THE OPEN MAPPING AND CLOSED GRAPH THEOREM

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by

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SCOPE AND CONTENTS : A topological group G is known as a $B(\mathcal{A})$ group if every continuous and almost open homomorphism from G onto a Hausdorff group is open. The permanence properties of the category of $B(\mathcal{A})$ groups are investigated and an internal characterization of such groups is established. Extensions of the closed graph and open mapping theorem are proved, employing this and related categories of groups. A similar concept is defined for topological semigroups, and further extensions of the open mapping and closed graph theorem are proved for them.

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INTRODUCTION

Our study has its origins in one of the most fundamental results of functional analysis, the Open Mapping and Closed Graph Theorem of Banach. Over the years, there have been many efforts to generalize this theorem for various classes of topological vector spaces and for topological groups.

This research has taken at least two distinct but supplementary lines. First and more obviously, broader classes of spaces have been sought for which a theorem of this type holds. Second, given a class of spaces in which an open mapping theorem is assumed to be true, what information does this yield about the spaces concerned? In particular, what does this assumption imply about the existence of a closed graph theorem in this class?

One fruitful observation in this connection is that every linear mapping of a metrizable topological vector space onto a Banach space is almost open; this is an important step in the proof of the classical Open Mapping Theorem. Presumably, then, by assuming the mapping to be almost open and continuous, one can obtain more general forms of the open mapping theorem and possibly of the closed graph theorem.

This was the avenue first explored by Ptak [15, 16], who defined a locally convex topological vector space E to be B-complete if every continuous and almost open linear map of E onto a Hausdorff locally convex space F is open, and B_r-complete if every such mapping which is also one-to-one is open. This work has been pursued further by Husain [9], Baker [1], and A.P. and W. Robertson [17]. A comprehensive bibliography of papers in this field can be found in [9].

There is also a "classical" open mapping theorem for topological groups, which states that a continuous homomorphism from a locally compact, σ-compact topological group onto a locally compact, Hausdorff group is open [14; Theorem 12]. Efforts have also been made to extend this result, principal among which is the paper of Pettis [13], in which he strives to reduce the restrictions on the groups by placing additional ones on the mappings.

Extending Ptak's idea, Husain [8] has defined a $B(\mathcal{G})$ group: if $\mathcal{G}$ is a class of Hausdorff topological groups, G is said to be a $B(\mathcal{G})$ group if every continuous and almost open homomorphism from G onto a group from $\mathcal{G}$ is open, a $B_r(\mathcal{G})$ group if every one-to-one homomorphism with these properties is open. The symbol $\mathcal{A}$ is reserved for the class of all Hausdorff topological groups. All these classes of groups figure prominently in what follows.

Chapter 0 contains those definitions and results from topology and analysis which are fundamental to later chapters. Included are a discussion of uniform spaces and the statement and proof of the classical Open Mapping and Closed

Graph Theorem for Banach spaces.

Chapter 1 concerns itself primarily with $B(\mathcal{A})$ and $B_r(\mathcal{A})$ groups, their internal properties and permanence properties as a subcategory of all topological groups. Husain [8; Theorem 31.3] has shown that every complete metrizable group is a $B(\mathcal{A})$ group; we show that locally complete metrizable groups also have this property. L.J. Sulley [22] has established a criterion relating the presence of the $B(\mathcal{A})$ property on an Abelian group with its presence on the completion; we extend this criterion to all completable groups. A condition on the group topologies of G equivalent to the $B(\mathcal{A})$ property is then given.

We explore the permanence properties of the categories of $B(\mathcal{A})$ and $B_r(\mathcal{A})$ groups, establishing that both categories are closed under retracts and open central subgroups, and supplying counterexamples for various other possible closure properties. An example of a $B(\mathcal{C})$ group which is not a $B(\mathcal{A})$ group is also produced, for the case where $\mathcal{C}$ is the category of first countable, Hausdorff groups.

In Chapter 2, we consider closed graph theorems in which $B_r(\mathcal{C})$ groups appear as codomains of the homomorphisms, where $\mathcal{C}$ is a category of Hausdorff groups having an additional permanence property. Investigations of this type have been carried on by Baker [1] and Husain [8, 10].

Baker's permanence property for $\mathcal{C}$ is somewhat intricate, involving inductive limit topologies; moreover, he concerns himself entirely with Abelian groups. This line will not be pursued in the sequel. For details, see [1].

Husain's assumption on $\mathcal{C}$ is that if G is a group in $\mathcal{C}$ and $f : G \longrightarrow H$ is a continuous, almost open homomorphism into a Hausdorff group, then H is in $\mathcal{C}$.

We retain Husain's assumption on $\mathcal{C}$, using the terminology, after Isbell [11], that $\mathcal{C}$ is right fitting with respect to continuous, almost open homomorphisms. We replace Husain's assumption [10] that the codomain of the homomorphism be Abelian with the requirement either that the range of the homomorphism lie in the centre of the codomain or else that the codomain have equal left and right uniform structures. We also prove a closed graph theorem of this type, where $\mathcal{C}$ is right fitting with respect to homomorphisms whose range is dense in the codomain. A list of important subcategories of Hausdorff groups having these right fitting properties is also developed.

In addition, we discuss an extension of the ultrabarrelled property to groups, introduced by S.O. Iyahen [12]. We note a flaw in Iyahen's work, and prove a corrected version of his results, as well as certain extensions.

In Chapter 3, we consider topological semigroups with the property of being embeddable as an open subset in a topological group. This class of semigroups has been investigated by Rothman [18], who developed an internal characterization for them. After defining a property for semigroups analagous to the B($\mathcal{A}$) property, we proceed to prove several versions of the open mapping and closed graph theorem for this class of semigroups.

CHAPTER 0

PRELIMINARIES

1. Topological semigroups, groups, and vector spaces

A topological group G is a group endowed with a topology such that the multiplication map $m : G \times G \longrightarrow G$ given by $(a,b) \longmapsto ab$ and the inversion map $i : G \longrightarrow G$ given by $a \longmapsto a^{-1}$ are continuous. Equivalently, G is a topological group iff the map $t : G \times G \longrightarrow G$ given by $(a,b) \longmapsto ab^{-1}$ is continuous.

A topological semigroup S is a semigroup with a topology such that the multiplication is a continuous map of $S \times S$ into S.

A topological vector space E is a vector space with underlying field K, either the reals or the complex numbers, with a topology such that the addition is a continuous map from $E \times E$ to E, and the scalar multiplication is a continuous map from $K \times E$ to E.

In practice, we shall frequently omit the adjective "topological" from the above phrases, if the noun is otherwise qualified. For instance, a "locally compact group" will be understood to be a topological group whose topology makes it a locally compact topological space.

In a topological semigroup S, the symbol $\mathcal{V}(S; x)$ will denote the set of all subsets B of S such that $x \in U$ and $U \subseteq B$, for some set U which is open in S. The collection $\mathcal{V}(S; x)$ will be known as the neighbourhood filter of x in S. The same notation will be used in groups and in vector spaces. The unit or identity element of a group G will be denoted by e_G or simply by e, if no confusion is likely to arise. We shall also denote $\mathcal{V}(G; e_G)$ as $\mathcal{V}(G)$.

The neighbourhood filter of the unit has certain properties which will be used frequently in the sequel. In particular, a filter $\mathcal{V}$ on a group G is the unit neighbourhood filter for a topology which is compatible with the group structure of G (i.e., with respect to which m and i are continuous maps) iff $\mathcal{V}$ satisfies the following three axioms:

(GV1) Given any $U \in \mathcal{V}$, there exists $V \in \mathcal{V}$ such that $V^2 \subseteq U$;

(GV2) Given any $U \in \mathcal{V}$, we have $U^{-1} \in \mathcal{V}$;

(GV3) For all $a \in G$ and all $V \in \mathcal{V}$, we have $(aVa^{-1}) \in \mathcal{V}$.

The above axioms appear in Chapter III, Section 1.2 of [2]. It follows at once from these that, if $U \in \mathcal{V}$, there exists $V \in \mathcal{V}$ such that $V \subseteq U$ and $V = V^{-1}$. Such a neighbourhood V is said to be symmetric.

For a subset A of a topological group G, we shall denote the topological closure of A by $Cl_G A$. If G has a topology, say, denoted by u, and if we wish to emphasize that the closure is taken with respect to this topology, we shall denote this set by $Cl_u A$. The same notation will be used in semigroups.

Similarly, if $A \subseteq G$, then $Int_G A$ will denote the interior of A in the topological space G, and the boundary of A, defined as $(Cl_G A) \cap (Cl_G \mathscr{C} A)$, will be denoted by $Bdy_G A$.

If G is a topological group and $A \subseteq G$, then

$$Cl_G A = \bigcap \{AU : U \in \mathcal{V}(G)\} = \bigcap \{UA : U \in \mathcal{V}(G)\}.$$

It follows from this that, for $U \in \mathcal{V}(G)$, $Cl_G U \subseteq U^2$. Hence, by (GV1), for every $U \in \mathcal{V}(G)$, there exists $V \in \mathcal{V}(G)$ such that $Cl_G V \subseteq U$.

2. Set-Theoretic and Topological Inclusions

We present here a series of inclusions of a set-theoretic and topological nature which will be used repeatedly in later chapters.

Let $f : X \longrightarrow Y$ be any map of sets, and let $\{B_i : i \in I\}$ be a collection of subsets of X. Then

$$f(\bigcap \{B_i : i \in I\}) \subseteq \bigcap \{f(B_i) : i \in I\}$$

$$f(\bigcup \{B_i : i \in I\}) = \bigcup \{f(B_i) : i \in I\}.$$

If $\{A_j : j \in J\}$ is a collection of subsets of Y, then

$$f^{-1}(\cap\{A_j : j \in J\}) = \cap\{f^{-1}(A_j) : j \in J\};$$

$$f^{-1}(\cup\{A_j : j \in J\}) = \cup\{f^{-1}(A_j) : j \in J\}.$$

If X and Y are topological spaces and f is a continuous map from X to Y, then, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$f(\mathrm{Cl}_X A) \subseteq \mathrm{Cl}_Y f(A) ,$$

and
$$\mathrm{Cl}_X f^{-1}(B) \subseteq f^{-1}(\mathrm{Cl}_Y B) .$$

Since multiplication in a topological group G is continuous, it follows, for subsets C and D of G, that

$$(\mathrm{Cl}_G C)(\mathrm{Cl}_G D) \subseteq \mathrm{Cl}_G(CD).$$

If $f : X \longrightarrow Y$ is an open map, then, for $A \subseteq X$, $f(\mathrm{Int}_X A) \subseteq \mathrm{Int}_Y f(A)$.

If $f : G \longrightarrow H$ is a homomorphism of topological groups, then, for subsets A and B of H,

$$f^{-1}(A)f^{-1}(B) \subseteq f^{-1}(AB) ,$$

whence $(\mathrm{Cl}_G f^{-1}(A))(\mathrm{Cl}_G f^{-1}(B)) \subseteq \mathrm{Cl}_G f^{-1}(AB)$.

Finally, it is convenient at this time to introduce some notation borrowed from universal algebra. If a map f from G to H is one-to-one, we may denote this by $f : G \rightarrowtail H$, and, if f is onto, by $f : G \twoheadrightarrow H$. If this notation is used, no further explicit mention of the injectivity and/or surjectivity of f will be made.

3. Uniform Spaces, Completions, and Uniformities on Topological Groups

If Y is any set, denote $Y \times Y$ by Y^2, and let $V \subseteq Y^2$. Then V^{-1} will denote the set of pairs (a,b) such that $(b,a) \in V$. If U, V are subsets of Y^2, then UV will denote the set of pairs $(a,b) \in Y^2$ such that $(a,c) \in U$ and $(c,b) \in V$ for some $c \in Y$. The set $\{(a,a) : a \in Y\}$ will be called the diagonal.

We define a uniform space to be a pair $(X, \mathcal{U})$, where X is a set and $\mathcal{U}$ is a filter on X^2 which satisfies the following three properties:

(U1) Each $U \in \mathcal{U}$ contains the diagonal;

(U2) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(U3) For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$.

The filter $\mathcal{U}$ is known as a uniformity on X.

A topological group G has two natural uniformities, which we now describe. For $U \in \mathcal{V}(G)$, define

$$L(U) = \{(a,b) \in G \times G : a^{-1}b \in U\},$$

$$R(U) = \{(a,b) \in G \times G : ba^{-1} \in U\}.$$

It is then easy to show that $\{L(U) : U \in \mathcal{V}(G)\}$ is a base for a uniformity on G, and similarly for $\{R(U) : U \in \mathcal{V}(G)\}$. The uniformities generated by these two collections are known as the left and right uniformities on G, respectively. In general, these two uniformities, which are denoted by $\mathcal{L}$

and $\mathcal{R}$, are distinct. If $\mathcal{L}=\mathcal{R}$, then G is said to have equal left and right uniformities, or simply to have equal uniformities.

Groups with this property have another interesting feature, namely that they are completable. To elucidate this property, we must first define an auxiliary concept.

A filter $\mathcal{F}$ on a uniform space $(X, \mathcal{U})$ is said to be a Cauchy filter if, for every $U \in \mathcal{U}$, there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$. The uniform space $(X, \mathcal{U})$ is said to be complete if every Cauchy filter on it converges.

We define a topological group G to be complete if $(G, \mathcal{L})$ and $(G, \mathcal{R})$ are complete uniform spaces. We then say that a topological group G is completable if it is isomorphic to a dense subgroup of a complete group G^*, and G^* will be known as the completion of G.

Without attempting a proof, we state the following exceedingly useful result, which will find many applications in the sequel.

Theorem 0.1 For a topological group G, the following implications hold : (2) $\Longrightarrow$(1), (2)$\Longleftrightarrow$(3)$\Longleftrightarrow$(4).

(1) G is completable;

(2) G has equal uniformities;

(3) For every $U \in \mathcal{V}(G)$, there exists $V \in \mathcal{V}(G)$ such that $xVx^{-1} \subseteq U$ for every $x \in G$;

(4) There exists a fundamental system of unit neighbourhoods in G which are fixed under all inner

automorphisms of G.

Statements (1)-(3) are an amalgam of Theorem 3.1, Chapter III of [2] and Section 4.14 of [6]. Statement (4) is trivially equivalent to (3), and is stated separately only because this is the form which has proved most useful in the sequel. Several other equivalent statements are known; we state only those used in later chapters.

4. Properties of Homomorphisms

We define a pair of properties for a homomorphism $f : G \longrightarrow H$ of topological groups which are weaker than continuity and openness, and which will be important in all that follows.

We say that f is almost continuous if, for every $V \in \mathcal{V}(H)$, we have $Cl_G f^{-1}(V) \in \mathcal{V}(G)$, and that f is almost open if, for every $U \in \mathcal{V}(G)$, we have $Cl_H f(U) \in \mathcal{V}(H)$.

Much of the work which follows will be directed toward determining conditions on G, H and f which will force f to be continuous or open. We state the following condition for an almost open homomorphism of topological groups to be open, which is new to the best of the author's knowledge.

Proposition 0.1 Let $f : G \longrightarrow H$ be an almost open homomorphism. Then f is open iff, for each $V \in \mathcal{V}(G)$, there is $B \in \mathcal{V}(H)$ such that $B \cap \mathrm{Bdy}_H f(V) = \emptyset$.

Proof: The "only if" part is trivial. For the "if" part, first let $V \in \mathcal{V}(G)$, and let $B \in \mathcal{V}(H)$ such that $B \cap \mathrm{Bdy}_H f(V) = \emptyset$. Since f is almost open, there exists $D \in \mathcal{V}(H)$ such that $D \subseteq \mathrm{Cl}_H f(V)$. Now,

$$\mathrm{Cl}_H f(V) = f(V) \cup \mathrm{Bdy}_H f(V) ,$$

whence $D \cap B \subseteq (f(V) \cup \mathrm{Bdy}_H f(V)) \cap B \subseteq f(V)$. Since $D \cap B \in \mathcal{V}(H)$, it follows that f is open.

In a virtually identical manner, one may prove the following:

<u>Proposition 0.2</u> Let $f : G \longrightarrow H$ be an almost continuous homomorphism. Then f is continuous iff, for each $V \in \mathcal{V}(H)$, there is $B \in \mathcal{V}(G)$ such that $B \cap \mathrm{Bdy}_G f^{-1}(V) = \emptyset$.

A map $f : A \longrightarrow B$ of topological spaces will be said to have the closed graph property or simply to have closed graph if the set $R(f) = \{(a, f(a)) : a \in A\}$ is a closed subset of A x B. Such maps are called "closed" by some authors [12, 17]. As this abbreviated terminology may cause confusion, we shall call a map f closed iff the image under f of each closed set is closed.

5. Quotient Groups

Given a topological group G and a normal subgroup H of G, let $n : G \twoheadrightarrow G/H$ be the natural map. The quotient topology on G/H is the topology consisting of all images under n of open sets from G. Unless specified otherwise, a quotient group will always be assumed to be endowed with this topology.

Let $f : G \longrightarrow K$ be a homomorphism of topological groups, $n : G \twoheadrightarrow G/\mathrm{Ker}\, f$ the natural map, and $g : G/\mathrm{Ker}\, f \rightarrowtail K$ the unique homomorphism such that $f = gn$. We then have the following result:

Proposition 0.3 For f, g as defined above:

(a) f is continuous iff g is;

(b) if f is almost continuous, then g is;

(c) f is open iff g is;

(d) f is almost open iff g is.

The above is Proposition 30.3 of [8]. The proof is omitted.

6. Baire Category Theorem

In a topological space X, we say that a subset A of X is nowhere dense if $\mathrm{Int}_X(\mathrm{Cl}_X A) = \emptyset$. A subset of X which is a countable union of nowhere dense sets is called a first category set; any subset of X which is not a first category set is said to be a second category set or to be of second category. If X is a second category subset of itself, it is said to be a Baire space.

This leads to the following important result from topology, known as the Baire Category Theorem:

Theorem 0.2 Every complete metric space and every locally compact Hausdorff space is a Baire space.

For proof, see [8; p. 4], [6; p. 42], or [26; p. 200].

7. The Classical Open Mapping and Closed Graph Theorem

The following deep result from functional analysis, due originally to Banach, has been the starting point for many generalizations. Because it is interesting and instructive to compare the methods used in topological vector spaces with those in topological groups, we prove this theorem in its entirety.

In this theorem, and throughout the dissertation, the symbol $\underline{N}$ will be reserved for the natural numbers.

Theorem 0.3 Let E and F be Banach spaces. Then any linear mapping $f : E \longrightarrow F$ with closed graph is continuous.

Proof: We first show that f is almost continuous.

Let $r > 0$, and consider the set $S_r = \{y \in F : \|y\| \leq r\}$, where $\|\cdots\|$ denotes the norm of F. Then

$$F = \bigcup\{nS_r : n \in \underline{N}\},$$

whence $E = \bigcup\{f^{-1}(nS_r) : n \in \underline{N}\} = \bigcup\{nf^{-1}(S_r) : n \in \underline{N}\}$.

By Theorem 0.2, there exists $k \in \underline{N}$ such that

$$\mathrm{Int}_E(\mathrm{Cl}_E kf^{-1}(S_r)) \neq \phi ,$$

whence $$\mathrm{Int}_E(\mathrm{Cl}_E f^{-1}(S_r)) \neq \phi .$$

Since r was an arbitrary positive number, it follows that, for some $p \in \mathrm{Cl}_E f^{-1}(S_{\frac{1}{2}r})$ and some $q > 0$,

$$p + T_q = \{p + x : \|x\| \leqslant q\} \subseteq \mathrm{Cl}_E f^{-1}(S_{\frac{1}{2}r}).$$

Then $$\begin{aligned} T_q &\subseteq (\mathrm{Cl}_E f^{-1}(S_{\frac{1}{2}r})) - p \\ &\subseteq (\mathrm{Cl}_E f^{-1}(S_{\frac{1}{2}r})) - (\mathrm{Cl}_E f^{-1}(S_{\frac{1}{2}r})) \\ &\subseteq 2(\mathrm{Cl}_E f^{-1}(S_{\frac{1}{2}r})) \subseteq \mathrm{Cl}_E f^{-1}(S_r) . \end{aligned}$$

Hence, $\mathrm{Cl}_E f^{-1}(S_r) \in \mathcal{V}(E)$, and f is almost continuous.

We now show that f is continuous by showing that $T_{\frac{1}{2}q} \subseteq f^{-1}(S_r)$. Let $x \in T_{\frac{1}{2}q}$. Then $x \in \mathrm{Cl}_E f^{-1}(S_r)$, whence there is $x_1 \in f^{-1}(S_{\frac{1}{2}r})$ such that $\|x - x_1\| \leqslant \frac{1}{4}q$. Now, this implies $(4x - 4x_1) \in T_q \subseteq \mathrm{Cl}_E f^{-1}(S_r)$, whence $(x - x_1) \in \mathrm{Cl}_E f^{-1}(S_{\frac{1}{4}r})$. Then there exists $x_2 \in f^{-1}(S_{\frac{1}{4}r})$ such that $\|x - x_1 - x_2\| \leqslant q/8$. Proceeding inductively, we obtain a sequence (x_n) in E such that $\|x - x_1 - \cdots - x_n\| \leqslant 2^{-n-1}q$. Then the partial sums of

$(f(x_n))$ form a Cauchy sequence in F, for

$$\|f(x_m) + \cdots + f(x_p)\| \leqslant \|f(x_m)\| + \cdots + \|f(x_p)\|$$
$$\leqslant r(2^{-m} + \cdots + 2^{-p}) ,$$

and this tends to zero if $p \geqslant m$ and m increases. Since F is complete,

$$\left(\sum_{n=1}^{m} f(x_n) \right)_{m \in \underline{N}}$$

has a limit, say y. By the same sort of limit consideration, we have

$$x = \sum_{i=1}^{\infty} x_i .$$

Then $(x,y) \in Cl_{E \times F} R(f)$, whence $y = f(x)$, for $R(f)$ is closed by hypothesis. Now

$$\|y\| \leqslant \sum_{n=1}^{\infty} \|f(x_n)\| \leqslant r \sum_{n=1}^{\infty} 2^{-n} = r .$$

Hence, $y \in S_r$, whence $x \in f^{-1}(S_r)$. Therefore, $T_{\frac{1}{2}q} \subseteq f^{-1}(S_r)$, and f is continuous. QED

The other part of the argument, the classical open mapping theorem, follows as a corollary.

Corollary If E and F are Banach spaces, any continuous linear mapping $f : E \twoheadrightarrow F$ is open.

Proof: It is clear that Ker f is a closed subspace of E, and that E/Ker f is a Banach space. Let n denote the natural map $E \longrightarrow E/\text{Ker } f$, and $g : E/\text{Ker } f \longrightarrow F$ be the unique one-to-one map such that $f = gn$. It follows from Proposition 0.3 that g is continuous; hence, its graph R(g) is closed. Since g is one-to-one, its inverse $g^{-1} : F \longrightarrow E/\text{Ker } f$ exists, and $R(g^{-1}) = R(g)$, which is closed. Hence, g^{-1} is continuous, by Theorem 0.3, and so g is open. Therefore, by Proposition 0.3, f is open. QED

8. Categories

We include this section not because categorical considerations play a large role in what follows, but rather to explain that categorical notions have provided a convenient language in which to phrase certain results, and to point out certain abuses of orthodox categorical terminology.

A concrete category $\mathcal{C}$ consists of a class $\mathcal{O}$ of sets, called the objects of $\mathcal{C}$, and, for each ordered pair (X,Y) of objects in $\mathcal{C}$, a set Hom(X,Y) of functions $f : X \longrightarrow Y$ called morphisms, such that

(a) the identity function on each object is a morphism;

(b) every function which is a composition of morphisms is a morphism.

This definition is essentially that of [11; p. ix].

In our work, the adjective "concrete" will be uniformly omitted, and such an entity will be called merely a category. Since all the categories considered will have as their objects topological groups or semigroups, and as their morphisms, continuous homomorphisms, we will also frequently commit the heresy of identifying a category with the class of its objects, and neglecting any mention of the morphisms.

CHAPTER 1

$B(\mathcal{A})$ AND $B_r(\mathcal{A})$ GROUPS

1. Definition and Elementary Properties

Definition If $\mathcal{C}$ is a category of Hausdorff topological groups, then G is said to be a $B(\mathcal{C})$ group if every continuous and almost open homomorphism from G onto a group H in $\mathcal{C}$ is open, a $B_r(\mathcal{C})$ group if every one-to-one homomorphism with these properties is open.

If $\mathcal{C}_1 \subseteq \mathcal{C}_2$ as classes of Hausdorff groups, then every $B(\mathcal{C}_2)$ group is a $B(\mathcal{C}_1)$ group, and similarly for the $B_r(\mathcal{C}_i)$ groups. Letting $\mathcal{A}$ represent the class of all Hausdorff topological groups, we then see that a $B(\mathcal{A})$ (or $B_r(\mathcal{A})$) group is a $B(\mathcal{C})$ (or $B_r(\mathcal{C})$) group, for every class $\mathcal{C}$ of Hausdorff groups.

The following result gives us one broad class of $B(\mathcal{A})$ groups.

Theorem 1.1 Every locally compact group is a $B(\mathcal{A})$ group.

Proof: Let G be a locally compact group, H a Hausdorff group, and $f : G \twoheadrightarrow H$ a continuous and almost open homomorphism. Let V be a compact unit neighbourhood in G. Then f(V) is compact, and so is closed in H. But $Cl_H f(V) \in \mathcal{V}(H)$, so $f(V) \in \mathcal{V}(H)$. Hence, f is open.

We then obtain the following corollary at once.

<u>Corollary</u> Every compact group is a $B(\mathcal{A})$ group.

<u>Examples</u> (1) Let R denote the additive group of real numbers. Then R^n, for any finite n, is locally compact and so a $B(\mathcal{A})$ group.

(2) The tori, T^n, are compact and so are $B(\mathcal{A})$ groups, for any cardinality n.

(3) Let K be the field of real or complex numbers. Then the additive group of n x n matrices with entries from K, denoted by $M_n(K)$, is locally compact, by Proposition 27.8 of [8], and so is a $B(\mathcal{A})$ group.

(4) The invertible elements of $M_n(K)$, denoted by $G_n(K)$, form an open subset of $M_n(K)$, by Proposition 27.9 of [8], and so are a locally compact group. Hence, $G_n(K)$ is a $B(\mathcal{A})$ group.

(5) Let C denote the field of complex numbers. Then the orthogonal groups $O_n(R)$ and $O_n(C)$ are closed in $M_n(C)$, and so are locally compact. Similarly, the unitary group U_n is locally compact. Hence, all of these are $B(\mathcal{A})$ groups.

(6) All discrete groups are locally compact, and are therefore $B(\mathcal{A})$ groups. This case is particularly

trivial, however, for, if there is an almost open homomorphism from a discrete group onto a Hausdorff group, then the codomain must also be discrete.

The following section gives another class of groups which has the $B(\mathcal{A})$ property.

2. Locally Complete Metrizable Groups

It is shown in Theorem 31.3 of [8] that every complete metrizable topological group is a $B(\mathcal{A})$ group. We now show that a slightly broader class of topological groups has this property. It is sufficient that each point of the group have a neighbourhood which, in the relative topology induced by the group, is a complete metrizable space. While not profound in itself, this generalization is useful in proving certain results in Chapter 3.

Definition A topological space X is said to be locally complete metrizable if every point of X has a fundamental system of neighbourhoods which are complete metrizable spaces in the relative topology induced by X.

Theorem 1.2 Every locally complete metrizable topological group is a $B(\mathcal{A})$ group.

Proof: Let G be a locally complete metrizable group, H a Hausdorff group, and $f : G \twoheadrightarrow H$ a continuous, almost open homomorphism. Let $U \in \mathcal{V}(G)$, and, without loss of generality, we may assume it is a complete metric space, closed, and symmetric. We then select a fundamental sequence $\{U_n\}$ of complete metric, closed, symmetric neighbourhoods of e_G such that $U_1^2 \subseteq U$ and $U_{n+1}^2 \subseteq U_n$, for each n, and such that $\bigcap U_n = \{e_G\}$. To find such a sequence, we first observe that, since U is a metric space, there exists a countable, fundamental system of closed unit neighbourhoods $\{V_n\}$ in U such that $V_{n+1} \subseteq V_n$ for each n. Let $\{Y_n\}$ be a sequence of closed unit neighbourhoods in G such that $Y_1^2 \subseteq U$ and $Y_{n+1}^2 \subseteq Y_n$, for each n. Let

$$U_n = (Y_n \cap V_n) \cap (Y_n \cap V_n)^{-1} ;$$

it is then easy to check that the resulting sequence has all the desired properties.

Let $W_n = Cl_H f(U_n)$. We now show that $\bigcap W_n = \{e_H\}$. Let $y \in \bigcap W_n$; then $V = V^{-1} = Cl_H V$ in $\mathcal{V}(H)$ implies there exists $x_n \in U_n$ such that $f(x_n) \in yV$, for each n. Since $\{U_n\}$ is a fundamental sequence of neighbourhoods of e_G in U, we conclude that $\{x_n\}$ converges to e_G. By the continuity of f, $\{f(x_n)\}$ then converges to $f(e_G) = e_H$. Since V is closed and $\{f(x_n)\} \subseteq yV$, we have $e_H \in yV$, whence $y \in V$, by the symmetry of V. Now, V was an arbitrary closed symmetric set from $\mathcal{V}(H)$, and since such neighbourhoods form a fundamental sy,tem of unit neighbourhoods in a topological group, it follows from the Hausdorff property of H that $y \in \bigcap \mathcal{V}(H) = \{e_H\}$. Hence, $y = e_H$, and our assertion is proved.

We now show that, for every k, $W_{k+1} \subseteq f(U_k)$. Let $y \in W_{k+1}$; then, since $f(U_{k+1})$ is dense in W_{k+1}, there exists $x_1 \in U_{k+1}$ such that $f(x_1) \in yW_{k+2}$. Then $y^{-1}f(x_1) \in W_{k+2}$, whence $f(x_1)^{-1}y \in W_{k+2}$, and so there is a point $x_2 \in U_{k+2}$ such that $f(x_2) \in f(x_1)^{-1}yW_{k+3}$. Thus, $f(x_1x_2) \in yW_{k+3}$.

Proceeding inductively, we construct a sequence x_n such that :

(a) $x_n \in U_{k+n}$, for each n, and

(b) $f(x_1x_2 \cdots x_n) \in yW_{k+n+1}$.

Now $x_nx_{n+1} \cdots x_{n+p} \in U_{k+n}U_{k+n+1} \cdots U_{k+n+p}$, but the latter product is a subset of U_{k+n-1}, by the initial conditions on the sequence $\{U_n\}$. Thus, for $p \geqslant 0$,

$$x_nx_{n+1} \cdots x_{n+p} \in U_{k+n-1} .$$

Since the $\{U_n\}$ are a fundamental sequence of neighbourhoods, it follows that, for any $V \in \mathcal{V}(G)$, we can find $n_o \in \underline{N}$ such that, for $p \geqslant 0$ and $n \geqslant n_o$, we have $x_n \cdots x_{n+p} \in V$.

Then, letting $S_n = x_1x_2 \cdots x_n$, we see that $\{S_n\}$ forms a Cauchy sequence; moreover, letting $n = 1$ in (a), we see that $S_n \in U_k$, for all n. Therefore, $\{S_n\}$ converges to some $x_o \in U_k$, since U_k is a complete metric space.

The proof is now concluded by showing that $y = f(x_o)$. First, observe that $\lim_{n \to \infty} f(x_1) \cdots f(x_n) = f(x_o)$. Now, for $m \geqslant n$, we have

$$y^{-1}f(x_1x_2 \cdots x_m) = y^{-1}f(x_1 \cdots x_n)f(x_{n+1}) \cdots f(x_m)$$

$$\subseteq W_{k+n+1}f(U_{k+n+1}) \cdots f(U_{k+m})$$

$$\subseteq W_{k+n+1}f(U_{k+n}) \subseteq Cl_H f(U_{k+n-1}) = W_{k+n-1} .$$

Then, $\lim_{n \to \infty} y^{-1}f(x_1 \cdots x_m) \in W_{k+n-1}$, since the latter is closed. Then

$$y^{-1}\cdot \lim_{m \to \infty} f(x_1 \cdots x_m) \in \bigcap\{W_{k+n-1} : n \in \underline{N}\} = \{e_H\} .$$

Thus, $y^{-1}f(x_o) = e_H$, whence $y = f(x_o) \in f(U_k)$. Hence, f is open.

<u>Corollary</u> Every complete metrizable group is a $B(\mathcal{A})$ group.

Proof: This is a special case of Theorem 1.2, since every complete metrizable group is locally complete metrizable.

Theorem 1.2 is a true generalization of the corollary, as the following example shows. Let Ω denote the first uncountable ordinal, and let G be that subset of $R^{[0,\Omega]}$ consisting of those elements which are non-zero in at most countably many entries. For each ordinal $\underline{a}$, denote by $G(\underline{a})$ the subgroup of G consisting of those elements $(x_{\underline{b}})$ such that $\underline{b} \geqslant \underline{a}$ and $\underline{b} \neq \Omega$ implies $x_{\underline{b}} = 0$. Since every countable subset of $[0,\Omega)$ has an upper bound [4; p. 54, Theorem 9.1], it follows that $G = \bigcup\{G(\underline{a}) : \underline{a} \in [0,\Omega)\}$. Let G be endowed with the subgroup topology induced by the $G(\underline{a})$; clearly this is a group topology. Moreover, each $G(\underline{a})$ is a complete metric space, by Theorem 2.5, p. 295 of [4], since it is isomorphic to $R^{\underline{a}+1}$. However, G is not first countable, and hence not metrizable.

The above corollary is Theorem 31.3 of [8]. This shows that every Banach space, for example, is a $B(\mathcal{a})$ topological group. One should note that this observation does not follow from the classical open mapping theorem, for not every homomorphism of a complex vector space need be a linear map. For instance, suppose a complex Banach space A has a basis $\{b_i\}$. Then each element $x \in A$ has a unique representation

$$x = \sum t_i b_i \quad ,$$

where $t_i \in C$. Define a map $f : A \longrightarrow A$ by

$$\sum t_i b_i \longmapsto \sum \bar{t}_i b_i \quad ,$$

the bar indicating complex conjugation. It is then trivial to check that this map is a homomorphism, but is not a C-linear map.

3. A Generalization of a Criterion of L. J. Sulley

Further discoveries which added to the list of known $B(\mathcal{A})$ groups were made by Sunyach [23] and Sulley [22]. The latter found criteria for dense subgroups of Abelian $B(\mathcal{A})$ and $B_r(\mathcal{A})$ groups to inherit the respective property. Theorems 1.3 - 1.6 are generalizations of Sulley's criteria to arbitrary completable groups. We must first prove the following two lemmas.

Lemma 1.1 Let E be a Hausdorff group, G a dense subgroup of E, H a closed normal subgroup of E, $q : E \twoheadrightarrow E/H$ the natural map. Then the restriction of q, $r : G \twoheadrightarrow q(G)/H$ is continuous and almost open. Furthermore, r is open iff $H \cap G$ is dense in H.

Proof: Clearly, r is continuous. To see that it is almost open, we first observe that

$$Cl_{q(G)} r(U \cap G) = q(G) \cap Cl_{E/H} q(U \cap G) ,$$

by Theorem 7.2, Chapter III of [4].

Now, $q^{-1}(Cl_{E/H}[q(U \cap G)])$ is closed in E, and is a unit neighbourhood, since it contains $Cl_E(U \cap G)$. Then

$Cl_{E/H}[q(U \cap G)] \in \mathcal{V}(E/H)$, since q is open, and so

$Cl_{q(G)}[r(U\cap G)] \in \mathcal{V}(E/H)$. Thus, r is almost open.

Next, suppose r is open. Let $U \in \mathcal{V}(E)$, and select $V \in \mathcal{V}(E)$ such that V is symmetric and $V^2 \subseteq U$. Then $r(V\cap G) = q(V\cap G)$ is in $\mathcal{V}(q(G))$. Then there exists $W \in \mathcal{V}(E/H)$ such that $W\cap q(G) \subseteq q(V\cap G)$, so that

$$q^{-1}(W)\cap GH \subseteq q^{-1}[q(V\cap G)] = (V\cap G)H .$$

Let $h \in H$; then there exists $g \in G$ such that $g \in (V\cap q^{-1}(W))h$, since G is dense in H, so that $gh^{-1} \in q^{-1}(W)\cap (GH) \subseteq (V\cap G)H$.

Then there exists $x \in V\cap G$, $k \in H$ such that $gh^{-1} = xk$, whence $h = k^{-1}x^{-1}g = k^{-1}(x^{-1}g)$. Now, $h \in H$, $k \in H$ together imply that $x^{-1}g \in H$; then $x \in G$, $g \in G$ imply $x^{-1}g \in G\cap H$. Thus, $k^{-1} = h(x^{-1}g)^{-1} = (hg^{-1})x$ is an element of $V^{-1}V = V^2 \subseteq U$. Therefore, $h = k^{-1}x^{-1}g \in U(G\cap H)$, and so $(G\cap H)$ is dense in H.

Conversely, suppose $G\cap H$ is dense in H, $U \in \mathcal{V}(E)$, and $V \in \mathcal{V}(E)$ such that V is symmetric and $V^2 \subseteq U$. Then $H \subseteq (G\cap H)V$. We show $V\cap(GH) \subseteq (U\cap G)H$. Let $x \in V$, $g \in G$, $h \in H$ such that $x = gh$. Then $h = g^{-1}x$; but $h \in (G\cap H)V$, by assumption, so $h = ky$, where $k \in G\cap H$, $y \in V$. It then follows that $x = gh = gky$, and $y = k^{-1}h \in H$. Also, $gk \in G$, and $gk = xy^{-1} \in VV^{-1} = V^2 \subseteq U$. Therefore, $x = gh = (gk)(k^{-1}h)$, and this last is a point of $(U\cap G)H$; hence, $V\cap (GH) \subseteq (U\cap G)H$.

Thus, $VH\cap GH \subseteq (U\cap G)H^2 = (U\cap G)H$; i.e., $q^{-1}(q(V))\cap q^{-1}(q(G)) \subseteq q^{-1}(q(U\cap G))$. Then, applying q to both sides, we obtain $q(V)\cap q(G) \subseteq q(U\cap G) = r(U\cap G)$. Thus, r is open. QED

The second lemma is proved in slightly greater generality than is necessary for our immediate purposes. The full strength of this lemma will, however, be invoked in Chapter 2.

<u>Lemma 1.2</u> If G is a group with equal uniformities and $f : G \longrightarrow H$ is a continuous, almost open homomorphism such that $f(G)$ is dense in H, then H has equal uniformities.

Proof: The sets of the form $Cl_H f(U)$, for $U \in \mathcal{V}(G)$, generate the topology of H, and it is enough to consider such sets. We show that $\bigcap\{bCl_H f(W)b^{-1} : b \in H\}$ is in $\mathcal{V}(H)$, for any $W \in \mathcal{V}(G)$.

Clearly, for $a \in G$, $f(a)Cl_H f(W)f(a)^{-1} = Cl_H f(aWa^{-1}) = Cl_H f(W)$, since G has equal uniformities. Now let $b \in H$, $W \in \mathcal{V}(G)$, and choose $V \in \mathcal{V}(G)$ such that V is symmetric and $V^3 \subseteq W$. Since $f(G)$ is dense in H, there exist $y \in Cl_H f(V)$ and $a \in G$ such that $by = f(a)$. Then,

$$\begin{aligned} b(Cl_H f(W))b^{-1} &\supseteq b(Cl_H f(V^3))b^{-1} \supseteq b(Cl_H f(V))^3 b^{-1} \\ &\supseteq byCl_H f(V)y^{-1}b^{-1} = f(a)Cl_H f(V)f(a)^{-1} \\ &= Cl_H f(V) . \end{aligned}$$

Thus, $\bigcap\{bCl_H f(W)b^{-1} : b \in H\} \supseteq Cl_H f(V)$, and so this intersection is again a unit neighbourhood. Therefore, H has equal left and right uniformities, by Theorem 0.1.

<u>Theorem 1.3</u> Let G be any Hausdorff group with equal uniformities, E its Hausdorff completion.

(a) If E is a $B(\mathcal{A})$ group and, for each closed normal subgroup H of E, $H \cap G$ is dense in H, then G is a $B(\mathcal{A})$ group.

(b) If E is a $B_r(\mathcal{A})$ group and the only closed normal subgroup H of E for which $H \cap G = \{e\}$ is $\{e\}$, then G is a $B_r(\mathcal{A})$ group.

Proof: (a) Let $f : G \twoheadrightarrow F$ be a continuous, almost open homomorphism. G has equal uniformities, and it follows from Lemma 1.2 that F also has this property. Hence, F has a completion, say F^*, and f extends to a continuous homomorphism $f^* : E \longrightarrow F^*$, by Proposition 3.5, Chapter III of [2].

Now let $U \in \mathcal{V}(E)$; then $Cl_{F^*}f^*(U) \supseteq Cl_{F^*}f(U \cap G) = Cl_{F^*}[Cl_F f(U \cap G)]$. Since f is almost open, $Cl_F f(U \cap G) \in \mathcal{V}(F)$, whence its closure is in $\mathcal{V}(F^*)$, by Proposition 4.7, Chapter III of [2]. Thus, f^* is almost open, and, since E is a $B(\mathcal{A})$ group, $f^* : E \twoheadrightarrow f^*(E)$ is open.

Let $H = \operatorname{Ker} f^*$, and define q, r as in Lemma 1.1. Let $f_1 : E/H \longrightarrow f^*(E)$ be the unique map such that $f_1 q = f^*$. Then f_1 is open, since f^* is; since f_1 is one-to-one, its restriction f_2 to $q(G)$ is open onto $f_1 q(G) = f^*(G)$. But $f = f_2 r$, so f is open.

(b) Let f, F, f^*, F^* be as in (a), but further assume that f is one-to-one. It follows as in (a) that

$f^* : E \twoheadrightarrow f^*(E)$ is almost open. Now Ker f^* is closed, and $G \cap \operatorname{Ker} f^* = \operatorname{Ker} f = \{e\}$; then Ker $f^* = \{e\}$, by hypothesis. Hence, f^* is open, since E is a $B_r(\mathcal{A})$ group. It then follows that f is open onto $f(G) = F$.

<u>Theorem 1.4</u> Let G be a dense subgroup of the Hausdorff topological group E.

(a) If G is a $B(\mathcal{A})$ group, then $H \cap G$ is dense in H for every closed normal subgroup H of E.

(b) If G is a $B_r(\mathcal{A})$ group, then the only closed normal subgroup H of G such that $H \cap G = \{e\}$ is $\{e\}$.

Proof: Let H be a closed normal subgroup of E, and let q and r be defined as in Lemma 1.1. By this lemma, r is continuous and almost open.

(a) If G is a $B(\mathcal{A})$ group, then r is open, and so, again by Lemma 1.1, $H \cap G$ is dense in H.

(b) If $H \cap G = \{e\}$, then r is one-to-one; then r is open, since G is a $B_r(\mathcal{A})$ group, whence $G \cap H = \{e\}$ is dense in H. But this implies $H = \{e\}$, since E is a Hausdorff group.

<u>Theorem 1.5</u> Let G be a dense subgroup of the topological group E.

(a) If G is a $B(\mathcal{A})$ group, then so is E.

(b) If G is a $B_r(\mathcal{A})$ group, then so is E.

Proof: (a) Let G be a $B(\mathcal{Q})$ group, and let $f : E \twoheadrightarrow F$ be a continuous, almost open homomorphism. Let $v : G \twoheadrightarrow f(G)$ be the restriction of f. We claim v is almost open. Let $U \in \mathcal{V}(E)$, so that

$$Cl_{f(G)}v(U \cap G) = [Cl_F v(U \cap G)] \cap v(G) ;$$

this follows from Theorem 7.2, Chapter III of [4]. Since f is continuous, $f^{-1}[Cl_F v(U \cap G)]$ is closed in E; denote this set by V, and note that $V \supseteq U \cap G$. Then $V \in \mathcal{V}(E)$, since G is dense in E. Now $Cl_F f(V) \in \mathcal{V}(F)$, since f is almost open; moreover, $Cl_F f(V) \subseteq Cl_F v(U \cap G)$, so the latter is in $\mathcal{V}(F)$. It then follows that $Cl_{f(G)} v(U \cap G) \in \mathcal{V}(f(G))$, and that v is almost open. Since G is a $B(\mathcal{Q})$ group, v is open.

Now we show f is open. Let $U \in \mathcal{V}(E)$, and choose $V \in \mathcal{V}(E)$ such that $V^2 \subseteq U$. Then $f(V) \supseteq W \cap f(G)$, for some open W in $\mathcal{V}(F)$, since $v = f|G$ is open. Thus,

$$f^{-1}(f(V)) \supseteq f^{-1}(W \cap f(G)) \supseteq f^{-1}(W) \cap G,$$

and so $f^{-1}(f(V))V \supseteq Cl_E[f^{-1}(f(V))] \supseteq Cl_E[f^{-1}(W) \cap G]$. Now $f^{-1}(W)$ is open in G, and G is dense in E, so $f^{-1}(f(V))V \supseteq f^{-1}(W)$. Hence, $f(U) \supseteq f(V^2) \supseteq f(V)^2$ $= f[f^{-1}(f(V))V] \supseteq f(f^{-1}(W))$, and this last is equal to W, since f is onto. Thus, f is open and (a) is proved.

(b) The proof is almost identical with (a), with certain simplifications resulting from the additional assumption that f is one-to-one.

We remark that Theorems 1.4 and 1.5 do not require that the group G be completable.

The following theorem summarizes our results for completable groups.

Theorem 1.6 Let G be any Hausdorff group with equal uniformities, E its Hausdorff completion.

(a) G is a $B(\mathcal{A})$ group iff E is a $B(\mathcal{A})$ group and, for each closed normal subgroup H of E, $H \cap G$ is dense in H.

(b) G is a $B_r(\mathcal{A})$ group iff E is a $B_r(\mathcal{A})$ group and G has non-trivial intersection with each non-trivial closed normal subgroup of E.

We conclude this section with a few applications of this criterion.

(a) The group Q of rational numbers under addition with the relative topology from R is not a $B_r(\mathcal{A})$ group. To see this, let b be an arbitrary irrational number. Then $Zb = \{nb : n \in Z\}$, where Z represents the group of integers, is a subgroup of R which is discrete and therefore closed. Furthermore, R is the completion of Q, and $Q \cap Zb = \{0\}$. Then, by Theorem 1.4(b), Q is not a $B_r(\mathcal{A})$ group.

(b) Let p be a fixed prime number, and let Z be endowed with the group topology having as its unit neighbourhood filter $\{p^n Z : n \in \underline{N}\}$. This group is totally bounded, and so has a compact completion, which we shall denote by Z_p. Now Z_p is compact, and therefore a $B_r(\mathcal{A})$ group. Furthermore, the only non-trivial closed subgroups of Z_p are the groups $\{p^n Z_p : n \in \underline{N}\}$, a fact we shall not prove here. Since, for all n, $p^n Z_p \cap Z \neq \{0\}$, we conclude from Theorem 1.4(b) that Z with the topology described above is a $B_r(\mathcal{A})$ group.

(c) Let U be the group of roots of unity, the torsion subgroup of the group T of complex numbers with unit modulus. Now, T is the completion of U, and, furthermore, every proper closed subgroup of T is also a subgroup of U [14; p. 246]. Since T is compact, it is a $B(\mathcal{A})$ group, and it follows from Theorem 1.3(a) that U is a $B(\mathcal{A})$ group.

Sulley and Sunyach both appeal to this example to show that a $B(\mathcal{A})$ group need not be topologically complete.

(d) The following example is due to Sulley [22], and shows that a $B_r(\mathcal{A})$ group need not be a $B(\mathcal{A})$ group. Let G be the subgroup of those elements of U which have squarefree order. Since every integer has some squarefree divisor, G intersects every non-trivial closed subgroup of T in a non-trivial manner. Hence, G is a $B_r(\mathcal{A})$ group, by Theorem 1.6(b). However, letting T_2 and T_4 represent the groups of elements of T of order 2 and 4, respectively, then $G \cap T_4 = T_2$, and this is not dense in T_4. Hence, by Theorem 1.6(a), G is not a $B(\mathcal{A})$ group.

(e) Finally, we display an example of a precompact group which is not a $B_r(\mathcal{A})$ group. For a fixed prime p, let U_p denote the group of all p-power order roots of unity. The completion of U_p is T. Now, if q is a prime distinct from p, then the group T_q of all q-th roots of unity is a closed subgroup of T which has trivial intersection with U_p. It follows from Theorem 1.6(b) that U_p is not a $B_r(\mathcal{A})$ group.

Some further applications of Sulley's exceedingly useful criterion occur in Section 5.

4. Internal Characterization of $B(\mathcal{A})$ Groups

The definition of a $B(\mathcal{A})$ group is entirely an external characterization of the objects in question, in that it concerns only the mappings defined on a group and makes no direct mention of the elements, substructures, or topology of the group itself. One internal characterization of $B(\mathcal{A})$ groups is provided in this section.

We first develop some notation. Let G be a topological group with topology u. If we write $v \leq u$, then v is a group topology on G such that every v-open set is u-open. If v is another group topology on G, then v(u) will denote the group topology whose unit neighbourhood filter is $\{Cl_v U : U \in \mathcal{V}(u)$.

The following result is Theorem 31.4 of [8]. Its proof is included here for completeness.

Lemma 1.3 If (G,u) is a $B(\mathcal{A})$ group, then, for any Hausdorff group topology v on G, $v \leq u$ and $v(u) = v$ together imply $u = v$. For $B_r(\mathcal{A})$ groups, this condition is both necessary and sufficient.

Proof: Let v be such a group topology, and consider the identity map $j : (G,u) \longrightarrow (G,v)$. Since $v \leq u$, j is continuous, and, since every $v(u)$-open set contains a v-open set, j is also almost open. The map is clearly onto, and so it is open, since G is a $B(\mathcal{A})$ group. Thus, $u \leq v$, and we conclude $u = v$.

Clearly, the above argument is also valid if G has the $B_r(\mathcal{A})$ property, for j is a one-to-one map.

Conversely, let $f : (G,u) \Vdash\!\!\longrightarrow\!\!\!\!\!\rightarrow (H,w)$ be a continuous and almost open homomorphism. Let v denote the group topology generated on G by $\{f^{-1}(W) : W \in \mathcal{V}(w)\}$. Since w is Hausdorff and f is one-to-one, it follows that v is Hausdorff. Furthermore, $v \subseteq u$, and, letting f_o be the map which goes from (G,v) to (H,w) and coincides with f pointwise, we see that f_o is continuous and open. Now, $v \subseteq u$ implies $v \subseteq v(u)$; we then show $v(u) \subseteq v$ by demonstrating that $U_o \in \mathcal{V}(u)$ implies there exists $V \in \mathcal{V}(v)$ such that $V \subseteq Cl_v U_o$.

$$
\begin{aligned}
\text{Now,}\quad Cl_v U_o &= \bigcap\{U_o f^{-1}(W) : W \in \mathcal{V}(w)\} \\
&= \bigcap\{f^{-1}[f(U_o f^{-1}(W))] : W \in \mathcal{V}(w)\} \\
&= \bigcap\{f^{-1}[f(U_o)W] : W \in \mathcal{V}(w)\}
\end{aligned}
$$

$$= f^{-1}\left[\bigcap\left\{f(U_o)W : W \in \mathcal{V}(w)\right\}\right]$$

$$= f^{-1}(Cl_w f(U_o)) .$$

But, $Cl_w f(U_o) \in \mathcal{V}(w)$, so $Cl_v U_o \in \mathcal{V}(v)$. Thus, $v(u) = v$. By our hypothesis, $u = v$, and so $f = f_o$. Therefore, f is open, and (G,u) is a $B_r(\mathcal{A})$ group.

Since $v \subseteq u$ implies $v \subseteq v(u)$, by Proposition 31.8 of [8], we can partially recast Lemma 1.3 as follows.

<u>Lemma 1.4</u> (G,u) is a $B_r(\mathcal{A})$ group iff, for every Hausdorff group topology v on G, $v(u) \subseteq v \subseteq u$ implies $v = u$.

For a topological group G, let $\mathcal{N}(G)$ denote the set of closed normal subgroups of G.

<u>Lemma 1.5</u> G is a $B(\mathcal{Q})$ group iff G/H is a $B_r(\mathcal{A})$ group for every $H \in \mathcal{N}(G)$.

Proof: The "only if" direction follows at once from Proposition 31.7 of [8].

Conversely, let $f : G \twoheadrightarrow A$ be a continuous, almost open homomorphism. Then f factors as $f = f^*n$, where n is the natural map $G \twoheadrightarrow G/\text{Ker } f$ and f^* is the unique one-to-one map $G/\text{Ker } f \twoheadrightarrow A$ which satisfies the above equation. By Proposition 0.3, f^* is continuous and almost open. Since $(\text{Ker } f) \in \mathcal{N}(G)$, $G/\text{Ker } f$ is a $B_r(\mathcal{A})$ group, and so f^* is open. Again by Proposition 0.3, f is open. Hence, G is a $B(\mathcal{A})$ group.

Now, we may apply the condition of Lemma 1.3 to the quotient groups G/H. For each $H \in \mathcal{N}(G)$, let u_H denote the usual quotient topology on G/H, n_H the natural map $G \twoheadrightarrow G/H$. Lemmas 1.4 and 1.5 can then be rewritten as:

(G,u) is a $B(\mathcal{A})$ group iff, for each $H \in \mathcal{N}(G)$ and for every Hausdorff group topology w on G/H, $w(u_H) \subseteq w \subseteq u_H$ implies $w = u_H$.

For w, a group topology on G/H, let $n_H^{-1}(w)$ denote the group topology on G generated by the inverse images under n_H of w-open sets. We let $uH = n_H^{-1}(u_H)$, the group topology generated by all sets of the form $\{UH : U \in \mathcal{V}(u)\}$.

Writing the above criterion in terms of neighbourhoods, and taking inverse images with respect to n_H, we obtain the following, for each $H \in \mathcal{N}(G)$: if, for every $U \in \mathcal{V}(u)$, there is a $W \in \mathcal{V}(w)$ such that $n_H^{-1}(W) \subseteq n_H^{-1}(Cl_w n_H(U))$,

and if, for every $W \in \mathcal{V}(w)$, there exists $U \in \mathcal{V}(u)$ such that $UH \subseteq n_H^{-1}(W)$, then $w = u_H$. We claim that

$$n_H^{-1}(Cl_w n_H(U)) = Cl_{n_H^{-1}(w)} UH = Cl_{n_H^{-1}(w)} U.$$

To show the first equality, we first observe that

$$\begin{aligned} n_H^{-1}(Cl_w n_H(U)) &= n_H^{-1}(\cap\{n_H(U)W : W \in \mathcal{V}(w)\}) \\ &= \cap\{n_H^{-1}(n_H(U)W) : W \in \mathcal{V}(w)\} \\ &\supseteq \cap\{(UH)n_H^{-1}(W) : W \in \mathcal{V}(w)\} \end{aligned}$$

$$= Cl_{n_H^{-1}(w)} UH \;.$$

We need now show only the reverse of the inclusion to have proved the first equality.

Let p be a point of $n_H^{-1}(n_H(U)W)$, for each $W \in \mathcal{V}(w)$. Then $n_H(p) \in n_H(U)W$, whence, for each such W, there exists $x \in U$, $y \in W$ such that $n_H(p) = n_H(x)y = n_H(x)n_H(t)$, for some $t \in n_H^{-1}(W)$. Thus, $p \in n_H^{-1}(n_H(xt)) = xtH = (xH)(tH) \subseteq UHn_H^{-1}(W)$. This proves the first equality.

The second equality is merely a reflection of the fact that H is a subset of every $n_H^{-1}(w)$-neighbourhood of the unit.

We now prove a result which enables us to dispense completely with references to topologies on the quotient groups G/H.

<u>Lemma 1.6</u> The group topologies on G coarser than uH are precisely the inverse images under n_H of group topologies on G/H coarser than u_H.

Proof: If w is a group topology on G coarser than uH, then $n_H(w)$ is coarser than $n_H(uH) = u_H$. Then the unit neighbourhood filter of $n_H^{-1}(n_H(w))$ is $\{WH : W \in \mathcal{V}(w)\}$. But each $W \in \mathcal{V}(w)$ is of the form $W = YH$, where $Y \in \mathcal{V}(u)$, since w is coarser than uH. Let $\mathcal{Y} = \{Y \in \mathcal{V}(u) : YH = W,$ some $W \in \mathcal{V}(w)\}$. Then,

$$\{WH : W \in \mathcal{V}(w)\} = \{YH^2 : Y \in \mathcal{Y}\} = \{YH : Y \in \mathcal{Y}\} = \mathcal{V}(w).$$

Hence, $w = n_H^{-1}(n_H(w))$.

Conversely, if w_H is a group topology on G/H coarser than u_H , its inverse image under n_H is clearly a group topology on G which is coarser than uH. QED

This lemma can now be combined with the equalities derived above to obtain the following condition for a topological group (G,u) and a closed normal subgroup H of G.

$P(H)$: If w is a group topology on G such that, for each $W \in \mathcal{V}(w)$, there exists $U \in \mathcal{V}(u)$ such that $UH \subseteq W$, and, for each $U \in \mathcal{V}(u)$, there exists $W \in \mathcal{V}(w)$ such that $W \subseteq Cl_w U$, then $w = uH$.

Combining the above results, we have proved:

Theorem 1.7 G is a $B(\mathcal{a})$ group iff $P(H)$ is satisfied for every $H \in \mathcal{N}(G)$.

5. Permanence Properties of $B(\mathcal{a})$ and $B_r(\mathcal{a})$ Groups

In this section, we examine $B(\mathcal{a})$ groups and $B_r(\mathcal{a})$ groups as subcategories of the category of all topological groups and continuous homomorphisms. We investigate the closure properties of these subcategories under the formation of products, subobjects, retracts, projective limits, and other categorical operations.

These categories are quite poorly behaved with respect to such "permanence" properties. We shall first prove theorems of a positive nature concerning retracts and a special case of subgroups, followed by several counterexamples to a number of attractive conjectures.

Definition A subgroup H of a topological group G is said to be a retract of G if there is a homomorphism $r : G \twoheadrightarrow H$ such that $r|H$ is the identity; r is then said to be a retraction.

A subgroup H of a topological group G is said to be a topological direct factor of G if there is a subgroup H' of G such that the multiplication map $m : H \times H' \twoheadrightarrow G$ is an isomorphism.

We now state the following result, which, to the best of the author's knowledge, is unpublished and due to B. Banaschewski.

Lemma 1.7 If H is a subgroup of G, then the following are equivalent:

(1) H is a topological direct factor of G;

(2) H is normal in G and a retract of G;

(3) H is the kernel of a retraction.

Proof: (1)$\Rightarrow$(2) Let $f : G \longrightarrow H \times H'$ be the inverse of the multiplication map, and let $p : H \times H' \longrightarrow H$ be the projection. Then $pf : G \longrightarrow H$ is a homomorphism whose restriction to H is the identity. Hence, H is a retract of G. Now, to see that H is normal in G, let $h \in H$, $a \in G$, and note that a can be expressed uniquely as a product $a = xy$, $x \in H$, $y \in H'$. Then $aha^{-1} = mf(aha^{-1}) = m[f(a)f(h)f(a)^{-1}] = m[(x,y)(h,e)(x^{-1},y^{-1})] = m(xhx^{-1},e) = xhx^{-1} \in H$.

(2)$\Longrightarrow$(3) Let H be normal in G, $f : G \longrightarrow H$ the retraction. Then $s \in \operatorname{Ker} f$, $t \in H$ implies $sts^{-1} \in H$, whence $sts^{-1} = f(sts^{-1}) = f(s)f(t)f(s)^{-1} = f(t) = t$. Hence, $ts = st$. Define $g : G \longrightarrow \operatorname{Ker} f$ by $x \longmapsto xf(x)^{-1}$. Then, $g(xy) = xyf(xy)^{-1} = xyf(y)^{-1}f(x)^{-1}$, and, since $f(x) \in H$ and $yf(y)^{-1} \in \operatorname{Ker} f$, it follows that $g(xy) = xf(x)^{-1}yf(y)^{-1} = g(x)g(y)$. Hence g is a homomorphism; since it is the composition of continuous maps, it is itself continuous. Then $g[g(x)] = g(xf(x)^{-1}) = xf(x)^{-1}f(x)f[f(x)^{-1}] = xf(x)^{-1}$, since $f[f(x)] = f(x)$. Hence, $gg = g$, and so g is a retraction.

Now $x \in \operatorname{Ker} g$ iff $g(x) = e$ iff $x = f(x)$ iff $x \in H$. Hence, H is the kernel of the retraction.

(3)$\Longrightarrow$(1) Let $H = \operatorname{Ker} g$, g a retraction, $g : G \longrightarrow H'$. As above, $s \in H$, $t \in H'$ implies $ts = st$, and it follows from this that the multiplication map $m : H \times H' \longrightarrow G$ is a homomorphism.

Define $h : G \longrightarrow H \times H'$ by $h(x) = (xg(x)^{-1}, g(x))$. Since $x \longmapsto xg(x)^{-1}$ is a continuous homomorphism, it follows that h is a continuous homomorphism.

Then $(mh)(x) = x$, and $(hm)(s,t) = h(st)$
$= (stg(st)^{-1}, g(st)) = (stg(t)^{-1}g(s)^{-1}, g(s)g(t)) = (stt^{-1}e, et)$
$= (s,t)$. Them m is invertible, and so is an isomorphism.

With the above result established, we prove the following:

<u>Theorem 1.8</u> If G is a $B_r(\mathcal{Q})$ group and if H is a normal subgroup of G and a retract of G, then H is a $B_r(\mathcal{Q})$ group.

Proof: By the definition and Lemma 1.7, the multiplication map $m : H \times H' \longrightarrow G$ is an isomorphism, and so m has an inverse j which is also an isomorphism. Let $f : H \longrightarrow A$ be continuous, almost open, one-to-one, and onto. Define $h : G \longrightarrow A \times H'$ by $h(s) = (f \times id_{H'})(j(s))$, so that this diagram commutes:

$$\begin{tikzcd}
H \times H' \arrow[r, shift left, "m"] \arrow[d, "f \times id_{H'}"'] & G \arrow[l, shift left, "j"] \arrow[dl, "h"] \\
A \times H' &
\end{tikzcd}$$

Clearly, h is one-to-one, onto, and continuous. Furthermore, $(f \times id_{H'})$ is almost open, for

$$Cl_{A \times H'}(f \times id_{H'})(V_1 \times V_2) = Cl_{A \times H'}(f(V_1) \times V_2)$$
$$\supseteq (Cl_A f(V_1)) \times V_2 .$$

Since j is open, $h = (f \times id_{H'})j$ is almost open. Since G is a $B_r(\mathcal{A})$ group, h is an open mapping. Now, $(f \times id_{H'}) = (f \times id_{H'})jm = hm$, but m is open, so $(f \times id_{H'})$ is open. It follows that f is open, and so H is a $B_r(\mathcal{A})$ group.

<u>Theorem 1.9</u> If G is a $B(\mathcal{A})$ group, and H is a normal subgroup of G and a retract of G, then H is a $B(\mathcal{A})$ group.

Proof: Same as above, dropping the assumption that f be one-to-one.

<u>Remark</u> The above results also hold for $B(\mathcal{B})$ and $B_r(\mathcal{B})$ groups, where $\mathcal{B}$ is any full subcategory of $\mathcal{A}$. In particular, in the case of Abelian groups, the normality assumption is also subsumed.

We now proceed to our second permanence property.

<u>Notation</u> We shall denote the centre of a topological group G by Cent G.

<u>Theorem 1.10</u> Let (G,u) be a $B_r(\mathcal{A})$ group, and H an open subgroup of G such that $H \subseteq \text{Cent } G$. Then H is a $B_r(\mathcal{A})$ group.

Proof: Let $u|H$ denote the relative topology on H induced by u, and let w be any Hausdorff group topology such that $w \subseteq u|H$ and $w(u|H) = w$. By Lemma 1.3, it suffices to show that $w = u|H$.

Define a new topology w_1 on G by taking $\mathcal{V}(w_1) = \mathcal{V}(w)$, recalling that H is open in G. Since H is in the centre of G, we observe that $xVx^{-1} = V$ for every $V \in \mathcal{V}(w)$ and $x \in G$; it follows that conjugation is continuous at e. The other axioms for a group topology follow trivially. Also, w_1 is Hausdorff, since w has this property and the unit neighbourhood filters coincide.

Now, $w_1 \subseteq u$, because H is open, and so $w_1 \subseteq w_1(u)$, by Proposition 31.8 of [8]. We show $w_1(u) \subseteq w_1$. Let $U \in \mathcal{V}(u)$, so $U \cap H \in \mathcal{V}(u|H)$. Since $w(u|H) = w$, there is some $W \in \mathcal{V}(w)$ such that $W \subseteq Cl_w(U \cap H)$. But this latter is precisely the w_1-closure of $U \cap H$, so $W \subseteq Cl_{w_1}(U \cap H) \subseteq Cl_{w_1} U$. Thus, $w_1(u) \subseteq w_1$.

Since G is a $B_r(\mathcal{a})$ group, this implies $w_1 = u$. Then $w = w_1 | H = u|H$, and so H is a $B_r(\mathcal{a})$ group, by Lemma 1.3.

By a similar method, we obtain an analagous permanence property for $B(\mathcal{a})$ groups.

<u>Theorem 1.11</u> If (G,u) is a $B(\mathcal{a})$ group and K is an open subgroup of G such that $K \subseteq \text{Cent } G$, then K is a $B(\mathcal{a})$ group.

Proof: We use the internal characterization of Theorem 1.7. Let $H \in \mathcal{N}(K)$. Since K is open in G, it is also closed; hence, H is closed in G. Since $H \subseteq \text{Cent } G$, it is a normal subgroup of G. Therefore, $H \in \mathcal{N}(G)$.

Let w be a group topology on K such that the two conditions of P(H) are satisfied (on K!). Let w_1 be the group topology on G generated by $\mathcal{V}(w)$ as unit neighbourhood filter. We show that w_1 satisfies the two conditions of P(H) on G.

First, if $W \in \mathcal{V}(w_1)$, then there exists $V \in \mathcal{V}(w)$ such that $V \subseteq W$, whence there exists $U \in \mathcal{V}(K)$ such that $UH \subseteq V$. But $U \in \mathcal{V}(G)$, since K is open, and so the first condition is satisfied.

Secondly, if $U \in \mathcal{V}(G)$, then $U \cap K \in \mathcal{V}(K)$, and so there exists $W \in \mathcal{V}(w)$ such that $W \subseteq Cl_w(U \cap K)$. Then $W \in \mathcal{V}(w_1)$ and $W \subseteq Cl_{w_1} U$, since $Cl_w A = Cl_{w_1} A$ for any subset A of K. Thus, the second condition is satisfied.

Since G is a $B(\mathcal{Q})$ group, it follows from Theorem 1.7 that $u = w_1$. Hence, the relative u-topology on K equals the relative w_1-topology on K, but the latter is precisely w. Therefore, K is a $B(\mathcal{Q})$ group, by Theorem 1.7.

We then have the following joint corollary of Theorems 1.10 and 1.11, which follows immediately from these theorems.

<u>Corollary</u> If G is an Abelian $B_r(\mathcal{Q})$ (resp., $B(\mathcal{Q})$) group and H is an open subgroup of G, then H is a $B_r(\mathcal{Q})$ (resp. $B(\mathcal{Q})$) group.

Several counterexamples follow, the first of which shows that, given a group G with two topologies, with respect to each of which G is a $B(\mathcal{A})$ group, then G need not even be a $B_r(\mathcal{A})$ group, when endowed with the join of the two topologies.

Example 1.1 Let (R,u) denote the reals with the usual topology, g a discontinuous automorphism of the reals [27; p. 49], and $(R,g(u))$ the reals, endowed with the topology consisting of the images under g of u-open sets.

We claim the identity map $j : (R, u \vee g(u)) \longrightarrow (R,u)$ is continuous and almost open. The continuity is clear. To see that j is almost open, it is sufficient to show that the image under g of any u-open set is u-dense in R, for then, if A,B are u-open sets, it follows that $Cl_u A = Cl_u(A \cap g(B))$.

To demonstrate this fact, we first observe that the image under g of any u-open set is unbounded. If not, then $g(V_o)$ is bounded for some open set V_o. Without loss of generality, we may assume $V_o = (-a,a)$, and that $g(V_o) \subseteq (-1,1)$. Let $p \in g^{-1}(-1,1)$, and let t be a rational number such that $0 \leq t \leq \frac{1}{2}\min\{1 - g(p), 1 + g(p)\}$. Then $g(p + (-at, at))$ $= g(p) + tg(-a,a) \subseteq g(p) + (-t,t) \subseteq (-1,1)$. Thus, $g^{-1}(-1,1)$ contains a u-neighbourhood of each of its points, and so is u-open. But this implies $g : (R,u) \longrightarrow (R,u)$ is continuous, a contradiction. Hence, $g(V)$ is unbounded for every u-open set V.

We now show that $g(V)$ is dense in R, with respect to the u-topology. Assume $0 \in V$, and that V is connected. Then, for each $x \in V$, $\left\{kx : 0 \leqslant k \leqslant 1,\ k \text{ rational}\right\} \subseteq V$, whence $\left\{kg(x) : 0 \leqslant k \leqslant 1,\ k \text{ rational}\right\} \subseteq g(V)$. For each x, the above set is u-dense in $[0, g(x)]$. But g(V) contains arbitrarily large real numbers. Thus, g(V) is u-dense in R. Hence, j is continuous and almost open, but it is obviously not open.

Now, (R,u) is locally compact, and so is (R,g(u)), since it is the continuous, open image of (R,u). Hence, both are $B(\mathcal{A})$ groups, but $(R, u \vee g(u))$ is not a $B_r(\mathcal{A})$ group, for we have constructed a continuous, almost open, one-to-one homomorphism from this group onto a Hausdorff group, which is not an open map.

Remark The above example also shows that the join of two locally compact group topologies need not be locally compact, for, if R with the join topology were locally compact, it would be a $B_r(\mathcal{A})$ group.

Our next example shows that a finite product of $B(\mathcal{A})$ groups need not be a $B_r(\mathcal{A})$ group.

Example 1.2 Let R be the reals with the usual topology, U the group of complex roots of unity. The latter was proved to be a $B(\mathcal{A})$ group in Example (c), Section 1.3, and, of course, R is locally compact. However, R x U is not a $B_r(\mathcal{A})$ group.

To see this, observe that the completion of R x U is R x T, where T is the circle group. Since R x T is locally compact, it is a $B(\mathcal{A})$ group. Let b be an arbitrary irrational number, and let

$$H = \{(n, \exp(2nb\pi i)) : n \in Z\}.$$

Then H is a discrete subgroup of R x T, and so is closed. However, $H \cap (R \times U) = \{(0,1)\}$, and so R x U is not a $B_r(\mathcal{A})$ group, by Theorem 1.6(b).

Remark Since products are trivial projective limits, we have also shown that $B(\mathcal{A})$ and $B_r(\mathcal{A})$ groups are not closed under projective limits. However, this is not a directed system, and the question of whether our categories are closed with respect to limits over such systems is still open.

Our next example concerns inductive limits.

Example 1.3 Let (R,u) and (R,d) denote the reals with the usual and discrete topologies, respectively. Let $G_1 = (R,u) \times (R,d)$, $G_2 = (R,d) \times (R,u)$, and let $f : G_1 \longrightarrow G_2$ by $(x,y) \longmapsto (y,x)$. Let this system be ordered by $1 \leqslant 2$. Its inductive limit in the category of topological spaces is then R endowed with the topology $(u \times d) \wedge (d \times u)$. It has been proved in [19], however, that this is not even a group topology, although the groups involved are locally compact and so $B(\mathcal{A})$ groups.

Finally, we consider the question of quotients of groups in our category.

<u>Example 1.4</u> Let G, T, T_2, T_4 be defined as in Section 1.3, Examples (c) and (d). It is shown in (d) that G is a $B_r(\mathcal{A})$ group which is not a $B(\mathcal{A})$ group. Let n be the natural map $T \twoheadrightarrow T/T_4$, $m : G \twoheadrightarrow n(G)$ its restriction. Then $\operatorname{Ker} m = G \cap \operatorname{Ker} n = G \cap T_4 = T_2$. We claim G/T_2 is not a $B_r(\mathcal{A})$ group.

Suppose it is. Then, by Lemma 1.1, m is continuous and almost open, whence its factorization $m^* : G/T_2 \twoheadrightarrow n(G)$ also has these properties. But, if G/T_2 is a $B_r(\mathcal{A})$ group, it follows that m^* is open, whence m is also open, by Proposition 0.3. However, Lemma 1.1 also states that m is open iff $G \cap T_4$ is dense in T_4, a condition which is clearly not satisfied. This contradiction proves that G/T_2 is not a $B_r(\mathcal{A})$ group.

<u>Remark</u> This counterexample shows that the portion of Proposition 31.7 of [8] which refers to $B_r(\mathcal{A})$ groups is false. Further reference to this fact will be made in Chapter 3.

<u>6. $B(\mathcal{C})$ Groups Which are not $B(\mathcal{A})$ Groups</u>

Several references have been made so far to $B(\mathcal{C})$

groups, but no example has been produced to show that a $B(\mathcal{C})$ group, for some class $\mathcal{C}$ of Hausdorff groups, need not be a $B(\mathcal{A})$ group. This defect will now be remedied by displaying such a class and such a group.

For this section only, $\mathcal{C}$ will represent the category of first countable Hausdorff groups.

Lemma 1.8 A continuous map of a countably compact space into a first countable Hausdorff space is closed.

Proof: Let $f : X \longrightarrow Y$ be such a map, A a closed subspace of X. Then A is countably compact, and so is f(A), by Theorem 3.6, Chapter XI of [4]. By the same theorem, a countably compact subspace of a first countable space is closed, so f(A) is closed in Y, whence f is a closed map.

Theorem 1.12 A continuous, almost open homomorphism f of a countably compact topological group G into any group H in $\mathcal{C}$ is open.

Proof: Let B be a closed unit neighbourhood in G. Then f(B) is closed, by Lemma 1.8; i.e., $f(B) = Cl_H f(B)$. But $Cl_H f(B)$ is a unit neighbourhood in H, since f is almost open. Hence, it follows that f is open.

Corollary Every countably compact topological group is a $B(\mathcal{C})$ group.

This is merely the special case of Theorem 1.12, where f is onto.

<u>Remark</u> The above results also hold if G is taken to be locally countably compact, a slight gain in generality.

To show that $B(\mathcal{B})$ groups, as a class, are strictly larger than $B(\mathcal{A})$ groups, we consider an example given by Pontryagin [14; p. 127-8].

<u>Example 1.5</u> Let A be any uncountable set, and let G be any non-trivial, compact, Hausdorff Abelian group. Let $G^* = G^A$, and define

$$P = \{(x_a) : x_a \neq e \text{ for at most countably many } a \in A\}.$$

Pontryagin shows that P is countably compact, but is also dense in the compact Hausdorff group G^*, and so is not compact.

Furthermore, P is not a $B_r(\mathcal{A})$ group. For each $g \in G$, let (g) be that element (x_a) of G^* such that $x_a = g$ for every $a \in A$. Let H denote the "diagonal" subgroup of G^* given by $H = \{(g) : g \in G\}$. It is easy to see that H is closed, and that $H \cap P = \{(e)\}$. It then follows from Theorem 1.6(b) that P is not a $B_r(\mathcal{A})$ group.

7. An Open Mapping Theorem for Non-surjective Homomorphisms

All our discussions up to this point have, directly or indirectly, been concerned with surjective homomorphisms. This brief section gives one example of an open mapping theorem in which this condition is at least slightly relaxed.

Theorem 1.13 Let G and H be locally compact groups, H Hausdorff, and let $f : G \longrightarrow H$ be a continuous, almost open homomorphism. If $f(G)$ is dense in H, then f is open.

Proof. Let $f_1 : G \twoheadrightarrow f(G)$ be the corestriction of f. Now f_1 is almost open, since, for $U \in \mathcal{V}(G)$,

$$Cl_{f(G)} f(U) = f(G) \cap Cl_H f(U) \in \mathcal{V}(f(G)),$$

by Theorem 7.2, Chapter III of [4]. It then follows from Theorem 26.4 of [8] that $f(G)$ is locally compact.

Now, G is a $B(\mathcal{A})$ group, so f_1 is open. Furthermore, by Exercise 6.95 of [27], $f(G)$ is open in H. Hence, the natural injection $j : f(G) \rightarrowtail H$ is open, and so $f = jf_1$ is open.

CHAPTER 2

THE CLOSED GRAPH THEOREM FOR TOPOLOGICAL GROUPS

1. Preliminaries

We seek to generalize the classical closed graph theorem to appropriate classes of topological groups. Since the classical result depends heavily on the fact that any linear mapping of a Banach space into a metrizable topological vector space is almost continuous, we ask for what classes of groups must a homomorphism which is almost continuous and has a closed graph be continuous. We also investigate how the $B(\mathscr{C})$ and $B_r(\mathscr{C})$ groups of the last chapter are involved in studies of this nature, and what other conditions on the homomorphisms may imply continuity.

Baker [1] and Husain [8, 10] have extended the closed graph theorem for $B(\mathscr{C})$ groups, where $\mathscr{C}$ is a category of Hausdorff groups which is assumed to have some additional permanence property. Baker imposes a condition involving inductive limit topologies on his groups, all of which are assumed to be Abelian. We shall not deal with this particular type of category here; for details, see [1].

Husain's condition is somewhat simpler, but, since we depart from his terminology, we must first state a definition, which is an adaptation from Isbell [11; p. 119].

<u>Definition</u> If $\mathcal{X}$ is a category, $\mathcal{Y}$ a subcategory of $\mathcal{X}$, and $\mathcal{C}$ a class of morphisms in $\mathcal{X}$, then $\mathcal{Y}$ is said to be right fitting with respect to $\mathcal{C}$ if $Y \in \mathcal{Y}$, $X \in \mathcal{X}$ and $f : Y \longrightarrow X$ in $\mathcal{C}$ together imply $X \in \mathcal{Y}$.

Throughout this chapter, the symbol $\mathcal{C}$ will be reserved for the class of morphisms in $\mathcal{A}$, the category of all Hausdorff topological groups and continuous homomorphisms, which are almost open. Similarly, $\mathcal{D}$ will denote those morphisms in $\mathcal{A}$, the image of whose domain is dense in the codomain.

Husain's condition is then that $\mathcal{B}$ be right fitting with respect to $\mathcal{C}$. We shall extend Husain's results, and also consider categories of groups which are right fitting with respect to $\mathcal{D}$, as well.

In addition, Husain [10] requires that the codomains of his maps be Abelian. We weaken this condition in two ways. First, we may simply require that the image of the domain group be contained in the centre of the codomain. Our results in this case are strict generalizations of those of Husain. Second, we may assume that the left and right uniform structures on the codomain coincide. Besides commutative groups, this embraces precompact and compact groups, as well as locally compact groups whose quotient group modulo the centre is compact [25]. In this second case, we must add some conditions to our maps, so these results extend, but do not strictly generalize, those of Husain.

2. Closed Graph Theorems

We first embark on an examination of a technique which will appear repeatedly in proving various versions of the closed graph theorem, namely the construction of a new topology on the codomain.

Consider a homomorphism $f : G \longrightarrow H$. For $U \in \mathcal{V}(H)$, let $\hat{U} = f(\mathrm{Cl}_G f^{-1}(U))$, $U^* = U\hat{U}$, and $\mathscr{W} = \{U^* : U \in \mathcal{V}(H)\}$. Let $\mathscr{W}$ be a subbasis of open neighbourhoods of e_H, and form a topology for the whole group by taking all translates of the sets in this subbasis. This topology will be denoted throughout this chapter by w.

Lemma 2.1 If $f(G) \subseteq \mathrm{Cent}\, H$, then w is a group topology.

Proof: We show that $\mathscr{W}$ satisfies the three axioms (GV1) - (GV3) for a group topology given in Section 0.1.

For (GV1), let $U \in \mathcal{V}(H)$ and let $V \in \mathcal{V}(H)$ such that

$V^2 \subseteq U$. Then $(V^*)^2 = (V\hat{V})^2 = V(\hat{V}V)\hat{V} = V^2(\hat{V})^2$, since $f(G)$ commutes with every element of H and $\hat{V} \subseteq f(G)$. Now,

$$
\begin{aligned}
(\hat{V})^2 &= (f[Cl_G f^{-1}(V)])^2 = f([Cl_G f^{-1}(V)]^2) \\
&\subseteq f(Cl_G[f^{-1}(V)^2]) \subseteq f(Cl_G[f^{-1}(V^2)]) \\
&\subseteq f(Cl_G[f^{-1}(U)]) = \hat{U} .
\end{aligned}
$$

Hence, $(V^*)^2 \subseteq V^2\hat{U} \subseteq U\hat{U} = U^*$.

As for (GV2), $U \in \mathcal{V}(H)$ implies $U^{-1} \in \mathcal{V}(H)$, and

$$
\begin{aligned}
(U^*)^{-1} &= (U\hat{U})^{-1} = (\hat{U})^{-1}U^{-1} \\
&= U^{-1}(\hat{U})^{-1} \text{ , since } \hat{U} \subseteq \text{Cent } H, \\
&= U^{-1}(U^{-1})^{\wedge} \text{ , by elementary calculations,} \\
&= (U^{-1})^* .
\end{aligned}
$$

Therefore, $(U^{-1})^* = (U^*)^{-1}$, and the latter is indeed in $\mathcal{W}$.

Finally, for (GV3), let $U \in \mathcal{V}(H)$ and $a \in H$. Pick $V \in \mathcal{V}(H)$ such that $aVa^{-1} \subseteq U$, and let $Y = V \cap U$. Then

$$
\begin{aligned}
aY^*a^{-1} &= (aYa^{-1})(a\hat{Y}a^{-1}) \\
&= (aYa^{-1})\hat{Y} \text{ , since } \hat{Y} \subseteq \text{Cent } H, \\
&\subseteq U\hat{U} = U^* .
\end{aligned}
$$

Therefore, w is a group topology for H.

Lemma 2.2 If H has equal left and right uniformities, $f(G)$ is dense in H, and f is almost open, then w is a group topology.

Proof: By Theorem 0.1, H has a fundamental system $\mathcal{U}$ of unit neighbourhoods which are fixed under all inner automorphisms of H. We now invoke the three axioms from Section 0.1 again.

Because of the property of $\mathcal{U}$ mentioned above, (GV1) and (GV2) follow easily, in a manner similar to that of the previous lemma.

To prove (GV3), we first show that, if $U \in \mathcal{U}$, then U^* is also invariant under all inner automorphisms of H. We begin by showing that $\hat{U}$ is invariant under conjugation by elements of $f(G)$.

Let $t \in G$, $U \in \mathcal{U}$. Then

$$\begin{aligned}\hat{U} = f\left[Cl_G f^{-1}(U)\right] &= f\left[Cl_G f^{-1}(f(t)Uf(t)^{-1})\right]\\ &\supseteq f\left[Cl_G(tf^{-1}(U)t^{-1}\right]\\ &\supseteq f(t)f\left[Cl_G f^{-1}(U)\right]f(t)^{-1}\\ &= f(t)\hat{U}f(t)^{-1}.\end{aligned}$$

Now, similarly, $\hat{U} \supseteq f(t)^{-1}\hat{U}f(t)$. Therefore,

$\hat{U} \supseteq f(t)\hat{U}f(t)^{-1} \supseteq f(t)\left[f(t)^{-1}\hat{U}f(t)\right]f(t)^{-1} = \hat{U}$, whence $\hat{U} = f(t)\hat{U}f(t)^{-1}$. It then follows easily that U^* is invariant under conjugation by elements of $f(G)$.

Now let a be an arbitrary element of H. Since f(G) is dense in H and since f is almost open, there exist $y \in G$ and $V \in \mathcal{V}(G)$ such that $a \in Cl_G f(yV)$. Given $U \in \mathcal{U}$, pick a symmetric $W \in \mathcal{U}$ such that $(W^*)^3 \subseteq U^*$. Then $(Wa) \cap f(yV) \neq \emptyset$, and $a = x^{-1}f(yt)$, where $x \in W$ and $t \in V$. Hence,

$$\begin{aligned} aW^*a^{-1} &= x^{-1}f(yt)W^*f(yt)^{-1}x \\ &= x^{-1}W^*x, \text{ by our previous observations,} \\ &\subseteq WW^*W \text{ , since W is symmetric,} \\ &\subseteq (W^*)^3 \subseteq U^* . \end{aligned}$$

Therefore, (GV1) - (GV3) being satisfied, w is a group topology.

<u>Lemma 2.3</u> If w is a group topology and if f has closed graph, then w is a Hausdorff topology.

Proof: Let R(f) be the graph of f, considered as a subset of G x H. We show that $\bigcap \{U^* : U \in \mathcal{V}(H)\} = \{e_H\}$.

Let $y \in \bigcap U^*$. Let $U \in \mathcal{V}(H)$, and let $W \in \mathcal{V}(H)$ such that W is symmetric and $W^2 \subseteq U$. Then $y \in W^*$, whence $y = xf(a)$, where $x \in W$ and $a \in Cl_G f^{-1}(W)$. Such a point a can be represented as $a = tz$, where $f(t) \in W$ and $z \in V$, V being an arbitrary unit neighbourhood in G. Then $az^{-1} = t \in f^{-1}(W)$, and so $f(a)f(z)^{-1} \in W$. Since W is symmetric, we have

$$f(z) \in Wf(a) = Wx^{-1}y \subseteq W^2y \subseteq Uy .$$

It follows that $(z, f(z)) \in (V \times Uy) \cap R(f)$. Since

U, V were arbitrary unit neighbourhoods, it then follows that $(e_G, y) \in Cl_{G \times H} R(f)$. Therefore, $(e_G, y) \in R(f)$, since the latter is closed by hypothesis, and so $y = f(e_G) = e_H$. It then follows from Theorem 21.4 of [8] that w is a Hausdorff topology.

We now proceed to various forms of the closed graph theorem, having these properties of the w-topology in hand.

<u>Theorem 2.1</u> Let $\mathcal{G}$ be a category of Hausdorff groups which is right fitting with respect to $\mathcal{L}$. Let $G \in \mathcal{G}$, H be a $B_r(\mathcal{G})$ group, and $f : (G,v) \longrightarrow (H,u)$ be an almost continuous, almost open homomorphism with closed graph. If w is a group topology, then $f \in \mathcal{L}$.

Proof: By Lemma 2.3, (H,w) is a Hausdorff topological group, and so is a T_3 topological space. We now apply Proposition 31.9 of [8] by showing that $w(u) = w$. Clearly, $w \subseteq u$, so $w \subseteq w(u) \subseteq u$, by Proposition 31.8 of [8]. Hence, it remains only to show that $w \subseteq w(u)$.

Let $U \in \mathcal{V}(H)$, and let $V \in \mathcal{V}(H)$ such that $V^2 \subseteq U$. We claim $V^* \subseteq Cl_w U$. Let $y \in V^*$. Then $y = sf(x)$, where $s \in V$ and $x \in Cl_G f^{-1}(V)$. Now, for any symmetric $W \in \mathcal{V}(H)$, $x \in f^{-1}(V) Cl_G f^{-1}(W)$, whence $t^{-1}x \in Cl_G f^{-1}(W)$, for some $t \in f^{-1}(V)$. Hence, $x^{-1}t \in Cl_G f^{-1}(W)$, since W is symmetric, and so $f(t) \in f(x) f\left[Cl_G f^{-1}(W)\right]$. Therefore,

$$sf(t) \in sf(x)W = y\hat{W} \subseteq yW^*.$$

Also, $sf(t) \in V^2 \subseteq U$, whence $sf(t) \in U \cap yW^* \neq \emptyset$. Since W was arbitrary, $y \in Cl_w U$, so $V^* \subseteq Cl_w U$. Therefore, $w(u) = w$.

Then the identity map $j : (H,u) \longrightarrow (H,w)$ is continuous and almost open. Furthermore, the map $g : (G,v) \longrightarrow (H,w)$ which coincides with f pointwise is continuous, for $g^{-1}(U^*) \supseteq g^{-1}(f\left[Cl_G f^{-1}(U)\right]) \supseteq Cl_G f^{-1}(U)$,

which is a unit neighbourhood since f is almost continuous. Hence, $g = jf \in \mathcal{L}$. Therefore, $(H,w) \in \mathcal{B}$, since $\mathcal{B}$ is right fitting with respect to $\mathcal{L}$. But then j is open, since (H,u) is a $B_r(\mathcal{B})$ group, and it follows that $w = u$. But we have proved that $g : G \longrightarrow (H,w)$ is in $\mathcal{L}$. Therefore, $f = g$ and $f \in \mathcal{L}$.

<u>Corollary 1</u> Let $\mathcal{B}$ be as above. Let $G \in \mathcal{B}$, H be a $B_r(\mathcal{B})$ group, and $f : G \longrightarrow H$ be an almost continuous, almost open homomorphism with a closed graph. If f(G) is contained in Cent H, then $f \in \mathcal{L}$.

Proof: By Lemma 2.1, w is a group topology, and so it is Hausdorff, by Lemma 2.3. It then follows from the above theorem that $f \in \mathcal{L}$.

<u>Corollary 2</u> Let $\mathcal{B}$ be as above. Let H be a $B_r(\mathcal{B})$

group, $G \in \mathcal{L}$, and assume H is Abelian. Then any homomorphism $f : G \longrightarrow H$ which is almost open and almost continuous and has a closed graph is continuous.

Proof: This is an immediate consequence of Corollary 1, for $H = \text{Cent } H$ if H is Abelian.

Corollary 2 is Theorem 1 of [10]. The following, however, is new.

<u>Corollary 3</u> Let $\mathcal{L}$ be as above. Let $G \in \mathcal{L}$, H be a $B_r(\mathcal{L})$ group, and assume that H has equal uniformities. Let $f : G \longrightarrow H$ be almost continuous and almost open, and have a closed graph. If $f(G)$ is dense in H, then $f \in \mathcal{L}$.

Proof: By Lemma 2.2, w is a group topology, and Hausdorff by Lemma 2.3. It then follows from the theorem that $f \in \mathcal{L}$.

We now consider closed graph theorems for the particular case of $B_r(\mathcal{a})$ groups.

<u>Theorem 2.2</u> Let G be a Hausdorff group, H a $B_r(\mathcal{a})$ group. Let $f : G \longrightarrow H$ be an almost continuous homomorphism with closed graph. If $f(G) \subseteq \text{Cent } H$, then f is continuous.

Proof: By Lemmas 2.1 and 2.3, w is a Hausdorff group topology on H. Hence, $(H,w) \in \mathcal{a}$, and the rest of the proof follows identically with Theorem 2.1.

<u>Corollary 1</u> Let G be a Hausdorff group, H an Abelian $B_r(\mathcal{A})$ group. Then any almost continuous homomorphism $f : G \longrightarrow H$ with closed graph is continuous.

The above corollary is Theorem 2 of [10].

<u>Remark</u> The category $\mathcal{A}$ is right fitting with respect to $\mathcal{L}$ by definition. However, even the continuous, open homomorphic image of a Hausdorff group need not be Hausdorff. For example, the group R/Q with its quotient topology is indiscrete, although the canonical map $R \longrightarrow R/Q$ is both continuous and open.

<u>Theorem 2.3</u> Let G be a Hausdorff group, H a $B_r(\mathcal{A})$ group with equal uniformities. Then an almost continuous, almost open homomorphism $f : G \longrightarrow H$ with closed graph is in $\mathcal{L}$ if f(G) is dense in H.

Proof: It follows from Lemmas 2.2 and 2.3 that w is a Hausdorff group topology, whence $(H,w) \in \mathcal{A}$. The result then follows identically with Theorem 2.1.

By strengthening the conditions on G, we can prove still another closed graph theorem for $B_r(\mathcal{A})$ groups. We must first state the following proposition.

<u>Proposition 2.1</u> If G is a Hausdorff topological group with the Baire property, H a separable (or Lindelöf)

topological group, then any homomorphism $f : G \longrightarrow H$ is almost continuous.

This is Proposition 32.11(b) of [8]. The proof is easy and is omitted.

<u>Theorem 2.4</u> Let G be a Hausdorff group with the Baire property, and H a separable (or Lindelöf) $B_r(\mathcal{A})$ group. Then a homomorphism $f : G \longrightarrow H$ with closed graph is continuous if (a) $f(G) \subseteq \text{Cent } H$, or (b) H has equal uniformities, f(G) is dense in H, and f is almost open.

Proof: By Proposition 2.1, f is almost continuous. Then f is continuous by Theorem 2.2 in case (a) and 2.3 in case (b).

We could now draw more corollaries of the type found in Section 32 of [8]. However, the presence of the assumptions (a) or (b) makes such corollaries weaker than known results. See, for example, Section 6.R of [26].

The foregoing considerations do enable us, however, to prove a form of the open mapping theorem for $B(\mathcal{A})$ groups.

<u>Theorem 2.5</u> Let G be a $B(\mathcal{A})$ group with equal uniformities, H a Hausdorff group. Then any almost continuous, almost open homomorphism $g : G \twoheadrightarrow H$ with closed graph is open.

Proof: Ket $K = \mathrm{Ker}\ g$; by Proposition 30.2 of [8], K is a closed normal subgroup of G. Let $n : G \twoheadrightarrow G/K$ be the natural map. Now, G/K is a $B_r(\mathcal{A})$ group, by Lemma 1.4. Let $f : G/K \twoheadrightarrow H$ be the unique map such that $g = fn$. We claim that R(f) is closed in $(G/K) \times H$; let $F = (G/K) \times H$.

Let $(z,y) \in Cl_F R(f)$. Then $(U \times V) \cap R(f) \neq \emptyset$, for every neighbourhood $U \times V$ of (z,y) in F. Now, $z = n(t)$ for some $t \in G$, so for every neighbourhood W of t in G,

$$(n(W) \times V) \cap R(f) \neq \emptyset .$$

Hence, there exists $a \in W$ such that $(fn)(a) = g(a) \in V$. Since V,W were arbitrary, this means $(W \times V) \cap R(g) \neq \emptyset$, whence $(t,y) \in Cl_{G \times H} R(g) = R(g)$. Hence, $y = g(t) = fn(t) = f(z)$, and so $(z,y) \in R(f)$.

Now, by Proposition 0.3, f is almost open and almost continuous, since g has these properties; since f is one-to-one, its inverse f^{-1} is defined and is also almost open and almost continuous. Furthermore, $R(f) = R(f^{-1})$, and so this set is closed. Also, by Lemma 1.2, G/K has equal uniformities. Hence, f^{-1} is continuous by Theorem 2.3, and f is open. It then follows from Proposition 0.3 that g is open.

We now turn to the other class of maps mentioned in Section 1, the class $\mathcal{G}$ of morphisms $f : G \longrightarrow H$ in $\mathcal{A}$ for which f(G) is dense in H.

Theorem 2.6 Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$ which is right fitting with respect to $\mathcal{G}$. Let $G \in \mathcal{B}$, and let H be a $B_r(\mathcal{B})$ group with equal uniformities. Then an almost continuous, almost open homomorphism $f : G \longrightarrow (H,u)$ with a closed graph, such that $f(G)$ is dense in H, is in $\mathcal{G}$.

Proof: By Lemmas 2.2 and 2.3, (H,w) is a Hausdorff topological group. Now let $j : (H,u) \longrightarrow (H,w)$ be the identity map. It follows as in Theorem 2.1 that the map $g = jf : G \longrightarrow (H,w)$ is continuous. Furthermore, since $w \subseteq u$, $g(G)$ is dense in (H,w). Hence, $g \in \mathcal{G}$, and $(H,w) \in \mathcal{B}$. The fact that j is continuous and almost open follows as in Theorem 2.1. Hence, j is open, since (H,u) is a $B_r(\mathcal{B})$ group, and so $w = u$. Therefore, $g = f$, and so f is continuous and in $\mathcal{G}$.

Remark The above theorem subsumes the results analagous to corollaries 1 & 2 of Theorem 2.1. If a homomorphism f has the properties that (1) $f(G) \subseteq \text{Cent } H$, and (2) $f(G)$ is dense in H, simultaneously, then H is Abelian, for Cent H is closed [8; Proposition 23.9], and so it must be the whole group. Of course, an Abelian group has equal left and right uniformities.

3. $\mathcal{E}$- and $\mathcal{G}$- Right Fitting Subcategories of $\mathcal{A}$

To indicate the breadth of application of the main results of Section 2, we establish in this section that some important subcategories of $\mathcal{A}$ are right fitting with respect to the classes of maps we have considered.

Theorem 2.7 The following categories of Hausdorff groups are right fitting with respect to $\mathcal{E}$:

(a) locally compact groups

(b) locally precompact groups

(c) first countable groups

(d) locally connected groups

(e) groups with an open subgroup.

Proof: (a) Let G be locally compact, H be Hausdorff, $f : G \longrightarrow H$ in $\mathcal{E}$, and V a compact neighbourhood of e_G. Then f(V) is compact, and so closed in H; i.e., $f(V) = Cl_H f(V)$. But the latter is a neighbourhood of e_H, since f is almost open. Since the neighbourhoods of e_H of the form $Cl_H f(U)$, where $U \in \mathcal{V}(G)$, are a fundamental system in $\mathcal{V}(H)$, it follows that H is locally compact.

(b) Let V be a precompact neighbourhood of e_G, f and H as in (a). Then f(V) is precompact, and $Cl_H f(V)$ is in $\mathcal{V}(H)$. Let $U \in \mathcal{V}(H)$, and $W \in \mathcal{V}(H)$ such that $W^2 \subseteq U$. Then $f(V) \subseteq XW$, for some finite subset X of H, whence

$Cl_H f(V) \subseteq f(V)W \subseteq XW^2 \subseteq XU$. Hence, $Cl_H f(V)$ is totally bounded, and so precompact. Hence, H is locally precompact.

(c) Let G be first countable, $\{U_i\}$ its countable fundamental system of unit neighbourhoods, f and H as before. Let V be a closed set in $\mathcal{V}(H)$. Then

$$V \supseteq Cl_H f\left[f^{-1}(V)\right] \supseteq Cl_H f(U_j),$$

for some j. But $Cl_H f(U_j) \in \mathcal{V}(H)$, since f is almost open. Hence, $\{Cl_H f(U_i)\}$ is a countable local base at e_H, and H is first countable.

(d) Let G be locally connected, V a connected unit neighbourhood in G, f and H as before. Then f(V) is connected, and so is $Cl_H f(V)$, by Theorems 1.4 and 1.6, Chapter V of [4]. But $Cl_H f(V)$ is in $\mathcal{V}(H)$, whence H is locally connected.

We point out that Theorem 2.7(a) has appeared as Theorem 26.4 of [8]. Its proof is included for completeness.

The above theorem enables us to state one further result of the same type as in Section 2.

Corollary 4 to Theorem 2.1 Let G be a first countable group, H a countably compact group. If $f : G \longrightarrow H$ is almost continuous and almost open, and has a closed graph, and if either of the following two conditions is satisfied:

(a) $f(G) \subseteq \text{Cent } H$;

(b) $f(G)$ is dense in H and H has equal uniformities,

then f is continuous.

Proof: By Corollary 1 to Theorem 1.12, H is a $B_r(\mathscr{F})$ group, where $\mathscr{F}$ is the class of all first countable, Hausdorff groups. By Theorem 2.7(c), $\mathscr{F}$ is right fitting with respect to $\mathscr{L}$. By Lemma 2.1 or 2.2, w is a group topology on H, so f is continuous by Theorem 2.1.

Before moving on to the next theorem, we recall from Section 0.8 that, for two Hausdorff groups A and B, we define Hom(A,B) to be the set of continuous homomorphisms with domain A and codomain B. This set is always non-empty; if it consists of only the map whose kernel is all of A, we say Hom(A,B) is trivial.

Theorem 2.8 The following categories of Hausdorff groups are right fitting with respect to $\mathscr{G}$:

(a) compact groups

(b) precompact groups

(c) Abelian groups

(d) connected groups

(e) separable groups

(f) groups G such that Hom(G,A) is trivial, for some fixed $A \in \mathcal{A}$.

Proof: (a) is trivial, for if G is compact and f(G) is dense in H, then f(G) = H and H is compact.

(b) Let G be precompact, $f: G \longrightarrow H$ in $\mathcal{G}$, $H \in \mathcal{A}$. Then f(G) is precompact, and so is $Cl_H f(G)$, as in Theorem 2.7(b). The latter set, however, is H.

(c) Let G be Abelian, f and H as before. Define $q : H \times H \longrightarrow H$ by $q(a,b) = aba^{-1}b^{-1}$. Then q is continuous, and $f(G) \times f(G) \subseteq \text{Ker } q$. Hence,

$$H \times H = Cl_{H \times H}(f(G) \times f(G)) \subseteq \text{Ker } q,$$

whence H is Abelian.

(d) Let G be connected, f and H as before. Then f(G) is connected, and so is its closure, as in Theorem 2.7(c). Therefore, H is connected.

(e) Let G be separable, X its countable dense subset, f and H as before. Then,

$$H = Cl_H f(G) = Cl_H f(Cl_G X) \subseteq Cl_H(Cl_H f(X)) = Cl_H f(X).$$

The reverse inclusion is trivial, so $H = Cl_H f(X)$. Hence, f(X) is dense in H, and this set is at most countable; therefore, H is separable.

(f) Let $G \in \mathcal{A}$ such that $\mathrm{Hom}(G,A)$ is trivial, f and H as before. Let $t \in \mathrm{Hom}(H,A)$. Then $tf \in \mathrm{Hom}(G,A)$, whence tf is trivial. Thus, $f(G) \subseteq \mathrm{Ker}\ t$, and it follows that $\mathrm{Ker}\ t = \mathrm{Cl}_H f(G) = H$. Hence, t is trivial, and H has the required property.

Before proceeding to the next theorem, we first state the following result without proof. The proof may be found in [6; p. 62, Theorem 7.7] or [8; Theorem 26.7].

<u>Lemma 2.4</u> Let G be a compact, zero-dimensional group. Then every neighbourhood of the identity contains a compact open normal subgroup. The converse also holds.

<u>Theorem 2.9</u> The following categories of Hausdorff groups are right fitting with respect to $\mathcal{L} \cap \mathcal{G}$:

(a) second countable groups

(b) compact, zero-dimensional, Abelian groups; i.e., Abelian profinite groups

(c) groups with equal left and right uniformities.

Proof: (a) Let G be second countable, $H \in \mathcal{A}$, $f\colon G \longrightarrow H$ in $\mathcal{L} \cap \mathcal{G}$, $\{U_i\}$ the countable neighbourhood base for the topology of G. Then $a \in H$, $V = \mathrm{Cl}_H V \in \mathcal{V}(H)$ together imply $f^{-1}(aV) \neq \emptyset$ and contains some U_j. Then $\mathrm{Cl}_H f(U_i) \subseteq \mathrm{Cl}_H f\left[f^{-1}(aV)\right] \subseteq \mathrm{Cl}_H(aV) = aV$, so $\{\mathrm{Cl}_H f(U_i)\}$ forms a neighbourhood base for the topology of H, whence H is second countable.

(b) Let G be compact, zero-dimensional, and Abelian, f and H as in (a). By Theorem 2.8, H is compact and Abelian. Let N be an open subgroup of G; then $Cl_H f(N)$ is an open subgroup of H. Moreover, every unit neighbourhood in H contains one such, for, if $B \in \mathcal{V}(H)$ and $A \in \mathcal{V}(H)$ such that $Cl_H A \subseteq B$, then it follows that $f^{-1}(A) \in \mathcal{V}(G)$. Hence, $f^{-1}(A)$ contains some open subgroup N, by Lemma 2.4. Then $Cl_H f(N) \subseteq Cl_H f[f^{-1}(A)] \subseteq Cl_H A \subseteq B$. Hence, H is compact, zero-dimensional and Abelian, by Lemma 2.4.

(c) This has already been proved in Lemma 1.2.

<u>Remark</u> Certain important categories of Hausdorff groups behave very badly in this connection; in particular, totally disconnected groups and metric groups are not right fitting with respect to $\mathcal{L} \cap \mathcal{B}$ and with respect to $\mathcal{B}$, respectively. As a counterexample to the first, one need only consider the natural injection of the rationals Q into the reals R. In the second case, let A be any uncountable set, and let $(Q^A)_d$ be the indicated product with its discrete topology. This is a metric space with the trivial metric. Consider the injection $(Q^A)_d \rightarrowtail R^A$, where the reals have the usual topology. This map is clearly in $\mathcal{B}$, but R^A is not a metric space, since it is not normal [6; Theorem 8.12].

4. Ultrabarrelled Groups

In an effort to extend the notion of an ultrabarrelled locally convex space (see [9], for example) to topological groups, S. O. Iyahen [12] has defined a g-ultrabarrelled group. From this, he derives an open mapping and closed graph theorem which appears to generalize Corollary 32.4 of [8] and our Theorem 2.4.

Unfortunately, Iyahen's proof contains a flaw which renders his result doubtful, although no counterexample is yet known. In this section, we point out the nature of this defect and suggest one corrected version.

Definition If (G,u) is a topological group, let V be a u-closed symmetric subset of G for which there exists a sequence $\{V_n\}$ of u-closed, symmetric sets such that:

(i) $V_1^2 \subseteq V$ and $V_{n+1}^2 \subseteq V_n$, for each n;

(ii) for any $x \in G$ and integer n, there exists an integer m such that $xV_m x^{-1} \subseteq V_n$.

If, under the group topology v having $\{V_n\}$ as a base of neighbourhoods, the group G is separable, then V is said to be a group ultrabarrel in (G,u), and $\{V_n\}$ is said to be a defining sequence for V. If (G,u) has the property that every group ultrabarrel is a neighbourhood of identity, then (G,u) is said to be a g-ultrabarrelled group.

Iyahen then proves that every topological group of second category in itself is g-ultrabarrelled, and provides a counterexample for the converse. Hence, the g-ultrabarrelled groups include the locally compact and complete metrizable groups.

Iyahen claims to have proved the following : any closed group homomorphism (i.e., having closed graph) from a Hausdorff g-ultrabarrelled space (i.e., group) E into a separable, complete metrizable topological group F is continuous, and any closed group homomorphism from F onto E is open.

The first statement depends on his assertion, not proved in [12], that any group homomorphism from a g-ultrabarrelled group E into a separable group F is almost continuous. It is here that the flaw mentioned above occurs. If one attempts to prove this statement by the straightforward method of taking an arbitrary symmetric unit neighbourhood U in F and a sequence $\{U_n\}$ of symmetric unit neighbourhoods such that $U_1^2 \subseteq U$ and $U_{n+1}^2 \subseteq U_n$ for each n, and claiming that $Cl_E f^{-1}(U)$ is a group ultrabarrel with $\{Cl_E f^{-1}(U_n)\}$ as defining sequence, one quickly discovers that condition (ii) for a group ultrabarrel is evidently not satisfied without some further hypothesis on the groups. For example, if E is Abelian, condition (ii) is trivial, and $Cl_E f^{-1}(U)$ is indeed a group ultrabarrel and so a unit neighbourhood. As another alternative, we prove the result for the case where F is separable and has equal left and right uniformities.

<u>Lemma 2.5</u> Every group homomorphism from a Hausdorff g-ultrabarrelled group G into a separable group (H,u) with equal uniformities is almost continuous.

Proof: Let U be a symmetric unit neighbourhood in H which is fixed under the inner automorphisms of H. Since the set of all such neighbourhoods is fundamental in $\mathcal{V}(H)$, we can select a sequence of neighbourhoods $\{U_n\}$ having this property such that $U_1^2 \subseteq U$ and $U_{n+1}^2 \subseteq U_n$ for each n. Let v be the group topology on H having $\{U_n\}$ as its unit neighbourhood basis. Since (H,u) is separable, so is (H,v), since $v \leq u$. Clearly, (H,v) is first countable, and so it is second countable. Hence, every subspace of (H,v) is separable, by Theorem 7.3, Chapter VIII of [4]. In particular, f(G) is separable in the relative topology of (H,v).

We now show that $Cl_G f^{-1}(U)$ is a group ultrabarrel on G with $\{Cl_G f^{-1}(U_n)\}$ as defining sequence. First, all $Cl_G f^{-1}(U_n)$ are closed and symmetric, since the U_n have this property. Secondly, $(Cl_G f^{-1}(U_n))^2 \subseteq Cl_G\, f^{-1}(U_n)^2$

$\subseteq Cl_G f^{-1}(U_n^2) \subseteq Cl_G f^{-1}(U_{n-1})$. Finally, since the U_n

are fixed under the inner automorphisms of H, for any integer n and $a \in G$, we have

$$a\left[\mathrm{Cl}_G f^{-1}(U_n)\right]a^{-1} = \mathrm{Cl}_G a f^{-1}(U_n)a^{-1} \subseteq \mathrm{Cl}_G f^{-1}\left[f(a)U_n f(a)^{-1}\right]$$

$$= \mathrm{Cl}_G f^{-1}(U_n).$$

Similarly, $a^{-1}\left[\mathrm{Cl}_G f^{-1}(U_n)\right]a \subseteq \mathrm{Cl}_G f^{-1}(U_n)$. Hence, $a\left[\mathrm{Cl}_G f^{-1}(U_n)\right]a^{-1} = \mathrm{Cl}_G f^{-1}(U_n)$. Therefore, $\mathrm{Cl}_G f^{-1}(U)$ is a group ultrabarrel on G; moreover, $\{\mathrm{Cl}_G f^{-1}(U_n)\}$ is a basis for a group topology, say w, on G.

Since (f(G),v) is separable, there is a countable set $X \subseteq G$ such that f(X) is v-dense in f(G); that is, $f(G) = f(X)\left[U_m \cap f(G)\right]$, for each integer m. Taking inverse images of both sides, we obtain

$$\begin{aligned} G = f^{-1}[f(G)] &= f^{-1}\left[f(X)(U_m \cap f(G))\right] \\ &= (X)(\mathrm{Ker}\ f)(f^{-1}(U_m)) \\ &\subseteq Xf^{-1}(U_m)^2\ , \text{ since Ker } f \subseteq f^{-1}(U_m) \text{ for all } m, \\ &\subseteq Xf^{-1}(U_{m-1})\ . \end{aligned}$$

Hence, $G = X\left[\mathrm{Cl}_G f^{-1}(U_m)\right]$, for any m, and it follows that (G,w) is separable. Since G is a g-ultrabarrelled group, it follows that $\mathrm{Cl}_G f^{-1}(U) \in \mathcal{V}(G)$, and so f is almost continuous.

The proof of the following is along lines dual to the above.

<u>Lemma 2.6</u> Every homomorphism from a separable group (H,u) with equal uniformities onto a Hausdorff g-ultrabarrelled group G is almost open.

With these results in hand, we can now state the following modified version of Iyahen's result.

<u>Theorem 2.10</u> Let G be a Hausdorff g-ultrabarrelled group, H a separable locally compact or complete metrizable group with equal uniformities. Then any homomorphism $f : G \longrightarrow H$ with a closed graph is continuous, and any homomorphism $g : H \twoheadrightarrow G$ with closed graph is open.

Proof: This follows at once from the two lemmas above, and from [26; p. 213, R(a) and R(c)].

Finally, we can apply our own earlier results to obtain:

<u>Theorem 2.11</u> Let G be a Hausdorff g-ultrabarrelled group, H a separable $B_r(\mathcal{A})$ group with equal uniformities. If $f : G \longrightarrow H$ is almost open and has a closed graph, and if $f(G)$ is dense in H, then f is continuous.

Proof: By Lemma 2.5, f is almost continuous. The result then follows from Theorem 2.3.

<u>Theorem 2.12</u> Let G be a $B(\mathcal{A})$ group with equal uniformities, H a Hausdorff g-ultrabarrelled group.

Then any homomorphism $f : G \longrightarrow H$ with a closed graph is open.

Proof: By Lemma 2.6, f is almost open, and the result follows from Theorem 2.5.

CHAPTER 3

THE OPEN MAPPING AND CLOSED GRAPH THEOREM FOR EMBEDDABLE TOPOLOGICAL SEMIGROUPS

1. Introduction to Embeddability

In this section, we consider the properties of a particular class of Hausdorff topological semigroups, namely those which can be embedded as an open subspace of a topological group. The question of when such an embedding exists has been investigated by a number of authors: Gelbaum, Kalisch and Olmsted [5], Christoph [3], and Rothman [18]. The last-named author gives a concise statement of the relevant elementary facts which will be quoted verbatim.

"An outline of the embedding of a commutative semigroup with cancellation in its group of quotients follows. Let S be such a semigroup; then the set S x S is again such a semigroup when the binary operation is defined coordinatewise. In S x S, define $(a,b)R(c,d)$, where $(a,b), (c,d) \in S \times S$, iff $ad = bc$. It follows easily that R is a congruence relation on S x S (the symbol R will be used to denote this relation). Let G be the collection of equivalence classes modulo R; then, G is a group and is called the group generated by S. Let $\pi : S \times S \longrightarrow G$ be the natural mapping which assigns to each $(a,b) \in S \times S$ the equivalence class in G containing the element (a,b) (*). It is easy to see that π is a homomorphism. For b, any element of S, define $P : S \longrightarrow G$ by $P(x) = \pi(xb,b)$. It follows that P is a well defined isomorphism of S into G and is independent of the choice of b. The function P is the embedding of S into the group generated by S....

"When S is a commutative (topological) semigroup with cancellation, let S x S have the product topology, and the group generated by S the quotient topology; that is, O is open in G iff $\pi^{-1}(O)$ is open in S x S. The semigroup S is said to be embeddable in G iff G is a Hausdorff topological group and P is a homeomorphism onto P(S), with the relative topology induced by G."

Several other concepts introduced by Rothman merit mention here. First, he obtains an internal characterization for a semigroup to be embeddable in a topological group in the manner described above.

<u>Definition</u> A topological semigroup S is said to have Property F if $x,y \in S$ and V an open set containing x together imply there is an open set W, with $y \in W$, such that $xy \in \bigcap\{Vy' : y' \in W\}$ and $yx \in \bigcap\{y'V : y' \in W\}$.

(*) The unit of G is clearly the image under π of the diagonal of S x S.

His fundamental result is the following:

Theorem 3.1 Let S be a commutative topological semigroup with cancellation. A necessary and sufficient condition that S be embeddable as an open subset of G, the group generated by S, is that S have Property F.

The proof is complicated, involving several lemmas, and will not be reproduced here. For details, see [18].

Finally, the open mapping and closed graph theorem for complete metric, separable, commutative semigroups with cancellation and a further convergence property was considered in [5; Theorem 17], and Rothman has also proved a theorem in this direction. Because he constructs a map which is of great importance in what follows, we state and prove the following result, which is Theorem 4.1 of [18]:

Theorem 3.2 Let S,T be commutative, cancellative topological semigroups, embeddable in the topological groups G,H, respectively. Assume that S is locally compact (locally complete metrizable) and separable, and that T is a second category subset of H, a topological (metric) group. If $f : S \twoheadrightarrow T$ is a continuous homomorphism, then f is open.

Proof: Let π_1 be the natural map of $S \times S$ onto G, the group generated by S, and let $\pi_2 : T \times T \longrightarrow H$

be defined by $\pi_2(t_1, t_2) = t_1 t_2^{-1}$. Define $(f \times f)$ from $S \times S$ to $T \times T$ in the natural way, and consider the diagram

$$\begin{tikzcd} G \arrow[r, "h"] & H \\ S \times S \arrow[u, "\pi_1"] \arrow[r, "f \times f"'] & T \times T \arrow[u, "\pi_2"'] \end{tikzcd}$$

in which $h(x) = \pi_2(f \times f)\pi_1^{-1}(x)$. This is, of course, an abuse of notation, in that $\pi_1^{-1}(x)$ is an equivalence class of the elements of $S \times S$. However, it is easy to check that the choice of a representative from within this class is immaterial, and that h is indeed well-defined. It then follows that h is a continuous homomorphism, and h(G) contains T, a second category subset of H. By Theorems 5 and 6 of [13], h is open. But S is open in G, and it follows that f is also open.

<u>Notation</u> We pause to make certain notational conventions. Throughout this chapter, the letters S and T will denote commutative, cancellative topological semigroups which are embeddable in the Hausdorff topological groups G and H, respectively. The extension of the semigroup homomorphism $f : S \longrightarrow T$ constructed above will be denoted as $h : G \longrightarrow H$. These assumptions underlie all the results

which follow, unless the contrary is stated, and no further explicit mention of them will be made.

2. B-Completeness and Preliminary Results

Definition A topological semigroup S will be called B-complete if every continuous and almost open homomorphism from S onto a Hausdorff topological semigroup T is open, B_r-complete if every such map which is also one-to-one is open.

The following result shows that the property of B-completeness is possessed by a wide class of topological semigroups, namely those which are locally compact.

Theorem 3.3 Let X be a locally compact topological space, Y a Hausdorff space, $f : X \twoheadrightarrow Y$ a continuous, almost open mapping. Then f is open.

Proof: Let $x \in X$, N be a compact neighbourhood of x. Then f(N) is compact, and so is closed in Y. Then $f(N) = Cl_Y f(N)$, but the latter is a neighbourhood of f(x), since f is almost open. Hence, f is open.

The fact that a locally compact semigroup is B-complete then follows, as a special case.

Another class of semigroups, those which are locally complete metrizable, are also B-complete. To show this, however, we must first establish some results of a technical nature.

We first show that certain useful properties of the homomorphism $f : S \longrightarrow T$ are inherited by its extension $h : G \longrightarrow H$.

Lemma 3.1 Let $f : S \longrightarrow T$ be a homomorphism. If f is (a) continuous, (b) open, (c) almost continuous, (d) almost open, (e) one-to-one, (f) onto, or (g) endowed with the closed graph property, then $h : G \longrightarrow H$ has the same property.

Proof: (a) This is embodied in the proof of Theorem 3.2.

(b) Let $V \in \mathcal{V}(G)$; then, for any $s_0 \in S$, we have $(s_0V \cap S) \in \mathcal{V}(S; s_0)$, whence $f(s_0V \cap S)$ is open in T and contains $f(s_0)$. Now, $f(s_0V \cap S)$ is open in H, since T is, and $h(V) \supseteq f(s_0)^{-1}f(s_0V \cap S)$. Hence, $h(V) \in \mathcal{V}(H)$, and it follows that h is open.

(c) Let $x \in G$, $B \in \mathcal{V}(H; h(x))$. Then $h(x) = h(s_1s_2^{-1})$ $= f(s_1)f(s_2)^{-1}$, where $s_1, s_2 \in S$. Hence, there are neighbourhoods C_1, C_2 of $h(s_1)$, $h(s_2)$, respectively, such that $C_1C_2^{-1} \subseteq B$. Then

$$\mathrm{Cl}_G h^{-1}(B) \supseteq \mathrm{Cl}_G h^{-1}(C_1C_2^{-1}) \supseteq \mathrm{Cl}_G h^{-1}(C_1)h^{-1}(C_2)^{-1}$$

$$\supseteq \mathrm{Cl}_G h^{-1}(C_1)(\mathrm{Cl}_G h^{-1}(C_2))^{-1}$$

$$\supseteq Cl_S f^{-1}(C_1 \cap T)\ Cl_S f^{-1}(C_2 \cap T)^{-1}$$

Since f is almost continuous, $Cl_S f^{-1}(C_i \cap T)$ is a neighbourhood of s_i in S, for i = 1, 2, and so a neighbourhood of s_i in G. Hence, $Cl_G h^{-1}(B)$ contains a neighbourhood of $x = s_1 s_2^{-1}$, and so h is almost continuous. (The converse is also true, for $x \in S$, $V \in \mathcal{V}(T;\ f(x))$ implies $Cl_S f^{-1}(V) = S \cap Cl_G h^{-1}(V) \in \mathcal{V}(S;\ x)$.)

(d) Let $V \in \mathcal{V}(G)$. Then $Cl_H h(V) = Cl_H h(a^{-1}aV)$,

for any $a \in S$, whence $Cl_H h(V) = h(a^{-1}) Cl_H h(aV)$, and this in turn contains $h(a^{-1}) Cl_T f(aV \cap S)$. But $Cl_T f(aV \cap S)$ is in $\mathcal{V}(T;\ f(a))$, whence $h(a^{-1}) Cl_T f(aV \cap S)$ is in $\mathcal{V}(H)$.

Furthermore, $h(a^{-1}) Cl_T f(aV \cap S) \subseteq Cl_H h(V)$, so h is almost open.

(e) Let f be one-to-one, and let $x, y \in G$. Then

$x = s_1 s_2^{-1}$, $y = a_1 a_2^{-1}$, for some $s_1, s_2, a_1, a_2 \in S$.

Then, if $h(s_1 s_2^{-1}) = h(x) = h(y) = h(a_1 a_2^{-1})$, it follows that

$h(s_1 a_2) = f(s_1 a_2) = f(a_1 s_2) = h(a_1 s_2)$. Since f is one-to-one, $s_1 a_2 = a_1 s_2$, whence $s_1 s_2^{-1} = a_1 a_2^{-1}$, and x = y.

Hence, h is one-to-one.

(f) This is trivial, for S, T generate G, H algebraically.

(g) Let f have the closed graph property, and let $R(f)$ and $R(h)$ be the graphs of f and h, respectively. Then $R(f) \subseteq S \times T$, $R(h) \subseteq G \times H$, and $R(f) \subseteq R(h)$.

Let $(a,y) \in Cl_{G\times H}R(h)$. Then, for any $U \times V$ in $\mathcal{V}(G \times H)$, $(aU \times yV) \cap R(h) \neq \emptyset$. Now let $s_o \in S$; then $(s_oU \times h(s_oa^{-1})yV) \cap R(h) \neq \emptyset$, for $(s,f(s)) \in (aU \times yV)$ implies $(s_oa^{-1}s, h(s_oa^{-1}s))$ is a point of

$$(s_oU \times h(s_oa^{-1})yV) \cap R(h).$$

Since S is open in G, we can assume that $U = s_o^{-1}B$, where $B \in \mathcal{V}(S;\ s_o)$; similarly, we may assume that $V = f(s_o)^{-1}C$, where $C \in \mathcal{V}(T;\ f(s_o))$. Then, for any such B and C,

$$(s_os_o^{-1}B \times h(s_oa^{-1})yf(s_o)^{-1}C) \cap R(h) \neq \emptyset,$$

whence $(B \times h(a^{-1})yC) \cap R(f) \neq \emptyset$. Then for each $B \times C$ in $\mathcal{V}(S;\ s_o) \times \mathcal{V}(T;\ f(s_o))$, there exists $s_{B\times C} \in B$ such that $f(s_{B\times C}) \in h(a)^{-1}yC$. This yields a net

$$\mathcal{B} = \left\{ s_{B\times C} : B \times C \in \mathcal{V}(S;\ s_o) \times \mathcal{V}(T;\ f(s_o)) \right\},$$

where $f(s_{B\times C}) \in h(a)^{-1}yC$. Since $\bigcap \mathcal{V}(T;\ f(s_o)) = \left\{f(s_o)\right\}$, we have $f(\mathcal{B})$ converging to $h(a)^{-1}yf(s_o)$. Clearly, $\mathcal{B}$ converges to s_o, and so $f(s_o) = h(a)^{-1}yf(s_o)$, since $R(f)$

is closed. Then $y = h(a)$, and so $R(h)$ is closed.

<u>Remarks</u> (1) Since all our semigroups are presumed commutative, the associated groups are also commutative, for, if $s_1, s_2 \in S$, then $s_1^{-1}s_2s_1 = s_1^{-1}s_1s_2 = s_2$, whence $s_1^{-1}s_2s_1s_1^{-1} = s_2s_1^{-1}$. Hence, $s_1^{-1}s_2 = s_2s_1^{-1}$, and it follows that G is Abelian.

(2) As for topological properties, we will wish to consider those of a certain nature; namely, if S is a topological semigroup with property (P) and if S is embeddable in a topological group G, then G also has property (P).

It suffices that the category of topological semigroups with property (P) be right fitting with respect to continuous, open mappings and closed with respect to finite products. Many such properties are of a "local" nature: first countability, local metrizability, local connectedness, local compactness. Others include compactness, connectedness, second countability, and separability.

(3) There is some inconsistency in the use of the term "topological invariant" with respect to complete metric spaces. Some authors, [7; p. 81] and [21; p..140], cite counterexamples to show that a homeomorphism does not preserve the complete metric property. Others [20; p. 37]

and [4; p. 295], for example, maintain that this property is indeed a topological invariant. The latter author is more explicit; we paraphrase his Theorem 2.5(1), Chapter XIV: if X is a complete metric space, Y a topological space, and there is a homeomorphism from X onto Y, then there is a metric on Y which generates the topology of Y and with respect to which Y is a complete metric space. Fortunately, this sense is sufficient for our purposes, and it is in this sense we shall use the term.

(4) Pursuant to Remark (2), it is evidently not the case that B-completeness is transmitted from a semigroup to its associated group, in general. This will necessitate the addition of certain assumptions to subsequent theorems. The best we can state in this direction is the following, whose proof is a trivial consequence of Lemma 3.1.

Lemma 3.2 Let S be a B-complete semigroup, T a Hausdorff semigroup, $f : S \twoheadrightarrow T$ a continuous, almost open homomorphism. Then the induced map $h : G \twoheadrightarrow H$ is open.

Having established these preliminaries, we proceed to prove that another large class of semigroups has the property of B-completeness.

<u>Theorem 3.4</u> A locally complete metrizable semigroup is B-complete.

Proof: Let S be such a semigroup, subject to the conventions outlined in Section 3.1. Since S is embeddable in its associated group G, it is open therein. Then, for any $s_o \in S$ and any complete metrizable neighbourhood V in $\mathcal{V}(S; s_o)$, $s_o^{-1}V$ is a complete metrizable unit neighbourhood in G, since translations are homeomorphisms [18; Lemma 3.1] and complete metrizability is a topological invariant in the sense described in Remark (3) above. Then G is a locally complete metrizable group, and so a $B(\mathcal{A})$ group, by Theorem 1.2.

Let $f : S \twoheadrightarrow T$ be a continuous, almost open homomorphism. It follows from Lemma 3.1 that the induced map $h : G \longrightarrow H$ is continuous, almost open, and onto. Hence, h is open. Since S, T are open in G, H, respectively, and $h|S = f$, it follows that f is open, and that S is B-complete.

In a manner analagous to groups, we may define a condition weaker than that of B-completeness for semigroups. If $\mathcal{L}$ is a category of (embeddable!) topological semigroups, we say that S is a $B(\mathcal{L})$ or $B_r(\mathcal{L})$ semigroup if every continuous and almost open homomorphism from S onto a semigroup in $\mathcal{L}$ is open. The following result illustrates this idea.

<u>Theorem 3.5</u> Let $\mathcal{C}_1$ be the category of first countable semigroups. Then every locally countably compact, and hence every countably compact, semigroup is a $B(\mathcal{C}_1)$ semigroup.

Proof: Let S be a locally countably compact, T a first countable semigroup, $f : S \longrightarrow\!\!\!\!\to T$ a continuous, almost open homomorphism. Let $x \in S$, and let V be a countably compact neighbourhood of x. Then f(V) is a countably compact subset of T, and so it is closed in T, by Theorem 3.6, Chapter XI of [4]. But $Cl_T f(V) \in \mathcal{V}(T; f(x))$, since f is almost open. Then $f(V) \in \mathcal{V}(T; f(x))$, and so f is open.

The above result is true, of course, even without our embeddability assumptions.

3. Open Mapping and Closed Graph Theorems

We now prove a result analagous to Theorem 1 of [10], which we will apply in a variety of situations. Let P be a property which is transmitted from an embeddable topological semigroup to its associated topological group. Let $\mathcal{C}$ be the category of Hausdorff topological semigroups with property P, and $\mathcal{C}^*$ that of topological groups with the same property. Further assume that $\mathcal{C}^*$ is right fitting with respect to continuous, almost open homomorphisms.

Theorem 3.6 Let $S \in \mathcal{B}$ and T be a $B_r(\mathcal{B})$ semigroup. Then if H is a $B_r(\mathcal{B}^*)$ group and $f : S \longrightarrow T$ is almost continuous, almost open, and has a closed graph, then f is continuous.

Proof: By Lemma 3.1, the extension of f, the homomorphism $h : G \longrightarrow H$ is also almost continuous, almost open, and has closed graph. By the assumptions above, $G \in \mathcal{B}^*$. Then, by Theorem 1 of [10], h is continuous, and so is its restriction $h|S = f$.

Theorem 3.7 Let S be any Hausdorff semigroup, T a B_r-complete semigroup. If H is a $B_r(\mathcal{A})$ group and if $f : S \longrightarrow T$ is almost continuous and has closed graph, then f is continuous.

Proof: By Lemma 3.1, h is almost continuous and its graph is closed. Then h is continuous, by Theorem 2 of [10], and so f is continuous.

Remark It has already been demonstrated by means of a counterexample (Example 1.4) that the portion of Proposition 31.7 of [8] which pertains to $B_r(\mathcal{B})$ groups is false. Theorems 32.8 and 32.9, which depend on this proposition, must therefore be amended. It suffices to make the slightly stronger assumption that E be a $B(\mathcal{A})$ group; this category is closed under quotients, and the arguments of [8] become valid.

<u>Theorem 3.8</u> Let S be a B-complete semigroup, T any Hausdorff semigroup. If $f : S \twoheadrightarrow T$ is almost open and has a closed graph, and if G is a $B(\mathcal{Q})$ group, then f is open.

Proof: Since H is Hausdorff and $h : G \twoheadrightarrow H$ is almost open and has a closed graph, it follows from the amended version of Theorem 32.8 of [8] that h is open. Therefore, f is open.

Before proceeding to open mapping and closed graph theorems of a more specialized type, we first establish some further preliminary results.

<u>Lemma 3.3</u> If a topological space X has the property that every point has a neighbourhood which is a Baire space, then X is a Baire space.

Proof: Suppose the contrary. Then, for some countable closed covering $\{A_n : n \in \underline{N}\}$ of X, $\mathrm{Int}_X A_n = \emptyset$, for all n. Let $x_0 \in X$, and let B be a neighbourhood of x_0 which is a Baire space. Then $\{A_n \cap B : n \in \underline{N}\}$ is a closed cover for B of the required cardinality. However, $\mathrm{Int}_B(A_n \cap B) = B \cap \mathrm{Int}_X A_n = \emptyset$, for all n. This contradicts the assumption that B is a Baire space. Hence, X is a Baire space.

The above is found in [4], Chapter XI, Exercise 10.4. The proof is included for completeness. From this, we obtain the following very easily.

Lemma 3.4 If S is a Baire semigroup, then the associated group G is also endowed with the Baire property.

Proof: Since S is open in G, for any $x \in G$, $s \in S$, we see that $xs^{-1}S$ is a neighbourhood of x which is a Baire space, since Baire spaces are invariant under continuous, open surjections. By Lemma 3.3, G is a Baire space.

Proposition 3.1 Any homomorphism f of a separable topological semigroup S onto a Baire semigroup T is almost open.

Proof: Since separability is preserved under finite products and continuous images, G is separable. By Lemma 3.4, H is a Baire group. Then, by Proposition 32.11 of [8], h is almost open. Now, if $B \in \mathcal{V}(S; x_0)$, then $Cl_T f(B) = T \cap Cl_H f(B) \in \mathcal{V}(T; f(x_0))$. Hence, f is almost open.

Proposition 3.2 Every homomorphism f of a Lindelöf topological semigroup S onto a Baire semigroup T is almost open.

Proof: Let $B \in \mathcal{V}(S; x_0)$. Then $x_0^{-1}B \in \mathcal{V}(G)$, and so $S = \bigcup\{S \cap sx_0^{-1}B : s \in S\}$. Since S is a Lindelöf space, there exists a countable subset $\{s_i : i \in \underline{N}\}$ of S such that

$S = \bigcup\{S \cap s_i x_o^{-1}B : i \in \underline{N}\}$. Hence, $T = \bigcup\{f(S \cap s_i x_o^{-1}B) : i \in \underline{N}\}$ $\subseteq \bigcup\{T \cap h(s_i x_o^{-1}B) : i \in \underline{N}\}$. Since the reverse inclusion is trivial, equality holds. Because T has the Baire property, $Cl_T(T \cap h(s_n x_o^{-1}B))$ has non-void interior, for some $n \in \underline{N}$. Hence, $[h(x_o s_n^{-1})Cl_T(T \cap h(s_n x_o^{-1}B))] \cap T \subseteq Cl_T h(B) = Cl_T f(B)$ contains an open set. Therefore, f is almost open.

<u>Proposition 3.3</u> Let S be a Baire semigroup, T a separable semigroup, and $f : S \longrightarrow T$ a homomorphism. Then f is almost continuous.

Proof: By Lemma 3.4, G is a Baire group; H is separable since T has this property. Hence, $h : G \longrightarrow H$ is almost continuous, by Proposition 2.1. Now, if $B \in \mathcal{V}(T; f(x_o))$, then $Cl_S f^{-1}(B) = S \cap Cl_G h^{-1}(B)$, and the latter is in $\mathcal{V}(S; x_o)$. Therefore, f is almost continuous.

<u>Proposition 3.4</u> Every homomorphism f of a Baire semigroup S into a Lindelöf semigroup T is almost continuous.

Proof: Let $a \in S$, $B \in \mathcal{V}(T; f(a))$. Then $f(a)^{-1}B \in \mathcal{V}(H)$. Let $V \in \mathcal{V}(H)$ such that $V^{-1}V \subseteq f(a)^{-1}B$. Since T is Lindelöf, it follows that $Cl_T f(S)$ also has this property; now,

$$\begin{aligned} Cl_T f(S) &= T \cap Cl_H f(S) \\ &= T \cap \bigcap\{f(S)W : W \in \mathcal{V}(H)\} \end{aligned}$$

$$= \bigcap \left\{ f(S)W \cap T \ : \ W \in \mathcal{V}(H) \right\}.$$

In particular, $Cl_T f(S) \subseteq f(S)V \cap T$. Since $Cl_T f(S)$ is Lindelöf, it follows that

$$Cl_T f(S) \subseteq \bigcup \left\{ f(s_i)V \cap T \ : \ i \in \underline{N} \right\},$$

for some countable subset $\{s_i\}$ of S.

Then $S \subseteq \bigcup \left\{ f^{-1}\left[f(s_i)V \cap T \right] \ : \ i \in \underline{N} \right\}$, and it follows from the fact that S has the Baire property that, for some $n \in \underline{N}$,

$$Int_S Cl_S f^{-1}\left[f(s_n)V \cap T \right] \neq \emptyset .$$

Hence, there exists $p \in S$ such that

$$Cl_S f^{-1}\left[f(s_n)V \cap T \right] \in \mathcal{V}(S; \ p) ,$$

whence there is some open $U \in \mathcal{V}(G)$ such that

$$U \subseteq p^{-1} Cl_S f^{-1}\left[f(s_n)V \cap T \right].$$

Then $U \subseteq (Cl_S f^{-1}\left[f(s_n)V \cap T \right])^{-1}(Cl_S f^{-1}\left[f(s_n)V \cap T \right])$

$$\subseteq Cl_S\left[(f^{-1}\left[f(s_n)V \cap T \right])^{-1}(f^{-1}\left[f(s_n)V \cap T \right]) \right]$$

$$\subseteq Cl_G h^{-1}\left[(f(s_n)V \cap T)^{-1}(f(s_n)V \cap T) \right]$$

$$\subseteq Cl_G h^{-1}(V^{-1}V) \subseteq Cl_G h^{-1}\left[f(a)^{-1}B \right].$$

Now, $(aU \cap S) \in \mathcal{V}(S; \ a)$, and

$$aU \cap S \subseteq (aCl_G h^{-1}\left[f(a)^{-1}B \right]) \cap S$$

$$\subseteq (Cl_G h^{-1}\left[f(a)f(a)^{-1}B \right]) \cap S$$

$$= Cl_S f^{-1}(B) .$$

This shows that f is almost continuous at the point a. Since this point was chosen arbitrarily, f is almost continuous.

We now turn to some versions of the open mapping theorem.

Corollary 3.1 Let S be a separable (or Lindelöf) B-complete semigroup, T a Baire semigroup. Then any continuous homomorphism $f : S \twoheadrightarrow T$ is open.

Proof: By Propositions 3.1 and 3.2, such a homomorphism is almost open, and so is open, since S is B-complete.

Corollary 3.2 Let S be separable (or Lindelöf) and locally complete metrizable, T a Baire semigroup. Then any continuous homomorphism $f : S \twoheadrightarrow T$ is open.

Proof: This is a consequence of Corollary 3.1, since every locally complete metrizable semigroup is B-complete, by Theorem 3.4.

Corollary 3.3 Let S be a separable (or Lindelöf), locally complete metrizable semigroup, T locally complete metrizable. Then any continuous $f : S \twoheadrightarrow T$ is open.

Proof: Since a complete metrizable space is a Baire space, by Theorem 0.2, T is locally a Baire space, and so is itself a Baire space, by Lemma 3.3. The result then follows from Corollary 3.2.

<u>Corollary 3.4</u> Let S be a locally compact, second countable semigroup, T a Baire semigroup. Then any continuous homomorphism $f : S \twoheadrightarrow T$ is open.

Proof: Every second countable space is Lindelöf; hence, f is almost open, by Proposition 3.2. Then f is open, since every locally compact semigroup is B-complete, by Theorem 3.3.

<u>Corollary 3.5</u> Let S and T be locally compact semigroups, with S second countable. Then any homomorphism $f : S \twoheadrightarrow T$ which is continuous is also open.

Proof: A locally compact space is a Baire space, by Theorem 0.2. This result then follows from Corollary 3.4.

<u>Corollary 3.6</u> Let S be a separable (or Lindelöf) B-complete semigroup, T a locally compact Hausdorff semigroup. Then any continuous homomorphism $f : S \twoheadrightarrow T$ is open.

Proof: This follows at once from Corollary 3.1, since T is locally compact and hence Baire.

<u>Corollary 3.7</u> Let S be a separable (or Lindelöf), locally complete metrizable semigroup, T a locally compact Hausdorff semigroup. Then any continuous homomorphism $f : S \twoheadrightarrow T$ is open.

Proof: This follows at once from Corollary 3.6 and Theorem 3.4.

<u>Corollary 3.8</u> Let S be a Baire semigroup, T a separable (or Lindelöf) semigroup such that H is a $B_r(\mathcal{A})$ group. If $f : S \longrightarrow T$ is a homomorphism with closed graph, then f is continuous.

Proof: Since S is a Baire space, so is G, and H is separable since T has that property. Then $h : G \longrightarrow H$ is almost continuous, by Propositions 3.3 and 3.4. Then, by Theorem 3.7, h is continuous, and so f is continuous.

<u>Corollary 3.9</u> Let S be a Baire semigroup, T a separable (or Lindelöf), locally complete metric semigroup. Then any homomorphism $f : S \longrightarrow T$ with a closed graph is continuous.

Proof: Since T is locally complete metric, H has this property, and so is a $B(\mathcal{A})$ group, by Theorem 1.2. The result then follows from Corollary 3.8.

<u>Corollary 3.10</u> Let S and T be locally complete metrizable semigroups, with T separable (or Lindelöf). Then any homomorphism $f : S \longrightarrow T$ with closed graph is continuous.

Proof: A locally complete metrizable space has the Baire property. Hence, this follows from Corollary 3.9.

<u>Corollary 3.11</u> Let S be a Baire semigroup, T a locally compact, second countable semigroup. Then any homomorphism $f : S \longrightarrow T$ with a closed graph is continuous.

Proof: Since T is locally compact, so is H, whence H is a $B(\mathcal{A})$ group. Furthermore, every second countable space is separable. It then follows from Corollary 3.8 that f is continuous.

<u>Corollary 3.12</u> Let S be a locally compact semigroup, T locally compact and second countable. Then any homomorphism $f : S \longrightarrow T$ with a closed graph is continuous.

<u>Corollary 3.13</u> Let S be a locally compact semigroup, T a separable semigroup such that H is a $B_r(\mathcal{A})$ group. Then any homomorphism $f : S \longrightarrow T$ with closed graph is continuous.

Proofs of Corollaries 3.12 and 3.13: These results follow from Corollaries 3.11 and 3.8, respectively, since every locally compact space is a Baire space.

<u>Remark</u> These results have essentially followed the pattern of Section 32 of [8]. However, our results gain nothing in generality by proceeding to consider the compact case, for a compact, cancellative topological semigroup is a topological group [24].

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