac{1}{8\pi^3} \int \psi_{\underline{k}}\, \xi_{\underline{k}}\, d\underline{k} \tag{4.1.8}$$

Then the familiar properties of the scattering operator P

$$\langle \psi\, P \xi \rangle = \langle \xi\, P \psi \rangle \tag{4.1.9}$$

and

$$\langle \psi\, P \psi \rangle \geq 0 \tag{4.1.10}$$

insure that the solution to (4.1.5) is the one that gives

$$\frac{\langle \psi^{I} P\, \psi^{I} \rangle}{\left[\langle \psi^{I} X \rangle\right]^2}$$

its minimum value. Now note that from (4.1.3)

$$\langle \psi^{I} X \rangle = \rho v_s \alpha \tag{4.1.11}$$

Thus the variational principle for (4.1.5) can be written as

$$\frac{1}{\alpha} \leq \rho v_s \frac{\langle \psi^{I} P \psi^{I} \rangle}{\left[\langle \psi^{I} X \rangle \right]^2}$$

$$= \frac{\rho v_s \dfrac{1}{8\pi^3} \displaystyle\int d\underline{k} \, \dfrac{\Omega \beta}{8\pi^3} \int d\underline{k}' \, \psi^{I}_{\underline{k}} \left(\psi^{I}_{\underline{k}} - \psi^{I}_{\underline{k}'} \right) P_{\underline{k}\underline{k}'}}{\left[\dfrac{1}{8\pi^3} \displaystyle\int d\underline{k} \, \left(q_s \cdot \underline{v}_{\underline{k}} - \omega_s \right) \left(\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{\underline{u}} \right) \psi^{I}_{\underline{k}} \dfrac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}} \right]^2}$$

$$= \frac{\tfrac{1}{2} \rho v_s \Omega \beta \displaystyle\int d\underline{k} \int d\underline{k}' \left(\psi^{I}_{\underline{k}} - \psi^{I}_{\underline{k}'} \right)^2 P_{\underline{k}\underline{k}'}}{\left[\displaystyle\int d\underline{k} \left(q_s \cdot \underline{v}_{\underline{k}} - \omega_s \right) \left(\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{\underline{u}} \right) \psi^{I}_{\underline{k}} \dfrac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}} \right]^2} \tag{4.1.12}$$

For longitudinal waves the Boltzmann equation can be written

$$\frac{\beta \Omega}{8\pi^3} \int \left(\psi_{\underline{k}} - \psi_{\underline{k}'} \right) P_{\underline{k}\underline{k}'} \, d\underline{k}' = - i \left(q_s \cdot \underline{v}_{\underline{k}} - \omega_s \right) \left[\psi_{\underline{k}} + \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{k} \cdot \hat{u} \right] \frac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}}$$

$$+ i \omega \frac{\partial E_F}{\partial N} n \frac{\partial f^{\circ}_{\underline{k}}}{\partial E_{\underline{k}}} \tag{4.1.13}$$

The real part is

$$- \frac{1}{u} \left. \frac{\partial f_{\underline{k}}}{\partial t} \right]^{R}_{\text{scatt}} = \frac{\beta \Omega}{8\pi^3} \int d\underline{k}' \left(\psi^{R}_{\underline{k}} - \psi^{R}_{\underline{k}'} \right) P_{\underline{k}\underline{k}'}$$

$$= (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\, \psi_{\underline{k}}^{I} \frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}} - \omega_s \frac{\partial E_F}{\partial N} \left(\frac{n}{u}\right)^{I} \frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}} \qquad (4.1.14)$$

The inequality (3.2.17) shows that the last term can be neglected. The imaginary part of (4.1.13) is

$$P\psi^{I} = -(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\left(\psi_{\underline{k}}^{R} + \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{u}\right) \frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}}$$

$$+ \left(\omega_s \frac{\partial E_F}{\partial N} \frac{n_0}{u} - (q_s \cdot \underline{v}_{\underline{k}} - \omega_s) \frac{\partial E_F}{\partial N} \left(\frac{n - n_0}{u}\right)^{R} \right) \frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}} \qquad (4.1.15)$$

The inequalities (3.2.15) and (3.2.16) show that the terms proportional to $\psi_{\underline{k}}^{R}$ and $\left(\frac{n-n_0}{u}\right)^{R}$ can be neglected. Thus (4.1.15) can be written in the form

$$P\psi^{I} = X \qquad (4.1.16)$$

where

$$X = -(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\, \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{u} \frac{\partial f_{\underline{k}}^{0}}{\partial E_{k}}$$

$$+ \omega_s \frac{\partial E_F}{\partial N} \frac{n_0}{u} \frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}} \qquad (4.1.17)$$

Now from (4.1.14)

$$\alpha = -\frac{1}{\rho v_s} \frac{1}{8\pi^3} \int d\underline{k}\, (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\, \psi_{\underline{k}}^{I} \frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}}\, \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{u}$$

$$= \frac{1}{\rho V_s} \langle X \psi^I \rangle + \frac{1}{8\pi^3 \rho V_s} \int d\underline{k} \; \omega_s \frac{\partial E_F}{\partial N} \left(\frac{n_0}{\omega} \right) \psi_{\underline{k}}^I \frac{\partial f_{\underline{k}}^0}{\partial E_{\underline{k}}} \qquad (4.1.18)$$

By (3.2.17) the last term in this expression is negligible and once again

$$\alpha = \frac{1}{\rho V_s} \langle X \psi^I \rangle \qquad (4.1.19)$$

and

$$\alpha^{-1} \leq \frac{\tfrac{1}{2} \rho V_s \Omega \beta \int d\underline{k} \int d\underline{k}' \left(\psi_{\underline{k}}^I - \psi_{\underline{k}'}^I \right)^2 P_{\underline{k}\underline{k}'}}{\left[\int d\underline{k} \left(\left(\underline{q}_s \cdot \underline{v}_{\underline{k}} - \omega_s \right) \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{\underline{u}} - \omega_s n_0 \frac{\partial E_F}{\partial N} \right) \psi_{\underline{k}}^I \frac{\partial f_{\underline{k}}^0}{\partial E_{\underline{k}}} \right]^2} \qquad \text{(4.1.20}$$

It is interesting to note that equations (4.1.12) and (4.1.20) demonstrate the validity of an equivalent of Matthiessen's rule for the ultrasonic attenuation in metals, i.e. for a particular sample

$$\frac{1}{\alpha(T)} = \frac{1}{\alpha_{res}} + \frac{1}{\alpha_{ideal}(T)} \qquad (4.1.21)$$

where α_{res} is the attenuation at zero temperature (when the electron free path is limited only by static lattice imperfections) and α_{ideal} is the attenuation in a perfect crystal of the metal in question (in which the electron free path is limited only by phonons). In practice equation (4.1.21) will not be satisfied exactly because the transition probabilities due to phonon and impurity scattering do not add in a simple way and because the addition of impurities into an ideal crystal will somewhat modify the functional form of the ψ^I that solves (4.1.5) or (4.1.16). An investigation of deviations from the equivalent of Matthiessen's rule (DEMR) will be undertaken in the next chapter.

Reasonable first order trial functions to insert into (4.1.12) and (4.1.19) are those obtained from the isotropic scattering time analysis of Chapter III, i.e.

$$\psi_{\underline{k}}^{F} \propto (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)(\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{\underline{u}}) \qquad \text{(transverse)}$$

$$\propto (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)(\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{u}) - \omega_s n_0 \frac{\partial E_F}{\partial N} \qquad \text{(longitudinal)} \qquad (4.1.22)$$

For a metal with a spherical fermi surface the first order trial function for longitudinal waves is

$$\psi_{\underline{k}}^{F} \propto (\hat{q}_s \cdot \hat{\underline{k}})^2 - 1/3 \qquad (4.1.23)$$

where terms of the order of v_s/v_F have been neglected. In order to obtain results for a randomly ordered polycrystal the value of

$$\int d\underline{k} \int d\underline{k}' \, (\psi_{\underline{k}}^{I} - \psi_{\underline{k}'}^{F})^2 \, P_{\underline{k}\underline{k}'}$$

must be replaced by its average over all orientations of $\underline{q}_s$. Intuitively one might also expect this replacement to produce quite good results for single crystals with cubic symmetry as well. The required average is evaluated in Appendix A. Making the substitutions (4.1.22) and (A.1) one obtains

$$\frac{1}{\alpha} \leq \frac{135\, \rho\, V_s\, \Omega\, \beta\, \hbar^2}{128\, \pi^2\, m^{*2} q_s^2\, V_F^2\, k_F^4} \int d\underline{k} \int d\underline{k}' \left(1 - \cos^2(\underline{k}\,\underline{k}')\right) \qquad \text{(4.1.2}$$

For transverse waves it is easy to show that

$$\alpha_{trans} = 3/4\ \alpha_{long} \times \frac{V_s^{long}}{V_s^{trans}} \qquad (4.1.25)$$

Equations (4.1.24) and (4.1.25) are the analogues of the standard first order variational formula for the electrical resistivity of a simple metal $^{(21)}$. If the scattering of electrons is isotropic then the integral over $\underline{k}$ in (4.1.24) amounts to a multiplication by a constant factor and it is easy to verify that the variational method outlined in this section reproduces the results of the Bhatia and Moore analysis.

4.2 The Residual Attenuation

In this section the zero temperature attenuation in a metal that contains dilute substitutional impurities or simple vacancies will be examined. The theory will be presented only for electrons describable by the one-OPW approximation but the generalisation to a metal with a non-spherical fermi surface is obvious. For a crystal containing a single impurity

$$P_{\underline{k}\underline{k}'} = f^{\circ}_{\underline{k}} \left(1 - f^{\circ}_{\underline{k}'} \right) Q_{\underline{k}\underline{k}'}$$

where $Q_{\underline{k}\underline{k}'}$ is given by (2.11) and includes the factor $\delta(E_{\underline{k}} - E_{\underline{k}'})$. The factors $f_{\underline{k}}(1 - f_{\underline{k}'})$ and $\delta(E_{\underline{k}} - E_{\underline{k}'})$ restrict scattering processes to electrons on the fermi surface and equation (4.1.24) can be written

$$\frac{1}{\alpha_{\mathrm{long}}} \leq \frac{135\, \rho V_s \Omega}{128\, \pi^2 m^{*2} q_s^2 V_F^4 k_F^4 N_T^2} \int_{FS} dS_{\underline{k}} \int_{FS'} dS_{\underline{k}'}$$

$$\times \frac{2\pi}{\hbar} \left(1 - \cos^2(\underline{k}\,\underline{k}') \right) \left| W(\underline{k} - \underline{k}') \right|^2 \qquad (4.2.1)$$

The standard formula

$$\int_{FS} dS_{\underline{k}} \int_{FS'} dS_{\underline{k}'} \, f(\underline{k} - \underline{k}') = 2\pi k_F^2 \int_{|q| < 2k_F} \frac{d^3q}{q} f(q) \qquad (4.2.2)$$

and the fact that

$$1 - \cos^2(\underline{k}\,\underline{k}') = \frac{q^2}{k_F^2} - \frac{q^4}{4 k_F^4} \qquad (4.2.3)$$

where

$$\underline{q} = \underline{k} - \underline{k}'$$

can be used to write (4.2.1) as

$$\frac{1}{\alpha} \leq \frac{135\,\rho v_s\,\Omega}{32\,\hbar\, m^{*2} q_s^2 v_F^4 k_F^4 N_t^2} \int_{|q|<2k_F} \frac{d^3\underline{q}}{q} \,|W(q)|^2 \left(\frac{q^2}{k_F^2} - \frac{q^4}{4k_F^4}\right) \qquad (4.2.4)$$

Multiplying by the number of impurities and changing the variable of integration to

$$\underline{\eta} = \underline{q} / k_F$$

yields

$$\frac{1}{\alpha} \leq \frac{135\,\rho v_s\, n_I}{32\, N_c\, m^{*2} q_s^2 v_F^4} \int_{|\eta|<2} d^3\underline{\eta}\, |W(k_F \underline{\eta})|^2 \left(\eta - \frac{\eta^3}{4}\right) \qquad (4.2.5)$$

where n_I is the impurity concentration (atomic fraction). For a metal with a parabolic conduction band this becomes

$$\frac{1}{\alpha} \leq \frac{135}{32} \left(\frac{\rho v_s}{m^* v_F^2 q_s^2 N_c}\right) \frac{n_I m^*}{\hbar^3 k_F^2} \int d^3\eta\, |W(k_F \underline{\eta})|^2 \left(\eta - \frac{\eta^3}{4}\right) \qquad (4.2.6)$$

This expression is in the same form as that used by Popovic et al.(30) to calculate the residual resistivity of dilute alkali alloys.

The interesting thing about (4.1.5) is that it weights scattering events with large momentum transfers much more heavily than does the comparable expression for the electrical resistivity (which has no η^3

term in the integrand). This means that a measurement of α reveals information about the pseudopotentials of the host and impurity atoms that very usefully complements any information available from measurements of the electrical resistivity ρ^*. Indeed it is easy for a theoretician to envisage a program of fitting several parameters specifying the form of the ion pseudopotentials to measurements of α and ρ^* in a large variety of host-impurity systems. It would also be most interesting to compare the results obtained using the pseudopotential formalism with those obtained by the use of realistic ion potentials for the impurities and the Friedel sum rule for the phase shifts of the scattered electrons. Such a program would seem to be impractical at the moment, however, because many of the resistivity measurements that have been made are wildly contradictory and because there are no measurements of the low temperature α in dilute alloys of known composition.

4.3 The Ideal Attenuation

In a perfect crystal the only contribution to the transition rates $P_{\underline{kk}'}$ comes from electron-phonon scattering. Substitution of the expression (2.9) for $P_{\underline{kk}'}$ produces a formula for α^{-1} that is proportional to

$$\int d\underline{k} \int d\underline{k}' \left(\psi^{I}_{\underline{k}} - \psi^{I}_{\underline{k}'} \right)^2 f^{0}_{\underline{k}} \left(1 - f^{0}_{\underline{k}'} \right) \frac{2\pi}{\hbar} \sum_{\lambda} \left| g_{\underline{k}\underline{k}'\lambda} \right|^2$$

$$\times \; n^{0}_{q\lambda} \Big[\delta\left(E_{\underline{k}'} - E_{\underline{k}} - \hbar\omega_\lambda(\underline{k} - \underline{k}') \right.$$

$$+ \; e^{\beta\hbar\omega_\lambda(\underline{k}-\underline{k}')} \, \delta\left(E_{\underline{k}'} - E_{\underline{k}} + \hbar\omega_\lambda(\underline{k} - \underline{k}') \right) \Big] \qquad (4.3.1)$$

This can be transformed into an integral over $dS_{\underline{k}}$, $dS_{\underline{k}'}$, $dE_{\underline{k}}$, and $d\omega$ where

$$\hbar\omega = E_{\underline{k}} - E_{\underline{k}'}$$

The result of such a procedure is well known (31) and can be written

$$4\pi \int d\omega \; R(\omega) \int_{FS} \frac{dS_{\underline{k}}}{\hbar |\underline{v}_{\underline{k}}|} \int_{FS'} \frac{dS_{\underline{k}'}}{\hbar |\underline{v}_{\underline{k}'}|}$$

$$\times \left[\psi^{I}_{\underline{k}} - \psi^{I}_{\underline{k}'} \right]^2 \frac{1}{\hbar} \sum_{\lambda} \left| g_{\underline{k}\underline{k}'\lambda} \right|^2 \delta\left(\omega - \omega_\lambda(\underline{k} - \underline{k}') \right) \qquad (4.3.2)$$

where

$$R(\omega) = \frac{\hbar\omega}{(e^{\beta\hbar\omega}-1)\,(1-e^{\beta\hbar\omega})} \qquad (4.3.3)$$

Thus, making use of the first order trial function (4.1.22), the variational expression for the attenuation of transverse waves is

$$\begin{aligned}
\frac{1}{\alpha} \leq\ & 2\pi\rho v_s \Omega \beta \int d\omega\, R(\omega) \int_{FS} \frac{dS_{\underline{k}}}{|\underline{v}_{\underline{k}}|} \int_{FS'} \frac{dS_{\underline{k}'}}{|\underline{v}_{\underline{k}'}|} \\
& \times \left[(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\,(\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{\underline{u}}) - (q_s \cdot \underline{v}_{\underline{k}'} - \omega_s)\,(\underline{v}_{\underline{k}'} \cdot \underline{\underline{M}}_{\underline{k}'} \cdot \hat{\underline{u}}) \right]^2 \\
& \times \frac{1}{\hbar} \sum_{\lambda} |g_{\underline{k}\underline{k}'\lambda}|^2\, \delta(\omega - \omega_\lambda(\underline{k}-\underline{k}')) \\
& \times \left[\int_{FS} \frac{dS_{\underline{k}}}{|\underline{v}_{\underline{k}}|} (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)^2 (\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{\underline{u}})^2 \right]^{-2} \qquad (4.3.4)
\end{aligned}$$

If one defines

$$\begin{aligned}
\beta_{ua}^2 F(\omega) =\ & \Omega \int_{FS} \frac{dS_{\underline{k}}}{|\underline{v}_{\underline{k}}|} \int_{FS'} \frac{dS_{\underline{k}'}}{|\underline{v}_{\underline{k}'}|} \\
& \times \left[(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)(\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{k} \cdot \hat{\underline{u}}) - (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)(\underline{v}_{\underline{k}'} \cdot \underline{\underline{M}}_{k} \cdot \hat{\underline{u}}) \right]^2 \\
& \times \frac{1}{\hbar} \sum_{\lambda} |g_{\underline{k}\underline{k}'\lambda}|^2\, \delta(\omega - \omega_\lambda(\underline{k}-\underline{k}')) \qquad (4.3.5)
\end{aligned}$$

then it becomes apparent that α^{-1} can be written as a constant factor times

$$\beta \int R(\omega)\, \beta_{ua}^{2} F(\omega)\, d\omega$$

The advantage of this formulation of the problem is that all the temperature dependence is contained in the factors β and $R(\omega)$. The time consuming calculation of $\beta_{ua}^{2} F(\omega)$ needs to be done only once and the evaluation of α at any temperature can be done by a simple integral over ω. Obviously the same procedure can be used to obtain a similar first order variational expression for the attenuation of longitudinal waves.

For a metal with a spherical fermi surface one can use (4.1.24) as a starting point and quickly arrive at

$$\frac{1}{\alpha} \leq \frac{135\, \rho V_s \Omega \beta}{128\, \pi^2 m^{*2} q_s^2 V_F^4} \int 4\pi\, R(\omega)\, d\omega$$

$$\times \int d\Omega_{\underline{k}} \int d\Omega_{\underline{k}'} \frac{1}{\hbar} \sum_{\lambda} |g_{\underline{k}\underline{k}'\lambda}|^2 \delta(\omega - \omega_\lambda(\underline{k} - \underline{k}'))$$

$$\times \left[1 - \cos^2(\underline{k}\,\underline{k}') \right] \qquad (4.3.6)$$

Define

$$\alpha_{ua}^{2} F(\omega) = \frac{\Omega\, m\, k_F}{2\pi^2 \hbar^2 (4\pi)^2} \int_{FS} d\Omega_{\underline{k}} \int_{FS'} d\Omega_{\underline{k}'}$$

$$\times \left[1 - \cos^2(\underline{k}\,\underline{k}') \right] \frac{1}{\hbar} \sum_{\lambda} |g_{\underline{k}\underline{k}'\lambda}|^2 \delta(\omega - \omega_\lambda(\underline{k} - \underline{k}')) \qquad (4.3.7)$$

then

$$\frac{1}{\alpha} \leq \frac{135\pi^3 \rho V_s \beta \hbar^2}{m^{*2} m q_s^2 V_F^4 k_F} \int R(\omega) \alpha_{ua}^2 F(\omega)\, d\omega \qquad (4.3.8)$$

For free electrons this is

$$\frac{1}{\alpha} \leq \frac{15 \rho V_s^3}{2 \omega_s^2 m V_F^2} \, 6\pi\beta \int R(\omega) \alpha_{ua}^2 F(\omega)\, d\omega \qquad (4.3.9)$$

and the appropriate value of τ in (3.2.20) is

$$\frac{1}{\tau} = 6\pi\beta \int R(\omega) \alpha_{ua}^2 F(\omega)\, d\omega \qquad (4.3.10)$$

The function $\alpha_{ua}^2 F(\omega)$ is very similar to the functions $\alpha^2 F(\omega)$ used in the theory of superconductivity and $\alpha_{tr}^2 F(\omega)$ familiar from the theory of electrical conductivity (32).

Since both the phonon frequencies and the pseudopotential form factors are volume dependent it would seem that a treatment of the pressure dependence of α would require a detailed recalculation of $\alpha_{ua}^2 F(\omega)$ at each lattice volume of interest. There is a much simpler scaling procedure, however, which can be used if certain assumptions are made about the physics of the metal considered. A similar method has been applied to scaling $\alpha^2 F(\omega)$ and $\alpha_{tr}^2 F(\omega)$ (32). To see how the method works one first of all uses (4.2.2) and (4.2.3) to write $\alpha_{ua}^2 F(\omega)$ as

$$\alpha^2_{ua}F(\omega) = C\frac{\Omega}{k_F^3}\int d^3q\left(q-\frac{q^3}{4k_F^2}\right)$$

$$\times\sum_\lambda |W(q)|^2\,\frac{|q\cdot\varepsilon_\lambda(q)|^2}{\omega_\lambda(q)}\,\delta(\omega-\omega_\lambda(q)) \tag{4.3.11}$$

where C is a constant. Then one assumes that a volume change scales up the frequencies of all the phonons by the same factor regardless of the wavevector or branch index, i.e.

$$[\omega_\lambda(q)]_\Omega = A\,[\omega_\lambda(q)]_{\Omega=V_0} \tag{4.3.12}$$

where

$$A = 1 + \left(\frac{V_0-\Omega}{V_0}\right)\gamma \tag{4.3.13}$$

and γ plays the role of an average Gruneisen constant. It is easy to show that if the form factors $W(\underline{q})$ were independent of volume then

$$\left[\alpha^2_{ua}F(A\omega)\right]_\Omega = \frac{1}{A^2}\left[\alpha^2_{ua}F(\omega)\right]_{\Omega=V_0} \tag{4.3.14}$$

To investigate the effects of changes in the pseudopotential form factors one assumes that any alteration in $\alpha^2_{ua}F(\omega)$ can be accounted for by a multiplication by a constant, i.e.

$$A^2\left[\alpha^2_{ua}F(A\omega)\right]_\Omega = S\left[\alpha^2_{ua}F(\omega)\right]_{\Omega=V_0} \tag{4.3.15}$$

Now it is useful to evaluate

$$\int [\alpha^2_{ua}F(\omega)]_\Omega \, \omega \, d\omega$$

$$= \frac{c\Omega}{k_F^3}\int d^3q\left(q-\frac{q^3}{4k_F^2}\right)|W(q)|^2\sum_\lambda |q\cdot\xi_\lambda(q)|^2 \qquad (4.3.16)$$

The normalization of the polarisation vector gives

$$\sum_\lambda (\xi_\lambda(q))^*_\alpha\,(\varepsilon_\lambda(q))_\beta = \delta_{\alpha\beta}$$

so

$$\int [\alpha^2_{ua}F(\omega)]_\Omega \, \omega \, d\omega = \frac{c\Omega}{k_F^3}\int d^3q\left(q^3-\frac{q^5}{4k_F^2}\right)|W(q)|^2 \qquad (4.3.17)$$

where k_F and $W(q)$ are evaluated for a lattice volume Ω. But this must also be equal to

$$\frac{S}{A^2}\int \left[\alpha^2_{ua}F\left(\frac{\omega}{A}\right)\right]_{\Omega=V_0}\omega\, d\omega = S\int [\alpha^2_{ua}F(\omega)]_{V_0}\,\omega\, d\omega \qquad (4.3.18)$$

So

$$S = \frac{\left[\frac{\Omega}{k_F^3}\int d^3q\left(q^3-\frac{q^5}{4k_F^2}\right)|W(q)|^2\right]_\Omega}{\left[\frac{V_0}{k_F^3}\int d^3q\left(q^3-\frac{q^5}{4k_F^2}\right)|W(q)|^2\right]_{V_0}} \qquad (4.3.19)$$

Since

$$[\Omega k_F^3]_\Omega = [V_0 k_F^3]_{V_0}$$

and since for many pseudopotentials of interest (such as the Ashcroft pseudopotential)

$$w(q) = W(q)$$

(4.3.19) can be simplified to

$$S = \frac{\left[\int_0^2 d\eta \left(\eta^5 - \frac{\eta^7}{4}\right) |W(k_F\eta)|^2\right]_\Omega}{\left[\int_0^2 d\eta \left(\eta^5 - \frac{\eta^7}{4}\right) |W(k_F\eta)|^2\right]_{V_0}} \qquad (4.3.20)$$

It has been found that a similar scaling procedure produces results for $\alpha^2_{tr}F(\omega)$ that are very close to those of a detailed recalculation even for very large volume changes (33).

Numerical calculations of the temperature and volume dependence of the low frequency ultrasonic attenuation in the alkali metals could be undertaken using equations (4.3.9) and (4.3.20). It has been found that calculations employing the one OPW approximation for the electrons and the Ashcroft pseudopotential to describe the electron-ion interaction can account quite well for other transport properties of the alkalis as long as a realistic description of the lattice dynamics is also used (34,35). For a numerical calculation of α in the alkalis the phonon frequencies and polarisation vectors could be taken from

Born-von Karmen models fit to measured phonon dispersion curves in Li (36), Na (37), K (38) and Rb (39). The core radii for the Ashcroft pseudopotentials could be fitted directly to experimentally measured values of α or they could be taken from reference (40) where they were fitted to electrical conductivity measurements (41) at selected temperatures. The numerical techniques for computer calculation of phonon distribution functions such as $\alpha_{uu}^2 F(\omega)$ are well developed (42).

4.4 Phonon Drag

Numerical calculations of the electrical resistivity of the alkali metals show that phonon drag effects are important below about 30 K in Li, 10 K in Na and K, and 2 K in Rb (43). Since the temperature range of present interest is about 0-20 K it is clear that any realistic treatment of ultrasonic attenuation due to interactions with conduction electrons must include phonon drag effects. In this section the variational formalism will be extended to allow for the consideration of phonon drag.

The description of phonon drag effects requires the use of two coupled Boltzmann equations for the electrons and phonons. If temperature gradients and phonon-phonon processes are ignored then the two (linearised) equations for transverse waves in the low frequency regime are (44)

$$\frac{\partial f^{0}_{\underline{k}}}{\partial E_{\underline{k}}}\left[-i\left(q_s\cdot\underline{v}_{k}-\omega_s\right)\phi_{\underline{k}}\right]$$

$$= \frac{\beta\Omega}{8\pi^{3}}\sum_{\lambda}\int d\underline{k}'\left[\left[\phi_{\underline{k}}-\tilde{\phi}_{\underline{k}}+\xi_{q\lambda}-(\phi_{\underline{k}'}-\tilde{\phi}_{\underline{k}'})\right]P^{\underline{k}'}_{\underline{k}q\lambda}\right.$$

$$\left.+\left[(\phi_{\underline{k}}-\tilde{\phi}_{\underline{k}}-\xi_{q\lambda}-(\tilde{\phi}_{\underline{k}}-\tilde{\phi}_{\underline{k}'})\right]P^{\underline{k}'q\lambda}_{\underline{k}}\right] \qquad (4.4.1)$$

and

$$\frac{\partial n^{0}_{q\lambda}}{\partial E_{q\lambda}}\left[-i\left(q_s\cdot\underline{v}_{\underline{k}\lambda}-\omega_s\right)\xi_{q\lambda}\right]$$

$$= \frac{\beta\Omega}{4\pi^3}\int \left[\phi_{\underline{k}} - \tilde{\phi}_{\underline{k}} - \xi_{q\lambda} - (\phi_{\underline{k}'} - \tilde{\phi}_{\underline{k}'})\right] P^{\underline{k}'}_{\underline{k}q\lambda}\, d\underline{k}' \qquad (4.4.2)$$

where $\underline{v}_{q\lambda}$ is the velocity of the phonon $(q\lambda)$,

$$q = \underline{k}' - \underline{k}$$

and where

$$n_{q\lambda} = n^0_{q\lambda} - \frac{\partial n^0_{q\lambda}}{\partial E_{q\lambda}}\,\xi_{q\lambda} \qquad (4.4.3)$$

The expressions for $P^{\underline{k}'}_{\underline{kq}}$ and $P^{k'q}_{\underline{k}}$ are

$$P^{\underline{k}'}_{\underline{k}q\lambda} = \frac{2\pi}{\hbar}\,|g_{\underline{k}\underline{k}'\lambda}|^2 f^0_{\underline{k}}(1-f^0_{\underline{k}'})\, n^0_{q\lambda}\, \delta(E_{\underline{k}'} - E_{\underline{k}} - \hbar\omega_\lambda(q)) \qquad (4.4.4)$$

$$P^{\underline{k}'q\lambda}_{\underline{k}} = \frac{2\pi}{\hbar}\,|g_{kk'\lambda}|^2 f^0_{\underline{k}}(1-f^0_{\underline{k}'})(n^0_{q\lambda}+1)\, \delta(E_{k'} - E_{\underline{k}} + \hbar\omega_\lambda(q)) \qquad (4.4.5)$$

A great simplification results if one makes the usual assumption

$$\xi_{q\lambda} = \xi_q \qquad (4.4.6)$$

It is also convenient to assume for the moment that

$$|v_{q\lambda}| = |\underline{v}_q| \approx v_s \qquad (4.4.7)$$

It will become clear later that assumption (3.4.8) has no effect on the final results. It is helpful to define

$$P_{\underline{k}q}^{\underline{k}'} = \sum_{\lambda} P_{\underline{k}q\lambda}^{\underline{k}'} \quad \text{and} \quad P_{\underline{k}}^{\underline{k}'q} = \sum_{\lambda} P_{\underline{k}}^{\underline{k}'q\lambda} \tag{4.4.8}$$

An estimate for the magnitude of ξ_q can be made by taking the limit of complete phonon drag (20). This means that the phonon distribution is displaced the same amount from its equilibrium in $\underline{q}$ space as the electron distribution is displaced from its equilibrium in $\underline{k}$ space. The displacement of the electron distribution is

$$\Delta K \sim \frac{m}{\hbar} \Delta V \sim \frac{m}{\hbar} q_s v_F \tau u \tag{4.4.9}$$

The phonon distribution function is

$$n_{q\lambda} = n_{q\lambda}^0 + \frac{\partial n_{q\lambda}^0}{\partial E_{q\lambda}} \frac{\partial E_{q\lambda}}{\partial q} \Delta K$$

$$\sim n_{q\lambda}^0 + \frac{\partial n_{q\lambda}}{\partial E_{q\lambda}} \hbar v_s \frac{m}{\hbar} (q_s v_F \tau) u$$

or

$$|\xi_q| \sim (q_s v_F \tau)\, m v_s u \tag{4.4.10}$$

Substituting (3.4.7) and (3.4.10) into (3.4.1) and (3.4.2), then dividing both equations by u and taking the imaginary part produces

$$-\frac{\partial f_{\underline{k}}^0}{\partial E_{\underline{k}}} \left[(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\, \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{\underline{u}} \right]$$

$$= \frac{\beta\Omega}{8\pi^3}\int\left([\psi_{\underline{k}}^{I} - \chi_q^{I} - \psi_{\underline{k}'}^{I}]\,\rho_{\underline{k}q}^{\underline{k}'} + [\psi_{\underline{k}}^{I} + \chi_q^{I} - \psi_{\underline{k}'}^{I}]\,\rho_{\underline{k}}^{\underline{k}'q}\right)d\underline{k}' \qquad (4.4.1$$

and

$$\frac{\partial n_q^0}{\partial E_{q\lambda}}\,[q_s v_s\,(q_s v_F \tau)\, m v_s]$$

$$= \frac{\beta\Omega}{4\pi^3}\int [\psi_{\underline{k}}^{I} - \chi_q^{I} - \psi_{\underline{k}'}^{I}]\,\rho_{\underline{k}q}^{\underline{k}'}\, d\underline{k}' \qquad (4.4.12)$$

where

$$\chi_q^{I} = \mathrm{Im}\left(\frac{\xi_q}{u}\right)$$

and

$$n_q^0 = \sum_{\lambda} n_{q\lambda}^0$$

The equations (4.4.11) and (4.4.12) can be written in the form

$$P\,\underline{Y} = \underline{X} \qquad (4.4.13)$$

where the vector functions $\underline{Y}$ and $\underline{X}$ are

$$\underline{Y} = \{Y_{\underline{k}}^1, Y_q^2\} = \{\psi_{\underline{k}}^{I}, \chi_q^{I}\}$$

$$\underline{X} = \{X_{\underline{k}}^1, X_q^2\} = \{(q_s \cdot \underline{v}_{\underline{k}} - \omega_s)(\underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{\underline{q}}),\ q_s v_s (q_s v_F \tau)\, m v_s\}$$

and

$$P(Y^1_{\underline{k}}, Y^2_{q}) = (P^1_{\underline{k}}, P^2_{q})$$

The expressions for $P^1_{\underline{k}}$ and $P^2_{\underline{q}}$ are

$$P^1_{\underline{k}} = \frac{\beta\Omega}{8\pi^3} \int [\psi^I_{\underline{k}} - \chi^I_q - \psi^I_{\underline{k}'}]\, P^{\underline{k}'}_{\underline{k}q} + [\psi^I_{\underline{k}} + \chi^I_q - \psi^I_{\underline{k}'}]\, P^{\underline{k}'q}_{\underline{k}}\, d\underline{k}'$$

$$P^2_q = \frac{\beta\Omega}{8\pi^3} \int [\psi^I_{\underline{k}} - \chi^I_q - \psi^I_{\underline{k}'}]\, P^{\underline{k}'}_{\underline{k}q}\, d\underline{k}'$$

The properties of the scattering operator insure that the solution $\underline{Y}$ of (4.4.13) is the function that gives

$$\frac{\langle \underline{Y} P \underline{Y} \rangle}{[\langle \underline{Y} \underline{X} Y \rangle]^2} \qquad (4.1.14)$$

its minimum value $^{(45)}$. Here the inner product is defined by

$$\langle \underline{X}\, \underline{Y} \rangle = \frac{1}{4\pi^3} \int X^1_{\underline{k}}\, Y^1_{\underline{k}}\, d\underline{k} + \frac{1}{8\pi^3} \int X^2_q\, Y^2_q\, dq \qquad (4.4.15)$$

where the integral over $\underset{\sim}{q}$ extends throughout the first Brillouin zone.

In the present case

$$\langle \underline{X}\, \underline{Y} \rangle = \frac{1}{4\pi^3} \int (q_s \cdot \underline{v}_{\underline{k}} - \omega_s)\, \underline{v}_{\underline{k}} \cdot \underline{\underline{M}}_{\underline{k}} \cdot \hat{u}\, \psi^I_{\underline{k}}\, \frac{\partial f^0_{\underline{k}}}{\partial E_{\underline{k}}}\, d\underline{k}$$

$$+ \frac{1}{8\pi^3}\int q_s v_s (q_s v_F \tau)\, m v_s\, \xi_q \frac{\partial n_q^0}{\partial E_q}\, dq \qquad (4.1.16)$$

By using the estimate (4.4.10) for ξ_q one can easily convince oneself that the second term on the right is negligible. It is thus easy to show that

$$\langle \underline{X}\, \underline{Y} \rangle = 2 \rho v_s \alpha \qquad (4.1.17)$$

It can also be shown that (45)

$$\langle \underline{Y}\, P\, \underline{Y} \rangle = \frac{1}{4\pi^3}\int d\underline{k}\, \frac{\Omega\beta}{8\pi^3}\int d\underline{k}' \left(\phi_{\underline{k}} - \chi_q - \phi_{\underline{k}'}\right)^2 P_{\underline{k}q}^{\underline{k}'} \qquad (4.1.18)$$

where it is understood that ($\underline{k}' - \underline{k}$) reduced to the first Brillouin zone is equal to $\underline{q}$.

The problem of calculating the attenuation of transverse waves including phonon drag has been reduced to one of minimising the expression (4.1.14). The only extra complication over the variational formalism without phonon drag is the necessity of minimising with respect to variations of <u>two</u> functions, ϕ_k and χ_q. A similar expression may be found for the attenuation of longitudinal waves; $\langle \underline{YPY} \rangle$ remains the same as in (4.4.18) but $(\langle \underline{XY} \rangle)^2$ becomes equal to four times the denominator of (4.1.20).

In the case of longitudinal waves reasonable first order trial functions for a metal with a spherical fermi surface are

$$\phi_{\underline{k}} = (\hat{\underline{k}} \cdot \hat{\underline{q}})^2 - 1/3$$

and

$$\chi_q = \gamma \left[\left[(\hat{\underline{k}} - \hat{\underline{k}}')_R \cdot \hat{\underline{u}} \right]^2 - 1/3 \right] \tag{4.4.19}$$

where

$$(\hat{\underline{k}} - \hat{\underline{k}}')_R$$

is k_F^{-1} times $(\underline{k} - \underline{k}')$ reduced to the first Brillouin zone and γ is to be determined by a minimisation. It can be shown that the result of this minimisation is

$$\frac{1}{\alpha} = \frac{1}{\alpha_0}\left(1 - \frac{P_{1L}^2}{P_{LL}\,P_{11}}\right) \tag{4.4.20}$$

where α_0 is the first order variational value for α without phonon drag,

$$P_{ij} = \int R(\omega)\, \alpha^2_{ua\,ij} F(\omega)\, d\omega \tag{4.4.21}$$

and

$$\alpha^2_{ua11} F(\omega) = \frac{4}{15} \alpha^2_{ua} F(\omega)$$

$$\alpha^2_{uaLL} F(\omega) = \frac{\Omega m k_F}{2\pi^2 \hbar^2 (4\pi)^2} \int_{FS} d\Omega_{\underline{k}} \int_{FS'} d\Omega_{\underline{k}'}$$

$$\times \left[\left((\hat{\underline{k}} - \hat{\underline{k}}')_R \cdot \hat{\underline{u}} \right)^2 - 1/3 \right]^2 \frac{1}{\hbar} \sum_\lambda |g_{\underline{k}\underline{k}'\lambda}|\, \delta(\omega - \omega_\lambda(\underline{k} - \underline{k}'))$$

$$\alpha^2_{ua1L} F(\omega) = \frac{\Omega m k_F}{2\pi^2 \hbar^2 (4\pi)^2} \int_{FS} d\Omega_{\underline{k}} \int_{FS'} d\Omega_{\underline{k}'}$$

$$\times \left[\left((\hat{\underline{k}} - \hat{\underline{k}}')_R \cdot \hat{\underline{u}} \right)^2 - 1/3 \right] \left[(\hat{\underline{k}} \cdot \hat{\underline{u}})^2 - 1/3 - \left((\hat{\underline{k}}' \cdot \hat{\underline{u}})^2 - 1/3 \right) \right]$$

$$\times \sum_{\lambda} \frac{1}{\hbar} \left| g_{\underline{k}\underline{k}'\lambda} \right|^2 \delta(\omega - \omega_\lambda(\underline{k} - \underline{k}'))$$

The integrand in the expression for $\alpha^2_{ua1L} F(\omega)$ can be written in terms of

$$q = \underline{k} - \underline{k}' \quad \text{and} \quad q_R = (\hat{\underline{k}} - \hat{\underline{k}}')_R$$

as follows. Note that

$$\left[(q_R \cdot \hat{\underline{u}})^2 - 1/3 \right] \left[(\hat{\underline{k}} \cdot \hat{\underline{u}})^2 - (\hat{\underline{k}}' \cdot \hat{\underline{u}})^2 \right]$$

$$= \left[(q_R \cdot \hat{\underline{u}})^2 - 1/3 \right] \left[(\hat{\underline{k}} - \hat{\underline{k}}') \cdot \hat{\underline{u}} \right] \left[(\hat{\underline{k}} - \hat{\underline{k}}' + 2\hat{\underline{k}}') \cdot \hat{\underline{u}} \right]$$

$$= \left[(q_R \cdot \hat{\underline{u}})^2 - 1/3 \right] (q \cdot \hat{\underline{u}})^2 + 2\left[(q_R \cdot \hat{\underline{u}})^2 - 1/3 \right] (q \cdot \hat{\underline{u}}) (\hat{\underline{k}}' \cdot \hat{\underline{u}}) \qquad (4.4.22)$$

Since $P_{\underline{k}\underline{k}'}$ depends only on $(\underline{k}' - \underline{k})$ and

$$P_{\underline{k}\underline{k}'}(q) = P_{\underline{k}\underline{k}'}(-q)$$

when the last term is inserted into the integral over $\Omega_{\underline{k}}$ it will make a zero contribution. Equation (4.2.2) now allows the expressions for $\alpha^2_{uu11} F(\omega)$, $\alpha^2_{ua1L} F(\omega)$ and $\alpha^2_{uaLL} F(\omega)$ to be written as

$$\alpha^2_{ua11}F(\omega) = \frac{\Omega m}{15\hbar^2 k_F^3 (4\pi^3)} \int_{|q|<2k_F} d^3q \left(\frac{q}{k_F^2} - \frac{q^3}{4k_F^4}\right)$$

$$\times \frac{1}{\hbar}\sum_\lambda |g_{\underline{k}\underline{k}'\lambda}|^2 \delta(\omega-\omega_\lambda(q)) \qquad (4.4.23)$$

$$\alpha^2_{ua22}F(\omega) = \frac{\Omega m}{\hbar^2 k_F^3 16\pi^3} \int d^3q \, \frac{1}{q} \left((q_r\cdot\hat{\underline{u}})^2 - 1/3\right)^2$$

$$\times \frac{1}{\hbar}\sum_\lambda |g_{\underline{k}\underline{k}'\lambda}|^2 \delta(\omega-\omega_\lambda(q)) \qquad (4.4.24)$$

and

$$\alpha^2_{ua12}F(\omega) = \frac{\Omega m}{\hbar^2 k_F^3 16\pi^3} \int d^3q \, \frac{1}{q} (q\cdot\hat{\underline{u}})^2 \left[(q_K\cdot\hat{\underline{u}})^2 - 1/3\right]$$

$$\times \frac{1}{\hbar}\sum_\lambda |g_{\underline{k}\underline{k}'\lambda}|^2 \delta(\omega-\omega_\lambda(q)) \qquad (4.4.25)$$

Equations (4.4.20, 21, 23, 24 and 25) allow α for longitudinal waves to be calculated with the inclusion of phonon drag effects. Similar equations could be found for the attenuation of transverse waves.

CHAPTER V

SCATTERING TIME APPROXIMATION

5.1 The Anisotropic Scattering Time Solution

The trial solutions given by (4.1.22) are clearly deficient in one important respect. They are taken from the isotropic scattering time approximation and ignore any anisotropy of the electron mean free paths over the fermi surface. In reality the phonon limited mean free path of an electron in the state $\underline{k}$ can depend on the position of $\underline{k}$ in Brillouin zone because
(i) the phonon frequencies $\omega_\lambda(q)$ which appear in the expression for $P_{\underline{kk}'}$ depend on the orientation of $\underline{q}$ as well as its magnitude
and (ii) the phonon wavevectors involved in Umklapp processes depend on the position of $\underline{k}$ with respect to the zone boundary. One approach to including this anisotropy would involve using as a trial function a linear expansion in terms of cubic harmonics and then minimising α^{-1} with respect to the expansion coefficients. Such a procedure would be laborious and probably not very edifying. Fortunately a much more transparent "anisotropic scattering time" solution has been proposed by Robinson and Dow (46) for the analogous problem in the theory of electrical conductivity. In what follows a similar "scattering time" solution to the Boltzmann equation will be derived for the ultrasonic attenuation in the low frequency regime. Attention will be restricted to metals with spherical fermi surfaces.

Suppose that the Bhatia and Moore solution for $\phi_{\underline{k}}$ applicable to transverse waves, given by (3.3.11), was modified to allow for anisotropic scattering times $\tau_1(\hat{\underline{k}})$ and $\tau_2(\hat{\underline{k}})$. Suppose further that these relaxation times have the symmetry of the Brillouin zone. Then

$$\phi_{\underline{k}} = m^* v_F u \left[(i q_s v_F)^2 \frac{\tau_1(\hat{\underline{k}})\,\tau_2(\hat{\underline{k}})}{5} + 1 \right] Y_{11}(\hat{\underline{k}})$$

$$+ \, i q v_F \tau_2(\hat{\underline{k}}) \, m^* v_F u \, Y_{21}(\hat{\underline{k}}) \qquad (5.1.1)$$

And

$$\left.\frac{\partial f_{\underline{k}}}{\partial t}\right]_{scatt} = \frac{m^* v_F u}{5} q_s^2 v_F^2 \frac{\Omega \beta}{8\pi^3} \int d\underline{k}' \, P_{\underline{k}\underline{k}'}$$

$$\times \left[\tau_1(\underline{k})\tau_2(\underline{k}) Y_{11}(\hat{\underline{k}}) - \tau_1(\hat{\underline{k}}')\tau_2(\hat{\underline{k}}') Y_{11}(\hat{\underline{k}}') \right]$$

$$- \frac{i m^* v_F u}{3} q_s v_F \frac{\Omega \beta}{8\pi^3} \int d\underline{k}' \, P_{\underline{k}\underline{k}'} \left[\tau_2(\hat{\underline{k}}) Y_{21}(\hat{\underline{k}}) - \tau_2(\hat{\underline{k}}') Y_{21}(\hat{\underline{k}}') \right] \qquad (5.1.2)$$

Now

$$\tau_1(\hat{\underline{k}})\tau_2(\hat{\underline{k}}) Y_{11}(\hat{\underline{k}}) - \tau_1(\hat{\underline{k}}')\tau_2(\hat{\underline{k}}') Y_{11}(\hat{\underline{k}}'')$$

$$= \tau_1(\hat{\underline{k}})\tau_2(\hat{\underline{k}}) \left[Y_{11}(\hat{\underline{k}}) - Y_{11}(\hat{\underline{k}}') \right] \qquad (5.1.3)$$

$$+ \, Y_{11}(\hat{\underline{k}}') \left[\tau_1(\hat{\underline{k}})\tau_2(\hat{\underline{k}}) - \tau_1(\hat{\underline{k}}')\tau_2(\hat{\underline{k}}') \right]$$

Since the $\tau_i(\hat{k})$ have the symmetry of the Brillouin zone the last term will pass through zero whenever $\underline{k}$ and $\underline{k}'$ are related by a symmetry operation. In a cubic crystal this will occur at a total of 48 points on the fermi surface. Thus it is reasonable to suppose that when inserted into the integral in (5.1.2) this term will make a negligible contribution. The same argument may be applied to the second integral in (5.1.2). Ignoring the small terms one obtains

$$\frac{\partial f_{\underline{k}}}{\partial t}\Big]_{scatt} = q_s^2\, v_F^2\, \frac{m^* v_F u}{5}\, \frac{\beta\Omega}{8\pi^3} \int d\underline{k}'\, P_{\underline{k}\underline{k}'}$$

$$\times\, \tau_1(\hat{\underline{k}})\,\tau_2(\hat{\underline{k}})\,\big(Y_{11}(\hat{\underline{k}}) - Y_{11}(\hat{\underline{k}}')\big)$$

$$- i q v_F\, \frac{m^* v_F u}{3}\, \frac{\beta\Omega}{8\pi^3} \int \tau_2(\hat{\underline{k}})\,\big(Y_{21}(\hat{\underline{k}}) - Y_{21}(\hat{\underline{k}}')\big)\quad P_{\underline{k}\underline{k}'}\, d\underline{k}' \qquad (5.1.4)$$

A substitution of the solution (5.1.1) into the right hand side of the Boltzmann equation (3.3.7) yields

$$\frac{\partial f_{\underline{k}}}{\partial t}\Big]_{scatt} = \Big[\, i q v_F\, m^* v_F u \big(1 - (qv)^2 (1/5\, \tau_1(\hat{\underline{k}})\, \tau_2(\hat{\underline{k}}))\big) \cos\theta\; Y_{11}(\hat{\underline{k}})$$

$$- q^2 v_F^2\, m^* v_F u\, (1/3\, \tau_2(\hat{\underline{k}}))\, \cos\theta\; Y_{21}(\hat{\underline{k}}) \Big] \frac{\partial f^0_{\underline{k}}}{\partial E_{\underline{k}}} \qquad (5.1.5)$$

The term proportional to $(q_s v_F)^2 \tau_1 \tau_2$ is clearly negligible and

$$\cos\theta\; Y_{11}(\hat{\underline{k}}) = \frac{i}{3}\, Y_{21}(\hat{\underline{k}})$$

So (5.1.5) can be written

$$\frac{\partial f_k}{\partial t}\Big]_{scatt} = \Big[iq v_F \frac{m^* v_F u}{3} Y_{21}(\hat{\underline{k}})$$

$$- q^2 v_F^2 m^* v_F u \frac{\tau_2(\hat{\underline{k}})}{3} \cos\theta \, Y_{21}(\hat{\underline{k}}) \Big] \frac{\partial f_{\underline{k}}^0}{\partial E_{\underline{k}}} \qquad (5.1.6)$$

Dividing each of the two expressions for $\frac{\partial f_k}{\partial t}\Big]_{scatt}$ ((5.1.4) and (5.1.6)) by u and equating the imaginary parts produces

$$\left(- \frac{\partial f_{\underline{k}}}{\partial E_{\underline{k}}} \right) \frac{Y_{21}(\hat{\underline{k}})}{\tau_2(\hat{\underline{k}})} = \frac{\beta \Omega}{8\pi^3} \int [Y_{21}(\hat{\underline{k}}) - Y_{21}(\hat{\underline{k}}')] P_{\underline{k}\underline{k}'} \, d\underline{k}' \qquad (5.1.7)$$

The $\tau_2(\hat{\underline{k}})$ defined by (5.1.7) does not precisely have the symmetry of the Brillouin zone. The equation Robinson and Dow found for their relaxation times is very similar and can be written

$$\left(- \frac{\partial f_{\underline{k}}^0}{\partial E_{\underline{k}}} \right) \frac{Y_{10}(\hat{\underline{k}})}{\tau(\hat{\underline{k}})} = \frac{\beta \Omega}{8\pi^3} \int [Y_{10}(\hat{\underline{k}}) - Y_{10}(\hat{\underline{k}}')] P_{\underline{k}\underline{k}'} \, d\underline{k}' \qquad (5.1.8)$$

Robinson and Dow replaced the integral in (5.1.8) by

$$\int [Y_{10}(\hat{\underline{k}}) - P_{10}(\hat{\underline{k}} \cdot \hat{\underline{k}}') Y_{10}(\hat{\underline{k}})] P_{\underline{k}\underline{k}'} \, d\underline{k}'$$

This is strictly correct only for isotropic scattering. In practice, however, there is very little difference between the results of calculations of the ideal resistivity using the Robinson and Dow approximation and those using the exact solution to (5.1.8) (47). Thus it seems reasonable to replace (5.1.7) by

$$\left(-\frac{\partial f_{\underline{k}}}{\partial E_{\underline{k}}}\right)\frac{1}{\tau_2(\underline{k})} = \frac{\beta\Omega}{8\pi^3}\int \left(1-P_{20}(\hat{\underline{k}}\cdot\hat{\underline{k}}')\right)P_{\underline{k}\underline{k}'}\,d\underline{k}' \qquad (5.1.9)$$

This $\tau_2(\hat{\underline{k}})$ does have the required symmetry.

The phonon limited $\tau_2(\hat{\underline{k}})$ can easily be determined from (5.1.9) by substituting expression (2.9) for $P_{\underline{kk}'}$ and integrating both sides with respect to $E_{\underline{k}}$. The energy integrals on the right hand side are then the same as those encountered in section 4.3 and so it is straightforward to obtain

$$\frac{1}{\tau_2(\hat{\underline{k}})} = \frac{\beta\Omega K_F^2}{2\pi^2\hbar V_F}\int R(\omega)\,d\omega\int d\Omega_{\underline{k}'}\left(1-P_{20}(\hat{\underline{k}}'\cdot\hat{\underline{k}})\right)$$

$$\times\sum_{\lambda}|g_{\underline{k}\underline{k}'\lambda}|^2\frac{1}{\hbar}\,\delta(\omega-\omega_\lambda(\underline{k}-\underline{k}')) \qquad (5.1.10)$$

Define

$$\alpha^2_{ua}F(\omega,\underline{k}) = \frac{\Omega m K_F}{8\pi^3\hbar V_F}\int d\Omega_{\underline{k}'}\left(1-\cos^2(\underline{k},\underline{k}')\right)$$

$$\times\frac{1}{\hbar}\sum_{\lambda}|g_{\underline{k}\underline{k}'\lambda}|^2\,\delta(\omega-\omega_\lambda(\underline{k}-\underline{k}')) \qquad (5.1.11)$$

Then the fact that

$$1-P_{20}(\hat{\underline{k}}\cdot\hat{\underline{k}}') = 3/2\left(1-\cos^2(\hat{\underline{k}}\,\hat{\underline{k}}')\right)$$

allows (5.1.10) to be written as

$$\frac{1}{\tau_2(\hat{\underline{k}})} = 6\pi\beta \, \frac{\hbar k_F}{m v_F} \int R(\omega)\, \alpha^2_{uc} \, F(\omega, \underline{k})\, d\omega \qquad (5.1.12)$$

The solution (5.1.1) can now be used to obtain an expression for the attenuation coefficient by a direct substitution into (4.1.11)

$$\alpha = \frac{1}{\rho v_s} \, \frac{q_s^2 v_F^2 m^{*2} k_F^2 v_F^2}{8\pi^3 \hbar v_F} \int_{4\pi} (\cos\theta \sin\theta \cos\phi)\, Y_{21}(\theta,\phi) \times \tau_2(\theta,\phi)\, d\Omega \qquad (5.1.13)$$

This reduces to

$$\alpha = \frac{m^{*2} v_F^3 k_F^2 q_s^2}{30\pi^2 \hbar v_s \rho} \langle \tau_2 \rangle \qquad (5.1.14)$$

where

$$\langle \tau_2 \rangle = \frac{15}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \left[\cos^2\theta \sin^2\theta \cos^2\phi \, \tau_2(\hat{\underline{k}}) \right]$$

The same approach may be used for longitudinal waves and the result is

$$\alpha = \frac{m^{*2} v_F^3 k_F^2 q_s^2}{30\pi^2 \hbar v_s \rho} \langle \tau_2 \rangle \qquad (5.1.15)$$

where

$$\langle \tau_2 \rangle = \frac{45}{16\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \left[(\cos^2\theta - 1/3) \, \tau_2(\underline{k}) \right]$$

Numerical calculations will have to be performed before it can be said with certainty that the scattering time solution presented in this section will produce smaller (and hence better) values for α^{-1} than the first order variational solution.

5.2 Attenuation in Dilute Alloys

The study of the finite temperature transport properties of alloys is, in general, a complicated subject. This is largely because inelastic scattering of electrons from impurities must be considered. Fortunately in very dilute alloys Kus and Carbotte (48) found that the electrical conductivity can be treated by a simple extension of the Robinson and Dow theory. In this section a completely analogous theory applicable to the ultrasonic attenuation is considered.

If one ignores inelastic scattering from impurities and any changes the impurities may cause in the spectrum of thermal phonons then the transition rates for the electrons may be written

$$P_{\underline{k}\underline{k}'} = P^{0}_{\underline{k}\underline{k}'} + P^{R}_{\underline{k}\underline{k}'}$$

where $P^{0}_{\underline{kk}'}$ is the ideal transition rate due to electron-phonon scattering and $P^{R}_{\underline{kk}'}$ arises from elastic scattering from impurities. The relaxation times of section 5.1 become

$$\frac{1}{\tau_2(\underline{k})} = \frac{1}{\tau_2^{0}(\underline{k})} + \frac{1}{\tau_2^{R}(\underline{k})} \qquad (5.2.1)$$

where $\tau_2^{0}(\hat{\underline{k}})$ is defined by (5.1.10) and

$$-\frac{\partial f^{0}_{\underline{k}}}{\partial E_{\underline{k}}}\,\frac{1}{\tau_2^{R}(\underline{k})} = \frac{\beta\Omega}{8\pi^3}\int \left(1 - P_{20}(\underline{k}\cdot\hat{\underline{k}}')\right) P^{R}_{\underline{k}\underline{k}'}\, d\underline{k}' \qquad (5.2.2)$$

The scattering probabilities $P^{R}_{\underline{kk}'}$ are, in general, somewhat anisotropic but Kus and Carbotte (48) found that quite good results can be obtained by ignoring this anisotropy. In the present case this amounts to setting

$$\frac{1}{\tau_2^R(\hat{\underline{k}})} = \frac{1}{\tau_2^R} \tag{5.2.3}$$

or equivalently

$$\tau_2(\hat{\underline{k}}) = \left\langle \frac{1}{\frac{1}{\tau_2^0(\hat{\underline{k}})} + \frac{1}{\tau_2^R}} \right\rangle = \left\langle \frac{\tau_2^0(\hat{\underline{k}})\,\tau_2^R}{\tau_2^0(\hat{\underline{k}}) + \tau_2^R} \right\rangle \tag{5.2.4}$$

The advantage of this assumption is that it allows τ_2^R to be determined from a measurement of the zero temperature attenuation. For transverse waves

$$\alpha_{res} = \frac{2 m^{*2} v_F^3 k_F^2 q_s^2}{45\pi^2 \hbar v_s \rho} \cdot \tau_2^R \tag{5.2.5}$$

and, of course, α_{res} for longitudinal waves is just 4/3 times (5.2.5).

Under the assumptions (5.2.1) and (5.2.3) the finite temperature attenuation for transverse waves is given by

$$\alpha = \frac{2 m^{*2} v_F^3 k_F^2 q_s^2}{45\pi^2 \hbar v_s \rho} \left\langle \frac{\tau_2^0(\underline{k})\,\tau_2^R}{\tau_2^0(\underline{k}) + \tau_2^R} \right\rangle \tag{5.2.6}$$

It is convenient to express this in terms of deviations from the equivalent of Matthiessen's rule (4.1.21), i.e.

$$\frac{1}{\alpha(T)} = \frac{1}{\alpha_{res}} + \frac{1}{\alpha_{ideal}(T)} + DEMR \qquad (5.2.7)$$

From (5.2.6)

$$DEMR = \frac{2m^{*2}V_F^3 k_F^2 q_s^2}{45\pi^2 \hbar V_s \rho}\left[\left\langle\frac{\tau_2^c(\underline{k})\,\tau_2^R}{\tau_2^c(\underline{k}) + \tau_2^R}\right\rangle - \frac{\tau_2^R}{1 + \left\langle\frac{\tau_2^R}{\tau_2^c(\underline{k})}\right\rangle}\right] \qquad (5.2.8)$$

Numerical calculations of the DEMR using (5.2.6) would provide an indication of how confidently one can use (4.2.21) to seperate out the ideal attenuation.

CHAPTER VI

ELECTRON-ELECTRON SCATTERING

Throughout the previous chapters of this thesis it has been tacitly assumed that electron-electron scattering processes can be neglected in the Boltzmann equation. It is well known (49) that the contribution of electron-electron collisions to the electrical resistivity varies as T^2. The extreme difficulty in finding any T^2 dependence of ρ^* experimentally is good evidence of the insignificance of electron collisions (49). The situation for the ultrasonic attenuation is rather different, however. Only Umklapp electron-electron processes contribute to ρ^* but, as will be shown below, Normal processes can contribute to α^{-1}. A rough theoretical calculation has been performed by Bhatia and Moore (11) who found that the effect of electron collisions may become important at temperatures below about 2°K in monovalent metals. Though this contribution to the attenuation may be significant over only a small temperature range it is interesting in its own right as a many-body effect in metals that is (in principle) observable. In fact a measurement of the T^{-2} dependence of α would essentially be a determination of the intrinsic viscosity of the conduction electron gas. In this chapter the theory of the contribution of electron-electron scattering to α will be presented. The treatment will be restricted to free electrons.

The effect of electron-electron scattering can be included in the

Bhatia and Moore analysis in a simple way $^{(11)}$. For transverse waves

$$\frac{1}{\alpha} = \frac{\rho v_s^3}{10\, m v_F^2 \omega_s^2}\left(\frac{1}{\tau} + \frac{1}{\tau_{ee}}\right) \tag{6.1}$$

where τ is the Bhatia and Moore τ_2 calculated by considering only electron-lattice scattering and where τ_{ee} is τ_2 evaluated with only electron-electron scattering. The attenuation coefficient for longitudinal waves is just 4/3 times that given by (6.1).

The expression for τ_{ee} can be written

$$\frac{\partial f_{\underline{k}}^{0}}{\partial E_{\underline{k}}} \frac{Y_{21}(\hat{\underline{k}})}{\tau_{ee}} = \frac{-\beta\,\Omega^3}{\hbar^3 v_F^3 (2\pi)^9} \int dE_{\underline{k}'} \int dE_{\underline{k}''} \int dE_{\underline{k}'''}$$

$$\int dS_{\underline{k}'} \int dS_{\underline{k}''} \int dS_{\underline{k}'''} \; f_{\underline{k}}^{0} f_{\underline{k}'}^{0} (1 - f_{\underline{k}''}^{0})(1 - f_{\underline{k}'''}^{0})$$

$$\times \left[Y_{21}(\hat{\underline{k}}) + Y_{21}(\hat{\underline{k}}') - Y_{21}(\hat{\underline{k}}'') - Y_{21}(\hat{\underline{k}}''')\right] \delta_{\underline{k}+\underline{k}',\, \underline{k}''+\underline{k}'''}\; Q_{\underline{k}\,\underline{k}'}^{\underline{k}''\underline{k}'''} \tag{6.2}$$

where $Q_{\underline{k}\underline{k}'}^{\underline{k}''\underline{k}'''}$ is the intrinsic transition rate of the two electron process

$$\underline{k} + \underline{k}' \rightarrow \underline{k}'' + \underline{k}'''$$

If both sides of (6.2) are integrated with respect to $E_{\underline{k}}$ the result is

$$\frac{1}{\tau_{ee}} = \frac{\beta\,\Omega^3}{\hbar^3 v_F^3 (2\pi)^9} \int dE_{\underline{k}} \int dE_{\underline{k}'} \int dE_{\underline{k}''} \int dE_{\underline{k}'''}$$

$$\int dS_{\underline{k}'} \int dS_{\underline{k}''} \int dS_{\underline{k}'''} \quad f^{0}_{\underline{k}}\, f^{0}_{\underline{k}'}\, (1-f^{0}_{\underline{k}''})(1-f^{0}_{\underline{k}'''})$$

$$\times \left[1 + P_{20}(\hat{\underline{k}}\cdot\hat{\underline{k}}') - P_{20}(\hat{\underline{k}}\cdot\hat{\underline{k}}'') - P_{20}(\hat{\underline{k}}\cdot\hat{\underline{k}}''') \right] \delta_{\underline{k}+\underline{k}';\,\underline{k}''+\underline{k}'''}\; Q^{\underline{k}''\underline{k}'''}_{\underline{k}\underline{k}'} \qquad (6.3)$$

where the fact that $Q^{\underline{k}''\underline{k}'''}_{\underline{k}\underline{k}'}$ depends only on the angle between $\underline{k}$ and $\underline{k}''$ has been used. The integrals appearing in (6.3) are very similar to those in reference (49) where it is shown that they can be reduced to

$$\frac{k_B^2 T^2 k_F^3 \Omega^2}{48 \pi^4 \hbar^3 v_F^3} \int d\Omega_{\underline{k}'} \int d\Omega_{\underline{k}''}$$

$$\times \left[1 + P_{20}(\hat{\underline{k}}\cdot\hat{\underline{k}}') - P_{20}(\hat{\underline{k}}\cdot\hat{\underline{k}}'') - P_{20}(\hat{\underline{k}}\cdot(\hat{\underline{k}}+\hat{\underline{k}}'-\hat{\underline{k}}'')) \right] W^{\underline{k}''\underline{k}'''}_{\underline{k}\underline{k}'} \qquad (6.4)$$

where

$$Q^{\underline{k}''\underline{k}'''}_{\underline{k}\underline{k}'} = \delta(\epsilon_{\underline{k}} - \epsilon_{\underline{k}'})\; W^{\underline{k}''\underline{k}'''}_{\underline{k}\underline{k}'}$$

and where it is understood that the vector

$$\underline{k} + \underline{k}' - \underline{k}''$$

must lie on the fermi surface.

The four dimensional integral in (6.4) can be further reduced to a two dimensional integral as follows. Figure 6.1 shows the vectors $\hat{\underline{k}}$, $\hat{\underline{k}}'$ and $\hat{\underline{k}}''_M$. $\hat{\underline{k}}''_M$ is the $\hat{\underline{k}}''$ which for given $\underline{k}$ and $\underline{k}'$ makes the largest angle θ_M. All possible vectors $-\hat{\underline{k}}''$ have their heads on the dashed circle shown

Fig. 6.1 The vectors $\hat{\underline{k}}$, $\hat{\underline{k}}'$ and $\hat{\underline{k}}''_M$.

Figure 6.1

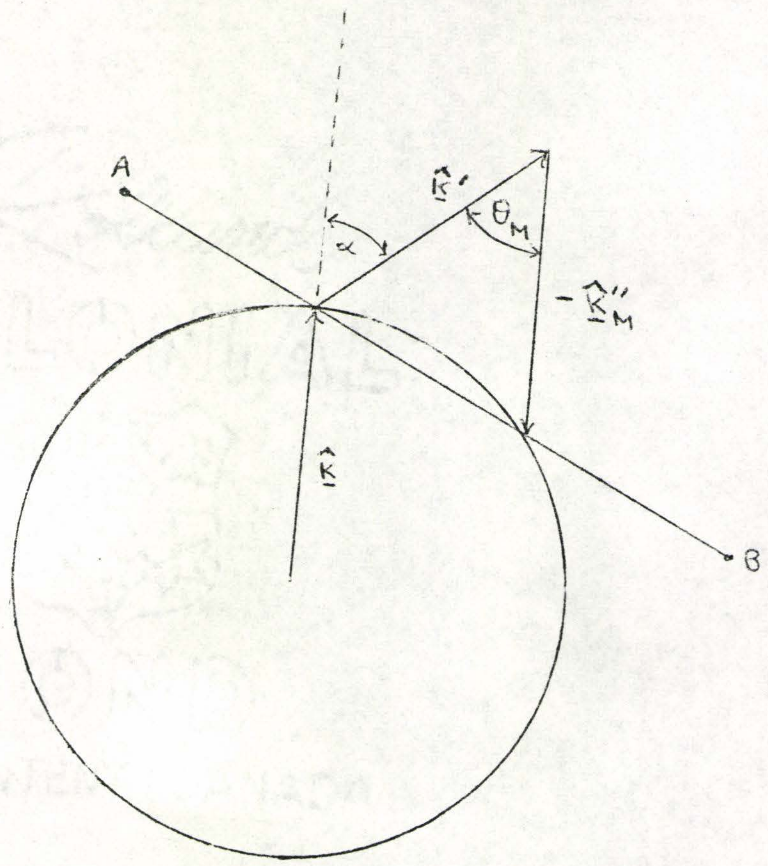
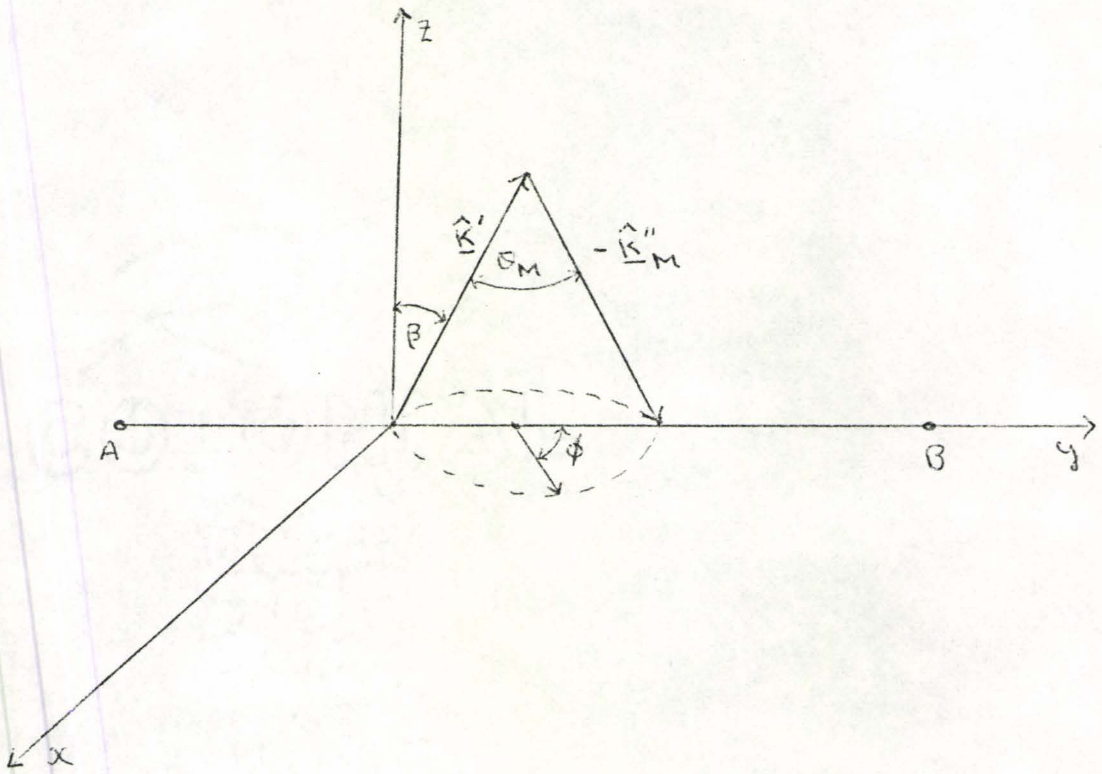

A
$\hat{\underline{k}}'$
α
θ_M
$-\hat{\underline{k}}''_M$
$\hat{\underline{k}}$
B

Fig. 6.2 The definition of the angles β and ϕ. The notation is the same as in Fig. 6.1.

Figure 6.2



in figure 6.2. Thus the integral over $\Omega_{\underline{k}'}$ is equivalent to one over α and the integral over $\Omega_{\underline{k}''}$ is equivalent to one over ϕ . More precisely

$$\int_{FS} d\Omega_{\underline{k}'} \longrightarrow 2\pi \int_0^{\pi} \sin\alpha \; d\alpha \qquad (6.5)$$

and

$$\int_{FS} d\Omega_{\underline{k}''} \rightarrow \sin\beta \int_0^{2\pi} d\phi \qquad (6.6)$$

The angle β can be found by noting that

$$\frac{1 + \cos\alpha}{2} = \frac{\hat{\underline{k}}' \cdot (\hat{\underline{k}} + \hat{\underline{k}}')}{2} = \cos\beta \qquad (6.7)$$

Now observe that in the (x,y,z) system of figure 6.2 the coordinates of $\hat{\underline{k}}$, $\hat{\underline{k}}'$ and $\hat{\underline{k}}''$ are

$$\hat{\underline{k}} = (0, \sin(\beta-\alpha), \cos(\beta-\alpha))$$

$$\hat{\underline{k}}' = (0, \sin\beta, \cos\beta)$$

$$\hat{\underline{k}}'' = (-\sin\beta \sin\phi, -\sin\beta\cos\phi, \cos\beta) \qquad (6.8)$$

so $\hat{\underline{k}}\cdot\hat{\underline{k}}'$ and $\hat{\underline{k}}\cdot\hat{\underline{k}}''$ can both be evaluated simply. Equations (6.5), (6.6), (6.7) and (6.8) allow (6.4) to be written as a complicated expression which, however, only involves a two dimensional integral.

To evaluate $W_{\underline{k}\underline{k}}^{\underline{k}''\underline{k}'''}$ the form of the electron-electron potential is needed. The collective coordinates method of treating many body effects in a charged fermi liquid produces a potential which may to a good approximation be written (50)

$$V(\underline{r}_1 - \underline{r}_2) = \frac{e^2}{4\pi\varepsilon_0 |r_1 - r_2|} \exp(-\xi |\underline{r}_1 - \underline{r}_2|) \qquad (6.9)$$

where

$$\xi = .398 \, (N_0 a_0^3)^{-1/2} \, k_F$$

and where a_0 is the Bohr radius. For electron densities of real metals the potential (6.9) is too strong for the Born approximation to be used (50). In fact the Born approximation may overestimate the total scattering cross-section by a factor of about 5 (51). It can be shown (52) that a proper partial wave analysis of the scattering gives

$$W_{\underline{k}\,\underline{k}'}^{\underline{k}''\,\underline{k}'''} = \frac{32\pi^3 \hbar^3}{m^2 \Omega^2} \, |f(\theta)| \qquad (6.10)$$

where θ is the angle between $\underline{k}$ and $\underline{k}''$ and

$$f(\theta) = k_F^{-1} \sum_{L=0}^{\infty} (2L+1) \, e^{i\delta_L} \sin\delta_L \, P_L(\cos\theta)$$

where the δ_L are the usual partial wave phase shifts (54) and the P_L are Legendre polynomials. The phase shifts can be determined from the potential (6.9) in the familiar fashion (54).

The results presented in this chapter would allow a numerical calculation of τ_{ee} as a function of electron density that would be very nearly exact for free electrons. Such a calculation would provide useful guidance to experimentalists who might wish to engage in a search for a T^{-2} dependence in the low temperature α.

Appendix A

In this appendix

$$\left\langle \left[Y_{21}(\hat{\underline{k}}) - Y_{21}(\hat{\underline{k}}') \right]^2 \right\rangle = \left\langle \left[Y_{20}(\hat{\underline{k}}) - Y_{20}(\hat{\underline{k}}') \right]^2 \right\rangle$$

$$= \left\langle \left[(\hat{q}\cdot\hat{\underline{k}})^2 - (\hat{q}\cdot\hat{\underline{k}}')^2 \right] \right\rangle$$

will be evaluated. The angular brackets indicate an average over all orientations of $\underline{q}$. Without loss of generality $\underline{k}$ can be taken to lie along the z-axis and $\underline{k}$ can be taken to lie in the xz-plane. The cartesian components of $\underline{k}$, $\underline{k}$ and $\underline{q}$ can be written as

$$\hat{\underline{k}} = (0, 0, 1)$$

$$\hat{\underline{k}}' = (\sin\alpha, 0, \cos\alpha)$$

$$\hat{q} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

where θ and ϕ are the polar coordinates of $\underline{q}$. The required average can thus be written

$$\langle \quad \rangle = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta$$

$$\times \left[\cos^2\theta - (\sin\alpha\sin\theta\cos\phi + \cos\alpha\cos\theta)^2 \right]^2$$

$$= \frac{4}{15}\left(1-\cos^2\alpha\right)$$

$$= \frac{4}{15}\left(1-\cos^2(\underline{k}\,\underline{k}')\right) \tag{A.1}$$

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