TOPICS ON THE NUCLEON-NUCLEON INTERACTION

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NUCLEON-NUCLEON INTERACTION

By

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SCOPE AND CONTENTS:

Some aspects of the nucleon-nucleon interaction are investigated. Phase-equivalent families of rank-two separable potentials are derived by inverting the on-shell phase-shifts. The off-shell properties of these potentials are examined and shown to be well behaved. These and additional pairs of phase-equivalent local and separable potentials are included in a proton-proton bremsstrahlung calculation. In particular, the off-shell dependence of the cross sections is shown to be small.

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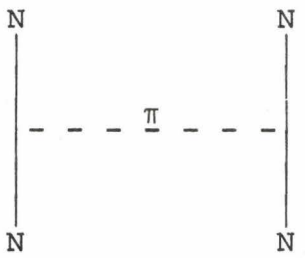
CHAPTER I

INTRODUCTION

An ultimate goal of theoretical nuclear physics is to understand nuclear structure and reactions in terms of a fundamental strong-interaction nucleon-nucleon potential. Thus in analogy with atomic and molecular physics, it was hoped that a knowledge of the two nucleon potential between all pairs of nucleons, incorporated into the Schroedinger equation, would in principle completely describe the world of nuclear physics. However, unlike atomic physics in which the basic interactions are of the well-understood electromagnetic nature, it soon became apparent that the strong-interaction was an extremely complex phenomenon, an understanding of which required prerequisite solutions of the more basic meson-nucleon interactions. In fact, it has been currently suggested that a description in terms of two-body forces alone may be somewhat optimistic, and that we may need three-body and possible higher order interactions. Because of the complexity of the problem, nuclear physicists soon adopted a more phenomenological approach and became somewhat divorced from the more fundamental considerations of the particle physicist. These two outlooks have coexisted over the past few decades with the nuclear physicist incorporating into his phenomenology any well established facts

from the realm of particle physics when possible.

After Yukawa (1935) it is generally believed that the nuclear force arises from the exchange of various mesons between the two nucleons. The simplest process is the one-pion-exchange mechanism, designated graphically as:



The famous one-pion-exchange potential (OPEP) is derived from the one-pion-exchange scattering amplitude using the Feynman rules for perturbative field theory, in the Born approximation. The resulting amplitude is a function of the momentum transfer between the nucleons. Taking the Fourier transform with respect to this variable produces the OPEP

$$V_{OPEP}(r) = f^2(m_\pi c^2)\underset{\sim}{\tau}_1\cdot\underset{\sim}{\tau}_2[\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2+S_{12}(1+\frac{3}{x}+\frac{3}{x^2})]e^{-x}/3x \quad \text{I-1.1}$$

where

$$x = m_\pi Cr/\hbar \quad \text{and} \quad f^2 \underset{\sim}{\approx} 0.08 \quad .$$

This has the asymptotic form

$$V_{OPEP}(r) \sim e^{-\mu r}/r \quad ,\text{I-1.2}$$

where $\mu = m_\pi C/\hbar$.

As was shown by Wick (1938), there is a relation between the range of the potential and the mass of the exchanged mesons:

$$\text{Range} \approx \hbar/[C\Sigma m] \qquad \text{I-1.3}$$

where Σm represents the total mass of the exchanged mesons. This is why the long range part of the potential is predomenantly determined by the OPEP, the pion with a mass of $\approx$ 138 MeV being the lightest meson. As the interaction distance is made shorter, exchanges of $2\pi(276)$, $3\pi(414)$, $\eta(549)$, $4\pi(452)$, ..., $\eta_{0}^{+}(700)$, $\rho(765)$, $\omega(784)$, ... will contribute. The numbers in parenthesis are the exchanged masses in MeV.

As has been indicated, the meson theoretic calculation for a one-meson exchange process is quite feasible, however the method becomes rapidly more complex for multi-meson exchanges. The two-pion-exchange potential has been investigated in detail by several authors (Miyazawa (1956), Cottingham and Vinh-Mau (1963)). The 3π exchange is prohibitively complex. An interesting feature of the multi-meson exchanges is the fact that the scattering amplitudes are generally a function of energy as well as momentum transfer. The resulting potentials are consequently non-local at short distances. The internucleon separation can be roughly divided into three regions according to the extent of theoretical reliability in each region (Tamagaki 1967): (1) the well-established outer region ($r \geq 2F$) where

the OPEP is dominant, (2) the intermediate region ($1F \lesssim r < 2F$) where one can get some qualitative insight from the field-theoretic two-pion-exchange and heavy meson exchanges, and (3) the innermost region ($r \lesssim 1F$) about which very little is known and which has been termed the core or phenomenological region.

An alternative approach which may be considered more phenomenological than fundamental, is the one-boson-exchange potential (OBEP). Here one considers the exchanges of only single mesons, namely, π, η, η_{0^+}, ρ, ω, ..., ϕ, ... and do not include any multi-meson exchanges. Here it is hoped that, for example in the 2π exchange, the pions are strongly correlated and can be simulated by single boson states. Hence the coupling constants for η_{0^+}, ρ, ω, etc. in this model should be regarded as phenomenological. How well this approach can be justified meson theoretically is of course open to question. However, several OBEP's, with a phenomenological short range repulsion, have been remarkably successful in fitting the two nucleon scattering data (Bryan and Scott 1967, Erkelenz et. al. 1969).

In the more phenomenological approaches to the nuclear potential, one assumes certain potential forms including a number of constants, and attempts to fit the two-body scattering and bound state data, by finding best values of the parameters. Of course, the potentials must satisfy the accepted invariance principles of; (1)

hermiticity, (2) translational and Galilean invariance; these require respectively that the potential be a function of the relative coordinate $\underset{\sim}{r} = \underset{\sim}{r}_1 - \underset{\sim}{r}_2$ and the relative momentum $\underset{\sim}{P} = \underset{\sim}{P}_1 - \underset{\sim}{P}_2$, (3) rotational invariance (conservation of angular momentum), (4) space reflection invariance (conservation of pairty), (5) time reversal invariance and (6) particle exchange invariance due to the indistinguishability of the two particles. Okubo and Marshak (1958) have shown that under their strong interaction invariances, the potential is restricted to the general form

$$V(r^2, L^2, P^2; \sigma_1, \sigma_2) = V_C + V_{SS}\, \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2 + V_T\, S_{12} + V_{LS}\, \underset{\sim}{L} \cdot \underset{\sim}{S}$$

$$+ V_Q\, Q_{12} + V_{PP} (\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{P})\ (\underset{\sim}{\sigma}_2 \cdot \underset{\sim}{P}) \quad .$$

Here r, $\underset{\sim}{P}$, $\underset{\sim}{L}$ are the interparticle distance, the momentum and the angular momentum operator. The spin operators S_{12}, $\underset{\sim}{L} \cdot \underset{\sim}{S}$ and Q_{12} are referred to as the tensor, spin orbit and quadratic spin orbit operators. The functions V_i in their most general form depend on r^2, L^2 and P^2. In many cases the L^2 and P^2 dependence is dropped, in which case the potential is referred to as being local.

In practice it is convenient to expand the Schroedinger equation into its partial waves, or angular momentum eigenstates. From semiclassical arguments it is easily shown that the ℓ'th partial wave is only sensitive to that region of the potential for which $r \gtrsim \frac{1}{2}\sqrt{\ell(\ell+1)}/k$. Thus at a given

energy the higher partial waves sense only the long range part of the interaction. This is commonly given by OPEP. As the energy increases each wave senses more and more of the short range part of the potential. At 10 MeV the scattering cross section is entirely dominated by the S-wave, while at 200 MeV it is necessary to include all waves $\ell \leq 5$.

An important feature of the phenomenological approach is the need for a strong short range repulsion. Very little is known about this core region since it corresponds roughly to region III of the internucleon separation. The core determines the higher energy phase-shifts but these are also unknown for energies above about 350 MeV. One of the first methods of assimilating the core was by means of an infinitely repulsive or hard-core for the short range portion of a local potential. By making the wave function zero inside and at the boundary of this core, it was ensured that the particles have zero probability of coming closer together than the hard-core radius. Some of the better known hard-core potentials are those of Brueckner, Gammel and Thaler (1958), Hamada-Johnston (1962) and the Yale potential (Lassila et. al. 1962). All these contain central, tensor and spin-orbit terms. The first of these has a hard-core radius of $\sim$ 0.4 F while the latter two have cores of $\sim$ 0.5 F with additional quadratic spin-orbit terms as well as the OPEP tail.

Several modifications to the hard-core have been made in an attempt to increase the binding energy of nuclear matter. These include the so-called soft-core potentials of Reid (1968) and Bressel-Kerman (1968) where the hard-cores are replaced respectively by Yukawas and finite repulsive square wells. Recently there have been the super-soft-core potentials of Sprung and Srivastava (1969).

Another type of interaction initiated to soften the core region is a special type of non-local interaction, the velocity dependent potential. These have been examined by various people including Green (1962) and Bhaduri and Preston (1964). Baker (1962) has shown the extent of equivalence between hard-core and velocity-dependent potentials.

The separable potential is another type of non-local interaction which has become increasingly popular in recent years. Although Lovelace (1964) has shown that the T-matrix is in fact separable in the vicinity of resonances and bound states, the only real reason for its use is its extreme simplicity. When placed in the partial wave Lippmann-Schwinger equations, the kernel is degenerate and the equations can be solved algebraically. Separable potentials are particularly convenient in the three-body problem, the Fadeev equations reducing to a set of one dimensional integral equations (Mitra and Bashin 1963). The Tabakin rank-two potential (1964) has been widely used

although not as good a fit to the data as some of its local counterparts. More recently Mongan (1968,1969) has fitted the data more accurately using different form factors. The objections most commonly raised to separable forms is that OPEP is not separable, not even non-local, and should represent the long range part of any realistic potential. It would probably be more appropriate to take only the short-range part of the interaction as being separable. Harrington (1965) has shown that one can superimpose the Coulomb potential on such a short range separable form and still solve the Lippmann-Schwinger equations in a closed form. Although the least fundamental of any of the potentials, it seems reasonable to assume that separable potentials are a valid computational convenience in so far as they reproduce the phase-shifts and have no strange off-shell behaviour. In this sense, they may be thought of as an approximation to some more realistic form for the interaction.

The end product of the phenomenological approach has been a host of potentials which all fit the experimental data, granted some better than others, but of radically different types. Even within a particular type there are variations in the size of the core and in the functional forms used to fit the data. From the point of view of two-body scattering all these potentials are very nearly equivalent. What happens if we incorporate several of these into a many-body calculation? Are they still equivalent?

It turns out, they can be quite different. In two-body scattering, the total energy and momentum before and after the collision are the same. This corresponds to a so-called on the energy shell process. The scattering amplitude is proportional to the T-matrix, $\langle \underline{p} | T(E) | \underline{k} \rangle$, where $|\underline{p}| = |\underline{k}| = \sqrt{E}$, that is the on-shell T-matrix. Many-body calculations however generally depend on fully off-shell matrix elements where $|\underline{p}| \neq |\underline{k}| \neq \sqrt{E}$. Further, once a particular potential has been adopted, the T-matrix is completely defined on- and off-shell. It is then not surprising that some of these potentials have an improper off-shell behaviour, being fitted solely to the on-shell data. It would be surprising if it were otherwise.

The need thus arises for examining the off-shell behaviour of a potential and fixing it experimentally if possible. To accomplish this, theorists turn to the many-body problem. Unfortunately, many-body calculations are fraught with the additional complications of the unsolvability of a quantum mechanical system of more than two bodies. Of course, the many-body problem is solved, but it is in terms of models and approximations, the precise consequences of which are hard to estimate. Possibly the best many-body system to approach in this manner is that of infinite nuclear matter. Here there are no boundary conditions or Coulomb effects and the theory as developed by Brueckner, Goldstone and Bethe is well established (Day 1967 ,

Rajaraman et. al. 1967), Sprung 1970). Even here, however, several approximations have been made. Possibly of more significance, a recent investigation by Bhaduri, Nogami and Ross (1970) estimates that a three-body force may be responsible for an appreciable contribution to the binding energy.

It was for the above reasons that physicists turned to nucleon-nucleon bremsstrahlung as a probe for the off-shell dependence of the scattering amplitude. These reactions

$$P+P \rightarrow P+P+\gamma$$

and

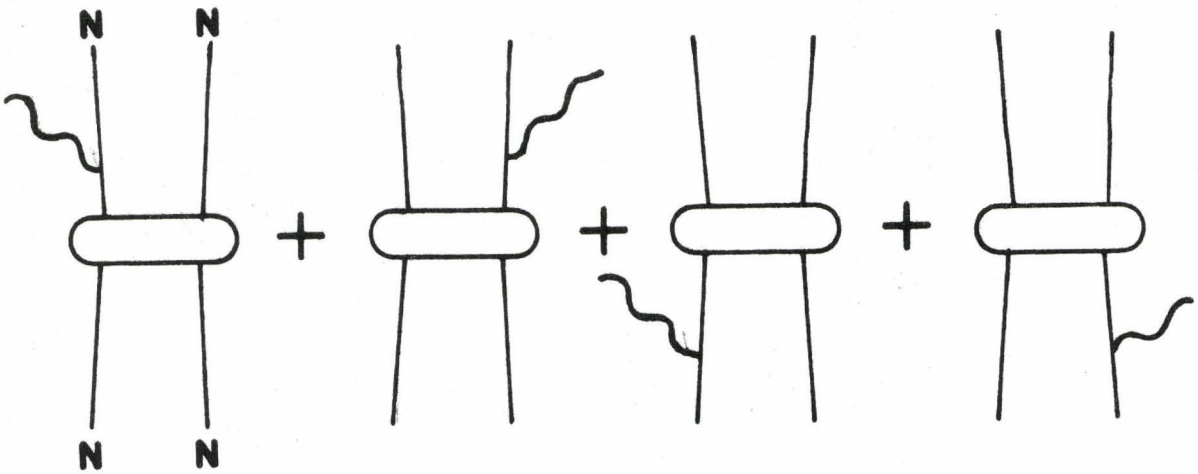
$$n+P \rightarrow n+P+\gamma \quad ,$$

represent the inelastic scattering of two nucleons with a photon being given off in the process. This system shows promise because there are no three-body nuclear interactions to worry about and the electromagnetic interaction is very well understood. Also the weakness of the electromagnetic interaction means that it can be treated to first order in a perturbative expansion. The bremsstrahlung cross section depends on half-off-shell matrix elements. This is quite general, since a fully off-shell element can be expanded in terms of half-off-shell elements.

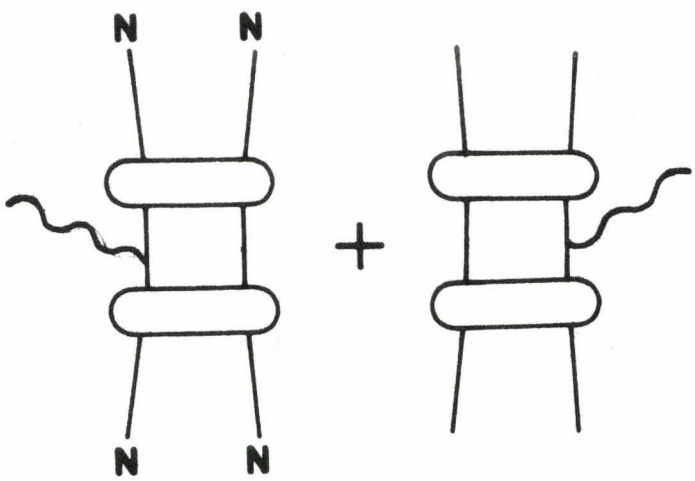
The first recent calculation was performed by Sobel and Cromer (1963). This was prior to any experimental results and gave cross sections for the Brueckner-Gammel-Thaler (BGT), and Yale potentials differing by as much as a

factor of 2.5. This was a bit disturbing, although it was known that BGT did not fit the scattering data that well. The geometry used in this calculation, and in by far the majority of the succeeding experimental and theoretical studies is one in which all the kinematic variables are coplanar, the two final state nucleons making equal angles with the direction of the incident beam. This is referred to in the literature as the Harvard geometry.

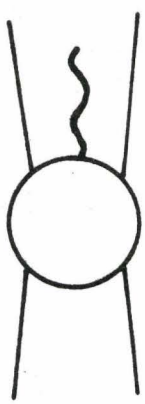
Theoretically, it has become conventional to classify the different terms in the total scattering amplitude, M, according to the corresponding Feynman diagrams which contribute. These are illustrated in figure 1.1. The simplest terms are the single scattering or pole terms, the 4 diagrams 1.1a, in which the photon is emitted from an external line. The internal scattering contribution consists of diagrams b, where the photon is emitted from a nucleon in an intermediate state and has become known as the rescattering contribution, plus diagram c which represents the exchange contribution, that is, the radiation emitted from mesons in internal lines. This last contribution in p-p-γ must arise from the exchange of at least two mesons and has been estimated (Ueda 1965) to be smaller than about 2% for protons with an energy of up to 200 MeV. This is in contrast to the situation in n-p-γ where apparently this radiation from exchanged charged mesons has to be taken into account (Baier et. al. 1969).



(a) Pole Contribution



(b) Rescattering Contribution



(c) Exchange Contribution

FIGURE I.1

CONTRIBUTIONS TO THE NUCLEON-NUCLEON SCATTERING AMPLITUDE

The first experimental data was published by the Harvard group of Gottschalk et. al. (1965) for an incident proton energy of 158 MeV. These results were smaller than the theoretical predictions by a factor of 4 for the Yale potential and by a factor of 10 for the BGT potential. This was a disturbing result, particularly for the Yale potential, and did much to stimulate even further interest in the field. Further experimental data by Warner (1965) at Manitoba for an energy of 48 MeV gave even poorer agreement, differing by factors of about 12 for any of the more reasonable potential models calculated by Signell and Marker (1966), using the theory of Sobel and Cromer.

Subsequently Signell and Marker (1967) uncovered a series of errors in the initial theory of Sobel and Cromer which reduced the cross sections at 158 MeV by approximately the factor 4 needed to agree with experiment. The results at 48 MeV were still out by a factor of 6. An independent calculation by Pearce et. al. (1967) however now gave a qualitative fit to all of the data. The dilemma was resolved by Signell and Marker (1967) and Drechsel and Maximon (1968). It has to do with the rescattering contribution which had been estimated at less than 10% by Sobel and Cromer, and subsequently ignored by the more recent calculations. The reason for Pearce's apparent success at low energies stemed from their choice of the centre of mass system in which to do the calculation. The difficulty is however more one of

gauge invariance than of a choice of frame in which to do the calculation. It is convenient to choose the transverse gauge in the frame in which the calculation is being carried out. It turns out that the rescattering contribution is small in the gauge which is transverse in the centre of mass. Although the Lorentz transformation taking us from the centre of mass system to the lab system does not change the contributions from each of the different diagrams, it no longer leaves the photon transversely polarized. Thus, choosing the transverse gauge in the lab frame is equivalent to a Lorentz transformation followed by a regauging. It is under the regauging that the contributions from the individual diagrams change and it is no longer proper to neglect the rescattering contribution, especially at low energies.

There have been some other less drastic improvements. Drechsel and Maximon (1968) have showed how to correct for non-coplanar events; for higher energies it is an improvement to include higher angular momentum states; Brown (1967, 1968) has derived a means for directly calculating the rescattering contribution for a potential model. This is however still an arduous task since unlike the pole terms one has to sum over all intermediate states. Signell and Marker (1968) have estimated the effect of including the Coulomb interaction and McGuire (1970) has investigated relativistic corrections. In general, the latest calculations using potential models are in disappointingly good agreement with

experiment. The experimental results are not of sufficient accuracy to distinguish between the different models in use.

The sole claim to distinguishability of off-shell effects comes from McGuire and Cromer (1969) who have calculated p-p-γ cross sections at 99 MeV for proton exit angles of 25°. They include only the pole terms and find that the calculated cross section where off-shell behaviour differs from that with the off-shell quantities replaced by their on-shell limits, differs by 4½ times the experimental errors of Sannes et. al. (1970) with experiment favouring the off-shell values. This is however not calculated on the basis of any potential model but directly from the experimental phase-shifts and the off-shell dependence taken from OPEP. This is an extreme test for off-shell dependence. He has neglected rescattering terms, contributions from exchange currents and Coulomb effects. The results are rather encouraging but the calculation is not precise enough to study purely off-shell effects.

Experimentally, there has been many results since the first data by Gottschalk, they are almost exclusively done in the Harvard geometry with the final state protons making angles of anywhere from 25° to 40° with the direction of the incident beam. The energies vary from an experiment at 3.2 MeV to the 204 MeV data of Rothe et. al. (1966,1967). The experimental errors are typically about 20%. The one real improvement has been the measured results of Sannes

et. al., mentioned above, who succeeded in reducing the error to about 5%. There are presently (McGuire 1970) experiments in progress at 46 MeV (Manitoba) and at 135 MeV (Harwell), with a comparable accuracy.

The underlying reason behind the apparent model independence of the p-p-γ cross sections is partially understood by means of Low's theorem (1958). He proved that for an expansion of the total bremsstrahlung amplitude, M, as a power series in the photon momentum K,

$$M = A/K + BK^{o} + CK + \ldots \quad ,$$

the term A is independent of off-shell effects and is derived entirely from the pole contributions. Further, off-shell effects from the pole and internal scattering contributions exactly cancel in B to make it an on-shell quantity. The off-shell effects consequently arise from terms of $0(K)$ or higher. This theorem has been useful as a check on the numerical validity of any calculation to $0(K^{o})$ and it permits the calculation of the internal scattering amplitude to lowest order in K, from a knowledge of the pole terms. Nyman (1967) has used Low's theorem to calculate p-p-γ cross sections to $0(K^{o})$. This type of evaluation is referred to as model independent, being independent of any off-shell terms. Although it has been shown that these results differ from experiment as one goes far off the energy shell (Signell and Marker 1968), Nyman gives surprisingly good

agreement with the data up to energies of about 160 MeV.

A problem obviously closely related to p-p-γ which we as yet haven't mentioned is that of n-p-γ.

A neutron-proton bremsstrahlung cross section has been less extensively studied than the p-p-γ. Experimentally, the cross sections are much more difficult to measure. The only direct measurement to date is that of Brady, Young and Badrinthan (1968) at 200 MeV. Although the cross sections are larger than for p-p-γ , the experimental error is about 25%. Potential model calculations have been done by Pearce et. al. (1967) and by Brown (1970). Pearce ignores all internal radiation effects and includes only the S, P, and D angular momentum states. Brown's calculation, using the Bryan-Scott and Hamada-Johnston potentials includes both the pole and rescattering terms, all evaluated by integrating the Schroedinger equation for all states $J \leq 4$. She has found that the rescattering contribution increases the cross section by a factor of 3 to 4. The model used is however not fully gauge invariant, due to the exchange character and momentum dependence of the nuclear potentials. All results are well within the rather large experimental error and do not distinguish in any way between the velocity dependent and hard core local potentials.

Other calculates of n-p-γ have been made by Nyman (1968), Baier et. al. (1968) and McGuire (1970). Nyman's result is based on the low energy theorem and does not contain any off-shell effects. The calculation of Baier et. al. is

covariant and gauge invariant; the nucleon interaction is included by means of a series of one-boson-exchange diagrams. It is not however unitary and does not fit the low energy data. McGuire's approach employs an off-shell extrapolation of the elastic phase shifts which is included through a modified one-pion exchange contribution. The internal scattering is included to lowest order in the photon momentum for which it occurs. Generally speaking, the n-p-γ cross section is more difficult to calculate since the distinguishability of the neutron and proton permits both isospin-singlet and triplet states. Also the photon angular distribution is no longer symmetric with respect to the beam direction and the importance of the internal scattering contribution makes it almost mandatory to evaluate this difficult term exactly. At present, the experimental data is scarce and imprecise. It appears hopeless to distinguish between potentials on this basis, at least until there is a considerable experimental improvement.

It appears then, that we are left with p-p-γ for which the cross section has been demonstrated to be nearly potential independent. Since it is a matter of small effects, any further theoretical calculations will have to be more exact to be meaningful. The need of separating the off-shell variations from the purely on-shell phase shift dependence has been mentioned by Signell (1968). That is, in order to distinguish purely off-shell effects it becomes necessary

to do a more accurate calculation in which the potentials are truly on-shell equivalent. It is thus desirable to have phase-equivalent potentials or classes of potentials of the various types; local, velocity dependent, separable,....; with as wide an off-shell variation as is feasible. Only then can we hope to distinguish the purely off-shell effects; and ideally if experiments and theory can be performed with sufficient precision, it would be possible to experimentally determine the complete potential, on- and off-shell. So far, no systematic analysis of this nature has been done.

This situation immediately suggests the so called inverse problem. Namely, one starts with a given phase shift, $\delta(k)$, and directly determines the potential which results in that phase shift. Here the potential is deduced solely from a knowledge of the bound state energies and phase shifts for all energies. Potentials derived in this manner can be as phase equivalent as desired and are independent of the amibiguities introduced by choosing a functional form for a potential and varying parameters to obtain a best fit to the experimental data. The fact that more than one potential may be obtained in this manner is related to the uniqueness of the inversion procedure. It has been shown that, in the absence of bound states, a purely local potential obtained in this manner is unique. There is however, an independent inversion procedure if one assumes a rank-one separable potential, (Gourdin and Martin 1958). Although

the potential is again unique, it differs from the corresponding local one. If we assume a rank-two seperable interaction, it is not unique (Fiedeldey 1969) and one has a wide choice in either the attractive or repulsive form factors (Fiedeldey 1969). Related to the inverse procedure for velocity dependent potentials is the transform technique of Baker (1962) which provides a great off-shell variation of phase equivalent velocity dependent potentials. The fact that our experimental knowledge of the phase shifts is usually restricted to energies less that about 350 MeV suggests another degree of freedom which may be exploited to produce phase-equivalence in a limited sense. Of course if the inverse procedure is unique, these higher energy phases determine the off-shell behaviour of the potential.

It should be mentioned that the inverse problem for a separable potential is much simpler to perform than is the case for a local potential, which can only be done approximately except in some simpler examples. Recently Srivastava and Sprung (1970) have developed phase equivalent pairs of local and separable potentials. Fuda (1970) has presented methods for accomplishing the same task. Calculations using phase equivalent potentials in the context of Nuclear Matter (Coester et. al. (1970)) have demonstrated interesting variations in the nuclear matter binding energy.

The general format of this thesis is as follows. Chapter II contains a brief formulation of the two body

scattering theory required in the succeeding chapters. We consider the inverse scattering problem in Chapter III. In particular the rank-two inversion procedure is reduced to an equivalent rank-one inversion and the technique used to generate families of phase-equivalent potentials. The off-shell properties of these potentials are examined and the phase equivalent pairs of separable and local potentials of Srivastava and Sprung (1970) are introduced. Chapter IV deals with the interpretation of velocity dependent potentials and possible ambiguities involved in going from a classical to a quantum mechanical system. In Chapter V we set up a calculation of the p-p-γ cross section in the centre-of-mass system and use it to examine the potentials developed in Chapter III. For comparison purposes we include calculations for the Reid soft core and Hamada-Johnston potentials as well.

CHAPTER II

II-1. INTRODUCTION

In this chapter we present the basic scattering theory as used in the following chapters. We define the off-shell T and R matrices, with their partial wave expansions. The quasi-phase and half-off-shell Kowalski-Noyes (Kowalski 1965, Noyes 1965) and reaction matrix elements are shown to give equivalent descriptions of the off-shell properties of a potential.

II-2. FORMALISM

The two-body Schroedinger equation in the centre of mass system for a general non-local potential may be written in the form

$$(\nabla^2+k^2)\ \psi_{\underset{\sim}{k}}(\underset{\sim}{r}) = \int V(\underset{\sim}{r},\underset{\sim}{r}')\ \psi_{\underset{\sim}{k}}(\underset{\sim}{r}')\,d\underset{\sim}{r}' \qquad .II\text{-}2.1$$

Here $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ are the relative coordinates, $\hbar^2k^2/\mu$ is the energy of the incident particle in the laboratory frame and μ is the reduced mass of the two particles, $\mu = m_1m_2/(m_1+m_2)$. The term $V(\underset{\sim}{r},\underset{\sim}{r}')$ is the non-local potential in units of $\hbar^2/2\mu$. This is referred to as non-local since the effective interaction at $\underset{\sim}{r}$ depends on the distribution at $\underset{\sim}{r}'$. The usual Schroedinger equation for a local potential is included as

the special case, $V(\underset{\sim}{r},\underset{\sim}{r}') = V(r)\ \delta(\underset{\sim}{r}-\underset{\sim}{r}')$.

It is convenient to write II-2.1 in the momentum representation, where

$$(k^2-p^2)\,\psi_{\underset{\sim}{k}}(\underset{\sim}{p}) = \int V(\underset{\sim}{p},\underset{\sim}{p}')\ \psi_{\underset{\sim}{k}}(\underset{\sim}{p}')\,d\underset{\sim}{p}' \qquad \text{.II-2.2}$$

If the kernel of II-2.2 is degenerate, we may write

$$V(\underset{\sim}{k},\underset{\sim}{k}') = \frac{1}{2\pi^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sum_{i=1}^{n_\ell}\sigma_{i\ell}\ g_{i\ell}(k)\ g_{i\ell}(k')\ P_\ell(\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{k}}') \qquad \text{.II-2.3}$$

This is known as a separable potential and is here written as rotationally invariant. The g's are called the form factors of the potential and the σ's are -1 or +1 respectively for an attractive or a repulsive contribution to the interaction.

One of the more useful solutions of II-2.1 for the purposes of scattering theory is the outgoing scattering solution, $\psi^+_{\underset{\sim}{k}}(\underset{\sim}{r})$, which has the asymptotic form

$$\psi^+_{\underset{\sim}{k}}(\underset{\sim}{r}) \sim e^{ikz} + f(\theta,\phi)\ \frac{e^{ikr}}{r} \qquad \text{.II-2.4}$$

It is convenient to incorporate this boundary condition and equation II-2.1 in the abstract integral equation

$$|\psi^+_{\underset{\sim}{k}}\rangle = |\underset{\sim}{k}\rangle + (k^2-H_0+i\varepsilon)^{-1}\ V|\psi^+_{\underset{\sim}{k}}\rangle \qquad \text{,II-2.5}$$

where H_0 is the kinetic energy operator in units of $2\mu/\hbar^2$. This is the well known Lippmann-Schwinger equation. We may

now define an operator $T(k^2)$, through

$$T(k^2)\,|\underset{\sim}{k}\rangle = V|\psi^+_{\underset{\sim}{k}}\rangle \qquad .II\text{-}2.6$$

The scattering amplitude, $f(\theta,\psi)$, of II-2.4 may now be expressed as

$$f(\theta,\phi) = -\frac{m}{4\pi}\langle k'|T(k^2)|k\rangle \qquad .II\text{-}2.7$$

For elastic scattering, $|k| = |k'|$, and we say the T-matrix **is** evaluated on-shell. Of course, II-2.5 and II-2.6 define the more general T operator, which satisfies

$$T(k^2) = V + V(k^2-H_0+i\varepsilon)^{-1}\,T(k^2) \qquad .II\text{-}2.8$$

This defines $T(k^2)$ fully off-shell, that is $\langle \underset{\sim}{p}|T(k^2)|\underset{\sim}{q}\rangle$ where $|\underset{\sim}{p}| \neq |\underset{\sim}{q}| \neq k$. This however may be expanded in terms of only half-off-shell elements where $|\underset{\sim}{p}| \neq |\underset{\sim}{q}| = k$.

In the absence of a tensor force, we are free to define the following partial wave expansion

$$\langle \underset{\sim}{r}|\psi^+_{\underset{\sim}{k}}\rangle = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)\,\frac{\psi_{\ell}(k,r)}{r}\,P_{\ell}(\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{r}}) \qquad .II\text{-}2.9$$

And

$$\langle \underset{\sim}{r}|\underset{\sim}{k}\rangle = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)\,j_{\ell}(kr)\,P_{\ell}(\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{r}}) \qquad .II\text{-}2.10$$

If we expand

$$V(\underset{\sim}{r},\underset{\sim}{r}') = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\,V(r,r')\,P_{\ell}(\hat{\underset{\sim}{r}}\cdot\hat{\underset{\sim}{r}}') \qquad II\text{-}2.11$$

then $\psi_\ell(k,r)$ satisfies the radial Schroedinger equation

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2\right)u_\ell(r) = \int V(r,r')rr'\, u_\ell(r')dr' \quad \text{.II-2.12}$$

and has the asymptotic value

$$\psi_\ell^+(kr) \sim e^{i\delta_\ell(k)} \sin(kr - \frac{\ell\pi}{2} + \delta(k))/k \quad \text{.II-2.13}$$

From II-2.6, we see that

$$\langle \underset{\sim}{k}' | T(k^2) | \underset{\sim}{k} \rangle = \frac{1}{2\pi^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, t_\ell(k',k;k^2)\, P_\ell(\hat{\underset{\sim}{k}}'.\hat{\underset{\sim}{k}}')/k \quad \text{II-2.14}$$

where

$$t_\ell(k',k;k^2) = \frac{1}{k}\int_0^\infty r\, j_\ell(k'\, r)V\, \psi_\ell^+(k,r)dr \quad \text{.II-2.15}$$

By defining

$$\Delta_\ell(k',k) = -\int_0^\infty j_\ell(k'\, r)V\, u_\ell(r)rdr \quad \text{II-2.16}$$

where $u_\ell(r)$ is a solution of II-2.12 which is asymptotic to $\sin(kr - \frac{\ell\pi}{2} + \delta_\ell(k))$, we may write

$$t_\ell(k',k;k^2) = -e^{i\delta_\ell(k)}\, \Delta_\ell(k',k) \quad \text{.II-2.17}$$

The function $\Delta_\ell(k',k)$ is the quasi-phase shift (Sobel and Cromer 1963) and provides a parameter for measuring the off-shell nature of the T-matrix. Since,

$$\sin \delta_\ell(k) = - \int_0^\infty j_\ell(kr) V u_{\ell k}(r) r dr \quad ,\text{II-2.18}$$

we may write the following on-shell limits

$$\Delta_\ell(k,k) = \sin \delta_\ell(k) \quad \text{II-2.19}$$

and

$$t_\ell(k,k;k^2) = -e^{-i\delta_\ell(k)} \sin \delta_\ell(k) \quad .\text{II-2.20}$$

Another parameter sometimes used to describe the off-shell nature of a potential is the Kowalski-Noyes (1965) half-off-shell matrix element, $f_\ell(k',k)$, defined as the ratio of the t-matrix off- and on-shell. That is

$$f_\ell(k',k) = \Delta_\ell(k',k)/\sin \delta_\ell(k) \quad .\text{II-2.21}$$

From II-2.19 this has the value 1 in the on-shell limit. This may be considered a more convenient measure of off-shell behaviour if the potentials being compared are not phase-equivalent.

For a separable potential as given by II-2.3, the Lippmann Schwinger equation for the ℓ'th partial wave is

$$\psi^+_\ell(k,p) = \delta(p-k) + \frac{2}{\pi} \int_0^\infty dq\, pq \sum_{i=1}^{n_\ell} \frac{\sigma_{i\ell}\, g_{i\ell}(p)\, g_{i\ell}(q)}{k^2-p^2+i\varepsilon} \psi_\ell(k,q)$$

$$.\text{II-2.22}$$

Following Gutkowski and Scalia (1968), we drop the index ℓ

and define

$$f_{ij}(k) = -\frac{2}{\pi}\int_0^\infty \sqrt{\sigma_i}\,\sqrt{\sigma_j}\; g_i(q)\; q_j(q) q^2 dq \qquad \text{II-2.23}$$

and the solution of II-2.22 may be written as

$$\psi_\ell^+(k,p) = \delta(p-k) + [\det|\delta_{ij}+f_{ij}(k)|]^{-1} \sum_{i,j=1}^{n_\ell} \times \frac{\sqrt{\sigma_i}\,\sqrt{\sigma_j}\; g_i(p)\; g_j(k)}{k^2-p^2+i\varepsilon}\, pk\, d\binom{j}{i} \qquad ,\text{II-2.24}$$

where $d\binom{j}{i}$ is the co-factor of the element $\delta_{ij}+f_{ij}(k)$ in the determinant. By observing that the scattered wave in momentum space is $t_\ell(p,k;k^2)/(k^2-p^2+i\varepsilon)$, we write

$$t_\ell(k',k;k^2) = \sum_{i,j=1}^{n_\ell} \frac{\sqrt{\sigma_{i\ell}}\,\sqrt{\sigma_{j\ell}}\; g_{i\ell}(k')\; g_{j\ell}(k)}{\det|\delta_{ij}+f_{ij\ell}(k)|}\, d\binom{j}{i}_\ell \qquad .\text{II-2.25}$$

The denominator is the Fredholm determinant of equation II-2.22, and is conventionally denoted as

$$D_\ell^+(k) = \det|\delta_{ij}+f_{ij\ell}(k)| \qquad .\text{II-2.26}$$

This quantity is of utmost importance in the inversion techniques employed to construct a separable potential. We note here that this is <u>not</u> the Jost function, as defined in the context of local potentials. It is customary to define the Jost function in terms of the Jost solutions of II-2.12,

which are defined as those solutions having the asymptotic behaviour

$$\lim_{r\to\infty} f(\pm k,r)\ e^{\pm ikr} = 0 \qquad \text{.II-2.27}$$

The Jost functions, $f(\pm k)$, are then

$$f(\pm k) = f(\pm k,0) \qquad \text{.II-2.28}$$

Although the Jost function turns out to be identical with the Fredholm determinant of the Lippmann Schwinger equation for a local potential, (Jost and Pais 1951), this is not true for a separable potential. However, in the recent literature, this has been called the Jost function. Possibly the Fredholm determinant is the more fundamental quantity, in which case it could be defined as the Jost function for the general non-local potential. The work of Gutkowski and Scalia (1968) indicates that this may be the case. In the remainder of this thesis we refer to the Fredholm determinant as the Jost function, but only in the sense here mentioned.

The explicit demonstration of the above is relegated to Appendix A.

To conclude this section we define the off-shell reaction or R-matrix. In analogy with II-2.6 and II-2.8 we define

$$R(k^2) = V + V\,\frac{P}{k^2-H_0}\,R(k^2) \qquad \text{II-2.29}$$

and the principal-value wave function $|\psi^s_{\underset{\sim}{k}}>$,

$$R(k^2)\,|\underset{\sim}{k}> = V|\psi^s_{\underset{\sim}{k}}> \qquad .II\text{-}2.30$$

Again in the absence of a tensor force we may write

$$<\underset{\sim}{p}|R(k^2)|\underset{\sim}{k}> = \frac{1}{2\pi^2}\sum_{\ell=0}^{\infty}(2\ell+1)\,R_\ell(q,k;k^2)\,P_\ell(\hat{\underset{\sim}{q}}.\hat{\underset{\sim}{k}}) \qquad .II\text{-}2.31$$

Using II-2.3, we have for the ℓ'th partial wave

$$R_\ell(p,k;k^2) = v_\ell(p,k) + \frac{2}{\pi}P\int_0^\infty v_\ell(p,q)\,\frac{R_\ell(q,k;k^2)}{k^2-q^2}\,q^2dq \qquad II\text{-}2.32$$

where we have taken

$$v_\ell(p,k) = \sum_{i=1}^{n_\ell}\sigma_{i\ell}\,g_{i\ell}(p)\,g_{i\ell}(k) \qquad .II\text{-}2.33$$

Equation II-2.30 is an inhomogeneous Fredholm integral equation of the second kind, with a degenerate kernel. The solution is straightforward (Hildebrand 1956) being algebraic. For a rank-one potential, i.e. n_ℓ=1, we have

$$R_\ell(p,k;k^2) = \frac{\sigma\,g(p)\,g(k)}{1+\sigma\frac{2}{\pi}P\int_0^\infty\frac{g^2(q)\,q^2}{q^2-k^2}\,dq} \qquad .II\text{-}2.34$$

The solution for n_ℓ=2 is given in section III-3.

The principal-value wave function follows from II-2.19 and II-2.30

$$|\psi^s_{\underset{\sim}{k}}> = |\underset{\sim}{k}> + \frac{P}{k^2-H_0}R(k^2)\,|\underset{\sim}{k}> \qquad .II\text{-}2.35$$

In analogy with II-2.5, we write

$$\langle \underset{\sim}{r} | \psi^s_{\underset{\sim}{k}} \rangle = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)\ \psi^s_{\ell}(k,r)\ P_{\ell}(\hat{\underset{\sim}{k}}.\hat{\underset{\sim}{r}}) \qquad \text{.II-2.36}$$

The partial wave $\psi^s_{\ell}(k,r)$ may be defined as the solution of

$$\frac{\psi^s_{\ell}(k,r)}{r} = j_{\ell}(kr) - \frac{2}{\pi} P \int_0^{\infty} j_{\ell}(pr)\ \frac{R_{\ell}(p,k;k^2)}{k^2-p^2} p^2 dp \qquad \text{II-2.37}$$

or as the solution of II-2.12, with the asymptotic behaviour

$$\psi^s_{\ell}(k,r) \sim \sin(kr - \frac{\ell\pi}{2} + \delta_{\ell}(k))/k \cos \delta_{\ell}(k) \qquad \text{.II-2.38}$$

To compare R(k',k) with the other half-off-shell elements, we need only multiply II-2.30 on the left by $\langle \underset{\sim}{k}' |$ and employ the partial wave expansions. The result is

$$R_{\ell}(k',k) = \int_0^{\infty} j_{\ell}(k'r) V\ \psi^s_{\ell}(k,r) r dr \qquad \text{.II-2.39}$$

From II-2.16 and II-2.38 it follows that

$$R_{\ell}(p,k) = -\Delta_{\ell}(p,k)/k \cos \delta_{\ell}(k) = -f_{\ell}(p,k) \tan \delta_{\ell}(k)/k \qquad \text{,II-2.40}$$

which has the on-shell limit

$$R_{\ell}(k,k) = -\tan \delta_{\ell}(k)/k \qquad \text{.II-2.41}$$

For convenience it is often best to work with the R-matrix which is real, rather than T which is complex.

Although two-body scattering problems are more directly related to the T-matrix, this is not a great hardship, the two being simply related through the Heitler damping equation (Rodberg and Thaler 1967).

CHAPTER III

THE INVERSE SCATTERING PROBLEM AND PHASE-EQUIVALENT POTENTIALS

III-1. INTRODUCTION

The basic aim of most scattering experiments is the determination of some aspect or another of the interparticle forces. The experimental quantities actually measured are cross sections, polarizations etc. These in turn are analyzed in terms of some convenient parameters such as the 'experimental' phase shifts, mixing parameters and binding energies. It thus becomes desirable to obtain, say, a Hamiltonian describing the interacting particles directly from these 'experimental' quantities. In contrast, what is usually done, is to assume a potential of a certain functional form with a number of parameters and solve the Schrödinger equation to obtain the phase shift. The parameters are adjusted so that a best fit to the observed data is achieved. In this way a large number of phenomenological or realistic forces have been proposed.

With the inverse problem there are no intermediate parameters. The prescription thus precludes the need for parameter fitting and the associated ambiguities incurred in the construction of a phenomenological potential. Of course if the inversion can be done uniquely then it is

more desirable.

Clearly connected with the uniqueness of the inverse problem is the concept of phase-equivalent potentials. By phase-equivalent is usually meant potentials which produce the same phase shifts and bound state energies. There are essentially three ways in which a lack of uniqueness can be exploited to produce phase-equivalence in families of potentials. First is our lack of knowledge of the high energy phases themselves. Experimentally they are known to about 350 MeV. Beyond this energy we are relatively free to vary the phases and still produce phase-equivalence in this limited sense. If the inversion procedure is unique, then this variation is equivalent to a variation of the off-shell properties of the potential. Possibly a more fundamental ambiguity is the choice of the type of potential. By type we mean local, velocity dependent, separable, or what have you. It is common to have independent unique inversions for a local and say a separable potential. The two potentials are, however, quite different. The third ambiguity arises when the inversion is not unique, even within a certain type of potential. This is the case with rank-two separable potentials.

Phase-equivalent potentials constructed from any of the above properties can vary only in their off-shell properties. Being the same on-shell, they present the natural approach for examining the dependence of a calculation,

say p-p bremsstrahlung, on the purely off-shell effects. This is the source of our interest in the problem. Before describing the present effort let us briefly review what others have done.

In the context of purely local potentials, Bargmann (1949) constructed explicit examples of phase-equivalent potentials. Since then, general methods have been presented for constructing the local potential directly from a knowledge of the on-shell parameters. Some of the better known methods are those of Gel'fand and Levitan (1951), Jost and Kohn (1952) and Marchenko (1950). A good general reference for this topic is R.G. Newton (1966). It has been shown that in the absence of a bound state, the local potential is uniquely determined by the phase shifts at all energies. However, the inversion of a system exhibiting n bound states requires a knowledge of the phases, and the binding energies and results in an n parameter family of potentials.

In the field of separable potentials, Gourdin and Martin (1957,1958) have presented a general but rather mathematical discussion of the rank-one inversion procedure. Again, in the absence of bound states, the potential is unique. However, the necessary measures taken to ensure a sign change in the high energy phase shifts may be physically unacceptable, as is discussed in Section III-3. Tabakin (1968,1969) and Bolsterli and Mackenzie (1965) have both rediscovered many of the results of Gourdin and Martin;

in a more physical but less general form. They both include sections on the coupled channel inversion. Chadan (1958) generalized the above results to include the superposition of a local and a one term separable interaction. The local potential is assumed known and the form factor of the separable term is determined from a knowledge of the S-matrix. More recently Fiedeldey (1969) has extended the theory to the case of a rank two separable potential. The total interaction is required to reproduce the phase shifts and a bound state energy. The attractive form factor is assumed, in which case it was shown that the repulsive form factor was determined uniquely from the inversion procedure. Triton calculations using their phase-equivalent potentials (Fiedeldey (1969A)) demonstrated interesting variations in the binding energies as a function of the potential. Similar calculations have been completed (Srivastava (1970A)) for infinite nuclear matter. Srivastava and Sprung (1970) have completed an investigation of phase-equivalent pairs of local and separable potentials. Fuda (1970) has presented methods for doing essentially the same thing and includes the possibility of assimilating a hard-core by the separable potential.

A type of non-local interaction not considered here is the velocity dependent potential. These were developed essentially as a method of assimilating the hard-core by something more amenable to perturbation calculations. A series of contributions including Bell (1962), Baker (1962)

and Mittelstaedt and Ristig (1966) illustrate transformation techniques which generate families of phase-equivalent potentials. Such potentials have demonstrated interesting variations in the binding energy and saturation density of nuclear matter (Ristig and Kistler (1968), Srivastava et. al. (1970) and Coester et. al. (1970)). These potentials are not so readily utilized in p-p-γ due to questions of guage invariance (Pearce et. al. (1967)).

In brief, Section III-2 describes the inversion procedure for rank-one separable potentials. This is included to illustrate the nature of the phase-shift sign change and because the rank-two inversion discussed in Section III-3 is reduced to an equivalent rank-one inversion. Section III-4 consists of a brief description of the phase-equivalent pairs of Srivastava (1970), and Section III-5 contains our investigation of the rank-two inversion for the $^1S_0$ interaction.

III-2. THE INVERSE PROBLEM FOR RANK-ONE SEPARABLE POTENTIALS

The half off-shell T-matrix for separable interactions, equation II-2.25, takes the following simple form for a rank-one separable potential,

$$T(p,k) = \sigma\, g(p)\, g(k)/D^+(k) \qquad .III-2.1$$

Here $g(p)$ is the form factor and $D^+(k)$ is the Jost function, in the sense described in Chapter II. The constant σ is

-1 for an attraction and +1 for a repulsion.

Defining

$$D(z) = 1 + \sigma \frac{2}{\pi} P \int_0^\infty dq \frac{q^2 g^2(q)}{q^2 - z^2} \qquad \text{III-2.2}$$

we may write

$$D^{\pm}(k) = D(k \pm i\epsilon) = D^{\mp}(-k) \qquad .\text{III-2.3}$$

We use z to denote a complex variable and k for a real variable.

From the on-shell expression for the T-matrix, II-2.20, the form factor is related to the phase-shift by the equation

$$-e^{i\delta(k)} \sin \delta(k) = \sigma g^2(k)/D^+(k) \qquad .\text{III-2.4}$$

This is the essential equation which is to be inverted to give the form factor from a knowledge of $\delta(k)$. A similar useful expression may be had from II-2.41, the on-shell expression for the R-matrix. In that case

$$\tan \delta(k) = k g^2(k)/D(k) \qquad .\text{III-2.5}$$

The inversion procedure followed here requires a knowledge of the Jost function in the complex energy plane. If $\omega = z^2$, then the upper half of the k-plane is mapped into the so-called physical sheet of the energy variable. In terms of this complex energy, equation III-2.2 becomes

$$D(\omega) = 1 + \sigma \frac{2}{\pi} P \int_0^\infty dq \frac{q^2 g^2(q)}{q^2 - \omega} \qquad \text{III-2.6}$$

and

$$D^{\pm}(k) = D(k^2 \pm i\varepsilon) \qquad \text{III-2.7}$$

It is easily shown that $D(\omega)$ has the property that

$$D(k^2+i\varepsilon) - D(k^2-i\varepsilon) = 2ik\,\sigma\, g^2(k) \qquad \text{.III-2.8}$$

This indicates that $D(\omega)$ has a branch cut on the real ω-axis. The discontinuity across this cut is proportional to the square of the potential form factor.

Let us review some additional properties of $D(\omega)$ for the separable potential. By considering III-2.6, it follows that there are no zeros of $D(\omega)$ on the physical sheet with the possible exception of $k = ik_B$ on the positive imaginary axis and that $D(\omega)$ obeys the Schwarz reflection principle,

$$D(\omega^*) = D^*(\omega) \qquad \text{.III-2.9}$$

From III-2.6 it is seen that a zero of $D(\omega)$ on the physical sheet may only occur for an attractive potential and that for a rank one potential there is at most one such zero. Gutkowski and Scalia (1968) have shown, under quite general conditions that zeros of $D(z)$ occuring on the imaginary positive axis correspond to bound states. They also show that the S-matrix may be written as the ratio

$$S(k) = e^{2i\delta(k)} = \frac{D^-(k)}{D^+(k)} \qquad \text{.III-2.10}$$

Returning to III-2.8 and invoking III-2.9, we get

$$\sigma\, k\, g^2(k) = \text{Im}\, D^{(+)}(k) \qquad \text{III-2.11}$$

To do the inverse problem we take the simplest case of D(z) analytic in the upper half plane with no zeros there. This represents a partial wave with no bound state such as the $^1S_0$. From III-2.2 it follows that D(z) goes to 1 for large values of z. Consider then ℓn D(z). This must also be analytic on the physical sheet but vanishes on the infinite semicircle in the upper half z-plane. These conditions are sufficient for Cauchy's theorem, whence

$$\ell n\, D(z) = \frac{1}{2\pi i} \int_c \frac{\ell n\, D(z')}{z'-z}\, dz' \qquad \text{III-2.12}$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ell n\, D^{(+)}(k')}{k'-z}\, dk' \qquad \text{III-2.13}$$

Here c is the closed semicircle in the upper half z-plane. Taking the real part and allowing z to approach k+iε,

$$\text{Re}\ \ell n\, D^+(k) = \frac{\mathcal{P}}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im}\, D^+(k')}{k'-k}\, dk' \qquad \text{III-2.14}$$

On using III-2.10 and III-2.9,

$$D^+(k) = |D(k)|\, e^{-i\delta(k)} \qquad \text{III-2.15}$$

and III-2.14 may now be written as

$$\ell n |D(k)| = -\Delta(k) \qquad \text{III-2.16}$$

where $\Delta(k)$ is the principle value integral,

$$\Delta(k) = \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{\delta(k')}{k'-k} dk' \qquad .III-2.17$$

The complete Jost function is now,

$$D^+(k) = e^{-[\Delta(k)+i\delta(k)]} \qquad .III-2.18$$

So III-2.11 becomes

$$-\sigma\, g^2(k) = e^{-\Delta(k)} \frac{\sin \delta(k)}{k} \qquad .III-2.19$$

This defines the form factor directly from the phase shifts.

The inclusion of a bound state, and hence a zero of $D^+(k)$ in the positive imaginary axis is straightforward (Tabakin (1969)). The resulting generalization of III-2.19 is

$$-\sigma\, g^2(k) = \left(\frac{k^2+k_B^2}{k^2}\right)^{(1-\sigma)/2} \frac{\sin \delta(k)}{k} e^{-\Delta(k)} \qquad III-2.20$$

where $D(z)$ has a zero at ik_B corresponding to a binding energy k_B^2. The inclusion of an attraction plus a repulsion in the same form factor is also possible. This feature is required if we are to describe phase shifts such as the $^1S_0$ and $^3P_0$ which change sign for high energies. There have been two methods proposed for a rank one sign change. Possibly the easiest (Bolsterli and MacKenzie (1965)) is to require σ to change sign at k_c, that value of k at which the phases change sign. This unfortunately makes the potential energy

dependent. An alternative approach by Gourdin and Martin (1957,1958) later used by Tabakin (1968,1969), is to require $D^+(k)$ to have zeros on the real axis at $k = \pm k_c$. What this accomplishes is best illustrated by III-2.5. The numerator is positive definite with possible zeros. Since the $^1S_0$ phase shift is $<\frac{\pi}{2}$ and >0 for $k<k_c$, the only way to ensure that δ pass through zero is to have a simultaneous zero of the numerator and denominator, the order of the former being greater. If the sign is to change as well, then the difference in the order of the zeros must be odd.

Tabakin (1957) has built an explicit one-term potential using the above trick. He showed that the scattering wave function had an additional node in the small radius portion of the scattering wave function. This extra zero in the wave function has the effect of reducing the integrated probability of finding the nucleons at short distances and it is in this sense that there is a short range repulsion. Although Tabakin has shown that the wave functions are well behaved, the condition $D^+(k_c) = 0$ is the condition for a bound state. This discrete state is of positive energy and is the 'positive energy degeneracy' referred to by Gourdin and Martin (1958). This potential is our case III of Section III-4. One consequence of this degeneracy is that the phase-shift asymptotically approaches $-\pi$ and consequently does not obey the standard Levinson's theorem. It does obey a modified Levinson's theorem (Martin (1958)),

$$\delta(0) - \delta(\infty) = (\nu+\sigma)\pi \qquad \text{III-2.21}$$

where ν is the number of bound states and σ the number of positive energy degeneracies.

III-3. THE INVERSE PROBLEM FOR RANK-TWO SEPARABLE POTENTIALS

We develop here a method for determining a rank-two separable potential from a knowledge of the phase-shifts. The total interaction is assumed to have no bound states. Fiedeldey (1969) has treated the case where the final potential has a bound state. He assumes the attractive form factor and deduces the repulsive one. It is less than obvious how to generalize that procedure to eliminate the bound state created by the attractive form factor so that none appears for the total interaction. This is the case where the total interaction has one less bound-state than the assumed initial part of the potential and where no positive energy degeneracies occur. This case has been especially mentioned by Chadan (1958) as one which he had not been able to solve. Our solution is to start with the repulsive form factor and thus essentially avoid the difficulty. The method is basically that of Chadan (1958).

We start with the $^1S_0$ phase-shifts which, ideally are determined from experiment, but for practical purposes will come from some well accepted potential. Our aim is to produce a rank-two separable potential which will reproduce these experimental phase-shifts. It is convenient to start

the inversion procedure by assuming a repulsive form factor $h(k)$, which when acting alone will produce the phase-shift $\delta_0(k)$ where for all k,

$$\delta_0(k) < 0 \qquad \text{.III-3.1}$$

Denoting the total phase-shift by $\delta(k)$, the inversion procedure requires that,

$$\delta_1(k) \equiv \delta(k) - \delta_0(k) > 0 \qquad \text{.III-3.2}$$

This is an obvious condition which simply means that the attraction form factor can not be expected to have repulsive properties. We have the liberty, in contrast to the rank-one inversion, to require the phases to obey Levinson's theorem. So,

$$\delta_0(0) = \delta(0) = 0 \quad \text{and} \quad \delta_0(\infty) = \delta(\infty) = 0 \qquad \text{.III-3.3}$$

The inversion process is now reduced to constructing a purely attractive form factor $g(k)$ which when combined with the repulsive form factor will reproduce the total phase shift $\delta(k)$. Denoting the total interaction by

$$V(k,k') = \frac{\hbar^2}{m} \frac{1}{2\pi^2} [-g(k)\, g(k') + h(k)\, h(k')] \qquad \text{,III-3.4}$$

and taking the Fourier transform, the resulting potential in coordinate space can be written as

$$V(r,r') = -v(r)\, v(r') + w(r)\, w(r') \qquad \text{III-3.5}$$

where

$$v(r) = \frac{2}{\pi} \int_0^\infty k\, g(k)\, \sin kr\, dk \qquad \text{.III-3.6}$$

The corresponding Schroedinger equation for the S-wave is,

$$\frac{d^2\psi}{dr^2} + k^2\psi = -\int_0^\infty v(r)\, v(r')\, \psi(r')dr' + \int_0^\infty w(r)\, w(r')\, \psi(r')dr' \qquad \text{.III-3.7}$$

Starting with only the repulsive interaction, we require solutions of

$$\frac{d^2\phi(k,r)}{dr^2} + k^2\phi(\mathbf{k},r) = \int_0^\infty w(r)\, w(r')\, \phi(k,r')dr' \qquad \text{.III-3.8}$$

Since we have chosen to work with the R-matrix, it is convenient to utilize the principle-value wave function, whose solution is

$$\phi(k^2,r) = \frac{\sin kr}{k} - \frac{2}{\pi} P \int_0^\infty R(p,k)\, \frac{\sin pr}{k^2-p^2}\, pdp \qquad \text{.III-3.9}$$

Here R(p,k) is the half off shell reaction matrix for a rank one separable potential. From II-2.34 ,

$$R(p,k) = h(k)\, h(p)/D_0(k) \qquad \text{III-3.10}$$

where

$$D_0(k) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{q^2(q)}{q^2-k^2}\, q^2 dq \qquad \text{.III-3.11}$$

We know from II-2.38 that ϕ has the asymptotic behaviour

$$\phi(k^2,r) \sim \frac{\sin\,(kr+\delta_0)}{k\,\cos\,\delta_0} \qquad \text{III-3.12}$$

and is normalized to

$$\int_0^\infty \phi(k^2,r)\;\phi(k'^2,r)\;dr = \frac{\pi\;\delta(k'-k)}{2k^2\;\cos^2\;\delta_0(k)} \equiv \frac{\delta(k'-k)}{c(k)} \qquad \text{III-3.13}$$

$$c(k) = 2k^2\;\cos^2\;\delta_0(k)/\pi \qquad .\text{III-3.14}$$

It has been shown by Ghirardi and Rimini (1964) that the eigenfunctions of a separable potential form a complete set. Since a repulsive potential can have no bound states, the completeness relation may be written as

$$\int_0^\infty \phi(k^2,r)\;\phi(k^2,r')\;c(k)dk = \delta(r-r') \qquad .\text{III-3.15}$$

The completeness of the above solutions enables us to follow the integral transform technique established by Chadan (1958). Thus for any square integrable function F(r), we may define, similar to a sine transform

$$F(k^2) = \int_0^\infty F(r)\;\phi(r^2,r)dr \qquad .\text{III-3.16}$$

Multiplying by $c(k)\;\phi(k^2,r')$ and using III-3.15 results in the inverse transform

$$F(r) = \int_0^\infty F(k^2)\;c(k)\;\phi(k^2,r)dk \qquad .\text{III-3.17}$$

To find a solution of III-3.7 we define

$$\tilde{\psi}(k^2,p) = \int_0^\infty \psi(k,r)\ \phi(p^2,r)dr \qquad .III-3.18$$

So

$$\psi(k^2,r) = \int_0^\infty \tilde{\psi}(k,p)\ c(p)\ \phi(p^2,r)dp \qquad ,III-3.19$$

$$\tilde{v}(p) = \int_0^\infty v(r)\ \phi(p^2,r)dr \qquad ,III-3.20$$

and

$$v(r) = \int_0^\infty v^2(p)\ c(p)\ \phi(p^2,r)dp \qquad .III-3.21$$

Substituting III-3.19 for $\psi(k,r)$ and changing the order of integration, III-3.7 becomes

$$\int_0^\infty \{\tilde{\psi}(k,p)\ c(p)\ [\frac{d^2}{dr^2} + k^2]\ \phi(p^2,r) - \int_0^\infty w(r)\ w(r')\ \phi(p^2,r')dr'\}dp$$

$$= -v(r)\ N(k) \qquad III-3.22$$

where

$$N(k) = \int_0^\infty v(r')\ \psi(k,r')dr' \qquad .III-3.23$$

Using III-3.8,

$$[\frac{d^2}{dr^2} + k^2]\ \phi(p^2,r) = (k^2-p^2)\ \phi(p^2,r) + \int_0^\infty w(r)w(r')\phi(p^2,r')dr' \qquad .III-3.24$$

So III-3.22 reduces to

$$\int_0^\infty \tilde{\psi}(k,p)\ c(p)(k^2-p^2)\ \phi(p^2,r)dp = -v(r)\ N(k) \qquad .III-3.25$$

Multiplying by $\phi(p'^2,r)$ and integrating over r,

$$\int_0^\infty dp\ \tilde{\psi}(k,p)\ c(p)(k^2-p^2) \int_0^\infty dr\ \phi(p'^2,r)\ \phi(p^2,r) = -N(k) \qquad III-3.26$$

$$x \int V(r)\ \phi(p'^2,r)dr$$

which upon using III-3.13 and III-3.20 becomes

$$\tilde{\psi}(k,p)(k^2-p^2) = -N(k)\ \tilde{v}(p) \qquad .III-3.27$$

The desired solution of III-3.7 must be a regular solution with the asymptotic behaviour

$$\psi(k,r) \sim A \sin(kr+\delta) \qquad .III-3.28$$

It also must reduce to $\phi(k^2,r)$ in the limit as $v(r) = 0$. For $v(r) = 0$, $\tilde{v}(p)$ is also zero and it is permissible to write the solution to the homogeneous equation III-3.27 as

$$\tilde{\psi}_1(k,p) = \frac{\delta(k-p)}{c(p)} \qquad .III-2.29$$

From the definition of the inverse transform III-3.19, it is immediately seen that

$$\psi_1(k,r) = \phi(k^2,r) \qquad .III-2.30$$

The required solution of the complete equation III-2.7 is thus

$$\tilde{\psi}(k,p) = \frac{\delta(k-p)}{c(p)} - N(k)\ P\ \frac{\tilde{v}(p)}{k^2-p^2} \qquad \text{III-2.31}$$

or, in coordinate space

$$\psi(k,r) = \phi(k^2,r) - N(k)\ P \int_0^\infty \frac{\tilde{v}(p)}{k^2-p^2}\, c(p)\ \phi(p^2,r)\,dp. \qquad \text{III-3.32}$$

To complete the solution it remains to solve for N(k). Substituting III-3.32 into III-3.23 yields

$$N(k) = \tilde{v}(k)/\left[1 + \int_0^\infty dr' v(r')\ P \int_0^\infty dp\ \frac{\tilde{v}(p)}{k^2-p^2}\, c(p)\ \phi(p^2,r')\right], \qquad \text{III-3.33}$$

and changing the order of integration

$$N(k) = \frac{\tilde{v}(k)}{1 + P \int_0^\infty \frac{\tilde{v}^2(p)\ c(p)}{k^2-p^2}\, dp}. \qquad \text{III-3.34}$$

To determine the complete phase shift it is necessary to examine $\psi(k,r)$ in the asymptotic limit, $r\to\infty$. From III-3.12, the second term of III-3.32 in this limit is

$$\sim -N(k)\ P \int_0^\infty \frac{\tilde{v}(p)\ c(p)}{k^2-p^2}\, \frac{\sin(pr+\delta_0(p))}{p\cos\delta_0(p)}\, dp \qquad \text{III-3.35}$$

$$= -N(k)\ \frac{2}{\pi}\ P \int_0^\infty \tilde{v}(p)\ \frac{p\cos\delta_0(p)}{k^2-p^2} \sin[pr+\delta_0(p)]\,dp. \qquad \text{III-3.36}$$

Since $\phi(k^2,r)$ is even in k, it follows that $\tilde{v}(p)$ is even. But $\delta_0(p)$ is an odd function; so III-3.36 becomes

$$-N(k)\ \frac{P}{\pi}\int_{-\infty}^{+\infty} \tilde{v}(p)\ \frac{p\ \cos\ \delta_0(p)}{k^2-p^2}\ \sin[pr+\delta_0(p)]dp \quad .III-3.37$$

By using the identity

$$\frac{1}{k^2-p^2} = -\ \frac{1}{2p}\ (\frac{1}{p-k} + \frac{1}{p+k}) \qquad III=3.38$$

and the symmetry of the trigonometric functions, III-3.37 becomes

$$N(k)\ \frac{i}{2\pi}\ P\int_{-\infty}^{+\infty} \tilde{v}(p)\ \cos\ \delta_0(p)\ e^{ipr}\ e^{i\delta_0(p)}\ (\frac{1}{p-k} + \frac{1}{p+k})dp \qquad .III-3.39$$

The integral may now be evaluated by closing the contour in the upper half plane and using the identity

$$\frac{P}{x-x_0} = \frac{1}{x-x_0+i\epsilon} + i\pi\ \delta(x-x_0) \qquad .III-3.40$$

This choice places the zeros of the denominator in the lower half plane. Any possible poles in the upper half plane will have zero contribution in the asymptotic limit due to the factor e^{ipr}. It is thus possible to replace

$$\frac{1}{p-k} + \frac{1}{p+k} \quad \text{by} \quad i\pi\ \delta(p-k) + i\pi\ \delta(p+k) \qquad III-3.41$$

and obtain

$$- \frac{N(k)}{2} [\tilde{v}(k) \cos \delta_0(k) \ (e^{ikr} e^{i\delta_0(k)} + e^{-ikr} e^{-i\delta_0(k)})$$

$$= -N(k) \tilde{v}(k) \cos \delta_0(k) \cos(kr+\delta_0(k)) \qquad .III-3.42$$

Upon collecting terms, we have

$$\psi(k,r) \sim \frac{1}{k \cos \delta_0(k)} [\sin(kr+\delta_0(k)) + \tan \delta_1(k) \cos(kr+\delta_0(k))] \qquad III-3.43$$

where $$\tan \delta_1(k) \equiv -N(k) \tilde{v}(k) k \cos^2 \delta_0(k) \qquad .III-3.44$$

That is

$$\psi(k,r) \sim \frac{1}{k \cos \delta_0(k) \cos \delta_1(k)} \sin(kr+\delta_0(k)+\delta_1(k)) \qquad III-3.45$$

and $\delta_1(k)$ is recognized to be the additional phase shift arising from the attractive interaction $-v(r) v(r')$.

At this stage we have a solution for the phase shift $\delta_0(k)+\delta_1(k)$ in terms of a rank two separable potential. Our wish is to insert the procedure and thus construct the attractive form factor from a knowledge of the 'experimental' phase shift δ and the phase shift δ_0 of our chosen repulsive form factor. More concisely, to determine $g(k)$ from a knowledge of $\delta_1(k) = \delta(k)-\delta_0(k)$. From III-3.44 and the definition of $N(k)$, it follows that

$$\frac{2}{\pi} k \tan \delta_1(k) = \frac{F(k^2)}{1-P \int_0^\infty \frac{F(p)}{p^2-k^2} dp} \qquad III-3.46$$

where

$$F(k^2) = \frac{2}{\pi} \tilde{v}^2(k)\ k^2 \cos^2 \delta_0(k) \qquad \text{.III-3.47}$$

If equation III-3.46 can be solved for F(k), then III-3.47 defines $\tilde{v}(k)$ and the attractive form factor g(k) follows from III-3.21 and III-3.6. Equation III-3.46 has been solved for various conditions.

It has become popular to quote Chadan's result (1958) which is deduced from a generalization of Muskhelishvili (1941) theory as developed by Omnes (1958). We prefer to reduce III-3.46 to the form of III-2.4 since we have here essentially a rank-one inversion problem. With this in mind, it becomes convenient to define

$$F(k^2) = \frac{2}{\pi} k^2\ u^2(k) \qquad \text{III-3.48}$$

or

$$u^2(k) = \tilde{v}^2(k) \cos^2 \delta_0(k) \qquad \text{III-3.49}$$

and III-3.46 becomes

$$\tan \delta_1(k) = \frac{k\ u^2(k)}{1 - \frac{2}{\pi} P \int_0^\infty \frac{p^2\ u^2(k)}{p^2-k^2}\, dp} \qquad \text{.III-3.50}$$

From III-3.40, for the interval $0 \leqslant p \leqslant \infty$,

$$\frac{P}{p^2-k^2} = \frac{1}{p^2-k^2-i\varepsilon} - \frac{i\pi}{2k} \delta(p-k) \qquad \text{III-3.51}$$

so

$$\frac{k\,u^2(k)}{\tan\,\delta_1(k)} = 1 - \frac{2}{\pi}\int_0^\infty \frac{p^2\,u^2(p)}{p^2-k^2-i\varepsilon}dp + ik\,u^2(k) \qquad \text{III-3.52}$$

which after some elementary trigonometry and algebra, is

$$-e^{i\delta_1(k)}\,\sin\,\delta_1(k) = -k\,u^2(k)/[1 - \frac{2}{\pi}\int_0^\infty \frac{p^2\,u^2(p)}{p^2-k^2-i\varepsilon}\,dp] \qquad \text{III-3.53}$$

This is exactly the basic inversion formula III-2.4 for the case of $\sigma = -1$ corresponding to an attraction. Also from the conditions III-3.2 and III-3.3 placed on $\delta_1(k)$, it is seen that this is again exactly the case of the phase-shift for an attractive potential with no bound states. Without further ado , we may use III-2.17, III-2.19 and write

$$u^2(k) = e^{-\Delta(k)}\,\frac{\sin\,\delta_1(k)}{k} \qquad \text{III-3.54}$$

where

$$\Delta(k) = \frac{P}{\pi}\int_{-\infty}^{+\infty} \frac{\delta_1(p)}{p-k}\,dp \qquad \text{III-3.55}$$

or

$$u^2(k) = \frac{\sin\,\delta_1(k)}{k}\,e^{-\frac{2}{\pi}P\int_0^\infty \frac{\delta_1(p)p}{p^2-k^2}\,dp} \qquad \text{III-3.56}$$

We note that III-3.56 is free of any irregular behaviour such as the occurence of positive energy degeneracies and $u(k)$ is the form factor of an attractive potential which when acting alone will produce the phase shift $\delta_1(k)$.

It is left only to separate g(k) from III-3.21 and III-3.47. So

$$g(k) = \int_0^\infty dr \, \frac{\sin kr}{k} \left[\int_0^\infty dp \, \tilde{v}(p) \, c(p) \, \phi(p,r) \right] \qquad \text{.III-3.57}$$

Using III-3.9,10,11 and changing the order of integration

$$g(k) = \tilde{v}(k) \cos^2 \delta_0(k) - \frac{2}{\pi} h(k) \, P \int_0^\infty \frac{\tilde{v}(p) p^2}{D_0(p)} \frac{\cos^2 \delta_0(p) h(p)}{p^2 - k^2} \, dp \qquad \text{III-3.58}$$

$$= u(k) \cos \delta_0(k) - h(k) \frac{2}{\pi} P \int_0^\infty \frac{p^2 \, u(p) \cos \delta_0(p) h(p)}{D_0(p) \, p^2 - k^2} \, dp \qquad \text{III-3.59}$$

where we have used III-3.49. Further simplification is possible from the on-shell limit of the R-matrix,

$$R_0(k,k) = - \frac{\tan \delta_0}{k} = \frac{h^2(k)}{D_0(k)} \qquad \text{.III-3.60}$$

And so

$$g(k) = u(k) \cos \delta_0(k) + h(k) \frac{2}{\pi} P \int_0^\infty \frac{p \sin \delta_0(p)}{h(p) \, (p^2 - k^2)} \, dp. \qquad \text{III-3.61}$$

This is the final result of the inversion. To summarize, an arbitrary repulsive form factor h(k) is chosen subject only to the conditions III-3.2 and III-3.3. This is used to calculate $\delta_0(k)$ and $\delta_1(k)$ from which equations III-3.56 and III-3.61 may be evaluated to yield the attractive form factor g(k).

The off-shell properties of the final potential may be examined by means of the half off-shell R-matrix. For a rank-two separable potential of the form III-3.4, equation II-2.32 can be readily solved to give

$$R(p,k) = \{- g(k)\, g(p)\, [1 + H(k)] + h(k)\, h(p)\, [1 - G(k)]$$

$$+ [g(k)\, h(p) + g(p)\, h(k)]\, M(k)\}/D(k^2) \qquad \text{.III-3.62}$$

Here,

$$D(k^2) = [1 - G(k)]\, [1 + H(k)] + M^2(k) \qquad \text{,III-3.63}$$

$$H(k) = \frac{2}{\pi} P \int_0^\infty \frac{h^2(p)p^2}{p^2-k^2}\, dp \qquad \text{,III-3.64}$$

$$M(k) = \frac{2}{\pi} P \int_0^\infty \frac{g(p)\, h(p)\, p^2}{p^2-k^2}\, dp \qquad \text{,III-3.65}$$

and

$$G(k) = \frac{2}{\pi} P \int_0^\infty \frac{g^2(p)p^2}{p^2-k^2}\, dp \qquad \text{.III-3.66}$$

The on-shell phase shifts may be calculated from

$$\tan \delta(k) = \frac{k\, g^2(k)[1+H(k)]-k\, h^2(k)[1-G(k)]-2k\, g(k)h(k)M(k)}{D(k^2)} \qquad \text{.III-3.67}$$

This may be compared with the input 'experimental' phases and provides a useful check on the numerical accuracy of the inversion process.

It is of interest to note that III-3.61 is considerably simpler than the corresponding result of Fiedeldey (1969) who started with the attractive form factor. This necessitated the inclusion of a bound state in the initial rank-one potential and the associated more complex completeness relationship. How to eliminate the bound state from the total interaction is not immediately apparent. The equivalent rank-one inversion would be for a phase-shift satisfying $\delta_1(0) - \delta_1(\infty) = -\pi$, which is a special case of a type which Chadan (1958) was unable to solve. On the other hand, assuming an initial repulsive form factor eliminates any consideration of a bound state if the physical system does not exhibit one. Even in the event of a bound state, it is felt that this would fit naturally into the present scheme of things. The equivalent rank-one inversion would be for a phase-shift obeying $\delta_1(0) - \delta_1(\infty) = +\pi$, which is just the standard rank-one inversion including a bound state but no sign change. The solution to this problem is given by equation II-2.20.

There have been several objections to Tabakin's rank-one type potentials. We feel that our present investigation has given us some insight into the problem.

Leung and Park (1969) have published some observations on this type of potential. They fit the S-wave α-α scattering data and then calculate the nuclear binding energy of α-α. This value of 3.6 MeV turns out to be higher

than the 2.8 MeV obtained using a Yamaguchi (1954) type form factor. They conclude that the 'repulsive' part of a Tabakin type potential helps increase the two-body binding energy. There is however no basis for this. From the point of view of the inverse problem, it is an accepted fact. (Newton (1966), Chong and Nogami (1970)) that the binding energy and phase shifts are independent quantities and hence any potential designed to fit the one will generally result in a fortuitous value for the other. In fact, apart from the inverse problem, in the explicit $^3S_1$ potential derived by Tabakin (1968) one of the parameters was used to fit the deuteron binding energy. Other comments made in this paper on the attractive or repulsive nature of the form factors make little sense as well. It is interesting that this same investigation yielded reasonable results for a three-body calculation of C^{12}. This is possibly peculiar in view of the 355 MeV binding energy obtained by Beam (1969) in a triton calculation using the same potential. Alessandrini and Canal (1968) experienced similar difficulties with the Tabakin triplet interaction. Beam recalculated using the Tabakin singlet and triplet potentials independently and concluded that both potentials exhibit peculiar effects. His explanation is that the potential doesn't really fit the high energy phases at all and that the phase-shift rather than passing through zero to negative values actually has a discontinuity of π at $k = k_c$. He claims that this is

necessary 'if the phase-shift is to be stable with respect to small variations in the interaction'. We feel that this isn't the proper explanation. In fact, for the precise value of the parameters as chosen by Tabakin (1968), the phase shift actually does pass through zero to negative values. Even if it did not, a discontinuity of π would not effect matters at all, the phase only being defined to within modulo π. The phase-shifts are, however, unstable with respect to small variations in the interaction, a fact first pointed out by Gourdin and Martin (1957). This is readily seen in the context of the rank-two inversion.

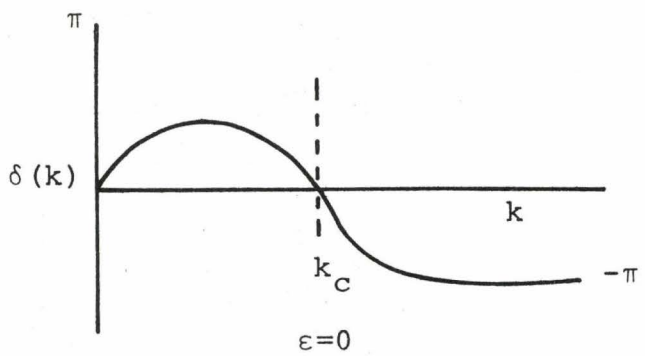
Consider equation III-2.5 in the vicinity of $k=k_c$. For the actual potential, this may be expanded as

$$\tan \delta(k) = \frac{-C(k-k_c)^2}{k-k_c} \qquad \text{III-3.68}$$

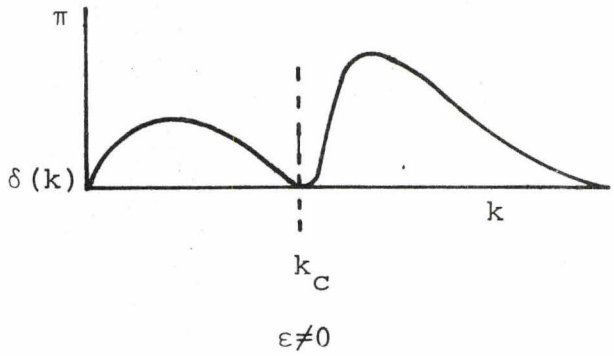
where C is some positive constant. For a small variation in the parameters of the potential, the expression will be

$$\tan \delta(k) = \frac{-C(k-k_c)^2}{k-k_c-\varepsilon} \qquad \text{.III-3.69}$$

Here we arbitrarily assume ε to be a positive quantity. The behaviour of the phase-shifts is illustrated in figures III-1.1 and III-1.2



III-1.1



III-1.2

It is thus critical that $\varepsilon=0$, since $\varepsilon\neq 0$ no matter how small will result in a resonance rather than a sign change. Now consider a potential V

$$V = \Delta V + V_0 \qquad \text{III-3.70}$$

where V_0 is a rank-one potential producing a phase-shift $\delta_0(k)$ with a sign change and ΔV is some arbitrarily small perturbation. If $\delta(k)$ is the phase-shift for the total interaction, then for stability, small changes in V should result in small changes in δ. However, the parameters of V_0 have been delicately chosen to get the proper behaviour of δ_0. Further, if δ is to undergo only small changes from the addition of ΔV, from our experience with the inverse problems for the superposition of two potentials, it follows that V_0 must have its parameters adjusted slightly. However, the actual values of the parameters is critical, in that

it represents the difference between a sign change and a resonance at $k=k_c$. It is obvious that if we superimpose just any ΔV on V_0 that it will result in a resonance effect for the total interaction. We feel that it is this instability of δ that is peculiar to potentials of this type and not that the potential doesn't give a proper fit to the data. Precisely how this will effect the triton binding energy is not immediately obvious. It should prove interesting to do neutron-deutron scattering calculation using Tabakin's rank-one potential for the nucleon-nucleon interaction. The total interaction experienced by the neutron will be a perturbation of the one-term potential and should give rise to a resonance effect. It will only be in the presence of the third particle that this effect will be present, the potential fitting the two-body data perfectly well. Of course, what has been said applies equally to any two-body problem if the interaction consists of two parts, one of which is given by a rank-one type potential.

III-4. PHASE EQUIVALENT PAIRS OF LOCAL AND SEPARABLE POTENTIALS

Srivastava (1970) has taken three well known separable potentials for the $^1S_0$ state and constructed, using the Marchenko (1950) procedure, the corresponding phase-equivalent local potentials. This was accomplished by obtaining the S-matrix from a knowledge of the Jost function

for the separable interaction. Allquantities can be calculated analytically and result in a rational S-matrix of the form

$$S(k) = \prod_{m=1}^{n_\alpha} \left[\frac{k-i\alpha_m}{k+i\alpha_m}\right] \prod_{n=1}^{n_\beta} \left[\frac{k+i\beta_n}{k-i\beta_n}\right] \qquad \text{III-4.1}$$

where the α's and β's are potential dependent constants. Application of the Marchenko inversion method yields a local potential of the form

$$V(x) = -2 \frac{d^2}{dx^2} [\log \Delta(x)] \qquad .\text{III-4.2}$$

These potentials, consisting of sums of negative exponentials in x, are of the generalized Bargmann type. The detailed description of the $\Delta(x)$'s is left to Appendix B. The reader is referred to Srivastava (1970) for a more detailed description of the construction procedure. The local potentials have been plotted and appear at the end of this section.

We now describe the three relevant separable potentials.

CASE I: The rank-two separable potential of Mongan (1969). This is his case II which he claims to be the best fit in his series of potentials. The fit has been made to the nucleon-nucleon scattering data of MacGregor et. al (1968). The potential is of the same general form as III-3.4 where both of the form factors are of the Yamaguchi (1954)

type

$$g(k) = \frac{\rho}{k^2+a^2} \qquad \text{.III-4.3}$$

For the attractive form factor

$$\rho_A = 5.319 \text{ fm}^{-3/2} \qquad a_A = 1.786 \text{ fm}^{-1}$$

and for the repulsive form factor

$$\rho_R = 58.776 \text{ fm}^{-3/2} \qquad a_R = 6.157 \text{ fm}^{-1} \quad .$$

CASE II: The two term separable potential of Tabakin (1964). This is again of the same form as III-3.4 . The attractive form factor is of the Yamaguchi type III-4.3 with

$$a_A = 1.1990 \text{ fm}^{-1} \text{ and } \rho_A = 1.8306 \text{ fm}^{-3/2} \quad ,$$

while the repulsive form factor is given by

$$h(k) = \frac{\rho_R \, k^2}{\{(k-d)^2+b^2\}\,\{(k-d)^2+b^2\}} \qquad \text{III-4.4}$$

with $b = 1.2484 \text{ fm}^{-1}$, $d = 1.4409 \text{ fm}^{-1}$ and $\rho_R = 2.6632 \text{ fm}^{-3/2}$.

This potential was specifically designed to replace the hard core of the local interactions so as to be practicable in nuclear Hartree-Fock calculations. It was constructed to fit the phase parameters of Breit et. al. (1962), have small off-shell matrix elements and to saturate nuclear

energy and density.

CASE III: This is the single term separable potential of Tabakin (1968) which contains both attraction and repulsion. This was developed as the simplest possible potential to fit the scattering data, mainly in view of simplifying three-body calculations. The measures taken to construct such a potential have aroused some controversy and have already been discussed in section III-3. The form factor is

$$g(k) = \frac{\rho(k_c^2-k^2)\ (k^2+d^2)}{(k^4+a^4)\ (k^2+b^2)} \qquad \text{III-4.5}$$

where $\rho^2 = 400.8434\ \text{fm}^{-3}$, $k_c = 1.7\ \text{fm}^{-1}$, $a = 4.05\ \text{fm}^{-1}$, $b = 1.08548\ \text{fm}^{-1}$ and $d = 1.683$.

III-5. RANK-TWO INVERSION FOR THE $^1S_0$ INTERACTION

In this section we describe our numerical investigation of phase equivalent potentials constructed on the basis of the theory developed in III-3. Briefly, we have taken the Mongan two-term potential as described in III-4 as our source of experimental phase shifts and keeping the repulsive form factor of the Yamagauchi type, have varied the strength and range parameters ρ_R and a_R from their initial values of $58.776\ \text{fm}^{-3/2}$ and $6.157\ \text{fm}^{-1}$ respectively. The experimental phase shifts were then inverted to produce the attractive form factor g(k).

Numerically, the repulsive form factor $h(k) = \rho_R/(k^2+a_R^2)$ was used to generate the phases $\delta_0(k)$ and thus

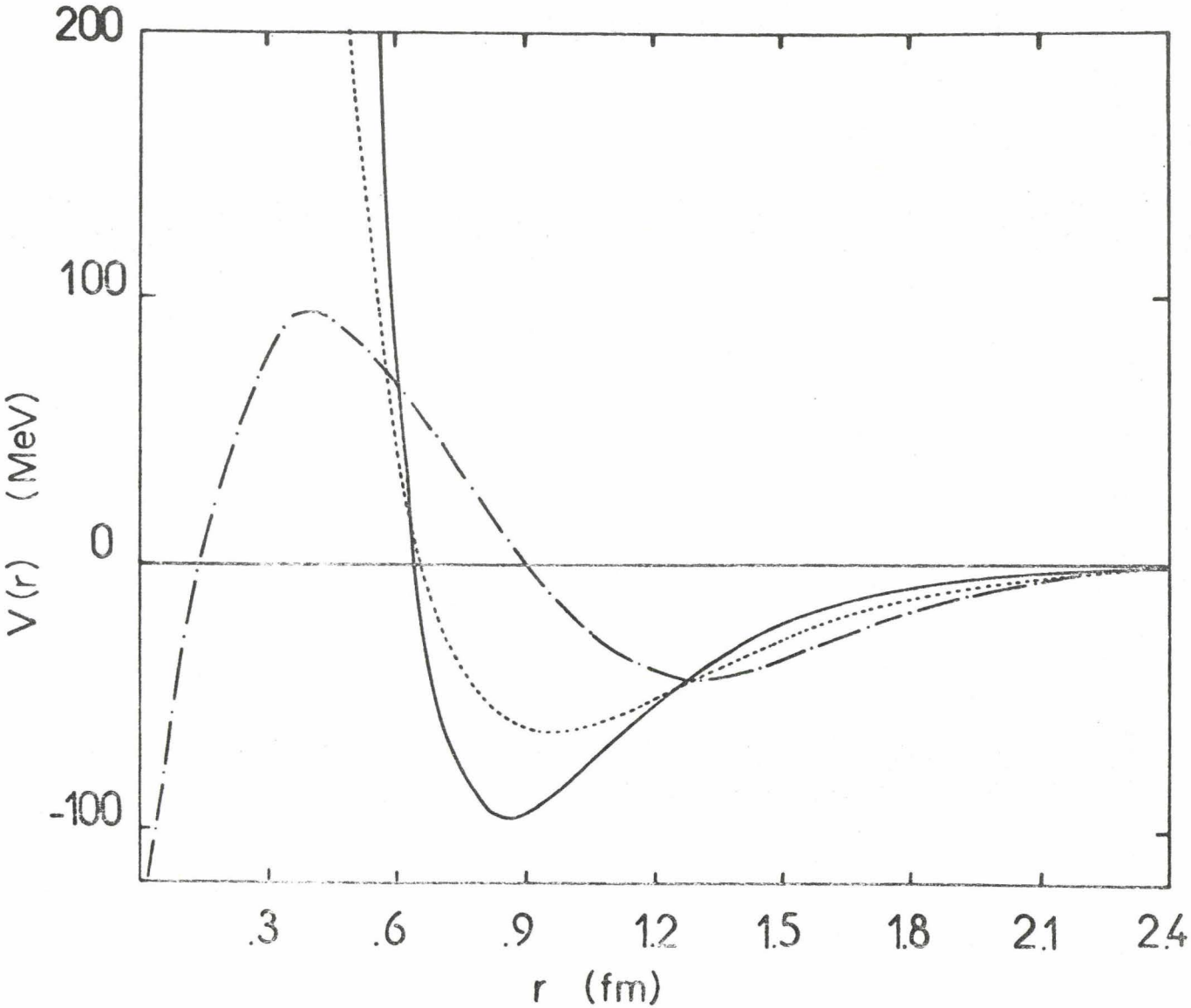


FIGURE III-4.1: EQUIVALENT LOCAL POTENTIALS OF CASE I (MONGAN, DOTTED LINE) AND CASE II (TABAKIN TWO TERM, DASH-DOT LINE) COMPARED TO REID SOFT CORE POTENTIAL (SOLID LINE).

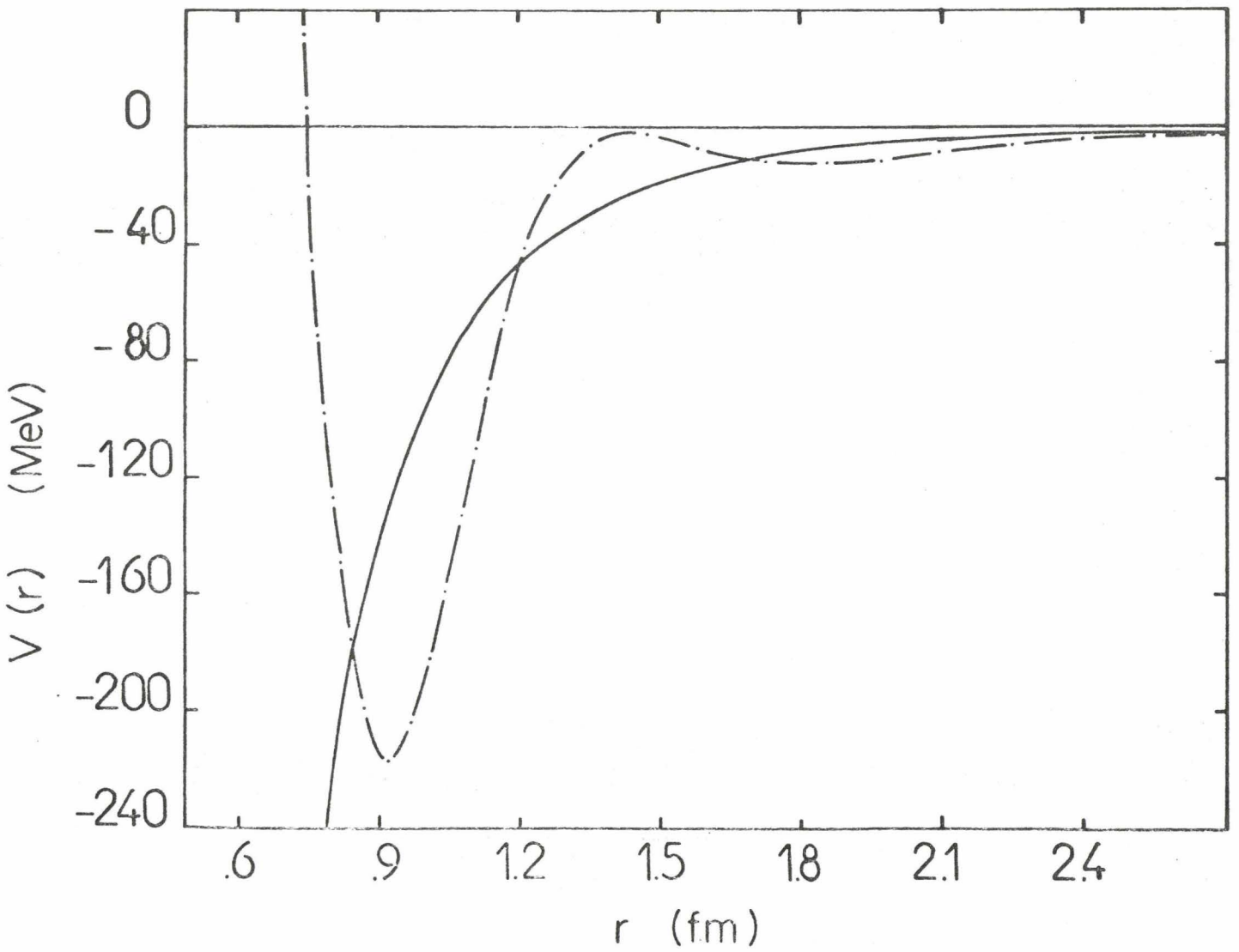


FIGURE III-4.2: EQUIVALENT LOCAL POTENTIAL OF CASE III (TABAKIN ONE TERM, DASH-DOT LINE) COMPARED TO HAMADA-JOHNSTON POTENTIAL (SOLID LINE). THE DISTANCE, IN fm, STARTS AT THE HAMADA-JOHNSTON HARD CORE RADIUS.

$\delta_1(k)$ from III-3.2. The attractive form factor follows from the two numerical integrations III-3.56 and III-3.61. The half-off-shell R-matrix, III-3.62, is given after the further integrations of III-3.63, to III-3.66. All of the above integrals are of the Cauchy principle value type and require careful numerical treatment. Integrals of this type were reduced to the form

$$\frac{2}{\pi} P \int_0^\infty \frac{f^2(p)}{p^2-k^2} p^2 dp = \frac{2}{\pi} \int_0^\infty \frac{f^2(p)p^2-f^2(k)k^2}{p^2-k^2} dp \qquad .III-5.1$$

This is the usual device which removes the singularity at p=k and improves the numerical accuracy of the integral. The integrations were done by the method of Gauss quadratures, the coefficients being taken from Abramowitz and Stegum (1965). This method requires a domain of integration from -1 to +1. It was found that the off-shell effects were much more sensitive to variations in the range parameter as opposed to the rather small effects obtained by varying the strength parameter. The numerical work greatly restricted parameter variation. For this reason we have limited ourselves to a variation in a_R for ρ_R equal to its initial value.

Of the calculations completed, it was found that the best results were obtained by breaking the range of integration into three regions, (0,a), (a,b) and (b,∞). The object here was to distribute the points of the integration so that more points fell in the regions with more structure.

This is the weak point of the numerical method used. A preliminary run using only one region of integration was found to be sensitive to the particular transformation used to gain the interval (-1,+1), but as the parameters were further varied the accuracy invariably dropped. The order of the quadrature was also increased, the highest considered being an 80 point effort. In retrospect, it would certainly be more efficient and probably more accurate to break the integration into a large number of intervals and keep the order of the quadrature low.

For the case of three intervals, the transformation used in the first interval was

$$\int_0^a dt = \int_{-1}^{+1} \frac{dt}{dz}\, dz \quad \text{where} \quad t = \frac{a}{2}(1+z) \qquad .$$

The corresponding transformations for the intervals (a,b) and (b,∞) were $t = \frac{a}{2}(1-z) + \frac{b}{2}(1+z)$ and $t = 2b/(1-z)$ respectively.

Practically our calculations were limited first by the requirement that δ_1, be > 0 and secondly by numerical accuracy. The first placed a restriction on the repulsive form factor in that it has to be nearly as repulsive as the original form factor. As a restriction on the numerical accuracy we required that the phases calculated from the potential obtained numerically agree with the experimental input phases to within 0.0002. This limit in the accuracy was reached for very low energies, the accuracy for higher

energies always being much greater. With this accuracy it was found possible to vary the range parameter from its original value of 6.157 fm^{-1} to a value of 1.0 fm^{-1}, keeping the strength parameter fixed at $\rho_R = 58.776\ fm^{-3/2}$. The phase-shifts δ_0, δ_1 and δ are plotted in figure III-5.1. We have kept only those examples with $a_R < 4.0\ fm^{-1}$, the larger values giving essentially the same off-shell values as the original potential. The limitations on numerical accuracy came mainly from the first integral III-3.56 over $\delta_1(k)$. In particular we found it necessary to vary the region of integration so that a and b fell on about the centers of the regions of greatest slope for the hump in the tail of the curve $\delta_1(k)$. Plots of the attractive and repulsive form factors are presented in figures III-5.2 and III-5.3. We note the great variation, particularly in the small k region. The repulsive and attractive form factors are seen to closely approximate each other as the range parameter is decreased.

The Kowalski-Noyes half-off-shell matrix elements, f(p,k), for the above potentials have been plotted for k values corresponding to laboratory energies of 20, 99, 158 and 300 Mev for values of p up to $5fm^{-1}$. These appear in figures III-5.4 through III-5.7 together with the phase equivalent local potential of section III-4. We note that f(k,k)=1 and that, in the vicinity of p=k for p>k, the slopes of all curves including the local potential are very nearly the same. As

p increases the matrix elements gradually separate until at $p=5fm^{-1}$ they differ by as much as a factor of 3, and are still diverging. For $p<k$ the elements vary more rapidly, especially if considered as a function of energy, to give a variation of about 15% at $p=0$. This is not as much as the difference between the local potential and the nearest separable curve, which are all characteristically larger. For high p the local matrix elements show a rapid decrease in magnitude sround $p=3.5fm^{-1}$, where the separable curves continue to increase in magnitude.

In figure III-5.5 we have included the Tabakin Two and One term separable potentials as well. This gives some indication of the variation possible from using different types of form factors. We should mention that all results show none of the drastic off-shell behaviour as found by Fiedeldey (1969) for the $^3S_1$ potentials. This is as expected, the large effects being closely associated with the deuteron binding energy.

Preliminary calculations varying the input 'experimental' phases for energies greater than 350 Mev indicate only small variations in off-shell properties. The inverse procedure commencing with the attractive form factor is more suitable for this type of calculation.

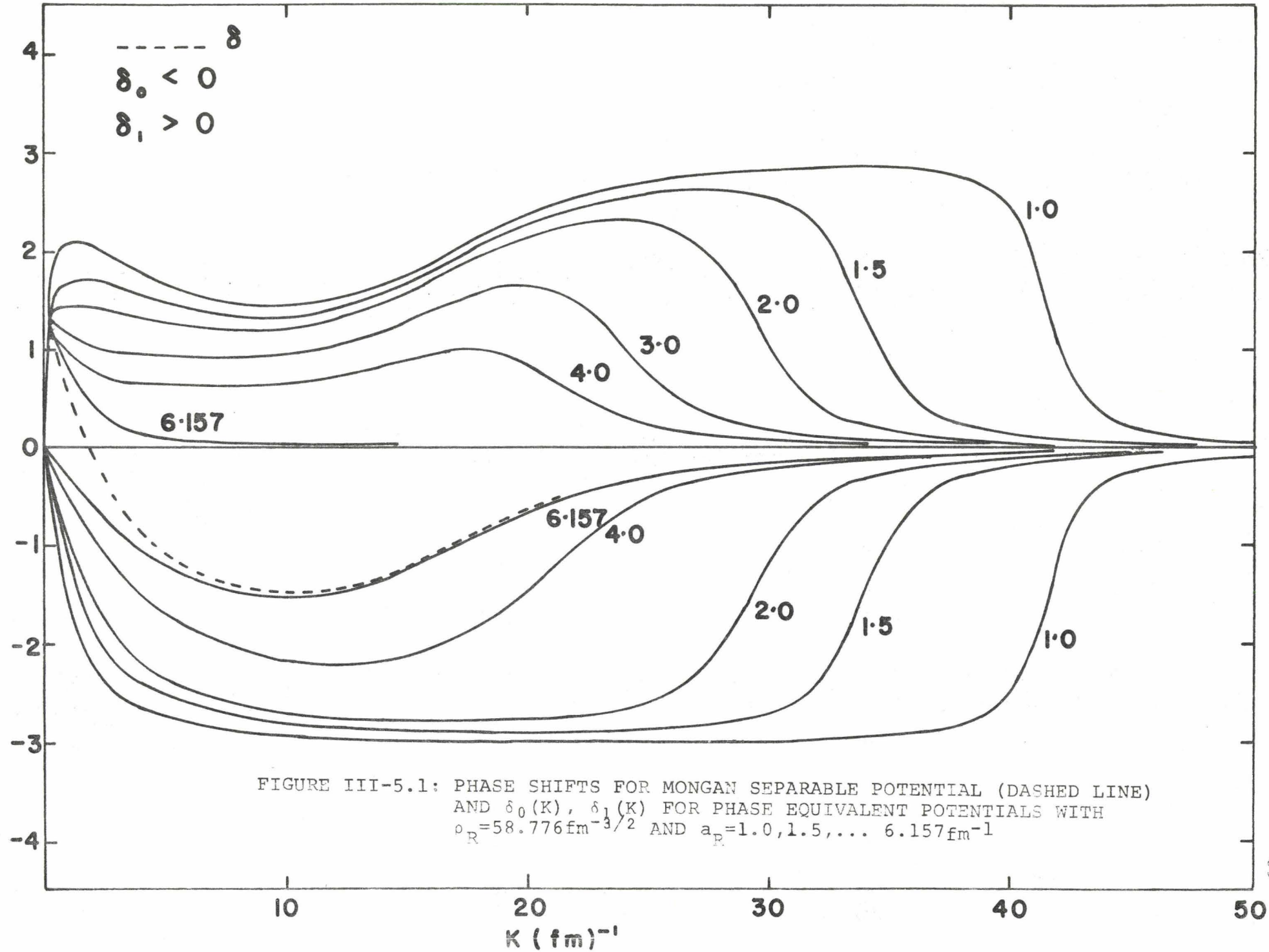


FIGURE III-5.1: PHASE SHIFTS FOR MONGAN SEPARABLE POTENTIAL (DASHED LINE) AND $\delta_0(K)$, $\delta_1(K)$ FOR PHASE EQUIVALENT POTENTIALS WITH $\rho_R = 58.776 fm^{-3/2}$ AND $a_R = 1.0, 1.5, \ldots 6.157 fm^{-1}$

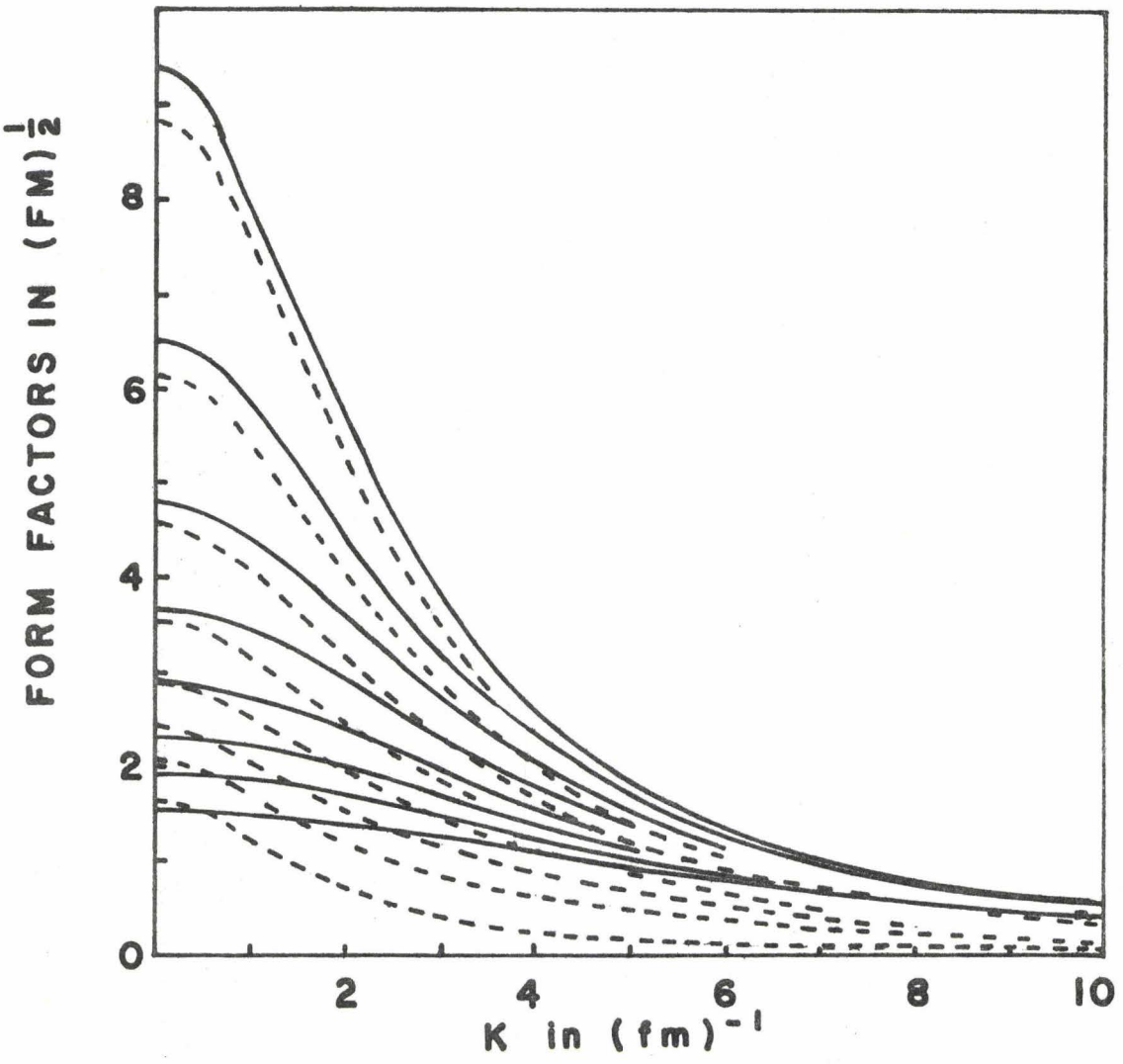


FIGURE III-5.2: ATTRACTIVE (DASHED LINES) AND REPULSIVE (SOLID LINES) FORM FACTORS FOR ρ_R=58.776fm$^{-3/2}$ AND a_R=2.5,3.0,3.5,4.0, 4.5,5.0,5.5 and 6.157fm^{-1} (TOP TO BOTTOM)

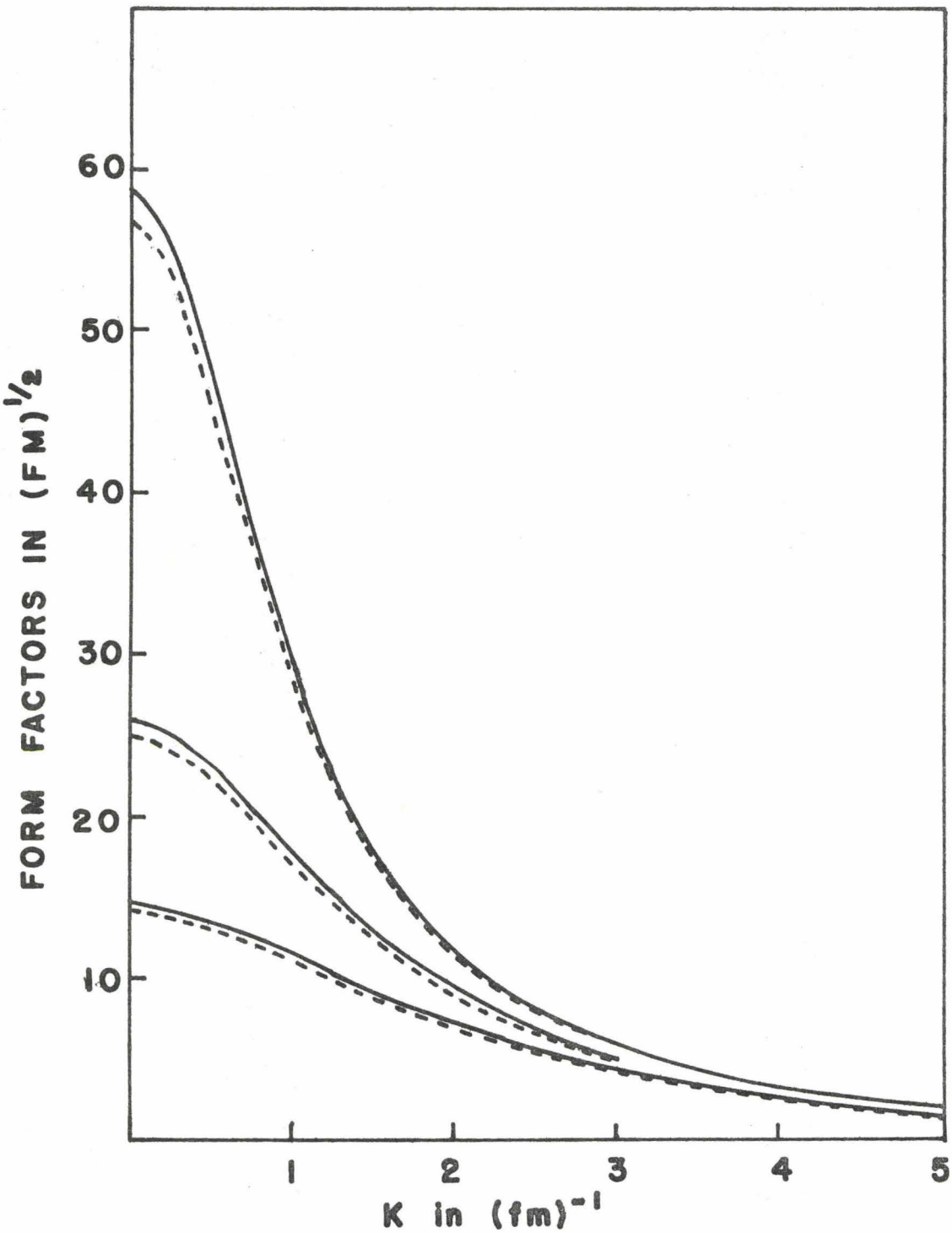


FIGURE III-5.3: ATTRACTIVE (DASHED LINES) AND REPULSIVE (SOLID LINES) FORM FACTORS FOR ρ_R=58.776fm$^{-3/2}$ AND a_R=1.0,1.5, 2.0fm^{-1} (TOP TO BOTTOM)

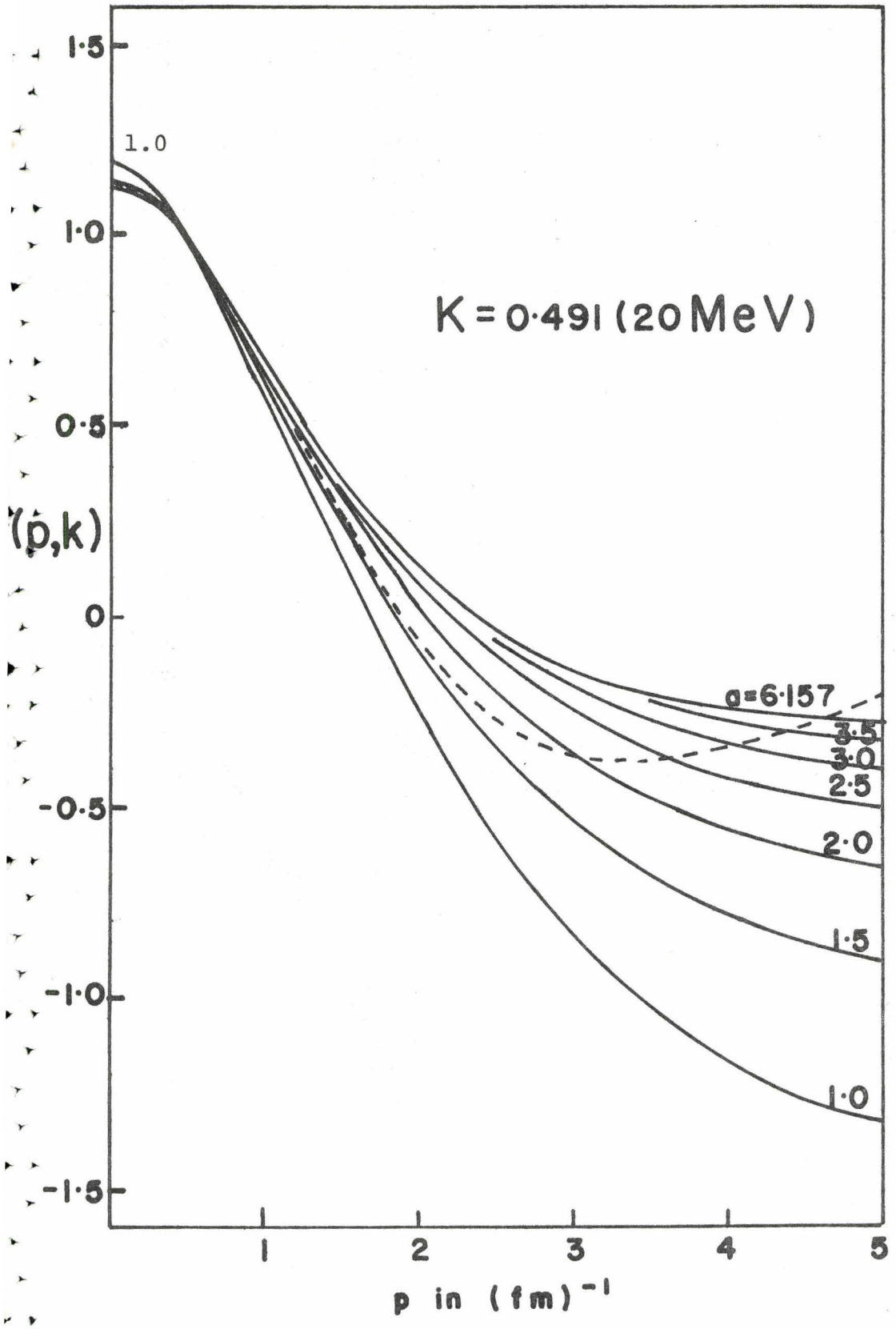


FIGURE III-5.4: KOWALSKI-NOYES HALF-OFF-SHELL MATRIX ELEMENTS FOR PHASE EQUIVALENT RANK-TWO SEPARABLE POTENTIALS (ρ_R=58.776fm$^{-3/2}$) AND THE PHASE EQUIVALENT LOCAL POTENTIAL OF SRIVASTAVA AND SPRUNG (DASHED LINE)

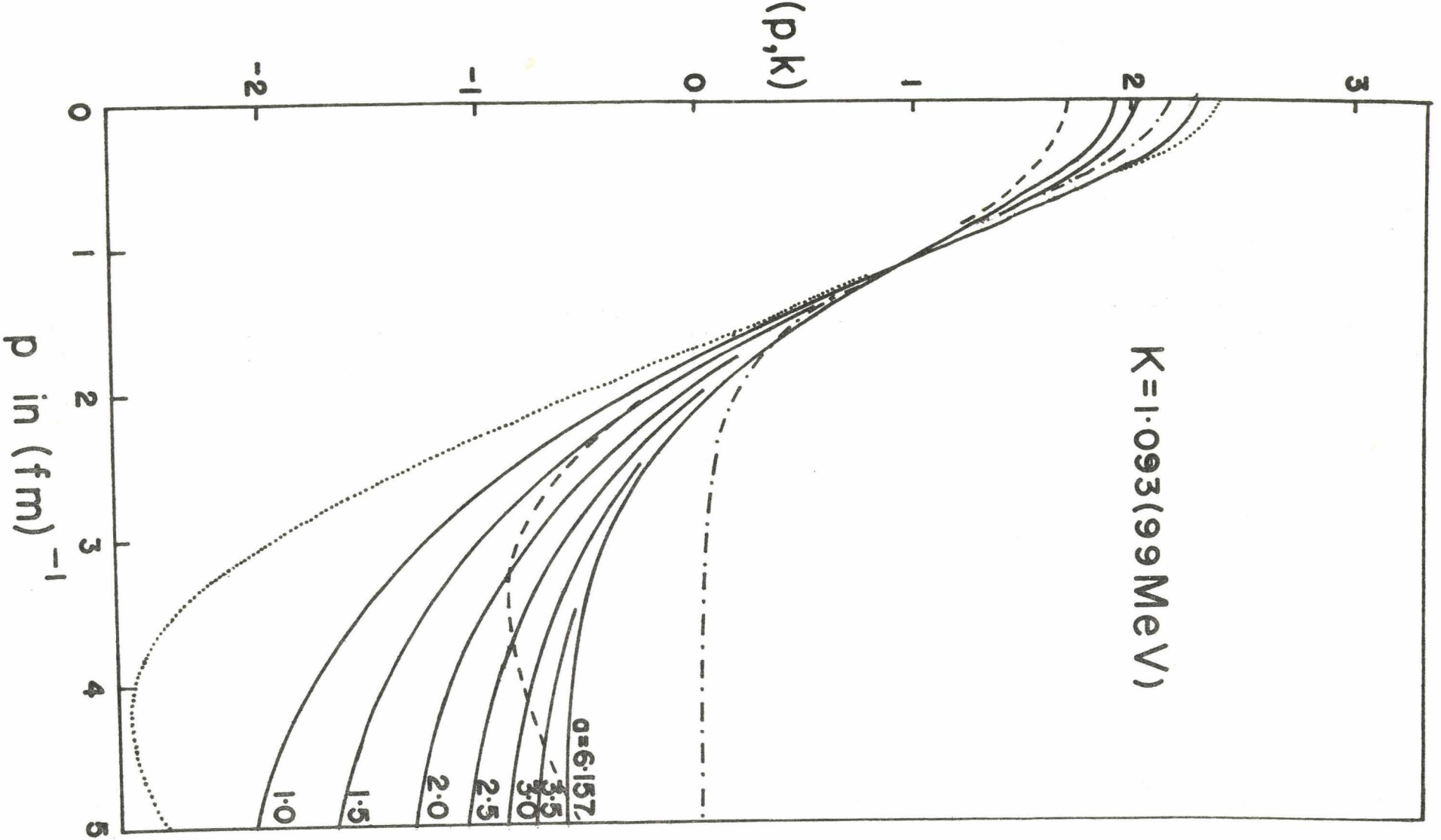


FIGURE III-5.5: KOWALSKI-NOYES HALF-OFF-SHELL MATRIX ELEMENTS FOR PHASE EQUIVALENT RANK-TWO SEPARABLE POTENTIALS (ρ_B=58.776fm$^{-3/2}$), THE PHASE EQUIVALENT LOCAL POTENTIAL OF SRIVASTAVA AND SPRUNG (DASHED LINE), AND THE TABAKIN TWO TERM (DASH DOT LINE) AND ONE TERM (DOTTED LINE) SEPARABLE POTENTIALS.

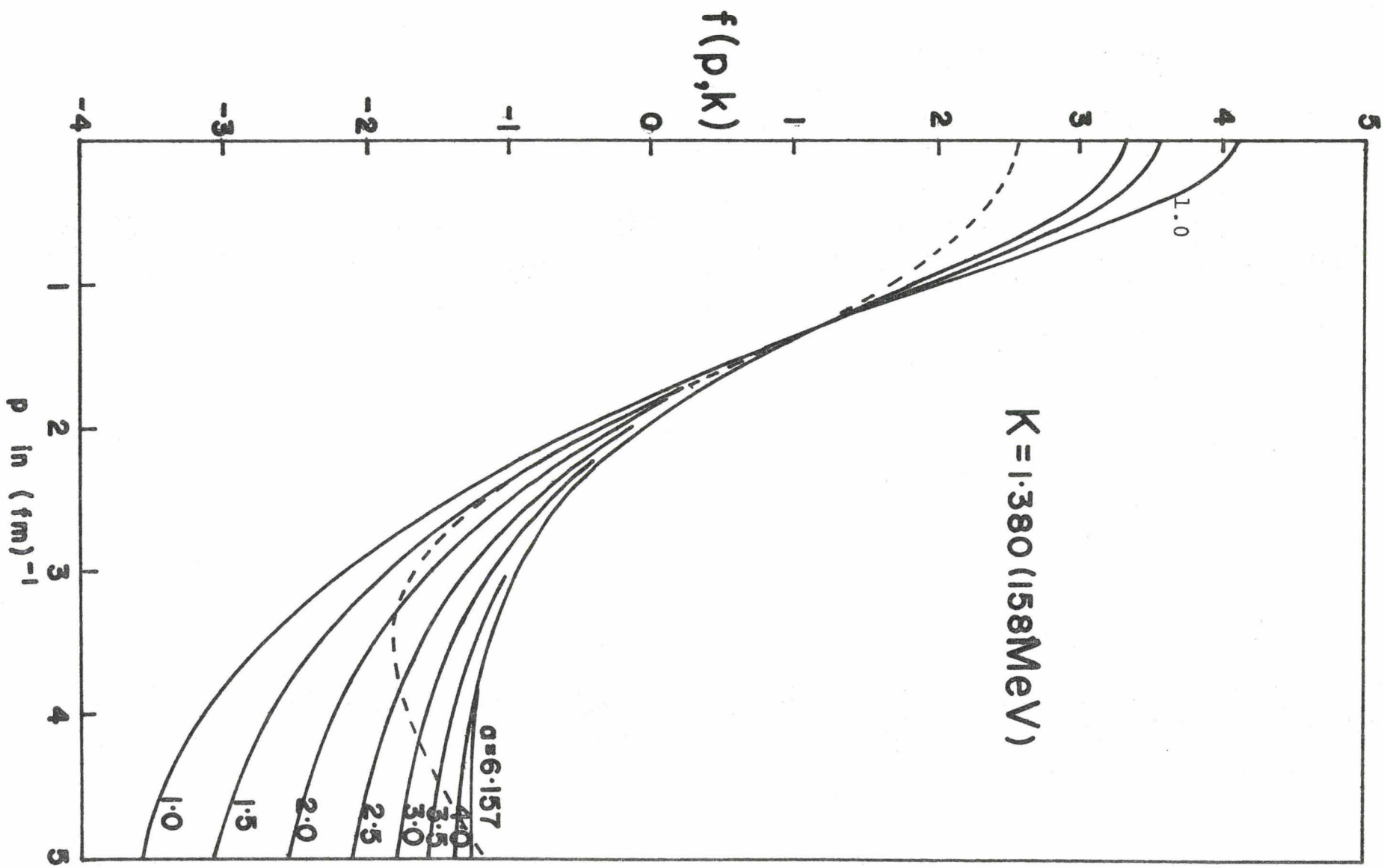


FIGURE III-5.6: KOWALSKI-NOYES HALF-OFF-SHELL MATRIX ELEMENTS FOR PHASE EQUIVALENT RANK-TWO SEPARABLE POTENTIALS (ρ_R=58.776fm$^{-3/2}$) AND THE PHASE EQUIVALENT LOCAL POTENTIAL OF SRIVASTAVA AND SPRUNG (DASHED LINE)

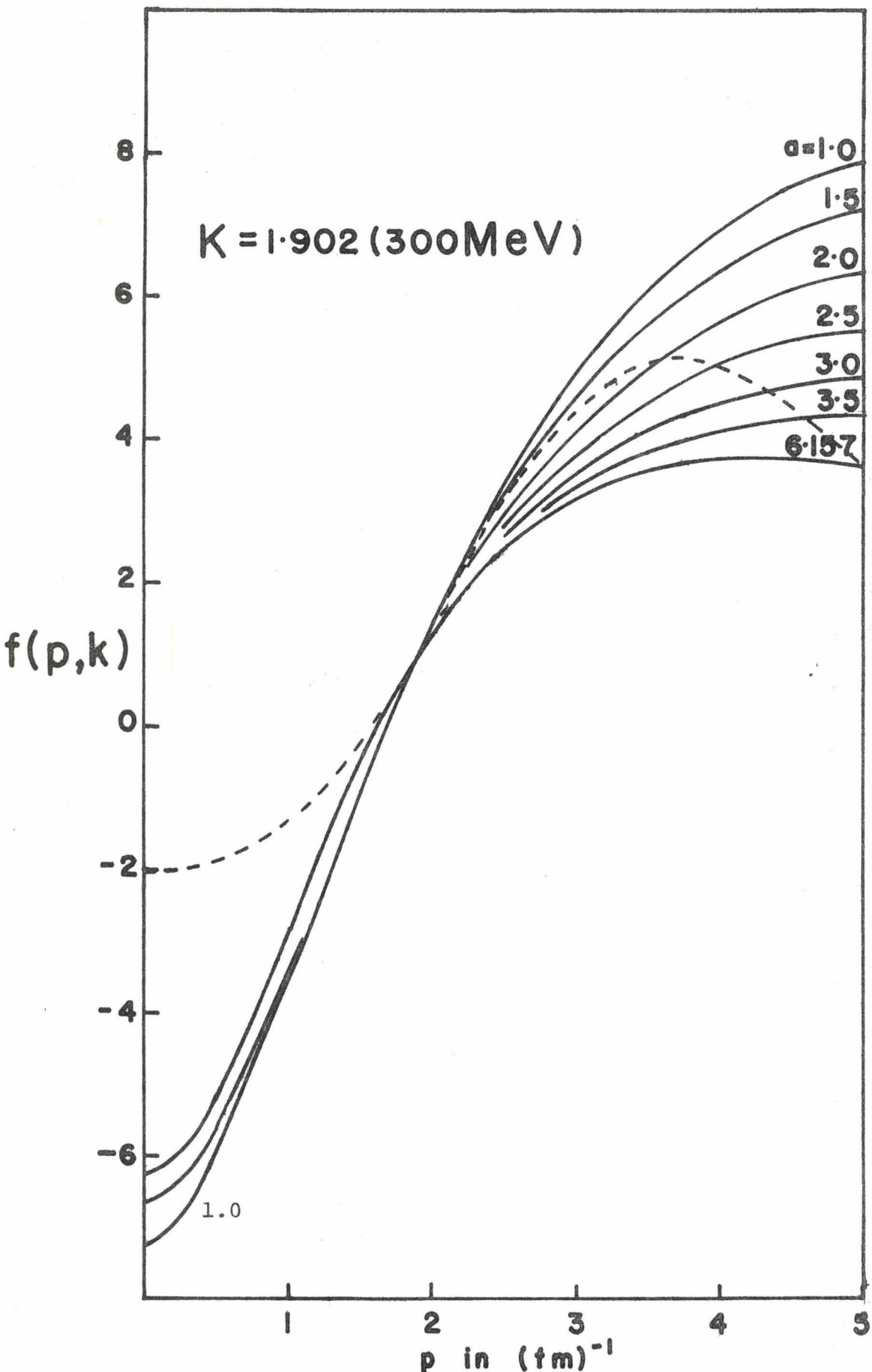


FIGURE III-5.7: KOWALSKI-NOYES HALF-OFF-SHELL MATRIX ELEMENTS FOR PHASE EQUIVALENT RANK-TWO SEPARABLE POTENTIALS (ρ_R=58.776fm$^{-3/2}$) AND THE PHASE EQUIVALENT LOCAL POTENTIAL OF SRIVASTAVA AND SPRUNG (DASHED LINE)

CHAPTER IV

ON THE VELOCITY DEPENDENT POTENTIAL

IV-1. INTRODUCTION

In deriving the nucleon-nucleon potential from meson theory, one obtains a local potential if one makes the static approximation in which the nucleon-recoil effect is ignored. The static approximation is well justified for the long-range part of the potential because the momentum transfer involved is small. At short distances, however, the validity of the static approximation is questionable. The nucleon recoil effect gives rise to a non-local or velocity dependent potential. In this sense, the nucleon-nucleon potential will be predominantly local at large distances while it becomes more non-local or velocity-dependent at shorter distances.

In the phenomenological approach to the nuclear potential, the strong repulsion at short distances, such as the hard core in the Hamada-Johnston potential, can be effectively replaced by a velocity dependent potential (Baker 1962). Various velocity dependent forces have been used with the hope that they may facilitate calculation of the many-body problem (Coester et. al. 1970).

Recently some questions have been raised concerning

the Hamiltonian and Lagrangian formulations with a velocity dependent potential (Razavy 1968, Kiang et. al. 1969). Razavy claims that for a class of velocity dependent potentials the Hamiltonian and energy of the system are not simply related, and, while the former is a constant of motion and does not depend on time explicitly, the latter quantity is time-dependent, and the Heisenberg equation of motion is not satisfied. On the other hand, Kiang et. al. claim that the difficulties pointed out by Razavy can be avoided by taking a different Hamiltonian operator. These arguments give the impression that there is some difficulty or inconsistency of a fundamental nature concerning the conventional method of dealing with velocity dependent potentials in quantum mechanics.

The purpose of this chapter is to summarize the arguments presented by Kiang et. al. and Razavy, and to point out some incompleteness in their arguments. We then indicate a possible method of resolving the difficulties within the framework of the Lagrangian formalism. We shall also point out that, if one adopts the second quantization representation, no difficulties arise in the Lagrangian formulation.

To begin with, let us review some relations between velocity dependent potentials and the general non-local interaction.

IV-2. RELATION BETWEEN A VELOCITY DEPENDENT AND A NON-LOCAL POTENTIAL

Let us rewrite equation II-2.1 , the Schrödinger equation for the general case of a non-local interaction.

$$(\nabla^2+K^2)\psi(\underset{\sim}{r}) = \int K(\underset{\sim}{r},\underset{\sim}{r}')\ \psi(\underset{\sim}{r}')d\underset{\sim}{r}' \qquad \text{IV-2.1}$$

The Taylor series expansion of $\psi(\underset{\sim}{r}')$ may be expressed as

$$\psi(\underset{\sim}{r}') = \psi(\underset{\sim}{r}) + (\underset{\sim}{r}'-\underset{\sim}{r})\ \nabla\psi(\underset{\sim}{r}) + \tfrac{1}{2}(\underset{\sim}{r}'-\underset{\sim}{r})^2\ \nabla^2\psi(\underset{\sim}{r}) + \ldots \qquad \text{IV-2.2}$$

to obtain

$$\begin{aligned}\int K(\underset{\sim}{r},\underset{\sim}{r}')\ \psi(\underset{\sim}{r}')d\underset{\sim}{r}' &= \int K(\underset{\sim}{r},\underset{\sim}{r}')d\underset{\sim}{r}'.\psi(\underset{\sim}{r}) + \int K(\underset{\sim}{r},\underset{\sim}{r}')(\underset{\sim}{r}'-\underset{\sim}{r})d\underset{\sim}{r}'.\nabla\psi(\underset{\sim}{r}) \\ &+ \tfrac{1}{2}\int K(\underset{\sim}{r},\underset{\sim}{r}')(\underset{\sim}{r}'-\underset{\sim}{r})^2 dr'.\nabla^2\psi(\underset{\sim}{r}) + \ldots \qquad \text{IV-2.3}\end{aligned}$$

$$= U_0(r)\ \psi(\underset{\sim}{r}) + U_1(r)\underset{\sim}{r}.\nabla\psi(\underset{\sim}{r}) + \tfrac{1}{2}U_2(r)r^2\nabla^2\psi(\underset{\sim}{r}) + \ldots \qquad \text{IV-2.4}$$

where

$$\begin{aligned}\int K(\underset{\sim}{r},\underset{\sim}{r}')d\underset{\sim}{r}' &= U_0(r) \\ \int K(\underset{\sim}{r},\underset{\sim}{r}')(\underset{\sim}{r}'-\underset{\sim}{r})d\underset{\sim}{r}' &= \underset{\sim}{r}\ U_1(r) \qquad \text{IV-2.5} \\ \int K(\underset{\sim}{r},\underset{\sim}{r}')(\underset{\sim}{r}'-\underset{\sim}{r})^2 d\underset{\sim}{r}' &= r^2 U_2(r)\end{aligned}$$

With a suitable rearrangement of terms, 2.4 can be rewritten in the form

$$\bar{U}_0(r)\ \psi(\underset{\sim}{r}) + \frac{1}{2}\bar{U}_1(\underset{\sim}{r})(\underset{\sim}{r}.\underset{\sim}{\nabla}+\underset{\sim}{\nabla}.\underset{\sim}{r})\ \psi(r)$$

$$+ \frac{1}{4}\bar{U}_2(r)[\underset{\sim}{r}.\underset{\sim}{\nabla}+\underset{\sim}{\nabla}.\underset{\sim}{r}]^2\ \psi(\underset{\sim}{r}) + \ldots$$

$$= [\bar{U}_0(r) + \frac{i}{2\hbar}\bar{U}_1(r)(\underset{\sim}{r}.\underset{\sim}{p}+\underset{\sim}{p}.\underset{\sim}{r}) + (\frac{i}{2\hbar})^2\bar{U}_2(r)(\underset{\sim}{r}.\underset{\sim}{p}+\underset{\sim}{p}.\underset{\sim}{r})^2+\ldots]\ \psi(\underset{\sim}{r}) \qquad \text{IV-2.6}$$

If we require time reversal invariance, $(\underset{\sim}{p}.\underset{\sim}{r}+\underset{\sim}{r}.\underset{\sim}{p})$ can only appear quadratically. Also, $(\underset{\sim}{p}.\underset{\sim}{r}+\underset{\sim}{r}.\underset{\sim}{p})^2$ can be expressed in terms of r^2, p^2 and $(\underset{\sim}{r}x\underset{\sim}{p})^2$. Hence the potential must be of the form

$$U(r^2,p^2,L^2) \qquad \text{IV-2.7}$$

The two most commonly used forms of the velocity-dependent potential are $\underset{\sim}{p}.f(r)\underset{\sim}{p}$ and $p^2g(r) + g(r)p^2$. These two forms are essentially equivalent in the sense that $p^2g(r) + g(r)p^2 = 2\underset{\sim}{p}.g(r)\underset{\sim}{p} - (2/r)\ g'(r) - g''(r)$. In the following we shall consider only the form $\underset{\sim}{p}.f(r)\underset{\sim}{p}$.

IV-3. COMPLICATIONS ARISING FROM A VELOCITY DEPENDENT POTENTIAL

To avoid uninteresting complications, let us confine ourselves to the one dimensional case with a coordinate q. Kiang et. al. start with the following Lagrangian in the

Heisenberg picture:

$$L(q,\dot{q}) = \frac{1}{2}\,\dot{q}\,f^{-1}(q)\dot{q} - V(q) \qquad \text{IV-3.1}$$

The canonical momentum for q is defined as

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{1}{2}\{\dot{q}\,f^{-1}(q) + f^{-1}(q)\dot{q}\} \qquad \text{IV-3.2}$$

where $\partial\dot{q}$ can be regarded as a c-number, so that the differentiation in (3.2) is done in the usual manner. The fundamental commutation relation is

$$[p,q] = -i\hbar \qquad \text{IV-3.3}$$

It follows that,

$$[\delta p,q] = [p,\delta q] = 0 \qquad \text{IV-3.4}$$

Hence, provided that p and q are independent variables, δp and δq can be regarded as c-numbers. From (3.2) and (3.3), the commutator of q and $\dot{q}$ is

$$[\dot{q},q] = -i\hbar\, f(q) \qquad \text{IV-3.5}$$

With the help of (3.5), $\dot{q}$ can be expressed in terms of p and q:

$$\dot{q} = \frac{1}{2}\{pf(q) + f(q)p\} \qquad \text{IV-3.6}$$

and the Hamiltonian corresponding to the Lagrangian (3.1)

is given by

$$K(q,p) = \frac{1}{2}(\dot{q}p+p\dot{q})-L$$

$$= \frac{1}{2}p\ f(q)p\ - \frac{1}{8}\ f^{-1}(q)[f'(q)]^2 + V(q) \quad \text{IV-3.7}$$

where

$$f'(q) \equiv df(q)/dq\ .$$

On the other hand, if we derive the Euler-Lagrange equation from the Lagrangian (3.1), or the Hamilton canonical equation from the Hamiltonian (3.7), by means of the variational principle, $\delta\dot{q}$ is no longer a c-number. This is because $\delta\dot{q}$, which is deduced from δq and δp through (3.6), is no longer commutable with q and p unless f(q) is a constant. For example, it is not proper to put $\delta\dot{q} = (d/dt)\ \delta(q)$ in

$$\delta I = \int_{t_1}^{t_2} [\frac{1}{2}\{(\delta\dot{q})p + \dot{q}\delta p + (\delta p)\dot{q} + p\delta\dot{q}\}$$

$$- (\frac{\partial K}{\partial q})\ \delta q - (\partial K/\partial p)\delta p]dt = 0\ . \quad \text{IV-3.8}$$

Hence the usual canonical equations of motion.

$$\dot{q} = \partial K/\partial p\ ,\quad \dot{p} = -\partial K/\partial q \quad \text{IV-3.9}$$

are inconsistent with the canonical commutation relation (3.3). A similar conclusion has been reached by Razavy (1968).

In order to avoid the above difficulty, Kiang et. al. propose to introduce a canonical transformation (q,p)→

(Q,P), where

$$q = g(Q) \ , \quad P = \frac{1}{2}\{g'(Q)p + pg'(Q)\} \qquad \text{IV-3.10}$$

and

$$g'(Q) = dg(Q)/dQ = dq/dQ = f^{1/2}(q) \ . \qquad \text{IV-3.11}$$

Equivalently, one may write

$$Q = \int^{q} f^{-1/2}(s)ds \qquad \text{IV-3.12}$$

$$P = \frac{1}{2}[p\ f^{1/2}(q) + f^{1/2}(q)p] \ . \qquad \text{IV-3.13}$$

The Lagrangian $L(q,\dot{q})$ can be re-expressed in terms of Q and $\dot{Q}$ as

$$L(Q,\dot{Q}) = \frac{1}{2}\dot{Q} - \frac{i}{4}[\dot{Q},g''(Q)(g'(Q))^{-1}] + \frac{1}{32}f'^{2}f^{-1} - V(q) \qquad \text{IV-3.14}$$

where

$$\dot{Q} = \frac{1}{2}[\dot{q}\ f^{-1/2}(q) + f^{-1/2}(q)\dot{q}] \qquad \text{IV-3.15}$$

The commutation relations for the new variables are now

$$[P,Q] = [\dot{Q},Q] = -i\hbar \qquad \text{IV-3.16}$$

where P has been defined by

$$P = \partial L/\partial \dot{Q} = \dot{Q} \qquad \text{IV-3.17}$$

With the new variables, there is no difficulty in the

variational procedure. The new Hamiltonian is given by

$$H(Q,P) = \frac{1}{2}(P\dot{Q}+\dot{Q}P) - L(Q,\dot{Q})$$

$$= \frac{1}{2}P^2 + \frac{1}{4}(\ln g'(Q))'' - \frac{1}{8}[(\ln g'(Q))']^2 + V(R) \qquad \text{IV-3.18}$$

which in terms of the old variables, is

$$H(Q,P) = K(q,p) + \frac{1}{16} f^{-1}(q)[f'(q)]^2 \qquad \text{IV-3.19}$$

By transforming from Q, P back to q, p one can confirm that

$$\dot{q} = \frac{\partial H(q,p)}{\partial p} \ , \ \dot{p} = -\partial H(q,p)/\partial q \qquad \text{IV-3.20}$$

This is what they call a "consistent" formulation.

Let us point out first that the variational procedure which Kiang et. al. have employed, with respect to the Lagrangian (3.1) is ill-defined. They apparently attempt to minimize (or maximize) the quantity

$$I = \int_{t_1}^{t_2} L(q,\dot{q})dt \qquad \text{IV-3.21}$$

which is not a c-number but a q-number. I is an operator, and hence it is not clear what is meant by minimizing it. Also, the Lagrangian $L(q,\dot{q})$ seems to be void of physical significance if one can not derive the equation of motion from it.

IV-4. FEYNMAN'S PRINCIPLE

The usual Hamiltonian presentation of quantum mechanics contains two distinct postulates: the commutation relations between generalized coordinates and momenta, and the equation of motion. In classical mechanics the Lagrangian formalism, which does not involve generalized momenta, can be replaced by the Hamiltonian theory. The laws of mechanics may then be expressed in a single postulate, which can alternatively assume the form of Lagrange's equations or of Hamilton's principle of least action. The obvious question then arises: Is it possible to obtain a similar Lagrangian formulation of quantum laws depending upon a single postulate only? This aim has been achieved in Feynman's formulation. However, we shall point out a possible difficulty in Feynman's theory when the velocity dependent potential is involved.

Before discussing Feynman's formulation, let us discuss the quantization of a classical system with a velocity dependent potential. The prescription originally given by Schrodinger is to replace p by $-i\hbar\,\partial/\partial q$ and the energy E by $i\hbar\,\partial/\partial t$ in the relation

$$H(q,p) = E \qquad \text{IV-4.1}$$

and apply it to the wave function $\psi(q,t)$. That is

$$H\left(q,-i\hbar\frac{\partial}{\partial q}\right)\psi(q,t) = i\hbar\frac{\partial\psi(q,t)}{\partial t} \qquad \text{IV-4.2}$$

This prescription is ambiguous if the classical Hamiltonian is of the form,

$$H(q,p) = \frac{1}{2}p^2 f(q) + V(q) \qquad \text{IV-4.3}$$

Another method of quantization is to postulate that the fundamental Poisson bracket $[q,p] = 1$ in classical mechanics be replaced by the commutation relation

$$[q,p] = i\hbar \qquad \text{IV-4.4}$$

We then obtain quantum mechanics in its Heisenberg representation. The equation of motion is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{i\hbar}[F,H] \qquad \text{IV-4.5}$$

where $F = F(q,p,t)$. This prescription is again ambiguous for the Hamiltonian 4.3.

In order to get rid of this ambiguity, let us propose the following prescription:

1) Perform a canonical transformation $(q,p) \rightarrow (Q,P)$ so that

$$H(q,p) = \frac{1}{2} p^2 f(q) + V(q)$$

becomes

$$K(Q,P) = \frac{1}{2}P^2 + W(Q) \qquad \text{IV-4.6}$$

Recall that in classical mechanics, the two descriptions, in terms of (q,p) and (Q,P), are completely equivalent.

2) Then quantize the system in terms of (Q,P). There now, is no ambiguity and the Schrodinger equation is

$$K(Q,P)\ \Psi(Q,t) = i\hbar \frac{\partial \Psi(Q,t)}{\partial t} \qquad \text{IV-4.7}$$

with $P = -i\hbar\partial/\partial Q$. The probability density is given by

$$|\Psi(Q,t)|^2 \qquad \text{IV-4.8}$$

3) Finally, one can return to (q,p) if it is desired. This transformation may be regarded as a simple variable transformation. The wave function will be transformed as

$$\psi(q,t) \equiv \Psi(Q(q),t)\,(dQ/dq)^{\frac{1}{2}} \qquad \text{IV-4.9}$$

so

$$|\psi(q,t)|^2 dq = |\Psi(Q(q),t)|^2 dQ \qquad \text{IV-4.10}$$

Let us carry out the above procedure explicitly for the Hamiltonian (4.3). The generating function

$$F(q,P) = \int^{q} \sqrt{f(s)}\ ds.P \qquad \text{IV-4.11}$$

generates the following canonical transformation:

$$Q = \int^{q} \sqrt{f(s)}\ ds\ , \quad P = p/\sqrt{f(q)} \qquad \text{IV-4.12}$$

and

$$K(Q,P) = H(q(Q,P), p(Q,P))$$

$$= \frac{1}{2} P^2 + W(Q) \qquad \text{IV-4.13}$$

where

$$W(Q) = V(Q(q)) \ . \qquad \text{IV-4.14}$$

Note that Q does not depend on p.

The Schrodinger equation is

$$-\frac{\hbar^2}{2} \frac{\partial^2 \Psi}{\partial Q^2} (Q,t) + W(Q)\ \Psi(Q,t) = i\hbar \frac{\partial \Psi(Q,t)}{\partial t} \qquad \text{IV-4.15}$$

To transform from (Q,P) back to (q,p), we note that

$$\frac{dQ}{dq} = \sqrt{f(q)} \ , \quad \frac{\partial}{\partial Q} = \frac{1}{\sqrt{f(q)}} \frac{\partial}{\partial q} \qquad \text{IV-4.16}$$

and

$$\Psi(Q,t) = \frac{1}{\sqrt[4]{f(q)}} \psi(q,t) \ . \qquad \text{IV-4.17}$$

Hence the Schrodinger equation in q becomes

$$\frac{-\hbar^2}{2}\left(\frac{1}{\sqrt[4]{f}} \frac{\partial}{\partial q} \frac{1}{\sqrt[4]{f}}\right)^2 \psi(q,t) + V(q)\ \psi(q,t) = i\hbar \frac{\partial}{\partial t} \psi(q,t) \ . \qquad \text{IV-4.18}$$

This illustrates that the quantum mechanical counterpart of the classical Hamiltonian (4.3) is

$$H(q,p)_{Q,m.} = \frac{1}{2}\left(\frac{1}{\sqrt[4]{f}}\ p\ \frac{1}{\sqrt[4]{f}}\right)^2 + V(q) \qquad \text{IV-4.19}$$

where $p = -i\hbar\ \partial/\partial q$.

Because $[Q,P] = i\hbar$, and $[q,p] = i\hbar$, the transformation

$(Q,P)\rightarrow(q,p)$ should be canonical. However there seems to be no simple operator relations between (Q,P) and (q,p). We note that equation (4.12) as such is not valid in quantum mechanics.

Feynman has formulated the quantum theory in a manner different from either that of Schrodinger or Heisenberg. He calls it the third formulation of quantum mechanics (Feynman and Hibbs 1965). Let us now examine the quantization of a system with a velocity dependent potential using this new formalism.

We start with the equation

$$\psi(q_2,t_2) = \int_{-\infty}^{+\infty} K(q_2t_2;q_1,t_1)\ \psi(q_1,t_1)dq_1 \qquad \text{IV-4.20}$$

where the kernel $K(2,1)$ is the contribution $\phi(q(t))$ from each path

$$K(2,1) = \sum_{\substack{\text{overall} \\ \text{paths from 1 to 2}}} \phi[q(t)] \qquad \text{IV-4.21}$$

The contribution from a path has a phase proportional to the action S.

$$\phi[q(t)] = \text{const}\ e^{(i/\hbar)\ S[q(t)]} \qquad \text{IV-4.22}$$

where

$$S = \int_{t_1}^{t_2} L(q,\dot{q},t)dt \qquad \text{IV-4.23}$$

and $L(q,\dot{q},t)$ is a classical Lagrangian.

By starting with the Lagrangian,

$$L(q,\dot{q},t) = \frac{m}{2}\dot{q}^2 - V(q) \qquad \text{IV-4.24}$$

it is possible (Feynman and Hibbs 1965, Chapter 4) to derive the Schrodinger equation,

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2} + V(q) = i\hbar\frac{\partial\psi}{\partial t} \qquad \text{IV-4.25}$$

A refreshing aspect of Feynman's quantization is the lack of any sudden changes such as replacing p by $-i\hbar\partial/\partial q$ or the poisson bracket of q and p by a commutator.

Feynman's prescription may be easily applied to our problem if we recognize that the Lagrangian remains numerically invariant under the transformation (4.12). The transformed Lagrangian is

$$L_Q(Q,\dot{Q}) = \tfrac{1}{2}\dot{Q}^2 - W(Q) \; . \qquad \text{IV-4.26}$$

Hence

$$S = \int_{t_1}^{t_2} L(q,\dot{q})dt = \int_{t_1}^{t_2} L_Q(Q,\dot{Q})dt \qquad \text{IV-4.27}$$

Then, following Feynman, we can derive the Schrodinger equation (4.15) in terms of Q which can be rewritten as an equation in q.

The above method is not very satisfactory, in the sense that it is required to do two transformations,

$(q,p) \rightarrow (Q,P) \rightarrow (g,p)$. Yourgrau and Mandelstam (1968) have shown that Lagrange's equation of motion

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \qquad \text{IV-4.28}$$

in operators can be derived from the Feynman principle. This equation yields the conventional equation of motion for a Lagrangian of the form, $L(q,\dot{q}) = \frac{1}{2}\dot{q}^2 - V(q)$, but leads to an inconsistency if a velocity dependent potential is involved. We have not been able to clarify the nature of this difficulty.

IV-5. LAGRANGIAN FORMALISM IN SECOND QUANTIZED FORM

In sections 3 and 4 we have indicated some difficulties concerning the Lagrangian formalism when the potential depended on velocity. We can show, however, that in the second-quantized form, there is no difficulty in the Lagrangian formulation.

As has been discussed by Schiff (1968), the Schrodinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + V(\underset{\sim}{r},t)\psi \qquad \text{IV-5.1}$$

can be derived from the Lagrangian density,

$$\underset{\sim}{L} = i\hbar\, \psi^* \dot{\psi} - \frac{\hbar^2}{2m} \underset{\sim}{\nabla}\psi^* . \underset{\sim}{\nabla}\psi - V(\underset{\sim}{r},t)\psi^*\psi \qquad \text{IV-5.2}$$

where $\dot{\psi} = \frac{\partial \psi}{\partial t}$. The Lagrangian is

$$L = \int \underset{\sim}{L}(\psi, \underset{\sim}{\nabla}\psi, \dot{\psi})\, d^3r \qquad \text{IV5.3}$$

which is required to satisfy

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \qquad \text{IV-5.4}$$

Equation (5.4) reduces to the Euler-Lagrange equation

$$\frac{\partial \underset{\sim}{L}}{\partial \psi} - \sum_{xyz} \frac{\partial}{\partial x} \frac{\partial L}{\partial (\frac{\partial \psi}{\partial x})} - \frac{\partial}{\partial t}(\frac{\partial \underset{\sim}{L}}{\partial \dot{\psi}}) = 0 \qquad \text{IV-5.5}$$

and the Hamiltonian is given by

$$H = \int \underset{\sim}{H} \, d^3r \ , \quad \underset{\sim}{H} = \pi\dot{\psi} - \underset{\sim}{L} \ , \quad \underset{\sim}{\pi} = \frac{\partial \underset{\sim}{L}}{\partial \dot{\psi}} \qquad \text{IV-5.6}$$

Now, replace $V(\underset{\sim}{r},t)$ in the above formulae by

$$V(\underset{\sim}{r},t) + \frac{1}{2}[p^2W(r) + W(r)p^2] \ , \qquad \text{IV-5.7}$$

and the Schrodinger equation becomes,

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2\psi + V(\underset{\sim}{r},t)\psi - \frac{\hbar^2}{2}[\nabla^2(W\psi) + W(\nabla^2\psi)] \qquad \text{IV-5.8}$$

This equation can be derived from the Lagrangian density

$$\underset{\sim}{L} = -i\hbar \, \psi^*\dot{\psi} \, \frac{-\hbar^2}{2m} \underset{\sim}{\nabla}\psi^* . \underset{\sim}{\nabla}\psi - \frac{\hbar^2}{2}[\nabla^2(W\psi) + W(\nabla^2\psi)] \qquad \text{IV-5.9}$$

We emphasize that it is the wave function ψ that is varied in this approach and not the operators q and p as was the case in section 3. It is possible to treat the general non-local potential $V(r,r')$ in an exactly similar manner.

The discussion given in this section does not really give a satisfactory answer to the question which we raised at the beginning of IV-4. It would be nicer if the difficulties could be resolved without going to the second quantized form. We feel that our discussion is relevant in view of a remark made by Razavy (1968). He claims that, in a problem like the nuclear photo effect, if the conventional Hamiltonian with a velocity dependent potential is used to obtain a sum rule, the resulting expression should be considered only as an approximation to the exact result. This is because of the questionable interpretation of the Hamiltonian, and consequently of the Schrodinger equation. Since there is nothing ambiguous or improper in the second quantization formulation, we expect that such sum rules can be derived exactly.

CHAPTER V

PROTON-PROTON BREMSSTRAHLUNG

V-1. INTRODUCTION

In this chapter we briefly develop the calculation for the proton-proton bremsstrahlung cross section following the formalism of Sobel and Cromer (1963,1966). We confine our attention to the Harvard geometry including only the S-wave contribution to the pole terms. The theory is extended to calculations in the center of mass.

Section V-2 contains the theory and V-3 the description of the calculations performed together with a discussion of the results.

V-2. THEORY

We wish to calculate the cross section for the inelastic scattering of a two proton initial state to a final state of two protons plus a photon. These states may be designated $|i\rangle \equiv |\underset{\sim}{p}_1,\underset{\sim}{p}_2,0\rangle$ and $|f\rangle \equiv |\underset{\sim}{p}_1',\underset{\sim}{p}_2',\underset{\sim}{k}\rangle$ respectively where the $\underset{\sim}{p}_i$'s and $\underset{\sim}{p}_i'$'s are initial and final proton momenta and $\underset{\sim}{k}$ is the momentum of the photon. The Hamiltonian for the free system is

$$H_0 = K_1+K_2+K_\gamma \qquad \text{V-2.1}$$

where K_1, K_2 and K_γ are the kinetic energy operators for protons 1 and 2 and the photon respectively. The total Hamiltonian of the system is

$$H = K_1+K_2+K_\gamma+V_N+V_{em} \qquad \text{V-2.2}$$

where V_N is the proton-proton interaction and V_{em} is the electromagnetic coupling of the γ-ray to the protons. It is convenient to write

$$H = H_N+K_\gamma+V_{em} \qquad \text{V-2.3}$$

where

$$H_N = K_1+K_2+V_N \qquad \text{V-2.4}$$

is the Hamiltonian for free proton-proton scattering. If $e(p) = p^2/2m$, we may write the total energy of the system as

$$E = e(p_1)+e(p_2) = e(p_1')+e(p_2')+k \equiv E'+k \qquad \text{V-2.5}$$

The T-matrix for transitions between eigenstates of H_0 is given by

$$T(E) = (V_N+V_{em}) + (V_N+V_{em}) [E-H_N-K_\gamma-V_{em}+i\varepsilon]^{-1}$$

$$x\ (V_N+V_{em}) \qquad \text{V-2.6}$$

By using the operator identity

$$(A-B)^{-1} = A^{-1} + (A-B)^{-1} BA^{-1} \qquad \text{V-2.7}$$

and keeping only terms to first order in V_{em}, we have

$$T(E) = V_N + V_{em} + V_N G_N V_N + V_{em} G_N V_N + V_N G_N V_{em} + V_N G_N V_{em} G_N V_N \qquad \text{V-2.8}$$

where

$$G_N = 1/[E-H_N-K_\gamma+i\varepsilon] \qquad \text{V-2.9}$$

Now, take the matrix element of T(E) between the photon states. Since V_N is diagonal in the photon states it follows that

$$\langle \underline{k}|T(E)|0\rangle = T_N(E)\ \delta^3(\underline{k}) + \langle \underline{k}|V_{em}|0\rangle + \langle \underline{k}|V_{em}|0\rangle\ G_0(E)\ T_N(E)$$

$$+ T_N(E')\ G_0(E')\ \langle \underline{k}|V_{em}|0\rangle + T_N(E')\ G_0(E')\ \langle \underline{k}|V_{em}|0\rangle$$

$$\times\ G_0(E)\ T_N(E) \qquad \text{V-2.10}$$

Here

$$G_0(E) = 1/[E-K_1-K_2+i\varepsilon] \qquad \text{V-2.11}$$

and

$$T_N(E) = V_N + V_N\ G_0(E)\ T_N(E) \qquad \text{V-2.12}$$

is the T-matrix for nucleon-nucleon scattering.

In equation V-2.10, the first term describes elastic p-p scattering and the second represents the emission of a photon without nucleon scattering. In the kinematics considered, both terms are zero. The final term describes a

photon emitted between two nuclear scatterings and is called the rescattering term, which won't be considered here. We are left with

$$\langle \underline{k}|T'(E)|0\rangle = \langle \underline{k}|V_{em}|0\rangle\, G_0(E)\, T_N(E) + T_N(E')\, G_0(E')\, \langle \underline{k}|V_{em}|0\rangle \qquad \text{V-2.13}$$

which on taking the matrix elements between the proton states becomes

$$\langle f|T'(E)|i\rangle = \int d^3\underline{k}_1\, d^3\underline{k}_2\, \{\langle \underline{p}_1'\, \underline{p}_2', \underline{k}|V_{em}|\underline{k}_1, \underline{k}_2, 0\rangle\, [E - e(k_1) - e(k_2) + i\varepsilon]^{-1}\, \langle \underline{k}_1, \underline{k}_2|T_N(E)|\underline{p}_1, \underline{p}_2\rangle + \langle \underline{p}_1', \underline{p}_2'|T_N(E')|\underline{k}_1$$

$$\times\, [E - e(k_1) - e(k_2) + i\varepsilon]^{-1}\, \langle \underline{k}_1, \underline{k}_2, \underline{k}|V_{em}|\underline{p}_1, \underline{p}_2, 0\rangle\} \qquad \text{V-2.14}$$

The electromagnetic contribution to the Hamiltonian may be derived from the rules of quantum electrodynamics (Feynman 1962), according to which the amplitude that a quantum system emit a photon during a process of transition from one state to another is exactly the same as the amplitude that the same transition will be made under the influence of a potential equal to that of a classical electromagnetic wave representing that photon. Thus by choosing the Coulomb gauge, this electromagnetic interaction can be represented by a vector potential $\underline{A}$. This is then properly introduced into the Hamiltonian in a gauge invariant manner by making the replacement $\underline{p}_i \rightarrow \underline{p}_i - \frac{e}{c}\underline{A}_i$.

If we choose the normalization properly and couple to the kinetic energy operators and include the magnetic dipole interaction, the result can be represented as

$$\langle \underline{k}_1, \underline{k}_2, \underline{k} | V_{em} | \underline{p}_1, \underline{p}_2, 0 \rangle = -e/2m \, \pi\sqrt{k} \, [a(\underline{p}_1) \, \delta(\underline{p}_2 - \underline{k}_2) \, \delta^3(\underline{p}_1 - \underline{k} - \underline{k}_1)$$

$$+ a(\underline{p}_2) \, \delta^3(\underline{p}_1 - \underline{k}_1) \, \delta^3(\underline{p}_2 - \underline{k} - \underline{k}_2)] \qquad \text{V-2.15}$$

where

$$a(\underline{p}_i) = [\underline{p}_i \cdot \hat{\underline{\varepsilon}} + \frac{i}{2} \mu \, \underline{\sigma}_i \cdot (\underline{k} \times \hat{\underline{\varepsilon}})] \qquad \text{V-2.16}$$

Here $e = 1/\sqrt{137}$ is the electronic charge, $\mu = -2.793$ is the magnetic moment of the proton in nuclear magnetons, $\hat{\underline{\varepsilon}}$ is the polarization vector of the photon and $\underline{\sigma}_i$ is the Pauli spin operator of the i'th nucleon. We note here that the above treatment omits any coupling to the angular momentum dependence of the nucleon-nucleon potential. The angular momentum operator $\underline{L}$ for instance should rigorously be replaced by $\underline{r} \times (\underline{p} - \frac{e}{c} \underline{A})$. Since this represents a coupling to the non-local part of the interaction, it may be considered the analogue of photons emitted by the intermediate charged mesons.

Using equation V-2.15, we may write V-2.14 as

$$\langle f | T' | i \rangle = -e/2m \, \pi\sqrt{k} \, \{a(\underline{p}_i') \, [E - e(\underline{p}_1' + \underline{k}) - e(\underline{p}_2')]^{-1} \, \langle \underline{p}_1' + \underline{k}, \underline{p}_2' | T_N(E) | \underline{p}$$

$$+ \langle \underline{p}_1', \underline{p}_2' | T_N(E') | \underline{p}_1 - \underline{k}, \underline{p}_2 \rangle \, [E' - e(\underline{p}_1 - \underline{k}) - e(\underline{p}_2)]^{-1}$$

$$\times a(\underline{p}_1) + a(\underline{p}_2') \, [E - e(\underline{p}_2' + \underline{k}) - e(\underline{p}_1')]^{-1}$$

$$x \langle \underline{p}_1', \underline{p}_2'+\underline{k} | T_N(E) | \underline{p}_1, \underline{p}_2 \rangle + \langle \underline{p}_1', \underline{p}_2' | T_N(E') | \underline{p}_1, \underline{p}_2 - \underline{k} \rangle$$

$$x \left[E - e(\underline{p}_1) - e(\underline{p}_2 - \underline{k})\right]^{-1} a(\underline{p}_2) \} \qquad .V-2.17$$

The four T_N matrices in this expression will be designated as terms a, b, c and d respectively. Collectively they are the pole contributions to the scattering amplitude represented graphically in figure I.1.

Now consider the nuclear matrix elements T_N. These can be expressed in terms of the more familiar two-body scattering matrices t, e.g.

$$\langle \underline{p}_1'+\underline{k}, \underline{p}_2' | T_N(E) | \underline{p}_1 \underline{p}_2 \rangle = \delta^3(\underline{p}_1'+\underline{p}_2'+\underline{k}-\underline{p}_1-\underline{p}_2) \langle \underline{k}_a' | t(e(k_a) | \underline{k}_a \rangle$$

$$\equiv -(2\pi^2 m)^{-1} M_a(\underline{k}_a', \underline{k}_a) \delta^3(\underline{p}_1'+\underline{p}_2'+\underline{k}-\underline{p}_1-\underline{p}_2)$$

$$.V-2.18$$

Here $\underline{k}_a = (\underline{p}_1 - \underline{p}_2)/2$ and $\underline{k}_a' = (\underline{p}_1'+\underline{k}-\underline{p}_2')/2$. The properties of the scattering matrix will be considered in more detail later on.

Using V-2.18 and the corresponding expressions for terms b, c and d, we may write

$$\langle f | T' | i \rangle = (e/4\pi^3 m^2 \sqrt{k}) M \delta^3(\underline{p}_1'+\underline{p}_2'+\underline{k}-\underline{p}_1-\underline{p}_2) \qquad V-2.19$$

where

$$M = a(\underline{p}_1')(M_a/\Delta E_a) + (M_b/\Delta E_b) a(\underline{p}_1) + a(\underline{p}_2')(M_c/\Delta E_c)$$

$$+ (M_d/\Delta E_d) a(\underline{p}_2) \qquad .V-2.20$$

Here the $a(P_i)$'s are the non relativistic energy denominators. For example,

$$\Delta E_a = E - e(\underline{P}_1' + \underline{k}) - e(\underline{P}_2') = |\underline{k}| - \frac{\underline{P}_1' \cdot \underline{k}}{2m_0} - \frac{k^2}{2m_0} \qquad \text{V-2.21}$$

It has become customary (Sobel 1966) to delete the small k^2 term and obtain the corresponding covariant quantity

$$\Delta E_a = (|\underline{k}|\,[\underline{P}_1'^2 + m_0^2] - \underline{k} \cdot \underline{P}_1')/m_0$$

$$= K \cdot P_1'/m_0 \qquad \text{V-2.21}$$

Here k and p_1' are relativistic four vectors. This form of the energy denominator has been recovered from a truly covariant calculation (McGuire 1969) and has been used in an attempt to obtain partial covariance of the essentially nonrelativistic formulation. The corresponding expressions for the remaining energy denominators are

$$\Delta E_b = -k \cdot P_1/m_0, \quad \Delta E_c = k \cdot P_2'/m_0 \text{ and } \Delta E_d = -k \cdot P_2/m_0 \qquad \text{V-2.22}$$

To calculate the cross section we need to evaluate $|\langle f|T|i\rangle|^2$. Since the incident beam and target are unpolarized and the final state spins and polarizations are unobserved, it is necessary to average over initial spins and sum over final state spins and polarizations. This

quantity is given by

$$(e^2/16\pi^6 m^4 k)\langle\tfrac{1}{2}\mathrm{tr}(M^+M)\rangle\ \delta^3(\underline{p}_1'+\underline{p}_2'+\underline{k}-\underline{p}_1-\underline{p}_2) \qquad \text{V-2.23}$$

where the $\langle\ \ \rangle$ represents the sum over photon polarizations.

Although the experimental variables are measured in the lab system, it is convenient to calculate the cross section, or at least part of the cross section, in another frame. We therefore use Moller's invariant form for the cross section (Goldberger and Watson 1964)

$$d\sigma = \left[\frac{d^3\underline{p}_1'\ d^3\underline{p}_2'\ d^3\underline{k}}{E_1'E_2'k}\ \delta^4(p_1'+p_2'+k-p_1-p_2)\right] \times$$

$$\left[\frac{(2\pi)^4}{E_1E_2|\underline{v}_1-\underline{v}_2|}\right]\left[\sum_{\substack{\text{spins}\\ \text{polarizations}}} |T_{Fi}|^2 E_1'E_2'kE_1E_2\right] \qquad \text{V-2.24}$$

Here the E's and k are the total relativistic energies and $\underline{v}_i$ is the initial velocity of proton i. On using V-2.23

$$d\sigma = \left[\frac{E_1'E_2'E_1E_2}{\pi^2 m^4}\ e^2\langle\frac{1}{4}\ \mathrm{tr}(M^+M)\rangle\right]$$

$$\times\ \left[\frac{1}{E_1E_2}\ \underline{v}_1-\underline{v}_2\ \right]\left[\frac{\delta^4(p_1'+p_2'+k-p_1-p_2)}{E_1'E_2'k}\ d^3\underline{p}_1'd^3\underline{p}_2'd^3\underline{k}\right] \quad .\text{V-2.25}$$

Each term in a square bracket is a Lorentz invariant.

The differential cross section of most interest is $d\sigma/d\Omega_1\ d\Omega_2 d\theta_\gamma$ giving the photon angular distribution for

fixed θ_1 and θ_2 in the Harvard geometry. In this geometry all kinematic variables are coplanar and are indicated in the following figure

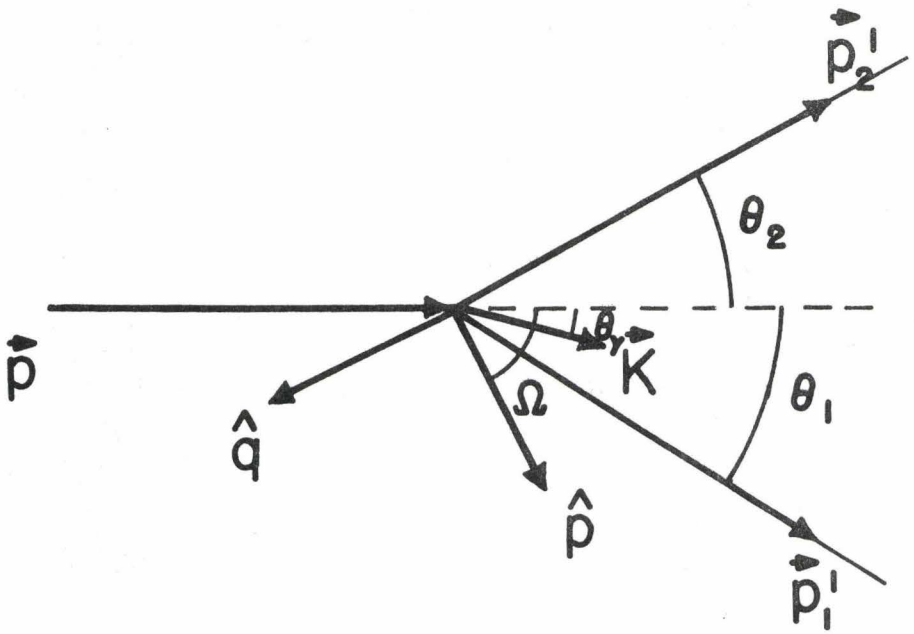


FIGURE V-2.1

To get the differential cross section from V-2.25 we must integrate over ϕ_γ, p_1' and p_2' using the momentum conserving delta function and over k using the energy delta function. The result is

$$\frac{d\sigma}{d\Omega_1 d\Omega_2 d\theta_\gamma} = [E_1' E_2' E_1 E_2 \, \langle \tfrac{1}{4} \text{ tr } M^+ M \rangle] \, [\frac{1}{E_1 E_2 |\underline{v}_1 - \underline{v}_2|}] \, \mathcal{F} \, \frac{e^2}{\pi^2 m^4} \quad \text{V-2.26}$$

where the phase space factor $\mathcal{F}$ is given by

$$\mathcal{F} = \int \frac{\delta^4(p_1'+p_2'+k-p_1-p_2)}{E_1'E_2'k} p_1'^2 p_2'^2 k^2 \sin\theta_\gamma dp_1' dp_2' d\psi_\gamma dk \qquad 102$$

$$= (p_1'^2 p_2'^2 k) \left|\sin(\theta_1+\theta_2) + v_2' \sin(\theta_\gamma-\theta_1) - v_1' \sin(\theta_2+\theta_\gamma)\right|^{-1}$$

V-2.27

Here $\theta_1 = \theta_2$ are on opposite sides of the incident beam and both taken to be positive. The angle θ_γ is taken as positive for k on the same side of p_1 as p_1' and negative when on the opposite side.

The kinematics for the cross section are calculated relativistically. Confining the reaction to coplanar events restricts the final state variables to six. There are three conservation laws, leaving three independent variables which can be chosen to be θ_1, θ_2 and θ_γ. From figure V-2.1, conservation of momentum yields,

$$p_1' + p_2' + k = p_1 + p_2 \qquad \text{V-2.28}$$

and from conservation of energy

$$\sqrt{p_1^2 + m_0^2} + m_0 = \sqrt{p_1'^2 + m_o^2} + \sqrt{p_2'^2 + m_0^2} + k \qquad \text{V-2.29}$$

These have a unique solution in terms of θ_1, θ_2 and θ_γ.

Although the complete T matrix is invariant as to which frame the calculation is carried out, the convenient choice of transverse gauge in that frame determines the relative contributions from the different terms. It has been shown that the effect of neglecting the rescattering contribution

is much less in the barycentric frame of the two initial protons (Signell and Marker 1967) than in the laboratory frame. In fact it has been pointed out (Signell 1968) that this term gives essentially a zero contribution if only the S state is considered and the calculation is done in the center of mass frame. We therefore calculate the first square bracket of V-2.26 in the center of mass system. This is accomplished by numerically transforming all kinematic variables to that frame.

Following Hagedorn (1957), if P and p are the 4-momenta of two particles in a reference frame G, and G' is the rest frame of $P = (E, \underline{P})$, then the transformation of $p = (e, \underline{p})$ from G to G' is given by

$$\underline{p}' = \underline{p} + \underline{\beta}\gamma \left(\frac{\gamma}{\gamma+1} \underline{\beta}\cdot\underline{p} - e\right) \qquad \text{V-2.30}$$

$$e' = \gamma \quad (e - \underline{p}\cdot\underline{\beta}) \qquad \text{V-2.31}$$

where

$$\underline{\beta} = \underline{P}/E \text{ and } \gamma = E/M_0 \qquad \text{V-2.32}$$

For our purposes, $P = p_1 + p_2$ and the variables transformed are the four momenta p_1, p_2, p_1', p_2' and k. The only alterations to the theory of Sobel and Cromer (1966) come from the fact that $\underline{p}_2 \neq 0$ in this frame.

To complete the evaluation of the cross section we have yet to evaluate $\langle 1/4 \text{ Tr } M^+ M \rangle$. Although a treatment of the $^1S_0$ contribution alone can be more simply calculated, my programs have been constructed to handle the higher partial waves as well. The $^1S_0$ cross sections were calculated by setting the appropriate terms equal to zero. We include here sufficient detail to describe how the actual calculation was done, omitting where possible, considerations of the higher partial waves.

In cases a and c, if we define $\hat{n}$ to be the unit vector in the direction $\underline{K}_i \times \underline{K}_i'$, $\hat{p}$ to be another arbitrary vector in the scattering plane and $\hat{q} = \hat{n} \times \hat{p}$, then we may use the Wolfenstein parameterization (Wolfenstein and Ashkin 1952) to write

$$M_i = A_i + B_i \underline{\sigma}_1 \cdot \hat{n} \underline{\sigma}_2 \cdot \hat{n} + C_i (\underline{\sigma}_1 \cdot \hat{n} + \underline{\sigma}_2 \cdot \hat{n}) + E_i \underline{\sigma}_1 \cdot \hat{q} \underline{\sigma}_2 \cdot \hat{q}$$
$$+ F_i \underline{\sigma}_1 \cdot \hat{p} \underline{\sigma}_2 \cdot \hat{p} + G_i (\underline{\sigma}_1 \cdot \hat{q} \underline{\sigma}_2 \cdot \hat{p} + \underline{\sigma}_1 \cdot \hat{p} \underline{\sigma}_2 \cdot \hat{q}) \qquad \text{V-2.33}$$

where the six amplitudes are the functions of K_i, K_i' and $\hat{K}_i \cdot \hat{K}_i'$. These amplitudes are expressed in the spin-spin repesentation. They can after considerable algebra be expressed in terms of the singlet-triplet M_{ij}'s of Stapp et al (1957) for the on-shell scattering amplitudes. The result is,

$$A = 1/4\ (2M_{11} + M_{00} + M_{ss})$$
$$B = 1/4\ (M_{00} - M_{ss} - 2M_{1-1})$$
$$C = i\ (M_{01} - M_{10})/\sqrt{8} \qquad \text{V-2.34}$$
$$F + E = 1/2(M_{11}+M_{1-1}-M_{ss})$$
$$F - E = 1/2[\sqrt{2}(M_{01}+M_{10})\sin 2\Omega+(M_{11}-M_{1-1}-M_{00})\cos 2\Omega]$$
$$G = 1/4[\sqrt{2}(M_{01}+M_{10})\cos 2\Omega-(M_{11}-M_{1-1}-M_{00})\sin 2\Omega]$$

Here Ω is the angle between $\hat{p}$ and the axis of quantization. The M_{ij}'s are given in table III of Stapp's paper. The $^1S_0$ state contributes only to the spin singlet term, M_{ss}, for the scattering of two protons. Explicitly,

$$M_{ss} = \frac{2}{ik} \sum_{\text{even } \ell} P_\ell(\theta) \frac{2\ell+1}{2} \alpha_\ell \qquad \text{V-2.35}$$

Sobel (1965) has shown how to generalize the on-shell amplitudes of Stapp et al for inelastic scattering. The generalized result requires that $\sin \delta_\ell(K)$ of V-2.33 be replaced by $\Delta_\ell(k', k)$, the quasi-phase as defined in II-2.16.

In cases b and d the M's are of the form

$$M_i = -2\pi^2 m \langle \underset{\sim}{k}_i | t(e_i') | \underset{\sim}{k}_i \rangle \qquad \text{V-2.36}$$

where e_i' is the energy of the final state. Cromer and Sobel (1966) have shown that this is also expressible in the form V-2.33, provided one makes allowance for a different angle Ω in V-2.34. This does not concern us, being limited to the $^1S_0$ state. In cases b and d, the phases appearing in

the formulae are for the final state energies, the vectors $\hat{n}$, $\hat{q}$ and $\hat{p}$ remain the same as for cases a and c so it is possible to write equation V-2.20 in terms of the spin matrices and the same unit vectors.

While the choice of $\hat{p}$ is arbitrary, the sum over photon polarization is simpler if $\hat{p}$ is taken to be in the direction of the photon momentum k. The polarization vector $\hat{\varepsilon}$ must lie in the $\hat{n}$-$\hat{q}$ plane so that

$$\hat{\varepsilon} = \hat{q} \cos \phi + \hat{n} \sin \phi \qquad \text{V-2.37}$$

and $$k \times \hat{\varepsilon} = |k| (\hat{n} \cos \phi - \hat{q} \sin \phi) \qquad \text{V-2.38}$$

where ϕ is the angle between $\hat{\varepsilon}$ and $\hat{q}$. On using V-2.37 and V-2.38, equation V-2.14 becomes

$$a(p_i) = p_i \cdot \hat{q} \cos \phi + \frac{i}{2} k\mu (\sigma_i \cdot \hat{n} \cos \phi - \sigma_i \cdot \hat{q} \sin \phi). \qquad \text{V-2.39}$$

Here $\hat{\sigma}_a = \hat{\sigma}_b = \hat{\sigma}_1$ and $\hat{\sigma}_c = \hat{\sigma}_d = \hat{\sigma}_2$.

From V-2.33 and V-2.39, equation V-2.20 becomes

$$M = \sin \phi \sum_{i=1}^{16} X_i O_i + \cos \phi \sum_{i=1}^{16} Y_i O_i \qquad \text{V-2.30}$$

where the O_i are the sixteen independent spin operators, 1, $\sigma_1 \cdot \hat{p}$, $\sigma_1 \cdot \hat{n}$, $\sigma_1 \cdot \hat{q}$, $\sigma_2 \cdot \hat{p}$, $\sigma_2 \cdot \hat{n}$, $\sigma_2 \cdot \hat{q}$, $\sigma_1 \cdot \hat{n}\ \sigma_2 \cdot \hat{n}$, $\sigma_1 \cdot \hat{p}\ \sigma_2 \cdot \hat{q}$, $\sigma_1 \cdot \hat{q}\ \sigma_2 \cdot \hat{p}$, $\sigma_1 \cdot \hat{p}\ \sigma_2 \cdot \hat{p}$, $\sigma_1 \cdot \hat{q}\ \sigma_2 \cdot \hat{q}$, $\sigma_1 \cdot \hat{q}\ \sigma_2 \cdot \hat{n}$, $\sigma_1 \cdot \hat{n}\ \sigma_2 \cdot \hat{q}$, $\sigma_1 \cdot \hat{n}\ \sigma_2 \cdot \hat{p}$, $\sigma_1 \cdot \hat{p}\ \sigma_2 \cdot \hat{n}$, which satisfy

$$\text{tr } O_i^+ O_j = 4\delta_{ij} \qquad \text{.V-2.31}$$

To sum over the photon polarization we integrate over the polarization angle ϕ, ie.

$$\langle \tfrac{1}{4} \text{ tr}(M^+M) \rangle = \frac{1}{\pi} \int_0^{2\pi} \tfrac{1}{4} \text{ tr}(M^+M)\, d\phi$$
$$= \sum |X_i|^2 + |Y_i|^2 \qquad \text{.V-2.32}$$

The last step follows from V-2.31. The constants X_i and Y_i are defined and appear in Table V-2.1. This completes the evaluation of the cross section.

V-3. THE CALCULATION

We have calculated the $^1S_0$ contribution to the p-p-γ cross section in the Harvard geometry for a series of potentials. These include the Hamada-Johnston (1962), Reid Soft Core (1966), phase-equivalent pairs of local and separable potentials of Srivastava and Sprung (1970) and the phase-equivalent family of rank-two separable potentials derived in Chapter III.

The Kowalski-Noyes half-off-shell matrix elements for the above potentials have been plotted for on-shell energy values of 99 and 158 MeV. These appear in figures V-3.1 and V-3.2 respectively. The bremsstrahlung cross sections although calculated for various energies and angles are presented for energies of 99 and 158 MeV with proton

TABLE V-2.1

THE COEFFICIENTS X_i AND Y_i OF EQUATION V-2.32. IN THIS TABLE THE AMPLITUDES A_x, . . . G_x ARE UNDERSTOOD TO BE DIVIDED BY THE CORRESPONDING ENERGY DENOMINATORS, ΔE_x AS DEFINED IN V-2.21 AND V-2.22. HERE $\xi = -\frac{1}{2}\mu k$.

i	O_i	X_i
1	1	$A_a\, \underline{p}_1' \cdot \hat{q} + A_b\, \underline{p}_1 \cdot \hat{q} + A_c\, \underline{p}_2' \cdot \hat{q} + i\xi(C_a + C_b + C_c + C_d)$
2	$\underline{\sigma}_1 \cdot \hat{n}$	$C_a\, \underline{p}_1' \cdot \hat{q} + C_b\, \underline{p}_1 \cdot \hat{q} + C_c\, \underline{p}_2' \cdot \hat{q} + i\xi(A_a + A_b + B_c + B_d)$
3	$\underline{\sigma}_2 \cdot \hat{n}$	$C_a\, \underline{p}_1' \cdot \hat{q} + C_b\, \underline{p}_1 \cdot \hat{q} + C_c\, \underline{p}_2' \cdot \hat{q} + i\xi(B_a + B_b + A_c + A_d)$
4	$\underline{\sigma}_1 \cdot \hat{n}\underline{\sigma}_2 \cdot \hat{n}$	$B_a\, \underline{p}_1' \cdot \hat{q} + B_b\, \underline{p}_1 \cdot \hat{q} + B_c\, \underline{p}_2' \cdot \hat{q} + i\xi(C_a + C_b + C_c + C_d)$
5	$\underline{\sigma}_1 \cdot \hat{p}\underline{\sigma}_2 \cdot \hat{q}$	$G_a\, \underline{p}_1' \cdot \hat{q} + G_b\, \underline{p}_1 \cdot \hat{q} + G_c\, \underline{p}_2' \cdot \hat{q} + \xi(E_a - E_b - F_c + F_d)$
6	$\underline{\sigma}_1 \cdot \hat{q}\underline{\sigma}_2 \cdot \hat{p}$	$G_a\, \underline{p}_1' \cdot \hat{q} + G_b\, \underline{p}_1 \cdot \hat{q} + G_c\, \underline{p}_2' \cdot \hat{q} - \xi(F_a - F_b - E_c + E_d)$
7	$\underline{\sigma}_1 \cdot \hat{p}\underline{\sigma}_2 \cdot \hat{p}$	$F_a\, \underline{p}_1' \cdot \hat{q} + F_b\, \underline{p}_1 \cdot \hat{q} + F_c\, \underline{p}_2' \cdot \hat{q} + \xi(G_a - G_b + G_c - G_d)$
8	$\underline{\sigma}_1 \cdot \hat{q}\underline{\sigma}_2 \cdot \hat{q}$	$E_a\, \underline{p}_1' \cdot \hat{q} + E_b\, \underline{p}_1 \cdot \hat{q} + E_c\, \underline{p}_2' \cdot \hat{q} - \xi(G_a - G_b + G_c - G_d)$
		Y_i
9	$\underline{\sigma}_1 \cdot \hat{q}$	$-i\xi(A_a + A_b + E_c + E_d)$
10	$\underline{\sigma}_2 \cdot \hat{q}$	$-i\xi(E_a + E_b + A_c + A_d)$
11	$\underline{\sigma}_1 \cdot \hat{p}$	$-i\xi(i(C_a - C_b) + G_c + G_d)$
12	$\underline{\sigma}_2 \cdot \hat{p}$	$-i\xi(G_a + G_b + i(C_c - C_d))$
13	$\underline{\sigma}_1 \cdot \hat{q}\underline{\sigma}_2 \cdot \hat{n}$	$-i\xi(C_a + C_b - i(G_c - G_d))$
14	$\underline{\sigma}_1 \cdot \hat{n}\underline{\sigma}_2 \cdot \hat{q}$	$-i\xi(-i(G_a - G_b) + C_c + C_d)$
15	$\underline{\sigma}_1 \cdot \hat{n}\underline{\sigma}_2 \cdot \hat{p}$	$-\xi(F_a - F_b - B_c + B_d)$
16	$\underline{\sigma}_1 \cdot \hat{p}\underline{\sigma}_2 \cdot \hat{n}$	$\xi(B_a - B_b - F_c + F_d)$

exit angles of 25^{o} and 30^{o} respectively and appear in figures V-3.3, 3.4 and 3.5. This choice was made on the basis of the experimental data. The 158 MeV geometry is that of Gottschalk et. al. (1967) while the more recent 99 MeV is that of Sannes et. al. (1970).

The phases and radial wave functions for the local potentials were obtained numerically by solving equation II-2.12 on the computer. The quasi phases followed after integrating equation II-2.16 over the potential and the wave function. There were numerous checks on the numerical methods used. The phase-shifts were compared with the published values where possible. The quasi phases were calculated numerically in their on-shell limit and compared with the sine of the corresponding phase-shift. Excellent agreement was obtained to six significant figures.

The phase-shifts and quasi phase parameters were obtained algebraically for the separable potentials by way of the R-matrix as discussed in Chapters II and III. The phase-equivalent pairs of separable and local potentials provided an additional check of our numerical work on local potentials. The actual cross sections have the proper quadrupole distribution and were compared with those of Brown (1969) for the $^{1}S_{0}$ contribution to the 158 MeV calculation. Agreement was as precise as could be discerned from her graph.

The off-shell properties of the various potential

models at 99 and 158 MeV are very similar as is evident from the graphs. In the on-shell limit f(p,k) is always equal to one and varies very little in the vicinity of this limit. That is, the slopes at p=k are all approximately the same. For p<k we note that all local potentials have very nearly the same values, for this reason only a few have been plotted. These are characteristically lower than the values for the separable models which show quite a spread in values, none of which is very close to that of the local potentials. For p>k there is a very large variation in values for large p. Here the separable models have variations which lie on either side of the results for local potentials. The marked similarity of results from the different local models indicates that attempts to observe purely off-shell effects with these potentials will be quite hopeless unless the matrix elements are very much off-shell. That is differences arising from functional form and cores will be rather small. It might prove interesting to see if the small p values for separable models can be reduced to those obtained for local potentials.

The bremsstrahlung cross sections of figures V-3.3, 3.4 and 3.5 are all of the same general form. Each graph includes the results for a phase-equivalent pair of separable and local potentials at the energies 99 and 158 MeV. For comparison, we have superimposed on all graphs the values calculated for the Hamada-Johnston potential. In addition

figure V-3.3 includes a plot of the two-term separable model, phase-equivalent to the Mongan two-term potential, but with a_R=1.0. This is the extreme case of the potentials derived in Chapter III and is the only one to give any significant variation in the cross section.

The curves for the local potentials are all characteristically higher than those for the separable models. The variations for the 158 MeV geometry are greater than those at 99 MeV. This is a general trend which has been observed at all energies calculated. Results taken at 61.7 MeV demonstrated very little variation while those at 20 MeV gave practically none. However, as the energy increases, the $^1S_0$ contribution becomes less as the total cross section grows larger. The curves at 99 MeV are of more interest for this reason and because the experimental data of Sannes et. al. is much more accurate than any other to date.

From our 99 MeV results, we observe that the cross sections as a function of θ_γ vary from about 2% to as much as 10%. The regions of large variation are peaked about the maxima in the quadrupole distribution. This is particularly true for the Mongan separable and phase-equivalent local potentials. The Tabakin two-term shows a more distributed variation but is still highly peaked in the vicinity of the maxima. The experimental error quoted by Sannes et. al. is about 6% for the integrated cross section. They then

deduce the photon distribution from the kinematics. The error bar indicated in their histogram shows an error of about 23%. It would seem that an experiment which detected the photon distribution as well should possibly give more reliable results. The calculated off-shell effects are certainly obscured in the integrated cross section, the effects being peaked about definite photon angles.

We might mention the calculation of McGuire and Cromer (1969). They have calculated using only the pole terms and find that the calculated cross section where off-shell behaviour differs from that with the off-shell quantities replaced by their on-shell limits, differs by more than four times the experimental error. This is an extreme measure of off-shell dependence and one can only say that experimental detection of off-shell properties is possible. This agrees with our findings.

Although our calculations have been for the $^1S_0$ contribution only, we feel, that short of a complete investigation, this is not a serious objection, since it is already known from previous calculations that the total cross sections will fall roughly within the experimental error. By a complete calculation is meant one which includes the Coulomb effects, the rescattering and exchange contributions and an examination of the off-shell effects by systematically including the higher partial waves phase-equivalently.

Signell and Marker (1968) have estimated that Coulomb effects account for 12% of the cross section at 20 MeV and 35^{o}. This of course decreases as the proton incident energy becomes larger and must be much less at 99 MeV. Brown (1969) is the only person to have calculated the rescattering contribution exactly. She has found that these effects contribute 0.2% at 62 MeV to 15% at 300 MeV of the total integrated cross section. McGuire and Cromer (1969) have estimated on the basis of one- and two-pion-exchange bremsstrahlung calculations that the contributions from exchange currents are at least as small as rescattering effects. Of course the higher partial waves must be included. The $^{1}S_{0}$ giving about 20% of the cross section at 158 MeV and upwards of 40% at 99 MeV.

Until these effects are properly included it will be difficult to reliably ascertain the true off-shell variation which, in principle, can then distinguish between the different potentials. In practice, the increasing experimental accuracy may well make this feasible.

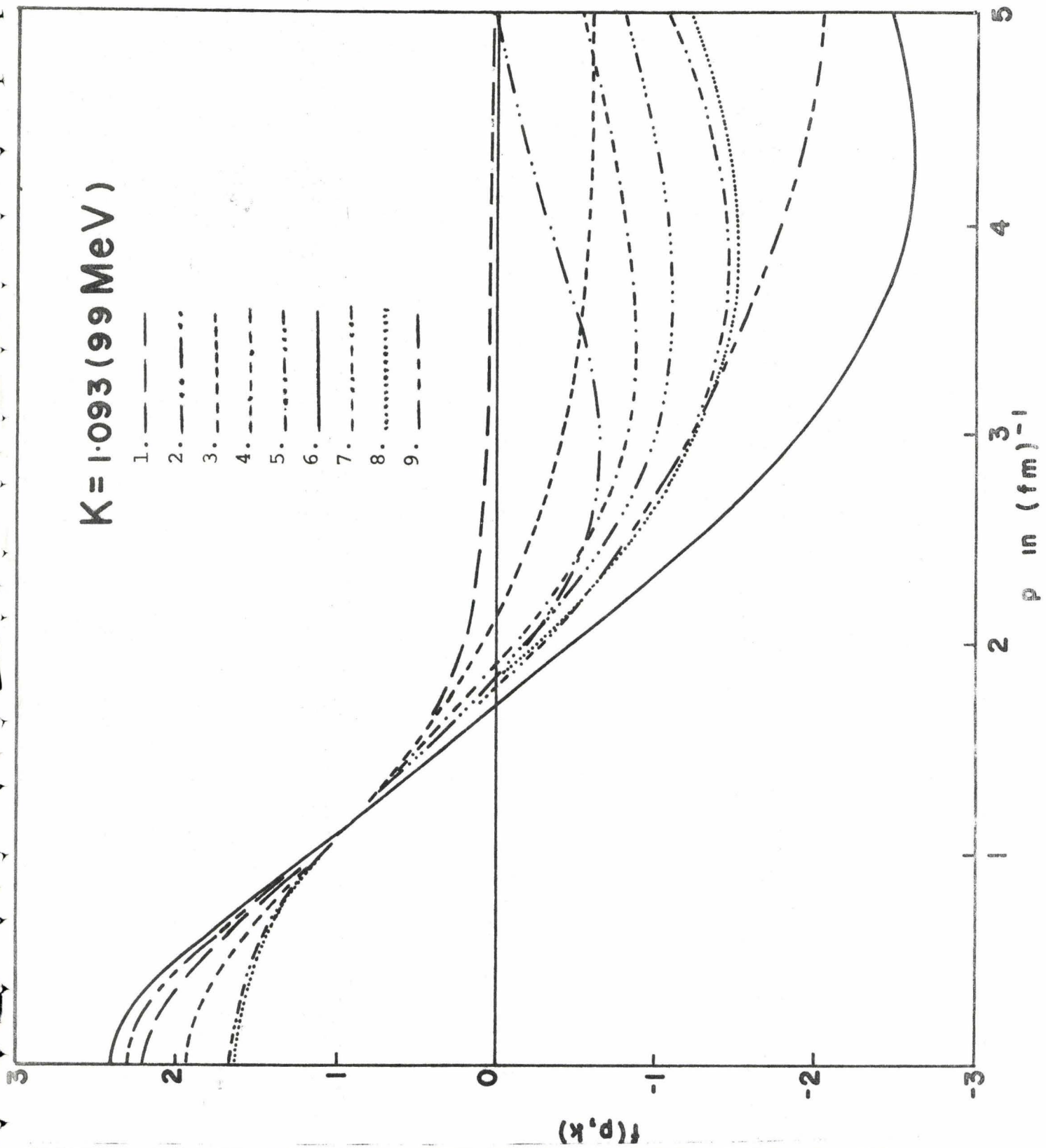


FIGURE V-3.1: KOWALSKI-NOYES HALF-OFF-SHELL MATRIX ELEMENTS FOR LOCAL AND SEPARABLE POTENTIALS, DENOTED BY L AND S RESPECTIVELY. THE POTENTIALS ARE: 1.TABAKIN TWO TERM S AND ITS 2.PHASE EQUIVALENT L(CASE II OF SECTION III-4),3.MONGAN TWO TERM S AND ITS 4.PHASE EQUIVALENT L(CASE I OF III-4), 5.REID SOFT CORE,6.TABAKIN ONE TERM S AND ITS 7.PHASE EQUIVALENT L (CASE III OF III-4),8.HAMADA-JOHNSTON AND 9.THE S POTENTIAL PHASE EQUIVALENT TO CASE 3 (a_R=1.0fm^{-1})OF III-5

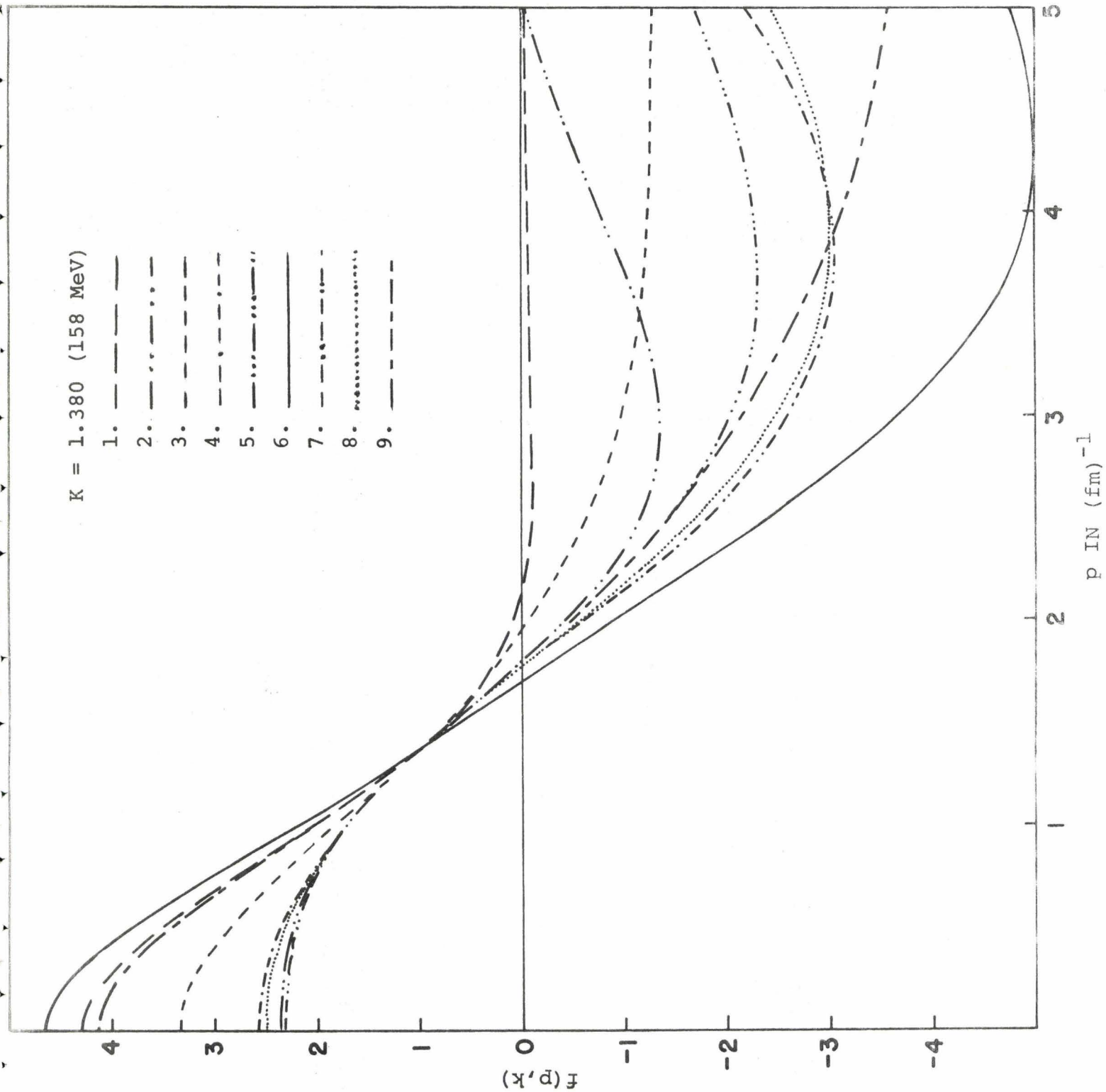


FIGURE V-3.2: SAME AS FIGURE V-3.1

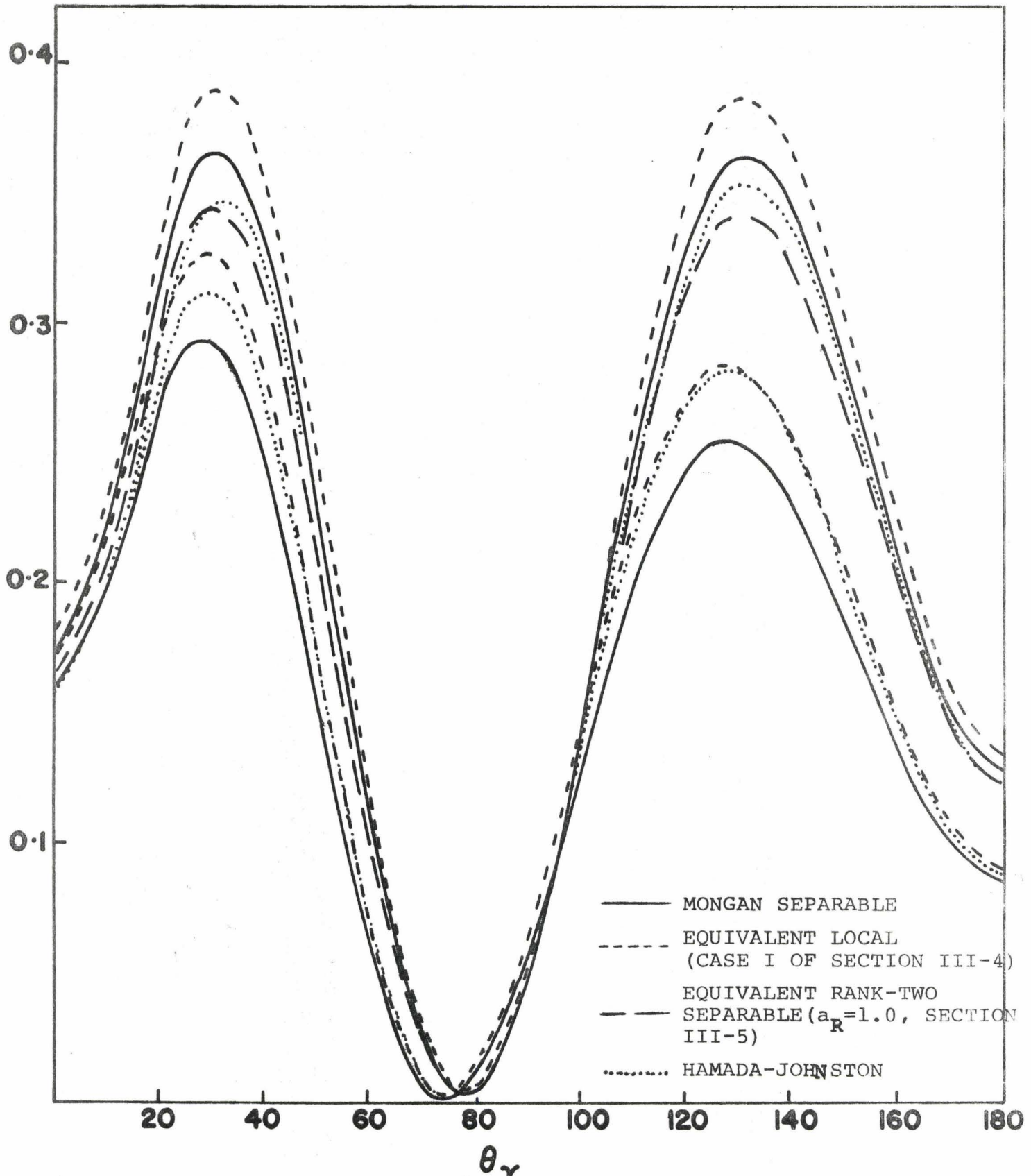


FIGURE V-3.3: THE CALCULATED $^1S_0$ CONTRIBUTION TO THE PROTON-PROTON BREMSSTRAHLUNG CROSS SECTION IN THE HARVARD GEOMETRY. THE UPPER CURVES ARE FOR A BEAM ENERGY OF 99 MeV. WITH PROTON EXIT ANGLES OF 25°, THE LOWER CURVES ARE AT 158 MeV. WITH EXIT ANGLES OF 30°. THE POTENTIALS ARE AS INDICATED.

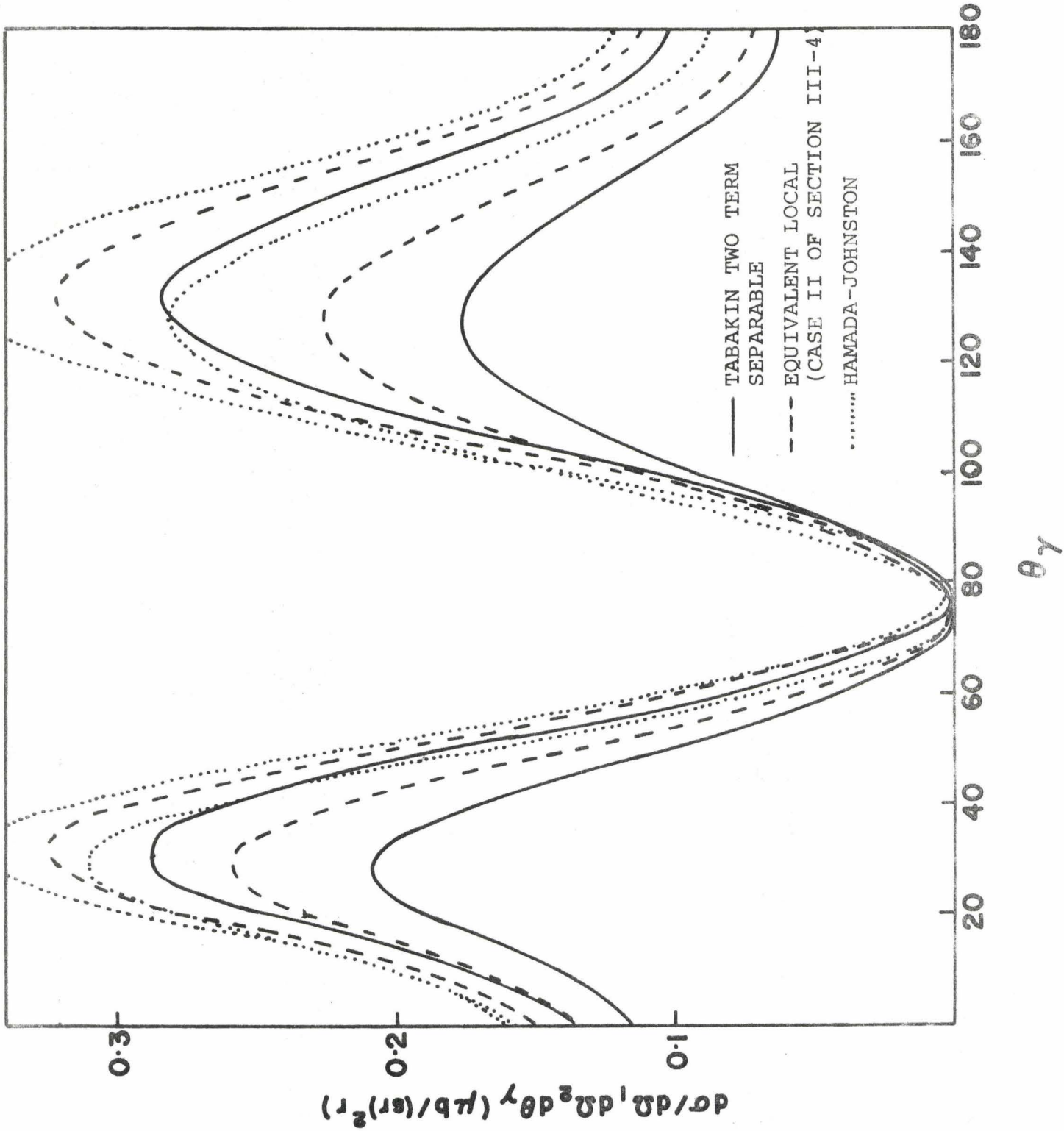


FIGURE V-3.4: SAME AS FIGURE V-3.3

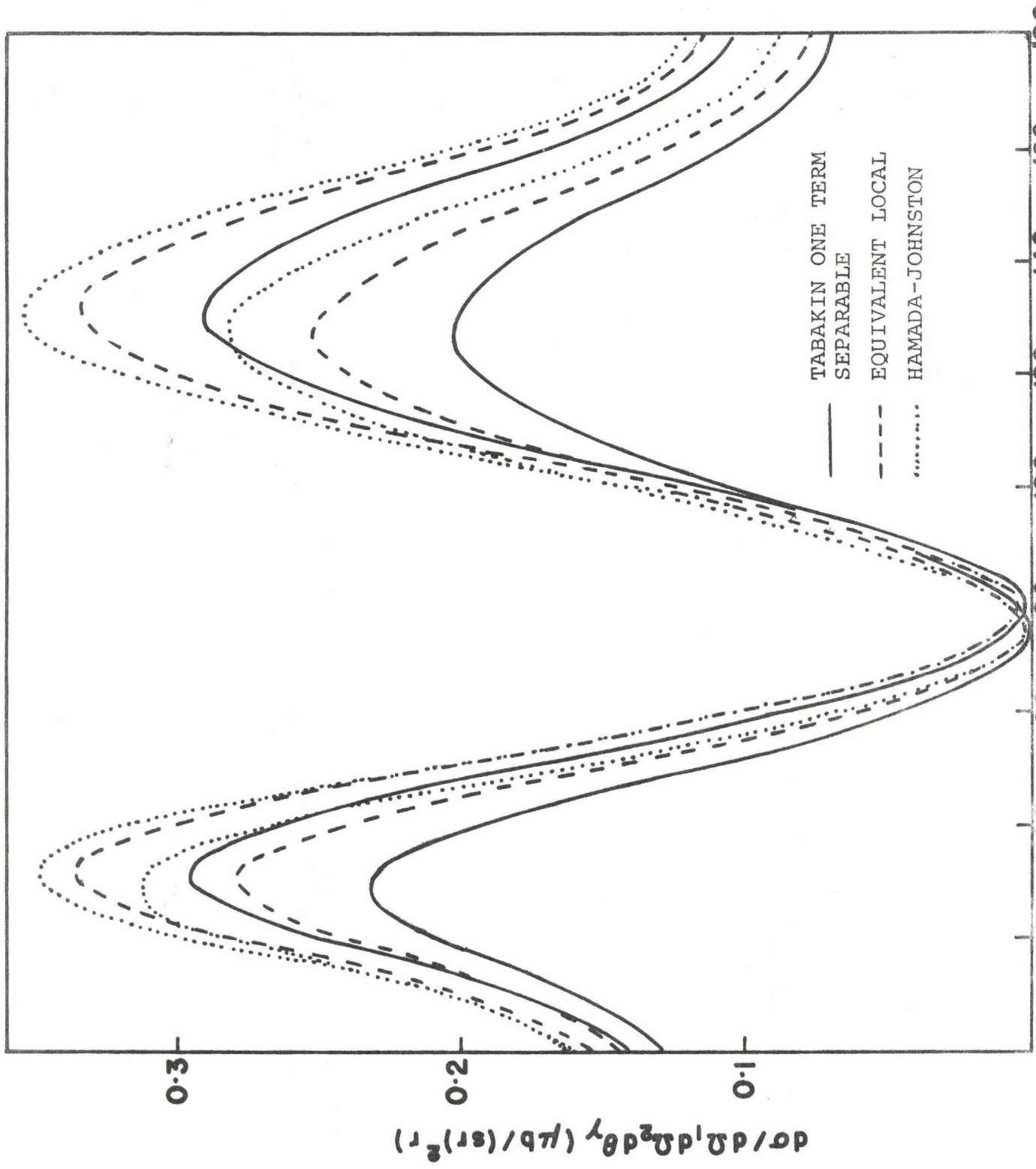


FIGURE V-3.5: SAME AS FIGURE V-3.3

CHAPTER VI

SUMMARY AND CONCLUSION

Because of the complex nature of the nucleon-nucleon interaction, nuclear physicists have been forced to adopt a phenomenological approach to the problem. They have developed a number of so-called realistic potentials which are fitted to the experimental two-body bound state and scattering data. Although these potentials vary greatly in type and in functional form, they all fit the two-body data, equally well, apart from small variations. However, when these potentials are used in a many-body calculation, they exhibit different properties. These are the off-shell effects, which simply means that the potential is evaluated between initial and final states of different energies. The off-shell effects are completely determined once a particular potential model is assumed.

To examine and possibly to fix experimentally this off-shell behaviour, physicists turn to the many-body problem. However, the quantum mechanical problem for more than two bodies is only solvable in terms of models and approximations. It is thus difficult to isolate purely off-shell effects. For this reason nucleon-nucleon bremsstrahlung was cited as an ideal testing ground, the third body being a photon which

only interacts via the electromagnetic interaction which is weak and well understood. Unfortunately it soon became apparent that in proton-proton bremsstrahlung the off-shell effects are small. It has thus been suggested (Signell 1968) that to examine this small off-shell dependence one should work with truly on-shell or phase-equivalent potentials. This has not been previously done.

The obvious approach to the generation of phase-equivalent potentials is the inverse problem. Here the potential is deduced directly from a knowledge of the phase-shifts at all energies. There is no parameter fitting as in the usual potential model and one can produce potentials as phase-equivalent as is desired. The fact that one can develop different potentials in this manner stems from the lack of uniqueness of the inversion procedure. Thus it is expected, for example, that the off-shell properties of potentials obtained from the local and separable potential inversion techniques will be quite different.

A method for inverting the $^1S_0$ phase-shifts to construct a rank-two separable potential has been developed. This has been utilized to construct a family of potentials phase-equivalent to the separable model of Mongan (1968). The off-shell properties of these potentials have been examined and exhibit a wide variation, the Kowalski-Noyes half-off-shell matrix elements, $f(p,k)$, differing by as much as a factor three for $p=5$ fm^{-1}.

An interesting outcome of our investigation of the inverse problem has been the classification of a distinction between the Jost function and the Fredholm determinant of the Lippmann-Schwinger equation in the case of rank-one separable potentials. The two are equivalent for local potentials.

The $^1S_0$ contribution to the proton-proton bremsstrahlung cross section, $d\sigma/d\Omega_1 d\Omega_2 d\theta_\gamma$, has been calculated in the center-of-mass frame using the formalism of Sobel and Cromer (1963,1966). The calculations were performed with the already mentioned rank-two separable potentials and the phase-equivalent pairs of separable and local potentials of Srivastava and Sprung (1970). This is the first calculation of this nature to be performed, differences in the cross sections being attributed solely to variations in the off-shell properties of different potential models. Due to cancellation effects and the limited distance off-shell sampled by the bremsstrahlung, these differences are not as large as might be hoped for. However, we can conclude that these differences do exist and have a definite dependence on the photon distribution. For the $^1S_0$ contribution they vary from about 2% to 10% and are strongly peaked about the maxima in the quadrupole photon distribution. It seems quite possible that with increasing experimental accuracy these differences may be detected.

A topic not directly related, but interest in which

stemmed from our main considerations is the problem of quantizing a classical system having a velocity dependent potential. We have pointed out that there are no difficulties if we work in the second quantized representation.

APPENDIX A

ON THE JOST FUNCTION FOR SEPARABLE POTENTIALS

As was mentioned in Chapter II, it is shown by Jost and Pais (1951) that the Jost function is identical to the Fredholm determinant of the Lippman-Schwinger equation for a local potential. We have found, however, that this is not the case with a non-local separable potential. In the following we show this for a rank-one separable potential.

The Jost solution f(k,r) which is subject to the asymptotic condition II-2.27 is obtained as a solution to the integral equation;

$$f(k,r) = e^{-ikr} - \frac{\sigma}{2\pi k} \int_r^\infty dr' \sin k(r-r') \, v(r') \int_0^\infty dr'' \, v(r'') \, x \, f(k,r'') \qquad A\text{-}1$$

where v(r) is related to V(r,r') of II-2.1 by

$$V(r,r') = (\sigma rr'/8\pi^2) \, v(r) \, v(r') \qquad .A\text{-}2$$

One can easily confirm that equation A-1 is equivalent to the original Schroedinger equation together with the boundary condition $\lim_{r\to\infty} e^{ikr} f(k,r) = 1$.

The integral equation A-1 can be easily solved to give

$$f(k,r) = e^{-ikr} - \frac{\sigma}{2\pi k} X(k) \int_r^\infty dr' \sin k(r-r') \, v(r') \qquad A\text{-}3$$

with

$$X(k) = \int_0^\infty dr\ v(r)\ f(k,r)$$

$$= \int_0^\infty dr\ v(r)\ e^{-ikr} \cdot [1 + \frac{\sigma}{2\pi k} \int_0^\infty dr\ v(r) \int_r^\infty dr'\ \sin k(r-r')$$

$$x\ v(r')]^{-1} \qquad \text{A-4}$$

The Jost function f(k) is then given by

$$f(k) \equiv f(k,0)$$

$$= 1 + \frac{\sigma}{2\pi k} X(k) \int_0^\infty dr\ \sin kr\ v(r) \qquad \text{A-5}$$

On the other hand the Fredholm determinant $D^+(k)$ was found in Chapter III to be

$$D^+(k) = 1 + \frac{\sigma}{2\pi} \int_0^\infty \frac{g^2(p)p^2dp}{p^2-k^2-i\varepsilon} \qquad \text{A-6}$$

Now we can easily see that $D^+(k) \neq f(-k,0)$ by comparing their imaginary parts; namely

$$\mathrm{Im}D^+(k) = -\frac{\sigma}{4}kg^2(k) \qquad \text{A-7}$$

while

$$\mathrm{Im}\ f(-k) = -\frac{\sigma}{4}kg^2(k) \cdot [\ 1 - Y(k^2)\ ]^{-1} \qquad \text{A-8}$$

where

$$Y(k^2) = 1 + \frac{\sigma}{2\pi k} \int_0^\infty dr\ v(r) \int_r^\infty dr'\ \sin k(r-r')v(r') \qquad \text{A-9}$$

which is a real, even function of k.

In fact we can show that

$$D^{+}(k) = f(-k)[\ 1 - Y(k^2)\ ] \qquad \text{A-10}$$

It is enough to show that Re $D^{+}(k)$ = Re $f(-k)[1-Y(k^2)]$.

Re $f(-k)[1-Y(k^2)]$

$$= [\ 1 + \frac{\sigma}{2\pi k} \text{Re } X(k) \int_0^\infty dr \sin krv(r)\ ] \cdot [\ 1 - Y(k^2)\ ]$$

$$= 1 + \frac{\sigma}{2\pi k} \int_0^\infty dr\ v(r)\ [\int_r^\infty dr' \sin k(r-r')v(r') + \int_0^\infty dr' \cos kr \sin kr' v(r')]$$

$$= 1 - \frac{\sigma}{2\pi} \int_0^\infty dr\ v(r) \int_0^\infty dr' v(r') G(r,r') \qquad \text{A-11}$$

where

$$G(r,r') = \begin{cases} -\frac{1}{k} \sin kr \cos kr' \\ -\frac{1}{k} \sin kr' \cos kr \end{cases} \quad \text{for} \quad \begin{cases} r'>r \\ r'<r \end{cases} \qquad \text{A-12}$$

Recalling

$$G(r,r') = -\frac{P}{2\pi} \int_0^\infty dq \frac{\sin qr \sin qr'}{q^2-k^2} \qquad \text{A-13}$$

and

$$\int_0^\infty v(r) \sin qr\ dr = (2\pi)^{3/2}\ (4\pi)^{-1}\ q\ g(q) \qquad \text{A-14}$$

we obtain

$$\text{Re } f(-k).[1-Y(k^2)] = 1 + \frac{\sigma}{2\pi} P \int_0^\infty \frac{g^2(q) q^2 dq}{q^2-k^2}$$

$$= \text{Re} D^{+}(k)$$

Q.E.D.

The difference we have discovered between $D^{\pm}(k)$ and $f(\mp k)$ is not entirely trivial. If $D^{+}(k)$ has a zero at $k = i\alpha$ $(\alpha>0)$, i.e. $D^{+}(i\alpha)=0$, then there is a bound state with the binding energy $(\hbar\alpha)^2/2\mu$. For a local potential, one can look for a zero of $f(-k)$ instead of $D^{+}(k)$. However, in the case of a nonlocal separable potential $f(-k)$ has redundant zeros which do not correspond to bound states. This can be illustrated with Yamaguchi's potential. Yamaguchi considered in momentum space, the potential,

$$V(k,k') = -\lambda g(k)g(k') \qquad \text{A-15}$$

with $g(k) = 1/(k^2+\beta^2)$. The Fredholm determinant is then given by

$$D^{+}(k) = 1 - \pi^2\lambda\{(\beta^2-k^2)/\beta + 2ik\} /(\beta^2+k^2)^2 \qquad \text{A-16}$$

The Jost solution is obtained as

$$f(k,r) = e^{-ikr} - \frac{2\pi^2\lambda(\beta-ik)e^{-\beta r}}{(\beta^2+k^2)(\beta^2+k^2+\pi^2\lambda/\beta)} , \qquad \text{A-17}$$

and hence

$$f(-k) = D^{+}(k)(\beta^2+k^2)/(\beta^2+k^2+\pi^2\lambda/\beta) \qquad \text{A-18}$$

The bound state can be found by solving

$D^{+}(i\alpha) = 0$ for $\alpha>0$, namely

$$1 - \pi^2\lambda\frac{(\beta^2+\alpha^2)/\beta - 2\alpha}{(\beta^2-\alpha^2)^2} = 0 \qquad \text{A-19}$$

which yields

$$\alpha = -\beta \pm \sqrt{\pi^2 \lambda/\beta} \qquad \text{A-20}$$

Therefore, there is one and only one bound state if $\pi^2\lambda/\beta^3 > 1$. Note that $f(-i\beta)=0$ but $D^+(i\beta)\neq 0$, hence $k=i\beta$ does not correspond to a bound state. This is the redundant zero of the Jost function.

Finally let us add a comment concerning a phase equivalent pair of local and separable potentials. If one starts with a given local potential, for which one can always solve the Schroedinger equation to obtain the phase shift, one can easily construct phase-equivalent separable potentials following the inversion prescriptions which we discussed in Chapter III. It is much more difficult, although possible, to obtain a local potential which is phase-equivalent to a given separable potential. The Marchenko equation can be analytically solved for some separable potentials of relatively simple forms (Srivastava 1970) but it is difficult to solve in general.

Very recently Coz et al (1970) have proposed an alternative method to obtain a local equivalent of a given nonlocal potential. Their method is remarkable in the sense that the problem is simply reduced to one of quadrature. Unfortunately, we can point out an inconsistency in their procedure.

They assume that the Jost solutions for a pair of phase-equivalent local and nonlocal potentials are related by

$$f_N(\pm k,r) = A(k,r) f_L(\pm k,r) \qquad \text{A-21}$$

where the indices N and L refer to the nonlocal and local potentials, respectively, and A(k,r) is a <u>real</u>, <u>even</u> function of k. Because of the common asymptotic condition $\lim_{r\to\infty} e^{\mp ikr} f_{N,L}(\pm k,r) = 1$, we require

$$\lim_{r\to\infty} A(k,r) = 1 . \qquad \text{A-22}$$

They then introduce the radial current

$$J(k,r) = \frac{1}{2ik}[f(k,r)f'(-k,r) - f'(k,r)f(-k,r)] . \qquad \text{A-23}$$

For the local potential, $J_L(k,r)=1$, whereas $J_N(k,r)$ is not a constant. However we can still show that

$$J_N(k,\infty) = J_N(k,0) = 1 . \qquad \text{A-24}$$

Substituing A-21 into A-23, we find that

$$J_N(k,r) = A^2(k,r) \qquad \text{A-25}$$

which implies that

$$A^2(k,\infty) = A^2(k,0) = 1 . \qquad \text{A-26}$$

Equation A-25 is an essential ingredient in their prescription which we are not going to reproduce here. It would be enough to point out that Eq. A-26 leads to

$$f_N(\pm k,0) = f_L(\pm k,0) \qquad \text{A-27}$$

which is inconsistent with our Eq. A-10.

The origin of the inconsistency can be found in their assumption on A(k,r) in equation A-21. We can show that A(k,r) is actually not a real, even function of k. For example, let us start with Yamaguchi's separable potential for which the Jost solution $f_N(k,r)$ is given by A-17. Following the Marchenko procedure we can obtain the Jost solution, $f_L(k,r)$, for a local potential which is phase-equivalent to that of Yamaguchi. Let us describe only an outline of the procedure. The Jost solution $f_L(k,r)$ is given by

$$f_L(k,r) = e^{-ikr} + \int_r^\infty A(r,s)\, e^{-iks}\, ds \qquad \text{A-28}$$

where A(r,s) has the form

$$A(r,s) = \sum_{i=1}^{2} M_i(r)\, e^{-\beta_i s} \qquad \text{A-29}$$

Here β_1 and β_2 are constants. Putting A-29 into A-28, we obtain

$$f_L(k,r) = e^{-ikr}\left[1 + \sum_{i=1}^{2} M_i(r)\, e^{-\beta_i r}/(\beta_i+ik)\right] \qquad \text{A-30}$$

which is to be compared with

$$f_N(k,r) = e^{-ikr}\left[1 - \frac{2\pi^2\lambda\, e^{-(\beta-ik)r}}{(\beta+ik)\,(\beta^2+k^2+\pi^2\lambda/\beta)}\right] \qquad \text{A-31}$$

Now if equation A-21 is valid, one has

$$\text{Re } f_N/\text{Im } f_N = \text{Re } f_L/\text{Im } f_L \qquad \text{A-32}$$

In view of the forms of f_L and f_N given above, it is easily shown that A-32 cannot be satisfied. A more detailed analysis of this problem will be reported elsewhere.

APPENDIX B

PARAMETERS FOR THE LOCAL POTENTIALS OF SRIVASTAVA AND SPRUNG

CASE I:

$$\Delta(x) = 1 + (c_1+c_2x)e^{-2\beta_1 x} + (c_3+c_4x)e^{-2\beta_2 x} + c_5\, e^{-4\beta_1 x}$$
$$+ c_6\, e^{-4\beta_2 x} + (c_7+c_8x+c_9x^2)e^{-2(\beta_1+\beta_2)x} + (c_{10}+c_{11}x)$$
$$x\, e^{-2(2\beta_1+\beta_2)x} + (c_{12}+c_{13}x)e^{-2(\beta_1+2\beta_2)x} + c_{14}\, e^{-4(\beta_1+\beta_2)x} \qquad .B\text{-}1$$

Here, $\beta_1 = 1.7860\ fm^{-1}$, $\beta_2 = 6.1570\ fm^{-1}$ and the remaining constants appear in Table B1.

CASES II AND III:

$$\Delta(x) = 1 + \sum_{i=1}^{6} a_i\, e^{-2\beta_i x} + \sum_{i<j}^{6} b_{ij}\, e^{-2(\beta_i+\beta_j)x} + \sum_{i<j<k}^{6}$$
$$x\, c_{ijk}\, e^{-2(\beta_i+\beta_j+\beta_k)x} + \sum_{i<j}^{6} d_{ij}\, e^{-2(\Sigma\beta-\beta_i-\beta_j)x} + \sum_{i=1}^{6}$$
$$x\, e_i\, e^{-2(\Sigma\beta-\beta_i)x} + f\, e^{-2(\Sigma\beta)x} \qquad .B\text{-}2$$

Here, $\Sigma\beta$ represents the sum of all six β's. For case II, $\beta_1 = 1.2000$, $\beta_2 = 1.1980$, $\beta_{3,4} = 1.2494 \mp 1.4411\ i$ and $\beta_{5,6} = 1.2474 \mp 1.4411\ i$. In case III, $\beta_1 = 1.0805$,

$\beta_2 = 1.0905$, $\beta_{3,4} = 2.8602\ (1\bar{+}i)$ and $\beta_{5,6} = 2.8673\ (1\bar{+}i)$.

.B-3

The remaining constants are defined in Table B2.

TABLE B1

NUMERICAL VALUES OF THE PARAMETERS IN EQ. B-1 FOR THE CASE I

Parameter	Value
C_1	0.493110
C_2	6.053838
C_3	- 0.048046
C_4	-15.011616
C_5	- 0.718093
C_6	- 0.371532
C_7	0.258126
C_8	- 9.259147
C_9	-27.520224
C_{10}	- 0.200245
C_{11}	0.988543
C_{12}	- 0.163112
C_{13}	- 0.206260
C_{14}	0.002244

TABLE B2

NUMERICAL VALUES OF THE PARAMETERS IN EQs. B-2 AND B-3 FOR THE CASES II AND III

Parameter	Case II		Case III	
	Real Part	Imaginary Part	Real Part	Imaginary Part
a_1	-3.756761 +2	0.0	3.451541 +1	0.0
a_2	3.765379 +2	0.0	-3.344541 +1	0.0
a_3	1.349527 +1	5.255720 +1	-3.095750 +2	-1.269441 +1
a_4	a_3^*		a_3^*	
a_5	-1.358397 +1	-5.253018 +1	3.094329 +2	1.214725 +1
a_6	a_5^*		a_5^*	
b_{12}	-9.774074 -2	0.0	-2.449323 -2	0.0
b_{13}	-4.046166 +3	3.340244 +3	-3.808427 +3	3.419870 +3
b_{14}	b_{13}^*		b_{13}^*	
b_{15}	4.037832 +3	-3.358981 +3	3.813791 +3	-3.423040 +3
b_{16}	b_{15}^*		b_{15}^*	
b_{23}	4.072374 +3	-3.338150 +3	3.641963 +3	-3.319780 +3
b_{24}	b_{23}^*		b_{23}^*	

TABLE B2 - CONTINUED

Parameter	Case II		Case III	
	Real Part	Imaginary Part	Real Part	Imaginary Part
b_{25}	-4.064064 +3	3.356967 +3	-3.647220 +3	3.322839 +3
b_{26}	b^*_{25}		b^*_{25}	
b_{34}	-3.915986 +3	0.0	-9.599783 +4	0.0
b_{35}	2.799282 -4	7.601045 -4	-1.457682 -1	-1.171859 -2
b_{36}	3.921977 +3	1.211275 +1	9.594656 +4	-3.062266 +2
b_{45}	b^*_{36}		b^*_{36}	
b_{46}	b^*_{35}		b^*_{35}	
b_{56}	-3.928017 +3	0.0	-9.589630 +4	0.0
c_{123}	3.174869 -1	1.515587 -1	1.017967 -1	-1.724350 0
c_{124}	c^*_{123}		c^*_{123}	
c_{125}	-3.190577 -1	-1.500371 -1	-1.059779 -1	1.729172 0
c_{126}	c^*_{125}		c^*_{125}	
c_{134}	9.745833 +4	0.0	-7.590705 +5	0.0
c_{135}	1.903767 -2	6.698608 -3	-1.244482 -1	1.151619 0

TABLE B2 - CONTINUED

Parameter	Case II		Case III	
	Real Part	Imaginary Part	Real Part	Imaginary Part
c_{136}	-9.771690 +4	-5.025020 +2	7.599662 +5	-3.934644 +3
c_{145}	c^*_{136}		c^*_{136}	
c_{146}	c^*_{135}		c^*_{135}	
c_{156}	-9.797877 +4	0.0	-7.608834 +5	0.0
c_{234}	-9.793739 +4	0.0	7.261035 +5	0.0
c_{235}	-1.908393 -2	-6.864338 -3	1.026247 -1	-1.103269 0
c_{236}	9.819690 +4	5.048611 +2	-7.269698 +5	3.778693 +3
c_{245}	c^*_{236}		c^*_{236}	
c_{246}	c^*_{235}		c^*_{235}	
c_{256}	-9.845972 +4	0.0	7.278568 +5	0.0
c_{345}	7.745994 -2	-7.998936 -3	4.526561 +1	2.000900 0
c_{346}	c^*_{345}		c^*_{345}	
c_{356}	-7.773877 -2	7.676729 -3	-4.525643 +1	-1.631969 0
c_{456}	c^*_{356}		c^*_{356}	

TABLE B2 - CONTINUED

Parameter	Case II				Case III			
	Real Part		Imaginary Part		Real Part		Imaginary Part	
d_{12}	2.066158	-6	0.0		2.138569	-2	0.0	
d_{13}	2.567335	-1	-4.344502	-1	1.202570	+2	1.107107	+2
d_{14}			d_{13}^*				d_{13}^*	
d_{15}	-2.584563	-1	4.309236	-1	-1.212267	+2	-1.093542	+2
d_{16}			d_{15}^*				d_{15}^*	
d_{23}	-2.566448	-1	4.308745	-1	-1.273929	+2	-1.155370	+2
d_{24}			d_{23}^*				d_{23}^*	
d_{25}	2.583444	-1	-4.273620	-1	1.284019	+2	1.141063	+2
d_{26}			d_{25}^*				d_{25}^*	
d_{34}	1.696956	0	0.0		1.225346	+2	0.0	
d_{35}	0.0		0.0		-1.848729	-4	2.232529	-5
d_{36}	-1.690495	0	1.216371	-2	-1.221732	+2	-8.776514	-1
d_{45}			d_{36}^*				d_{36}^*	
d_{46}			d_{35}^*				d_{35}^*	

TABLE B2 - CONTINUED

Parameter	Case II				Case III			
	Real Part		Imaginary Part		Real Part		Imaginary Part	
d_{56}	1.684146	0	0.0		1.218192	+2	0.0	
e_1	3.439914	-6	0.0		-3.670860	-2	0.0	
e_2	-3.414180	-6	0.0		3.887291	-2	0.0	
e_3	-1.260843	-6	-1.849740	-6	7.112685	-4	1.315192	-2
e_4			e_3^*				e_3^*	
e_5	1.238232	-6	1.850472	-6	-8.680922	-4	-1.310391	-2
e_6			e_5^*				e_5^*	
f	0.0		0.0		-1.415755	-6	0.0	

[1]Each entry is followed by its exponent to the base 10. The values less than 10^{-6} have been put equal to zero. The asterisk indicates the complex conjugate.

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