ON THE GENERALIZED DIRICHLET PROBLEM

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Paul Douglas Haines, B.Sc.

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AUTHOR: Paul Douglas Haines, B.Sc. (McMaster University)

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SCOPE AND CONTENTS:

In this thesis, we shall solve the classical Dirichlet problem for a ball in n-dimensional Euclidean space, and then point out that the classical Dirichlet problem is not always solvable. Following Wiener and Brelot, we then introduce a generalized Dirichlet problem for any bounded region in n-dimensional Euclidean space and establish necessary and sufficient conditions for its solution. We show that the solution of the generalized Dirichlet problem coincides with the solution of the classical Dirichlet problem whenever the latter exists. Finally, we characterize those regions for which the classical Dirichlet problem is solvable by considering the boundary behaviour of those functions for which the generalized problem is solvable.

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P. D. H.

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Introduction

From a mathematical point of view, the Dirichlet problem, also commonly referred to as "the boundary value problem of the first kind", belongs to the theory of harmonic functions. For the time being, we shall restrict ourselves to real valued functions defined on some region R (open, connected subset) of three dimensional Euclidean space E^3, where a function U on R is said to be harmonic on R if it possesses continuous partial derivatives up to and including second order in all variables, and satisfies everywhere on R the partial differential equation of Laplace: $\sum_{i=1}^{3} \frac{\partial^2 U}{\partial x_i^2} = 0$. The classical Dirichlet problem in its simplest form consists of trying to find a harmonic function on a bounded region R, which is continuous on the closure of R (R union its boundary in E^3) and which coincides with a given continuous function on the boundary of R.

For over a century, this problem has attracted the attention of numerous mathematicians including H. A. Schwarz, H. Poincaré, H. Lebesgue, O. Perron, N. Wiener, and M. Brelot to name only a few, each of whom has directed considerable effort to the problem and its solution. The problem itself has physical origins of fundamental significance, some of which were recognized before the birth of the German mathematician, whose name the problem now bears in nearly all cases.

We offer the following physical problem which helped to provide historical motivation for consideration of the Dirichlet problem: Consider a spherical shaped ball of uniform density which has a high degree of thermal conductivity and which is at a given initial temperature. Let us apply to the surface of the ball a continuous temperature function f, which is independent of time, and let T be the temperature inside the ball at a time t . Then T(x, y, z, t) satisfies the following partial differential equation:

$$\sum_{i=1}^{3} \frac{\partial^2 T}{\partial x_i^2} = K \frac{\partial T}{\partial t} \quad ([15] , p.78).$$

The function U defined to be $(x, y, z) = \lim_{t \to \infty} T(x, y, z, t)$ is independent of time and hence is called the steady state temperature corresponding to the boundary function f, and since U is independent of time t, it follows that U satisfies Laplace's partial differential equation in the interior of the ball and it turns out that U is continuous on the closure, and coincides with f on the boundary. In view of our original statement of the Dirichlet problem, the function U is a solution of this problem for the ball whose boundary function is f.

The theory of harmonic functions and the associated Dirichlet problem can be applied to an endless variety of disciplines within the fields of mathematics and physics. We refer the reader to the recent book by Duff and Naylor ([10] , p.134) for more information on these matters.

For many years, it was generally conjectured that the Dirichlet problem was solvable for any bounded region, and that limitations of generality were inherent in methods, rather than in the problem itself. However, every attempt to construct a general solution invariably had to presume some restrictions on the boundary and it was not until 1911 that Zaremba ([24], p.310) published a resolution of this conjecture, by pointing out that there did exist regions for which the problem was not solvable, such as the deleted unit ball. A modified version of his example will be considered in Chapter III.

Other examples of non-solvable Dirichlet problems soon followed, one of which is the so called "spine of Lebesgue" published in 1913 ([21] , pp.12-13). The basic difference between the Lebesgue example and that of Zaremba, lies in the fact that the boundary of Lebesgue's region is a one-to-one continuous image of the sphere, whereas in Zaremba's example, the boundary consists of the union of the unit sphere and a singleton set whose member is the origin.

Because of the fact that it is possible to construct regions for which the classical Dirichlet problem is not solvable, Wiener ([22]) was induced to define a generalized Dirichlet problem whose solution always coincided with that of the classical Dirichlet problem, whenever the latter was solvable. It turns out that the generalized Dirichlet problem is always solvable for any continuous function defined on the boundary of any bounded region in E^3.

In this thesis we shall be chiefly concerned with the establishment of necessary and sufficient conditions for which the generalized Dirichlet problem is solvable relative to a bounded region and characterize those regions for which the classical Dirichlet problem is solvable in n-dimensional Euclidean space.

The methods employed here will generally follow those of Brelot and it is to his works (especially [4]) that we shall constantly refer. Finally, we shall consider the boundary behaviour of the solution of the generalized Dirichlet problem.

Since the mathematical machinery required to accomplish these ends is considerable, our first chapter, which is the largest, is devoted to the development of this machinery.

Whereas much reference shall be made to the Dirichlet problem itself, we shall find it convenient to call this problem the "D" problem.

I. SOME FUNDAMENTAL CONCEPTS OF ANALYSIS

§1 Some basic concepts and results in point set topology and n-dimensional Euclidean space.

It is convenient to first introduce some standard definitions and results from point set topology before considering the basic theory of n-dimensional Euclidean space. We shall assume acquaintance with standard set theoretic definitions and will generally follow the notation of Bourbaki.

Definition 1.1.1: Let X be a set and $\mathcal{B}(X)$ its set of subsets or power set. If $\mathcal{T} \subset \mathcal{B}(x)$, then $\mathcal{T}$ is called a topology on X if and only if $\mathcal{T}$ satisfies the following axioms:

(i) $\emptyset \in \mathcal{T}$

(ii) $X \in \mathcal{T}$

(iii) If $\{A_\alpha\}$ is any subset of $\mathcal{T}$, then $\bigcup_\alpha A_\alpha \in \mathcal{T}$.

(iv) If $\{A_i\}$ is a finite subset of $\mathcal{T}$, then $\bigcap_i A_i \in \mathcal{T}$.

Definition 1.1.2: We define a topological space to be a set X endowed with a topology $\mathcal{T}$ on X and denote it by $(X, \mathcal{T})$.

Definition 1.1.3: If $(X, \mathcal{T})$ is a topological space then $O \subset X$ is said to be open if and only if $O \in \mathcal{T}$.

Definition 1.1.4: If $(X, \mathcal{T})$ is a topological space, then $F \subset X$ is said to be closed if and only if $(X - F) \in \mathcal{T}$.

Definition 1.1.5: Let $A \subset X$ where $(X, \mathcal{T})$ is a topological space. Then $\{O_\alpha\} \subset \mathcal{T}$ is called an open covering of A if and only if $\bigcup_\alpha O_\alpha \supset A$.

Definition 1.1.6: Let $(X, \mathcal{T})$ be a topological space, $A \subset X$ and $\theta = \{O_\alpha\}$ an open covering of A. We define $\theta' \subset \theta$ to be a subcovering of A (relative to θ) if θ' is itself an open covering of A.

Definition 1.1.7: Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then the set A is defined to be compact with respect to $\mathcal{T}$ if for any open covering of A, there exists a finite subcovering (that is a subcovering possessing only a finite number of members) of A.

Remark: We note that the above definition of compactness coincides with Bourbaki's definition of quasi-compactness. Bourbaki reserves the term compact for special spaces having the property of definition 1.1.7, namely the T_2 spaces to be defined later.

Definition 1.1.8: A topological space $(X, \mathcal{T})$ is said to be compact if the set X is compact with respect to the topology $\mathcal{T}$.

Definition 1.1.9: Let $(X, \mathcal{T})$ be a topological space and $x \in X$. We define $V(x) \subset X$ to be a neighbourhood of x relative to $\mathcal{T}$ if and only if there exists $O \in \mathcal{T}$ such that $x \in O$ and $O \subset V(x)$.

Remark: A neighbourhood may or may not be an open set. Those that are open are usually called open neighbourhoods. Usually we shall follow the convention of requiring any neighbourhood to be open.

<u>Definition 1.1.10</u>: Let $A \subset X$ and $(X, \mathcal{J})$ be a topological space. Let $\{F_\alpha\}$ be the family of closed sets in $(X, \mathcal{J})$ such that each $F_\alpha \supset A$. Then $\bigcap_\alpha F_\alpha$ is closed, is called the <u>closure</u> of A, and is denoted by $\bar{A}$.

<u>Definition 1.1.11</u>: Let $(X, \mathcal{J})$ be a topological space and $A \subset X$. Let $\{O_\alpha\}$ be the family of all open sets in $(X, \mathcal{J})$ such that each $O_\alpha \subset A$. Then $\bigcup_\alpha O_\alpha$ is open, and is called the <u>interior</u> of A. It is denoted by $\mathring{A}$.

<u>Remark</u>: For any $A \subset X$, it is always true that $\mathring{A} \subset A \subset \bar{A}$.

<u>Definition 1.1.12</u>: Let $(X, \mathcal{J})$ be a topological space and $A \subset X$. The <u>boundary</u> of A denoted by ∂A, is defined to be the set $\bar{A} - \mathring{A}$.

<u>Remark</u>: For any $A \subset X$, it is always true that ∂A is closed for a given topology since $\partial A = \bar{A} - \mathring{A} = \bar{A} \cap (X - \mathring{A})$ which is the intersection of two closed sets.

<u>Definition 1.1.13</u>: Let f be a function (single-valued relation) whose domain is the set X and whose range is a subset of Y where $(X, \mathcal{J}_x)$ and $(Y, \mathcal{J}_Y)$ are topological spaces. Then f is said to be <u>continuous</u> if and only if for any $O \in \mathcal{J}_Y$ it is always true that $f^{-1}(O) \in \mathcal{J}_x$ where $f^{-1}(O)$ denotes the inverse image of O under f.

<u>Theorem 1.1.1</u>: Let f be a function from the topological space $(X, \mathcal{J}_x)$ to the topological space $(Y, \mathcal{J}_Y)$. For any $x \in X$, let $y \in Y$ be the image of x under f (or $y = f(x)$). Then f is continuous if and only if it follows that for any open neighbourhood $V(y)$ relative to $\mathcal{J}_Y$ then $f^{-1}(V(y))$ is an open neighbourhood of x relative to $\mathcal{J}_x$.

Proof: Let f be continuous according to definition 1.1.13 and let $y = f(x)$. Then $x \in f^{-1}(y)$ or $\{x\} \subset f^{-1}(\{y\})$. Since $V(y) \supset \{y\}$, therefore $f^{-1}(V(y)) \supset f^{-1}(\{y\}) \supset \{x\}$, and hence $x \in f^{-1}(V(y))$. Since $V(y)$ is open, therefore $f^{-1}(V(y))$ is open which implies that $f^{-1}(V(y))$ is a neighbourhood of x. Proceeding in the other direction, we let $O \in \mathcal{T}_Y$ and $A = f^{-1}(O)$. Since O is an open neighbourhood of every $y \in O$, therefore A is an open neighbourhood of every x in A. It follows that A is open.

Remark: The truth of the above theorem is not affected when an open neighbourhood is replaced by any neighbourhood.

Definition 1.1.14: Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be homeomorphic or topologically equivalent if and only if there exists a function f whose domain is X and whose range is Y which satisfies the following conditions:

(i) f is one-to-one (or f^{-1} is a function).

(ii) f is continuous.

(iii) f^{-1} is continuous.

Definition 1.1.15: Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Let $\mathcal{T} = \{O_\alpha\}$ and define $\mathcal{T}_A = \{ B \subset X : B \in \mathcal{T}_A$ if and only if there exists an $O_\alpha \in \mathcal{T}$ such that $B = A \cap O_\alpha \}$. Then $\mathcal{T}_A$ satisfies the axioms for a topology on A, and $\mathcal{T}_A$ is called the relative topology of $\mathcal{T}$ on A, or we say that $\mathcal{T}$ is relativized to A.

<u>Definition 1.1.16</u>: Let $(X, \mathfrak{J})$ be a topological space and $A \subset X$. Then A is said to be <u>disconnected</u> in $(X, \mathfrak{J})$ if and only if there exist B and C where $A = B \cup C$ and where $B \cap C = \emptyset$, $B \neq \emptyset$, $C \neq \emptyset$, and $B, C \in \mathfrak{J}_A$.

<u>Definition 1.1.17</u>: A set $A \subset X$ is said to be <u>connected</u>, relative to a topology $\mathfrak{J}$, if and only if A is not disconnected in the space $(X, \mathfrak{J})$.

<u>Theorem 1.1.2:</u> Let $(X, \mathfrak{J})$ be a topological space and A a compact subset of X. Then $(A, \mathfrak{J}_A)$ is a compact topological space.

<u>Proof</u>: Let θ_A be an open covering of A with respect to $\mathfrak{J}_A$. Then $\theta_A = \{O_\alpha \cap A\}$ where each $O_\alpha \in \mathfrak{J}$ and $\theta = \{O_\alpha\}$ is an open covering for A relative to $\mathfrak{J}$. Since A is compact with respect to $\mathfrak{J}$, therefore we can extract from $\{O_\alpha\}$ a finite subcovering of A denoted by $\{O_{\alpha(i)}\}$, $1 \leqslant i \leqslant n$. Since $A \subset \bigcup_{i=1}^{n} (O_{\alpha(i)})$, therefore $A \subset \bigcup_{i=1}^{n} (O_{\alpha(i)} \cap A)$ and hence $\{O_{\alpha(i)} \cap A\}$, $1 \leqslant i \leqslant n$, is a finite subcovering of θ_A from which the theorem follows.

<u>Definition 1.1.18</u>: A topological space is said to be <u>locally compact</u> if and only if for every $x \in X$, there exists a compact set $K \subset X$ which contains an open neighbourhood of x.

<u>Definition 1.1.19</u>: Let $(X, \mathfrak{J})$ be a topological space and $A \subset X$. Then A is said to be <u>relatively compact</u> in $(X, \mathfrak{J})$ if and only if $\bar{A}$ is compact.

<u>Definition 1.1.20</u>: Let $(X, \mathfrak{J})$ be a topological space and $x \in X$, $y \in X$ where $x \neq y$, and both are arbitrary. Then $(X, \mathfrak{J})$ is said to be <u>Hausdorff or T_2</u> if and only if there exists a neighbourhood $V(x) \ni x$ and a neighbourhood $V(y) \ni y$ such that $V(x) \cap V(y) = \emptyset$.

Remark: In future, all topological spaces considered will be T_2.

Theorem 1.1.3: In a T_2 space $(X, \mathcal{T})$, every compact subset is closed.

Proof: Let $K \subset X$ be compact. We shall show that $(X - K)$ is open. Let $p \in (X - K)$ be fixed and let $x \in K$ be arbitrary. There exist disjoint neighbourhoods of p and x denoted by $V_x(p)$ and $V(x)$ respectively such that $V_x(p) \cap V(x) = \emptyset$. The set of neighbourhoods $\{V(x)\}$ form an open covering of K, and $\{V_x(p)\}$ forms a collection of neighbourhoods of p such that $V_x(p) \cap V(x) = \emptyset$ for every $x \in K$. Since K is compact we can extract from $\{V(x)\}$ a finite subcovering of K denoted by $\{V(x_i)\}$, $1 \leq i \leq n$. Let $V(p) = \bigcap_{i=1}^{n} (V_{x_i}(p))$, and note that $V(p) \cap (\bigcup_{i=1}^{n} V(x_i)) = \emptyset$. Then $V(p)$ is a neighbourhood of p such that $V(p) \subset (X - K)$ and the theorem follows.

Theorem 1.1.4: Let $(X, \mathcal{T})$ be a topological space, and K a compact subset of X. For any closed $F \subset K$ it follows that F is compact.

Proof: Let $\{O_\alpha\}$ be an open covering of F, and note that $(X-F)$ is open. Then $\{O_\alpha\} \cup \{(X-F)\}$ is an open covering of K. Since K is compact we can extract out a finite subcovering denoted by $\{O_{\alpha(i)}\} \cup \{(X-F)\}$ where $1 \leq i \leq n$. Since $(\bigcup_{i=1}^{n} O_{\alpha(i)}) \cup (X-F) \supset K$ and since $(X-F) \cap F = \emptyset$, therefore $\bigcup_{i=1}^{n} O_{\alpha(i)} \supset F$.

Definition 1.1.21: A topological space $(X, \mathcal{T})$ is said to be a <u>subspace</u> of $(Y, \mathcal{T}')$ if and only if $X \subset Y$ and $\mathcal{T}$ is the topology $\mathcal{T}'$ <u>relativized to X.</u>

Definition 1.1.22: Let $(X, \mathcal{T})$ be a T_2 space and $A \subset B \subset X$. Then A is said to be <u>dense in B</u> relative to $\mathcal{T}$ if $\bar{A} \supset B$.

<u>Definition 1.1.23</u>: Let $(X, \mathfrak{J}_X)$ be a T_2 space, and $(Y, \mathfrak{J}_Y)$ a compact T_2 space. Then $(Y, \mathfrak{J}_Y)$ is said to be a <u>compactification</u> of $(X, \mathfrak{J}_X)$ if and only if $(X, \mathfrak{J}_X)$ is homeomorphic to a dense subspace of $(Y, \mathfrak{J}_Y)$.

<u>Definition 1.1.24</u>: Let $(X, \mathfrak{J})$ be a T_2 space and $x \in X$. A set $\mathcal{V} = \{V_\alpha\} \subset \mathfrak{J}$ is said to be a <u>fundamental system of neighbourhoods of x</u> if and only if for any neighbourhood V of x, there exists $V_\alpha \in \mathcal{V}$ such that $V_\alpha \subset V$ where every V_α is a neighbourhood of x.

<u>Theorem 1.1.5</u>: Let $(X, \mathfrak{J})$ be a locally compact T_2 space which is not compact. Let $X^* = X \cup \{\infty\}$ where ∞ is a new element adjoined to the set X. Let $\mathcal{A}^* \subset \mathcal{B}(X^*)$ be defined as follows:

(i) If $A^* \subset X$ then $A^* \in \mathcal{A}^*$ if and only if $A^* \in \mathfrak{J}$.

(ii) If $\infty \in A^*$ then $A^* \in \mathcal{A}^*$ if and only if $A^* \cap X = X - K$

where K is a compact subset of X with respect to $\mathfrak{J}$.

Then $\mathcal{A}^*$ is a topology on X^*.

<u>Proof</u>: (i) Since $\emptyset \subset X \subset X^*$ and $\emptyset \in \mathfrak{J}$, therefore $\emptyset \in \mathcal{A}^*$.

(ii) Since $X \subset X$ and $X \in \mathfrak{J}$, therefore $X \in \mathcal{A}^*$.

(iii) Let $\{A^*_\alpha\} \subset \mathcal{A}^*$, and let $A^* = \bigcup_\alpha A^*_\alpha$. In the case where every $A^*_\alpha \subset X$, then $A^*_\alpha \in \mathfrak{J}$ also, and hence $A^* = \bigcup (A^*_\alpha) \in \mathfrak{J}$. In general $A^* \in \mathcal{A}^*$ for otherwise $X \cap A^* = \bigcup_\alpha (X \cap A^*_\alpha)$ and hence $\complement(X \cap A^*) = \bigcap_\alpha \complement(X \cap A^*_\alpha)$ where $\complement$ denotes the complement of a set taken with respect to X. Since $A^*_\alpha \cap X$ is always open for any α, therefore $\complement(X \cap A^*)$ is closed, and since $\complement(X \cap A^*_\alpha)$ is compact for at least one α, therefore $\complement(X \cap A^*)$ is a closed subset of a compact set, and hence compact by theorem 1.1.4. Thus $\mathcal{A}^*$ is closed under arbitrary union.

(iv) Let $\{A^*_i\} \subset \mathcal{A}^*$ where $1 \leqslant i \leqslant n$, and let $A^* = \bigcap_{i=1}^{n} A_i$. If $A^* \subset X$ then $A^* = \bigcap_{i=1}^{n} (A^*_i \cap X)$ where $(A^*_i \cap X) \in \mathfrak{T}$ for each i, and hence $A^* \in \mathfrak{T}$. Thus $A^* \in \mathcal{A}^*$ for otherwise $\infty \in A^*$, and hence $\infty \in A^*_i$ for all i. Then $A^* \cap X = \bigcap_{i=1}^{n} (A^*_i \cap X)$ and $\complement(A^* \cap X) = \bigcup_{i=1}^{n} \complement(A^*_i \cap X) = \bigcup_{i=1}^{n} K_i$ where each K_i is compact. Hence $A^* \cap X = X - K$ where $K = \bigcup_{i=1}^{n} K_i$ is compact. Therefore $A^* \in \mathcal{A}^*$.

(i), (ii), (iii) and (iv) consequently fulfil the four axioms of a topology and our theorem is proved.

Thus $\mathcal{A}^*$ is a topology on X^* and we may write $\mathcal{A}^* = \mathfrak{T}^*$.

<u>Theorem 1.1.6</u>: Let $(X, \mathfrak{T})$ be a locally compact T_2 space, and $(X^*, \mathfrak{T}^*)$ defined as in theorem 1.1.5. Then $(X^*, \mathfrak{T}^*)$ is a compact T_2 space.

<u>Proof</u>: We first show that $(X^*, \mathfrak{T}^*)$ is T_2. If $x \in X \subset X^*$, $y \in X \subset X^*$, $x \neq y$, there exists a $V(x) \in \mathfrak{T}$, $V(y) \in \mathfrak{T}$ such that $x \in V(x)$, $y \in V(y)$ and $V(x) \cap V(y) = \emptyset$ because $(X, \mathfrak{T})$ is T_2. If $x \in X$ and $y = \infty$, there exists by the local compactness of $(X, \mathfrak{T})$ a compact $K \subset X$ such that $x \in V(x) \subset K$ where $V(x) \in \mathfrak{T}$ and hence $V(x) \in \mathfrak{T}^*$ since $\mathfrak{T} \subset \mathfrak{T}^*$ always. But $(X^* - K)$ is a neighbourhood of $y = \infty$ in $\mathfrak{T}^*$ denoted by $V(y)$ say. It follows that $V(x) \cap V(y) = \emptyset$, and hence $(X^*, \mathfrak{T}^*)$ is T_2.

In order to show the compactness of $(X^*, \mathcal{T}^*)$ we let $\{O^*_\alpha\}$ be an open covering of X^* relative to $\mathcal{T}^*$. Then ∞ is a member of at least one of the sets in $\{O^*_\alpha\}$. Let $\infty \in O^*_{\alpha(1)}$, and note that for every α, it follows that $X \cap O^*_\alpha \in \mathcal{T}$. Then $X \cap O^*_{\alpha(1)} = X - K$ where K is compact with respect to $\mathcal{T}$ and the following must be true: $\{X \cap O^*_\alpha\}$ is an open covering for K and we can extract out a finite subcovering of K denoted by $\{X \cap O^*_{\alpha(i)}\}$ where $2 \leq i \leq n$, and hence $\{O^*_{\alpha(i)}\}$ is a finite covering of K where $2 \leq i \leq n$. Hence $\{O^*_{\alpha(i)}\}$, $1 \leq i \leq n$ is an open covering of X^* with respect to $\mathcal{T}^*$.

<u>Theorem 1.1.7</u>: If $(X, \mathcal{T})$ is a locally compact T_2 space which is not compact and $(X^*, \mathcal{T}^*)$ is defined as in theorem 1.1.5, then X is dense in $(X^*, \mathcal{T}^*)$.

<u>Proof</u>: If X were closed in $(X^*, \mathcal{T}^*)$ then X would be a compact subset of X^*, which contradicts the hypothesis. Thus X fails to be closed and therefore $\bar{X} = X^*$.

<u>Definition 1.1.25</u>: The results of theorems 1.1.6 and 1.1.7 show that $(X^*, \mathcal{T}^*)$ is a compactification of the locally compact T_2 space $(X, \mathcal{T})$. The compact T_2 space $(X^*, \mathcal{T}^*)$ is called the <u>one point-compactification</u> or Alexandroff compactification of $(X, \mathcal{T})$.

<u>Definition 1.1.26</u>: A <u>metric</u> d on a set Y is defined to be a function whose domain is the cartesian product set Y x Y and whose range is

a subset of the non-negative reals such that:

(i) $d(x,y) = 0$ if and only if $x=y$ where $x \in Y$, $y \in Y$.

(ii) $d(x,y) = d(y,x)$ for every $x, y \in Y$.

(iii) $d(x,y) \leq d(x,z) + d(z, y)$ for every $x, y, z \in Y$,

which is often referred to as the triangle inequality.

Definition 1.1.27: A metric space is defined to be a set X endowed with a metric d, and is designated by the symbol (X,d).

Definition 1.1.28: If (X,d) is a metric space, $x_o \in X$ and r a real number greater than zero, then $\{ x \in X : d(x,x_o) < r \}$ is defined to be the open ball of centre x_o and radius r, denoted by $B_r(x_o)$.

Definition 1.1.29: If (X,d) is a metric space, $x_o \in X$ and r a real number greater than or equal to zero, then $\{ x \in X : d(x,x_o) \leq r \}$ is defined to be the closed ball of centre x_o and radius r, denoted by $\overline{B}_r(x_o)$.

Definition 1.1.30: If (X,d) is a metric space, $x_o \in X$ and r a real number greater than or equal to zero, then $\{ x \in X : d(x,x_o) = r \}$ is defined to be the sphere of radius r and centre x_o, denoted by $S_r(x_o)$.

Definition 1.1.31: Let (X,d) be a metric space, and $\mathcal{B} \subset \mathcal{B}(X)$ the set of all open balls in (X,d) i.e. $\mathcal{B} = \{ B_r(x_o) \}$ for all $x_o \in X$ and all real $r > 0$. Then the least topology on X (relative to set inclusion in $\mathcal{B}(x)$) which contains $\mathcal{B}$ is known as the topology deduced from the metric d and is called the <u>metric topology</u> $\mathcal{T}_d$ associated with d, or simply $\mathcal{T}$ if no confusion arises as to what metric is associated with it.

<u>Remark</u>: The topology $\mathcal{T}_d = \bigcap_i (\mathcal{T}_i)$ where $\mathcal{T}_i$ is any topology on X such that $\mathcal{T}_i \supset \mathcal{B}$. It is a consequence of the distributive laws of set theory relative to the operations of set union and intersection respectively which permits us to consider any $O \in \mathcal{T}_d$ to be either the arbitrary union of sets each one of which is the intersection of a finite number of open balls in (X,d) or the finite intersection of sets each one of which is the union of an arbitrary number of open balls in (X,d) ([2] , p.107, Proposition 8).

<u>Theorem 1.1.8</u>: Let (X,d) be a metric space and $\mathcal{T}$ the topology on X which is deduced from d. For any $x_o \in X$ it follows that $\{B_{1/n}(x_o)\}$, for all natural numbers n, is a fundamental system of neighbourhoods of x_o with respect to the topology $\mathcal{T}$.

<u>Proof</u>: Let $V(x_o)$ be any open neighbourhood of x_o with respect to $\mathcal{T}$. Since $V(x_o)$ is a non empty open set such that $x_o \in V(x_o)$ therefore it follows from the above remark that $V(x_o)$ contains a set O which is the intersection of a finite number of open balls and such that $x_o \in O$. Then $x_o \in O = \bigcap_{i=1}^{n} B_{r_i}(x_i)$ where each $B_{r_i}(x_i)$ is an open ball in (X,d) and $O \subset V(x_o)$.

We first show that if $x_o \in B_{r_1}(x_1)$ then there exists an $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x_o) \subset B_{r_1}(x_1)$. Let $d(x_1, z) = r_1$ and note that from the triangle inequality, $r_1 \leq d(x_1,x_0) + d(x_0,z)$ or $d(x_0,z) \geq r_1 - d(x_1,x_0)$. Let $r_1 - d(x_1,x_0) = 2\varepsilon_1$, which is necessarily greater than zero since $x_0 \in B_{r_1}(x_1)$. Hence $B_{\varepsilon_1}(x_0) \subset B_{r_1}(x_1)$) and similarly we can construct a finite sequence of ε_i's, $1 \leq i \leq n$, such that $\varepsilon_i > 0$ and $B_{\varepsilon_i}(x_0) \subset B_{r_i}(x_i)$. Then $B_\varepsilon(x_0) = \bigcap_{i=1}^{n} B_{\varepsilon_i}(x_0)$ where $\bigcap_{i=1}^{n} B_{\varepsilon_i}(x_0) \subset \bigcap_{i=1}^{n} B_{r_i}(x_i)$ and hence the ball $B_\varepsilon(x_0) \subset O \subset V(x_0)$ where $\varepsilon > 0$. We now choose n_0 such that $\frac{1}{n_0} < \varepsilon$ so that $B_{1/n_0}(x_0) \subset B_\varepsilon(x_0)$. The theorem follows.

Remark: We may now think of any metric space (X,d) as being a topological space $(X, \mathcal{T})$ as well, where X is endowed with the metric topology. It is always true that any metric space is a T_2 space.

Discussion

We are now prepared to consider the Euclidean spaces. Let E denote the set of real numbers, and d_1 the usual metric on E which we define by $d_1(x,y) = |x-y|$ for any $x, y \in E$. Then (E,d_1) is a metric space which we denote by E' and which is also a topological space where the topology is deduced from the metric d_1. We refer to this topology as the usual topology or E' topology and often refer to the space E' as the real line and the metric d_1 as the one dimensional Euclidean metric.

We now consider the cartesian product set $E \times E$ which is the set of ordered pairs of real numbers, that is $E \times E = \{ (x,y) : x,y \in E \}$.

We now define a metric d_2 on E x E as follows: If $p_1=(x_1,y_1) \in E\times E$ and $p_2 = (x_2, y_2) \in E \times E$, then $d_2 (p_1,p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The function d_2 is analogously referred to as the two dimensional Euclidean metric and the metric space (E x E, d_2) denoted by E^2, is called the two dimensional Euclidean space, or Euclidean plane. Its topology is deduced from d_2 and is analogously referred to as the usual topology or E^2 - topology.

In a similar way, we define the n-dimensional Euclidean metric function on the n^{th} cartesian product $\overset{n}{\underset{i=1}{X}} E_i$ to be

$$d_n(p_1,p_2) = \left(\sum_{i=1}^{n} (x_{1i} - x_{2i})^2 \right)^{1/2} \text{ where } p_1 = (x_{11}, x_{12}, \ldots, x_{1n})$$

and $p_2 = (x_{21}, x_{22} \ldots, x_{2n})$. Then d_n is a metric on $\overset{n}{\underset{i=1}{X}} E_i$ and the resulting metric space $(\overset{n}{\underset{i=1}{X}} E_i, d_n)$ shall be denoted by E^n, with the resulting topology deduced from d_n again called the usual topology or E^n- topology.

We shall often find it convenient to endow E^n with the usual <u>vector</u> (linear) structure ([11] , p.35) and the resulting <u>vector space</u>, which is often referred to as n-dimensional Euclidean vector space, shall also be denoted by E^n. When thought of as vectors, if $\vec{x} = (x_1, \ldots, x_n) \in E^n$ and $\vec{y} = (y_1, \ldots, y_n)$, then $\vec{x}+\vec{y} = (x_1+y_1, \ldots, x_n + y_n) \in E^n$ and $r\vec{x} = (rx_1, \ldots, rx_n)$ for any real number r. We shall then write $|\vec{x}|$ in place of $d_n(\vec{0},\vec{x})$, where $\vec{0}$ is the origin, and hence $|\vec{x} - \vec{y}|$ in place of $d_n(\vec{x}, \vec{y})$.

<u>Definition 1.1.32</u>: Let $\bar{I} = [0,1]$ and $\mathcal{T}_{\bar{I}}$ the usual topology of E' relativized to $\bar{I}$, and $\vec{f}$ a continuous function from $(\bar{I}, \mathcal{T}_{\bar{I}})$ into E^n. Such a function is called a <u>curve</u> in E^n.

<u>Remark</u>: $\vec{f} = (f_1, .., f_n)$ possesses the property that each f_i is a continuous function from $(\bar{I}, \mathcal{T}_{\bar{I}})$ into E'. We say that the curve $\vec{f}$ joins $\vec{f}(0)$ to $\vec{f}(1)$.

<u>Definition 1.1.33</u>: The image of $\bar{I}$ under $\vec{f}$, i.e. $\vec{f}(\bar{I}) \subset E^n$, is called the <u>image</u> of the curve $\vec{f}$.

<u>Definition 1.1.34</u>: The curve $\vec{f}$ is called an <u>arc</u> if and only if $\vec{f}(0) \neq \vec{f}(1)$.

<u>Definition 1.1.35</u>: An arc $\vec{f}$ is called a simple arc or <u>Jordan arc</u> if and only if the function $\vec{f}$ is one-to-one.

<u>Definition 1.1.36</u>: A curve is called a simple closed curve or <u>Jordan curve</u> if and only if $\vec{f}(0) = \vec{f}(1)$ and $\vec{f}$ is one-to-one on the half open interval $[0,1)$.

<u>Remark</u>: The image of a Jordan arc in the relative topology of E^n is homeomorphic to the closed interval I endowed with the relative topology and the image of a Jordan curve in the relative topology of E^n is homeomorphic to a circle (sphere in E^2 according to definition 1.1.30) endowed with the relative topology in E^2.

<u>Definition 1.1.37</u>: A set $X \subset E^n$ is said to be <u>arcwise connected</u> if and only if any two points of X can be joined by an arc.

Theorem 1.1.9: ([1], p.178, 179) An open set $O \subset E^n$ is connected (see definition 1.1.17) if and only if it is arcwise connected.

Definition 1.1.38: A region in E^n is defined to be an open connected subset of E^n.

Definition 1.1.39: A component of a set $X \subset E^n$ is defined to be a maximal (with respect to set inclusion) connected subset of X.

Theorem 1.1.10: ([1], p.182) Any open set in E^n has a countable number of components.

Definition 1.1.40: A set $X \subset E^n$ is said to be bounded if and only if there exists some $r > 0$ such that $B_r(\vec{0}) \supset X$.

Theorem 1.1.11: Any compact set K in E^n is closed and bounded.

Proof: Since E^n is a T_2 space, therefore K is closed (theorem 1.1.3). Suppose K were unbounded, then $\{B_n(\vec{0}) : n = 1, 2, \ldots\}$ is an open covering of K which has no finite subcover, which contradicts the supposition that K is compact.

Theorem 1.1.12: (Heine-Borel Theorem ([1], p.53). Any set $X \subset E^n$ which is closed and bounded is also compact.

Remark: From theorems 1.1.11 and 1.1.12, the compact sets in E^n are characterized by those which are closed and bounded.

Definition 1.1.41: A region $R \subset E^n$ is said to be a Jordan region if and only if it is homeormorphic to the ball $B_1(\vec{0}) \subset E^n$.

Definition 1.1.42: A set $M \subset E^n$ is said to be the image of a Jordan manifold if and only if $(M, \mathcal{J}_M)$ is homeormorphic to the sphere $S_1(\vec{0}) \subset E^n$.

Theorem 1.1.13: ([23] , p.63) (Jordan Curve Theorem) Let C be the image of a Jordan curve in E^2. Then $E^2 - C$ consists of two non-empty components one of which is bounded and called the interior of C and the other is unbounded and called the exterior of C, and C is the common boundary of its interior and its exterior.

Remark: The term "interior" as used in the last theorem has an entirely different meaning from that of its previous use.

Theorem 1.1.14:([23] , p.63) (n-Dimensional Analogue of Jordan Curve Theorem). Let M be the image of an (n-1) Jordan manifold in E^n. Then $E^n - M$ consists of two non-empty components one of which is called the interior of M, and the other the exterior of M. The interior of M is a Jordan region whose boundary is M.

Remark: By an abuse of language, we shall simply speak of (n-1) Jordan manifolds rather than the image of such a manifold.

§2 Some essentials of measure and integration theory

The concept of a measure can be introduced via two different methods. One view is to consider a measure as an extended real valued function defined on a certain class of sets as is done in Halmos [14] and the other is to think of a measure as a positive linear functional defined on a topological vector space. The latter approach is taken by Bourbaki [3] and the two views can be welded together by the Riesz representation theorem to be considered later. We shall find it convenient to adopt one point of view in certain cases and the other one for different situations. Since good reference material is readily available for most of the results of this section, which are quite standard and basic to the theory of measure and integration we shall omit specific references to a large extent. Initially, we shall follow the approaches of Halmos and Royden [17]. Our first definitions will follow from some of the original historical results.

Let E' be the real line and $I = (a,b)$ be an open interval in E'.

Definition 1.2.1: We define the classical (Lebesgue) measure of $I = (a,b)$ to be $m_1(I) = b - a$.

Remark: If a is real and $b = +\infty$, then we follow the convention of putting $m_1(I) = +\infty$.

<u>Definition 1.2.2:</u> Let $O = \bigcup_{i=1}^{\infty} (I_i)$ be an open set in E', where each I_i is a <u>component set</u> of O. We define the classical measure of O to be $m_1(O) = \sum_{i=1}^{\infty} m_1(I_i)$ provided the series converges in the ordinary sense. Otherwise we formally define $m_1(O)$ to be $+\infty$.

<u>Definition 1.2.3:</u> Let K be a compact subset of E' and I = (a,b), (where a and b are both real), have the property that $I \supset K$. Then we define the <u>classical measure</u> of K to be $m_1(K) = m_1(I) - m_1(I-K)$.

<u>Definition 1.2.4:</u> Let $A \subset E'$ or $A \in \mathcal{B}(E')$, and let $\theta = \{O_\alpha\}$ be the family of all open sets in E', each member of which contains A, i.e. $A \subset O_\alpha$ for all α. Now consider the set of positive real numbers $\{m_1(O_\alpha)\}$. We define $\inf_\alpha \{m_1(O_\alpha)\}$ to be the <u>outer classical measure</u> of A, and designate it by the symbol $(m_1)^*(A)$.

<u>Definition 1.2.5:</u> Let $A \subset E'$, and $\mathcal{K} = \{K_\alpha\}$ be the family of all compact sets in E', each member of which is contained in A, i.e. $K_\alpha \subset A$ for all α. Define $(m_1)_*(A) = \sup_\alpha \{m_1(K_\alpha)\}$ to be the <u>inner classical measure</u> of A.

<u>Theorem 1.2.1:</u> It is always true that $0 \leq (m_1)_*(A) \leq (m_1)^*(A) \leq +\infty$.

<u>Remark:</u> We note that both $(m_1)^*$ and $(m_1)_*$ are functions from $\mathcal{B}(E')$ into the non-negative extended reals.

<u>Definition 1.2.6:</u> A bounded set $A \subset E'$ is said to be m_1-<u>measurable</u> if and only if $(m_1)_*(A) = (m_1)^*(A)$ and the common value is called the classical m_1-measure of A, denoted by $m_1(A)$.

<u>Definition 1.2.7</u>: An arbitrary set $A \subset E'$ is said to be m_1-measurable if for any $I_n = (-n, +n)$, the set $A \cap I_n$ is m_1-measurable.

<u>Theorem 1.2.2</u>: Every open set $O \subset E'$ and every closed set $F \subset E'$ is m_1-measurable.

<u>Theorem 1.2.3</u>: A set $A \subset E'$ is m_1-measurable if and only if the following condition holds for any $X \subset E'$: $m_1{}^*(X) \geq m_1{}^*(X \cap A) + m_1{}^*(X \cap \complement A)$. This condition is often referred to as the <u>Carathéodory criterion</u> of measurability.

<u>Remark</u>: The statement of theorem 1.2.3 is sometimes taken to be the definition of m_1-measurability in which case our definition 1.2.6 turns into a theorem.

<u>Theorem 1.2.4</u>: If $A \subset E'$ is m_1-measurable, then $\complement A$ is also m_1-measurable, where $\complement A$ denotes the complement of A.

<u>Theorem 1.2.5</u>: Let $\{A_i\}$, i = 1, 2, 3, ... be a sequence of m_1-measurable subsets of E'. Then $A = \bigcup_{i=1}^{\infty} A_i$ is also m_1-measurable.

<u>Theorem 1.2.6</u>: If $A \subset E'$ has the property that $m_1^*(A) = 0$, then A is m_1-measurable.

<u>Theorem 1.2.7</u>: (<u>Complete additivity property</u>). If $\{A_i\}$ is a sequence of m_1-measurable sets in E' such that $A_i \cap A_j = \emptyset$ if $i \neq j$, then

$$m_1 \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} m_1(A_i).$$

<u>Theorem 1.2.8</u>: Let $\{A_i\}$, $i = 1, 2, \ldots$ be a sequence of m_1-measurable sets in E'. Then $A = \bigcap_{i=1}^{\infty} A_i$ is m_1-measurable.

<u>Proof</u>: By DeMorgan's laws for set theory, $\complement A = \bigcup_{i=1}^{\infty} (\complement A_i)$. Since each A_i is m_1-measurable, so is $\complement A_i$ by theorem 1.2.4 and hence $\complement A$ is m_1-measurable by theorem 1.2.5. Then A is m_1-measurable by theorem 1.2.4.

<u>Definition 1.2.8</u>: Let $\mathcal{S} \subset \mathcal{B}(X)$ for a given set X which possesses the following properties:

(i) If $A \in \mathcal{S}$, then $\complement A \in \mathcal{S}$.

(ii) Let $\{A_i\}$ be a sequence of sets such that if each $A_i \in \mathcal{S}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$ (closure under countable union).

Then $\mathcal{S}$ is defined to be a <u>Boolean σ-algebra</u> on X.

<u>Remark</u>: It follows immediately that both X and $\emptyset$ are members of any Boolean σ-algebra $\mathcal{S}$ on X.

At this stage we make two observations. Firstly, the class of sets on E' on which the measure m_1 is defined, satisfies the axioms for a Boolean σ-algebra or simply a σ-algebra, and secondly, the measure m_1 itself possesses the property of <u>countable additivity</u> (theorem 1.2.7). These considerations will motivate our next set of definitions.

<u>Definition 1.2.9</u>: Let X be a set and $\mathcal{S} \subset \mathcal{B}(X)$ a σ-algebra on X. Then the pair $(X, \mathcal{S})$ shall be defined to be a <u>measurable space</u>.

<u>Definition 1.2.10:</u> Let $(X, \mathcal{S})$ be a measurable space and μ a function from $\mathcal{S}$ into the non-negative extended reals such that

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\bigcup_{i=1}^{\infty} (A_i)) = \sum_{i=1}^{\infty} \mu(A_i)$ where $\{A_i\}$ is

any sequence of elements contained in $\mathcal{S}$, and where $A_i \cap A_j = \emptyset$ if $i \neq j$.

Then μ is called a measure on the measurable space $(X, \mathcal{S})$ and the triple $(X, \mathcal{S}, \mu)$ is called a <u>measure space</u>.

<u>Definition 1.2.11:</u> A measure space $(X, \mathcal{S}, \mu)$ is called <u>σ-finite</u> if $\{X_i\}$ is a sequence such that

(i) $X_i \in \mathcal{S}$ for all $i = 1, 2, \ldots$

(ii) $\mu(X_i) < +\infty$

(iii) $\bigcup_{i=1}^{\infty} X_i = X$.

<u>Definition 1.2.12:</u> A measure space $(X, \mathcal{S}, \mu)$ is called <u>finite</u> if $\mu(X) < +\infty$.

<u>Definition 1.2.13:</u> Let $(X, \mathcal{S}, \mu)$ be a measure space such that for any $A \in \mathcal{S}$, such that $\mu(A) = 0$, then B is μ-measurable for any $B \subset A$. Then μ is said to be a <u>complete</u> measure on $\mathcal{S}$, i.e. a measure space is complete if and only if every subset of a set of measure zero is measurable.

<u>Theorem 1.2.9:</u> Let $\{\mathcal{S}_\alpha\}$ be a family of σ-algebras on X, then $\mathcal{S} = \bigcap_\alpha \mathcal{S}_\alpha$ is a σ-algebra on X.

Proof: (i) Let $A \subset X$ be such that $A \in \mathcal{S}$. Then $A \in \mathcal{S}_\alpha$ for every $\mathcal{S}_\alpha \in \{\mathcal{S}_\alpha\}$. Hence $\complement A \in \mathcal{S}_\alpha$ for every $\mathcal{S}_\alpha \in \{\mathcal{S}_\alpha\}$. Hence $\complement A \in \mathcal{S} = \bigcap_\alpha \mathcal{S}_\alpha$. Therefore $\mathcal{S}$ is closed under complementation.

(ii) Let $\{A_i\}$ be a sequence of sets such that $A_i \in \mathcal{S}$ for each i = 1, 2, ... Then $A_i \in \mathcal{S}_\alpha$ for each α, or $\{A_i\} \subset \mathcal{S}_\alpha$ for each α. Let $A = \bigcup_{i=1}^{\infty} A_i$ and note that $A \subset \mathcal{S}_\alpha$ for every α. Hence $A \in \mathcal{S}$, and it follows that $\mathcal{S}$ is closed under countable union. The theorem follows.

Remark: If $\mathcal{Z} \subset \mathcal{B}(X)$, theorem 1.2.9 indicates that there always exists a smallest σ-algebra which contains $\mathcal{Z}$.

Definition 1.2.14: Let θ be the open sets of E', and $\mathcal{B}$ the smallest σ-algebra which contains θ. Then $\mathcal{B}$ is called the class of Borel sets or sometimes Baire sets on E'.

Definition 1.2.15: When m_1 is restricted to $\mathcal{B}$, we refer to it as the classical Borel measure.

Definition 1.2.16: When a measure μ is defined on $(E', \mathcal{B})$, it is called

(i) a Borel measure.

(ii) a Baire measure.

(iii) a Radon measure.

Theorem 1.2.10: The measure space $(X, \mathcal{B}, m_1)$ is not complete.

<u>Definition 1.2.17</u>: Let $\mathcal{N} \subset \mathcal{B}(E')$ be defined such that $N \in \mathcal{N}$ if and only if $N \subset A$ for some $A \in \mathcal{B}$ such that $m_1(A) = 0$. The least σ-algebra $\mathcal{L}$ which contains $\mathcal{B} \cup \mathcal{N}$ is called the <u>completion</u> of $\mathcal{B}$.

<u>Theorem 1.2.11:</u> The family $\mathcal{L}$ which is the completion of $\mathcal{B}$ is the σ-algebra of m_1-measurable sets defined in definitions 1.2.6 and 1.2.7.

<u>Remark</u>: $\mathcal{L}$ is called the σ-algebra of Lebesgue measurable sets. In general we shall be more interested in the <u>Borel</u> or <u>Radon</u> measures than in the actual Lebesgue measures.

<u>Definition 1.2.18</u>: Let $(X, \mathcal{S})$ and $(Y, \mathcal{S}')$ be measurable spaces and let $X \times Y$ be the Cartesian product of X and Y. Now consider the set $\{A \times B : A \in \mathcal{S} \text{ and } B \in \mathcal{S}'\}$. We call $A \times B$ a <u>measurable rectangle</u> and define $\mathcal{S} \times \mathcal{S}'$ to be the least σ-algebra which contains the set $\{(A \times B)\}$ of all measurable rectangles. Then $(X \times Y, \mathcal{S} \times \mathcal{S}')$ is a measurable space called the <u>product space</u> of $(X, \mathcal{S})$ and $(Y, \mathcal{S}')$.

<u>Definition 1.2.19</u>: Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{S}', v)$ be two measure spaces. We can define a measure λ on the product space $(X \times Y, \mathcal{S} \times \mathcal{S}')$ as follows:

(i) If $A \times B$ is a measurable rectangle, define $\lambda(A \times B) = \mu(A)v(B)$ where we do not permit $\mu(A) = 0$ and $v(B) = +\infty$ simultaneously.

(ii) If $z \in \mathcal{S} \times \mathcal{S}'$, we let $\{C_i\} = \mathcal{C}_\alpha$ be a countable covering of z by measurable rectangles, and define $\lambda_\alpha = \sum_{i=1}^{\infty} \lambda(C_i)$. We let $\lambda^*(z) = \inf_\alpha \{\lambda_\alpha\}$ and make use of the fact ([17] , p.230) that λ^* restricted to $\mathcal{S} \times \mathcal{S}'$ is a measure on $\mathcal{S} \times \mathcal{S}'$ called the <u>product measure</u> of μ and v denoted by $\lambda = \mu \times v$. Then the space $(X \times Y, \mathcal{S} \times \mathcal{S}', \lambda)$ is called the <u>product measure space</u>.

<u>Remark</u>: We define m_2, the classical measure in E^2, to be $m_1 \times m_1$ and can extend this process into higher dimensions to get m_n, the classical measure in E^n, to be $m_n = m_1 \times m_{n-1}$ where $n \geq 2$.

<u>Definition 1.2.20</u>: Let $(X, \mathcal{S}, \mu)$ and $(X, \mathcal{S}, v)$ be two measure spaces on the same measurable space. We say that v is <u>absolutely continuous</u> with respect to μ if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $v(A) < \varepsilon$ if $\mu(A) < \delta$ for any $A \in \mathcal{S}$.

<u>Definition 1.2.21</u>: A measure space $(X, \mathcal{S}, \mu)$ is said to have a property <u>almost everywhere</u> on X denoted by a.e. on X if it has this property everywhere on X, except on a set of μ-measure zero.

<u>Theorem 1.2.12</u>: Let $(X, \mathcal{S}, \mu)$ be a measure space and let f be an extended real valued function on X. Then the following four statements are equivalent:

(i) For any real r, $f^{-1}(r, +\infty] \in \mathcal{S}$

(ii) For any real r, $f^{-1}[r, +\infty] \in \mathcal{S}$

(iii) For any real r, f^{-1} $[-\infty, r) \in \mathcal{S}$

(iv) For any real r, f^{-1} $[-\infty, r] \in \mathcal{S}$

The previous 4 statements imply the 5th:

(v) For any extended real r, $f^{-1}\{r\} \in \mathcal{S}$

<u>Definition 1.2.21</u>: Let $(X, \mathcal{S}, \mu)$ be a measure space and f an extended real valued function on X. Then we define f to be <u>measurable</u> (or μ-measurable) if it satisfies any one of the first four assertions of theorem 1.2.12.

<u>Remark</u>: It is immediate that if f is measurable on $(X, \mathcal{S}, \mu)$, then $f^{-1}(I) \in \mathcal{S}$ where I is any interval in the extended reals.

<u>Theorem 1.2.13</u>: If f and g are measurable on $(X, \mathcal{S}, \mu)$, then $(f + g)$, rf and $|f|$ are also measurable on $(X, \mathcal{S}, \mu)$ for r any real number.

We now consider some integration theory, restricing ourselves to bounded real valued functions.

<u>Definition 1.2.22</u>: Let $(X, \mathcal{S}, \mu)$ be a measure space and $\{A_i\}$, $1 \leqslant i \leqslant m$, a finite subset of $\mathcal{S}$. The function $\varphi = \sum_{i=1}^{m} r_i \chi_{A_i}$ where χ_{A_i} is the <u>characteristic function</u> on A_i

$$\text{i.e. } \chi_{A_i} \equiv 1 \text{ on } A_i$$
$$\equiv 0 \text{ on } X - A_i$$

and r_i a finite sequence of real numbers, is called a simple function on $(X, \mathcal{A}, \mu)$.

Definition 1.2.23: If $(X, \mathcal{A}, \mu)$ is a measure space and $A_i \in \mathcal{A}$, we define the integral of χ_{A_i} with respect to μ to be $\int_X \chi_{A_i} \, d\mu = \mu(A_i)$.

Definition 1.2.24: If $(X, \mathcal{A}, \mu)$ is a measure space and $\varphi = \sum_{i=1}^{m} r_i \chi_{A_i}$ is a simple function on X, we define the integral of φ with respect to μ to be $\int_X \varphi \, d\mu = \sum_{i=1}^{m} r_i \mu(A_i)$.

Definition 1.2.25: Let f be a bounded real valued function on the measure space $(X, \mathcal{A}, \mu)$, and let Φ be the family of simple functions such that $\varphi \in \Phi$ if and only if $\varphi \geq f$ on X. We define the upper integral of f with respect to μ to be $\bar{\int_X} f \, d\mu = \inf_{\varphi \in \Phi} \left\{ \int_X \varphi \, d\mu \right\}$.

Definition 1.2.26: Let f be defined as in definition 1.2.25. We define Ψ to be the family of simple functions on $(X, \mathcal{A}, \mu)$ such that $\psi \in \Psi$ if and only if $\psi \leq f$ on X. We define the lower integral of f with respect to μ to be $\underline{\int}_{(X)} f d\mu = \sup_{\psi \in \Psi} \left\{ \int_X \psi \, d\mu \right\}$.

Theorem 1.2.14: It is always true that $\underline{\int}_{(X)} f \, d\mu \leq \bar{\int_X} f \, d\mu$.

Definition 1.2.27: If f is bounded on the measure space $(X, \mathcal{A}, \mu)$ then we say that f is integrable or summable with respect to μ if and only if $\underline{\int}_{(X)} f \, d\mu = \bar{\int_X} f d\mu$ and denote the common value by $\int_X f d\mu$. We call the value of $\int_X f d\mu$ the integral of f with respect to μ.

<u>Theorem 1.2.15</u>: A bounded function f on $(X, \mathcal{S}, \mu)$ is integrable with respect to μ if and only if it is measurable with respect to μ.

<u>Definition 1.2.28</u>: Let f be an extended real valued function where $f \geq 0$ on the measure space $(X, \mathcal{S}, \mu)$, and let $f_n = f \wedge n = \inf \{f, n\}$ on X. We define $\int_X f d\mu = \lim_{n \to \infty} \int_X f_n d\mu$ provided that f_n is μ-integrable for each n.

<u>Remark</u>: In an analogous manner to definition 1.2.28, we define the integral of a negative function. We allow the possibility that $\int_X f du = +\infty$ for $f \geq 0$. Some authors do not permit this.

<u>Definition 1.2.29</u>: Let f be an extended real valued function on $(X, \mathcal{S}, \mu)$ and denote $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. We define $\int_X f d\mu$ to exist if and only if $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ both exist and are not both infinite simultaneously. We then define

$$\int_X f du = \int_X f^+ du - \int_X f^- du.$$

<u>Remark</u>: A bounded function f is μ-integrable if and only if it is μ-measurable, and this is also true of a positive function. But in general, the class of μ-summable (integrable) functions are a proper subclass of the class of μ-measurable functions.

<u>Theorem 1.2.16</u>: If f and g are μ-summable on $(X, \mathcal{S}, \mu)$ then so is f+g and rf for r a real number. It follows that

$$\int_X (f + g)\, d\mu = \int_X f d\mu + \int_X g d\mu \quad \text{and} \quad \int_X r\, f d\mu = r \int_X f d\mu.$$

Theorem 1.2.17: (Radon-Nikodym Theorem). Let $(X, \mathcal{S}, \mu)$ and $(X, \mathcal{S}, v)$ be σ-finite measure spaces and assume that v is absolutely continuous with respect to μ. Then for any $A \in \mathcal{S}$, it follows that $v(A) = \int_A f\, du$ where f is some μ-measurable function on X. The function f is unique up to sets of μ-measure zero.

Remark: The function f need only be defined a.e. on X with respect to μ.

Definition 1.2.30: If $(X, \mathcal{S}, \mu)$ and $(X, \mathcal{S}, v)$ are σ-finite measure spaces where $v(A) = \int_A f\, d\mu$ for f defined a.e. on X for each $A \in \mathcal{S}$, then we say that f is the Radon-Nikodym derivative of v with respect to μ. We also call f a density function of v with respect to μ.

Throughout the remainder of this section we shall only consider measures on compact T_2 spaces. We require first some definitions from functional analysis.

Definition 1.2.31: Let (X, d) be a metric space and $\{x_n\}$ a sequence of points in X. We define $\{x_n\}$ to be a Cauchy sequence if and only if for any $\varepsilon > 0$, there exists an $n_o(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m > n_o$.

Definition 1.2.32: A metric space (X, d) is called complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.

<u>Definition 1.2.33</u>: Let W be a vector (linear) space over the reals and let f be a function from W into the non-negative reals such that

(i) $f(\vec{x}) = 0$ if and only if $\vec{x} = \vec{0}$

(ii) $f(\vec{x} + \vec{y}) \leq f(\vec{x}) + f(\vec{y})$

(iii) $f(r\vec{x}) = |r| f(\vec{x})$ for r a real number

Such a function on W is called a <u>norm</u>.

<u>Definition 1.2.34</u>: A <u>normed linear space</u> (W, f) is a vector space W endowed with a norm f.

<u>Remark</u>: A normed linear space (W, f) may be thought of as a metric space by defining $d(\vec{x},\vec{y}) = f(\vec{x}-\vec{y})$. The metric d is said to be <u>deduced</u> from the <u>norm</u> f.

<u>Definition 1.2.35</u>: If a normed linear space (W, f) is complete with respect to the metric d deducible from the norm f, we call it a <u>Banach space</u>.

<u>Definition 1.2.36</u>: Let W be a vector space over the reals, and x' a function from W into the reals such that

(i) $x'(\vec{x}+\vec{y}) = x'(\vec{x}) + x'(\vec{y})$ and

(ii) $x'(r\vec{x}) = rx'(\vec{x})$

for any real number r, then x' is called a <u>linear functional</u> on W.

<u>Remark</u>: The collection of linear functionals on W may be made into a vector space by defining $(x' + y') = z'$ to be $z'(\vec{x}) = x'(\vec{x}) + y'(\vec{x})$

for each $\vec{x} \in W$, and by defining $(rx') = z''$ to be $z''(\vec{x}) = r(x'(\vec{x}))$, r a real number, for each $\vec{x} \in W$.

Definition 1.2.37: The collection of linear functionals on W when thought of as a vector space, is called the (algebraic) dual space and denoted by W'.

Definition 1.2.38: If (W, f) is a Banach space, then $x' \in W'$ is said to be a continuous linear functional if and only if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $|x'(\vec{x})| < \varepsilon$ whenever $f(\vec{x}) < \delta$.

Definition 1.2.39: Let (W, f) be a Banach space. The continuous linear functionals form a subspace of W' denoted by W*, and is called the topological dual space of W with respect to the norm f.

Definition 1.2.40: Let $(X, \mathcal{T})$ be a compact T_2 space and g a continuous function from X into the reals. The uniform norm of g, denoted by $\|g\|$ is defined to be $\|g\| = \sup_{x \in X} \{|g(x)|\}$.

Definition 1.2.41: Let $(X, \mathcal{T})$ be a compact T_2 space. Define C to be the set of continuous functions from X into E' such that C is endowed with a vector structure and the uniform norm. C then becomes a normed linear space.

Theorem 1.2.18 ([1], p. 395). The normed linear space C that was defined in definition 1.2.41 is a Banach space.

Theorem 1.2.19: Let $(X, \mathcal{T})$ be a compact T_2 space and $(X, \mathcal{S}, \mu)$ be a measure space where $\mathcal{S}$ constitutes the σ-algebra of Borel sets on X.

Then any continuous extended real valued function f on X is a μ-measurable function.

Proof: Since f is continuous, it follows that $f^{-1}(r, +\infty)$ is an open subset of X for any real r and hence $f^{-1}(r, +\infty) \in \mathcal{S}$. From definition 1.2.21, f is a μ-measurable function.

Remark: For any finite Radon (Borel) measure μ on the compact space $(X, \mathcal{T})$, it is not hard to see that it is a member of C*, the topological dual of C with respect to the uniform norm. What is not so obvious, is the converse.

Definition 1.2.42: A linear functional $\lambda \in C^*$ is said to be positive if and only if $\lambda(f) \geq 0$, where $f \in C$ and $f \geq 0$ on X.

Remark: A positive linear functional is always continuous.

Theorem 1.2.20: (Riesz Representation Theorem). Let $(X, \mathcal{T})$ be a compact T_2 space, C the Banach space of continuous real valued functions on X, and C* the topological dual of C. Let $\lambda \in C^*$ be a positive linear functional on C. Then there is a Borel measure μ on X such that

$$\lambda(f) = \int_X f \, d\mu \quad \text{for every } f \in C.$$

Remark: It is common to identify a positive linear functional with its representation measure, and since any continuous function is measurable with respect to this measure, it follows that the open sets are also measurable, and hence the σ-algebra of measurable sets contains at least the Borel sets. It is for this reason that Bourbaki is able to define a Radon measure on X to be a member of C*. We further remark that there is a Riesz representation theorem for all of C*, but in this more general case, the representation measures may be <u>signed measures</u>. However, this level of generality is not required here.

§3 Aspects of the Green calculus of n-dimensional manifolds

The material of this section properly belongs to the general theory of integration on manifolds, and the modern theory of differential forms. We shall not develop a detailed account of this theory, but shall merely mention a few results which are necessary for our later work in harmonic functions. Our chief references will be M. Spivak [18] and H. Flanders [13]. We seek an n-dimensional representation of the divergence theorem of Gauss.

Let us first introduce some definitions and notation.

Definition 1.3.1: Let $\vec{\nabla}$ denote the vector operator $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ relative to E^n.

Definition 1.3.2: A function f from E^n into E' is said to be a member of C^K if and only if f possesses continuous partial derivatives of all orders up to and including K.

Definition 1.3.3: If f is a C' function from E^n into E', then $\vec{\nabla} f = (f_{x_1}, \ldots, f_{x_n})$ which is a function from E^n into E^n, is called the gradient of f.

Definition 1.3.4: If $\vec{g} = (g_1, \ldots, g_n)$ is a C' function from E^n into E^n (which means that $g_i \in C'$ for all i, $1 \leqslant i \leqslant n$), then $\vec{\nabla} \cdot \vec{g} = \sum_{i=1}^{n} (\frac{\partial g_i}{\partial x_i})$ is called the divergence of $\vec{g}$. We note that $\vec{\nabla} \cdot \vec{g}$ is a function from E^n into E'.

<u>Definition 1.3.5</u>: If $\vec{\alpha} \in E^n$, where $|\vec{\alpha}| = 1$, then we replace $\vec{\alpha}$ by the symbol $\hat{\alpha}$.

<u>Definition 1.3.6</u>: If f is a C' function on a region $R \subset E^n$, then we define the <u>directional derivative</u> of f at $\vec{x}_o \in R$ in the direction of $\hat{\alpha}$ to be $\lim_{h\to 0^+} \dfrac{f(\vec{x}_o + h\hat{\alpha}) - f(\vec{x}_o)}{h}$, which is equal to $(\vec{\nabla} f(\vec{x}_o))\cdot\hat{\alpha}$ and often written $(\frac{\partial f}{\partial \hat{\alpha}})(\vec{x}_o)$.

<u>Definition 1.3.7</u>: The operator $\vec{\nabla}\cdot\vec{\nabla}$ or $\nabla^2 = \sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ is called the <u>Laplacian operator</u>, which is of fundamental importance in our later definitions of harmonic functions.

<u>Definition 1.3.8</u>: Let $M \subset E^n$ be a Jordan manifold such that $M = \{\vec{p} \in E^n : g(\vec{p}) = 0\}$ where $g \in C'$ in E^n. Let $\vec{q} \in M$, then by analogy with classical E^2 and E^3 theory, we define the vector $(\vec{\nabla} g)_{\vec{q}}$ to be a <u>normal vector</u> to the manifold M at the point $\vec{q}$, denoted by $\vec{n}_{\vec{q}}$.

<u>Definition 1.3.9</u>: A Jordan manifold $M \subset E^n$ such that $M = \{\vec{p} \in E^n : g(\vec{p}) = 0\}$ where $g \in C'$ in E^n is called <u>smooth</u>.

<u>Definition 1.3.10</u>: The set of vectors of the form $\vec{V} = \vec{p} - \vec{q}$ where $\vec{V}\cdot\vec{n}_{\vec{q}} = 0$ are called the set of tangent vectors to M at $\vec{q}$. The set of such tangent vectors are referred to as the <u>tangent space</u> to M at $\vec{q}$. The tangent space is a copy of E^{n-1}.

<u>Definition 1.3.11</u>: Let $M \subset E^n$ be a smooth Jordan manifold and T be its projection function into an (n-1) subspace E^{n-1}. We assume that M and E^{n-1} are so chosen that T is a one-to-one function, and let $M' = T(M)$. Let $\hat{N}$ be the unit normal vector to E^{n-1} and $\hat{n}_{\vec{q}}$ be the unit normal vector to M at $\vec{q}$. We require that $\hat{n}_{\vec{q}} \cdot \hat{N} \neq 0$ for any $\vec{q} \in M$. We define a measure on M, denoted by σ_{n-1}, whose density function at $\vec{q} \in M$ with respect to the classical measure m_{n-1} on E^{n-1} is $\dfrac{1}{|\hat{n}_{\vec{q}} \cdot \hat{N}|}$. In other words $\int_M d\sigma_{n-1} = \int_{M'} \dfrac{dm_{n-1}(\vec{q}\,')}{|(\hat{n}_{\vec{q}} \cdot \hat{N})|}$ where $\vec{q}\,' = T(\vec{q})$.

<u>Remark:</u> In E^2, σ_1 is the measure for <u>arc length</u> of a curve. If $M = \{(x, y) : y = f(x) \in C', 0 \leq x \leq b\}$, then

$$\frac{1}{|\hat{n} \cdot \hat{j}|} = \sqrt{1 + f'(x)^2} \text{ and therefore } \int_M d\sigma_1 = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

In E^3, σ_2 is the measure for surface area. If $M = \{(x, y, z) : z = f(x,y) \in C' \text{ on } R \subset E^2\}$, then $\dfrac{1}{|\hat{n} \cdot \hat{k}|} = \sqrt{1 + f_x^2 + f_y^2}$, and $\int_M d\sigma_2 = \int_R \sqrt{1 + f_x^2 + f_y^2}\, dm_2$. In the modern terminology of measure theory, the density function $\dfrac{1}{|\hat{n}_{\vec{q}} \cdot \hat{N}|}$ is called the <u>Radon-Nikodym</u> derivative of σ_{n-1} with respect to m_{n-1}.

Before proceeding further with the Green calculus, we require a few definitions from combinatorial topology.

Definition 1.3.12: Let $\{\vec{x}_1, \ldots, \vec{x}_p\} = A_p$ denote a finite set of points in E^n of which there are p members. The closed convex hull of A_p, denoted by $[A_p]$, is then defined by the set

$$[A_p] = \left\{ \vec{x} \in E^n : \vec{x} = \sum_{i=1}^{p} t_i \vec{x}_i \text{ where } t_i \geqslant 0 \text{ and } \sum_{i=1}^{p} t_i = 1 \right\}.$$

Remark: $\vec{x} \in [A_p]$ may be regarded as the centre of mass of the unit mass distribution μ on A_p, where $\mu\{x_i\} = t_i$, $1 \leqslant i \leqslant p$.

Definition 1.3.13: A set $A_{p+1} = \{\vec{x}_o, \vec{x}_1, \ldots, \vec{x}_p\} \subset E^n$ is said to be independent if the vectors $\{V_i\}$, $1 \leqslant i \leqslant p$, defined by $\vec{V}_i = (\vec{x}_i - \vec{x}_o)$ are linearly independent in E^n.

Definition 1.3.14: Let $A_{p+1} = \{\vec{x}_o, \vec{x}_1, \ldots, \vec{x}_p\} \subset E^n$ be an ordered set of (p+1) independent points in E^n where $p \leqslant n$. Then the closed convex hull $[A_{p+1}] = [x_o, x_1, \ldots, x_p]$ is called a p-simplex in E^n.

Remark: A 0-simplex is a singleton set.

A 1-simplex is a directed closed line segment.

A 2-simplex is a closed ordered triangle.

The set A_p is called the vertices of the simplex $[A_p]$.

Definition 1.3.15: A specific ordering of the vertices of $[A_p]$ is said to produce an orientation on $[A_p]$.

Definition 1.3.16: A permutation on the vertices of $[A_p]$ is said to be simple if any two adjacent vertices are permuted.

<u>Definition 1.3.17</u>: Two orientations on A_p are said to be <u>equivalent</u> if one can be transformed into the other by an even number of simple permutations.

<u>Remark</u>: There are only two different orientations on a p-simplex each of which is an equivalence class of orderings on the vertices. In the case of E^2, we say that C is a Jordan curve of <u>positive</u> orientation if the interior of C is to the left as one moves along the curve.

We now return to the Green calculus and shall consider only manifolds which have simple properties. Such manifolds shall be sufficient for our purposes. We now state without proof a version of Green's theorem for the plane.

<u>Theorem 1.3.1</u>: (Green's Theorem ([1] , p.289)). Let $\vec{f} = f_1\hat{\imath} + f_2\hat{\jmath}$ be a C' function from E^2 into E^2 and let $R \subset E^2$ be a Jordan region whose boundary ∂R is a smooth Jordan curve with positive orientation. We parameterize ∂R to be the function $\vec{\alpha}$ defined on $\bar{I} = [0, 1]$ so that $\partial R = \vec{\alpha}(\bar{I})$, and let $\hat{t}$ be the unit tangent vector on ∂R, denoted by $\hat{t} = \left(\frac{d\vec{\alpha}}{dt}\right) / \left|\frac{d\vec{\alpha}}{dt}\right|$. Then $\int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dm_2 = \int_{\partial R} (\vec{f}.\hat{t})\, d\sigma_1$.

We shall find it convenient to consider also the so-called divergence form of Green's theorem. Let $\hat{n}$ be the outer normal relative to ∂R and focus our attention on $\int_{\partial R} (\vec{f}.\hat{n})\ d\sigma_1$.

Theorem 1.3.2: Let $\vec{f}$, R, ∂R be defined as in theorem 1.3.1. Then $\int_{\partial R} (\vec{f}.\hat{n})\, d\sigma_1 = \int_R (\vec{\nabla}.\vec{f})\, dm_2$ (where $\int_{\partial R} (\vec{f}.\hat{n})\, d\sigma_1$ denotes the total flux across the boundary ∂R of $\vec{f}$.)

Proof: We briefly consider the relationship between $\hat{t}$ and $\hat{n}$. If $\hat{t} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$, then $\hat{n} = \cos(\theta - \frac{\pi}{2})\hat{\imath} + \sin(\theta - \frac{\pi}{2})\hat{\jmath} = \sin\theta\hat{\imath} - \cos\theta\hat{\jmath}$, and therefore $\vec{f}.\hat{n} = (f_1\hat{\imath} + f_2\hat{\jmath})\cdot(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath})$

$$= f_1 \sin\theta - f_2\cos\theta = \vec{F}.\hat{t} \text{ where}$$

$\vec{F} = -f_2\hat{\imath} + f_1\hat{\jmath}$ and called the <u>conjugate</u> of $\vec{f}$. Hence

$$\int_{\partial R} (\vec{f}.\hat{n})\, d\sigma_1 = \int_{\partial R} (\vec{F}.\hat{t})\, d\sigma_1 = \int_R \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) dm_2 \text{ by theorem 1.3.2.}$$

The theorem follows because $\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) = \vec{\nabla}\cdot\vec{f}$.

Theorem 1.3.3: Let $R(r_1, r_2) = R$ be the region denoting the annulus between the circles $S_1 = \{\vec{x} \in E^2 : |\vec{x}| = r_1\}$ and $S_2 = \{\vec{x} \in E^2 : |x| = r_2\}$ and consider $\bar{R} = R \cup S_1 \cup S_2$. Let $\vec{f} = f_1\hat{\imath} + f_2\hat{\jmath}$ be a C' function on $\bar{R}$ and assume that S_1 and S_2 are both oriented in the positive direction. Then $\int_{\bar{R}} (\vec{\nabla}.\vec{f})\, dm_2 = \int_{S_1} (\vec{f}.\hat{n})\, d\sigma_1 + \int_{S_1} (\vec{f}.\hat{n})\, d\sigma_1 = \int_{\partial R} (\vec{f}.\hat{n})\, d\sigma_1$, where $\hat{n}$ is the outer normal to $\bar{R}$.

Proof: We cut $\bar{R}$ into two pieces by the x-axis:

let $\bar{R}_1 = \{\vec{x} = (x,y) \in \bar{R} : y \geq 0\}$

$\bar{R}_2 = \{\vec{x} = (x,y) \in \bar{R} : y \leq 0\}$

$$\bar{I}_1 = \{\vec{x} = (x,y)\in\bar{R} : y = 0 \text{ and } r_1 \leqslant x \leqslant r_2\}$$

$$\bar{I}_2 = \{\vec{x} = (x,y)\in\bar{R} : y = 0 \text{ and } -r_2 \leqslant x \leqslant -r_1\}$$

$$S_{11} = \{\vec{x} = (x,y)\in S_1 : y \geqslant 0\}$$

$$S_{12} = \{\vec{x} = (x,y)\in S_1 : y \leqslant 0\}$$

$$S_{21} = \{\vec{x} = (x,y)\in S_2 : y \geqslant 0\}$$

$$S_{22} = \{\vec{x} = (x,y)\in S_2 : y \leqslant 0\}$$

Then $\int_{\bar{R}} (\vec{\nabla}.\vec{f})\, dm_2 = \int_{\bar{R}_1} (\vec{\nabla}.\vec{f})\, dm_2 + \int_{\bar{R}_2} (\vec{\nabla}.\vec{f})\, dm_2.$

But $\int_{\bar{R}_1} (\vec{\nabla}.\vec{f})\, dm_2 = \int_{\partial R_1} (\vec{f}.\hat{n})\, d\sigma_1$

$$= \int_{S_{21}} (\vec{f}.\hat{n})\, d\sigma_1 + \int_{\bar{I}_2} (\vec{f}.\hat{n})\, d\sigma_1 + \int_{S_{11}} (\vec{f}.\hat{n})\, d\sigma_1 + \int_{\bar{I}_1} (\vec{f}.\hat{n})\, d\sigma_1$$

and $\int_{\bar{R}_2} (\vec{\nabla}.\vec{f})\, dm_2 = \int_{\bar{I}_1} (\vec{f}.\hat{n})\, d\sigma_1 + \int_{S_{12}} (\vec{f}.\hat{n})\, d\sigma_1 + \int_{\bar{I}_2} (\vec{f}.\hat{n})\, d\sigma_1 + \int_{S_{22}} (\vec{f}.\hat{n})\, d\sigma_1.$

Since the outer normal to $\bar{R}_1$ is the inner normal to $\bar{R}_2$ along I_2 or I_1, therefore, the integrals along the $\bar{I}$'s all cancel and we get

$$\int_{\bar{R}_1} (\vec{\nabla} \cdot \vec{f})\, dm_2 + \int_{\bar{R}_2} (\vec{\nabla} \cdot \vec{f})\, dm_2 =$$

$$\int_{S_{21}} (\vec{f}\cdot\hat{n})\, d\sigma_1 + \int_{S_{11}} (\vec{f}\cdot\hat{n})\, d\sigma_1 + \int_{S_{12}} (\vec{f}\cdot\hat{n})\, d\sigma_1 + \int_{S_{22}} (\vec{f}\cdot\hat{n})\, d\sigma_1$$

$$= \int_{S_1} (\vec{f}\cdot\hat{n})\, d\sigma_1 + \int_{S_2} (\vec{f}\cdot\hat{n})\, d\sigma_1 .$$

Hence $\int_{\bar{R}} (\vec{\nabla} \cdot \vec{f})\, dm_2 = \int_{S_2} (\vec{f}\cdot\hat{n})\, d\sigma_1 + \int_{S_1} (\vec{f}\cdot\hat{n})\, d\sigma_1$ and the theorem is proved.

<u>Remark</u>: In the statement of theorem 1.3.3, the unit normal vector is always exterior to $\bar{R}$. Hence $\int_{S_2} (\vec{f}\cdot\hat{n}) d\sigma_1 = \int_{S_2} (\vec{f}\cdot\hat{n}_e)\, d\sigma_1$, where $\hat{n}_e$ is the outer normal to the ball $\bar{B}_{r_2}(\vec{0})$. But $\int_{S_1} (\vec{f}\cdot\hat{n})\, d\sigma_1 = - \int_{S_1} (\vec{f}\cdot\hat{n}_e)\, d\sigma_1$ where $\hat{n}_e$ is the outer unit normal to the ball $\bar{B}_{r_1}(\vec{0})$.

We now state the n-dimensional divergence theorem for a ball. More general regions could be considered, but the ball and annular regions between two spheres will be sufficient for our purposes.

<u>Theorem 1.3.4</u>: ([18] , p.123) (<u>n-dimensional divergence theorem</u>) Let f be a C' function from $\bar{B}_{r_o}(\vec{0}) = \bar{B} \subset E^n$ into E^n and $S = \partial B$.

Then $\int_{\bar{B}} (\vec{\nabla} . \vec{f})\, dm_n = \int_S (\vec{f}.\hat{n})\, d\sigma_{n-1}$ where $\hat{n}$ is the outer normal vector to $\bar{B}$ and S is endowed with the proper orientation.

<u>Theorem 1.3.5</u>: Let $\bar{R} = \bar{R}_1' \cap \bar{R}_2$ where $\bar{R}_2 \subset E^n$ is the interior of the sphere S_2 union S_2 itself, and $\bar{R}_1' \subset E^n$ is the exterior of the sphere S_1 union S_1 itself, where $S_1 \subset R_2$. Let $\vec{f}$ be a C' function from the annular region R into E^n. Then

$$\int_{\bar{R}} (\vec{\nabla} . \vec{f}) dm_n = \int_{S_1} (\vec{f}.\hat{n}) d\sigma_{n-1} + \int_{S_2} (\vec{f}.\hat{n})\, d\sigma_{n-1} = \int_{\partial R} (\vec{f}.\hat{n})\, d\sigma_{n-1}$$

where $\hat{n}$ is taken to be exterior to R in all cases.

<u>Proof</u>: The theorem follows by reasoning similar to that in the proof of theorem 1.3.3.

<u>Remark</u>: Again, as in the statement of theorem 1.3.5, the unit normal vector is always exterior to R. Hence $\int_{S_2} (\vec{f}.\hat{n}) d\sigma_{n-1} = \int_{S_1} (\vec{f}.\hat{n}_e) d\sigma_{n-1}$ where $\hat{n}_e$ is the outer normal to the ball $\bar{B}_2$ whose boundary is S_2. But $\int_{S_1} (\vec{f}.\hat{n})\, d\sigma_{n-1} = - \int_{S_1} (\vec{f}.\hat{n}_e)\, d\sigma_{n-1}$ where $\hat{n}_e$ is the outer normal to the ball $\bar{B}_1$ whose boundary is S.

<u>Theorem 1.3.6</u>: Let u and $\hat{V}$ be C^2 functions from E^n into E'. Then $\vec{\nabla} . (u \vec{\nabla} V) = u (\nabla^2 V) + (\vec{\nabla} u) \cdot (\vec{\nabla} V)$.

<u>Proof:</u> The left side gives $\vec{\nabla}\cdot(u\frac{\partial V}{\partial x_1},\ldots, u\frac{\partial V}{\partial x_n}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\frac{u\partial V}{\partial x_i})$

But $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\frac{u\partial V}{\partial x_i}) = \sum_{i=1}^{n}(\frac{u\partial^2 V}{\partial x_i^2} + \frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i}) = u\sum_{i=1}^{n}\frac{\partial^2 V}{\partial x_i^2} + \sum_{i=1}^{n}(\frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i})$

= the right side.

<u>Theorem 1.3.7</u>: (Green's first identity). Let $\bar{R}$ be either a ball or annular region in E^n. Then

$$\int_{\partial R} u\left(\frac{\partial V}{\partial \hat{n}}\right) d\sigma_{n-1} = \int_{R}\left(u(\nabla^2 V) + \vec{\nabla}u\cdot\vec{\nabla}V\right) dm_n .$$

<u>Proof:</u> If $f \in C'$ on $\bar{R}$, then $\int_{\bar{R}}(\vec{\nabla}\cdot\vec{f})\, dm_n = \int_{\partial R}(\vec{f}\cdot\hat{n})\, d\sigma_{n-1}$ where $\hat{n}$ is the outer normal to $\bar{R}$. If u and V are C^2 functions, let $\vec{f} = u(\vec{\nabla}V)$. Then $\int_{\partial R} u(\vec{\nabla}V)\cdot\hat{n}\, d\sigma_{n-1} = \int_{\bar{R}}\left(u(\nabla^2 V) + \vec{\nabla}u\cdot\vec{\nabla}V\right) dm_n$.

<u>Theorem 1.3.8</u>: (Green's second identity). Let u and V be C^2 functions on $\bar{R} \subset E^n$, where $\bar{R}$ is either a closed ball or annular domain. Then

$$\int_{\partial R}\left[u\left(\frac{\partial V}{\partial \hat{n}}\right) - V\left(\frac{\partial u}{\partial \hat{n}}\right)\right] d\sigma_{n-1} \quad \int_{R}\left[u(\nabla^2 V) - V(\nabla^2 u)\right] dm_n$$

<u>Proof:</u> From theorem 1.3.7, we have

$$\int_{\partial R} u\left(\frac{\partial V}{\partial \hat{n}}\right) d\sigma_{n-1} = \int_{R}\left[u(\nabla^2 V) + \vec{\nabla}u\cdot\vec{\nabla}V\right] dm_n \text{ and also}$$

$$\int_{\partial R} V\left(\frac{\partial u}{\partial \hat{n}}\right) d\sigma_{n-1} = \int_{R}\left[V(\nabla^2 u) + \vec{\nabla}V\cdot\vec{\nabla}u\right] dm_n.$$

Subtracting the last equation from the first, our theorem follows.

§4 Convergence and envelope theory

In this section we shall find it convenient to formally introduce a compactification on E' called the two point compactification and denote it by $E^{\#}$. We shall also refer to $E^{\#}$ as the extended real line.

Definition 1.4.1: Let the underlying set of the extended real line be $E^{\#} = E' \cup \{+\infty\} \cup \{-\infty\}$ where $+\infty$ and $-\infty$ are two new elements added to E'.

Definition 1.4.2: Let ϕ be a function from $\bar{I} = [-\pi/2, \pi/2]$ onto $E^{\#}$ defined as follows:

$$\phi(x) = \tan x \text{ if } x \in (-\pi/2, \pi/2)$$

$$\phi(\pi/2) = +\infty$$

$$\phi(-\pi/2) = -\infty.$$

Then ϕ is a one-to-one mapping of $\bar{I}$ onto $E^{\#}$. We now define a topology $\mathcal{J}^{\#}$ on $E^{\#}$ such that $O \in \mathcal{J}^{\#}$ if and only if $\phi^{-1}(O) \in \mathcal{J}_{\bar{I}}$ where $\mathcal{J}_{\bar{I}}$ is the usual topology of E' relativized to $\bar{I}$. The topological space $(E^{\#}, \mathcal{J}^{\#})$ is called the extended real line.

Remark: The topological spaces $(\bar{I}, \mathcal{J}_{\bar{I}})$ and $(E^{\#}, \mathcal{J}^{\#})$ are homeomorphic. We stress that the elements $+\infty$ and $-\infty$ have only topological value, and cannot be treated algebraically. In future, many of our functions will have their range in $E^{\#}$ rather than in E'.

<u>Definition 1.4.3</u>: Let $\{f_i\}$, $1 \leqslant i \leqslant n$, be a finite family of functions from a set X into the extended real line $E^{\#}$. The function f defined on X such that $f(x) = \sup_i \{f_i(x)\}$, $1 \leqslant i \leqslant n$, for each $x \in X$, is called the <u>upper envelope</u> of the family $\{f_i\}$, $1 \leqslant i \leqslant n$, and is denoted by the symbol $f = \bigvee_{i=1}^{n} f_i$.

<u>Definition 1.4.4</u>: Let $\{f_\alpha\}$ be any family of functions from a set X into $E^{\#}$. We define f the <u>upper envelope</u> of $\{f_\alpha\}$ in such a way that $f(x) = \sup_\alpha \{f_\alpha(x)\}$ for each $x \in X$.

<u>Definition 1.4.5</u>: Let f be a function from a topological space $(X, \mathcal{J})$ into $E^{\#}$ and let $x_o \in X$. Then $r \in E^{\#}$ is said to be a <u>cluster point</u> of f at x_o if $r \in \overline{f(V')}$ for every deleted neighbourhood V' of x_o.

<u>Definition 1.4.6</u>: Let f be a function from a topological space $(X, \mathcal{J})$ into $E^{\#}$ and $x_o \in X$. The collection of cluster points of f at x_o is called the <u>cluster set</u> of f at x_o. We note that the closure of f(V') i.e. $\overline{f(V')}$, is relative to the two point compactification.

<u>Definition 1.4.7</u>: Let f be a function from the topological space $(X, \mathcal{J})$ into $E^{\#}$, and $x_o \in X$. We define $\limsup_{x \to x_o} f(x) = \overline{\lim_{x \to x_o}} f(x)$ to be the supremum (in $E^{\#}$) of the cluster set of f at x_o.

<u>Remark</u>: Any function from a topological space $(X, \mathcal{J})$ into $E^{\#}$ always possesses an upper limit in $E^{\#}$ at any $x_o \in X$. Similarly we may define $\underline{\lim}_{x \to x_o} f(x)$ to be the infimum of the cluster set of f at x_o.

<u>Definition 1.4.8</u>: A function f from a topological space $(X, \mathcal{J})$ into $E^{\#}$ is said to be <u>upper semi-continuous</u> at $x_o \in X$ if and only if $f(x_o) \geq \overline{\lim}_{x \to x_o} f(x)$.

<u>Definition 1.4.9</u>: A function f from $(X, \mathcal{J})$ into $E^{\#}$ is said to be <u>upper semi-continuous</u> on X if and only if it is upper semi-continuous at every point of X.

<u>Remark</u>: We may similarly define a function f to be <u>lower semi-continuous</u> at $x_o \in X$ if and only if $f(x_o) \leq \underline{\lim}_{x \to x_o} f(x)$. A function f is defined to be lower semi-continuous on $(X, \mathcal{J})$ if and only if it is lower semi-continuous at every point of X.

<u>Theorem 1.4.1</u>: A function f from $(X, \mathcal{J})$ into $E^{\#}$ is continuous if and only if it is both <u>upper semi-continuous</u> and <u>lower semi-continuous</u> on X.

<u>Proof</u>: If f is continuous and $x_o \in X$, then $f(x_o) = \lim_{x \to x_o} f(x)$ and hence f is both upper semi-continuous and lower-semicontinuous at x_o. If f is both upper semi-continuous and lower-semicontinuous at x_o, then $f(x_o) \geq \overline{\lim}_{x \to x_o} f(x)$ and $\underline{\lim}_{x \to x_o} f(x) \geq f(x_o)$. It is always true that $\underline{\lim}_{x \to x_o} f(x) \leq \overline{\lim}_{x \to x_o} f(x)$, and the combined conditions of upper and lower semicontinuity at x_o, force $\overline{\lim}_{x \to x_o} f(x) \leq f(x_o) \leq \underline{\lim}_{x \to x_o} f(x)$.

Hence $\underline{\lim}_{x \to x_o} f(x) = f(x_o) = \overline{\lim}_{x \to x_o} f(x)$ and f is continuous at x_o.

The argument extends over X itself.

<u>Theorem 1.4.2</u>: Let $\{f_\alpha\}$ be a family of continuous functions on $(X, \mathcal{T})$, and let f be the upper envelope of $\{f_\alpha\}$. Then f is lower semicontinuous on X.

<u>Proof</u>: For any $x_0 \in X$, we shall show that f is lower semicontinuous at x_0. Since $f \geqslant f_\alpha$ for any α, it follows that $\underline{\lim}_{x \to x_0} f(x) \geqslant \underline{\lim}_{x \to x_0} f_\alpha(x)$. But $\underline{\lim}_{x \to x_0} f_\alpha(x) = \lim_{x \to x_0} f_\alpha(x) = f_\alpha(x_0)$ since f_α is continuous at x_0. It follows that $\underline{\lim}_{x \to x_0} f(x) \geqslant f_\alpha(x_0)$ for each α and therefore is an upper bound for $\{f_\alpha(x_0)\}$. Hence $\underline{\lim}_{x \to x_0} f(x) \geqslant f(x_0)$ because $f(x_0)$ is the least upper bound of $\{f_\alpha(x_0)\}$.

<u>Corollary</u>: The lower envelope of $\{f_\alpha\}$ is upper semi-continuous.

<u>Theorem 1.4.3</u>: Let f and g be functions from $(X, \mathcal{T})$ into $E^{\#}$ and $x_0 \in X$. Then $\overline{\lim}_{x \to x_0} (f + g)(x) \leqslant \overline{\lim}_{x \to x_0} f(x) + \overline{\lim}_{x \to x_0} g(x)$.

<u>Proof</u>: Let $\overline{\lim}_{x \to x_0} f(x) = L_1$ and $\overline{\lim}_{x \to x_0} g(x) = L_2$, and choose $\epsilon > 0$. Then there exists a deleted neighbourhood V_1 of x_0 such that $f(x) < L_1 + \epsilon/2$ for any $x \in V_1$, and also a deleted neighbourhood V_2 of x_0 such that $g(x) < L_2 + \epsilon/2$ for any $x \in V_2$. It follows that $h(x) = f(x) + g(x) < L_1 + L_2 + \epsilon$ for any $x \in V_1 \cap V_2$ which is also a deleted neighbourhood of x_0. Hence $\overline{\lim}_{x \to x_0} h(x) \leqslant L_1 + L_2$ and the theorem follows.

<u>Corollary</u>: $\underline{\lim}_{x \to x_0} (f+g)(x) \geqslant \underline{\lim}_{x \to x_0} f(x) + \underline{\lim}_{x \to x_0} g(x)$.

<u>Theorem 1.4.4</u>: For a function f from X into $E^{\#}$, it follows that

$$-\sup_{x \in X} \{f(x)\} = \inf_{x \in X} \{-f(x\} .$$

<u>Proof</u>: Let $S = \sup_{x \in X} \{f(x)\}$. Then $S \geqslant f(x)$ for all $x \in X$ and therefore $(-S) \leqslant -f(x)$ for all $x \in X$. Hence $(-S)$ is a lower bound for $\{-f(x)\}$ and hence $-S \leqslant \inf_{x \in X} \{-f(x)\}$. Let $S' = \inf_{x \in X} \{-f(x)\}$; then $S' \leqslant -f(x)$ and hence $(-S') \geqslant f(x)$ or $-S'$ is an upper bound for $\{f(x)\}$. Hence $(-S') \geqslant S$ or $S' \leqslant (-S)$.

<u>Corollary</u>: When the argument of theorem 1.4.4 is applied locally, we obtain $-\overline{\lim}_{x \to x_o} (f(x)) = \underline{\lim}_{x \to x_o} (-f(x))$.

<u>Theorem 1.4.5</u>: Let f be any function from a topological space $(X, \mathcal{J})$ into the extended real line $E^{\#}$. We define a new function g on X such that $g(x_o) = \overline{\lim}_{x \to x_o} f(x)$ for every $x_o \in X$. Then g is upper semi-continuous on $(X, \mathcal{J})$.

<u>Proof</u>: Suppose the theorem is false. In such an event, there would exist some $x_o \in X$ such that $g(x_o) < \overline{\lim}_{x \to x_o} g(x) = L$ say. We put $L - g(x_o) = 2\epsilon > 0$. For any open neighbourhood $V(x_o)$, there exists $x_1 \in V(x_o), x_1 \neq x_o$ such that $g(x_1) > L - \epsilon/2$ or $g(x_o) + \frac{3}{2}\epsilon < g(x_1)$. But $V(x_o)$ is also an open neighbourhood of x_1, and hence there exists $x_2 \in V(x_o)$ such that $x_2 \neq x_1$ and $f(x_2) > g(x_1) - \epsilon/2$, (because $g(x_1) = \overline{\lim}_{x \to x_1} f(x)$). Since $g(x_1) - \epsilon/2 > g(x_o) + \epsilon$ therefore

$f(x_2) > g(x_o) + \epsilon$. We have shown that for any neighbourhood $V(x_o)$, there exists $x_2 \in V(x_o)$, $x_2 \neq x_o$, such that $f(x_2) > g(x_o) + \epsilon$ and hence $\overline{\lim_{x \to x_o}} f(x) \geq g(x_o) + \epsilon$. This contradicts the definition of g. It follows that g is upper semi-continuous.

Corollary: If $g(x_o) = \underline{\lim_{x \to x_o}} f(x)$ for every $x_o \in X$, then g is lower semi-continuous.

Theorem 1.4.6: A function f from a topological space $(X, \mathcal{J})$ into $E^{\#}$ is lower semi-continuous if and only if $f^{-1}(r, +\infty] \in \mathcal{J}$ where r is any extended real and $(r, +\infty]$ means $\{x: r < x \leq +\infty\}$.

Proof: Let f be lower semi-continuous, and let $V = f^{-1}(r, +\infty]$ for any given r. For any $x_o \in V$ it follows that $f(x_o) > r$ and hence for sufficiently small $\epsilon > 0$ it follows that $x \in V$ if $f(x) > f(x_o) - \epsilon$. Hence V is a neighbourhood of x_o and since x_o is arbitrary it follows that V is a neighbourhood of each of its points. Hence V is open. Now suppose $f^{-1}(r, +\infty]$ is open for any $r \in E^{\#}$, and let $x_o \in X$. Choose $\epsilon > 0$ and let $r_o = f(x_o)$. Then $f^{-1}(r_o - \epsilon, +\infty]$ constitutes an open neighbourhood of x_o, and it follows that $\underline{\lim_{x \to x_o}} f(x) > r_o - \epsilon$. Since $\epsilon > 0$ was arbitrary, therefore $\underline{\lim_{x \to x_o}} f(x) \geq f(x_o)$ and the lower semi-continuity of f follows.

Corollary: A function f from $(X, \mathcal{J})$ into $E^{\#}$ is upper semi-continuous if and only if $f^{-1}[-\infty, r) \in \mathcal{J}$ for any $r \in E^{\#}$.

<u>Theorem 1.4.7</u>: Let f be a lower semi-continuous function from $(X, \mathcal{J})$ into $E^{\#}$ and K a compact subset of X. Let $S = \inf_{x \in X} \{f(x)\}$. Then there exists $x_o \in K$ such that $f(x_o) = S$.

<u>Proof:</u> We first construct a sequence $\{x_n\}$ on K such that $\lim_{n \to \infty} f(x_n) = S$. Since K is compact, the sequence $\{x_n\}$ possesses a subsequence $\{x_{n(i)}\}$ which converges to a point $x_o \in K$. Hence $\lim_{i \to \infty} f(x_{n(i)}) = S$ and $\lim_{i \to \infty} x_{n(i)} = x_o$. For a general f, $S \leqslant f(x_o)$, but since f is lower semi continuous $f(x_o) \leqslant \lim_{\to \infty} f(x_{n(i)}) = S$

and the theorem follows.

<u>Corollary</u>: An upper semi-continuous function attains its supermum on a compact subset of a topological space $(X, \mathcal{J})$.

We now consider a fundamental structure theorem pertaining to lower semicontinuous functions, ([19] , p.36).

<u>Theorem 1.4.8</u>: Let $f \geqslant 0$ be a lower semi-continuous function on a metric space (X,d). Then there exists a monotone non decreasing sequence $\{\varphi_n\}$ of continuous functions from (X, d) into E^1 such that $f(x) = \lim_{n \to \infty} \varphi_n(x)$ for every $x \in X$.

<u>Proof:</u> We shall assume that $f \not\equiv +\infty$. For each $x \in X$, we define $\varphi_n(x) = \inf_{y \in X} \{f(y) + nd(x,y)\}$, and note that $\varphi_n(x)$ is finite

for every $x \in X$, and that $\varphi_{n+1}(x) \geq \varphi_n(x)$ for every $x \in X$. We shall first show that φ_n is a continuous function. Let $x \in X$, $x' \in X$ such that $d(x,x') = \delta$. Then there exists $y_o(\delta) \in X$ such that $\varphi_n(x) + \delta > f(y_o) + nd(x,y_o)$ by definition of $\varphi_n(x)$. But $\varphi_n(x') < f(y_o) + nd(x',y_o) < f(y_o) + n(d(x',x)+d(x,y_o))$ by the triangle inequality. Hence $\varphi_n(x') < f(y_o) + nd(x,y_o) + n\delta < (\varphi_n(x) + \delta) + n\delta$. It follows that $(\varphi_n(x') - \varphi_n(x)) < (n+1)\, d(x,x')$. By a similar reasoning process we can show that $(\varphi_n(x) - \varphi_n(x')) < (n+1)\, d(x,x')$ and hence $|\varphi_n(x) - \varphi_n(x')| < (n+1)\, d(x,x')$. The continuity of φ_n follows and hence $\{\varphi_n\}$ is a monotone sequence of continuous functions. It remains to be shown that $f(x) = \lim_{n\to\infty} \varphi_n(x)$ for each $x \in X$. Since $\varphi_n(x) \leq (f(y) + nd(x,y))$ for any $y \in X$. It follows that

$$\varphi_n(x) \leq f(x) + nd(x,x), \text{ or } \varphi_n(x) \leq f(x) \text{ for } n = 1, 2, \ldots$$

Since f is lower semi-continuous at $x \in X$, it follows that for any $\epsilon > 0$, $f(x') > f(x) - \epsilon$ where x' lies in some ρ-neighbourhood of x where ρ depends on ϵ. Hence $f(x') > f(x) - \epsilon$ if $d(x',x) < \rho$ and we note that $f(x') + nd(x,x') > n\rho$ if $d(x,x') \geq \rho$. For fixed $x \in X$, we can choose n sufficiently large so that $n\rho > f(x) - \epsilon$. For such an n it follows that $f(x') + nd(x,x') > f(x) - \epsilon$, for every $x' \in X$, and hence $\varphi_n(x) \geq f(x) - \epsilon$ because $\varphi_n(x) = \inf_{x' \in X} \{f(x') + nd(x,x')\}$. Since $\epsilon > 0$ is arbitrary, therefore, $\lim_{n\to\infty} \varphi_n(x) \geq f(x)$, but f(x) is an upper bound for $\{\varphi_n(x)\}$ and therefore $f(x) \geq \lim_{n\to\infty} \varphi_n(x)$. It follows that $f(x) = \lim_{n\to\infty} \varphi_n(x)$ and the theorem is proved.

The remaining theorems of this section will involve convergence theory applied to measure and integration. We observe from theorem 1.4.6 that if f is the characteristic function on a set $A \subset X$ where $(X, \mathcal{J})$ is a topological space, then $f = \chi_A$ is lower semi-continuous if and only if A is open, and f is upper semi-continuous if and only if A is closed.

Theorem 1.4.9: ([14] , p.112) (Monotone convergence theorem)
Let $(X, \mathcal{S}, \mu)$ be a measure space and $\{f_n\}$ a monotone non-decreasing sequence of extended real valued functions each of which is integrable. Then $f = \lim_{n\to\infty} f_n$ is integrable where convergence is point wise and $\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu$ where we allow the possibility that $\int_X f\,d\mu = +\infty$.

Theorem 1.4.10:([14] , p.110) (Dominated convergence theorem)
Let $(X, \mathcal{S}, \mu)$ be a measure space and $\{f_n\}$ a sequence of μ-integrable functions which converges to f almost everywhere with respect to μ. If there exists a μ-integrable function denoted by g such that $|f_n(x)| \leq |g(x)|$ almost everywhere for all n, then f is integrable and $\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu$.

Remark: Let A be a bounded subset of E' and m_1 the Lebesgue measure on E'. A set $B \subset E'$ is m_1-measurable if and only if its characteristic function χ_B is m_1-measurable and it follows that $m_1(B) = \int_{E'} \chi_{(B)}\, dm_1$, where χ_B is the characteristic function of B and the outer Lebesgue measure of A is $m_1^*(A) = \overline{\int_{E'}} \chi_A\, dm_1$ where $\overline{\int_{E'}} \chi_A dm_1 = \inf_\alpha \left\{ \int_{E'} (\chi_{B_\alpha})\, dm_1 \right\}$ where $\{B_\alpha\}$ is the family of m_1-measurable sets such that $B_\alpha \supset A$ and hence $\chi_{B_\alpha} \geqslant \chi_A$. But $\overline{\int_{E'}} \chi_A\, dm_1 = \inf_\alpha \left\{ \int_{E'} (\chi_{O_\alpha})\, dm_1 \right\}$ where $\{O_\alpha\}$ is the family of open sets such that each $O_\alpha \supset A$, and hence $\chi_{O_\alpha} \geqslant \chi_A$. Since every χ_{O_α} is lower semi-continuous and since every χ_B which is lower semi-continuous is the characteristic function of an open set, the following theorem holds.

Theorem 1.4.11: If $A \subset E'$, it follows that $m_1^*(A) = \overline{\int_{E'}} (\chi_A)\, dm_1 = \inf_\alpha \left\{ \int_{E'} \varphi_\alpha\, dm_1 \right\}$ where $\{\varphi_\alpha\}$ is the family of lower semi-continuous functions each of which dominates χ_A, that is $\varphi_\alpha \geqslant \chi_A$ for every α.

Proof: The family $\{\varphi_\alpha\}$ contains the family $\{\chi_{O_\alpha}\}$ where O_α is any open set containing A, and therefore $\inf_\alpha \left\{ \int_{E'} \varphi_\alpha\, dm_1 \right\} \leqslant \inf_\alpha \left\{ \int_{E'} (\chi_{O_\alpha}) dm_1 \right\} = m_1^*(A)$.

For any φ_α we define φ'_α as follows:

$$\varphi'_\alpha(x) = 1, \text{ if } \varphi_\alpha(x) \geqslant 1.$$

$$\varphi'_\alpha(x) = 0, \text{ if } \varphi_\alpha(x) < 1.$$

Since $[1, +\infty] = \bigcap_{n=1}^{\infty} (1 - \frac{1}{n}, +\infty]$, therefore $\varphi_\alpha^{-1}[1 + \infty] = \bigcap_{n=1}^{\infty} \varphi_\alpha^{-1} (1 - \frac{1}{n}, +\infty]$ is a G_δ set and hence φ'_α is a simple, Borel measurable function. Since $\varphi'_\alpha \leq \varphi_\alpha$, it follows that $m_1^*(A) = \inf_\alpha \left\{ \int_{E'} \varphi'_\alpha \, dm_1 \right\} \leq \inf_\alpha \left\{ \int_{E'} \varphi_\alpha \, dm_1 \right\}$. If we combine the two inequalities the theorem follows.

<u>Corollary</u>: Let f be a simple function relative to m_1. Then $\int_{E'} f dm_1 = \inf_\alpha \left\{ \int_{E'} \varphi_\alpha dm_1 \right\}$ where $\{\varphi_\alpha\}$ is the family of lower semi-continuous functions each member of which dominates f.

<u>Corollary</u>: Let f be any function from E' into $E^{\#}$. Then $\overline{\int_{E'}} f dm_1 = \inf_\alpha \left\{ \int_{E'} \varphi_\alpha \, dm_1 \right\}$ where $\{\varphi_\alpha\}$ is the family of lower semi-continuous functions each member of which dominates f.

General Remarks for this Section

Let $(X, \mathfrak{J})$ be a compact T_2 space, and C the Banach space of continuous real valued functions on X. Then every positive linear functional on C, which necessarily is a member of C* also, may by virtue of the Riesz representation theorem be identified with a measure μ on X, which Bourbaki calls a Radon measure. Since the continuous functions on X are μ-integrable and hence μ-measurable it follows that the open sets in $(X, \mathfrak{J})$ are in the σ-algebra $\mathcal{S}$ of the domain of μ, and hence $\mathcal{S}$ is either the Borel sets with respect to $(X, \mathfrak{J})$ or some σ-algebra containing the Borel sets. Hence, any lower semi-continuous function f from $(X, \mathfrak{J})$ into $E^{\#}$ is μ-measurable and if $f \geq 0$, then f is μ-integrable allowing the possibility

$\int_X f d\mu = +\infty$. In fact $\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$ where $\{f_n\}$ is a monotone non-decreasing sequence of continuous functions which converges pointwise to f. Finally if g is any function from X into $E^{\#}$, one can define $\bar{\int_X} g\, d\mu = \inf_{\alpha} \left\{\int_X \varphi_\alpha\, d\mu\right\}$ where $\{\varphi_\alpha\}$ is the family of lower semi-continuous functions each of which dominates g on X, and $\underline{\int}_{(x)} g d\mu = \sup_{\alpha} \left\{ \int_X \psi_\alpha\, d\mu \right\}$ where $\{\psi_\alpha\}$ is the family of upper semi-continuous functions each of which is less than or equal to g on X. These definitions can be made consistent with the earlier ones in section 2 of this chapter by extending the argument of theorem 1.4.11 into more general spaces. Then the function g is μ-integrable if and only if $\underline{\int} g\, d\mu = \bar{\int} g\, d\mu$ and if g is μ-integrable, it is μ-measurable. In the case where g is bounded, then g is μ-integrable if and only if it is μ-measurable.

II. SOME FUNDAMENTAL RESULTS OF HARMONIC AND SUBHARMONIC FUNCTION THEORY

§1 Aspects of harmonic function theory

Definition 2.1.1: A function $u \in C^2$ defined on a region $R \subset E^n$ and satisfying the equation $\nabla^2 u = 0$ at all points of R is said to be harmonic there.

Laplace's partial differential equation is a fundamental object of study in this thesis. Indeed, the class of solutions of it defined in $E^n - \{0\}$ which depend only on the distance r from the origin, play a major role in our development. We prove the following theorem.

Theorem 2.1.1: Let u depend on r alone and satisfy Laplace's partial differential equation on $E^n - \{0\}$. Then $u = a \log r + b$ for $n = 2$ where a and b are real constants, and $u = \frac{c}{r^{n-2}} + d$ for $n \geq 3$ where where c and d are real constants.

Proof: Since $\nabla^2 u = 0$ in $E^n - \{0\}$, $n \geq 2$, and u is a function of r alone, where $r^2 = \sum_{i=1}^{n} x_i^2$, we have $2r \frac{\partial r}{\partial x_i} = 2 x_i$ and hence

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \text{for } i = 1, \ldots, n.$$

Therefore $\frac{\partial u}{\partial x_i} = (\frac{\partial u}{\partial r})(\frac{\partial r}{\partial x_i}) = u'(r)(\frac{x_i}{r})$ since $\frac{\partial u}{\partial r} = \frac{du}{dr}$,

and $\frac{\partial^2 u}{\partial x_i^2} = u''(\frac{x_i}{r})^2 + u'(\frac{r - x_i(\frac{x_i}{r})}{r^2})$ for $i = 1, .., n$

$$= u''(\frac{x_i^2}{r^2}) + \frac{u'}{r} - \frac{u'x_i^2}{r^3}.$$

Then $\nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{u''}{r^2}(\sum_{i=1}^{n} x_i^2) + \frac{nu'}{r} - \frac{u'}{r^3}(\sum_{i=1}^{n} x_i^2)$

$$= u''(\frac{r^2}{r^2}) + \frac{nu'}{r} - \frac{u'}{r} = u'' + \frac{(n-1)u'}{r}.$$

If $\nabla^2 u = 0$, then $u'' + \frac{(n-1)u'}{r} = 0$ which is a second order ordinary differential equation. If we let $u' = V$, then $V' + \frac{(n-1)V}{r} = 0$ is a first order linear differential equation. Multiplying the last equation by the integrating factor $e^{\int \frac{n-1}{r} dr} = r^{n-1}$, we have $V' r^{n-1} + (n-1) r^{n-2} V = 0$ and hence $(Vr^{n-1})' = 0$ which implies $V = \frac{a}{r^{n-1}}$ if $n \geq 2$, <u>a</u> a real constant. Thus $u'(r) = \frac{a}{r^{n-1}}$ if $n \geq 2$.

<u>Case 1</u>: $n = 2$: then $u'(r) = \frac{a}{r}$ and hence $u(r) = a \log r + b$, b a real constant.

<u>Case 2</u>: $n \geq 3$ then $u'(r) = ar^{1-n}$ and hence $u(r) = a \int r^{1-n} dr$

$$= \frac{a}{1-n} r^{2-n} + d$$

$$= \frac{c}{r^{n-2}} + d \text{ where}$$

c and d are real constants. The theorem follows.

Returning to our Green calculus and its relationship to harmonic functions, we prove the following theorem.

Theorem 2.1.2: Let R be a Jordan region in E^n whose boundary ∂R is a Jordan manifold which is smooth almost everywhere (σ_{n-1} measure), and let $u \in C^2$. Then $\int_R (\nabla^2 u)\, dm_n = \int_{\partial R} (\frac{\partial u}{\partial \hat{n}})\, d\sigma_{n-1}$ where we recall that σ_{n-1} is the classical hypersurface measure on ∂R.

Proof: We recall that Green's second identity states that

$$\int_R (u\nabla^2 V - V\nabla^2 u)\, dm_n = \int_{\partial R} (u(\frac{\partial V}{\partial \hat{n}}) - V(\frac{\partial u}{\partial \hat{n}}))\, d\sigma_{n-1}$$

where u and V are both C^2 functions on $R \cup \partial R$. If $V \equiv 1$ on R, then $\int_R (\nabla^2 u)\, dm_n = \int_{\partial R} (\frac{\partial u}{\partial \hat{n}})\, d\sigma_{n-1}$.

The following corollary is then immediate:

Corollary: Under the hypothesis of theorem 2.1.2, if u is harmonic, then $\int_{\partial R} (\frac{\partial u}{\partial \hat{n}})\, d\sigma_{n-1} = 0$.

Theorem 2.1.3: (Green's third identity for E^2). Let $u \in C^2$ and be defined on a Jordan region $R \subset E^2$ where $\vec{0} \in R$, and where ∂R is a rectifiable Jordan curve which is smooth a.e(σ_1). Then

$$2\pi\, u(\vec{0}) = \int_{\partial R} \left[(\log \frac{1}{r})(\frac{\partial u}{\partial \hat{n}}) - u \frac{\partial(\log \frac{1}{r})}{\partial \hat{n}}\right] d\sigma_1 - \int_R \log (\frac{1}{r})(\nabla^2 u)\, dm_2 .$$

Proof: Let $\bar{B}_\varepsilon = \bar{B}_\varepsilon(\vec{0}) = \{\vec{x} : |\vec{x}| \leq \varepsilon\}$, $S_\varepsilon = \{\vec{x} : |\vec{x}| = \varepsilon\}$ and $R' = R - \bar{B}_\varepsilon$ where $\bar{B}_\varepsilon \subset R$. Then $\partial R' = \partial R \cup S_\varepsilon$. From Green's second identity we have $\int_{R'} (u\nabla^2 V - V\nabla^2 u)\, dm_2 = \int_{\partial R'} \left[u(\frac{\partial V}{\partial \hat{n}}) - V(\frac{\partial u}{\partial \hat{n}}) \right] d\sigma_1$

where $\hat{n}$ is outer normal relative to R' and u and V are both C^2 functions on R' $\partial R'$. But

$$\int_{\partial R'} \left[u(\frac{\partial V}{\partial \hat{n}}) - V(\frac{\partial u}{\partial \hat{n}}) \right] d\sigma_1 = \int_{\partial R} \left[u(\frac{\partial V}{\partial \hat{n}}) - V\frac{\partial u}{\partial \hat{n}} \right] d\sigma_1 + \int_{S_\varepsilon} \left[u(\frac{\partial u}{\partial \hat{n}}) - V(\frac{\partial u}{\partial \hat{n}}) \right] d\sigma_1 .$$

Therefore

$$\int_{R'} (u\nabla^2 V - V\nabla^2 u)\, dm_2 = \int_{\partial R} \left[u(\frac{\partial V}{\partial \hat{n}}) - V(\frac{\partial u}{\partial \hat{n}}) \right] d\sigma_1 - \int_{S_\varepsilon} \left[u(\frac{\partial V}{\partial r}) - V(\frac{\partial u}{\partial r}) \right] d\sigma_1$$

because outer normal relative to R' on S_ε is inner normal to B_ε on S_ε.

The function V = log r is harmonic on R' and therefore

$$\int_{R'} 0 - (\log r)\,(\vec{\nabla}^2 u)\, dm_2 = \int_{\partial R} \left[u\,\frac{\partial(\log r)}{\partial \hat{n}} - (\log r)\,\frac{\partial u}{\partial \hat{n}} \right] d\sigma_1 - \int_{S_\varepsilon} \left[u\,\frac{1}{\varepsilon} - (\log \varepsilon)(\frac{\partial u}{\partial r}) \right] d\sigma_1 .$$

We analyze the term $\int_{S_\varepsilon} (\frac{u}{\varepsilon})\, d\sigma_1 = \frac{1}{\varepsilon}\int_0^{2\pi} u\,(\vec{x}_\varepsilon)\,\varepsilon\, d\theta$ where $|x| = \varepsilon$.

Then $\int_0^{2\pi} u(\vec{x}_\varepsilon)\,d\theta = \int_0^{2\pi} \left[u(\vec{x}_\varepsilon) - u(\vec{0}) \right] d\theta + \int_0^{2\pi} u(\vec{0})\, d\theta$ and hence

$$\left| \int_0^{2\pi} u(\vec{x}_\varepsilon)\, d\theta - 2\pi u(\vec{0}) \right| \leqslant \int_0^{2\pi} \left| u(\vec{x}_\varepsilon) - u(\vec{0}) \right| d\theta \leqslant M_\varepsilon \int_0^{2\pi} d\theta \text{ where}$$

$$M_\varepsilon = \sup_{\vec{x}_\varepsilon \in S_\varepsilon} \left| u\,(\vec{x}_\varepsilon) - u(\vec{0}) \right| .$$

Since u is continuous at $\vec{0}$, therefore $\lim_{\varepsilon \to 0} M_\varepsilon = 0$, and hence

$$2\pi u(\vec{0}) = \lim_{\varepsilon \to 0} \left\{ \int_{S_\varepsilon} \frac{u}{\varepsilon}\, d\sigma_1 \right\} .$$

Similarly, we consider the term $\int_{S}(\log\varepsilon)(\frac{\partial u}{\partial r})\,d\sigma_1$.

From Green's second identity with $V \equiv 1$, we have

$(\log\varepsilon)\int_{S_\varepsilon}(\frac{\partial u}{\partial r})\,d\sigma_1 = \log\varepsilon\int_{B_\varepsilon}(\nabla^2 u)\,dm_2$ and since $(\nabla^2 u)$ is continuous and hence bounded on $\bar{B}_\varepsilon$, therefore $|\nabla^2 u| < M$ on $\bar{B}_\varepsilon$.

It follows that $\left|\log\varepsilon\int_{S_\varepsilon}\frac{\partial u}{\partial r}\,d\sigma_1\right| \leq M\left|\log\varepsilon\cdot\pi\varepsilon^2\right|$ for $\varepsilon > 0$

and since $\lim_{\varepsilon\to 0}\varepsilon^2\log\varepsilon = \lim_{\varepsilon\to 0}\left(\frac{\log\varepsilon}{\frac{1}{\varepsilon^2}}\right) = \lim_{\varepsilon\to 0}\frac{\frac{1}{\varepsilon}}{\left(-\frac{2}{\varepsilon^3}\right)} = 0$ by L'Hôpital's rule, therefore $\lim_{\varepsilon\to 0}\int_{S_\varepsilon}(\log\varepsilon)(\frac{\partial u}{\partial r})\,d\sigma_1 = 0$.

Hence

$$\lim_{\varepsilon\to 0}\int_{R'}-(\log r)(\nabla^2 u)\,dm_2 = \int_{\partial R}\left[u\left(\frac{\partial(\log r)}{\partial\hat{n}}\right)-\log r\left(\frac{\partial u}{\partial\hat{n}}\right)\right]d\sigma_1 - \lim_{\varepsilon\to 0}\int_{S_\varepsilon}\frac{u}{\varepsilon}\,d\sigma_1 + \lim_{\varepsilon\to 0}\int_{S_\varepsilon}\log\varepsilon\left(\frac{\partial u}{\partial r}\right)d\sigma_1$$

and therefore

$$\int_{R}\log\left(\frac{1}{r}\right)(\nabla^2 u)\,dm_2 = \int_{\partial R}\left[\log\left(\frac{1}{r}\right)\left(\frac{\partial u}{\partial\hat{n}}\right) - u\left(\frac{\partial\log(\frac{1}{r})}{\partial\hat{n}}\right)\right]d\sigma_1 - 2\pi u(\vec{0})$$

or

$$2\pi u(\vec{0}) = \int_{\partial R}\left[\log\left(\frac{1}{r}\right)\left(\frac{\partial u}{\partial\hat{n}}\right) - u\left(\frac{\partial\log(\frac{1}{r})}{\partial\hat{n}}\right)\right]d\sigma_1 - \int_{R}\log\left(\frac{1}{r}\right)(\nabla^2 u)\,dm_2 .$$

The following corollary is then immediate.

<u>Corollary:</u> Under the hypothesis of theorem 2.1.3, if u is also harmonic, then

$$u(\vec{0}) = \frac{1}{2\pi}\int_{\partial R}\left[\log\left(\frac{1}{r}\right)\left(\frac{\partial u}{\partial\hat{n}}\right) - u\left(\frac{\partial\log(\frac{1}{r})}{\partial n}\right)\right]d\sigma_1 .$$

Theorem 2.1.4: ([13] , p.74-75) Let $\overline{B}_1(\vec{0})$ be the closed unit ball in E^n and $S_1(\vec{0})$ its boundary. Then $\int_{\overline{B}_1(\vec{0})} dm_n = V_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$

and $\int_{S_1(\vec{0})} d\sigma_{n-1} = s_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ where $\Gamma(x) = \int_0^\infty e^{-t}\, t^{(x-1)}\, dt$

for $x > 0$ and $\Gamma(n+1) = n!$ if n is a natural number.

We consider now, the n-dimensional analogue of theorem 2.1.3. for $n \geq 3$.

Theorem 2.1.5:(Green's third identity for E^n, $n \geq 3$). Let $u \in C^2$ be defined on a Jordan region $R \subset E^n$ where $\vec{0} \in R$ and where ∂R is an (n-1) Jordan manifold which is sufficiently smooth and where $\int_{\partial R} d\sigma_{n-1} < +\infty$.

Then

$$(n-2)\, s_{n-1}\, u(\vec{0}) = \int_{\partial R}\left[\left(\frac{1}{r^{n-2}}\right)\left(\frac{\partial u}{\partial \hat{n}}\right) - u\,\frac{\partial\left(\frac{1}{r^{n-2}}\right)}{\partial \hat{n}}\right] d\sigma_{n-1} - \int_R \left(\frac{1}{r^{n-2}}\right)(\nabla^2 u)\, dm_n.$$

Proof: We employ similar notation to that used in theorem 2.1.3.

Then $\int_{R'} \left[u(\nabla^2 V) - V(\nabla^2 u)\right] dm_n = \int_{\partial R'}\left[u\left(\frac{\partial V}{\partial \hat{n}}\right) - V\left(\frac{\partial u}{\partial \hat{n}}\right)\right] d\sigma_{n-1}$ where $\hat{n}$ is the outer unit normal relative to R' and where

$$\int_{\partial R'}\left[u\left(\frac{\partial V}{\partial \hat{n}}\right) - V\left(\frac{\partial u}{\partial \hat{n}}\right)\right] d\sigma_{n-1} = \int_{\partial R}\left[u\left(\frac{\partial V}{\partial \hat{n}}\right) - V\left(\frac{\partial u}{\partial \hat{n}}\right)\right] d\sigma_{n-1} - \int_{S_\epsilon}\left[u\left(\frac{\partial V}{\partial r}\right) - V\left(\frac{\partial u}{\partial r}\right)\right] d\sigma_{n-1}.$$

Let $V = \frac{1}{r^{n-2}}$ which is harmonic in R' by theorem 2.1.1.

Then

$$\int_{R'}\left[(u)(0) - \left(\frac{1}{r^{n-2}}\right)(\nabla^2 u)\right] dm_n = \int_{\partial R}\left[u\,\frac{\partial(r^{2-n})}{\partial \hat{n}} - (r^{2-n})\left(\frac{\partial u}{\partial \hat{n}}\right)\right] d\sigma_{n-1} - \int_{S_\epsilon}\left[u\,\frac{\partial(r^{2-n})}{\partial r} - (r^{2-n})\left(\frac{\partial u}{\partial r}\right)\right]_{r=\epsilon} d\sigma_{n-1}$$

Consider the first term

$$\int_{S_\varepsilon}\left[r^{2-n}\left(\frac{\partial u}{\partial r}\right)\right]_{r=\varepsilon} d\sigma_{n-1} = \frac{1}{\varepsilon^{n-2}}\int_{S_\varepsilon}\left(\frac{\partial u}{\partial r}\right) d\sigma_{n-1} = \frac{1}{\varepsilon^{n-2}}\int_{\bar{B}_\varepsilon}(\nabla^2 u)\,dm_n$$

by theorem 2.1.2.

Since $\nabla^2 u$ is continuous on $\bar{B}_\varepsilon$ and hence its absolute value is bounded there by M say, therefore

$$\frac{1}{\varepsilon^{n-2}}\left|\int_{\bar{B}_\varepsilon}(\nabla^2 u)\,dm_n\right| \leqslant \frac{M\varepsilon^n}{\varepsilon^{n-2}}V_n = MV_n\,\varepsilon^2.$$

It follows that $\lim\limits_{\varepsilon\to 0}\int_{S_\varepsilon}\left[r^{2-n}\left(\frac{\partial u}{\partial r}\right)\right]_{r=\varepsilon} d\sigma_{n-1} = \lim\limits_{\varepsilon\to 0} MV_n\varepsilon^2 = 0.$

Now consider term $\int_{S_\varepsilon}\left[u\,\frac{\partial(r^{2-n})}{\partial r}\right]_{r=\varepsilon} d\sigma_{n-1} = \int_{S_\varepsilon} u(2-n)\,\varepsilon^{1-n}\,d\sigma_{n-1}$

where $\vec{x}_\varepsilon \in S_\varepsilon$.

Then

$$\frac{1}{\varepsilon^{n-1}}\int_{S_\varepsilon} u(\vec{x}_\varepsilon)\,d\sigma_{n-1} = \frac{1}{\varepsilon^{n-1}}\int_{S_1}\varepsilon^{n-1}\,u(\vec{x}_\varepsilon)\,d\sigma_{n-1} =$$

$$= \int_{S_1}\left[u(\vec{x}_\varepsilon) - u(\vec{0})\right] d\sigma_{n-1} + \int_{S_1} u(\vec{0})\,d\sigma_{n-1} = (s_{n-1})u(\vec{0}) + \int_{S_1}\left[u(\vec{x}_\varepsilon) - u(\vec{0})\right] d\sigma_{n-1}\,.$$

Since u is continuous, therefore $|u(\vec{x}_\varepsilon) - u(\vec{0})| \leqslant M_\varepsilon$ where $\lim\limits_{\varepsilon\to 0} M_\varepsilon = 0$,

and hence

$$\left|\int_{S_1}\left[u(\vec{x}_\varepsilon) - u(\vec{0})\right] d\sigma_{n-1}\right| \leqslant \int_{S_1}\left|u(\vec{x}_\varepsilon) - u(\vec{0})\right| d\sigma_{n-1} \leqslant (M_\varepsilon)(s_{n-1})$$ and hence

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{n-1}} \left(\int_{S_\varepsilon} u \, d\sigma_{n-1} \right) = (s_{n-1}) \, u(\vec{0}).$$

Thus

$$\lim_{\varepsilon \to 0} \int_{R'} \left[0 - \left(\frac{1}{r^{n-2}}\right)(\nabla^2 u) \right] dm_n =$$

$$\int_{\partial R} \left[u \frac{\partial(r^{2-n})}{\partial \hat{n}} - (r^{2-n}) \left(\frac{\partial u}{\partial \hat{n}}\right) \right] d\sigma_{n-1} - \lim_{\varepsilon \to 0} \int_{S_\varepsilon} u \left(\frac{\partial r^{2-n}}{\partial r}\right) d\sigma_{n-1} + \lim_{\varepsilon \to 0} \int_{S_\varepsilon} r^{2-n} \left(\frac{\partial u}{\partial r}\right) d\sigma_{n-1}$$

Therefore

$$(n-2) \, s_{n-1} \, u(\vec{0}) = \int_{\partial R} \left[(r^{2-n}) \left(\frac{\partial u}{\partial \hat{n}}\right) - u \left(\frac{\partial r^{2-n}}{\partial \hat{n}}\right) \right] d\sigma_{n-1} - \int_R (r^{2-n}) (\nabla^2 u) \, dm_n$$

and the theorem follows.

The following corollary is then immediate.

<u>Corollary</u>: Under the hypothesis of theorem 2.1.5, if u is also harmonic, then

$$u(\vec{0}) = \frac{1}{(n-2) s_{n-1}} \int_{\partial R} \left[r^{2-n} \left(\frac{\partial u}{\partial \hat{n}}\right) - u \left(\frac{\partial r^{2-n}}{\partial \hat{n}}\right) \right] d\sigma_{n-1} \, .$$

<u>Theorem 2.1.6</u>: (Mean value theorem). Let u be harmonic on the ball $\bar{B}_\rho(\vec{0}) = \bar{B}_\rho$. Then $u(\vec{0}) = \frac{1}{s_{n-1}} \int_{S_1} u(\vec{x}_\rho) \, d\sigma_{n-1}$ for $\vec{x}_\rho \in S_\rho$.

<u>Proof</u>: <u>Case 1</u>: $n = 2$, in which case $s_1 = 2\pi$

From the corollary of theorem 2.1.3 it follows that

$$u(\vec{0}) = \frac{1}{s_1} \int_{S_\rho} \left[\log\left(\frac{1}{r}\right)\left(\frac{\partial u}{\partial r}\right) - u \left(\frac{\partial \log\left(\frac{1}{r}\right)}{\partial r}\right) \right]_{r=\rho} d\sigma_1$$

$$= \frac{1}{s_1} \left(\log \frac{1}{\rho}\right) \int_{S_\rho} \left(\frac{\partial u}{\partial r}\right) d\sigma_1 + \frac{1}{s_1} \int_{S_\rho} u \left(\frac{\partial \log r}{\partial r}\right)\Bigg|_{r=\rho} d\sigma_1 .$$

By the corollary of theorem 2.1.2, it follows that

$$\int_{S_\rho} \left(\frac{\partial u}{\partial r}\right) d\sigma_1 = 0 \text{ and therefore } u(\vec{0}) = \frac{1}{\rho s_1} \int_{S_\rho} u(\vec{x}_\rho)\, d\sigma_1$$

$$= \frac{1}{\rho s_1} \int_{S_1} u(\vec{x}_\rho)\rho\, d\sigma_1$$

$$= \frac{1}{s_1} \int_{S_1} u(\vec{x}_\rho)\, d\sigma_1 \quad \text{which}$$

is often called the <u>linear mean</u> of u on S_ρ .

<u>Case 2</u>: $n \geq 3$.

From the corollary of theorem 2.1.5 it follows that

$$u(\vec{0}) = \frac{1}{(n-2)s_{n-1}} \int_{S_\rho} \rho^{2-n} \left(\frac{\partial u}{\partial r}\right) d\sigma_{n-1} - \frac{1}{(n-2)s_{n-1}} \int_{S_\rho} u(\vec{x}_\rho)(2-n)\, \rho^{1-n}\, d\sigma_{n-1} .$$

But by the corollary of theorem 2.1.2, the first term on the right of the last equation is zero. Therefore cancelling the (n-2) factor, we get

$$u(\vec{0}) = \frac{1}{\rho^{n-1} s_{n-1}} \int_{S_1} u(\vec{x}_\rho)\, \rho^{n-1}\, d\sigma_{n-1} = \frac{1}{s_{n-1}} \int_{S_1} u(\vec{x}_\rho)\, d\sigma_{n-1}$$

and the theorem follows.

Throughout the remainder of this thesis we shall denote the integral $\frac{1}{s_{n-1}} \int_{S_1} u(\vec{x}_\rho)\, d\sigma_{n-1}$ by $L(u, \vec{0}, \rho)$. We will observe that the origin point $\vec{0}$ was taken as the centre of S, in the above discussions, but this was merely done so for convenience as any such point $\vec{x}_0 \in E^n$ would suffice by a translation and S_ρ would then be defined by $S_\rho = \left\{ \vec{x} : \left| \vec{x} - \vec{x}_o \right| = \rho \right\}$. We should also observe that $\frac{1}{s_{n-1}} \int_{S_1} u(\vec{x}_\rho)\, d\sigma_{n-1}$ may be replaced by the equivalent expression

$$\frac{\int_{S_\rho} u(\vec{x}_\rho)\, d\sigma_{n-1}}{\int_{S_\rho} d\sigma_{n-1}}$$

which is a more apt notation for mean value.

We come now to a stronger form of theorem 2.1.6.

<u>Theorem 2.1.7</u>: (Extended form of the mean value theorem).

Let u be a harmonic on the ball $B_\rho(\vec{O}) = B_\rho$ and continuous on $\bar{B}_\rho(\vec{O}) = B_\rho \cup S_\rho$. Then $u(\vec{O}) = L(u, \vec{O}, \rho)$.

<u>Proof</u>: For any $r < \rho$, it follows that $u(\vec{O}) = L(u, \vec{O}, \rho)$ by theorem 2.1.6. We shall show that $L(u, \vec{O}, \rho) = \lim_{\substack{r \to \rho \\ r < \rho}} L(u, \vec{O}, r)$ or $\lim_{r \to \rho^-} L(u, \vec{O}, r)$.

Now $L(u, \vec{O}, \rho) = \frac{1}{s_{n-1}} \int_{S_1} u(\vec{x}_\rho)\, d\sigma_{n-1}$ where $\vec{x}_\rho \in S_\rho$ and

$L(u, \vec{O}, r) = \frac{1}{s_{n-1}} \int_{S_1} u(\vec{x}_r)\, d\sigma_{n-1}$ where $\vec{x}_r \in S_r$ and $r < \rho$.

Then $L(u, \vec{O}, \rho) - L(u, \vec{O}, r) = \frac{1}{s_{n-1}} \int_{S_1} \left[u(\vec{x}_\rho) - u(\vec{x}_r) \right] d\sigma_{n-1}$.

If $\vec{x}_\rho$ and $\vec{x}_r$ are on the same ray, then $\left| \vec{x}_\rho - \vec{x}_r \right| = \rho - r$ and since u is uniformly continuous on $\bar{B}_\rho(\vec{O})$, it follows that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\left| u(\vec{x}_\rho) - u(\vec{x}_r) \right| < \varepsilon$ if $\left| \vec{x} - \vec{x}_r \right| < \delta$. If $(\rho - r) < \delta$, then

$$\left| L(u, \vec{O}, \rho) - L(u, \vec{O}, r) \right| < \frac{1}{s_{n-1}} (\varepsilon) \left(\int_{S_1} d\sigma_{n-1} \right) < \varepsilon .$$

Hence, $\lim_{r \to \rho^-} L(u, \vec{0}, r) = u(\vec{0}) = L(u, \vec{0}, \rho)$ and the theorem follows.

<u>Theorem 2.1.8</u>: (Special form of the maximum principle for a ball):
Let u be harmonic on the closed ball $\bar{B}_\rho(\vec{0})$. Then u cannot assume its maximum value relative to $\bar{B}_\rho(\vec{0})$ at 0 unless u is constant.

<u>Proof</u>: From theorem 2.1.6 we have
$$u(\vec{0}) = \frac{\int_{S_r} u(\vec{x}_r)\, d\sigma_{n-1}}{\int_{S_r} d\sigma_{n-1}}$$
for any $r \leqslant \rho$.
If u attains its maximum at $\vec{0}$, then $\int_{S_r} u(\vec{0})\, d\sigma_{n-1} = \int_{S_r} u(\vec{x}_r) d\sigma_{n-1}$ for any $r \leqslant \rho$ and therefore $\int_{S_r} \left[u(\vec{0}) - u(\vec{x}_r) \right] d\sigma_{n-1} = 0$ for any $r \leqslant \rho$ where $\left[u(\vec{0}) - u(\vec{x}_r) \right] \geqslant 0$ on S_r. Since $(u(\vec{0}) - u(\vec{x}_r))$ is continuous on S_r, therefore $u(\vec{0}) - u(\vec{x}_r) \equiv 0$ on S_r.
Since $u(\vec{x}_r) = u(\vec{0})$ for any $\vec{x}_r \in S_r$ and for any $r \leqslant \rho$, therefore $u(\vec{x}) \equiv u(\vec{0})$ on $\bar{B}_\rho(\vec{0})$.

<u>Theorem 2.1.9</u>: (Maximum principle for a region (first form)).
Let u be harmonic on a region $R \subset E^n$ and suppose u attains its maximum value at $\vec{x}_0 \in R$. Then u is a constant function on R.

<u>Proof</u>: We first consider a topological space $(R, \mathcal{J}_R)$ where $\mathcal{J}_R$ is the usual topology of E^n relativized to R. Let $G \subset R$ be defined so that $G = \{ \vec{x} \in R : u(\vec{x}) = u(\vec{x}_0) \}$. Then G is open in the usual topology of E^n by theorem 2.1.8, and hence in $\mathcal{J}_R$ because R is open in E^n. Since u is continuous, therefore $G \subset R$ is a closed subset of

R in relation to $\mathcal{T}_R$ because the singleton set $\{u(\vec{x}_o)\}$ is closed in E' and if u is continuous, then the inverse image of a closed set is closed. Also we note that $G \neq \emptyset$. Now consider the set $G' = R-G$. Since G is closed in $(R, \mathcal{T}_R)$, then G' is open in $(R, \mathcal{T}_R)$ and therefore $R = G \cup G'$ where both G and G' are open in relation to $\mathcal{T}_R$ and $G \cap G' = \emptyset$. Since R is connected, therefore $G' = \emptyset$ for otherwise by definition 1.1.16, R would have to be disconnected. The theorem follows.

Corollary 1: If u is a non-constant harmonic function on a bounded region $R \subset E^n$, which is continuous on $R \cup \partial R$, then u attains its supremum on and only on the boundary ∂R.

Proof: Since u is continuous on the compact set $R \cup \partial R$, therefore u attains its supremum on $R \cup \partial R$. Since u is a non-constant harmonic function, therefore u cannot attain its supremum on R by theorem 2.1.8. Hence u attains its supremum on ∂R.

Corollary 2: (Minimum Principle for a region (first form))
Let u be harmonic on a region $R \subset E^n$ and suppose u attains its minimum value at $\vec{x}_o \in R$. Then u is a constant function on R.

Proof: Similar to theorem 2.1.9.

Corollary 3: If u is a non-constant harmonic function on a bounded region $R \subset E^n$ which is continuous on $R \cup \partial R$, then u attains its infimum on and only on the boundary ∂R.

Proof: Similar to corollary 1.

Theorem 2.1.10: (Maximum principle for a region (second form)

Let R be a bounded region in E^n, and u a harmonic function on R, with the additional condition that

$$\overline{\lim_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R \\ \vec{x}^*\in \partial R}}} u(\vec{x}) \leq M \text{ for every } \vec{x}^*\in \partial R.$$

Then $u(\vec{x}) \leq M$ for all $\vec{x}\in R$.

Proof: Choose any $\varepsilon > 0$ and $\vec{x}^* \in \partial R$. Then there exists a neighbourhood $V(\vec{x}^*)$ of $\vec{x}^*$ such that $u < M + \varepsilon$ in $V(\vec{x}^*) \cap R$. The set $\{V(\vec{x}^*)\}$ form an open covering of ∂R and since ∂R is compact, we can extract out a finite subcovering of $\{V(\vec{x}^*)\}$ denoted by $\{V(\vec{x}^*_i)\}$, $1 \leq i \leq n$. Let $\overline{R}_\varepsilon = R - \bigcup_{i=1}^{n} V(x^*_i)$ and note that the distance between $\partial(\overline{R}_\varepsilon)$ and ∂R is greater than zero. Since $\overline{R}_\varepsilon \subset R$ is compact, therefore u is harmonic on $\overline{R}_\varepsilon$ and $u < M + \varepsilon$ on $\partial\overline{R}_\varepsilon$. By corollary 1 of theorem 2.1.9, $u < M + \varepsilon$ throughout $\overline{R}_\varepsilon$. But $u < M + \varepsilon$ in $(\bigcup_{i=1}^{n} V(\vec{x}^*_i) \subset R$ and hence $u < M + \varepsilon$ throughout R. Since $\varepsilon > 0$ is arbitrary, $u \leq M$ throughout R.

§2 Some elements of subharmonic function theory

General Remarks:

In section 1 of this chapter, we learned that a harmonic function u on a region $R \subset E^n$ possesses on R the mean value property, i.e. if $\overline{B}_{r_o}(\vec{x}_o) \subset R$, then $u(\vec{x}_o) = L(u, \vec{x}_o, r_o)$. In chapter III, we shall show that if u is continuous on R and satisfies the mean value property everywhere on R, then u is harmonic on R. Thus, a harmonic function on R may be characterized in terms of properties involving its integral mean rather than in terms of a solution of Laplace's partial differential equation. For a subharmonic function, we shall find it convenient to employ the integral mean as a fundamental tool for defining such. Our chief source of reference for this section will be Radó's book [16]. Also in this section, all integrals will be regarded as taken with respect to Radon measures.

Definition 2.2.1: Let u be an upper semi-continuous function from a region $R \subset E^n$ into $E^{\#}$. We shall allow u to take on the value $-\infty$ but not $+\infty$. Suppose that for any $\overline{B}_{r_o}(\vec{x}_o) \subset R$, it follows that $u(\vec{x}_o) \leq L(u, \vec{x}_o, r_o)$. Then u is said to be subharmonic on R provided that $u \not\equiv -\infty$.

Definition 2.2.2: A function u from a region $R \subset E^n$ into $E^{\#}$ is defined to be superharmonic if and only if $(-u)$ is subharmonic.

<u>Theorem 2.2.1</u>: Let u be subharmonic on $\overline{B}_{r_o}(\vec{x}_o)$, and suppose that u attains its supremum at $\vec{x}_o$. Then u is constant on $\overline{B}_{r_o}(x_o)$.

<u>Proof</u>: Since $u(\vec{x}_o) \leqslant \frac{1}{n-1} \int_{S_r} u \, d\sigma_{n-1}$, $r \leqslant r_o$ hence $\frac{1}{s_{n-1}} \int_{S_r} \left[u - u(\vec{x}_o) \right] d\sigma_{n-1} \geqslant 0$. If u on $\overline{B}_r(\vec{x}_o)$ attains its supremum at $\vec{x}_o$, then $u(\vec{x}_o) - u(\vec{x}) \geqslant 0$ for any $\vec{x} \in S_r(\vec{x}_o)$, and hence $\frac{1}{s_{n-1}} \int_{S_r} \left[u(\vec{x}_o) - u \right] d\sigma_{n-1} \geqslant 0$. Combining the two inequalities, it follows that $\int_{S_r} \left[u(\vec{x}_o) - u \right] d\sigma_{n-1} = 0$ where $u(\vec{x}_o) - u \geqslant 0$ on S_r. Let $u(\vec{x}_o) - u(\vec{x}) = V(\vec{x})$ on S_r, and note that $V(\vec{x}) \geqslant 0$ is superharmonic on S_r. Then ([14] , p.104), $V(\vec{x}) \equiv 0$ a.e. on S_r with respect to the σ_{n-1} measure, and in particular, the set where V = 0 is dense in S_r. Now let $\vec{x}_1 \in S_r$ and note that $V(\vec{x}_1) \leqslant \underline{\lim}_{\substack{\vec{x} \to \vec{x}_1 \\ \vec{x} \in S_r}} V(\vec{x})$.

Hence $\underline{\lim}_{\substack{x \to x_1 \\ x \in S_r}} V(x) = 0$ since $V(\vec{x}) \geqslant 0$ and the subset of S_r on which V = 0 is dense in S_r. Since V is lower semi-continuous and $V(\vec{x}_1) \geqslant 0$, it follows that $V(\vec{x}_1) = 0$, and since $\vec{x}_1$ is arbitrary, then $V \equiv 0$ on S_r for any $r \leqslant r_o$. Since r is arbitrary, $V \equiv 0$ on $\overline{B}_{r_o}(\vec{x}_o)$ and the theorem is proved.

Theorem 2.2.2: Let u be subharmonic on a region $R \subset E^n$ and suppose that u attains its supremum at $\vec{x}_o \in R$. Then $u \equiv u(\vec{x}_o)$ on R.

Proof: Let $A \subset R$ such that $\vec{x} \in A$ if and only if $u(\vec{x}) = u(\vec{x}_o)$. We shall show that A is open. Let $\vec{x}_1 \in A$ and $\bar{B}_{r_1}(\vec{x}_1) \subset R$. Since u attains its supremum at $\vec{x}_1$, therefore $u \equiv u(\vec{x}_1)$ in $B_{r_1}(\vec{x}_1)$ by theorem 2.2.1 and hence $B_{r_1}(\vec{x}_1) \subset A$. It follows that A is an open subset of R with respect to the relative topology $\mathcal{T}_R$. Now let $B = R - A$. We claim that B is open in $\mathcal{T}_R$ also. Otherwise, if $\vec{x}_1 \in B$, there exists a sequence $\{\vec{x}'_n\}$ in A such that $\lim_{n\to\infty} \vec{x}'_n = \vec{x}_1$ and since u is upper semi-continuous, therefore $u(\vec{x}_1) \geq \lim_{n\to\infty} u(\vec{x}'_n) = u(\vec{x}_o)$. Hence $\vec{x}_1 \in A$ which is a contradiction. It follows that B is open with respect to $\mathcal{T}_R$. Hence $R = A \cup B$ where $A, B \in \mathcal{T}_R$ and $A \cap B = \emptyset$. Since $A \neq \emptyset$, it follows that $B = \emptyset$ since R is connected.

Theorem 2.2.3: Let u be subharmonic in $R \subset E^n$. Then

$$u(\vec{x}_o) = \overline{\lim_{\substack{\vec{x}\to\vec{x}_o \\ \vec{x}\in R}}}\ u(\vec{x}) \text{ for any } \vec{x}_o \in R.$$

Proof: Since u is upper semi-continuous, therefore $u(\vec{x}_o) \geq \overline{\lim}_{\substack{\vec{x}\to\vec{x}_o \\ \vec{x}\in R}} u(\vec{x})$

let $\overline{\lim}_{\substack{\vec{x}\to\vec{x}_o \\ \vec{x}\in R}} u(\vec{x}) = L$. If $u(\vec{x}_o) > L$, then $u(\vec{x}_o) = L + \varepsilon$ where $\varepsilon > 0$.

Since $\overline{\ell im}_{\substack{\vec{x}\longrightarrow\vec{x}_o \\ \vec{x}\in R}} u(\vec{x}) = L$, there exists $B_r(\vec{x}_o) \subset R$ such that $u(\vec{x}) \leq u(\vec{x}_o)$, for any $\vec{x} \in B_r(\vec{x}_o) - \{\vec{x}_o\}$. By theorem 2.2.1, it follows that $u(\vec{x}) \equiv u(\vec{x}_o)$ in $B_r(\vec{x}_o)$ which is a contradiction to the assumption that $u(\vec{x}_o) > \overline{\ell im}_{\substack{\vec{x}\longrightarrow\vec{x}_o \\ \vec{x}\in R}} u(\vec{x})$.

Remark: Theorem 2.2.2 is a form of the maximum principle for sub-harmonic functions. There is an analogous minimum principle for super-harmonic functions; for harmonic functions, both the maximum and minimum principles are valid.

Theorem 2.2.4: If u_1 and u_2 are subharmonic on $R \subset E^n$, then $V = u_1 + u_2$ is subharmonic on R.

Proof: Let $\vec{x}_o \in R$, and note that $u_1(\vec{x}_o) = \overline{\ell im}_{\vec{x}\to\vec{x}_o} u_1(\vec{x})$, $u_2(\vec{x}_o) = \overline{\ell im}_{\vec{x}\to\vec{x}_o} u_2(\vec{x})$. Then $V(\vec{x}_o) = \overline{\ell im}_{\vec{x}\to\vec{x}_o} u_1(\vec{x}) + \overline{\ell im}_{\vec{x}\to\vec{x}_o} u_2(\vec{x}) \geq \overline{\ell im}_{\vec{x}\to\vec{x}_o} (u_1(\vec{x})+u_2(\vec{x}))$.

by theorem 1.4.3, and hence $V(\vec{x}_o) \geq \overline{\ell im}_{x\to x_o} V(\vec{x})$ from which it follows that V is upper semi-continuous at $\vec{x}_o$ and hence on R. For any $\bar{B}_r(\vec{x}_o) \subset R$, it follows that $u_1(\vec{x}_o) \leq L(u_1, \vec{x}_o, r)$ and $u_2(\vec{x}_o) \leq L(u_2, \vec{x}_o, r)$. Hence $V(\vec{x}_o) \leq L(u_1, \vec{x}_o, r) + L(u_2, \vec{x}_o, r) = L(u_1 + u_2, \vec{x}_o, r)$ or $V(\vec{x}_o) \leq L(V, \vec{x}_o, r)$ and the theorem follows.

<u>Theorem 2.2.5</u>: Let u_1, u_2 be subharmonic on R. Then V defined so that $V(\vec{x}) = \sup\{u_1(\vec{x}), u_2(\vec{x})\}$ for each $\vec{x} \in R$, is subharmonic on R.

<u>Proof</u>: We note that $V(\vec{x}_o) = \sup\{u_1(\vec{x}_o), u_2(\vec{x}_o)$. Hence $V(\vec{x}_o) \geq \overline{\lim}_{\vec{x} \to \vec{x}_o} u_1(\vec{x})$ and $V(\vec{x}_o) \geq \overline{\lim}_{x \to x_o} u_2(\vec{x}_o)$. It follows that $V(\vec{x}_o) \geq \overline{\lim}_{x \to x_o} (u_1 \vee u_2)$, so that V is upper semi-continuous. For any $\overline{B}_r(\vec{x}_o) \subset R$, it follows that $u_1(\vec{x}_o) \leq L(u_1, \vec{x}_o, r)$, $u_2(\vec{x}_o) \leq L(u_2, \vec{x}_o, r)$ and therefore $L(V, \vec{x}_o, r)$ is an upper bound for both $u_1(\vec{x}_o)$ and $u_2(\vec{x}_o)$. Hence $V(\vec{x}_o) \leq L(V, \vec{x}_o, r)$ and the theorem follows.

<u>Theorem 2.2.6</u>: Let $\{u_n\}$ be a monotone non-increasing sequence of subharmonic functions on $R \subset E^n$. The pointwise limit function $V = \lim_{n \to \infty} (u_n)$ is either subharmonic on R or else is $\equiv -\infty$.

<u>Proof</u>: We first show that V is upper semi-continuous on R. Let $\vec{x}_o \in R$, and note that $u_n(\vec{x}_o) = \overline{\lim}_{x \to x_o} u_n(\vec{x})$. Since $u_n(\vec{x}) \geq V(\vec{x})$ on R, it follows that $\overline{\lim}_{\vec{x} \to \vec{x}_o} u_n(\vec{x}) \geq \overline{\lim}_{\vec{x} \to \vec{x}_o} V(\vec{x})$. Hence $\overline{\lim}_{\vec{x} \to \vec{x}} V(\vec{x}) \leq V(\vec{x}_o)$ because $V(\vec{x}_o)$ is the greatest lower bound of $\{u_n(\vec{x}_o)\}$. Let $\overline{B}_r(\vec{x}_o) \subset R$ and recall that $u_n(\vec{x}_o) \leq L(u_n, \vec{x}_o, r)$. Then $V(\vec{x}_o)$ is a lower bound of $\{L(u_n, \vec{x}_o, r)\}$ and by the monotone convergence theorem (theorem 1.4.9) it follows that $L(V, \vec{x}_o, r)$ exists and $L(V, \vec{x}_o, r) = \lim_{n \to \infty} L(u_n, \vec{x}_o, r)$, and hence $V(\vec{x}_o) \leq L(V, \vec{x}_o, r)$. The theorem follows.

General Discussion

By application of the Green calculus, one can show that any C^2 function u on $R \subset E^n$ such that $\nabla^2 u \geq 0$ on R is a subharmonic function in the sense of definition 2.2.1. Such subharmonic functions are called smooth and the locally uniform limit of a sequence of smooth subharmonic functions is a continuous subharmonic function. The limit of a monotone decreasing sequence of continuous subharmonic functions will be a general subharmonic function or $\equiv -\infty$, but not necessarily a continuous subharmonic function. A virtue of the general definition of subharmonic function lies in the fact that for any monotone non-increasing sequence of such functions, their pointwise limit function is also subharmonic. Finally, we point out that for any theorem that can be proved about subharmonic functions, an analogous one can be proved about superharmonic functions. We close this section by stating a strengthened form of the maximum principle for subharmonic functions which is analogous to that for harmonic functions (theorem 2.1.10).

Theorem 2.2.7: Let R be a bounded region in E^n, and u a subharmonic function on R, with the additional condition that

$$\overline{\lim_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R \\ \vec{x}^*\in \partial R}}}\ u(\vec{x}) \leq M \text{ for every } \vec{x}^* \in \partial R.$$

Then $u(\vec{x}) \leq M$ for all $\vec{x} \in R$. The proof is analogous to that of theorem 2.1.10.

III. THE CLASSICAL DIRICHLET PROBLEM

§1 Classical Dirichlet Problem for the n-ball and its solution.

Throughout this section we shall generally concentrate on the solution of the "D" problem for a ball of radius 1 and centre $\vec{O}$. We shall first introduce a special kernel function which will be useful, called the Green's function for the Laplacian operator. It will often be necessary to treat the case of n = 2 and $n \geqslant 3$ separately.

Definition 3.1.1: Let $B \subset E^2$ be the open unit disk of centre the origin, and let $\vec{p} \in B$ be fixed, though arbitrary. The Green's function for B with pole at p is defined to be the function $G_{\vec{p}}(\vec{x}) = \log \frac{1}{|\vec{p}-\vec{x}|} + h(\vec{x})$ where h is a harmonic function on B with the property $\lim_{\vec{x} \to \vec{x}^*} G_{\vec{p}}(\vec{x}) = 0$ for every $\vec{x}^* \in \partial B = S$.

Definition 3.1.2: Let $B \subset E^n$, $n \geqslant 3$, be the open n-ball of centre the origin, and let $\vec{p} \in B$ be fixed, though arbitrary. The Green's function for B with pole at $\vec{p}$ is defined to be the function

$$G_{\vec{p}}(\vec{x}) = \frac{1}{|\vec{p} - \vec{x}|^{n-2}} + h(\vec{x})$$

where h is a harmonic function on B with the property $\lim_{x \to x^*} G_{\vec{p}}(\vec{x}) = 0$ for every $x^* \in \partial B = S$.

Remark: In order to construct the Green's function of B, $n \geq 2$, we shall have to make use of the geometrical inverse of a point with respect to a sphere.

Definition 3.1.3: Let $\vec{x} \in B$ where $|\vec{x}| = r$ and $\vec{x} = (x_1, \ldots, x_n)$. Then the geometrical inverse of $\vec{x}$ with respect to the unit sphere, denoted by $(\vec{x})^{-1}$, is by definition the point possessing the properties that

(i) $(\vec{x})^{-1} = \lambda x$ where $\lambda > 0$ is a real number.

(ii) $|(\vec{x})^{-1}|\,|\vec{x}| = 1.$

Remark: Combining properties (i) and (ii), it follows that $\lambda r^2 = 1$ and hence $\lambda = \frac{1}{r^2}$. Thus if $\vec{x} = (x_1, \ldots, x_n)$, then $(\vec{x})^{-1} = (\frac{x_1}{r^2}, \ldots, \frac{x_n}{r^2})$.

Theorem 3.1.1: If $\vec{p} \in B$ such that $|\vec{p}| = r$ and $\vec{q} \in S$ (i.e. $|\vec{q}| = 1$), then

$$\left|\frac{\vec{q} - \vec{p}}{\vec{q} - (\vec{p})^{-1}}\right| = r.$$

Proof: We shall show that $|\vec{q} - \vec{p}|^2 = r^2 |\vec{q} - (\vec{p})^{-1}|^2$ if $|\vec{q}| = 1$. Left side:

$|\vec{q} - \vec{p}|^2 = |\vec{q}|^2 + |\vec{p}|^2 - 2(\vec{q}.\vec{p}) = 1 + r^2 - 2r\cos\phi$ where ϕ is the angle between the vectors $\vec{q}$ and $\vec{p}$.

Right side:

$$r^2 |\vec{q} - (\vec{p})^{-1}|^2 = r^2 (|\vec{q}|^2 + |(\vec{p})^{-1}|^2 - 2\vec{q}.((\vec{p})^{-1}))$$
$$= r^2 (1 + \frac{1}{r^2} - \frac{2}{r} \cos\phi) = r^2 + 1 - 2r\cos\phi$$

The theorem follows.

<u>Theorem 3.1.2</u>: For $B \subset E^2$, the Green's function with pole at $\vec{p} \in B$ is $G_{\vec{p}}(\vec{x}) = \log \frac{1}{|\vec{p} - \vec{x}|} + \log |(\vec{p})^{-1} - \vec{x}| + \log |\vec{p}|$

$$= \log \frac{|\vec{p}||(\vec{p})^{-1} - \vec{x}|}{|\vec{p} - \vec{x}|} \quad \text{if } \vec{p} \neq \vec{0}$$

$$= \log \frac{1}{|\vec{x}|} \quad \text{if } \vec{p} = \vec{0}$$

<u>Proof</u>: The case for $\vec{p} = \vec{0}$ is immediate and if $\vec{p} \neq \vec{0}$ then from theorem 3.1.1, $\frac{|\vec{p} - \vec{x}^*|}{|(\vec{p})^{-1} - \vec{x}^*|} = |\vec{p}|$ if $|\vec{x}^*| = 1$.

Hence $G_{\vec{p}}(\vec{x}) = \log |\vec{p}| + \log \frac{|(\vec{p})^{-1} - \vec{x}|}{|\vec{p} - \vec{x}|}$ since

$$\lim_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} G_{\vec{p}}(\vec{x}) = \log |\vec{p}| + \lim_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} \log \left| \frac{(\vec{p})^{-1} - \vec{x}}{\vec{p} - \vec{x}} \right| = \log |\vec{p}| + \log \left| \frac{(\vec{p})^{-1} - \vec{x}^*}{\vec{p} - \vec{x}^*} \right|$$

$$= \log |\vec{p}| + \log \frac{1}{|\vec{p}|} = 0 \text{ for any } \vec{x}^* \in S.$$

<u>Theorem 3.1.3</u>: Let $B \subset E^n$, $n \geq 3$, $\vec{p} \in B$ with $|\vec{p}| = r$.

Then

$$G_{\vec{p}}(\vec{x}) = \frac{1}{|\vec{x}|^{n-2}} - 1 \text{ if } \vec{p} = 0$$

$$= \frac{1}{|\vec{x} - \vec{p}|^{n-2}} - \frac{1}{r^{n-2}|\vec{x} - (\vec{p})^{-1}|^{n-2}} \text{ if } \vec{p} \neq 0.$$

<u>Proof</u>: For $\vec{p} = \vec{0}$, the theorem is immediate. Otherwise we note that $h(\vec{x}) = -\frac{1}{r^{n-2}|\vec{x} - (\vec{p})^{-1}|^{n-2}}$ is harmonic on B, due to theorem 2.1.1. Also $\frac{|\vec{x}^* - \vec{p}|}{|\vec{x}^* - (\vec{p})^{-1}|} = r$ if $|\vec{x}^*| = 1$.

Hence $\lim_{\vec{x}\to\vec{x}^*} h(\vec{x}) = -\left(\frac{1}{r^{n-2}}\right)\left(\frac{1}{|\vec{x}^* - (\vec{p}^*)^{-1}|^{n-2}}\right) = -\frac{1}{r^{n-2}} \frac{r^{n-2}}{|\vec{x}^* - \vec{p}|^{n-2}}$.

Therefore $\lim_{\vec{x}\to\vec{x}^*} G_{\vec{p}}(\vec{x}) = \frac{1}{|\vec{x}^* - \vec{p}|^{n-2}} - \frac{1}{|\vec{x}^* - \vec{p}|^{n-2}} = 0$

for any $\vec{x}^* \in S$, and the theorem is proved.

<u>Theorem 3.1.4</u>: Let u be harmonic on the closed ball $\bar{B} \subset E^2$ of radius 1, $\vec{p} \in B$, and $\rho = |\vec{x} - \vec{p}|$. Then $u(\vec{p}) = -\frac{1}{2\pi}\int_S u\left(\frac{\partial G_{\vec{p}}}{\partial r}\right) d\sigma_1$ where $S = \{\vec{x} : |\vec{x}| = 1\}$ and S is oriented in a positive direction, and where $G_{\vec{p}} = \log\left(\frac{1}{\rho}\right) + h$ is the Green's function with pole at $\vec{p}$.

<u>Proof</u>: Let $\bar{B}' = \bar{B} - B_\varepsilon$ where $B_\varepsilon = \{\vec{x} : |\vec{x} - \vec{p}| < \varepsilon\}$ and let $G_{\vec{p}}$ be denoted by V. Then applying Green's second identity as in theorem 2.1.3 and denoting by S_ε the set $\{\vec{x} : |\vec{x} - \vec{p}| = \varepsilon\}$, we have

$$\int_{B'} [u\nabla^2 V - V(\nabla^2 u)]\, dm_2 = \int_S \left[u\left(\frac{\partial V}{\partial r}\right) - V\left(\frac{\partial u}{\partial r}\right)\right] d\sigma_1 - \int_{S_\varepsilon}\left[u\frac{\partial V}{\partial \rho} - V\frac{\partial u}{\partial \rho}\right] d\sigma_1 .$$

Both terms vanish on the left side because u and V are both harmonic in B'. Also $\int_S V\left(\frac{\partial u}{\partial r}\right) d\sigma_1 = 0$ because $V = G_{\vec{p}}$ vanishes on S.

Therefore we have

$$0 = \int_S u\left(\frac{\partial V}{\partial r}\right) d\sigma_1 - \int_{S_\varepsilon} u\,\frac{\partial\left(\log\frac{1}{\rho} + h\right)}{\partial \rho}\, d\sigma_1 + \int_{S_\varepsilon}\left(\log\frac{1}{\rho} + h\right)\frac{\partial u}{\partial \rho}\, d\sigma_1 .$$

Again by Green's second identity,

$$\int_{S_\varepsilon}\left[h\frac{\partial u}{\partial \rho} - u\frac{\partial h}{\partial \rho}\right] d\sigma_1 = \int_{B_\varepsilon}(h\nabla^2 u - u\nabla^2 h)\, dm_2 = 0 \text{ and therefore}$$

$$0 = \int_S u \left(\frac{\partial V}{\partial r}\right) d\sigma_1 - \int_{S_\epsilon} \left[\frac{u \partial \log\left(\frac{1}{\rho}\right)}{\partial \rho}\right]_{\rho=\epsilon} d\sigma_1 + \int_{S_\epsilon} \left[\left(\log \frac{1}{\epsilon}\right) \left(\frac{\partial u}{\partial \rho}\right)\right]_{\rho=\epsilon} d\sigma_1 .$$

From Green's second identity,

$$\int_{S_\epsilon} \left(\log \left(\frac{1}{\rho}\right) \left(\frac{\partial u}{\partial \rho}\right)\right)_{\rho=\epsilon} d\sigma_1 = 0,$$

and $\lim_{\epsilon \to 0} \int_{S_\epsilon} \left(u \frac{\partial \log \rho}{\partial \rho}\right)_{\rho=\epsilon} d\sigma_1 = 2\pi u(\vec{p})$ (see proof of theorem 2.1.3)

and it follows that $2\pi u(\vec{p}) = - \int_S u \left(\frac{\partial G_{\vec{p}}}{\partial r}\right) d\sigma_1$.

<u>Theorem 3.1.5</u>: Let u be harmonic on the closed unit ball $\bar{B} \subset E^n$, $n \geq 3$, and $\vec{p} \in B$. Then $u(\vec{p}) = \frac{-1}{(n-2)\, s_{n-1}} \int_S u \left(\frac{\partial G_{\vec{p}}}{\partial r}\right) d\sigma_{n-1}$,

where $S = \partial B$ and is positively oriented.

<u>Proof</u>: Employing results of theorem 2.1.5, we note that if $V = G_{\vec{p}}$ on $\bar{B}' = \bar{B} - B_\epsilon$ where $B_\epsilon = \{ \vec{x} : | \vec{x} - \vec{p} | < \epsilon \}$, then

$$\int_{B'} (u \nabla^2 V - V \nabla^2 u)\, dm_n = \int_S \left(u \frac{\partial V}{\partial r} - V \frac{\partial u}{\partial r}\right) d\sigma_{n-1} - \int_{S_\epsilon} \left(u \frac{\partial V}{\partial \rho} - V \frac{\partial u}{\partial \rho}\right)_{\rho=\epsilon} d\sigma_{n-1} .$$

Hence

$$0 = \int_S u \left(\frac{\partial V}{\partial r}\right) d\sigma_{n-1} - \int_{S_\epsilon} \left.\frac{u \partial\left(\frac{1}{\rho^{n-2}}\right)}{\partial \rho}\right|_{\rho=\epsilon} d\sigma_{n-1} + \int_{S_\epsilon} \left(\frac{1}{\epsilon^{n-2}}\right) \frac{\partial u}{\partial \rho}\, d\sigma_{n-1}$$

$$+ \int_{S_\epsilon} \left(h \frac{\partial u}{\partial \rho} - u \frac{\partial h}{\partial \rho}\right) d\sigma_{n-1}$$

Therefore

$$\int_{S_\epsilon} \frac{(2-n)}{\epsilon^{n-1}}\, u\, d\sigma_{n-1} = \int_S u \left(\frac{\partial V}{\partial r}\right) d\sigma_{n-1},$$ and by the same

reasoning as in theorem 2.1.5,

$$\lim_{\varepsilon \to 0} \int_S \frac{(2-n)}{\varepsilon^{n-1}} \, u \, d\sigma_{n-1} = (2-n) \, s_{n-1} \, u(\vec{p})$$

Hence $$u(\vec{p}) = \frac{-1}{(n-2) \, s_{n-1}} \int_S u \left(\frac{\partial G_{\vec{p}}}{\partial r}\right) d\sigma_{n-1} .$$

<u>Remark</u>: Theorems 3.1.4 and 3.1.5 suggest that the normal derivative of the Green's function constitutes an important aid in establishing a relationship between the value of a harmonic function inside a ball and its values on the boundary.

<u>Theorem 3.1.6</u>: If $G_{\vec{p}}$ is the Green's function for $B = \{ \vec{x} : |\vec{x}| < 1 \}$ in E^2, then the outer normal derivative for

$$G_{\vec{p}} \text{ at } \vec{q} \in S = \{ \vec{x} : |\vec{x}| = 1 \} \text{ is } \frac{\partial G_{\vec{p}}}{\partial r} = \frac{|\vec{p}|^2 - 1}{|\vec{p} - \vec{q}|^2} .$$

<u>Proof</u>: We first note that $\frac{\partial}{\partial r} \log |\vec{x} - \vec{p}| = \frac{1}{|\vec{x}-\vec{p}|} \frac{\partial}{\partial r} |\vec{x} - \vec{p}|$,

and if $|\vec{x} - \vec{p}| = \rho$, then $\rho^2 = (x_1 - p_1)^2 + (x_2 - p_2)^2$ where

$\vec{x} = (x_1, x_2)$, $\vec{p} = (p_1, p_2)$ and $\frac{\partial \rho}{\partial x_i} = \frac{x_i - p_i}{\rho}$, $i = 1, 2$.

Hence $\vec{\nabla}(\log \rho) = \frac{1}{\rho^2} \left((x_1 - p_1)\hat{i} + (x_2 - p_2)\hat{j} \right)$. If $|\vec{x}| = 1$, then

$$\frac{\partial \log \rho}{\partial r} = (\vec{\nabla}(\log \rho)) \cdot \vec{x} = \frac{1}{\rho^2} \left(x_1^2 - p_1 x_1 \right) + \left(x_2^2 - p_2 x_2 \right)$$

$$= \frac{1}{\rho^2} (1 - \vec{p} \cdot \vec{x}).$$

Since $G_{\vec{p}}(\vec{x}) = -\log|\vec{x}-\vec{p}| + \log|\vec{x}-(\vec{p})^{-1}| + \log|\vec{p}|$, therefore

$$\frac{\partial G_{\vec{p}}(\vec{x})}{\partial r} = (\vec{\nabla}(G_{\vec{p}}(\vec{x}))\,.\,\vec{x} \text{ if } |\vec{x}| = 1$$

$$= (-\vec{\nabla}(\log \rho))\cdot\vec{x} + (\vec{\nabla}(\log \rho'))\cdot\vec{x} \text{ where } \rho' = \left|(\vec{x} - \frac{\vec{p}}{|\vec{p}|^2})\right|$$

$$= -\frac{1}{\rho^2}(1 - \vec{p}.\vec{x}) + \frac{1}{(\rho')^2}(1 - \frac{\vec{p}.\vec{x}}{|\vec{p}|^2}).$$

We recall that $\frac{\rho}{\rho'} = |\vec{p}|$ by theorem 3.1.1.

Hence

$$\frac{\partial G_{\vec{p}}(\vec{x})}{\partial r} = -\frac{1}{\rho^2}(1) + \frac{\vec{p}.\vec{x}}{\rho^2} + \frac{|\vec{p}|^2}{\rho^2} - \frac{\vec{p}.\vec{x}}{\rho^2} = \frac{|\vec{p}|^2-1}{|\vec{p}-\vec{x}|^2}$$

and the theorem follows.

<u>Theorem 3.1.7</u>: If $G_{\vec{p}}$ is the Green's function for $B \subset E^n$, $n \geqslant 3$, then $\frac{\partial G_{\vec{p}}}{\partial r} = -(n-2)\frac{(1-|\vec{p}|^2)}{|\vec{p}-\vec{q}|^n}$ at $\vec{q} \in S = \{\vec{x} : |\vec{x}| = 1\}$.

<u>Proof</u>: Let $\rho = |\vec{x}-\vec{p}|$ and $\rho' = \left|\vec{x} - \frac{\vec{p}}{|\vec{p}|^2}\right|$ as in theorem 3.1.6. Then $\frac{\partial\rho}{\partial r} = (\vec{\nabla}\rho)\,.\,\vec{x}$ if $|\vec{x}| = 1$.

Now $\frac{\partial\rho}{\partial x_i} = (\frac{x_i-p_i}{\rho})$ for $i = 1, \ldots, n$, $\vec{x} = (x_1, \ldots, x_n)$, $\vec{p} = (p_1, \ldots, p_n)$

Therefore $(\vec{\nabla}\rho)\,.\,\vec{x} = \frac{1}{\rho}(1 - \vec{p}.\vec{x})$.

Similarly $\frac{\partial\rho'}{\partial r} = (\vec{\nabla}\rho')\,.\,\vec{x} = \frac{1}{\rho'}(1 - \frac{\vec{p}.\vec{x}}{|\vec{p}|^2})$.

Therefore

$$\frac{\partial G_{\vec{p}}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{\rho^{n-2}}\right) - \frac{\partial}{\partial r}\left(\frac{1}{|\vec{p}|^{n-2}(\rho')^{n-2}}\right)$$

$$= (2-n)\left(\frac{1}{\rho^{n-1}}\left(\frac{\partial \rho}{\partial r}\right) - \frac{1}{|\vec{p}|^{n-2}(\rho')^{n-1}}\frac{\partial \rho'}{\partial r}\right).$$

Since $\rho = |\vec{p}|\,\rho'$ therefore

$$\frac{\partial G_{\vec{p}}}{\partial r} = \frac{(2-n)}{\rho^{n-1}}\left(\frac{\partial \rho}{\partial r} - |\vec{p}|\left(\frac{\partial \rho'}{\partial r}\right)\right) = \frac{(2-n)}{\rho^{n-1}}\left(\frac{(1-\vec{p}.\vec{x})}{\rho} - \frac{|\vec{p}|}{\rho'}\left(1 - \frac{\vec{p}.\vec{x}}{|\vec{p}|^2}\right)\right)$$

$$= \frac{(2-n)}{\rho^{n}}\left(1 - \vec{p}.\vec{x} - |\vec{p}|^2 + \frac{|\vec{p}|^2}{|\vec{p}|^2}\,\vec{p}.\vec{x}\right) = \frac{(2-n)}{\rho^n}\left(1 - |\vec{p}|^2\right)$$

$$= \frac{-(n-2)\,(1-|\vec{p}|^2)}{|\vec{p} - \vec{x}|^n}$$

<u>Definition 3.1.4</u>: If $B \subset E^2$, we define $-\dfrac{\partial G_{\vec{p}}}{\partial r} = \dfrac{1-|\vec{p}|^2}{|\vec{q} - \vec{p}|^2}$ to be the <u>Poisson kernel function</u> of B with pole at $\vec{q}$. We denote the Poisson kernel function with pole at $\vec{q}$ to be $K_{\vec{q}}(p)$.

<u>Definition 3.1.5</u>: If $B \subset E^n$, $n \geq 3$, we define $-\dfrac{\partial G_{\vec{p}}}{(n-2)\partial r} = \dfrac{1-|\vec{p}|^2}{|\vec{q} - \vec{p}|^n}$ to be the Poisson kernel function of B with pole at $\vec{q}$. Again we employ the symbol $K_{\vec{q}}(\vec{p})$ to represent it.

<u>Remark</u>: By direct computation, $K_{\vec{q}}$ is harmonic on B. Also by theorems 3.1.4 and 3.1.5, if $u \equiv 1$ on $\bar{B}$, then $\dfrac{1}{s_{n-1}}\displaystyle\int_S K_{\vec{q}}\, d\sigma_{n-1} = 1.$ Let f be a continuous function on S and construct the function

$$u(\vec{p}) = \frac{1}{s_{n-1}} \int_S f(\vec{q}) \; K_{\vec{q}}(\vec{p}) \; d\sigma_{n-1} \; (\vec{q}).$$ We shall show that

u is the solution of the classical "D" problem for B with boundary function f.

<u>Theorem 3.1.8</u>: The function u, defined such that

$$u(\vec{p}) = \frac{1}{s_{n-1}} \int_S f(\vec{q}) \; K_{\vec{q}}(\vec{p}) \; d\sigma_{n-1}(\vec{q}),$$ is harmonic on B.

<u>Proof</u>: By the dominated convergence theorem (theorem 1.4.10)

$$\frac{\partial u(\vec{p})}{\partial x_i} = \frac{\partial}{\partial x_i} \left\{ \frac{1}{s_{n-1}} \int_S f(\vec{q}) \; K_{\vec{q}}(\vec{p}) \; d\sigma_{n-1} \right\}$$

$$= \frac{1}{s_{n-1}} \int_S f(\vec{q}) \; \frac{\partial}{\partial x_i} (K_{\vec{q}}(\vec{p})) \; d\sigma_{n-1}$$

and hence $\nabla^2 u = \frac{1}{S_{n-1}} \int_S f(\vec{q}) \left(\nabla^2 K_{\vec{q}}(\vec{p}) \right) d\sigma_{n-1} = 0$ because $K_{\vec{q}}$ is harmonic on B.

<u>Theorem 3.1.9</u>: Let f be continuous on $S = \left\{ \vec{x} : |\vec{x}| = 1 \right\}$ and

$$u(\vec{p}) = \frac{1}{s_{n-1}} \int_S f(\vec{q}) \; K_{\vec{q}}(\vec{p}) \; d\sigma_{n-1}.$$

Then $\lim_{\substack{\vec{p} \to \vec{q}_o \\ \vec{p} \in B}} u(\vec{p}) = f(\vec{q}_o)$ where $\vec{q}_o \in S$ is fixed.

<u>Proof</u>: Without loss of generality, we may assume that $f(\vec{q}_o) = 0$.

We shall show that $\lim_{\vec{p} \to \vec{q}_o} u(\vec{p}) = 0$.

For any $\varepsilon > 0$, we may choose $\delta_1(\varepsilon)$ such that if $|\vec{q} - \vec{q}_o| < \delta_1$ then $|f(q)| < \varepsilon$ since f is continuous. Let $S_1 = \{ \vec{q} \in S : |\vec{q}-\vec{q}_o| < \delta_1 \}$, and note that $I_1(\vec{p}) = \frac{1}{s_{n-1}} \int_{S_1} f(\vec{q}) K_{\vec{q}}(\vec{p})\, d\sigma_{n-1}$ has the property that $|I_1(\vec{p})| < \frac{1}{s_{n-1}} \int_{S_1} (\varepsilon) K_{\vec{q}}(\vec{p})\, d\sigma_{n-1} = \frac{\varepsilon}{s_{n-1}}$ for any $\vec{p} \in B$.

Let $S_2 = S - S_1$ and let $I_2(\vec{p}) = \frac{1}{s_{n-1}} \int_{S_2} f(\vec{q}) K_{\vec{q}}(\vec{p})\, d\sigma_{n-1}$.

We note that $I_1(\vec{p}) + I_2(\vec{p}) = u(\vec{p})$. Then

$$|I_2(\vec{p})| \leqslant \frac{1}{s_{n-1}} \int_{S_2} |f(\vec{q})| \frac{(1-|\vec{p}|^2)}{(\delta_1{}^n)}\, d\sigma_{n-1} \leqslant \frac{M}{s_{n-1}(\delta_1)^n} \int_{S_2} (1-|\vec{p}|^2)\, d\sigma_{n-1}$$

where $M = \sup_{\vec{q} \in S} |f(\vec{q})|$. Now choose $\delta_2 < \delta_1$ where $|\vec{p} - \vec{q}_o| < \delta_2$. Then

$1 - |\vec{p}| < |\vec{p} - \vec{q}_o| < \delta_2$ and it follows that

$I_2(\vec{p}) < \frac{M}{s_{n-1}} (\delta_2)(2)(s_{n-1})$. We may choose δ_2 such that

$|I_2(\vec{p})| < \varepsilon$ and hence $|u(\vec{p})| < (\frac{\varepsilon}{s_{n-1}} + \varepsilon)$. The theorem follows.

We have thus solved the classical "D" problem for the n-dimensional unit ball and by the maximum principle, the solution is unique. By means of a simple dilation process, our theory is applicable to an n-dimensional ball of any finite radius, say r_o, for let $\vec{p} \in B_1(\vec{0})$ and $\vec{p}' \in B_{r_o}(\vec{0})$; then $\vec{p}' = r_o\vec{p}$ or $p = \frac{1}{r_o}(p')$ is a dilation transformation. Thus if $h(\vec{p})$ is a harmonic function on $B_1(\vec{0})$, then so is $h(\frac{\vec{p'}}{r_o}) = H(p')$ on $B_{r_o}(\vec{0})$. By means of a simple translation process, our theory is still applicable to any n-ball with finite radius, whose centre point is other than $\vec{0}$.

§2 Convergence theory associated with harmonic functions

In this section we shall consider some special properties of harmonic functions, and shall focus our attention on certain convergence theorems.

Theorem 3.2.1: (Converse of the mean value theorem)
Let u be continuous on a region $R \subset E^n$ and possess the property that $u(\vec{p}) = L(u, \vec{p}, \delta)$ for each $\vec{p} \in R$ and every $\delta > 0$ such that $\bar{B}_\delta(\vec{p}) \subset R$. Then u is harmonic on R.

Proof: Let $\vec{p}_o \in R$, and fix $\delta > 0$ so that $\bar{B}_\delta(\vec{p}_o) \subset R$. Since u is continuous on $S_\delta(\vec{p}_o)$, we can solve the "D" problem for $B_\delta(\vec{p}_o)$ whose boundary function is $u \mid S_\delta(\vec{p}_o)$ i.e. u restricted to $S_\delta(\vec{p}_o) = \{\vec{p} : |\vec{p} - \vec{p}_o| = \delta\}$. Let the solution of this "D" problem be denoted by h, and consider the function $V = (u-h)$ on $B_\delta(\vec{p}_o)$ Since the functions u and h satisfy the mean value property on $B_\delta(\vec{p}_o)$, so does the function V, and hence V satisfies both the maximum and minimum principle on $B_\delta(\vec{p}_o)$.
Since $\lim_{\substack{\vec{p} \to \vec{q} \\ \vec{p} \in B_\delta(\vec{p}_o)}} V(\vec{p}) = 0$ for every $\vec{q} \in S_\delta(\vec{p}_o)$, it follows that $V \leq 0$, $V \geq 0$

and hence $V \equiv 0$ on $B_\delta(\vec{p}_o)$. Therefore $u \equiv h$ on $B_\delta(\vec{p}_o)$ and is therefore harmonic at $\vec{p}_o$. Thus u is harmonic on R since $\vec{p}_o$ was arbitrary.

Theorem 3.2.2: (Harnack's inequality in n-dimensional form)

Let u be a harmonic function on the unit ball $B = \{\vec{x} : |\vec{x}| < 1\}, B \subset E^n$ and continuous on its closure. Then for any $\vec{p} \in B$ such that $|\vec{p}| = r < 1$, it follows that

$$\frac{(1 - r^2)}{(1 + r)^n} u(\vec{0}) \leq u(\vec{p}) \leq \frac{(1 - r^2)}{(1 - r)^n} u(\vec{0}).$$

Proof: By theorem 3.1.7, it follows that $u(\vec{p}) = \frac{1}{s_{n-1}} \int_S u(\vec{q}) \frac{1 - |\vec{p}|^2}{|\vec{p} - \vec{q}|^n} d\sigma_{n-1}$ and note that $u(\vec{0}) = \frac{1}{s_{n-1}} \int_S u(\vec{q})\, d\sigma_{n-1}$. For $\vec{p}$ fixed, $K_{\vec{q}}(\vec{p})$ attains its maximum value when $\vec{q} = \lambda\vec{p}$ with $\lambda > 0$ for in that case $\vec{q} - \vec{p} = (1-r)$ and hence the maximum of $K_{\vec{q}}(\vec{p})$ is $\frac{1-r^2}{(1-r)^n}$. Also $K_{\vec{q}}(\vec{p})$ attains its minimum when $\vec{q} = \lambda\vec{p}$ with $\lambda < 0$ in which case $|\vec{q} - \vec{p}| = 1 + r$ and hence the minimum of $K_{\vec{q}}(\vec{p})$ is $\frac{1 - r^2}{(1+r)^n}$. It follows that

$$\frac{1}{s_{n-1}} \int_S u(\vec{q}) \frac{(1-r^2)}{(1+r)^n} d\sigma_{n-1} \leq u(\vec{p}) \leq \frac{1}{s_{n-1}} \int_S u(\vec{q}) \frac{(1-r^2)}{(1-r)^n} d\sigma_{n-1}$$

and hence $u(\vec{0}) \frac{(1-r^2)}{(1-r)^n} < u(\vec{p}) < \frac{u(\vec{0})(1-r^2)}{(1-r)^n}$ or

$$\frac{u(\vec{0})(1 - r)}{(1 + r)^{n-1}} \leq u(\vec{p}) \leq \frac{u(\vec{0})(1 + r)}{(1-r)^{n-1}}.$$

Corallary: If $B = B_{r_o}(\vec{0})$ for $r_o \neq 1$, and if u is harmonic on B and continuous on its closure, then for any $\vec{p} \in B$ such that $|\vec{p}| = r_o < r$, $r > 0$ and finite, it follows that

$$\frac{u(\vec{0})\,(r_o^2 - r^2)(r_o^{n-2})}{(r_o + r)^n} \leq u(\vec{p}) \leq \frac{u(\vec{0})(r_o^2-r^2)(r_o^{n-2})}{(r_o - r)^n}$$

Proof: Let $\vec{p} = \dfrac{\vec{p}'}{r_o}$ be the dilation transformation of $\vec{p} \in B_1(\vec{0})$ to $\vec{p}' \in B_{r_o}(\vec{0})$. Then $K_{\vec{q}'}(\vec{p}') = \dfrac{\left(1 - \left(\frac{|\vec{p}'|}{r_o}\right)^2\right)}{|\vec{q} - \vec{p}'|^n} = \dfrac{r_o^2 - |\vec{p}'|^2\, r_o^{(n-2)}}{|\vec{q}' - \vec{p}'|^n}$

By reasoning similar to that of theorem 3.2.2, the corollary follows.

Definition 3.2.1: A family of continuous functions $\{f_\alpha\}$ defined on a common region $R \subset E^n$ is said to be equicontinuous at $\vec{x}_o \in R$ if and only if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon, \vec{x}_o)$ such that $|f_\alpha(\vec{x}) - f_\alpha(\vec{x}_o)| < \varepsilon$ provided that $|x - x_o| < \delta$. (We emphasize that δ is independent of the α's.

Theorem 3.2.3: Let $\{u_\alpha\}$ be a family of harmonic functions on the open unit ball $B \subset E^n$. Let $\{u_\alpha\}$ be also uniformly bounded above by M and below by -M where $M > 0$. Then $\{u_\alpha\}$ is equicontinuous at $\vec{0}$.

Proof: From theorem 3.2.2 we have that for any $\vec{p} \in B$

$$\frac{(1-r)}{(1+r)^{n-1}} u_\alpha(\vec{0}) \leq u_\alpha(\vec{p}) \leq \frac{(1+r)}{(1-r)^{n-1}} u_\alpha(\vec{0}) \quad .$$

Hence

$$u_\alpha(\vec{p}) - u_\alpha(\vec{0}) \leq u_\alpha(\vec{0}) \left(\frac{1+r}{(1-r)^{n-1}} - 1\right) = u_\alpha(\vec{0}) \left(\frac{1 + r - (1-r)^{n-1}}{(1 - r)^{n-1}}\right)$$

$$= \frac{u_\alpha(\vec{0})}{(1-r)^{n-1}} \left(1 + r - (1-r)^{n-1}\right)$$

If we let $0 < r \leq ½$, where $|\vec{p}| = r$, then

$$u_\alpha(\vec{p}) - u_\alpha(\vec{0}) \leq \frac{u_\alpha(\vec{0})}{(½)^{n-1}} (1 + r - (1 - r)^{n-1}).$$

Let $f(r) = 1 + r - (1 - r)^{n-1}$ where $0 < r < \delta$. By the mean value theorem of elementary calculus, $f(r) = f(0) + f'(\xi) r$, where $0 < \xi < r$, and hence $f(r) = 0 + (1 + (n-1)(1-\xi)^{n-2}) r < nr$. Hence $u_\alpha(\vec{p}) - u_\alpha(\vec{0}) \leq u_\alpha(\vec{0}) (2^{n-1}) nr < M (2^{n-1}) nr$. For any $\epsilon > 0$, it follows that one can find δ_1 such that $0 < \delta_1 < ½$ and if $r < \delta_1$, then $u_\alpha(\vec{p}) - u_\alpha(\vec{0}) < \epsilon$. Similarly

$$u_\alpha(\vec{p}) - u_\alpha(\vec{0}) \geq u_\alpha(\vec{0}) \left(\frac{1 - r}{(1+r)^{n-1}} - 1 \right) \quad \text{or}$$

$$u_\alpha(\vec{0}) - u_\alpha(\vec{p}) \leq u_\alpha(\vec{0}) \left(1 - \frac{1-r}{(1+r)^{n-1}} \right) = \frac{u_\alpha(\vec{0})}{(1+r)^{n-1}} ((1+r)^{n-1} - 1 + r)$$

$$= u_\alpha(\vec{0})((n-1)(1+\xi)^{n-2}+1)r \text{ if } 0 < \xi < r.$$

If $r < ½$, $u_\alpha(\vec{0}) - u_\alpha(\vec{p}) \leq u_\alpha(\vec{0}) (1 + (n-1) (\frac{3}{2})^{n-2}) r \leq M (1+(n-1)(\frac{3}{2})^{n-2}) r$.

We may choose δ_2 such that if $r < \delta_2$, then

$u_\alpha(\vec{0}) - u_\alpha(\vec{p}) < \epsilon$ (or $u_\alpha(\vec{p}) - u_\alpha(\vec{0}) > -\epsilon$) for any $\epsilon > 0$.

Choose $\delta = \min(\delta_1, \delta_2)$. If $r < \delta$, then $|u_\alpha(\vec{p}) - u_\alpha(\vec{0})| < \epsilon$ and the theorem is proved.

<u>Definition 3.2.2</u>: A family of continuous functions $\{f_\alpha\}$ on $R \subset E^n$ is defined to be equicontinuous on R if and only if it is equicontinuous at every point of R.

<u>Corollary 1</u>: Under the hypothesis of theorem 3.2.3 with the unit n-ball replaced by the n-ball of any finite radius $r_o \neq 1$, equicontinuity of $\{u_\alpha\}$ at the origin holds.

<u>Corollary 2</u>: Under the hypothesis of theorem 3.2.3 with the unit n-ball replaced by the n-ball of radius $r_o \neq 1$, $\{u_\alpha\}$ is equicontinuous on B if and only if $\{u_\alpha\}$ is equicontinuous at every point of B.

<u>Theorem 3.2.4</u>: Let $\{u_n\}$ be a sequence of harmonic functions on a bounded region $R \subset E^n$ each of which has a continuous extension onto ∂R. Let $\{u_n\}$, when restricted to ∂R, converge uniformly to a limit function f on ∂R. Then $\{u_n\}$ converges uniformly to a limit function u on R as well.

<u>Proof</u>: Since $\{u_n\}$ converges to f uniformly on ∂R, therefore $\{u_n\}$ is a uniform Cauchy sequence on ∂R, i.e. for any $\varepsilon > 0$, there exists $n_o(\varepsilon)$ such that $|u_n(\vec{x}^*) - u_m(\vec{x}^*)| < \varepsilon$ if n, m are both greater than n_o, and n_o is independent of $\vec{x}^* \in \partial R$. Thus for any n, m, $(u_n - u_m)$ is harmonic on R and by the maximum principle, $|u_n(\vec{x}) - u_m(\vec{x})| < \varepsilon$ if n, m are both greater than n_o, where $\vec{x}$ is any element of R. Hence $\{u_n\}$ is a uniform Cauchy sequence on R also and hence converges uniformly to a limit function u on R.

<u>Theorem 3.2.5</u>: Let $\{u_n\}$ be a sequence of harmonic functions on a region $R \subset E^n$ such that $\{u_n\}$ converges uniformly to a limit function u on R. Then u is harmonic on R.

Proof: Let $\vec{x}_o \in R$ and $\bar{B}_\delta(\vec{x}_o) \subset R$. Then $u_n(\vec{x}_o) = L(u_n, \vec{x}_o, \delta)$ for each n. Since $U.\ell im_{n\to\infty}\ u_n = u$ on R and hence on $S_\delta(\vec{x}_o)$, therefore $\ell im_{n\to\infty}\ L(u_n, \vec{x}_o, \delta) = L(u, \vec{x}_o, \delta)$ by theorem 1.4.10 and hence $u(\vec{x}_o) = L(u, \vec{x}_o, \delta)$. Since u is continuous on R ([1], p.396 theorem13.8), the harmonicity of u follows.

Theorem 3.2.6: (Harnack's theorem of uniform convergence for the n-ball) Let $\{u_m\}$ be a monotone non-decreasing sequence of harmonic functions on the unit ball $B \subset E^n$. Then $\ell im_{n\to\infty}\ u_m(\vec{x}) \equiv +\infty$ or else $\ell im_{n\to\infty}\ u_m(\vec{x}) = u(\vec{x})$ for every $\vec{x} \in B$ where u is a harmonic function on B, and the convergence is uniform on any $\bar{B}_\delta \subset B$.

Proof: We recall by theorem 3.2.2 that

$$u_m(\vec{0})\,\frac{(1-r)}{(1+r)^{n-1}} \leqslant u_m(\vec{p}) \leqslant u_m(\vec{0})\,\frac{(1+r)}{(1-r)^{n-1}} \quad \text{for any } \vec{p} \in B$$

Case 1: Suppose $\ell im_{m\to\infty}\ u_m(\vec{0}) = +\infty$.

Then $u_m(\vec{p}) \geqslant \left(\frac{(1-r)}{(1+r)^{n-1}}\right) u_m(\vec{0})$ and hence $\ell im_{m\to\infty}\ u_m(\vec{p}) = +\infty$.

It follows that $\ell im_{m\to\infty} u_m(\vec{p}) = +\infty$ on B.

Case 2: Suppose $\ell im_{m\to\infty}\ u_m(\vec{0})$ is finite.

Since $u_m(\vec{p}) \leqslant u_m(\vec{0})\,\frac{(1+r)}{(1-r)^{n-1}}$, therefore $\{u_m(\vec{p})\}$ is bounded above and hence $\ell im_{m\to\infty}\ u_m(\vec{p}) = u(\vec{p})$ exists as a finite number for each $\vec{p} \in B$.

Now let $B_\delta \subset B$ and let $\vec{p} \in \bar{B}_\delta$.

Denote $u_{mi} = u_m - u_i$ which is a harmonic function on B for m and i fixed. If $m > i$, then $0 \leqslant u_{mi}(\vec{p}) = u_m(\vec{p}) - u_i(\vec{p})$ and

$$u_m(\vec{p}) - u_i(\vec{p}) \leqslant u_{mi}(\vec{0}) \frac{(1+r)}{(1-r)^{n-1}} \leqslant u_{mi}(\vec{0}) \frac{(2)}{(1-\delta)^{n-1}} .$$

The sequence $\{u_i(\vec{0})\}$ is a Cauchy sequence and therefore $\{u_m(\vec{p})\}$, $\vec{p} \in \bar{B}_\delta$, is a uniform Cauchy sequence. Hence $\{u_m\}$ converges to u uniformly on $\bar{B}_\delta$ for any $\delta < 1$. Hence u is harmonic in $\bar{B}_\delta$ by theorem 3.2.5. Since $\delta < 1$ is arbitrary, therefore u is harmonic in B itself.

<u>Corollary</u>: A similar theorem holds for an n-ball of any finite radius.

<u>Theorem 3.2.7</u>: (Harnack's theorem of monotone convergence)

Let $\{U_m\}$ be a monotone non-decreasing sequence of harmonic functions on a region $R \subset E^n$. Then $\lim_{m\to\infty} U_m(\vec{x}) \equiv +\infty$ for every $\vec{x} \in R$, or else $\lim_{m\to\infty} U_m(\vec{x}) = U(\vec{x})$ for every $\vec{x} \in R$ where U is a harmonic function, and convergence is uniform on all compact subsets of R.

<u>Proof</u>: Let $A \subset R$ such that $\vec{x} \in A$ if and only if $\lim_{m\to\infty} U_m(\vec{x}) = +\infty$. We shall show that A is open. Let $B_\delta(\vec{x}) \subset R$. By the corollary of theorem 3.2.6, it follows that $\lim_{m\to\infty} U_m(\vec{p}) = +\infty$ for any $\vec{p} \in B_\delta(\vec{x})$. Hence A is an open subset of R. Now let $B \subset R$ be so defined that $\vec{x} \in B$ if and only if $\lim_{m\to\infty} U_m(\vec{x}) = U(\vec{x})$ is finite.

We shall show that B is also open. For any $B_\delta(\vec{x}) \subset R$, it follows by the corollary of theorem 3.2.6, that $\lim_{m\to\infty} U_m(\vec{p})$ is finite for every $\vec{p} \in B_\delta(\vec{x})$. Hence $B \subset R$ is also open in both the E^n topology and the relative topology $\mathcal{J}_R$. But $R = A \cup B$ where $A \cap B = \emptyset$ and where $A \in \mathcal{J}_R$ and $B \in \mathcal{J}_R$. It follows that either $A = \emptyset$ or else $B = \emptyset$ since R is connected. In the case where $\lim_{m\to\infty} U_m(\vec{x}) = U(\vec{x})$ and $B_\delta(\vec{x}) \subset R$ it follows that U_m converges to U uniformly on $\bar{B}_{\delta/2}(\vec{x})$ and therefore U is harmonic at $\vec{x}$, and hence harmonic everywhere on R. Now let $K \subset R$ be compact, and $\vec{x} \in R$. Then there exists an open neighbourhood of $\vec{x}$ denoted by $V(\vec{x})$ such that U_m converges to U uniformly on $V(\vec{x})$ i.e. $|U(\vec{p}) - U_m(\vec{p})| < \varepsilon$ for any $m > M(\vec{x}, \varepsilon)$.

The family $\{V(\vec{x})\}$, $\vec{x} \in K$ form an open covering for K and we can extract out a finite subcovering denoted by $\{V(\vec{x}_i)\}$, $1 < i < \ell$. Let $M_i = M(\vec{x}_i, \varepsilon)$ and define $M = \max\{M_i\}$, $1 < i < \ell$. Then $|U(\vec{p}) - U_m(\vec{p})| < \varepsilon$ for $m > M$ independently of $\vec{p} \in \bigcup_{i=1}^{\ell} V(\vec{x}_i)$ and hence independently of $\vec{p} \in K$. The theorem follows.

<u>Remark</u>: The Harnack convergence principle allows us to solve an extended "D" problem for a ball whose boundary function is bounded above and upper semi-continuous.

<u>Definition 3.2.3:</u> Let φ be an upper semi-continuous function on $S_r(\vec{x}_o)$ which is also bounded above. Then there exists a monotone

decreasing sequence $\{f_n\}$ of continuous functions on $S_r(\vec{x}_o)$ (theorem 1.4.8) such that $\varphi(\vec{x}) = \lim_{n\to\infty} f_n(\vec{x})$ for any $\vec{x} \in S_r(\vec{x}_o)$. Let $H_{f_n}^{B_r}$ be the solution of the "D" problem for $B_r(\vec{x}_o)$ whose boundary function is f_n. Then $\lim_{n\to\infty} H_{f_n}^{B_r} = H_{\varphi}^{B_r}$ is a harmonic function on $B_r(\vec{x}_o)$ by Harnack's convergence principle. We define $H_{\varphi}^{B_r}$ to be the solution of the "D" problem for $B_r(\vec{x}_o)$ whose boundary function is φ.

§3 Examples of insolvability of the classical Dirichlet Problem

In this section, we shall consider an example which shows that the classical "D" problem is not always solvable.

Let $B' = \{ \vec{x} : 0 < |\vec{x}| < 1 \}$ be the deleted unit ball in E^3.

Let $f \equiv 0$ on $S_1(\vec{0})$.

$= 1$ at $\vec{x} = \vec{0}$.

We claim that it is impossible to solve the classical "D" problem for B' relative to the boundary function f, for let $B'_n = \{ \vec{x} : \frac{1}{n} < |\vec{x}| < 1 \}$, and U_n the solution of the "D" problem for B'_n whose boundary function is

$f_n \equiv 0$ on $S_1(\vec{0})$.

$\equiv 1$ on $S_{\frac{1}{n}}(\vec{0})$.

It turns out that $U_n = \frac{1}{n-1}(\frac{1}{r} - 1)$. Let us assume now that U is the solution of the "D" problem for B' whose boundary function is f as defined above. By the maximum principle, $U \leqslant U_n$ on B'_n for every n. Now let $\vec{x}_1 \in B'$, and note that $U(\vec{x}_1) \leqslant U_n(\vec{x}_1)$ for all n which are sufficiently large. If we let $r_1 = |\vec{x}_1|$, then $\lim_{n\to\infty} U_n(\vec{x}_1) = \lim_{n\to\infty} (\frac{1}{n})(\frac{1}{r_1} - 1) = 0$ and hence $U(\vec{x}_1) = 0$. It follows that $U \equiv 0$ on B'. But $\lim_{x\to 0} u(\vec{x}) \neq 1$, and hence U cannot be a classical solution of the D problem for B' with boundary function f.

Another example of a non-solvable "D" problem, was introduced by H. Lebesgue ([21] , p.12). The region considered consists of a deformed ball whose boundary is homeomorphic to the unit sphere. Later authors have referred to this as the <u>spine of Lebesgue</u>.

IV. RESOLUTIVITY OF THE GENERALIZED DIRICHLET PROBLEM

§1 Continuous and bounded theory

In this section we restrict f to be a real-valued bounded function on ∂R where $R \subset E^n$ is a bounded region. If we are able to associate with f a certain harmonic function on R constructed according to a formal process, denoted by H_f^R, we shall say that this function H_f^R, sometimes called the Wiener function, is the solution on R of the generalized D problem whose boundary function is f.

We first let $M = \sup_{\vec{x}^* \in \partial R} \left\{ f(\vec{x}^*) \right\}$ and $m = \inf_{\vec{x}^* \in \partial R} \left\{ f(\vec{x}^*) \right\}$.

<u>Definition 4.1.1</u>: Let $F_i(f) = \left\{ u_\alpha \right\}$ be the family of continuous subharmonic functions on R such that $u_\alpha \in F_i(f)$ if and only if

$$\overline{\lim_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}}} \; u_\alpha(x) \leq f(x^*) \text{ for every } \vec{x}^* \in \partial R.$$

<u>Remark</u>: The family $F_i(f)$ is non-empty because it contains all constant functions less than or equal to m. We shall be concerned with the upper envelope function of $F_i(f)$, i.e. the pointwise supremum of $\left\{ u_\alpha \right\}$ and shall denote this function by $\underline{H}_f^R$. By the maximum principle for subharmonic functions it follows that $\underline{H}_f^R(\vec{x}) \leq M$ for any $\vec{x} \in R$. Thus the upper envelope $\underline{H}_f^R$ is bounded, i.e. $m \leq \underline{H}_f^R \leq M$.

<u>Definition 4.1.2:</u> Let $F'_i(f) \subset F_i(f)$ such that $u'_\alpha \in F'_i(f)$ if and only if $u'_\alpha = \sup(u_\alpha, k)$ on R, denoted by $u'_\alpha = u_\alpha \vee k$, where $k \leqslant m$ is a fixed constant function, and $u_\alpha \in F_i(f)$.

<u>Remark:</u> The family $F'_i(f)$ is uniformly bounded below (by the constant function k). If $u'_{\alpha 1} \in F'_i(f)$ and $u'_{\alpha 2} \in F_i(f)$, then $u'_{\alpha 1} \vee u'_{\alpha 2} \in F'_i(f)$.

<u>Theorem 4.1.1:</u> Let $(\underline{H}^R_f)'$ be the upper envelope of $F'_i(f)$. Then $(\underline{H}^R_f)'(\vec{x}) = \underline{H}^R_f(\vec{x})$ for every $\vec{x} \in R$.

<u>Proof:</u> For any $u_\alpha \in F_i(f)$ it follows that $u'_\alpha = u_\alpha \vee k$ has the property that $u'_\alpha(\vec{x}) \geqslant u_\alpha(\vec{x})$ for every $\vec{x} \in R$. Therefore $\sup_{u'_\alpha \in F'_i(f)} \{u'_\alpha(\vec{x})\} \geqslant \sup_{u_\alpha \in F_i(f)} \{u_\alpha(\vec{x})\}$ for each $\vec{x} \in R$ or $(\underline{H}^R_f)' \geqslant \underline{H}^R_f$ on R.

But $F'_i(f) \subset F_i(f)$ and therefore $(\underline{H}^R_f)' \leqslant \underline{H}^R_f$ on R.

The equality of the two functions follows.

<u>Let $\bar{B} \subset R$ be a fixed closed ball whose radius is greater than zero.</u>

<u>Definition 4.1.3:</u> Define $F''_i(f) \subset F'_i(f)$ so that $u''_\alpha \in F''_i(f)$ if and only if u''_α is the Poisson modification of u'_α on B.

<u>Remark:</u> We recall that u''_α is harmonic in B and since $u''_\alpha - u'_\alpha$ is superharmonic in B and identically equal to zero on ∂B, therefore, by the minimum principle of superharmonic functions $u''_\alpha - u'_\alpha \geqslant 0$ on $\bar{B}$ and therefore on R.

<u>Theorem 4.1.2:</u> Let $(\underline{H}_f^R)''$ be the upper envelope of $F''_i(f)$. Then $(\underline{H}_f^R)''(\vec{x}) = (\underline{H}_f^R)'(\vec{x})$ for every $\vec{x} \in R$.

<u>Proof:</u> By the previous remark, $u''_\alpha \geq u'_\alpha$ on R for each $u'_\alpha \in F'_i(f)$, therefore $(\underline{H}_f^R)'' \geq (\underline{H}_f^R)'$ on R. But $F_i''(f) \subset F'_i(f)$ and therefore $(\underline{H}_f^R)'' \leq (\underline{H}_f^R)'$ on R. The equality of the two functions follows.

<u>Remark:</u> By theorems 4.1.1 and 4.1.2, we therefore have

$$(\underline{H}_f^R)'' = (\underline{H}_f^R)' = \underline{H}_f^R \text{ on R.}$$

We introduce the notation $h_\alpha = u''_\alpha$ restricted to the open ball B and define $u = \sup_\alpha \{h_\alpha\}$ noting that $u = \underline{H}_f^R$ restricted to B.

<u>Theorem 4.1.3:</u> The upper envelope of $\{h_\alpha\}$ is a continuous function on B.

<u>Proof:</u> Since $\{h_\alpha\}$ is a uniformly bounded family of harmonic functions it is by theorem 3.2.3 equicontinuous. Let $\vec{x}_o \in B$ be a fixed point and $\vec{x} \in B$ an arbitrary point. For any given $\varepsilon > 0$, we note that

$$\left|u(\vec{x}) - u(\vec{x}_o)\right| \leq \left|u(\vec{x}) - h_\alpha(\vec{x})\right| + \left|h_\alpha(\vec{x}) - h_\alpha(\vec{x}_o)\right| + \left|h_\alpha(\vec{x}_o) - u(\vec{x}_o)\right|$$

for any h_α function. Since $\{h_\alpha\}$ is equicontinuous, there exists a $\delta(\varepsilon) > 0$ such that $\left|h_\alpha(\vec{x}) - h_\alpha(\vec{x}_o)\right| < \varepsilon/3$ if $\left|\vec{x} - \vec{x}_o\right| < \delta$ and we emphasize that δ is independant of α. We now fix $\vec{x}$ so that $\left|\vec{x} - \vec{x}_o\right| < \delta$ and note that there exists an α_1 such that $u(\vec{x}_o) - h_{\alpha_1}(\vec{x}_o) < \varepsilon/3$ as well as an α_2 such that $u(x_o) - h_{\alpha_2}(x_o) < \varepsilon/3$ by definition of u.

We recall that h_{α_1} is u''_{α_1} restricted to B. Since the function $u''_{\alpha_1} \vee u''_{\alpha_2} \in F'_i(f)$ and its Poisson modification, denoted by u''_{α_3}, is a member of $F''_i(f)$, then defining $h_{\alpha_3} = u''_{\alpha_3}$ restricted to B, we have that $\left| h_{\alpha_3}(\vec{x}) - u(\vec{x}) \right| < \varepsilon/3$ and $\left| h_{\alpha_3}(\vec{x}_o) - u(\vec{x}_o) \right| < \varepsilon/3$. It follows that

$$\left|u(\vec{x}) - u(\vec{x}_o)\right| \leq \left|u(\vec{x}) - h_{\alpha_3}(\vec{x})\right| + \left|h_{\alpha_3}(\vec{x}) - h_{\alpha_3}(\vec{x}_o)\right| + \left|h_3(\vec{x}_o) - u(\vec{x}_o)\right| < \varepsilon$$

for any $\vec{x} \in B$ such that $\left|\vec{x} - \vec{x}_o\right| < \delta$.

<u>Remark</u>: Since continuity is a local condition and since B R is arbitrary therefore $\underline{H}_f^R$ is continuous everywhere on R.

<u>Theorem 4.1.4</u>: u is subharmonic on B, for u and B as defined in theorem 4.1.3.

<u>Proof</u>: Let $\vec{x}_o \in B$ be so chosen that $\bar{B}_\delta(\vec{x}_o) \subset B$. Then $h_\alpha(\vec{x}_o) = L(h_\alpha, \vec{x}_o, \delta)$. Since $h_\alpha(\vec{x}) \leq u(\vec{x})$ for all $\vec{x} \in B$, therefore $L(h_\alpha, \vec{x}_o, \delta) \leq L(u, \vec{x}_o, \delta)$; therefore $L(u, \vec{x}_o, \delta)$ is an upper bound for the set $\{h_\alpha(\vec{x}_o)\}$. But $u(\vec{x}_o)$ is the least upper bound for $\{h_\alpha(x_o)\}$. Therefore $u(\vec{x}_o) \leq L(u, \vec{x}_o, \delta)$ and hence u is subharmonic at $\vec{x}_o$. But $\vec{x}_o \in B$ was arbitrary. Thus u is subharmonic in B. The next two theorems will establish the fact that u is also superharmonic on B.

<u>Theorem 4.1.5</u>: For the compact ball $\bar{B} \subset R$ and for any $\varepsilon > 0$, there exists a function $u'_\alpha \in F'_i(f)$ such that $u'_\alpha(\vec{x}) > \underline{H}_f^R(x) - \varepsilon$ uniformly on $\bar{B}$.

Proof: Choose any $\vec{x} \in \bar{B}$ and $\varepsilon > 0$. Then there exists an $\alpha(\vec{x}, \varepsilon)$, abbreviated to $\alpha(\vec{x})$ such that $u_{\alpha(\vec{x})} \in F'_i(f)$ and $u_{\alpha(\vec{x})}(\vec{x}) > \underline{H}_f^R(\vec{x}) - \varepsilon/3$. Since $u_{\alpha(\vec{x})}$ is continuous, there exists a $\delta(\vec{x}, \varepsilon)$ such that if $\vec{y} \in V_\delta(\vec{x}) = \{\vec{y} : |\vec{y} - \vec{x}| < \delta,\}$ then $|u_{\alpha(\vec{x})}(\vec{y}) - u_{\alpha(\vec{x})}(\vec{x})| < \varepsilon/3$. Since $\underline{H}_f^R$ is continuous on $\bar{B}$, therefore

$$\left|\underline{H}_f^R(\vec{y}) - u_{\alpha(\vec{x})}(\vec{y})\right| \leq \left|\underline{H}_f^R(\vec{y}) - \underline{H}_f^R(\vec{x})\right| + \left|\underline{H}_f^R(\vec{x}) - u_{\alpha(\vec{x})}(\vec{x})\right| + \left|u_{\alpha(\vec{x})}(\vec{x}) - u_{\alpha(\vec{x})}(\vec{y})\right| .$$

Thus $\left|\underline{H}_f^R(\vec{y}) - u_{\alpha(\vec{x})}(\vec{y})\right| < \varepsilon$ for $\vec{y} \in V_{\delta'}(\vec{x})$ where $\delta' < \delta$ is such that $\left|\underline{H}_f^R(\vec{y}) - \underline{H}_f^R(\vec{x})\right| < \varepsilon/3$ if $|\vec{y} - \vec{x}| < \delta'$. We emphasize that δ' depends on both $\vec{x}$ and ε. The family $\{V_{\delta'}(\vec{x})\}$ forms an open covering of $\bar{B}$, and by the Heine-Borel theorem we can extract out a finite subcovering $\{V_{\delta'_i}(\vec{x}_i)\}$, $1 \leq i \leq n$. We let $\alpha_i = \alpha(x_i)$, and note that $u_{\alpha_i} \in F'_i(f)$ for each i, $1 \leq i \leq n$. The function $u'_\alpha = \bigvee_{i=1}^{n} u_{\alpha_i}$ satisfies the requirements of our theorem.

Theorem 4.1.6: u is superharmonic on B.

Proof: Let $\vec{x}_o \in B$ and $\bar{B}_\delta(\vec{x}_o)$ B and $\varepsilon > 0$. By theorem 4.1.5, there exists an h_α such that $h_\alpha(\vec{x}) > u(\vec{x}) - \varepsilon$ uniformly on $\bar{B}$. Then $h_\alpha(\vec{x}_o) = L(h_\alpha, \vec{x}_o, \delta) \geq L(u - \varepsilon, \vec{x}_o, \delta) = L(u, \vec{x}_o, \delta) - \varepsilon L(1, \vec{x}_o, \delta) > L(u, \vec{x}_o, \delta) - \varepsilon$ because the constant function 1 is harmonic and satisfies the mean value condition. Therefore $u(\vec{x}_o) \geq h_\alpha(\vec{x}_o) \geq L(u, \vec{x}_o, \delta) - \varepsilon$ for any $\varepsilon > 0$ and hence $u(\vec{x}_o) \geq L(u, \vec{x}_o, \delta)$. Since δ and $\vec{x}_o$ can be made arbitrary, therefore u is superharmonic on B.

We are now able to establish the following important and fundamental result.

<u>Theorem 4.1.7</u>: $\underline{H}_f^R$ is harmonic on R.

<u>Proof</u>: Since $\underline{H}_f^R$ is continuous on B by theorem 4.1.3.,

$\underline{H}_f^R$ is subharmonic on B by theorem 4.1.4.,

$\underline{H}_f^R$ is superharmonic on B by theorem 4.1.6.,

therefore $\underline{H}_f^R$ is harmonic on B. Since $B \subset R$ was arbitrary, and the property of being harmonic is a local condition, therefore, $\underline{H}_f^R$ is harmonic on R.

<u>Remark</u>: In a similar way, we define the family $F_s(f) = \{V_\alpha\}$ to be the family of continuous superharmonic functions on R with the property that $v_\alpha \in F_s(f)$ if and only if $\underline{\lim}_{\substack{\vec{x} \to \vec{x}^* \\ x \in R}} V_\alpha(\vec{x}) \geq f(\vec{x}^*)$ for every $\vec{x}^* \in \partial R$.. This family is also not empty since it contains all constant functions greater than or equal to M. We shall denote the lower envelope function of $F_s(f)$ by $\overline{H}_f^R$, observing that by the minimum principle for superharmonic functions, that $\overline{H}_f^R(\vec{x}) \geq m$ for any $\vec{x} \in R$. By analogous reasoning taken in the process of showing $\underline{H}_f^R$ to be harmonic on R, we may also show $\overline{H}_f^R$ to be harmonic on R.

Properties of $\overline{H}_f^R$, $\underline{H}_f^R$:

Theorem 4.1.8: $\underline{H}_f^R \leq \overline{H}_f^R$ on R.

Proof: Let $V \in F_s(f)$ and $u \in F_i(f)$. Then $V + (-u)$ is superharmonic on R.

By theorem 1.4.3,

$$\liminf_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} (V - u)(\vec{x}) \geq \liminf_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} V(\vec{x}) + \liminf_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} (-u)(\vec{x}) \text{ for any } \vec{x}^* \in \partial R, \text{ but } \partial R,$$

but by theorem 1.4.4,

$$\liminf_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} V(\vec{x}) + \liminf_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} (-u)(\vec{x}) = \liminf_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} V(\vec{x}) - \limsup_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} u(\vec{x})$$

Since $\limsup_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} u(\vec{x}) \leq f(\vec{x}^*) \leq \liminf_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} V(\vec{x})$ for any $\vec{x} \in \partial R$, therefore

$\liminf_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} V(\vec{x}) - \limsup_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} U(\vec{x}) \geq 0$ everywhere on ∂R. Hence by the minimum principle for superharmonic functions $V - u \geq 0$ on R i.e. $V \geq u$ on R.

Since $V \in F_s(f)$ was arbitrary, therefore $\underline{H}_f^R \leq \overline{H}_f^R$ on R.

Theorem 4.1.9: Let f and g be bounded functions on ∂R. Then $\underline{H}_f^R + \underline{H}_g^R \leq \underline{H}_{f+g}^R$ and $\overline{H}_f^R + \overline{H}_g^R \geq \overline{H}_{f+g}^R$.

<u>Proof</u>: Let $u_f \in F_i(f)$ and $u_g \in F_i(g)$ which means $\overline{\ell im}_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} u_f \leq f(\vec{x}^*)$

and $\overline{\ell im}_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} u_g(\vec{x}^*) \leq g(\vec{x}^*)$ on ∂R. Then by theorem 1.4.3,

$$\overline{\ell im}_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} (u_f + u_g)(\vec{x}) \leq \overline{\ell im}_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} u_f(\vec{x}) + \overline{\ell im}_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} u_g(\vec{x}) \leq f(\vec{x}^*) + g(\vec{x}^*)$$

and hence $u_f + u_g \in F_i(f + g)$. Therefore $u_f + u_g \leq \underline{H}_{f+g}^R$.
Thus for fixed u_g, $\underline{H}_{f+g}^R - u_g$ is an upper bound for $F_i(f)$.
Hence $\underline{H}_f^R + u_g < \underline{H}_{f+g}^R$. Letting u_g vary, we have
$\underline{H}_f^R + \underline{H}_g^R \leq \underline{H}_{f+g}^R$. By similar reasoning we have $\overline{H}_f^R + \overline{H}_f^R \geq \overline{H}_{f+g}^R$.

<u>Theorem 4.1.10</u>: $\underline{H}_k^R = k = \overline{H}_k^R$ for k a constant function.

<u>Proof</u>: $k \in F_i(k)$ and $k \in F_s(k)$ and hence $\overline{H}_k^R \leq k \leq \underline{H}_k^R$. But $\underline{H}_k^R \leq \overline{H}_k^R$ and the theorem follows.

<u>Definition 4.1.4</u>: When $\underline{H}_f^R = \overline{H}_f^R$, then f is said to be resolutive and their common value is denoted by H_f^R, commonly called the Wiener function, and is the solution of the <u>Generalized "D" problem</u>.

<u>Remark</u>: By theorem 4.1.9, if f and g are resolutive, it follows that $f+g$ is resolutive also and that $H_{f+g}^R = H_f^R + H_g^R$. By similar reasoning, one may also show $H_{\lambda f}^R = \lambda H_f^R$ on R for f bounded and defined on ∂R and λ a constant function.

We come now to a fundamental result of resolutivity due to Norbert Wiener ([22]) and which will be based on the following theorems.

<u>Theorem 4.1.11</u>: If f is continuous on $R \cup \partial R$ and subharmonic on R, then $f|\partial R$ is resolutive.

<u>Proof</u>: f continuous on $R \cup \partial R$ implies $\lim\limits_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} f(\vec{x}) = f(\vec{x}^*)$ for all $\vec{x}^* \in \partial R$. Since f is subharmonic in R, therefore $f \in F_i(f)$ and hence $\underline{H}^R_{f\,|\partial R} \geq f$ for all $\vec{x} \in R$.

Thus

$$\lim_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}^*\in\partial R}} \underline{H}^R_{f\,|\partial R}(\vec{x}) \geq \lim_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}^*\in\partial R}} f(\vec{x}) = \lim_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}^*\in\partial R}} f(\vec{x}) = f(\vec{x}^*)$$

Therefore $\underline{H}^R_{f\,|\partial R} \in F_s(f)$ and hence $\underline{H}^R_{f\,|\partial R} \geq \overline{H}^R_{f\,|\partial R}$ on R.

Since $\underline{H}^R_{f\,|\partial R} \leq \overline{H}^R_{f|\partial R}$ on R always, we have $\underline{H}^R_{f\,|\partial R} = \overline{H}^R_{f|\partial R}$ on R.

Hence $f\,|\,\partial R$ is resolutive.

<u>Theorem 4.1.12</u>: Let $g(\vec{x}) = \prod\limits_{i=1}^{n} (x_i)^{m_i}$ where $\vec{x} = (x_{11}\ \ldots,\ x_n) \in (E^n)^+ = \{\vec{x} : x_i > 0,\ 1 \leq i \leq n\}$. Then g is subharmonic in $(E^n)^+$ and any polynomial in $(E^n)^+$ is the difference of two subharmonic polynomials.

<u>Proof</u>: By direct computation, we have $\dfrac{\partial^2 g}{\partial x_i^2} = \left\{\dfrac{m_i(m_i-1)}{x_i^2}\right\} g$ and therefore $\nabla^2 g = g\left[\sum\limits_{i=1}^{n} \left(\dfrac{m_i(m_i-1)}{x_i^2}\right)\right]$. Since $g > 0$ in $(E^n)^+$

and $m_i \geq 0$, therefore $\nabla^2 g \geq 0$ and the subharmonicity of g follows. Let $P(\vec{x})$ be a polynomial restricted to $(E^n)^+$. Then $P(\vec{x}) = P_1(\vec{x}) - P_2(\vec{x})$ where all the coefficients of both $P_1(\vec{x})$ and $P_2(\vec{x})$ are positive. Since ∇^2 is a linear operator and $\nabla^2 P_1(\vec{x}) \geq 0$ and $\nabla^2 P_2(\vec{x}) \geq 0$, therefore P is the difference of two subharmonic polynomials.

__Theorem 4.1.13__: (Wiener's Theorem ([22]): Let $R \subset (E^n)^+$, then any real valued continuous function f defined on ∂R is resolutive.

__Proof__: By theorem 4.1.12, any polynomial is the difference of two subharmonic polynomials. __Since__ the Stone-Weierstrass theorem ([17], p.150) says that any continuous function on a compact set can be uniformly approximated by a polynomial, it follows that any continuous function on $\partial R \subset (E^n)^+$ can be uniformly approximated by the difference of two subharmonic polynomials. By theorem 4.1.11, subharmonic polynomials restricted to ∂R are resolutive. By theorem 4.1.9, the sum of two resolutive functions is again resolutive, and it follows then that any continuous function f on ∂R can be uniformly approximated by a resolutive function P on ∂R. Thus for any $\varepsilon > 0$, $P - \varepsilon < f < P + \varepsilon$ on ∂R. Therefore $\overline{H}^R_{P-\varepsilon} \leq \overline{H}^R_f \leq \overline{H}^R_{P+\varepsilon}$ on R. Since $P + \varepsilon$ and $P - \varepsilon$ are resolutive, therefore $H^R_P - \varepsilon \leq \overline{H}^R_f \leq H_P + \varepsilon$ on R. Similarly $H^R_P - \varepsilon \leq \underline{H}^R_f \leq H^R_P + \varepsilon$ on R. Thus for fixed $\vec{x}_0 \in R$, we have $\left| \overline{H}^R_f(\vec{x}_0) - H^R_P(\vec{x}_0) \right| \leq \varepsilon$. Similarly $\left| \underline{H}^R_f(\vec{x}_0) - H^R_P(\vec{x}_0) \right| \leq \varepsilon$ and we have that

$$\left| \overline{H}_f^R(\vec{x}_o) - \underline{H}_f^R(\vec{x}_o) \right| \leq \left| \overline{H}_f^R(\vec{x}_o) - H_p^R(\vec{x}_o) \right| + \left| H_p^R(\vec{x}_o) - \underline{H}_f^R(\vec{x}_o) \right| \leq 2\varepsilon > 0.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\overline{H}_f^R(\vec{x}_o) = \underline{H}_f^R(\vec{x}_o)$.

Since $\vec{x}_o$ was arbitrarily chosen in R, it follows that $\overline{H}_f^R = \underline{H}_f^R$

and hence f is resolutive.

§2 Resolutivity of semi-continuous functions

Throughout most of this section R will be a bounded region though we allow f on ∂R to take on any extended real value and we shall consider general subharmonic functions.

Theorem 4.2.1: Let f be a function from ∂R into $E^{\#}$ where we recall that $E^{\#}$ is the two point compactification of E'. Let $G_i(f)$ be the set of all subharmonic functions on R such that $u \in G_i(f)$ if and only if $\overline{\lim}_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} u(\vec{x}) \leqslant f(\vec{x}^*)$ for any $\vec{x}^* \in \partial R$. If $G_i(f) \neq \emptyset$, then the upper envelope of $G_i(f)$, denoted by $\underline{\mathcal{H}}_f^R$, is either identically equal to $+\infty$ or else is a harmonic function in R.

Proof: Let $\vec{x}_o \in R$ and $\bar{B} = \bar{B}_\delta(\vec{x}_o)$ such that $\bar{B} \subset R$ and let $u_o \in G_i(f)$ and u'_o its Poisson modification on B. Then u'_o is bounded in $\bar{B}_{\delta/2}(\vec{x}_o)$. For any $u_\alpha \in G_i(f)$ we define $u'_\alpha = u_\alpha \vee u'_o$, and $G'_i(f) = \{u'_\alpha\}$. By analogy to theorem 4.1.11, we remark that $G'_i(f)$ and $G_i(f)$ have the same upper envelope. Now define $G''_i(f) \subset G'_i(f)$ in such a way that $u''_\alpha \in G''_i(f)$ if and only if u''_α is the Poisson modification of $u'_\alpha \in G'_i(f)$ on B and remark that again by analogy to theorem 4.1.12, $G''_i(f)$ and $G'_i(f)$ have the same upper envelope on R, denoted by $\underline{\mathcal{H}}_f^R$. If $\underline{\mathcal{H}}_f^R(\vec{x}_o) = +\infty$, then $\underline{\mathcal{H}}_f^R \equiv +\infty$ in B by the

Harnack convergence principle, and if $\mathcal{H}_f^R(\vec{x}_o)$ is finite, then $\mathcal{H}_f^R$ is also finite throughout B. Hence $\mathcal{H}_f^R$ is finite, throughout R or else identically $+\infty$ in R since if R_1 equals the set of all points in R where $\mathcal{H}_f^R$ is finite and R_2 equals the set of all points in R where $\mathcal{H}_f^R$ is $+\infty$, then we have that

$$R = R_1 \cup R_2, \; R_1 \cap R_2 = \emptyset \text{ and } R_1, R_2 \text{ are both open.}$$

Hence either $R_1 = \emptyset$ or $R_2 = \emptyset$ by definition of connectedness. If $\mathcal{H}_f^R(\vec{x}_o)$ is finite, then $\{u''_\alpha\}$ is a uniformly bounded family of harmonic functions in $B_{\delta/2}(\vec{x}_o)$ and hence by theorem 3.2.3 $\{u''_\alpha\}$ is equicontinuous at $\vec{x}_o$ and thus everywhere on $B_{\delta/2}(\vec{x}_o)$, it follows that $\mathcal{H}_f^R$ is continuous in $B_{\delta/2}(\vec{x}_o)$, and also subharmonic by theorem 4.1.4. By theorem 4.1.6, it also follows that $\mathcal{H}_f^R$ is superharmonic in $B_{\delta/2}(\vec{x}_o)$. Since $\mathcal{H}_f^R$ is harmonic at $\vec{x}_o$ and $\vec{x}_o \in R$ was arbitrary, it follows that $\mathcal{H}_f^R$ is harmonic throughout R.

For our next result, we need to borrow a fundamental theorem from combinatorial topology which is phrased in a form suitable for our requirements.

Theorem 4.2.2: Let $R \subset E^n$. There exists a sequence of n-simplexes, denoted by $\{[B_p]\}$, p = 1, 2, ... such that $B_i \cap B_j = \emptyset$, $i \neq j$, where B_p is the interior of $[B_p]$ p = 1, 2, ... and such that

$$R = \bigcup_{i=1}^{\infty} [B_i] .$$

This theorem may be rephrased to state that R may be <u>triangulated</u>.

We shall carry out the argument of the next theorem for E^2, realizing that a modified argument could be introduced in E^n.

<u>Theorem 4.2.3</u>: Let R be a bounded region in E^2, and let f be an extended real valued real valued function defined on ∂R. Let $G_i(f)$ be defined as in theorem 4.2.1, and let $F_i(f) \subset G_i(f)$. Then $\underline{H}_f^R = \underline{\mathcal{H}}_f^R$, where $\underline{H}_f^R$ is the upper envelope of $F_i(f)$ and $\underline{\mathcal{H}}_f^R$ is the upper envelope of $G_i(f)$.

<u>Proof</u>: Since $F_i(f) \subset G_i(f)$, it is evident that $\underline{H}_f^R \leqslant \underline{\mathcal{H}}_f^R$. Now let $\{[B_i]\}$ be a triangulation on R, and let $u \in G_i(f)$. We shall show that there exists a $V \in F_i(f)$ such that $V \geqslant u$ on R. For each $[B_i]$, (a closed triangle), we may solve the "D" problem where the boundary function is u restricted to $\partial[B_i]$, and denote this solution by $H_u^{[B_i]}$. We now define u_1 as follows:

$$u_1 \equiv H_u^{[B_i]} \text{ in } B_i \text{ for each } [B_i] \in \{[B_i]\}$$

$$\equiv u \text{ elsewhere on R.}$$

Then $u_1 \geqslant u$ on R, u_1 is subharmonic on R and may possess discontinuities only in $\bigcup^{\infty} \partial[B_i]$. Now let $\{[B_i']\}$ be another

triangulation of R such that each member $[B_i']$ $\in \{[B_i']\}$, i = 1, 2, ...
has no edges in common with any member of the first triangulation $\{[B_i]\}$. Define u_2 on R as follows:

$$u_2 \equiv \mathcal{H}_{u_1}^{[B_i']} \text{ in } B_i' \text{ for each } [B_i'] \in \{[B_i']\}$$

$$\equiv \quad u_1 \text{ elsewhere on } R.$$

Then $u_2 \geqslant u_1$ on R, u_2 is subharmonic on R and is continuous except on the set A consisting of the common points of the edges of the two triangulations. Let $A = \bigcup_{i=1}^{\infty} (\partial B_i) \cap (\partial B_i')$, and note that u_2 is continuous except on A, where A is countable. We define again a third triangulation $\{[B_i'']\}$ on R such that every member $a \in A$ is in the interior of some $[B_i'']$. Now define

$$V = \mathcal{H}_{u_2}^{[B_i'']} \text{ in } B_i'' \text{ for each } [B_i''] \in \{[B_i'']\}$$

$$= \quad u_2 \text{ elsewhere on } R.$$

Then $V \geqslant u_2$, V is subharmonic on R and continuous on R. Hence $\underline{\mathcal{H}}_f^R \geqslant \underline{\mathcal{H}}_f^R$, and the theorem follows.

__Remark:__ We have an analogous theorem with respect to the family $G_s(f)$ of lower semi-continuous superharmonic functions whose lower limit is greater than or equal to f on ∂R. It should be noted that if $G_i(f)$ and $G_s(f)$ are simultaneously non-empty, then every function of $G_s(f)$ does not necessarily majorize every function $G_i(f)$. To see this, consider the ball $\{B_1 - \{0\}\} \subset E^n$. Let $f \equiv 1$ on the sphere $S_1 \subset E^n$ and equal $+\infty$ at the origin. Let $u_k = \left(\frac{k}{|\vec{x}|^{n-2}} - k + 1\right)$ where $k > 0$ is a constant function and $n \geq 3$. Then u_k is harmonic in $\{B_1 - \{0\}\}$ and $u_k \in G_i(f) \cap G_s(f)$ since u_k, a harmonic function, is both subharmonic and superharmonic on $R \subset E^n$. Then $u_1 \in G_i(f)$ and $u_{1/2} \in G_s(f)$ where u_1 is strictly greater than $u_{1/2}$.

In view of this example, it is not suitable to use the envelopes of the families $G_i(f)$ and $G_s(f)$ for the solution of the generalized D-problem. We, therefore, restrict $G_i(f)$ and $G_s(f)$ according to some process.

Definition 4.2.1: Let $G_i'(f) \subset G_i(f)$ be defined such that $u \in G_i'(f)$ if and only if u is bounded above.

Remark: Each $u \in G_i'(f)$ is bounded above, but the family $G_i'(f)$ is not necessarily uniformly bounded above. We similarly define $G_s'(f) \subset G_s(f)$. Henceforth we define $\underline{H}_f^R$ to be the upper envelope of $G_i'(f)$ and $\overline{H}_f^R$ to be the lower envelope of $G_i'(f)$.

Theorem 4.2.4: It is always true that $\underline{H}_f^R \leq \overline{H}_f^R$.

Proof: Let $V \in G_s'(f)$ and $u \in G_s'(f)$. Then $V-u$ is superharmonic and $\underline{\lim}_{\vec{x}\to\vec{x}^*} (V-u) \geq \underline{\lim}_{\vec{x}\to\vec{x}^*} V - \overline{\lim}_{\vec{x}\to\vec{x}^*} u \geq 0$ because $\overline{\lim}_{\vec{x}\to\vec{x}^*} u$ is finite.

Hence $V \geq u$ by the minimum principle and the theorem follows.

Definition 4.2.2: We define f on ∂R to be resolutive if and only if $\underline{H}_f^R = \overline{H}_f^R$ and denote the common function by H_f^R.

Remark: By the minimum principle, $\underline{H}_f^R = \overline{H}_f^R$ if and only if there exists a point $\vec{x}_o \in R$ such that $\underline{H}_f^R(\vec{x}_o) = \overline{H}_f^R(\vec{x}_o)$. Our definition of resolutivity given in definition 4.2.4, includes the earlier definition of resolutivity given in § 1 of this chapter.

Remark: We recall from § 1, that $H_f^R \geq 0$ where $f \geq 0$ is continuous on the boundary of R, and therefore for a fixed $\vec{x}_o \in R$, the function $H_f^R(\vec{x}_o)$ is a positive linear functional on the Banach space of continuous real valued functions on ∂R. The functional $H_f^R(\vec{x}_o)$ is therefore a Radon measure on ∂R, denoted by $\mu^{(\vec{x}_o)}$.

Theorem 4.2.5: Let f be bounded above and upper semi-continuous on ∂R. Then f is resolutive.

Proof: Since f is upper semi-continuous and bounded above, there exists a monotone non-increasing sequence $\{f_n\}$ of continuous functions on ∂R whose pointwise limit on ∂R is f (Theorem 1.4.8). Since $f \leq f_n$ for all n, it follows that $\overline{H}_f^R \leq \overline{H}_{f_n}^R$ or rather $\overline{H}_f^R \leq H_{f_n}^R$ since f_n is resolutive for all n by theorem 4.1.13. Hence $\overline{H}_f^R(\vec{x}_o) \leq H_{f_n}^R(\vec{x}_o)$ and therefore $\overline{H}_f^R(\vec{x}_o) \leq \lim_{n\to\infty} \int_{\partial R} f_n \, d\mu^{(\vec{x}_o)}$ or $\overline{H}_f^R(\vec{x}_o) \leq \int_{\partial R} f \, d\mu^{(\vec{x}_o)}$ because $\int_{\partial R} f \, d\mu^{(\vec{x}_o)} = \lim_{n\to\infty} \int_{\partial R} f_n \, d\mu^{(\vec{x}_o)}$.

<u>Case 1:</u> If $\int_{\partial R} f \, d\mu^{(\vec{x}_o)} = -\infty$, then $\overline{H}_f^R(\vec{x}_o) = -\infty$ and hence f is resolutive where $H_f^R \equiv -\infty$ on R.

<u>Case 2:</u> Assume that $\int_{\partial R} f \, d\mu^{(\vec{x}_o)}$ is finite. We shall show that for any $\varepsilon > 0$, it follows that $\underline{H}_f^R(\vec{x}_o) > \overline{H}_f^R(\vec{x}_o) - \varepsilon$. We first construct $u_1 \in G_i'(f_1)$ such that $u_1(\vec{x}_o) > H_{f_1}^R(x_o) - \varepsilon_1$ for given $\varepsilon_1 > 0$. We then construct $u_2 \in G_i'(f_2 - f_1)$ such that $u_2(\vec{x}_o) > H_{(f_2 - f_1)}^R(\vec{x}_o) - \varepsilon_2$ for given $\varepsilon_2 > 0$ and note that $u_2 \leqslant 0$ because $f_2 - f_1 \leqslant 0$. We continue to construct a sequence $\{u_n\}$ of subharmonic functions such that $u_n \in G_i'(f_n - f_{n-1})$ if $n \geqslant 2$ and $u_n(\vec{x}_o) \geqslant H_{(f_n - f_{n-1})}^R(\vec{x}) - \varepsilon_n$ for given $\varepsilon_n > 0$. We note that $u_n \leqslant 0$, $n \geqslant 2$, since $f_n - f_{n-1} \leqslant 0$. We now define a new sequence of subharmonic functions $\{V_n\}$ such that $V_n = \sum_{i=1}^{n} u_i$. Then $\{V_n\}$ is a monotone decreasing sequence of subharmonic functions whose limit function, V say, must also be subharmonic ([16] , p.14). Also

$$\overline{\lim_{x \to x^*}} V_n(\vec{x}) \leqslant \sum_{i=1}^{n} \overline{\lim_{x \to x^*}} u_i(\vec{x})$$

$$\leqslant f_1(\vec{x}^*) + \sum_{i=2}^{n} (f_i - f_{i-1})(\vec{x}^*)$$

$$\leqslant f_n(\vec{x}^*)$$ and hence $V_n \in G_i'(f_n)$ for each n.

Also $V_n(\vec{x}_o) = \sum_{i=1}^{n} u_i(\vec{x}_o) \geq (H^R_{f_1}(\vec{x}_o) - \varepsilon_1) + \sum_{i=2}^{n}\left[H^R_{(f_i-f_{i-1})}(\vec{x}_o) - \varepsilon_i\right]$

$$\geq H^R_{f_n}(\vec{x}_o) - \sum_{i=1}^{n} \varepsilon_i.$$

Since $V_n \in G_i'(f_n)$, and $V \leq V_n$, for all n, on R, therefore

$\overline{\lim}_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}^*\in \partial R}} V(\vec{x}) \leq f_n(\vec{x}^*)$ for all n and hence $\overline{\lim}_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in \partial R}} V(\vec{x}) \leq f(\vec{x}^*)$. It follows that $V \in G_i'(f)$ and $V(\vec{x}_o) \geq \lim_{n\to\infty} H_{f_n}(\vec{x}_o) - \sum_{i=1}^{\infty} \varepsilon_i$ or $V(\vec{x}_o) \geq \int_{\partial R} f d\mu^{(\vec{x}_o)} - \sum_{i=1}^{\infty} \varepsilon_i$ provided that $\sum_{i=1}^{\infty} \varepsilon_i$ converges.

Now choose $\varepsilon > 0$, and $\varepsilon_i = (\frac{\varepsilon}{2^i})$ for every i, and then construct $\{V_i\}$ accordingly. It follows that $V \in G_i'(f)$ and hence $V(\vec{x}_o) \leq \underline{H}^R_f(\vec{x}_o)$. But $V(\vec{x}_o) \geq \int_{\partial R} f d\mu^{(\vec{x}_o)} - \varepsilon$ because $\varepsilon = \sum_{i=1}^{\infty} \varepsilon_i$, and hence $V(\vec{x}_o) \geq \overline{H}^R_f(\vec{x}_o) - \varepsilon$. Therefore $\underline{H}^R_f(\vec{x}_o) \geq \overline{H}^R_f(\vec{x}_o) - \varepsilon$ for every $\varepsilon > 0$, and hence $\underline{H}^R_f(\vec{x}_o) \geq \overline{H}^R_f(\vec{x}_o)$. Hence $\underline{H}^R_f(\vec{x}_o) = \overline{H}^R_f(\vec{x}_o)$ and the resolutivity of f follows.

§3 General resolutivity

Let $R \subset E^n$ be a bounded region. We shall characterize those functions on ∂R for which the generalized Dirichlet problem is solvable. From §1 we know that a real valued continuous function on ∂R is resolutive, and from §2 we know that an upper semi-continuous function on ∂R which is bounded above is also resolutive.

Theorem 4.3.1: For any f on ∂R, it follows that $\underline{H}_f^R = \sup_{\varphi \leq f} \{ H_\varphi^R \}$ where every φ is bounded above and upper semi-continuous.

Proof: We recall that $\underline{H}_f^R$ is the upper envelope of the family $G_i'(f)$. For any $\varphi \leq f$ on ∂R, it follows that $G_i'(\varphi) \subset G_i'(f)$ and therefore $\underline{H}_\varphi^R \leq \underline{H}_f^R$. Since every φ is bounded above and upper semi-continuous, therefore every such φ is resolutive and therefore $\underline{H}_\varphi^R = \overline{H}_\varphi^R = H_\varphi^R$. For every $\vec{x} \in R$, $\underline{H}_f^R(\vec{x})$ is an upper bound for $\{ H_\varphi^R(\vec{x}) \}$. Therefore $\underline{H}_f^R(\vec{x}) \geq \sup_{\varphi \leq f} \{ H_\varphi^R(\vec{x}) \}$.

It follows that $\underline{H}_f^R \geq \sup_{\varphi \leq f} \{ H_\varphi^R \}$. Now let $u \in G_i'(f)$ and define V on ∂R such that $V(\vec{x}^*) = \overline{\lim_{\vec{x} \to \vec{x}^*}} u(\vec{x}) \leq f(\vec{x}^*)$ for any $\vec{x}^* \in \partial R$.

Then V is upper semi-continuous on ∂R by theorem 1.4.5, and is bounded above.

It follows that $u \in G_i^!(V)$ and therefore $u \leqslant \overline{H}_V^R$. Hence the upper envelope of $G_i^!(f)$ is less than or equal to $\sup_{\varphi \leqslant f}\{H_\varphi^R\}$, φ upper semi-continuous and bounded above. Combining the two inequalities, it follows that $\underline{H}_f^R = \sup_{\varphi \leqslant f}\{H_\varphi^R\}$ where each φ is upper semi-continuous and bounded above.

<u>Theorem 4.3.2</u>: For any f on ∂R, it follows that $\underline{H}_f^R = \lim_{n \to \infty} (H_{\varphi_n}^R)$ where $\{\varphi_n\}$ is a monotone non decreasing sequence of upper semi-continuous functions on ∂R each of which is bounded above and is less than or equal to f on ∂R.

<u>Proof</u>: Choose $\vec{x}_o \in R$, and note that $\underline{H}_f^R(\vec{x}_o) = \sup_{\varphi \in \Phi}\{H_\varphi^R(\vec{x}_o)\}$ where Φ is the set of all upper semi-continuous functions each of which is bounded above and dominated by f on ∂R. We choose a sequence $\{\psi_n\} \subset \Phi$ such that $\lim_{n\to\infty} H_{\psi_n}^R(\vec{x}_o) = \underline{H}_f^R(\vec{x}_o)$ and then define $\varphi_n = \bigvee_{i=1}^{n} \psi_i$. Each $\varphi_n \in \Phi$, and is a monotone non decreasing sequence. Since $\varphi_n \geqslant \psi_n$, it follows that $\lim_{n\to\infty} H_{\varphi_n}^R(\vec{x}_o) \geqslant \underline{H}_f^R(\vec{x}_o)$. Since $\{\varphi_n\} \subset \Phi$, therefore $\underline{H}_f^R(\vec{x}_o) \geqslant \sup_{\varphi_n \in \Phi}\{H_{\varphi_n}^R(\vec{x}_o)\} = \lim_{n\to\infty} H_{\varphi_n}^R(\vec{x}_o)$. Combining the two inequalities, it follows that $\underline{H}_f^R(\vec{x}_o) = \lim_{n\to\infty} H_{\varphi_n}^R(\vec{x}_o)$. In general, $\underline{H}_f^R \geqslant \lim_{n\to\infty} H_{\varphi_n}^R$ on R since $\{\varphi_n\} \subset \Phi$.

By the Harnack convergence principle (theorem 3.2.7), $\lim_{n\to\infty} H^R_{\varphi_n} = H$ is a harmonic function on R. Since $\underline{H}^R_f \geq H$ on R and since $\underline{H}^R_f(\vec{x}_o) = H(\vec{x}_o)$, therefore $\underline{H}^R_f = H$ everywhere on R by the minimum principle. The theorem follows.

<u>Remark</u>: For the Banach space of continuous functions on ∂R, the mapping $H^R_f(\vec{x}_o)$ is a positive linear functional from C into the reals and hence can be identified with a Radon measure on ∂R by the Riesz representation theorem (theorem 1.2.20). We can represent $H^R_f(\vec{x}_o)$ by $\int_{\partial R} f d\mu^{(\vec{x}_o)}$ where $\mu^{(\vec{x}_o)}$ is the representation measure of the linear functional H^R_f evaluated at $\vec{x}_o$. For any upper semi-continuous function φ on ∂R which is bounded above, H^R_φ exists and $H^R_\varphi(\vec{x}_o) = \lim_{n\to\infty} H^R_{f_n}(\vec{x}_o)$ where $\{f_n\}$ is a monotone non increasing sequence of continuous functions converging pointwise to φ. Hence

$$H^R_\varphi(\vec{x}_o) = \lim_{n\to\infty} \int_{\partial R} f_n \, d\mu^{(\vec{x}_o)}$$

$$= \int_{\partial R} \varphi \, d\mu^{(\vec{x}_o)} .$$

by the monotone convergence theorem.

<u>Theorem 4.3.3</u>: For any f from ∂R into $E^{\#}$ and $\vec{x}_o \in R$, it follows that

$$\underline{H}^R_f(\vec{x}_o) = \underline{\int_{\partial R}} f \, d\mu^{(\vec{x}_o)} .$$

Proof: Since $\mu^{(\vec{x}_o)}$ is a Radon measure whose σ- algebra domain of definition necessarily contains the open sets, it follows from theorem 1.4.11 and its corollaries, that

$\underline{\int}_{(\partial R)} f\, d\mu^{(\vec{x}_o)} = \sup_{\varphi \in \Phi} \left\{ \int_{\partial R} d\mu^{(\vec{x}_o)} \right\}$ where Φ consists of

the family of upper semi-continuous functions each one of which is bounded above and dominated by f. But $H_{\varphi}^{R}(\vec{x}_o) = \int_{\partial R} \varphi\, d\mu^{(\vec{x}_o)}$ and

$\underline{H}_f^R(\vec{x}_o) = \sup_{\varphi \in \Phi} \left\{ H_{\varphi}^{R}(\vec{x}_o) \right\}$ by theorem 4.3.1. It follows that

$$\underline{H}_f^R(\vec{x}_o) = \underline{\int}_{(\partial R)} f d\mu^{(\vec{x}_o)} .$$

Corollary: $\overline{H}_f^R(\vec{x}_o) = \overline{\int}_{\partial R} f\, d\mu^{(\vec{x}_o)}$.

Theorem 4.3.4: The function f from ∂R into $E^{\#}$ is resolutive if and only if it is $\mu^{(\vec{x}_o)}$ — summable for a given $\vec{x}_o \in R$.

Proof: Suppose f is resolutive. Then $\underline{H}_f^R(\vec{x}_o) = \overline{H}_f^R(\vec{x}_o)$ and the common value is denoted by $H_f^R(\vec{x}_o)$. But $\overline{H}_f^R(\vec{x}_o) = \overline{\int}_{\partial R} f d\mu^{(\vec{x}_o)}$ and $\underline{H}_f^R(\vec{x}_o) = \underline{\int}_{(\partial R)} f d\mu^{(\vec{x}_o)}$. Therefore resolutivity of f implies the existence of $\int_{\partial R} f d\mu^{(\vec{x}_o)}$ or the $\mu^{(\vec{x}_o)}$ - summability of f.

If f is $\mu^{(\vec{x}_o)}$- summable, then $\underline{\int}_{(\partial R)} f d\mu^{(\vec{x}_o)} = \overline{\int}_{\partial R} f d\mu^{(\vec{x}_o)}$, and therefore $\underline{H}_f^R(\vec{x}_o) = \overline{H}_f^R(\vec{x}_o)$. Since $\overline{H}_f^R \geqslant \underline{H}_f^R$ on R, it follows that $\underline{H}_f^R = \overline{H}_f^R$ everywhere on R by the minimum principle.

Remark: If a function f on ∂R is $\mu^{(\vec{x}_0)}$— summable for a given $\vec{x}_0 \in R$, it is $\mu^{(\vec{x}_1)}$— summable for any other $\vec{x}_1 \in R$, and hence it is customary to simply say μ-summability without reference to any given point in R. For a bounded function f, μ-summability is equivalent to μ-measurability. Following Brelot ([4]), we shall refer to the measure μ as harmonic measure. We caution the reader to note that harmonic measure is only a measure in the usual sense when it is taken with respect to a specific reference point.

The theory of the generalized Dirichlet problem as developed in this chapter can be extended to unbounded regions. For any region $R \subset E^n$, one can introduce the one point compactification on $R \cup \partial R$ and consider a generalized Dirichlet problem for R whose boundary function is on $\partial R \cup \{\infty\}$, a compact subset of the compact T_2 space $R \cup \partial R \cup \{\infty\}$. We refer the reader to the work of Brelot ([6]) for results of this extension.

V. RESOLUTIVITY OF THE CLASSICAL DIRICHLET PROBLEM AND BOUNDARY BEHAVIOUR

In this chapter we shall characterize the regions for which the classical Dirichlet problem is solvable and analyze the boundary behaviour of the functions $\underline{H}_f^R$ and $\overline{H}_f^R$ in such regions.

Definition 5.1: Let R be a bounded region and $\vec{p}^* \in \partial R$. Then a function ω defined on R is said to be a barrier for R at $\vec{p}^*$ if the following conditions are satisfied:

(i) ω is harmonic on R.

(ii) $\lim_{\vec{x}\to\vec{p}^*} \omega(\vec{x}) = 0, \ \vec{x} \in R.$

(iii) $\lim_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} \omega(\vec{x}) > 0$ for any $\vec{x}^* \in \partial R$ where $\vec{x}^* \neq \vec{p}^*$.

Definition 5.2: A boundary point which admits a barrier function is called regular.

Definition 5.3: A boundary point which does not admit a barrier function is called irregular.

<u>Theorem 5.1</u>: Let V be harmonic on a bounded region R such that

$$\lim_{\substack{\vec{x}\to\vec{x}^* \\ \vec{x}\in R}} V(\vec{x}) > 0 \text{ for any } \vec{x}^* \in \partial R.$$

Then there exists a $k > 0$ such that $V(\vec{x}) \geq k$ on R.

<u>Proof</u>: $V(\vec{x}) > 0$ by the minimum principle, for if there exists an $\vec{x}_o^* \in R$ such that $V(\vec{x}_o) = 0$, then $V(k) \equiv 0$ on all R. Suppose the theorem is false; then the real number zero is an accumulation point of the range set $\{V(\vec{x})\}$ which implies there exists a sequence $\{\vec{x}_n\} \subset R$ such that $\lim_{n\to\infty} V(\vec{x}_n) = 0$. By the Bolzano-Weierstrass theorem, $\{\vec{x}_n\}$ has cluster points in $R \cup \partial R$. Let $\vec{x}_o$ be any cluster point in R. Then $V(\vec{x}_o) = \lim_{k\to\infty} V(\vec{x}_{n(k)}) = 0$ and hence $V(\vec{x}) \equiv 0$ on R, which is a contradiction of hypothesis. If $\vec{x}_o^*$ is a cluster point in ∂R, then $\lim_{\vec{x}_{n(k)}\to\vec{x}_o^*} V(\vec{x}_{n(k)}) = 0$ implies $\lim_{\vec{x}\to\vec{x}_o^*} V(\vec{x}) \leq 0$ which is a contradiction of hypothesis.

<u>Theorem 5.2</u>: Let f be defined on ∂R and bounded above, where R is a bounded region, and let $\vec{x}_o$ be a regular boundary point for R. If f is upper semi-continuous at $\vec{x}_o^*$, then $\overline{\lim}_{\substack{\vec{x}\to\vec{x}_o^* \\ \vec{x}\in R}} \overline{H}_f^R(\vec{x}) \leq f(\vec{x}_o^*)$.

<u>Proof:</u> The fact that f is upper semi-continuous at $\vec{x}_o^*$ implies

$$f(x_o^*) \geq \overline{\lim_{\vec{x}^* \to \vec{x}_o^*}} f(\vec{x}^*).$$

If V is a barrier function at x_o^* then:

(1) $V > 0$ on R

(2) $\lim_{\substack{\vec{x} \to \vec{x}_o^* \\ \vec{x} \in R}} V(\vec{x}) = 0$

(3) $\underline{\lim}_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} V(\vec{x}) > 0, \begin{cases} \vec{x}^* \in \partial R \\ \vec{x}^* \neq \vec{x}_o^* \end{cases}$.

(4) For a sufficiently small ρ-ball B_ρ of $\vec{x}_o^*$, V is continuous on $B_\rho(\vec{x}_o^*) \cup \partial B_\rho(\vec{x}_o^*)$.

(5) V is harmonic on $B_\rho(\vec{x}_o{}^*)$.

We choose an $\epsilon > 0$ and then find $\rho(\epsilon) > 0$ such that if $|\vec{x}_o^* - \vec{x}^*| < \rho$, then $f(\vec{x}^*) < f(\vec{x}_o{}^*) + \epsilon$. Since f is upper semi-continuous, the existence of such a $\rho(\epsilon) > 0$ is always assured.

Let $k = \text{Inf} \left\{ k' : k' = \underline{\lim}_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in (R - \bar{B}_\rho) \\ \vec{x}^* \in \partial(R - \bar{B}_\rho)}} V(\vec{x}) \quad |\vec{x} - \vec{x}_o^*| \geq \rho \right\}$.

By theorem 5.1, $k > 0$.

Let $U(\vec{x}) = f(\vec{x}_o{}^*) + \epsilon + \dfrac{V(\vec{x})}{k} (M_f - f(\vec{x}_o^*))$ where $M_f = \sup_{\vec{x}^* \in \partial R} f(\vec{x}^*)$

We will show that $U \in F_s(f)$.

<u>Case 1:</u> $\vec{x}^* = \vec{x}_o^*$.

We note that $\lim_{\vec{x} \to \vec{x}_o^*} U(\vec{x}) = f(\vec{x}_o^*) + \epsilon > f(\vec{x}^*)$.

Case 2: $\vec{x}^* \neq \vec{x}_o^*$, $|\vec{x}^* - \vec{x}_o^*| < \rho$.

Since

$$\underset{\vec{x}\to\vec{x}^*}{\underline{\lim}} U(\vec{x}) = f(\vec{x}_o^*) + \epsilon + \left[\frac{1}{k} \underset{\vec{x}\to\vec{x}^*}{\underline{\lim}} V(\vec{x})\right](M_f - f(\vec{x}_o^*))$$

$$\geq f(x_o^*) + \epsilon > f(x^*).$$

Case 3: $\vec{x}^* \neq \vec{x}_o^*$, $|\vec{x}^* - \vec{x}_o^*| \geq \rho$.

Since

$U(\vec{x}) = f(\vec{x}_o^*) + \epsilon + \lambda(M_f - f(\vec{x}_o^*)$ where $\lambda \geq 1$ since $V(\vec{x}) \geq k$

therefore $\underset{\vec{x}\to\vec{x}^*}{\underline{\lim}} U(\vec{x}) \geq f(\vec{x}_o^*) + \epsilon + (M_f - f(\vec{x}_o^*)) = \epsilon + M_f \geq f(\vec{x}^*)$.

In all cases we have shown that $\underset{\vec{x}\to\vec{x}^*}{\underline{\lim}} U(\vec{x}) \geq f(x^*)$ and hence

$U \in F_s(f)$.

Therefore $\overline{H}_f^R(\vec{x}) \leq U(\vec{x})$ for any $\vec{x} \in R$.

Hence $\underset{\vec{x}\to\vec{x}_o^*}{\overline{\lim}} \overline{H}_f^R(\vec{x}) \leq \underset{\vec{x}\to\vec{x}_o^*}{\overline{\lim}} U(\vec{x}) \leq f(x_o^*) + \epsilon$

But $\epsilon > 0$ was arbitrary. Thus $\underset{\vec{x}\to\vec{x}_o^*}{\overline{\lim}} \overline{H}_f^R(\vec{x}) \leq f(\vec{x}_o^*)$.

Corollary: If f is lower semi-continuous at $\vec{x}_o^*$, then

$$\underset{\substack{\vec{x}\to\vec{x}_o^* \\ \vec{x}\in R}}{\underline{\lim}} \underline{H}_f^R(x) \geq f(x_o^*).$$

Theorem 5.3: The classical Dirichlet problem is solvable for any bounded region R in E^n such that every boundary point of R is regular.

Proof: Let R be a bounded region in E^n and let f be any continuous function on ∂R, where every point of ∂R is regular. Let $\vec{x}_o{}^*$ be any element of ∂R; then by theorem 5.2 and its corollary

$$\lim_{\substack{\vec{x} \to \vec{x}_o^* \\ \vec{x} \in R}} H_f^R(\vec{x}) \leq f(\vec{x}_o^*) \quad \text{and} \quad \lim_{\substack{\vec{x} \to \vec{x}_o^* \\ \vec{x} \in R}} H_f^R(\vec{x}) \geq f(x_o^*)$$

Hence

$$\lim_{\substack{\vec{x} \to \vec{x}_o^* \\ \vec{x} \in R}} H_f^R(\vec{x}) = f(\vec{x}_o^*)$$ and the theorem follows.

We may thus conclude by the above results, the following

Theorem 5.4: A boundary point $\vec{x}^*$ of R a bounded region in E^n is regular only if the generalized solution H_f^R corresponding to f tends to $f(\vec{x}^*)$ for any continuous f.

Thus the origin point of the sphere in Zaremba's example of a non-solvable classical "D" problem (chapter III, § 3) is an irregular boundary point of the deleted sphere.

We state now without proof, a modified version of the Kellogg-Evans theorem ([8] , p.2), a fundamental result in the theory of irregular boundary points.

Theorem 5.5: Let R be a bounded region in E^n and let $\{\vec{x}^*\} \subset \partial R$ be the set of all irregular boundary points. Then there exists a function $V > 0$ which is harmonic in R and such that $\lim_{\vec{x} \to \vec{x}^*} V(\vec{x}) = +\infty$.

<u>Concluding Remarks:</u> From the Kellogg-Evans theorem one can show ([4]) that H_f^R is independent of the values of f at the irregular boundary points. If $H_f^R \not\equiv -\infty$, there exists some subharmonic functions in R, general or continuous or harmonic, each of which is bounded above and of upper limit less than or equal to f at all regular boundary points; and H_f^R is the upper envelope of each of these three families. If $H_f^R \equiv -\infty$, there do not exist any functions belonging to any of the three families. From these results it follows that a boundary point $\vec{x}^* \in \partial R$ is regular if $\lim_{\substack{\vec{x} \to \vec{x}^* \\ \vec{x} \in R}} H_f^R(x) = f(\vec{x}^*)$ for every continuous f, and hence the regions for which the classical "D" problem is solvable are those and only those whose boundaries consist only of regular points. In other words, a classical "D" problem cannot in general be solved for a region R if ∂R possesses any irregular boundary points.

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