REPRESENTATION THEORY

OF

PARTIALLY ORDERED VECTOR SPACES

By

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SCOPE AND CONTENTS: The major results of this work concern perfect ideals of ordered vector spaces, and a representation theory for ordered vector spaces. Perfect ideals are characterized by the property that their annihilators in the order dual are ideals. We obtain a number of conditions for an ordered vector space which are equivalent to the intersection of the set of perfect maximal ideals being 0. We also obtain conditions which permit an ordered vector space to be represented as a subspace of the sections of a vector bundle. This generalizes the representation theory for odered vector spaces with unit.

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INTRODUCTION

The major results of this work concern perfect ideals of ordered vector spaces, and a representation theory for ordered vector spaces. Perfect ideals are characterized by the property that their annihilators in the order dual are ideals. We obtain a number of conditions for an ordered vector space which are equivalent to the intersection of the set of perfect maximal ideals being 0. We also obtain conditions which permit an ordered vector space to be represented as a subspace of the sections of a vector bundle. This generalizes the representation theory for ordered vector spaces with unit.

Let E be an ordered vector space (OVS), and let Θ be a set of order ideals of E. The map $\langle , \rangle : E \times \Theta \to \bigcup_\Theta E/I$, defined by $\langle a,I \rangle = a_I \in E/I$, for $a \in E$, $I \in \Theta$, where a_I is the image of a in E/I, for the natural projection $E \to E/I$, induces a canonical representation of E into a subspace of the product of the family of OVS $(E/I)_{I \in \Theta}$. A map $q: \bigcup_\Theta E/I \to F$, where q restricted to each E/I is an order isomorphism of E/I, and an OVS F, is called a trivialization of $\bigcup_\Theta E/I$. Clearly, for $\bigcup_\Theta E/I$ to have a trivialization, it is necessary that all the E/I be isomorphic. If q is a trivialization of $\bigcup_\Theta E/I$, then the map $a \to q\hat{a}$, for $a \in E$, defines a representation of E as a subspace of $\mathfrak{F}(\Theta,F)$ the set of functions from Θ to F. More precise representations of an OVS E may be obtained from a system $\langle , \rangle : E \times \Theta \to \bigcup_\Theta E/I$ if Θ and $\bigcup_\Theta E/I$ can be topologized, and if $\bigcup_\Theta E/I$ has continuous local or global trivializations.

Let E be an Archimedean OVS with an order unit e. The intersection of $\Omega(E)$ the set of maximal ideals of E is 0. The OVS E/M is order isomorphic to R the OVS of real numbers, for any M in Ω. Since e is not in any M, e_M generates E/M as a vector space. E is order isomorphic to a subspace of $\Pi_\Omega E/M$ by the natural representation. The unit e may be used to trivialize $\bigcup_\Omega E/M$. The map $q_e : \bigcup_\Omega E/M \to R$, defined by $q_e(x_M)$ equal the unique real number λ such that $x_M = \lambda e_M$, is a trivialization. Since Ω may be given the weakest topology such that the functions $g\hat{a}:\Omega \to R$, for $a \in E$, are continuous, E may be represented as a subspace of the space of continuous functions from Ω to R. The set Ω , endowed with this topology is topologically isomorphic to B_e, the set of positive linear functionals on E for which $h(e) = 1$, endowed with the weak topology of the order dual of E. This approach to the well known representation theory of Archimedean OVS with units forms the basis of our generalization.

An ideal I of an OVS E is called modular if E/I has an order unit, and a positive element a of E is called a mod unit of I if a_I is a unit of E/I. The representation theory for OVS with units may be generalized to OVS which have sufficiently many modular ideals. These ideals will be used to construct local trivializations of $\bigcup_\Omega E/M$.

If $\langle E,F \rangle$ is a dual system of (real) vector spaces, where E and F are ordered, and if $\langle x,y \rangle \geq 0$, for any positive x in E, and any positive y in F, we call $\langle E,F \rangle$ a dual system of OVS. If $\langle E,F \rangle$ is a dual system of OVS, and I is an order ideal of E, we obtain a number of lemmas relating the order dual of E/I to I^o, the annihilator of I in F. Preservation of order and topological properties by quotients of OVS is disclosed.

The annihilator of an ideal of an OVS E is not necessarily an ideal of E^*, the order dual of E. We study a class of ideals which are called perfect. These were introduced by Bonsall for OVS with units in (1) and by Kist for directed OVS in (8). We characterize perfect ideals of directed OVS as those whose annihilators in the order deal are ideals. The annihilators of perfect maximal ideals are one-dimensional ideals in the order dual. If an OVS E has a unit e, then this fact may be used to correspond perfect maximal ideals of E to the extreme points of the set B_e of E^*. If E is an Archimedean OVS with a unit e, then $\langle E,E^*\rangle$ is a dual system of normed OVS. The Krein-Milman Theorem may be applied to assert that B_e is the weak-closure of the convex hull of the extreme points of B. We show that this assertion is equivalent to the perfect maximal ideals of E having 0 intersection, and this is proved without using the Krein-Milman Theorem. When $\langle E,E^*\rangle$ is a dual system of OVS in which E has a weakly closed cone, we obtain conditions on E and E^* which are equivalent to the perfect maximal ideals of E having 0 intersection. This generalizes the results for the case that E is Archimedean with a unit. If E is an Archimedean lattice OVS with unit, then the HK-closure on the set of perfect maximal ideals is shown to be a topological closure.

The first chapter will introduce the basic terminology and contain the definitions and results from the theory of topological vector spaces which will be needed in future chapters. Use of the contents of Chapter I will sometimes be made without explicit reference.

The second chapter will introduce the basic concepts of the theory of ordered vector spaces, and develop the algebraic aspect of the representation theory. Although most of the results of this chapter are known, our development of the theory permits the simplification of many proofs.

In the third chapter we will introduce a type of OVS E, called an absolute OVS, whose order is induced from a map $||:E \to E$, which satisfies some of the properties that the map $x \to |x| = x \vee - x$ in a lattice OVS E satisfies. For absolute OVS we have natural definitions of homomorphisms and ideals. An absolute ideal is called prime if $x \wedge y = 0$ imply x or y is in P. The hull-kernel closure generated by the prime ideals is a topological closure.

Chapter four will contain the results concerning dual systems of OVS, those concerning perfect ideals, and the representation theory.

CHAPTER I

PRELIMINARIES

This chapter introduces the basic terminology and contains the definitions and results from the theory of topological vector spaces which are needed in future chapters.

1. Terminology

Let X,Y be sets. We use the standard notations $x \in X$, $x \notin X$, $X \subseteq Y$, $X = Y$, $X \cap Y$, $X \cup Y$, and $\{..:....\}$. For sets X and Y, the set $\{x \in X : x \notin Y\}$ will be denoted by $X \sim Y$. The empty set will be denoted by $\emptyset$; the power set of a set X by $\mathcal{P}(X)$; and the set whose elements are ordered pairs of elements of X and Y by $X \times Y$. If $A \subseteq X \times Y$, then A^{-1} will denote the set $\{(y,x) \in Y \times X : (x,y) \in A\}$.

A function or map between sets X and Y will be denoted by $f:X \to Y$. The notation $x \to F(x)$ may be used sometimes to define functions. If $f:X \to Y$ is a function, then $A \to f(A) = \{f(x) \in Y : x \in A\}$, for $A \subseteq X$, specifies a function. The notation $(x_i)_{i \in I}$, where $x_i \in X$ will also be used sometimes to denote a function from I to X and when it is used, the function is called a <u>family</u>.

If $(X_i)_{i \in I}$ is a family of sets, $\bigcup_I X_i$ will denote the set $\{x : x \in X_i$ for some $i \in I\}$ and $\bigcap_I X_i$ will denote the set $\{x : x \in X_i$, for all $i \in I\}$. If $(X_i)_{i \in I}$ is a disjoint family of sets, that is, such that $X_i \cap X_j = \emptyset$, for any $i \neq j$, then there is a natural surjection $p: \bigcup_I X_i \to I$ defined by $p(x) = i$, for the unique $i \in I$, such that $x \in X_i$. Conversely, if $p:X \to I$ is a surjective map, then $(X_i = p^{-1}(i))_{i \in I}$ is a disjoint family of sets. If $(X_i)_{i \in I}$ is a family of sets, then $(X_i \times \{i\})_{i \in I}$ is a disjoint family, and $\dot\bigcup_I X_i = \bigcup_I X_i \times \{i\}$ is called the disjoint union of the family $(X_i)_{i \in I}$. The set $\Pi X_i = \{f:I \to \dot\bigcup_I X_i : pf:I \to \dot\bigcup_I X_i \to I$ is the identity map on $I\}$ is called the product of the family $(X_i)_{i \in I}$.

If S is an equivalence relation on a set X, the set of equivalence classes, called the <u>quotient set of</u> X <u>by</u> S, will be denoted by X/S, and $(x)_S$, or briefly, x_S will denote the equivalence class which contains $x \in X$.

A set X endowed with reflexive, transitive antisymmetric relation is called an <u>ordered set</u>. Let $(X,\le)$ be an ordered set. For any $a,b \in X$, $[a,b]$ will denote the set $\{x \in X : a \le x \le b\}$ and $]a,b[$ will denote the set $\{x \in X : a < x < b\}$. The map from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ defined by $A \to [A] = \bigcup\{[x,y] : x,y \in A\}$ is a closure operator.

An ordered set D is said to be <u>directed</u> (up) if, for any $a,b \in D$, there exists an upper bound of the set $\{a,b\}$. A function $(a_i)_{i \in D}$ from a directed set D to a set A is called a <u>net</u>. A <u>subnet</u> of a net $(a_i)_{i \in D}$ is the restriction of the function to some subset D' of D which is also directed. A subnet $(a_i)_{i \in D'}$ of a net $(a_i)_{i \in D}$ is <u>confinal</u>, if D' is confinal in D, i.e., for each $i \in D$ there is an $i' \in D'$ such that $i' \ge i$.

Let X be an ordered set. A net $(a_i)_{i \in D}$ in X is said to order converge to a limit $a \in X$ if $a = \bigvee_i \bigwedge_{j>i} a_j = \bigwedge_i \bigvee_{j>i} a_j$, and we write $a_i \overset{o}{\to} a$, or $\text{o-}\lim_D a_i = a$. If $(a_i)_{i \in I}$ is an o-convergent net in X, then any cofinal subnet also converges to the same limit. A net $(a_i)_{i \in D}$ is increasing (decreasing) if the function is monotone (antimonotone). We write $(a_i\uparrow)_{i \in D}$ for an increasing net in X and $(a_i\downarrow)_{i \in D}$ for a decreasing net in X. An ordered set X is called conditionally $\uparrow$ (conditionally $\downarrow$) complete if any bounded increasing (bounded decreasing) net in X converges.

Let S and T be ordered sets. A pair of antimonotone maps $\sigma: S \to T$ and $\tau: T \to S$ such that $x < \tau\sigma(x)$, for any $x \in S$, and $y < \sigma\tau(y)$, for any $y \in T$, is called a Galois connection between S and T. Let A and B be sets, and let Φ be a subset of $A \times B$. The maps $A \supseteq V \to V^\sigma = \{b \in B : (a,b) \in \Phi, \text{ for all } a \in V\}$, $B \supseteq U \to U^\tau = \{a \in A : (a,b) \in \Phi, \text{ for all } b \in U\}$ define a Galois connection between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ called a polarity. The maps $\tau\sigma : \mathcal{P}(A) \to \mathcal{P}(A)$ and $\sigma\tau: \mathcal{P}(B) \to \mathcal{P}(B)$ are closure operators. The sets $\{V \in \mathcal{P}(A) : V = (V^\sigma)^\tau\}$ and $\{U \in \mathcal{P}(B) : U = (U^\tau)^\sigma\}$ are the closed sets of $\mathcal{P}(A)$ and $\mathcal{P}(B)$ respectively. The closed sets form complete lattices in the ordering by inclusion, in which greatest lower bound means intersection, and the lattices are dually isomorphic.

Let X be a set and $\mathcal{J}$ a collection of subsets of X. The maps

(1) $X \supseteq U \to H(U) = \{I \in \mathcal{J} : U \subseteq I\}$ and

(2) $\mathcal{J} \supseteq V \to K(V) = \bigcap \{I \in \mathcal{J} : I \in V\}$

define a polarity between $\mathcal{P}(X)$ and $\mathcal{P}(\mathcal{J})$, called the <u>hull-kernel</u> polarity generated by the collection $\mathcal{J}$ in X. If $\mathcal{J}$ is closed under intersections, then the elements of $\mathcal{J}$ are KH-closed in $\mathcal{P}(X)$. The sets in $\mathcal{P}(\mathcal{J})$ of the form $U_A = \{I \in \mathcal{J} : A \not\subseteq I\}$, for $A \in \mathcal{P}(X)$, are called <u>HK-open</u> sets. The sets U_a, $a \in X$, are a basis for the HK-open sets in the sense that $U_A = \bigcup_{a \in A} U_a$. The HK-closed sets may be expressed in the form $\sim U_A$, for some $A \subseteq X$, since $\sim U_A = H(A) = HKH(A)$ is closed, and conversely, if V is HK-closed, then $V = HK(V) = \{I \in \mathcal{J} : K(V) \subseteq I\} = \sim U_{K(V)}$.

A HK-closure generated by a set X and a collection of subsets $\mathcal{J}$ of X is a topological closure if, for any $I \in \mathcal{J}$, $U, V \subseteq \mathcal{J}$, $K(U) \cap K(V) \subseteq I$ implies that either $K(U) \subseteq I$, or $K(V) \subseteq I$. It suffices to show that for any $U, V \subseteq \mathcal{J}$, $HK(U \cup V) = HK(U) \cup HK(V)$. If $I \in HK(U)$, then $K(U \cup V) \subseteq K(U) \subseteq I$, which implies $I \in HK(U \cup V)$. If $I \in HK(U \cup V)$, then $K(U) \cap K(V) = K(U \cup V) \subseteq I$. Hence $K(U) \subseteq I$, or $K(V) \subseteq I$.

2. <u>Vector Spaces</u>

All vector spaces considered in the following will have the real numbers as scalar field. All unquantified small Greek letters will be real numbers, R will be the vector space of real numbers, and ^{+}R the set $\{x \in R : x \geq 0\}$.

Let E be a vector space. A <u>vector subspace</u> (briefly, <u>subspace</u>) of E is a non-empty subset M of E closed under addition and scalar multiplication, that is, such that $M + M \subseteq M$ and $RM \subseteq M$. The set of all subspaces of E is closed under arbitrary intersections. If A is a subset of E, the linear hull of A is (A), the intersection of all subspaces of E that contain A; (A) is also called the subspace of E generated by A and can be characterized as the set of all linear combinations of elements of E.

If M is a subspace of a vector space E, the relation $x \tilde{} y$ if $x - y \in M$ is an equivalence relation on E. The quotient set will be denoted by E/M, and (x_M), or briefly, x_M will denote the equivalence class generated by $x \in E$, which is the set $x + M$. The set E/M becomes a vector space by the definitions $x_M + y_M = (x + y)_M$, $\lambda x_M = (\lambda x)_M$.

Let E,F be vector spaces. A function $f:E \to F$ is called a linear map if $f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2)$, for $x_1, x_2 \in E$. The set $L(E,F)$ of all linear maps of E into F, becomes a vector space with the definitions $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ and $\lambda f_1(x) = f_1(\lambda x)$, for $f_1 f_2 \in L(E,F)$, $x \in E$. The vector space $E^{\#} = L(E,R)$ is called the linear dual of E, and its elements are called linear forms (linear functionals) on E. The vector spaces E and F are called (linearly) isomorphic if there exists a linear bijection $f:E \to F$; such a map is called an isomorphism of E onto F.

If $f:E \to F$ is a linear map, the subspace $\ker(f) = f^{-1}(0)$ of E is called the null space (kernel) of f, and f defines an isomorphism f' of $E/\ker(f)$ onto $\operatorname{im}(f) = f(E)$. If p is the quotient map $E \to E/\ker(f)$ and q is the inclusion map $\operatorname{Im}(f) \to F$, then $qf'p$ is called the canonical decomposition of f.

Let E be a vector space. A subset $M \subseteq E$ is called a linear

<u>variety</u> (<u>flat</u>) if $M = x_o + M'_o$ where M'_o is a subspace of E and $x \varepsilon E$. A variety H of E is called a <u>hyperplane</u> of E if $H = x + H'$ where H' is a maximal subspace of E. For any $h \varepsilon E^{\#}$, $h \neq 0$, and any $\lambda \varepsilon R$, the set $h^{-1}(\lambda)$ is a hyperplane of E. Conversely, for any hyperplane H in E, there exists $h \varepsilon E^{\#}$, $h \neq 0$, and $\lambda \varepsilon R$, such that $H = h^{-1}(\lambda)$, and H is a subspace if and only if $\lambda = 0$. If $H = h_1^{-1}(\lambda_1) = h_2^{-1}(\lambda_2)$, for $h_1, h_2 \varepsilon E^{\#}$, $h_1 \neq h_2$, then there exists $\mu \varepsilon R, \mu \neq 0$, such that $h_1 = \mu h_2$ and $\lambda_1 = \mu\lambda_2$.

Let E be a vector space. For any $x,y \varepsilon E$, the set $xy = \{\alpha x + \beta y \ \varepsilon E : \alpha, \beta > 0, \alpha + \beta = 1\}$ is called the <u>segment</u> between x and y. An <u>open segment</u> is a segment with its end points deleted. A set $S \subseteq E$ is called <u>convex</u> if, for any $x,y \varepsilon S$, the segment xy is contained in S. The intersection of a family of convex sets is convex. Hence each subset $S \subseteq E$ is contained in a smallest convex set conv(S), called the <u>convex hull</u> of S. For a non-empty convex set $C \subseteq E$, a set $S \subseteq C$ is called <u>extreme</u> in C if $C \neq \emptyset$ and each segment in C having an interior point, i.e., a point of the open segment, in S is contained in S. A variety M of E is called a <u>supporting</u> variety for C, a convex set in E, if $M \cap C$ is extreme.

Let E be a linear space. A set $A \subseteq E$ is called <u>circled</u> (<u>balanced</u>) if $[-1,1]A \subseteq A$. The intersection of any family of circled sets is circled. The set Cir(A), intersection of all circled sets containing a set $A \subseteq E$, is called the <u>circled hull</u> of A. If sets A,B are contained in E, A <u>absorbs</u> B if there exists $\alpha > 0$ such that $B \subseteq \lambda A$ for $\lambda \geq \alpha$, equivalently, there exists $\alpha > 0$ such that, for any $\mu, \mu \neq 0$, if $\mu \leq \alpha$, then $\mu B \subseteq A$. A set $A \subseteq E$ is absorbing (radial) if A absorbs each finite subset of E. The set of absorbing subsets of E is closed under finite intersections. A set $A \subseteq E$ is a radial at a point $x \varepsilon E$ if and only if $A^{\sim}x$ absorbs each set $\{y\}$, $y \varepsilon E$. Let $h:E \to F$ be a linear map, A a circled subset of E, and B a circled subset of F, then $h(A)$ and $h^{-1}(B)$ are circled. If C is an absorbing subset of F, then $h^{-1}(C)$ is absorbing. If A is an absorbing subset of E and h is surjective, then $h(A)$ is absorbing. A circle set $A \subseteq E$ is absorbing if and only if for any $x \varepsilon E$, there exists $\alpha, \alpha \neq 0$, such that $\alpha x \varepsilon A$.

Let E be a vector space. A function p from E to R is called subadditive if $p(x + y) \leq p(x) + P(y)$, for all x,y in E; p is positive homogeneous if $p(\lambda x) = \lambda p(x)$, for $\lambda > 0$ and x in E; p is sublinear if it is both subadditive and positive homogeneous; p is absolutely homogeneous if $p(\lambda x) = \lambda p(x)$, for all λ and all $x \varepsilon E$; p is a semi-norm if it is both subadditive and absolutely homogeneous. A seminorm p is a norm if $p(x) = 0$, if and only if, $x = 0$, for $x \varepsilon E$. If $p:E \to R$ is sublinear, then $p(0) = 0$ and $-p(-x) \leq p(x)$. If $p:E \to R$ is a seminorm, then $p(x) \geq 0$, for all x in E, and $\ker(p) = \{x \varepsilon E : p(x) = 0\}$ is a linear subspace of E. There is a bijection between the set of seminorms on E and the set of subsets of E which are convex, circled, absorbing and contain 0. This correspondence is given by $p \to M_p = \{x \varepsilon E : p(x) \leq 1\}$ and by $M \to p_M(x) = \inf\{\alpha \varepsilon R : \alpha > 0, 1/\alpha x \varepsilon M\}$. The seminorm p_M is called the (Minskowski) gauge determined by the set M.

Let $p:E \to R$ be a sublinear functional on a linear space E

and let $K_p = \{h \in E^{\#} : h(x) \leq p(x),$ for $x \in E\}$, then the set $Ex(K_p)$ of extreme points of K_p is nonempty and, for each $x \in E$, there exists $h \in Ex(K_p)$ such that $h(x) = p(x)$. This is a result of F. Bonsall which in part, contains the result of Banach that K_p is nonempty.

3. Topological Vector Spaces

Let E be a vector space. A topology $\mathcal{U}$ on E is called a vector topology for E if:

(VT_1) the map $+: E \times E \to E$, defined by $(x,y) \to x + y$, is continuous, and

(VT_2) the map $\cdot: R \times E \to E$, defined by $(\lambda,x) \to \lambda x$, is continuous where E is endowed with $\mathcal{U}$, and $E \times E$ and $R \times E$ are endowed with the product topologies.

The pair $(E, \mathcal{U})$ is called a topological vector space (TVS) and is denoted by $E_{\mathcal{U}}$. If $\mathcal{U}$ is a vector topology for E, then 0 possesses a neighborhood base $\mathcal{B}$ satisfying

(1) for each $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ such that $U + U \subseteq V$,

(2) every $V \in \mathcal{B}$ is circled an absorbing.

A filter base $\mathcal{B}$ in E satisfying (1) and (2) is a neighborhood base of 0 for a unique vector topology $\mathcal{U}$ on E, and will be called a base for a topology on E. A TVS $E_{\mathcal{U}}$ is a Hausdorff space if and only if $\bigcap \mathcal{B} = \{0\}$, where $\mathcal{B}$ is any neighborhood base of 0 in E, equivalently, if, for any $x \in E$, $x \neq 0$, there exists a neighborhood U of 0 such that $x \notin U$. A subset B of a TVS $E_{\mathcal{U}}$ is bounded if it is absorbed by all neighborhoods of 0 in E.

A TVS $E_{\mathcal{U}}$ is called locally convex (l.c.) and $\mathcal{U}$ is called a locally convex (vector topology) if the convex neighborhoods of 0 form a base at 0. If E is a linear space, a filter base $\mathcal{B}$ in E consisting of radial, convex, circled sets satisfying, $V \in \mathcal{B}$ implies $1/2V \in \mathcal{B}$, is a base for a unique l.c. topology. A family $(p_i)_{i \in I}$ of seminorms on E determines a l.c. topology on E as follows: the collection $\mathcal{B} = \{^1/nU\}$, for n, a natural number, and for U, an intersection of finitely many $U_i = \{x \in E : p_i(x) \leq 1\}$, is a base for a l.c. topology on E, the topology generated by $(p_i)_{i \in I}$. Conversely, every l.c. topology on E is generated by a family of seminorms; it suffices to take the gauge functions of a family of convex, circled, neighborhoods of 0 whose positive multiples form a subbase at 0. Every member of a family $(p_1 : E \rightarrow R)_{i \in I}$ of seminorms is continuous in the topology generated by the family, and the topology is Hausdorff if and only if, for any $x \in E$, $x \neq 0$, there exists $i \in I$ such that $p_i(x) > 0$.

Let $E_{\mathcal{U}}$, $F_{\mathcal{V}}$ be TVS. A linear map $h : E \rightarrow F$ is continuous if and only if, it is continuous at 0 in E. If the topologies $\mathcal{U}, \mathcal{V}$ are l.c. and $(p_i)_{i \in I}$ is a family of seminorms generating $\mathcal{U}$, then h is continuous if and only if, for each continuous seminorm $q : F \rightarrow R$, there exists a finite subset $N \subseteq I$ and $\lambda > 0$ such that $q(h(x)) \leq \lambda \sup_N p_i(x)$, for all $x \in E$.

Let $E_{\mathcal{U}}$ be a TVS. The vector space $E'_{\mathcal{U}}$ of all continuous linear functionals on E is called the topological dual of E.

Let $E_{\mathcal{U}}$ be a TVS, N a (vector) subspace of E. The quotient topology $\bar{\mathcal{U}}$ for E/N is the finest topology on E/N for which the projection $p:E \to E/N$ is continuous. This is a vector topology and $(E/N)_{\bar{\mathcal{U}}}$ is called the quotient space of E over N. A base for $\bar{\mathcal{U}}$ may be given by $\mathcal{B}_N = \{ V_N = V + N : V \in \mathcal{B}\}$ where $\mathcal{B}$ is a base for $\mathcal{U}$. The projection p is an open map. If $F_{\mathcal{V}}$ is a TVS, a linear map $h:E/N \to F$ is continuous if and only if, $h:E \to F$ is continuous on E. The quotient topology $\bar{\mathcal{U}}$ is Hausdorff if and only if, N is closed in E and $\bar{\mathcal{U}}$ is l.c. if $\mathcal{U}$ is l.c.

4. Dual Systems of Sets

A pairing of sets A,B to a set C is a map $\langle , \rangle : A \times B \to C$ which satisfies the separation properties:

(1) for any $a_1, a_2 \in A$, there exists $b \in B$ such that $\langle a_1, b\rangle \neq \langle a_2, b\rangle$ and

(2) for any $b_1, b_2 \in B$, there exists $a \in A$ such that $\langle a, b_1\rangle \neq \langle a, b_2\rangle$.

A pairing $\langle , \rangle : A \times B \to C$ will be denoted by $\langle A,B;C\rangle$ and will also be called a dual system of sets or a duality.

Let $\langle A,B;C\rangle$ be a duality, let $\mathcal{F}(A;C)$ be the set of all functions from A to C, and let $\mathcal{F}(B;C)$ be the set of all functions from B to C. The map $\hat{}:A \to \mathcal{F}(B;C)$ defined by $\hat{a}(b) = \langle a, B\rangle$, for $a \in A$, $b \in B$, is an injection called the canonical representation of A determined by the duality $\langle A,B;C\rangle$. The canonical representation $\hat{}:B \to \mathcal{F}(A,C)$ is defined analogously. The images of A and

B will be denoted by $\hat{A}$ and $\hat{B}$, respectively.

If $\langle A,B;R\rangle$ is a dual system of sets paired to R, the set of real numbers, we abbreviate $\langle A,B;R\rangle$ to $\langle A,B\rangle$. The sets $\mathfrak{F}(A;R)$ and $\mathfrak{F}(B;R)$ are vector spaces and have an order relation under the pointwise definition of these operations and relations. The order relation may be subduced, by restriction, to $\hat{A}$ and $\hat{B}$ from $\mathfrak{F}(B;R)$ and $\mathfrak{F}(A;R)$, respectively. Similarly the linear operations may be subduced to at least partially defined operations on $\hat{A}$ and $\hat{B}$. Since the maps $\hat{}:A \to \hat{A} \subseteq \mathfrak{F}(B;R)$ and $\hat{}:B \to \hat{B} \subseteq \mathfrak{F}(A;R)$ are bijections, the structure on $\hat{A}$ and $\hat{B}$ may be transferred to A and B, respectively. The pairing function $\langle,\rangle:A \times B \to R$ is bilinear with respect to these induced operations, whenever they are defined.

Let $\langle A,B\rangle$ be a dual system of sets (paired to R). The <u>weak topology</u> $(w(A,B))$ for A, determined by $\langle A,B\rangle$, is the weakest topology on A for which the functions in $\hat{B}$ are continuous. A neighborhood base for a point $a \in A$ in the $w(A,B)$-topology is given by $N(a,W,\lambda) = \{a' \in A : |\langle a',b_i\rangle - \langle a,b_i\rangle| \leq \lambda \ , b_i \in W\}$, for every finite subset W of B, and for every $\lambda \in R$. The topology $w(A,B)$ is Hausdorff, since, for any $a_1, a_2 \in A$, $a_1 \neq a_2$, there exists $b \in B$ such that $\langle a_1,b\rangle \neq \langle a_2,b\rangle$; let $\lambda = |\langle a_1,b\rangle - \langle a_2,b\rangle| /2$, then $N(a_1,b,\lambda) \cap N(a_2 b,\lambda) = \emptyset$. A net $(a_i)_{i \in D}$ in A converges to an element $a \in A$ in the $w(A,B)$-topology if and only if, $\langle a_i,b\rangle$ converges to $\langle a,b\rangle$, for each $b \in B$.

Polarities arise naturally in dual systems of sets. Let $\langle A,B;R\rangle$ be a duality, and let $\mathcal{P}(A)$, $\mathcal{P}(B)$ be the power sets of A

and B respectively. The maps ${}^{o}:\mathcal{P}(A) \to \mathcal{P}(B)$, defined by $U \to U^{o} = \{b \in B : \langle a,b\rangle = o,\ \text{for } a \in A\}$, and ${}^{o}:\mathcal{P}(B) \to \mathcal{P}(A)$, defined analogolously, are a polarity, the polarity of <u>annihilation</u>, and U^{o} is called the <u>annihilator</u> of U.

5. <u>Dual Systems of Vector Spaces</u>

A duality $\langle E,F\rangle$ is called a <u>dual system of linear spaces</u> if E and F are linear spaces and $\langle,\rangle : E \times F \to R$ is a bilinear functional. If $\langle E,F\rangle$ is a linear duality, the image of E under the canonical representation is a subspace of $F^{\#}$, and the image of F is a subspace $E^{\#}$. The $w(E,F)$-topology on E is a l.c. Hausdorff vector topology. A linear map $f:E \to R$ is $w(E,F)$-continuous if and only if, there exists $y \in F$ such that $f(x) = \langle x,y\rangle$, for all $x \in E$. Thus $F_{w(F,E)}$ is isomorphic to $E'_{w(E,F)}$ as a TVS.

If $E_{\mathcal{U}}$ is a l.c. Hausdorff TVS, then, as a consequence of the Banach Theorem, $E'_{\mathcal{U}}$ separates points of E. Thus the bilinear map $\langle,\rangle : E_{\mathcal{U}} \times E'_{\mathcal{U}} \to R$ defined by $\langle a,h\rangle = h(a)$, for any $a \in E$, $h \in E'$ is a natural pairing of $E_{\mathcal{U}}$ and $E'_{\mathcal{U}}$. If $\langle E,E'\rangle$ is a dual system of linear spaces, then $\mathcal{U}$ is a l.c. Hausdorff topology, since $\mathcal{U}$ is finer than $w(E,E')$ and $w(E,E')$ is l.c. Hausdorff.

Let $\langle E,F\rangle$ be a dual system of linear spaces. The maps ${}^{\bullet}:\mathcal{P}(E) \to \mathcal{P}(F)$, defined by $U \to U^{\bullet} = \{b \in F : |\langle a,b\rangle| \leq 1,\ \text{if } a \in U\}$, and ${}^{\bullet}:\mathcal{P}(F) \to \mathcal{P}(E)$, defined analogolously, are a polarity between $\mathcal{P}(E)$ and $\mathcal{P}(F)$. The polar $U^{\bullet}$ of a subset $U \subseteq E$ is a $w(E,F)$-closed convex subset of F containing 0. If N is a subspace of E, then $N^{\bullet} = N^{o}$ and N^{o} is a subspace of F. The following results are consequences of the Banach Theorem. For any subset $U \subseteq E$, the bipolar $U^{\circ\circ}$ is the $w(E,F)$-closed, convex hull of $U \cup \{0\}$. This is called the Bipolar Theorem. If $(U_i)_{i \varepsilon I}$ is a family of $w(E,F)$-closed convex subsets of E, each containing 0, then the polar of $U = \bigcap_I U_i$ is the $w(E,F)$-closed convex hull of $\bigcup_I U_i$. If N is a subspace of E, then $N = N^{oo}$ if and only if, N is $w(E,F)$-closed. The map $N \to N^o$ is a dual isomorphism of the lattice of $w(E,F)$-closed subspaces onto the lattice of $w(F,E)$-closed subspaces of F, the lattice operations being defined by $N_1 \wedge N_2 = N_1 \cap N_2$ and $N_1 \vee N_2 = \overline{N_1 + N_2}$, w-closure of $N_1 + N_2$.

Let $E_{\mathcal{U}}$ be a l.c. Hausdorff TVS, let N be a subspace of E, and let $p: E \to F = E/N$ be the projection map. The linear map $(E/N)'_{\mathcal{U}} \to N^o \subseteq E'$, defined by $g \to gp$, is a linear isomorphism and $w(F,F'_{\mathcal{U}})$ is the quotient topology of $w(E,E'_{\mathcal{U}})$.

Let $F_{\mathcal{U}}$ be a l.c. Hausdorff TVS, let $M_{\mathcal{V}}$ be a subspace of F endowed with the topology induced from $F_{\mathcal{U}}$, and let $q: M \to F$ be the canonical injection map. The linear map $F'_{\mathcal{U}} \to M'_{\mathcal{V}}$, defined by $f \to fq$, induces a linear isomorphism between $M'_{\mathcal{V}}$ and $F'_{\mathcal{V}}/M^o$, and the $w(M,M')$ topology is the topology induced from $F_{w(F,F'_{\mathcal{U}})}$.

Let $\langle F,G \rangle$ be a dual system of linear spaces. A l.c. topology $\mathcal{U}$ on F is consistent with the duality if the dual of F is G. A topology on F consistent with $\langle F,G \rangle$ is finer than $w(F,G)$, hence Hausdorff. The closure of a convex subset $C \subseteq F$ is the same for all l.c. topologies on F consistent with $\langle F,G \rangle$. The families of bounded subsets of F are identical for all l.c. topologies on F consistent with $\langle F,G \rangle$. There exists a finest l.c.

topology on F consistent with $\langle F,G\rangle$ which is called the Mackey-topology on F with respect to $\langle F,G\rangle$.

Let $E_{\mathcal{U}}$ be a l.c. Hausdorff TVS. The convex hull of a finite family of compact, convex subsets is compact. The Krein-Milman Theorem asserts that each compact, convex set A in E coincides with the closed convex hull of the set Ex(A) of extreme points of A. Furthermore, if K is a compact subset of E whose closed convex hull C is compact, then every extreme point of C is in K.

Let $\langle E,F\rangle$ be a dual system of vector spaces, and if A is a convex, circled set in E containing 0, then A° is w(F,E)-compact in F.

6. Normed Vector Spaces

Let E be a linear space with a norm p. The pair (E,p) is called a normed vector space. The topology generated by a norm is l.c. and Hausdorff. A complete normed space is called a Banach space. A TVS $E_{\mathcal{U}}$ whose topology can be generated by a norm is normable. Two norms p and q on E are equivalent if they generate the same topology. It is necessary and sufficient for p and q to be equivalent that there exist $\alpha, \beta \in R$ such that $q(x) \leq \alpha p(x) \leq \beta q(x)$, for all $x \in E$. The set $\{x \in E : p(x) \leq 1\}$ is called the unit sphere of a normed space E with norm p.

The quotient space of a normable (and complete) TVS $E_{\mathcal{U}}$ over a closed subspace N of E is normable (and complete). If p is a norm which generates $\mathcal{U}$, then $x_N \to \bar{p}(x_N) = \inf \{ p(x) : x \in x_N \}$ is a norm which generates the quotient topology on E/N.

If E is a normed space with a norm p, and $\mathcal{U}$ is the topology generated by p, then $h \to q(h) = \sup\{ |h(x)| : p(x) \leq 1\}$ is a norm on $E'_{\mathcal{U}}$, and $E'_{\mathcal{U}}$ is complete in this norm. The unit sphere of $(E'_{\mathcal{U}},q)$ is the polar S° of the unit sphere S in (E,p), and S° is $w(E',E)$-compact.

CHAPTER II

ORDERED VECTOR SPACES

This chapter is concerned with the general properties of ordered vector spaces, and with the algebraic aspects of their ideal and representation theory.

1. Basic Definitions

Let E be a linear space. A set $K \subseteq E$ is called a wedge if:

(C_1) $0 \in K$,

(C_2) $K + K \subseteq K$,

(C_3) $\lambda K \subseteq K$, for all $\lambda > 0$.

A wedge K is called a cone if

(C_4) $K \cap -K = \{0\}$.

A cone K in E induces an order $\leq$ on E by $a \leq b$ if $b - a \in K$. Moreover, this order satisfies

(O_1) $a \leq b$ implies $\lambda a \leq \lambda b$, for $\lambda > 0$, and $\lambda b \leq \lambda a$, for all $\lambda < 0$,

(O_2) $a \leq b$ and $c \leq d$ imply $a + c \leq b + d$.

An order $\leq$ on a linear space E which satisfies (O_1) and (O_2) is called a vector ordering for E. If $\leq$ is a vector ordering on E, then $K = \{x \in E : 0 \leq x\}$ is a cone in E. A pair $(E, {}^{+}E)$, where E is a linear space and ${}^{+}E$ is a cone in E, is called an ordered vector space (OVS).

(1.1) Lemma: For any a,b in an OVS E, $a \vee b$ exists if and only if, there exist $c \in E$ such that $(a + {}^{+}E) \cap (b + {}^{+}E) = c + {}^{+}E$.

Proof: Suppose there exists $c \in E$ such that $(a + {}^{+}E) \cap (b + {}^{+}E) = c + {}^{+}E$. We show that $c = a \vee b$. Since $c \in c + {}^{+}E$, we have $c \in a + {}^{+}E$ and $c \in b + {}^{+}E$. This implies $c = a + x = b + y$, for some $x,y \in {}^{+}E$, which implies $c \geq a,b$. If $u \geq a,b$, then $u = a + (u - a) = b + (u - b)$, which implies $u \in a + {}^{+}E$ and $u \in b + {}^{+}E$, which implies $u \in c + {}^{+}E$, which implies $u = c + w$, for some $w \in {}^{+}E$. Thus, $c = a \vee b$. Conversely, if $a \vee b$ exists, we show that $(a + {}^{+}E) \cap (b + {}^{+}E) = a \vee b + {}^{+}E$. If $u \in a \vee b + {}^{+}E$, then $u = a \vee b + w$, for some $w \geq 0$, which implies $u \geq a + w$ and $u \geq b + w$. Hence, $u = a + w + w'$ and $u = b + w + w''$, for suitable w', w'' in ${}^{+}E$. Thus, $u \in (a + {}^{+}E) \cap (b + {}^{+}E)$. If $u \in (a + {}^{+}E) \cap (b + {}^{+}E)$, then $u = a + z = b + z'$ which implies $u \geq a,b$, which implies $u \geq a \vee b$, which implies $u = a \vee b + z''$, for some $z'' \in {}^{+}E$. Thus, $u \in a \vee b + {}^{+}E$.

The following identities are valid in an OVS E, if the required meets and joins exist on one side of the identity:

(1) $(a \wedge b) + c = (a + c) \wedge (b + c)$; $(a \vee b) + c = (a + c) \vee (b + c)$,

(2) $\lambda(a \wedge b) = \lambda a \wedge \lambda b$; $\lambda(a \vee b) = \lambda a \vee \lambda b$, for $\lambda > 0$,

(3) $-(a \wedge b) = -a \vee -b$; $-(a \vee b) = -a \wedge -b$.

Furthermore, if $a \vee b$ exists, then $a \wedge b = a + b -(a \vee b)$ since, $a + b -(a \vee b) = a + b + (-a \wedge -b) = (a + b -a) \wedge (a + b -b) = a \wedge b$.

Let E be an OVS. For any subset $A \subseteq E$, we have $[A] = \bigcup \{[x,y]: x,y \in A\} = (A + {}^+E) \cap (A - {}^+E)$. A subset $B \subseteq E$ is called (order) saturated if $B = [B]$; it is immediate that, for any $A \subseteq E$, $[A]$ is the intersection of all saturated sets containing A, and $[A]$ is called the saturated hull of A. Furthermore, $[A]$ is convex if A is convex and $[A]$ is circled if A is circled. If $\mathfrak{F}$ is a filter base in E, then $\{[F]: F \in \mathfrak{F}\}$ is a filter base in E, the corresponding filter will be denoted by $[\mathfrak{F}]$.

Let K be a cone in a linear space E. If $E = K + (-K)$, then K is called reproducing. The order induced by K is directed if and only if, K is reproducing. To show that a vector ordering is directed, it suffices to show that any $x \in E$ has a positive upper bound. A vector ordering is called almost Archimedean if whenever $x,y \in E$, are such that $-\lambda y \leq x \leq \lambda y$, for all $\lambda > 0$, then $x = 0$. A vector ordering is called Archimedean if, whenever $x,y \in E$ are such that $0 \leq y$ and $x \leq \lambda y$, for all $\lambda > 0$, then $x \leq 0$. Every Archimedean ordering is almost Archimedean.

2. Order Ideals

If F is a linear subspace of an OVS E, then F becomes an OVS when it is paired with the subduced cone ${}^+F = {}^+E \cap F$.

A subspace I of an OVS E, which satisfies any of the following equivalent conditions, will be called an (order) ideal of E.

(1) $I = [I]$ i.e., $x,z \in I$ and $y \in [x,z]$ imply $y \in I$,

(2) $x \in {}^+I$, $y \in [-x,x]$ imply $y \in I$,

(3) $x \in {}^{+}I$, $y \in [0,x]$ imply $y \in I$.

The proofs that (1) implies (2), and (2) implies (3) are trivial. To see (3) implies (1), suppose $x,z \in I$ and $y \in [x,z]$. Thus, $0 \leq y - x \leq z - x$ and $z - x \in {}^{+}I$, which implies $y - x \in I$, and so $y = (y - x) + x$ is an element of I.

An OVS E, is trivially ordered if and only if, ${}^{+}E = \{0\}$ and E is fully ordered if and only if, $E = {}^{+}E \cup (-{}^{+}E)$.

Subspaces of an OVS E, which are trivially ordered in the subduced order are clearly ideals.

(2.1) <u>Lemma</u>: (Edwards (4)). A linear subspace I, of an OVS E, is an ideal if and only if, $I \cap {}^{+}E$ is extremal, or $I \cap {}^{+}E = \emptyset$.

<u>Proof</u>: If I is an ideal of E, then $A = I \cap {}^{+}E$ is convex and contains zero. Suppose $x \in A$ where $x = \lambda p + (1 - \lambda)q$, for $p,q \in {}^{+}E$, and $0 < \lambda < 1$. We have $0 \leq \lambda p \leq x$, which implies $\lambda p \in I$, and so $p \in I \cap {}^{+}E$. Similarly, $q \in I \cap {}^{+}E$. Since A is convex, the segment pq is contained in A; thus, A is extremal. Conversely, suppose $I \cap {}^{+}E = A$ is extremal. If $0 < x < y$, for $y \in I$, then $1/2y = 1/2(y - x) + 1/2x$ and $1/2y \in I \cap {}^{+}E$. Since A is extremal, $x \in I \cap {}^{+}E$, and so I is an ideal.

3. Order Homomorphisms

Let E and F be OVS, and let L(E,F) be the vector space of linear maps from E to F. An element h in L(E,F) is called <u>o-bounded</u> if for any order interval [x,y] in E, h ([x,y]) is contained in some order interval in F; h is called <u>positive</u> (an <u>order homomorphism</u>) if $h({}^{+}E) \subseteq {}^{+}F$. The map h is positive if and only if, h is isotone. If h is positive, then h is o-bounded, since $h([x,y]) = [h(x),h(y)]$. A map h is called an <u>embedding</u> if h is an injection and $h(x) \geq 0$, for $x \in E$, implies $x \geq 0$. An order <u>isomorphism</u> is a linear bijection $h:E \to F$ such that h and h^{-1} are positive.

A map $h:E \to F$, in the category of OVS and positive (linear) maps, is a monomorphism if and only if, it is injective. First, we know that since h may be regarded as a map in the category of linear spaces and linear maps, h is injective if and only if, $\ker(h) = \{0\}$. If $g:E' \to E$ and $f:E' \to E$ are maps such that $hf = hg$, and if h is injective, then $h(f(x) - g(x)) = 0$, for $x \in E'$, and since $\ker(h) = \{0\}$, we have $f(x) - g(x) = 0$, for all $x \in E$, i.e., $f = g$. Conversely, if h is monomorphic, then ker(h) is an OVS. The zero map and the injection map from ker(h) to E are positive maps and $h0 = h\,inj$. Thus, since h is monomorphic, $0 = inj$ and so $\ker(h) = \{0\}$.

Surjective maps in the category of OVS and positive maps are epimorphisms. If $h:E \to F$ is a surjective positive map, and if $j:F \to G$ and $k:F \to G$ are positive maps such that $jh = kh$, then $j(x) = k(x)$ for $x \in h(E) = F$ and so $j = k$.

A map in the category of OVS and positive maps may be both an epimorphism and a monomorphism without being an isomorphism. If E has a non-trivial cone ${}^{+}E$, then the identity map $i: (E,\{0\}) \to (E,{}^{+}E)$ is bijective, hence it is both a monomorphism and an epimorphism.

However, it is not an isomorphism.

Let E and F be OVS. The kernel of a positive map $h:E \to F$ is an ideal, because $\ker(h)$ is a subspace and if $x \in \ker(h)$, $0 \leq x$, $y \in [0,x]$, then $0 = h(0) \leq h(y) \leq h(x) = 0$, which implies $y \in \ker(h)$. If I is an ideal of E, then the quotient E/I can be given a vector ordering so that the projection $E \to E/I$ is positive. We show that $E_I = \{x + I \in E/I : x \in {}^+E\}$ is a cone in E/I. It is clearly a wedge. If $a + I = -b + I$, for some $a,b \in {}^+E$, then $a + b = i$ for some $i \in I$. Hence, a and b are in $[0,i]$, which implies $a,b \in I$, which implies $a + I = b + I = I$. Thus, ${}^+E_I \cap (-{}^+E_I) = 0$. We have that $x_I \geq 0$ if and only if, there exists a $y \in x_I$ such that $y \geq 0$. If E is directed, then E/I is also directed, for any ideal I in E, for if $x + I \in E/I$, then $x + I = y - z + I = y + I - z + I$, for some $y,z \in {}^+E$, and so $x + I$ and $z + I$ are positive.

Let E and F be OVS. If a positive map $h:E \to F$ is surjective and if $h({}^+E) = {}^+F$, then the map $h':E/\ker(h) \to F$, defined by $h'(x_I) = h(x)$, is an order isomorphism. We have that h' is a linear bijection. If $x_I \in {}^+E_I$, then there exist $y \in x_I$ such that $y \in {}^+E$ and $h'(x_I) = h(y) \in {}^+F$. If $z \in {}^+F$, then there exists $x \in {}^+E$ such that $h(x) = z$, and so $h'^{-1}(z) = x_I \in {}^+E_I$. Hence, h' and h'^{-1} are positive.

Let $h:E \to F$ be a positive map between OVS E and F. If I is an ideal in F, then $h^{-1}(I)$ is an ideal in E, since $h^{-1}(I)$ is a subspace, and if $x \in {}^+E$, $y \in {}^+(h^{-1}(I))$, and $x \leq y$, then $h(x) \leq h(y)$

implies $h(x)$ is in I and so $y \in H^{-1}(I)$.

Let $h:E \to F$ be a positive map between OVS E and F. If I is an ideal of E which is contained in $\ker(h)$, then there exists a unique positive map $h^*:E/I \to F$ obtained by composing the projection $E/I \to E/\ker(h)$ and the map $h':E/\ker(h) \to F$ induced by h.

Let E and F be OVS, and let ${}^+H(E,F)$ be the set of positive linear maps in $L(E,F)$. The set ${}^+H$ is always a wedge in $L(E,F)$. However, in general ${}^+H$ is not a cone. If E is directed, then ${}^+H$ is a cone. To verify that ${}^+H \cap -{}^+H = \{0\}$, suppose that $h \in {}^+H \cap -{}^+H$. We have $h({}^+E) \subseteq {}^+F$, $-h({}^+E) \subseteq {}^+F$, and so $h({}^+E) = 0$. Since E is directed, every $x \in E$ is in some $[-a,a]$, for $a \in {}^+E$. Hence $h(x) = 0$, for all $x \in E$. Thus ${}^+H$ is a cone.

Let E and F be OVS, such that ${}^+H(E,F)$ is a cone in $L(E,F)$. This provides a natural order for $L(E,F)$. If F is Archimedean, then any subspace M of $L(E,F)$ is also Archimedean in the subduced ordering. Suppose $f \in M$, $g \in {}^+M$ and $f \leq \lambda g$, for all $\lambda > 0$, then for any $x \in {}^+E$, we have $f(x) \leq \lambda g(x)$, which implies $f(x) \leq 0$; thus, $f \leq 0$.

If E is a directed OVS, then the directed OVS $E^* = {}^+H(E,R) + (-{}^+H(E,R))$ is called the <u>order dual</u> of E.

4. Direct Products and Sums of Ordered Vector Spaces

Let $(E_i)_{i \in I}$ be a family of OVS. The set $\Pi_I E_i$ becomes an OVS with the linear structure defined by the pointwise operations, and with the cone ${}^+(\Pi_I E_i) = \{f \in \Pi_I E_i : f(i) \in {}^+E_i \text{ for } i \in I\}$. The projections $p_i : \Pi_I E_i \to E_i$, for $i \in I$, defined by $p_i(f) = f(i)$, are positive maps. The subspace $\oplus_I E_i = \{f \in \Pi_I E_i : f(i) = 0$, for all but finitely many $i \in I\}$, with the subduced ordering is an OVS. The injections $q_i : E_i \to \oplus_I E_i$, defined by $[q_i(x)](j)$ equals x if $i = j$; and equals 0 if $i \neq j$, are positive maps. The OVS $\Pi_I E_i$ and $\oplus_I E_i$ are called the <u>(order) product</u> and <u>(order) sum</u> respectively, of the family of OVS $(E_i)_{i \in I}$. Clearly, finite sums and products are isomorphic.

(4.1) <u>Lemma</u>: If an OVS E, is the sum $\oplus_A E_i$ of a family $(E_i)_{i \in A}$ of OVS, then the family $(I_i)_{i \in A}$, where $I_i = q_i(E_i)$, the canonical image of E_i in E, is a disjoint family of ideals of E, and $E = \sum_A I_i$, and $^+E = \sum_A {}^+I_i$. Conversely, if $(I_i)_{i \in A}$ is a disjoint family of ideals of an OVS E, such that $E = \sum_A I_i$ and $^+E = \sum_A {}^+I_i$, then E is isomorphic to $\oplus_A I_i$.

<u>Proof</u>: Suppose $f,g \in {}^+E$, $g \in I_{i'}$, and $f \leq g$, for $i' \in A$, then $f(i) \leq g(i)$, for all $i \in A$, and $g(i) = 0$, for $i \neq i'$, implies $f(i) = 0$, for $i \neq i'$, and so $f \in I_{i'}$. Clearly $I_i \cap I_j = \{0\}$, for $i \neq j$. If $f \in E$, then $f = \sum_N q_i p_i(f)$, for a finite subset N of A, where $p_i : \oplus_A E_i \to E_i$ and $q_i : E_i \to \oplus_A E_i$ are the natural projections and injections; furthermore, if $f \in {}^+E$, then $q_i p_i(f) \in {}^+I_i$ since the composition of positive maps is positive. Conversely, if $(I_i)_{i \in A}$ is a disjoint family of ideals of an OVS E, for which $E = \sum_A I_i$ and $^+E = \sum_A {}^+I_i$, then each f in E has a unique decomposition $f = \sum_N f_i$, where $f_i \in I_i$ and N is a finite subset of A. The map $f \to \hat{f}$ where $\hat{f}(i) = f_i$, for $f \in E$, $i \in A$, is a vector isomorphism

between E and $\bigoplus_A I_i$, and since $f \in {}^+E$, if and only if, $f = \sum_N f_i$ where $f_i \in {}^+I_i$, this mapping is an order isomorphism.

5. Projective and Injective Limits of Ordered Vector Spaces

A projective (inverse) family of OVS is a net $(E_i)_{i \in D}$ of OVS together with a family of positive maps $(h_i^j E_j \to E_i)_{i,j \in D, i \leq j}$ which satisfies:

(1) h_i^i is the identity map on E_i, for $i \in D$,

(2) $h_i^k = h_i^j h_j^k : E_k \to E_i$, for $i \leq j \leq k$ in D.

A morphism of a projective family of OVS, $(h_i^j : E_j \to E_i)_{i,j \in D, i \leq j}$ is a family of positive maps $(g_i : H \to E_i)_{i \in D}$ which satisfies

$$g_i = h_i^j g_j, \text{ if } i \leq j.$$

A projective limit of a projective family, $(h_i^j E_j \to E_i)_{i,j \in D, i \leq j}$, is a morphism $p_i : \text{proj lim}_D E_i \to E_i$ for which, given any morphism $(g_i : H \to E_i)_{i \in D}$, there exists a unique positive map $g' : H \to \text{proj lim}_D E_i$ such that $p_i g' = g_i$, for all $i \in D$.

If $(h_i^j : E_j \to E_i)_{i,j \in D, i \leq j}$, is a projective family of OVS, then the projective limit of this family may be constructed as follows. The set $E = \{a \in \Pi_D E_i : a(i) = h_i^j(a(j))$, whenever $i \leq j\}$ is a linear subspace of $\Pi_D E_i$, because, if $a, b \in E$, then $\mu a(i) + \tau b(i) = h_i^j(\mu a(j)) + h_i^j(\tau b(j)) = h_i^j(\mu a(j) + \tau b(j))$, for $i \leq j$, implies $\mu a + \tau b \in E$. With the subduced order, E is an OVS. The maps $p_i : E \to E_i$ defined by $p_i(x) = x(i)$ are positive and $p_i(x) = x(i) = h_i^j(x(j)) = h_i^j p_j(x)$, if $i \leq j$. If $(g_i : F \to E_i)_{i \in D}$ is a family of positive maps which satisfy $g_i = h_i^j g_j$, for $i \leq j$, then the map $g : F \to E$ defined by $g(x)(i) = g_i(x)$, for $i \in D$, for $x \in F$, is positive, and $p_i g = g_i$, since, for

$x \in F$, $p_i(g(x)) = g(x)(i) = g_i(x)$. The map $g:F \to E$ is unique, for if $f:F \to E$ satisfies $p_i f = g_i$, then, for $x \in F$, $p_i(f(x)) = g_i(x)$, hence, $f(x)(i) = g_i(x)$ and so $f = g$. Hence, $E = \text{proj lim}_D E_i$.

We note further that $g:F \to \text{proj lim}_D E_i$ is an isomorphism if and only if:

(a) if $g_i(b) = g_i(b')$, for all $i \in D$, $b, b' \in F$, then $b = b'$,

(b) for any $a \in \text{proj lim}_E E_i$, there exists $b \in F$ such that $g_i(b) = a(i)$, for all $i \in D$,

(c) for any $a \in {}^+(\text{proj lim}_D E_i)$, there exists $b \in {}^+F$ such that $g_i(b) = a(i)$.

An <u>injective (direct) family</u> of OVS is a net $(E_i)_{i \in D}$ of OVS together with a family of positive maps $(h_i^j:E_i \to E_j)_{i,j \in D, i \leq j}$ which satisfies:

(1) h_i^i is the identity map on E_i, for $i \in D$,

(2) $h_i^j = h_j^k h_i^j : E_i \to E_k$, for $i \leq j \leq k$ in D.

A <u>morphism</u> of an injective family of OVS is a family of positive maps $(g_i:E_i \to H)_{i \in D}$, which satisfies :

$$g_i = g_j h_i^j \quad \text{if} \quad i \leq j.$$

An <u>injective limit</u> of an injective family, $(h_i^j:E_i \to E_j)_{i,j \in D, i \leq j}$ is a morphism $q_i:E_i \to \text{inj lim}_D E_i$ for which, given any morphism $q_i:E_i \to H$, $i \in D$, there exists a unique positive map $g':\text{inj lim}_D E_i \to H$ such that $g'q_i = g_i$, for all $i \in D$.

If $(h_i^j:E_i \to E_j')_{i,j \in D, i \leq j}$ is an injective family of OVS, then the injective limit of the family may be constructed as follows. The relation $\sim$ in the disjoint union, $\dot{\bigcup}_D E_i$, defined by $x_i \sim y_j$, where

$x_i \in E_i$, $y_j \in E_j$, if there exists $k \in D$ such that $i,j \leq k$ and $h_i^k(x_i) = h_j^k(y_j)$, is an equivalence relation. The set E of equivalence classes may be given the structure of a linear space. If $x,y \in E$, then, for any $x_i \in X$ and $y_j \in y$, there exists $k \geq i,j$ such that $h_i^k(x_i)$, $h_j^k(y_j) \in E_k$ and $h_i^k(x_i) \in x$, $h_j^k(y_j) \in y$, and so $\tau x + \mu y = (\tau h_i^k(x_i) + \mu h_j^k(x_j))_\sim$ defines linear operations on E. The set ${}^+E = \{x \in E :$ there exists $x_i \in x$ with $x_i \geq 0\}$ is a cone in E. If $x,y \in {}^+E$, then there exist $x_i \in x$, $y_j \in y$ with $x_i,y_j \geq 0$, and there exist $k \in D$ such that $x_k = h_i^k(x_i)$ and $y_k = h_j^k(x_j)$ are positive, hence, $x + y = (h_i^k(x_i) + h_j^k(x_j))_\sim \in {}^+E$. If $x \in {}^+E \cap -{}^+E$, then there exist $x_i,x_j \in x$ such that $x_i \in {}^+E_i$, $x_j \in -{}^+E_j$, and there exists $k \in D$ with $k > i,j$ such that $h_i^k(x_i) \in {}^+E_k$, $h_j^k(x_k) \in -{}^+E_k$, and $h_i^k(x_i) = h_j^k(x_j) = 0$; hence, $x = 0$. Clearly $0 \in {}^+E$ and ${}^+R\,{}^+E \subseteq {}^+E$. The maps $u_i : E_i \to E$, defined by composing the canonical injections $E_i \to \dot{\bigcup}_D E_i \to \dot{\bigcup}_D E_i/\sim$ are positive and $u_i = u_j h_i^j$, if $i \leq j$. Suppose $f_i : E_i \to F$ is a morphism of $(h_i^j : E_i \to E_j)_{i,j \in D,\, i \leq j}$. If x_i, $y_j \in x$, then there exist $k \in D$ with $k > i,j$ and $h_i^k(y_j) = h_j^k(y_j)$; hence, $f_j(y_j) = f_k h_j^k(y_j) = f_k h_i^k(x_i) = f_i(x_i)$. For $x \in E$, there exist $x_i \in E_i$ such that $u_i(x_i) = x$. The correspondence $x \to f(x) = f_i(x_i)$ is a map, since $f_j(x_j) = f_i(x_i)$, for any other $x_j \in x$, and f is positive, since $x \in {}^+E$ implies there exists $x_i \in x$ with $x_i \in {}^+E_i$ and $f_i(x_i) = f(x) \geq 0$. If $f(u_i(x_i) = f_i(x_i)$, then $fu_i = f_i$. If $gu_i = f_i$, then $g(u_i(x_i) = f_i(x_i)$ and so $g(x) = f(x)$. Hence, $E = \text{inj lim}_D E_i$.

(5.1) <u>Lemma</u>: If an OVS $E = \dot{\bigcup}_D E_i$, where $(E_i)_{i \in D}$ is a family of subspaces of E, each ordered with the ordering subduced from E, and D is directed and ordered by $i \leq j$ if $E_i \subseteq E_j$, then the family $(q_i^j : E_i \to E_j)_{i,j \in D, i \leq j}$ is an injective family, where q_i^j are the canonical inclusion maps and $(q_i : E_i \to E)_{i \in D}$ is the injective limit of this family.

<u>Proof</u>: The family $(q_i : E_i \to E)$ satisfies $q_i = q_j h_i^j$ if $i \leq j$, hence, there exists a unique positive map $f : \text{inj lim}_D E_i \to E$ for which $fu_i = q_i$, where u_i is the injection $u_i : E_i \to \text{inj lim}_D E_i$. Since $x \in E$ implies there exists $i \in D$ such that $x = q_i(x_i) = f(u_i(x_i))$, f is a surjection, and furthermore, if $x \in {}^+E$, then $u_i(x_i) \in {}^+(\text{inj lim}_D E_i)$. We have f is an injection, since $f(x) = 0$ implies $f(u_i(x)) = q_i(x_i) = 0$ and $q_i(x_i) = 0$ if and only if, $x_i = 0$, which implies $x = 0$. Hence, f is bijective, f^{-1} is positive, and so f is an isomorphism.

6. <u>The Lattice of Order Ideals</u>

Let E be an OVS. The complete lattice of KH-closed sets on $\mathcal{P}(E)$ determined by the polarity generated by $\mathcal{I}(E)$, the set of ideals of E, coincides with $\mathcal{I}(E)$, since E is an ideal and the intersection of any family of ideals is an ideal. We write 0 for $\{0\}$ the least element of $\mathcal{I}(E)$ and, for any set $S \subseteq E$, $\langle S \rangle = KH(S)$ is called the <u>ideal generated</u> by S.

(6.1) <u>Lemma</u>: For any set S contained in an OVS E, we have $\langle S \rangle = [(S)]$.

<u>Proof</u>: Clearly, $(S) \subseteq \langle S \rangle$ and $[\langle S \rangle] = \langle S \rangle$ imply $[(S)] \subseteq \langle S \rangle$.

To show equality, it suffices to show $[(S)]$ is an ideal, and for this it suffices to show $[(S)]$ is a linear subspace. If $x,y \in [(S)]$, then $x \in [a,b]$ and $y \in [c,d]$, for some a,b,c,d in (S). We have that $x + y \in [a + c, b + d]$, $-x \in [-a,-b]$, and $\lambda x \in [\lambda a, \lambda b]$. These intervals are contained in $[(S)]$ and hence $x + y$, $-x$, and λx are in $[(S)]$.

In particular, if $a \in E$ is not comparable to 0, then $\langle a \rangle = (a)$, and if $a \in {}^{+}E$, then $\langle a \rangle = \bigcup_{\lambda > 0} [-\lambda a, \lambda a]$. An element $e \in {}^{+}E$ is called a unit of E if $E = \langle a \rangle$. E is directed if and only if, $E = \langle {}^{+}E \rangle$ since $\langle {}^{+}E \rangle = [({}^{+}E)] = [{}^{+}E - {}^{+}E] = {}^{+}E - {}^{+}E$.

(6.2) Theorem: If E is a directed OVS, then E is the injective limit of the family $(q_a^b : \langle a \rangle \to \langle b \rangle)_{a,b \in {}^{+}E;\, a \leq b}$.

Proof: Since E is directed, $E = \bigcup_{{}^{+}E} \langle a \rangle$ and $(\langle a \rangle)_{a \in {}^{+}E}$ is directed by inclusion, as $a + b \geq a,b$; thus, $\langle a + b \rangle \supseteq \langle a \rangle, \langle b \rangle$. Thus, the result follows from (5.1).

7. Fully Ordered OVS

(7.1) Lemma: An OVS E, is fully ordered if and only if, E contains no non-trivial trivially ordered ideals.

Proof: If E is fully ordered and I is a trivially ordered ideal of E, then $x \in I$ is comparable to 0, only when $x = 0$, and so $I = \{0\}$. Suppose E is not fully ordered, then there exists $x \in E$ such that $x \notin {}^{+}E \cup -{}^{+}E$. If x is incomparable to 0, then so is any scalar multiple of x. Hence, $(x) \cap {}^{+}E = \{0\}$ and so (x), the vector subspace spanned by x, is a trivially ordered ideal.

An OVS E, is called simple if $E \neq 0$ and E has no non-trivial ideals.

(7.2) Theorem: (Bonsall(3)). If E is a simple OVS, then either E is isomorphic to R, or E is isomorphic to (R,0).

Proof: If E is not linearly ordered, there exists an $x \in E$ such that not both x and $-x$ are in ^+E. Hence, $(x) = \langle x \rangle$ and E simple, implies $E = (x)$; that is, E is isomorphic to (R,0). If E is linearly ordered, there exists an $x > 0$; hence, $^+E \neq 0$. Any $e \in {}^+E$ is a unit of E. The map $p_e : E \to R$ defined by $p(x) = \inf\{\alpha \in R : x \leq \alpha e\}$ is finite valued, since any $x \in E$ is in $[-\lambda e, \lambda e]$, for some $\lambda > 0$. If $x \leq p(x)e$, then $y = (p(x)e - x) \geq 0$. If $y > 0$, then $\langle y \rangle = E$; hence, there exists $\lambda \in R$ such that $y > \lambda e$. Thus, $p(x)e - x \geq \lambda e$. But $(p(x) - \lambda)e \geq x$ contradicts the definition of p; hence, $y = 0$, for each $x \in E$. Thus, $p : E \to R$ is an isomorphism.

8. Maximal Ideals and Modular Ideals

Let E be an OVS. An ideal M of E is called maximal if E/M is simple. A proper ideal I of E is called modular if there exists an a in ^+E such that $\langle a_I \rangle = E/I$. Such an element $a \in {}^+E$ is called a mod unit of I. If a is a mod unit of an ideal I, then $a \notin I$, for $a \in I$ implies $a_I = 0_I$, which implies $E/I = \langle 0_I \rangle$; this contradicts the propriety of I. If M is a maximal ideal, then any $a \notin I$, $a > 0$, is a mod unit of M. Thus, if E is a directed OVS, all maximal ideals are modular. If M is a maximal ideal and M is modular, then E/M is isomorphic to R, otherwise E/M is trivially ordered.

(8.1) Lemma: If I is a modular ideal of an OVS E, with mod unit a, if J is an ideal of E for which $a \notin J$, and if I is contained in J, then a is a mod unit for J.

Proof: If there exists $x_J \in E/J$ such that $x_J \notin \langle a_J \rangle$, then $p^{-1}(x_J)$ is not contained in $p^{-1}(\langle a_J \rangle)$, where p is the projection of E/I to E/J. But $p^{-1}(\langle a_J \rangle) = E/I$. Hence, there does not exist an $x_J \in E/J$ such that $x_J \notin \langle a_J \rangle$.

(8.2) Lemma: (Bonsal (2)). Each modular ideal of an OVS E is contained in a maximal ideal which is modular.

Proof: If I is a modular ideal and a is a mod unit of I, then the set $\{I' \in \mathcal{J}(E) : I' \supseteq I, a \notin I\}$ is inductive. Hence, there exists $M \in \mathcal{J}$ which is maximal with respect to $I \subseteq M$ and $a \notin M$. If E/M is not simple, then for any non-zero ideal N in E/M, $p^{-1}(N)$, the inverse of N, under the projection map $p: E \to E/M$, is an ideal in E. Since a_M is a unit of E/M, $a_M \notin N$, and so $a \notin p^{-1}(N)$. Also $p^{-1}(N) \supsetneq M$. This contradicts the maximality of M.

9. OVS with Units

Let E be an OVS for which $e \in E$ is a unit of E. The map $P: E \to R$ defined by $p(x) = \inf \{\lambda > 0 : -\lambda e < x < \lambda e\}$ is a seminorm, and a norm if and only if, E is almost Archimedean. If E is

Archimedean, then $[-e,e]$ is $\{x \in E : p(x) < 1\}$. The maps $u:E \to R$ defined by $u(x) = \inf\{\alpha \in R : \alpha e > x\}$ and $\ell:E \to R$ defined by $\ell(x) = \sup\{\beta \in R : \beta e < x\}$ are sublinear functions, and if E is almost Archimedean, then, for $x \in E$, $x \neq 0$, $u(x)$ and $\ell(x)$ are not both 0. These facts are well known and their verification is routine.

(9.1) <u>Lemma</u>: (Giles (6)). Let E be an OVS with a unit e. A subspace I of E is a non-trivially ordered ideal if and only if, I supports $[0,e]$.

<u>Proof</u>: If I is an ideal of E such that $I \cap {}^{+}E \neq 0$, then $I \cap [0,e] \neq \emptyset$. If $x = \alpha y + (1 - \alpha)y'$, where $x \in I \cap [0,e]$, $y,y' \in [0,e]$ and $0 < \alpha < 1$, then $0 \leq \alpha y$, $(1 - \alpha)y' \leq x$ imply $y,y' \in I$. Thus I supports $[0,e]$. Conversely, suppose I supports $[0,e]$, $x,y \in {}^{+}I$, $x \leq y$. Without loss of generality $y \leq e$. Since $1/2y = 1/2x + 1/2(y - x)$, where x, $y - x \in [0,e]$, and I supports $[0,e]$, we have $x \in I$. Thus I is an ideal of E.

The following lemma is proved in (7) under the additional assumption that E is Archimedean.

(9.2) <u>Lemma</u>: (Kadison (7)). Let E be an almost Archimedean OVS with a unit e. For any $x \in E$, $x \neq 0$, there exists an ideal I of E such that $x_I = u(x)e_I$ or $x_I = \ell(x)e_I$ in E/I. In part, x_I or $-x_I$ is a unit for E/I.

<u>Proof</u>: If $x \in E$, $x \neq 0$, then either $\ell(x)$ or $u(x)$ is not zero.

Suppose $u(x) \neq 0$. If $x \in I = \langle u(x)e - x \rangle$, then there exists $\lambda \in {}^{+}R$ such that $x \leq \lambda(u(x)e - x)$. Thus, $(1 + \lambda)x = x + \lambda x \leq \lambda u(x)e$, which implies $x \leq (\lambda/1 + \lambda)u(x)e < u(x)e$, and since $(\lambda/1 + \lambda)u(x) < u(x)$, this contradicts the definition of $u(x)$; hence, $x \notin \langle u(x)e - x \rangle$ and $x_I = u(x)e_I$. If $\ell(x) \neq 0$, and $x \in \langle \ell(x)e - x \rangle = I$, then there exists $\lambda \geq 0$ such that $-\lambda(\ell(x)e - x) \leq x$. Without loss of generality $\lambda \geq 1$. Hence we have $-\lambda\ell(x)e \leq (1 - \lambda)x$ which implies $\ell(x)e < (-\lambda/1 - \lambda)\ \ell(x)e \leq x$. This contradicts the definition of $\ell(x)$; hence, $x \notin \langle \ell(x)e - x \rangle$ and so $x_I = \ell(x)e_I$.

(10.3) Theorem: (Kadison (7)). If E is an almost Archimedean OVS with a unit e, then, for any $x \in E$, $x \neq 0$, there exists a (modular) maximal ideal M such that $x_M = u(x)e_M$ or $x_M = \ell(x)e_M$ in E/M.

Proof: By the preceding lemma, for any $x \in E$, $x \neq 0$, there exists an ideal I of E such that $x_I = u(x)e_I$, or $x_I = (x)e_I$. The set $\mathcal{J}_I = \{J \in \mathcal{J}(E) : J \supseteq I, x \notin J\}$ is inductive; hence by Zorn's Lemma, $\mathcal{J}_I$ has maximal elements. If M is a maximal element of $\mathcal{J}_I$, then E/M is simple, for otherwise, if there exist ideal N in E/M, $N \neq 0$, then $p^{-1}(N)$, where $p: E \to E/M$ is the projection map, is an ideal in E which contains M, and since $x_M \notin N$, because e_M is a unit for E/M and $x_M = \lambda e_M$, for some $\lambda \in R$, we have $x \in p^{-1}(N)$. This contradicts the maximality of M in $\mathcal{J}_I$. Thus M is a maximal ideal of E.

10. OVS with Bases

Let E be an OVS with a non-trivial cone ${}^{+}E$. A nonempty convex subset B of ^{+}E is called a <u>base</u> for ^{+}E if each nonzero element $x \varepsilon {}^{+}E$ has a unique representation $x = \lambda b$, for $b \varepsilon B, \lambda > 0$. The element 0 is not in a base B, for if $0 \varepsilon B$, we have $\lambda b \varepsilon B$, for $b \varepsilon B$, and $0 < \lambda < 1$, since B is convex, and this implies that the representation for elements of ^{+}E is not unique.

The following two lemmas are well known and their proof may be found in (Peressini (9)).

(10.1) <u>Lemma</u>: If B is a base for the cone of an OVS E, and if $\sum_i^n \lambda_i b_i = 0$, for $b_i \varepsilon B$, $\lambda_i \varepsilon R, i = 1,..,n$, then $\sum_i^n \lambda_i = 0$.

(10.2) <u>Lemma</u>: A subset B of an OVS E is a base for ^{+}E if and only if, there exists a strictly positive linear functional $e: E \to R$ such that $e^{-1}(1) \cap {}^{+}E = B$. The representation of elements of ^{+}E by elements of B is given by $a = e(a)(a/e(a))$, for $a \varepsilon {}^{+}E$. Furthermore if E is directed, then e is unique.

(10.3) <u>Lemma</u>: (Edwards (4)). If F is an OVS with a base B for ^{+}F, then a linear subspace I of F is an ideal if and only if, I supports B, or $I \cap B = \emptyset$.

<u>Proof</u>: Suppose I is an ideal of E and $I \cap B \neq \emptyset$. If $h \varepsilon I \cap B$, $h = \alpha b + (1 - \alpha)b'$, for $0 < \alpha < 1$; $b, b' \varepsilon B$, then $\alpha b \leq h$ and $(1 - \alpha)b' \leq h$ imply $b, b' \varepsilon I$. Thus I supports B. If $I \cap B = \emptyset$, then I is a trivially ordered ideal of E. If I supports B and $x,y \varepsilon {}^{+}F$, $x \leq y$, $y \varepsilon I$, then there exist $b \varepsilon I \cap B$ and $\lambda > 0$ such that $0 \leq \lambda x \leq b$, furthermore $\lambda x = \mu b$, for $0 \leq \mu \leq 1$ and $b - \lambda x =$

δb_2, for $\delta \varepsilon R; b_1, b_2 \varepsilon B$. Since $e(b) = e(\mu b_1) + e(\delta b_2)$ where e is a strictly positive functional induced by B and I supports B, we have $b_1 \varepsilon I$ which implies $x \varepsilon I$. Thus I is an ideal of E.

(10.4) Lemma: (Edwards (4)). Let F be a directed OVS with a base B for ^+F. The convex set conv$(B \cup -B)$ is circled and absorbing and the Minskowski gauge g of conv$(B \cup -B)$, restricted to ^+F, coincides with the strictly positive functional $e:F \to R$ induced by B. Hence g is additive on ^+F and $g(B) = 1$.

Proof: We have $S = \text{conv}(B \cup -B) = \{\lambda b - \lambda' b' : b,b' \varepsilon B, \lambda,\lambda' \geq 0, \lambda + \lambda' = 1\}$. Since $0 \varepsilon S$, $\lambda b \varepsilon S$, for any $b \varepsilon B$, $0 < \lambda < 1$, we have $S = \{\lambda b - \lambda' b' : b,b' \varepsilon B, \lambda,\lambda' > 0, \lambda + \lambda' < 1\}$. Clearly S is circled. Any $x \varepsilon E$ has a representation $x = \lambda b - \lambda' b'$ where $\lambda,\lambda' > 0$ and so $(1/\lambda + \lambda')x \varepsilon S$. Thus, S is absorbing. If $x \varepsilon {}^+F$, then $x = e(x)b$, which implies $g(x) = \inf \{\alpha > 0 : 1/\alpha x \varepsilon S\} = e(x)$.

11. Extremal Ideals

Let E be an OVS. An element $x \varepsilon {}^+E$ is called indecomposable if it satisfies the following equivalent conditions:

(1) if $y \varepsilon {}^+E$, $y < x$, then $x = \lambda y$, for some $\lambda > 0$,

(2) if $y,z \varepsilon {}^+E$, $x = y + z$, then $y,z \varepsilon \langle x \rangle$,

(3) $\langle x \rangle$ is a one dimensional subspace of E.

An ideal I of E is called indecomposable (extremal) if it is a one dimensional subspace of E.

(11.1) Lemma: If E is an OVS with a base B, then $x \varepsilon B$ is indecomposable if and only if, x is an extreme point of B.

Proof: Suppose $x \varepsilon B$ is indecomposable and $x = \alpha x_1 + (1 - \alpha)x_2$, where $x_1, x_2 \varepsilon B$, and $0 < \alpha < 1$. Since $0 < \alpha x_1 < x$, we have $x = \lambda x_1$, for some $\lambda > 0$ and by the uniqueness of the representation of x by elements of B, we have $\lambda = 1$. Thus x is an extreme point of B. Conversely, if x is an extreme point of B and $0 < x_1 < x$, then $x = (x - x_1) + x_1$. Let e be the positive functional induced by B. Hence, $h(x) = 1 = h(x - x_1) + h(x_1)$, $x = h(x - x_1)$ $(x - x_1)/h(x - x_1) + h(x_1)\, x_1/h(x_1)$, and $(x - x_1)/h(x - x_1)$, $x_1/h(x_1) \varepsilon B$. Since x is an extreme point of B, $x = x_1/h(x_1)$ and so x is indecomposable.

12. Algebraic Representation Theory

Let E be an OVS, let Θ be a set of ideals of E, and let HK and KH be the closure operators of the polarity induced by Θ. There is a canonical positive map of E into $\Pi_\Theta E/I$, the product of the family $(E/I)_{I \varepsilon \Theta}$, defined by $a \to \hat{a}(I) = a_I \varepsilon E/I$, for any $a \varepsilon E$, $I \varepsilon \Theta$. The image of E, $E(X)$, is an OVS in the subduced product ordering. For any $a \varepsilon E$, $s(a) = \{I \varepsilon \Theta : \hat{a}(I) \neq 0\}$ is called the support of $\hat{a}$, We note that $s(a) = U_a{}' = \{I \varepsilon \Theta : a \notin I\}$, for $a \varepsilon E$. If HK is a topological closure on Θ, then clearly its open sets are those generated by $\{s(a) : a \varepsilon E\}$,and HK is sometimes called the Zariski topology determined by E.

If U and V are sets of ideals of an OVS E such that $V \subseteq U$, then the restriction map $r_V^U : E(U) \to E(V)$, defined by $\hat{x} \to \hat{x}|V$, for $\hat{x} \in E(U)$, satisfies;

(1) $E(\emptyset) = 0$,

(2) r_V^U is the identity map if $V = U$,

(3) if $W \subseteq V \subseteq U$, then $r_W^U = r_W^V r_V^U$.

Let E be a directed OVS, let Θ be a set of proper ideals of E, and let HK be the closure operator generated by Θ. The set $\mathcal{U} = \{U_a : a \in {}^+E\}$ of HK basic open sets, covers Θ, since, for any proper ideal I, there exists $x \in E$, $x \notin I$, and there exists $a \in {}^+E$, $a \geq x$ with $a \notin I$. Hence $I \in U_a$. Furthermore, $\mathcal{U}$ is directed by inclusion, since, for any $a,b \in {}^+E$, $a + b \geq a,b$ and therefore $U_a, U_b \subseteq U_{a+b}$.

(12.1) Lemma: Let E be a directed OVS and let Θ be a set of ideals of E. Then $(r_a^b : E(U_b) \to E(U_a))_{a,b \in {}^+E,\ a \leq b}$ is a projective family of OVS and $(r_a : E(\Theta) \to E(U_a))_{a \in {}^+E}$ is the projective limit of this family.

Proof: Clearly $(r_a^b : E(U_b) \to E(U_a))_{a,b \in {}^+E,\ a \leq b}$, is a projective family of OVS. The family of positive maps $(r_a : E(X) \to E(U_a))_{a \in {}^+E}$ satisfies $r_a = r_a^b r_b$ if $a \leq b$; hence, there exists a unique positive map $g : E(\Theta) \to \text{proj lim}_{{}^+E}\, E(U_a)$. We verify that g is an isomorphism. If $r_a(b) = r_a(b')$, for $b,b' \in E(\Theta)$, for all $a \in {}^+E$, we have $\hat{b}|U_a = \hat{b}'|U_a$, for all $a \in {}^+E$, and since $\mathcal{U} = \{U_a : a \in {}^+E\}$ covers Θ, $b(I) = b'(I)$, for all $I \in \Theta$, thus $b = b'$. For $x \in \text{proj lim}_{{}^+E}\, E(U_a)$, there exists $\hat{x} \in E(\Theta)$ such that $r_a(\hat{x}) = x(a) = \hat{x}|U_a$, for all $a \in {}^+E$, and furthermore, if $x \in {}^+(\text{proj lim}_{{}^+E}\, E(U_a))$, then $x(a) \in {}^+E(U_a)$, for all $a \in {}^{+}E$, and so $\hat{x} \in {}^{+}E(\Theta)$.

Let E be an OVS, let Θ be a set of ideals of E and let HK and KH be the closure operators generated by Θ. For any set $S \subseteq E$, the kernel of the map $\hat{}:E \to E(H(S)) \subseteq \Pi_{H(S)} E/I$ is $KH(S)$, since $x_I = 0$, for all $I \in H(A)$, if and only if, $x \in KH(I)$. Furthermore, in the canonical (linear) decomposition $\hat{}:E \overset{p}{\to} E/KH(S) \overset{\hat{}'}{\to} E(H(S)) \overset{q}{\to} {}_{H(S)}E/I$, the linear bijection $\hat{}':E/KH(S) \to E(H(S))$ is positive. Suppose $x_{KH(S)} \in E/KH(S)$ and $x_{KH(S)} \geq 0$, then there exists $y \in x + KH(S)$ such that $y \geq 0$, and so $y \in x + I$, for all $I \in H(S)$. This implies $x_I \geq 0$, for all $I \in H(S)$, thus $\hat{x}' \geq 0$.

(12.2) Lemma: Let Θ be a set of ideals of an OVS E for which E/I is almost archimedean, for any $I \in \Theta$, then E/J is almost archimedean, for any KH-closed ideal J in E.

Proof: The OVS $E(H(J))$ is almost archimedean in the induced product order and the canonical map $E/KH(J) \to E(H(J))$ is a bijection. If $-\lambda y \leq x \leq \lambda y$, for $x,y \in E$, $\lambda > 0$, then $-\lambda y_I \leq x_I \leq \lambda y_I$, for all $I \in H(J)$. Since E/I is almost archimedean, for any I, we have $x_I = 0$, for all $I \in H(J)$. Thus, $\hat{x}$ is 0, which implies x is 0, which implies $E/KH(J)$ is almost archimedean.

The set $\Omega(E)$ of modular maximal ideals of an OVS E may be used to induce a polarity between $\mathcal{P}(E)$ and $\mathcal{P}(\Omega)$. The HK-closure on Ω will not in general be a topological closure. Singleton sets of Ω are closed because $HK(\{M\}) = \{M' \in \Omega : M' \supseteq M\} = \{M\}$.

(12.3) Lemma: If an OVS E has a unit e, then Ω is a compact element in the family of open sets of Ω.

Proof: Suppose $\Omega \subseteq \bigcup_{i \varepsilon I} U_{A_i}$, for some family $(U_{A_i})_{i \varepsilon I}$ of open sets. Since $\bigcup_I U_{A_i} = U_{\bigcup_I A_i}$, we have $\Omega = U_a \subseteq \{M \varepsilon \Omega : \langle \bigcup_I A_i \rangle \not\subseteq M\}$. If $a \notin \langle \bigcup_I A_i \rangle$, then the set of ideals $\{J \varepsilon \mathcal{J}(E) : a \notin J, \langle \bigcup_I A_i \rangle \subseteq I\}$ is non-empty, and so the set is inductive. Because E has a unit, all ideals are modular. Hence, this set contains a maximal element which is a maximal modular ideal. This contradicts the fact that if $a \notin M$, then $\bigcup_I A_i \not\subseteq M$. Hence, $a \varepsilon \langle \bigcup_I A_i \rangle$. Thus, $a \varepsilon \langle \bigcup_1^n A_i \rangle$, for some finite subfamily of I. If $a \varepsilon \langle \bigcup_1^n A_i \rangle$, then $U_a = \quad \subseteq U_{\bigcup_1^n A_i} = \bigcup_1^n U_{A_i}$.

(12.4) Theorem: For a modular ideal I of an OVS E, E/I is almost Archimedean if and only if, I is KH-closed, where KH is the closure operator generated by the set Ω of modular maximal ideals of E.

Proof: By a preceding lemma, if I is KH-closed, then E/I is almost Archimedean. Suppose E/I is almost Archimedean. If $x \varepsilon E$, $x \notin I$, then $x_I \neq 0$ in E/I, and since E/I is almost Archimedean with a unit, there exists a maximal ideal M in E/I such that $x_I \notin M$. This implies $x \notin p^{-1}(M)$, where $p: E \to E/I$ is the natural projection, and $p^{-1}(M)$ is a modular maximal ideal in E. Hence, $x \notin KH(I)$ and so $I = KH(I)$.

(12.5) Corollary: Let E be an OVS, if the intersection of the set

(E), ideals of E which are modular and for which E/I is almost Archimedean is 0, then the intersection of $\Omega(E)$ is 0.

Proof: For any $x \in E$, there exists an ideal I for which $x \notin I$ and E/I is almost Archimedean. Furthermore, there exists a maximal ideal M of E/I such that $x_M \notin M$, hence, $p^{-1}(M)$ is a maximal ideal in E and $x \notin p^{-1}(M)$, where $p:E \to E/I$ is the natural projection.

Let $(E_i)_{i \in X}$ be a disjoint family of OVS such that E_i is isomorphic to E_j, for any $i,j \in X$, and let $p: \mathcal{E} = \bigcup_X E_i \to X$ be the natural projection. A map $h:\mathcal{E} \to E$, where $h|E_i$ is an isomorphism of E_i with an OVS E, is called a trivialization of $p:\mathcal{E} \to X$. If $h: \mathcal{E} \to E$ is a trivialization of $p: \mathcal{E} \to X$, then the map from $\Pi_X E_i$ to E^X, the set of all functions from X to E, defined by $f \to hf$, for $f \in \Pi_X E_i$, is an isomorphism.

Let X be a set of modular maximal ideals of an OVS E. The family $(E_I = E/I)_{I \in X}$ is disjoint and E_I is isomorphic to R, for any $I \in X$. For any $a_I \in {}^+E_I$, $a_I \neq 0$, the map $p_{a_I}:E_I \to R$, defined by $p_{a_I}(X_I) = \inf\{\alpha \in R : X_I < \alpha a_I\}$, for $x_I \in E_I$, is an isomorphism. For any set $A \subseteq E$ such that $(U_a)_{a \in A}$ covers X and such that $a_I = b_I$, for any $I \in U_a \cap U_b$, and for any $a,b \in A$, the map $p_A: \mathcal{E} = \bigcup_X E_I \to R$, defined by $p_A(x_I) = p_{a_I}(x_I)$, for $x_I \in \mathcal{E}$ and for $a \in A$, is well defined, since $p_{a_I}(x_I) = p_{b_I}(x_I)$ whenever $I \in U_a \cap U_b$, and is a trivialization of $p: \mathcal{E} \to X$. In particular, if E has a unit e, then $p_e: \mathcal{E} \to R$ is a trivialization. Since a trivialization of $p: \mathcal{E} \to R$ induces an isomorphism between $\Pi_X E_I$ and R^X, it induces a representation of E into R^X.

CHAPTER III

RIESZ ORDERED VECTOR SPACES AND ABSOLUTE ORDERED VECTOR SPACES

The present chapter introduces OVS, which satisfy the Riesz decomposition property, and OVS E, whose order is induced from a map $| \ |: E \to E$, which is called an absolute valuation.

1. Join Ideals

An ideal I of an OVS E is called a join-ideal ($\vee$-ideal), if whenever $a \vee b$ exists, for $a, b \in I$, then $a \vee b \in I$. The set of all $\vee$-ideals of E is a complete lattice in the ordering by inclusion. A linear map $h: E \to F$ between OVS E and F is called strongly positive if whenever $a \vee b$ exists, for $a, b \in E$, then $h(a) \vee h(b) = h(a \vee b)$, or equivalently, if whenever $a \wedge b$ exists, for $a, b \in E$, then $h(a \wedge b) = h(a) \wedge h(b)$, since $a \wedge b = a + b - (a \vee b)$. Clearly, a strongly positive map is positive.

A linear map $H: E \to F$ between OVS is strongly positive if $a \wedge b = 0$, for $a, b \in E$, implies $h(a) \wedge h(b) = 0$. To see that this is sufficient, suppose $a \wedge b$ exists, for $a, b \in E$, then $(a - (a \wedge b)) \wedge (b - (a \wedge b)) = 0$, and so $(h(a) - h(a \wedge b)) \wedge (h(b) - h(a \wedge b)) = 0$, which implies $h(a) \wedge h(b) = h(a \wedge b) = 0$.

If $h: E \to F$ is a strongly positive map between OVS, then $\ker(h)$ is a $\vee$-ideal, because, if $a, b \in \ker(h)$ and $a \vee b$ exists, then $0 = h(a) \wedge h(b) = h(a \wedge b) = h(a) + h(b) - h(a \vee b)$, and $h(a) + h(b) = 0$, imply $h(a \vee b) = 0$, and so $a \vee b \in \ker(h)$.

If I is a $\vee$-ideal of an OVS E, then the projection $E \to E/I$ is strongly positive, because if $a \vee b$ exists, for $a,b \in E$, then $(a \vee b) + c = (a + c) \vee (b + c)$ implies $a_I \vee b_I = (a \vee b)_I$.

If E is an OVS, $\mathcal{D}(E)$ will be the set of ideals of E which are directed in their order. Directed ideals are strongly positive so the canonical projection preserves meets and joins for directed ideals. There is a bijection between $\mathcal{D}(E)$ and $\mathcal{S}(^+E) = \{K \subseteq {}^+E : K$ is a cone, and if $x,y \in {}^+E$, $x \leq y$, $y \in K$, then $x \in K\}$ the set of semi ideals of ^+E. The correspondence is given by $I \to I \cap {}^+E$ and $K \to K - K$.

2. Riesz OVS

A directed OVS E satisfies the Riesz decomposition property if:

(D) if $a, b_1,..,b_n \in {}^+E$, $a \leq b_1 +..+ b_n$, then there exist $a_i \in {}^+E$, such that $a = a_1 +..+ a_n$ and $a_i \leq b_i$, for $i = 1,.., n$,

and E is called a Riesz OVS.

(2.1) Theorem: (Fuchs (5)). For a directed OVS E these conditions are equivalent:

(1) E is a Riesz OVS,

(2) if $a,b \in {}^+E$, then $[0,a] + [0,b] = [0,a + b]$,

(3) if $a_1,a_2,b_1,b_2 \in E$, $a_i \leq b_j$, for $i = 1,2$ and $j = 1,2$, then there exist $c \in \dot{E}$ such that $a_i \leq c \leq b_j$. The proof of their equivalence is routine.

A lattice OVS is a Riesz OVS. Let E be a lattice OVS, $a, b \in {}^{+}E$. Clearly, $[0,a] + [0,b] \subseteq [0,a+b]$. If $c \in [0,a+b]$, let $u = c \wedge a$ and $v = c - u$. Since $V = c - (c \wedge a) = c + (-c \vee -a) = 0 \vee (c - a) \leq 0 \vee (a + b - a) = b$. Thus, $c = u + v$ where $u \in [0,a]$ and $v \in [0,b]$.

(2.2) Lemma: (Fuchs (5)). If I is a directed ideal of a Riesz OVS E, then E/I is a Riesz OVS.

Proof: Since E is directed, E/I is a directed OVS. For this proof we write $\bar{a}$ for the coset a_M, where $a \in E$. If $\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2 \in E/I$, $\bar{a}_i \leq \bar{b}_j$, for $i = 1,2;\ j = 1,2$, there exist $b_{ji} \in \bar{b}_j$ such that $a_i \leq b_{ji}$, for $i = 1,2\ ;\ j = 1,2$. The cosets mod I are directed, hence there are $b_j \in \bar{b}_j$ such that $b_{j1} \leq b_j$ and $b_{j2} \leq b_j$, for $j = 1,2$. Hence, there exists $c \in E$ such that $a_i \leq c \leq b_j$, for all i,j, and so $\bar{a}_i \leq \bar{c} \leq \bar{b}_j$, for all i,j.

(2.3) Theorem: (Fuchs (5)). The set of directed ideals of a Riesz OVS E is a distributive sublattice of the lattice of all vector subspaces of E.

Proof: To prove that $I \cap J$ is a directed ideal, for directed ideals I and J of E, we need only to show that $I \cap J$ is directed. If $x,y \in I \cap J$, then some $a \in I$, $b \in J$ satisfy $x,y \leq a$ and $x,y \leq b$. By the interpolation property (3) of (2.1), there exists $c \in E$, such that $x \leq c \leq a$ and $y \leq c \leq b$. Hence, c is an upper bound for $\{x,y\}$ in $I \cap J$.

The subspace $I + J$ is directed in the subduced order. To prove that $I + J$ is an ideal, suppose $0 \leq x \leq a + b$, where $x \in E$, $a \in {}^+I$, $b \in {}^+J$. By (2.1) there are $a', b' \in E$ such that $0 \leq a' \leq a$; $0 \leq b' \leq b$ and $x = a' + b'$, hence $x \in I + J$.

To verify the distributive law $I \cap (J + L) = (I \cap J) + (I \cap L)$ for directed ideals I, J, L of E, it suffices to prove the inclusion $\subseteq$ only, and even this, only for the positive elements of $I \cap (J + L)$, in view of the bijection between directed ideals of E, and semi-ideals of ${}^+E$. Let $a = b + c$, where $a \in {}^+I$ and $b \in J$, $c \in L$. By the directedness of J and L, there are positive elements b_1 and c_1, such that $b \leq b_1$ and $c \leq c_1$. Since $0 \leq a \leq b_2 + c_1$, there exist b_2, c_2 with $0 \leq b_2 \leq b_1$; $0 \leq c_2 \leq c_1$ and $a = b_2 + c_2$. Furthermore, since $b_2 \in J$, $c_2 \in L$ and b_2, $c_2 \leq a$, we have $b_2 \in I \cap J$ and $c_2 \in I \cap L$. Thus, $a \in (I \cap J) + (I \cap L)$, as desired.

3. <u>Absolute OVS</u>

Let E be a linear space. An <u>absolute valuation</u> on E is a map $|\,|: E \to E$ satisfying:

(A_1) $||a|| = |a|$,

(A_2) $|a| = 0$ if and only if $a = 0$,

(A_3) $|\lambda||a| = |\lambda a|$, for all $\lambda \in R$,

(A_4) $|a| + |b| = ||a| + |b||$,

(A_5) $a = |a| = -a$, if and only if, $a = 0$,

(A_6) $||a| - a| = |a| - a$ and $||a| + a| = |a| + a$,

(A_7) $|a + b| \leq |a| + |b|$.

The set ${}^{+}E = \{a \varepsilon E : a = |a|\}$ is clearly a cone in E and (A_6) implies $a, -a < |a|$. Define $a^{+} = (|a| + a)/2$ and $a^{-} = (|a| - a)/2$. Since $a \leq |a|$, we have that $0 \leq |a| - a$ and hence that $a^{-} = (|a| - a)/2$ is positive. Similarly, a^{+} is positive. We have also that:

(1) $|a| = a^{+} + a^{-}$,

(2) $a = a^{+} - a^{-}$,

(3) $a^{-} = (-a)^{+}$.

In particular ${}^{+}E$ is reproducing.

Since $|x| = |(x - y) + y| \leq |x + y| + |-y| = |x + y| + |y|$, we also have:

(4) $||a| - |y|| \leq |x + y|$.

A linear space E, together with an absolute valuation $||$, will be called an <u>absolute ordered vector space</u> (AOVS).

Let E be an AOVS. The binary relation $\perp$ in E defined by $a \perp b$, if and only if, $|a| \wedge |b| = 0$ satisfies:

(5) $a \perp b$, if and only if, $b \perp a$,

(6) $a \perp a$, if and only if, $a = 0$,

(7) $a \perp b$, if and only if, $|a| \vee |b| = |a| + |b|$,

(8) $a \perp b$ and $|b| \geq |c|$, imply $a \perp c$.

Let E be an AOVS. If $a \vee 0 = a^{+}$ whenever $a \vee 0$ exists, then whenever $a \vee 0$ exists, we have $a \vee b = (b + 0) \vee ((a - b) + b) = ((a - b) \vee 0) + b = (a - b)^{+} + b$. We also have

(A_8) $x \perp y$ implies $||y| - |x|| = |y| + |x|$,

since $|x| + |y| = |x| \vee |y| = |x| + (|y| - |x|)^{+} = |x| + 1/2\{||y| -$

$|x|| + |y| - |x|\} = 1/2\{||y| - |x|| + |y| + |x|\}$.

If E is a lattice OVS, then with the map $a \to |a| = a \vee -a$, E is an AOVS.

Let E be an AOVS. A sequence $(a_v)_{v = 1,2,\ldots}$ in E <u>converges</u> to an <u>absolute limit</u> a in E, if there exists a sequence $(\ell_v)_{v = 1,2,\ldots}$ in E decreasing to 0 in E and an increasing sequence of natural numbers $(\mu_v)_{v = 1,2,\ldots}$, such that $|a_\mu - a| < \ell_v$, for every $\mu \geq \mu_v$, $v = 1,2,\ldots$, and this is denoted by $ab - \lim_{v \to \infty} a_v \doteq a$. An absolute limit $a \in E$ is uniquely determined if it exists. This follows, because, if there exists $b \in E$, a sequence $(k_v)_{v = 1,2,\ldots}$ decreasing to 0 in E, and an increasing sequence of natural numbers $(\delta_v)_{v = 1,2,\ldots}$ such that $|a_\mu - b| \leq k_v$, for every $\mu \geq \delta_v$, $v = 1,2,\ldots$, then, for every $\mu > \text{maximum}\,\{\mu_v, \delta_v\}$, we have $|a_\mu - a| \leq \ell_v$ and $|a_\mu - b| \leq k_v$, and so we have $|a - b| = |a_\mu - a - a_\mu + b| \leq |a_\mu - a| + |-a_\mu + b| \leq \ell_v + k_v$, for every $v = 1,2,\ldots$ Since $\ell_v + k_v \downarrow_{v=1}^{\infty} 0$, we have $|a - b| = 0$ and so $a = b$. Clearly, $\text{ab-lim}_{v \to \infty} a_v = a$, if and only if, $\text{ab-lim}_{v \to \infty} |a_v - a| = 0$.

Also, if $\text{ab-lim}_{v \to \infty} a_v = a$, then $\text{ab-lim}_{v \to \infty} a_{\mu_v} = a$, for any subsequence $(a_{\mu_v})_{v = 1,2,\ldots}$.

(3.1) <u>Lemma</u>: If a sequence $(a_v)_{v = 1,\ldots}$ in an AOVS E increases or decreases to $a \in E$, then $\text{ab-lim}_{v \to \infty} a_v = a$.

<u>Proof</u>: If $(a_v)_{v = 1,2,\ldots}$ increases or decreases to a, then $(a - a_v)_{v = 1,2,\ldots}$ or $(a_v - a)_{v = 1,2\ldots}$ to 0. Thus $|a_\mu - a| \leq |a_v - a|$, for all $\mu \geq v$, $v = 1,2,\ldots$ and so $(a_v)_{v = 1,2,\ldots}$ converges

absolutely to a.

(3.2) Lemma: If a sequence $(a_v)_{v=1,2,..}$ in an AOVS E is increasing (decreasing) and $\text{ab-lim}_{v \to \infty} a_v = a$, then $(a_v)_{v=1,2,..}$ increases (decreases) to a.

Proof: If $\text{ab-lim}_{v \to \infty} a_v = a$, then there exists a sequence $(\ell_v)_{v=1,2,...}$ in E decreasing to 0, and a sequence of natural numbers $(\mu_v)_{v=1,2,...}$, such that $|a_\mu - a| \leq \ell_v$, for every $\mu \geq \mu_v$, $v = 1,2,....$ Hence, $a_\mu - a \leq |a_\mu - a| < \ell_v$ and $a - a_\mu \leq |a_\mu - a| \leq \ell$ imply $a - \ell_v \leq a_\mu \leq a + \ell_v$, for every $\mu \geq \mu_v$, $v = 1,2,....$ Since $(a_v)_{v=1,2,...}$ increases, we have $a_\mu \leq a + \ell_v$, for every $v = 1,2,...$, and so $a_\mu \leq \bigwedge_{v=1}^{\infty} (a + \ell_v) = a$. If $x \geq a_\mu$, for every $\mu = 1,2,...$, then $x \geq a - \ell_v$, for every $v = 1,2,...$, and hence, $x \geq \bigvee_{v=1}^{\infty} (a - \ell_v) = a$. Therefore, $(a_v)_{v=1,2,...}$ increases to a. The case where $(a_v)_{v=1,2,..}$ decreases may be derived similarly.

(3.3) Lemma: If $\text{ab-lim}_{v \to \infty} a_v = a$ and $\text{ab-lim}_{v \to \infty} b_v = b$, for sequences $(a_v)_{v=1,2...}$ and $(b_v)_{v=1,2,...}$ in an AOVS E, then, for any $\alpha, \beta \in R$, $\text{ab-lim}_{v \to \infty} (\alpha a_v + \beta b_v) = \alpha a + \beta b$.

Proof: By assumption, there exist $(\ell_v)_{v=1,2...}$ and $(k_v)_{v=1,2...}$ in E decreasing to 0, and $(\mu_v)_{v=1,2...}$, $(\delta_v)_{v=1,2...}$, such that

$$|a_\mu - a| \leq \ell_v, \text{ for all } \mu \geq \mu_v,\ v = 1,2,...., \text{ and}$$

$$|b_\mu - b| \leq k_v, \text{ for all } \mu \geq \mu_v,\ v = 1,2,....$$

If $x_v = \text{maximum } \{\mu_v, \delta_v\}$, then

$$|(\alpha a_\mu + \beta b_\mu) - (\alpha a + \beta b)| \leq |\alpha|\ell_v + |\beta|k_v,$$

for every $\mu \geq x_v$, $v = 1,2,\ldots$

A linear subspace I of an AOVS E is called an absolute ideal (ab-ideal) if $|x| \leq |y|$, and $y \in I$ imply $x \in I$. Absolute ideals are directed order ideals as both x^+, x^- are in I if $x \in I$.

(3.4) Lemma: The set $\mathcal{A}(E)$ of all ab-ideals of an AOVS E ordered by inclusion is a complete lattice.

Proof: If $I = \bigcap_A I_i$ where $I_i \in \mathcal{A}$, for $i \in A$, $|x| \leq |y|$, and $y \in I$, then $y \in I_i$, for all $i \in A$, so that $x \in I_i$, for all i, which implies $x \in I$. If $I_i \in \mathcal{A}$, for $i \in A$, then $\bigvee_A I_i = \bigcap \{J \in \mathcal{A} : J \supseteq \bigcup_A I_i\}$.

(3.5) Lemma: Let E be an AOVS, then, for any subset $A \subseteq E$, $|A|$, the ab-ideal generated by A is the set $I = \{x \in E : |x| < \sum_i^n |\lambda_i||a_i|$, for some $i \in A\}$.

Proof: Suppose $|x| \leq \sum_1^n |\lambda_i||a_i|$ and $|y| \leq \sum_1^m |\lambda_i||a_i|$, then $|x + y| \leq |x| + |y| \leq \sum_1^n |\lambda_i||a_i| + \sum_1^m |\lambda_i||a_i|$. Clearly, $-x$, $\lambda x \in I$, if $x \in I$. If $|x| \leq |y|$ and $y \in I$, then $|x| \leq |y| \leq \sum_i^n |\lambda_i||a_i|$ implies $x \in I$. Thus, I is an ab-ideal which contains A and is clearly contained in every ab-ideal which contains A.

Let E,F be two AOVS. A linear map $h:E \to F$ is called an absolute map (ab-map) if $h(|x|) = |h(x)|$, for $x \in E$. Hence, h is an order homomorphism for the induced orders on E and F.

(3.6) Lemma: The kernel of an ab-map $h:E \to F$ is an ab-ideal.

Proof: Suppose $|x| \leq |y|$ and $y \in \ker(h)$, then $0 \leq h(|x|) \leq h(|y|) = |h(y)| = 0$ implies $|h(x)| = h(|x|) = 0$, which implies $h(x) = 0$.

(3.7) Lemma: Let E be an AOVS and let I be an ab-ideal of E, then the projection $E \to E/I$ is an ab-map.

Proof: We show that E/I is an AOVS with absolute valuation $|\,|: E/I \to E/I$, defined by $x + I \to |x + I| = |x| + I$. Then, by definition, the projection is an ab-map.

(A_1) $||x + I|| = ||x| + I| = ||x|| + I = |x| + I = |x + I|$

(A_2) $|x + I| = 0 + I$, if and only if, $|x| + I = 0 + I$, if and only if, $|x| \in I$ if and only if $x \in I$, if and only if, $x + I = 0 + I$.

(A_3) $|\lambda||x + I| = |\lambda||x| + I = |\lambda x| + I = |\lambda x + I|$

(A_4) $|x + I| + |y + I| = |x| + I + |y| + I = |x| + |y| + I = ||x| + |y| + I = ||x| + |y| + I| = ||x| + I + |y| + I|$.

(A_5) $x + I = |x + I| = -x + I$, if and only if, $2x + I = 0 + I$, if and only if, $x + I = 0 + I$.

(A_6) $||x + I| - x + I| = ||x| + I| - x + I| = ||x| - x + I| = ||x| - x| + I = |x| - x + I = |x| + I + (-x + I)$.

(A_7) $||x| + I + |y| + I - |x + y| + I| = ||x| + |y| - |x + y| + I| = ||x| + |y| - |x + y| + I = |x| + |y| - |x + y| + I$.

(3.8) Lemma: For any subset A in an AOVS E, $A^{\perp} = \{x \in E : x \perp a$, for any $a \in A\}$ is an ab-ideal of E.

Proof: Suppose that $x, y \in A^{\perp}$ and $a \in A$. To show that $x + y \perp a$, it suffices to show $(|x| + |y|) \wedge |a| = 0$. The set $\{|x| + |y|, |a|\}$ is bounded below by 0. If $0 \leq u \leq |x| + |y|, |a|$, then $u \leq (|x| + |y|) \wedge (|x| + |a|) = |x| + (|y| \wedge |a|) = |x|$. Thus, $u \leq |x| \wedge |a| = 0$ and so $(|x| + |y|) \wedge |a| = 0$. For any $\alpha \in R$, there exists an integer n, such that $n \geq |\alpha|$. Thus, $|\alpha x| \wedge |a| = (|\alpha||x|) \wedge |a| \leq n|x| \wedge n|a| = n(|x| \wedge |a|) = 0$. Hence, $A^{\perp}$ is a linear subspace of E. Since $|x| \leq |y|$ and $y \perp a$ clearly imply that $x \perp a$, $A^{\perp}$ is an ideal of E.

(3.9) Lemma: Let E be an AOVS which satisfies (A_8). If M is a linear subspace of E such that $E = M + M^{\perp}$, then $M = M^{\perp\perp}$ and and E is the order direct sum of M and $M^{\perp}$.

Proof: Since $M \cap M^{\perp} = \{0\}$, it follows that E is the linear direct sum of M and $M^{\perp}$. Suppose $z \in M^{\perp\perp}$ but $z \notin M$, then $z = x + y$ where $x \in M$, $y \in M^{\perp}$. For any $w \in M^{\perp}$, we have $0 = |z| \wedge |w| = |x + y| \wedge |w| \geq ||x| - |y|| \wedge |w| = (|x| + |y|) \wedge |w| \geq |x - y| \wedge |w|$, since $x \perp y$; hence, $x - y \in M^{\perp\perp}$. It follows that $y \in M^{\perp\perp}$, since $x + y \in M^{\perp\perp}$. This contradicts the definition of y; hence, $M = M^{\perp\perp}$. If $z \in E$ and $z = z + y$ where $x \in M$, $y \in M^{\perp}$, then $|z| = |x + y| \leq |x| + |y|$. Also $|z| = |x + y| \geq ||x| - |y|| = |x + y|$, since $x \perp y$; hence $|z| = |x| + |y|$.

(3.10) Lemma: Let E be an AOVS. The map $\perp : \mathcal{P}(E) \to \mathcal{P}(E)$ defines a polarity. Hence, $\mathcal{P}_{\perp}(E)$, the closed sets in $\mathcal{P}(E)$ form a complete lattice with least element $0 = \{0\}$ and greatest element E. The map

$A \to A^{\perp}$ in $\mathcal{P}^{\perp}(E)$ is a complementation. Furthermore, $A \wedge B = A \cap B$ in $\mathcal{P}^{\perp}(E)$.

Proof: Since the relation $\perp$ on E is symmetric, the map $A \to A^{\perp}$ in $\mathcal{P}^{\perp}(E)$ is an involution, that is, $A^{\perp\perp} = A$, $(A \wedge B)^{\perp} = A^{\perp} \vee B^{\perp}$ and $(A \vee B)^{\perp} = A^{\perp} \wedge B^{\perp}$. The fact that $x \perp x$ implies $x \perp y$, for all $y \in E$, implies $A \wedge A^{\perp} = \{0\}$ and $A \vee A^{\perp} = E$. These facts hold for arbitrary polarities and may be found in (Birkhoff (11)). To show that $A \cap B = (A \cap B)^{\perp\perp}$, it suffices to show that $(A \cap B)^{\perp\perp} \subseteq A \cap B$. For any $x \in (A \cap B)^{\perp\perp}$, we have $x \perp y$, for any $y \in (A \cap B)^{\perp}$, which implies $x \perp b$, for any $b \in B^{\perp}$, and $x \perp a$, for any $a \in A^{\perp}$, which implies $x \in B^{\perp\perp} = B$ and $x \in A^{\perp\perp} = A$. Thus, $x \in A \cap B$.

An AOVS E is called prime if $x, y \in E$, $x \perp y$ implies $x = 0$, or $y = 0$.

(3.11) Lemma; A prime AOVS E is fully ordered, if and only if, for any $a \in E$, $a^{+} \wedge a^{-} = 0$.

Proof: If E is fully ordered, then $a = a^{+}$, or $a = a^{-}$, which implies $a^{+} \wedge a^{-} = 0$. Conversely, if $a^{+} \wedge a^{-} = 0$, then a^{+}, or $a^{-} = 0$, which implies $a = a^{+}$, or $a = a^{-}$. Thus, E is fully ordered.

(3.12) Lemma: E is a prime AOVS, if and only if, for any $I, J \in \mathcal{A}(E)$, $I \cap J = \{0\}$ implies $I = 0$, or $J = 0$.

Proof: Suppose E is prime and $I \cap J = 0$. If $I \neq 0$, there exists

$x \varepsilon I$ such that $x \neq 0$, and $x \perp y$, for any $y \varepsilon J$. This implies $y = 0$ and so $J = 0$. Conversely, if $x,y \varepsilon E$, $x \perp y$, then $\langle x \rangle \cap \langle y \rangle = 0$. Hence, $\langle x \rangle$ or $\langle y \rangle$ is 0, and so x or y is 0.

An ab-ideal P of an AOVS E is called <u>prime</u> if $x,y \varepsilon E$, $x \perp y$ implies x or $y \varepsilon P$.

(3.13) <u>Lemma</u>: An ab-ideal P of E is prime, if and only if, for $I, J \varepsilon \mathcal{A}(E)$, $I \cap J = 0$ implies $I \subseteq P$ or $J \subseteq P$.

<u>Proof</u>: Suppose P is prime, $I \cap J = 0$, and $I \not\subseteq P$, then there exists $a \varepsilon I$ such that $a \notin P$. Since $I \cap J = 0$, $a \perp b$, for any $b \varepsilon J$, and $a \notin P$, we have $J \subseteq P$. Conversely, if $x,y \varepsilon E$, $x \perp y$, then $\langle x \rangle \cap \langle y \rangle = 0$, which implies x or $y \varepsilon P$. Thus P is prime.

(3.14) <u>Corollary</u>: The set $\Pi(E)$ of prime ab-ideals of an AOVS E is a topological space under the HK-operator.

<u>Proof</u>: The proof follows immediately from (3.13).

(3.15) <u>Lemma</u>: Let E be an AOVS, I an ab-ideal of E. Then E/I is prime, if and only if, I is prime.

<u>Proof</u>: We note that since I is a directed o-ideal, the projection $E \to E/I$ preserves meets and joins. Suppose E/I is prime, $a \perp b$ and $a \notin I$. Since $0 + I = |a| \wedge |b| + I = (|a| + I) \wedge (|b| + I)$ and $|a| + I \neq 0 + I$, then $|b| + I = 0 + I$, which implies $b \varepsilon I$. Conversely, $(|a| + I) \wedge (|b| + I) = 0 + I$ implies $|a| \wedge |b| + I = 0 + I$, which implies $|a| \wedge |b| \varepsilon I$, which implies $|a|$ or $|b|$ is in I, which implies $|a| + I$ or $|b| + I$ is $0 + I$.

CHAPTER IV

This chapter is concerned with dual systems $\langle E,F \rangle$ of OVS, and with the relations between the ideals of E and those of F. Several results concerning the preservation of topological and order properties by quotients of OVS are given. A class of ideals, called perfect ideals, introduced by Bonsall in (1) and by Kist in (8), is discussed. Perfect ideals are characterized by the property that their annihilators in the order dual are ideals. The anniliators of perfect maximal ideals are one dimensional ideals in the order dual. If an OVS E has a unit e, then this fact may be used to correspond perfect maximal ideals of E to the extreme points of $B_e = \{h \in {}^{+}E^* : h(e) = 1\}$. If E is an Archimedean OVS with a unit e, then $\langle E,E^* \rangle$ is a dual system of normed OVS. The Krein-Milman Theorem may be applied to assert that B_e is the $w(E^*,E)$ - closure of the convex hull of the extreme points of B_e. We show that this assertion is equivalent to the perfect maximal ideals of E having 0 intersection, and this is proved without using the Krein-Milman Theorem. When $\langle E,E^* \rangle$ is a dual system of OVS where E has a weakly closed cone, we obtain conditions on E and E^* which are equivalent to the perfect maximal ideals of E having 0 intersection. This generalizes the results for the case that E is Archimedean with a unit. If E is an Archimedean lattice OVS with unit, then the HK-closure on the set of perfect maximal ideals is shown to be a topological closure.

If E is an Archimedean OVS with a unit, then it is well known that E is isomorphic to a subspace of a space $\mathcal{C}(x)$ of continuous real valued functions on a compact Hausdorff space. This representation depends on choosing a unit e of E. Then the space X is B_e, endowed with the $w(E^*,E)$-topology, or some subspace of B_e. In a roundabout way this representation may be obtained as follows: Since the intersection of $\Omega(E)$ the set of maximal ideals of E is 0, E is isomorphic to a subspace of $\Pi_\Omega E/M$ by the subdirect representation. The map $p_e : \bigcup_\Omega E/M \to R$, defined by $x_M \to p_{e_M}(x_M)$, where $p_{e_M} : E/M \to R$ is the isomorphism induced by the unit e_M of E/M, trivializes $p : \bigcup_\Omega E/M \to \Omega$. Hence, the map $x \to p_e\hat{x}$, for $x \in E$, represents E as a subspace of functions from Ω to R, which may be given the weak topology. This is the same representation as choosing X to be B_e, since there is a bijection between B_e and $\Omega(E)$. We generalize this approach to represent OVS which do not necessarily have units. If the intersection of $\Omega(E)$ is 0, for an OVS E, then we have the subdirect representation $\hat{} : E \to \Pi_\Omega E/M$. If $(I_a)_{a \in A}$ is a family of weakly closed modular perfect ideals of E, then they may be used to give $\Pi_\Omega E/M \to \Omega$ a vector bundle structure. Hence E is represented as a subspace of the OVS of sections of this bundle.

1. Ideals and Positive Linear Maps

(1.1) Lemma: Let E be a directed OVS. There is a bijection between $\Omega(E)$, the set of (modular) maximal ideals of E, and the one-dimensional rays contained in ${}^+E^*$ given by $M \to M^o \cap {}^+E^*$, where $M^o = \{h \in E^* : h(x) = 0, \text{ for any } x \in M\}$.

<u>Proof</u>: For any $M \in \Omega$, $M^o \cap {}^+E^*$ is non empty, since for any unit $a_M \in E/M$, the projection map $E \to E/M$, composed with $p_{a_M} : E/M \to R$, is a positive linear functional on E with kernel M. Clearly $M^o \cap {}^+E^*$ is a ray. Conversely, if H is a ray in ${}^+E^*$, then $M = H^o$ is a maximal ideal in E.

(1.2) <u>Lemma</u>: For any ideal I of an OVS E, there is a bijection between ${}^+I^o$ in E^* and ${}^+(E/I)^*$.

<u>Proof</u>: For any $h \in {}^+I^o$, $\ker(h) \supseteq I$. Hence, there exists a unique positive map $\bar{h}: E/I \to R$ such that $h(x) = \bar{h}(x_I)$, for $x \in E$. If $h_1 \neq h_2$ for $h_1, h_2 \in {}^+I^o$, then there exists $x \in E$, $x \notin I$ such that $h_1(x) \neq h_2(x)$. Hence, $\bar{h}_1(x_I) \neq \bar{h}_2(x_I)$. For any $h \in {}^+(E/I)^*$, the projection $E \to E/I$ composed with h, defines a positive functional h' on E, such that $\ker(h') \supseteq I$ and $h'(x) = h(x_I)$, for $x \in E$.

(1.3) <u>Lemma</u>: If I is a directed ideal of an OVS E, then I^o is an ideal in E^*.

<u>Proof</u>: Suppose $h, g \in {}^+I^o$, $h \leq g$, then for any $x \in I$, we have $x = x_1 - x_2$, $x_1, x_2 \in {}^+I$, and so $0 \leq h(x_i) \leq g(x_i) = 0$, for $i = 1,2$. Thus, $h(x) = h(x_1) - h(x_2) = 0$; hence $h \in I^o$.

(1.4) <u>Corollary</u>: If M is a maximal ideal of E which is directed, then M^o is a one-dimensional ideal in E^*.

(1.5) <u>Lemma</u>: Let E be a directed OVS. For any $a \in {}^+E$, there is a bijection between $U_a = \{M \in \Omega(E) : a \notin M\}$ and $V_a = \{h \in {}^+E^* : h(a) = 1\}$, and if V_a is nonempty, then it is a base for a cone in E*. In particular, if a is an order unit of E, then V_a is a base for $^+E^*$.

Proof: For $M \in U_a$, let h_M be the positive linear functional on E, which is the composition of the projection $E \to E/M$ and $p_{a_M}:E/M \to R$, where $p_{a_M}(x_M) = \inf\{\alpha \in R : x_M \leq \alpha a_M\}$. Since $h_M(a) = p_{a_M}(a_M) = 1$, we have $h_M \in V_a$. If $h \in V_a$, then $\ker(h) \in U_a$ and $h = h_M$, since the canonical linear isomorphism $h:E/M \to R$ induced by h, is p_{a_M}, since $h'(a_M) = 1$. If $M_1, M_2 \in U_a$, $M_1 \neq M_2$, then there exists an x in M_1, $x \notin M_2$, hence $h_{M_1}(x) = 0$ and $h_{M_2}(x) \neq 0$ and so $h_{M_1} \neq h_{M_2}$. Thus, the map $M \to h_M$ is a bijection. Suppose that V_a is nonempty. Let K_a be the cone $\{h \in {}^+E^*: h(a) > 0 \text{ or } h = 0\}$ and let $E^*_a = (K_a)$ be the linear space in E* generated by K_a. The map $\hat{a}:E^*_a \to R$ defined by $\hat{a}(h) = h(a)$ is strictly positive on E^*_a, $a^{-1}(1) = V_a$, and so V_a is a base for E^*_a.

2. Dual Systems of OVS

(2.1) Let <E,F> be a dual system of linear spaces. If ^+E is a cone in E, then the set $^+E^* = \{h \in F : \langle a,h\rangle \geq 0, \text{ for all } a \in {}^+E\}$ is a wedge in F. Furthermore, $^+E^*$ is a cone if and only if, $E = \overline{{}^+E - {}^+E}$, where $\overline{{}^+E - {}^+E}$ is the w(E,F)-closure of $^+E - {}^+E$. We verify that $^+E^*$ is a cone when $\overline{{}^+E - {}^+E} = E$. Suppose $h \in {}^+E^* \cap -{}^+E^*$, then $h(x) = 0$, for any $x \in {}^+E$. Hence, for any $x \in E = \overline{{}^+E - {}^+E}$. To verify the converse, suppose $^+E^*$ is a cone in F. If there exists

$x \in E$, $x \notin \overline{{}^+E - {}^+E}$, then there exists $h \in F$, such that $h(x) \neq 0$, and $\ker(h) \supseteq \overline{{}^+E - {}^+E}$. Hence, $h \in {}^+E^* \cap - {}^+E^*$ and $h \neq 0$. Thus, by way of contradiction, we have $E = \overline{{}^+E - {}^+E}$. Also, ${}^+E^* = (-{}^+E)^{\bullet}$, and so is $w(F,E)$-closed. From the Bipolar theorem, ${}^+E$ is $w(E,F)$-closed if and only if, ${}^+E = ({}^+E)^{\bullet\bullet}$.

The following lemma is well known, and its proof follows from the observations in the preceding paragraph.

(2.2) Lemma: Let $\langle E,F \rangle$ be a dual system of linear spaces where E is an OVS. Then:

(a) ${}^+E$ is closed for a topology $\mathcal{U}$ consistent with $\langle E,F \rangle$, if and only if, $x \in {}^+E$ precisely when $\langle x,f \rangle \geq 0$, for all $f \in {}^+E^*$.

(b) the $\mathcal{U}$-closure $\overline{{}^+E}$ of ${}^+E$ is a cone in E, if and only if, ${}^+E^x - {}^+E^*$ is $w(F,E)$-dense in F.

A dual system of OVS is a dual system of linear spaces $\langle E,F \rangle$ where E and F are OVS, and where $\langle x,f \rangle \geq 0$, for any $f \in {}^+F$ and for any $x \in {}^+E$. If $\langle E,F \rangle$ is a dual system of OVS, then the canonical representations $E \to \hat{E} \subseteq F^{\#}$ and $F \to \hat{F} \subseteq E^{\#}$, are positive maps, where $F^{\#}$ and $E^{\#}$ are ordered by the cones ${}^+F^*$ and ${}^+E^*$ respectively. Furthermore, if E and F are directed, then $\hat{E} \subseteq E^*$ and $\hat{F} \subseteq F^*$.

(2.3) Lemma: If $\langle E,F \rangle$ is a dual system of directed OVS, then E and F are almost Archimedean.

Proof: If $x \in [-\lambda y, \lambda y]$, for $x \in E$, $y \in {}^+E$, $\lambda > 0$, then $\hat{x}(f) =$

$\langle x,f \rangle = 0$, for all $f \in {}^{+}F$, since R is Archimedean. Since F is directed, $\hat{x}(f) = 0$, for all $f \in F$. Thus, $x = 0$, since the kernel of $E \to \hat{E} \subseteq F^*$ is 0.

(2.4) A dual system of directed OVS may be obtained from a dual system of sets paired to ${}^{+}R$. Let $\langle A,B;{}^{+}R \rangle$ be a dual system of sets. The linear space $E = (A)$, generated by $\hat{A}$ in $\mathfrak{F}(B,R)$, and the linear space $F = (B)$, generated by $\hat{B}$ in $\mathfrak{F}(A,R)$, form a linear duality. E and F are directed OVS when endowed respectively, with the cones ${}^{+}E = \{a \in E : a = \sum_i^n \lambda_i a_i$, for some $\lambda_i \geq 0$, $a_i \in A\}$ and ${}^{+}F = \{h \in F : h = \sum_i^n \mu_i b_i$, for some $b_i \in B$, $\lambda_i \geq 0\}$. For $g \in E$, $g = \sum_i^n \lambda_i a_i$ where $\lambda_i \in R$, $a_i \in A$, let P be the set of indices for which $\lambda_i > 0$, and let N be the set of indices for which $\lambda_i < 0$; then $g = g_1 - g_2$ where $g_1 = \sum_p \lambda_i a_i$, and $g_2 = \sum_N -\lambda_i a_i$. If $g \in {}^{+}E$, then $g = \sum \lambda_i a_i$ where $\lambda_i > 0$, $a_i \in A$, and so $\langle g,b \rangle = \langle \sum \lambda_i a_i, b \rangle = \sum \lambda_i \langle a_i,b \rangle \geq 0$, for any $b \in B$. If $f \in {}^{+}F$, then $f = \sum \mu_i b_i$ where $\mu_i \geq 0$, $b_i \in B$, which implies $\langle g,f \rangle = \langle g, \sum \mu_i b_i \rangle = \sum \mu_i \langle g,b_i \rangle \geq 0$. Thus, $\langle E,F \rangle$ is a dual system of directed OVS.

A dual system of OVS $\langle E,F \rangle$ is said to be <u>full</u> if ${}^{+}E$ and ${}^{+}F$ are w-closed. By (2.2), $x \in {}^{+}E$ if and only if $\langle x,h \rangle \geq 0$, for all $h \in {}^{+}F$. If $\langle E,F \rangle$ is a full dual system of OVS, then the canonical representations $E \to F^{\#}$ and $F \to E^{\#}$ are embeddings.

(2.5) <u>Lemma</u>: If $\langle E,F \rangle$ is a full dual system of OVS, then E and F are Archimedean ordered.

Proof: If $x,y \in E$, $0 < y$, and $x \leq \lambda y$, for all $\lambda > 0$, then $\langle y,h \rangle \geq 0$, and $\langle x,h \rangle \leq \lambda \langle y,h \rangle$, for all $\lambda > 0$ and for all $h \in {}^{+}F$. Since R is Archimedean, we have $\langle x,h \rangle \leq 0$, for all $h \in {}^{+}F$. This implies $x \leq 0$, since $E \to \hat{E}$ is an isomorphism.

(2.6) Lemma: Let $\langle E,E^* \rangle$ be a full dual system of OVS. If I is an ideal of E, then ${}^{+}(E/I)^* = {}^{+}(E/I)^{\prime}_{\bar{w}(E,E^*)}$.

Proof: Clearly, ${}^{+}(E/I)^{\prime}_{\bar{w}(E,E^*)} = \{h \in (E/I)^{\prime}_{\bar{w}} : h(x) \geq 0$, for all $x \in {}^{+}(E/I)\}$ is contained in ${}^{+}(E/I)^*$. We show that ${}^{+}(E/I)^* \subseteq {}^{+}(E/I)^{\prime}_{\bar{w}}$. If $u:E/I \to R$ is a positive linear map, then $up:E \to E/I \to R$ is a positive linear map; hence up is $w(E,E^*)$-continuous. Since $\bar{w}$ is the inductive topology induced by the projection $p:E \to E/I$, u is $\bar{w}$-continuous on E/I. Thus, ${}^{+}(E/I)^* = {}^{+}(E/I)^{\prime}_{\bar{w}(E,E^*)}$.

(2.7) Lemma: Let $\langle E,F \rangle$ be a full dual system of OVS. If I is an ideal of E, then I^o, with the subduced ordering, is isomorphic to $((E/I)^{\prime}_{\bar{w}(E,F)}, (E/I)^{\prime}_{\bar{w}} \cap {}^{+}(E/I)^*)$, and $I^o = \overline{{}^{+}I^o - {}^{+}I^o}$, the $w(F,E)$-closure of ${}^{+}I^o - {}^{+}I^o$. If I is w-closed, then $\langle E/I, (E/I)^{\prime}_{\bar{w}} \rangle$ is a full dual system of OVS.

Proof: Let $p:E \to E/I$ be the natural projection and let $G = E/F$. I^o is linearly isomorphic to $H = G^{\prime}_{\bar{w}} = G^{\prime}_{w(G,H)}$, by the correspondence $h \to hp \in I^o$, for $h \in H$. We show that this correspondence and its inverse are positive maps. If $g \in {}^{+}I^o = I^o \cap {}^{+}F$, then $\ker(\hat{g})$ is an ideal containing I. Thus, $\hat{g}^{\prime}:E/I \to E/\ker(\hat{g}) \to R$, the inverse of g under the linear isomorphism, is in ${}^{+}(E/I)^{\prime}_{\bar{w}} = (E/I)^{\prime}_{\bar{w}} \cap {}^{+}(E/I)^*$.

If $h \in {}^{+}G'_{w}$, then $hp: E \to E/I \to R$ is in ${}^{+}E^{*} \cap \hat{F}$. Thus $\widehat{hp}^{-1}$, the functional in F corresponding to hp is positive since $F \to \hat{F}$ is an isomorphism. The cone ${}^{+}G$ is $\bar{w}(E,F)$-closed, since ${}^{+}E$ is $w(E,F)$-closed and the projection is an open surjection. Since ${}^{+}G'_{w}$ is the dual cone of ${}^{+}G$, G'_{w} is the $\bar{w}$-closure of ${}^{+}G'_{w} - {}^{+}G'_{w}$, by (2.1). Thus $I^{o} = {}^{+}I^{o} - {}^{+}I^{o}$. If I is w-closed, then clearly $\langle E/I, (E/I)'_{w}\rangle$ is a full dual system of OVS.

3. Topology and Order

An order seminorm on an OVS E, is a seminorm $p: E \to R$ for which, if $x, y \in {}^{+}E$, $x \leq y$, then $p(x) \leq p(y)$. The kernel of an order seminorm is an ideal of E.

(3.1) Let E be an OVS. A vector topology $\mathcal{U}$ for E is called order normal, if it satisfies the following equivalent conditions:

(1) $\mathcal{V} = [\mathcal{V}]$, where $\mathcal{V}$ is the neighborhood filter of 0.

(2) $\mathcal{U}$ has a base $\mathcal{B}$, such that $V \in \mathcal{B}$ implies $[V \cap {}^{+}E] \subseteq V$.

(3) $\mathcal{U}$ is generated by a family of order seminorms.

For a proof of the equivalence of these conditions see (10) page 215.

(3.2) Theorem: (Krein) If E is a directed OVS with an order norm topology $\mathcal{U}$, then $E'_{\mathcal{U}}$ is directed.

Proof: For the proof see (10) page 218.

The following theorem may be found in (10) page 219. However, the proof given here is more algebraic.

(3.3) Theorem: If E is a directed OVS with a Hausdorff order normal topology $\mathcal{U}$, then $E'_{\mathcal{U}}$ is directed, and hence $\langle E,E'_{\mathcal{U}}\rangle$ is a dual system of directed OVS.

Proof: Let $\mathcal{P}$ be a family of order seminorms generating $\mathcal{U}$. If $h \in E'$, then there exists $\lambda > 0$ and $p_i \in \mathcal{P}$, for $i = 1,..,n$, such that $|h(x)| < \lambda \sup_i p_i(x)$. Furthermore, $p = \lambda \sup_i p_i$ is an order seminorm on E, and the OVS $E/\ker(p)$ is order normable with $\hat{x} \to \bar{p}(\hat{x}) = p(x)$, for $x \in \hat{x}$; $\hat{x} \in E/\ker(p)$. Since $\ker(h) \supseteq \ker(p)$, there exist a unique linear functional $\bar{h}:E/\ker(p) \to R$, such that $h = \bar{h}q$ where $q:E \to E/\ker(p)$ is the natural projection. Thus, h is bounded by $\bar{p}$; hence, continuous. By (3.2), $\bar{h} = \bar{h}_1 - \bar{h}_2$ where $\bar{h}_i$ are positive, and so $h = \bar{h}_1 q - \bar{h}_2 q$. Thus $E'_{\mathcal{U}}$ is directed.

(3.4) Theorem: Let $\langle E,E^*\rangle$ be a dual system of directed OVS. Then $E_{w(E,E^*)}$ is order normal.

Proof: For the proof see (10) page 220.

(3.5) Lemma: Let E be a Riesz OVS and I a directed ideal of E. If $\mathcal{U}$ is an order normal topology for E, then $\bar{\mathcal{U}}$ the quotient topology for E/I is order normal.

Proof: Let $\mathcal{B}$ be a base of $\mathcal{U}$, such that $V \in \mathcal{B}$ implies $[V \cap {}^+E] \subseteq V$. The set $\{ V_I : V \in \mathcal{B}\}$ is a base for $\bar{\mathcal{U}}$ in E/I. If $x_I \in [V_I \cap {}^+(E/I)]$, then $0 \leq x_I \leq y_I$, for some $y \in V \cap {}^+E$, where $x \in {}^+E$. Hence, $x + m \leq y + n$, for some $m,n \in I$, which implies $x \leq y + m - n \leq y + p$, for some

$p \in {}^{+}I$, since I is directed. Since E is a Riesz OVS, $x = x_1 + x_2$ where $x_1 \leq y$, $x_2 \leq p$. Hence, $x_2 \in I$, which implies $(x_1)_I = x_I$, and $x_1 \in V$. Thus, $x_I \in V_I$, and so $\overline{\mathcal{U}}$ is order normal.

The set E^b of all linear functionals on an OVS E which are bounded on the order intervals of E, with the subduced ordering, is an OVS called the <u>order bounded</u> dual of E. It is clear that $E^* \subseteq E^b$.

Let E,F be OVS. The <u>order topology</u> $\mathcal{U}_o$ for E is the finest locally convex topology $\mathcal{U}$, for which every order bounded set is $\mathcal{U}$-bounded. A base for $\mathcal{U}_o$ is given by the collection of all convex circled sets which absorb all order bounded sets. If $h:E \to F$ is an order bounded linear map, then h is continuous for the respective order topologies on E and F. The dual $E'_{\mathcal{U}_o}$ of E coincides with E^b, the OVS of order bounded linear functionals on E. If $E_{\mathcal{U}_o}$ is Hausdorff, then $\mathcal{U}_o$ is the Mackey topology of the duality $\langle E, E'_o \rangle$.

(3.6) <u>Theorem</u>: (Riesz) If E is a Riesz OVS, then $E^* = E^b$.

<u>Proof</u>: For a proof see Perressini (9) page 24.

(3.7) <u>Theorem</u>: Let $\langle E, E^* \rangle$ be a full dual system of OVS, where E is a Riesz OVS, and let I be a directed ideal of E. Then I^o in E^* is (order) isomorphic to $(E/I)^*$, and the quotient topology on E/I of the order topology $\mathcal{U}_o$ on E is the order topology of E/I.

<u>Proof</u>: The quotient topology on E/I of the $w(E,E^*)$-topology is normal by (3.5). Thus $(E/I)'_w = (E/I)^*$ by (2.6) and (2.7), and so

I^o is isomorphic to $(E/I)^*$. The quotient topologies $\bar{w}(E,E^*)$ and $\mathcal{U}_o$, and the order topology on E/I all determine the same duality. Thus $\mathcal{U}_o$ is coarser than the order topology on E/I. Since the projection $p:E \to E/I$ is continuous with respect to the order topologies on E and E/I, and the quotient topology is the finest l.c. topology for which the projection is continuous, the quotient topology $\bar{\mathcal{U}}_o$ is finer than the order topology. Thus, they coincide.

4. Archimedean OVS with Units

(4.1) Let E be an Archimedean OVS, and let $e \in E$ be a unit. The norm $p(x) = \inf\{\lambda > 0 : x \in [-\lambda e, \lambda e]\}$ is clearly an order norm. Hence E_p' is directed. Any positive linear map $h:E \to R$ is norm continuous, since $|h(x)| \leq h(e)$, for any $x \in [-e,e]$. Hence, $E_p' = E^*$. The order unit norm topology is the order topology $\mathcal{U}_o$. The order topology is finer than the norm topology, since the norm topology is l.c. and $[-e,e]$ absorbs order intervals, and the norm topology is finer than $\mathcal{U}_o$, since $[-e,e]$ is $\mathcal{U}_o$-bounded. The base $B_e = \{h \in {}^+E^* : h(e) = 1\}$ of ${}^+E^*$ is $w(E^*,E)$-compact, since B is the intersection of the w-closed hyperplane $\{h \in E^* : h(e) = 1\}$ and the w-compact set $[-e,e]^\bullet$. Suppose $x \in E$, $h(x) \geq 0$, for all $h \in {}^+E^*$. There exists a positive functional $h' \in {}^+E^*$, such that $h'(x) = u(x) = \sup\{\lambda \in R : \lambda e \leq x\}$, which implies $u(x) \geq 0$, which implies $x \geq 0$. Thus ${}^+E$ is $w(E,E')$-closed, and so it follows that $< E,E^* >$ is a full dual system of directed OVS.

(4.2) Lemma: Let E be an Archimedean OVS, with an order unit e.

The polar $[-e,e]^{\circ}$ in E^* is $\text{conv}(B \cup -B)$ where B is the base in E^*, induced by $\hat{e}:E^* \to R$, defined by $\hat{e}(h) = h(e)$.

Proof: Let $S = \text{conv}(B \cup -B)$. First, we show $S^{\circ} = [-e,e]$. If $x \in S^o$, then $| <x,b> | \leq 1$, for any $b \in B$. This implies $| <x,b> | \leq <e,b>$, for any $b \in B$, which implies $| <x,h> | \leq <e,h>$, for any $h \in {}^+E^*$, which implies $x \in [-e,e]$. If $x \in [-e,e]$, then for $h = \alpha b - \alpha' b' \in S$, where $\alpha + \alpha' = 1$, we have $<x,h> = \alpha <x,b> - \alpha' <x,b'> \leq \alpha - \alpha'$. Thus $| <x,h> | \leq 1$, and so $x \in S^{\circ}$. The Bipolar theorem implies that $S^{\circ\circ}$ is the $w(E^*,E)$-closure of S. However, since B and $-B$ are w-compact, convex sets, the convex hull is compact, and so $S = S^{\circ\circ}$.

(4.3) Lemma: Let E be an Archimedean OVS, and let $e \in E$ be a unit of E. Then $E^{**} = E^{*\prime}$.

Proof: A linear functional $h:E^* \to R$ is in $E^{*\prime}$, if and only if, there exists $\beta > 0$, such that $|h(x)| \leq \beta g(x)$, for all $x \in E^*$, where g is the Minskowski gauge of $\text{conv}(B \cup -B)$. If $h \in E^{*\prime}$, $x \in {}^+E^*$, then $-\beta < e,x> = -\beta g(x) \leq h(x) \leq \beta g(x) = \beta <e,x>$, which implies $h \in [-\beta e, \beta e]$. Thus $h \in E^{**}$. If $h \in E^{**}$, then $\hat{h} = h|B$ is continuous in the $w(E^*,E)$-topology on B. Since B is $w(E^*,E)$-compact, $\hat{h}$ is bounded; that is, there exists $\lambda > 0$ such that $|h(b)| \leq \lambda = \lambda e(b)$, for $b \in B$. To show $h \in E^{**}$ is norm bounded, it suffices to show that $|h(x)|$ is bounded on the unit sphere $\text{conv}(B \cup -B) = \{x \in E^* : g(x) \leq 1\}$. If $g(x) \leq 1$, then $\alpha x = b_1 + (1-\alpha)b_2$, for $b_1,b_2 \in B \cup -B$. Hence, $|h(x)| = |\alpha h(b_1) + (1-\alpha)H(b_2)| \leq \alpha\lambda + (1-\alpha)\lambda = \lambda$.

(4.4) Theorem: Let E be an OVS and I a modular ideal of E. Then E/I is archimedean if and only if, E/I is isomorphic to $E(H(I))$, where $H(I) = \{M \varepsilon \Omega : I \subseteq M\}$.

Proof: If E/I is isomorphic to $E(H(I))$, then E/I is Archimedean, since $E(H(I))$ is Archimedean. If E/I is Archimedean, then $I = KH(I)$ and the canonical map $\hat{}:E/I \to E(H(I))$ is a bijective homomorphism. Thus, it suffices to show $\hat{}$ is an embedding. If $\hat{a} \varepsilon {}^{+}E(H(I))$, then $h(a_I) \geq 0$, for all $h \varepsilon {}^{+}(E/I)^*$. Since E/I is Archimedean, and has a unit, $\langle E/I, (E/I)^* \rangle$ is a full dual system. Thus, $a_I \varepsilon {}^{+}(E/I)$, and so E/I is isomorphic to $E(H(I))$.

(4.5) Corollary: (Kadison) Let E be an Archimedean OVS with a unit e. Then E is isomorphic to $E(\Omega)$.

(4.6) Theorem: Let $\langle E,E^* \rangle$ be a full dual system of OVS, for which $E^* = E^b$. If I is a modular ideal of E, then E/I is Archimedean, if and only if, I is $w(E,E^*)$-closed.

Proof: If I is $w(E,E^*)$-closed, then $\langle E/I, (E/I)_{\overline{w(E,E)}} \rangle$ is a full dual system, by (2.7), which implies E/I is Archimedean. Conversely, if E/I is Archimedean and has a unit, the order topology of E/I is a l.c. Hausdorff topology. The projection $p:E \to E/I$ is continuous for the respective order topologies. This implies that the quotient topology $\overline{\mathcal{U}}_o$ on E/I is finer than the order topology, which in turn implies that the quotient topology is Hausdorff. Thus I is closed in the quotient topology and hence in the $w(E,E^*)$-topology.

5. Perfect Ideals

Let E be an OVS. For any linear subspace V of E, and for any $a \in {}^{+}E$, let $H_a(V) = \{x \in E :$ for each $\lambda > 0$, there exists $v_\lambda \in V$, for which $-(\lambda a + v_\lambda) < x < \lambda a + v_\lambda\}$. An ideal I of E is <u>perfect</u>, if $I \subseteq H_a(I)$, for some $a \in {}^{+}E$.

Verification of the following two lemmas is routine.

(5.1) <u>Lemma</u>: If V is a subspace of an OVS E, then $H_a(V)$ is an ideal of E, for any $a \in {}^{+}E$.

(5.2) <u>Lemma</u>: If V is a subspace of an OVS E, and $V \subseteq H_a(V)$, for some $a \in {}^{+}E$, then $H_a(V)$ is a perfect ideal of E.

(5.3) <u>Lemma</u>: If M is a maximal ideal of an OVS E, and V is a subspace of M, then $H_a(V) \subseteq M$, for any $a \in {}^{+}E$.

<u>Proof</u>: Let $h:E \to R$ be a positive linear functional with $M = \ker(h)$. If $x \in H_a(V)$, then for any $\lambda > 0$, there exists $v_\lambda \in V$, such that $-(\lambda h(a) + h(v_\lambda)) \leq h(x) \leq \lambda h(a) + h(v_\lambda)$. This implies $-\lambda h(a) \leq h(x) \leq \lambda h(a)$, for any $\lambda > 0$, which implies $h(x) = 0$. Thus $H_a(V) \subseteq M$.

(5.4) <u>Theorem</u>: Let E be a directed OVS. An ideal I of E is perfect, if and only if, I^o is an ideal of E^*.

<u>Proof</u>: Suppose I is a perfect ideal of E, and $I \subseteq H_a(I)$, for some $a \in {}^{+}E$. If $h,g \in {}^{+}E^*$, $g \leq h$, $h \in I^o$, and $x \in I$, then, for any $\lambda > 0$, there exists $y_\lambda \in I$, for which $-(\lambda a + y_\lambda) \leq x \leq \lambda a + y_\lambda$. This implies $|g(x)| \leq g(\lambda a + y_\lambda) \leq h(\lambda a + y_\lambda) = \lambda h(a)$, for all $\lambda > 0$, and so $g(x) = 0$.

Thus, $g \in I^o$.

Suppose I^o is an ideal of E^*. For any $f \in {}^+I^o$, for any $x \in E$, let $q_f(x) = \inf \{ f(y) : y \in {}^+E, y \geq x\}$. Since E is directed, $q_f(x) \in R$. It is easily verified that $q_f : E \to R$ satisfies the properties:

(1) q_f is sublinear,

(2) for any $x,y \in E$, if $x \leq y$, then $q_f(x) \leq q_f(y)$,

(3) $q_f(x) \geq 0$, for all $x \in E$,

(4) if $x \in -{}^+E$, then $q_f(x) = 0$,

(5) for any $x \in E$, $q_f(x) \leq f(x)$,

(6) if $x \in {}^+E$, then $q_f(x) = f(x)$.

Let $H = \{h \in E^{\#} : h(x) \leq q_f(x)$, for all $x \in E$. The Banach Theorem implies $q_f(x) = \sup \{h(x) : h \in H\}$, for $x \in E$. If $h \in H$, $x \in {}^+E$, then $h(-x) < q(-x) = 0$ implies $h(x) \geq 0$, thus $0 \leq h \leq f$. Since I^o is an ideal of E^*, $h \in I^o$. Thus, we have

(7) $q_f(x) = 0$, for any $x \in I$.

There exists $f \in {}^+I^o$, such that $f \neq 0$, and so there exists $a \in {}^+E$, such that $f(a) = 1$. For any $x \in I$, and for any $\lambda > 0$, there exists $y_\lambda \in {}^+E$, such that $x \leq y_\lambda$ and $\beta = f(y_\lambda) \leq \lambda/2$, since $q_f(x) = 0$. If $v'_\lambda = y_\lambda - \beta a$, then $f(v'_\lambda) = 0$, and so $v'_\lambda \in I$. Moreover, $v'_\lambda + (\lambda/2)a \geq v'_\lambda + \beta a = y_\lambda$. Similarly, there exists $v''_\lambda \in I$, such that $v''_\lambda + (\lambda/2)a$ is in ${}^+E$ and is greater than, or equal to, $-x$. Thus, $\pm x \leq \lambda a + v_\lambda$, where $v_\lambda = v'_\lambda + v''_\lambda$. Thus $I \subseteq H_a(I)$, i.e., I is perfect.

(5.5) <u>Corollary</u>: If E is a directed OVS, any directed ideal is perfect.

Proof: If I is a directed ideal of E, then I^o is an ideal of E^*.

(5.6) <u>Corollary</u>: Let E be a directed OVS. A maximal ideal M of E is perfect, if and only if, M^o is an indecomposable ideal of E.

<u>Proof</u>: Since M^o is a 1-dimensional subspace of E^*, M^o is an ideal, if and only if, it is indecomposable.

(5.7) <u>Theorem</u>: Let E be a lattice OVS. If an ideal of E is perfect and $w(E,E^*)$-closed, then it is an absolute (lattice) ideal.

<u>Proof</u>: Let I be a perfect w-closed ideal of E. For any $x \in I$, and for any $\lambda > 0$, there exist $a \in {}^+E$ and $x_\lambda \in I$, such that $= x \leq x_\lambda + \lambda a$. Hence $0 \leq |x| \leq x_\lambda + \lambda a$. If $h \in I^o$, then $0 \leq h(|x|) \leq h(x_\lambda + \lambda a) = \lambda h(a)$, for all $\lambda > 0$. Thus, $h(|x|) = 0$, and, since I is w-closed, $|x| \in I$. This shows that I is an absolute ideal, for, suppose $|x| \leq |y|$, $y \in I$, then $-|y| \leq x \leq |y|$ implies $y \in I$.

(5.8) <u>Corollary</u>: If E is a lattice OVS, a maximal ideal of E is perfect, if and only if, it is a lattice ideal.

<u>Proof</u>: Perfect maximal ideals of E are $w(-E,E^*)$-closed.

(5.9) <u>Theorem</u>: Let E be a directed OVS, for which E^* is a lattice OVS. If I is a perfect, $w(E^*,E)$-closed ideal of E^*, then it is an absolute ideal.

<u>Proof</u>: The proof is analogolous to (5.7).

(5.10) <u>Theorem</u>: (Bonsall (1)). Let E be an OVS with a unit e.

If E has no nontrivial perfect ideals, then E is isomorphic to R.

Proof: The map $u:E \to R$, defined by $u(x) = \inf \{\lambda : \lambda e \geq x\}$, satisfies:

(1) u is sublinear,

(2) $u(e) = -u(-e) = 1$,

(3) $u(x) \leq 0$ whenever $x \leq 0$.

Given $x \in E$, let $y = u(x)e - x$. By the Banach Theorem, there exists a linear functional $h:E \to R$, such that $h(x) = u(x)$, and $h(z) \leq u(z)$, for $z \in E$. It follows from (2) and (3) that h is positive and $h(e) = 1$. Also, $h(y) = u(y) - h(x) = 0$. By definition of u, we have, for each $\lambda > 0$, $(\lambda e/2) + y \geq 0$. Thus $-(\lambda e + y) \leq y \leq \lambda e + y$. Hence, $Y = (y) \subseteq H_e(Y)$, $H_e(Y)$ is a non zero perfect ideal and $H_e(Y) \subseteq \ker(h)$. Thus, $H_e(Y)$ is proper. Thus, $y = 0$, $x = u(x)e$ and $u:E \to R$ is the desired isomorphism.

(5.11) Lemma: If I is a perfect ideal of an OVS E, and if J is a perfect ideal of E/I, then $p^{-1}(J)$ is perfect in E, where $p:E \to E/I$ is the natural projection.

Proof: Since $I \subseteq p^{-1}(J)$, $(p^{-1}(J))^o \subseteq I^o$. If $h \in {}^+(p^{-1}(J))^o$ and $0 \leq g \leq h$, then $h,g \in {}^+I^o$, which implies g corresponds to a unique $\bar{g} \in {}^+(E/I)^*$ and h corresponds to a unique $\bar{h} \in {}^+N^o \subseteq {}^+(E/I)^*$. Thus, since $\bar{g} \leq \bar{h}$ and J^o is an ideal of $(E/I)^*$, $\bar{g} \in J^o$, and so $g \in (p^{-1}(J))^o$.

(5.12) Lemma: Let E be an Archimedean OVS which has a unit e. For any $x \in E$, there exists a perfect ideal I of E, such that x_I is a unit of E/I.

Proof: The ideal $\langle u(x)e - x \rangle$, or the ideal $\langle \ell(x)e - x \rangle$ is directed, (II. 9.2) hence, perfect.

(5.13) Theorem: Let E be an Archimedean OVS which has a unit e. The intersection of the set $\Delta(E)$ of perfect maximal ideals of E is 0.

Proof: For any $x \varepsilon E$, $x \neq 0$, there exists an ideal I of E, such that $x_I = u(x)e_I$, or $x_I = \ell(x)e_I$ in E/I, by Kadison's Lemma. Furthermore, since $I = \langle u(x)e - x \rangle$ or $I = \langle \ell(x)e - x \rangle$, I is directed, hence, perfect. The set $\mathcal{J}_I = \{J \varepsilon \mathcal{J}(E) : J \supseteq I, x \notin J$ and J is perfect$\}$ is inductive, for if $(J_i)_{i \varepsilon C}$ is a chain in $\mathcal{J}_I$, then $x \notin J = \bigcup_C J_i$ and $J^o = \bigcap_C J_i^o$ is an ideal in E*, hence, J is perfect. Thus, $\mathcal{J}_I$ has maximal elements. If M is a maximal element of $\mathcal{J}_I$, then x_M or $-x_M$ is a unit of E/M and so x_M is not in any perfect ideals of E/M. If N is a perfect ideal of E/M, then $p^{-1}(N)$ is a perfect ideal in E, by (5.11). This contradicts the maximality of M. Thus E/M has no proper perfect ideals and so E/M is isomorphic to R, by (5.10).

(5.14) Lemma: Let <E,F> be a full dual system of directed OVS. If I is a perfect ideal of E, then I^o and I^{oo} are ideals.

Proof: The proof that I^o is an ideal of F is analogolous to (5.4). If $x,y \varepsilon {}^+E$, $x \leq y$, $y \varepsilon I^{oo}$, then $\langle x,h \rangle = 0$, for any $h \varepsilon {}^+I^o$.

Thus $\langle x,h \rangle = 0$, for any $h \in I^o$, since I^o is the $w(F,E)$-closure of $^+I^o - {}^+I^o$ (2.7). Thus I^{oo} is an ideal of E.

(5.15) <u>Corollary</u>: Let $\langle E,F \rangle$ be a full dual system of directed OVS, let $\mathcal{J}'(E) = \{I \in \mathcal{J}(E) : I^o$ is an ideal of $F\}$ and let $\mathcal{J}'(F) = \{I \in \mathcal{J}(F) : I^o$ is an ideal of $E\}$. The annihilation maps $I \to I^o$ are a Galois connection between $\mathcal{J}'(E)$ and $\mathcal{J}'(F)$.

(5.16) <u>Lemma</u>: Let $\langle E,E^* \rangle$ be a full dual system of directed OVS, let KH_1 be the closure generated by $\Delta(E)$, the set of perfect maximal ideals of E, and let KH_2 be the closure generated by the $w(E,E^*)$-closed modular perfect ideals of E. Then KH_1 and KH_2 coincide.

<u>Proof</u>: For any set $S \subseteq E$, we have $KH_2(S) \subseteq KH_1(S)$. If $x \in KH_1(S)$, then $x \in M$, for any $M \in \Delta$ such that $S \subseteq M$. Suppose there exist $P \in H_2(S)$, such that $x \notin P$. E/I has a unit, $x_I \neq 0$ and by (4.6) E/I is Archimedean. Hence, by (5.13) there exists a perfect maximal ideal M of E/I, such that $x_I \notin M$ and by (5.11) $p^{-1}(M)$ is a perfect maximal ideal in E which does not contain x. This contradicts the fact that $x \in KH_1(S)$. Hence, $KH_1(S) = KH_2(S)$.

(5.17) <u>Theorem</u>: Let $\langle E,E^* \rangle$ be a full dual system of directed OVS. These conditions are equivalent:

(1) The intersection of the set of $w(E,E^*)$-closed modular perfect ideals of E is 0.

(2) The intersection of the set of perfect maximal ideals of E is 0.

(3) The $w(E^*,E)$-closure of the convex hull of the union extremal ideals of E^* is E^*.

(4) The $w(E^*,E)$-closure of the sum of the annilators of perfect ideals of E is E^*.

Proof: (1) implies (2). This follows from (5.16). The proof of (2) implies (3) follows from the Bipolar theorem. The proof of (3) implies (4) is obvious. The proof of (4) implies (1) follows from the Bipolar theorem.

(5.18) Theorem: Let E be an Archimedean Riesz OVS with a unit e. If I,J are directed ideals of E, then $(I \cap J)^o = I^o + J^o$. Hence, $I \cap J$ is perf

Proof: We have I^o and J^o are $w(E,E^*)$-closed directed ideals of E^* by (III. 2.3) and by (3.7). Thus, $I^o + J^o$ is a directed ideal of E^*. From the Bipolar theorem, we have $(I \cap J)^o = \overline{(I^o + J^o)}$, the w-closure of $I^o + J^o$. Thus, it suffices to show that $I^o + J^o$ is w-closed. Since I^o is w-closed, $I^o \cap S$ is w-compact where S is the unit sphere of E^*. Since I is a directed OVS with a base norm (restricted from E^*), each $h \in I^o$ has a representation $h = \lambda s$ where $s \in I^o \cap S$, the unit sphere of I^o, and $\lambda \in {}^+R$. Analogous statements hold for J^o. Clearly, $\text{conv}((I^o \cap S) \cup (J^o \cap S)) \subseteq (I^o + J^o) \cap S$. If $h \in (I^o + J^o) \cap S$, then $h = \lambda_1 s_1 + \lambda_2 s_2$, where $s_1 \in I^o \cap S$, $s_2 \in J \cap S$, and $\lambda \in {}^+R$. Since $q(h) = q(s_1) = q(s_2) = 1$, where $q: E^* \to R$ is the base norm, we have $\lambda_1 + \lambda_2 = 1$, and so $h \in C = \text{conv}(I^o \cap S) \cup (J^o \cap S))$. Thus, $C = (I^o + J^o) \cap S$. Furthermore, C is $w(E^*,E)$-compact, since the convex hull of two compact convex sets is compact. Every element of the directed normed ideal $I^o + J^o$ is in some scalar multiple of $(I^o + J^o) \cap S$. Thus, $I^o + J^o$ is $w(E^*,E)$-closed. Hence, $(I \cap J)^o = I^o + J^o$ and $I \cap J$ is perfect.

Let E be a directed OVS, and let KH be the closure on $\mathcal{P}(E)$ generated by $\mathcal{J}'(E)$ the set of perfect ideals of E. If $\mathcal{J}'(E)$ is closed under intersection, then $\mathcal{J}'(E)$ coincides with the set of KH-closed elements of $\mathcal{P}(E)$, and hence is a complete lattice.

(5.19) Theorem: Let E be an Archimedean lattice OVS with a unit e. The set of perfect ideals of E is closed under intersection.

Proof: Let $(I_i)_{i \in A}$ be a family of perfect ideals of E. We have that I_i^{oo}, for $i \in A$, are w-closed perfect ideals by (5.14) and that I_i^{oo}, for $i \in A$, are absolute ideals by (5.7). Thus, $\bigcap_A I_i^{oo}$ is an absolute ideal and so $(\bigcap_A I_i)^o = (\bigcap_A I_i^{oo})^o$ is an ideal in E^*. Thus $\bigcap_A I_i$ is perfect.

(5.20) Theorem: Let E be an Archimedean Lattice OVS with a unit e. The HK-closure generated by $\Delta(E)$, the set of perfect maximal ideals of E, is a topological closure.

Proof: To show HK is a topological closure, it suffices to show that for any $M \in \Delta$, for any $U,V \subseteq \Delta$, if $K(U) \cap K(V) \subseteq M$, then $K(U) \subseteq M$, or $K(V) \subseteq M$. Since the set of perfect ideals is a complete lattice, $I = K(U)$ and $J = K(V)$ are $w(E,E^*)$-closed perfect ideals. Hence, $M^o \subseteq I^o + J^o$. Suppose $h \in {}^+M^o$, $h = h_1 + h_2$, where $h_1 \in I^o$, $h_2 \in J^o$.

Since I^o, J^o are directed ideals, there exist $h_1' \in {}^+I$, $h_2' \in {}^+J^o$, such that $h_1 \leq h_1'$, $h_2 \leq h_2'$. Thus, $h \leq h_1' + h_2'$. Since E* is a Riesz OVS, $h = h_1'' + h_2''$, where $h_1'' \in {}^+I^o$, $h_2'' \in {}^+I^o$. Since h is indecomposable, $h_1'' = \alpha h_2''$, for some $\alpha \in R$. Thus, $h_i'' \in I^o$, or $h_i'' \in J^o$. A similar argument applies to $h \in -{}^+M^o$. Thus $M^o \subseteq I^o$ or $M^o \subseteq J^o$. Thus, $M \supseteq I$, or $M \supseteq J$.

6. Fields of OVS

A <u>field</u> of OVS is a surjective topological map $p: \mathcal{E} \to X$, where $E_x = p^{-1}(x)$, the <u>fiber</u> over $x \in X$ is an OVS. The space X is called the <u>base space</u>, and the space $\mathcal{E}$ is called the <u>total</u> space of the field. Since p is a surjection, $(E_x)_{x \in X}$ is a disjoint family. A <u>section</u> of $p: \mathcal{E} \to X$ is a continuous map $f: X \to \mathcal{E}$, such that $pf: X \to X$ is the identity. For any subspace $U \subseteq X$, the restriction of p to $\mathcal{E}|U = p^{-1}(U)$ is a field which we denote by $p: \mathcal{E}|U \to U$. The set $\Gamma(U)$ of all sections of $p: \mathcal{E}|U \to U$ inherits the structure of an OVS in a natural way.

A field of OVS $p: \mathcal{E} \to X$ is called an <u>ordered real bundle</u> (ORB), if every $x \in X$ has a neighborhood U, such that $\mathcal{E}|U$ is <u>trivial</u>; i.e., there exists a map $g: \mathcal{E}|U \to U \times R$, called a <u>trivialization</u>, which is a topological isomorphism of $\mathcal{E}|U$ with the topological product $U \times R$, and $g|E_x \to R$ is an (order) isomorphism, for any $x \in U$. If $\mathcal{E}|U$ is trivial, then $\Gamma(U)$ is isomorphic to $\mathcal{C}(U)$, the OVS of all continuous maps from X to R. If $g: \mathcal{E}|U \to U \times R$ is a trivialization of $p: \mathcal{E}|U \to U$ and $p_2: U \times R \to R$ is the natural projection, then $f \to p_2 gf$, for $f \in \Gamma(U)$ is an isomorphism of $\Gamma(U)$ and $\mathcal{C}(U)$.

(6.1) Ordered real bundles may be obtained by the following construction. (1) Let $(E_x)_{x \varepsilon X}$ be a disjoint family of OVS, each of which is isomorphic to R, and let $p: \mathcal{E} = \bigcup_X E_x \to X$ be the natural projection. (2) Let $(g_a : X_a \to X)_{a \varepsilon A}$ be a family of injections where $(X_a)_{a \varepsilon A}$ is a family of topological spaces, for which, (a) the family $(\hat{X}_a = g_a(X_a))_{a \varepsilon A}$ is a cover of X, and (b) for any $a,b \varepsilon A$, the topology on $\hat{X}_a \cap \hat{X}_b$, induced from X_a, coincides with the topology on $\hat{X}_a \cap \hat{X}_b$, induced from X_b, and $\hat{X}_a \cap \hat{X}_b$ is closed in $\hat{X}_a$ and in $\hat{X}_b$. (3) Let $(h_a : \hat{X}_a \times R \to \mathcal{E}|X_a)_{a \varepsilon A}$ be a family of injections where $\hat{X}_a \times R$ is endowed with the product topology, and, for any $x \varepsilon X_a$, $h_a|\{x\} \times R \to \mathcal{E}|\hat{X}_a$ is an isomorphism of R onto E_x. It is easily verified that $(h_a)_{a \varepsilon A}$ satisfies conditions (a) and (b).

Thus, X and $\mathcal{E}$ may be endowed with the inductive topology determined respectively by the families $(g_a : X_a \to X)_{a \varepsilon A}$ and $(h_a : X_a\ E \to \mathcal{E})_{a \varepsilon A}$. This is the finest topology on X for which each g_a is continuous. It is characterized by the property that a topological map $f: X \to Y$ is continuous, if and only if, $fg_a : X_a \to Y$ is continuous, for each $a \varepsilon A$. Analogolous statements hold for $\mathcal{E}$. For each $x \varepsilon X$, x is in $\hat{X}_a$, for some $a \varepsilon A$, since $(\hat{X}_a)_{a \varepsilon A}$ covers X, and $g_a^{-1}: \mathcal{E}|\hat{X}_a \to X_a \times R$ is a trivialization of $\hat{X}_a$. Thus, $p: \mathcal{E} \to X$ endowed with the respective inductive topologies is an ordered real bundle.

(6.2) <u>Theorem</u>: Let E be a directed OVS, such that the intersection of $\Omega(E)$ the set of all maximal ideals of E is 0. Any family $(I_a)_{a \varepsilon A}$ of $w(E,E^*)$-closed modular perfect ideals of E, for which, given any $M \varepsilon \Omega$, there exists $a \varepsilon A$, such that $I_a \subseteq M$, induces an ordered vector bundle structure on the natural projection $p:\mathcal{E} = \bigcup_\Omega E/M \to \Omega$, and E is isomorphic to a subspace of $\Gamma(\Omega)$.

Proof: Without loss of generality, we may suppose each $a \in A$ is a mod unit for I_a. The sets $B_a = \{h \in {}^+I_a^o : h(a) = 1\}$, for $a \in A$, are $w(E^*,E)$-compact and the maps $g_a:B_a \to \Omega$, defined by $g_a(h) = \ker(h)$, for $h \in B_a$, are injections with images $g(B_a) = \hat{B}_a = \{M \in \Omega : I \subseteq M\} = \{M \in \Omega : I \subseteq M, a \notin M\}$, by (1.5). The family $(\hat{B}_a)_{a \in A}$ covers Ω, since for any $M \in \Omega$, there exists $a \in A$, such that $I_a \subseteq M$ and since a is a mod unit for I_a, $a \notin M$. The maps $k_a : B_a \times R \to \mathcal{E}|\hat{B}_a$, defined by $k_a(h,\lambda) = \lambda a_{\ker(h)}$, are injections and $k_a : \{h\} \times R \to E_{\ker(h)}$ is an isomorphism, for any $a \in A$, $h \in B_a$. The sets B_a, for $a \in A$, endowed with the $w(E^*,E)$-topology are topologically isomorphic to $\bar{B}_a = \{\bar{h} \in (E/I_a)^*: \bar{h}(a_I) = 1\}$ endowed with the $w((E/I_a)^*, E/I_a)$-topology. Furthermore $E/KH(I_a)$ is order isomorphic to $E(\hat{B}_a) = E(H(I_a))$, for $a \in A$. We have that $E(\hat{B}_a \cap \hat{B}_b)$ is an OVS with units $\hat{a}$ and $\hat{b}$, for any $a,b \in A$. Hence, $\hat{a}$ and $\hat{b}$ induce equivalent order norms on $E(\hat{B}_a \cap \hat{B}_b)$. Hence, the topology on $\hat{B}_a \cap \hat{B}_b$, induced from B_a, coincides with the topology induced from B_b, and $\hat{B}_a \cap \hat{B}_b$ is closed in $\hat{X}_a$ and in $\hat{X}_b$, for any $a,b \in A$. Thus, $p:\mathcal{E} \to \Omega$ endowed with the respective inductive topologies is an ordered real bundle, by (6.11).

(6.3) Let E be a directed OVS with a family $(I_a)_{a \in A}$ of modular perfect $w(E,E\,)$-closed ideals of E indexed by mod units as above, which induces a field structure on $p:\mathcal{E} = \bigcup_\Omega E/M \to \Omega$. If every $M \in \Omega$ contains all but finitely many $a \in A$, then $g:\mathcal{E} \to \Omega \times R$

defined by $g(x_M) = \sum_A g_\alpha^{-1}(x_M)$ is a (global) trivialization of $p: \mathcal{E} \to \Omega$. Since $g_\alpha^{-1}(x_M) = 0$, for all but finitely many $a \in A$, g is well defined, and $g|E_M \to R$ is an isomorphism, since $g|E_M = \sum_A g_o^{-1}|E_M$, i. e., $g|E_M$ is the sum of a (finite) set of isomorphisms.

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