ARCS OF CYCLIC ORDER THREE IN THE CONFORMAL PLANE
ARCS OF CYCLIC ORDER THREE IN THE CONFORMAL PLANE

By

NEERA GUPTA

A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Master of Science

McMaster University
November, 1966
This thesis is concerned with the properties of arcs of cyclic order three in the conformal plane. It establishes necessary and sufficient conditions for the union of two arcs of cyclic order three to be again an arc of cyclic order three, and for an arc of cyclic order three to be extensible to a larger arc of cyclic order three.
PREFACE

The first chapter of this thesis summarizes some properties of the conformal plane and, in particular, of Mobius transformations.

The second chapter deals with the differentiability properties of general arcs in the conformal plane.

In the third chapter, properties of arcs of cyclic order three are discussed.

In the final chapter, we give necessary and sufficient conditions for the union of two arcs of cyclic order three to be again an arc of cyclic order three. We also give conditions under which an arc of cyclic order three can be extended to a larger arc of cyclic order three.
ACKNOWLEDGMENTS

The author wishes to express her sincere appreciation to her research director Dr. N. D. Lane for his willing assistance and guidance, and for the generosity with which he has given his valuable time during the course of this research.

Appreciation is also expressed to McMaster University for financial assistance.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Foundations of the Conformal Geometry</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>Conformal Differentiability</td>
<td>27</td>
</tr>
<tr>
<td>III</td>
<td>Arcs of Cyclic Order Three in the Conformal Plane</td>
<td>45</td>
</tr>
<tr>
<td>IV</td>
<td>Union and Extension of the Arcs of Cyclic Order Three</td>
<td>86</td>
</tr>
</tbody>
</table>
1.1. The Conformal Plane.

In the Euclidean plane we can introduce an 'ideal' point $\infty$ which we call the "point at infinity". The set of points in the Euclidean plane together with the point at infinity is called the conformal plane. We shall assume that every straight line passes through the point at infinity. Every point in the conformal plane can be represented by a complex number $z = x + iy$ or by $\infty$.

1.2. Stereographic Projection.

Consider the unit sphere $S$ whose equation in the Euclidean three dimensional space is

$$x^2 + y^2 + u^2 = 1.$$ 

With every point on $S$, except $N:(0,0,1)$, we can associate the complex number

$$(1.2-1) \quad z = \frac{x + iy}{1 - u}$$

and this correspondence is one-to-one. The correspondence can be completed by letting the point at infinity correspond to $(0,0,1)$. 

1
Thus we can regard the sphere as a representation of the conformal plane.

Let $P(x,y,u)$ be any point other than $N$ on the sphere. Then the line $NP$ meets the plane in the point $P'$, given by $(1.2-1)$. This mapping of the sphere onto the $(x,y)$-plane is a central projection from the centre $(0,0,1)$. It is called a stereographic projection (Fig. 1).

1.2.1. Theorem. Under stereographic projection circles on the sphere are mapped into circles and straight lines of the plane and vice-versa.

Proof. A circle on the sphere $S$ is the intersection of the sphere with a plane

$$(1.2-2) \quad Ax + By + Cu = D$$

with $A^2 + B^2 + C^2 > D^2$ to ensure actual intersection. Let $(\xi, \eta, \zeta)$ be any point on the sphere corresponding to the point $(x,y,0)$ in the $(x,y)$-plane. By $(1.2-1)$ we have

$$z = \frac{\xi + i \eta}{1 - \zeta}, \text{ where } z = x + iy.$$ 

Thus

$$(1.2-3) \quad x = \frac{\xi}{1 - \zeta}, \quad y = \frac{\eta}{1 - \zeta}.$$ 

Now $|z|^2 = \frac{\xi^2 + \eta^2}{(1 - \zeta)^2}$. But $\xi^2 + \eta^2 + \zeta^2 = 1$.

$$|z|^2 = \frac{1 + \xi}{1 - \zeta}.$$
Solving this for $\zeta$, and then computing $\zeta$ and $\eta$, we obtain

$$\zeta = \frac{|z|^2 - 1}{|z|^2 + 1}, \quad \eta = \frac{z + \overline{z}}{|z|^2 + 1} \quad \text{and} \quad \eta = \frac{i(z - \overline{z})}{|z|^2 + 1}.$$

Thus if $(\zeta, \eta, \zeta)$ also lies on the plane given by equation (1.2-2.) we obtain

$$A(z + \overline{z}) + B(\overline{z} - z) + C(|z|^2 - 1) = D(|z|^2 + 1).$$

or

(1.2-4.) \hspace{1cm} (D - C)(x^2 + y^2) = 2Ax - 2By + C + D = 0.

For $D \neq C$, this is the equation of a circle, and for $D = C$ it represents a straight line. In the latter case, the plane given by (1.2-2.), and hence also the circle on the sphere, passes through $(0, 0, 1)$.

To prove the converse, we start with the equation (1.2-4.) with $A^2 + B^2 + C^2 > D^2$ and retrace our steps. Equation (1.2-4.) represents all lines and circles in the $(x, y)$-plane. Using formulas (1.2-3.) to express $x$ and $y$ in terms of $(\zeta, \eta, \zeta)$ we see that the latter point on the sphere must lie in the plane (1.2-2.).

Thus the stereographic projection of a circle or a straight line in the plane onto the sphere is always a circle, and this circle passes through $N: (0, 0, 1)$ if and only if the pre-image is a straight line.

1.3. The Möbius Plane.

In general by a conformal transformation we mean an angle-preserving mapping. There are conformal representations in which...
circles are not necessarily transformed into circles, but we shall not consider these in this discussion.

Rather, we shall restrict our attention to the mappings $T$ of the form

$$w = \frac{az + b}{cz + d},$$

whose coefficients $a,b,c,d$ are complex numbers. We assume that $ad - bc \neq 0$, so that $w$ is not independent of $z$. This also makes $w$ well-defined, except when $c \neq 0$ and $z = -\frac{d}{c}$. A mapping of this form is called a "Mobius Transformation." (It is also called a circular transformation and a linear (fractional) transformation.) Certain geometrical properties will remain invariant under these transformations.

**Definition.** (Mobius) Conformal Geometry is the study of the properties which remain invariant under the conformal mappings (1.3-1.)

The equation (1.3-1.) can be solved with respect to $z$ and yields

$$z = \frac{dw - b}{-cw + a}.$$ 

The resulting transformation is inverse to $T$; we denote it by $T^{-1}$. The existence of an inverse shows that the correspondence defined by $T$ is one-to-one.

1.4. Cross-ratio.

Given any three mutually distinct points $z_2, z_3, z_4$ in the conformal plane, there exists a Mobius transformation $T$ which carries these points into $0, 1, \infty$. In particular, if none of the given points is $\infty$, $T$ will be given by
\[ Tz = \frac{z - z_2}{z - z_4} \cdot \frac{z_3 - z_2}{z_3 - z_4} \cdot \frac{z - z_2}{z - z_4} \cdot \frac{z_3 - z_2}{z_3 - z_4} \]

If \( z_2, z_3 \) or \( z_4 = \infty \), the transformation reduces to

\[ \frac{z_3 - z_4}{z - z_4}, \frac{z - z_2}{z - z_4}, \frac{z - z_2}{z_3 - z_2} \]

respectively.

**Definition.** The **cross-ratio** \((z_1, z_2, z_3, z_4)\) is the image of \( z_1 \) under a **Möbius transformation** \( T \) which carries \( z_2, z_3, z_4 \) into \( 0, 1, \infty \).

**1.4.1. Theorem.** Under a Möbius transformation \( U \), the cross-ratio of any four distinct complex numbers \( z_1, z_2, z_3, z_4 \) is invariant; i.e.,

\[ (Uz_1, Uz_2, Uz_3, Uz_4) = (z_1, z_2, z_3, z_4). \]

**Proof.** If \( Tz_1 = (z_1, z_2, z_3, z_4) \), then \( TU^{-1} \) carries \( Uz_2, Uz_3, Uz_4 \) into \( 0, 1, \infty \). By definition, we have

\[ (Uz_1, Uz_2, Uz_3, Uz_4) = TU^{-1} (Uz_1) = Tz_1 = (z_1, z_2, z_3, z_4). \]

**1.5. Angles in the conformal plane.**

Let \( z_1, z_2, z_3 \) and \( z_4 \) be the complex numbers \( \lambda \) (with finite coefficients), 0, 1 and \( \infty \) respectively. Then \( (z_1, z_2, z_3, z_4) = \lambda \), and hence

\[ \text{amp}(z_1, z_2, z_3, z_4) = \text{amp}(\lambda), \]

(Fig. 2).

The relation (1.5-1.) is unaltered if we interchange simultaneously \( z_1 \) and \( z_4 \) and \( z_2 \) and \( z_3 \). It is likewise invariant under a Möbius trans-
FIGURE 2
formation on account of the relation (1,4-1.).

Let $C_1$ and $C_2$ be two circles intersecting at two points $P$ and $R$ (Fig. 3). Let $Q$ be a point on $C_1$ and $S$ a point on $C_2$, $Q \neq P \neq S$ and $Q \neq R \neq S$. By a Mobius transformation, we can map $P$, $Q$, $R$ into $0,1,\infty$ respectively. Let $S$ be mapped into a complex number $\lambda$. Then $C_1$ and $C_2$ will be mapped into the two straight lines through $0$ and $1$ and $0$ and $\lambda$ respectively (Fig. 4). Let these lines intersect at an angle $\theta$. Then

$$\theta = \text{amp}(\lambda).$$

On account of the relation (1,5-2.) and Theorem 1,4,1,

$$\theta = \text{amp} \frac{(P-Q)(R-S)}{(P-S)(R-Q)}.$$

Thus we can define an angle between two circles in the conformal plane as follows.

If two circles $C_1$ and $C_2$ intersect at $P$ and $R$, and if $Q$ and $S$ lie on $C_1$ and $C_2$ respectively, where $P,Q,R,S$ are mutually distinct, then the angle between $C_1$ and $C_2$ is the amplitude of the cross-ratio $(P,Q,R,S)$, where $P,Q,R,S$ are complex numbers.

In particular, if the amplitude of the cross-ratio is $\frac{\pi}{2}$, then we say that $C_1$ and $C_2$ are orthogonal.

Since a Mobius transformation preserves the cross-ratio of four points, it preserves the angle between two circles.
We define two circles to be tangent to each other if they have a single point in common.

Remark. We observe that if $R$ tends to $P$ and $S \neq P \neq Q$, then

$$\lim_{{R \to P, S \neq P \neq Q}} \text{amp}(P, Q, R, S) = \text{amp}(1) = 0,$$

and hence

$$\lim \angle (C_1, C_2) = 0$$

whether or not the circles $C_1 = C(P, Q, R)$ and $C_2 = C(P, R, S)$ themselves converge. Thus we may define the angle between two tangent circles to be zero.

1.6. Orientation of a circle.

An orientation of a circle $C$ is determined by an ordered triple of mutually distinct points $z_1, z_2, z_3$ on $C$. With respect to this orientation a point $z$ not on $C$ is said to lie to the left of $C$ if $\text{Im}(z, z_1, z_2, z_3) > 0$ and to the right of $C$ if $\text{Im}(z, z_1, z_2, z_3) < 0$. If we orient the circle $C$, then the region lying to the left of the oriented circle is called the "interior" of $C$, and the region to the right is called the "exterior" of $C$. Thus any proper circle $C$ (i.e., not a point-circle) divides the conformal plane into two open regions, the "interior" $C_1$ of $C$ and the "exterior" $C_2$ of $C$.

We now show that there are only two different orientations; i.e., the distinction between left and right is the same for all triples, while the meanings may be reversed.

Because of the invariance of the cross-ratio it is sufficient to consider the case where $C$ is a real-axis. We have then to examine
\[ \text{Im}(z, z_1, z_2, z_3). \] Writing \( (z, z_1, z_2, z_3) = \frac{az + b}{cz + d} \) with real coefficients, we obtain

\[ \text{Im}(z, z_1, z_2, z_3) = \frac{ad - bc}{|cz + d|^2} \text{Im}z. \]

Hence the distinction between left and right is identical with the distinction between the upper and lower half-plane.

Since a M"obius transformation \( T \) carries the real axis into a circle which we orient through the triple \( Tz_1, Tz_2, Tz_3 \), hence from the invariance of the cross-ratio it follows that the left and right of the real axis will correspond to the left and right of the image circle.

In general there is no way or reason to compare the orientations of two circles. An exception occurs when the circles are tangent to each other. In this case they can be transformed into parallel lines, and the circles are said to be equally oriented if they correspond to lines with the same direction. Another exception occurs when we consider the circles through three points of an arc of cyclic order three (cf. 3.3.2).

1.6.1. Theorem. A M"obius transformation preserves the angles between oriented circles.

Proof. Let \( C_1 \) and \( C_2 \) be any two oriented circles. Suppose \( C_1 \) and \( C_2 \) intersect at two points \( P \) and \( R \). If \( Q \in C_1 \) and \( S \in C_2 \) then we have defined the angle between \( C_1 \) and \( C_2 \) as the amplitude of the cross-ratio \( (P, Q, R, S) \) (cf. Theorem 1.4.1). Hence we conclude that the amplitude of the cross-ratio is preserved, i.e., angle between the oriented circles is preserved.
In the geometric representation the orientation \( z_1, z_2, z_3 \)
can be indicated by an arrow which points from \( z_1 \) over \( z_2 \) to \( z_3 \)
(Fig. 5).

When the unextended complex plane (Euclidean plane) is
considered as part of the extended plane, (conformal plane) the point
at infinity is distinguished. We can therefore define an absolute
positive orientation of all finite circles by the requirement that point
at infinity should lie to the right of the oriented circles.

On a Riemann sphere there is no reason to call one side of a
circle the interior.

1.7. Reflection

1.7.1. The points \( z \) and \( \overline{z} \) are symmetric with respect to the
real axis. A Möbius transformation with real coefficients carries the
real axis into itself and \( z, \overline{z} \) into points which are again symmetric.
More generally, if a Möbius transformation \( T \) carries the real axis into
a circle \( C \), we shall say that the points \( w \) and \( w' \) defined by \( w = Tz \) and
\( w' = T\overline{z} \) are symmetric with respect to \( C \).

The relation \( T^{-1}w = T^{-1}w' \) between \( w \) and \( w' \) and \( C \) does not depend
on the particular choice of \( T \). For if \( S \) is another transformation which
carries the real axis into \( C \), then \( S^{-1}T \) is a real transformation and
hence
\[
S^{-1}w = S^{-1}Tz = \overline{\zeta} \quad \text{and} \quad S^{-1}w' = S^{-1}T\overline{z} = \overline{\zeta}
\]
are also conjugate. Thus \( w = S\overline{\zeta} \) and \( w' = S\overline{\zeta} \) and so \( S^{-1}w' = \overline{S^{-1}w} \).
FIGURE 5
Suppose $T$ carries $0, 1, \infty$ into $w_1, w_2, w_3$ on the circle $C$. The transformation defined by

$$w \mapsto (w, w_1, w_2, w_3)$$

maps $w_1, w_2, w_3$ into $0, 1, \infty$ respectively. Hence

$$(w, w_1, w_2, w_3) = T^{-1}w_*$$

Thus

$$T^{-1}w' = T^{-1}w \iff (w', w_1, w_2, w_3) = (w, w_1, w_2, w_3).$$

Then symmetry can be defined in the following terms:

Definition. The points $w$ and $w'$ are said to be symmetric with respect to the circle $C$ through $w_1, w_2, w_3$ if and only if

$$(w', w_1, w_2, w_3) = (w, w_1, w_2, w_3).$$

The points on $C$ and any only those are symmetric to themselves.

The mapping which carries $w$ into $w'$ is a one-to-one correspondence and is called reflection with respect to $C$. A reflection is isogonal but not conformal, i.e., it preserves angles in magnitude but reverses the signs. Two reflections will evidently result in a M"obius transformation.

1.7.2. In this subsection we would revert to the notation $z, z', z_1, z_2, z_3$ instead of $w, w', w_1, w_2, w_3$ and investigate the geometric significance of symmetry. Suppose first that one of the points $z_1, z_2, z_3$ is the point at infinity. Then choosing $z_3 = \infty$, the condition for symmetry becomes
Taking absolute values, we obtain,

\[ |z' - z_1| = |z - z_1| \]

Here \( z_1 \) can be any finite point on \( C \), which in this case is a straight line. Thus we conclude that \( z \) and \( z' \) are equidistant from all points on \( C \). By the relation (1.7-1) we have further

\[ \text{Im} \frac{z' - z}{z_2 - z_1} = -\text{Im} \frac{z - z_1}{z_2 - z_1}. \]

Hence \( z \) and \( z' \) are in different half-planes determined by \( C \).

Now we consider the case where \( z_1, z_2 \) and \( z_3 \) all are finite. Let \( a \) be the centre and \( R \) the radius of the circle \( C \) through \( z_1, z_2, z_3 \).

Since cross-ratio is invariant under a Mobius transformation we have

\[
\frac{z_1, z_{1}', z_{2}, z_{3}'}{z_1, z_2, z_3} = \frac{(z-a, z_{1}-a, z_{2}-a, z_{3}-a)}{(z-a, z_{1}-a, z_{2}-a, z_{3}-a)}
\]

\[
= \left( \frac{z_{1}-a}{z_1-a}, \frac{z_{2}-a}{z_2-a}, \frac{z_{3}-a}{z_3-a} \right)
\]

\[
= \left( \frac{\frac{1}{z-a}}{z-\bar{a}}, \frac{\frac{R^2}{z_1-a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}}{\frac{R^2}{z_{1}-a}, \frac{R^2}{z_{2}-a}, \frac{R^2}{z_{3}-a}} \right)
\]

\[
= \left( \frac{\frac{R^2}{z-\bar{a}} + a, z_1-a, z_2-a, z_3-a}{\frac{R^2}{z_{1}-a}, \frac{R^2}{z_{2}-a}, \frac{R^2}{z_{3}-a}} \right).
\]
This equation shows that the symmetric point of $z$ is $z' = \frac{R^2}{z-a} + a$ or that $z$ and $z'$ satisfy the relation

$$(z'-a)(\overline{z-a}) = R^2,$$

from which we obtain

$$|z' - a| = |z - a| = R,$$

which shows that $z$ and $z'$ lie in different regions determined by $C$.

Also the ratio $\frac{z'-a}{z-a}$ is real and positive, say $z' - a = k(z-a)$ which means that the amplitude of $z' - a$ is equal to the amplitude of $z - a$ and hence $z$ and $z'$ are situated on the same half line from the centre $a$. There is a simple geometric construction of the symmetric point of $z$ (Fig. 6). We note that the symmetric point of $a$ is $\infty$.

**Theorem.** If a Möbius transformation $T$ carries a circle $C_1$ into a circle $C_2$, then it transforms any pair of symmetric points with respect to $C_1$ into a pair of symmetric points with respect to $C_2$, i.e., Möbius transformation preserves symmetry.

**Proof.** If $C_1$ or $C_2$ is the real axis the theorem follows from the definition of symmetry. In the general case the assertion follows by the use of an intermediate transformation $U$ which carries $C_1$ into the real axis. Thus $TU^{-1}$ takes the points $Uz$ and $Uz'$ which are symmetric with respect to the real axis into the symmetric points $Tz$ and $Tz'$ on $C_2$. 
1.7.3. Consider a Möbius transformation of the form

\[ w = k \frac{z - a}{z - b}, \]

where \( k \neq 0 \) is a constant, say \( k = 1 \). Here \( z = a \) corresponds to \( w = 0 \) and \( z = b \) to \( w = \infty \). It follows that the straight lines through the origin of the w-plane are images of the circles through \( a \) and \( b \) in the z-plane. These circles in the z-plane have equations of the form

\[ \text{amp} \frac{z - a}{z - b} = \theta, \quad -\pi \leq \theta \leq \pi. \]

On the other hand, the concentric circles about the origin in the w-plane, \( |w| = \rho \) (where \( \rho > 0 \) is a constant) correspond to the circles in the z-plane with the equation

\[ \left| \frac{z - a}{z - b} \right| = \rho. \]

These are called circles of "Apollonius", with limit points \( a \) and \( b \).

Denote by \( C_1 \) the circles through \( a \) and \( b \) and by \( C_2 \) the circles of "Apollonius" with these limit points (Fig. 7 for w-plane and Fig. 8 for z-plane).

The configuration formed by all the circles \( C_1 \) and \( C_2 \) is called the Steiner Configuration determined by \( a \) and \( b \). This Steiner Configuration has many interesting properties.

(i) There is exactly one \( C_1 \) and one \( C_2 \) through each point \( \neq a, b \), in the plane.

Proof. Any point of the plane, together with the point \( a \) and \( b \), determines a unique circle \( C_1 \). Any finite point \( c \) in the plane
FIGURE 7
determines a unique \[ f = \left| \frac{z - a}{z - b} \right| \] and the unique circle \( C_2 \):

\[ \left| \frac{z - a}{z - b} \right| = f. \]

(ii) Every \( C_1 \) meets every \( C_2 \) at right angles.

Proof. This follows from the fact that a Möbius transformation preserves the angles between two circles.

(iii) Reflection in a \( C_1 \) transforms every \( C_2 \) into itself and every \( C_1 \) into another \( C_1 \). Reflection in a \( C_2 \) transforms every \( C_1 \) into itself and every \( C_2 \) into another \( C_2 \).

Proof. Let \( z \) be any point on \( C_2 \) and \( z_2 \) be on \( C_1 \) (Fig. 9). Let \( z' \) be the image of \( z \) with respect to a reflection in \( C_1 \). Then the following relation holds.

\[ (z', a, z_2, b) = (z_2, a, z, b) \]

\[ \frac{z' - a}{z' - b} = \frac{z_2 - a}{z_2 - b}. \]

Hence

\[ \left| \frac{z' - a}{z' - b} \right| = \left| \frac{z_2 - a}{z_2 - b} \right|, \]

and

\[ \left| \frac{z' - a}{z' - b} \right| = \left| \frac{z - a}{z - b} \right| = f, \text{ say.} \]

This shows that \( z' \) lies on \( C_2 \).

Since points on \( C_1 \) are always symmetric to themselves with respect to \( C_1 \) hence by reflection the point \( a \) is mapped into itself and \( b \) into itself. Thus a circle through the point \( a \) and \( b \) is transformed into a circle through the point \( a \) and \( b \).
For reflection in a $C_2$, consider the $w$-plane. Then the circles $C_2$ are the concentric circles about the origin and $C_1$ are the straight lines through the origin. Let $z$ by any point on $C_1$. Then by reflection in a $C_2$, $z$ is mapped into the symmetric point $z'$. But by the discussion in the subsection 1.7.2 we know that $z$ and $z'$ are situated on the same half-line from the centre of the circle $C_2$. Hence $z'$ lies on $C_1$. Thus $C_1$ is mapped onto itself by a reflection in a $C_2$. By a Mobius transformation we can extend the property (iii) to the Steiner configuration.

(iv) The limit points are symmetric with respect to each $C_2$, but not with respect to any other circle.

Proof. Let the point $a$ be symmetric to $b$ with respect to a circle $C$. Then for any points $z_1, z_2, z_3$ on $C$ we have

$$(a, z_1, z_2, z_3) = (b, z_1, z_2, z_3).$$

Or

$$\frac{a - z_1}{a - z_3} / \frac{z_2 - z_1}{z_2 - z_3} = \frac{b - z_1}{b - z_3} / \frac{z_2 - z_1}{z_2 - z_3}.$$ 

Hence

$$\left| \frac{a - z_1}{a - z_3} \right| / \left| \frac{z_2 - z_1}{z_2 - z_3} \right| = \left| \frac{b - z_1}{b - z_3} \right| / \left| \frac{z_2 - z_1}{z_2 - z_3} \right|$$

and

$$\left| \frac{a - z_1}{a - z_3} \right| = \left| \frac{b - z_1}{b - z_3} \right|.$$

Finally

$$\left| \frac{a - z_1}{b - z_1} \right| = \left| \frac{a - z_3}{b - z_3} \right| = 0,$$ say.
But this relation holds if and only if \( z_1 \) and \( z_3 \) lie on the \( C_2 \) given by

\[
\left| \frac{z - a}{z - b} \right| = f;
\]

cf. 1.7.3.

Hence the point \( a \) is symmetric to \( b \) with respect to \( C_2 \) but not with respect to any other circle.

The points \( a \) and \( b \) are called the fundamental points of the pencil \( C_1 \), and \( C_1 \) is called a pencil of the first kind. \( C_2 \) is called a pencil of the third kind (Fig. 8).

A limiting case of a pencil of the first kind as \( a \) tends to \( b \) is a pencil of the second kind. It possesses only one fundamental point and is identical with the set of those circles that touch a given circle at that point (Fig. 10).

1.8. The closure property of the conformal plane.

We know that the conformal plane may be represented on the surface of a sphere in three dimensional Euclidean space. Hence every infinite sequence of points in the conformal plane possesses at least one accumulation point, which also lies in the plane. Thus the conformal plane is closed and compact.

Theorem. Every infinite sequence of circles in the conformal plane possesses at least one limit circle.

Proof. Let \( \{ C_n \} \) be an infinite sequence of circles in the conformal plane. Then there exists a subsequence, \( \{ C'_m \} \subset \{ C_n \} \) of circles which contains an infinite sequence of points possessing an
FIGURE 10
accumulation point. Again there is a subsequence \( \{ C''_q \} \subseteq \{ C'_m \} \) of circles which contains a different sequence of points possessing an accumulation point. Finally there is a subsequence \( \{ C''_r \} \subseteq \{ C''_q \} \) of circles which contains yet another sequence of points possessing an accumulation point. Thus we have a sequence \( C''' \) of circles which possesses a limiting circle, the circle determined by three accumulation points.

We call a limit circle an accumulation circle of the original sequence.

1.9. Convergence.

A sequence of points \( P_1, P_2, \ldots \) is said to be convergent to \( P \), if there exists a number \( n = n(C) \) for every circle \( C \) with \( P \subseteq C \), such that \( y < C \) if \( y > n \).

In the same way, convergence of circles to a point is defined. Such a point is called a point-circle.

A sequence of circles \( C_1, C_2, \ldots \) in the conformal plane is said to be convergent to \( C \) if there exists a number \( n = n(C', C'') \), for every pair of circles with \( C' \subseteq C \) and \( C'' \subseteq C \), such that \( C' \subseteq C \) and \( C'' \subseteq C \) for every \( y > n \).
CHAPTER II

CONFORMAL DIFFERENTIABILITY

2.1. Arcs.

An arc $A$ is the continuous image of a real closed interval. Thus if a sequence of points of that parameter interval converges to a point $p$, then their image points converge to the image of $p$. We shall use the same letters $p, t, \ldots$ to denote both the parameters and their images on $A$. The end (interior) points of $A$ are the images of the end (interior) points of the parameter interval.

A neighbourhood of $p$ on $A$ is the image of a neighbourhood of the parameter $p$ on the parameter interval. If $p$ is an interior point $A$, this neighbourhood is decomposed by $p$ into two (open) one-sided neighbourhoods.

From the definition, different points of $A$, i.e., points with different parameters, may coincide with the same point of the conformal plane. However, we shall assume that each point $p$ of $A$ has a neighbourhood such that no other point of that neighbourhood coincides with $p$ (cf. condition I, Sec. 2.3).

2.2. Support and Intersection.

Let $p$ be an interior point of $A$ in the conformal-plane. Then $p$ is called a point of support (intersection) with respect to a circle $C$, if a sufficiently small neighbourhood of $p$ in $A$ is decomposed by $p$.
into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by $C$. The circle $C$ is then called a 
**supporting (intersecting) circle** of $A$ at $p$. Thus $C$ supports $A$ at $p$ even if $p \not\in C$. By definition, the point circle $p$ always supports $A$ at $p$
(Fig. 11).

It is possible that every neighbourhood of $p$ has points $\not\in C$ in common with $C$. Then $C$ **neither supports nor intersects** $A$ at $p$.  

2.3. Differentiable points of an arc in the conformal plane.

Suppose $p$ is a fixed point of an arc $A$ and $t$ a variable point. If $P \neq Q$ are points different from $p$, the unique circle through these points will be denoted by $C(P,Q,p)$.

An arc $A$ is called **once conformally differentiable** at $p$ if it satisfies the following:

**Condition I.** To every point $Q \neq p$ and to every sequence of points $t \to p$, $t \neq p$, $t \in A$ there exists a circle $C'$ such that

$C(t,p,Q) \to C'$.

The limit circle $C'$ is called a **tangent circle** at $p$ and is denoted by $C(r,Q)$. By our definition this limit tangent circle is independent of the choice of the sequence of points $t$. The point $p$ itself is the **tangent-point circle** of $A$ at $p$. The family of tangent circles together with point circle $p$ will be denoted by $r$.

We call $A$ **conformally differentiable** at $p$ if it satisfies

**Condition I** and
Condition II. \( \lim_{t \to p} C(\tau, t) \) exists.

The limit osculating circle is denoted by \( C(p) \).

Remark. Condition I does not imply Condition II.
(c.f. 4, Sec. 7, P. 516).

2.3.1. Theorem. The set \( \pi = \pi(p) \) of all the tangent circles of \( A \) at \( p \) is a pencil of the second kind with the fundamental point \( p \).

Proof. Let \( P, Q, R \) be three mutually distinct points. If the point \( R' \neq R \) converges to \( R \), then the angle between the circles \( C(R', R, P) \) and \( C(R', R, Q) \) converges to zero; cf. 1.5. We choose \( R = p \) and \( R' = t \). Since the angle between two circles depends on them continuously, we conclude that any two tangent circles at \( p \) touch each other at that point. Thus two tangent circles of \( A \) at \( p \) that have another point in common are identical. In particular, there exists one and only one tangent circle at \( p \) through each point different from \( p \).

2.3.2. Theorem. Suppose \( A \) is once conformally differentiable at \( p \). Let \( \pi \) be a pencil of second kind with \( p \) as its fundamental point; \( \pi \neq \tau \). If the points \( t \neq p \) converge to \( p \) then \( C(\pi, t) \to p \).

Proof. Let us assume that the statement in the theorem is false. Then there exists a circle \( C \) such that \( p \in C \) and a sequence of points \( t \to p, t \neq p \) such that \( C(\pi, t) \nsubseteq C \) for each \( t \). Let \( C_1 \) and \( C_2 \) be two circles of \( \pi \) that touch \( C \). We may assume that \( \pi \) is
oriented such that $C$ lies in the closure of $C_1^* \cap C_2^*$. Then this
closed domain also contains the circles $C(n, t)$ and therefore the points
t (Fig. 12).

Let $Q$ be any point of $C_1$; $Q \neq p$. If a sequence of points $R$
converges to $p$ through the above domain, then the circles $C(p, Q, R)$
converge to $C_1$. Choosing $R = t$, we obtain $C_1 = C(t, Q)$, while $t \neq \pi$.
This is a contradiction.

2.4. In the following $p$ can be either an interior point or
an end point of $A$.

We call $C$ a general tangent circle of an arc $A$ at a point $p$
if there exists a sequence of triplets of mutually distinct points
$t, u, Q$ such that

$$\lim_{t, u \to p} C(t, u, Q) = C.$$  \hfill (2.4-1)

If in addition, $Q \in A$ also converges to $p$, then we call $C$ a general
osculating circle of $A$ at $p$.

$A$ is said to be once strongly conformally differentiable at
$p$ if the following condition is satisfied:

Condition I'. Let $R \neq p$, $Q \to R$ and $t, u$ be two distinct
points on $A$. Then

$$\lim_{t, u \to p} C(t, u, Q) \text{ exists.}$$

$$Q \to R$$
FIGURE 12
2.4.1 Lemma. Condition I' implies Condition I.

Proof. Condition I' implies that the limit circle given by the relation (2.4-1.) depends on p and R but not on the choice of the particular sequences t and u. Specializing \( Q = R \) and \( u = p \), we see that Condition I' implies Condition I and therefore,

\[
\lim_{u,v \to p, Q \to R} C(t,u,Q) = C(\tau,R).
\]

Thus the general tangent circles of a point which satisfies Condition I' are identical with its ordinary ones.

We call A strongly conformally differentiable at p if it satisfies Condition I' and

\text{Condition II'}. Let \( t,u,v \) be three mutually distinct points on A. Then

\[
\lim_{t,u,v \to p} C(t,u,v) \text{ exists.}
\]

2.4.2. Lemma. Conditions II' and I imply Condition II.

Proof. Suppose a sequence of points \( \{ u_n \} \) converges on A to p. By choosing a suitable subsequence of \( \{ u_n \} \) we may assume that the sequence \( \{ C(\tau,u_n) \} \) converges. Each \( C(\tau,u_n) \) can be approximated by a circle \( C(p,u_n,v_n) \) such that the sequence \( \{ C(p,u_n,v_n) \} \) has the same limit circle and such that the sequence \( \{ v_n \} \) also converges to p. On account of Condition II', \( \lim C(t,u,v) \) and in particular, \( \lim C(p,u_n,v_n) \) is independent of the choice of the sequences \( t,u,v \) converging to p. Hence the same will hold true of \( \lim C(\tau,u_n) \).
Condition II is satisfied, and we have
\[
\lim_{t,u,v \to p} C(t,u,v) = \lim_{u,v \to p} C(p,u,v) = \lim_{u \to p} C(t,u) = C(p).
\]

Thus strong differentiability implies ordinary differentiability and \(C(p)\) is one and only one general osculating circle.

2.4.3. Theorem. Condition II' with \(\lim C(t,u,v) \neq p\), or with \(p\) an end-point or with Condition I implies Condition I'.

Proof. Assume Condition II' and let \(R \neq p\).

Case I: \(\lim_{t,u,v \to p} C(t,u,v) \neq p\).

Choose a point \(S\) on \(C(t,u,v)\) such that \(S\) does not tend to \(p\) as \(t,u,v\) converge to \(p\); thus \(C(t,u,v) = C(t,u,S)\). Then the angle between \(C(Q,t,u)\) and \(C(S,t,u)\) is given by the amplitude of the cross-ratio \((u,Q,t,S)\). This amplitude tends to zero as \(t\) and \(u\) converge to \(p\).

Hence any accumulation circle of the circle \((Q,t,u)\) is the unique circle through \(R\), which is tangent to \(C\) at \(p\). Thus \(A\) satisfies Condition I' at \(p\), (Fig. 13).

Case II: \(\lim_{t,u,v \to p} C(t,u,v) = p\).

In this case we can choose a subarc \(B\) of \(A\) with \(p \in B\) such that \(R \notin C(t,u,v)\) for every choice of \(t,u,v\) on \(B\). This implies that \(C(t,u,R)\) does not meet \(B\) elsewhere; thus \(B\) has "R-order" two, i.e., no circle through \(R\) meets \(B\) more than twice.
We first prove that if \( p \) is an end-point of \( B \) then \( B \) and hence \( A \) satisfies Condition I at \( p \).

Suppose Condition I is not satisfied. Then there are two sequences of points \( s_{2k} \) and \( s_{2k+1} \) different from \( p \) and converging on \( B \) to \( p \) such that the circles \( C_{2k} = C(s_{2k}, p, R) \) and \( C_{2k+1} = C(s_{2k+1}, p, R) \) converge to different limit circles \( C_0 \) and \( C_1 \) respectively. Since \( p \) is an end-point of \( B \) we may assume that \( s_{n+1} \) lies between \( p \) and \( s_n \). If \( k \) is large \( C_{2k} \) will lie close to \( C_0 \). Let \( C \) and \( C' \) be two circles through \( p \) and \( R \) which separate \( C_0 \) and \( C_1 \). Then \( C \cup C' \) will separate \( C_n \) and \( C_{n+1} \) and therefore also \( s_n \) and \( s_{n+1} \) for every large \( n \). Hence the subarc of \( B \) bounded by \( s_n \) and \( s_{n+1} \) will meet \( C \cup C' \) in at least one-point. Thus \( B \) will meet \( C \cup C' \) infinitely many times. This is impossible. Hence Condition I holds at \( p \).

We now prove that case 2 and Condition I imply Condition I', whether or not \( p \) is an end-point of \( A \).

Since \( B \) has "R-order" two, as \( x \) moves continuously and monotonically from \( p \) to \( v \) \( C(R, t, x) \) moves continuously and monotonically from \( C(R, p, t) \) to \( C(R, t, v) \). Thus

\[
C(R, t, u) \subseteq \left[ C(R, p, t) \cap C(R, t, v) \right] \cup \left[ C(R, p, t) \cap C(R, t, v) \right] \cup R \cup t, \tag{Fig. 14}
\]

(cf. 3, 2.7.)
FIGURE 14
Let $D$ denote any accumulation circle of the $C(R,t,u)$. Letting $t,u \to p$ we conclude that $D$ lies in the closure of

$$\left[ C(\tau,R) \cap C(R,p,v) \right] \cup \left[ C(\tau,R) \cap C(R,p,v) \right]$$

for each choice of $v$ on $A$. Hence letting $v \to p$ we obtain

$$\lim_{t,u \to p} C(R,t,u) = C(\tau,R).$$

Thus Condition I' is satisfied.

Remark. If Condition II' holds with $\lim C(t,u,v) = p$ and if $p$ is an interior point of $A$, then Condition I' need not hold.

For example, consider the arc $A = A_3 \cup p \cup A_3$ where $A_3$ and $A_3'$ are given by

$$x = t^2, \quad y = t^3, \quad 0 \leq t \leq \frac{1}{2},$$

and

$$x = -t^3, \quad y = t^2, \quad -\frac{1}{2} \leq t < 0,$$

respectively (Fig. 15).

Now $A$ satisfies Condition II' at the origin $p$, and

$$\lim_{t,u,v \to p} C(t,u,v) = p.$$

However Condition I, and hence Condition I' does not hold.

Proof. Since $A_3$ and $A_3'$ are of cyclic order three they are strongly differentiable at $p$; cf. Theorem 3.3.2.

Let $\tau$ and $\tau'$ denote families of tangent circles of $A_3$ and $A_3'$ respectively.

Hence

$$\lim_{s \to p} C(s,t,u) = \lim_{s \to p} C(\tau,s).$$
FIGURE 15

\[ C(\tau \cdot R) \]

\[ A' \]

\[ A \]

\[ p \]

\[ C(\tau', R) \]
But \( C(t,s) \) is given by equation

\[(2.4-2.) \quad x^2 + y^2 - (s-s^3)y = 0.\]

Thus as \( s \to p \) (i.e., \( s \to 0 \)) this circle tends to the circle

\[x^2 + y^2 = 0,\]

which is the point-circle \( p \).

Hence

\[
\lim_{s,t,u \to p} C(s,t,u) = \lim_{s \to p} C(t,s) = p.
\]

Let \( R \not\in p \) be a fixed point. We first show that no circle through \( R \) can meet \( A_3 \) and \( A'_3 \) twice each near \( p \).

A circle through \( R \) and two points of \( A_3 \) near \( p \) is close to \( C(t,R) \), while a circle \( C' \) through \( R \) and two points of \( A'_3 \) near \( p \) is close to \( C(t,R') \), where \( t \) and \( t' \) denote the family of tangent circles of \( A_3 \) and \( A'_3 \) at \( p \), respectively. But \( t \neq t' \), because, in fact, circles belonging to \( t \) are orthogonal to circles belonging to \( t' \). Hence we obtain that \( C \neq C' \).

Suppose

\[
\lim_{t,u,t',u' \to p} C(t,u,t',u') = C_0 \neq p.
\]

Take \( S \in C_0, S \neq p \). Let \( Q \in C(t,u,v), Q \to S \) as \( t,u,v \to p \). Then

\[C(t,u,t',u') = C(t,u,Q) = C(t',u',Q)\]

which contradicts the fact that \( t \neq t' \). Hence it follows that a circle through two points of \( A_3 \) near \( p \) and two points of \( A'_3 \) near \( p \) is close to the point-circle \( p \).
It is readily verified (cf. eq. 2.4-2.) that $C(\tau, R)$ supports $A = A_3 \cup p \cup A'_3$ at $p$. Hence the end-points of a small neighbourhood $M = N \cup p \cup N'$ of $p$ lie on the same side of $C(\tau, R)$. Hence a circle $C_1$ through $R$ and two points of $N \subset A_3$ will meet $M$ with an even multiplicity and hence twice or four times. From the above, $C_1$ cannot meet $M$ four times. Hence $C_1$ meets $N$ exactly twice, i.e., $C_1$ meets $N$ twice and $C_1 \cap N' = \emptyset$. Symmetrically a circle $C'_1$ through $R$ and two points of $N' \subset A'_3$ will meet $N'$ exactly twice and $C'_1 \cap N = \emptyset$.

Finally, a circle through three points of a sufficiently small neighbourhood of $p$ on $A$ cannot pass through a point $Q$ near $R$ if $R \neq p$. As above, this implies that

$$\lim_{t,u,v \to p} C(t,u,v) = p.$$  

$$t,u,v \in A$$

\textbf{2.5. Non-tangent circles.}

Let $p$ be an interior point of $A$. Suppose that $p$ satisfies Condition I (cf. 2.3).

\textbf{2.5.1. Theorem.} Every non-tangent circle either supports or intersects $A$ at $p$.

\textbf{Proof.} If a circle $C$ neither supports nor intersects $A$ at $p$, then $p \in C$ and there exists a sequence of points $t \to p$ such that $t \in A \cap C$ and $t \neq p$. Let $P \in C$, $P \neq p$. Then $C = C(t,p,P)$ for each $t$, and Condition I implies that $C = C(\tau,P)$, which is a contradiction to the fact that $C$ is a non-tangent circle.
2.5.2. Theorem. Non-tangent circles through $p$ all intersect or all support.

Proof. Let $C_1$ and $C_2$ be two non-tangent circles through $p$.

Suppose that $C_1$ and $C_2$ have another point $P$ in common. Let $C_1$ intersect and $C_2$ support $A$ at $p$. Thus $A \cap C_1^*$ and $A \cap C_2^*$ are non-void. We may assume that $A \subset C_2^*$ (Fig. 16).

If $t \in A \cap C_2^*$, then

$$C(p,t,P) \subset (C_1^* \cap C_2^*) \cup (C_1^* \cap C_2^*) \cup P \cup p.$$ 

By having $t \rightarrow p$, we conclude that

$$(2.5-1.) \quad C(\tau,P) \subset (C_1^* \cap C_2^*) \cup (C_1^* \cap C_2^*) \cup P \cup p.$$ 

Considering now a sequence of points, $t' \rightarrow p$, where $t' \in A \cap C_1^*$ we obtain symmetrically the relation

$$(2.5-2.) \quad C(\tau;P) \subset (C_1^* \cap C_2^*) \cup (C_1^* \cap C_2^*) \cup P \cup p.$$ 

Hence by the relations (2.5-1.) and (2.5-2.) we see that $C(\tau,P)$ lies in the intersection $C_1 \cup C_2$ of these two domains, i.e., $C(\tau,P)$ is either $C_1$ or $C_2$, contrary to our assumption. Thus $C_1$ and $C_2$ either both intersect or both support.

If $C_1$ and $C_2$ meet only at $p$, then they touch at that point.

Choose any non-tangent circle $C_3$ through $p$ that does not belong to the pencil through $C_1$ and $C_2$. From above, $C_1$ and $C_3$, and also $C_3$ and $C_2$, either both support or both intersect. Hence our statement
FIGURE 16
remains valid for $C_1$ and $C_2$ also in this case.

**Definition:** We call an interior point of $A$ which satisfies Condition I a *cusp point* if the non-tangent circles of $A$ at $q$ all support $A$ at $q$. 
CHAPTER III
ARCS OF CYCLIC ORDER THREE

3.1. Arcs of finite cyclic order.

An arc $A$ is said to be of finite cyclic order if it has only a finite number of points in common with any circle. If the least upper bound of these numbers is finite, then it is called the cyclic order of $A$, and $A$ is said to be of bounded cyclic order. The order of a point $p$ of $A$ then is the minimum of the orders of all the neighbourhoods of $p$ on $A$.

3.1.1. Lemma. Let $A$ be an arc of finite cyclic order, and let a circle $C$ intersect $A$ at a point $p$. Then any circle $C'$, sufficiently close to $C$, also intersects $A$, and does so in an odd number of points close to $p$.

Proof. Since $C$ intersects $A$ at $p$, the end-points of a sufficiently small neighbourhood $N$ of $p$, lie in opposite regions with respect to $C$. Hence they lie on opposite sides of $C'$. Since $C'$ meets $N$ a finite number of times, it must intersect $N$ an odd number of times.
3.1.2. Lemma. Let $A$ be an arc of finite cyclic order. If the parameter $t_n$ tends to one of the end-points of the parameter interval, then the corresponding sequence of points $t_n$ on $A$ converges.

Proof. Let $\lim_{y \to \infty} t_{2y} = p$ and $\lim_{y \to \infty} t_{2y+1} = q$ be any two accumulation points of the sequence $t_n$. We may assume that $t_{n+1}$ lies between $t_n$ and $t_{n+2}$ for all $n$. If $p \neq q$, let $C$ be a circle separating these two points. Thus there is a number $N = N(C)$ such that $t_{2y}$ and $t_{2y+1}$ are separated for all $y > N$. But this implies that $A$ meets $C$ an infinite number of times, which is not true. Hence $p = q$.

3.1.3. Theorem. Let $p$ be an end-point of an arc $A$ of finite cyclic order. Then the arc $A$ is conformally differentiable at $p$.

Proof. Suppose Condition I of section 2.3 is not satisfied. Let $t_{2k}$ and $t_{2k+1}$ be two sequences of points converging to $p$ such that some point $R \neq p$, $C_{2k} = C(R, t_{2k}, p) \to C_0$ and $C_{2k+1} = C(R, t_{2k+1}, p) \to C_1$, $C_1 \neq C_0$. We may assume that $t_{n+1}$ lies between $t_n$ and $t_{n+2}$ on $A$.

If $k$ is large, $C_{2k} \left[C_{2k+1}\right]$ will lie close to $C_0 \left[C_1\right]$. Let $C$ and $C'$ be two circles through $p$ and $R$ which separate $C_0$ and $C_1$, (Fig. 17).

Then, for each $n$ sufficiently large, $C$ and $C'$ separate $C(R, t_n, p)$ and $C(R, t_{n+1}, p)$. Hence the sub-arc of $A$ bounded by $t_n$ and $t_{n+1}$ will meet $C \cup C'$ in at least one point. Thus $A$ will meet $C \cup C'$ infinitely many times. This is impossible. Thus Condition I holds.

Let us now suppose that Condition II of section 2.3 does not hold. Let $t_{2k}$ and $t_{2k+1}$ be two sequences of points converging to $p$
on A such that \( C(\tau, t_{2k}) \rightarrow C_1 \) and \( C(\tau, t_{2k+1}) \rightarrow C_1 \neq C_0 \). As before we assume that \( t_{n+1} \) lies between \( t_n \) and \( t_{n+2} \) on A. Both of the circles \( C_0 \) and \( C_1 \), being the limit of sequences of tangent circles are themselves tangent circles, and since family of tangent circles at \( p \) form a pencil of second kind, they touch at \( p \).

Suppose first of all, that \( C_0 \) and \( C_1 \) are both proper circles. We may assume that \( C_1 \subset C_0 \cup p \) and \( C_0 \subset C_1 \cup p \). Consider a circle \( C \subset \tau, C \neq p, C \subset (C_0 \cap C_1) \cup p \). We may assume \( C_1 \subset C_0 \cup p \) and \( C_0 \subset C_1 \cup p \) (Fig. 18). Then for sufficiently large \( k \), \( C(\tau, t_{2k+1}) \subset C_0 \cup p \), and \( C(\tau, t_{2k}) \subset C_1 \cup p \). Here again the arc \( A \) crosses \( C \) an infinite number of times, which is impossible.

If now, \( C_1 \) is a point circle \( p \), consider two circles of \( \tau, C \) and \( C', C \subset C_0 \cup p \) and \( C' \subset C_0 \cup p \). Also we may assume that \( C_0 \subset (C_0 \cap C_1') \cup p \) (Fig. 19). Then for sufficiently large \( k \), \( C(\tau, t_{2k}) \subset (C_0 \cap C_1') \cup p \) while \( C(\tau, t_{2k+1}) \subset (C_0 \cup C_1') \cup p \). Since these two regions are separated by \( C \) and \( C' \), one or both of these circles will meet \( A \) between \( t_{2k} \) and \( t_{2k+1} \). Thus \( C \cup C' \) will meet \( A \) an infinite number of times. Since this too is impossible by our hypothesis, Condition II holds, and the point \( p \) is differentiable.

3.2. Arcs of Cyclic Order Three.

Since any three distinct points determine a circle, the cyclic order of any arc is at least three. An arc \( A \) is said to have cyclic order three if no circle meets \( A \) more than three times. Let \( A_3 \) denote an open arc of cyclic order three.
FIGURE 18

FIGURE 19
3.2.1. **Multiplicities.**

We introduce *multiplicities* by counting an end-point \( p \) of \( A_3 \) twice on any non-osculating tangent circle at \( p \), and three times on \( C(p) \). We count an interior point \( q \) of \( A_3 \) three times on any general osculating circle at \( q \) and twice on any other general tangent circle at \( q \).

Let \( p \) and \( e \) be the end-points of \( A_3 \). \( A_3 = p \cup A_3 \cup e \). Let \( e \) denote the family of tangent circles of \( A_3 \) at \( e \). The goal of this section is to prove that:

If \( C(\tau, e) \neq C(p, \tau_e) \), then no circle meets the closure of \( A_3 \) more than three times. Thus the inclusion of the end-points of \( A_3 \) and the introduction of multiplicities do not alter the cyclic order of \( A_3 \) (cf. Theorem 3.2.12).

**Remark.** Any circle through \( q \in A_3 \) will either support or intersect \( A_3 \) there, because of the finiteness of the cyclic order of \( A_3 \).

3.2.2. **Lemma.** A general osculating circle at an interior point \( q \) of \( A_3 \) intersects \( A_3 \) at \( q \) while any other general tangent circle of \( A_3 \) at \( q \) supports \( A_3 \) there.

**Proof.** Let \( q \) be an interior point of \( A_3 \). Let \( C \) be a general osculating circle at \( q \). Then for some triplets \( t, u, v \),

\[
\lim_{t, u, v \to q} C(t, u, v) = C.
\]
Consider a neighbourhood $N$ of the point $q$. If $t, u, v$ are sufficiently close to $q$, then, the end-points of $N$ lie in opposite regions with respect to $C(t, u, v)$; otherwise the arc $A_3$ meets $C(t, u, v)$ once more, because any circle meets an arc an even number of times if and only if the end-points of the arc lie in the same region with respect to the circle. Hence the end-points of $N$ lie in opposite regions with respect to the circle $C$.

Let $R \to q \neq q$ and let $t, u \to q$. Choose any neighbourhood $N$ of $q$ on $A_3$. Let $C_1$ be a general tangent circle of $A_3$ at $q$ which is not an osculating circle of $A_3$ at $q$;

$$\lim_{t, u \to p} C(t, u, q) = C_1.$$ 

If $t, u$ are sufficiently close to $q$, then the end-points of $N$ will lie in the same region with respect to $C(t, u, q)$ and hence will lie in the same region with respect to $C_1$. Hence $C_1$ supports $A_3$ at $q$.

### 3.2.3. Lemma. No circle through $p$ and two points of $A_3$ intersects $A_3$ at another point.

**Proof.** Suppose a circle $C$ through $p$ intersects $A_3$ at $q$ and meets $A_3$ in two more points $r$ and $s$. Choose disjoint neighbourhoods $N$ of $p$ and $M$ of $q$ which do not contain $r$ or $s$. If $t \in N$ and $t \to p$, then $C(t, r, s) \to C$. By Lemma 3.1.1 $C(t, r, s)$ separates the end-points of $M$ if $t$ is sufficiently close to $p$. Thus $C(t, r, s)$ meets $A_3$ again in the neighbourhood of $q$ (Fig. 20). Thus $C(t, r, s)$ meets $A_3$ in not less than four points, which is impossible.
Figure 20
3.2.4. Lemma. A circle through three points of $A_3 \cup p$ does not support $A_3$ at any of them.

Proof. If a circle $C$ supports $A_3$ at $q$ and also meets $A_3 \cup p$ at $r$ and $s$, then a suitable circle near $C$ through $r$ and $s$ intersects $A_3$ twice near $q$. This is impossible by Lemma 3.2.3 and the definition of $A_3$.

3.2.5. Lemma. No circle meets $A_3 \cup p$ in four points.

Proof. Lemmas 3.2.3 and 3.2.4.

3.2.6. Lemma. No tangent circle at $p$ meets $A_3$ in more than one point.

Proof. Suppose a tangent circle $C$ at $p$ intersects $A_3$ at $q$ and meets $A_3$ also at $r \neq q$. Then there will be a circle through $p$ and $r$ which intersects $A_3$ near $p$ and also near $q$. Thus we have a circle which meets $A_3 \cup p$ in four points, which is impossible by Lemma 3.2.5.

Suppose the circle $C$ tangent at $p$ supports $A_3$ at $q$. Then we have another tangent circle $C'$ at $p$ which intersects $A_3$ twice near $q$. This contradicts the above.

3.2.7. Lemma. $C(p)$ does not meet $A_3$.

Proof. $C(p)$ being the limit circle of tangent circles at $p$ is also tangent circle at $p$. By Lemma 3.2.6, $C(p)$ can meet $A_3$ only once and that point is a point of intersection. Suppose that $C(p)$ intersects $A_3$ at $q$. Let $N$ and $M$ be disjoint neighbourhoods of $p$ and $q$ respectively, and let $t \in N$, $t \to p$. Then $C(t, t)$, when close to $C(p)$
will meet N, this contradicts Lemma 3.2.6.

3.2.8. Lemma. No circle supports $A_3$ at two distinct points.

Proof. Suppose a circle $C$ supports $A_3$ at two distinct points $q$ and $r$. We may assume $A_3 \subset C \cup C^*$. Let $M$ and $N$ be disjoint neighbourhoods on $A_3$ of $q$ and $r$ respectively. Choose a circle $D$ in $C^*$ and sufficiently close to $C$ (Fig. 21). Since the end-points of $M$ and $N$ lie in $C^*$, they will also lie in $D^*$. On the other hand $C \subset D^*$ implies $q \in D^*$ and $r \in D^*$. Thus $D$ separates $q$ from the end-points of $N$, $D$ will intersect $N[N]$ in not less than two points, and thus $D \cap A_3$ contains more than three points, which is impossible by the definition of $A_3$.

3.2.9. Lemma. Let $C$ be a general tangent circle of $A_3$ at $q$ but not a general osculating circle there. Then $C$ meets $A_3 \cup p$ at most once outside $q$ and that point is not a point of support.

Proof. By Lemma 3.2.2, $C$ supports $A_3$ at $q$. By Lemma 3.2.8 any other point of $A_3 \cap C$ is a point of intersection. By Lemma 3.2.4 $C$ meets $A_3 \cup p$ at most once outside $q$.

3.2.10. Lemma. A general osculating circle at an interior point of $A_3$ does not meet $A_3 \cup p$ elsewhere.

Proof. Let $C$ be a general osculating circle of $A_3$ at an interior point $q$. Thus

$$ C = \lim_{t,u,v \to q} C(t,u,v). $$

$$ t,u,v \in A_3 $$
FIGURE 21
Suppose $C$ meets $A_3 \cup p$ at a point $r \neq q$. Then the orthogonal circle of $C$ through $q$ and $r$ will intersect $C(t,u,v)$ at a point $R$ converging to $r$. (Fig. 22). Thus

$$C(t,u,v) = C(t,u,R).$$

The circles $C(t,u,r)$ will not meet $A_3 \cup p$ elsewhere by Lemma 3.2.5, and they will intersect $A_3$ at $t$ and $u$, by Lemma 3.2.4. Thus the end-points of any small neighbourhood of $q$ will lie on the same side of $C(t,u,r)$, if this circle is close enough to $C$. Hence any limit circle $D$ of $C(t,u,r)$ will support $A_3$ at $q$. By the Remark of 1.5,

$$\lim \xi [C(t,u,R), C(t,u,r)] = 0.$$

Since the angle between two circles depends on them continuously, it follows that $\xi [C,D] = 0$. Since $C$ and $D$ have points $q$ and $r$ in common, this implies $C = D$. However $D$ supports but $C$ intersects (by Lemma 3.2.1) $A_3$ at $q$. Hence $C$ does not meet $A_3 \cup p$ outside $q$.

3.2.11. Combining the Lemmas from 3.2.5 to 3.2.10, we obtain

Lemma. No circle meets $A_3 \cup p$ more than three times.

3.2.12. Theorem. If $p$ and $e$ are the end-points of $A_3$ then no circle meets $A_3 = e \cup A_3 \cup p$ more than three times provided that $C(\tau, p) \neq C(\tau, e)$. 
Proof. By 3.2.11 we know that \( A_3 \cup p \) is of cyclic order three counting multiplicities. Thus to prove that \( \bar{A}_3 \) is of cyclic order three we need to show that \( C(e, t, p) \) where \( t \in A_3 \), does not support \( A_3 \) at \( t \) and does not meet \( A_3 \) elsewhere and that \( C(\tau, e) \) and \( C(\tau_e, p) \) do not meet \( A_3 \) again, and the osculating circle at one end-point does not pass through the other end-point.

Assume that \( C = C(e, t, p) \) meets \( A_3 \) in \( q \) (Fig. 23). By 3.2.4 \( t \) and \( q \) are points of intersection of \( A_3 \). Then there exists a circle \( C' \) through \( p \) and \( t \) close to \( C \) which intersects the arc \( A_3 \) in neighbourhoods of the points \( q \) and \( e \). Thus \( C' \) meets \( A_3 \) four times. This contradicts 3.2.11. Thus no circle through \( p \) and \( e \) intersects \( A_3 \) at two distinct points.

Suppose \( C(p, t, e) \) supports \( A_3 \) at \( t \). Then a suitable circle through \( e \) and \( p \) would meet \( A_3 \) at two points near \( q \). This is impossible. Hence no circle through \( p \) and \( e \) supports \( A_3 \) at any point.

Let \( C(\tau, e) \) intersect \( A_3 \) at a point \( r \). Then there is a circle \( C' \) through \( p \) and \( e \) and close to \( C(\tau, e) \) which meets \( A_3 \) near \( p \) and \( r \) (Fig. 24), contrary to the above. Thus \( C(\tau, e) \) does not meet \( A_3 \).

Similarly \( C(\tau_e, p) \) does not meet \( A_3 \).

Suppose \( C(p) \) goes through the other end-point \( e \). Then there is a circle \( C' \), close to \( C(p) \), which is a tangent circle of \( A_3 \) at \( p \) and intersects \( A_3 \) near \( e \) and \( p \) (Fig. 25). This contradicts Lemma 3.2.11. Hence \( C(p) \) does not pass through \( e \). Similarly \( C(e) \) does not pass through \( p \).
FIGURE 25
Remark. There exists an open arc $A_3$ of cyclic order three whose closure $\overline{A}_3$ is not of cyclic order three. We shall construct an open arc $A_3$ of cyclic order three with the end-points $p$ and $e$ of $A_3$ such that $C(\tau, e) = C(\tau, p)$. Then $\overline{A}_3 = pvA_3u e$ will not be of cyclic order three.

**Proof.** Let $p(-1,0)$ and $e(1,0)$ be the end-points of an open arc $A$, passing through the third and first quadrant, of the lemniscate given by

$$(3.2-1.) \quad (x^2 + y^2)^2 = x^2 + y^2 = 0, \quad (\text{Fig. 26}).$$

Claim: The arc $A$ is of cyclic order three.

(i) **Tangent circles at $p$ or $e$ meet the arc $A$ at most once.**

**Proof.** Different tangent circles at $e$ are given by

$$(x-h)^2 + y^2 = (1-h)^2, \quad \text{for different values of } h.$$  

Or

$$(3.2-2.) \quad x^2 + y^2 - 2hx + 2h - 1 = 0.$$  

Solving $(3.2-1.)$ and $(3.2-2.)$ for $x$ we get

$$(2hx-2h+1)^2 - 2x^2 + 2hx - 2h + 1 = 0.$$  

This equation being of second degree in $x$ has at most two distinct roots. Hence the circles given by $(3.2-2.)$ meet the lemniscate at most two times. Because of symmetry, the circles $(3.2-3)$ meet the arc $A$ at most once. The symmetric result holds for $p$.  

\[ \mathcal{C}(\tau, e) = \mathcal{C}(\tau_e, p) \]
(ii) **Circles through the end-points** p and e **meet the arc** A
at most once outside p and e.

**Proof.** Circles passing through p and e are given by

\[(3.2-3.) \quad x^2 + y^2 + 2fy - 1 = 0.\]

Solving (3.2-1.) and (3.2-3.) for y we have,

\[(1-2fy)^2 + 2y^2 + 2fy - 1 = 0.\]

This being a second degree equation in y has at most two distinct roots. Hence the circles through p and e meet the lemniscate at most twice. By symmetry, only one of these points lie on the arc A. Thus the above circles (3.2-3.) meet A at most once outside p and e.

(iii) **Circles through p or e** meet the arc in at most two points.

**Proof.** Let r, s be two points interior to the arc A. Suppose C(e, r, s) meets the arc A in t. Let e and r be fixed and let s move from e to p. Clearly t cannot coincide with e, because then we have C(e, r, s) \(\in \tau_e\), contrary to (i). Also t does not coincide with p, as this would contradict (ii). Finally t cannot drop out as a point of support, because in that case the circle would meet the lemniscate five times and we know that any circle meets the lemniscate at most four times. Hence C(e, r, s) always meets A at another point t. Let s \(\to p\). Then we have a circle through both the end-points of A which meets the arc in two interior points this contradicts (ii). By symmetry C(p, r, s) does not meet A again.
(iv) No circle through three interior points of $A$ meets $A$ elsewhere.

**Proof.** Consider $C(r, s, t)$, where $r, s, t$ are three interior points of $A$. Suppose $C(r, s, t)$ meets $A$ in a fourth point say $x$. Let $r, s$ be fixed and let $t$ move on $A$. Then $x$ cannot coincide with either of the end-points, as this would contradict (iii). Also $x$ cannot drop out as a point of support, otherwise, $C(r, s, t)$ would meet $A$ five times. Thus $x \in C(r, s, t)$, has to remain an interior point of $A$. Now let $t \to p$. Then we obtain a contradiction to (iii).

Thus $A$ is of cyclic order three. But the circle $x^2 + y^2 = 1$ is a tangent circle of $A$ at both the end-points. Thus

$$C(r, e) = C(r, p).$$

Hence $\bar{A} = p \cup A \cup e$ is not of cyclic order three.

### 3.3. Strong differentiability of arcs of cyclic order three.

#### 3.3.1. Theorem. Let $A_3$ be an open arc of cyclic order three. Then every point of $A_3$ satisfies Condition I'; cf. 2.4.

**Proof.** Let $q, r \in A_3$ and $q \neq r$. Choose two disjoint one-sided neighbourhoods $N_1 \supseteq q$ and $N_2 \supseteq q$ such that $r \notin M = N_1 \cup q \cup N_2$. Let $C_1$ and $C_2$ meet $A_3$ at least twice at $q$ and altogether at least three times. Hence $C_i$ ($i=1, 2$) meets $A_3$ exactly twice at $q$, once at $r$ and nowhere else (cf. Lemma 3.2.9). In particular, $C_i$ ($i=1, 2$) supports $A_3$ at $q$ (cf. Lemma 3.3.1). Without loss of generality we may assume $N_1 \cup N_2 \subseteq C_1 \cap C_2$. 
Suppose $C_1 \neq C_2$. Then there is a third circle $C_3$ through $q$ and $r$ which does not meet $C_1 \cap C_2$. Thus $C_3$ will support $A_3$ at $q$. We may assume that $N_1 \cup N_2 \subset C_3^\ast$ (Fig. 27). By Theorem 3.1, the arcs $N_1 \cup q$ and $N_2 \cup q$ satisfy Condition I at $q$. Thus they possess two well-defined tangent circles at $q$ through $r$. At least one of the circles $C_1, C_2, C_3$ say the circle $C_1$ is different from them. Let $\pi$ be the pencil of the second kind of the circles which touch $C$ at $q$.

Let $s \in N_1 \cup N_2$. Thus $s \in C_\pi$ and hence $C(\pi, s) \subset C_\pi \cup q$. Also if $s$ approaches $q$ through $N_1$ or $N_2$,

$$\lim_{s \to q} C(\pi, s) = q;$$

cf. Theorem 2.3.2.

Since $C(\pi, s)$ depends continuously on $s$, there are circles in $\pi$ which are arbitrarily small and meet both $N_1$ and $N_2$ near $q$. Thus they meet $M$ not less than three times. On the other hand, the end-points of $M$ will lie on the same side of such a small circle. Hence it will meet $M$ with an even multiplicity and therefore not less than four times. This being impossible we obtain $C_1 = C_2$. Thus the general tangent circle at $q$ through $r$ is unique.

Let $C'$ and $C''$ be the two one-sided tangent circles of $A_3$ at through a point $R$ of $A_3$. Since
\* [c', c_1] = 0 \\

and \\
\* [c'', c_2] = 0,

it is true that \\
\* [c', c'] = 0.

Since C' and C'' have the point R \neq q in common they coincide; cf. Theorem 2.3.1. Hence the tangent circle of \( A_3 \) at q through R is determined.

\[ \text{3.3.2. Theorem. Let } p \text{ be an end-point of an open arc } A_3 \]

of cyclic order three. Then \( A_3 \cup p \) is strongly differentiable at p.

Proof. Let \( A_3 = p \cup A_3 \cup \in \), and \( p, q, r, s, u, e \) be mutually distinct points on \( A_3 \) in the indicated order. We may assume that \( e \in C(p)^* \). Thus

\( A_3 \subset C(p)^* \cap C(r,e)^* \).

As q moves continuously and monotonically from p to v on \( A_3 \), \( C(q,r,s) \) moves continuously and monotonically from \( C(p,r,s) \) to \( C(r,s,v) \). We orient \( C(q,r,s) \) continuously. Thus

\[ (3.3-1.) \ C(q,r,s) \subset [C(p,r,s)^* \cap C(r,s,v)^*] \cup [C(p,r,s)^* \setminus C(r,s,v)^*] \]

\( \cup r \cup s \)

and

\[ (3.3-2.) \ C(q,r,s) \supset C(p,r,s)^* \cap C(r,s,v)^* \]
Since \( p \) is an end-point of \( Z \cup p \) is conformally differentiable at \( p \). It suffices to show that \( p \) satisfies Condition II', in particular
\[
\lim_{\mathbf{q}, \mathbf{r}, \mathbf{s} \to \mathbf{p}} C(\mathbf{q}, \mathbf{r}, \mathbf{s}) = C(p);
\]
cf. Lemma 2.4.2.

Letting \( \mathbf{r} \to \mathbf{p} \) in the relation (3.3-1.) we have
\[
C(\mathbf{q}, \mathbf{r}, \mathbf{s}) \subset \left[ C(\mathbf{r}, \mathbf{s})_* \cap C(\mathbf{p}, \mathbf{s}, \mathbf{v})* \right] \cup \left[ C(\mathbf{r}, \mathbf{s})* \cap C(\mathbf{p}, \mathbf{s}, \mathbf{v})_* \right] \cup \mathbf{p} \cup \mathbf{s}.
\]

Let \( D_1 \) denote an accumulation circle of the circle \( C(\mathbf{q}, \mathbf{p}, \mathbf{s}) \). By choosing a suitable subsequence of the sequence \( \mathbf{q}, \mathbf{r}, \mathbf{s} \) we may assume that
\[
\lim_{\mathbf{q}, \mathbf{s} \to \mathbf{p}} C(\mathbf{p}, \mathbf{q}, \mathbf{s}) = D_1.
\]
Hence
\[
D_1 \subset \left[ C(\mathbf{p})* \cap C(\mathbf{r}, \mathbf{v})* \right] \cup \left[ C(\mathbf{p})* \cap C(\mathbf{r}, \mathbf{v})* \right] \cup C(\mathbf{p}) \cup C(\mathbf{r}, \mathbf{v}).
\]
This holds for every choice of \( \mathbf{v} \) on \( Z \cup e \) while \( D_1 \) is independent of \( \mathbf{v} \). Letting \( \mathbf{v} \to \mathbf{p} \) we obtain
\[
D_1 = C(p).
\]

Thus
\[
\lim_{\mathbf{q}, \mathbf{s} \to \mathbf{p}} C(\mathbf{p}, \mathbf{q}, \mathbf{s}) = C(p).
\]
Hence
\[
\lim_{\mathbf{q}, \mathbf{s} \to \mathbf{p}} C(\mathbf{p}, \mathbf{r}, \mathbf{s}) = C(p).
\]

As in the relation (3.3-1.)
\[
C(\mathbf{q}, \mathbf{r}, \mathbf{v}) \subset \left[ C(\mathbf{p}, \mathbf{q}, \mathbf{v})* \cap C(\mathbf{q}, \mathbf{s}, \mathbf{v})* \right] \cup \left[ C(\mathbf{p}, \mathbf{q}, \mathbf{v})* \cap C(\mathbf{q}, \mathbf{s}, \mathbf{r})_* \right]
\]
\[
\cup \mathbf{q} \cup \mathbf{v}.
\]
Let $D_2$ be an accumulation circle of the $C(q,r,v)$ as $q$ and $r$ tend to $p$. Hence

$$D_2 \subset \left[ C(\tau,v) \cap C(p,s,v) \right] \cup \left[ C(\tau,v) \cap C(p,s,v) \right] \cup C(\tau,v) \cup C(p,s,v).$$

This holds for every choice of $s$ on $A_3$ while $D_2$ is independent of $s$. Letting $s \to p$ we have $D_2 \subset C(\tau,v)$. Since $D$ passes through $p$ and $v$ we have

$$D_2 = C(\tau,v).$$

Hence also

$$\lim_{r,s \to p} C(r,s,v) = C(\tau,v).$$

Let $D$ be an accumulation circle of the circles $C(q,r,s)$. By choosing the suitable subsequence of the sequences $q,r,s$, we may assume

$$\lim_{q,r,s \to p} C(q,r,s) = D.$$ 

By letting $r$ and $s$ tend to $p$ in the relation (3.3-1) and recalling that

$$\lim_{s \to p} C(\tau,s) = \lim_{r,s \to p} C(p,r,s) = C(p),$$

we obtain

$$D \subset \left[ C(p) \cap C(\tau,v) \right] \cup \left[ C(p) \cap C(\tau,v) \right] \cup C(p) \cup C(\tau,v).$$

But

$$C(p) \subset C(\tau,v),$$

therefore

$$C(p) \cap C(\tau,v) = \emptyset.$$
Hence D lies in the closure of \([\mathbf{C}(p)^* \cap \mathbf{C}(\tau, \nu)^*]\). In particular, D lies in the closure of \(\mathbf{C}(\tau, \nu)^*\). Since \(p \in D\) we have \(D \in \tau\).

From the relation (3.3-2.)

\[D \supset \mathbf{C}(p)^* \cap \mathbf{C}(\tau, \nu)^*\]

Let \(v \to p\).

If \(\mathbf{C}(p) \neq p\), then \(D \neq p\) and \(D = \mathbf{C}(p)\).

If \(\mathbf{C}(p) = p\), then \(D = p\).

This implies that \(D = \mathbf{C}(p)\), whether \(\mathbf{C}(p) = p\) or not.

2.3.3. Let \(\tau^e\) denote the pencil of tangent circles of \(A_3\) at \(e\).

Lemma. Let \(p\) and \(e\) be the end-points of an open arc \(A_3\); thus \(\overline{A_3} = p \cup A_3 \cup e\). We may assume that \(e \in \mathbf{C}(p)^*\). Then

\[\mathbf{C}(q, r, s)^* \supset \mathbf{C}(p)^* \cap \mathbf{C}(p, \tau^e)^*\]

where \(q, r, s \in A_3\) in the indicated order.

Proof. By our assumptions, we have

\[A_3 \subset \mathbf{C}(p)^* \cap \mathbf{C}(\tau, e)^* \cap \mathbf{C}(p, \tau^e)^* \cap \mathbf{C}(e)^*\]

If \(x\) moves continuously and monotonically on \(A_3\) from \(p\) to \(e\), the circle \(\mathbf{C}(x, r, s)\) moves continuously and monotonically from \(\mathbf{C}(p, r, s)\) to \(\mathbf{C}(r, s, e)\). Hence for any choice of \(x\) on \(A_3\) we have

(3.3-3.) \[\mathbf{C}(x, r, s)^* \supset \mathbf{C}(p, r, s)^* \cap \mathbf{C}(r, s, e)^*\]

Hence putting \(x = q\), we have
Let \( r \rightarrow p \) and then put \( x = r \) in (3.3-3.). This yields

\[(3.3-5.)\quad C(r,p,s) \supseteq C(r,s) \cap C(p,s,e) \]

Similarly let \( r \rightarrow e \) and replace \( x \) by \( r \). Then

\[(3.3-6.)\quad C(r,e,s) \supseteq C(p,e,s) \cap C(s,\tau) e \]

using (3.3-5.) and (3.3-6.) we have from (3.3-4.)

\[
C(q,r,s) \supseteq \left[ C(r,s) \cap C(p,s,e) \right] \cup \left[ C(p,s,e) \cap C(s,\tau) e \right],
\]

or

\[(3.3-7.)\quad C(q,r,s) \supseteq C(r,s) \cap C(p,s,e) \cap C(s,\tau) e \]

Let \( r \) and \( s \) both tend to \( p \) in (3.3-3.) and then take \( x = s \).

Then

\[(3.3-8.)\quad C(\tau,s) \supseteq C(p) \cap C(\tau,e) \]

Let \( r \rightarrow p \), \( s \rightarrow e \) in (3.3-3.) and then put \( x = s \). We obtain

\[(3.3-9.)\quad C(s,p,e) \supseteq C(\tau,e) \cap C(p,\tau) e \]

Let \( r,s \rightarrow e \) in (3.3-9.) and put \( x = s \). Then we have

\[(3.3-10.)\quad C(s,\tau) e \supseteq C(p,\tau) e \cap C(e) \]

Using (3.3-8.), (3.3-9.) and (3.3-10.) in (3.3-7.) we get

\[
C(q,r,s) \supseteq \left[ C(p) \cap C(\tau,e) \right] \cap \left[ C(\tau,e) \cap C(p,\tau) e \right] \cap \left[ C(p,\tau) e \cap C(e) \right]
\]
(3.3-11.) \[ C(q,r,s) \supset C(p) \cap C(e) \cap C(p,e) \cap C(e) \]

Since \( C(p) \subset C(e) \) and \( C(p,e) \subset C(e) \), relation (3.3-11.) reduces to

\[ C(q,r,s) \supset C(p) \cap C(p,e) \]

3.4. Let \( q \in A_3 \). Let \( A_3 = B_3 \cup q \cup B_3' \) such that if \( p \) and \( e \) are the end-points of \( A_3 \), then \( B_3 \) is bounded by \( p \) and \( q \) and \( B_3' \) by \( q \) and \( e \). Let \( C \) denote a general osculating circle of \( A_3 \) at \( q \) and \( C(q) \) and \( C'(q) \) the unique osculating circles of \( B_3 \) and \( B_3' \) at \( q \) respectively.

3.4.1. Lemma. If \( C(q) \subset C \), then \( B_3 \subset C(q) \), and symmetrically if \( C(q) \supset C \), then \( B_3 \supset C(q) \) (Fig. 28).

Proof. Since both \( C \) and \( C(q) \) are general osculating circles of \( A_3 \) by Lemma 3.2.10,

\[ B_3 \cap C = B_3 \cap C(q) = q. \]

Also by Lemma 3.2.2, \( C \) and \( C(q) \) both intersect \( A_3 \) at \( q \). The general tangent circles of \( A_3 \) at \( q \) form a pencil \( \tau_q \) (cf. Lemma 2.3.1 and 2.4.1); thus \( C \in \tau_q \), \( C(q) \in \tau_q \), where \( \tau_q \) denotes the family of tangent circles of \( A_3 \) at \( q \).

Suppose that \( B_3 \subset C(q) \). Then

(3.4-1.) \[ B_3 \subset C(q) \cap C; \]

otherwise, \( C(\tau_q,s) \) could not converge to \( C(q) \) as \( s \) tends to \( q \) on \( B_3 \).
FIGURE 28
Now (3.4-1.) implies that $C(q)$ and $C$ cannot both intersect $A_3$ at $q$. Thus $B_3 \subset C(p)_*$.

The following theorem is a consequence of the above lemma.

3.4.2. Theorem. If $q$ is an interior point of $A_3$, then any general osculating circle of $A_3$ at $q$ lies between the two one-sided osculating circles of $A_3$ at $q$ in the pencil $\tau_q$.

Proof. Let $C(q)_* \subset C_*$. Then $B_3 \subset C(q)_* \subset C_*$. Since $A_3$ intersects both $C(q)$ and $C$ at $q$, we obtain

$$B_3^t \subset C^* \subset C(q)^*.$$ 

By Lemma 3.4-1 applied to $B_3^t$, if $C(q)_* \subset C_*$, then

$$B_3^t \subset C'(q)_* \subset C_*.$$ 

where $C'(q)$ is the osculating circle of $B_3^t$ at $q$, (Fig. 29). This is a contradiction. Hence $C'(q)_* \subset C^*$. We note that then

$$B_3^t \subset C'(q)^*.$$ 

Thus

$$B_3 \subset C(q)_* \text{ and } B_3^t \subset C'(q)^*$$

and we obtain $C(q)_* \subset C_* \subset C'(q)_*$.

3.4.3. Theorem. If $A_3$ satisfies Condition II at an interior point $q$, then $A_3$ satisfies Condition II' at $q$. Thus if $A_3$ is differentiable at an interior, then it is strongly differentiable there.
Proof. Let \( q \in A_3 \) and let \( A_3 \) satisfy Condition II at \( q \). Then from Theorem 3.4.2 any general osculating circle of \( A_3 \) at \( q \) satisfies

\[
C(q) \subseteq C_e \subseteq C'(q).
\]

But \( C(q) = C'(q) \). Hence \( C = C(q) = C'(q) \). Thus there is a unique general osculating circle of \( A_3 \) at \( q \). Therefore Condition II' holds at \( q \).

3.4.4. Theorem. Two general osculating circles at distinct points of \( A_3 \) have no points in common.

Proof. Let \( q, r \in A_3 \) and \( B_3 \) be a subarc of \( A_3 \) such that \( q \) and \( r \) are end-points of \( B_3 \). Thus \( B_3 \) has uniquely defined osculating circles \( C(q) \) and \( C(r) \) at \( q \) and \( r \), respectively. We may assume that \( C(q) \neq q \) and \( C(r) \neq r \). Let \( \tau_q \) and \( \tau_r \) denote the families of tangent circles at \( q \) and \( r \), respectively, cf. Theorem 3.3.1. Let \( s, t, u \) be mutually distinct points on \( A_3 \) in the indicated order. Let \( B_3 \subset C(q)_e \).

Thus

\[
C(\tau_q, r) \subseteq C(q)_e \quad \text{and} \quad C(\tau_r, q) \subseteq C(r)_e.
\]

(3.4-1.)

\[
B_3 \subset C(q) \cap C(\tau_q, r) \cap C(\tau_r, q) \cap C(r).
\]

Since \( C(\tau_q, r) \neq C(\tau_r, q) \), \( C(\tau_r, q) \) intersects \( C(\tau_q, r) \) at \( q \) and \( r \). Hence \( C(\tau_r, q) \) also intersects \( C(q) \) at \( q \) and at another point. Similarly, \( C(\tau_q, r) \) intersects \( C(r) \) at \( r \) and one other point, \( R \), say. The points \( r \) and \( R \) decompose \( C(r) \) into two arcs \( C' \) and \( C'' \), such that
\[ C \subseteq C(q, r)_* \cap C(q, r)_* \]

while
\[ C'' \subseteq C(q, r)_* \cap C(q, r)_* \] (Fig. 30.)

Since \( C(q, r)_* \subseteq C(q)_* \), we obtain
\[ C' \subseteq C(q)_*. \]

Suppose \( C'' \) meets \( C(q) \); thus \( C'' \) meets \( C(q) \cap C(q, r)_* \).

Then \( C'' \) decomposes the region
\[ C(q)_* \cap C(q, r)_* \cap C(q, r)_*, \]
into three disjoint regions. Two of these lie in the set
\[ C(q, r)_* \cap C(r)_* \cap C(q)_* \]

and their boundaries have at most a single point in common which lies on \( C(q) \). The region of \( S \) whose boundary includes an arc of \( C(q, r)_* \), contains points of \( B_3 \) close to \( q \), and the region of \( S \) whose boundary includes an arc of \( C(q, r)_* \) contains points of \( B_3 \) close to \( r \). But then the continuity of \( B_3 \) and the relation (3.4-1.) imply that these two regions are connected. Hence \( C'' \subseteq C(q)_* \), and the whole of \( C(r) = C' \cup C'' \cup \{r, R\} \) lies in \( C(q)_* \).

Thus \( C(q) \) and \( C(r) \) do not meet.

Remark. The following alternative method of proving that \( C'' \subseteq C(q)_* \) is shorter and direct, but it requires the full Jordan curve theorem.

As above \( C'' \subseteq C(q, r)_* \cap C(q, r)_* \).
Since $C(r)$ does not meet $A_3$, $C''$ even lies in the region in $C(q, r)$, bounded by $A_3$ and $C(q, r)$. Hence $C'' \subset C(q)$.

3.4.5. Theorem. All but a countable number of points of $A_3$ are strongly conformally differentiable.

Proof. Let $p$ and $e$ be the end-points of $A_3$. If $C(p) = p$ and $C(e) = e$, (cf. Remark at the end of this subsection) we can decompose $A_3$ into two subarcs $B_3$ and $B_3'$ such that $A_3 = B_3 \cup q \cup B_3'$, and consider $B_3$ and $B_3'$ separately. Thus there is no loss of generality if we assume that $C(p) \neq p$ and $A_3 \subset C(p)$. By taking the point at infinity in $C(p)$, we can introduce a local coordinate system keeping $A_3 \subset C(p)$ and use the standard metric function $d$ on $\mathbb{R}^2$; thus if $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are points in $\mathbb{R}^2$, then $d(a, b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.

By choosing this coordinate system suitably, we may even assume that $C(p)$ is a circle of area 1. In fact, a suitable translation will move the origin to the centre of $C(p)$ and if $C(p)$ has radius $r$, then

$$x' = \frac{x}{r\sqrt{\pi}}, \quad y' = \frac{y}{r\sqrt{\pi}}$$

will transform the equation of $C(p)$ into the form $x'^2 + y'^2 = \frac{1}{\pi}$.

Let $s \in A_3$ be a point at which $A_3$ is not strongly conformally differentiable; then $A_3$ does not satisfy Condition II at $s$ (cf. Theorem 3.4.3.). Let $C(s)$ and $C'(s)$ be the one-sided osculating circles of $A_3$ at $s$. We may assume that $C(s) \subset C'(s)$. Let $f(s)$ be the area between $C(s)$ and $C'(s)$ (Fig. 31). By Theorem 3.4.4, the regions $C(s) \cap C'(s)$ and $C(t) \cap C'(t)$ are disjoint if $s \neq t$, and they lie
FIGURE 31
in \( C(p) \). Since the area of \( C(p) \) is 1,

(i) the class of points \( s \) for which

\[ 1 > f(s) \geq \frac{1}{2}, \]

has at most 2 members.

(ii) The class of points \( s \) for which,

\[ \frac{1}{2} > f(s) \geq \frac{1}{4}, \]

has at most four i.e., \( 2^2 \) members.

(iii) The class of points \( s \) for which,

\[ \frac{1}{4} > f(s) \geq \frac{1}{8}, \]

has at most eight i.e., \( 2^3 \) members.

Thus in general, the class of points for which

\[ \frac{1}{2^{n-1}} > f(s) \geq \frac{1}{2^n} \quad (n=1,2,3\ldots) \]

does not have more than \( 2^n \) members.

Since every point \( s \in A \) with \( f(s) > 0 \) is included in exactly one of these classes, there is only a countable set of points \( s \) with \( f(s) > 0 \).

Remark. There are arcs of cyclic order three which have point-osculating circle at both the end-points.

For example consider the open arc \( A \) given by

\[ x = t^2, \quad y = t^3, \]
for $0 < t < \infty$, with $(0,0)$ and $\infty$ as the end-points $p$ and $e$ respectively, (Fig. 32).

First we show that the arc $A$ is of cyclic order three. Let

\begin{equation}
(3.4-2.) \quad x^2 + y^2 + 2gx + 2fy + c = 0
\end{equation}

be any circle. Then the points of $A$ which are common with a circle (3.4-2.) are the roots of the equation

\begin{equation}
(3.4-3.) \quad t^6 + t^4 + 2ft^3 + 2gt^2 + c = 0.
\end{equation}

Now there can be at most three variations in the signs of the coefficients in the equation (3.4-3.). Hence by Descarte's rule it can have at most three real positive roots. Thus any circle meets the arc at most three times. Hence $A$ is of cyclic order three.

The tangent circle of $A$ at $p$ through a point $(s^2, s^3)$ is given by

\begin{equation*}
x^2 + y^2 - (s-s^3)y = 0.
\end{equation*}

As $s \to 0$ this circle tends to the point-circle

\begin{equation*}
x^2 + y^2 = 0.
\end{equation*}

Thus $C(p) = p$.

The circle $C(t,u,e)$, $t,u \in A$, which is a straight line, is given by

\begin{equation*}
y - u^3 = \frac{t^3 - u^3}{t^2 - u^2} (x-u^2).
\end{equation*}
FIGURE 32
As $u \to e$ (i.e., as the parameter $u \to \infty$) this becomes a straight line through the point $t$ and parallel to the $y$-axis,

$$x = t^2.$$  \hspace{1cm} (3.4-5)

Thus the circle $C(v_e, t)$ is given by (3.4-5). Hence

$$\lim_{t \to e} C(v_e, t) = C(e) = \infty.$$  Thus $C(e)$ is also a point-circle.
CHAPTER IV

UNION AND EXTENSION OF ARCS OF CYCLIC ORDER THREE.

4.1. Union of Arcs of cyclic order three.

Let $A_3$ and $A'_3$ be open arcs of cyclic order three with a common end-point $p$, and let $e$, $e'$ be the other end-points respectively. Put

$$\bar{A}_3 = e' \cup A_3 \cup p, \quad \bar{A}_3 = e \cup A_3 \cup p$$

and

$$A = A_3 \cup p \cup A'_3; \quad \bar{A} = e' \cup A \cup e.$$

Let $\tau, \tau_e$ denote the pencil of tangent circles of $A_3$ at $p$ and $e$ and $\tau', \tau_{e'}$ of $A'_3$ at $p$ and $e'$. Assume that $\bar{A}_3$ and $\bar{A}'_3$ are also of cyclic order three.

Thus

$$C(p, \tau_e) \neq C(\tau, e) \text{ and } C(p, \tau_{e'}) \neq C(\tau', e').$$

Let $C(p)$ and $C'(p)$ denote the osculating circles of $A_3$ and $A'_3$ respectively at $p$. We may assume that $e \in C(p)^*.$

If $A$ has cyclic order three, the following conditions will hold.

(1) $A$ satisfies Condition I at $p$ (cf. Theorem 3.3.1.).

Thus the two pencils $\tau$ and $\tau'$ coincide. We denote this common pencil by $\tau$.
(ii) $C(\tau, e')_* \subset C'(p)_* \subset C(p)_* \subset C(\tau, e)_*$.

Thus

$$A_3' \subset C'(p)_* \subset C(p)_* \text{ and } A_3 \subset C(p)^* \subset C'(p)^* \text{ (Fig. 33).}$$

(iii) $A_3' [A_3']$ does not meet $C(p, \tau_{e, e'}) \cap C(p, \tau_{e'}) \text{ (Fig. 34).}$

(iv) $A_3 \cup p [A_3' \cup p]$ does not meet $C(\tau_{e, e'}) \cap C(\tau_{e'}, \tau_{e}) \text{ (Fig. 35).}$

Our goal is to show that Conditions (i) - (iv) are not only necessary but are also sufficient for $A$ to have cyclic order three.

We observe that $A$ will also have cyclic order three if we add the condition

$$C(\tau_{e, e'}) \neq C(\tau_{e}, e).$$

4.2. Remark. It is clear that Condition (ii) implies Condition (i). However Conditions (ii), (iii) and (iv) are independent as we shall now prove.

4.2-1. Conditions (i), (ii) and (iii) do not imply Condition (iv).

Proof. Let $e' (-1, 0)$ and $q (1, 0)$ be the end-points of the open arc $B$ of the lemniscate

$$(x^2 + y^2) - x^2 + y^2 = 0,$$

consisting of the origin and the subarcs $A_3'$ and $B_3$ which lie in the third and first quadrant respectively. Then $B$ is of cyclic order three (cf. Remark on Theorem 3.2.2.). Clearly $A_3'$ and $B_3$ are of cyclic order three.
FIGURE 33
FIGURE 34
FIGURE 35
Now, we show that $C(q)$, the osculating circle of $B^3_3$ at $q$ is not a point-circle. The radius of curvature $R$ of the lemniscate in the polar coordinates is given by

\[ R = \frac{r^2 + r_1^2}{r^2 + 2r_1^2 - rr''}. \]

At $q$, $R = \frac{1}{3}$. Since $R \neq 0$ at $q$, $C(q) \neq q$. Hence $B^3_3$ can be extended through $q$ to a larger arc of cyclic order three (cf. Theorem 4.6-5.).

Let $B^3_3$ be extended through $q$ to $e$ such that the closure $\bar{A}_3$ of the open arc $A_3$ with the end-points $p$ and $e$ is also of cyclic order three.

Let $A = A'_p \cup p \cup A'_e$. By (4.2-1.) we see that the radius of curvature at $p$ is infinite and hence $C(p)$ is a straight line. We may take the direction of $C(p)$ such that $e \in C(p)^*$. Since $A$ is conformally differentiable at $p$, Condition (i) automatically holds. Also $e' \in C(p)_*$ hence $C(p)$ separates $C(\tau, e)$ and $C(\tau, e')$, i.e., $C(\tau, e')_* \subset C(p)_* \subset C(\tau, e)_*$, where $\tau$ denotes the family of tangent circles of $A$ at $p$. Thus Condition (ii) is satisfied.

Now $C(p, \tau_1^e)$ does not meet the right half-plane $x > 0$, hence does not meet $A' _3$. Since $C(p, \tau_e)$ will lie in the union of the right half plane, the upper half plane, and $C(p)^*$, $C(p, \tau_e)$ does not meet $A'_3$. Hence Condition (iii) also holds.

But we see that Condition (iv) does not hold. Because $e \in C(q)^*$ (cf. Theorem 4.6.), $C(\tau_1^e, q) = C(\tau_1^e, e')$ we obtain

\[ C(\tau_1^e, e')_* \subset C(\tau_1^e, q)_* \]

and hence $C(\tau_1^e, e)$ intersects $A'_3$ (Fig. 36).
FIGURE 36
4.2.2. If $A$ has a cusp at $p$, then the Conditions (i), (ii), and (iv) do not imply Condition (iii).

Consider $A'_3$ to be the arc of the lemniscate
$$(x^2+y^2)^2 - 2xy = 0,$$
in the half-plane $x - y > 0$, with end-points $p = (0,0)$ and $e' = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Let $A_3$ be the arc of the lemniscate
$$(x^2+y^2)^2 + 2xy = 0$$
in the half-plane $x + y > 0$ with end-points $p = (0,0)$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Put
$$A = A_3 \cup p \cup A'_3.$$
Then $A$ has a cusp at $p$ (Fig. 37).

Obviously $C(p)$, the osculating circle of $A$ at $p$ is the straight line $x = 0$. Let $e \in C(p)$ and hence $e' \notin C(p)$. Thus $C(p)$ separates $C(p,e')$ and $C(p,e)$. Hence Condition (ii), and therefore Condition (i) holds.

Now the equation of $C(p,e') = C(p,e')$ is
$$x^2 + y^2 = 1,$$
which clearly does not meet $A_3 \cup p$ or $A'_3 \cup p$. Therefore Condition (iv) is satisfied.

But shall see that Condition (iii) does not hold. The circle $C(p,e'_3)$ is given by the equation
$$x^2 + y^2 - \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y = 0.$$
Thus the centre $(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$ lies on the line $y = x$, and the radius is $\frac{1}{2}$. Also $C(p,e'_3)$ being a non-tangent circle, supports $A$ at the cusp point $p$. Since
FIGURE 37
we obtain that $C(p, r_e)$ meets $A_3$ at some point. Thus Condition (iii) is violated.

4.2.3. Conditions (i), (iii) and (iv) do not imply Condition (ii).

Let $A_3$ be the arc of the ellipse

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \neq b, \]

with end-points $e(0, b)$ and $p(a, 0)$; and $A_3'$ be the arc of the same ellipse with end-points $e'(0, b)$ and $p(a, 0)$. Thus let

\[ A = A_3 \cup p \cup A_3', \quad (\text{Fig. 38}). \]

First of all we show that $A_3$ and $A_3'$ are of cyclic order three.

The circle $C(p, r_e)$ has the equation

\[ x^2 + y^2 + \frac{a^2 - b^2}{b} y - a^2 = 0. \]

Clearly this is not a tangent circle of $A_3$ at $p$, which shows that $A_3$ is of cyclic order three.

The circle (4.2-3.) and the ellipse (4.2-2.) meet at two points $(\pm a, 0)$ and touch at $(0, b)$. Hence they do not meet elsewhere. Thus $C(p, r_e)$ does not meet $A_3'$. Similarly $C(p, r_e')$ does not meet $A_3$. Thus Condition (iii) holds.

Since $p$ is a conformally differentiable point of the ellipse (and hence of $A$), Condition (i) is satisfied.
Finally, \( C(\tau_e, e') = C(\tau_1, e) \) is given by
\[
x^2 + y^2 = b^2.
\]

Now \( C(\tau_e, e') \) does not meet \( \Lambda \) since \( a \neq b \). Hence \( C(\tau_e, e') \) does not meet \( \Lambda \). Thus Condition (iv) holds.

From symmetry with respect to the \( x \)-axis
\[
C(\tau, e) = C(\tau, e').
\]
This implies that Condition (ii) does not hold.

4.3. Lemma. Assume Conditions (i), (ii) and (iv). Then Condition (iii) is equivalent to \( \Lambda \) having no cusp at \( p \).

Proof. The following discussion is easiest to follow if we designate \( p \) as the point at infinity.

By (ii)
\[
A_3 \subseteq C(p)^* \cap C(\tau, e)^* \quad \text{and} \quad A_3^! \subseteq C'(p)^* \cap C(\tau, e')^*.
\]
Thus
\[
A_3 \cup A_3^! \subseteq C(\tau, e)^* \cap C(\tau, e')^* = R \quad \text{say}.
\]

Since \( C(\tau_e, p) \neq C(\tau_e, e) \) they intersect at \( e \). Hence \( C(\tau_e, e') \) also intersects \( C(\tau, e) \) at \( e \). Furthermore, since \( C(\tau_e, e') \) does not meet \( A_3^! \cup p \) and since \( A_3^! \) is of cyclic order three, \( C(\tau_e, e') \) will intersect \( C(\tau, e') \) at \( e' \). Symmetrically \( C(\tau_1, e) \) intersects \( C(\tau, e') \) at \( e' \) and \( C(\tau, e) \) at \( e \).
Orient $C(\tau_e, e') [C(\tau_{e'}, e)]$ such that $A_3 \subset C(\tau_e, e')$.

Thus
\[ A_3 \subset C(\tau_e, e') \cap C(\tau_{e'}, e) \cap C'(p). \]

and
\[ A_3 \subset C(\tau_{e'}, e) \cap C(\tau_e, e) \cap C(p). \]

Hence $A_3 \cup A_3$ has no points in common with
\[ C(\tau_e, e') \cap C(\tau_{e'}, e) \cap R = R_0 \text{ say}. \]

The boundary of $R_0$ decomposes $R$ into three disjoint regions of which $R_0$ is one. Let $R_1$ and $R_2$ be the other two; thus

\[ A_3 \cup A_3 \subset R_1 \cup R_2. \]

Case I: A has a cusp at $p$.

Then $A_3$ and $A_3$ both lie in $R_1$ or both of them lie in $R_2$; say in $R_1$. In this case, both $e$ and $e'$ will lie on the boundary of $R_1$. Since $C(p, \tau_e)$ and $C(e', \tau_e)$ are tangent circles at $e$, $C(p, \tau_e)$ will decompose $R_1$ into two disjoint regions and $A_3$ will have points in both of them. Hence $C(p, \tau_e)$ will intersect $A_3$ (Fig. 39).

Case II: A has no cusp at $p$.

Here $A_3$ lies in $R_1$, say, and $A_3$ lies in $R_2$. Thus $e[e']$ lies on the boundary of $R_1 \setminus R_2$. Then the circular arc $C(p, \tau_e) \cap R$ lies in $R_1$ and the arc $C(\tau_{e'}, \tau_e) \cap R$ lies in $R_2$. Hence $C(p, \tau_e) \cap C(\tau_{e'}, \tau_e)$ does not meet $A_3 \setminus A_3$ (Fig 40 and 41).
FIGURE 39
Corollary. Conditions (i) - (iv) imply that \( A \) has no cusp at \( p \).

Remark. We observe that if the Conditions (i), (ii) and (iv) hold and \( A \) has a cusp at \( p \), then \( C(\tau_e, e') \) will have to coincide with \( C(\tau_e', e) \).

Thus Conditions (i), (ii), (iv) and \( C(\tau_e, e') \neq C(\tau_e', e) \) imply Condition (iii), and hence that \( A \) has cyclic order three.

4.4. Lemma. If Conditions (i) - (iv) hold and \( q, r, s \in A, q', r', s' \in A' \) then

\[
A \cup q' \subset C(q, r, s) * \text{ and } A' \cup q' \subset C(q', r', s') *.
\]

These results remain valid if two or all of \( q, r, s \), coincide with one another or with \( p \) or \( e \).

Proof. Since \( e \in C(p)* \), Lemma 3.3.3 implies that

\[
C(q, r, s) * \supset C(p) * \cap C(p, \tau_e)*.
\]

Since \( A = A \cup p \cup A' \) satisfies Conditions (i) - (iv), \( A \) has no cusp at \( p \) (cf. corollary of Lemma 4.1.1). Also \( C(p, \tau_e) \neq \tau \). Hence \( C(p, \tau_e) \) intersects \( A \) at \( p \). Now \( A \subset C(p, \tau_e)* \) and by Condition (iii) \( A' \) does not meet \( C(p, \tau_e) \). Condition (iv) implies that \( e' \neq C(p, \tau_e) \).

Thus \( A \cup e' \) does not meet \( C(p, \tau_e) \). Hence

\[
A' \cup e' \subset C(p, \tau_e)*.
\]
By Condition (ii), \( C(p) \) separates \( C(\tau, e) \) and \( C(\tau, e') \). Also \( e \in C(p)^* \). Hence \( e' \in C(p)^* \) and \( A_{3}^{1} \cup e' \subseteq C(p)^* \). Altogether,
\[
A_{3}^{1} \cup e' \subseteq \left[ C(p)^* \cap C(p, \tau_e) \right] \subseteq C(q, r, s)^*.
\]

4.5. Theorem. Conditions (ii) - (iv) are not only necessary but are also sufficient for \( A \) to have cyclic order three.

Proof. Let \( t, u \in A_{3} \cup p, t', u' \in A_{3}^{1} \cup p \). Using the Lemma 4.4, we prove successively that \( C(\tau_e, t) \) and symmetrically \( C(t', \tau_e) \); \( C(e', t, e) \) and \( C(e', t', e) \); \( C(e', t', t) \) and \( C(t', t, e) \); \( C(e', t, u) \) and \( C(e, t', u') \); \( C(t', t, u) \) do not meet \( A \) elsewhere.

\[
C_{1}(t) = \begin{cases} 
C(\tau_e, t) & \text{if } i = 1 \\
C(e', t, e) & \text{if } i = 2 \\
C(t', t, e) & \text{if } i = 3 \\
C(e, t', u') & \text{if } i = 4 \\
C(t', t, u) & \text{if } i = 5
\end{cases}
\]

Now \( C_{1}(p) \) does not meet \( A \) again. If \( t \) moves continuously on \( A_{3} \) from \( p \) to \( e \), \( C_{1}(t) \) cannot pass through \( p \), cannot increase the multiplicity with which it meets \( e \) or \( e' \), and cannot support \( A_{3}^{1} \cup A_{3}^{1} \) at a new point. Hence \( C_{1}(t) \) does not meet \( A \) elsewhere.
4.6. Extension of an arc of cyclic order three.

In this section we wish to prove that an open arc $A_3$ can be extended through an end-point $p$ to a larger arc of cyclic order three if and only if $C(p) \neq p$ and $\overline{A}_3$, the closure of $A_3$, is of cyclic order three.

Let $e$ be the other end-point of $A_3$ and let $\tau$ and $\tau_e$ denote the family of tangent circles of $A_3$ at $p$ and $e$ respectively.

4.6.1. We know that a reflection in a circle followed by a reflection in an orthogonal circle is a conformal transformation which leaves both these circles invariant; cf. section 1.6. Let $A_3$ be an open arc of cyclic order three with an end-point $p$, such that $C(p) \neq p$ and $\overline{A}_3$ is also of cyclic order three,

\[ \text{i.e., } C(p, \tau_e) \neq C(\tau, e). \]

Let $D$ be any circle through $p$ and orthogonal to $C(p)$, such that $D$ does not meet $A_3 \cup e$. We construct an arc $A_3'$ by first reflecting $A_3$ in $C(p)$ and then reflecting the resulting arc in the circle $D$. Then we choose a suitable subarc $B_3 \subseteq A_3$ with image $B_3' \subseteq A_3'$, and take

\[ A = B_3' \cup p \cup A_3', \quad (\text{Fig. 42}). \]

To achieve our goal we show that the arc $A$ is also of cyclic order three. It suffices to show that Conditions (i) -- (iv) of section 4.1 hold for $A$. 
4.6.2. Since both reflections leave $C(p)$ invariant, the arc $A$ is conformally differentiable at $p$. Thus Conditions (i) and (ii) will hold.

Let $e, e', f$ and $f'$ be the end-points $\neq p$ of $A_3, A'_3, B_3$ and $B'_3$ respectively. We may assume that $e \in D^*$. Thus

$$A_3 \cup e \subset C(p)^* \cup D^*,$$

and

$$A'_3 \cup e' \subset C(p)^* \cup D^*.$$

4.6.3. Condition (iii) is satisfied.

Let $F$ be the circle orthogonal to the family of circles, $T$, through $p$ and any point of $A_3$. Let $F'$ be its image under the reflection in $C(p)$ followed by the reflection in $D$. Thus

$$F' \subset D \subset F^*.$$

Choose

$$B_3 \subset A_3 \cap F^*,$$

and

$$B'_3 = p \cup B_3 \cup f, f \in F;$$

thus

$$B'_3 \subset A'_3 \cap F^* \text{ and } B'_3 = p \cup B'_3 \cup f'; f' \in F'.$$

Since $A_3 \subset C(p)^* \cup D^*$, while

$$C(\tau_f, p) \subset [C(\tau_f') \cup F^* \cup f' \cup p] \cup C(p) \cup D \cup p,$$

$A_3$ does not meet $C(\tau_f, p)$ (Fig. 43).
FIGURE 43
By shortening $B'_3$ if necessary, (e.g., choosing $B_3$ in $C(\tau_e', p)$*) we can assume that $B'_3$ does not meet $C(p, \tau_e)$, (Fig. 44).

4.6.4. Condition (iv) also holds for the arc $A$.

In the following it is convenient to take $e$ as the point at infinity. If $B'_3$ is chosen small enough, then $C(f', \tau_e)$ will be close to $C(p, \tau_e)$, while a circle through $e$ and two points of $B'_3$ will be close to $C(\tau, e)$. Since $C(p, \tau_e) \neq C(\tau, e)$, hence, $C(f', \tau_e)$ does not meet $B'_3 \cup p$. We may assume that

$$B'_3 \cup p \subset C(f', \tau_e)^+ .$$

Next, $C(f', p, e)$ is close to $C(\tau, e)$, while a circle which meets $B'_3$ again. Since

$$f' \in C(\tau, e)^+ \cap C(p, \tau_e)^+ ;$$

we have

$$C(f', p, e) \subset \left[ C(\tau, e)^+ \cap C(p, \tau_e)^+ \right] \cup \left[ C(\tau, e)^+ \cap C(p, \tau_e)^+ \right] \cup p \cup e ;$$

(Fig. 45, 46).

But

$$A_3 \subset C(\tau, e)^+ \cap C(p, \tau_e)^+ ,$$

therefore $C(f', p, e)$ does not meet $A_3$. We may assume that

$$A_3 \subset C^*(f', p, e)^+ ;$$

thus

$$B'_3 \subset C(f', p, e)^+ .$$
FIGURE 44
FIGURE 46
Now $C(\tau_{f'}, e)$ is close to $C(\tau, e)$, while $C(\tau_{f'}, p)$ is close to $C(p)$. Hence

$$C(\tau_{f'}, e) \neq C(\tau_{f'}, p).$$

Since

$$B_3 \subseteq C(f', p, e)^* \cap C(f', \tau_e)^*,$$

we obtain

$$C(\tau_{f'}, e) \subseteq \left[ C(f', p, e)^* \cap C(f', \tau_e)^* \right] \cup \left[ C(f', p, e)^* \cap C(f', \tau_e)^* \right] \cup C(f', \tau_e)^*.$$

As

$$A_3 \subseteq C(f', p, e)^* \cap C(f', \tau_e)^*,$$

we obtain $C(\tau_{f'}, e)$ does not meet $A_3 \cup p$.

4.6.5. Combining the subsections from 4.6.1 to 4.6.5 we obtain the following Theorem.

Theorem. An open arc $A_3$ can be extended through an end-point $p$ to a larger arc of cyclic order three if and only if $C(p) \neq p$ and $A_3$ is of cyclic order three.
REFERENCES


5. N. D. Lane and Peter Scherk, Characteristic and order of differentiable points in the conformal plane, Trans. Amer. Math. Soc. 81 (1956), 358-378.