ARCS OF CYCLIC ORDER THREE IN THE CONFORMAL PLANE

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By

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TITLE: Arcs of Cyclic Order Three In The Conformal Plane

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SCOPE AND CONTENTS:

This thesis is concerned with the properties of arcs of cyclic order three in the conformal plane. It establishes necessary and sufficient conditions for the union of two arcs of cyclic order three to be again an arc of cyclic order three, and for an arc of cyclic order three to be extensible to a larger arc of cyclic order three.

PREFACE

The first chapter of this thesis summarizes some properties of the conformal plane and, in particular, of Möbius transformations.

The second chapter deals with the differentiability properties of general arcs in the conformal plane.

In the third chapter, properties of arcs of cyclic order three are discussed.

In the final chapter, we give necessary and sufficient conditions for the union of two arcs of cyclic order three to be again an arc of cyclic order three. We also give conditions under which an arc of cyclic order three can be extended to a larger arc of cyclic order three.

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CHAPTER I

FOUNDATIONS OF CONFORMAL GEOMETRY

1.1. The Conformal Plane.

In the Euclidean plane we can introduce an 'ideal' point ∞ which we call the "point at infinity". The set of points in the Euclidean plane together with the point at infinity is called *the conformal plane*. We shall assume that every straight line passes through the point at infinity. Every point in the conformal plane can be represented by a complex number $z = x+iy$ or by ∞.

1.2. Stereographic Projection.

Consider the unit sphere S whose equation in the Euclidean three dimensional space is

$$x^2 + y^2 + u^2 = 1.$$

With every point on S, except N:(0,0,1), we can associate the complex number

$$z = \frac{x + iy}{1 - u} \tag{1.2-1}$$

and this correspondence is one-to-one. The correspondence can be completed by letting the point at infinity correspond to (0,0,1).

Thus we can regard the sphere as a representation of the conformal plane.

Let $P(x,y,u)$ be any point other than N on the sphere. Then the line NP meets the plane in the point P', given by (1.2-1). This mapping of the sphere onto the (x,y)-plane is a central projection from the centre $(0,0,1)$. It is called a stereographic projection (Fig. 1).

1.2.1. Theorem. Under stereographic projection circles on the sphere are mapped into circles and straight lines of the plane and vice-versa.

Proof. A circle on the sphere S is the intersection of the sphere with a plane

$$Ax + By + Cu = D \tag{1.2-2.}$$

with $A^2 + B^2 + C^2 \geqslant D^2$ to ensure actual intersection. Let (ξ, η, ζ) be any point on the sphere corresponding to the point $(x,y,0)$ in the (x,y)-plane. By (1.2-1.) we have

$$z = \frac{\xi + i\eta}{1 - \zeta}, \text{ where } z = x + iy.$$

Thus

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}. \tag{1.2-3.}$$

Now $|z|^2 = \dfrac{\xi^2 + \eta^2}{(1-\zeta)^2}$. But $\xi^2 + \eta^2 + \zeta^2 = 1$.

$$|z|^2 = \frac{1+\zeta}{1-\zeta}.$$

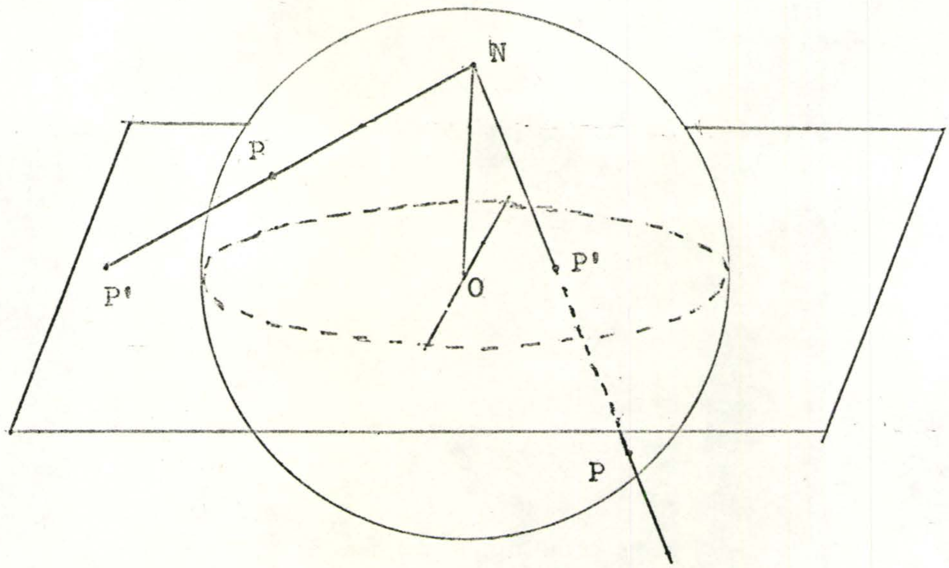

FIGURE 1

Solving this for ζ , and then computing ξ and η , we obtain

$$\zeta = \frac{|z|^2 - 1}{|z|^2 + 1}, \quad \xi = \frac{z + \bar{z}}{|z|^2 + 1} \text{ and } \eta = \frac{i(\bar{z} - z)}{|z|^2 + 1}.$$

Thus if (ξ, η, ζ) also lies on the plane given by equation (1.2-2.) we obtain

$$A(z+\bar{z}) + Bi(\bar{z}-z) + C(|z|^2-1) = D(|z|^2+1).$$

or

(1.2-4.) $$(D-C)(x^2+y^2) - 2Ax - 2By + C + D = 0.$$

For $D \neq C$, this is the equation of a circle, and for $D = C$ it represents a straight line. In the latter case, the plane given by (1.2-2.), and hence also the circle on the sphere, passes through (0,0,1).

To prove the converse, we start with the equation (1.2-4.) with $A^2 + B^2 + C^2 \geqslant D^2$ and retrace our steps. Equation (1.2-4.) represents all lines and circles in the (x,y)-plane. Using formulas (1.2-3.) to express x and y in terms of (ξ, η, ζ) we see that the latter point on the sphere must lie in the plane (1.2-2.).

Thus the stereographic projection of a circle or a straight line in the plane onto the sphere is always a circle, and this circle passes through N:(0,0,1) if and only if the pre-image is a straight line.

1.3. The Möbius Plane.

In general by a conformal transformation we mean an angle-preserving mapping. There are conformal representations in which circles are not necessarily transformed into circles, but we shall not consider these in this discussion.

Rather, we shall restrict our attention to the mappings T of the form

(1.3-1.) $$w = \frac{az + b}{cz + d},$$

whose coefficients a,b,c,d are complex numbers. We assume that $ad - bc \neq 0$, so that w is not independent of z. This also makes w well-defined, except when $c \neq 0$ and $z = -\frac{d}{c}$. A mapping of this form is called a Möbius Transformation. (It is also called a circular transformation and a linear (fractional) transformation.) Certain geometrical properties will remain invariant under these transformations.

Definition. (Möbius) Conformal Geometry is the study of the properties which remain invariant under the conformal mappings (1.3-1.)

The equation (1.3-1.) can be solved with respect to z and yields

$$z = \frac{dw - b}{-cw + a}.$$

The resulting transformation is inverse to T; we denote it by T^{-1}. The existence of an inverse shows that the correspondence defined by T is one-to-one.

1.4. Cross-ratio.

Given any three mutually distinct points z_2, z_3, z_4 in the conformal plane, there exists a Möbius transformation T which carries these points into $0, 1, \infty$. In particular, if none of the given points is ∞, T will be given by

$$Tz = \frac{z - z_2}{z - z_4} \Big/ \frac{z_3 - z_2}{z_3 - z_4} .$$

If z_2, z_3 or $z_4 = \infty$, the transformation reduces to

$$\frac{z_3 - z_4}{z - z_4} , \frac{z - z_2}{z - z_4} , \frac{z - z_2}{z_3 - z_2} ,$$

respectively.

<u>Definition.</u> <u>The cross-ratio</u> (z_1, z_2, z_3, z_4) <u>is the image of</u> z_1 <u>under a Möbius transformation</u> T <u>which carries</u> z_2, z_3, z_4 <u>into</u> $0, 1, \infty$.

<u>1.4.1.</u> <u>Theorem.</u> <u>Under a Möbius transformation</u> U, <u>the cross-ratio of any four distinct complex numbers</u> z_1, z_2, z_3, z_4 <u>is invariant</u>; i.e.,

$$(Uz_1, Uz_2, Uz_3, Uz_4) = (z_1, z_2, z_3, z_4). \tag{1.4-1.}$$

<u>Proof.</u> If $Tz_1 = (z_1, z_2, z_3, z_4)$, then TU^{-1} carries Uz_2, Uz_3, Uz_4 into $0, 1, \infty$. By definition, we have

$$(Uz_1, Uz_2, Uz_3, Uz_4) = TU^{-1}(Uz_1) = Tz_1 = (z_1, z_2, z_3, z_4).$$

<u>1.5. Angles in the conformal plane.</u>

Let z_1, z_2, z_3 and z_4 be the complex numbers λ (with finite coefficients), 0, 1 and ∞ respectively. Then $(z_1, z_2, z_3, z_4) = \lambda$, and hence

$$\operatorname{amp}(z_1, z_2, z_3, z_4) = \operatorname{amp}(\lambda), \tag{1.5-1.}$$

(Fig. 2).

The relation (1.5-1.) is unaltered if we interchange simultaneously z_1 and z_4 and z_2 and z_3. It is likewise invariant under a Möbius trans-

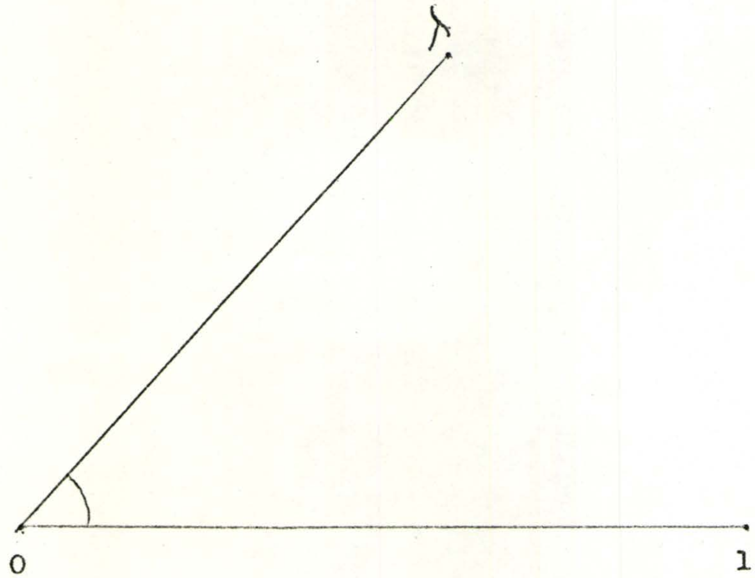

FIGURE 2

formation on account of the relation (1.4-1.).

Let C_1 and C_2 be two circles intersecting at two points P and R (Fig. 3). Let Q be a point on C_1 and S a point on C_2, $Q \neq P \neq S$ and $Q \neq R \neq S$. By a Möbius transformation, we can map P, Q, R into $0, 1, \infty$ respectively. Let S be mapped into a complex number μ. Then C_1 and C_2 will be mapped into the two straight lines through 0 and 1 and 0 and μ respectively (Fig. 4). Let these lines intersect at an angle θ. Then

$$\theta = \operatorname{amp}(\mu).$$

On account of the relation (1.5-2.) and Theorem 1.4.1,

$$\theta = \operatorname{amp} \frac{(P-Q)(R-S)}{(P-S)(R-Q)} .$$

Thus we can define an angle between two circles in the conformal plane as follows.

If two circles C_1 and C_2 intersect at P and R, and if Q and S lie on C_1 and C_2 respectively, where P,Q,R,S are mutually distinct, then the angle between C_1 and C_2 is the amplitude of the cross-ratio (P,Q,R,S), where P,Q,R,S are complex numbers.

In particular, if the amplitude of the cross-ratio is $\frac{\pi}{2}$, then we say that C_1 and C_2 are orthogonal.

Since a Möbius transformation preserves the cross-ratio of four points, it preserves the angle between two circles.

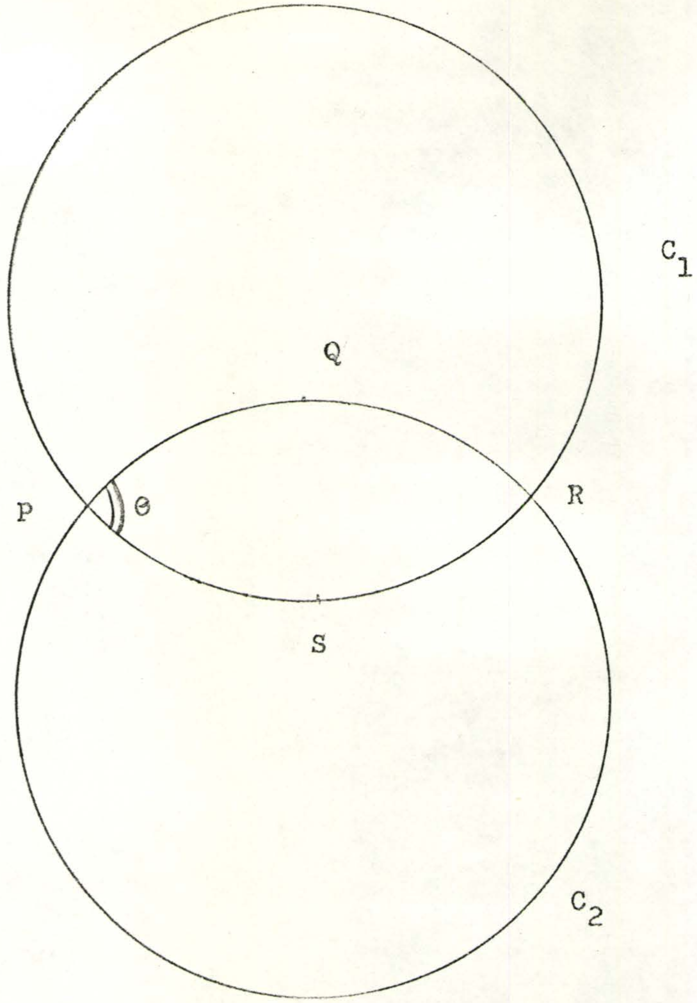

FIGURE 3

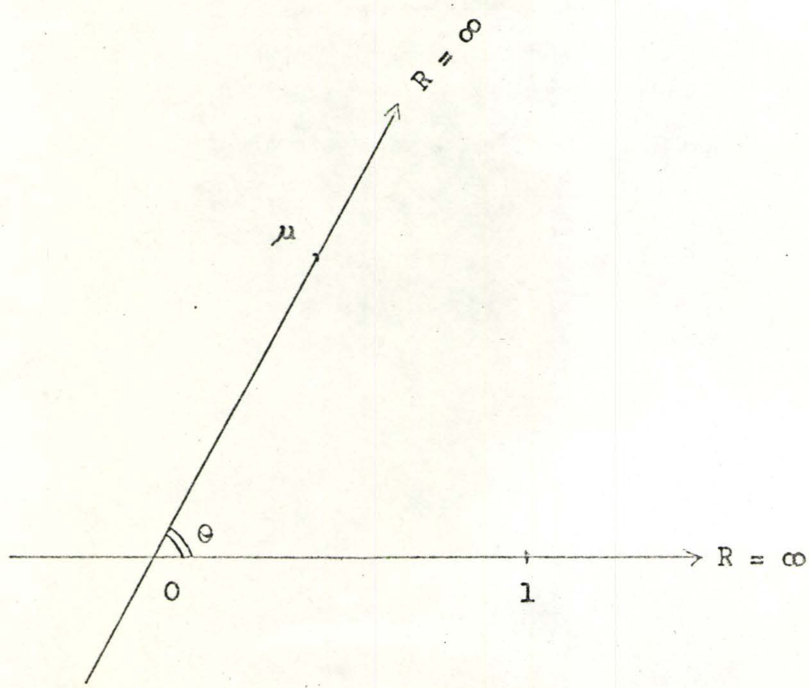

FIGURE 4

We define two circles to be tangent to each other if they have a single point in common.

Remark. We observe that if R tends to P and $S \neq P \neq Q$, then

$$\lim_{\substack{R \to P \\ S \neq P \neq Q}} \operatorname{amp}(P,Q,R,S) = \operatorname{amp}(1) = 0,$$

and hence

$$\lim \measuredangle (C_1, C_2) = 0$$

whether or not the circles $C_1 = C(P,Q,R)$ and $C_2 = C(P,R,S)$ themselves converge. Thus we may define the angle between two tangent circles to be zero.

1.6. Orientation of a circle.

An orientation of a circle C is determined by an ordered triple of mutually distinct points z_1, z_2, z_3 on C. With respect to this orientation a point z not on C is said to lie to the left of C if $\operatorname{Im}(z, z_1, z_2, z_3) > 0$ and to the right of C if $\operatorname{Im}(z, z_1, z_2, z_3) < 0$. If we orient the circle C, then the region lying to the left of the oriented circle is called the "interior" of C, and the region to the right is called the "exterior" of C. Thus any proper circle C (i.e., not a point-circle) divides the conformal plane into two open regions, the "interior" C_* of C and the "exterior" C^* of C.

We now show that there are only two different orientations; i.e., the distinction between left and right is the same for all triples, while the meanings may be reversed.

Because of the invariance of the cross-ratio it is sufficient to consider the case where C is a real-axis. We have then to examine

$\text{Im}(z,z_1,z_2,z_3)$. Writing $(z,z_1,z_2,z_3) = \frac{az+b}{cz+d}$ with real coefficients, we obtain

$$\text{Im}(z,z_1,z_2,z_3) = \frac{ad-bc}{|cz+d|^2}\,\text{Im}z.$$

Hence the distinction between left and right is identical with the distinction between the upper and lower half-plane.

Since a Möbius transformation T carries the real axis into a circle which we orient through the triple Tz_1, Tz_2, Tz_3, hence from the invariance of the cross-ratio it follows that the left and right of the real axis will correspond to the left and right of the image circle.

In general there is no way or reason to compare the orientations of two circles. An exception occurs when the circles are tangent to each other. In this case they can be transformed into parallel lines, and the circles are said to be equally oriented if they correspond to lines with the same direction. Another exception occurs when we consider the circles through three points of an arc of cyclic order three (cf. 3.3.2).

<u>1.6.1</u>. <u>Theorem</u>. <u>A Möbius transformation preserves the angles between oriented circles</u>.

<u>Proof</u>. Let C_1 and C_2 be any two oriented circles. Suppose C_1 and C_2 intersect at two points P and R. If $Q \in C_1$ and $S \in C_2$ then we have defined the angle between C_1 and C_2 as the amplitude of the cross-ratio (P,Q,R,S)(cf. Theorem 1.4.1). Hence we conclude that the amplitude of the cross-ratio is preserved, i.e., angle between the oriented circles is preserved.

In the geometric representation the orientation z_1, z_2, z_3 can be indicated by an arrow which points from z_1 over z_2 to z_3 (Fig. 5).

When the unextended complex plane (Euclidean plane) is considered as part of the extended plane, (conformal plane) the point at infinity is distinguished. We can therefore define an absolute positive orientation of all finite circles by the requirement that point at infinity should lie to the right of the oriented circles.

On a Riemann sphere there is no reason to call one side of a circle the interior.

1.7. Reflection

1.7.1. The points z and $\bar{z}$ are symmetric with respect to the real axis. A Möbius transformation with real coefficients carries the real axis into itself and $z, \bar{z}$ into points which are again symmetric. More generally, if a Möbius transformation T carries the real axis into a circle C, we shall say that the points w and w' defined by $w = Tz$ and $w' = T\bar{z}$ are symmetric with respect to C.

The relation $\overline{T^{-1}w} = T^{-1}w'$ between w and w' and C does not depend on the particular choice of T. For if S is another transformation which carries the real axis into C, then $S^{-1}T$ is a real transformation and hence

$$S^{-1}w = S^{-1}Tz = \zeta \text{ and } S^{-1}w' = S^{-1}T\bar{z} = \bar{\zeta}$$

are also conjugate. Thus $w = S\zeta$ and $w' = S\bar{\zeta}$ and so $S^{-1}w' = \overline{S^{-1}w}$.

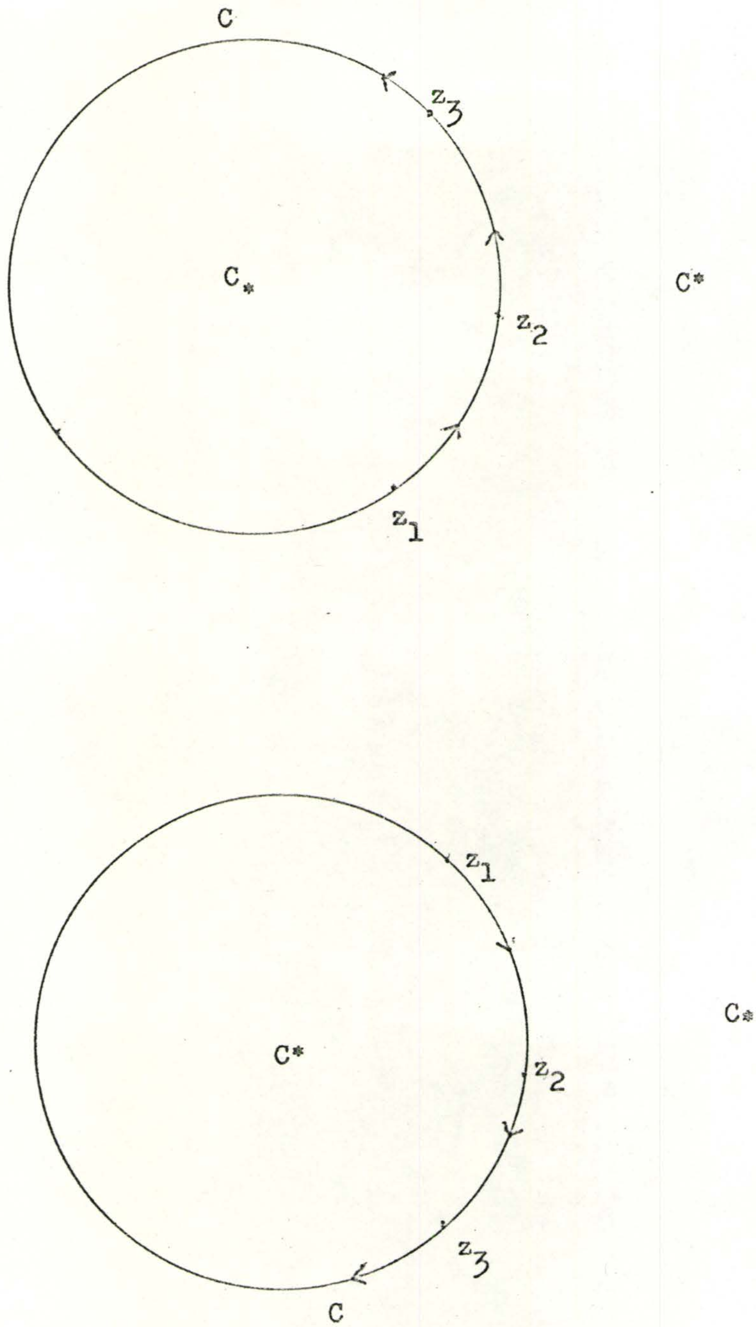

FIGURE 5

Suppose T carries $0, 1, \infty$ into w_1, w_2, w_3 on the circle C. The transformation defined by

$$w \to (w, w_1, w_2, w_3)$$

maps w_1, w_2, w_3 into $0, 1, \infty$ respectively. Hence

$$(w, w_1, w_2, w_3) = T^{-1}w.$$

Thus

$$T^{-1}w' = \overline{T^{-1}w} \Leftrightarrow (w', w_1, w_2, w_3) = \overline{(w, w_1, w_2, w_3)}.$$

Then symmetry can be defined in the following terms:

Definition. The points w and w' are said to be symmetric with respect to the circle C through w_1, w_2, w_3 if and only if $(w', w_1, w_2, w_3) = \overline{(w, w_1, w_2, w_3)}$.

The points on C and any only those are symmetric to themselves.

The mapping which carries w into w' is a one-to-one correspondence and is called reflection with respect to C. A reflection is isogonal but not conformal, i.e., it preserves angles in magnitude but reverses the signs. Two reflections will evidently result in a Möbius transformation.

1.7.2. In this subsection we would revert to the notation z, z', z_1, z_2, z_3 instead of w, w', w_1, w_2, w_3 and investigate the geometric significance of symmetry. Suppose first that one of the points z_1, z_2, z_3 is the point at infinity. Then choosing $z_3 = \infty$, the condition for symmetry becomes

$$\text{(1.7-1)} \qquad \frac{z' - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} .$$

Taking absolute values, we obtain,

$$|z' - z_1| = |z - z_1|.$$

Here z_1 can be any finite point on C, which in this case is a straight line. Thus we conclude that z and z' are equidistant from all points on C. By the relation (1.7-1.) we have further

$$\operatorname{Im} \frac{z' - z}{z_2 - z_1} = - \operatorname{Im} \frac{z - z_1}{z_2 - z_1} .$$

Hence z and z' are in different half-planes determined by C.

Now we consider the case where z_1, z_2 and z_3 all are finite. Let a be the centre and R the radius of the circle C through z_1, z_2 and z_3. Since cross-ratio is invariant under a Möbius transformation we have

$$\begin{aligned}
\overline{(z, z_1, z_2, z_3)} &= \overline{(z-a, z_1-a, z_2-a, z_3-a)} \\
&= (\bar{z}-\bar{a}, \bar{z}_1-\bar{a}, \bar{z}_2-\bar{a}, \bar{z}_3-\bar{a}) \\
&= (\bar{z}-\bar{a}, \frac{(\bar{z}_1-\bar{a})(z_1-a)}{z_1-a}, \frac{(\bar{z}_2-\bar{a})(z_2-a)}{(z_2-a)}, \frac{(\bar{z}_3-\bar{a})(z_3-a)}{(z_3-a)}) \\
&= (\bar{z}-\bar{a}, \frac{R^2}{z_1-a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}) \\
&= (\frac{1}{\bar{z}-\bar{a}}, \frac{z_1-a}{R^2}, \frac{z_2-a}{R^2}, \frac{z_3-a}{R^2}) \\
&= (\frac{R^2}{\bar{z}-\bar{a}}, z_1-a, z_2-a, z_3-a) \\
&= (\frac{R^2}{\bar{z}-\bar{a}} + a, z_1, z_2, z_3).
\end{aligned}$$

This equation shows that the symmetric point of z is $z' = \frac{R^2}{\bar{z}-\bar{a}} + a$ or that z and z' satisfy the relation

$$(z'-a)(\bar{z}-\bar{a}) = R^2,$$

from which we obtain

$$|z' - a|\,|z - a| = R^2,$$

which shows that z and z' lie in different regions determined by C. Also the ratio $\frac{z'-a}{z-a}$ is real and positive, say $z' - a = k(z-a)$ which means that the amplitude of $z' - a$ is equal to the amplitude of $z - a$ and hence z and z' are situated on the same half line from the centre a. There is a simple geometric construction of the symmetric point of z (Fig. 6). We note that the symmetric point of a is ∞.

Theorem. If a Möbius transformation T carries a circle C_1 into a circle C_2, then it transforms any pair of symmetric points with respect to C_1 into a pair of symmetric points with respect to C_2, i.e., Möbius transformation perserves symmetry.

Proof. If C_1 or C_2 is the real axis the theorem follows from the definition of symmetry. In the general case the assertion follows by the use of an intermediate transformation U which carries C_1 into the real axis. Thus TU^{-1} takes the points Uz and Uz' which are symmetric with respect to the real axis into the symmetric points Tz and Tz' on C_2.

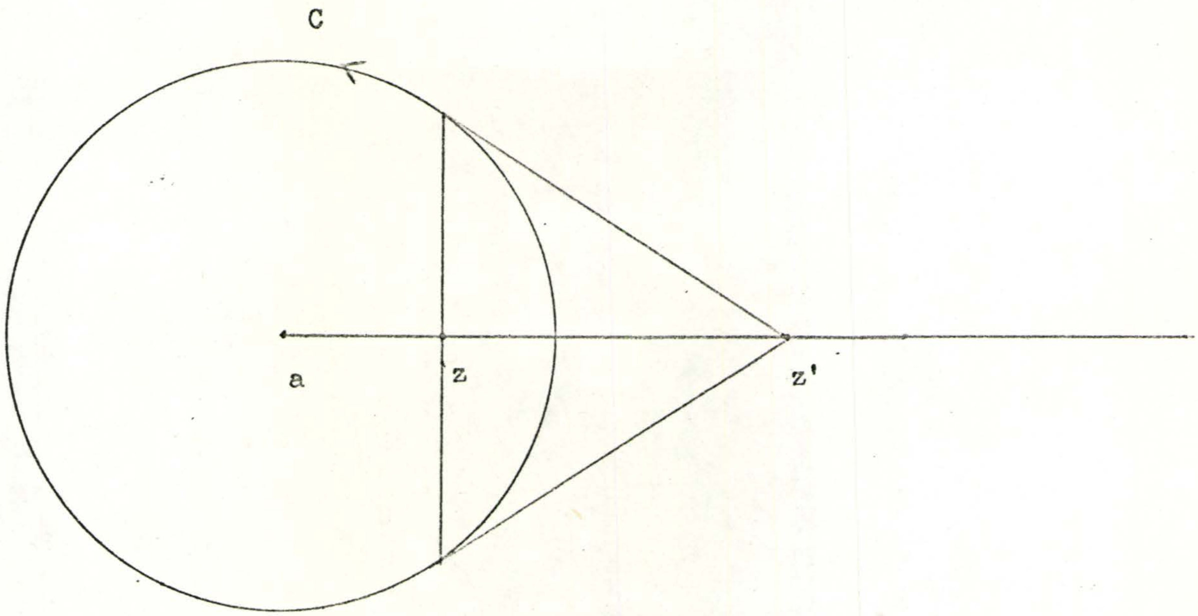

FIGURE 6

1.7.3. Consider a Möbius transformation of the form

$$w = k\,\frac{z - a}{z - b},$$

where $k \neq 0$ is a constant, say $k = 1$. Here $z = a$ corresponds to $w = 0$ and $z = b$ to $w = \infty$. It follows that the straight lines through the origin of the w-plane are images of the circles through a and b in the z-plane. These circles in the z-plane have equations of the form

$$\operatorname{amp}\frac{z - a}{z - b} = \theta, \quad -\pi \leqslant \theta \leqslant \pi.$$

On the other hand, the concentric circles about the origin in the w-plane, $|w| = \rho$ (where $\rho > 0$ is a constant) correspond to the circles in the z-plane with the equation

$$\left|\frac{z - a}{z - b}\right| = \rho .$$

These are called circles of "Apollonius", with limit points a and b.

Denote by C_1 the circles through a and b and by C_2 the circles of "Appolonius" with these limit points (Fig. 7 for w-plane and Fig. 8 for z-plane).

The configuration formed by all the circles C_1 and C_2 is called the Steiner Configuration determined by a and b. This Steiner Configuration has many interesting properties.

(i) There is exactly one C_1 and one C_2 through each point $\neq a,b$, in the plane.

Proof. Any point of the plane, together with the point a and b determines a unique circle C_1. Any finite point c in the plane

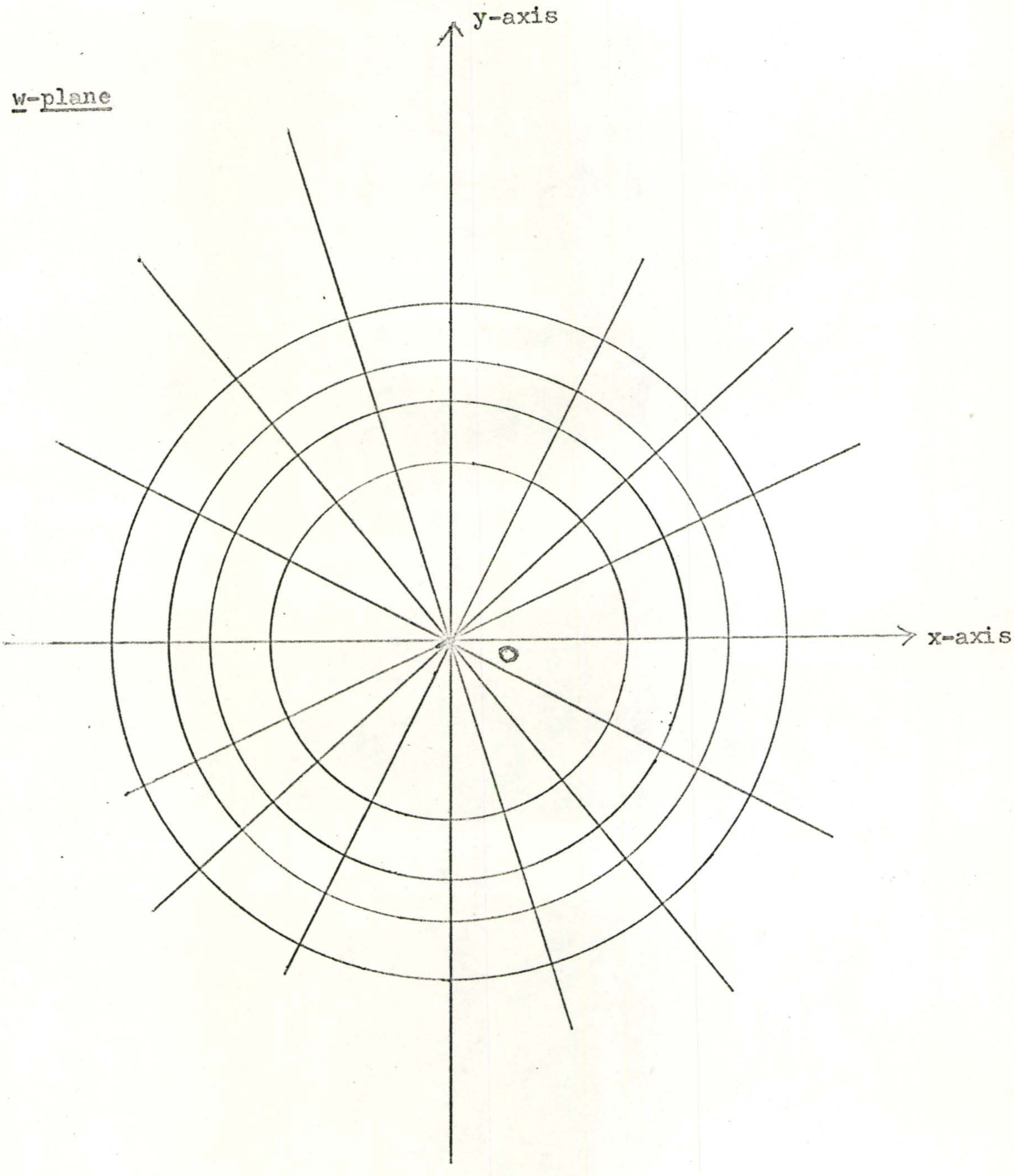

FIGURE 7

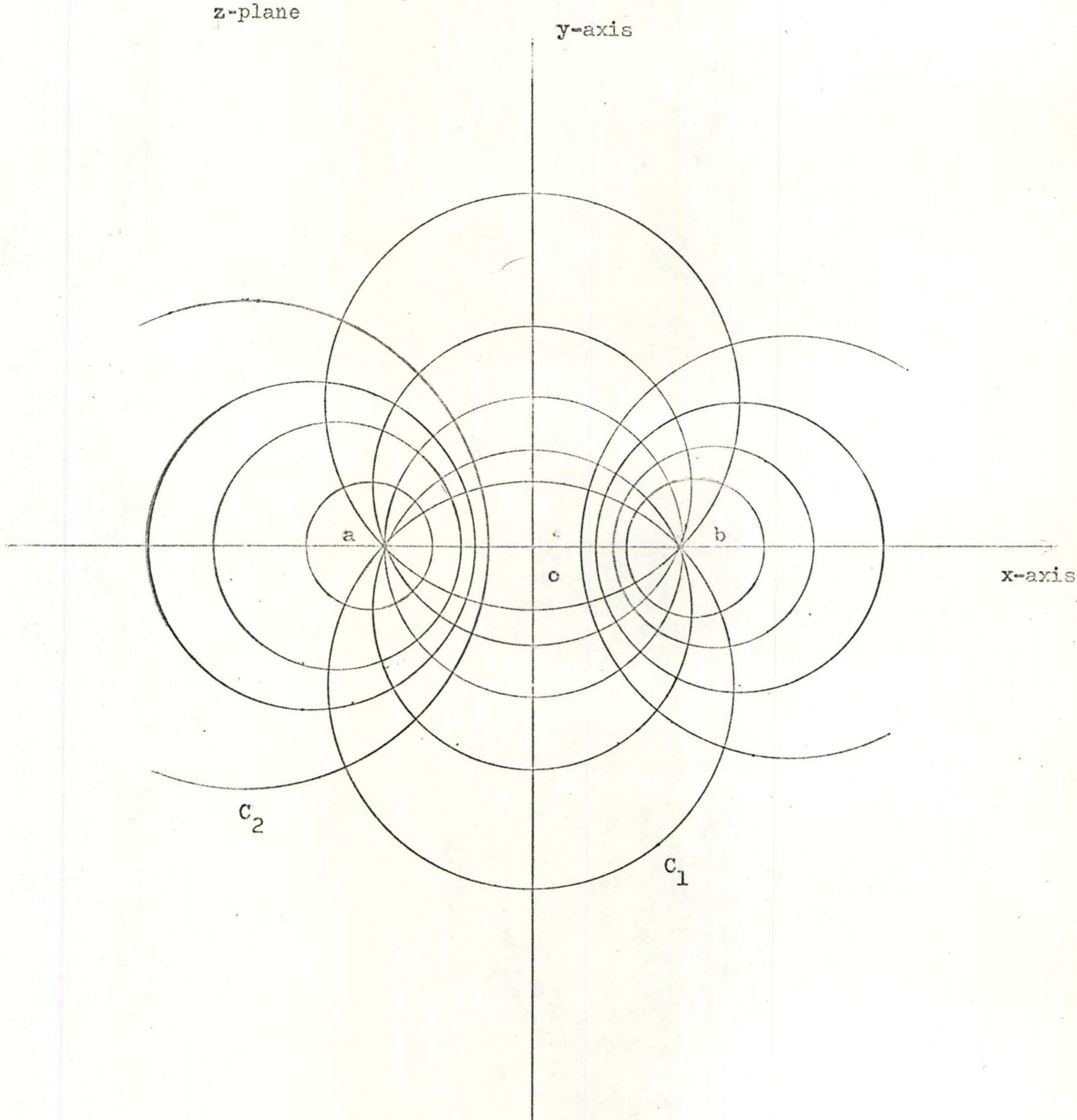

FIGURE 8

determines a unique $\rho = \left|\frac{c - a}{c - b}\right|$ and the unique circle C_2: $\left|\frac{z - a}{z - b}\right| = \rho$.

(ii) <u>Every C_1 meets every C_2 at right angles.</u>

<u>Proof.</u> This follows from the fact that a Möbius transformation preserves the angles between two circles.

(iii) <u>Reflection in a C_1 transforms every C_2 into itself and every C_1 into another C_1. Reflection in a C_2 transforms every C_1 into itself and every C_2 into another C_2.</u>

<u>Proof.</u> Let z be any point on C_2 and z_2 be on C_1 (Fig. 9). Let z' be the image of z with respect to a reflection in C_1. Then the following relation holds.

$$(z', a, z_2, b) = \overline{(z, a, z_2, b)}$$

$$\frac{z' - a}{z' - b} \Big/ \frac{z_2 - a}{z_2 - b} = \overline{\left(\frac{z - a}{z - b} \Big/ \frac{z_2 - a}{z_2 - b}\right)}.$$

Hence

$$\left|\frac{z' - a}{z' - b} \Big/ \frac{z_2 - a}{z_2 - b}\right| = \left|\frac{z - a}{z - b} \Big/ \frac{z_2 - a}{z_2 - b}\right|,$$

and

$$\left|\frac{z' - a}{z' - b}\right| = \left|\frac{z - a}{z - b}\right| = \rho \text{, say.}$$

This shows that z' lies on C_2.

Since points on C_1 are always symmetric to themselves with respect to C_1 hence by reflection the point a is mapped into itself and b into itself. Thus a circle through the point a and b is transformed into a circle through the point a and b.

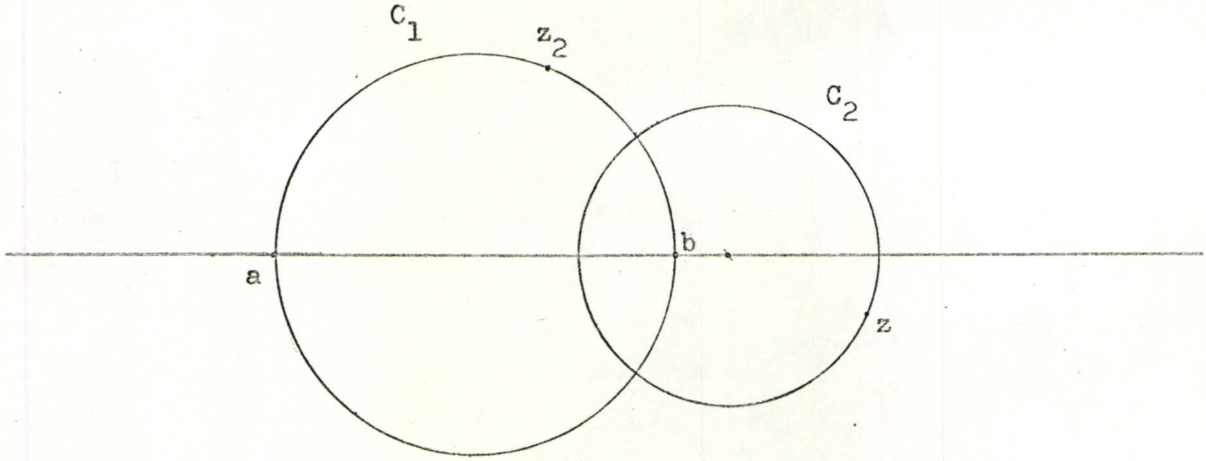

FIGURE 9

For reflection in a C_2 consider the w-plane. Then the circles C_2 are the concentric circles about the origin and C_1 are the straight lines through the origin. Let z by any point on C_1. Then by reflection in a C_2, z is mapped into the symmetric point z'. But by the discussion in the subsection 1.7.2 we know that z and z' are situated on the same half-line from the centre of the circle C_2. Hence z' lies on C_1. Thus C_1 is mapped onto itself by a reflection in a C_2. By a Möbius transformation we can extend the property (iii) to the Steiner configuration.

(iv) <u>The limit points are symmetric with respect to each C_2, but not with respect to any other circle.</u>

<u>Proof.</u> Let the point a be symmetric to b with respect to a circle C. Then for any points z_1, z_2, z_3 on C we have

$$(a, z_1, z_2, z_3) = \overline{(b, z_1, z_2, z_3)}.$$

Or

$$\frac{a - z_1}{a - z_3} \Big/ \frac{z_2 - z_1}{z_2 - z_3} = \overline{\left(\frac{b - z_1}{b - z_3} \Big/ \frac{z_2 - z_1}{z_2 - z_3} \right)}.$$

Hence

$$\left|\frac{a - z_1}{a - z_3}\right| \Big/ \left|\frac{z_2 - z_1}{z_2 - z_3}\right| = \left|\frac{b - z_1}{b - z_3}\right| \Big/ \left|\frac{z_2 - z_1}{z_2 - z_3}\right|$$

and

$$\frac{|a - z_1|}{|a - z_3|} = \frac{|b - z_1|}{|b - z_3|} .$$

Finally

$$\left|\frac{a - z_1}{b - z_1}\right| = \left|\frac{a - z_3}{b - z_3}\right| = \rho , \text{ say.}$$

But this relation holds if and only if z_1 and z_3 lie on the C_2 given by

$$\left|\frac{z - a}{z - b}\right| = \rho ;$$

cf. 1.7.3.

Hence the point a is symmetric to b with respect to C_2 but not with respect to any other circle.

The points a and b are called the <u>fundamental points</u> of the pencil C_1, and C_1 is called a <u>pencil of the first kind</u>. C_2 is called a <u>pencil of the third kind</u> (Fig. 8).

A limiting case of a pencil of the first kind as a tends to b is a <u>pencil of the second kind</u>. It possesses only one fundamental point and is identical with the set of those circles that touch a given circle at that point (Fig. 10).

<u>1.8. The closure property of the conformal plane.</u>

We know that the conformal plane may be represented on the surface of a sphere in three dimensional Euclidean space. Hence every infinite sequence of points in the conformal plane possesses at least one accumulation point, which also lies in the plane. Thus the conformal plane is closed and compact.

<u>Theorem.</u> <u>Every infinite sequence of circles in the conformal plane possesses at least one limit circle.</u>

<u>Proof.</u> Let $\{C_n\}$ be an infinite sequence of circles in the conformal plane. Then there exists a subsequence, $\{C'_m\} \subset \{C_n\}$ of circles which contains an infinite sequence of points possessing an

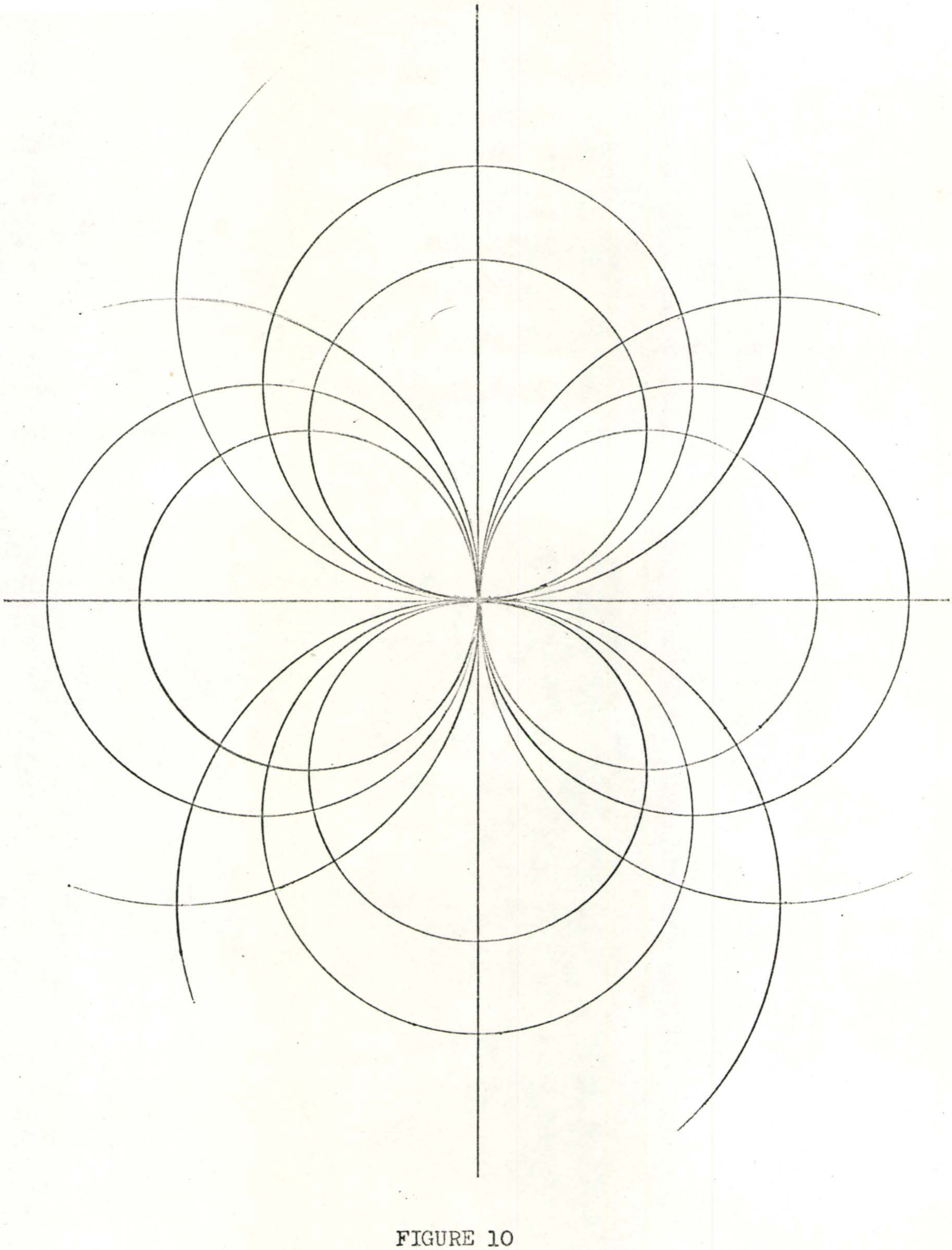

FIGURE 10

accumulation point. Again there is a subsequence $\{C''_q\} \subset \{C'_m\}$ of circles which contains a different sequence of points possessing an accumulation point. Finally there is a subsequence $\{C'''_r\} \subset \{C''_q\}$ of circles which contains yet another sequence of points possessing an accumulation point. Thus we have a sequence C''' of circles which possesses a limiting circle, the circle determined by three accumulation points.

We call a limit circle an accumulation circle of the original sequence.

1.9. Convergence.

A sequence of points $P_1, P_2 \ldots$ is said to be convergent to P, if there exists a number $n = n(C)$ for every circle C with $P \subset C_*$ such that $P_y \subset C_*$ if $y > n$.

In the same way, convergence of circles to a point is defined. Such a point is called a point-circle.

A sequence of circles $C_1, C_2 \ldots$ in the conformal plane is said to be convergent to C if there exists a number $n = n(C', C'')$, for every pair of circles with $C' \subset C_*$ and $C'' \subset C^*$, such that $C' \subset C_{y*}$ and $C'' \subset C^*_y$ for every $y > n$.

CHAPTER II

CONFORMAL DIFFERENTIABILITY

2.1. Arcs.

An arc A is the continuous image of a real closed interval. Thus if a sequence of points of that parameter interval converges to a point p, then their image points converge to the image of p. We shall use the same letters p,t,... to denote both the parameters and their images on A. The end (interior) points of A are the images of the end (interior) points of the parameter interval.

A neighbourhood of p on A is the image of a neighbourhood of the parameter p on the parameter interval. If p is an interior point A, this neighbourhood is decomposed by p into two (open) one-sided neighbourhoods.

From the definition, different points of A, i.e., points with different parameters, may coincide with the same point of the conformal plane. However, we shall assume that each point p of A has a neighbourhood such that no other point of that neighbourhood coincides with p (cf. condition I, Sec. 2.3).

2.2. Support and Intersection.

Let p be an interior point of A in the conformal-plane. Then p is called a point of support (intersection) with respect to a circle C, if a sufficiently small neighbourhood of p in A is decomposed by p into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by C. The circle C is then called a <u>supporting</u> (<u>intersecting</u>) <u>circle</u> of A at p. Thus C supports A at p even if $p \notin C$. By definition, the point circle p always supports A at p (Fig. 11).

It is possible that every neighbourhood of p has points $\neq p$ in common with C. Then C <u>neither supports nor intersects</u> A at p.

<u>2.3. Differentiable points of an arc in the conformal plane.</u>

Suppose p is a fixed point of an arc A and t a variable point. If $P \neq Q$ are points different from p, the unique circle through these points will be denoted by $C(P,Q,p)$.

An arc A is called <u>once conformally differentiable</u> at p if it satisfies the following:

<u>Condition I.</u> <u>To every point</u> $Q \neq p$ <u>and to every sequence of points</u> $t \to p$, $t \neq p$, $t \in A$ <u>there exists a circle</u> C' <u>such that</u> $C(t,p,Q) \to C'$.

The limit circle C' is called a <u>tangent circle</u> at p and is denoted by $C(\tau,Q)$. By our definition this limit tangent circle is independent of the choice of the sequence of points t. The point p itself is the <u>tangent-point circle</u> of A at p. The family of tangent circles together with point circle p will be denoted by τ.

We call A <u>conformally differentiable</u> at p if it satisfies Condition I and

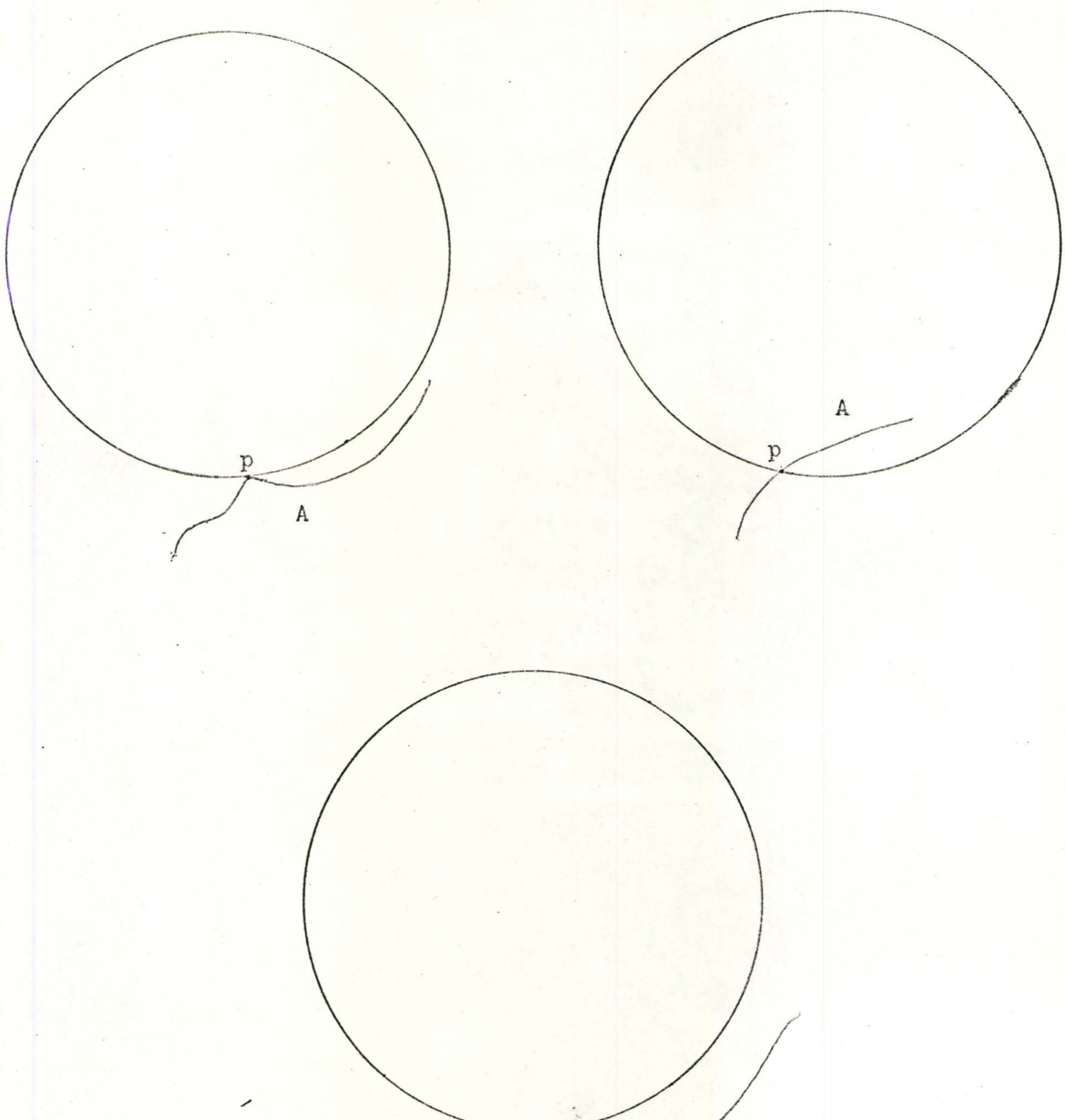

FIGURE 11

Condition II. $\lim_{\substack{t \to p \\ t \neq p}} C(\tau,t)$ exists.

The limit osculating circle is denoted by $C(p)$.

Remark. Condition I does not imply Condition II.

(cf. 4, Sec. 7, P. 516).

2.3.1. Theorem. The set $\tau = \tau(p)$ of all the tangent circles of A at p is a pencil of the second kind with the fundamental point p.

Proof. Let P,Q,R be three mutually distinct points. If the point $R' \neq R$ converges to R, then the angle between the circles $C(R',R,P)$ and $C(R',R,Q)$ converges to zero; cf. 1.5. We choose $R = p$ and $R' = t$. Since the angle between two circles depends on them continuously, we conclude that any two tangent circles at p touch each other at that point. Thus two tangent circles of A at p that have another point in common are identical. In particular, there exists one and only one tangent circle at p through each point different from p.

2.3.2. Theorem. Suppose A is once conformally differentiable at p. Let π be a pencil of second kind with p as its fundamental point; $\pi \neq \tau$. If the points $t \neq p$ coverge to p then $C(\pi,t) \to p$.

Proof. Let us assume that the statement in the theorem is false. Then there exists a circle C such that $p \in C_*$ and a sequence of points $t \to p$, $t \neq p$ such that $C(\pi,t) \not\subset C_*$ for each t. Let C_1 and C_2 be two circles of π that touch C. We may assume that π is oriented such that C lies in the closure of $C_1^* \cap C_{2*}$. Then this closed domain also contains the circles $C(\pi,t)$ and therefore the points t (Fig. 12).

Let Q be any point of C_1; $Q \neq p$. If a sequence of points R converges to p through the above domain, then the circles $C(p,Q,R)$ converge to C_1. Choosing $R = t$, we obtain $C_1 = C(\tau,Q)$, while $\tau \neq \pi$. This is a contradiction.

2.4. In the following p can be either an interior point or an end point of A.

We call C a general tangent circle of an arc A at a point p if there exists a sequence of triplets of mutually distinct points t,u,Q such that

$$\lim_{\substack{t,u \to p \\ t \neq u \neq Q}} C(t,u,Q) = C. \tag{2.4-1}$$

If in addition, $Q \in A$ also converges to p, then we call C a general osculating circle of A at p.

A is said to be once strongly conformally differentiable at p if the following condition is satisfied:

Condition I'. Let $R \neq p$, $Q \to R$ and t,u be two distinct points on A. Then

$$\lim_{\substack{t,u \to p \\ Q \to R}} C(t,u,Q) \text{ exists.}$$

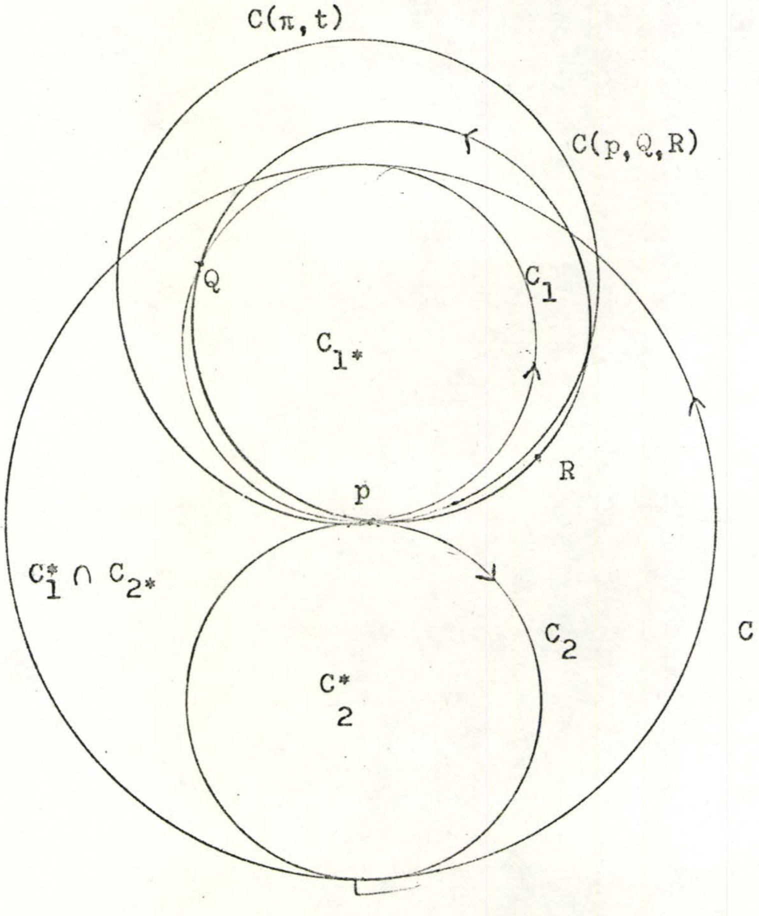

FIGURE 12

2.4.1 Lemma. Condition I' implies Condition I.

Proof. Condition I' implies that the limit circle given by the relation (2.4-1.) depends on p and R but not on the choice of the particular sequences t and u. Specializing Q = R and u = p, we see that Condition I' implies Condition I and therefore,

$$\lim_{\substack{u,v \to p \\ Q \to R}} C(t,u,Q) = C(\tau,R).$$

Thus the general tangent circles of a point which satisfies Condition I' are identical with its ordinary ones.

We call A strongly conformally differentiable at p if it satisfies Condition I' and

Condition II'. Let t,u,v be three mutually distinct points on A. Then

$$\lim_{t,u,v \to p} C(t,u,v) \text{ exists.}$$

2.4.2. Lemma. Conditions II' and I imply Condition II.

Proof. Suppose a sequence of points $\{u_n\}$ converges on A to p. By choosing a suitable subsequence of $\{u_n\}$ we may assume that the sequence $\{C(\tau,u_n)\}$ converges. Each $C(\tau,u_n)$ can be approximated by a circle $C(p,u_n,v_n)$ such that the sequence $\{C(p,u_n,v_n)\}$ has the same limit circle and such that the sequence $\{v_n\}$ also converges to p. On account of Condition II', lim C(t,u,v) and in particular, $\lim C(p,u_n,v_n)$ is independent of the choice of the sequences t,u,v converging to p. Hence the same will hold true of $\lim C(\tau,u_n)$, Condition II is satisfied, and we have

$$\lim_{t,u,v \to p} C(t,u,v) = \lim_{u,v \to p} C(p,u,v) = \lim_{u \to p} C(\tau,u) = C(p).$$

Thus strong differentiability implies ordinary differentiability and $C(p)$ is one and only one general osculating circle.

2.4.3. Theorem. Condition II' with $\lim C(t,u,v) \neq p$, or with p an end-point or with Condition I implies Condition I'.

Proof. Assume Condition II' and let $R \neq p$.

Case I: $\lim_{t,u,v \to p} C(t,u,v) \neq p.$

Choose a point S on $C(t,u,v)$ such that S does not tend to p as t,u,v converge to p; thus $C(t,u,v) = C(t,u,S)$. Then the angle between $C(Q,t,u)$ and $C(S,t,u)$ is given by the amplitude of the cross-ratio (u,Q,t,S). This amplitude tends to zero as t and u converge to p.

Hence any accumulation circle of the circle (Q,t,u) is the unique circle through R, which is tangent to C at p. Thus A satisfies Condition I' at p, (Fig. 13).

Case II: $\lim_{t,u,v \to p} C(t,u,v) = p.$

In this case we can choose a subarc B of A with $p \in B$ such that $R \notin C(t,u,v)$ for every choice of t,u,v on B. This implies that $C(t,u,R)$ does not meet B elsewhere; thus B has "R-order" two, i.e., no circle through R meets B more than twice.

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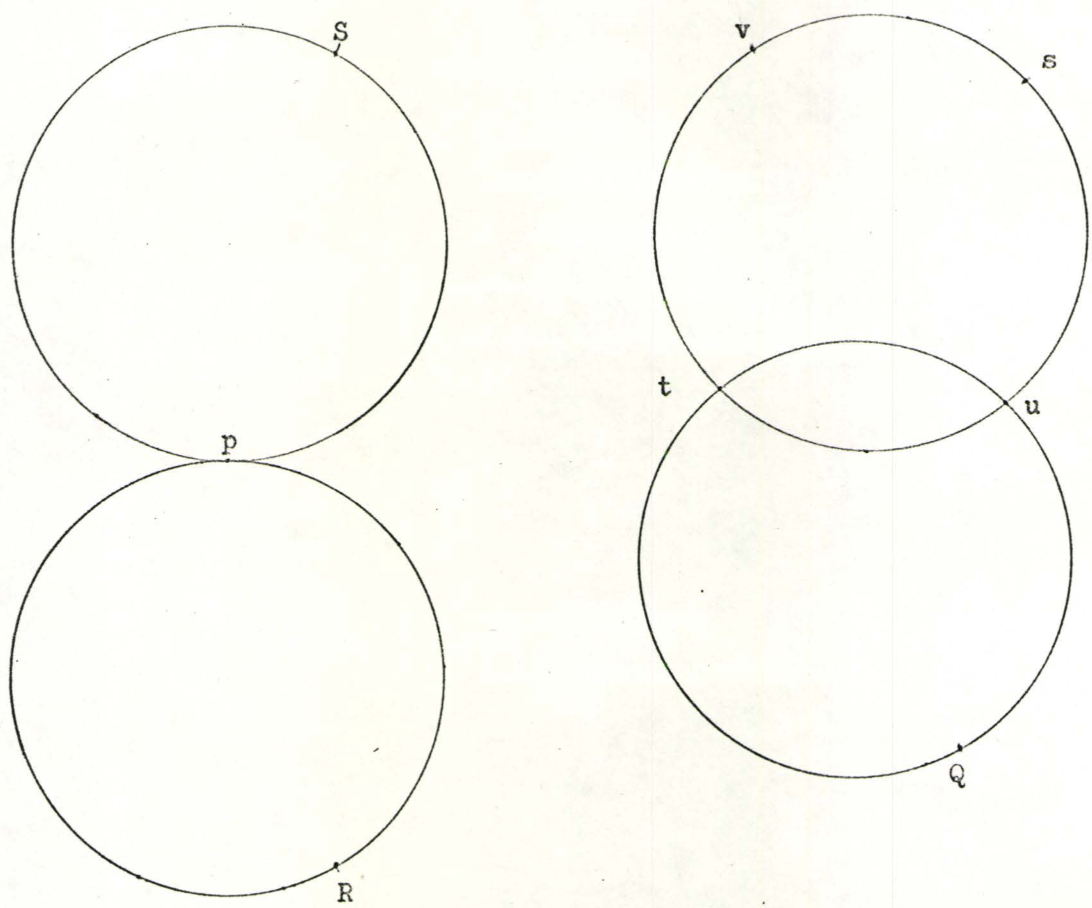

FIGURE 13

We first prove that if p is an end-point of B then B and hence A satisfies Condition I at p.

Suppose Condition I is not satisfied. Then there are two sequences of points s_{2k} and s_{2k+1} different from p and converging on B to p such that the circles $C_{2k} = C(s_{2k}, p, R)$ and $C_{2k+1} = C(s_{2k+1}, p, R)$ converge to different limit circles C_o and C_1 respectively. Since p is an end-point of B we may assume that s_{n+1} lies between p and s_n. If k is large C_{2k} $[C_{2k+1}]$ will lie close to C_o $[C_1]$. Let C and C' be two circles through p and R which separate C_o and C_1. Then $C \cup C'$ will separate C_n and C_{n+1} and therefore also s_n and s_{n+1} for every large n. Hence the subarc of B bounded by s_n and s_{n+1} will meet $C \cup C'$ in at least one-point. Thus B will meet $C \cup C'$ infinitely many times. This is impossible. Hence Condition I holds at p.

We now prove that case 2 and Condition I imply Condition I' whether or not p is an end-point of A.

Since B has "R-order" two, as x moves continuously and monotonically from p to v $C(R,t,x)$ moves continuously and monotonically from $C(R,p,t)$ to $C(R,t,v)$. Thus

$$C(R,t,u) \subset \left[C(R,p,t)_* \cap C(R,t,v)^*\right] \cup \left[C(R,p,t)^* \cap C(R,t,v)_*\right] \cup R \cup t,$$

(Fig. 14).

(cf. 3, 2.7.)

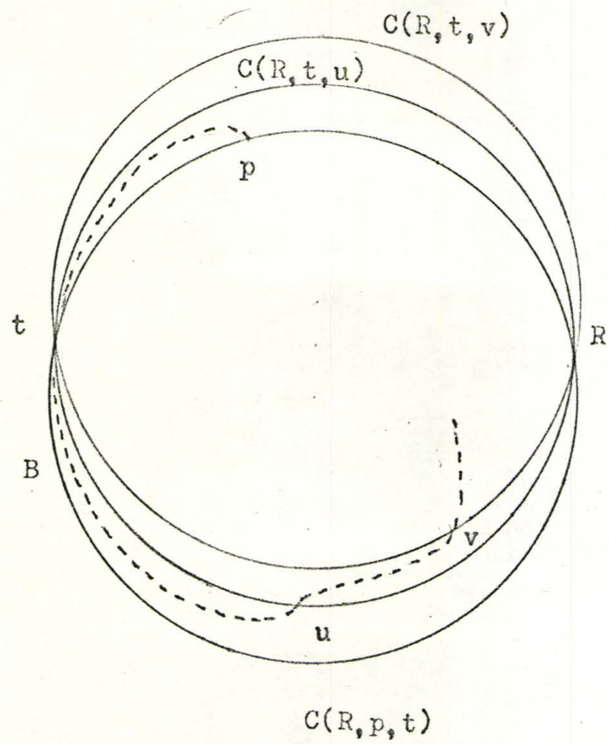

FIGURE 14

Let D denote any accumulation circle of the $C(R,t,u)$. Letting $t,u \to p$ we conclude that D lies in the closure of

$$\left[C(\tau,R)_* \cap C(R,p,v)^*\right] \cup \left[C(\tau,R)^* \cap C(R,p,v)_*\right]$$

for each choice of v on A. Hence letting $v \to p$ we obtain

$$\lim_{t,u \to p} C(R,t,u) = C(\tau,R).$$

Thus Condition I' is satisfied.

<u>Remark.</u> <u>If Condition</u> II' <u>holds with</u> $\lim_{t,u,v \to p} C(t,u,v) = p$ <u>and if</u> p <u>is an interior point of A</u>, <u>then Condition</u> I' <u>need not hold</u>.

For example, consider the arc $A = A_3' \cup p \cup A_3$ where A_3 and A_3' are given by

$$x = t^2,\ y = t^3, \quad 0 \leqslant t \leqslant \tfrac{1}{2},$$

and

$$x = -t^3,\ y = t^2, \quad -\tfrac{1}{2} \leqslant t < 0,$$

respectively (Fig. 15).

Now A satisfies Condition II' at the origin p, and

$$\lim_{t,u,v \to p} C(t,u,v) = p.$$

However Condition I, and hence Condition I' does not hold.

<u>Proof.</u> Since A_3 and A_3' are of cyclic order three they are strongly differentiable at p; cf. Theorem 3.3.2.

Let τ and τ' denote families of tangent circles of A_3 and A_3' respectively.

Hence $$\lim_{\substack{s,t,u \in A \\ s,t,u \to p}} C(s,t,u) = \lim_{s \to p} C(\tau,s).$$

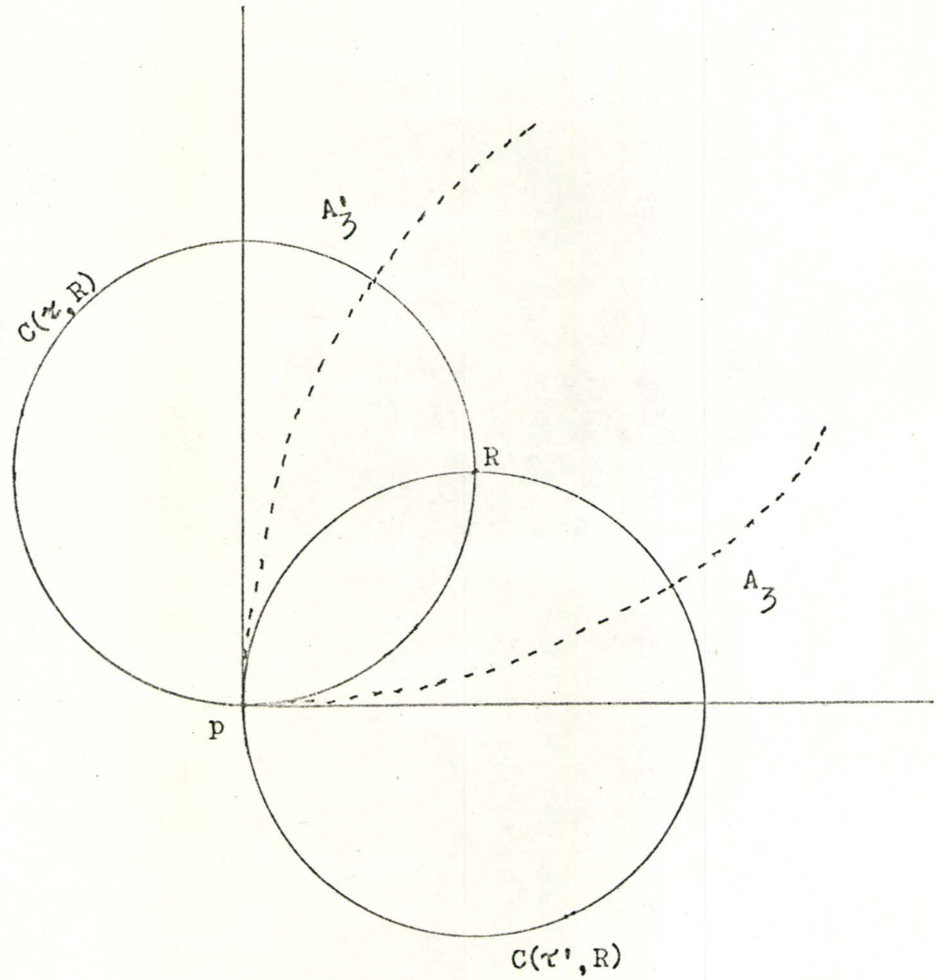

FIGURE 15

But $C(\tau,s)$ is given by equation

$$x^2 + y^2 - (s-s^3)y = 0. \tag{2.4-2.}$$

Thus as $s \to p$ (i.e., $s \to 0$) this circle tends to the circle

$x^2 + y^2 = 0$, which is the point-circle p.

Hence

$$\lim_{\substack{s,t,u \to p \\ s,t,u \in A_3}} C(s,t,u) = \lim_{s \to p} C(\tau,s) = p.$$

Let $R \neq p$ be a fixed point. We first show that <u>no circle through</u> R <u>can meet</u> A_3 <u>and</u> A_3' <u>twice each near</u> p.

A circle through Q near R and two points of A_3 near p is close to $C(\tau,R)$, while a circle C' through R and two points of A_3' near p is close to $C(\tau',R)$, where τ and τ' denote the family of tangent circles of A_3 and A_3' at p, respectively. But $\tau \neq \tau'$, because, in fact, circles belonging to τ are orthogonal to circles belonging to τ'. Hence we obtain that $C \neq C'$.

Suppose

$$\lim_{t,u,t',u' \to p} C(t,u,t',u') = C_0 \neq p.$$

Take $S \in C_0$, $S \neq p$. Let $Q \in C(t,u,v)$, $Q \to S$ as $t,u,v \to p$. Then

$$C(t,u,t',u') = C(t,u,Q) = C(t',u',Q)$$

which contradicts the fact that $\tau \neq \tau'$. Hence it follows that a circle through two points of A_3 near p and two points of A_3' near p is close to the point-circle p.

It is readily verified (cf. eq. 2.4-2.) that $C(\tau,R)$ supports $A = A_3 \cup p \cup A_3'$ at p. Hence the end-points of a small neighbourhood $M = N \cup p \cup N'$ of p lie on the same side of $C(\tau,R)$. Hence a circle C_1 through R and two points of $N \subset A_3$ will meet M with an even multiplicity and hence twice or four times. From the above, C_1 cannot meet M four times. Hence C_1 meets M exactly twice, i.e., C_1 meets N twice and $C_1 \cap N' = \emptyset$. Symmetrically a circle C_1' through R and two points of $N' \subset A_3'$ will meet N' exactly twice and $C_1 \cap N = \emptyset$.

Finally, a circle through three points of a sufficiently small neighbourhood of p on A cannot pass through a point Q near R if $R \neq p$. As above, this implies that

$$\lim_{\substack{t,u,v \to p \\ t,u,v \in A}} C(t,u,v) = p.$$

2.5. Non-tangent circles.

Let p be an interior point of A. Suppose that p satisfies Condition I (cf. 2.3).

2.5.1. Theorem. Every non-tangent circle either supports or intersects A at p.

Proof. If a circle C neither supports nor intersects A at p, then $p \in C$ and there exists a sequence of points $t \to p$ such that $t \in A \cap C$ and $t \neq p$. Let $P \in C$, $P \neq p$. Then $C = C(t,p,P)$ for each t, and Condition I implies that $C = C(\tau,P)$, which is a contradiction to the fact that C is a non-tangent circle.

2.5.2. Theorem. Non-tangent circles through p all intersect or all support.

Proof. Let C_1 and C_2 be two non-tangent circles through p. Suppose that C_1 and C_2 have another point P in common. Let C_1 intersect and C_2 support A at p. Thus $A \cap C_{1*}$ and $A \cap C_1^*$ are non-void. We may assume that $A \subset C_2^*$ (Fig. 16).

If $t \in A \cap C_{1*}$, then

$$C(p,t,P) \subset (C_{1*} \cap C_2^*) \cup (C_1^* \cap C_{2*}) \cup P \cup p.$$

By having $t \to p$, we conclude that

$$(2.5\text{-}1.) \qquad C(\tau,P) \subset (C_{1*} \cap C_2^*) \cup (C_1^* \cap C_{2*}) \cup P \cup p.$$

Considering now a sequence of points, $t' \to p$, where $t' \in A \cap C_1^*$ we obtain symmetrically the relation

$$(2.5\text{-}2.) \qquad C(\tau;P) \subset (C_{1*} \cap C_{2*}) \cup (C_1^* \cap C_2^*) \cup P \cup p.$$

Hence by the relations (2.5-1.) and (2.5-2.) we see that $C(\tau,P)$ lies in the intersection $C_1 \cup C_2$ of these two domains, i.e., $C(\tau,P)$ is either C_1 or C_2, contrary to our assumption. Thus C_1 and C_2 either both intersect or both support.

If C_1 and C_2 meet only at p, then they touch at that point. Choose any non-tangent circle C_3 through p that does not belong to the pencil through C_1 and C_2. From above, C_1 and C_3, and also C_3 and C_2, either both support or both intersect. Hence our statement

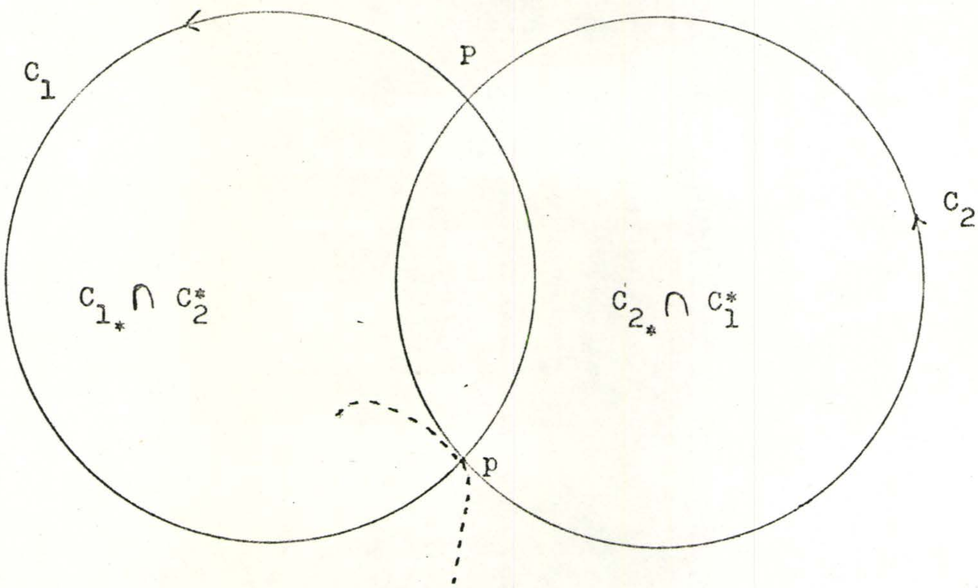

FIGURE 16

remains valid for C_1 and C_2 also in this case.

<u>Definition</u>: We call <u>an interior point of</u> A <u>which satisfies Condition I a cusp point if the non-tangent circles of</u> A <u>at</u> q <u>all support</u> A <u>at</u> q.

CHAPTER III

ARCS OF CYCLIC ORDER THREE

3.1. Arcs of finite cyclic order.

An arc A is said to be of finite cyclic order if it has only a finite number of points in common with any circle. If the least upper bound of these numbers is finite, then it is called the cyclic order of A, and A is said to be of bounded cyclic order. The order of a point p of A then is the minimum of the orders of all the neighbourhoods of p on A.

3.1.1. Lemma. Let A be an arc of finite cyclic order, and let a circle C intersect A at a point p. Then any circle C', sufficiently close to C, also intersects A, and does so in an odd number of points close to p.

Proof. Since C intersects A at p, the end-points of a sufficiently small neighbourhood M of p, lie in opposite regions with respect to C. Hence they lie on opposite sides of C'. Since C' meets M a finite number of times, it must intersect M an odd number of times.

3.1.2. Lemma. *Let A be an arc of finite cyclic order. If the parameter t_n tends to one of the end-points of the parameter interval, then the corresponding sequence of points t_n on A converges.*

Proof. Let $\lim\limits_{\gamma \to \infty} t_{2\gamma} = p$ and $\lim\limits_{\gamma \to \infty} t_{2\gamma+1} = q$ be any two accumulation points of the sequence t_n. We may assume that t_{n+1} lies between t_n and t_{n+2} for all n. If $p \neq q$, let C be a circle separating these two points. Thus there is a number $N = N(C)$ such that $t_{2\gamma}$ and $t_{2\gamma+1}$ are separated for all $\gamma > N$. But this implies that A meets C an infinite number of times, which is not true. Hence $p = q$.

3.1.3. Theorem. *Let p be an end-point of an arc A of finite cyclic order. Then the arc A is conformally differentiable at p.*

Proof. Suppose Condition I of section 2.3 is not satisfied. Let t_{2k} and t_{2k+1} be two sequences of points converging to p such that some point $R \neq p$, $C_{2k} = C(R,t_{2k},p) \to C_o$ and $C_{2k+1} = C(R,t_{2k+1},p) \longrightarrow C_1$, $C_1 \neq C_o$. We may assume that t_{n+1} lies between t_n and t_{n+2} on A. If k is large, $C_{2k}\left[C_{2k+1}\right]$ will lie close to $C_o\left[C_1\right]$. Let C and C' be two circles through p and R which separate C_o and C_1, (Fig. 17). Then, for each n sufficiently large, C and C' separate $C(R,t_n,p)$ and $C(R,t_{n+1};p)$. Hence the sub-arc of A bounded by t_n and t_{n+1} will meet $C \cup C'$ in at least one point. Thus A will meet $C \cup C'$ infinitely many times. This is impossible. Thus Condition I holds.

Let us now suppose that Condition II of section 2.3 does not hold. Let t_{2k} and t_{2k+1} be two sequences of points converging to p

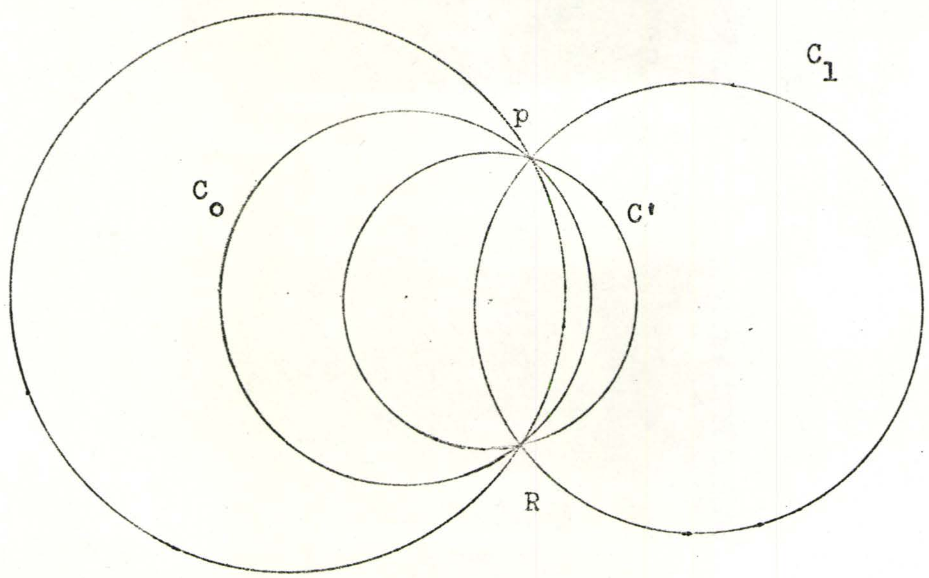

FIGURE 17

on A such that $C(\tau, t_{2k}) \to C_o$ and $C(\tau; t_{2k+1}) \to C_1 \neq C_o$. As before we assume that t_{n+1} lies between t_n and t_{n+2} on A. Both of the circles C_o and C_1, being the limit of sequences of tangent circles are themselves tangent circles, and since family of tangent circles at p form a pencil of second kind, they touch at p.

Suppose first of all, that C_o and C_1 are both proper circles. We may assume that $C_1 \subset C_{o*} \cup p$ and $C_o \subset C_1^* \cup p$. Consider a circle $C \in \tau$, $C \neq p$, $C \subset (C_{o*} \cap C_1^*) \cup p$. We may assume $C_1 \subset C_* \cup p$ and $C_o \subset C^* \cup p$ (Fig. 18). Then for sufficiently large k, $C(\tau, t_{2k+1}) \subset C_* \cup p$, and $C(\tau, t_{2k}) \subset C^* \cup p$. Here again the arc A crosses C an infinite number of times, which is impossible.

If now, C_1 is a point circle p, consider two circles of τ, C and C', $C \subset C_{o*} \cup p$ and $C' \subset C_o^* \cup p$. Also we may assume that $C_o \subset (C^* \cap C'_*) \cup p$ (Fig. 19). Then for sufficiently large k, $C(\tau, t_{2k}) \subset (C^* \cap C'_*)$ p while $C(\tau, t_{2k+1}) \subset (C_* \cup C'^*) \cup p$. Since these two regions are separated by C and C', one or both of these circles will meet A between t_{2k} and t_{2k+1}. Thus $C \cup C'$ will meet A an infinite number of times. Since this too is impossible by our hypothesis, Condition II holds, and the point p is differentiable.

3.2. Arcs of Cyclic Order Three.

Since any three distinct points determine a circle, the cyclic order of any arc is at least three. An arc A is said to have cyclic order three if no circle meets A more than three times. Let A_3 denote an open arc of cyclic order three.

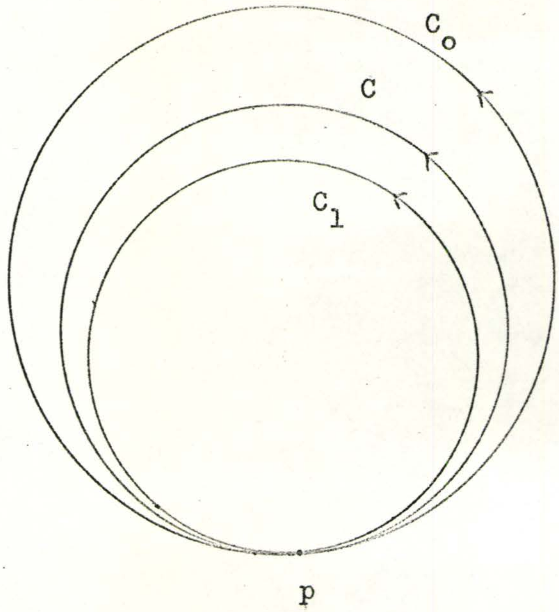

FIGURE 18

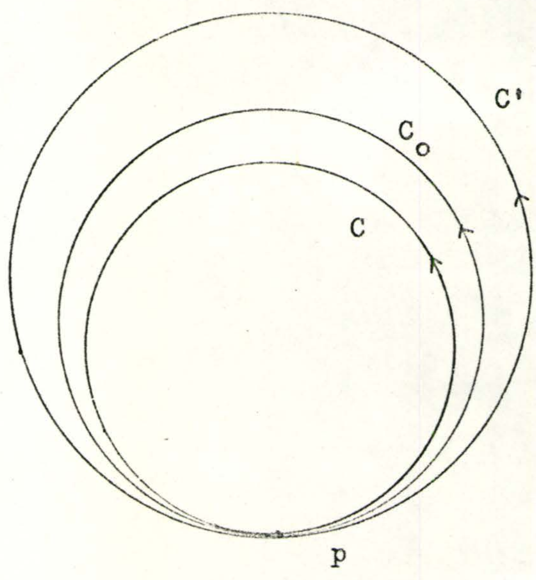

FIGURE 19

3.2.1. Multiplicities.

We introduce multiplicities by counting an end-point p of A_3 twice on any non-osculating tangent circle at p, and three times on C(p). We count an interior point q of A_3 three times on any general osculating circle at q and twice on any other general tangent circle at q.

Let p and e be the end-points of A_3. $\bar{A}_3 = p \cup A_3 \cup e$. Let τ_e denote the family of tangent circles of A_3 at e. The goal of this section is to prove that:

If $C(\tau, e) \neq C(p, \tau_e)$, then no circle meets the closure of A_3 more than three times. Thus the inclusion of the end-points of A_3 and the introduction of multiplicities do not alter the cyclic order of A_3 (cf. Theorem 3.2.12).

Remark. Any circle through $q \in A_3$ will either support or intersect A_3 there, because of the finiteness of the cyclic order of A_3.

3.2.2. Lemma. A general osculating circle at an interior point q of A_3 intersects A_3 at q while any other general tangent circle of A_3 at q supports A_3 there.

Proof. Let q be an interior point of A_3. Let C be a general osculating circle at q. Then for some triplets t,u,v,

$$\lim_{t,u,v \to q} C(t,u,v) = C.$$

Consider a neighbourhood N of the point q. If t,u,v are sufficiently close to q, then, the end-points of N lie in opposite regions with respect to $C(t,u,v)$; otherwise the arc A_3 meets $C(t,u,v)$ once more, because any circle meets an arc an even number of times if and only if the end-points of the arc lie in the same region with respect to the circle. Hence the end-points of N lie in opposite regions with respect to the circle C.

Let $R \to Q \neq q$ and let $t,u \to q$. Choose any neighbourhood N of q on A_3. Let C_1 be a general tangent circle of A_3 at q which is not an osculating circle of A_3 at q;

$$\lim_{t,u \to p} C(t,u,Q) = C_1.$$

If t,u are sufficiently close to q, then the end-points of N will lie in the same region with respect to $C(t,u,Q)$ and hence will lie in the same region with respect to C_1. Hence C_1 supports A_3 at q.

3.2.3. Lemma. No circle through p and two points of A_3 intersects A_3 at another point.

Proof. Suppose a circle C through p intersects A_3 at q and meets A_3 in two more points r and s. Choose disjoint neighbourhoods N of p and M of q which do not contain r or s. If $t \in N$ and $t \to p$, then $C(t,r,s) \to C$. By Lemma 3.1.1 $C(t,r,s)$ separates the end-points of M if t is sufficiently close to p. Thus $C(t,r,s)$ meets A_3 again in the neighbourhood of q (Fig. 20). Thus $C(t,r,s)$ meets A_3 in not less than four points, which is impossible.

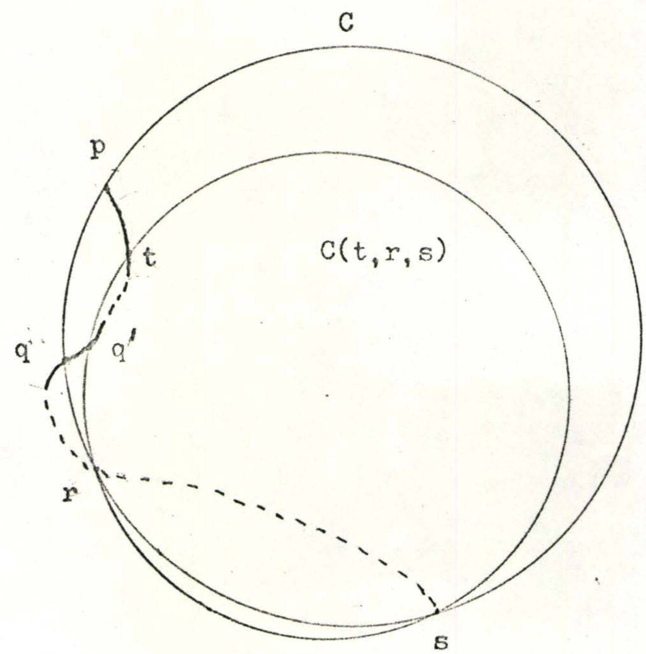

FIGURE 20

<u>3.2.4.</u> <u>Lemma.</u> <u>A circle through three points of</u> $A_3 \cup p$ <u>does not support</u> A_3 <u>at any of them.</u>

<u>Proof.</u> If a circle C supports A_3 at q and also meets $A_3 \cup p$ at r and s, then a suitable circle near C through r and s intersects A_3 twice near q. This is impossible by Lemma 3.2.3 and the definition of A_3.

<u>3.2.5.</u> <u>Lemma.</u> <u>No circle meets</u> $A_3 \cup p$ <u>in four points.</u>

<u>Proof.</u> Lemmas 3.2.3 and 3.2.4.

<u>3.2.6.</u> <u>Lemma.</u> <u>No tangent circle at</u> p <u>meets</u> A_3 <u>in more than one point.</u>

<u>Proof.</u> Suppose a tangent circle C at p intersects A_3 at q and meets A_3 also at $r \neq q$. Then there will be a circle through p and r which intersects A_3 near p and also near q. Thus we have a circle which meets $A_3 \cup p$ in four points, which is impossible by Lemma 3.2.5.

Suppose the circle C tangent at p supports A_3 at q. Then we have another tangent circle C' at p which intersects A_3 twice near q. This contradicts the above.

<u>3.2.7.</u> <u>Lemma.</u> C(p) <u>does not meet</u> A_3.

<u>Proof.</u> C(p) being the limit circle of tangent circles at p is also tangent circle at p. By Lemma 3.2.6, C(p) can meet A_3 only once and that point is a point of intersection. Suppose that C(p) intersects A_3 at q. Let N and M be disjoint neighbourhoods of p and q respectively, and let $t \in N$, $t \to p$. Then $C(t,\tau)$, when close to C(p)

will meet M, this contradicts Lemma 3.2.6.

<u>3.2.8.</u> <u>Lemma.</u> <u>No circle supports</u> A_3 <u>at two distinct points.</u>

<u>Proof.</u> Suppose a circle C supports A_3 at two distinct points q and r. We may assume $A_3 \subset C \cup C^*$. Let M and N be disjoint neighbourhoods on A_3 of q and r respectively. Choose a circle D in C* and sufficiently close to C (Fig. 21). Since the end-points of M and N lie in C*, they will also lie in D*. On the other hand $C \subset D_*$ implies $q \in D_*$ and $r \in D_*$. Thus D separates q [r] from the end-points M [N], D will intersect M[N] in not less than two points, and thus $D \cap A_3$ contains more than three points, which is impossible by the definition of A_3.

<u>3.2.9.</u> <u>Lemma.</u> <u>Let</u> C <u>be a general tangent circle of</u> A_3 <u>at</u> q <u>but not a general osculating circle there.</u> <u>Then</u> C <u>meets</u> $A_3 \cup p$ <u>at most once outside</u> q <u>and that point is not a point of support.</u>

<u>Proof.</u> By Lemma 3.2.2, C supports A_3 at q. By Lemma 3.2.8 any other point of $A_3 \cap C$ is a point of intersection. By Lemma 3.2.4 C meets $A_3 \cup p$ at most once outside q.

<u>3.2.10.</u> <u>Lemma.</u> <u>A general osculating circle at an interior point of</u> A_3 <u>does not meet</u> $A_3 \cup p$ <u>elsewhere.</u>

<u>Proof.</u> Let C be a general osculating circle of A_3 at an interior point q. Thus

$$C = \lim_{\substack{t,u,v \to q \\ t,u,v \in A_3}} C(t,u,v).$$

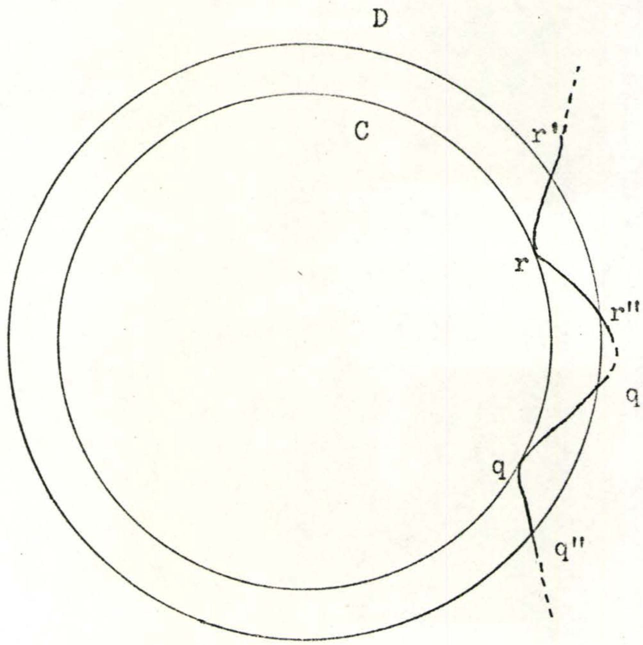

FIGURE 21

Suppose C meets $A_3 \cup p$ at a point $r \neq q$. Then the orthogonal circle of C through q and r will intersect $C(t,u,v)$ at a point R converging to r. (Fig. 22). Thus

$$C(t,u,v) = C(t,u,R).$$

The circles $C(t,u,r)$ will not meet $A_3 \cup p$ elsewhere by Lemma 3.2.5, and they will intersect A_3 at t and u, by Lemma 3.2.4. Thus the end-points of any small neighbourhood of q will lie on the same side of $C(t,u,r)$, if this circle is close enough to C. Hence any limit circle D of $C(t,u,r)$ will support A_3 at q. By the Remark of 1.5,

$$\lim \measuredangle \left[C(t,u,R),\ C(t,u,r)\right] = 0.$$

Since the angle between two circles depends on them continuously, it follows that $\measuredangle\left[C,D\right] = 0$. Since C and D have points q and r in common, this implies $C = D$. However D supports but C intersects (by Lemma 3.2.1) A_3 at q. Hence C does not meet $A_3 \cup p$ outside q.

<u>3.2.11</u>. Combining the Lemmas from 3.2.5 to 3.2.10, we obtain

<u>Lemma</u>. <u>No circle meets</u> $A_3 \cup p$ <u>more than three times</u>.

<u>3.2.12</u>. <u>Theorem</u>. <u>If</u> p <u>and</u> e <u>are the end-points of</u> A_3 <u>then no circle meets</u> $\bar{A}_3 = e \cup A_3 \cup p$ <u>more than three times provided that</u> $C(\tau_e, p) \neq C(\tau, e)$.

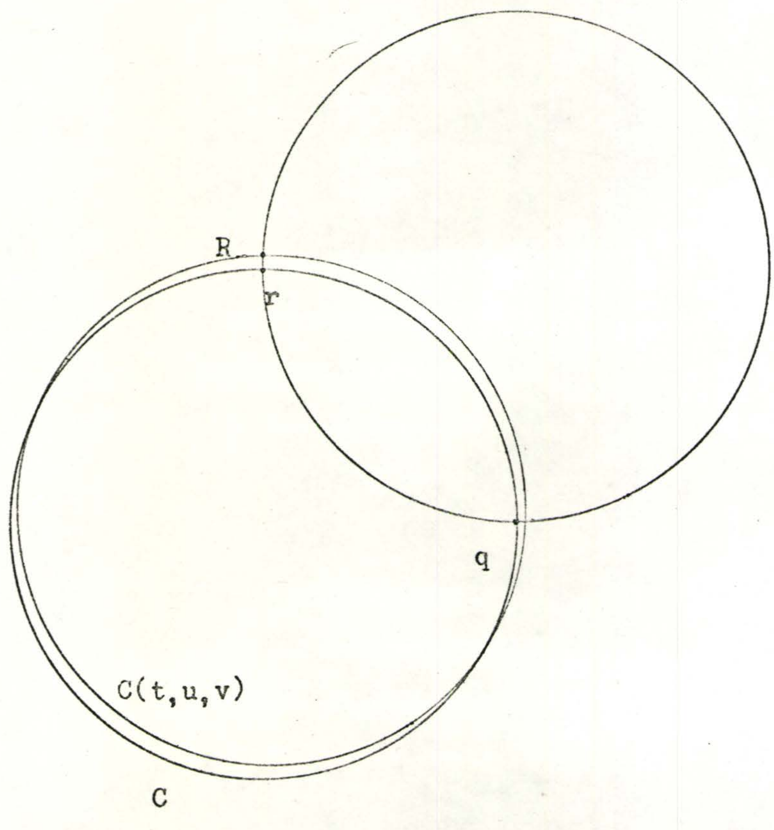

FIGURE 22

Proof. By 3.2.11 we know that $A_3 \cup p$ is of cyclic order three counting multiplicities. Thus to prove that $\bar{A}_3$ is of cyclic order three we need to show that $C(e,t,p)$ where $t \in A_3$, does not support A_3 at t and does not meet A_3 elsewhere and that $C(\tau,e)$ and $C(\tau_e,p)$ do not meet A_3 again, and the osculating circle at one end-point does not pass through the other end-point.

Assume that $C = C(e,t,p)$ meets A_3 in q (Fig. 23). By 3.2.4 t and q are points of intersection of A_3. Then there exists a circle C' through p and t close to C which intersects the arc A_3 in neighbourhoods of the points q and e. Thus C' meets $A_3 \cup p$ four times. This contradicts 3.2.11. Thus no circle through p and e intersects A_3 at two distinct points.

Suppose $C(p,t,e)$ supports A_3 at t. Then a suitable circle through e and p would meet A_3 at two points near q. This is impossible. Hence no circle through p and e supports A_3 at any point.

Let $C(\tau,e)$ intersect A_3 at a point r. Then there is a circle C' through p and e and close to $C(\tau,e)$ which meets A_3 near p and r (Fig. 24), contrary to the above. Thus $C(\tau,e)$ does not meet A_3. Similarly $C(\tau_e,p)$ does not meet A_3.

Suppose $C(p)$ goes through the other end-point e. Then there is a circle C', close to $C(p)$, which is a tangent circle of A_3 at p and intersects A_3 near e and p (Fig. 25). This contradicts Lemma 3.2.11. Hence $C(p)$ does not pass through e. Similarly $C(e)$ does not pass through p.

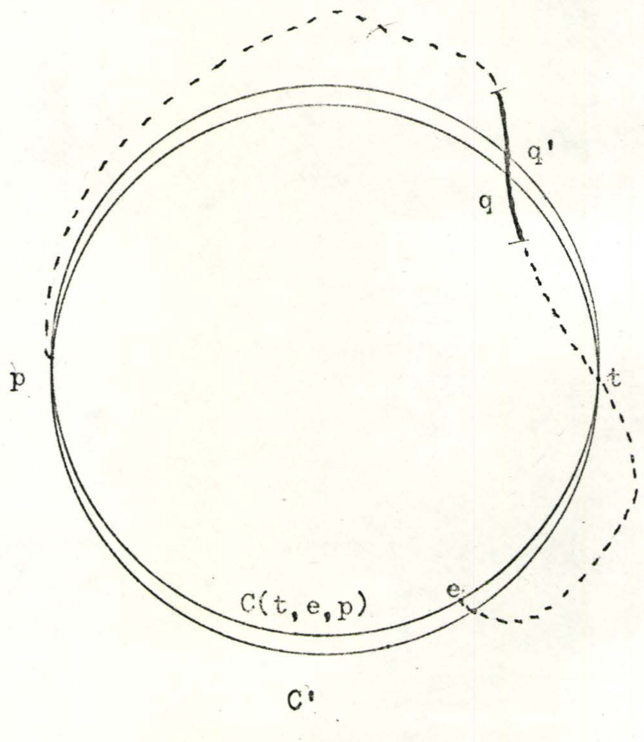

FIGURE 23

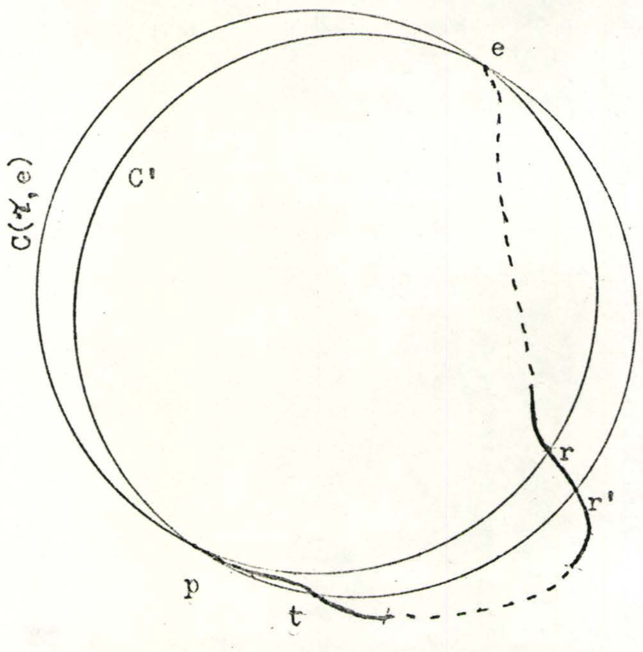

FIGURE 24

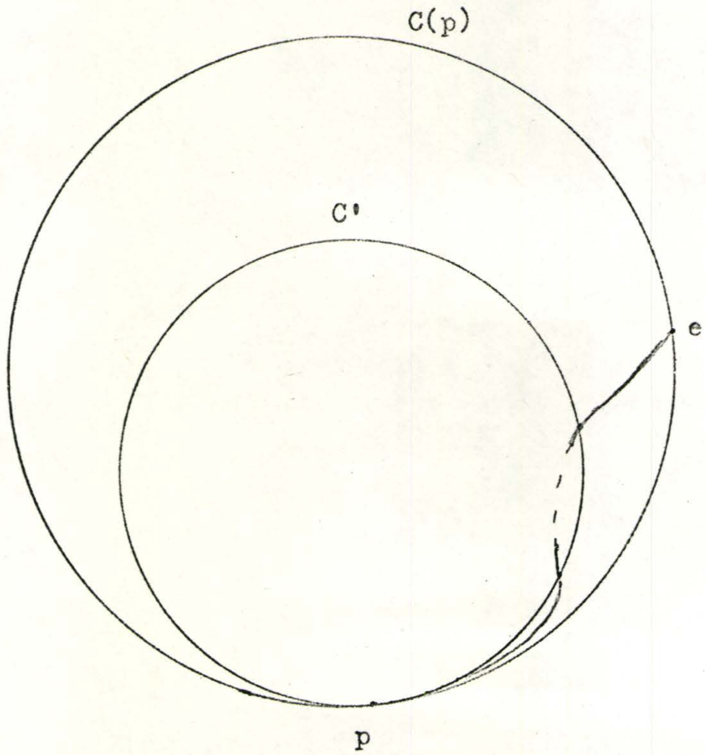

FIGURE 25

Remark. There exists an open arc A_3 of cyclic order three whose closure $\bar{A}_3$ is not of cyclic order three. We shall construct an open arc A_3 of cyclic order three with the end-points p and e of A_3 such that $C(\tau,e) = C(\tau_e,p)$. Then $\bar{A}_3 = p \cup A_3 \cup e$ will not be of cyclic order three.

Proof. Let p(-1,0) and e(1,0) be the end-points of an open arc A, passing through the third and first quadrant, of the lemniscate given by

(3.2-1.) $$(x^2+y^2)^2 - x^2 + y^2 = 0 \text{ , (Fig. 26).}$$

Claim: The arc A is of cyclic order three.

(i) Tangent circles at p or e meet the arc A at most once.

Proof. Different tangent circles at e are given by

$$(x-h)^2 + y^2 = (1-h)^2, \text{ for different values of h.}$$

Or

(3.2-2.) $$x^2 + y^2 - 2hx + 2h - 1 = 0.$$

Solving (3.2-1.) and (3.2-2.) for x we get

$$(2hx-2h+1)^2 - 2x^2 + 2hx - 2h + 1 = 0.$$

This equation being of second degree in x has at most two distinct roots. Hence the circles given by (3.2-2.) meet the lemniscate at most two times. Because of symmetry, the circles (3.2-3) meet the arc A at most once. The symmetric result holds for p.

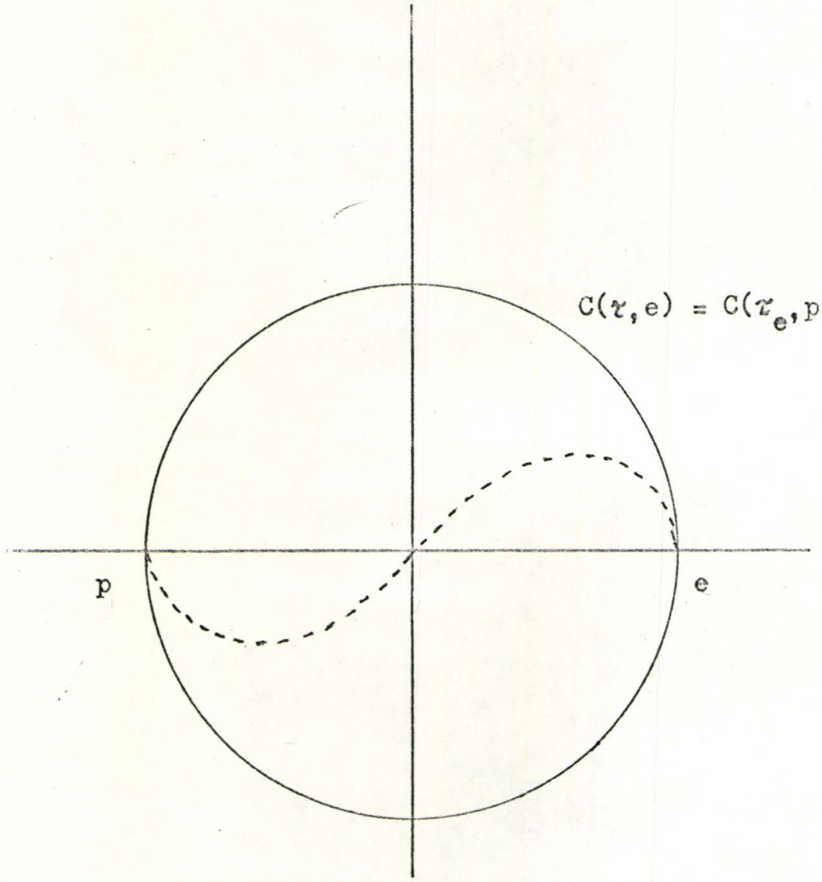

FIGURE 26

(ii) Circles through the end-points p and e meet the arc A at most once outside p and e.

Proof. Circles passing through p and e are given by

$$x^2 + y^2 + 2fy - 1 = 0. \tag{3.2-3.}$$

Solving (3.2-1.) and (3.2-3.) for y we have,

$$(1-2fy)^2 + 2y^2 + 2fy - 1 = 0.$$

This being a second degree equation in y has at most two distinct roots. Hence the circles through p and e meet the lemniscate at most twice. By symmetry, only one of these points lie on the arc A. Thus the above circles (3.2-3.) meet A at most once outside p and e.

(iii) Circles through p or e meet the arc in at most two points.

Proof. Let r,s be two points interior to the arc A. Suppose C(e,r,s) meets the arc A in t. Let e and r be fixed and let s move from e to p. Clearly t cannot coincide with e, because then we have C(e,r,s) ϵ τ_e, contrary to (i). Also t does not coincide with p, as this would contradict (ii). Finally t cannot drop out as a point of support, because in that case the circle would meet the lemniscate five times and we know that any circle meets the lemniscate at most four times. Hence C(e,r,s) always meets A at another point t. Let $s \to p$. Then we have a circle through both the end-points of A which meets the arc in two interior points this contradicts (ii). By symmetry C(p, r,s) does not meet A again.

(iv) No circle through three interior points of A meets A elsewhere.

Proof. Consider $C(r,s,t)$, where r,s,t are three interior points of A. Suppose $C(r,s,t)$ meets A in a fourth point say x. Let r,s be fixed and let t move on A. Then x cannot coincide with either of the end-points, as this would contradict (iii). Also x cannot drop out as a point of support, otherwise, $C(r,s,t)$ would meet A five times. Thus $x \in C(r,s,t)$, has to remain an interior point of A. Now let $t \to p$. Then we obtain a contradiction to (iii).

Thus A is of cyclic order three. But the circle $x^2 + y^2 = 1$ is a tangent circle of A at both the end-points. Thus

$$C(\tau,e) = C(\tau_e,p).$$

Hence $\bar{A} = p \cup A \cup e$ is not of cyclic order three.

3.3. Strong differentiability of arcs of cyclic order three.

3.3.1. Theorem. Let A_3 be an open arc of cyclic order three. Then every point of A_3 satisfies Condition I'; cf. 2.4.

Proof. Let $q,r \in A_3$ and $q \neq r$. Choose two disjoint one-sided neighbourhoods N_1 and N_2 of q such that $r \notin M = N_1 \cup q \cup N_2$. Let C_1 and C_2 meet A_3 at least twice at q and altogether at least three times. Hence C_i $(i=1,2)$ meets A_3 exactly twice at q, once at r and nowhere else (cf. Lemma 3.2.9). In particular, C_i $(i=1,2)$ supports A_3 at q (cf. Lemma 3.3.1). Without loss of generality we may assume $N_1 \cup N_2 \subset C_{1*} \cap C_{2*}$.

Suppose $C_1 \neq C_2$. Then there is a third circle C_3 through q and r which does not meet $C_{1*} \cap C_{2*}$. Thus C_3 will support A_3 at q. We may assume that $N_1 \cup N_2 \subset C_{3*}$ (Fig. 27). By Theorem 3.1, the arcs $N_1 \cup q$ and $N_2 \cup q$ satisfy Condition I at q. Thus they possess two well-defined tangent circles at q through r. At least one of the circles C_1, C_2, C_3 say the circle C_1 is different from them. Let π be the pencil of the second kind of the circles which touch C at q.

Let $s \in N_1 \cup N_2$. Thus $s \in C_*$ and hence $C(\pi, s) \subset C_* \cup q$. also if s approaches q through N_1 or N_2,

$$\lim_{s \to q} C(\pi, s) = q;$$

cf. Theorem 2.3.2.

Since $C(\pi, s)$ depends continuously on s, there are circles in π which are arbitrarily small and meet both N_1 and N_2 near q. Thus they meet M not less than three times. On the other hand, the end-points of M will lie on the same side of such a small circle. Hence it will meet M with an even multiplicity and therefore not less than four times. This being impossible we obtain $C_1 = C_2$. Thus the general tangent circle at q through r is unique.

Let C' and C'' be the two one-sided tangent circles of A_3 at through a point R A_3. Since

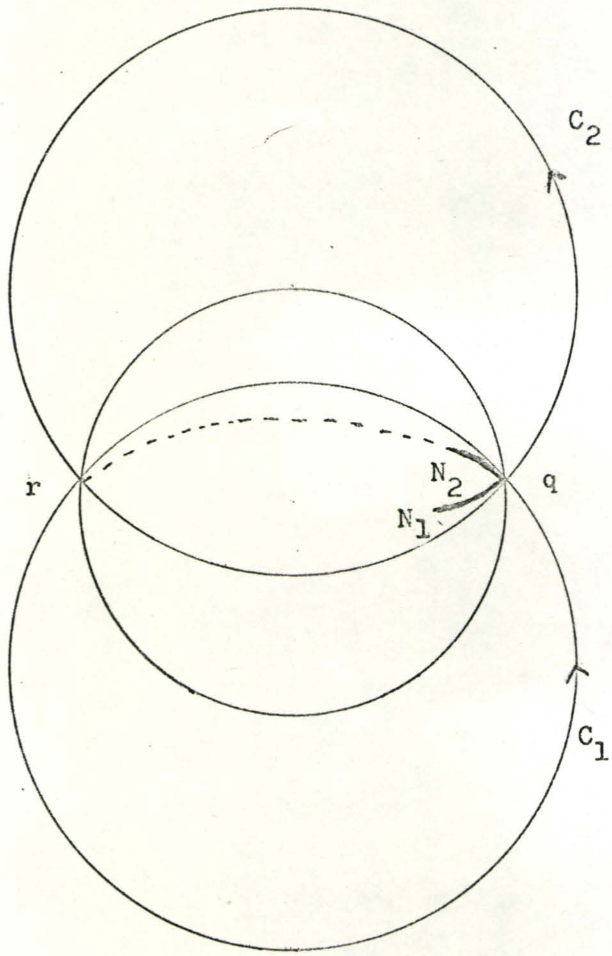

FIGURE 27

$$\sphericalangle\left[C', C_1\right] = 0$$

and

$$\sphericalangle\left[C'', C_2\right] = 0,$$

it is true that

$$\sphericalangle\left[C', C''\right] = 0.$$

Since C' and C'' have the point $R \neq q$ in common they coincide; cf. Theorem 2.3.1. Hence the tangent circle of A_3 at q through R is determined.

3.3.2. Theorem. Let p be an end-point of an open arc A_3 of cyclic order three. Then $A_3 \cup p$ is strongly differentiable at p.

Proof. Let $\bar{A}_3 = p \cup A_3 \cup e$, and p,q,r,s,u,e be mutually distinct points on $\bar{A}_3$ in the indicated order. We may assume that $e \in C(p)^*$. Thus

$$A_3 \subset C(p)^* \cap C(\tau,e)_*.$$

As q moves continuously and monotonically from p to v on A_3, $C(q,r,s)$ moves continuously and monotonically from $C(p,r,s)$ to $C(r,s,v)$. We orient $C(q,r,s)$ continuously. Thus

$$(3.3\text{-}1.)\qquad C(q,r,s) \subset \left[C(p,r,s)_* \cap C(r,s,v)^*\right] \cup \left[C(p,r,s)^* \quad C(r,s,v)_*\right] \cup r \cup s$$

and

$$(3.3\text{-}2.)\quad C(q,r,s)_* \supset C(p,r,s)_* \cap C(r,s,v)_*$$

Since p is an end-point of A_3, $A_3 \cup p$ is conformally differentiable at p. It suffices to show that p satisfies Condition II', in particular

$$\lim_{q,r,s \to p} C(q,r,s) = C(p);$$

cf. Lemma 2.4.2.

Letting $r \to p$ in the relation (3.3-1.) we have

$$C(q,r,s) \subset \left[C(\tau,s)_* \cap C(p,s,v)^*\right] \cup \left[C(\tau,s)^* \cap C(p,s,v)_*\right] \cup p \cup s.$$

Let D_1 denote an accumulation circle of the circle C(q,p,s). By choosing a suitable subsequence of the sequence q,r,s we may assume that

$$\lim_{q,s \to p} C(p,q,s) = D_1.$$

Hence

$$D_1 \subset \left[C(p)_* \cap C(\tau,v)^*\right] \cup \left[C(p)^* \cap C(\tau,v)_*\right] \cup C(p) \cup C(\tau,v).$$

This holds for every choice of v on $A_3 \cup e$ while D_1 is independent of v. Letting $v \to p$ we obtain

$$D_1 = C(p).$$

Thus

$$\lim_{q,s \to p} C(p,q,s) = C(p).$$

Hence

$$\lim_{q,s \to p} C(p,r,s) = C(p).$$

As in the relation (3.3-1.)

$$C(q,r,v) \subset \left[C(p,q,v)_* \cap C(q,s,v)^*\right] \cup \left[C(p,q,v)^* \cap C(q,s,r)_*\right] \cup q \cup v.$$

Let D_2 be an accumulation circle of the $C(q,r,v)$ as q and r tend to p. Hence

$$D_2 \subset \left[C(\tau,v)_* \cap C(p,s,v)^*\right] \cup \left[C(\tau,v)^* \cup C(p,s,v)_*\right] \cup C(\tau,v) \cup C(p,s,v).$$

This holds for every choice of s on A_3 while D_2 is independent of s. Letting $s \to p$ we have $D_2 \subset C(\tau,v)$. Since D passes through p and v we have

$$D_2 = C(\tau,v).$$

Hence also

$$\lim_{r,s \to p} C(r,s,v) = C(\tau,v).$$

Let D be an accumulation circle of the circles $C(q,r,s)$. By choosing the suitable subsequence of the sequences q,r,s, we may assume

$$\lim_{q,r,s \to p} C(q,r,s) = D.$$

By letting r and s tend to p in the relation (3.3-1) and recalling that

$$\lim_{s \to p} C(\tau,s) = \lim_{r,s \to p} C(p,r,s) = C(p),$$

and

$$\lim_{s \to p} C(p,s,v) = \lim_{r,s \to p} C(r,s,v) = C(\tau,v),$$

we obtain

$$D \subset \left[C(p)_* \cap C(\tau,v)^*\right] \cup \left[C(p)^* \cap C(\tau,v)_*\right] \cup C(p) \cup C(\tau,v).$$

But

$$C(p)_* \subset C(\tau,v)_*,$$

therefore

$$C(p)_* \cap C(\tau,v)^* = \emptyset.$$

Hence D lies in the closure of $\left[C(p)^* \cap C(\tau,v)_*\right]$. In particular, D lies in the closure of $C(\tau,v)_*$. Since $p \in D$ we have $D \in \tau$.

From the relation (3.3-2.)

$$D_* \supset C(p)_* \cap C(\tau,v)_*.$$

Let $v \to p$.

If $C(p) \neq p$, then $D \neq p$ and $D = C(p)$.

If $C(p) = p$, then $D = p$.

This implies that $D = C(p)$, whether $C(p) = p$ or not.

<u>3.3.3.</u> Let τ_e denote the pencil of tangent circles of A_3 at e.

<u>Lemma.</u> <u>Let</u> p <u>and</u> e <u>be the end-points of an open arc</u> A_3; <u>thus</u> $\bar{A}_3 = p \cup A_3 \cup e$. <u>We may assume that</u> $e \in C(p)^*$. <u>Then</u>

$$C(q,r,s)_* \supset C(p)_* \cap C(p,\tau_e)_*,$$

<u>where</u> $q,r,s \in A_3$ <u>in the indicated order.</u>

<u>Proof.</u> By our assumptions, we have

$$A_3 \subset C(p)^* \cap C(\tau,e)_* \cap C(p,\tau_e)^* \cap C(e)_*.$$

If x moves continuously and monotonically on A_3 from p to e, the circle $C(x,r,s)$ moves continuously and monotonically from $C(p,r,s)$ to $C(r,s,e)$. Hence for any choice of x on A_3 we have

$$C(x,r,s)_* \supset C(p,r,s)_* \cap C(r,s,e)_*. \tag{3.3-3.}$$

Hence putting $x = q$, we have

(3.3-4.) $$C(q,r,s)_* \supset C(p,r,s)_* \cap C(r,s,e)_*$$

Let $r \rightarrow p$ and then put $x = r$ in (3.3-3.). This yields

(3.3-5.) $$C(r,p,s)_* \supset C(\tau,s)_* \cap C(p,s,R)_*$$

Sinilarly let $r \rightarrow e$ and replace x by r. Then

(3.3-6.) $$C(r,e,s)_* \supset C(p,e,s)_* \cap C(s,\tau_e)_*$$

using (3.3-5.) and (3.3-6.) we have from (3.3-4.)

$$C(q,r,s)_* \supset \left[C(\tau,s)_* \cap C(p,s,e)_*\right] \cup \left[C(p,s,e)_* \cap C(s,\tau_e)_*\right],$$

or

(3.3-7.) $$C(q,r,s)_* \supset C(\tau,s)_* \cap C(p,s,e)_* \cap C(s,\tau_e)_*.$$

Let r and s both tend to p in (3.3-3.) and then take $x = s$. Then

(3.3-8.) $$C(\tau,s)_* \supset C(p)_* \cap C(\tau,e)_*.$$

Let r p, $s \rightarrow e$ in (3.3-3.) and then put $x = s$. We obtain

(3.3-9.) $$C(s,p,e)_* \supset C(\tau,e)_* \cap C(p,\tau_e)_*$$

Let $r,s \rightarrow e$ in (3.3-9.) and put $x = s$. Then we have

(3.3-10.) $$C(s,\tau_e)_* \supset C(p,\tau_e)_* \cap C(e)_*$$

Using (3.3-8.), (3.3-9.) and (3.3-10.) in (3.3-7.) we get

$$C(q,r,s)_* \supset \left[C(p)_* \cap C(\tau,e)_*\right] \cap \left[C(\tau,e)_* \cap C(p,\tau_e)_*\right] \cap \left[C(p,\tau_e)_* \cap C(e)_*\right],$$

or

(3.3-11.) $$C(q,r,s)_* \supset C(p)_* \cap C(\tau,e)_* \cap C(p,\tau_e)_* \cap C(e)_*.$$

Since $C(p)_* \subset C(\tau,e)_*$ and $C(p,\tau_e)_* \subset C(e)_*$, relation (3.3-11.) reduces to

$$C(q,r,s)_* \supset C(p)_* \cap C(p,\tau_e)_*.$$

<u>3.4.</u> Let $q \in A_3$. Let $A_3 = B_3 \cup q \cup B_3'$ such that if p and e are the end-points of A_3, then B_3 is bounded by p and q and B_3' by q and e. Let C denote a general osculating circle of A_3 at q and $C(q)$ and $C'(q)$ the unique osculating circles of B_3 and B_3' at q respectively.

<u>3.4.1.</u> <u>Lemma.</u> <u>If</u> $C(q)_* \subset C_*$ <u>then</u> $B_3 \subset C(q)_*$ <u>and symmetrically if</u> $C(q)^* \subset C^*$ <u>then</u> $B_3 \subset C(q)^*$ (Fig. 28).

<u>Proof.</u> Since both C and $C(q)$ are general osculating cirlces of A_3 by Lemma 3.2.10,

$$B_3 \cap C = B_3 \cap C(q) = q.$$

Also by Lemma 3.2.2, C and $C(q)$ both intersect A_3 at q. The general tangent circles of A_3 at q form a pencil τ_q (cf. Lemma 2.3.1 and 2.4.1); thus $C \in \tau_q$, $C(q) \in \tau_q$, where τ_q denotes the family of tangent circles of A_3 at q.

Suppose that $B_3 \subset C(q)^*$. Then

(3.4-1.) $$B_3 \subset C(q)^* \cap C_*;$$

otherwise, $C(\tau_q,s)$ could not converge to $C(q)$ as s tends to q on B_3.

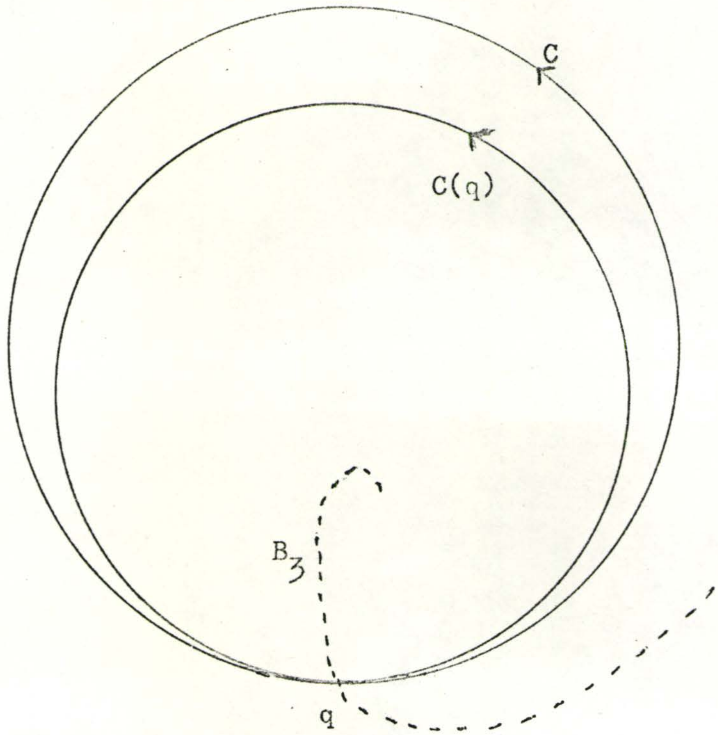

FIGURE 28

Now (3.4-1.) implies that $C(q)$ and C cannot both intersect A_3 at q. Thus $B_3 \subset C(p)_*$.

The following theorem is a consequence of the above lemma.

3.4.2. Theorem. *If q is an interior point of A_3, then any general osculating circle of A_3 at q lies between the two one-sided osculating circles of A_3 at q in the pencil τ_q.*

Proof. Let $C(q)_* \subset C_*$. Then $B_3 \subset C(q)_* \subset C_*$. Since A_3 intersects both $C(q)$ and C at q, we obtain

$$B_3' \subset C^* \subset C(q)^*.$$

By Lemma 3.4-1 applied to B_3', if $C(q)_* \subset C_*$, then

$$B_3' \subset C'(q)_* \subset C_*.$$

where $C'(q)$ is the osculating circle of B_3' at q, (Fig. 29). This is a contradiction. Hence $C'(q)^* \subset C^*$. We note that then

$$B_3' \subset C'(q)^*.$$

Thus

$$B_3 \subset C(q)_* \text{ and } B_3' \subset C'(q)^*$$

and we obtain $C(q)_* \subset C_* \subset C'(q)_*$.

3.4.3. Theorem. *If A_3 satisfies Condition II at an interior point q, then A_3 satisfies Condition II' at q.* Thus if A_3 is differentiable at an interior, then it is strongly differentiable there.

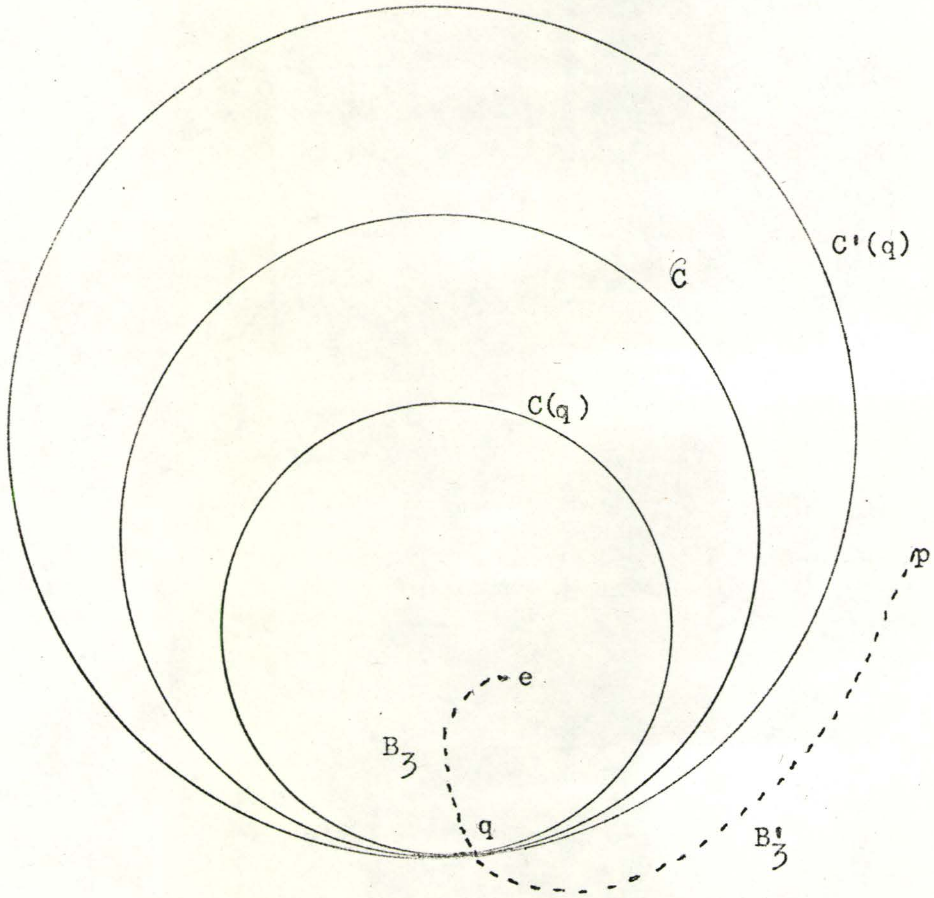

FIGURE 29

<u>Proof</u>. Let $q \in A_3$ and let A_3 satisfy Condition II at q. Then from Theorem 3.4.2 any general osculating circle of A_3 at q satisfies

$$C(q)_* \subset C_* \subset C'(q)_*.$$

But $C(q) = C'(q)$. Hence $C = C(q) = C'(q)$. Thus there is a unique general osculating circle of A_3 at q. Therefore Condition II' holds at q.

<u>3.4.4.</u> <u>Theorem</u>. <u>Two general osculating circles at distinct points of A_3 have no points in common.</u>

<u>Proof</u>. Let $q, r \in A_3$ and B_3 be a subarc of A_3 such that q and r are end-points of B_3. Thus B_3 has uniquely defined osculating circles $C(q)$ and $C(r)$ at q and r, respectively. We may assume that $C(q) \neq q$ and $C(r) \neq r$. Let τ_q and τ_r denote the families of tangent circles at q and r, respectively, cf. Theorem 3.3.1. Let s,t,u be mutually distinct points on A_3 in the indicated order. Let $B_3 \subset C(q)_*$. Thus

$$(3.4\text{-}1.) \qquad \begin{gathered} C(\tau_q, r)_* \subset C(q)_* \text{ and } C(\tau_r, q)^* \subset C(r)^*. \\ B_3 \subset C(q)_* \cap C(\tau_q, r)^* \cap C(\tau_r, q)_* \cap C(r)^*. \end{gathered}$$

Since $C(\tau_q, r) \neq C(\tau_r, q)$, $C(\tau_r, q)$ intersects $C(\tau_q, r)$ at q and r. Hence $C(\tau_r, q)$ also intersects $C(q)$ at q and at another point. Similarly, $C(\tau_q, r)$ intersects $C(r)$ at r and one other point, R, say. The points r and R decompose $C(r)$ into two arcs C' and C", such that

$$C' \subset C(q,\tau_r)_* \cap C(\tau_q,r)_*$$

while

$$C'' \subset C(q,\tau_r)_* \cap C(\tau_q,r)^* \text{(Fig. 30.)}$$

Since $C(\tau_q,r)_* \subset C(q)_*$, we obtain

$$C' \subset C(q)_*.$$

Suppose C'' meets $C(q)$; thus C'' meets $C(q) \cap C(q,\tau_r)_*$. Then C'' decomposes the region

$$C(q)_* \cap C(q,\tau_r)_* \cap C(\tau_q,r)^*,$$

into three disjoint regions. Two of these lie in the set

$$C(q,\tau_r)_* \cap C(r)^* \cap C(q)_*, = S \text{ say},$$

and their boundaries have at most a single point in common which lies on $C(q)$. The region of S whose boundary includes an arc of $C(\tau_q,r)$, contains points of B_3 close to q, and the region of S whose boundary includes an arc of $C(\tau_r,q)$ contains points of B_3 close to r. But then the continuity of B_3 and the relation (3.4-1.) imply that these two regions are connected. Hence $C'' \subset C(q)_*$, and the whole of $C(r) = C' \cup C'' \cup \{r,R\}$ lies in $C(q)_*$.

Thus $C(q)$ and $C(r)$ do not meet.

Remark. The following alternative method of proving that $C'' \subset C(q)_*$ is shorter and direct, but it requires the full Jordan curve theorem.

As above $C'' \subset C(q,\tau_r)_* \cap C(\tau_q,r)^*$.

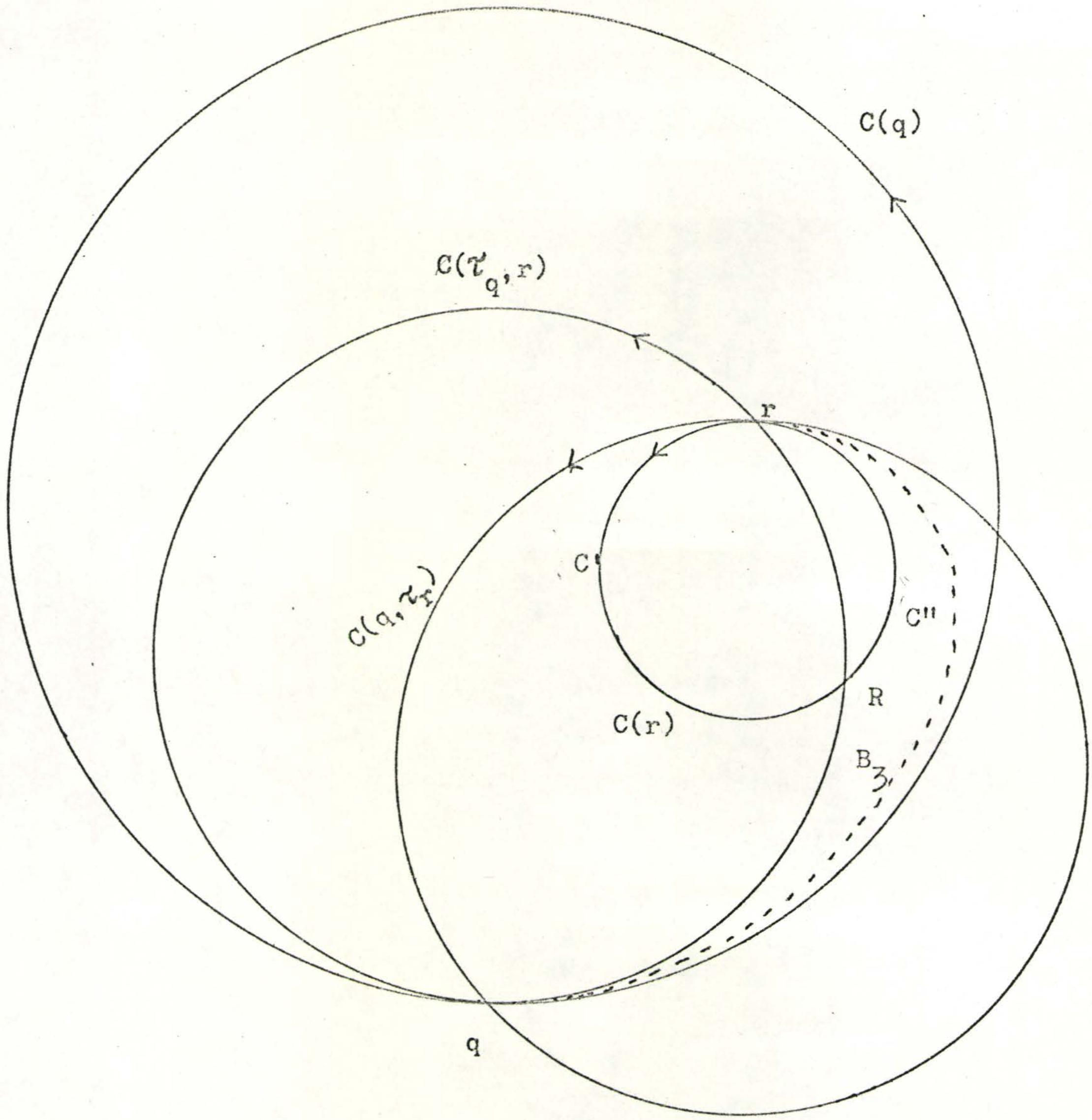

FIGURE 30

Since $C(r)$ does not meet A_3, C'' even lies in the region in $C(q,\tau_r)_*$ bounded by A_3 and $C(\tau_q,r)$. Hence $C'' \subset C(q)_*$.

3.4.5. Theorem. All but a countable number of points of A_3 are strongly conformally differentiable.

Proof. Let p and e be the end-points of A_3. If $C(p) = p$ and $C(e) = e$, (cf. Remark at the end of this subsection) we can decompose A_3 into two subarcs B_3 and B_3' such that $A_3 = B_3 \cup q \cup B_3'$, and consider B_3 and B_3' separatly. Thus there is no loss of generality if we assume that $C(p) \neq p$ and $A_3 \subset C(p)_*$. By taking the point at infinity in $C(p)^*$, we can introduce a local coordinate system keeping $A_3 \subset C(p)_*$ and use the standard metric function d on R^2; thus if $a = (a_1,a_2)$ and $b = (b_1,b_2)$ are points in R^2, then $d(a,b) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$. By choosing this coordinate system suitably, we may even assume that $C(p)$ is a circle of area 1. In fact, a suitable translation will move the origin to the centre of $C(p)$ and if $C(p)$ has radius r, then $x' = {}^x/r\sqrt{\pi}$, $y' = {}^y/r\sqrt{\pi}$ will transform the equation of $C(p)$ into the form $x^2 + y^2 = {}^1/\pi$.

Let $s \in A_3$ be a point at which A_3 is not strongly conformally differentiable; then A_3 does not satisfy Condition II at s (cf. Theorem 3.4.3.). Let $C(s)$ and $C'(s)$ be the one-sided osculating circles of A_3 at s. We may assume that $C(s)_* \subset C'(s)_*$. Let $f(s)$ be the area between $C(s)$ and $C'(s)$ (Fig. 31). By Theorem 3.4.4, the regions $C(s)^* \cap C'(s)_*$ and $C(t)^* \cap C'(t)_*$ are disjoint if $s \neq t$, and they lie

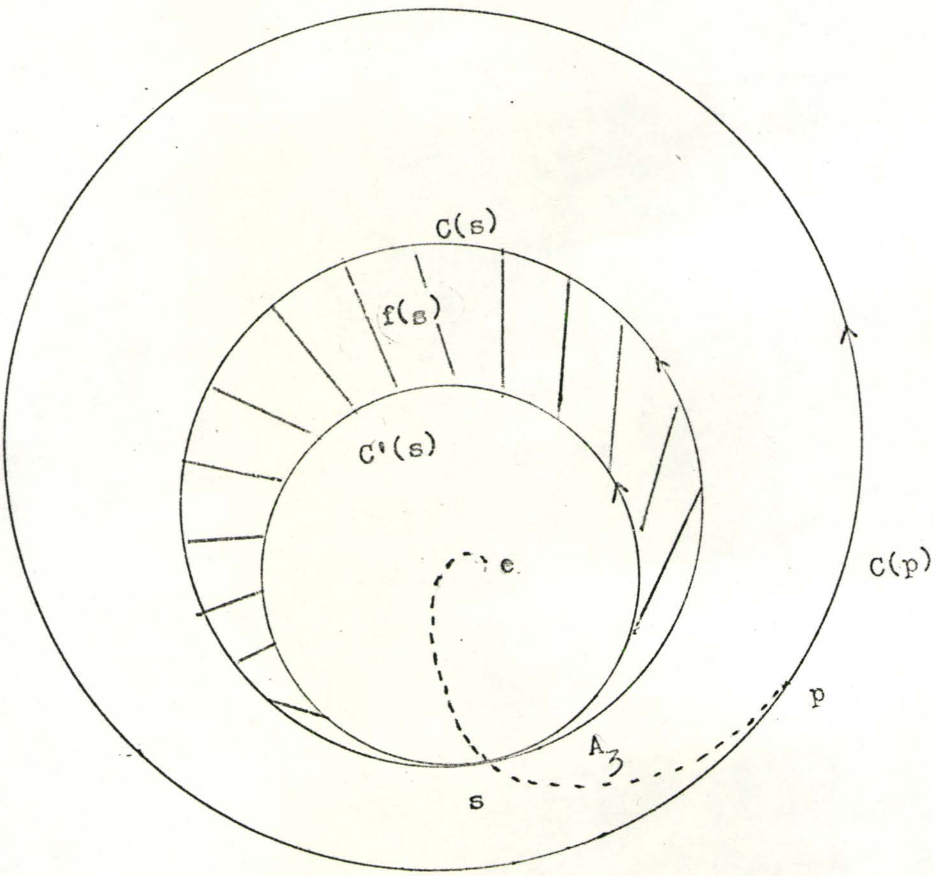

FIGURE 31

in $C(p)_*$. Since the area of $C(p)$ is 1,

(i) the class of points s for which

$$1 > f(s) \geqslant \frac{1}{2},$$

has at most 2 members.

(ii) The class of points s for which,

$$\frac{1}{2} > f(s) \geqslant \frac{1}{4},$$

has at most four i.e., 2^2 members.

(iii) The class of points s for which,

$$\frac{1}{4} > f(s) \geqslant \frac{1}{8}$$

has at most eight i.e., 2^3 members.

Thus in general, the class of points for which

$$\frac{1}{2^{n-1}} > f(s) \geqslant \frac{1}{2^n} \qquad (n=1,2,3\ldots)$$

does not have more than 2^n members.

Since every point $s \in A_3$ with $f(s) > 0$ is included in exactly one of these classes, there is only a countable set of points s with $f(s) > 0$.

<u>Remark.</u> <u>There are arcs of cyclic order three which have point-osculating circle at both the end-points.</u>

For example consider the open arc A given by

$$x = t^2, \; y = t^3,$$

for $0 < t < \infty$, with $(0,0)$ and ∞ as the end-points p and e respectively, (Fig. 32).

First we show that the arc A is of cyclic order three. Let

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{3.4-2.}$$

be any circle. Then the points of A which are common with a circle (3.4-2.) are the roots of the equation

$$t^6 + t^4 + 2ft^3 + 2gt^2 + c = 0. \tag{3.4-3.}$$

Now there can be at most three variations in the signs of the coefficients in the equation (3.4-3.). Hence by Descarte's rule it can have at most three real positive roots. Thus any circle meets the arc at most three times. Hence A is of cyclic order three.

The tangent circle of A at p through a point (s^2, s^3) is given by

$$x^2 + y^2 - (s - s^3)y = 0.$$

As $s \to 0$ this circle tends to the point-circle

$$x^2 + y^2 = 0.$$

Thus $C(p) = p$.

The circle $C(t,u,e)$, $t,u \in A$, which is a straight line, is given by

$$y - u^3 = \frac{t^3 - u^3}{t^2 - u^2}(x - u^2).$$

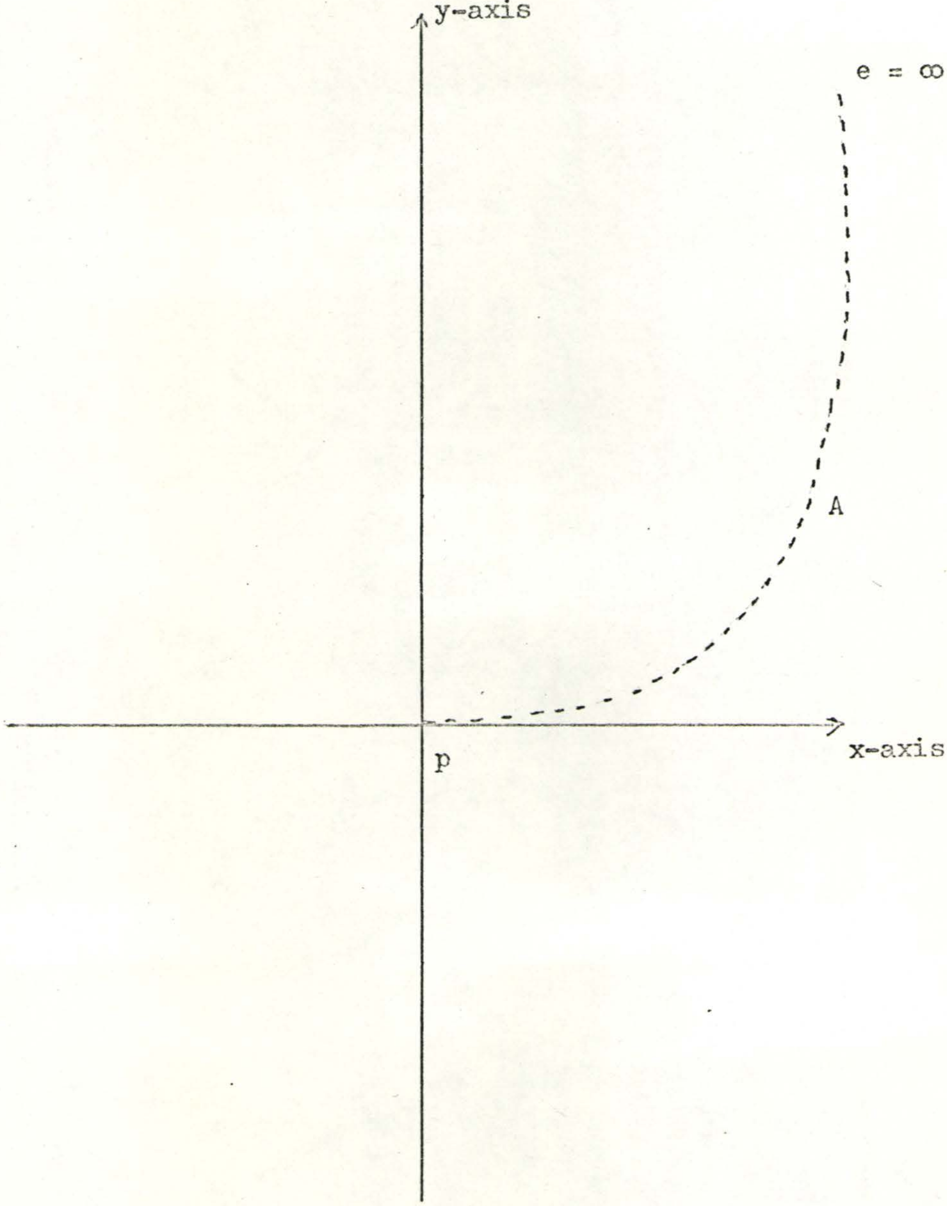

FIGURE 32

As $u \to e$ (i.e., as the parameter $u \to \infty$) this becomes a st. line through the point t and parallel to the y-axis,

$$x = t^2. \tag{3.4-5}$$

Thus the circle $C(\tau_e, t)$ is given by (3.4-5). Hence $\lim_{t \to e} C(\tau_e, t) = C(e) = \infty$. Thus $C(e)$ is also a point-circle.

CHAPTER IV

UNION AND EXTENSION OF ARCS OF CYCLIC ORDER THREE.

4.1. Union of Arcs of cyclic order three.

Let A_3 and A_3' be open arcs of cyclic order three with a common end-point p, and let e, e' be the other end-points respectively. Put

$$\bar{A}_3' = e' \cup A_3' \cup p, \quad \bar{A}_3 = e \cup A_3 \cup p$$

and

$$A = A_3 \cup p \cup A_3'; \ \bar{A} = e' \cup A \cup e.$$

Let τ, τ_e denote the pencil of tangent circles of A_3 at p and e and $\tau', \tau_{e'}$ of A_3' at p and e'. Assume that $\bar{A}_3$ and $\bar{A}_3'$ are also of cyclic order three.

Thus

$$C(p,\tau_e) \neq C(\tau,e) \text{ and } C(p,\tau_{e'}) \neq C(\tau',e').$$

Let C(p) and C'(p) denote the osculating circles of A_3 and A_3' respectively at p. We may assume that $e \in C(p)^*$.

If A has cyclic order three, the following conditions will hold.

(i) A satisfies Condition I at p (cf. Theorem 3.3.1.). Thus the two pencils τ and τ' coincide. We denote this common pencil by τ.

(ii) $C(\tau,e')_* \subset C'(p)_* \subset C(p)_* \subset C(\tau,e)_*$.

Thus

$$A_3' \subset C'(p)_* \subset C(p)_* \text{ and } A_3 \subset C(p)^* \subset C'(p)^* \quad \text{(Fig. 33).}$$

(iii) $A_3 \left[A_3'\right]$ <u>does not meet</u> $C(p,\tau_{e'}) \left[C(p,\tau_e)\right]$ (Fig. 34).

(iv) $A_3 \cup p \left[A_3' \cup p\right]$ does not meet $C(\tau_{e'},e) \left[C(e',\tau_e)\right]$, (Fig. 35).

Our goal is to show that Conditions (i) - (iv) are not only necessary but are also sufficient for A to have cyclic order three.

We observe that $\bar{A}$ will also have cyclic order three if we add the condition

$$C(\tau_e,e') \neq C(\tau_{e'},e).$$

<u>4.2.</u> <u>Remark.</u> It is clear that Condition (ii) implies Condition (i). However Conditions (ii), (iii) and (iv) are independent as we shall now prove.

<u>4.2-1.</u> <u>Conditions</u> (i), (ii) <u>and</u> (iii) <u>do not imply Condition</u> (iv).

<u>Proof.</u> Let e' (-1,0) and q(1,0) be the end-points of the open arc B of the lemniscate

$$(x^2+y^2) - x^2 + y^2 = 0,$$

consisting of the origin and the subarcs A_3' and B_3 which lie in the third and first qudrant respectively. Then B is of cyclic order three (cf. Remark on Theorem 3.2.2.). Clearly $\bar{A}_3'$ and $\bar{B}_3$ are of cyclic order three.

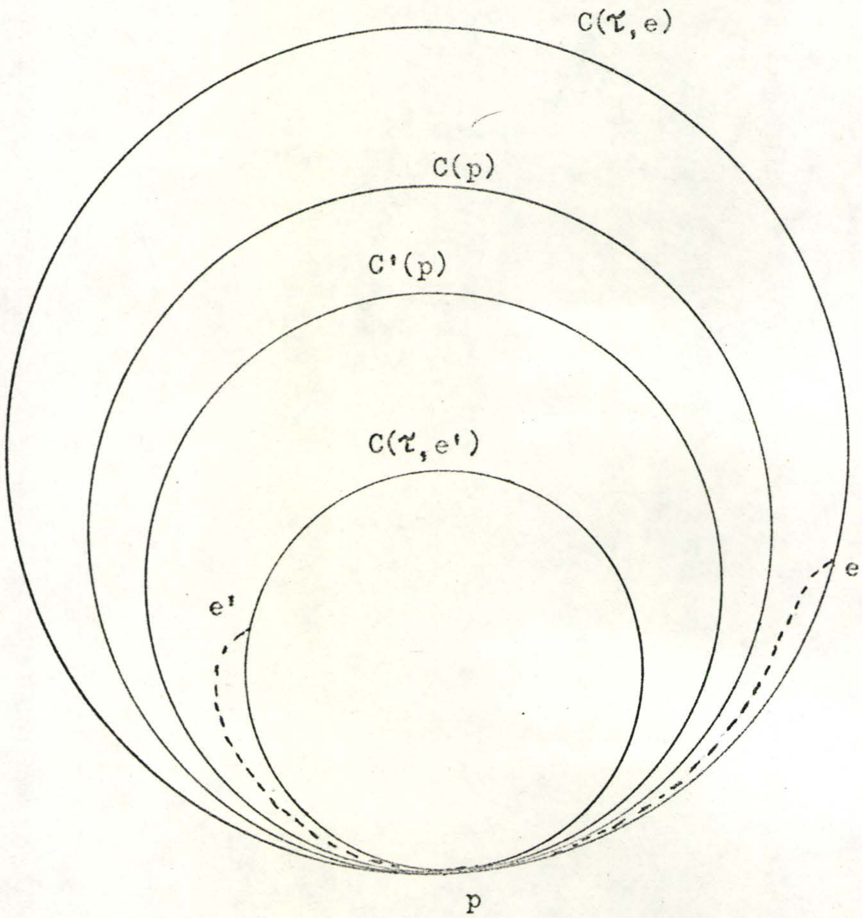

FIGURE 33

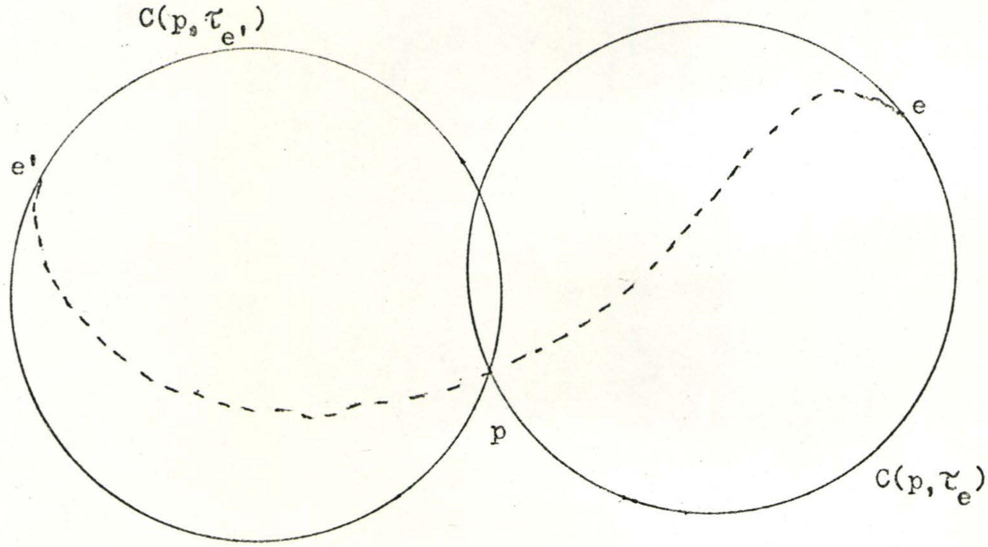

FIGURE 34

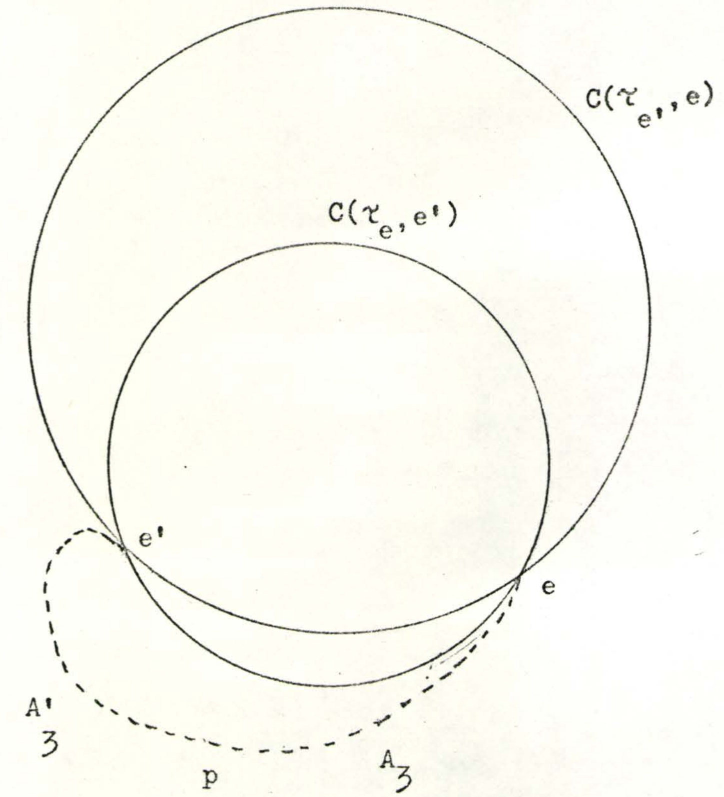

FIGURE 35

Now, we show that $C(q)$, the osculating circle of B_3' at q is not a point-circle. The radius of curvature R of the lemniscate in the polar coordinates is given by

$$(4.2\text{-}1.) \qquad R = \frac{(r^2+r'^2)^{3/2}}{r^2+2r'^2-rr''} .$$

At q, $R = \frac{1}{3}$. Since $R \neq 0$ at q, $C(q) \neq q$. Hence B_3 can be extended through q to a larger arc of cyclic order three (cf. Theorem 4.6-5.). Let B_3 be extended through q to e such that the closure $\bar{A}_3$ of the open arc A_3 with the end-points p and e is also of cyclic order three.

Let $A = A_3' \cup p \cup A_3$. By (4.2-1.) we see that the radius of curvature at p is infinite and hence $C(p)$ is a straight line. We may take the direction of $C(p)$ such that $e \in C(p)^*$.

Since A is conformally differentiable at p, Condition (i) automatically holds. Also $e' \in C(p)_*$ hence $C(p)$ separates $C(\tau,e)$ and $C(\tau,e')$, i.e., $C(\tau,e')_* \subset C(p)_* \subset C(\tau,e)_*$, where τ denotes the family of tangent circles of A at p. Thus Condition (ii) is satisfied.

Now $C(p,\tau_{e'})$ does not meet the right half-plane $x > 0$, hence does not meet A_3. Since $C(p,\tau_e)$ will lie in the union of the right half plane, the upper half plane, and $C(p)^*$, $C(p,\tau_e)$ does not meet A_3'. Hence Condition(iii) also holds.

But we see that Condition (iv) does not hold. Because $e \in C(q)^*$ (cf. Theorem 4.6.), $C(\tau_{e'},q) = C(\tau_q,e')$ we obtain

$$C(\tau_{e'},e)_* \subset C(\tau_{e'},q)_*$$

and hence $C(\tau_{e'},e)$ intersects A_3 (Fig. 36).

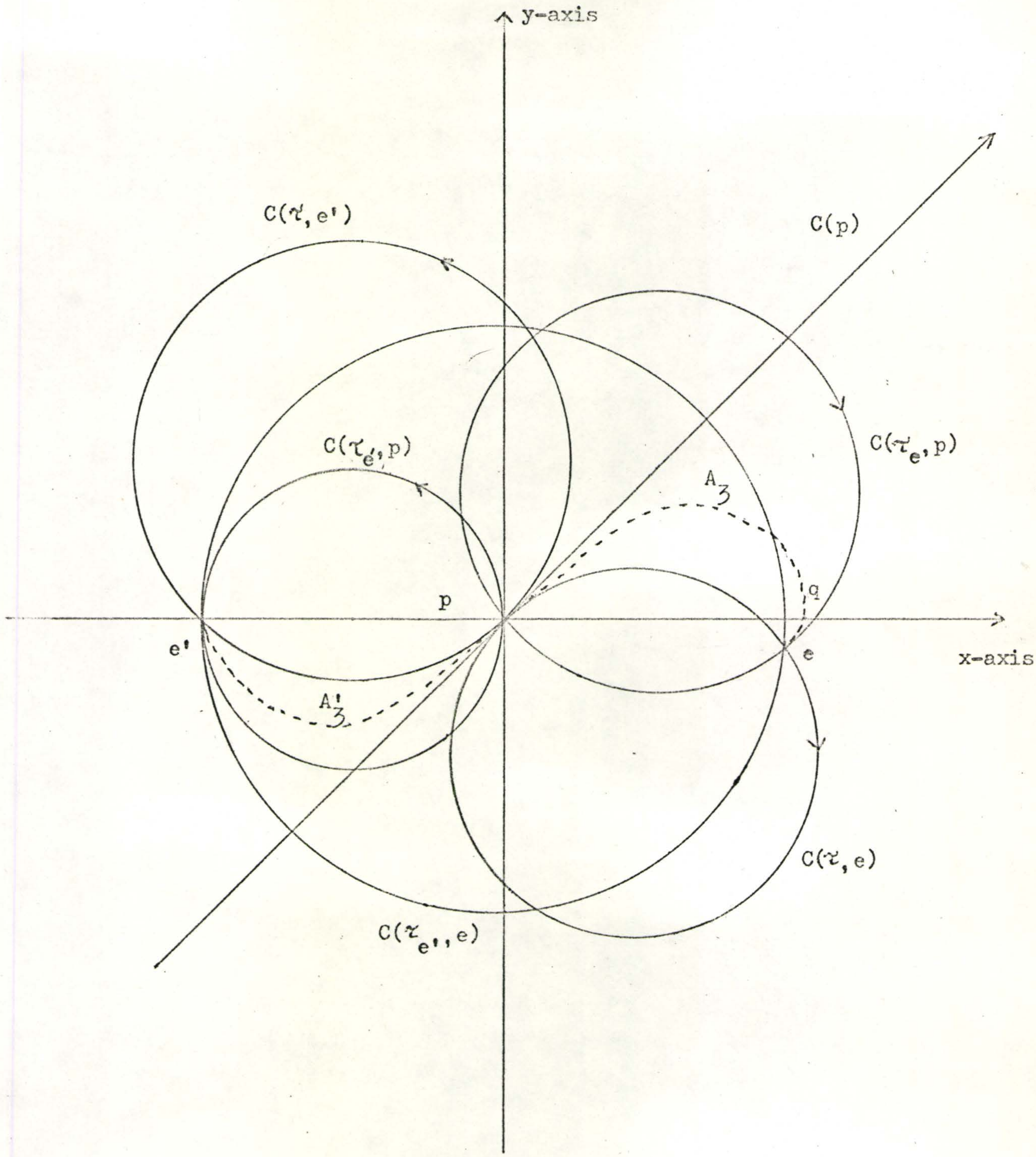

FIGURE 36

<u>4.2.2.</u> <u>If</u> A <u>has a cusp at</u> p, <u>then the Conditions</u> (i), (ii), <u>and</u> (iv) <u>do not imply Condition</u> (iii).

Consider A_3' to be the arc of the lemniscate

$$(x^2+y^2)^2 - 2xy = 0,$$

in the half-plane $x - y > 0$, with end-points $p = (0,0)$ and $e' = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Let A_3 be the arc of the lemniscate

$$(x^2+y^2)^2 + 2xy = 0$$

in the half-plane $x + y > 0$ with end-points $p = (0,0)$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

Put

$$A = A_3 \cup p \cup A_3'.$$

Then A has a cusp at p (Fig. 37).

Obviously C(p), the osculating circle of A at p is the straight line $x = 0$. Let $e \in C(p)^*$ and hence $e' \in C(p)_*$. Thus C(p) separates $C(\tau, e')$ and $C(\tau, e)$. Hence Condition (ii), and therefore Condition (i) holds.

Now the equation of $C(\tau_{e'}, e) = C(\tau_e, e')$ is

$$x^2 + y^2 = 1,$$

which clearly does not meet $A_3 \cup p$ or $A_3' \cup p$. Therefore Condition (iv) is satisfied.

But shall see that Condition (iii) does not hold. The circle $C(p, \tau_{e'})$ is given by the equation

$$x^2 + y^2 - \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y = 0.$$

Thus the centre $(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$ lies on the line $y = x$, and the radius is $\frac{1}{2}$. Also $C(p, \tau_{e'})$ being a non-tangent circle, supports A at the cusp point p. Since

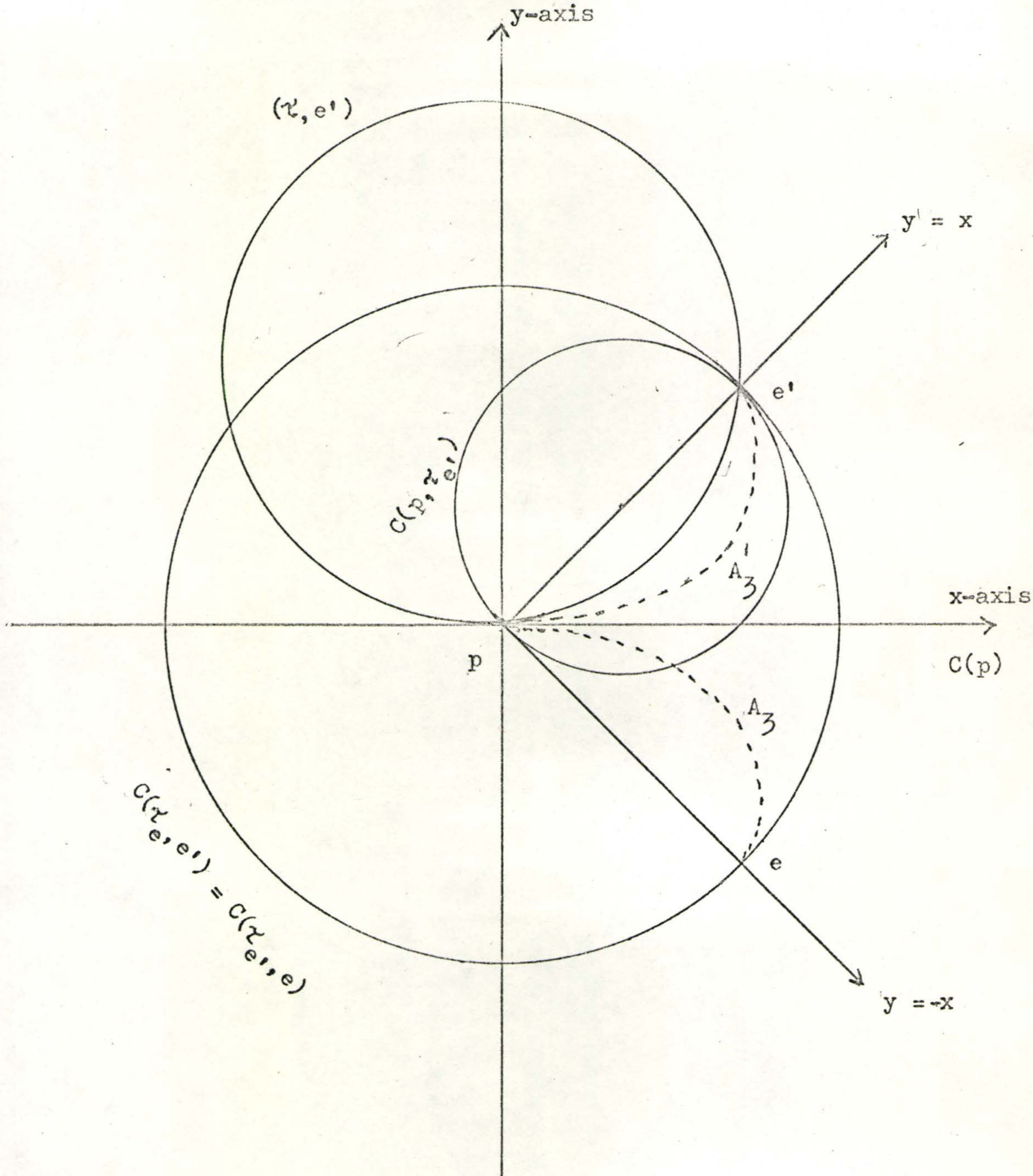

FIGURE 37

$$A_3 \subset C(p,\tau_e) \text{ while } e \in C(p,\tau_e)^*,$$

we obtain that $C(p,\tau_{e'})$ meets A_3 at some point. Thus Condition (iii) is violated.

4.2.3. Conditions (i), (iii) and (iv) do not imply Condition (ii).

Let A_3 be the arc of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \neq b, \tag{4.2-2.}$$

with end-points $e(0,b)$ and $p(a,0)$; and A_3' be the arc of the same ellipse with end-points $e'(0,b)$ and $p(a,0)$. Thus let

$$A = A_3 \cup p \cup A_3', \text{ (Fig. 38).}$$

First of all we show that $\bar{A}_3$ and $\bar{A}_3'$ are of cyclic order three. The circle $C(p,\tau_e)$ has the equation

$$x^2 + y^2 + \frac{a^2 - b^2}{b} y - a^2 = 0. \tag{4.2-3.}$$

Clearly this is not a tangent circle of A_3 at p, which shows that $\bar{A}_3$ is of cylcic order three.

The circle (4.2-3.) and the ellipse (4.2-2.) meet at two points $(\pm a,0)$ and touch at $(0,b)$. Hence they do not meet elsewhere. Thus $C(p,\tau_e)$ does not meet A_3'. Similarly $C(p,\tau_{e'})$ does not meet A_3. Thus Condition (iii) holds.

Since p is a conformally differentiable point of the ellipse (and hence of A), Condition (i) is satisfied.

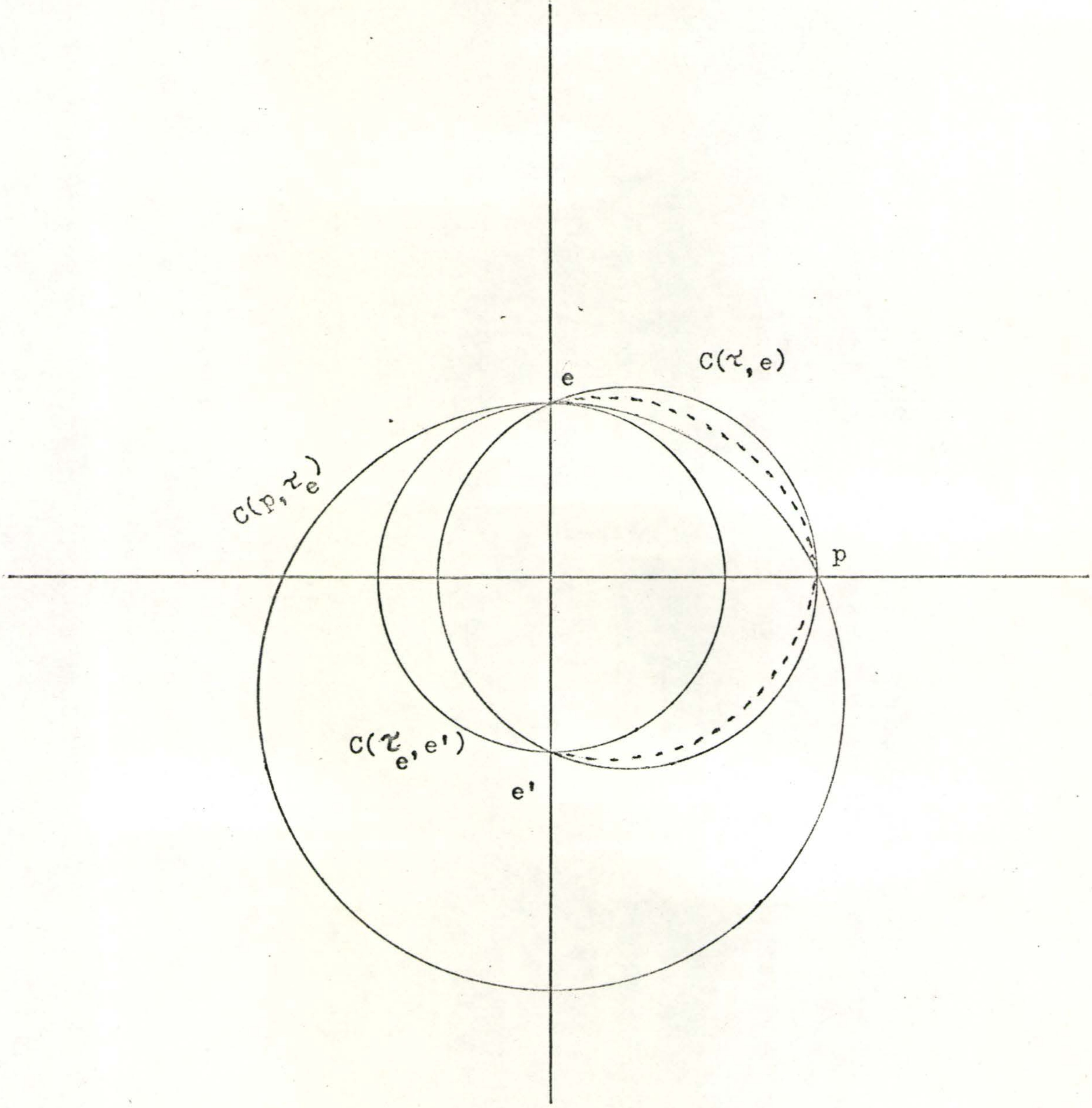

FIGURE 38

Finally, $C(\tau_e, e') = C(\tau_{e'}, e)$ is given by

$$x^2 + y^2 = b^2.$$

Now $C(\tau_e, e')$ does not meet A since $a \neq b$. Hence $C(\tau_e, e')$ does not meet A. Thus Condition (iv) holds.

From symmetry with respect to the x-axis

$$C(\tau, e)\check{} = C(\tau, e').$$

This implies that Condition (ii) does not hold.

4.3. Lemma. Assume Conditions (i), (ii) and (iv). Then Condition (iii) is equivalent to A having no cusp at p.

Proof. The following discussion is easiest to follow if we designate p as the point at infinity.

By (ii)

$$A_3 \subset C(p)^* \cap C(\tau, e)_* \text{ and } A_3' \subset C'(p)_* \cap C(\tau, e')^*.$$

Thus

$$A_3 \cup A_3' \subset C(\tau, e)_* \cap C(\tau, e')^* = R \text{ say.}$$

Since $C(\tau_e, p) \neq C(\tau, e)$ they intersect at e. Hence $C(\tau_e, e')$ also intersects $C(\tau, e)$ at e. Furthermore, since $C(\tau_e, e')$ does not meet $A_3' \cup p$ and since $\bar{A}_3'$ is of cyclic order three, $C(\tau_e, e')$ will intersect $C(\tau, e')$ at e'. Symmetrically $C(\tau_{e'}, e)$ intersects $C(\tau, e')$ at e' and $C(\tau, e)$ at e.

Orient $C(\tau_e,e')$ $[C(\tau_{e'},e)]$ such that $A_3' \subset C(\tau_e,e')^*$ $[A_3 \subset C(\tau_{e'},e)^*]$.

Thus

$$A_3' \subset C(\tau_e,e')^* \cap C(\tau,e')^* \cap C'(p)_*$$

and

$$A_3 \subset C(\tau_{e'},e)^* \cap C(\tau,e)_* \cap C(p)^*$$

Hence $A_3 \cup A_3'$ has no points in common with

$$C(\tau_e,e')_* \cap C(\tau_{e'},e)_* \cap R = R_o \text{ say.}$$

The boundary of R_o decomposes R into three disjoint regions of which R_o is one. Let R_1 and R_2 be the other two; thus $A_3 \cup A_3' \subset R_1 \cup R_2$.

Case I: A has a cusp at p.

Then A_3 and A_3' both lie in R_1 or both of them lie in R_2; say in R_i. In this case, both e and e' will lie on the boundary of R_i. Since $C(p,\tau_e)$ and $C(e',\tau_e)$ are tangent circles at e, $C(p,\tau_e)$ will decompose R_i into two disjoint regions and A_3' will have points in both of them. Hence $C(p,\tau_e)$ will intersect A_3' (Fig. 39).

Case II: A has no cusp at p.

Here A_3 lies in R_1, say, and A_3' lies in R_2. Thus e [e'] lies on the boundary of R_1 $[R_2]$. Then the circular arc $C(p,\tau_e) \cap R$ lies in R_1 and the arc $C(\tau_{e'},p) \cap R$ lies in R_2. Hence $C(p,\tau_e)$ $[C(\tau_{e'},p)]$ does not meet A_3' $[A_3]$ (Fig 40 and 41).

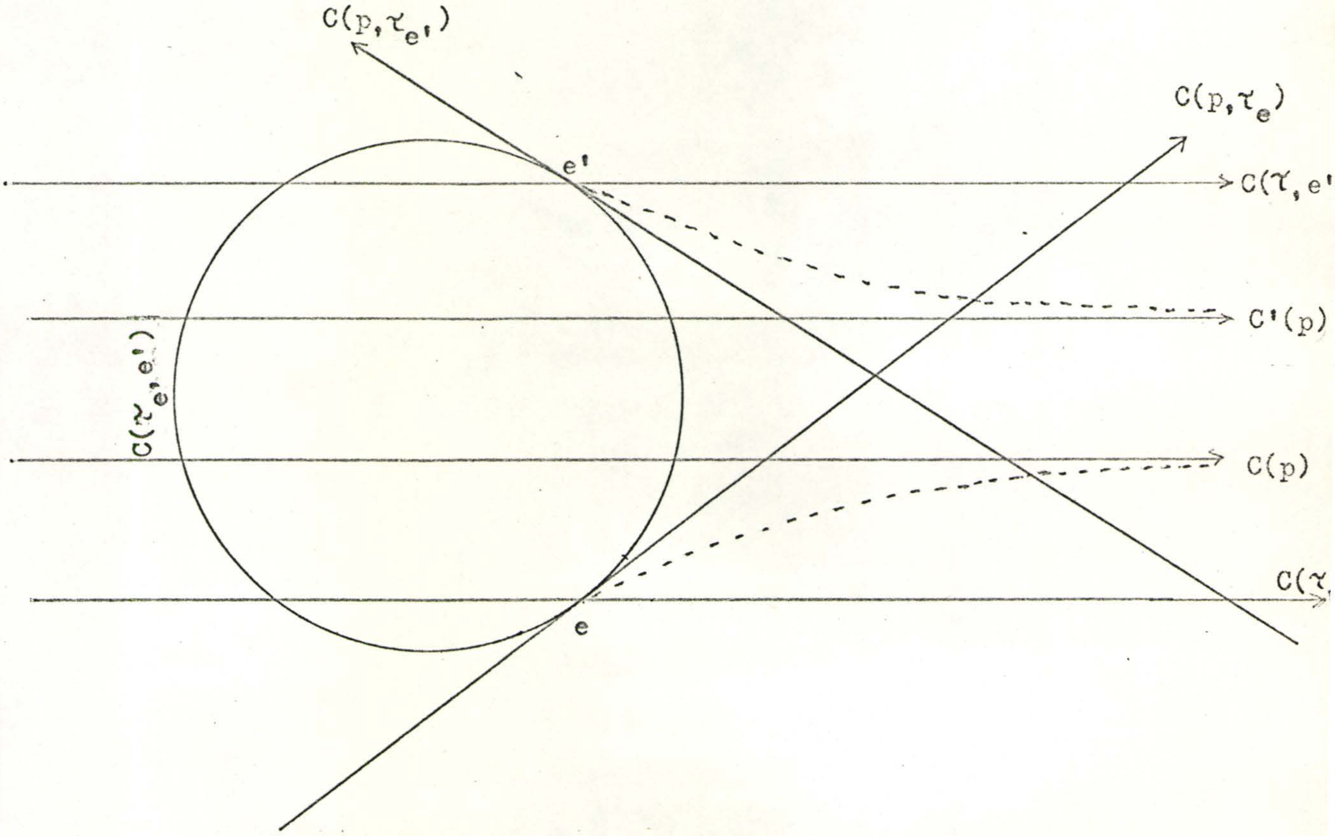

FIGURE 39

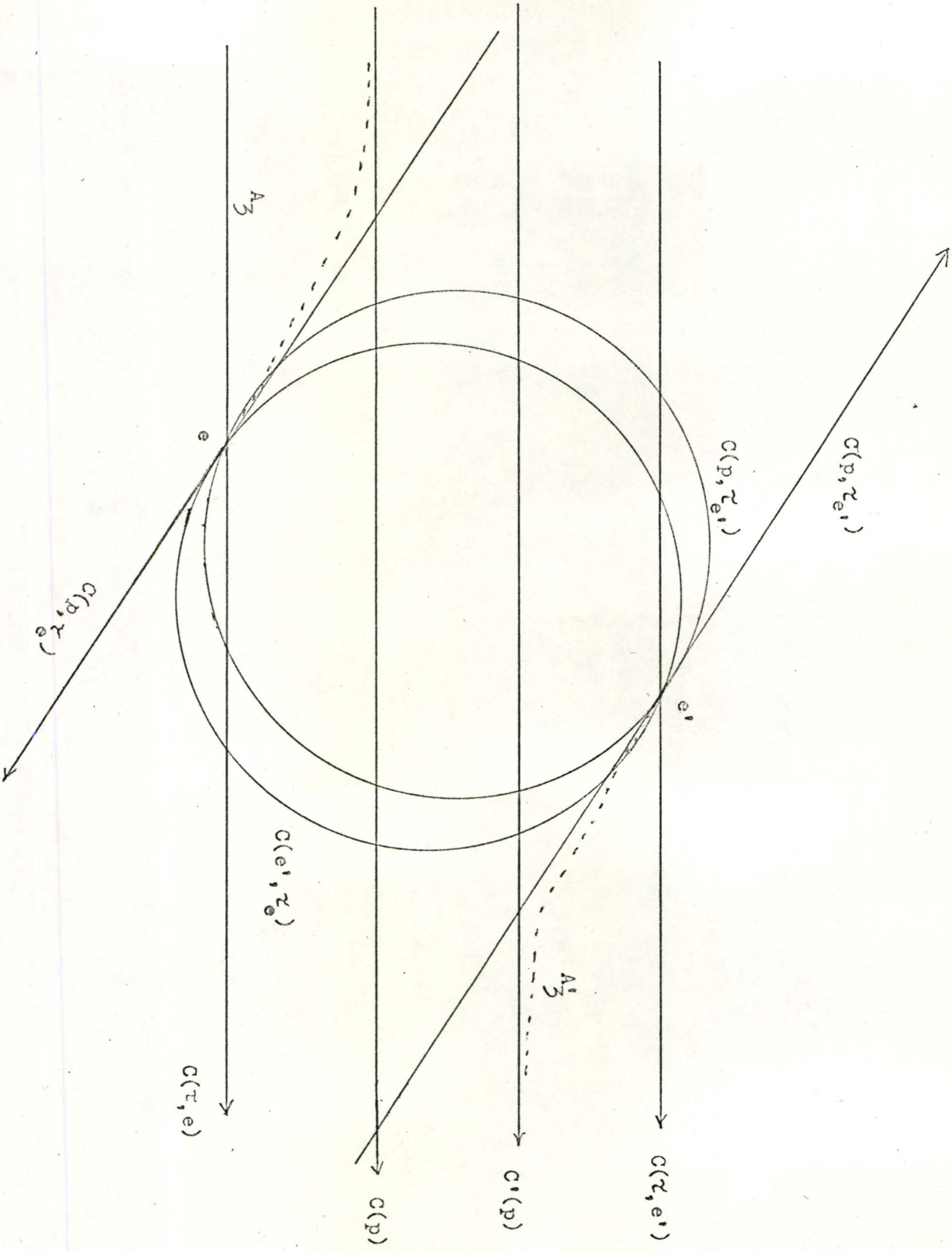

FIGURE 40

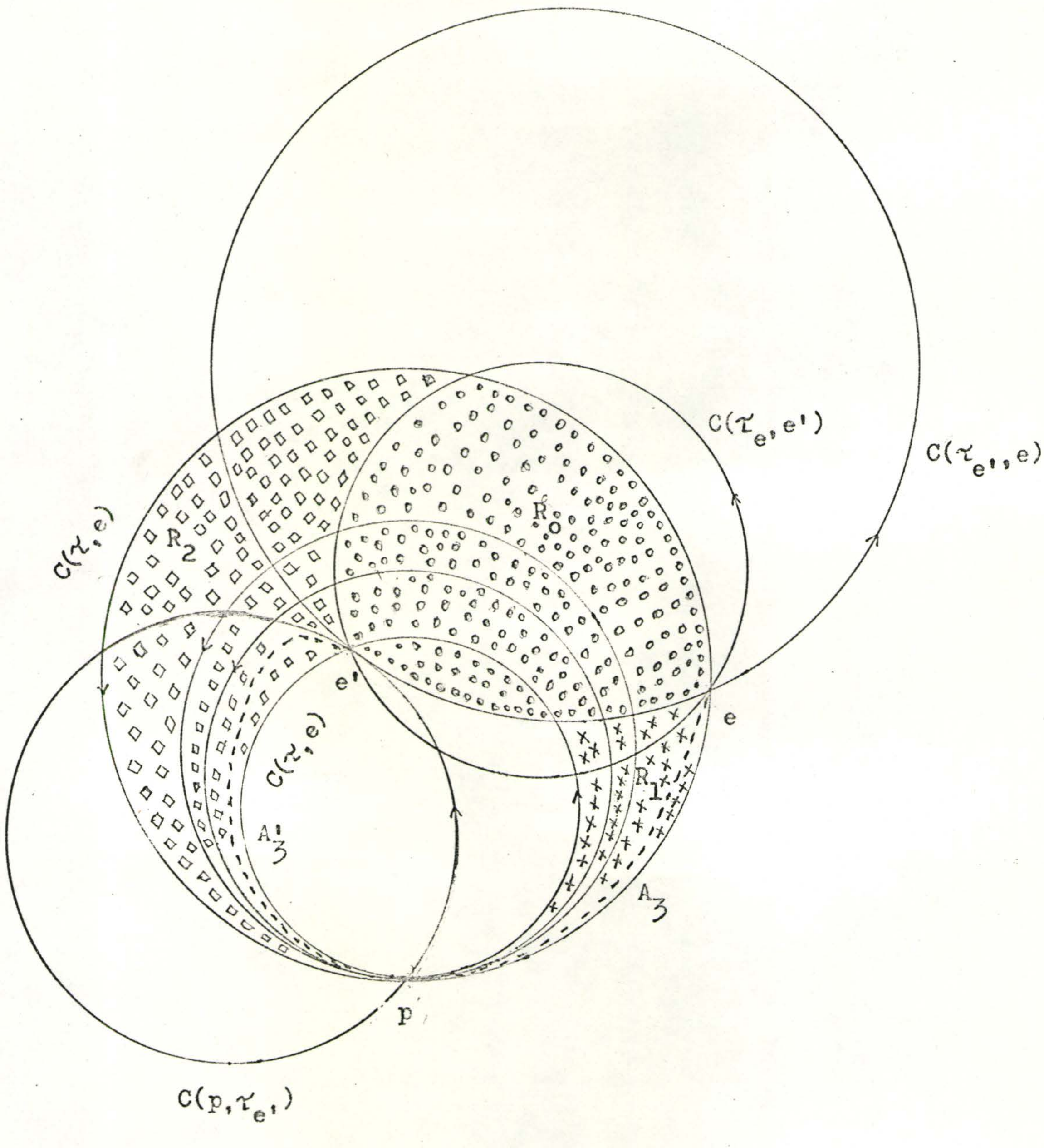

FIGURE 41

Corollary. Conditions (i) - (iv) *imply that* A *has no cusp at* p.

Remark. We observe that if the Conditions (i), (ii) *and* (iv) *hold and* A *has a cusp at* p, *then* $C(\tau_e, e')$ *will have to coincide with* $C(\tau_{e'}, e)$.

Thus Conditions (i), (ii), (iv) *and* $C(\tau_e, e') \neq C(\tau_{e'}, e)$ *imply Condition* (iii), *and hence that* $\bar{A}$ *has cyclic order three*.

4.4. *Lemma. If Conditions* (i) - (iv) *hold and* $q, r, s \in A_3$, $q', r', s' \in A_3'$ *then*

$$A_3' \cup e' \subset C(q,r,s)_* \text{ and } A_3 \cup e \subset C(q',r',s')^*.$$

These results remain valid if two or all of q, r, s, *coincide with one another or with* p *or* e.

Proof. Since $e \in C(p)^*$, Lemma 3.3.3 implies that

$$C(q,r,s)_* \supset C(p)_* \cap C(p, \tau_e)_*.$$

Since $A = A_3 \cup p \cup A_3'$ satisfies Conditions (i) - (iv), A has no cusp at p (cf. corollary of Lemma 4.1.1). Also $C(p, \tau_e) \not\subset \tau$. Hence $C(p, \tau_e)$ intersects A at p. Now $A_3 \subset C(p, \tau_e)^*$ and by Condition (iii) A_3' does not meet $C(p, \tau_e)$. Condition (iv) implies that $e' \notin C(p, \tau_e)$. Thus $A_3' \cup e'$ does not meet $C(p, \tau_e)$. Hence

$$A_3' \cup e' \subset C(p, \tau_e)_*.$$

By Condition (ii), $C(p)$ separates $C(\tau,e)$ and $C(\tau,e')$. Also $e \in C(p)^*$. Hence $e' \in C(p)_*$ and $A_3' \cup e' \subset C(p)_*$. Altogether,

$$A_3' \cup e' \subset [C(p)_* \cap C(p,\tau_e)_*] \subset C(q,r,s)_*$$

4.5. <u>Theorem. Conditions</u> (ii) - (iv) <u>are not only necessary but are also sufficient for</u> A <u>to have cyclic order three.</u>

<u>Proof.</u> Let $t,u \in A_3 \cup p$, $t',u' \in A_3' \cup p$. Using the Lemma 4.4, we prove successively that $C(\tau_{e'},t)$ and symmetrically $C(t',\tau_e)$; $C(e',t,e)$ and $C(e',t',e)$; $C(e',t',t)$ and $C(t',t,e)$; $C(e',t,u)$ and $C(e,t',u')$; $C(t',t,u)$ do not meet A elsewhere.

$$C_i(t) = \begin{cases} C(\tau_{e'},t) \\ C(e',t,e) \\ C(t',t,e) & \text{if } i \\ C(e,t',u') \\ C(t',t,u) \end{cases} \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{cases}$$

Now $C_i(p)$ does not meet A again. If t moves continuously on A_3 from p to e, $C_i(t)$ cannot pass through p, cannot increase the multiplicity with which it meets e or e', and cannot support $A_3 \cup A_3'$ at a new point. Hence $C_i(t)$ does not meet A elsewhere.

4.6. Extension of an arc of cyclic order three.

In this section we wish to prove that an open arc A_3 can be extended through an end-point p to a larger arc of cyclic order three if and only if $C(p) \neq p$ and $\bar{A}_3$, the closure of A_3, is of cyclic order three.

Let e be the other end-point of A_3 and let τ and τ_e denote the family of tangent circles of A_3 at p and e respectively.

4.6.1. We know that a reflection in a circle followed by a reflection in an orthogonal circle is a conformal transformation which leaves both these circles invariant; cf. section 1.6. Let A_3 be an open arc of cyclic order three with an end-point p, such that $C(p) \neq p$ and $\bar{A}_3$ is also of cyclic order three,

$$\text{i.e.,} \quad C(p,\tau_e) \neq C(\tau,e).$$

Let D be any circle through p and orthogonal to C(p), such that D does not meet $A_3 \cup e$. We construct an arc A_3' by first reflecting A_3 in C(p) and then reflecting the resulting arc in the circle D. Then we choose a suitable subarc $B_3 \subset A_3$ with image $B_3' \subset A_3'$, and take

$$A = B_3' \cup p \cup A_3, \quad \text{(Fig. 42)}.$$

To achieve our goal we show that the arc A is also of cyclic order three. It suffices to show that Conditions (i) — (iv) of section 4.1 hold for A.

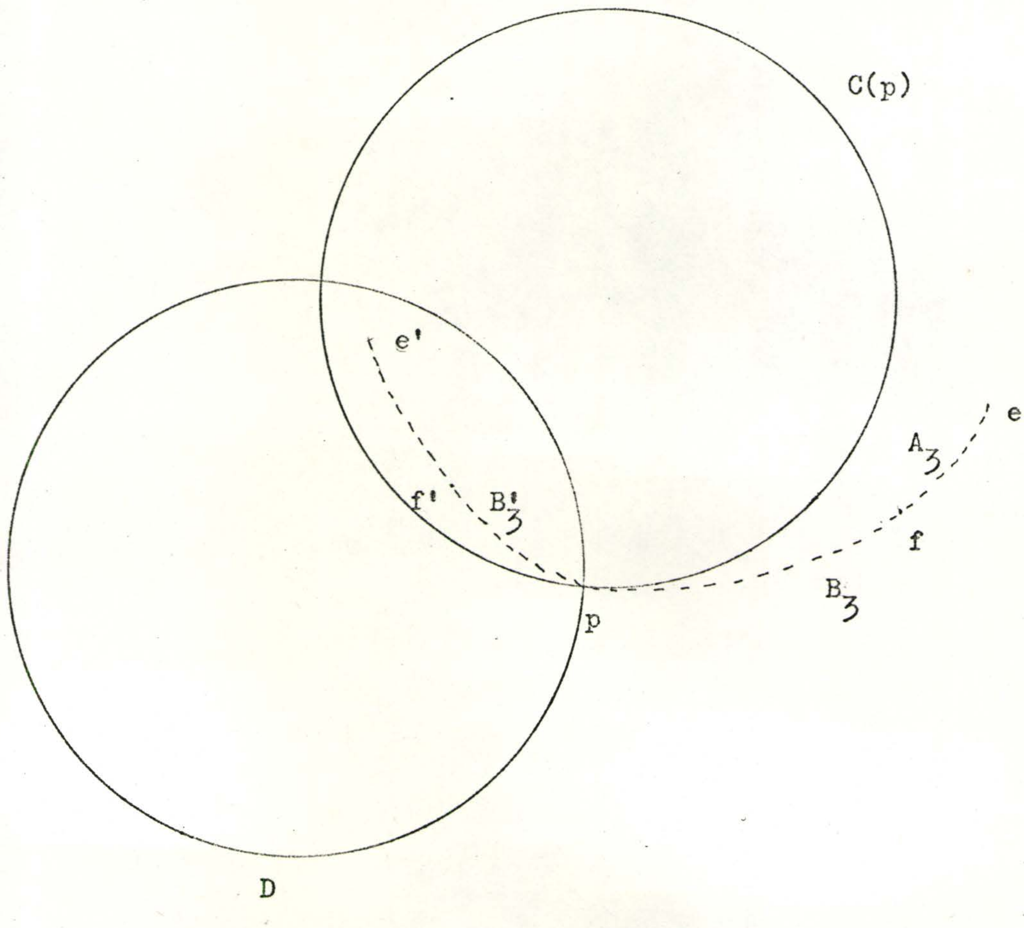

FIGURE 42

4.6.2. Since both reflections leave $C(p)$ invariant, the arc A is conformally differentiable at p. Thus Conditions (i) and (ii) will hold.

Let e, e', f and f' be the end-points $\neq p$ of A_3, A_3', B_3 and B_3' respectively. We may assume that $e \in D^*$. Thus

$$A_3 \cup e \subset C(p)^* \cap D^*,$$

and

$$A_3' \cup e' \subset C(p)_* \cap D_*.$$

4.6.3. Condition (iii) is satisfied.

Let F be the circle orthogonal to the family of circles, τ, through p and any point of A_3. Let F' be its image under the reflection in $C(p)$ followed by the reflection in D. Thus

$$F_*' \subset D_* \subset F_*.$$

Choose

$$B_3 \subset A_3 \cap F^*,$$

and

$$\bar{B}_3 = p \cup B_3 \cup f, \quad f \in F;$$

thus

$$B_3' \subset A_3' \cap F_*' \text{ and } \overline{B_3'} = p \cup B_3' \cup f'; \quad f' \in F'.$$

Since $A_3 \subset C(p)^* \cap D^*$, while

$$C(\tau_{f'}, p) \subset \left[C(\tau, f')_* \cup F_*' \cup f' \cup p\right] \subset C(p)_* \cup D_* \cup p,$$

A_3 does not meet $C(\tau_{f'}, p)$ (Fig. 43).

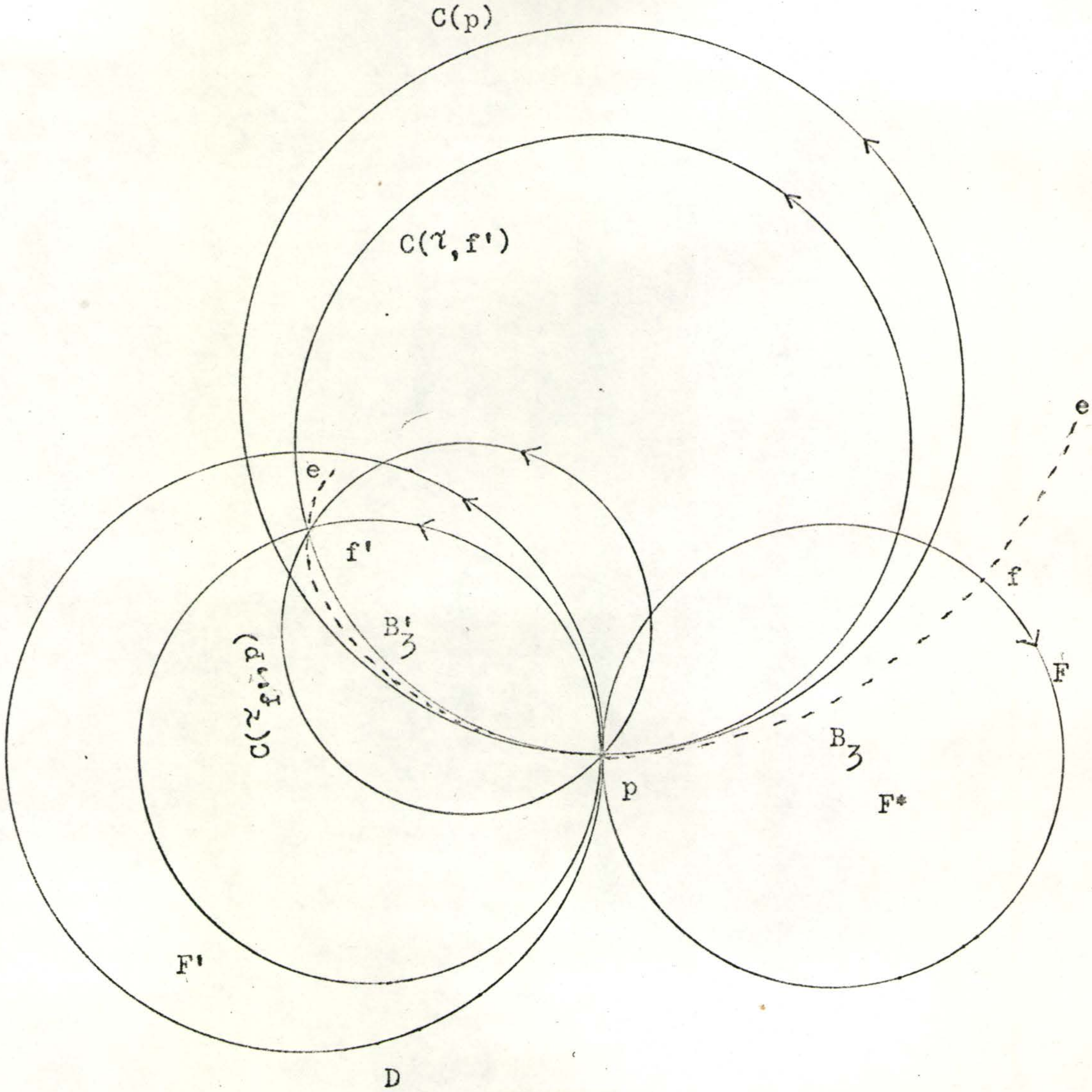

FIGURE 43

By shortening B_3' if necessary, (e.g., choosing B_3 in $C(\tau_{e'},p)^*$) we can assume that B_3' does not meet $C(p,\tau_e)$, (Fig. 44).

4.6.4. Condition (iv) also holds for the arc A.

In the following it is convenient to take e as the point at infinity. If B_3' is chosen small enough, then $C(f',\tau_e)$ will be close to $C(p,\tau_e)$, while a circle through e and two points of $\overline{B_3'}$ will be close to $C(\tau,e)$. Since $C(p,\tau_e) \neq C(\tau,e)$, hence, $C(f',\tau_e)$ does not meet $B_3' \cup p$. We may assume that

$$B_3' \cup p \subset C(f',\tau_e)^*.$$

Next, $C(f',p,e)$ is close to $C(\tau,e)$, while a circle which meets $\overline{B_3'}$ again. Since

$$f' \in C(\tau,e)_* \cap C(p,\tau_e)_*,$$

we have

$$C(f',p,e) \subset \left[C(\tau,e)_* \cap C(p,\tau_e)_*\right] \cup \left[C(\tau,e)^* \cap C(p,\tau_e)^*\right] \cup p \cup e;$$

(Fig. 45, 46).

But

$$A_3 \subset C(\tau,e)_* \cap C(p,\tau_e)^*,$$

therefore $C(f',p,e)$ does not meet A_3. We may assume that

$$A_3 \subset C^*(f',p,e)_*;$$

thus

$$B_3' \subset C(f',p,e)^*.$$

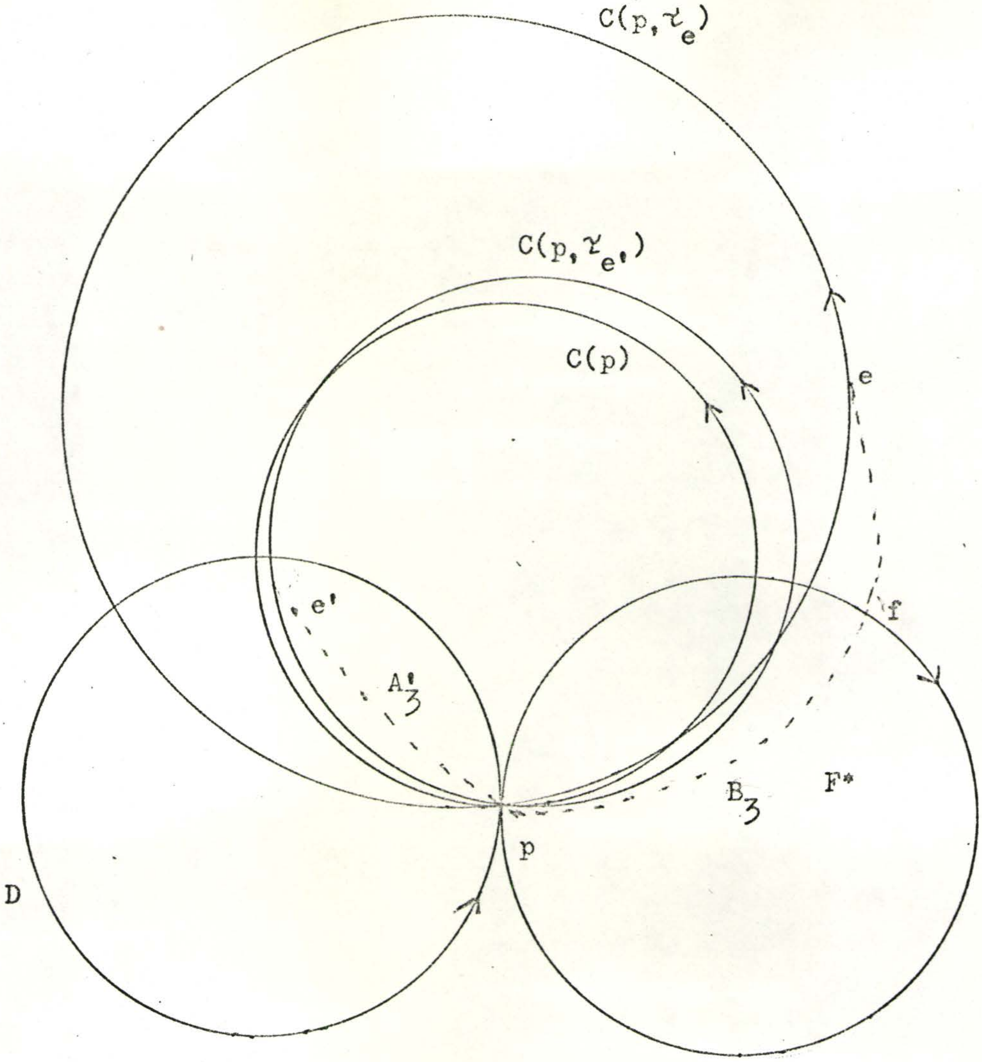

FIGURE 44

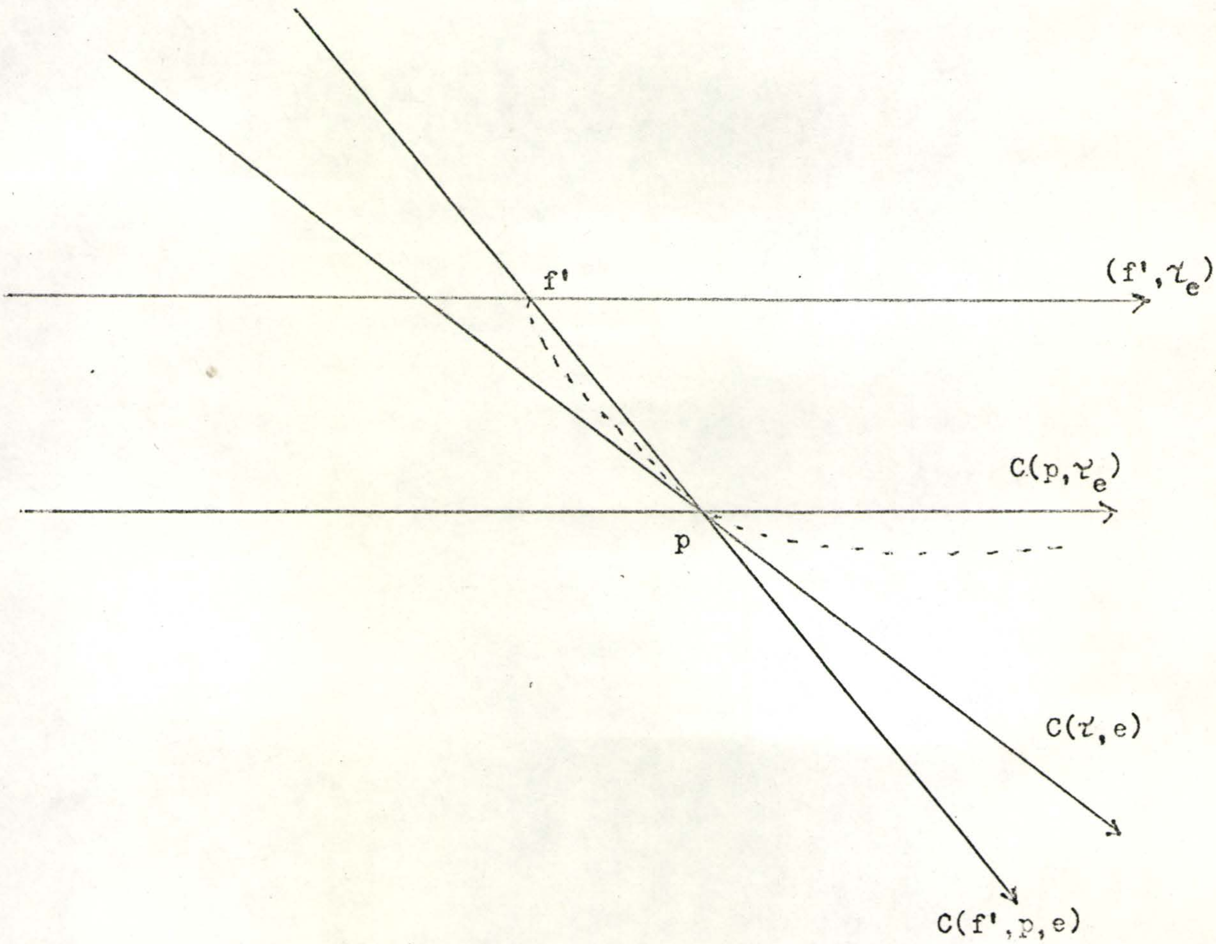

FIGURE 45

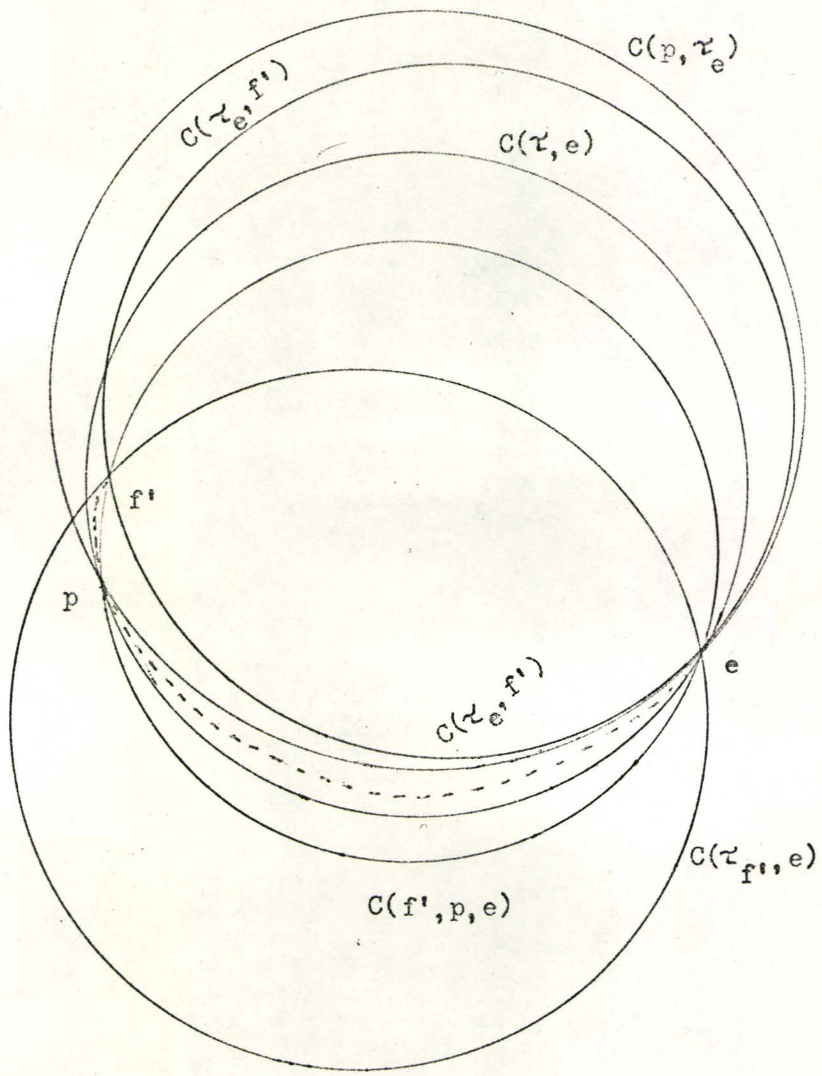

FIGURE 46

Now $C(\tau_{f'},e)$ is close to $C(\tau,e)$, while $C(\tau_{f'},p)$ is close to $C(p)$. Hence

$$C(\tau_{f'},e) \neq C(\tau_f,p).$$

Since

$$B'_3 \subset C(f',p,e)^* \cap C(f',\tau_e)^*,$$

we obtain

$$C(\tau_{f'},e) \subset \left[C(f',p,e)^* \cap C(f',\tau_e)^*\right] \cup \left[C(f',p,e)_* \cap C(f',\tau_e)_*\right] \cup C(f',\tau_e).$$

As

$$A_3 \subset C(f',p,e)_* \cap C(f',\tau_e)^*,$$

we obtain $C(\tau_{f'},e)$ does not meet $A_3 \cup p$.

<u>4.6.5</u>. Combining the subsections from 4.6.1 to 4.6.5 we obtain the following Theorem.

<u>Theorem</u>. <u>An open arc A_3 can be extended through an end-point</u> p <u>to a larger arc of cyclic order three if and only if</u> $C(p) \neq p$ <u>and</u> A_3 <u>is of cyclic order three</u>.

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