CHARACTERISTIC AND ORDER FOR POLYNOMIAL DIFFERENTIABILITY

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FOR
POLYNOMIAL DIFFERENTIABILITY

By

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SCOPE AND CONTENTS: A definition of polynomial differentiability of an arc in the real affine plane at a point is given. The differentiable points are classified with respect to the intersection and support properties of certain families of osculating polynomials. For a given point of an arc, these properties are used to define a certain n-tuple of integers, the characteristic of that point. It is shown that the polynomial order of polynomially differentiable interior point of an arc is at least as great as the sum of the digits of its characteristic.

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INTRODUCTION

In [14], P. Scherk introduced order characteristics for the interior differentiable points of arcs in projective n-space. The characteristic of a differentiable point p of an arc was a certain n-tuple of integers defined by the intersection and support properties of various families of osculating hyperplanes through p. It was shown that the sum of these numbers was associated with the order of the arc in a small neighbourhood of p. Later, in [8] and [9], Lane and Scherk developed an analogous theory of characteristics for differentiable points in the conformal plane. A similar discussion was carried on in [10], [11] and [12], [15] by Lane and Singh in the case of parabolically and conically differentiable points in the real affine plane and in the projective plane, respectively. These papers, Popoviciu's discussion of polynomial convexity in [13] and O. Haupt's work on higher convexity in [1] - [5] naturally led to the study of characteristic and order for polynomially differentiable points of arcs.

In the following, the term "polynomial" will also be used to denote the point set $\{(x, f(x)) | x \in \mathbb{R}\}$, where f is a polynomial over $\mathbb{R}$, i.e., the graph of a polynomial function will also be called a polynomial.

In Chapter I, polynomials are regarded as subsets of the plane and a notion of pointwise convergence is used to obtain certain kinds of degenerate polynomials. The set of all non-degenerate polynomials of degree at most n will be denoted by $\mathcal{P}_n$ and the union of $\mathcal{P}_n$ with the set of degenerate ones will be denoted by $\overline{\mathcal{P}}_n$. An orientable polynomial and orientable pointwise convergence is defined. Multiple point, multiple component and a notion of multiplicity pointwise convergence is also introduced. The concept of support and intersection of an orientable polynomial of $\overline{\mathcal{P}}_n$ with respect to a polynomial of $\mathcal{P}_n$ is defined and discussed.

In Chapter II, three topologies $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are introduced, each finer than the preceding one, and it is shown that the $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ convergences are equivalent to pointwise, orientable pointwise and multiplicity pointwise convergences respectively.

Chapter III contains a discussion of some families of polynomials of $\mathcal{P}_n$ and particularly of the family Φ_h. It is shown that the three convergences introduced in Chapter I and Chapter II are equivalent on $\overline{\Phi}_h$.

In Chapter IV, arcs are introduced and polynomially differentiable points of an arc are defined. Also the intersection and support properties of the various osculating polynomials at a differentiable point of the arc are discussed.

Chapter V is concerned with the definition of a characteristic of a polynomially differentiable point of an arc in the real affine plane. Examples of the different types of points which are differentiable with respect to the family of polynomials of degree at most n are given. We use a sequence of characteristics for an infinitely polynomially differentiable point to construct an infinite characteristic for that point.

In Chapter VI it is shown that the polynomial order of a differentiable interior point p of an arc A is at least as great as sum of the digits of the characteristic of p.

CHAPTER I

Polynomials and Pointwise Convergence

1.1. Notation. Let G denote the real affine plane. The letters p, q, Q, ... usually denote the points in the plane, with the small italics indicating points of arcs. Gothic letters $\mathfrak{L}$, $\mathfrak{Y}$, ... denote lines. $\mathfrak{L} \nparallel \mathfrak{M}$ indicates that $\mathfrak{L}$ and $\mathfrak{M}$ are not parallel. $\mathfrak{L}(Q, \mathfrak{M})$ will denote the line parallel to a line $\mathfrak{M}$ through a point Q.

Let $\mathfrak{K}$ denote the family of polynomials (i.e., polynomial curves) K of degree $\leq$ n which can be represented by an equation of the form

$$ay = a_0 + a_1 x + \ldots + a_n x^n,$$

where $a \neq 0$ and $a, a_i \in \mathbb{R}$; i = 0, 1, ..., n.

We shall denote the line corresponding to x = 0 by $\mathfrak{Y}$ and put $\mathfrak{L}(Q) = \mathfrak{L}(Q, \mathfrak{Y})$.

1.2. Pointwise convergence. A neighbourhood of a point P is the interior of an ellipse which contains p in its interior.

A sequence of points $\{P_i\}$ is defined to be *convergent* to a point P if every neighbourhood of P contains P_i for all but a finite number of i.

A point P is defined to be an *accumulation point* [*a limit point*] of a sequence $\{S_i\}$ of sets S_i if every neighbourhood of P contains points of S_i for infinitely many i [for all but a finite number of i]. In particular, this holds when each S_i is a polynomial K_i.

A sequence $\{S_i\}$ of sets [$\{K_i\}$ of polynomials] is defined to be *pointwise convergent* if it has at least one accumulation point and every accumulation point is a limit point.

1.3. The set $\overline{\mathcal{K}}$ *of polynomials*. Let $\overline{\mathcal{K}}$ denote the set of all polynomials of $\mathcal{K}$ together with the limit sets of convergent sequences of polynomials of $\mathcal{K}$.

Let $K \varepsilon \overline{\mathcal{K}}$ be the limit set of a pointwise convergent sequence $\{K_i\}$ of polynomials of $\mathcal{K}$.

1.3.1. If $P \varepsilon K$ *and* N *is a neighbourhood of* P *with the boundary* B(N), *then*

$$B(N) \cap K \neq \emptyset.$$

<u>Proof</u>. Since $P \varepsilon K$, K_i will have points in N for all sufficiently large i. Thus K_i has points in common with B(N) for all large i. These points will have an accumulation point in $B(N) \cap K$, i.e., $B(N) \cap K \neq \emptyset$.

<u>Remark</u>. The above statement and a similar proof is valid even if K is just a non-empty set of accumulation points of $\{K_i\}$ and $\{K_i\}$ is not a convergent sequence, thus the set of accumulation points of a sequence does not have an isolated point.

<u>1.3.2. Suppose $P_1 \varepsilon K$ and $P_2 \varepsilon K$ are separated by a line $\mathcal{L} \nparallel \mathcal{Y}$. Then the intersection of $\mathcal{L}$ with the closed strip bounded by $\mathcal{L}(P_1)$ and $\mathcal{L}(P_2)$ contains a point of K.</u>

<u>Proof</u>. Choose neighbourhoods N_1 and N_2 of P_1 and P_2 respectively, such that

$$N_1 \cap \mathcal{L} = \emptyset = N_2 \cap \mathcal{L} .$$

Since P_1 and P_2 are in K, K_i has points P_{i1} and P_{i2} in N_1 and N_2 respectively, for all but a finite number of i; i.e., K_i has points on both the sides of $\mathcal{L}$. Since each K_i is a continuous function, it intersects $\mathcal{L}$ at a point between the lines $\mathcal{L}(P_{i1})$ and $\mathcal{L}(P_{i2})$.

Therefore these points of intersection are bounded on $\mathcal{L}$ and they have an accumulation point on $K \cap \mathcal{L}$ which lies between $\mathcal{L}(P_1)$ and $\mathcal{L}(P_2)$ or on $\mathcal{L}(P_1)$ or $\mathcal{L}(P_2)$.

Corollary. If $P_1 \varepsilon K$, $P_2 \varepsilon K$ and $\mathcal{L}(P_1) = \mathcal{L}(P_2)$, then the segment $P_1 P_2$ belongs to K.

1.3.3. Let $\{K_i\}$ be a pointwise convergent sequence of polynomials of $\mathcal{D}$. Let K be the set of limit points of $\{K_i\}$.

Now each K_i can be represented by an equation of the form

(1.3.3-1) $$a_i y = a_{0i} + a_{1i} x + \ldots + a_{ni} x^n;$$

$a_i \neq 0$ and $a_i, a_{ji} \varepsilon \mathbb{R}$; $j = 0, \ldots, n$. Hence we can associate with each K_i the point $(a_i, a_{0i}, a_{1i}, \ldots, a_{ni})$ of the real projective $n + 1$ space. Let $(b, b_0, \ldots, b_n)$ be an accumulation point of the sequence $\{(a_i, a_{0i}, \ldots, a_{ni})\}$. It can be easily seen that if $P \varepsilon K$, then the coordinates of P satisfy the equation

(1.3.3-2) $$by = b_0 + b_1 x + \ldots + b_n x^n.$$

Thus if K' is the solution set of the equation (1.3.3-2), then $K \subseteq K'$.

1.3.4. The set K is either a polynomial of $\mathcal{L}$ or it consists of at most n components each of which is a line or a ray parallel to $\mathcal{Y}$.

Proof. (i) Let $b \neq 0$ in (1.3.3-2). Let $P \in K'$, say $P = (x_0, y_0)$. Then we wish to show that $P \in K$ and therefore $K' \subseteq K$.

Let $P_i = (x_0, y_i)$, where y_i is defined by

$$a_i y_i = a_{0i} + a_{1i} x_0 + \dots + a_{ni} x_0^n.$$

Then $P_i \in K_i$ and y_i tends to y_0 as i tends to infinity suitably. Thus P is a limit point of the sequence $\{P_i\}$ and therefore of the sequence $\{K_i\}$. Hence $P \in K$. Therefore $K' \subseteq K$.

Also by 1.3.3, $K \subseteq K'$. Hence if $b \neq 0$, then $K = K'$ is a polynomial of $\mathcal{L}$.

(ii) Let $b = 0$. Then K' consists of at most n components, each of which is a line parallel to $\mathcal{Y}$. Therefore each component C of K is a line parallel to $\mathcal{Y}$ or a (connected) component of such a line. Suppose C is a segment or an isolated point. Then there exists an ellipse B(N) containing C such that

$$(N \cup B(N)) \cap K \setminus C = \emptyset.$$

This is impossible by 1.3.1. Hence every component of K is a line or a ray, parallel to $\mathcal{V}$.

Also, by the Corollary of 1.3.2, no two disjoint rays of K lie on the same line parallel to $\mathcal{V}$.

1.3.5. Since each $K \varepsilon \overline{\mathcal{K}}$ can be obtained as the limit set of a sequence $\{K_i\}$ of polynomials of $\mathcal{K}$, we could have defined $\overline{\mathcal{K}}$ to be the family of limit sets of pointwise convergent sequences of $\mathcal{K}$.

If $K \varepsilon \mathcal{K}$ is a polynomial of degree m, $0 \leq m \leq n$, we call it an m-regular polynomial.

If $K \varepsilon \overline{\mathcal{K}} \setminus \mathcal{K}$, i.e., if every component of K is a line or ray, then we call K a degenerate polynomial.

Remark. Since the members of $\overline{\mathcal{K}}$ are also subsets of G, the notion of pointwise convergence is valid for sequences of degenerate polynomials. We note that if a sequence $\{K_i\}$ of polynomials of $\overline{\mathcal{K}}$ converge to a regular polynomial, then, for sufficiently large i, K_i are also regular polynomials.

1.3.6. A regular polynomial K_1 supports [intersects] a polynomial K of $\mathcal{K}$ at a point $Q \varepsilon K_1$

if for a sufficiently small neighbourhood N of Q, $K \cap N$ is decomposed by Q into two disjoint open sets which lie in the same region [in different regions] bounded by K_1.

Remark. Above definition of support and intersection is symmetric in K and K_1; cf. 1.8.2.

1.3.7. Let K_1 and K_2 be two regular polynomials given by the equations, say $f_1(x) = \sum_{i=0}^{n} a_i x^i$ and $f_2(x) = \sum_{i=0}^{n} b_i x^i$ respectively. Then K_1 and K_2 meet each other at least (exactly) r times or meet with multiplicity r if the equation $f_1(x) - f_2(x) = 0$ has at least (exactly) r real roots. In particular, K_1 and K_2 have at least (exactly) r-point contact at $Q = (x_0, y_0)$ if x_0 is a root of $f_1(x) - f_2(x) = 0$ of at least (exact) multiplicity r, equivalently, if f_1 and f_2 have same derivatives at Q up to the (r - 1)-th order.

We note that the number of roots of a polynomial equation $f(x) = 0$ between a and b counted according to their multiplicity is odd or even according as f(a) and f(b) have opposite or like signs.

The above discussion and *1.3.6* yields the following result.

Let K *and* L *be two regular polynomials which have exactly* r-*point contact at* Q. *Then* K *supports* [*intersects*] L *at* Q *if and only if* r *is even* [*odd*].

1.3.8. Let $\{K_i\}$ *converge to* K, $K \varepsilon \overline{\mathcal{D}} \setminus \mathcal{D}$ *and let* C *be a component of* K. *Let* $Q_j \varepsilon C$; N_j *be a neighbourhood of* Q_j *such that* $(K \cap N_j) \setminus C = \emptyset$, *and let* $\mathcal{L}_j \nparallel \mathcal{W}$ *be a line through* Q_j; j = 1, 2. *Then* K_i *meets both* $\mathcal{L}_1 \cap N_1$ *and* $\mathcal{L}_2 \cap N_2$ *with an even multiplicity or both of them with an odd multiplicities.*

Proof. Let B(N) be an ellipse containing N_1 and N_2 in its interior N such that $(N \cap K) \setminus C = \emptyset$. Let B(M) be a convex quadrilateral inscribed in N such that two opposite sides are parallel to $\mathcal{W}$ and Q_1 and Q_2 are interior points of the other two sides which are segments of $\mathcal{L}_1$ and $\mathcal{L}_2$. Let P be a point on C between Q_1 and Q_2 and let $N_0 \subset M$ be a neighbourhood of P. Then N_0 contains points of K_i for all sufficiently large i. Now K_i will meet B(M) with even multiplicity and for sufficiently large i, the points of $B(M) \cap K_i$ will lie on the sides of B(M) through Q_1 and Q_2. Moreover, the accumulation set of $\{B(M) \cap K_i\}$ is $\{Q_1, Q_2\}$. Hence either K_i meets both $\mathcal{L}_1 \cap N_1$ and $\mathcal{L}_2 \cap N_2$ with an even multiplicity or both of them with an odd multiplicity.

1.3.9. *Let* $\{K_i\}$ *converge to* $K \in \overline{\mathcal{K}} \setminus \mathcal{K}$ *and let a component* C *of* K *be a ray. Let* $Q \in C$ *and* N_0 *be a sufficiently small neighbourhood of* Q *such that* $N_0 \cap K/C = \emptyset$. *Then for any line* $\mathcal{L} \nparallel \mathcal{Y}$ *through* Q, K_i *meets* $\mathcal{L} \cap N_0$ *an even number of times.*

Proof. Let P be the vertex of C and let $Q \neq P$. Again construct an ellipse B(N) containing P and N_0 in its interior N such that $N \cap K \setminus C = \emptyset$. Let B(M) be a quadrilateral in N with two sides parallel to $\mathcal{Y}$ and such that one of the other two sides is a segment of $\mathcal{L}$ containing Q in its interior and P lies in the interior of B(M). Then for a sufficiently large i, $K_i \cap B(M) = K_i \cap \mathcal{L} \cap N_0$. Hence K_i and $\mathcal{L} \cap N_0$ meet with an even multiplicity for $Q \neq P$.

By 1.3.8, this K_i and a line through P will also meet with an even multiplicity in a small neighbourhood of P.

1.4. *Orientations of polynomials of* $\overline{\mathcal{K}}$.

1.4.1. A regular polynomial K divides G into two disjoint regions denoted by K^1 and K^{-1}. Each of the ordered pairs (K^1, K^{-1}) and (K^{-1}, K^1) is called an *orientation* of K.

1.4.2. Let $K \in \overline{\mathcal{K}}$ be the limit set of a convergent sequence $\{K_i\}$ of polynomials of $\mathcal{K}$. Let

$P \notin K$. Then $P \notin K_i$ for all sufficiently large i. We can then choose a fixed $\alpha \varepsilon \{1, -1\}$ and assign to K_i the orientation such that $P \varepsilon K_i^{\alpha}$ for all large i. Define

$$(1.4.2) \qquad K^{\alpha} = \{Q | Q \notin K \text{ and } Q \varepsilon K_i^{\alpha} \text{ for all large i}\}.$$

Now, it can happen that for each point $Q \notin K$, either $Q \varepsilon K_i^{\alpha}$ for all large i, or $Q \varepsilon K_i^{-\alpha}$ for all large i; cf. (1.6.3-1). In such a case, every point in the plane belongs to one of the three sets K, K^1 and K^{-1}. We then say that K is <u>orientable</u> and call each of the ordered pairs (K^1, K^{-1}) and (K^{-1}, K^1) an <u>orientation</u> of K.

One of K^1 and K^{-1} may be void. For example, let K_m be given by

$$y = mx^2; \; m = 1, 2, \ldots\ldots .$$

Then the sequence $\{K_m\}$ converges to the double ray $x^2 = 0$, $y \geq 0$, cf. <u>1.6.2</u>. If we take $(1, 0) \varepsilon K_m^{-1}$ for all large m, then $K^{+1} = \emptyset$.

<u>1.4.3</u>. <u>Let</u> $\{K_i\}$ <u>be pointwise convergent to</u> K. <u>Let</u> L^{α} <u>be the set of limit points of the sequence</u> $\{K_i^{\alpha}\}$, $\alpha = \pm 1$. <u>Then</u>

$$L^{\alpha} = K^{\alpha} \cup K; \text{ cf. } \underline{1.4.2}.$$

Proof. (i) Let $P\varepsilon K$. Then every neighbourhood N of P is met by K_i for all large i. Let $Q\varepsilon N \cap K_i$. Then for any neighbourhood N_i of Q, $N_i \subset N$ there are points $Q_{i\alpha}\varepsilon N_i \cap K_i^\alpha$, $\alpha = \pm 1$. Thus every neighbourhood N of P contains points $Q_{i\alpha}$ of K_i^α. Hence P is a limit point of K_i^α, $\alpha = \pm 1$.

(ii) Let $P\varepsilon K^\alpha$. Thus $P\varepsilon K_i^\alpha$ for all large i and $P \notin K$. Hence P is a limit point of $\{K_i^\alpha\}$. Thus $K \cup K^\alpha \subseteq L^\alpha$.

(iii) Let $P\varepsilon L^\alpha$, $P\notin K$. Let N be a neighbourhood of P such that $N \cap K = \emptyset$. Then $N \cap K_i = \emptyset$ for all large i and N meets K_i^α for all large i. Also $P\notin K$ implies that $P\notin K_i$ for all large i. Hence $P\varepsilon K_i^\alpha$ for all large i. Thus $P\varepsilon K^\alpha$. Therefore $K^\alpha \cup K \supseteq L_\alpha$.

1.5. Orientably pointwise convergence. A sequence $\{K_i\}$ of oriented polynomials of $\overline{\mathcal{R}}$ is orientably pointwise convergent to $K\varepsilon \overline{\mathcal{R}}$, if $\{K_i\}$ is pointwise convergent to K and K is also orientable; i.e., for all $P\notin K$ either $P\varepsilon K_i^1$ for all large i or $P\varepsilon K_i^{-1}$ for all large i; cf. 1.4.2. For example, the sequence in (1.6.3-1) is orientably convergent.

1.5.1. Let $\{K_i\}$ be orientably convergent to K. Then $\{K_i^\alpha\}$ is a convergent sequence of sets K_i^α and the limit set is $K^\alpha \cup K$; cf. 1.4.3; i.e.,

$$\lim_{i\to\infty} K_i^\alpha = K^\alpha \cup K = L^\alpha.$$

Proof. Let P be an accumulation point of $\{K_i^\alpha\}$, $P \notin K$. Then $P \in K_i^\alpha$ for infinitely many i. Since $\{K_i\}$ is orientably convergent, $P \in K_i^\alpha$ for all large i. Hence P is a limit point of $\{K_i^\alpha\}$ and $P \in L^\alpha = K^\alpha \cup K$.

1.5.2. Let $\{K_i\}$, $K_i \in \mathfrak{K}$, be orientably convergent to $K \in \overline{\mathfrak{K}} \setminus \mathfrak{K}$. Then each component C of K is associated with a multiplicity m(C), $m(C) \equiv 1$ or 0 (mod 2); i.e., each K_i will meet a line $\mathcal{L} \nparallel C$ with an even multiplicity for all large i or with an odd multiplicity for all large i in a sufficiently small neighbourhood of $\mathcal{L} \cap C$.

1.5.3. A component C of an orientable polynomial K, $K \in \overline{\mathfrak{K}} \setminus \mathfrak{K}$, is decomposing [non-decomposing] if for $P \in C$ and for any sufficiently small neighbourhood N of P such that $N \cap K \setminus C = \emptyset$, neither [exactly one] of the sets $K^1 \cap N$ and $K^{-1} \cap N$ is void.

1.5.4. If $m(C) \equiv 1$ [$\equiv 0$] (mod 2), then C is a decomposing [non-decomposing] component.

Proof. Let $P \in C$ and let N be a neighbourhood of P such that $N \cap K \setminus C = \emptyset$. Let $B(N) \subseteq N$ be an ellipse containing P in its interior N_0. Let $\mathcal{L}$ be a line through P, $\mathcal{L} \pitchfork C$. Let $B(N_0) \cap \mathcal{L} = \{Q_1, Q_2\}$. Then to prove the statement, it suffices to show that the line segments $\mathcal{L}(P, Q_1) \setminus \{P\}$ and $\mathcal{L}(P, Q_2) \setminus \{P\}$ lie in different sides [on the same side] of K.

Now each K_i and $\mathcal{L}$ meet in N_0 with an odd [even] multiplicity for all large i. Hence Q_1 and Q_2 lie on different sides [on the same side] of K_i for all large i. Thus if $m(C) \equiv 1$ [$\equiv 0$] and $Q_1 \varepsilon K_i^{\alpha}$ then $Q_2 \varepsilon K_i^{-\alpha}$ [$Q_2 \varepsilon K_i^{\alpha}$] for large i.

1.6. Multiplicities

1.6.1. Let $\{K_i\}$ be a sequence of regular polynomials converging to $K \varepsilon \overline{\mathfrak{K}}$. Let $\mathcal{L}$ be a line, $\mathcal{L} \pitchfork \mathfrak{W}$, through a point $P \varepsilon K$. Then $\mathcal{L}$ <u>meets</u> K <u>at</u> P <u>at least</u> r-<u>times</u> if there exists a sequence $\{\mathcal{L}_i\}$ of lines converging to $\mathcal{L}$ such that $\mathcal{L}_i$ and K_i meet r times in a sufficiently small neighbourhood of P.

If $\mathcal{L}$ meets K at least r-times at Q and does not meet it at least (r + 1)-times at Q we say that $\mathcal{L}$ <u>meets</u> K <u>exactly</u> r-<u>times</u>.

It can be verified that the above definition implies the one given in 1.3.7.

The following statement is an immediate consequence of the above definition.

Let a sequence $\{K_i\}$ *of polynomials of* $\tilde{\mathcal{K}}$ *converge to* $K \varepsilon \tilde{\mathcal{K}}$. *Let* P_i *and* Q_i *converge to* P; $P_i \varepsilon K_i$, $Q_i \varepsilon K_i$; $P_i \neq Q_i$ (thus $P \varepsilon K$). *Then the line* $P_i\, Q_i$ *converges to a line which meets* K *at* P *at least twice and is the tangent of* K *at* P.

1.6.2. Let $\{K_i\}$ be convergent to a degenerate polynomial K. A point P of K *has multiplicity* r $m(P) = r$, if there exists a line $\mathcal{L}$, $\mathcal{L} \nparallel \mathcal{Y}$ through P which meets K at P exactly r-times and no such line through P meets K, $(r + 1)$-times at P.

If all but a finite number of points of a component C of K are counted with the same multiplicity r, then C *is counted* r *times*; cf. (1.6.3-1) for $r = 1$.

1.6.3. We look at the multiplicities of the components of several degenerate polynomials K, for $n = 3$, i.e., when the sequence $\{K_i\}$ consists of cubics.

In the following, let m be a positive integer.

We consider the sequence of cubics given by

$$y = mx^3. \qquad (1.6.3\text{-}1)$$

Let m tend to infinity. Then the limit set K is the line given by $x^3 = 0$, which by our definitions in 1.6.2 is a single line on $\mathfrak{Y}$. The multiplicity of the origin, however, is three.

On the other hand, if we examine the limit set K of a sequence given by

$$(1.6.3\text{-}1) \qquad y = -m^2 x + m^4 x^3$$

as m tends to infinity, then K is again a line given by $x^3 = 0$, but we interpret it as a triple line on $\mathfrak{Y}$, i.e., a line counted three times on $\mathfrak{Y}$.

Again we consider the sequence given by

$$(1.6.3\text{-}3) \qquad y = m^3 x^2 + m^4 x^3,$$

and let m tend to infinity. The limit set K is still given by $x^3 = 0$, but the points (0, y) with $y \geq 0$ [with $y < 0$] have the multiplicity three [one]. We interpret K as the union of $\mathfrak{Y}$ with a double ray on $\mathfrak{Y}$ with the vertex at the origin.

Now consider the sequence given by

$$(1.6.3\text{-}4) \qquad y = mx^2 + mx^3.$$

Let m tend to infinity. Then the equation (1.6.3-4) tends to the equation $x^3 + x^2 = 0$ and the limit set K consists of a double ray on $\mathfrak{Y}$ with vertex at the origin and the line given by $x = -1$.

If the sequence is given by

$$(1.6.3\text{-}5) \qquad y = -m + m^2 x^2 + (m^2 - m)x^3,$$

the limit equation is again $x^3 + x^2 = 0$, but now the component of K through the origin is a double line.

If the sequence is given by

$$(1.6.3\text{-}6) \qquad y = 3m^2 x^2 + 2m^3 x^3,$$

the limit equation is $x^3 = 0$. Here K consists entirely of points of $\mathfrak{Y}$, these points being counted three times between $y = 0$ and $y = 1$ and once elsewhere. In this example, K may be interpreted as the union of $\mathfrak{Y}$ and a double segment on $\mathfrak{Y}$.

1.7. *Multiplicity pointwise convergence.* A sequence $\{K_i\}$ of polynomials of $\overline{\mathfrak{K}}$ is *multiplicity pointwise convergent* to $K \varepsilon \overline{\mathfrak{K}}$ if $\{K_i\}$ is orientably pointwise convergent to K and each point $P \varepsilon K$ has a multiplicity m(P); cf. *1.6.2*.

1.7.1. Let $\{K_i\}$ be a sequence of regular polynomials which is multiplicity pointwise convergent to $K \in \overline{\mathcal{R}} \setminus \mathcal{R}$. Thus K_i has equation of the type (1.3.3-1) and K an equation

$$(1.7.1\text{-}1) \qquad b_0 + b_1 x + \ldots + b_n x^n = 0$$

of the type (1.3.3-2) with b = 0. At least one of $b_1, \ldots, b_n$, say b_h, is not zero and therefore $a_{ih} \neq 0$ in (1.3.3-1) for all large i. Hence we can normalize (1.3.3-1) and (1.7.1-1) by taking $b_h = a_{ih} = 1$.

Let P = (c, d) be a point on K. Suppose that a line $\mathcal{L}$: y = mx + e through P meets K_i at r distinct points $P_j(x_{ij}, y_{ij})$ which converge to P; j = 1, ...,r. Then the equation

$$(1.7.1\text{-}2) \quad a_{ni}x^n + \ldots + a_{2i}x^2 + (a_{1i} - m\, a_i)x + (a_{0i} - e\, a_i) = 0$$

contains the factor

$$(1.7.1\text{-}3) \qquad (x - x_{i1}) \ldots (x - x_{ir}).$$

As i tends to infinity (1.7.1-2) tends to (1.7.1-1) and the factor (1.7.1-3) tends to $(x - c)^r$. Thus if $P \varepsilon K$, $P = (c, d)$ and $m(P) = r$, then $(x - c)^r$ is a factor of (1.7.1-1). Hence we have the following lemma.

Lemma. Let $K \varepsilon \overline{\mathcal{R}} \setminus \mathcal{R}$, $K = C_1 \cup C_2 \cup \ldots \cup C_\ell$, C_i a component of K, $1 \leq \ell \leq n$, be a limit set of a multiplicity convergent sequence. Let $P_i \varepsilon C_i$; $1 \leq i \leq \ell$. Then $\sum_1^\ell m(P_i) \leq n$. In particular, if $\ell = n$, then for any $P_i \varepsilon C_i$, $m(P_i) = 1$. If a C_i is a ray, then $\ell \leq n - 1$.

1.7.2. The sections 1.3.8 and 1.3.9 yield the following result.

Theorem 1. Let $K \varepsilon \overline{\mathcal{R}} \setminus \mathcal{R}$ be the limit set of a multiplicity pointwise convergent sequence. Then a component of K is one of the following types.

(i) a line parallel to $\mathcal{Y}$ on which all the points are counted with the same multiplicity mod 2,

(ii) a ray parallel to $\mathcal{Y}$ on which all the points are counted with an even multiplicity.

1.8. Support and Intersection.

1.8.1. Let $K_1 \varepsilon \mathcal{R}$. Let $K_2 \varepsilon \overline{\mathcal{R}}$ be an orientable polynomial. Then K_2 intersects [supports] K_1 at a point P if for a sufficiently small neighbourhood N of P neither [one] of the sets $K_1 \cap N \cap K_2^{\pm 1}$ is empty; cf. 1.4.2.

Since a regular polynomial is always orientable, the above definition with $K_2 \in \mathcal{L}$ is equivalent to that given in 1.3.6.

We note that if K_2 is degenerate and the component of K_2 through P is decomposable [non-decomposable], then K_2 intersects [supports] K_1 at P; cf. 1.5.3.

1.8.2. Let K_1 and K_2 be two regular polynomials. Then the definition in 1.8.1 is equivalent to saying that K_1 and K_2 intersect [support] each other at a point P if none [exactly one] of the sets $K_1^{\pm 1} \cap N \cap K_2^{\pm 1}$ is empty, for every sufficiently small neighbourhood N of P.

Proof. We wish to verify that

(1) $K_1 \cap N \cap K_2^{\alpha} \neq \emptyset$, for both $\alpha = 1$ and $\alpha = -1$

$\Updownarrow$

(2) none of the four sets $K_1^{\pm 1} \cap N \cap K_2^{\pm 1}$ is void

and

(3) $K_1 \cap N \cap K_2^{\beta} = \emptyset$, for either $\beta = 1$ or $\beta = -1$

$\Updownarrow$

(4) exactly one of the four sets $K_1^{\pm 1} \cap N \cap K_2^{\pm 1}$ is void.

Claim (i). (1) implies (2) and (4) implies (3). Suppose that

$$K_1^{\alpha} \cap N \cap K_2^{\beta} = \emptyset \text{ for some } \alpha \text{ and } \beta \text{ in } \{1, -1\}.$$

Then

$$(5) \qquad K_2^{\beta} \cap N \subset (K_1^{-\alpha} \cup K_1) \cap N.$$

Taking the interiors of the sets in (5) and noting that $K_2^{\beta} \cap N$ is an open set in G, we have

$$K_2^{\beta} \cap N \subset K_1^{-\alpha} \cap N.$$

Hence

$$K_1 \cap K_2^{\beta} \cap N \subset K_1 \cap K_1^{-\alpha} \cap N = \emptyset.$$

Claim (ii). (2) implies (1). Also (3) implies that at least one of the four sets $K_1^{\pm 1} \cap N \cap K_2^{\pm 1}$ is void.

Suppose, e.g., that

$$K_1 \cap N \cap K_2^{\beta} = \emptyset.$$

Then

$$K_2^{\beta} \cap N \subset (K_1^1 \cup K_1^{-1}) \cap N = (K_1^1 \cap N) \cup (K_1^{-1} \cap N).$$

Since $K_2^\beta \cap N$ is connected and $K_1^1 \cap N$ and $K_1^{-1} \cap N$ are disjoint, we obtain

$$K_2^\beta \cap N \subset K_1^\alpha \cap N \text{ for either } \alpha = 1 \text{ or } \alpha = -1.$$

Hence

$$K_1^{-\alpha} \cap (K_2^\beta \cap N) \subset K_1^{-\alpha} \cap (K_1^\alpha \cap N) = \emptyset.$$

Claim (iii). (3) implies (4). It remains to prove that only one of the four sets $K^{\pm 1} \cap N \cap K_2^{\pm 1}$ can be void. Assume, for example

$$K_1^\alpha \cap N \cap K_2^1 = \emptyset = K_1^\gamma \cap N \cap K_2^{-1}; \; \alpha = \pm 1, \; \gamma = \pm 1.$$

Then by claim (i)

$$K_1 \cap N \cap K_2^1 = \emptyset = K_1 \cap N \cap K_2^{-1}.$$

Hence

$$N \cap K_1 \subset N \setminus \{(K_2^1 \cup K_2^{-1}) \cap N\} = N \cap K_2.$$

Since $N \cap K_1 \cap K_2 = \{P\}$, this is impossible.

More precisely,

$$(3) \implies \begin{cases} K_2^{\beta} \cap N \cap K_1^{\alpha} = \emptyset \text{ for exactly one } \alpha = \pm 1; \\ K_2^{-\beta} \cap N \cap K_1^{\alpha} \neq \emptyset \text{ for } \alpha = 1, -1. \end{cases}$$

Chapter II

Topologies on $\overline{\mathcal{R}}$

2.1. Introduction. In this section our goal is to introduce three topologies $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ on the set $\overline{\mathcal{R}}$, each finer than its predecessor. We shall do this by introducing a neighbourhood filter at each polynomial $K \in \overline{\mathcal{R}}$. The neighbourhood system in $\mathcal{J}_2$ will enable us to distinguish between a line associated with an odd multiplicity and a line associated with an even multiplicity; cf. 1.5.2. The topology $\mathcal{J}_3$ will allow us to distinguish between a multiple line [ray] and a single line [ray].

2.2. The space ($\overline{\mathcal{R}}$, $\mathcal{J}_1$).

2.2.1. A base for a neighbourhood filter of a regular polynomial. Let $K_0 \varepsilon \mathcal{R}$. Let S_1 and S_2 be two finite subsets of the plane separated by K_0. Define

(2.2.1) $N(K_0) = N(K_0;S_1,S_2) = \{K \varepsilon \mathcal{R} \mid S_1$ and S_2 are separated by $K\}$.

Let (K_0^1, K_0^{-1}) be an orientation of K_0. Then each $K \varepsilon N(K_0)$ can be oriented such that if

$$S_1 \varepsilon K_0^{\alpha} \text{ and } S_2 \varepsilon K_0^{-\alpha},$$

then

$$S_1 \varepsilon K^{\alpha} \text{ and } S_2 \varepsilon K^{-\alpha};\ \alpha \varepsilon \{1, -1\}.$$

Put

$$U_{K_0} = \{N(K_0;\ S_1,\ S_2)\},$$

where S_1 and S_2 run over all pairs of finite subsets of the plane which are separated by K_0.

It can be easily verified that U_{K_0} is a filter base.

2.2.2. A base for a neighbourhood filter of a degenerate polynomial. Let $K_0 \varepsilon \overline{\mathcal{K}} \setminus \mathcal{K}$ be the limit of a pointwise convergent sequence $\{K_i\}$ of polynomials of $\mathcal{K}$. Suppose that K_0 has k distinct components, say $A_1, \ldots, A_k$; $1 \leq k \leq n$. Thus each A_λ is a ray or a line parallel to $\mathcal{Y}$.

Let $\{\mathcal{M}_{\lambda j}\}$ be a finite set of closed line segments $\mathcal{M}_{\lambda j}$ each of which meets A_λ, does not pass through the vertex of A_λ if A_λ is a ray and is not parallel to $\mathcal{Y}$; $\lambda = 1, \ldots, k$. Also let $\mathcal{M}_{\lambda j} \cap \mathcal{M}_{\lambda \ell} = \emptyset$ if $j \neq \ell$. Let $\{\mathcal{N}_h\}$ be a finite set of closed line segments $\mathcal{N}_h \nparallel \mathcal{Y}$, none of which meets any A_λ; $\lambda = 1, \ldots, k$. Define

(2.2.2) $\quad N(K_0) = N(K_0; \{\mathcal{M}_{\lambda j}\}, \{\mathcal{N}_h\})$

be the set of all those polynomials of $\overline{\mathcal{R}}$ which meet each $\mathcal{M}_{\lambda j}$ and do not meet any of the $\mathcal{N}_h$.

Let U_{K_0} denote the family of all such $N(K_0)$. Then it can be easily shown that U_{K_0} is a filter base.

<u>2.2.3</u>. For each $K \varepsilon \overline{\mathcal{R}}$ let $\mathcal{F}_K$ be the filter generated by U_K; cf. <u>2.2.1</u> and <u>2.2.2</u>. Then it can be easily verified that for all $V \varepsilon \mathcal{F}_K$ there is a $W \varepsilon \mathcal{F}_K$ such that $W \subset V$ and $V \varepsilon \mathcal{F}_{K'}$ for each $K' \varepsilon W$. Hence there is a topology $\mathcal{J}_1$ on $\overline{\mathcal{R}}$ such that $\mathcal{F}_K$, $K \varepsilon \overline{\mathcal{R}}$, is precisely the neighbourhood system of K with respect to the topology $\mathcal{J}_1$; cf. [6].

<u>2.2.4</u>. The space $(\overline{\mathcal{R}}, \mathcal{J}_1)$ has the following property.

$(\overline{\mathcal{R}}, \mathcal{J}_1)$ <u>satisfies the first axiom of countability</u>. We can verify this by determining the sets and segments involved in the U_K's by a finite collection of rational numbers.

2.2.5. A sequence $\{K_i\}$ of polynomials of $\overline{\mathcal{K}}$ is globally convergent or $\mathcal{J}_1$-convergent to a polynomial K in $\overline{\mathcal{K}}$ if and only if every neighbourhood of K contains all but finitely many K_i.

2.2.6. The following results are standard; cf. [6].

(i) If K is an accumulation polynomial of a sequence $\{K_i\}$ of polynomials in $\overline{\mathcal{K}}$, then there is a subsequence of $\{K_i\}$ converging to K.

(ii) $(\overline{\mathcal{K}}, \mathcal{J}_1)$ is a Hausdorff space.

2.3. Equivalence of pointwise and global or $\mathcal{J}_1$-convergence.

2.3.1. Let a sequence $\{K_i\}$ of polynomials of $\overline{\mathcal{K}}$ be pointwise convergent to K_0. Then $\{K_i\}$ is globally convergent to K_0.

Proof. (i) Let K_0 be regular. Then it has a normalized equation of the form

$$y = b_0 + b_1x + \dots + b_nx^n.$$

Since K_0 is regular, K_i are regular for all large i;

cf. Remark 1.3.5. If the equation of K_i is of the form

$$y = a_{i0} + a_{i1}x + \dots + a_{in}x^n,$$

then

$$\lim_{i \to \infty} a_{ij} = b_j;\ j = 0, 1, \dots, n;$$

cf. 1.3.3, 1.3.4.

Let $Q \notin K_0$ and let (x_0, y_0) be the coordinates of Q. Then

$$b_0 + b_1x_0 + \dots + b_nx_0^n - y_0 \neq 0.$$

Suppose, for instance, that

$$b_0 + b_1x_0 + \dots + b_nx_0^n - y_0 > 0.$$

Then, for all sufficiently large i,

$$a_{i0} + a_{i1}x_0 + \dots + a_{in}x_0^n - y_0 > 0.$$

From this it follows that if two points P and Q are separated by K_0, then they are also separated by

K_i for all large i. Hence $\{K_i\}$ is globally convergent to K_0.

(ii) Let $K_0 \varepsilon \overline{\mathcal{K}} \setminus \tilde{\mathcal{K}}$. Let $N(K_0; \{\mathfrak{M}_{\lambda j}\}, \{\mathfrak{N}_h\})$ be a neighbourhood of K_0; cf. 2.2.2.

The curve K_i cannot meet the closed segment $\mathfrak{N}_h$ for all large i, otherwise $\{K_i\}$ would have a limit point on $\mathfrak{N}_h$.

Let $P = \mathfrak{M}_{\lambda j} \cap A_\lambda$. Let N be a neighbourhood of P in G such that $\mathfrak{M}_{\lambda j}$ decomposes N into two disjoint regions. Choose Q_1 and Q_2 in $A_\lambda \cap N$ on opposite sides of $\mathfrak{M}_{\lambda j}$. Then there exist points $Q_{1\lambda}$ and $Q_{2\lambda}$ in $K_i \cap N$ lying in opposite sides of $\mathfrak{M}_{\lambda j}$. Hence K_i meets $\mathfrak{M}_{\lambda j} \cap N$.

2.3.2. <u>A sequence $\{K_i\}$ of polynomials of $\overline{\mathcal{K}}$ which is globally convergent to a polynomial $K_0 \varepsilon \overline{\mathcal{K}}$ is also pointwise convergent to K_0.</u>

<u>Proof</u>. (i) Let $K_0 \varepsilon \tilde{\mathcal{K}}$; $P \varepsilon K_0$. Let N(P) be a neighbourhood of P in G. Choose a neighbourhood $N(K_0) = N(K_0; S_1, S_2)$ of K_0 in $\tilde{\mathcal{K}}$ such that

$$N(P) \cap S_1 \neq \emptyset \text{ and } N(P) \cap S_2 \neq \emptyset.$$

Then all $K_i \varepsilon N(K_0)$ meet $N(P)$. Hence P is a limit point of the sequence $\{K_i\}$.

Next, suppose that $Q \notin K_0$. Then there is a neighbourhood $N(Q)$ of Q in G such that

$$N(Q) \cap K_0 = \emptyset.$$

Now the points of K_0 satisfy a normalized equation of the form

$$a_0 + a_1 x + \ldots + a_n x^n - y = 0. \tag{2.3.3}$$

Since $Q \notin K_0$, the coordinates of Q say, (x_0, y_0) do not satisfy the equation (2.3.3). Thus, say,

$$a_0 + a_1 x_0 + \ldots + a_n x_0^n - y_0 = \delta > 0.$$

Hence there is a $\delta_1 > 0$ such that

$$a_0 + a_1 x_1 + \ldots + a_n x_1^n - y_1 > \delta_1$$

for $(x_1, y_1) = Q_1 \varepsilon N(Q)$ if $N(Q)$ is sufficiently small.

Choose points $P_0, \ldots, P_n$ on K_0 and neighbourhoods $N(P_0), \ldots, N(P_n)$. Choose points R_j and T_j

in $N(P_j)$ such that R_j and T_j are separated by K_0 and all the points R_j lie on the same side of K_0; $j = 0, \ldots, n$.

Put

$$S_1 = \bigcup_j R_j, \ S_2 = \bigcup_j T_j.$$

Then $K_i \varepsilon N \ (K_0; S_1, S_2)$ for all large i. Hence K_o will meet each $N(P_j)$.

By choosing $N(P_j)$ sufficiently small we can ensure that the coefficients

$$a_{i0}, \ \ldots, \ a_{in}$$

in the normalized equation for K_i will be close to

$$a_0, \ \ldots, \ a_n$$

respectively. Hence

$$a_{i0} + a_{i1}x_1 + \ldots + a_{in}x_1^n - y_1 > 0.$$

Thus the points of N(Q) do not lie on K_i for sufficiently large i. Hence Q is not an accumulation point of $\{K_i\}$. Thus any accumulation point of $\{K_i\}$ belongs to K_0 and therefore is a limit point.

(ii) Let $K_0 \varepsilon \overline{\mathcal{K}} \setminus \mathcal{K}$. Let $P \varepsilon K_0$, P not a vertex of a ray of K_0. Let N(P) be a neighbourhood of P. Consider a neighbourhood $N(K_0; \{ \mathfrak{M}_{\lambda j}\}, \{ \mathfrak{N}_h\})$ of K_0 such that one of the $\mathfrak{M}_{\lambda j}$ passes through P and is contained in N(P). Since $\{K_i\}$ is globally convergent to K, K_i meets this $\mathfrak{M}_{\lambda j}$ and therefore N(P) for all large i. Hence P is a limit point.

Now, let K be the set of all accumulation points of $\{K_i\}$. Then K is the union of limit sets of all convergent subsequences of $\{K_i\}$; cf. [7]. Also any convergent subsequence of $\{K_i\}$ converges pointwise to a degenerate polynomial. For if some subsequence of $\{K_i\}$ converges pointwise to a regular polynomial K', then by 2.3.1, it converges globally to K'. Hence a component of K is a line parallel to $\mathcal{Y}$ or part of such a line; cf. 1.3.4. We wish to show that $K = K_0$.

Let $Q \varepsilon K \setminus K_0$. Then there is a neighbourhood N(Q) of Q such that $N(Q) \cap K_0 = \emptyset$. Let $N(K_0; \{ \mathfrak{M}_{\lambda j}\}, \{ \mathfrak{N}_h\})$ be a neighbourhood of K_0 such that one of the $\mathfrak{N}_h$ is contained in N(Q). Then K_i does not meet this $\mathfrak{N}_h$ for all large i. Hence K does not have points on both the sides of $\mathfrak{N}_h$ in N(Q). Therefore Q cannot be an interior point of the component of K through Q. Thus all the interior points of the components of K belong to K_0, hence are limit points of $\{K_i\}$.

Since K and K_0 do not have isolated points (cf. Remark 1.3.1), end-points of components of K and K_0 are also limit points of $\{K_i\}$,

Thus $\{K_i\}$ converges to K_0 pointwise.

2.4. The space $(\overline{\mathfrak{K}}, \mathfrak{T}_2)$.

2.4.1. Let K_0 be the limit of an orientably pointwise convergent sequence $\{K_i\}$ of polynomials. Let the component A_λ of K_0 be assigned the multiplicity $m_\lambda = m(A_\lambda) \equiv 0$ or 1 (mod 2); cf. 1.5.2.

If K_0 is degenerate we replace (2.2.2) by the set

$$N(K_0) = N(K_0; \{\mathfrak{M}_{\lambda j}\}, m_\lambda, \{\mathfrak{N}_h\}) \tag{2.4.1}$$

which consists of K_0 and those polynomials of $\overline{\mathfrak{K}}$ which meet each $\mathfrak{M}_{\lambda j}$ with a multiplicity $\equiv m_\lambda$ (mod 2) and which do not meet any of the $\mathfrak{N}_h$.

If K_0 is regular, $N(K_0)$ is still defined by (2.2.1).

Let $\mathfrak{T}_2$ be the topology with respect to the neighbourhood basis in (2.4.1). Then the statements in sections 2.2.3 up to 2.2.6 still hold if $\mathfrak{T}_1$

is replaced by $\mathcal{T}_2$.

Remark. Clearly $\mathcal{T}_2$-convergence implies $\mathcal{T}_1$-convergence. Hence $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

2.4.2. The proof of the following statement follows directly from 2.3 and the definitions of $\mathcal{T}_2$-convergence and orientably pointwise convergence.

A sequence $\{K_i\}$ of polynomials of $\overline{\mathcal{R}}$ is $\mathcal{T}_2$-convergent to K_0 if and only if $\{K_i\}$ is orientably pointwise convergent to K_0.

2.5. The space ($\overline{\mathcal{R}}$, $\mathcal{T}_3$).

2.5.1. Let K_0 be the limit of a multiplicity pointwise convergent sequence of polynomials of $\overline{\mathcal{R}}$; cf. 1.7. Thus any point Q of K_0 has a multiplicity m(Q). If Q_1 and Q_2 are points of the same component C of K_0, then $m(Q_1) \equiv m(Q_2) \pmod 2$; cf. 1.7.2, Theorem 1.

If K_0 is degenerate, then we define a set

$$N(K_0) = N(K_0; \{P_{\lambda j}\}, m_{\lambda j}, \{ \mathcal{M}_h\}); \; m_{\lambda j} = m(P_{\lambda j}), \tag{2.5.1}$$

which consists of K_0 and all those polynomials of $\overline{\mathcal{R}}$ which meet a suitable closed line segment $\mathcal{M}_{\lambda j}$

through $P_{\lambda j}$, $\mathfrak{M}_{\lambda j} \nVdash \mathfrak{Y}$, with the multiplicity $m_{\lambda j}$ and do not meet any of the closed line segments $\mathfrak{M}_h$; cf. 2.2.2.

If K_0 is regular, then $N(K_0)$ is still defined by (2.2.1).

Again the family of all such $N(K_0)$ form a filter base and induces a topology $\mathcal{T}_3$ on $\overline{\mathfrak{K}}$ such that the statements in sections 2.2.3 to 2.2.6 still hold if we replace $\mathcal{T}_1$ by $\mathcal{T}_3$.

Remark. Clearly $\mathcal{T}_3$-convergence implies $\mathcal{T}_2$-convergence, i.e., $\mathcal{T}_3$ is finer than $\mathcal{T}_2$.

2.5.2. The following statement can be easily verified with the help of 1.7 and 2.4.2.

A sequence $\{K_i\}$ of polynomials of $\overline{\mathfrak{K}}$ is $\mathcal{T}_3$-convergent to K_0 if and only if it is multiplicity pointwise convergent to K_0.

2.5.3. It may occur to the reader that the neighbourhood system which has been introduced in this chapter could be replaced by the topology defined by regarding the polynomials

$$ay = a_0 + a_1 x + \ldots + a_n x^n$$

as points of a projective (n+1)-space:

$$(a, a_0, \ldots, a_n) \neq (0, 1, 0, \ldots, 0).$$

This correspondence however, does not take care of the double rays, for instance. In particular, as a tends to zero the points (a, 0, 0, 1, 0, ..., 0) in the projective (n+1)-space converge to the unique point (0, 0, 0, 1, 0, ... 0) which we might normally associate with the double line $x^2 = 0$. On the other hand, if a tends to zero, $a > 0$, then the parabolas $ay = x^2$ converge to the double ray $x^2 = 0$, $y \geq 0$, but if a tends to zero, $a < 0$, they converge to the opposite double ray $x^2 = 0$, $y \leq 0$.

2.6. Two lemmas. Let $K_1 \varepsilon \overline{\mathcal{K}}$ be an orientable polynomial and K_2 be a regular polynomial. Then the following two lemmas can be easily verified with the help of the neighbourhood system described earlier.

2.6.1. *Let K_1 intersect K_2 at a point P; cf. 1.8.1. If K_1' and K_2' are sufficiently close to K_1 and K_2 respectively, then K_1' intersects K_2' at a point P' close to P.*

2.6.2. *Let K_1 intersect [support] K_2 at a point P. Then any K sufficiently close to K_1 meets $K_2 \cap N$*

<u>with an odd multiplicity</u> [<u>with an even multiplicity or does not meet at all</u>] <u>for a sufficiently small neighbourhood</u> N <u>of</u> P.

<u>2.7</u>. <u>Continuous orientation</u>. Let $\mathfrak{A} \subseteq (\bar{\mathcal{K}}, \mathcal{J}_2)$ be a family of polynomials with no isolated members. We call $\mathfrak{A}$ <u>continuously oriented at</u> $K_0 \varepsilon \mathfrak{A}$ if for every $P \varepsilon K_0^{\alpha}$, $\alpha = 1$ or -1, we have $P \varepsilon K^{\alpha}$ for all $K \varepsilon \mathfrak{A}$ sufficiently close to K_0. The family $\mathfrak{A}$ is <u>continuously oriented</u> if $\mathfrak{A}$ is continuously oriented at every $K \varepsilon \mathfrak{A}$.

Thus the oriented family $N(K_0)$ defined in (2.2.1), (2.4.1) and (2.5.1) is continuously oriented at K_0.

CHAPTER III

Some families of polynomials of $\mathcal{K}$.

3.1. The family $\mathcal{K}_1(P)$.

3.1.1. Let $\mathcal{K}_1(P)$ denote the family of all the polynomials of $\mathcal{K}$ through a fixed point P.

The $\mathcal{J}_1$-closure of the family $\mathcal{K}_1(P)$, denoted by $\overline{\mathcal{K}}_1(P)$, is obtained by adding to $\mathcal{K}_1(P)$ all its limit polynomials. Clearly $\overline{\mathcal{K}}_1(P)$ is $\mathcal{J}_1$-closed. We note that P is a limit point of every sequence of $\overline{\mathcal{K}}_1(P)$; cf. 1.2.

Now, since $\overline{\mathcal{K}}_1(P)$ is closed, by the Generalized Bolzano-Weierstrass theorem it can be seen that every sequence of polynomials of $\overline{\mathcal{K}}_1(P)$ contains a convergent subsequence whose limit belongs to $\overline{\mathcal{K}}_1(P)$; cf. [7]. Hence $\overline{\mathcal{K}}_1(P)$ is a countably compact set of ($\overline{\mathcal{K}}$, $\mathcal{J}_1$).

3.1.2. Let P, Q_1, ..., Q_m be mutually distinct points. If no two of them lie on the same line parallel to $\mathcal{Y}$, then there is a unique regular polynomial of degree $\leq$ m of $\mathcal{K}_1(P)$ through them. It will be denoted by $K(P, Q_1, \ldots, Q_m)$.

If a degenerate polynomial K is the limit of a sequence of m-regular polynomials, then K is called a <u>degenerate polynomial of degree</u> m.

If exactly two of the m + 1 distinct points P, Q_1, ..., Q_m lie on the same line parallel to $\mathcal{Y}$, say $\mathcal{L}(Q_{m-1}, Q_m) \parallel \mathcal{Y}$, then there is exactly one degenerate polynomial of $\overline{\mathcal{R}}_1(P)$ of degree m through them, namely

$$\mathcal{L}(P) \cup \mathcal{L}(Q_1) \cup \ldots\ldots \cup \mathcal{L}(Q_{m-1}).$$

If the set $\{\mathcal{L}(P), \mathcal{L}(Q_1), \ldots\ldots, \mathcal{L}(Q_m)\}$ consists of fewer than m mutually distinct lines, then there are infinitely many degenerate polynomials of degree m of $\overline{\mathcal{R}}_1(P)$.

<u>3.2</u>. <u>The family</u> $\Psi_h(P)$. Put

$$\Psi_0 = \mathcal{R} \text{ and } \Psi_1 = \mathcal{R}_1(P).$$

Then for any polynomial $K_0 \varepsilon \Psi_1$ we define

$$\Psi_h = \Psi_h(K_0);\ h = 2, \ldots, n,$$

to be a family of Ψ_1 which consists of K_0 and all those regular polynomials which have at least h point contact with K_0 at P. Thus a polynomial $K \varepsilon \Psi_h(K_0)$ of degree $\leq m$ is determined by $m - h + 1$ of its points all of which are distinct and different from P, say $Q_1, \ldots, Q_{m-h+1}$. Such a polynomial will be denoted by $K = K(\Psi_h; Q_1, \ldots, Q_{m-h+1})$.

3.3. The family $\Phi_h(P)$.

3.3.1. Let K_0 be a polynomial of $\mathcal{K}_1(P)$. Let $Q_1, \ldots, Q_{m-h}$ be $m - h$ points such that the lines $\mathcal{L}(P)$, $\mathcal{L}(Q_1)$, ..., $\mathcal{L}(Q_{m-h})$ are mutually distinct. Let

$$\Phi_h = \Phi_h(P, Q_1, \ldots, Q_{m-h})$$

be the family of those polynomials of $\Psi_h(K_0)$ which pass through the fixed points $P, Q_1, \ldots, Q_{m-h}$ and have degree $\leq m$. The family Φ_h has the property that through each point of $G \setminus \{P, Q_1, \ldots, Q_{m-h}\}$ there is at most one member of Φ_h. Such a family will be called a one-family.

Let P be the origin and let the equation of $\mathcal{L}(Q_i)$ be $x - b_i = 0$, $i = 1, \ldots, m - h$. Let K_1 be the unique polynomial of $\Psi_h(K_0)$ of degree $\leq m - 1$ through the points $Q_1, \ldots, Q_{m-h}$. Suppose K_1 has the equation

$$g_1(x) = a_2x^2 + \ldots + a_{m-1}x^{m-1},$$

thus, the x-axis is taken along the tangent of K_1 at P.

Define

$$g_2(x) = x^h(x - b_1) \ldots (x - b_{m-h}).$$

Then the family Φ_h is given by

$$f(x, \lambda) = g_1(x) + \lambda g_2(x); \; -\infty < \lambda < \infty.$$

<u>3.3.2</u>. Let $y = mx + c$ be a line $\mathcal{L} \nparallel \mathcal{Y}$. Then we shall show that for sufficiently large λ the equation

$$(3.3.2) \qquad f(x, \lambda) - mx - c = 0$$

does not have more than one root in a sufficiently small neighbourhood of any of the b_i's.

Suppose that (3.3.2) has more than one root near b_i. Then the equation

$$f'(x, \lambda) - m = 0$$

has at least one such root. It can be easily seen, however, that for any $m \neq 0$ we can choose a λ_0 such that

$$f'(x, \lambda) - m \neq 0 \text{ for all } |\lambda| > |\lambda_0|.$$

Thus the equation (3.3.2) has at most one root close to b_i, $i = 1, \ldots, m - h$ for sufficiently large λ.

3.3.3. Next, let us consider the roots $x > 0$ of the equation (3.3.2) in a neighbourhood of the point $R(0, c)$, $c \neq 0$, say $c > 0$. Let N be a sufficiently small neighbourhood of R. If $(x, y) \varepsilon N \cap \mathcal{L}$, then x is small and $f(x, \lambda)$ is close to c.

Now for x small $g_1(x)$ is small and hence $f(x, \lambda)$ is close to $\lambda g_2(x)$ which is close to $\lambda x^h(-b_1)\ldots(-b_{m-h})$. Hence $\lambda x^h(-b_1)\ldots(-b_{m-h})$ is close to c. Therefore the dominating term of $f'(x, \lambda)$; i.e., of $\lambda g_2'(x)$, which is $\lambda h x^{h-1}(-b_1)\ldots(-b_{m-h})$ is large for sufficiently small x. Hence

$$f'(x, \lambda) - m \neq 0$$

for $(x, y) \varepsilon N \cap \mathcal{L}$, $x > 0$. Thus the equation (3.3.2) has at most one root $x > 0$ with $(x, y) \varepsilon N \cap \mathcal{L}$.

Similarly, it has at most one root $x < 0$, $(x, y) \varepsilon N \cap \mathcal{L}$.

Remark 1. If h is odd and x is small, then $\lambda x^h(-b_1) \ldots (-b_{m-h})$ cannot be close to c for both $x > 0$ and $x < 0$. Hence for an odd h the equation (3.3.2) has at most one root x with $(x, y) \varepsilon N$. However, if h is even and $\lambda x^h(-b_1) \ldots (-b_{m-h})$ is close to c for $x > 0$ then it is so for $x < 0$.

By applying Descarte's rule of signs we obtain the following.

Remark 2. If for a sufficiently large $\lambda_0 > 0$ [sufficiently small $\lambda_0 < 0$] the equation (3.3.2) has one root $x > 0$, then it will have exactly one such root for all $\lambda > \lambda_0 [\lambda < \lambda_0]$.

3.3.4. Let $h \geq 2$. First we wish to show that the multiplicity of P with respect to the lines through P different from the tangent $\mathcal{T}$ of K_1 is at most three.

Let $\mathcal{L}$: $y = mx$, $m \neq 0$ be any line through P. We consider the non-zero roots of the equation

$$(1) \qquad f(x, \lambda) - mx = 0.$$

Division by x yields

$$\sum_{i=2}^{m-1} a_i x^{i-1} + \lambda x^{h-1}(x - b_1) \dots (x - b_{m-h}) - m = 0.$$

Now for $x > 0$ [$x < 0$], x close to zero, $\lambda x^{h-1}(x-b_1)\dots(x-b_{m-h})$ is close to $\lambda x^{h-1}(-b_1) \dots (-b_{m-h})$ and $m - \sum_{i=2}^{m-1} a_i x^{i-1}$ is close to m. Hence for x close to zero $\lambda x^{h-1}(-b_1)\dots(-b_{m-h})$ is close to m. Therefore $f''(x, \lambda)$ which is close to $2a_2 + \lambda h(h - 1)x^{h-2}(-b_1) \dots (-b_{m-h})$ when x is close to zero (and thus λ is sufficiently large), is close to $2a_2 + mh(h-1)/x$ i.e., for sufficiently large λ, $f''(x, \lambda) \neq 0$ for $x \neq 0$, x close to zero. Hence $f'(x, \lambda) - m = 0$ has at most one positive root [at most one negative root] near zero for sufficiently large λ. Thus the equation (1) has at most one positive (negative) root near zero for sufficiently large λ. Thus by Remark 2 of 3.3.3, we conclude that the point P has multiplicity at most three with respect to lines different from $\mathcal{T}$.

Next, we wish to show that K_λ meets $\mathcal{T}$ outside P at most once on each side of the origin in a sufficiently small neighbourhood of P.

We may assume that $a_i \neq 0$ for some $i < h$, where $g_1(x) = \sum_{i=2}^{m-1} a_i x^i$. Let a_r be the first non-zero coefficient in $g_1(x)$. Hence the non-zero roots of $f(x,\lambda)$ are precisely the roots of the equation

$$a_r + a_{r+1}x + \ldots + a_{m-1}x^{m-r-1} + \lambda x^{h-r}(x - h) \ldots (x - b_{m-h}) = 0.$$

Now for x sufficientlyclose to zero, $x \neq 0, \lambda x^{h-r}(x - b_1) \ldots (x - b_{m-h})$ is close to $\lambda x^{h-r}(-b_1) \ldots (-b_{m-h})$ and $a_r + a_{r+1}x + \ldots + a_{m-1}x^{m-r-1}$ is close to a_r and hence $\lambda x^{h-r}(-b_1) \ldots (-b_{m-h})$ is close to a_r. Therefore for x close to zero (and thus for sufficiently large λ)

$|f^{(r+1)}(x, \lambda)|$ which is close to $|(r+1)!a_{r+1} + \lambda h(h-1) \ldots (h-r)x^{h-r-1}(-b_1) \ldots (-b_{m-h})|$ is large.

Now, suppose that $f(x, \lambda) = 0$ has two positive roots near zero, say x_1 and x_2. Then $f'(x, \lambda)$ has a positive root x_2' between x_1 and x_2 and $f'(x, \lambda)$ has also a positive root x_1' between 0 and x_1. But $f''(0, \lambda) = 0$. Hence $f''(x, \lambda)$ has a positive root x_1'' between 0 and x_1'. Since

$$f^{(r-1)}(0, \lambda) = \ldots. = f'(0, \lambda) = f(0, \lambda) = 0,$$

we can continue in this fashion and deduce that $f^{(r)}(x,\lambda) = 0$ has two positive roots close to 0. Hence $f^{(r+1)}(x, \lambda) = 0$ has a positive root close to 0; a contradiction.

Again by 3.3.3, Remark 2, the point P has multiplicity.

3.3.5. The discussion in 3.3.2 to 3.3.4 shows that if K is a degenerate polynomial of $\bar{\Phi}_h$, then each point Q of K has a multiplicity m(Q) and m(Q) = 1 if $Q \varepsilon \mathcal{L}(Q_i)$, $1 \leq i \leq m - h$ and m(Q) = 1 or 2 for $Q \varepsilon C(P)$, $Q \neq P$ according as h is odd or even.

Also, let $\{K_\lambda\}$ be $\mathcal{J}_1$-convergent to K. Let $R = (x_0, y_0)$, $R \notin K$. Then $(x_0, y_0) \notin K_\lambda$ for sufficiently large λ, i.e., $f(x_0, \lambda) \neq 0$ for sufficiently large λ.

Now if $f(x_0, \lambda_0) > 0$ $[f(x_0, \lambda_0) < 0]$ for one sufficiently large λ_0, i.e., $|\lambda_0| > \left|\frac{g_1(x_0)}{g_2(x_0)}\right|$ then $f(x_0, \lambda) > 0$ $[f(x_0, \lambda) < 0]$ for all $|\lambda| \geq |\lambda_0|$. Hence $\{K_\lambda\}$ is $\mathcal{J}_2$-convergent to K.

Thus it follows that if $\{K_\lambda\}$ is $\mathcal{J}_1$-convergent to K, then it is in fact $\mathcal{J}_3$-convergent to K. Therefore the $\mathcal{J}_1$-closure $\overline{\Phi}_h$ of Φ_h is actually the $\mathcal{J}_3$-closure. Hence all the three topologies $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ coincide on $\overline{\Phi}_h$, therefore from now on we can simply say that a sequence of $\overline{\Phi}_h$ is convergent without referring to a topology $\mathcal{J}_1$, $\mathcal{J}_2$ or $\mathcal{J}_3$.

Also, we note that $\overline{\Phi}_h$ is a closed subset of $\overline{\mathcal{K}}_1(p)$ and $\overline{\mathcal{K}}_1(p)$ is $\mathcal{J}_1$-compact, therefore $\overline{\Phi}_h$ is $\mathcal{J}_1$-compact, hence $\overline{\Phi}_h$ is compact; cf. [7].

3.3.6. Let K be a degenerate polynomial of $\overline{\Phi}_h$. Then

$$K = C(P) \cup \mathcal{L}(Q_1) \cup \cdots \cup \mathcal{L}(Q_{m-h}),$$

where C(P) is the single line $\mathcal{L}(P)$ or a double ray with vertex P according as h is odd or even and $\mathcal{L}(Q_i)$ are single lines, $1 \leq i \leq m - h$.

Proof. Let $\{K_i\}$ be a sequence of polynomials of Φ_h which converge to a degenerate polynomial K.

First we show that if h is even, then the component C of K through P is not a line or a ray of which P is an interior point. Suppose that C = $\mathcal{L}$ (P) or such a ray. Let K_1 be the unique polynomial of $\Psi_h(K_0)$ of degree $\leq$ m - 1 through the points Q_1, ..., Q_{m-h}. Since P is an interior point of C, we have, for any neighbourhood N of P in G,

$$C \cap K_1^{\pm 1} \cap N \neq \emptyset.$$

Therefore, for sufficiently large i,

$$K_i \cap K_1^{\pm 1} \cap N \neq \emptyset.$$

Hence K_1 intersects K_i at P. But K_1 and K_i belong to $\Psi_h(K_0)$ and h is even, Thus we obtain a contradiction; cf. <u>1.3.7</u>. Hence C is a ray with the vertex P and therefore a double ray; cf. <u>3.3.5</u>.

Now we shall show that if h is odd, then the component C of K through P is not a ray. Suppose that C is a ray. Define K_1 as above. Then K supports K_1 at P; cf. <u>1.8.1</u>. Hence for sufficiently large i, K_i meets K_1 an even number of times in a small neighbourhood N of P in G; cf. <u>2.6.2</u>. Since h is odd, K_i and K_1 meet at least once more in N. Thus for sufficiently large i,

K_i and K_1 have at least m - h + 1 points in common outside P, hence they coincide. Therefore $K = K_1$; which is impossible. Thus if h is odd, then the component through P is the line $\mathcal{L}(P)$. By the Remark 1 of 3.3.3, $\mathcal{L}(P)$ is a single line. Again by 3.3.2 the components of K through $Q_1, \ldots, Q_{m-h}$ are single lines $\mathcal{L}(Q_1), \ldots, \mathcal{L}(Q_{m-h})$.

3.5.7. Let N be a small neighbourhood of P in G.

(i) *If h is even*, then K_1 decomposes Φ_h into two disjoint subfamilies, say $\Phi_{h,1}$ and $\Phi_{h,-1}$ such that any member of $\Phi_{h,\alpha}$ passes through $N \cap K_1^{\alpha}$, $\alpha = 1, -1$. Furthermore, $\Phi_{h,\alpha}$ is bounded by K_1 and the degenerate polynomial through $P, Q_1, \ldots, Q_{m-h}$ whose component through P is a double ray passing through $N \cap K_1^{\alpha}$. *Thus $\overline{\Phi}_h$ is homeomorphic to a real closed interval and is bounded by two degenerate polynomials.*

(ii) *If h is odd*, then K_1 decomposes Φ_h into two disjoint families $\Phi_{h,1}$ and $\Phi_{h,-1}$ such that any member of $\Phi_{h,\alpha}$ passes through

$$(N \cap \mathcal{L}(P)^{\alpha} \cap K_1^{\alpha}) \cup (N \cap \mathcal{L}(P)^{-\alpha} \cap K_1^{-\alpha}),$$

for suitable orientations of $\mathcal{L}(P)$ and K_1.

Here each $\Phi_{h,\alpha}$ is bounded by K_1 and

$$K = \mathcal{L}(P) \cup \mathcal{L}(Q_1) \cup \cdots \cup \mathcal{L}(Q_{m-h}).$$

Thus $\overline{\Phi}_h$ is homeomorphic to a circle.

CHAPTER IV

Polynomial Differentiability of an Arc

4.1. Arcs.

4.1.1. An arc A is defined as the one-to-one continuous image in the real affine plane G of a real parameter interval. Thus if a sequence of points of the parameter interval converges to a point p, the corresponding sequence of image points is defined to be convergent to the image of p. The same letters, p, t, ... denote the points of the parameter interval and their images on A. The end-points (interior points) of A are the respective images of the end-points (interior points) of the parameter interval.

A neighbourhood of p on A is the image of a neighbourhood of the parameter p on the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) one-sided neighbourhoods.

4.1.2. A polynomial $K \varepsilon \mathcal{K}$ meets A at p at least r-times if there exists a sequence $\{K_i\}$ of polynomials of $\mathcal{K}$ converging to K such that each K_i and A have r mutually distinct points in common which converge to p.

A polynomial $K\varepsilon\, \mathfrak{K}$ meets A at p exactly r-times if K meets A at p at least r-times but not at least (r + 1)-times.

4.1.3. The $\mathfrak{K}$ -order. If no polynomial of $\mathfrak{K}$ meets an arc A in more than a finite number of points, then A has finite polynomial order or finite $\mathfrak{K}$ -order. If the least upper bound of these numbers is finite, it is called the $\mathfrak{K}$-order of A. If A has finite $\mathfrak{K}$ -order and for any given integer m there is always a polynomial K of $\mathfrak{K}$ which meets A in more than m points, then the $\mathfrak{K}$-order of A is unbounded.

The $\mathfrak{K}$ -order of a point p of A is the $\mathfrak{K}$ -order of a sufficiently small neighbourhood of p on A.

From now on we shall assume that the point p of the arc A has finite polynomial order.

4.1.4. Support and intersection. An orientable polynomial $K\varepsilon\, \overline{\mathfrak{K}}$ intersects [supports] the arc A at a point $p\varepsilon A$ if for every sufficiently small neighbourhood N of p in G

$$N \cap A \cap K = \{p\}$$

and neither [one] of the sets $A \cap N \cap K^{\pm 1}$ is empty; cf. 1.4.2.

In particular, if p is a point of a ray of a polynomial $K \varepsilon \overline{\mathcal{R}}$, then one of the sets $N \cap K^{\pm 1}$ is void if N is small. Hence K supports A at p if $N \cap A \cap K = \{p\}$ when N is sufficiently small.

4.2. Tangent polynomials of an arc.

4.2.1. Let p be a point on the arc A. Since A is a 1-1 continuous image of a real interval, the line K(p, s) will be uniquely determined if $s \neq p$, $s \varepsilon A$. From now on the point s will always be assumed to lie on A and be different from p.

The arc A is called once polynomially differentiable at p if the following condition is satisfied.

Condition 1. The line K(p, s) converges if s tends to p, i.e.,

$$\lim_{s \to p} K(p, s) \text{ exists.}$$

We shall call this limit $K(p^2)$ or $\mathcal{T}$. It is the ordinary tangent of A at p.

Remark. It can be easily verified that if p is an end-point of an arc A of finite order, then A satisfies Condition 1 at p; cf. [9].

4.2.2. Let $p \varepsilon A$. If $\mathcal{L}(p)$ has infinitely many points in common with A, then a line $\mathcal{L}$ sufficiently close to $\mathcal{L}(P)$, $\mathcal{L} \nparallel \mathcal{M}$, also has infinitely many points in common with A. Since p has finite $\mathfrak{K}$-order, $s \notin \mathcal{L}(p)$ for s sufficiently close to p.

Let $\mathcal{L}(P)$, $\mathcal{L}(Q_1)$, ..., $\mathcal{L}(Q_{m-1})$ be mutually distinct and for $s \varepsilon A \setminus \{p\}$ let

$$K(s) = K(p, s, Q_1, \ldots, Q_{m-1});\ 1 \leq m \leq n ,$$

be a unique polynomial of degree $\leq m$ of $\overline{\mathfrak{K}}_1(p)$. Since p has finite $\mathfrak{K}$-order, K(s) is not a regular polynomial of degree $< m$ and also from the above it is not degenerate of degree $\leq m$ for s sufficiently close to p. Thus for s sufficiently close to p, K(s) is m-regular.

In the next three sections K(s) will mean the m-regular polynomial $K(p, s, Q_1, \ldots, Q_{m-1})$. Now $K(s) \varepsilon \Phi_1(p, Q_1, \ldots, Q_{m-1}) = \Phi_1$. Since $\overline{\Phi}_1$ is compact, any sequence $\{K(s)\}$ has an accumulation polynomial in $\overline{\Phi}_1$ and if it has only one accumulation polynomial K, then it converges to K.

4.2.3. *Let K(s) and K(p, s) converge to K and $\mathcal{L}$ respectively. Here we let s range through a certain sequence of points converging to p. Then K is* *regular if and only if* $\mathcal{L} \nparallel \mathcal{W}$.

Proof. If m = 1, then clearly above statement is true. Hence let $m \geq 2$.

Suppose that $\mathcal{L} \nparallel \mathcal{W}$ and K is degenerate. Thus K consists of the single lines $\mathcal{L}(p)$, $\mathcal{L}(Q_1)$, ..., $\mathcal{L}(Q_{m-1})$; cf. 3.3.6. Hence K intersects $\mathcal{L}$ at p and at m - 1 other points, say R_1, ..., R_{m-1}. Therefore for s sufficiently close to p, K(s) will meet the line K(p, s) at p, s and m - 1 other points close to R_1, ..., R_{m-1} respectively; cf. 2.6.1. Therefore K(s) = K(p, s); a contradiction, because K(s) is m-regular, $m \geq 2$.

Next, we observe that if K is regular, then $\mathcal{L}$ will be tangent of K; cf. 1.6.1. Hence $\mathcal{L} \nparallel \mathcal{W}$.

4.2.4. *The arc* A *satisfies Condition* 1 *at* p *if and only if the unique polynomial* K(s), cf. 4.2.2, *converges as* s *tends to* p.

Proof. (i). Let A satisfy Condition 1 at p. Let K be an accumulation polynomial of the K(s).

If $\mathcal{T} \nparallel \mathcal{W}$, then K is the unique regular polynomial of degree $\leq$ m which has two point contact with $\mathcal{T}$ at p and passes through the points Q_1, ..., Q_{m-1}; cf. 4.2.3.

If $\mathcal{T} \parallel \mathcal{W}$, then K is the unique degenerate polynomial consisting of the lines $\mathcal{T}$, $\mathcal{L}(Q_1)$, ..., $\mathcal{L}(Q_{m-1})$.

(ii). Let $K = \lim_{s \to p} K(s)$. Then we shall show that A satisfies Condition 1 at p.

If K is regular, then the line K(p, s) converges to the tangent $\mathcal{T}$ of K at p; cf. 1.6.1. If K is degenerate, then the line K(p, s) converges to $\mathcal{L}(p)$, by 4.2.3.

4.2.5. Let A satisfy Condition 1 at p. The limit polynomial of the K(s) will be denoted by $K(p^2, Q_1, \ldots, Q_{m-1})$.

Let $\mathcal{K}_2(p)$ be the set of all polynomials which can be obtained as a limit of the polynomial of the type

$$K(s) = K(p, s, Q_1, \ldots, Q_{m-1})$$

for any m, $1 \leq m \leq n$.

By 4.2.3, if $K(p^2)$ is regular, i.e., if $\mathcal{T} \nparallel \mathcal{W}$, then $\mathcal{K}_2(p)$ consists of regular polynomials. Thus in this case

$$\mathcal{K}_2(p) = \Psi_2(K(p^2)); \text{ cf. } 3.2.$$

If $K(p^2) = \mathcal{L}(p)$, then $\tilde{\mathcal{K}}_2(p)$ is the set of degenerate polynomials each member K of which consists of m lines parallel to $\mathcal{Y}$ exactly one of which is $\mathcal{L}(p)$; $1 \leq m \leq n$.

By 3.3.6 each of the components of K is a single line; i.e., a line counted once. The members of $\tilde{\mathcal{K}}_2(p)$ will be called tangent polynomials of A at p.

4.2.6. Let p, $Q_1, \ldots, Q_{m-1}$ be mutually distinct points such that

$$\mathcal{L}(p),\quad \mathcal{L}(Q_1),\ \ldots,\ \mathcal{L}(Q_{m-2}) = \mathcal{L}(Q_{m-1})$$

are mutually distinct. By 4.2.2 we have, for s close to p,

$$s \notin \mathcal{L}(p) \cup \mathcal{L}(Q_1) \cup \cdots \cup \mathcal{L}(Q_{m-2}).$$

Hence there is a unique degenerate polynomial of degree m , namely

$$\begin{aligned} K(s) &= K(p, s, Q_1, \ldots, Q_{m-1}) \\ &= \mathcal{L}(p) \cup \mathcal{L}(s) \cup \mathcal{L}(Q_1) \cup \cdots \cup \mathcal{L}(Q_{m-2}) \in \overline{\tilde{\mathcal{K}}}_1(p). \end{aligned}$$

Hence any accumulation polynomial K of the K(s) as s tends to p consists of the double line through p, and m - 2 other lines, all parallel to $\mathcal{T}$.

4.2.7. Let p be an interior point of the arc A. Suppose that A satisfies Condition 1 at p. Then A satisfies the following standard lemma; cf. [10].

The lines different from $\mathcal{T}$ *through* p *either all support* A *at* p *or all of them intersect* A *at* p.

4.2.8. The non-tangent polynomials K *of* A *at* p *either all support* A *at* p *or all of them intersect* A *at* p.

Proof. If K is regular, this lemma is the special case of *4.5.1* in which h = 1.

If K is degenerate it follows from *4.2.7*.

4.2.9. Let A satisfy Condition 1 at the interior point p.

Then A *has a cusp* at p if the non-tangent polynomials of A through p support A at p. It is a *cusp of the first kind* if the tangent $\mathcal{T}$ of A at p *intersects* A at p and it is a *cusp of the second kind* if $\mathcal{T}$ *supports* A at p.

4.3. Osculating Polynomials.

4.3.1. Suppose that the arc A is once differentiable at p. Let $\mathfrak{T} = K(p^2)$ be the tangent line of A at p, $\mathfrak{T} \nparallel \mathfrak{Y}$. Then $s \notin \mathfrak{T}$ if s sufficiently close to p, cf. 4.2.2. Hence by 4.2.5, there is a unique tangent 2-regular polynomial $K(p^2, s)$ (i.e., an ordinary tangent parabola of A at p through s).

We say that the arc A *is twice differentiable* at p, if the following condition holds.

Condition 2. The polynomial $K(p^2, s)$ converges as s tends to p, i.e.,

$$\lim_{s \to p} K(p^2, s) \text{ exists.}$$

We denote this limit by $K(p^3)$ and call it the *osculating parabola of* A *at* p. If $K(p^3)$ is degenerate, then, by 3.3.6, it is a double ray with the vertex p.

4.3.2. Remark. If the osculating parabola $K(p^3)$ happens to coincide with the tangent $\mathfrak{T}$ of A at p, this is of no special significance in our theory. For a suitable non-linear transformation of the plane will map the family of the regular tangent parabolas of A at p into the same set of curves such that the new osculating parabola will be 2-regular. This will not affect the intersection and support properties of these curves at p with respect to the arc A.

4.3.3. Let $\mathcal{T} \parallel \mathcal{V}$. Since p has finite $\mathcal{K}$-order, $s \notin \mathcal{T}$ for s sufficiently close to p. By 3.1.2 and 4.2.4, the unique degenerate tangent polynomial $K(p^2, s)$ of degree 2 of A at p through s consists of the pair of distinct single lines $\mathcal{T}$ and $\mathcal{L}(s)$. As s tends to p, $K(p^2, s)$ converges to the double line on $\mathcal{V}$. Thus in this case Condition 2 is automatically satisfied. This limit polynomial will also be denoted by $K(p^3)$.

4.4. The h-osculating polynomials.

4.4.1. Suppose that the differentiability of the arc A with respect to polynomials of $\mathcal{K}$ has been defined up to the order h - 1 and A is h - 1 times differentiable at p. Thus the family $\mathcal{K}_r(p)$ of (r - 1)-osculating polynomials has been defined and exists; $1 \leq r \leq h$. In particular, $K(p^r, s)$ has been defined and exists when s is sufficiently close to p and

$$\lim_{s \to p} K(p^r, s) = K(p^{r+1}) \text{ exists; } 1 \leq r \leq h - 1.$$

Suppose that $K(p^r, s)$ is r-regular for $1 \leq r \leq h - 1$.

4.4.2. Let us assume at first that $K(p^h)$ is regular. Thus $K(p^r)$ is also regular; $1 \leq r \leq h - 1$. Define $\mathfrak{K}_h(p)$ to be the family $\Psi_h(K(p^h))$ of the (h - 1)-osculating polynomials of A at p; cf. 3.2. Then for s sufficiently close to p, there is a unique regular polynomial $K(p^h, s)$ of $\mathfrak{K}_h(p)$ of degree $\leq h$ through s. In fact, $K(p^h, s)$ is h-regular, otherwise it would coincide with $K(p^h)$, which leads to a contradiction of the assumption that p has finite polynomial order.

We say that the arc A is h *times differentiable at* p if the following condition holds.

Condition h. The h-regular polynomial $K(p^h, s)$ converges as s tends to p; i.e.,

$$\lim_{s \to p} K(p^h, s) \text{ exists.}$$

We denote this limit polynomial by $K(p^{h+1})$ and call it the h-*osculating polynomial* of A at p.

If $K(p^{h+1})$ is degenerate, then by 3.3.6, it is one of the two double rays with the vertex p [the line $\mathcal{L}(p)$] if h is even [odd].

4.4.3. We return to 4.4.1 and assume now that $K(p^{h-1})$ is regular but $K(p^h)$ is degenerate, $3 \leq h \leq n + 1$. Define the family $\mathfrak{K}_h(p)$ of $(h - 1)$-osculating polynomials of A at p to be the set of the degenerate polynomials

$$K(p^h) \cup \mathcal{L}(Q_1) \cup \cdots \quad \cup \mathcal{L}(Q_{m-h})$$

where $\mathcal{L}(p)$, $\mathcal{L}(Q_1)$, ..., $\mathcal{L}(Q_{m-h})$ are mutually distinct and $h \leq m \leq n$. In particular, the unique polynomial $K(p^h, s)$ of $\mathfrak{K}_h(p)$ consists of a double ray on $\mathcal{L}(p)$ with vertex p [the single line $\mathcal{L}(p)$] and $\mathcal{L}(s)$ if h is odd [even]; cf. 3.3.6. Hence as s tends to p, $K(p^h, s)$ converges to this double ray together with the line $\mathcal{L}(p)$ [the double line on $\mathcal{L}(p)$] if h is odd [even]. Thus Condition h is satisfied automatically in this case.

More generally, if $K(p^h)$ is degenerate but $K(p^{h-1})$ is regular, then all the Conditions h, h + 1, ..., n are satisfied automatically.

If h is odd, then $K(p^r)$, $r \geq h$, consists of a double ray on $\mathcal{L}(p)$ with the vertex p together with the line $\mathcal{L}(p)$ counted r - h times. If h is even, then $K(p^r)$ consists of the line $\mathcal{L}(p)$ counted r - h + 1 times.

4.4.4. The members of $\mathfrak{K}_{h+1}(p)$ will be called h-osculating polynomials of A at p.

As in 4.4.2, we note that if $K(p^{h+1})$ is regular, then $\mathfrak{K}_{h+1}(p)$ is the family $\Psi_{h+1}(K(p^{h+1}))$; cf. 3.2. Thus each $K \varepsilon \mathfrak{K}_{h+1}(p)$ will have at least h + 1 point contact with $K(p^{h+1})$ at p. If K is an m-regular polynomial of $\mathfrak{K}_{h+1}(p)$ and if $Q_i \varepsilon K$, $Q_i \neq p$, $Q_i \neq Q_j$, $i \neq j$, $1 \leq i \leq m - h$, $1 \leq j \leq m - h$, then K will be denoted by $K(p^{h+1}, Q_1, \ldots, Q_{m-h})$.

4.4.5. Suppose that the arc A is h - 1 times differentiable at p and suppose that $K(p^h)$ is regular. Suppose the straight lines $\mathcal{L}(p)$, $\mathcal{L}(Q_1)$, ..., $\mathcal{L}(Q_m)$ are mutually distinct; $0 \leq m \leq n - h$. Then for s sufficiently close to p, there is a unique regular polynomial

$$K(s) = K(p^h, s, Q_1, \ldots, Q_m)$$

of degree $\leq m + h$ in $\mathfrak{K}_h(p)$. Since A has finite polynomial order, K(s) is (m + h)-regular. Also $K(s) \varepsilon \Phi_h(p, Q_1, \ldots, Q_m)$ and $\overline{\Phi}_h$ is compact; cf. 3.3.5. Hence any sequence {K(s)} has an accumulation polynomial and if it has only one accumulation polynomial K, then it converges to K.

4.4.6. Let A satisfy Condition h at p. Then $K(s)$ converges to a polynomial K' such that $K' \varepsilon \Psi_{h+1}(K(p^{h+1}))$ [$K' = K(p^{h+1}) \cup \mathcal{L}(Q_1) \cup \ldots \cup \mathcal{L}(Q_m)$] if $K(p^{h+1})$ is regular [degenerate].

Proof. Let K' be an accumulation polynomial of the $K(s)$. Let $\{s_i\}$ be a subsequence of $\{s\}$ converging to p such that K' is the limit of $K(s_i)$.

(i) Let $K(p^{h+1})$ be regular. If K' is degenerate then it intersects $K(p^{h+1})$ in m points, say $R_1, \ldots, R_m$; $R_i \varepsilon \mathcal{L}(Q_i)$, $1 \leq i \leq m$. Hence $K(p^h, s_i)$ and $K(s_i)$ will intersect each other at m points close to the R_i's, for s_i sufficiently close to p; cf. 2.6.1. Thus $K(p^h, s_i)$ and $K(s_i)$ meet altogether $m + h + 1$ times, hence they coincide. Therefore $K(p^{h+1}) = K'$, which is impossible. Clearly $K' \varepsilon \Psi_{h+1}(K(p^{h+1}))$. Also K' is a unique regular polynomial of degree $\leq m + h$, hence $K(s)$ converges to K'.

(ii) Let $K(p^{h+1})$ be degenerate. First, we shall show that K' is degenerate by using induction on m. Clearly the statement is true for $m = 0$. Assume that we have already proved that if $K(p^h, s, Q_1, \ldots, Q_{m-1})$ converges then it converges to a degenerate polynomial K_0.

Let $Q_m \notin K(p^h, s, Q_1, \ldots, Q_{m-1})$ and $Q_m \notin \mathcal{L}(p) \cup \mathcal{L}(Q_1) \cup \ldots \cup \mathcal{L}(Q_{m-1})$. Let

$K(p^h, s, Q_1, \ldots, Q_{m-1}, Q_m)$ converge and suppose that it converges to a regular polynomial K. Then K_0 supports [intersects] K at p, if h is even [odd]; cf. 1.3.7. Let N be a sufficiently small neighbourhood of p in G. Then for $s' \varepsilon N$ sufficiently close to p on A, $K(p^h, s', Q_1, \ldots, Q_{m-1})$ and $K(p^h, s', Q_1, \ldots, Q_m)$ will meet each other in N with an even [odd] multiplicity, if h is even [odd]; cf. 2.6.2. Hence $K(p^h, s', Q_1, \ldots, Q_{m-1})$ and $K(p^h, s', Q_1, \ldots, Q_m)$ will meet once more in N and therefore they will coincide. Hence $Q_m \varepsilon K(p^h, s', Q_1, \ldots, Q_{m-1})$; a contradiction. This proves that an accumulation polynomial K' is degenerate if $K(p^{h+1})$ is degenerate.

Next, we shall show that

$$K' = K(p^{h+1}) \cup \mathcal{L}(Q_1) \cup \ldots \cup \mathcal{L}(Q_m).$$

If h is odd it is clearly true by 3.3.6. Now let h be even. If $\{s_{2i}\}$ and $\{s_{2i+1}\}$ are two subsequences of $\{s\}$ converging to p such that $K(s_{2i})$ and $K(s_{2i+1})$ converge to K_1 and K_0 respectively, then s_{2i} and s_{2i+1} must lie on the same side of $K(p^h, Q_1, \ldots, Q_m)$, otherwise $K(p^h, s_{2i})$ and $K(p^h, s_{2i+1})$ will converge to two different degenerate polynomials lying on opposite sides of $K(p^h, Q_1, \ldots, Q_m)$. Thus the double ray $K(p^{h+1})$ belongs to both K_1 and K_2 and hence

$$K_1 = K_2 = K(p^{h+1}) \cup \mathcal{L}(Q_1) \cup \ldots . \cup \mathcal{L}(Q_m).$$

Thus K(s) converges to its unique accumulation polynomial K'.

4.4.7. *Let* K(s) *converge to a polynomial* K *as* s *converges to* p. *Then* A *satisfies Condition* h, i.e., $K(p^h, s)$ *converges*.

Proof. Let K' be any accumulation polynomial of the $K(p^h, s)$. Let $\{s_i\}$ be a subsequence of {s} such that K' is the limit of $K(p^h, s_i)$.

If K' is degenerate, then as in 4.4.6, K is also degenerate and K' is the component of K through p. Hence K' is the only accumulation polynomial of $K(p^h, s)$. Thus $K(p^h, s)$ converges to K' which is the single line $\mathcal{L}(p)$ or a double ray with vertex p according as h is odd or even.

If K' is regular, then so is K. Also K' is a polynomial of degree $\leq$ h and has (h + 1)-point contact with K. Hence K' is unique. Thus again $K(p^h, s)$ converges to K'.

4.4.8. *Theorem 2. Let the arc* A *be* h *times differentiable at* p. *Suppose that* $K(p^h)$ *is regular*.

Then $\tilde{\mathcal{R}}_{h+1}(p)$ is one of the following subsets of $\tilde{\mathcal{R}}_h(p)$.

(a) $\tilde{\mathcal{R}}_{h+1}(p)$ is a family Ψ_{h+1}; cf. 3.2.

(b) $\tilde{\mathcal{R}}_{h+1}(p)$ consists of those degenerate polynomials of $\overline{\tilde{\mathcal{R}}}$ whose component through p is the double ray belonging to $K(p^{h+1})$ with the vertex p [the line $\mathcal{L}(p)$] if h is even [odd].

4.4.9. Remark. The following example shows that Condition h does not imply Condition h + 1.

Consider the arc A given by

$$y = x^2 + x^{h+1} \sin \frac{1}{x}.$$

Then A satisfies Condition h, but not Condition h + 1 at the point p = (0, 0).

In this example, the polynomial order of p is not finite.

4.4.10. We call the arc A $\tilde{\mathcal{R}}$-differentiable at the point p if it is n times differentiable there.

4.5. Support and intersection properties of the polynomials of $\tilde{\mathcal{R}}_h(p)$; $1 \leq h \leq n + 1$.

In this section let A be m-times differentiable at an interior point p, $1 \leq m \leq n$. From now on we put $\mathfrak{K}_i = \mathfrak{K}_i(p)$.

<u>4.5.1. Theorem 3. The polynomials of $\mathfrak{K}_h \setminus \mathfrak{K}_{h+1}$; $h \leq m$, either all support A at p or all intersect A at p.</u>

<u>Proof</u>. Since p has finite $\mathfrak{K}$-order, any polynomial $K \varepsilon \mathfrak{K}_h$ either intersects or supports A at p. Let K_1 and K_2 be two polynomials of $\mathfrak{K}_h \setminus \mathfrak{K}_{h+1}$.

(i) Let $K(p^h)$ be regular. Thus K_1 and K_2 are regular.

Suppose that K_1 intersects and K_2 supports A at p. Let N be a sufficiently small neighbourhood of p in G such that

$$K_1 \cap K_2 \cap N = \{p\}$$

and

$$A \cap N \cap K_1 = \{p\} = A \cap N \cap K_2.$$

Let K_2 be oriented such that

$$A \cap N \subset K_2^{-1} \cup \{p\}.$$

Suppose at first that K_1 and K_2 have $n - h$ points, say $Q_1, \ldots, Q_{n-h}$ in common outside p. Consider the one parameter family of polynomials of $\mathcal{K}_h$ through $Q_1, \ldots, Q_{n-h}$; cf. <u>3.3</u>. Let $s \varepsilon A \cap N$ and Let K(s) be the member of this family through s. Then K(s) has no points in common with K_1 and K_2 outside p, $Q_1, \ldots, Q_{n-h}$ and

$$K(s) \cap K_1 \cap N = \{p\} = K(s) \cap K_2 \cap N.$$

Since K_1 intersects A at p, we have

$$A \cap K_1^{\pm 1} \cap N \neq \emptyset; \text{ cf. } \underline{4.1.4}.$$

Also if h is odd [even], then K(s) intersects [supports] K_1 and K_2; cf. <u>1.3.7</u>. Hence if h is odd [even] and $s \varepsilon A \cap N \cap K_1^1$, then

$$(1)\ K(s) \cap N \subset (K_1^1 \cap K_2^{-1}) \cup (K_1^{-1} \cap K_2^1) \cup \{p\} \left[(K_1^1 \cap K_2^{-1}) \cup \{p\}\right].$$

Similarly if $s \varepsilon A \cap N \cap K_1^{-1}$, then

$$(2)\ K(s) \cap N \subset (K_1^{-1} \cap K_2^{-1}) \cup (K_1^1 \cap K_2^1) \cup \{p\} \left[(K_1^{-1} \cap K_2^{-1}) \cup \{p\}\right].$$

Let s tend to p. By 4.4.6, K(s) converges to

$$K_0 = K(p^{h+1}, Q_1, \ldots, Q_{n-h}) \in \mathcal{K}_{h+1}.$$

From (1) and (2)

$$K_0 \cap N \subset K_1 \cup K_2.$$

Since K_1 and K_2 are regular, K_0 cannot be degenerate and thus K_0 is also regular. Hence either

$$K_0 = K_1 \text{ or } K_0 = K_2.$$

Therefore either K_1 or K_2 is a polynomial of $\mathcal{K}_{h+1}$; a contradiction.

Next suppose that K_1 and K_2 have fewer than n - h points in common outside p. Choose n - h + 1 distinct points $R_1, \ldots, R_{n-h+1}$ such that their x-coordinates are in increasing order of magnitude and such that R_j and R_{j+1} lie on opposite sides of both K_1 and K_2; $1 \le j \le n - h$. Let

$$K_3 = K(p^h, R_1, \ldots, R_{n-h+1}).$$

Then K_1 and K_3 [K_2 and K_3] intersect each other at n - h points outside p. Hence, from the above, K_1 and K_3 [K_2 and K_3] either both support or both intersect A at p.

Thus either both K_1 and K_2 intersect A at p or both of them support A at p.

(ii). Let $K(p^r)$ be degenerate. Then it is clear that for $h \geq r$ the degenerate polynomials of

$$\mathcal{K}_h \setminus \mathcal{K}_{h+1} = \mathcal{K}_h$$

either all intersect A at p or all of them support A at p.

4.5.2. Let $K(p^i)$ be regular and let $K(p^{i+1})$ be degenerate; $i > 1$. Let $h > i$.

Then the component of $K \varepsilon \mathcal{K}_h \setminus \mathcal{K}_{h+1}$ through p is the line $\mathcal{L}(p)$ counted h - i times if i is odd and a double ray with vertex p together with the line $\mathcal{L}(p)$ counted (h - i - 1)-times if i is even; cf. 4.4.3.

In particular, if A has a cusp at p, then $K \varepsilon \mathcal{K}_h \setminus \mathcal{K}_{h+1}$ always supports A at p and if A has no cusp at p, then Kε $\mathfrak{K}_h \setminus \mathfrak{K}_{h+1}$ will intersect [support] A at p if h is even [odd]; $1 < i < h$.

Let $K(p^2) = \mathcal{L}(p)$, $h > 1$. If A has a cusp of the first kind at p, then Kε $\mathfrak{K}_h$ intersects or supports A at p according as h is even or odd. If A has a cusp of the second kind, then Kε $\mathfrak{K}_h$ always supports A at p.

<u>4.5.3.</u> <u>Let</u> $h \leq m$ <u>be an even integer.</u> <u>If</u> $K(p^h)$ <u>is regular, then all the polynomials of</u> $\mathfrak{K}_h \setminus \mathfrak{K}_{h+1}$ <u>support</u> A <u>at</u> p. <u>If</u> $K(p^h)$ <u>is degenerate, then</u> Kε $\mathfrak{K}_h$ <u>supports or intersects</u> A <u>at</u> p <u>according as</u> A <u>has or does not have a cusp at</u> p, if $K(p^2) \neq \mathcal{L}(p)$.

<u>Proof</u>. Let $K(p^h)$ be regular and let Kε $\mathfrak{K}_h \setminus \mathfrak{K}_{h+1}$. Let $Q_1, \ldots, Q_{n-h}$ be n - h mutually distinct points on K all different from p. Suppose that K intersects A at p. Let

$$K(s) = K(p^h, s, Q_1, \ldots, Q_{n-h})$$

be the unique regular polynomial of $\mathfrak{K}_h$ through s, s sufficiently close to p on A. Since p has finite order, we can choose s so close to p, that $s \notin K$ and K(s) is n-regular. Also, since K and K(s) have exactly h-point contact and h is even, they will support each other at p;

cf. 1.3.7. Hence if N is a sufficiently small neighbourhood of p in G and $s \in A \cap N \cap K^1$, then

$$K(s) \cap N \subset (K^1 \cup \{p\}) \cap N.$$

Let s tend to p and put

$$K_0 = \lim_{s \to p} K(s) = K(p^{h+1}, Q_1, \ldots, Q_{n-h}) \varepsilon \mathcal{K}_{h+1}.$$

Hence

$$K_0 \cap N \subset (K^1 \cup K) \cap N.$$

Similarly, for $s \varepsilon A \cap N \cap K^{-1}$,

$$K(s) \cap N \subset (K^{-1} \cup \{p\}) \cap N.$$

Hence

$$K_0 \cap N \subset (K^{-1} \cup K) \cap N.$$

Thus

$$K_0 \cap N \subset K \cap N.$$

Since K is regular, $K_0 = K$. Hence $K \varepsilon \mathcal{R}_{h+1}$; contradiction.

If $K(p^h)$ is degenerate, and $K(p^2) \neq \mathcal{L}(p)$ then the assertion follows from 4.5.2.

Corollary. Let i be even. Let $K(p^i)$ be regular but $K(p^{i+1})$ be degenerate. Then $K(p^i)$ supports A at p.

4.5.4. Let $h \leq m$ be an odd integer. Let $K(p^{h+1})$ be regular. Then each $K \varepsilon \mathcal{R}_h \setminus \mathcal{R}_{h+1}$ supports or intersects A at p according as A has or does not have a cusp at p.

Proof. If $h = 1$, then the statement is true by the definition of a cusp; cf. 4.2.9. Hence let $h \geq 2$.

Let A have a cusp [no cusp] at p. Thus $\mathcal{L} = \mathcal{L}(p)$ supports [intersects] A at p. Let $K \varepsilon \mathcal{R}_h \setminus \mathcal{R}_{h+1}$. Let $Q_1, \ldots, Q_{n-h}$ be mutually distinct points on K, all different from p. Let

$$K(s) = K(p^h, s, Q_1, \ldots, Q_{n-h})$$

be the unique regular polynomial of $\mathcal{R}_h$ through s, $Q_1, \ldots, Q_{n-h}$. Then for s sufficiently close to p, $s \notin K$ and K(s) will be n-regular.

Suppose that K intersects [supports] A at p.

Let N be a sufficiently small neighbourhood of p in G such that

$$K(s) \cap K \cap N = \{p\}$$

and

$$A \cap \mathcal{L} \cap N = \{p\} = A \cap K \cap N.$$

Now K and K(s) are both regular, hence

$$K \cap \mathcal{L} = \{p\} = K(s) \cap \mathcal{L} .$$

Let $\mathcal{L}$ [K] be oriented such that

$$A \cap N \subset \mathcal{L}^{-1} \cup \{p\} [A \cap N \subset K^{-1} \cup \{p\}].$$

Since K and K(s) have exactly h-point contact at p and h is odd, they will intersect each other at p; cf. <u>1.3.7</u>. Also $\mathcal{L}$ intersects K(s) at p. Hence if

$$s \epsilon A \cap N \cap \mathcal{L}^{-1} \cap K^{-1} [s \epsilon A \cap N \cap \mathcal{L}^{1} \cap K^{-1}],$$

then we have

$$K(s) \cap N \subset (K^{-1} \cap \mathcal{L}^{-1}) \cup (K^{1} \cap \mathcal{L}^{1}) \cup \{p\}$$
$$\left[(K^{-1} \cap \mathcal{L}^{1}) \cup (K^{1} \cap \mathcal{L}^{-1}) \cup \{p\}\right].$$

Similarly, for

$$s \varepsilon A \cap N \cap \mathcal{L}^{-1} \cap K^{1} \left[s \varepsilon A \cap N \cap \mathcal{L}^{-1} \cap K^{-1} \right]$$

we have

$$K(s) \cap N \subset (K^{1} \cap \mathcal{L}^{-1}) \cup (K^{-1} \cap \mathcal{L}^{1}) \cup \{p\}$$
$$\left[(K^{-1} \cap \mathcal{L}^{-1}) \cup (K^{1} \cap \mathcal{L}^{1}) \cup \{p\}\right].$$

Let s tend to p and put

$$K_0 = \lim_{s \to p} K(s) = K(p^{h+1}, Q_1, \ldots, Q_{n-h}) \varepsilon \, \mathfrak{K}_{h+1}.$$

Then we conclude that

$$K_0 \cap N \subset K \cup \mathcal{L} .$$

Since K_0 is regular, $K_0 \cap N \subset K$ and therefore $K_0 = K$; a contradiction.

<u>Remark</u>. In <u>4.5.4</u> if $K(p^h)$ is regular but $K(p^{h+1})$ is degenerate, then the assertion need not be true; cf. <u>5.2.4</u>.

<u>4.5.5</u>. <u>Let the arc</u> A <u>have a cusp at</u> p. <u>Suppose that</u> $K \varepsilon \, \mathfrak{K}_h \setminus \mathfrak{K}_{h+1}$ <u>intersects</u> A <u>at</u> p. <u>Then</u> $K(p^{h+1})$ <u>is degenerate</u>. <u>Thus if</u> $K(p^{m+1})$ <u>is regular</u>, <u>then</u> $K \varepsilon \, \mathfrak{K}_h \setminus \mathfrak{K}_{h+1}$ <u>supports</u> A <u>at</u> p; $1 \leq h \leq m$.

Proof. By *4.2.9*, $h \geq 2$ and by *4.5.3*, h is odd.

Suppose that $K(p^{h+1})$ is regular. Since h is odd and p is a cusp, we have, by *4.5.4*, $K \varepsilon \mathcal{K}_h \setminus \mathcal{K}_{h+1}$ supports A at p; a contradiction to the assumption.

4.5.6. *Let* A *have a cusp of the first kind at* p. *Let* $K(p^i)$ *be regular but* $K(p^{i+1})$ *be degenerate*. *Then* $K(p^h) = \mathfrak{F}$ *for* $2 \leq h \leq i$ *and* i *is either odd or equal* to $m + 1$.

Proof. We first prove that $K(p^h) = \mathfrak{F}$ for $h = 2, 3, \ldots, i$. This assertion is trivial for $h = 2$. Suppose it has been proved up to h; $h < i$.

Let $\Phi = \Phi_h$ be the one-parameter family of regular polynomials of degree $\leq h$ which have h-point contact with $\mathfrak{F} = K(p^h)$ at p; cf. *3.3*.

If h is even Φ is decomposed by $\mathfrak{F}$ into two disjoint subfamilies $\Phi_\alpha \subset \mathfrak{F}^\alpha$ bounded by $\mathfrak{F}$ and a double ray with vertex p; $\alpha = \pm 1$; cf. *3.3.7*. If $s \varepsilon \mathfrak{F}^\alpha$, then

$$K(p^h, s) \varepsilon \Phi_\alpha.$$

Hence

$$\lim_{s \to p} K(p^h, s) = K(p^{h+1}) = \mathcal{J} .$$

If h is odd, then $K(p^h) = \mathcal{J}$ and $\mathcal{L} = \mathcal{L}(p)$ decompose Φ into two disjoint families Φ_α, $\alpha = \pm 1$, each of which is bounded by $K(p^h)$ and $\mathcal{L}$. If $s_\alpha \varepsilon\ A \cap \mathcal{J}^\alpha \cap \mathcal{L}^1$, then

$$K(p^h, s_1) \subset (\mathcal{J}^1 \cap \mathcal{L}^1) \cup (\mathcal{J}^{-1} \cap \mathcal{L}^{-1}) \cup \{p\},$$

$$K(p^h, s_{-1}) \subset (\mathcal{J}^{-1} \cap \mathcal{L}^1) \cup (\mathcal{J}^1 \cap \mathcal{L}^{-1}) \cup \{p\}.$$

Hence

$$K(p^{h+1}) = \lim_{s \to p} K(p^h, s) \subset \mathcal{J} \cup \mathcal{L} .$$

Since $h < i$, we have $K(p^{h+1}) = \mathcal{J}$.

Now, suppose that $i < m + 1$ is even. Let

$$K(s) = K(p^i, s)$$

be the unique i-regular polynomial of $\mathcal{R}_i$ through s, s sufficiently close to p. Since i is even K(s) supports $\mathcal{J}$ at p. Hence if $s_\alpha \varepsilon\ \mathcal{J}^\alpha$ then

$$K(s_1) \subset \mathcal{J}^1 \cup \{p\} \text{ and } K(s_{-1}) \subset \mathcal{J}^{-1} \cup \{p\}.$$

Hence

$$K(p^{i+1}) = \mathcal{F};$$

a contradiction.

4.6. The degeneracy index. The degeneracy index $I(p)$ of a differentiable point p is defined as follows.

(1) $I(p) = i$, $1 \leq i \leq n$, if and only if i is the smallest integer such that $K(p^{i+1})$ is degenerate; or equivalently, such that $\tilde{\mathcal{K}}_{i+1}$ consists of degenerate polynomials.

(2) $I(p) = n + 1$, if $K(p^{n+1})$ is regular.

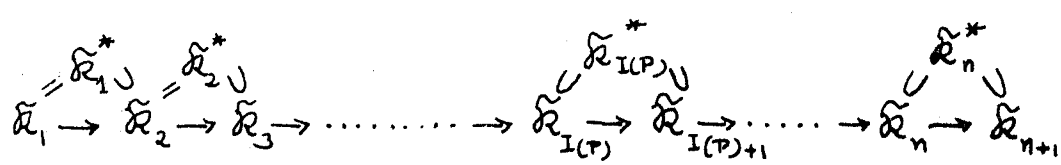
4.7. Relations between the families $\tilde{\mathcal{K}}_h$ and $\tilde{\mathcal{K}}_{h+1}$. Define

$$\tilde{\mathcal{K}}_j^* = \tilde{\mathcal{K}}_j \cup \tilde{\mathcal{K}}_{j+1}; \; j = 1, \ldots, n.$$

The following diagram shows how the families $\tilde{\mathcal{K}}_1, \ldots, \tilde{\mathcal{K}}_{n+1}$ are related.

We observe that

$$\tilde{\mathcal{K}}_h \cap \tilde{\mathcal{K}}_j = \emptyset \text{ if } j > h \geq I(p).$$



CHAPTER V

A Characteristic of a Polynomially Differentiable Point

5.1. Characteristics. With each interior differentiable point p of type i, $i = I(p)$, $1 \leq i \leq n + 1$, of the arc A, we associate a characteristic

$$(a_0, a_1, \ldots, a_n; i).$$

The numbers a_j are equal to 1 or 2; $0 \leq j \leq n$. We define them inductively as follows:

$$\sum_{j=0}^{h-1} a_j \text{ is even [odd]}$$

if the polynomials of $\mathcal{P}_h \setminus \mathcal{P}_{h+1}$ all support [intersect] A at p; $1 \leq h \leq n$. Thus a_0 is even, i.e., $a_0 = 2$, if and only if A has a cusp at p. The number

$$\sum_{j=0}^{n} a_j \text{ is even [odd]}$$

if the polynomial $K(p^{n+1})$ supports [intersects] A at p.

5.2. Characteristics of non-cusp points. Throughout this section we assume that A does not have a cusp at p.

5.2.1. Let $I(p) = i$, $1 \leq i \leq n$. If i is even, then $a_i = 2$ and $a_j = 1$ for $j \neq i$. Let i be odd. Then $a_i = 2$ and $a_j = 1$, $j \neq i$ [$a_{i-1} = 2$ and $a_j = 1$, $j \neq i - 1$] if $K\varepsilon \mathfrak{R}_i$ intersects [supports] A at p.

Proof. If $1 \leq h < i$, then by 4.5.3 and 4.5.4, $K\varepsilon \mathfrak{R}_h \setminus \mathfrak{R}_{h+1}$ supports or intersects A at p according as h is even or odd. Hence $a_{h-1} = 1$ for $h = 1, \ldots, i - 1$.

Now by 4.5.2,

(1) $\sum_{j=0}^{h-1} a_j \equiv h - 1 \pmod 2$ if $i < h \leq n + 1$.

This yields, in particular,

$$\sum_{j=0}^{i} a_j = \sum_{j=0}^{i-2} a_j + a_{i-1} + a_i = (i - 1) + a_{i-1} + a_i \equiv i \pmod 2,$$

(2) i.e., $a_i + a_{i-1} \equiv 1 \pmod 2$.

If i is even, then 4.5.3 still yields $a_{i-1} = 1$ and therefore (2) implies that

$$a_i \equiv 0 \pmod 2, \text{ i.e., } a_i = 2.$$

If i is odd, then by (2), either $a_{i-1} = 1$ and $a_i = 2$ or $a_{i-1} = 2$ and $a_i = 1$. Obviously $a_{i-1} = 2[a_{i-1} = 1]$ if $K \varepsilon \mathfrak{K}_i$ supports [intersects] A at p.

Now it can be easily seen that for both i even and i odd (1) implies that

$$a_{i+1} = \ldots\ldots = a_n = 1.$$

<u>5.2.2</u>. If i = 1, then <u>4.5.2</u> implies that the characteristic of p is

$$(1, 1, 2, 1, \ldots. 2; 1) \text{ or } (1, 2, 1, \ldots., 1; 1)$$

according as $\mathfrak{J}$ supports or intersects A at p.

<u>5.2.3</u>. If i = n + 1, then <u>4.5.3</u> and <u>4.5.4</u> imply that $a_0 = a_1 = \ldots. = a_{n-1} = 1$. Thus p has characteristic

$$(1, 1, \ldots., 1, 2; n + 1) \text{ or } (1, 1, \ldots., 1; n + 1)$$

according as $K(p^{n+1})$ supports or intersects A at p.

5.2.4. The following examples show that for $i = n + 1$ and for each odd i, $3 \leq i \leq n$, both of the types of differentiable non-cusp point discussed in 5.2.1 and 5.2.3 exist, and that there exists a differentiable non-cusp for each even i, $2 \leq i \leq n$.

In these examples, we take the point p given by $s = 0$ on the indicated arcs. Consider

$$x = s^k, \ y = s^\ell + s^m, \ -\delta \leq s \leq \delta.$$

Here $\delta > 0$ is sufficiently small. The indices k, ℓ, m are positive integers such that $k > 1$, $m > \ell$, k is odd, $(i - 1)k < \ell < ik$.

If i is even and $2 \leq i \leq n$, let ℓ be even. Then p has the characteristic $(1, \ldots, 1, a_i = 2,1,\ldots,1;i)$.

If i is odd and $3 \leq i \leq n$, then p has characteristic

$$(1, \ldots., 1, a_i = 2, 1, \ldots., 1; i)$$

or

$$(1, \ldots., 1, a_{i-1} = 2, 1, \ldots., 1; i)$$

according as ℓ is odd or even.

If $i = n + 1$, we obtain examples of the characteristics

$$(1, \ldots., 1, 2; n + 1) \text{ or } (1, \ldots., 1; n + 1)$$

by letting $\ell = kn$.

5.2.5. From 5.2.1, 5.2.2 and 5.2.3, we obtain the following result.

If A does not have a cusp at p, then the characteristic $(a_0, a_1, \ldots, a_n; i)$ of p is one of $\frac{3}{2}n + 2\left[\frac{3n + 5}{2}\right]$ different types if n is even [odd].

5.3. *Characteristics of cusp points. Throughout this section we assume that A has a cusp at p.*

5.3.1. *Let $I(p) = i$, $1 \leq i \leq n$. Then the characteristic of p is*

$$(2, 2, \ldots, 2; i)$$

if i is even. If i is odd, then the characteristic of p is

$$(2, 2, \ldots, 2, 1, 1, 2, \ldots, 2; i), \; a_{i-1} = a_i = 1$$

or

$$(2, 2, \ldots, 2; i),$$

<u>according as</u> $K(p^i)$ <u>intersects or supports</u> A <u>at</u> p.

<u>Proof</u>. For $1 \le h < i$, by <u>4.5.3</u> and <u>4.5.4</u>,

$$K \in \mathfrak{K}_h \setminus \mathfrak{K}_{h+1}$$

all support A at p. Hence

$$a_j = 2, \ 0 \le j \le i - 2.$$

By <u>4.5.2</u>, $K \varepsilon \mathfrak{K}_h$ supports A at p if $i < h \le n + 1$. Therefore,

$$\sum_0^{h-1} a_j \equiv 0 \pmod 2.$$

If i is even, then by <u>4.5.3</u>, $K \varepsilon \mathfrak{K}_i$ supports A at p. Hence

$$\sum_0^{i-1} a_j \equiv 0 \pmod 2.$$

Thus if i is even, then $a_j = 2$, $0 \le j \le n$ and p has the characteristic

$$(2, 2, \ldots, 2; i).$$

If i is odd, then by 4.5.2

$$\sum_0^i a_j \equiv a_{i-1} + a_i \equiv 0 \pmod 2.$$

Thus

$$a_{i-1} = a_i = 1 \text{ or } 2.$$

Obviously, $a_{i-1} = 1$ if and only if $\sum_0^{i-1} a_j$ is odd; i.e., if and only if A is intersected by $K(p^i)$.

5.3.2. The following examples show that for odd i, i > 1 both types of differentiable cusp of the second kind exist.

Let the arc A be given by

$$(5.3.2) \qquad x = s^k, \; y = s^\ell + s^m,$$

where k and ℓ are even, m is odd, $m > \ell$, and $(i - 1)k \leq \ell < ik$, $3 \leq i \leq n$. The point p is the origin.

If $\ell = (i - 1)k$, then the characteristic of p is

$$(2, 2, \ldots, 2, 1, 1, 2, \ldots, 2; i), \; a_{i-1} = a_i = 1,$$

otherwise it is

$$(2, 2, \ldots, 2; i).$$

5.3.3. The following result is an immediate consequence of 4.5.2.

If $I(p) = i = 1$, *then the characteristic of* p *is* $(2, 1, 1, \ldots, 1; 1)$ *or* $(2, 2, \ldots, 2; 1)$ *according as* p *is a cusp of the first or second kind.*

5.3.4. The next statement is a corollary of 4.5.5.

If $I(p) = n + 1$, *then a cusp point* p *has characteristic*

$$(2, 2, \ldots, 2; n + 1) \text{ or } (2, 2, \ldots, 2, 1; n + 1)$$

according as $K(p^{n+1})$ *supports or intersects* A *at* p.

Examples of both types are given by (5.3.2) with $nk < \ell$, k and ℓ even, m odd and $m > \ell$.

5.3.5. From 5.3.1, 5.3.2 and 5.3.4, we obtain the following result.

Let p be a cusp of the second kind. Then the characteristic of p is one of $\frac{3n}{2} + 1 \left[\frac{3n+3}{2}\right]$ _different types if n is even [odd; n > 1]._

5.3.6. Let p be a cusp of the first kind. If $1 < i = I(p) \leq n$, _then the characteristic of p is_

$$(2, 2, \ldots., 2, 1, 1, 2, \ldots., 2; i),$$

where $a_{i-1} = a_i = 1$. _If_ $i = n + 1$, _then p has the characteristic_ $(2, 2, \ldots, 2, 1; n + 1)$.

Proof. By _4.5.6_, $K(p^i) = \mathfrak{Y}$ and i is odd for $1 < i \leq n$. Now $\mathfrak{Y}$ intersects the arc A at p. Hence if $1 < i \leq n$, then by _5.3.1_, the characteristic of p is $(2, 2, \ldots, 2, 1, 1, 2, \ldots., 2; i)$.

If $i = n + 1$, then again by _4.5.6_, $K(p^{n+1}) = \mathfrak{Y}$. Hence by _5.3.4_, the characteristic of p is $(2, 2, \ldots., 2, 1,;n+1)$.

Remark. Examples of these types are given by the arc in (5.3.2) if k is even, ℓ is odd, $m > \ell$ and $(i - 1)k \leq \ell < ik \, [ik \leq \ell]$ if $i \leq n \, [i = n + 1]$.

_5.3.7._ Let $n = 1$, i.e., the polynomials of $\mathfrak{S}$

are straight lines. If A has a cusp of the first kind at p, then the characteristic of p is (2, 1; 1) or (2, 1; 2). If p is a cusp of the second kind, then p has the characteristic (2, 2; 1) or (2, 2; 2).

5.3.8. From 5.3.1 to 5.3.7, we obtain the following.

Let A have a cusp of the first kind at p. Then the characteristic of p is one of $\frac{n}{2} + 1$ $[\frac{n+3}{2}]$ *different types if n is even [odd].*

5.3.9. From 5.2.5, 5.3.5 and 5.3.8, the number of types of differentiable points is $\frac{7n}{2} + 4$ $[\frac{7n+11}{2}]$ if n is even [odd].

5.4. Infinitely differentiable points.

5.4.1. We shall define a characteristic of a point p of an arc A which is n times differentiable at p with respect to the family of polynomials of degree at most n for all positive integers n.

The family $\mathcal{R}$ of regular polynomials of degree at most n shall be denoted by $\mathcal{R}^{(n)}$ and the families $\mathcal{R}_h^{(n)}(p)$, $1 \leq h \leq n + 1$, by $\mathcal{R}_h^{(n)}$. Also, the characteristic $(a_0, a_1, \ldots, a_n; i)$ shall be denoted by $(a_0^{(n)}, a_1^{(n)}, \ldots, a_n^{(n)}; i_n)$.

We observe that $\mathfrak{K}^{(n)}$-differentiability implies $\mathfrak{K}^{(n-1)}$-differentiability. Thus if A is n-times differentiable at p, it is also (n - 1)-times differentiable at p and has a second characteristic

$$(a_0^{(n-1)}, \ldots, a_{n-1}^{(n-1)}; i_{n-1}).$$

Each $\mathfrak{K}_h^{(n)}$ is the union of $\mathfrak{K}_h^{(n-1)}$ with a family $\Phi(p^h; Q_1, \ldots, Q_{n-h+1})$ of polynomials of degree $\leq n$; $h = 1, \ldots, n$. This readily yields

$$(1) \quad \mathfrak{K}_h^{(n-1)} \setminus \mathfrak{K}_{h+1}^{(n-1)} \subset \mathfrak{K}_h^{(n)} \setminus \mathfrak{K}_{h+1}^{(n)}; \; h = 1, \ldots, n - 1.$$

However, the one curve of $\mathfrak{K}_n^{(n-1)}$ may be equal to the curve of $\mathfrak{K}_{n+1}^{(n)}$. Thus it need not lie in $\mathfrak{K}_n^{(n)} \setminus \mathfrak{K}_{n+1}^{(n)}$. Hence by 4.5.1,

$$a_h^{(n-1)} = a_h^{(n)} = a_h, \text{ say, for } h = 0, 1, \ldots, n - 2,$$

but not necessarily for h = n - 1. Also

$$i_{n-1} = i_n \text{ if } i_{n-1} \leq n - 1.$$

Now let A be n times differentiable at p for each n = 1, 2, ... Then we have an infinite sequence

$$(a_0, a_1^{(1)}; i_1), (a_0, a_1, a_2^{(2)}; i_2), (a_0, a_1, a_2, a_3^{(3)}; i_3), \ldots$$
$$\ldots, (a_0, a_1, \ldots, a_{n-2}, a_{n-1}, a_n^{(n)}; i_n), \ldots$$

of characteristics of p.

If there exists a least integer m such that

$$i_m < m + 1,$$

then

$$i_n = i_m \text{ for all } n \geq m,$$

and we define

$$(a_0, a_1, \ldots; i_m)$$

to be the characteristic of p with respect to all polynomials and call i_m the index of p.

If however,

$$i_n = n + 1 \text{ for all } n,$$

then the characteristic of p shall be denoted by

$$(a_0, a_1, \ldots, a_n, \ldots; \infty).$$

5.4.2. Examples. (i) Consider the arc A given by

$$x = s^k, \; y = s^{k+\ell},$$

where $k > 1$ and ℓ are odd integers. Let $(i - 2)k < \ell < (i-1)k$. Then the point $p = (0, 0)$ has finite index i_i for $i = 2, \ldots, n + 1$. Moreover, p has the characteristic $(a_0, \ldots, a_n; i_i)$ where $a_i = 2$ and $a_j = 1$, $j \neq i$ $[a_{i-1} = 2$ and $a_j = 1$, $j \neq i - 1]$ if i is even [odd].

(ii) The point $p = (0, 0)$ on the arc

$$x = s, \; y = \begin{cases} e^{-\frac{1}{s^2}} & s \neq 0 \\ 0 & s = 0 \end{cases}$$

has the characteristic of the type $(a_0, a_1, \ldots, a_n, \ldots; \infty)$.

CHAPTER VI

The Order of a Differentiable Point.

6.1. Introduction. Recall that the $\mathfrak{K}$-order of a point p of A is the $\mathfrak{K}$-order of a sufficiently small neighbourhood of p on A; cf. 4.1.3.

In this chapter our aim is to prove the following theorem.

Theorem 4. *Let p be an interior differentiable point of the arc A. Let p have the characteristic $(a_0, a_1, \ldots, a_n; i)$. Then the $\mathfrak{K}$-order of p is not less than* $\sum_{j=0}^{n} a_j$.

6.2. Certain pencils λ_j in $\overline{\mathfrak{K}}$. It will be convenient to introduce certain pencils λ_j in $\overline{\mathfrak{K}}$; $j = n + 1, n, \ldots, 1, 0$. Let

$$\lambda_{n+1} = \mathfrak{K}_{n+1} = \{K(p^{n+1})\}.$$

Put

$$K(p^{n+1}) = K_{n+1}.$$

Since the order of p is finite, there is a neighbour-hood N_n of p on A such that

$$K_{n+1} \cap N_n = \{p\}.$$

Recall that s always denotes a point on $A \setminus \{p\}$; cf. 4.2.1. Also since A is differentiable at p, $K(p^n, s)$ exists if $s \in N_n$ and

$$K(p^n, s) \in \mathcal{K}_n \setminus \{K_{n+1}\}.$$

Let

$$\lambda_n = \{K(p^n, s) \mid s \in N_n\}$$

and put

$$(2)_n \qquad K(p^n, s) = K(\lambda_n, s).$$

Then, for $s \in N_n$, we have

$$(3)_n \qquad s \notin K_{n+1} \text{ and thus } K(\lambda_n, s) \neq K_{n+1}$$

$$(4)_n \qquad \lim_{s \to p} K(\lambda_n, s) = K_{n+1},$$

(5)$_n$ $\lambda_n = \{K(p^n, s) | s N_n\} \subset \mathcal{K}_n \setminus \mathcal{K}_{n+1}$.

Choose

(6)$_n$ $s_n \varepsilon N_n$ and put $K(\lambda_n, s_n) = K_n$.

Again, since the order of p is finite, there is a neighbourhood M_{n-1} of p on A such that

$$K_n \cap M_{n-1} = \{p\} .$$

Choose

(1)$_{n-1}$ $Q_1 \varepsilon K_n \setminus \mathcal{L}(p)$.

Thus

(7)$_n$ $K_n = K(\lambda_n, Q_1) = K(p^n, Q_1)$.

By 4.4.5, for a sufficiently small neighbourhood M'_{n-1} of p on A and $s \varepsilon M'_{n-1}$, there exists a unique polynomial

$$K(p^{n-1}, s, Q_1) \varepsilon \mathcal{K}_{n-1}.$$

Let

$$N_{n-1} = M_{n-1} \cap M'_{n-1} \cap N_n.$$

Let

$$\lambda_{n-1} = \{K(p^{n-1}, s, Q_1) \mid s \varepsilon N_{n-1}\}.$$

Put

$(2)_{n-1}$ $\quad K(p^{n-1}, s, Q_1) = K(\lambda_{n-1}, s).$

Then for $s \varepsilon N_{n-1}$, we have

$(3)_{n-1}$ $\quad s \notin K_n$ and thus $K(\lambda_{n-1}, s) \neq K_n$,

$(4)_{n-1}$ $\quad \lim_{s \to p} K(\lambda_{n-1}, s) = K_n$,

$(5)_{n-1}$ $\quad \lambda_{n-1} = \{K(\lambda_{n-1}, s) \mid s \varepsilon N_{n-1}\} \varepsilon \mathfrak{K}_{n-1} \setminus \mathfrak{K}_n.$

Choose

$(6)_{n-1}$ $\quad s_{n-1} \varepsilon N_{n-1}$ and put $K_{n-1} = K(\lambda_{n-1}, s_{n-1})$

Let $j \leq n - 1$. Assume we have already defined the neighbourhoods

$$N_n \supset N_{n-1} \supset \cdots \supset N_j$$

of p on A, the points $Q_1, \ldots, Q_{n-j}$, the families $\lambda_n, \ldots, \lambda_j$, the points $s_n, \ldots, s_j$, and the curves $K_n, \ldots, K_j$ such that the following conditions are satisfied:

$(1)_j$ $\mathcal{L}(p)$, $\mathcal{L}(Q_1)$, ..., $\mathcal{L}(Q_{n-j})$ are mutually distinct

and for $h = n, n - 1, \ldots, j$

$(2)_h$ $K(\lambda_h, s) = K(p^h, s, Q_1, \ldots, Q_{n-h})$ } if $s \varepsilon N_h$

$(3)_h$ $K(\lambda_h, s) \neq K_{h+1}$ }

$(4)_h$ $\lim_{s \to p} K(\lambda_h, s) = K_{h+1}$.

$(5)_h$ $\lambda_h = \{K(\lambda_h, s) | s \varepsilon N_h\} \subset \mathfrak{L}_h \setminus \mathfrak{L}_{h+1}$.

$(6)_h$ $s_h \varepsilon N_h$, $K_h = K(s_h, s_h)$.

Then we define λ_{j-1} as follows.

Since the order of p is finite, there exists a neighbourhood $M_{j-1} \subset N_j$ of p on A such that

$$K_j \cap M_{j-1} = \{p\}.$$

Choose Q_{n-j+1} on K_j such that $(1)_{j-1}$ is satisfied.. Thus

$$(7)_j \qquad K_j = K(p^j, Q_1, \ldots, Q_{n-j+1}).$$

By 4.4.5, there exists a neighbourhood $N_{j-1} \subset M_{j-1}$ such that $K(p^{j-1}, s, Q_1, \ldots, Q_{n-j+1}) \in \tilde{\mathcal{L}}_{j-1}$ is unique for each $s \in N_{j-1}$. Define λ_{j-1} through $(2)_{j-1}$ and the left half of $(5)_{j-1}$. Then $(3)_{j-1}$-$(5)_{j-1}$ will hold true. Define s_{j-1} and K_{j-1} through $(6)_{j-1}$.

6.3. A lemma. The main tool which we shall use to prove our theorem is the following lemma.

Lemma. Let M be any neighbourhood of p on A. Then λ_j contains polynomials K arbitrarily close to K_{j+1} which intersect $M \setminus \{p\}$ in not less than a_j points; $0 \leq j \leq n$.

The proof is given in 6.3.1-6.3.3.

6.3.1. Clearly, if $j < I(p)$ [$j > I(p)$], then all the polynomials of λ_j and the polynomial K_{j+1} are regular [degenerate]. If $j = I(p)$ then all the polynomials of λ_j are regular but K_{j+1} is degenerate.

By $(2)_j$ and $(4)_j$,

$$\{p, Q_1, \ldots, Q_{n-j}\} \subseteq K \cap K_{j+1} \text{ for all } K \varepsilon \lambda_j.$$

Actually, we have equality for $j \leq I(p)$. For if $j < I(p)$, then K and K_{j+1} are distinct, have $Q_1, \ldots, Q_{n-j}$ in common and have j-point contact with each other at p. If $j = I(p)$, then a regular polynomial K of λ_j has a point in common with each of the components of the degenerate polynomial K_{j+1}.

If $j > I(p)$, then the components of K and K_{j+1} through $Q_1, \ldots, Q_{n-j}$ are identical. However, the component of K_{j+1} through p is $K(p^j) \cup \mathcal{L}(p)$, while K contains $\mathcal{L}(s)$ which does not belong to K_{j+1}.

<u>6.3.2</u>. The case $j > I(p)$ is straightforward, since then the families $\mathcal{K}_{j+1}(p)$ and $\mathcal{K}_j(p)$ are both degenerate. We recall that $\mathcal{K}_j(p) \cap \mathcal{K}_{j+1}(p) = \emptyset$ and $\mathcal{K}_{j+1}(p) \cap \mathcal{K}_{j+2}(p) = \emptyset$.

By $(2)_{j+1}$, $(6)_{j+1}$ and $(7)_{j+1}$,

$$\begin{aligned} K_{j+1} &= K(\lambda_{j+1}, s_{j+1}) \\ &= K(p^{j+1}, s_{j+1}, Q_1, \ldots, Q_{n-j-1}) \\ &= K(p^{j+1}, Q_1, \ldots, Q_{n-j}) \end{aligned}$$

$$= K(p^{j+1}) \cup \mathcal{L}(Q_1) \cup \dots \quad \cup \mathcal{L}(Q_{n-j})$$

$$\varepsilon \ \mathcal{K}_{j+1}(p).$$

If s is sufficiently close to p on A, then

$$K(\lambda_j, s) = K(p^j, s, Q_1, \dots, Q_{n-j})$$

$$= K(p^j) \cup \mathcal{L}(s) \cup \mathcal{L}(Q_1) \cup \dots \cup \mathcal{L}(Q_{n-j})$$

$$\varepsilon \ \mathcal{K}_j(p).$$

We observe that $K(\lambda_j, s)$ and K_{j+1} cannot both inters ect A at p. The following statements can be verified directly.

$$a_j = 1 \ [\ a_j = 2\ ]$$

$$\Updownarrow$$

One of $K(\lambda_j, s)$ and K_{j+1} intersects A at p, while the other supports [$K(\lambda_j, s)$ and K_{j+1} both support A at p].

$$\Updownarrow$$

$\mathcal{L}(p)$ intersects [supports] A at p.

$$\Updownarrow$$

$K(\lambda_j, s)$ intersects A close to p in at least one [two] points.

6.3.3. Let $j \leq I(p)$. Choose a neighbourhood

$$N^{(j)} = N_1^{(j)} \cup \{p\} \cup N_2^{(j)}$$

of p in $M \cap N_j$ such that

$$K_{j+1} \cap N^{(j)} = \{p\}.$$

Let $s_\alpha \varepsilon N_\alpha^{(j)}, \alpha = 1, 2$ and put

$$U_\alpha = \{K(\lambda_j;\ s_\alpha) | s_\alpha \varepsilon N_\alpha^{(j)}\};\ \alpha = 1, 2.$$

Now the family

$$\overline{U_1 \cup \{K_{j+1}\} \cup U_2} \subset \lambda_j \cup \{K_{j+1}\}$$

is homeomorphic to a closed interval (cf. 3.3.7). Hence there exists an end-polynomial C_α of U_α other than K_{j+1}. We may assume that C_α does not meet $N_1^{(j)}$ or $N_2^{(j)}$.

We may assign a continuous orientation to the one parameter family $U_1 \cup \{K_{j+1}\} \cup U_2$. Then the polynomials of U_α all pass through the open set E_α of points in G, defined by

$$E_\alpha = \{C_\alpha^1 \cap K_{j+1}^{-1}\} \cup \{C_\alpha^{-1} \cap K_{j+1}^{i}\}.$$

In the case where $j = n = I(p)$ and n is even, K_{n+1} is a double ray and hence one of the open sets K_{n+1} and K_{n+1}^{-1} is empty. Therefore, one of the sets

$$\{C_\alpha^1 \cap K_{j+1}^{-1}\} \text{ and } \{C_\alpha^{-1} \cap K_{j+1}^1\}$$

in each E_α is empty; $\alpha = 1, 2$.

We have

$$N^{(j)} \subset E_1 \cup \{p\} \cup E_2.$$

Now let $N(K_{j+1})$ be a neighbourhood of K_{j+1}. Then for s sufficiently close to p, we have

$$K(\lambda_j, s) \varepsilon N(K_{j+1}).$$

Hence there is an interval $V_\alpha \subset U_\alpha \cap N(K_{j+1})$. Let $e_\alpha \varepsilon N_\alpha^{(j)}$ be such that $K(\lambda_j, e_\alpha)$ is the end-polynomial different from K_{j+1} of V_α. As s_α moves continuously and monotonically on $N_\alpha^{(j)}$ from e_α to p, then $K(\lambda_j, s_\alpha)$ moves continuously in U_α from $K(\lambda_j, e_\alpha)$ to K_{j+1}. Hence the polynomials $K(\lambda_j, s_\alpha)$ omit none of the polynomials of V_α, i.e., every polynomial of V_α meets $N_\alpha^{(j)}$.

Let $K \varepsilon V_\alpha$. Thus K lies between $K(\lambda_j, e_\alpha)$ and K_{j+1} in V_α. Since p has finite order, there is a neighbourhood N' of p in $N_\alpha^{(j)}$ such that if $s_\alpha \varepsilon N_\alpha^j \cap N'$ is sufficiently close to p, then $s_\alpha \varepsilon K$ and K will also lie between $K(\lambda_j, e_\alpha)$ and $K(\lambda_j, s_\alpha)$ in V_α. Since $e_\alpha \varepsilon K$ and $\{s_\alpha, e_\alpha\} \in N^{(j)} \subset E_\alpha$, they will be separated by K. In particular, at least one point of $K \cap N_\alpha^{(j)}$ is a point of intersection of K and $N_\alpha^{(j)}$; cf. 4.1.4.

Now, if $a_j = 1$, then one of the polynomials K_{j+1} and C_α intersects $N^{(j)}$ at p, while the other supports $N^{(j)}$ there. Hence $N_2^{(j)} \not\subset E_1$; C_1 and C_2 lie on opposite sides of K_{j+1} in λ_j; and U_1 and U_2 are disjoint.

If $a_j = 2$, then K_{j+1} and C_α either both intersect $N^{(j)}$ at p or both support $N^{(j)}$. Hence $N_2^{(j)} \subset E_1$ and C_1, C_2 lie on the same side of K_{j+1} and one of U_1 and U_2 will be contained in the other.

Thus the polynomials $K(\lambda_j, s_1)$ in $V_1 \subset U_1$ and the polynomials $K(\lambda_j, s_2)$ in $V_2 \subset U_2$ lie on the opposite sides of K_{j+1} or the same side of K_{j+1} according as $a_j = 1$ or $a_j = 2$. This proves the lemma.

6.4. We can now complete the proof of our Theorem 4. We first approximate $K(p^{n+1})$ by a polynomial K_n in λ_n, then K_n by a K_{n-1} in λ_{n-1} and so on until we finally approximate K_1 by a K_0 in λ_0. Thus K_0 does not pass through p.

Let M_n be a neighbourhood of p on A. By Lemma 6.3, there exists a K_n in λ_n close to K_{n+1}

which intersects $M_n \setminus \{p\}$ in at least a_n points, say s_n.

In the sequel, we put $j = n - 1, \ldots, 0$, in turn. In M_{j+1} we construct neighbourhoods A_{j+1} of the s_{j+1} and M_j of p which are all mutually disjoint. By Lemma 6.3, there exists a polynomial K_j of λ_j close to K_{j+1} which intersects $M_j \setminus \{p\}$ in at least a_j points s_j and which also intersects each of the $\sum_{k=j+1}^{n} a_k$ arcs $A_{j+1}, \ldots, A_n$.

Altogether, K_0 is close to $K(p^{n+1})$ and intersects $M_n \setminus \{p\}$ in at least $a_0 + a_1 + \ldots + a_n$, distinct points, all of which are different from p.

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