A GENERAL STUDY OF RELIABILITY IN DESIGN

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IN DESIGN

By

KALYAN KUMAR GHOSH, B.Tech.(Hons.)

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AUTHOR: Kalyan Kumar Ghosh, B.Tech.(Hons.) (I.I.T., Kharagpur, India)

SUPERVISOR: Professor J.N. Siddall

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SCOPE AND CONTENTS: A general study of theory of reliability has been made. The mechanism of failure of mechanical components, the properties of mechanical components, and the interaction of properties responsible for failure have been investigated. Elements of probability and statistics pertinent to reliability theory have been reviewed in brief. The various testing methods for determining component reliability using exponential, normal and Weibull distributions have been investigated. Acceptance sampling procedures for satisfying the necessary reliability requirements have been presented. Use of statistical methods in predicting fatigue life of mechanical components in general and rolling contact bearings in particular have been discussed.

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1. DEFINITIONS AND INTRODUCTORY CONCEPTS

1.1 Definition of Reliability

Reliability has been variously defined by different people. For engineering equipment, reliability is used to mean the capability of an equipment not to break down in operation. A better definition of reliability uses the statistical concept of probability and is stated as follows by Bazovsky (Ref. A.1.10) -

> "The Reliability of a component is its conditional probability of performing its function within specified performance limits at a given age for the period of time intended and under the operating stress conditions encountered."

Hence this definition implies that reliability is the probability that the device will not fail to perform as required for a certain length of time. So here, we define reliability as the probability of survival. Similarly, probability of failure may be defined as unreliability.

1.2 Concept

From the advent of industrial age, the need for reliability consideration was felt. Initially, it was common to take intuitive approach. The practice of using the factor of safety also originated due to the need for higher reliability. A well-designed, well-engineered, thoroughly tested and properly maintained component should never fail in operation. But, experience shows that even the best design, manufacture and maintenance cannot eliminate failure. This can be explained from the statistical nature of reliability. Absolute perfection cannot be attained in practice and one can only try to reduce the probability of failure to some small amount.

1.3 Historical Development

Reliability studies seem to have been made first on ball and roller bearings. The life of bearings was associated with a definite probability of survival. In the last century, a lot of studies were also made on railroad equipment because of the high rate of fatigue failure. But the science of reliability, as we know now, developed from the advent of aeroplanes and it assumed crucial importance with the advent of space vehicles. Space vehicles, complex systems as they are, are extremely prone to failures unless proper consideration has been given to reliability from the beginning. Reliability studies on aircraft were made in U.S.A. and U.K. in the 1930's and extensive work on reliability was done in Germany during the Second World War on rockets. After the war, U.S.A. and U.S.S.R. led the world in reliability studies.

1.4 Reliability and Quality Control - a Comparison

Reliability control and Quality control sometimes use very similar techniques. Yet there is a great difference between them. Quality is generally used to mean good performance and longevity. Reliability is also used to mean the same thing. But quality control measures only instantaneous performance and its variation from specimen to specimen. In quality control, the performance of a product as it leaves the production line is checked. But no consideration is given to the way the performance deteriorates when the part is put in service. So quality control is concerned with the "as is" performance of materials, parts and products etc., when the manufacture is complete. In reliability control, the performance measurements from the instantaneous "as is" time independent domain are extended to operating-time and life domain. In reliability, not only does it matter what the initial number of defectives is and the variation of performance characteristics is, but it also matters how long a product will maintain its original characteristics when in operation, how the variations spread with time etc. The statistical techniques used in reliability testing are very similar to those of the conventional quality control methods, with time added as a new dimension.

1.5 Strength

We may now examine the term strength a little more closely. In general, strength is the value of the external load at which failure occurs. Generally, in the handbooks, typical values of strengths of various materials are listed. Strictly speaking, this sort of approach is not proper. Due to various practical reasons and imperfections, the same type of strength of the same material shows an appreciable amount of scatter. For example, the tensile strength of Low Carbon Steel is listed as 63,000 psi. But actually, this listed value is associated with a certain probability and it would be more proper to specify the probability value that the tensile strength would be higher than 63,000 psi.

In general, the measured values of strengths are found to be normally distributed. Again the strength of a certain product is influenced by manufacturing methods and design, in addition to the material properties. For example, the strength of a shaft with sharp corners at the shoulders would be less than a similar shaft with smoother fillet radius. The strength of a product has been termed as Load Capability by Johnson (Ref. G.1.5). He states that the typical distribution curve for predicted load capability of a manufactured element is as shown in Fig. 1-1.

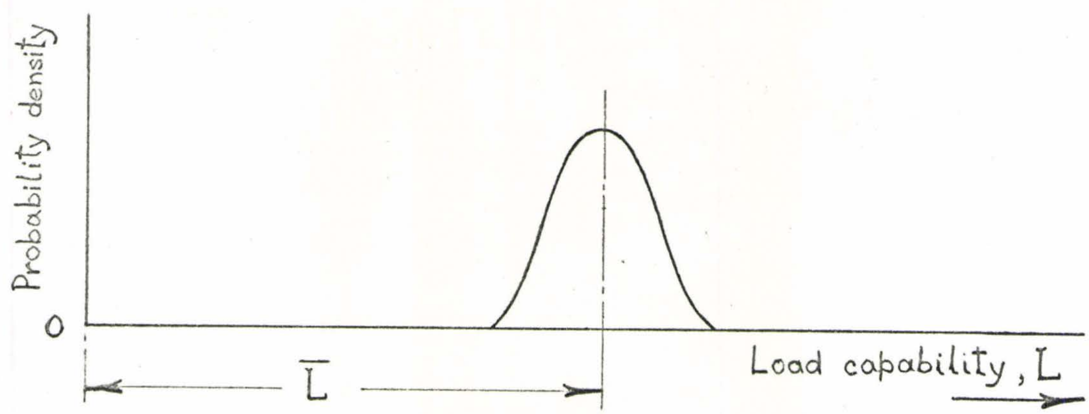


Fig. 1-1 Typical Statistical Distribution Curve for Load Capability of a Manufactured Mechanical Element

$\bar{L}$ = mean value of load capability

Haviland (Ref. A.1.36) mentions the use of a method of calculating the strength of a component by considering the crystal structure. As an example, we can consider the case of a shearing force applied to a perfect cubic crystal. As a result of the applied force, there is a displacement of the lattice planes of the crystal. If x is the shear displacement of a lattice plane with respect to the adjacent displaced plane and σ is the shear stress, then

$$x = \frac{d\sigma}{G} \tag{1-1}$$

where d is the spacing of the lattice plane and G is the shear modulus. Fig. 1-2 is a diagrammatic representation and x_i is the shear displacement.

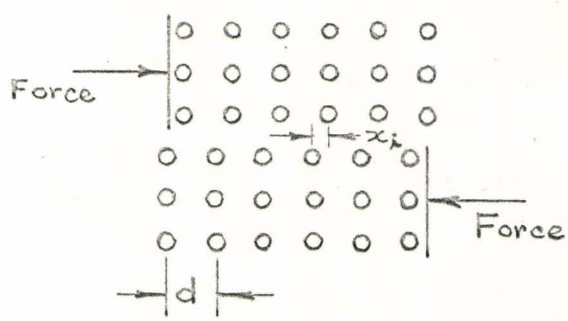


Fig. 1-2 Shear Force Applied to a Cubic Crystal

Since the lattice binding forces are electrostatic (coulomb force), they vanish by symmetry at a displacement of $d/2$. At some smaller displacement than this, of the order of $d/4$, an increase in stress will cause slippage of one lattice plane with respect to its adjacent one. Thus the maximum shear stress σ_m is

$$\sigma_m \approx \frac{G}{4} \tag{1-2}$$

which shows the point at which the body ceases to absorb energy linearly by this particular form of storage.

Next, we should consider the change of strength and performance of a component with time. Some objects such as concrete gain strength with time for a certain period (Ref. A.1.36, Fig. 3-15). This is also true for age hardening of Cast Iron. For some other objects, such as steel under static loading at atmospheric temperature, the strength remains essentially constant with time. Finally, the strength starts to reduce due to wear out.

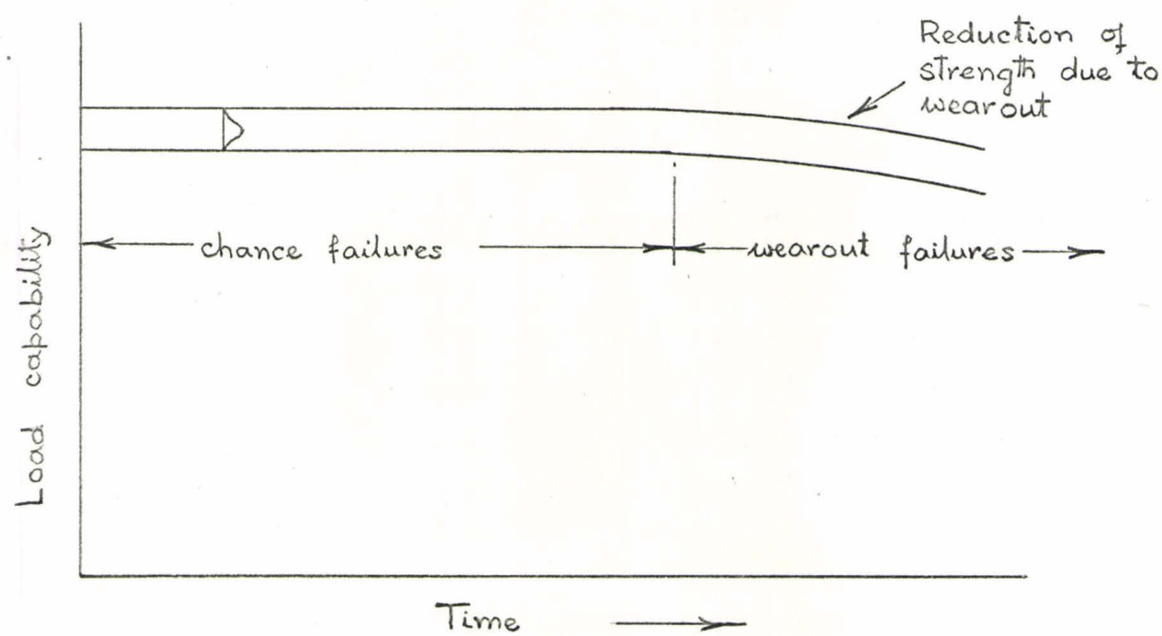


Fig. 1-3 Variation of Load Capability With Time for Mechanical Components Under Static Loading

In Fig. 1-3, the load capability of a certain statically loaded member was constant for a certain length of time. Finally, due to wear out, probably the cross-sectional area was reduced and load capability started to drop.

Dynamic loading produces fatigue and this is one of the most important causes of deterioration of strength of mechanical components. Consideration of fatigue is further complicated due to the fact that fatigue strength depends on the size of the specimen and is very sensitive to physical shapes due to stress-concentration effects. Ferrous metals and some other alloys show a definite endurance limit, and the specimen is capable of withstanding infinite number of stress reversals below this limit. For non-ferrous metals and alloys and non-metallic materials, a fatigue limit is not observed. The test results of fatigue testing show a lot of scatter and statistical techniques have to be used to obtain a meaningful conclusion. The method suggested by Peterson (Ref. G.2.23 and G.2.11) is based on the following assumptions:

(1) The population is normally dispersed with respect to stress.

(2) Results obtained at one lifetime can be extrapolated to another lifetime.

Epremian and Mehl (Ref. G.2.3) suggest (for data taken so that several samples are tested at a single stress level) a calculation based upon the idea that values of log N are normally distributed about a mean value. In general, the normal curve provides a good fit, if the number of data points are large and this method is used widely.

Another method due to Weibull is used extensively for estimation of lifetime of ball and roller bearings.

To sum up, the variation of strength with age can be expressed in a meaningful manner, only if associated probabilities are stated as shown in Fig. 1-4.

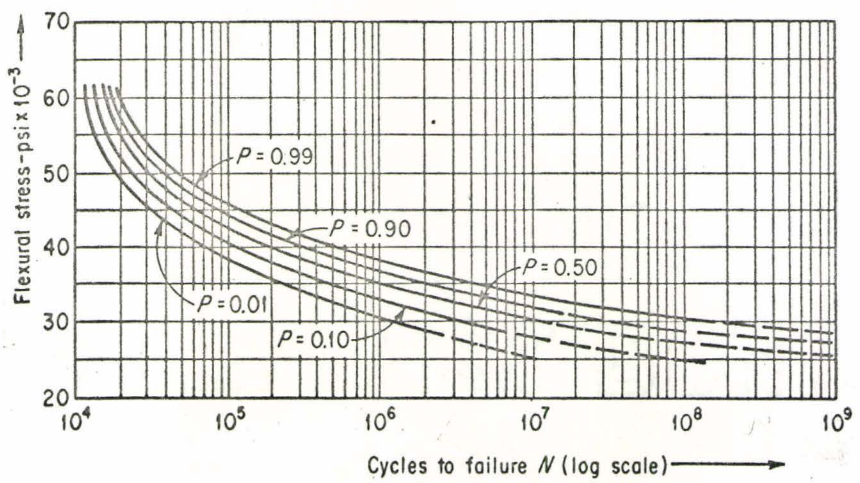


Fig. 1-4 Fatigue Strength - Cycle Relations for Various Probabilities of Failure

Another cause of variation of strength of mechanical components is creep. Face-centred cubic metals such as copper show gradual increase in yield tensile strength and gradual reduction of ductility as temperature is lowered, whereas for non-face-centred cubic metals such as Iron, these effects are more pronounced.

Under static external loading conditions, one of the important influences of constant exposure to high temperature is to produce a continuous creep deformation. Excessive creep deformation may cause malfunction of a component and it may even cause failure at lower stress levels.

1.6 Load

Having briefly reviewed strength of components and its deterioration, we will now consider the variation of external load. For practical objects, it is very difficult to state any particular value of external load. For an aircraft, we can express the load in form of a stress spectrum as shown in Fig. 1-5.

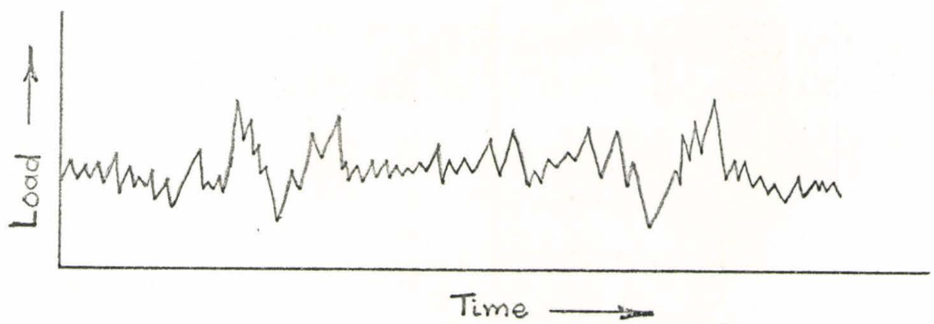


Fig. 1-5 Stress Spectrum for an Aircraft

The load peaks up at random intervals, probably due to gusts of wind or turbulence. Some investigators assume the load to be normally distributed as shown in Fig. 1-6. The numerical values in many cases can only be determined by experimentation.

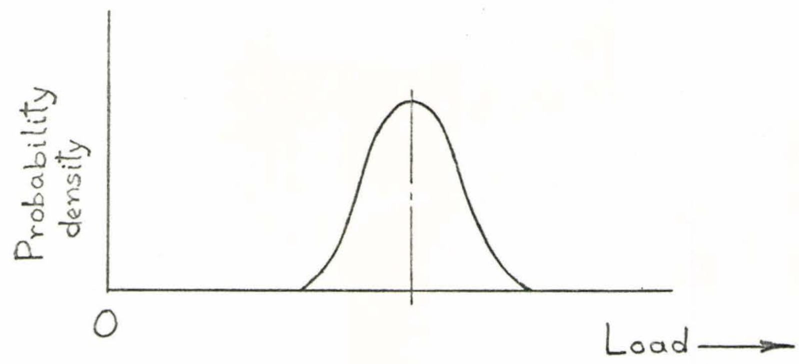


Fig. 1-6 Typical Statistical Distribution Curve for Load on a Mechanical Element in Usage

Then it is possible to state a value of load associated with a certain probability of occurrence and use this information in design. One can draw an envelope on the highest peaks of load spectrum and consider that as the highest expected load. Haviland (Ref. A.1.36) suggests the use of the theory of extreme values for the calculation of the expected value of the heaviest external load. This theory has been used with success for many natural and artificial phenomena. The gusts experienced by aircraft are distributed according to this relationship. Similar relations have been found for earthquakes, for weather and for strength of materials (Ref. F.1.29 and A.1.65).

1.7 Estimation of Lifetime Considering Strength and Load

If we consider both strength and external load to be normally distributed, the situation looks as shown in Fig. 1-7.

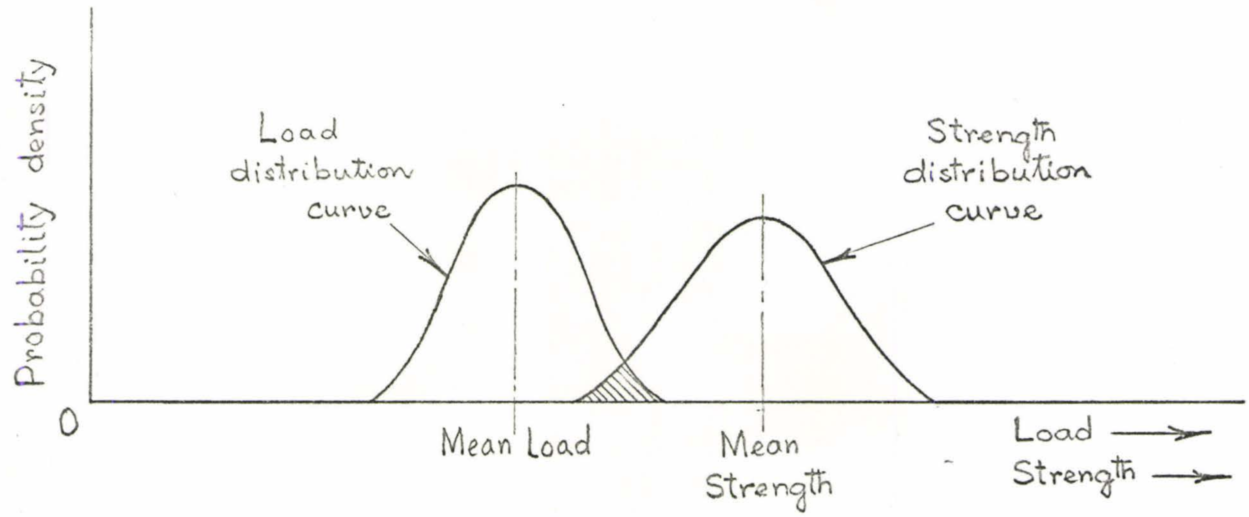


Fig. 1-7 Relationship of Typical Distribution Curves for Load and Strength for a Mechanical Element

The hatched part of the curve then signifies failure. If we have an idea of mean values of strength and load and if standard deviation can be estimated by experiment or previous experience, the probability of failure can be estimated.

A graph may also be plotted using time as horizontal axis (Fig. 1-8).

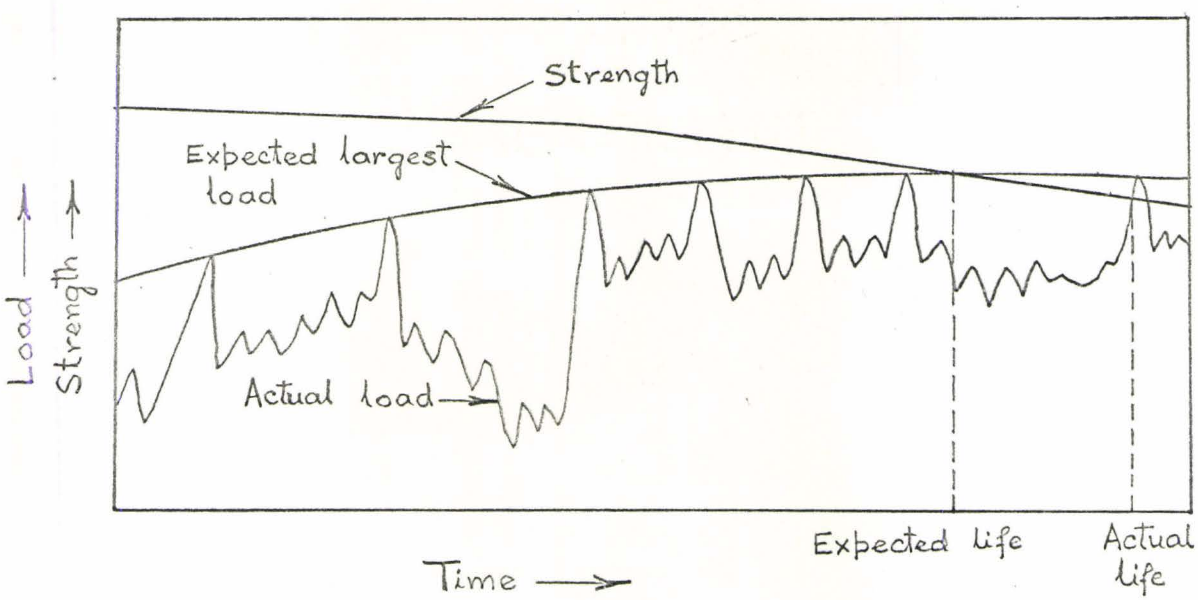


Fig. 1-8 Relationship Between Strength and Load for a Mechanical Element as the Operating time Increases

In case of cyclic loading causing fatigue, the situation may be as follows (Fig. 1-9) -

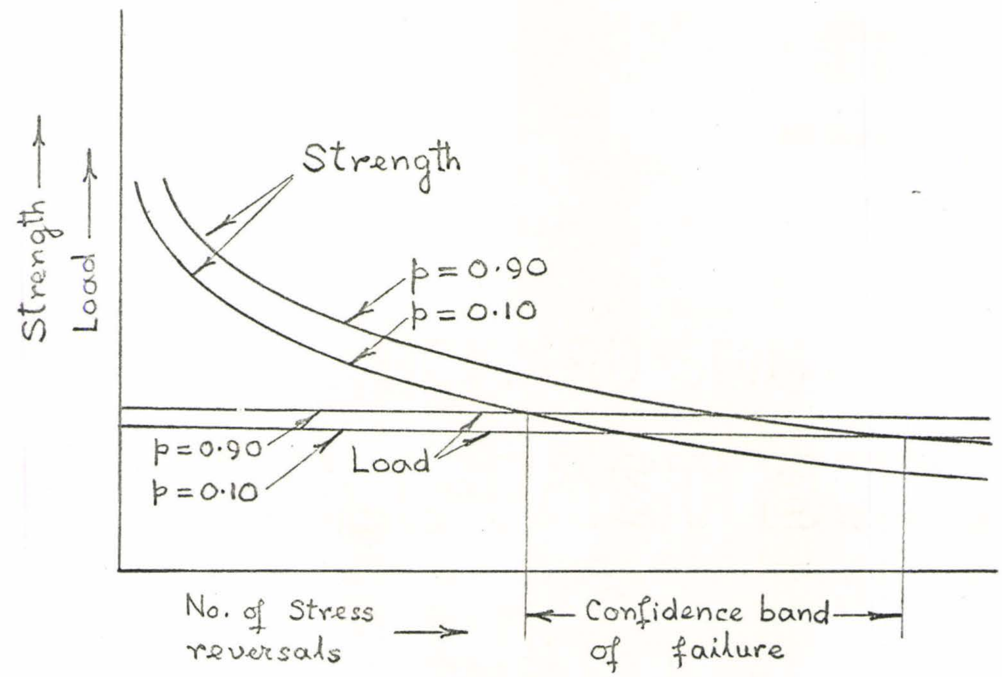


Fig. 1-9 Relationship Between Fatigue Strength and Cyclic Load for a Mechanical Element (confidence band of failure and associated probabilities are shown)

1.8 Environmental Conditions

Environment plays a very important role in reliability engineering since reliability is the probability of failure free operation for a specified time under the operating conditions. Hence the results of reliability tests conducted in the laboratory are apt to be misleading, unless sufficient care was exercised in simulating the operating conditions. Environment, by definition, includes all factors external to the object other than the particular load being considered. As such, the environment includes a set of materials and a set of energies. The material environment includes all materials that surround an object and are not part of it. The material environment must be considered with respect to two characteristics. The first of these is the nature of materials which determines the activation energy of the deterioration process. The second is the concentration of the material available for reactions. The practical environmental factors are items such as the composition of the atmosphere, the amount of dust present and similar factors.

The energy environment includes all forms of energy that may flow into or out of an object. This energy environment includes energies that are always associated with environment materials and other energies that are independent of the materials. An example of an associated energy is the ambient temperature. The free energy includes the gravitational, electrostatic, magnetic and electromagnetic fields. The energy environment must be considered in two ways. First, it is a factor in the deterioration process, so the energy available must be determined. Second, the energy must be considered as a load, although this consideration may be neglected if it is determined that the load applied by a given energy is small compared to some other energy source.

In few cases, it is also necessary to consider the materials as a factor in the determination of load. This arises if an energy field is present. The most common example of this is the ice and snow loads on structures where the gravitational field acting on the materials produces an added load.

It is common to consider the environment as composed of two classes, the free environment and the constrained environment. The environment is established by natural processes, and is beyond control. The constrained environment is established by other manufactured objects and so is controllable within limits that are established by the external free environment and by the nature of the associated objects. For example, a building is subjected to a free environment, whereas a computer installed within the building is subjected to a constrained environment. The variations in environmental conditions fall into limited number of patterns, which are: constant value, cyclic variation, strong persistence and irregular pattern.

For example, if we are designing an I.C. engine for a generator to operate in Arabia, we must note the annual cyclic variation of the cooling water temperature and also the mean value. Environmental conditions would be vastly different for such a plant working in Yukon territory of Canada.

Sometimes the term microenvironment is used to describe the environment surrounding a specific object. Microenvironment is always a constrained environment since the object itself has introduced a constraint on the variation possible. Constraints are also found due to the influence of additional objects. An example of the importance of microenvironment is the temperature distribution in a "black box" normally used to contain electronic equipment.

Statistical techniques are sometimes used in analyzing the effect of environmental conditions. Suppose a new object is being designed, which will move from one area to another. Environment for each area will be known. The movement plan of the object or the time it spends in each area is then set up. The expected pattern of variation is established for each area. Samples are then drawn from the area patterns in proportion to the time spent in that area by Monte Carlo technique. These samples are then tabulated and analyzed for the mean, effective and the largest values as needed.

2. ELEMENTS OF PROBABILITY AND STATISTICS PERTINENT TO RELIABILITY ENGINEERING

Reliability engineering is based on theory of probability and statistics and some important principles and laws are reviewed here.

2.1 Definitions

Mathematical (or a priori) definition of probability - If there are n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to an event A, the probability of the happening of A is defined as the ratio $^{m}/_{n}$. (Ref. F.1.45.)

Statistical (or empirical) definition of probability - If trials be repeated a great number of times under essentially the same conditions, then the limit of the ratio of the number of times that an event happens to the total number of trials as the number of trials increases indefinitely is called the probability of the happening of that event. It is assumed that the ratio approaches a finite and unique limit.

Independent Events - Events A and B are said to be independent if the information that A happened does not influence the probability of B.

Dependent Events - Events are dependent on one another if the occurrence of any one has an effect on the occurrence of another.

2.2 Theorem of Compound Probability

The probability that A and B happen is the probability that A happens times the probability that B then happens (Ref. F.2.9). If A and B are any events, then

$$p(AB) = p(A)p_A(B) \tag{2-1}$$

For independent events,

$$p_A(B) = p(B)$$

hence $$p(AB) = p(A).p(B) \tag{2-2}$$

The notationsused above are as follows -

$p(A)$ = probability that event A happens

$p(B)$ = probability that event B happens

$p_A(B)$ = probability that event B happens, provided that event A has happened

$p(AB)$ = probability that events A and B both happen

$p(A+B)$ = probability that event A happens or event B happens or both A and B happen

We will illustrate the application of the above theorem by an example. In Fig. 2-1, if a device will fail due to failure of either component X or Y, and for a given life the probability of survival of component X is R(X) and the probability of survival of component Y is R(Y), then probability of survival of both X and Y for the life of the device is R(X).R(Y), which is the reliability of the chain model.

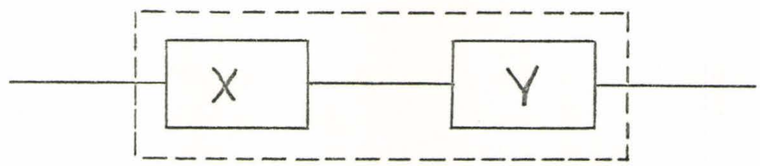


Fig. 2-1 A Device With Two Components Connected in Series

Here we assumed that the performance of one component does not affect the performance of the other.

As an example of dependent events, we can consider the probability of survival or reliability of an aircraft from time t_1 to t_2, after it has operated from time t_o to t_1. Then the probability that the aircraft survives from t_o to t_2

= (Probability that it survives from t_o to t_1)

X (Probability that it then survives from t_1 to t_2).

2.3 Theorem of Total Probability

When A and B are any events, then

$$p(A+B) = p(A) + p(B) - p(AB) \qquad (2\text{-}3)$$

The probability that event A happens or B happens or both A and B happen is given by the probability that A happens and probability that B happens less the probability of A and B happening simultaneously (Ref. F.2.9).

For mutually exclusive events, A and B cannot both happen simultaneously and hence

$$p(AB) = 0$$

Hence in this case,

$$p(A+B) = p(A) + p(B) \tag{2-4}$$

An application of this theorem is found in case of parallel redundant systems.

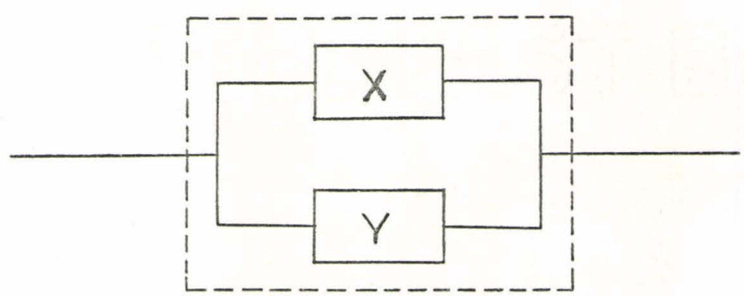


Fig. 2-2 Black Box Containing Two Components Connected in Parallel

Fig. 2-2 shows a black box containing two components X and Y in parallel. The system operates if either one of the components or both of them operate.

Then, the reliability of the black box

$$R = R(X) + R(Y) - R(XY) \tag{2-5}$$

We can arrive at the same result using the theorem of compound probability for independent events.

reliability of component X = R(X)

reliability of component Y = R(Y)

Then unreliability of component X = 1 - R(X) = Q(X) and unreliability of component Y = 1 - R(Y) = Q(Y). The black box fails if both the components fail.

So unreliability of black box

$$Q = Q(X).Q(Y)$$
$$= [1 - R(X)]\,[1 - R(Y)]$$
$$= 1 - R(X) - R(Y) + R(XY) \qquad (2\text{-}6)$$

Hence the reliability of the black box

$$R = 1 - Q(X).Q(Y)$$
$$= R(X) + R(Y) - R(XY) \qquad (2\text{-}7)$$

2.4 Bayes' Probability Theorem

If A is an event which depends on one of two mutually exclusive events B_i and B_j of which one must necessarily occur, then the probability of the occurrence of A is given by

$$P(A) = P(A, \text{ given } B_i).P(B_i)+P(A, \text{ given } B_j).P(B_j) \qquad (2\text{-}8)$$

Applied in case of reliability engineering, we can state the following rule (Ref. A.1.10) -

The probability of system failure equals the probability of systems failure given that a specified component in the system is good, times the probability that the component is good, plus the probability of system failure given that the said component is bad, times the probability that the component is bad.

We can also state the rule as

P(system failure) = P(system failure if component X is good) x P(X is good) + P(system failure if X is bad) x P(X is bad)

As an example, we will use this theorem to calculate the reliability of the system shown in block diagram in Fig. 2-3.

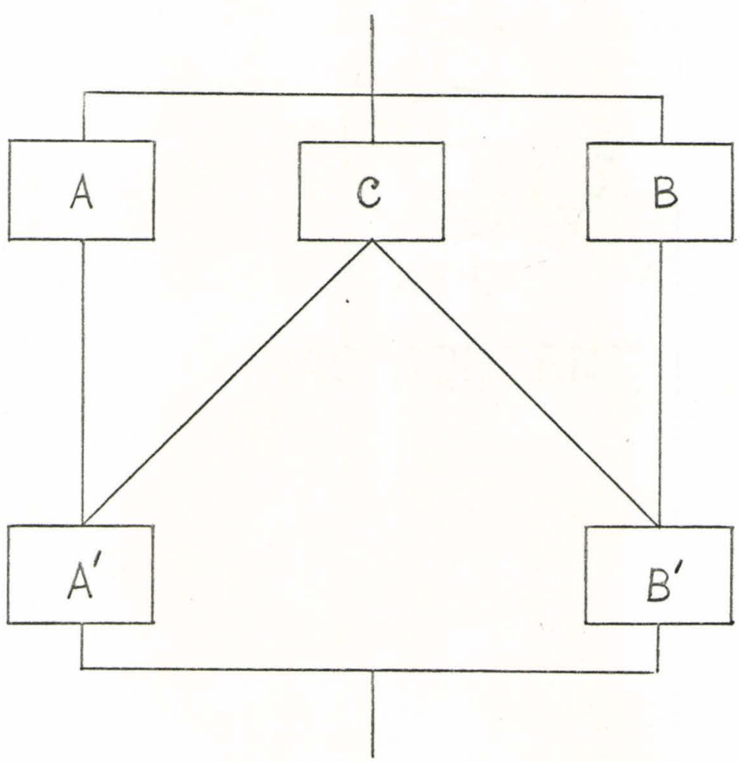


Fig. 2-3 Schematic Block Diagram Showing the Mode of Connection of Components

Two equal paths A-A' and B-B' operate in parallel. To improve the reliability further, another component C is connected to both A' and B'. Hence the system may operate using a combination of components as follows -

A-A', B-B', C-A', C-B'.

Let R_S and Q_S be system reliability and unreliability respectively

R_A be reliability of component A

R_B be reliability of component B

R_C be reliability of component C

Q_C be unreliability of component C.

Using the rule,

$$Q_S = Q_S(\text{if C is good}).R_C + Q_S(\text{if C is bad}).Q_C$$

Now we will calculate Q_S if C is good. If component C is good, the system will fail only if both A' and B' fails.

So

$$Q_S(\text{if C is good}) = (1 - R_{A'})(1 - R_{B'})$$

Next if C is bad, the system will fail only if both parallel paths A-A' and B-B' fail.

Hence

$$Q_S(\text{if C is bad}) = (1 - R_A R_{A'})(1 - R_B R_{B'})$$

Unreliability of the whole system now becomes

$$Q_S = (1 - R_{A'})(1 - R_{B'}).R_C + (1 - R_A R_{A'})(1 - R_B R_{B'}).Q_C \qquad (2\text{-}9)$$

Reliability of the system $R_S = 1 - Q_S$ (2-10)

2.5 Statistical Distributions

A number of statistical distributions find wide use in reliability and quality control and they are briefly enumerated here.

2.5.1 Discrete Distributions

We shall discuss briefly some discrete distributions used in reliability theory (Ref. F.1.24).

2.5.1(a) Binomial Distribution

It applies to situations often referred to as repeated trials. This is particularly useful when we are dealing with attributes, so that the outcome of a trial is either success or failure as the component is either good or bad. So it is a case of complementary and mutually exclusive events. The assumptions which underlie the binomial distribution are (i) the probability of a success is the same for each trial

(ii) the trials are independent.

In a sequence of n independent trials, let the probability of success on each trial be p. We denote the number of successes by the random variable S_n. The random variable S_n may assume any one of the discrete integer values K = 0, 1, 2, ..., n.

The associated probability distribution is (Ref. A.3.28)

$$Pr(S_n = K) = \binom{n}{K} p^K (1-p)^{n-K}$$

$$K = 0, 1, 2, \ldots\ldots, n. \qquad (2\text{-}11)$$

The mean of the random variable is

$$E(S_n) = np \qquad (2\text{-}12)$$

and the variance is

$$\mathrm{Var}(S_n) = np(1-p) \tag{2-13}$$

2.5.1(b) Multinomial Distribution

The binomial distribution can be generalized to the case of n repeated independent trials where each trial can have one of r outcomes. We denote the possible outcomes of each trial by E_1, E_2, ..., E_r and let the probability of the realization of E_i in each trial be $p_i (i = 1, 2, ..., r)$, where in general, p_i is only subject to the condition

$$p_1 + p_2 + \ldots\ldots + p_r = 1, \quad p_i \geqslant 0$$

The probability that in n trials E_1 occurs K_1 times, E_2 occurs K_2 times, etc. is

$$\frac{n!}{K_1!\, K_2! \cdots\cdots K_r!}\, p_1^{K_1}\, p_2^{K_2} \cdots\cdots p_r^{K_r} \tag{2-14}$$

where the K_i's are non-negative integers subject to

$$K_1 + K_2 + \ldots\ldots + K_r = n.$$

If r = 2, then this distribution reduces to the binomial form with $p_1 = p$, $p_2 = 1 - p$, $K_1 = K$ and $K_2 = n - K$.

The mean number of occurrences of the event E_j in N trials is Np, and its variance is $Npj(1 - p_j)$. The covariance between the number of occurrences of E_i and E_j is $-Np_jp_i$.

2.5.1(c) Geomatric Distribution

Let us consider a sequence of Bernoulli trials, for which the probability of success on each trial is p. We define the random variable Y_1 as the number of trials up to first success. It can be shown readily that Y_1 is a discrete random variable which may assume any one of the discrete integer values 1, 2, 3, ...

The probability distribution of Y_1 is

$$Pr(Y_1 = j) = p(1 - p)^{j-1}, \quad j = 1, 2, 3, \ldots \qquad (2\text{-}15)$$

The mean of Y_1 is

$$E(Y_1) = {}^1/_p \qquad (2\text{-}16)$$

and the variance of Y_1 is

$$Var(Y_1) = \frac{1 - p}{p^2} \qquad (2\text{-}17)$$

2.5.1(d) Pascal Distribution

This is a generalization of the geomatric distribution. The random variable of interest is Y_K, the number of trials until the K^{th} success occurs. Y_K is a discrete random variable which may assume any one of the discrete integer values K, K+1, K+2, ...

The probability distribution of Y_K is

$$Pr(Y_K=j) = \left(\frac{j-1}{K-1}\right) p^K q^{j-K}, \quad j = K, K+1, \ldots \qquad (2\text{-}18)$$

The mean of Y_K is

$$E(Y_K) = {}^{K}/_{p} \tag{2-19}$$

and the variance of Y_K is

$$\mathrm{Var}(Y_K) = \frac{K(1 - p)}{p^2} \tag{2-20}$$

The Geometric and Pascal distributions are frequently called discrete waiting time distributions. If each trial takes one second to perform, Y_1 is the number of seconds until the first success occurs and Y_K is the number of seconds until the K^{th} success occurs.

2.5.1(e) Poisson Distribution

The Poisson distribution can be obtained from the binomial distribution by simultaneously letting $n \rightarrow \infty$ and $p \rightarrow 0$, in such a way that the product $np = \lambda$, where λ is a preassigned positive constant. If this is done, the random variable S_n converges to a random variable S with associated probability distribution

$$\Pr\ (S=K) = \frac{e^{-\lambda}\lambda^K}{K!}\ ,\ K = 0,\ 1,\ 2,\ \ldots.. \tag{2-21}$$

The mean of S is given by

$$E(S) = \lambda \tag{2-22}$$

and the variance of S by

$$\mathrm{Var}(S) = \lambda$$

The Poisson distribution is used extensively in industry in quality control work (Ref. F.1.65 and F.1.5). For example, in an attributes sampling plan, a random sample is drawn from the submitted lot.

Let C = number of defects,

n = sample size,

p = lot fraction defective

and $P(C)$ = probability of any number of defects (C) in the sample.

Using Binomial distribution, we can write

$$P(C) = \binom{n}{C} p^C (1 - p)^{n-C}$$

But in most cases of quality control, the lot size is large relative to the sample size, and the fraction defective in a lot is small. Hence the Poisson distribution can be used, since the requisites are approximately satisfied.

We can write

$$P(C) = \frac{e^{-pn} (pn)^C}{C!} \tag{2-24}$$

Here pn represents the expected number of defectives per sample.

Sometimes, the probability of getting C defects or less is needed. Hence we must use the cumulative form of the Poisson distribution, which is

$$P(C \text{ or less}) = \sum_{C=0}^{C} \frac{e^{-(pn)} (pn)^C}{C!} \tag{2-25}$$

Cumulative probability curves of the Poisson exponential are available (Ref. F.3.3) and the probability of occurrence of C or less defects can be directly obtained. Tables given in Ref. F.3.7 are also very useful.

Poisson distribution is used in a variety of other cases such as hourly traffic loading, frequency of radioactive disintegration etc.

2.5.2 Continuous Distributions

2.5.2(a) Exponential Distribution

This is the most widely used distribution in reliability theory. Usually it is given in the following forms -

$$f_T(t) = \frac{1}{\theta}\, e^{-\frac{t}{\theta}} \tag{2-26}$$

$$f_T(t) = \lambda\, e^{-\lambda t} \tag{2-27}$$

where λ and θ are constants.

The mean is given by

$$E(T) = \int_0^\infty t\, f_T(t)\, dt$$

$$= \int_0^\infty \frac{t}{\theta}\, e^{-\frac{t}{\theta}}\, dt$$

$$= \theta = \frac{1}{\lambda} \tag{2-28}$$

and the variance is given by

$$\mathrm{Var}(T) = \theta^2 = \frac{1}{\lambda^2} \tag{2-29}$$

In reliability work, the parameters are designated as follows -

λ = chance failure rate

θ = mean time between failure (MTBF)

MTBF is also commonly designated by m

t is a variable signifying life time and is reckoned from any arbitrary instant.

The usefulness of exponential distribution for reliability studies becomes apparent, if we examine the variation of the failure rate during the lifetime of a component shown in Fig. 2-4.

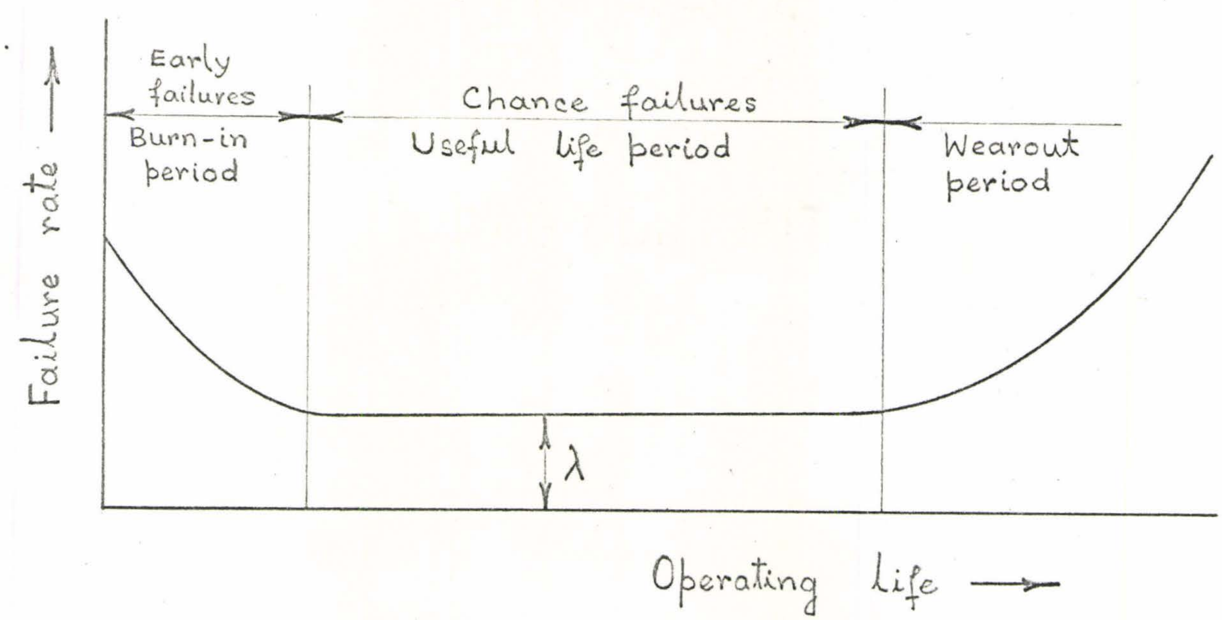


Fig. 2-4 Component Failure Rate as a Function of Operating Life

We see that in the useful life period, the failure rate λ is approximately constant and hence the requisite for using the exponential distribution is satisfied. This is a very convenient distribution to use in practice because as long as the failure rate remains constant, the age of the component is immaterial to the question of whether or not it will survive the next increment of time. Computationally also, the exponential distribution is easy to use and only a table of exponential function e^x is needed (Ref. F.3.11).

One often refers to the exponential distribution as corresponding to a purely random failure pattern (Ref. C.2.12). It is implied that whatever is causing the failure occurs according to a Poisson process (random, rare events) with some rate λ (Ref. F.2.2).

Let T be the random variable associated with the time interval between successive events (failures)

then $Pr(T > t) = Pr$ [no event occurs in the interval $(0, t)$],

where $t = 0$ is the time when the most recent event occurred.

From the Poisson assumption (Ref. A.1.65),

$$Pr(T > t) = e^{-\lambda t} \tag{2-30}$$

$$\text{Thus } Pr(T \leqslant t) = 1 - e^{-\lambda t} \tag{2-31}$$

The probability density function is then given by

$$f(t) = \lambda e^{-\lambda t}$$

The relevance of this sort of distribution to a real life situation can be explained as follows -

Imagine a situation where a device under test is being subjected to an environment E, which is some sort of random process. Let us imagine that this random process has peaks distributed in a Poisson manner and that it is only these peaks that can affect the device, in the sense that the device will fail if a peak occurs and will not fail otherwise. If this is the situation and if the peaks in the stochastic process describing the environment occur with Poisson rate λ, then the failure distribution of the device under test will be exponential, and the p.d.f. will be given by

$$f(t) = \lambda e^{-\lambda t}$$

This failure distribution, in reality describes the frequency of severe shocks in the environment. Hence, what is actually meant is that the device fails, if and only if a peak occurs and not otherwise.

In case of complex mechanisms, times between failure result from a superposition of failure patterns of the individual parts and Cox and Smith (Ref. F.1.10 and F.1.11) have shown that this gives rise to an exponential distribution of the times between successive breakdowns.

It will be instructive to derive the exponential distribution function from the definition of reliability. We have defined reliability as the probability of survival for a certain length of operating time under specified operating conditions. If we test a fixed number of components N_o, then suppose after time t, N_S components survive the test and N_f components fail. Then by definition,

$$\text{Reliability } R(t) = \frac{N_S}{N_o} = \frac{N_o - N_f}{N_o} = 1 - \frac{N_f}{N_o} \tag{2-32}$$

Differentiating, we obtain

$$\frac{dR}{dt} = -\frac{1}{N_o} \cdot \frac{dN_f}{dt} \tag{2-33}$$

or rearranging

$$\frac{dN_f}{dt} = -N_o \frac{dR}{dt} \tag{2-34}$$

Dividing either side by N_S,

$$\frac{1}{N_S} \cdot \frac{dN_f}{dt} = -\frac{N_o}{N_S} \frac{dR}{dt} \tag{2-35}$$

The term $\frac{1}{N_S} \cdot \frac{dN_f}{dt}$ is the instantaneous probability of failure per one component, which is called failure rate λ.

Then

$$\lambda = -\frac{N_o}{N_S} \frac{dR}{dt} = -\frac{1}{R} \cdot \frac{dR}{dt} \tag{2-36}$$

or

$$\lambda \, dt = -\frac{dR}{R} \tag{2-37}$$

Integrating,

$$\ln R = -\int_0^t \lambda \, dt \tag{2-38}$$

During the useful life of the component, the failure rate is approximately constant. Then taking λ as constant,

$$R(t) = e^{-\lambda t} \tag{2-39}$$

The failure density function $f_T(t)$ is the distribution of failures in time on a per component basis or the failure frequency curve per component basis.

$$\text{Then } f(t) = \frac{1}{N_o} \frac{dN_f}{dt}$$

$$= - \frac{dR}{dt}$$

$$= - \frac{d}{dt}\left(e^{-\lambda t}\right)$$

$$\text{or } f(t) = \lambda e^{-\lambda t}$$

This is the exponential distribution function developed before.

The unreliability Q(t) is the cumulative probability function and can be obtained by integrating the probability of failure distribution function.

$$Q(t) = \int_0^t f(t)\,dt \tag{2-40}$$

$$\text{Then reliability } R(t) = 1 - \int_0^t f(t).dt$$

$$= \int_t^\infty f(t).dt. \tag{2-41}$$

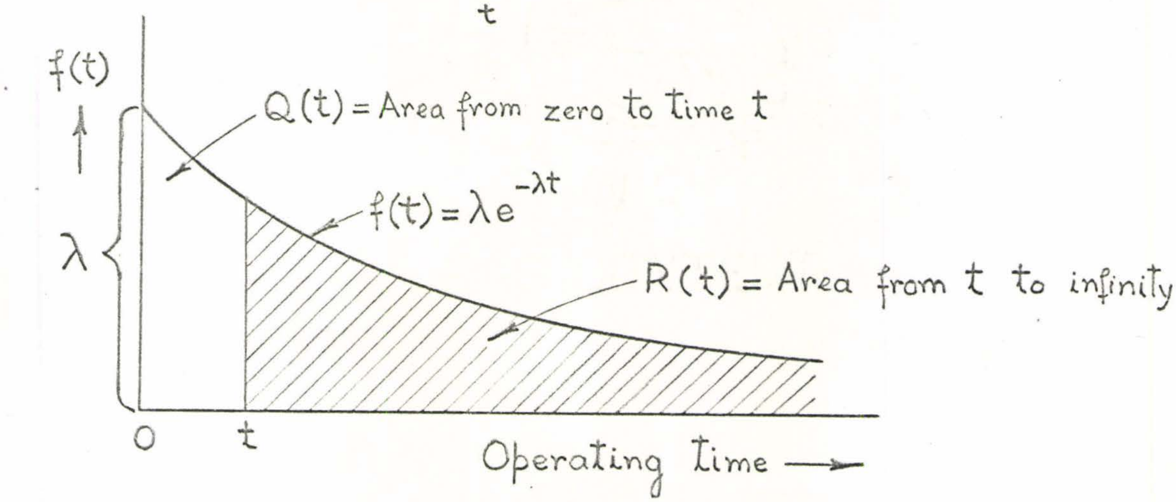


Fig. 2-5 shows the exponential probability density function.

The exponential distribution works reasonably well in many cases and agrees closely with empirical facts. Computational simplicity obtained by using this distribution in such cases as series or parallel circuits is also another reason for its popularity. There are certain drawbacks in using this distribution (Ref. B.1.51). The most important one is due to the fact that a clear physical distinction has not yet been found to satisfactorily delineate between failures in burn-in period and wear out failures on the one hand and the random failures of the mid-period (Ref. Fig. 2-4). Another disturbing feature is that, in some cases the failure rate never really remains constant, but goes on increasing with age throughout the lifetime of the component. Such a situation can be met with under fatigue conditions.

2.5.2(b) Mixed Exponential Distribution

This is a generalization of the exponential distribution where it is assumed that a piece of equipment can fail with probability p from one of two sources, each of them being exponentially distributed. The usual form of density function is given by

$$f(t) = p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t} \tag{2-42}$$

The mean is given by

$$E(T) = \frac{p}{\lambda_1} + \frac{(1-p)}{\lambda_2} \tag{2-43}$$

and the variance is given by

$$\mathrm{Var}(T) = \frac{p}{\lambda_1^2} + \frac{(1-p)}{\lambda_2^2} \tag{2-44}$$

2.5.2(c) Gamma Distribution

Gamma distribution is useful distribution in fatigue and wear-out studies. It has also a very important relationship to the exponential distribution, namely the sum of n identically distributed random variables, each an exponential distribution with parameter θ, is gamma distribution with parameters n and θ (Ref. A.1.66). The probability density function of failure in general form is given by

$$f_T(t) = \frac{t^{\alpha-1} e^{-\frac{t}{\theta}}}{\alpha!\ \theta^{\alpha}} \tag{2-45}$$

The mean is given by

$$E(T) = \alpha^{\theta} \tag{2-46}$$

and the variance by

$$\mathrm{Var}(T) = \alpha\theta^2 \tag{2-47}$$

α is also called shaping parameter and θ the scaling parameter.

Gamma distribution is a two-parameter-type statistical distribution, whereas the exponential distribution admits one. Two parameters permit greater flexibility in curve fitting, and hence a better fit to empirical data can be obtained than with a single parameter model. If the value of the shaping parameter α is chosen as 1, the gamma distribution density function reduces to the familiar exponential form. Choices of α less than 1 produce probability density functions more convex (when viewed from the origin) than does the exponential law. Choices of α greater than 1 produce a humped probability density function which can be considered to be a not too unsatisfactory approximation to a Gaussian error function. A special case is obtained when we put $\alpha = \frac{1}{2}$ and $\theta = 2$. Chi-square distribution then results. If under the same physical assumptions of random failure processes as for the exponential law, the failure data were plotted against total aggregate test time for all components, a gamma distribution arises (Ref. B.1.51).

2.5.2(d) Normal Distribution

The normal distribution is one of the most familiar forms of statistical distributions.

The density function is given by

$$f(T) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(T-M)^2}{2\sigma^2}} \tag{2-48}$$

The mean is given by

$$E(T) = M \tag{2-49}$$

and variance as

$$Var(T) = \sigma^2 \tag{2-50}$$

Normal distribution approximates the wearout phenomena quite well (Ref. A.1.10) and has been widely used for this purpose. Failures cluster around the mean wearout life M and standard deviation σ provides a measure of scatter. This distribution is convenient from a computational point of view since tables of areas under the curve and ordinates are readily available. One disadvantage of using this distribution in approximating wearout phenomena stems from the fact that the normal probability density function curve stretches from $-\infty$ to $+\infty$, and obviously it is quite impossible to have a value of component age T at failure less than zero; neither can any component be expected to last infinitely. But since the area under the curve at both ends is very small, the error introduced is negligible. For instance, probability of a component failing due to wearout at an age less than $M-3\sigma$ is only 0.0013. Hence to eliminate any significant part of the distribution lying on the negative axis, one can state as a general rule that this distribution may be used to life test situations if its mean is positive and the ratio of the mean to the standard deviation is greater than 2.5 (Ref. A.3.28).

Another restriction is that the failure effect contributions must be additive.

The cumulative probability of failure can be obtained from the density function by integration.

$$Q(T) = \int_{-\infty}^{T} f(T).dT \qquad (2\text{-}51)$$

But using lower limit of - makes no sense and we can alternatively write

$$Q(T) = 1 - \int_{T}^{+\infty} f(T)dT \qquad (2\text{-}52)$$

The cumulative reliability is then

$$R(T) = \int_{T}^{+\infty} f(T)\,dT \qquad (2\text{-}53)$$

Truncated normal distribution has been discussed in Sec. 3.3.

2.5.2(e) Logarithmic Normal Distribution

This distribution has been used by some investigators in describing wearout failures. It is generally obtained by taking the logarithm of a set of values, which follow a normal distribution.

The probability density function is of the form

$$f(T) = \frac{1}{\sigma T\sqrt{2\pi}}\, e^{-\frac{(\log T - M)^2}{2\sigma^2}} \qquad (2\text{-}54)$$

Here the variable log T is treated as normally distributed and σ and M are the standard deviation and mean of log T.

Bazovsky (Ref. A.1.10) advocates use of logarithmic normal distribution to approximate wearout phenomena when $M < 3\sigma$, because of the advantage of having f(T) = 0 at T = 0.

3. RELIABILITY TESTING

3.1 Introduction

In reliability testing, also called life testing, a number of components or assemblies are operated under some desired operating conditions, and the lives or times to failure are measured. In most cases, single components or sub-assemblies are tested in the laboratory at simulated stress conditions and reliability data is obtained. The designer then uses this data in designing a machine as a system which is required to have a certain specified reliability. Or in other words, the system reliability is predicted from the component reliability data using mathematical methods. The designer can then try various configurations of components, such as series, parallel, standby etc. to obtain the specified reliability at minimum cost.

Since reliability is defined as the probability of survival for a certain time under operating conditions, it is very important that, for reliability testing in laboratory, the operating environmental conditions are closely simulated. Thus, before undertaking any life tests, one must have enough data about the operating environment. Environmental tests are used to this end and various statistical methods are used to obtain meaningful data. Another method of reliability testing is measurement at actual service stress levels. Components are observed during the actual service use of the equipment in which they are installed and failure data is recorded. Then information can be obtained about the reliability of the components. But this is a post factum approach and data are collected for equipments which have already been designed. So if any modification is recommended based on this data, redesign as costly modifications may be necessary, which are expensive and time consuming. But, on the other hand, such historical data are very useful for future design as development. This type of testing has been used by the automobile industry for a long time and has resulted in reliability improvement in future models.

In laboratory reliability measurements, failure times of components are noted. To analyze this data and to obtain useful reliability information, it is convenient to use some statistical distribution and assume it is a mathematical model adequately representing the behaviour of components. In reality, no distribution is exactly followed, but still some distributions may be chosen which approximate the failure data to a reasonable accuracy. Choice of a particular distribution depends a great deal on the past experience with the process. Exponential distribution has been widely used in the chance failure region of the life of the component and normal distribution has been used for wearout region. Recently, the Weibull distribution is also finding widespread acceptance. One should be careful in choosing a distribution, since if the data do not follow the assumed distribution, any conclusions drawn will be largely invalid. There are some statistical procedures, known as nonparametric methods, which do not depend on the nature of the distribution of the population from which the data is drawn. Here no assumption is made concerning the distribution for the operating life of the equipment. Of course, there is some loss in statistical efficiency when a nonparametric method is used and hence, in general, the component life is assumed to follow some distribution.

3.2 Life Testing Assuming Exponential Distribution

3.2.1 Fixed Failure Truncated Life Test

The probability density function is given by

$$f(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}} \tag{3-1}$$

or

$$f(t) = \lambda e^{-\lambda t} \tag{3-2}$$

θ is Mean time between failures (MTBF) and λ is failure rate. Thus parameters are constant for exponential distribution.

Then Reliability $R(t) = e^{-\lambda t}$ (3-3)

Exponential distribution is used during the useful life period of a component, after early failures have been eliminated and wearout has not set in. Failures are assumed to occur due to environmental peaks, which are considered as random rare events. Cox and Smith (Ref. F.1.10 and F.1.11) have shown that in case of complex mechanisms, times between failure result from a superposition of failure patterns of the individual parts, and this gives rise to an exponential distribution of the times between successive breakdowns.

In life testing under exponential distribution, one has to be careful to see that only debugged components are used (after eliminating early failures) and the test is terminated before wearout sets in. We are interested in obtaining an estimate of θ, the Mean time between failures from the failure data of the test pieces. Knowing MTBF, reliability for any mission time can immediately be calculated.

Ideally, we can put n components or assemblies on test and note the time to failure t_i of each.

$$\text{Then estimate of MTBF } \hat{\theta} = \frac{\Sigma t_i}{n} \tag{3-4}$$

But in practice, this is almost impossible, since we have to run the test for a great length of time and the expense incurred would be prohibitive. The method usually followed is to put n components to test. As the components fail, an ordered set of failure times t_1, t_2, t_3 ... are obtained, such that $t_1 < t_2 < t_3$ ... . The test is discontinued as soon as the r^{th} component fails. This method is known as Fixed Failure Truncated Life Test. Then the best estimate of MTBF is given by

$$\hat{\theta} = \frac{t_1 + t_2 + t_3 + \ldots + t_r + (n-r)\, t_r}{r}$$

$$= \frac{\sum_{i=1}^{r} t_i + (n-r)\, t_r}{r} \tag{3-5}$$

Epstein and Sobel (Ref. C.2.13) has shown that it is the best estimate of parameter θ in the sense that it is maximum likelihood, unbiased, minimum variance, efficient and sufficient. This is valid for a non-replacement test. In case the failed units are replaced by new units so that the sample size remains constant at n, Roberts (Ref. A.1.65) states that

$$\hat{\theta} = \frac{n\, t_r}{r} \tag{3-6}$$

We have assumed the exponential distribution to hold true in the chance failure region of the operating life of the components. But wearout life of a component cannot be assumed to follow the same distribution. Hence if any of the r failures are considered to be due to wearout, they must be censored out in estimating MTBF (Ref. A.1.10).

If failure of K components are deemed to be due to wearout or any cause other than chance, then

$$\hat{\theta} = \frac{\sum_{i=1}^{r} t_i + (n-r)\, t_r}{r-K} \tag{3-7}$$

By stopping the test after the r^{th} component fails, the remaining (n-r) components can be put to service use, since they are as good as new (assuming exponential distribution). This results in saving of testing expense. Testing time is also reduced substantially, as can be seen from Table 3.1 as given by Epstein and Sobel (Ref. C.2.13).

Here

$E(t_{r,n})$ = average waiting time to observe first r failures from a sample of size n $(n > r)$

and

$E(T_{r,r})$ = average waiting time to observe all r failures from a sample of size r.

TABLE 3-1

Ratio of the expected waiting time to observe the r^{th} failure in samples of size n and r respectively

$$\frac{E(t_{r,n})}{E(t_{r,r})} = \alpha_{r,n}$$

r \ n	1	2	3	4	5	10	15	20
1	1	0.50	0.33	0.25	0.20	0.10	0.067	0.050
2		1	0.56	0.39	0.30	0.14	0.092	0.068
3			1	0.59	0.43	0.18	0.12	0.087
4				1	0.62	0.23	0.14	0.104
5					1	0.28	0.18	0.125
10						1	0.35	0.23

So we see that if we take a sample of 10 and discontinue to test after 5^{th} failure, the time necessary would be only 0.28 of the time needed for all 10 components to fail.

Now let us examine the confidence interval of the estimate of MTBF. The calculated value of $\hat{\theta}$ is a point estimate of the true parameter θ and to obtain an interval estimate, we must have the confidence limits. Epstein (Ref. C.2.18) has shown that the ratio $2r \frac{\hat{\theta}}{\theta}$ has a chi-square distribution with 2r degrees of freedom when the test from which the estimate θ was obtained was terminated after the r^{th} failure. For a two sided confidence level $(1-\alpha)$, we can write

$$P\left(\chi^2_{1-\alpha/2,\,2r} \leq \frac{2r\hat{\theta}}{\theta} \leq \chi^2_{\alpha/2,\,2r}\right) = (1-\alpha) \qquad (3\text{-}8)$$

This means that there is a probability $(1-\alpha)$ that the value of the ratio $\frac{2r\hat{\theta}}{\theta}$ will be within the interval given by two percentage points of the chi-square distribution.

By rearrangement, we can write

$$\hat{\theta}\,\frac{2r}{\chi^2_{\alpha/2,\,2r}} \leq \theta \leq \hat{\theta}\,\frac{2r}{\chi^2_{1-\alpha/2,\,2r}} \qquad (3\text{-}9)$$

Two-sided lower confidence limit is then

$$L = \frac{2r}{\chi^2_{\alpha/2,\,2r}}\,\hat{\theta} \qquad (3\text{-}10)$$

and upper confidence limit

$$U = \frac{2r}{\chi^2_{1-\alpha/2,\,2r}}\,\hat{\theta} \qquad (3\text{-}11)$$

Sometimes, only one sided confidence limits are desired. We want to know that the true value of the parameter θ exceeds a certain minimum life with a probability $(1-\alpha)$.

Then one-sided lower confidence limit

$$C_L = \frac{2r\hat{\theta}}{\chi^2_{\alpha,2r}} \tag{3-12}$$

MTBF confidence limits can also be obtained from the graph shown in Fig. 3-1 (from Ref. A.1.13). The graph provides two-sided confidence limits. To obtain one-sided confidence limits, the graph may be used with following conversions-

Two-sided confidence level (%)	One-sided confidence level (%)
60	80
80	90
90	95
95	97.5
99	99.5

We shall now solve a typical example. Suppose a manufacturing firm produces some mass produced appliance such as a washing machine. The machine uses a gearbox subassembly in the drive train. In order to improve the reliability and gain more competitive advantage in the market, the manufacturer decides to do some life tests on the gearbox subassembly. The life distribution is assumed to be exponential and it is decided to truncate the test after a fixed number of failures. Initially, a sample size of 10 is chosen and it is decided to obtain 3 failures.

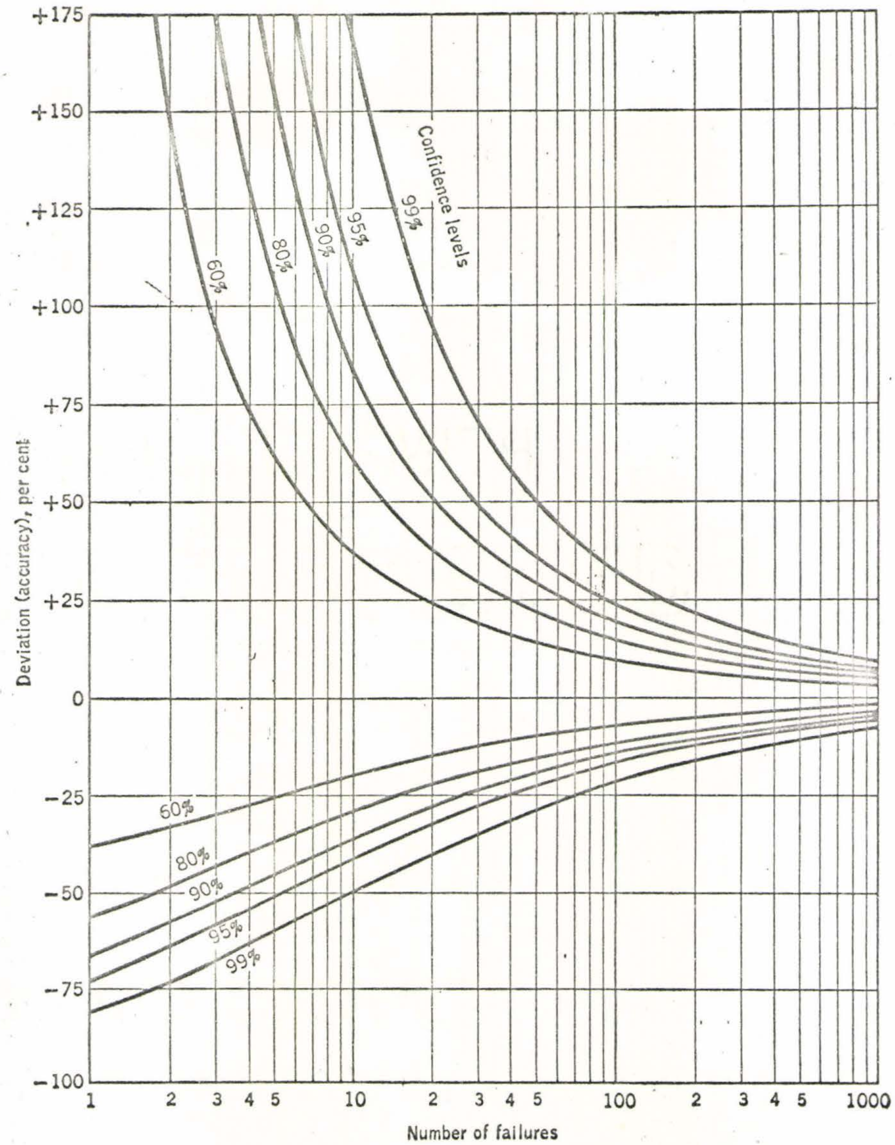


Fig. 3-1 Confidence Limits for Measurement of Mean Time Between Failures

From Table 3.1, it was seen that

$$\frac{E(t_{r,n})}{E(t_{r,r})} = 0.18$$

for n = 10 and r = 3

But it was then realized that the test time could be reduced by about 50% by putting 20 gearboxes on test.

$$\text{for } n = 20 \text{ and } r = 3, \quad \frac{E(t_{r,n})}{E(t_{r,r})} = 0.087$$

Increasing the number of specimens in the test reduced the test time and the variation among the specimens was also considered. Now the test is run in the laboratory at simulated stress conditions. The following data was obtained.

First failure after 833 hours

2nd failure after 838 hours

3rd failure after 896 hours

The components were debugged before being put on test to remove early failures and on examining the failed specimens, it was found that the failures were not due to wearout.

Using equation (3-5) we obtain

$$\hat{\theta} = \frac{833 + 838 + 896 + (20-3)\ 896}{3}$$

$$= \frac{17799}{3} = 5933 \textbf{ hours.}$$

Now we will calculate the confidence limits.

If we choose $\alpha = 0.10$, $\chi^2_{\alpha/2, 2r} = \chi^2_{0.05, 6} = 12.592$

$$\chi^2_{1-\alpha/2, 2r} = \chi^2_{0.95, 6} = 1.635$$

$$\chi^2_{\alpha, 2r} = \chi^2_{0.10, 6} = 10.645$$

From equation (3-10)

$$L = \frac{2r\hat{\theta}}{\chi^2_{\alpha/2, 2r}}$$

$$= \frac{2 \times 3 \times 5933}{12.592} = 2822 \text{ hours}$$

from equation (3-11)

$$U = \frac{2r\hat{\theta}}{\chi^2_{1-\alpha/2, 2r}}$$

$$= \frac{2 \times 3 \times 5933}{1.635} = 21780 \text{ hours}$$

from equation (3-12)

$$C_L = \frac{2r\hat{\theta}}{\chi^2_{\alpha, 2r}}$$

$$= \frac{2 \times 3 \times 5933}{10.645} = 3340 \text{ hours}$$

So with 90% confidence, we can state that the true value of MTBF lies between 2822 hours and 21780 hours, and is greater than 3340 hours.

We had conducted this test with a sample of 20 specimens and truncated the test after 3rd failure. Let us examine the case, if we continue the test till we obtain 5 failures.

Then $n = 20,\ r = 5$

From Table 3-1, $\dfrac{E(t_{r,n})}{E(t_{r,r})} = 0.125.$

In previous case, this ratio was 0.087. So the testing time would increase considerably. Suppose that now with $n = 20$ and $r = 5$, we obtain $\theta = 5933$ hours, which is the same as in previous case. We shall now examine the confidence intervals.

As before, we choose $\alpha = 0{\cdot}10$,

$$\chi^2_{\alpha/2,2r} = \chi^2_{0\cdot05,10} = 18{\cdot}307$$

$$\chi^2_{1-\alpha/2,2r} = \chi^2_{0\cdot95,10} = 3{\cdot}940$$

$$\chi^2_{\alpha,2r} = \chi^2_{0\cdot10,10} = 15{\cdot}987$$

Using equation (3-10)

$$L = \frac{2r\hat{\theta}}{\chi^2_{\alpha/2,2r}}$$

$$= \frac{2 \times 5 \times 5933}{18.307} = 3220 \text{ hours}$$

equation (3-11) gives

$$U = \frac{2r\hat{\theta}}{\chi^2_{1-\alpha/2,\,2r}}$$

$$= \frac{2 \times 5 \times 5933}{3.940} = 15030 \text{ hours}$$

and from equation (3-12)

$$C_L = \frac{2r\hat{\theta}}{\chi^2_{\alpha,\,2r}}$$

$$= \frac{2 \times 5 \times 5933}{15.987} = 3715 \text{ hours.}$$

So we see that by running a longer test and obtaining 5 failures, the confidence interval has been considerably narrowed down for the same level of significance α. But a longer test would obviously be more expensive. Hence, we have to reach some compromise between the cost of testing and precision obtained. It is of interest to note that sample size n does not affect the calculation of confidence interval. It is of importance only for the waiting time of test truncation.

For reliability testing, often one works in the reverse order. We may need to know with $100(1-\alpha)$ per cent confidence that the true reliability R is larger than $\exp(-t/C_L)$. So the level of significance α and the lower confidence limit C_L for a one-sided test have been specified. We must decide a value for r, the number of failures at test truncation.

Equation (3-12) can then be written as

$$\hat{\theta} \geqslant C_L \frac{\chi^2_{\alpha,2r}}{2r}$$

To satisfy the specified reliability requirements, the estimated value of MTBF should exceed this value.

Writing T = total observed operating time = $\hat{\theta}r$, we can state that in T hours, not more than r failures should occur. The value of sample size n can now be chosen from the consideration of waiting time to r^{th} failure.

Epstein (Ref. C.2.15) treats this problem from a different viewpoint. He defines the quantile x_p as being that life such that a proportion p of the items live for at least time x_p. Accordingly,

$$Pr\ (t \geqslant x_p) = p \tag{3-13}$$

The p.d.f. for an exponential distribution is

$$f(t) = \frac{1}{\theta}\, e^{\frac{-t}{\theta}}$$

Substituting, we get

$$x_p = \theta \ln \frac{1}{p} \tag{3-14}$$

The one-sided 100(1-α) per cent confidence interval for x_p is given by

$$x_p > \frac{2r\hat{\theta} \ln 1/p}{\chi^2_{\alpha,2r}} \tag{3-15}$$

This means that we can be $100(1-\alpha)$ per cent confident of the assertion that the fraction of items surviving

$$\tau = \frac{2r\hat{\theta}\ln 1/p}{\chi^2_{\alpha,2r}}$$

or more time units is greater than or equal to p. If we define

$$K(r,\alpha,p) = \frac{2r\ln 1/p}{\chi^2_{\alpha,2r}}, \tag{3-16}$$

then we can say that the multipliers $K(r, \alpha, p)$ are such that there is a probability $(1-\alpha)$ that at least 100p per cent of the population has life exceeding $K(r, \alpha, p)\hat{\theta}_{r,n}$. Epstein (Ref. C.2.15) has tabulated values of $K(r, \alpha, p)$ for various values of r, α and p. Table 3.2 has been reproduced from Epstein's paper.

If we choose $\alpha = 0.10$, $r = 3$ and $p = 0.50$, then $K(r, \alpha, p) = 0.391$. If $\hat{\theta}_{r,n}$ as estimated from test is 5933 hours, then $K(r, \alpha, p)\hat{\theta}_{r,n} =$

$$= \tau = 0.391 \times 5933 = 2320 \text{ hours.}$$

Hence we can assert with 90% confidence that at least 50% of the population have lives exceeding 2320 hours.

TABLE 3.2

Values of K(r,0.01,p)

r \ p	.50	.75	.90	.95	.99
1	.150	.0624	.0229	.0111	.0022
2	.209	.0866	.0317	.0154	.0030
3	.247	.103	.0376	.0183	.0036
4	.276	.115	.0419	.0204	.0040
5	.299	.124	.0454	.0221	.0043
6	.317	.132	.0483	.0235	.0046
7	.333	.138	.0506	.0246	.0048
8	.347	.144	.0527	.0257	.0050
9	.358	.149	.0545	.0265	.0052
10	.369	.153	.0561	.0273	.0053
11	.378	.157	.0575	.0280	.0055
12	.387	.161	.0588	.0286	.0056
13	.395	.164	.0601	.0292	.0057
14	.402	.167	.0611	.0298	.0058
15	.408	.169	.0621	.0302	.0059
16	.414	.172	.0630	.0307	.0060
17	.420	.174	.0639	.0311	.0061
18	.426	.177	.0647	.0315	.0061
19	.430	.179	.0655	.0319	.0062
20	.435	.181	.0662	.0322	.0063
25	.455	.189	.0692	.0337	.0066
30	.470	.195	.0716	.0348	.0068
40	.493	.205	.0750	.0365	.0071
50	.510	.212	.0776	.0378	.0074
75	.539	.223	.0818	.0398	.0078
100	.556	.231	.0845	.0411	.0080

Values of K(r,0.05,p)

r \ p	.50	.75	.90	.95	.99
1	.231	.0961	.0352	.0171	.0033
2	.292	.121	.0445	.0216	.0042
3	.330	.137	.0495	.0241	.0048
4	.358	.148	.0544	.0264	.0052
5	.378	.157	.0575	.0280	.0055
6	.396	.164	.0602	.0293	.0057
7	.410	.170	.0623	.0303	.0059
8	.421	.175	.0641	.0312	.0061
9	.432	.180	.0658	.0320	.0062
10	.442	.183	.0671	.0327	.0064
11	.450	.187	.0684	.0333	.0065
12	.457	.190	.0695	.0338	.0066
13	.464	.192	.0705	.0343	.0067
14	.469	.195	.0714	.0347	.0068
15	.475	.197	.0722	.0351	.0069
16	.480	.199	.0730	.0355	.0069
17	.485	.201	.0738	.0359	.0070
18	.489	.203	.0744	.0362	.0071
19	.493	.205	.0750	.0365	.0071
20	.497	.206	.0756	.0368	.0072
25	.514	.213	.0781	.0380	.0074
30	.526	.218	.0800	.0389	.0076
40	.544	.226	.0827	.0403	.0079
50	.557	.231	.0847	.0412	.0080
75	.579	.240	.0880	.0428	.0084
100	.593	.246	.0901	.0439	.0086

Values of K(r,0.10,p)

r \ p	.50	.75	.90	.95	.99
1	.301	.125	.0457	.0223	.0043
2	.356	.148	.0542	.0264	.0051
3	.391	.162	.0594	.0289	.0056
4	.415	.172	.0631	.0307	.0060
5	.434	.180	.0660	.0321	.0063
6	.448	.186	.0682	.0332	.0065
7	.461	.191	.0701	.0341	.0067
8	.471	.196	.0717	.0349	.0068
9	.480	.199	.0730	.0356	.0069
10	.488	.203	.0742	.0361	.0070
11	.495	.205	.0753	.0366	.0071
12	.501	.208	.0762	.0371	.0072
13	.507	.210	.0770	.0375	.0073
14	.512	.212	.0778	.0379	.0074
15	.516	.214	.0785	.0382	.0075
16	.521	.216	.0792	.0385	.0075
17	.525	.218	.0798	.0388	.0076
18	.529	.220	.0803	.0391	.0076
19	.532	.221	.0808	.0393	.0077
20	.535	.222	.0814	.0396	.0077
25	.549	.228	.0835	.0406	.0079
30	.559	.232	.0850	.0413	.0081
40	.574	.238	.0873	.0425	.0083
50	.585	.243	.0890	.0433	.0084
75	.602	.250	.0916	.0446	.0087
100	.613	.255	.0933	.0454	.0089

Values of K(r,0.20,p)

r \ p	.50	.75	.90	.95	.99
1	.430	.179	.0655	.0319	.0062
2	.463	.192	.0704	.0343	.0067
3	.486	.202	.0739	.0360	.0070
4	.502	.209	.0764	.0372	.0073
5	.516	.214	.0784	.0382	.0074
6	.526	.218	.0800	.0389	.0076
7	.534	.222	.0813	.0396	.0077
8	.542	.225	.0824	.0401	.0078
9	.548	.228	.0834	.0406	.0079
10	.554	.230	.0842	.0410	.0080
11	.559	.232	.0850	.0413	.0081
12	.563	.234	.0856	.0417	.0081
13	.567	.235	.0862	.0420	.0082
14	.570	.237	.0867	.0422	.0082
15	.574	.238	.0873	.0425	.0083
16	.577	.239	.0877	.0427	.0083
17	.579	.241	.0881	.0429	.0084
18	.582	.242	.0885	.0431	.0084
19	.584	.243	.0889	.0432	.0084
20	.586	.243	.0892	.0434	.0085
25	.596	.247	.0906	.0441	.0086
30	.603	.250	.0917	.0446	.0087
40	.613	.255	.0933	.0454	.0089
50	.621	.258	.0944	.0460	.0090
75	.633	.263	.0962	.0468	.0091
100	.640	.266	.0973	.0473	.0092

Values of K(r,0.25,p)

r \ p	.50	.75	.90	.95	.99
1	.500	.207	.0760	.0370	.0072
2	.515	.214	.0783	.0381	.0074
3	.530	.220	.0806	.0392	.0077
4	.543	.225	.0825	.0402	.0078
5	.552	.229	.0840	.0409	.0080
6	.560	.232	.0852	.0415	.0081
7	.567	.235	.0862	.0420	.0082
8	.573	.238	.0871	.0424	.0083
9	.577	.240	.0878	.0427	.0083
10	.582	.241	.0884	.0430	.0084
11	.585	.243	.0891	.0433	.0085
12	.589	.245	.0896	.0436	.0085
13	.592	.246	.0900	.0438	.0085
14	.595	.247	.0904	.0440	.0086
15	.597	.248	.0908	.0442	.0086
16	.600	.249	.0912	.0444	.0087
17	.602	.250	.0916	.0446	.0087
18	.604	.251	.0919	.0447	.0087
19	.606	.251	.0921	.0448	.0087
20	.608	.252	.0924	.0450	.0088
25	.615	.255	.0936	.0456	.0089
30	.621	.258	.0944	.0460	.0090
40	.629	.261	.0957	.0466	.0091
50	.635	.264	.0965	.0470	.0092
75	.645	.268	.0980	.0477	.0093
100	.651	.270	.0990	.0482	.0094

Values of K(r,0.50,p)

r \ p	.50	.75	.90	.95	.99
1	1.000	.415	.152	.0740	.0144
2	.826	.343	.126	.0611	.0119
3	.778	.323	.118	.0576	.0112
4	.755	.313	.115	.0559	.0109
5	.742	.308	.113	.0549	.0107
6	.733	.304	.112	.0543	.0106
7	.728	.302	.111	.0539	.0105
8	.723	.300	.110	.0535	.0104
9	.719	.299	.109	.0532	.0104
10	.717	.297	.109	.0530	.0103
11	.715	.297	.109	.0529	.0103
12	.713	.296	.108	.0527	.0103
13	.711	.295	.108	.0526	.0103
14	.710	.295	.108	.0525	.0102
15	.709	.294	.108	.0524	.0102
16	.708	.294	.108	.0524	.0102
17	.707	.293	.108	.0523	.0102
18	.706	.293	.107	.0523	.0102
19	.706	.293	.107	.0522	.0102
20	.705	.293	.107	.0522	.0102
25	.702	.291	.107	.0520	.0101
30	.701	.291	.107	.0519	.0101
40	.699	.290	..106	.0517	.0101
50	.698	.290	.106	.0517	.0101
75	.696	.289	.106	.0515	.0101
100	.695	.289	.106	.0515	.0101

3.2.2 Fixed Time Truncated Life Test

Frequently it is convenient to terminate a life test at some pre-assigned total life T, whether or not a failure occurs exactly at that time. Items under test may or may not be replaced. Suppose during test time T, r failures are observed. Epstein (Ref. C.2.16) gives the following relationships for the confidence interval of the MTBF.

The two-sided 100(1-α) per cent confidence interval for θ is given by

$$\frac{2T}{\chi^2_{\alpha/2,\,2r+2}} \leqslant \theta \leqslant \frac{2T}{\chi^2_{1-\alpha/2,\,2r}} \tag{3-17}$$

One-sided 100(1-α) per cent confidence interval for θ is

$$\theta > C_L = \frac{2T}{\chi^2_{\alpha,\,2r+2}} \tag{3-18}$$

Alternately, one sided 100(1-α) per cent confidence interval for the quantity $x_p = \theta \log \frac{1}{p}$ is given by

$$x_p > \frac{2T \ln 1/p}{\chi^2_{\alpha,\,2r+2}} \tag{3-19}$$

We define

$$\tau = \frac{2T \ln 1/p}{\chi^2_{\alpha, 2r+2}} \tag{3-20}$$

Then we can assert with 100(1-α) per cent confidence that at least 100p per cent of the items survive for a length of time τ.

If we define

$$\tilde{\theta} = \frac{T}{r+1} \tag{3-21}$$

one-sided 100(1-α) per cent confidence interval for x_p is given by

$$x_p > \frac{2(r+1)\tilde{\theta} \ln 1/p}{\chi^2_{\alpha, 2r+2}} \tag{3-22}$$

We can now define

$$K(r+1, \alpha, p) = \frac{2(r+1) \ln 1/p}{\chi^2_{\alpha, 2r+2}} \tag{3-23}$$

and use tables of Epstein (Ref. C.2.15) obtain the appropriate value of the multiplier K. Then we obtain

$$\tau = K\tilde{\theta} \tag{3-24}$$

To consider a specific example, we assume that we put 20 gearboxes on test and the test was truncated after 900 hours, and 3 failures were obtained in this interval. The test was a replacement type, so that as soon as a gearbox failed, it was replaced by a new one.

Total life T = 20 x 900 = 18000 hours.

and number of failures r = 3.

Using equation (3-18) the 90% one-sided lower confidence limit for MTBF is

$$C_L = \frac{2T}{\chi^2_{\alpha,\,2r+2}} = \frac{2 \times 18000}{\chi^2_{0.10,\,8}}$$

$$= \frac{2 \times 18000}{13.362} = 2697 \text{ hours.}$$

So we can state with 90% confidence that the true value of the parameter θ is higher than 2697 hours.

Suppose we want to find the value of τ for p = 0.50.

From equation (3-21)

$$\tilde{\theta} = \frac{T}{r+1} = \frac{18000}{4} = 4500 \text{ hours.}$$

From Table 3-2,

$$K(r+1,\ \alpha,\ p) = K(4,\ 0.10,\ 0.50)$$

$$= 0.415.$$

$$\tau = K(r+1,\ \alpha,\ p)\tilde{\theta} = 0.415 \times 4500$$

$$= 1868 \text{ hours.}$$

So we can state with 90% confidence that at least 50% of the gearboxes will survive for 1868 hours.

It is interesting to note that in case no failure occurs before the truncation time, point estimates of MTBF cannot be obtained. But interval estimates can still be obtained and values of γ can also be calculated.

Simonds (Ref. C.2.47) has provided tables and graphs to obtain MTBF confidence limits. Table 3.3 has been reproduced from the paper by Simonds. He defines a nominal test MTBF as

$$\theta_T = \frac{T}{r} \tag{3-25}$$

If U and L are upper and lower confidence limits for MTBF, Simonds defines

$$\text{Upper MTBF multiplication factor} = \frac{U}{\theta_T} \tag{3-26}$$

$$\text{Lower MTBF multiplication factor} = \frac{L}{\theta_T} \tag{3-27}$$

He has tabulated the values for these multiplication factors for various confidence levels and numbers of test failures. Graphs have also been plotted to obtain these multiplication factors.

In our example, we had

$$T = 1800 \text{ hours}$$

$$r = 3$$

$$\text{and } \alpha = 0.10$$

TABLE 3.3

MTBF Multiplication Factors for One-Sided Lower Confidence Limit

Number of Test Failures	Confidence Level (%)																						
	1.0	2.5	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	97.5	99.0
1	6.757	4.132	2.809	1.880	1.547	1.214	1.062	0.911	0.819	0.727	0.662	0.596	0.546	0.495	0.452	0.410	0.372	0.333	0.296	0.258	0.211	0.179	0.151
2	4.598	3.236	2.451	1.817	1.560	1.302	1.174	1.045	0.960	0.875	0.812	0.748	0.695	0.641	0.596	0.551	0.508	0.466	0.422	0.377	0.317	0.277	0.238
3	3.641	2.757	2.193	1.720	1.514	1.307	1.196	1.085	1.010	0.934	0.876	0.817	0.769	0.721	0.676	0.630	0.586	0.543	0.496	0.449	0.387	0.342	0.299
4	3.125	2.461	2.030	1.643	1.468	1.294	1.197	1.100	1.032	0.964	0.910	0.857	0.810	0.762	0.720	0.678	0.638	0.597	0.548	0.500	0.437	0.390	0.345
5	2.796	2.271	1.911	1.587	1.434	1.280	1.194	1.107	1.045	0.983	0.932	0.882	0.838	0.794	0.750	0.706	0.664	0.622	0.572	0.521	0.455	0.407	0.359
6	2.574	2.132	1.828	1.542	1.405	1.268	1.188	1.109	1.051	0.993	0.946	0.899	0.858	0.816	0.778	0.739	0.699	0.659	0.615	0.571	0.507	0.458	0.412
7	2.410	2.025	1.757	1.503	1.379	1.255	1.182	1.109	1.055	1.001	0.956	0.912	0.872	0.833	0.797	0.761	0.722	0.684	0.640	0.595	0.534	0.486	0.438
8	2.279	1.945	1.703	1.472	1.358	1.245	1.176	1.108	1.058	1.007	0.965	0.923	0.885	0.847	0.814	0.780	0.742	0.705	0.661	0.617	0.556	0.508	0.461
9	2.179	1.875	1.657	1.447	1.341	1.235	1.171	1.107	1.059	1.011	0.971	0.931	0.894	0.857	0.823	0.789	0.754	0.720	0.677	0.634	0.573	0.526	0.479
10	2.095	1.822	1.620	1.425	1.325	1.225	1.165	1.105	1.059	1.013	0.975	0.937	0.902	0.866	0.836	0.805	0.769	0.733	0.691	0.649	0.590	0.544	0.497
11	2.028	1.773	1.589	1.406	1.312	1.217	1.160	1.103	1.060	1.103	1.060	1.016	0.980	0.943	0.842	0.811	0.778	0.745	0.704	0.664	0.603	0.559	0.512
12	1.968	1.735	1.559	1.388	1.300	1.211	1.156	1.102	1.060	1.018	0.983	0.948	0.914	0.879	0.852	0.824	0.790	0.757	0.716	0.674	0.615	0.573	0.524
13	1.919	1.698	1.535	1.374	1.289	1.204	1.152	1.099	1.059	1.019	0.985	0.951	0.922	0.893	0.861	0.829	0.795	0.761	0.724	0.688	0.627	0.584	0.540
14	1.874	1.667	1.515	1.359	1.278	1.198	1.148	1.098	1.059	1.020	0.987	0.954	0.926	0.897	0.865	0.833	0.802	0.771	0.734	0.697	0.639	0.594	0.549
15	1.838	1.642	1.497	1.348	1.271	1.194	1.146	1.097	1.059	1.021	0.989	0.957	0.929	0.901	0.870	0.840	0.810	0.780	0.742	0.704	0.649	0.605	0.560
16	1.801	1.617	1.478	1.337	1.263	1.189	1.142	1.095	1.058	1.022	0.991	0.960	0.932	0.905	0.876	0.846	0.817	0.788	0.750	0.711	0.659	0.615	0.570
17	1.769	1.594	1.462	1.327	1.256	1.184	1.138	1.093	1.058	1.023	0.992	0.962	0.935	0.908	0.880	0.852	0.824	0.795	0.756	0.718	0.668	0.625	0.580
18	1.742	1.575	1.448	1.318	1.248	1.179	1.135	1.091	1.057	1.024	0.994	0.965	0.938	0.911	0.884	0.858	0.829	0.800	0.762	0.724	0.676	0.633	0.589
19	1.715	1.555	1.433	1.309	1.242	1.174	1.132	1.089	1.056	1.024	0.995	0.966	0.940	0.913	0.888	0.864	0.835	0.805	0.768	0.731	0.683	0.642	0.597
20	1.693	1.540	1.422	1.301	1.236	1.171	1.130	1.088	1.056	1.024	0.996	0.968	0.942	0.916	0.891	0.866	0.838	0.810	0.774	0.737	0.689	0.649	0.605
21	1.670	1.525	1.411	1.294	1.230	1.167	1.127	1.087	1.056	1.024	0.996	0.969	0.944	0.918	0.893	0.868	0.841	0.814	0.778	0.743	0.693	0.656	0.613
22	1.651	1.510	1.401	1.286	1.225	1.164	1.124	1.085	1.055	1.024	0.997	0.971	0.946	0.920	0.896	0.871	0.845	0.819	0.784	0.750	0.700	0.662	0.620
23	1.634	1.497	1.391	1.280	1.220	1.161	1.123	1.085	1.055	1.025	0.998	0.972	0.947	0.921	0.899	0.876	0.850	0.823	0.790	0.756	0.706	0.666	0.626
24	1.606	1.484	1.381	1.273	1.216	1.158	1.121	1.084	1.054	1.025	0.999	0.973	0.948	0.923	0.902	0.881	0.854	0.828	0.795	0.762	0.711	0.671	0.632
25	1.602	1.473	1.373	1.268	1.212	1.155	1.118	1.082	1.053	1.024	0.999	0.974	0.950	0.925	0.904	0.882	0.856	0.829	0.798	0.766	0.717	0.676	0.636
30	1.537	1.425	1.337	1.245	1.194	1.143	1.110	1.077	1.050	1.024	1.001	0.978	0.955	0.932	0.910	0.889	0.861	0.843	0.813	0.783	0.737	0.699	0.662
35	1.489	1.388	1.310	1.227	1.180	1.133	1.103	1.073	1.048	1.024	1.003	0.982	0.963	0.944	0.922	0.900	0.877	0.854	0.826	0.798	0.756	0.718	0.709
40	1.451	1.360	1.288	1.212	1.169	1.126	1.098	1.070	1.047	1.024	1.004	0.983	0.965	0.947	0.926	0.905	0.888	0.870	0.839	0.808	0.769	0.736	0.699
45	1.420	1.338	1.270	1.199	1.159	1.119	1.094	1.066	1.045	1.024	1.004	0.985	0.968	0.950	0.932	0.914	0.894	0.873	0.846	0.818	0.779	0.748	0.711
50	1.394	1.318	1.256	1.188	1.151	1.114	1.089	1.064	1.044	1.023	1.005	0.987	0.970	0.952	0.935	0.917	0.896	0.876	0.854	0.832	0.792	0.756	0.723
60	1.355	1.286	1.230	1.171	1.138	1.104	1.081	1.058	1.040	1.022	1.006	0.989	0.972	0.955	0.942	0.928	0.908	0.887	0.864	0.841	0.808	0.776	0.746
70	1.324	1.262	1.212	1.159	1.128	1.097	1.076	1.055	1.038	1.021	1.006	0.991	0.974	0.958	0.944	0.930	0.914	0.897	0.874	0.851	0.822	0.789	0.760
80	1.301	1.244	1.196	1.147	1.119	1.091	1.071	1.051	1.036	1.020	1.006	0.992	0.980	0.968	0.950	0.933	0.920	0.906	0.883	0.860	0.831	0.804	0.773
90	1.281	1.229	1.186	1.139	1.112	1.086	1.068	1.050	1.034	1.019	1.006	0.993	0.981	0.970	0.956	0.942	0.925	0.908	0.888	0.868	0.839	0.811	0.786
100	1.265	1.206	1.176	1.132	1.107	1.082	1.064	1.047	1.032	1.018	1.006	0.993	0.982	0.971	0.957	0.943	0.930	0.917	0.897	0.877	0.847	0.819	0.793
150	1.211	1.174	1.141	1.107	1.087	1.067	1.053	1.039	1.028	1.017	1.006	0.995	0.984	0.974	0.964	0.955	0.942	0.929	0.912	0.895	0.872	0.850	0.828
200	1.180	1.149	1.121	1.092	1.075	1.058	1.046	1.034	1.024	1.014	1.006	0.997	0.986	0.976	0.966	0.957	0.948	0.939	0.926	0.913	0.889	0.873	0.851
250	1.160	1.132	1.109	1.083	1.068	1.053	1.042	1.031	1.022	1.013	1.005	0.997	0.992	0.986	0.976	0.967	0.958	0.949	0.936	0.922	0.897	0.881	0.866
300	1.147	1.122	1.101	1.081	1.065	1.049	1.039	1.029	1.021	1.013	1.005	0.998	0.992	0.987	0.978	0.968	0.960	0.953	0.940	0.926	0.904	0.892	0.876
400	1.125	1.104	1.085	1.066	1.054	1.042	1.034	1.025	1.018	1.011	1.005	0.999	0.993	0.988	0.979	0.970	0.964	0.959	0.946	0.934	0.919	0.908	0.892
500	1.110	1.092	1.075	1.062	1.050	1.037	1.030	1.023	1.016	1.010	1.005	0.999	0.994	0.988	0.983	0.978	0.969	0.960	0.951	0.942	0.933	0.916	0.899
750	1.104	1.087	1.072	1.059	1.048	1.036	1.029	1.022	1.016	1.010	1.005	1.000	0.995	0.990	0.985	0.980	0.971	0.962	0.954	0.946	0.937	0.921	0.905
1000	1.098	1.082	1.068	1.056	1.045	1.034	1.028	1.021	1.016	1.010	1.005	1.000	0.996	0.991	0.986	0.981	0.972	0.964	0.957	0.950	0.941	0.925	0.910

Using equation (3-12), $\theta_T = \frac{T}{r} = \frac{18000}{3} = 6000$ hours.

From Table 3.3, we find, for a one-sided confidence limit, multiplication factor = 0.449.

So the lower confidence limit for MTBF (one-sided) = $0.449 \times \theta_T$

$$= 0.449 \times 6000$$

$$= 2694 \text{ hours.}$$

It agrees closely with the value that we obtained earlier. The same result can be obtained by using the graphs. One disadvantage of this method is that the tables or the graphs cannot be used if no failures are obtained before test truncation time.

We can make some comments regarding the sample size. The larger the sample size, the larger is the total test time T for a specified testing interval. If T is larger, we expect to get more failures and then the confidence limits are closer. Hence by obtaining more test failures, our estimate of MTBF is more precise. This can be appreciated from Fig. 3-1.

Another method of MTBF and reliability estimation has been suggested by Epstein (Ref. C.2.18). Here n items are placed under test and straight test duration time t* is known. At the end of the test time, the number of failed items are counted and failed items are not replaced. Then, it can be stated nonparametrically with 100(1-α) per cent confidence that at least 100b per cent of the population survives for a length of time t*, the value of b being given by

$$b = \frac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\alpha,\,2r+2,\,2n-2r}} \tag{3-28}$$

Values of F_{α, f_1, f_2} can be obtained from the tables of F distribution.

This method being nonparametric is independent of the underlying distribution. Equation (3-28) can also be interpreted as the lowest estimate of Reliability for t* hours with 100(1-α) per cent confidence is b. In the particular case where the underlying distribution is exponential, one-sided 100(1-α) per cent confidence interval for MTBF is given by

$$\theta \geqslant \frac{t^*}{\ln\left[1 + \left(\frac{r+1}{n-r}\right) F_{\alpha, 2r+2, 2n-2r}\right]} \tag{3-29}$$

If no failure occurs during the test time t*, equation (3-29) becomes

$$\theta \geqslant \frac{t^*}{\ln\left[1 + \frac{1}{n} F_{\alpha, 2, 2n}\right]} \tag{3-30}$$

and equation (3-28) can be written as

$$R(t^*) \geqslant \frac{1}{1 + \frac{1}{n} F_{\alpha, 2, 2n}} \tag{3-31}$$

We will use this method in our example. Suppose the test was run for 900 hours with 20 components and 3 failures were obtained. Failed components were not replaced.

So $\quad t^* = 900$ hours

$\quad\quad n = 20$

$\quad\quad r = 3$

If we choose $\alpha = 0.10$, $F_{0.10,\ 8,\ 34} = 1.86$

$$\theta \geqslant \frac{900}{\ln\left[1 + \left(\frac{3+1}{20-3}\right) F_{0.10, 8, 34}\right]}$$

$$\geqslant \frac{900}{\ln\left[1 + \frac{4 \times 1.86}{17}\right]}$$

$$\geqslant 2472 \text{ hours}$$

Similarly

$$R(t^*) \geqslant \frac{1}{1 + 0.438}$$

$$\geqslant 0.696$$

3.2.3 Acceptance Sampling Plans for Reliability Testing

In acceptance sampling, the supplier of components supplies a lot which is required to have a certain minimum MTBF or reliability. Samples are drawn at random from the lot, tests conducted and depending on the test results, a decision is made whether to accept or reject the lot. Here, a quantity C, called acceptance number is defined. If in a test with a sample of n components for a test duration t, C or fewer units fail, the lot from which the sample was drawn is accepted and, if more than C units fail, the lot is rejected; this decision being made at a specified level of confidence. If the population has MTBF of θ_o, then probability of failure for time t is

$$F_o(t) = 1 - e^{-t/\theta_o}$$

With n units on test, the probability of C or fewer items failing is obtained by summing up the terms of the Binomial expansion.

If the desired confidence level is $100(1-\beta)$, we can write

$$\sum_{i=0}^{C} \frac{n!}{i!\,(n-i)!} \left[F_o(t)\right]^{i} \left[1 - F_o(t)\right]^{n-i} \leqslant \beta \qquad (3\text{-}32)$$

From the table of Cumulative Binomial Distribution (Ref. F.3.8), the smallest integral values of n satisfying this inequality can be obtained. The results were tabulated by Sobel and Tischendorf (Ref. C.2.48) and can be conveniently used in planning sampling plans (Table 3.4).

We will demonstrate the use of the table by solving a typical example in which the minimum acceptable specified value of mean life is 4500 hours. We wish to run the test for 900 hours. For a confidence level of 90%, we wish to fix the value of C and n.

$$t = 900 \text{ hours}$$

$$\theta_o = 4500 \text{ hours}$$

$$\frac{t}{\theta_o} = \frac{900}{4500} = 0.2$$

TABLE 3.4

Minimum Size of Sample to be Tested for a Time t to Assure a Minimum Mean Life of θ_0 with Confidence P* when C is the Acceptance Number

$P^* = 100(1 - \beta) = 90$ per cent

c	t/θ_0												
	1.0	0.5	0.2	0.1	0.05	0.02	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
0	3	5	12	24	47	116	231	461	1,152	2,303	4,606	11,513	23,026
1	5	9	20	40	79	195	390	778	1,946	3,891	7,780	19,450	38,898
2	7	12	28	55	109	266	533	1,065	2,662	5,323	10,645	26,612	53,223
3	9	15	35	69	137	333	668	1,337	3,341	6,681	13,362	33,404	66,808
4	11	19	42	83	164	398	798	1,599	3,997	7,994	15,988	39,968	79,936
5	13	22	49	97	190	462	927	1,855	4,638	9,275	18,549	46,374	92,747
6	15	25	56	110	217	528	1,054	2,107	5,267	10,533	21,064	52,661	105,322
7	16	28	63	123	243	589	1,178	2,355	5,886	11,771	23,542	58,855	117,710
8	18	31	70	136	269	648	1,300	2,599	6,498	12,995	25,990	64,974	129,948
9	20	34	76	149	294	709	1,421	2,842	7,103	14,206	28,412	71,030	142,060
10	22	37	83	161	319	770	1,541	3,082	7,704	15,407	30,814	77,034	154,068
11	23	40	89	174	344	830	1,660	3,320	8,300	16,598	33,197	82,991	165,982
12	25	42	95	187	369	888	1,779	3,557	8,891	17,782	35,564	88,908	177,816
13	27	45	102	199	393	947	1,896	3,792	9,479	18,958	37,916	94,790	189,580
14	29	48	108	212	417	1,007	2,013	4,026	10,064	20,128	40,256	100,640	201,280

$P^* = 100(1 - \beta) = 95$ per cent

c	t/θ_0												
	1.0	0.5	0.2	0.1	0.05	0.02	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
0	3	6	15	31	60	149	298	600	1,498	2,996	5,992	14,979	29,958
1	6	11	25	49	97	236	473	949	2,372	4,744	9,488	23,720	47,439
2	8	14	33	65	129	314	630	1,260	3,148	6,296	12,592	31,479	62,958
3	10	18	41	80	159	386	775	1,551	3,877	7,754	15,508	38,769	77,537
4	12	21	48	95	189	456	914	1,831	4,577	9,154	18,307	45,768	91,535
5	14	24	56	109	217	527	1,052	2,103	5,257	10,514	21,027	52,566	105,131
6	16	27	63	123	245	597	1,183	2,369	5,922	11,843	23,685	59,212	118,424
7	18	31	70	137	274	656	1,315	2,630	6,575	13,149	26,297	65,741	131,481
8	19	34	77	150	299	722	1,444	2,887	7,218	14,435	28,870	72,174	144,347
9	21	37	84	164	327	785	1,571	3,142	7,853	15,706	31,411	78,526	157,052
10	23	40	91	178	353	846	1,697	3,393	8,482	16,963	33,925	84,811	169,622
11	25	43	98	190	379	908	1,821	3,642	9,104	18,208	36,416	91,038	182,076
12	27	46	105	205	405	971	1,945	3,889	9,722	19,443	38,886	97,213	194,426
13	29	49	111	218	431	1,034	2,067	4,134	10,335	20,669	41,338	103,343	206,686
14	30	52	117	231	457	1,095	2,189	4,378	10,944	21,887	43,773	109,433	218,865

From Table 3.4, we can choose the following alternatives -

C	n
0	12
1	20
2	28
3	35
4	42
5	49

So we see that the tables provide a useful method of devising a suitable sampling plan. The tables, strictly speaking, are valid for infinite lot sizes only. But the result obtained from the tables, when the lot size is finite, is always conservative and the error is towards the safer side. If the sample size is less than 15% of the lot size, the error is negligible.

The usefulness of these tables is immense. Nevertheless, the sampling plans set up using these tables have some disadvantages (Ref. B.1.30). A large sample size is required. The alternative to a large sample size is for the true reliability to be much better than the requirement.

Operating characteristic curves, which are very common in quality control work are also used in reliability testing. An operating characteristic curve is a plot of the probability that a given quality level will be accepted by a sampling plan versus the value of the quality level. The problem has been analysed by Epstein and Sobel (Ref. C.2.13), Peterson (Ref. C.2.41), Goldsmith (Ref. C.2.20) and Altman (Ref. C.1.1). The proportion of units tested that can be expected to fail at any time T is given by

$$p = F(T) = 1 - e^{-T/\theta_o} \tag{3-33}$$

So a sample of size n is taken, tested for T hours and number of failing units are determined on an attribute basis. To obtain the O.C. curve, we compute the probabilities of acceptance as a function of failure rate of the lot and acceptance number C. We shall illustrate the computations using an example.

Let n = 50 and T = 1000 hours. If the failure rate λ for the lot is 0.00001 per hour or 1% per 1000 hours, then

$$\theta_o = \frac{1}{\lambda} = 100000 \text{ hours}$$

$$\frac{T}{\theta_o} = \frac{1000}{100000} = 0.01$$

and

$$e^{-T/\theta_o} = 0.99.$$

Using equation (3-33), the proportion of items expected to fail by 1000 hours = 1 - 0.99,

$$\text{or } p = 0.01$$

So the expected number of failures per sample is

$$pn = 0.01 \times 50 = 0.50$$

The cumulative form of Poisson distribution is given by

$$P_{cum} = \sum_{C=0}^{c} \frac{e^{-(pn)}(pn)^c}{C!} \tag{3-34}$$

This gives us the probability of having C or less failures, and hence this also gives us the probability P_a that the sample will accept the lot.

To determine P_a, we may use the tables by Burr (Ref. F.1.9) or the charts of Dodge and Romig (Ref. F.3.3) or those given by Bowman and Fetter (Ref. F.1.5).

The calculations for the O.C. curve are shown below.

n = 50, C = 1, T = 1000 hours.

Lot failure rate λ (per cent per 1000 hours)	$\theta = 1/\lambda$ (in hours)	T/θ_o	$p=1-e^{-T/\theta_o}$	pn	P_a
0		0.00	0.000	0.00	1.00
1	100,000	0.01	0.010	0.50	0.60
2	50,000	0.02	0.020	1.00	0.37
3	33,333	0.03	0.030	1.50	0.22
4	25,000	0.04	0.039	1.95	0.14
5	20,000	0.05	0.049	2.45	0.09
6	16,667	0.06	0.058	2.90	0.05

The resulting O.C. curve is shown in Fig. 3-1.

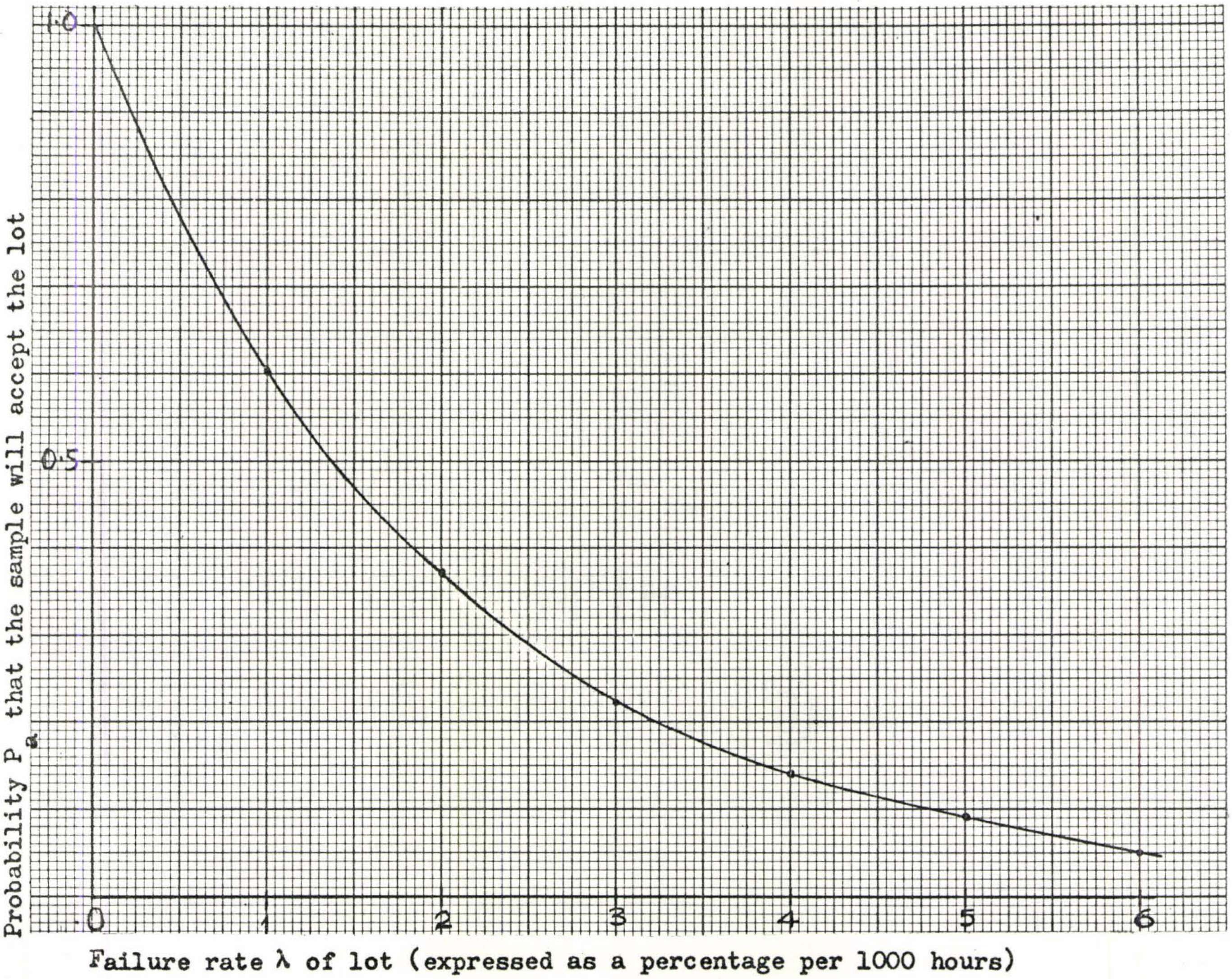
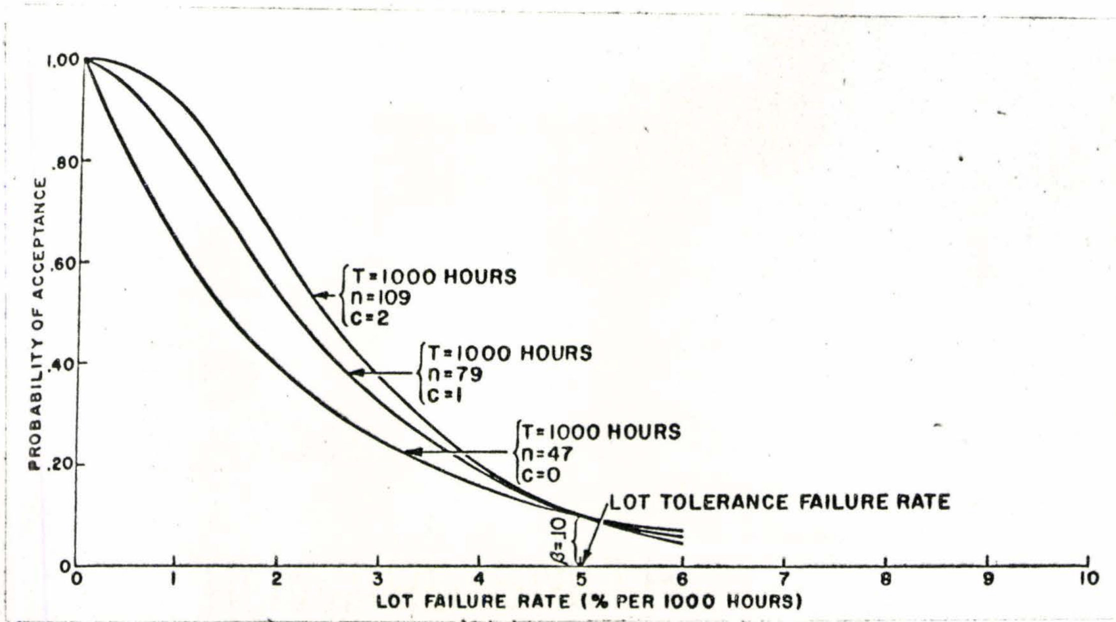
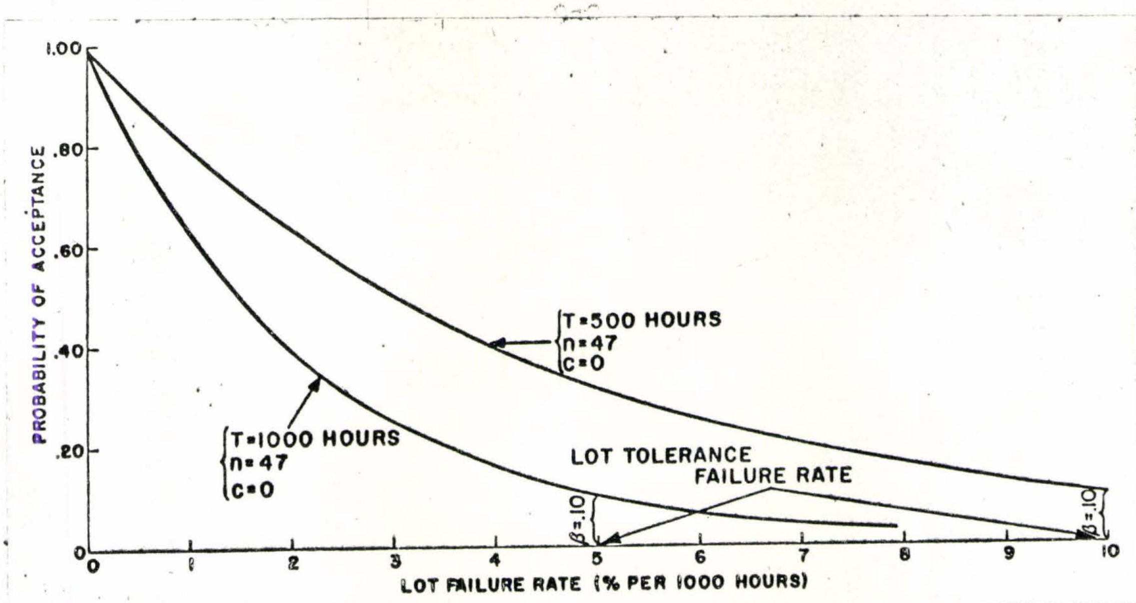


Fig. 3-1   O.C. Curve ($n = 50$, $C = 1$, $T = 1000$ hours)

Hence, this sampling plan will accept lots with $\lambda = 0.00001$ per hour 60% of the time, and will accept lots with $\lambda = 0.00006$ per hour 5% of the time. Reducing the acceptance number C makes the curve more discriminating, if test time T and sample size n are kept constant. Similarly, if C and n are held constant, a longer test time makes the curve more discriminating. A larger sample also yields a steeper curve, keeping C and T constant. These characteristics are shown graphically in Fig. 3-2 (Ref. C.1.1).



(A)



(B)

Fig. 3-2 O.C. Curves for Different Life Test Sampling Plans

(A) Effect of Acceptance Number

(B) Effect of Test Time

A sampling plan may be designed to suit some specified requirements. This can be demonstrated by the following example. Let

R_1 = Acceptable Reliability Level

= reliability level of submitted lots at which it is desired to set P_a equal to $(1-\alpha)$

= a failure rate of 0.00002 per hour.

R_2 = Lot Tolerance Failure Rate

= reliability level of the submitted lots at which it is desired to set P_a equal to β

= a failure rate of 0.00005 per hour.

α = producer's risk or probability with which lots of reliability R_1 will be rejected by the plan.

= 0.10

β = consumer's risk or probability with which lots of reliability R_2 will be accepted by the plan.

= 0.10

We now need to know three quantities, test time T, acceptance number C and sample size n. We are to decide on the value of one of these and the other two can then be determined. Suppose we assume that we will run the test for 500 hours only. Thus T = 500 hours. From the Poisson Chart, we obtain various values of p_1n and p_2n corresponding to the respective probability of acceptance.

C	Values of $p_1 n$ for $P_a = 0.90$	Values of $p_2 n$ for $P_a = 0.10$	Ratio $\frac{p_2}{p_1}$
0	0.107	2.3	21.45
1	0.538	3.9	7.23
2	1.09	5.3	4.86
3	1.74	6.7	3.85
4	2.41	8.0	3.32
5	3.15	9.2	2.92

Here we have ARL = R_1 = 0.00002 failures/hour.

$$\theta_0^{(1)} = \frac{1}{0.00002} = 50000 \text{ hours}$$

$$\frac{T}{\theta_0^{(1)}} = \frac{500}{50000} = 0.01$$

$$p_1^* = 1 - e^{-T/\theta_0^{(1)}}$$

$$= 1 - e^{-0.01}$$

$$= 1 - 0.99$$

$$= 0.01$$

Again RTFR = R_2 = 0.00005 failures/hour.

$$\theta_o^{(2)} = \frac{1}{0.00005} = 2000 \text{ hours}$$

$$\frac{T}{\theta_o^{(2)}} = \frac{500}{2000} = 0.25$$

$$p_2^* = 1 - e^{-T/\theta_o^{(2)}} = 1 - e^{-0.25} = 0.221$$

So the required ratio is $\frac{p_2^*}{p_1^*} = \frac{0.221}{0.010} = 22.1$

Comparing with the ratios computed earlier, we see that a plan with C = 0 is most suitable.

Then $p_1 n = 0.107$

Substituting $p_1 = 0.01$, $n = \frac{0.107}{0.010} = 10.7 \approx 11$

So the required sampling plan is as follows -

Test duration T = 500 hours

Sample size n = 11

Acceptance number C = 0

If no failure occurs during the test, we accept the lot, otherwise we reject it, the risks in making this decision being α and β. Alternative plans can also be designed by taking a different value of T.

Bonis (Ref. B.1.9) suggested drawing O.C. curves with a non-dimensional quantity called Normalized Time as the horizontal axis. Normalized Time is the ratio of total testing time in hours and the true MTBF in hours.

$$\text{Normalized time } d = \frac{\text{total test time } T}{\text{true MTBF}}$$

Hence, this is the expected number of failures for true MTBF. The chart is shown in Fig. 3-3. In the chart, a represents acceptance number.

Suppose for some components, true MTBF = 500 hours.

Total test duration T = 4500 hours (this takes sample size into account)

Then

$$d = \frac{4500}{500} = 9.$$

If acceptance number is 15, from Fig. 3-3 we get P_S = 98%. Or in other words, probability of observing an MTBF of $\frac{4500}{15}$ = 300 hours is at least 0.98, when true MTBF is 500 hours.

3.2.4 Sequential Testing in Reliability

Sequential testing plans allow the number of items inspected or the total testing time to be determined by the cumulative results of the inspection process. Pioneering work on sequential testing for quality control work was done by Wald (Ref. F.1.66). This method has been adopted for reliability work by Epstein and Sobel (Ref. C.2.14), Lieberman (Ref. C.2.37), Brewington and Tiger (Ref. B.1.14), Aroian (Ref. C.2.3), Eagle (Ref. C.2.8) and others.

The major advantage of sequential testing plans is that on the average they require less testing than other plans. This is particularly true for very poor or very good material. As in acceptance sampling plans, interval estimates of reliability or MTBF are of importance and with a specified margin of error (or risk), it is ascertained whether the

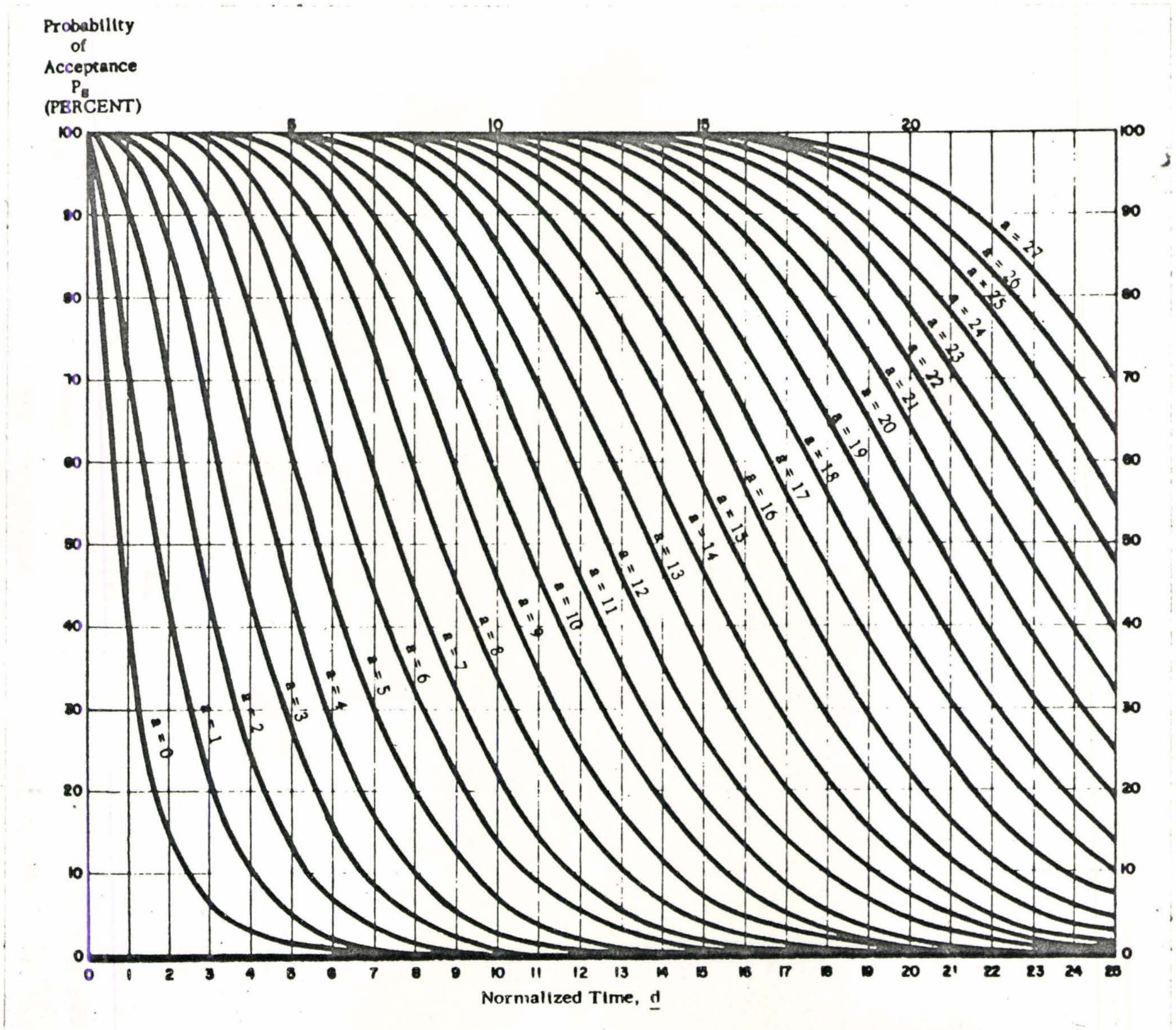


Fig. 3-3 Reliability Operating Characteristic Curves

reliability of the items to be tested is at least as good as specified. If the material is accepted by sequential testing, we still do not know by how much the equipment is better.

We define the following terms -

t_o = acceptable value of MTBF

R_U = acceptable reliability

t_1 = unacceptable value of MTBF

R_L = unacceptable level of reliability

α = producer's risk - probability of rejecting a lot with MTBF t_o or better (Reliability R_U or better)

β = consumer's risk - probability of accepting a lot with MTBF t_1 or worse (Reliability R_L or worse)

Roberts (Ref. A.1.65) suggests a testing procedure, where the data is obtained as cumulative successes or cumulative failures. An item performing the specified operation is counted as a success, otherwise it is counted as a failure.

Let F = cumulative number of failures

S = cumulative number of successes

From simple probabilistic concepts, the equations for accept and reject lines can be derived as follows -

$$F \ln \frac{1 - R_L}{1 - R_U} + S \ln \frac{R_L}{R_U} = \ln A \tag{3-35}$$

and

$$F \ln \frac{1 - R_L}{1 - R_U} + S \ln \frac{R_L}{R_U} = \ln B, \tag{3-36}$$

where $$A = \frac{1 - \beta}{\alpha} \tag{3-37}$$

and $$B = \frac{\beta}{1 - \alpha} \tag{3-38}$$

The above lines are graphically represented in Fig. 3-4. The region between the two lines is the area of no decision. As the test proceeds, the results are plotted on the graph and testing is continued till the plotted line meets either the accept or reject line. So it is clear that number of samples to be tested depends on the cumulative result of the previous tests and cannot be decided in advance. But the average or expected sample size can be computed using the following relationship and is useful for planning purposes

$$\text{ASN} = \frac{(1 - \alpha)\ \ln B + \alpha\ \ln A}{(1-R_U)\ln\left[\frac{(1-R_L)}{(1-R_U)}\right] + R_U\ \ln\left(\frac{R_L}{R_U}\right)} \tag{3-39}$$

where ASN = average sampling number.

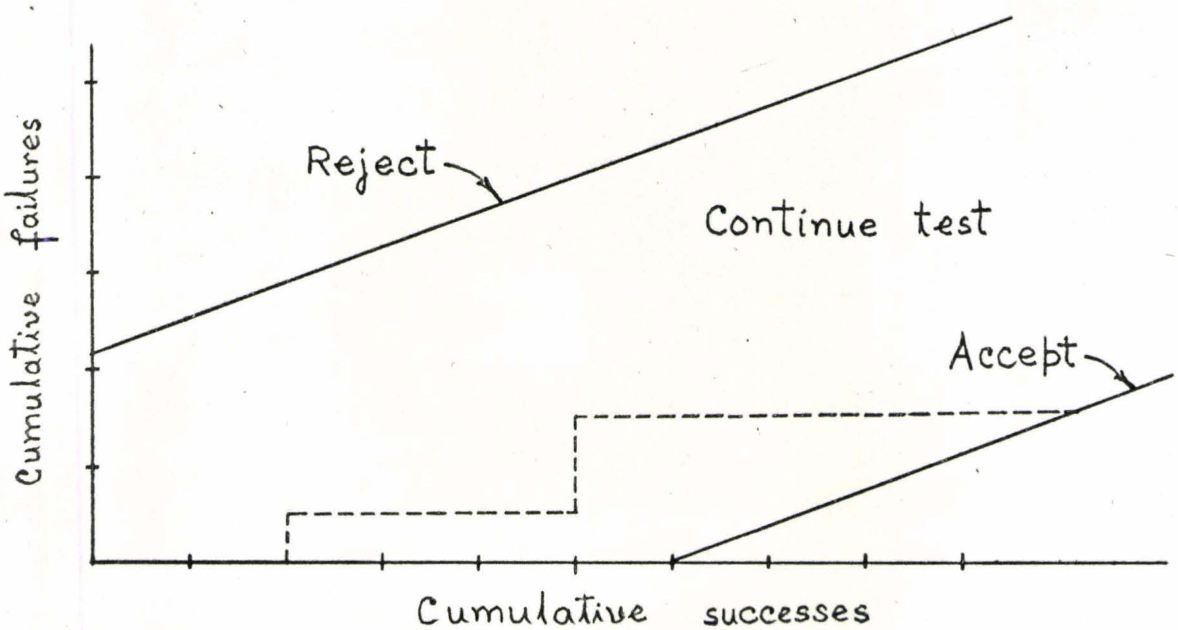


Fig. 3-4 Graphical Representation of Sequential test When Data is Available as Cumulative failure or Cumulative success

This method is quite simple to use. However, it is based on attributes, or in other words, for the specified mission time we test the components and the results are reported as failures or successes. It is felt that this limits the generality of the method. The approach of Epstein and Sobel (Ref. C.2.14) appears to be more powerful.

We consider n items are drawn at random (when underlying p.d.f. is exponential) and placed on life test. We wish to test the hypothesis $H_o : t = t_o$ against hypothesis $H_1 : t = t_1$ with associated type I error of α and type II error of β as before.

Let

$V(T)$ = Total accumulated operating time

T = Straight test time

Then in the replacement case,

$$V(T) = nT \tag{3-40}$$

and in the non-replacement case,

$$V(T) = \sum_{i=1}^{r} x_i + (n-r)(T-x_r) \tag{3-41}$$

where r = number of failures observed during the test

and x_i = time of i^{th} failure.

If t is the true value of MTBF, then from the Poisson distribution, the probability of getting r failures in test duration T is

$$P(r) = \left[\frac{V(T)}{t}\right]^r \frac{e^{-V(T)/t}}{r!} \tag{3-42}$$

If the MTBF of the components is exactly equal to t_1,

$$P_1(r) = \left[\frac{V(t)}{t_1}\right]^r \frac{e^{-V(T)/t_1}}{r!} \tag{3-43}$$

and if MTBF is exactly equal to t_o,

$$P_2(r) = \left[\frac{V(t)}{t_o}\right]^r \frac{e^{-V(T)/t_o}}{r!} \tag{3-44}$$

We form a probability ratio p(r), so that

$$p(r) = \frac{P_1(r)}{P_o(r)} = \left(\frac{t_o}{t_1}\right)^r \exp\left\{-\left[\frac{1}{t_1}-\frac{1}{t_o}\right]V(t)\right\} \tag{3-45}$$

So our decision criterion will be

$$B < p(r) < A \tag{3-46}$$

If at any stage of the test, $p(r) \leqslant B$, we make an accept decision, and if $p(r) \geqslant A$, we reject the lot. For intermediate values of p(r), the test is continued.

Substituting for p(r) from equation (3-45) in equation (3-46), we get

$$B < \left(\frac{t_o}{t_1}\right)^r \exp\left\{-\left[\frac{1}{t_1}-\frac{1}{t_o}\right]V(t)\right\} < A \tag{3-47}$$

Taking logarithms and re-arranging, we get

$$\left[\frac{-\ln A}{\frac{1}{t_1}-\frac{1}{t_0}}+\frac{r\ln\frac{t_0}{t_1}}{\frac{1}{t_1}-\frac{1}{t_0}}\right] < V(T) < \left[\frac{-\ln B}{\frac{1}{t_1}-\frac{1}{t_0}}+\frac{r\ln\frac{t_0}{t_1}}{\frac{1}{t_1}-\frac{1}{t_0}}\right] \quad (3\text{-}48)$$

To plot the data continuously in time, we can write the above equation as

$$-h_1 + r\,S < V(T) < h_0 + r\,S \quad (3\text{-}49)$$

Equation of the accept line is

$$V(T) = h_0 + r\,S \quad (3\text{-}50)$$

and equation of the reject line is

$$V(T) = -h_1 + r\,S \quad (3\text{-}51)$$

h_0, h_1 and S are positive constants given by

$$h_0 = \frac{-\ln B}{\frac{1}{t_1}-\frac{1}{t_0}} \quad (3\text{-}52)$$

$$h_1 = \frac{\ln A}{\frac{1}{t_1}-\frac{1}{t_0}} \quad (3\text{-}53)$$

and $$S = \frac{\ln(t_0/t_1)}{\frac{1}{t_1}-\frac{1}{t_0}} \quad (3\text{-}54)$$

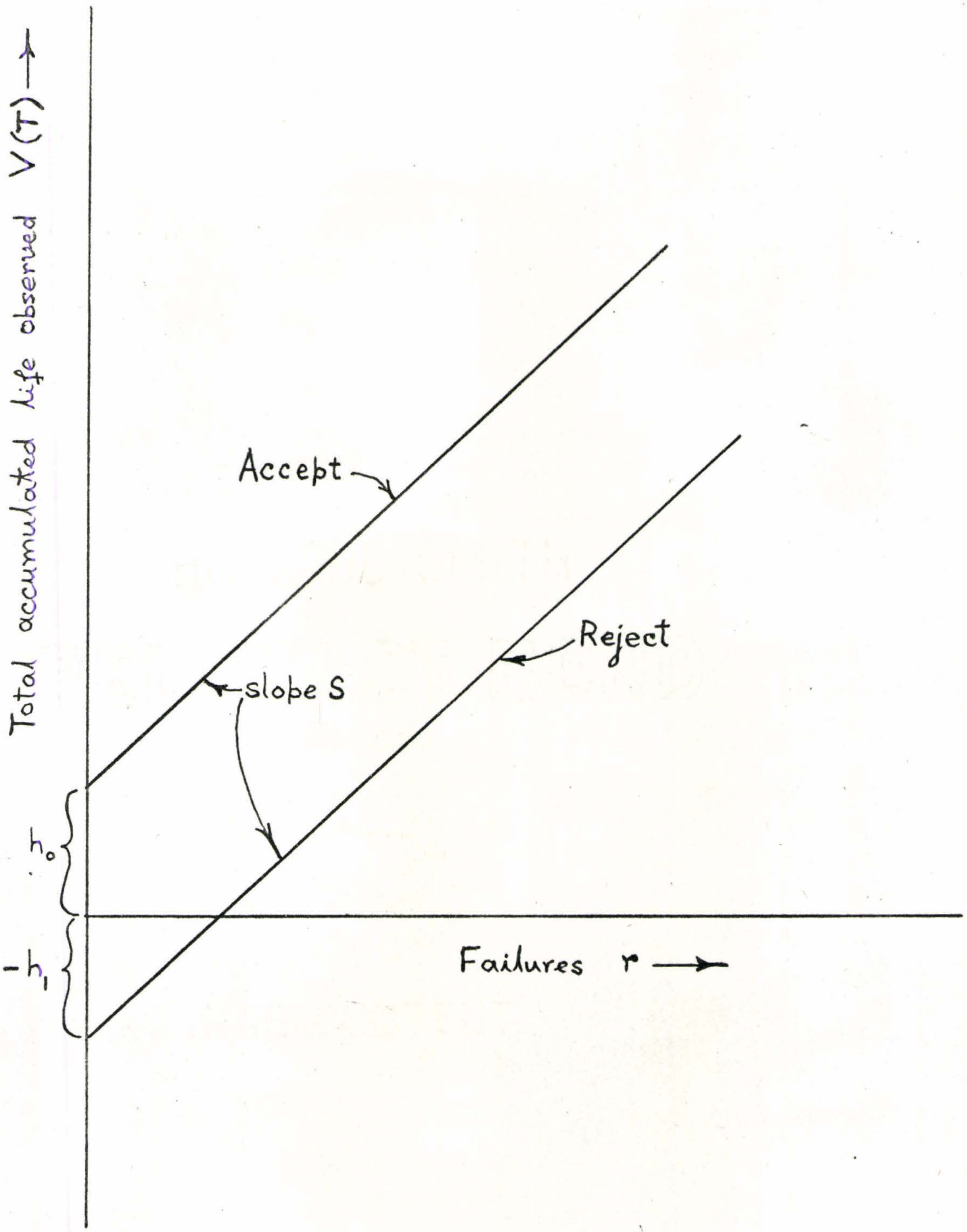


Fig. 3-5 shown above is a graphical representation of this method.

h_o is the intercept of the accept line on time axis, $-h_1$ is the intercept of the reject line on time axis, and S is the slope of both the lines.

If L(t) = probability of accepting H_o when t is true MTBF, Wald (Ref. F.1.66, pp. 48-50) gives the following relationships for determining L(t) for any value of t by assigning different values to h.

$$L(t) = \frac{A^h - 1}{A^h - B^h} \tag{3-55}$$

and

$$t = \frac{\left(\frac{t_o}{t_1}\right)^h - 1}{h\left(\frac{1}{t_1} - \frac{1}{t_o}\right)} \tag{3-56}$$

This enables us to obtain an O.C. curve. Some points $[L(t), t]$ on the O.C. curve are $[0, 0]$, $[B, t_1]$, $\left[\frac{\ln A}{\ln A - \ln B}, S\right]$, $[(1-\alpha), t_o]$ and $[1, \infty]$.

Epstein and Sobel (Ref. C.2.14) give some formulas to compute ASN approximately. This information is very useful in planning the tests in advance.

Denoting ASN by $E_t(r)$,

$$E_t(r) \approx \frac{h_1 - L(t)(h_o + h_1)}{S-t} \qquad \text{for } t \neq S \tag{3-57}$$

$$\approx \frac{h_o\, h_1}{S^2} \qquad \text{for } t = S \tag{3-58}$$

Lieberman (Ref. C.2.37) has provided a number of tables and charts for sequential testing without using graphical methods. These tables are based on the above formulas.

We will now demonstrate the use of this method by solving a typical example. A manufacturer of household appliances obtains certain gearboxes in large lots from various suppliers. The desired value of MTBF is 4000 hours, and components with MTBF lower than 2000 hours are not acceptable.

Hence $t_o = 4000$ hours

and $t_1 = 2000$ hours

It was agreed to have $\alpha = \beta = 0.10$.

from equation (3-37),

$$A = \frac{1-\beta}{\alpha} = \frac{1 - 0.1}{0.1} = \frac{0.9}{0.1} = 9$$

and equation (3-38) gives

$$B = \frac{\beta}{1-\alpha} = \frac{0.1}{1-0.1} = \frac{0.1}{0.9} = 0.111$$

from equation (3-52)

$$h_o = \frac{-\ln B}{\frac{1}{t_1} - \frac{1}{t_o}} = \frac{-\ln 0.111}{\frac{1}{2000} - \frac{1}{4000}} = 8792 \text{ hours.}$$

From equation (3-53),

$$h_1 = \frac{\ln A}{\frac{1}{t_1} - \frac{1}{t_o}} = \frac{\ln 9}{\frac{1}{2000} - \frac{1}{4000}} = 8788 \text{ hours.}$$

from equation (3-54),

$$S = \frac{\ln \left({}^{t_o}/_{t_1}\right)}{\frac{1}{t_1} - \frac{1}{t_o}} = \frac{\ln 2}{\frac{1}{2000} - \frac{1}{4000}} = 2772 \text{ hours.}$$

Equation of accept line is

$$V(T) = 8792 + 2772\, r$$

and that of reject line is

$$V(T) = -8788 + 2772r$$

These lines are shown in Fig. 3-6. Using equations (3-55) and (3-56), an O.C. curve can now be drawn. The following points on the O.C. curve were obtained -

Probability of acceptance L(t)	Average MTBF t(hours)
0	0
0.10	2000
0.50	2772
0.90	4000
1	∞

O.C. curve has been plotted in Fig. 3-7.

Now we can investigate the variation of ASN with Average MTBF of lot. We use equations (3-57) and (3-58) and obtain following values

Average MTBF t (hours)	ASN
0	3.2
2000	9.1
2772	10.1
4000	5.7
∞	0

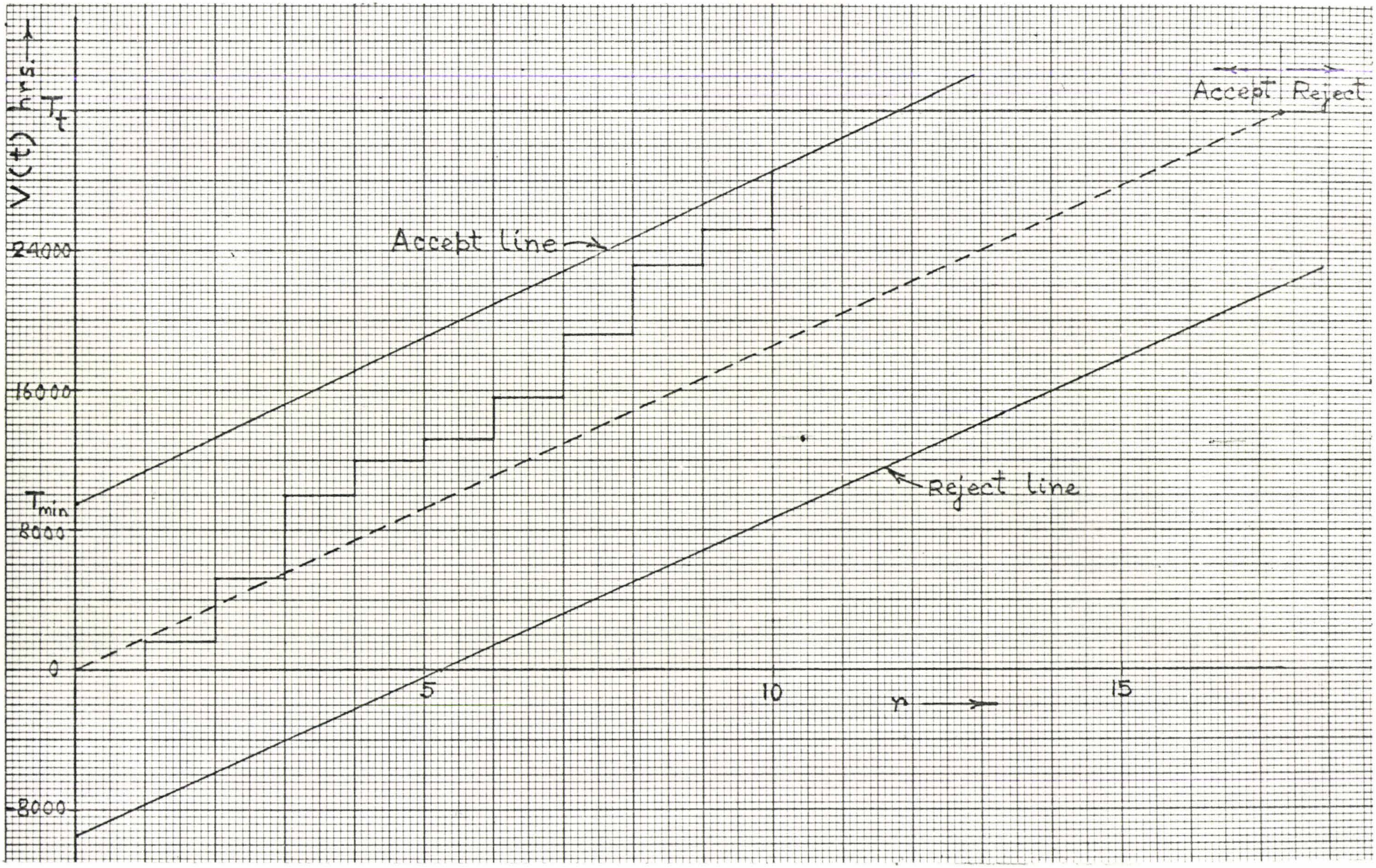


Fig. 3-6 Graphical Representation of Sequential Reliability Testing

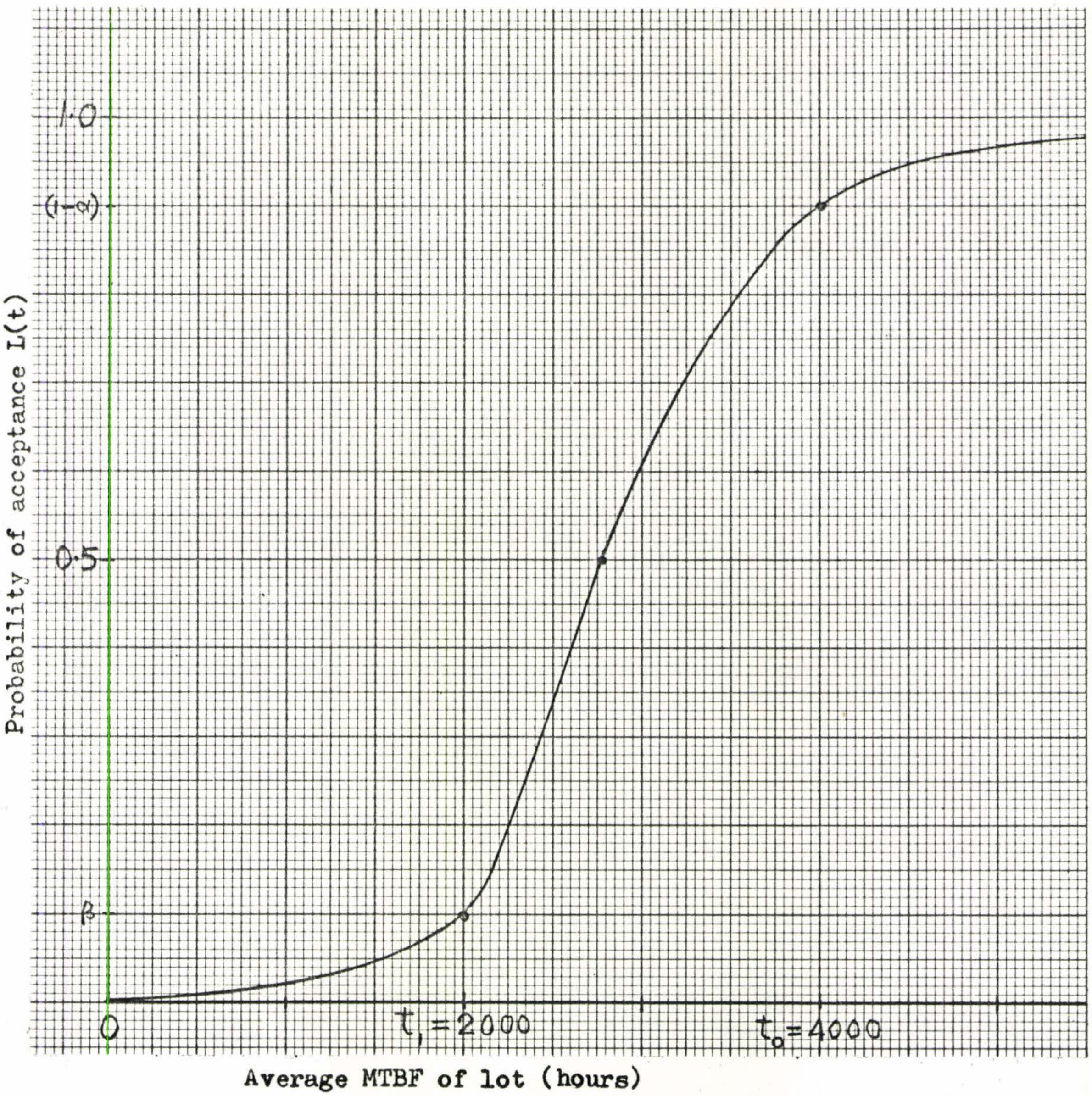


Fig. 3-7 O.C. Curve

A curve of ASN vrs. MTBF (true value of population) has been plotted in Fig. 3-8. We would expect on the average, a sample size of 10 should be sufficient, though in some cases, as many as 20 or 30 items need be tested before a decision can be made.

In this particular case, we will run a test of replacement type and replace components as they fail. The results are as follows -

No. of failures r	Life of individual components (hours)	Accumulated operating time (hours)
1	1600	1600
2	3600	5200
3	4800	10000
4	2000	12000
5	1200	13200
6	2400	15600
7	3600	19200
8	4000	23200
9	2000	25200
10	3600	28800

An accept decision was made after the 10^{th} failure. The procedure has been shown graphically in Fig. 3-6.

The same results can be obtained by using tables given by Lieberman (Ref. C.2.37). He provides limits of time for accepting or rejecting

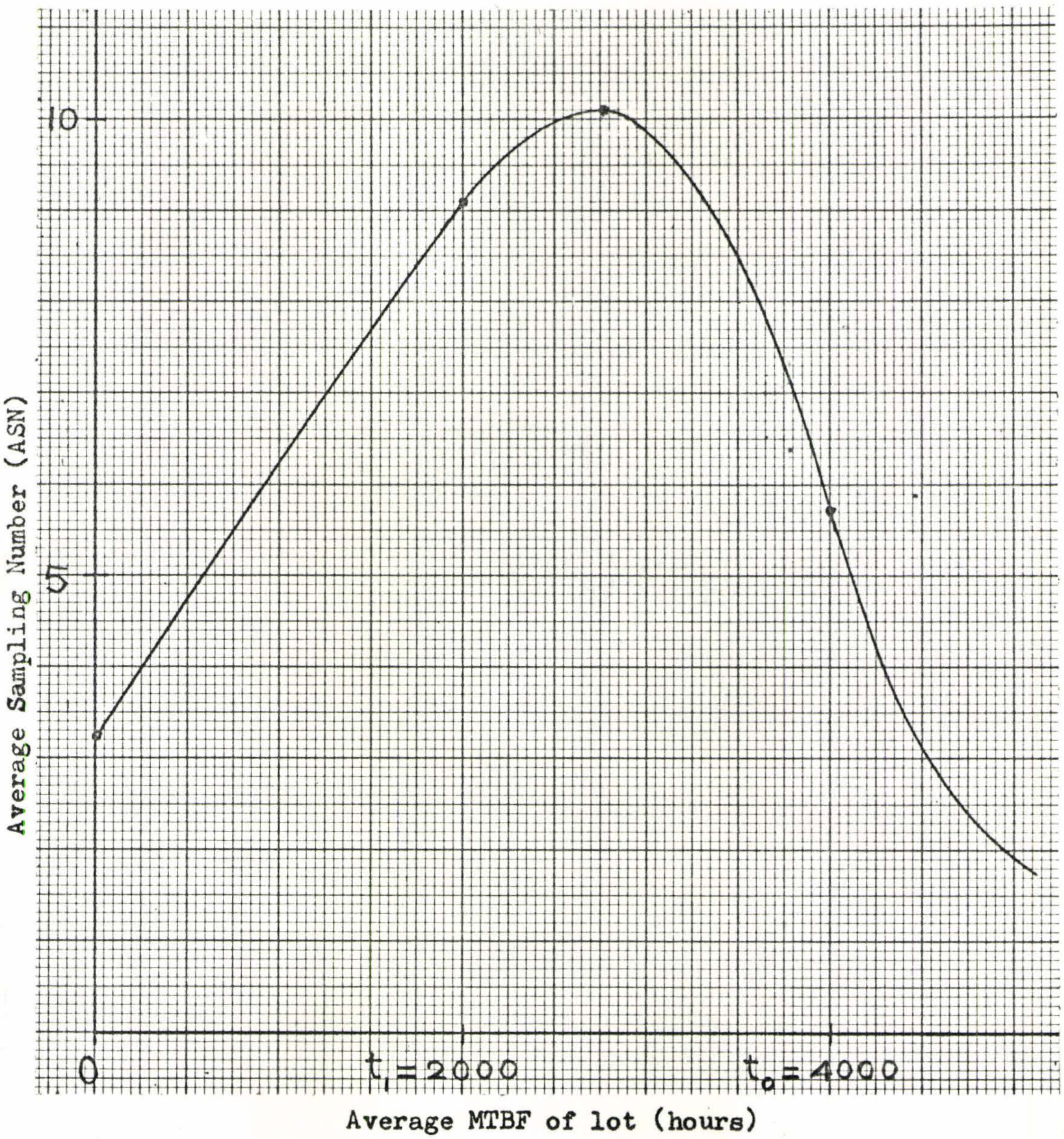


Fig. 3-8 Variation of ASN with MTBF

after each failure, and the test is terminated as soon as either of the two limiting values is reached.

In summary, we can make the following remarks regarding the number of samples needed to reach a decision -

(i) ASN increases with decreasing α and β.

This is because the decision must be made with a smaller margin of error.

(ii) ASN increases as the ratio $\frac{t_o}{t_1}$ decreases.

A smaller $\frac{t_o}{t_1}$ ratio will require that the test be more discriminating and hence more testing is necessary.

(iii) If the true value of MTBF of a lot is much higher than t_o or much lower than t_1 a decision is reached sooner. Or in other words, very good or bad lots need less testing.

Another important point which needs to be considered is test truncation. Tests continued for very long duration tend to be expensive and hence sometimes, decisions are made to truncate the tests after a predetermined test time. Bazovsky (Ref. A.1.10) suggests that a line be drawn through the origin parallel to the accept and reject lines (dashed line in Fig. 3-6). If no decision has been made before the truncation time T_t, then an accept decision is made if r(t) step function at T_t is towards the left of the dashed line, otherwise a reject decision is made. If truncation time T_t is sufficiently large compared to T_{min}, the minimum time for acceptance, the error introduced in this process is small. Eagle (Ref. C.2.8) suggests that tests may be terminated at $T_t = 10\ t_o$. Various methods have been suggested to estimate the error introduced by truncation and have been discussed by Ginsburg and Shaffer (Ref. C.2.19), Aroian (Ref. C.2.3) and others. But it is felt that the method suggested above is sufficient for most engineering purposes.

In conclusion, we may mention that Eagle (Ref. C.2.8) recently proposed a new method of drawing sequential testing charts, where the accept and reject boundaries are parabolic arcs instead of straight lines. But Aroian (Ref. C.2.3) criticized this approach since the probability of rejection of a lot is much higher than the specified value. This is because Eagle assumed sample points on sequential testing charts to be independent of each other, which is in fact not true. Each point in sequential life test depends on the previous point.

Sequential testing is a very useful method of reliability testing when large number of components are to be tested and the assurance that the reliability of the components is higher than a specified limit is more important than exact determination of reliability.

3.3 Life Testing Assuming Normal or Logarithmic Normal Distribution

Normal distribution has been used widely as a model for the wearout life of a component. In most cases, the observed data fits the normal p.d.f. reasonably well. The theoretical normal curve extends from $-\infty$ to $+\infty$. In practice, it is inconceivable to have a negative value of component life, and hence some investigators, such as Bazovsky (Ref. A.1.10) suggest using Logarithmic normal distribution as an appropriate model. But in the normal distribution, the probability at the two ends of the curve is so small that, for engineering purposes, very little error is introduced by using normal distribution.

In testing for wearout life, the sample size need not be large, but the testing time is usually long to obtain sufficient number of wearout failures. As the test proceeds, some chance failures may occur, which are caused before the onset of wearout due to chance. Such failures may be recognized by physical examination and eliminated from further consideration. Statistical methods, such as determination of skewness or any of the standard methods for checking of outliers using Extreme Value theory may be used. Some of these methods are those proposed by Irwin (Ref. F.1.36), Grubbs (Ref. F.1.26) and Dixon (Ref. F.1.14 and F.1.15).

If n components are put on wearout life test and if r_w wearout failures were obtained, estimates of mean and standard deviation are

$$\hat{M} = \frac{\sum_{i=1}^{r_w} t_{iw}}{r_w} \tag{3-59}$$

and

$$\sigma_1 = \sqrt{\frac{\sum_{i=1}^{r_w} (t_{iw} - \hat{M})^2}{r_w}} \tag{3-60}$$

where t_{iw} is the operating time to wearout failure of i^{th} component. Hence $(n-r_w)$ failures have been considered to be caused by factors other than wearout and were eliminated from further consideration.

In reliability work, the standard deviation calculated from the sample is used as an approximation for the true parameter, if the number of wearout failures obtained during the test is at least 25 (Ref. A.1.10).

We have reviewed some methods of obtaining point estimates of mean wearout life and its standard deviation. But for reliability work, interval estimates are of great importance and confidence intervals are to be obtained. The estimate of standard deviation obtained in equation (3-60) is a biased one.

Unbiased estimate of standard deviation of the universe

$$\hat{\sigma} = \sqrt{\sigma_1^2 \frac{r_\omega}{r_\omega - 1}} \tag{3-61}$$

The standard error of the mean can be calculated as

$$\sigma_M = \frac{\hat{\sigma}}{\sqrt{r_\omega}} \tag{3-62}$$

Here we assume that the lot size is much larger than r_w.

The upper and lower confidence limits of the mean wearout life can now be calculated. If α = level of significance, then in two-sided case, lower confidence limit

$$L = \hat{M} - Z_{\alpha/2} \sigma_M \tag{3-63}$$

and upper confidence limit

$$U = \hat{M} + Z_{\alpha/2} \sigma_M \tag{3-64}$$

If only the lower limit of mean wearout life is desired, we use the one-sided test, and lower confidence limit

$$C_L = \hat{M} - Z_\alpha \sigma_M \tag{3-65}$$

associated with confidence level 100(1-α). Z_α is the percentage point for level α and can be obtained from Tables of normal p.d.f. So if in a wearout test, the estimate of wearout life obtained is $\hat{M}$, we can be 100(1-α) per cent confident that the true wearout life of the lot is at least C_L.

The estimate of standard deviation $\hat{\sigma}$ has been obtained from the sample itself. It is possible to obtain an idea of the error involved in this estimation. If the wearout lives of the items in the lot is considered to be normally distributed, Stockton (Ref. F.1.61) states that the standard error of standard deviation is

$$\sigma_s = \frac{\hat{\sigma}}{\sqrt{2 r_w}} \tag{3-66}$$

It can be readily seen that this error is small unless the number of wearout failures obtained during the test r_w is small.

When the sample size or more precisely the number of wearout failures obtained is small (less than about 25), it is more appropriate to use t distribution in computing the confidence limits. For two-sided test, upper confidence limit

$$U = \hat{M} + t_{\alpha/2, r_w - 1}\, \sigma_M \tag{3-67}$$

The lower confidence limit is

$$L = \hat{M} - t_{\alpha/2, r_w - 1}\, \sigma_M \tag{3-68}$$

and for one-sided test, lower confidence limit

$$C_L = \hat{M} - t_{\alpha, r_w - 1}\, \sigma_M \tag{3-69}$$

For the sake of accuracy, it is advisable to take the standard error of the standard deviation into consideration, since here we are dealing with small sample sizes.

More frequently, the test cannot be continued till all the components have failed due to enormous time required and due to economic reasons. Hence, the test has to be truncated before all the components have failed.

Let

t_o = truncation time

n = total number of items put to test

a = number of items which have not failed up to time t_o

then $(n-a)$ = number of items failed due to wearout before test truncation

t_i = times to failure of $(n-a)$ components, $i = 1, 2, \ldots, (n-a)$.

Here we are dealing with a truncated normal distribution and Hald (Ref. F.1.30) gives the following method for estimating the mean wearout life M and standard deviation.

We compute an estimate of degree of truncation as

$$h = \frac{a}{n} \tag{3-70}$$

Then from (n-a) observations, we compute

$$y = \frac{(n-a)\sum_{i=1}^{n-a}(t_o - t_i)^2}{2\left[\sum_{i=1}^{n-a}(t_o - t_i)\right]^2} \tag{3-71}$$

Now we refer to Table X of Hald (Ref. F.3.5) and obtain an estimate of the standardized point of truncation as $Z = f(h, y)$.

Knowing Z, we can obtain from the same table the parameter $\psi'(Z)$ and calculate

$$g(h,z) = \frac{n-a}{a\,\psi'(z) - (n-a)z} \tag{3-72}$$

Then an estimate of standard deviation can be computed as

$$S = \frac{\sum_{i=1}^{n-a} (t_o - t_i)}{n-a}\, g(h,z) \tag{3-73}$$

and an estimate of mean wearout life as

$$M = t_o + ZS \tag{3-74}$$

For truncated normal distribution, Hald (Ref. F.1.30) states that the standard error of the mean can be obtained by using a correction factor $\mu_{11}(z)$, so that the standard error of the mean is

$$\sigma_M = \sqrt{\frac{S^2}{n}\,\mu_{11}(z)} \tag{3-75}$$

Then one-sided lower confidence limit for mean wearout life is given by

$$C_L = \hat{M} - z_\alpha\,\sigma_M \tag{3-76}$$

We will illustrate the above by solving an example. In a wearout life test of 25 gear boxes, the times to wearout failures were as follows -

Gear box No. (i)	Time (t_i) in hours
1	6000
2	6800
3	6980
4	7200
5	7500
6	8000
7	8200
8	8345
9	8420
10	8500
11	8520
12	8528
13	8588
14	8623
15	8646
16	8672
17	8735
18	8780
19	8858
20	8940

The test was truncated after 20^{th} failure at 9000 hours.

Using the previous notation,

$$a = 5$$

$$n = 25$$

$$n-a = 20$$

from equation (3-70), degree of truncation

$$h = \frac{a}{n} = \frac{5}{25} = 0.2$$

We now compute

$$\sum_{i=1}^{20} (t_o - t_i)^2 \quad \text{and} \quad \sum_{i=1}^{20} (t_o - t_i)$$

i	$(t_o - t_i)$	$(t_o - t_i)^2$
1	3000	9000000
2	2200	4840000
3	2020	4080400
4	1800	3240000
5	1500	2250000
6	1000	1000000
7	800	640000
8	655	429025
9	580	336400
10	500	250000
11	480	230400
12	472	222784
13	412	169744
14	377	142129
15	354	125316
16	328	107584
17	265	70225
18	220	48400
19	142	20164
20	60	3600
	17165	27206171

We have

$$\sum_{i=1}^{20} (t_o - t_i) = 17165$$

and

$$\sum_{i=1}^{20} (t_o - t_i)^2 = 27206171$$

Then from equation (3-71)

$$y = \frac{(n-a)\sum_{i=1}^{n-a}(t_o - t_i)^2}{2\left[\sum_{i=1}^{n-a}(t_o - t_i)\right]^2}$$

or

$$y = \frac{20 \times 27206171}{2 \times 17165 \times 17165} = 0.923$$

From Table X of Hald (Ref. F.3.5), we have

$$Z = f(h, y) = f(0.2, 0.923) = -0.614$$

and $\psi'(Z) = 1.2230$

Then using equation (3-72)

$$g(h,z) = \frac{n-a}{a\,\psi'(z) - (n-a)z}$$

or $$g(h, Z) = \frac{20}{[5 \times 1.223] - [20 \times (-0.614)]} = 1.087$$

We can now compute the estimate of standard deviation using equation (3-73)

$$S = \frac{\sum_{i=1}^{n-a} (t_o - t_i)}{n-a} \cdot g(Z)$$

$$\text{or} \quad = \frac{17165 \times 1.087}{20} = 932.918 \text{ hours.}$$

Finally, estimate of mean wearout life from equation (3-74) is

$$\hat{M} = t_o + Z\,S$$

$$= 9000 - (0.614 \times 932.918)$$

$$= 8427.19 \text{ hours.}$$

Having obtained a point estimate of the parameter, we may now proceed to compute the confidence limits.

From tables,

$$\mu_{11}(Z) = 1.110$$

Using equation (3-74), standard error of the mean

$$\sigma_M = \sqrt{\frac{S^2}{n}\,\mu_{11}(Z)}$$

$$= \sqrt{\frac{(932.918)^2 \times 1.110}{25}}$$

$$= 195.913 \text{ hours.}$$

If the desired level of confidence is 95% in one-sided case,

$$\alpha = 0.05.$$

From tables of normal p.d.f.,

$$Z_\alpha = 1.645$$

Hence, using equation (3-76), the lower confidence limit

$$C_L = \hat{M} - Z_\alpha \sigma_M$$

$$= 8427.19 - (1.645 \times 195.913)$$

$$= 8104.91 \text{ hours.}$$

Hence from the results of this test, we can assert with 95% confidence that the mean wearout life of the gearboxes is above 8104.91 hours, assuming that the wearout life is represented by normal distribution.

Various investigators have used the normal distribution to represent the fatigue life of components. Pope (Ref. G.2.24) suggests a method where the specimens are tested at a single stress level and the variable N, the number of reversals to failure is treated to be normally distributed. Using the methods described earlier, the number of reversals a specimen is expected to survive at a certain stress level can be predicted with a specified level of confidence.

Epremain and Mehl (Ref. G.2.3) used logarithmic normal distribution and based their calculations upon the idea that the values of log N are normally distributed about a mean value (for data obtained at a fixed stress level).

In contrast to the previous methods, Peterson (Ref. G.2.23) uses data obtained at different stress levels. The procedure is difficult to justify statistically, but this is probably the only method that can be used when data have been taken without previous planning for statistical analysis. This method provides a measure of standard deviation in stress for a specified life time measured in number of stress reversals.

3.4 Nonparametric Methods of Reliability Testing

There are some methods to analyze and evaluate life test data of equipments, without making any assumption concerning the distribution of the operating life of the equipments.

One such method has been suggested by Virene (Ref. C.2.51) using the procedure suggested by Harris (Ref. F.1.33) and Gumbel and Von Schelling (Ref. F.1.27). A life test is conducted with a sample size n. Then, the probability that, in a large future sample, at most a fraction K of the future units will fail at a life time less than the shortest recorded in the trial sample n is given by

$$W = 1 - (1 - K)^n \tag{3-77}$$

TABLE 3.5

Minimum Sample Sizes Required to Provide W% Probability that Fewer than K% of Future Units will Fail in Time Less than Shortest Recorded in n

Maximum Present of Future Units Failing K%	Probability* W%				
	99.9	99	95	75	50
	Sample Sizes, n				
1	688	459	299	138	69
2	342	228	149	69	35
3	227	152	99	46	23
4	170	113	74	34	17
5	135	90	59	28	14
10	66	44	29	14	7
15	43	29	19	9	5
20	31	21	14	7	4
25	25	17	11	5	3
30	20	13	9	4	2
35	17	11	7	4	2
40	14	10	6	3	2
45	12	8	6	3	2
50	10	7	5	2	1

*When rounding up small sample sizes to integers, W may actually be considerably above the given value.

Table 3.5 has been constructed based on this formula. The usefulness of this table can be demonstrated by the following example.

A life test with a sample 90 gearboxes was run and of all the units tested, 1875 hours was the shortest life time. Then from the table, we can assert with 99% confidence that not more than 5 units will fail at a shorter operating life. The table lists only a few typical values. Equation (3-77) can be used for cases not tabulated.

Dixon and Massey (Ref. F.1.16) describes a number of non-parametric testing methods and Roberts (Ref. A.1.24) has demonstrated the use of these methods in reliability testing. Some of these methods are Rank-sum test, Run test, Exceedance test and Maximum-deviation test. Rank-sum test is a very sensitive test for testing of hypothesis in testing of environmental effects. We shall illustrate the use of this test by the following example.

We put a sample of size n = 8 components on life test under normal operating environment and note the times to failure of each component. We call this sample Control Sample. Let the mean life of these components be θ_o. A similar sample of 8 components are subjected to life test under a different environment E. This sample is called exposure sample. Let the mean life of the components under the new environment be θ_E.

We wish to know whether the mean life of the components have changed significantly due to exposure to the different environment. Or in other words, we wish to test the null hypothesis

$$H_o : \theta_E = \theta_o$$

against an alternate hypothesis

$$H_1 : \theta_E \neq \theta_o$$

We need to make an assumption that the exposure to the new environment does not change the dispersion of the component lives.

We set a level of significance $\alpha = 0.05$, so that our risk of incurring type I error (rejecting H_o, when it is in fact true) is 0.05.

In life testing, the data is obtained in an ordered manner, so that the times to failure are arranged in increasing order. This is very important for this test. The data for control sample is as follows -

time to failure x: $x_1 < x_2 < \cdots\cdots < x_n$

20.1

28.2

30.5

34.3

36.8

38.7

39.9

40.8

The data for exposure sample are as follows -

time to failure y: $y_1 < y_2 < \cdots\cdots < y_n$

16.3

18.2

25.6

28.9

30.7

35.4

37.8

39.1

The data is now combined and ordered in increasing order, and ranks are assigned to each entry.

Z: $Z_1 < Z_2 < \cdots\cdots < Z_{16}$

Rank	Data
1	16.3
2	18.2
3	20.1
4	25.6
5	28.2
6	28.9
7	30.5
8	30.7
9	34.3
10	35.4
11	36.8
12	37.8
13	38.7
14	39.1
15	39.9
16	40.8

Then the ranking order of the control sample x is:

3, 5, 7, 9, 11, 13, 15, 16

and that of exposure sample y is:

1, 2, 4, 6, 8, 10, 12, 14

Next the sum of the ranks of two samples are computed.

$$(\Sigma \text{ ranks})_x = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 16 = 79$$

$$(\Sigma \text{ ranks})_y = 1 + 2 + 4 + 6 + 8 + 10 + 12 + 14 = 57$$

Wilcoxon (Ref. F.1.69) has listed Rank-sum-test significance criteria S for various sample sizes and levels of significance. The smaller of sums of the ranks are compared with the appropriate value of S and if it is less than or equal to S, H_o is rejected. Here the smaller rank sum is 57. From Table 3.6, for $n = 8$ and $\alpha = 0.05$, we get $S = 49$.

Hence we accept H_o, so that with 0.05% risk, we can state that exposure to new environment has not changed the mean life of components significantly.

One disadvantage of this test is that all components of the sample must be tested to failure and no test truncation is possible.

Run test also allows no test truncation and has the added disadvantage of not being as sensitive. But it is simple and little computation need be done. Tables by Swed and Eisenhart (Ref. F.1.62) are used

TABLE 3.6

Rank-Sum Test Significance Criteria

n	$\alpha = 0.05$	$\alpha = 0.02$	$\alpha = 0.01$
5	18	16	15
6	27	24	23
7	37	34	32
8	49	46	43
9	63	59	56
10	79	74	71
11	97	91	87
12	116	110	105
13	137	130	125
14	160	152	147
15	185	176	170
16	212	202	196
17	241	230	223
18	271	259	252
19	303	291	282
20	338	324	315

for this test. Exceedance test and Maximum-deviation test allow test truncation without having to wait for all items to fail. Tables by Epstein (Ref. F.1.20) are used for Exceedance test and results of the work by Tsao (Ref. F.1.64), Massey (Ref. F.1.53) and Smirnov (Ref. F.1.59) are useful for Maximum-deviation test.

4. WEIBULL DISTRIBUTION AND ITS ROLE IN RELIABILITY TESTING

4.1 Introduction

Weibull distribution is a member of extreme value family of distributions (Ref. F. 1.38) and has recently found extensive use in reliability work. This distribution is one of the limiting type to which the distribution of the smallest member of a sample, under general conditions, tends as the sample size is increased indefinitely (Ref. G.2.18), and is the third asymptotic distribution of smallest values (Ref. G.2.12). The distribution in cumulative form, as given by Weibull (Ref. F. 1. 37) in 1951 is as follows -

$$F(x) = 1 - e^{-\frac{(x-x_u)^m}{x_o}} \quad ; \; x \geqslant x_u, \; x > 0, m > 0$$

$$F(x) = 0 \qquad \text{otherwise} \tag{4-1}$$

where x_u is the location parameter,

x_o is the scale parameter,

and m is the shape parameter.

For any given x, F(x) is the proportion of x-values less than or equal to x.

The Weibull probability density function can be obtained by differentiating F(x) with respect to x.

$$f(x) = \frac{m(x - x_u)^{m-1}}{x_0} e^{-\frac{(x-x_u)^m}{x_0}} \qquad x \geqslant x_u, x_0 > 0, m > 0$$

$$f(x) = 0 \qquad \text{otherwise} \qquad (4\text{-}2)$$

The Weibull distribution, since it is a type having three parameters, can be fitted to a greater variety of experimental data than can other distributions. For reliability work, it is usual to set x_u equal to zero, since failures can occur as soon as the experiment has been started.

If we fix the value of x_o and let m = 1, we get the exponential probability density function. As m increases, the distribution approaches the normal distribution more and more closely and for m = 4, the normal distribution and Weibull distribution are almost indistinguishable. This shows the versatility of Weibull distribution and the failure characteristics of a component during the various phases of its lifetime can be represented by a single distribution by the proper choice of the parameters for each phase. Fig. 4-1 shows the

Weibull p.d.f. for various values of parameter m.

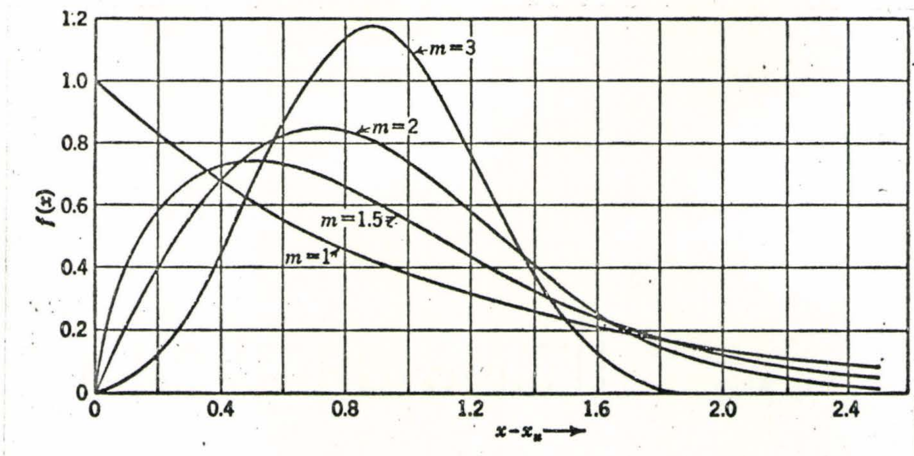


Fig. 4-1 Weibull Probability Density Function with Fixed x_o and Various m

An expression for reliability can now be derived. We will start from the fundamental probability concept. For any continuous p.d.f. f(x), the probability that the chance variable X will be between a and b is

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx = \int_{-\infty}^b f(x)\,dx - \int_{-\infty}^a f(x)\,dx$$
$$= F(b) - F(a) \qquad (4\text{-}3)$$

Here F(X) is the c.d.f. and hence

$$dF(X) = f(x)dx.$$

If the distribution is a failure age distribution starting at the origin, we have

$$P(X \leqslant b) = \int_0^b f(x)\,dx = F(b) \qquad (4\text{-}4)$$

and F(b) is the probability of failure at age b. Then $[1 - F(b)]$ is the probability of nonfailure at age b, which is also the reliability by definition. So in general,

Reliability $R(x) = 1 - F(x)$

Substituting for F(x) from eqn. (4-1), we get

$$R(x) = e^{-\frac{(x - x_u)^m}{x_o}} \qquad (4\text{-}5)$$

We can now derive an expression for Weibull instantaneous failure rate or hazard rate. Probability G(h,x) of failure in a finite period h, given that no failure has occured up to age x is given by

$$\mathcal{G}(h,x) = \frac{\int_x^{x+h} f(y)\,dy}{1 - F(x)} = \int_x^{x+h} \frac{f(y)}{R(x)}\,dy \qquad (4\text{-}6)$$

This equation was obtained using the theorem of compound probability (Section 2.2). If we let h approach zero, then the instantaneous failure rate or hazard rate Z(x) is given by

$$Z(x) = \frac{f(x)}{R(x)} = \frac{f(x)}{1 - F(x)} \tag{4-7}$$

So hazard rate is the number of failures per unit time at a time x ratioed to the number still operative at that time. Substituting from eqns. (4-1) and (4-2), we get

$$Z(x) = \frac{m\,(x - x_u)^{m-1}}{x_o} \tag{4-8}$$

This was termed "conditional density function" by Davis (Ref. A.3.10.), but now the term hazard rate is universally used.

In reliability testing, theoretically it is possible for a component to fail as soon as the test has been started and so Z(x) must be positive for all values of x greater than zero. This requires that the location parameter x_u be zero. In all future work in this chapter, x_u will be considered to be zero, since we are interested in using the Weibull distribution in life testing only.

If, in eqn. (4-8), we set $x_u = 0$ and m = 1, we get

$$Z(x) = \frac{1}{x_o} \tag{4-9}$$

This is the failure rate λ in the exponential distribution and is independent of component age. This is one of the major differences between the Weibull distribution and the exponential distribution. Hazard rate in Weibull distribution increases with x for $m > 1$ and decreases with x for $m < 1$ and is independent of x for $m = 1$.

During the "burn-in" period of component life, we would expect the hazard rate to reduce with time as components of sub-standard quality are eliminated and this should be followed by a period of constant hazard rate (when exponential distribution model can be used) and finally in the wear out stage, hazard rate should increase rapidly with time. But from numerous experiments, it has been demonstrated that very rarely does a component show a constant hazard rate during any part of its lifetime, which proves that the exponential distribution is not a good model in most cases. The ability of the Weibull distribution to fit empirical data to a much better degree partically accounts for its wide acceptance in reliability work in recent times.

4.2 Different Testing Procedures

As with any other type of testing, the criteria for judging failure of a component should be clarified before the tests are planned. In case of some mechanical equipment, such as anti-friction bearings, gears etc, this may present considerable difficulties, since a gear may continue to function even when the flanks of the teeth have been badly pitted and the operation is noisy. In such a case, the gear may be considered to have failed. A clear and unambigious statement of failure criteria is essential to obtain meaningful information from test data.

We should next consider the case of test truncation . It is quite satisfactory if testing can be continued till all items fail. But limitations of testing time and economic considerations generally call for terminating the test before all items have failed. The tests are usually truncated in two different ways (Ref. F.1.43) -

(a) Item truncation - The test is stopped when the r^{th} item out of a sample of n items fails. In such a case, the precise failure ages of each individual item are usually obtained. If x_i is the age at failure of i^{th} item, then the data is obtained in an ordered manner, such that $0 < x_1 < x_2 < \ldots < x_r$.

For tests lasting for long duration, it may be necessary to arrange for automatic monitoring and recording apparatus, so that continuous recording is possible. This type of data is referred to as "ungrouped life testing data".

(b) Time truncation - The test is stopped after a certain test time z_k regardless of the number of failures that have occurred during the test.

During testing, the number of failed components are counted periodically after a certain fixed time. These times of inspection z_j (j = 1,2, ..., k) are chosen conveniently when the tests are planned. The failure data is obtained as f_j, which is the number of failures that occurred during the period z_{j-1} and z_j. Or in other words, the observations are pairs of numbers z_j, f_j (for j = 1, 2,....,k). These paired ordered observations are referred to as "grouped life testing data". For life tests which require a long time to obtain an adequate number of failures, this is a more convenient and economical method. The items on test require no attention between inspection times, sonce the precise time of failure of a single item is not needed. But in this method, intragroup information in the data is lost and the estimate made from such data are liable to be inaccurate, if inspections are done at long intervals.

Many engineering devices are designed for a long life under design load and operating conditions. Accelerated testing has been attempted to reduce testing time and cost. Failures are induced sooner by increasing the load and the severity of the environmental complex. But the most difficult thing is to extrapolate the accelerated data back to normal use conditions. No general rule can be laid out and the way the component life is affected by varying the load or any environmental condition is purely a characteristic of the particular component. A few relationships have been proposed, but they vary widely for different component types and for different environments. To do any accelerated testing, the experimenter must investigate the behaviour of the component life with varying load or environment and must devise appropriate correlations for extrapolation. Weisenberg (Ref. C.2.52) has reported that accelerated testing is done at the Harrison Radiator Division of GM Corporation on automotive radiators by subjecting the radiators to elevated temperatures and cyclically fluctuating pressure. Levenbach (Ref. C.2.36) and Kimmel (Ref. C.2.35) have attempted to provide some relationships for accelerated testing of paper dielectric capacitors. Guild (Ref. C.2.28) studied the burnout times of heaters of a certain type of vacuum tube under normal and accelerated conditions. Cary and Thomas (Ref. C.2.5) proposed a particularly useful method, in which accelerated testing is treated as a special case of general model theory. The model is subjected to a more severe environment than prevails in normal use and then failure data are extrapolated to normal operating conditions.

4.3 Estimation of Parameters of Weibull Distribution

In Sec. 4.1, it was mentioned that in reliability work, the location parameter is assumed to be zero. Eqns. (4-1), (4-2), (4-5), and (4-8) then take the following form

$$F(x) = 1 - e^{-\frac{x^m}{x_o}} \quad ; \ x \geqslant 0, x_o > 0, m > 0$$

$$F(x) = 0 \qquad \text{elsewhere} \tag{4-10}$$

$$f(x) = \frac{m x^{m-1}}{x_o} e^{-\frac{x^m}{x_o}} \quad ; \ x \geqslant 0, x_o > 0, m > 0$$

$$f(x) = 0 \qquad \text{elsewhere} \tag{4-11}$$

$$R(x) = e^{-\frac{x^m}{x_o}} \tag{4-12}$$

and $$Z(x) = \frac{m x^{m-1}}{x_o} \tag{4-13}$$

To use the Weibull distribution in reliability work, the parameters m and x_o are estimated from test data. We shall now briefly review some of the methods available for this purpose.

4.3.1 Graphical Method Using Weibull Probability Paper

The Weibull c.d.f. as given in eqn. (4-10) is

$$F(x) = 1 - e^{-\frac{x^m}{x_o}}$$

This can be rewritten as

$$\frac{1}{1-F(x)} = e^{\frac{x^m}{x_o}}$$

Taking natural logarithm twice, we get

$$\ln \ln \left[\frac{1}{1-F(x)}\right] = -\ln x_o + m \ln x \qquad (4\text{-}14)$$

This is the equation of a straight line with independent variable ln x and dependent variable $\ln \ln [1/\{1-F(x)\}]$. Hence, if a set of data following Weibull distribution is plotted with principal abscissa ln x and principal ordinate $\ln \ln [1/\{1-F(x)\}]$, a straight line is obtained with intercept ($-\ln x_o$) and slope m. The graphical estimation of x_o and m is done using Weibull probability paper.

If we have a set of values of x and corresponding values of F(x), we first plot the values of ln x and $\ln \ln [1/\{1-F(x)\}]$. If a good fit is obtained by one straight line, we can the proceed to estimate m and x_o. But sometimes, one encounters mixed Weibull distribution, where one straight line fits a part of the points and another straight line with a different slope fits the remaining points. This is a situation where the total population is composed of two segments of proportions P_1 and P_2 and each segment is itself a Weibull distribution. Such a situation may be expected, if the failure characteristics of a component are different during the normal operating period and during the wearout period, which is quite normal. In case of a mixed Weibull distribution, the parameters m and x_o are estimated seperately for each segment. The great value of the graphical method is that the presence of a mixed distribution can immediately be detected by examining the plotted points.

If a good straight line fit is obtained on Weibull probability paper, the parameters m and x_o are estimated as follows -

(i) To estimate shape parameter m - We obtain the value of $\ln \ln[1/\{1-F(x)\}]$ at $\ln x = 0$ and then we obtain the value of $\ln x$ at $\ln \ln[1/\{1-F(x)\}] = 0$.

Then

$$m = \frac{-\ln\ln\left[\frac{1}{1-F(x)}\right]\Bigg|_{\text{at } \ln x = 0}}{\ln x\Bigg|_{\text{at } \ln\ln\left[\frac{1}{1-F(x)}\right] = 0}} \tag{4-15}$$

(ii) To estimate scale parameter x_o - We read off the value of $\ln \ln [1/\{ 1-F(x)\}]$ at $\ln x = 0$

Then from eqn. (4-14), we get

$$\ln x_o = -\ln\ln\left[\frac{1}{1-F(x)}\right]\Bigg|_{\text{at } \ln x = 0}$$

$$\text{or} \quad x_o = e^{-\ln\ln\left[\frac{1}{1-F(x)}\right]} \tag{4-16}$$

Weibull probability paper has been modified by Kao to include some extra scales, so that these parameters can be obtained directly from the graph.

Several conventions are in use to estimate F(x) from experimental data, each with its own statistical nature. For any given x, F(x) is the proportion of x-values less than or equal to x. For ungropued data, Weibull used the mean rank as an unbiased estimate and the most likely or expected value (Ref. G.2.24, p. 130 and F.1.3). Then

$$F(x_r) = \frac{r}{n + 1} \tag{4-17}$$

where n = sample size

and r = number of x-values less than or equal to x_r.

The larger the sample size, the more precise is the estimate of $F(x_r)$. One disadvantage of this approach is that no quantitative information is obtained about the confidence level. On the other hand, Johnson (Ref. F.1.39) uses median ranks calculated from the table of the incomplete Beta function. Values of median ranks up to n=50 have been tabulated (Ref. F.1.39, Table I). Johnson has also tabulated 5% and 95% ranks of Order Statistics for sample sizes up to 50 (Ref. F.1. 39, Tables II and III). These were computed from the incomplete Beta function Using these tables, a 90% confidence band can be plotted on the Weibull probability paper, so that one can assert with 90% confidence that the plotted points would lie within this band. For grouped data, Kao (Ref. F.1.43) states that

$$F(z_j) = \frac{F_j}{n} \tag{4-18}$$

where n = sample size

and $F_j = \sum_{i=1}^{j} f_i$

= cumulative number of failures occuring on or before the inspection time z_j.

Kao states that this is an unbiased minimum variance estimate of $F(z_j)$. We will now illustrate the use of this method by solving an example.

A random sample of 43 step motors were tested. A power supply furnished electrical pulses to each motor and a motor was considered to have failed if the motor failed to make a step or index even though an electrical impulse was provided. The data obtained in grouped form is given in Table 4.1 (Ref. F.1.3). Here mean ranks were used in estimating F(x). The data was plotted on Weibull probability paper and the graph is shown in Fig. 4-2. It is clear that this is a case of mixed Weibull distribution. We will estimate the parameters for each segment seperately.

For the first segment,

$$\ln \ln \left[\frac{1}{1-F(x)}\right]\Bigg|_{\text{at } \ln x = 0} = -2.48$$

and

$$\ln x \Bigg|_{\text{at } \ln \ln \left[\frac{1}{1-F(x)}\right] = 0} = 3.9$$

TABLE 4.1

Number of Steps to Failure x 10^{-3} x	Cumulative Percent Failure F(x)
0.11	2.32
0.61	6.96
2.37	13.92
6.16	23.20
11.73	30.16
20.47	51.04
29.68	76.56
35.00	83.52
46.00	92.80

Then from eqn. (4-15),

$$m^{(1)} = \frac{2.48}{3.9} = 0.64$$

From eqn. (4-16),

$$x_o^{(1)} = e^{2.48} = 11.94$$

For the second segment,

$$\ln \ln \left[\frac{1}{1-F(x)}\right]\Bigg|_{\text{at } \ln x = 0} = -5.5$$

and

$$\ln x \Bigg|_{\text{at } \ln \ln \left[\frac{1}{1-F(x)}\right]} = 3.5$$

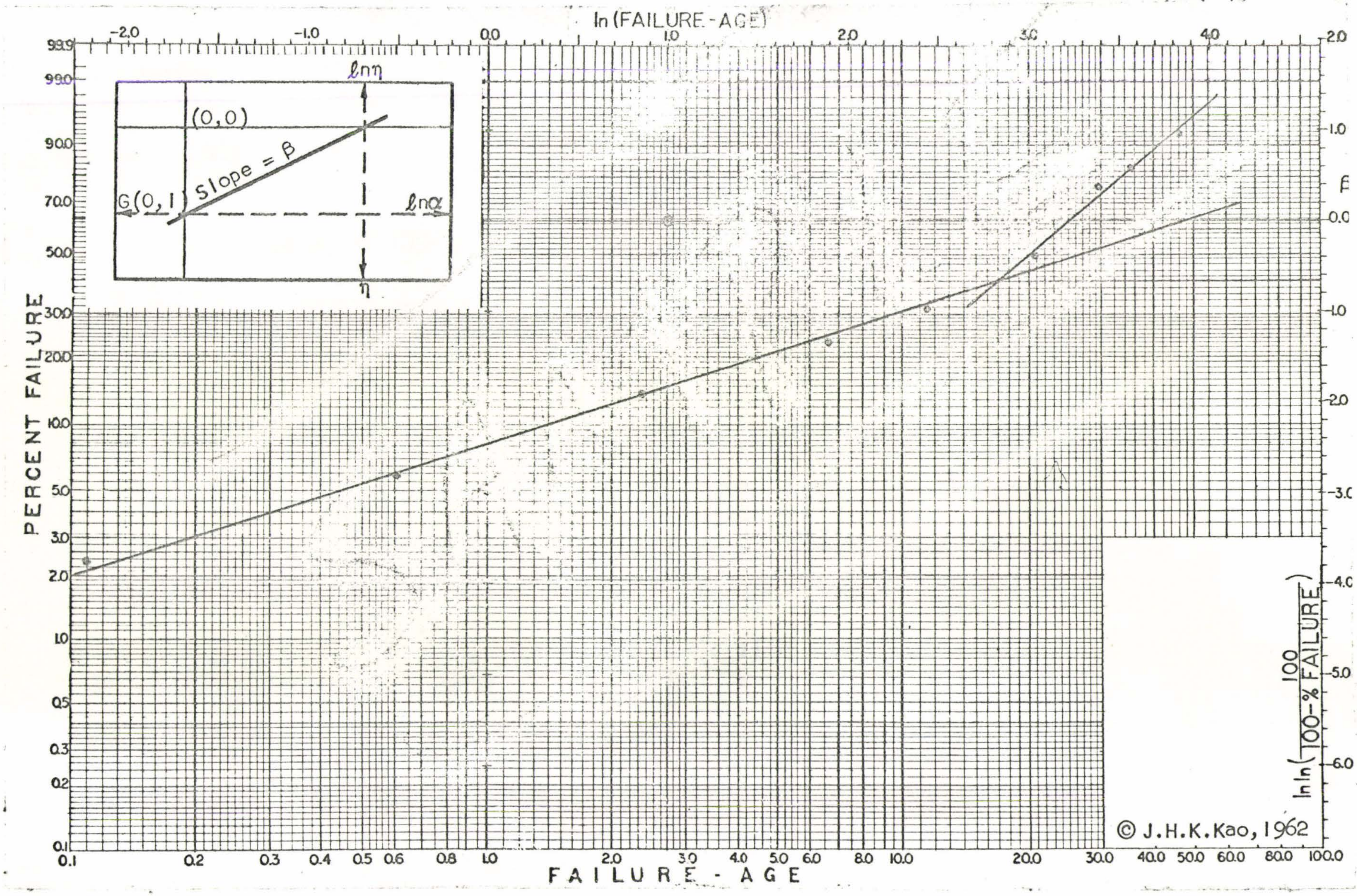


Fig. 4-2 Data of Step Motor Failure Plotted on Weibull Probability Paper

From eqn. (4-15),

$$m^{(2)} = \frac{5.5}{3.5} = 1.57$$

and from eqn (4-16),

$$x_o^{(2)} = e^{5.5} = 244.69$$

So initially, the failure data follows the Weibull distribution

$$F(x) = 1 - e^{-\frac{x^{0.64}}{11.94}} \quad \text{for } 0 \leq x \leq 17$$

and then the data follows the distribution

$$F(x) = 1 - e^{-\frac{x^{1.57}}{244.69}} \quad \text{for } x \geq 17$$

Here the unit of x is number of steps to failure x 10^{-3} .

The expression for reliability is then

$$R(x) = e^{-\frac{x^{0.64}}{11.94}} \quad \text{for } 0 \leq x \leq 17$$

and

$$R(x) = e^{-\frac{x^{1.57}}{244.69}} \quad \text{for } x \geq 17$$

The hazard rate for Weibull distribution was given in eqn. (4-13) as

$$Z(x) = \frac{mx^{m-1}}{x_o}$$

The hazard rate for the first segment is then

$$Z_1(x) = 0.054x^{-0.36} \quad \text{for } 0 \leq x \leq 17$$

and the hazard rate for the second segment is

$$Z_2\ (x) = 0.0064\ x^{0.57} \quad \text{for } x \geq 17$$

The variation of the hazard rate can also be demonstrated graphically. Taking the logarithm of eqn. (4-13), we get

$$\ln Z(x) = (m-1) \ln x + (\ln m - \ln x_o)$$

which is an equation of a straight line.

For the first segment,

$$\ln Z(x) = - 0.36 \ln x - 2{\cdot}926 \quad \text{for } 0 \leq x \leq 17 \quad \text{for } 0 \leq x \leq 17$$

and for the second segment,

$$\ln Z(x) = 0.57 \ln x - 5.049 \quad \text{for } x \geq 17$$

The values of Z (x) for various values of x were computed and tabluated in Table 4.2.

A graph of Z(x) against x has been plotted on logarithmic graph paper and is shown in Fig. 4-3. It can be seen that the hazard rate continues to reduce up to $x = 17 \times 10^{-3}$ steps and then increases. So it appears that the failure pattern of the motors change from infant mortality stage to wearout stage without any intervening transition period of random

TABLE 4.2

x	ln x	ln Z(x)	Z(x)
0.11	-2.207	-2.131	0.119
0.61	-0.494	-2.748	0.064
2.37	0.863	-3.237	0.039
6.16	1.818	-3.580	0.028
11.73	2.460	-3.812	0.022
20.47	3.020	-3.328	0.036
29.68	3.911	-3.116	0.044
35.00	3.555	-3.023	0.049
46.00	3.829	-2.866	0.057

failures with a constant failure rate. Hence it is clear that an exponential distribution would have been a very poor model to represent the failure pattern.

We will now review another example where automotive radiators were tested for reliability (Ref. C.2.52). The radiators were operated at a constant elevated temperature and a solution of water and ethylene glycol was circulated. The pressure was varied cyclically, one cycle consisting of increasing the pressure for zero to the maximum

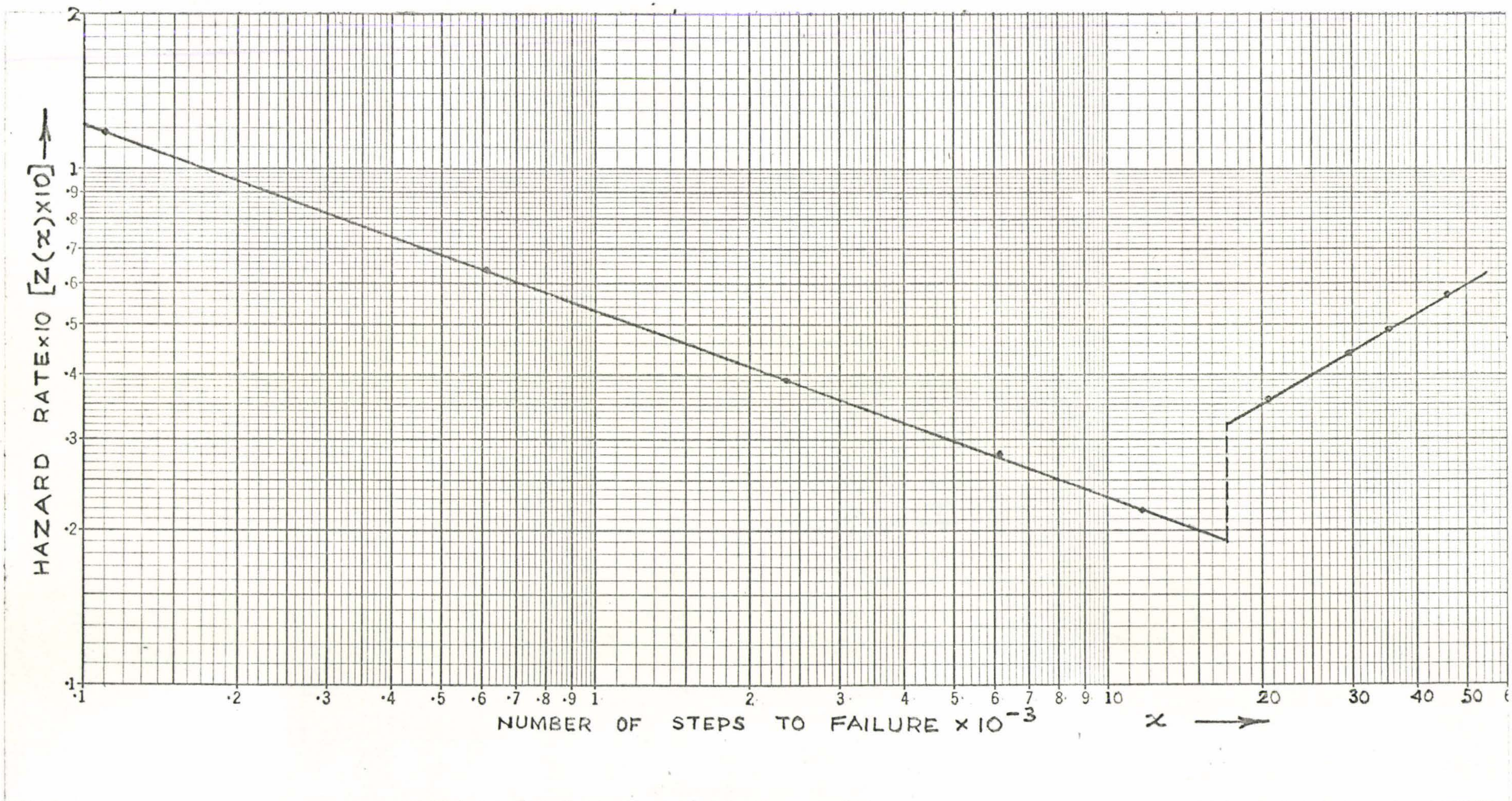


Fig. 4-3. Variation of Hazard Rate with Age for Step Motor (Here the Unit of Hazard Rate is Number of Failures per 1000 Steps)

value and returning to zero. A random sample of 9 radiators were tested and failure age in cycles was defined as the last cycle observed prior to the cycle in which failure was observed. The data was of ungrouped form and median rank was used in preference to mean rank for plotting on Weibull probability paper. For each point plotted, an upper confidence limit at 95 percent rank and a lower confidence limit at 5 percent rank were obtained (Ref. F.1.39, Tables I, II, III). So it was possible to obtain a 90 percent confidence band for the straight line plotted. The experimental data and the ranks computed are given in Table 4.3.

TABLE 4.3

Radiator Failure Number	Failure Age In Cycles	Mean Rank	Median Rank	5% Rank	95% Rank
1	24,432	o.10	0.074	0.006	0.283
2	36,365	0.20	0.181	0.041	0.429
3	36,490	0.30	0.287	0.098	0.550
4	49,000	0.40	0.394	0.169	0.655
5	55,700	0.50	0.500	0.251	0.749
6	69,690	0.60	0.606	0.345	0.831
7	77,150	0.70	0.713	0.450	0.903
8	89,785	0.80	0.819	0.571	0.959
9	96,302	0.90	0.926	0.717	0.994

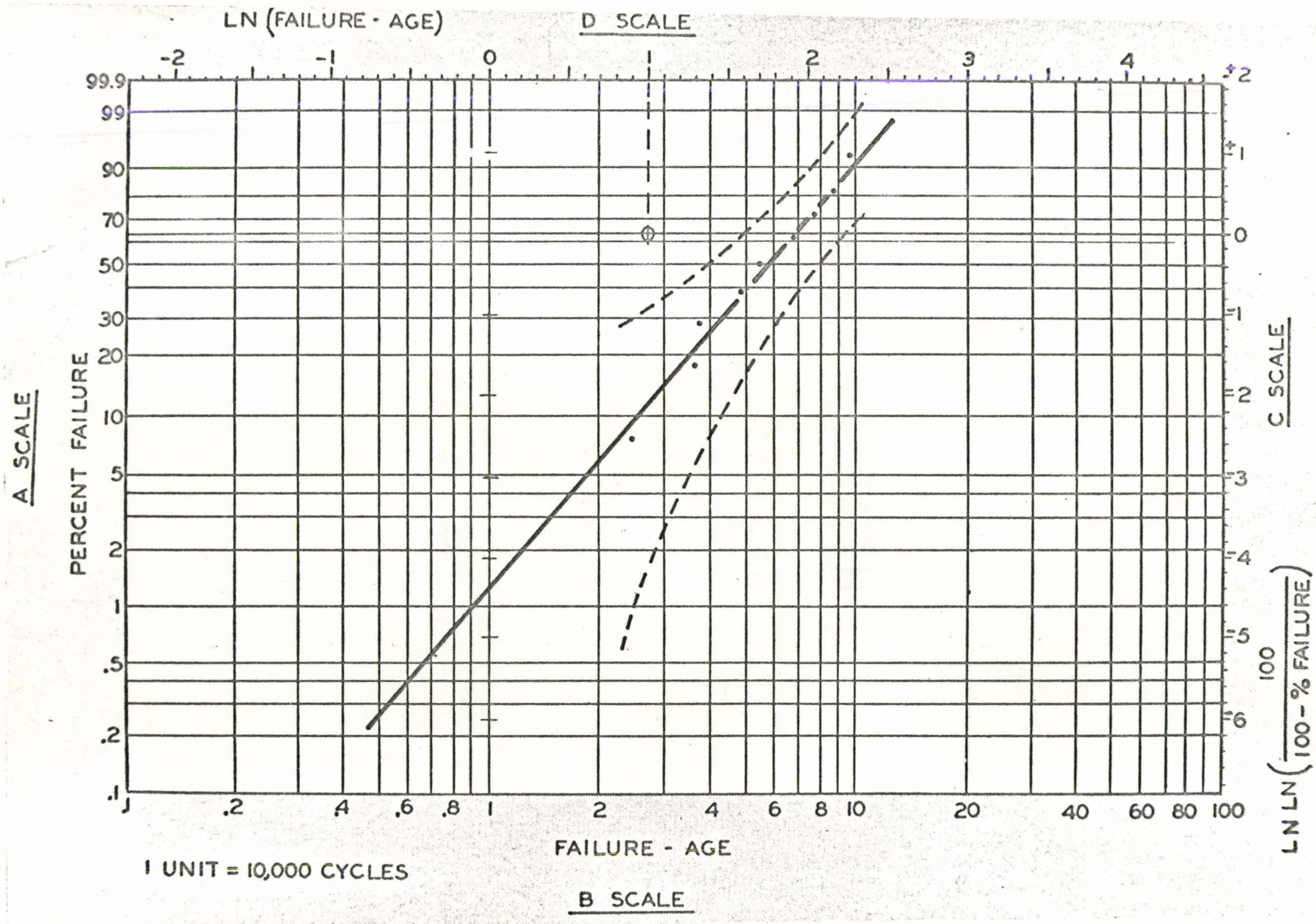


Fig. 4-4 Data of Automotive Radiator Failure Plotted on Weibull Probability Paper. The Dotted Lines Enclose the 90% Confidence Band.

The graph, drawn on Weibull probability paper is shown in Fig. 4-4.
The estimated values of Weibull parameters are

Shape parameter $m = 2.35$

and scale parameter $x_o = 2.272 \times 10^{11}$ cycles.

The reliability of the radiators is then given by

$$R(x) = e^{-\left(\frac{x^{2.35}}{2.272 \times 10^{11}}\right)}$$

4.3.2 The Method of Least Squares on Transformed Data

Eqn. (4-14) was

$$\ln \ln \left[\frac{1}{1 - F(x)}\right] = - \ln x_o + m \ln x$$

In the previous section, a straight line was fitted to the points plotted on Weibull probability paper visually. The method of least squares may also be used to obtain a better fit.

Eqn. (4-14) can also be written as

$$a = d + mb \tag{4-19}$$

where

$$a = \ln \ln \left[\frac{1}{1 - F(x)} \right]$$

$$d = - \ln x_o$$

$$b = \ln x$$

Here d and m are constants and a series of values for a and b are obtained from experimental data as a_i and b_i, where $i = 1,2,\ldots,r$ We then obtain the normal equations for the least squares line as

$$\Sigma a = dr + m \, \Sigma \, b$$

$$\Sigma ab = d \, \Sigma \, b + m \, \Sigma \, b^2$$

Solving these equations simultaneously, we obtain

$$d = \frac{(\Sigma a)(\Sigma b^2) - (\Sigma b)(\Sigma ab)}{r \, \Sigma b^2 - (\Sigma b)^2} \tag{4-20}$$

and

$$m = \frac{r \, \Sigma ab - (\Sigma b)(\Sigma a)}{r \, \Sigma b^2 - (\Sigma b)^2} \tag{4-21}$$

The shape parameter m is obtained directly from eqn. (4-21) and the scale parameter x_o is obtained as

$$x_o = \exp \, (-d) \tag{4-22}$$

To obtain an idea of the goodness of fit, the coefficient of correlation must be calculated. We calculate the mean values of variables a and b as

$$\bar{a} = \frac{\sum_{i=1}^{r} a_i}{r}$$

and

$$\bar{b} = \frac{\sum_{i=1}^{r} b_i}{r}$$

The deviations from the mean values are

$$A_i = a_i - \bar{a}$$

$$B_i = b_i - \bar{b}$$

Then the coefficient of correlation is given as

$$R^* = \frac{\Sigma AB}{\sqrt{(\Sigma A^2)(\Sigma B^2)}} \tag{4-23}$$

The better the least squares line fits the data points, the closer is the value of R^* to unity. A smaller value of R^* would indicate that the data does not follow Weibull distribution closely or it may indicate the presence of a mixed Weibull distribution.

The graphical method or least squares method of estimating Weibull parameters are the simplest of all available methods. Nevertheless, these two methods have several weaknesses. Kao (Ref. F.1.43) states that these methods do not necessarily guarantee the best fit of the raw data in the cartesian scale. Another disadvantage is the selection of the proper plotting position. Gumbel (Ref. F.1.29) and Weibull have advocated the use of mean rank, whereas some other investigators like Johnson (Ref. F.1.39) advocate the use of median rank. Lieblein and Zelen (Ref. G. 2.18) have pointed out that the method of least squares as usually used fails to take adequate account of the items which have not failed when the test has been stopped.

The greatest advantage of these two methods is that an approximate estimate of Weibull parameters can be obtained quite easily and can be used as an initial estimate for more precise methods.

We will now illustrate the use of the method of least squares by solving an example (Ref. F.1.56). A random sample of 20 relays was put on life test and the number of actuations made by a relay before failure was noted. Hence the data was obtained in ungrouped form and the test was truncated after 17 relays had failed. Mean ranks were used in computing the cumulative percent failure. The test data has been tabulated in Table 4.4. A computer program was set up for obtaining the Weibull parameters and is enclosed in Appendix A-1. The results obtained from the computer are as follows -

Coefficient of correlation $R^{*} = 0.99554$

Shape parameter $m = 1.83797$

Scale parameter $x_o = 0.11073 \times 10^{12}$

TABLE 4.4

Relay Failure Number	Cummulative Percent Failures F(x)	Actuations to Failure x
1	0.0476	190000
2	0.0952	334000
3	0.1429	365000
4	0.1905	420000
5	0.2381	472000
6	0.2857	589000
7	0.3333	610000
8	0.3810	662000
9	0.4286	792000
10	0.4762	840000
11	0.5238	850000
12	0.5714	900000
13	0.6190	960000
14	0.6667	1102000
15	0.7143	1195000
16	0.7619	1240000
17	0.8095	1303000

Since the coefficient of correlation is close to unity, we infer that the experimental data follows Weibull distribution quite closely. The reliability of the relays is given by

$$R(x) = \exp\left[- \frac{x^{1.838}}{0.111 \times 10^{12}} \right]$$

4.3.3 The Method of Maximum Likelihood For Ungrouped Data

Modern statistical practice makes considerable use of the likelihood function introduced by Gauss and developed by R. A. Fisher (Ref. F.1.18, F.1.45 and F.2.1). For a sample of n values $x_1, x_2, \ldots, x_n$ drawn at random from a continuous distribution with p.d.f. f(x,d), the likelihood function is defined by

$$L(x_1, x_2, \ldots, x_n, \alpha) = \prod_{i=1}^{n} f(x_i, \alpha)$$

$$= f(x_1,\alpha)\, f(x_2,\alpha) \,\ldots\ldots\, f(x_n,\alpha).$$

The maximum likelihood method consists in choosing, as an estimate of the unknown population parameter of α, the value that renders the likelihood function as large as possible. For Weibull p.d.f. given by eqn. (4-11), the likelihood function for ungrouped life testing data $x_1, x_2, \ldots\ldots x_r$, from a sample of size n where $r \leq n$ is given by Kao (Ref. F.1.43) as

$$L = \frac{n!}{(n-r)!}\left(\frac{m}{x_o}\right)^r \prod_{i=1}^{r} x_i^{m-1} \exp\left\{-\frac{1}{x_o}\left[\sum_{i=1}^{r} x_i^m + (n-r)x_r^m\right]\right\}$$

Putting the partial derivations of ln L with respect to m and x_o equal to zero, we get

$$x_o = \frac{1}{r}\left[\sum_{i=1}^{r} x_i^m + (n-r)x_r^m\right] \qquad (4\text{-}24)$$

$$x_o = \frac{\sum_{i=1}^{r} x_i^m \ln x_i + (n-r)x_r^m \ln x_r}{\frac{r}{m} + \sum_{i=1}^{r} \ln x_i} \qquad (4\text{-}25)$$

These equations can be solved simultaneously by trial and error to yield the maximum likelihood estimates of the parameters m and x_o for ungrouped data. These two simultaneous transcendental equations can also be solved by numerical methods, such as Newton's approximation. The approximate estimate obtained by graphical means or by least squares method is used as the initial trial value and these equations are then solved to obtain a more accurate value.

The example solved in Sec. 4.3.2 was solved by this method. A computer program was set up and is enclosed in Appendix A-2. From the estimate of the shape parameter m obtained by the least squares method, various trial values of m were used and eqns.(4-24) and (4-25) were solved.

The following results were obtained from the computer.

Estimate of shape parameter m = 1.8362

Estimate of scale parameter $x_o = 0.10089 \times 10^{12}$

Using trial values of m at closer intervals, a more precise estimate can be obtained.

4.3.4 The Method of Maximum Likelihood for Grouped Data

The Likelihood function for the Weibull p.d.f., when data is obtained in grouped form, has been given by Kao (Ref. F.1.43) as

$$L = e^{-\frac{Z_K^m}{x_o}\left(n - \sum_{j=1}^{K} f_j\right)} \prod_{j=1}^{K} \left[e^{-\frac{Z_{j-1}^m}{x_o}} - e^{-\frac{Z_j^m}{x_o}} \right]^{f_j}$$

Taking natural logarithum of the above equation, we get

$$\ln L = -\frac{Z_K^m}{x_o}\left(n - \sum_{j=1}^{K} f_j\right) + \sum_{j=1}^{K} f_j \ln\left[e^{-\frac{Z_{j-1}^m}{x_o}} - e^{-\frac{Z_j^m}{x_o}} \right]$$

The function ln L and hence L can now be maximized with respect to m and x_o.

Setting the partial derivatives $\frac{\partial(\ln L)}{\partial m}$ and $\frac{\partial(\ln L)}{\partial x_o}$ equal to zero, we get

$$\sum_{j=1}^{K} f_j \left[\left\{ \frac{1}{e^{-a} - e^{-b}} \right\} \left\{ (b \ln z_j) e^{-b} - (a \ln z_{j-1}) e^{-a} \right\} \right]$$

$$- \frac{z_K^m \ln z_K}{x_o} \left(n - \sum_{j=1}^{K} f_j \right) = 0 \qquad (4\text{-}26)$$

$$\sum_{j=1}^{K} f_j \left[\left\{ \frac{1}{e^{-a} - e^{-b}} \right\} \left\{ \frac{a}{x_o} e^{-a} - \frac{b}{x_o} e^{-b} \right\} \right]$$

$$+ \frac{z_K^m}{x_o^2} \left(n - \sum_{j=1}^{K} f_j \right) = 0 \qquad (4\text{-}27)$$

where

$$a = \frac{z_{j-1}^m}{x_o}$$

and

$$b = \frac{z_j^m}{x_o}$$

Eqns. (4-26) and (4-27) can be solved by trial and error on the computer to yield m and x_o.

Since equations (4-26) and (4-27) are quite complex and difficult to solve, equations (4-24) and (4-25) have been modified to yield approximate estimates of m and x_o for grouped data. These equations are as follows -

$$x_o = \frac{1}{r}\left\{ \sum f_j x_j^m + (n-r)\, x_r^m \right\} \tag{4-28}$$

$$x_o = \frac{\sum f_j x_j^m \ln x_j - (n-r)\, x_r^m \ln x_r}{\frac{r}{m} + \sum f_j \ln x_j} \tag{4-29}$$

The grouped data obtained in the form z_j, f_j ($j = 1,2,\ldots.,k$) has been converted to ungrouped form $x_1, x_2 \ldots\ldots, x_r$ by assuming that (a) all f_j items which failed between z_{j-1} and z_j have a failure age of $x_j = \frac{1}{2}(z_{j-1} + z_j)$ and (b) the last inspection time z_k equal to x_r. Both these approximations are not unreasonable if $(z_j - z_{j-1})$ are small for all j. Again equations (4-28) and (4-29) may be solved by trial and error or by numerical methods. If the inspection periods are long and number of failures per inspection period are not small, equations (4-26) and (4-27) must be used to obtain adequate accuracy.

4.3.5 The Method of Minimized Chi-Squares for Grouped Data (Ref. F. 1. 43)

For grouped life data obtained in the form z_j, f_j $(j = 1, 2 \ldots\ldots k)$, let p_j be the probability that any item will fail in the time interval z_{j-1} to z_j. Then

$$p_j = F(z_j) - F(z_{j-1})$$

and in particular,

$$p_1 = F(z_1) - F(0) = F(z_1)$$

and

$$p_{k+1} = F(\infty) - F(z_k) = 1 - (z_k)$$

Then np_j $(j = 1, 2, \ldots., k + 1)$ will be the expected number of failures between inspection times z_{j-1} and z_j. For large n, and if $np_j \geqslant 5$ and $k + 1 \geqslant 5$, the following quantity has a chi-square distribution with k degrees of freedom (Ref. F. 1.34) -

$$\chi^2 = \sum_{j=1}^{k+1} \frac{(f_j - np_j)^2}{np_j} = \sum_{j=1}^{k+1} \frac{f_j^2}{np_j} - n \qquad (4\text{-}30)$$

The smaller the chi-square value, the better is the goodness of fit. An estimate of Weibull parameters x_o and m can be obtained by minimizing the chi-square value with respect to x_o and m.

For a Weibull distribution,

$$\sum_{j=1}^{K+1} \frac{f_j^{\,2}}{p_j} = \left(n - \sum_{j=1}^{K} f_j\right) e^{\frac{z_K^m}{x_o}} + \frac{\sum_{j=1}^{K} f_j}{\left[e^{-\frac{z_{j-1}^m}{x_o}} - e^{-\frac{z_j^m}{x_o}}\right]} \tag{4-31}$$

By minimizing the right hand side of eqn. (4-31) with respect to x_o and m, these parameters may be estimated.

4.3.6 Method Suggested by Gumbel for Ungrouped Data (Ref. G. 2.12)

Gumbel makes use of extremal probability paper to plot the failure data. The mean rank is used for plotting position, so that

$$F(x_m) = \frac{m}{N + 1}$$

where N is the sample size

Then a reduced variable y is defined as

$$y = \ln\left[-\ln\left\{1 - F(x)\right\}\right] \tag{4-32}$$

so that

$$F(x) = 1 - e^{-e^{y}} \tag{4-33}$$

Comparing with equation (4-10), we see

$$\frac{x^{m}}{x_{o}} = e^{y}$$

and therefore $y = m \ln x - \ln x_{o}$ (4-34)

So a straight line plot is obtained on extremal probability paper, when y is plotted as ordinate and log x as abscissa. Of course, a straight line can only be obtained, if the failure data follows Weibull distribution. For $x = 1$, $y = - \ln x_{o}$

and hence

$$x_{o} = e^{-y} \qquad \text{at } x = 1.$$

For $y = 0$, $m \ln x = \ln x_{o}$

and $m = \left.\dfrac{\ln x_{o}}{\ln x}\right|_{\text{at } y = 0}$

Gumbel has also provided tables and charts to estimate the parameters. He defines

$$\lambda = \frac{1}{m} \tag{4-35}$$

TABLE 4.5

λ	$E(x)/\sigma$	λ	$E(x)/\sigma$	λ	$E(x)/\sigma$
0.01	78.534	0.15	5.688	0.60	1.621
.02	39.543	.20	4.366	.65	1.508
.03	26.542	.25	3.564	.70	1.408
.04	20.039	.30	3.024	.75	1.320
.05	16.135	.35	2.634	.80	1.242
.06	13.531	.40	2.337	.85	1.172
.07	11.669	.45	2.104	.90	1.109
.08	10.271	.50	1.913	.95	1.052
.09	9.183	.55	1.755	1.00	1.000
.10	8.312				

Table 4.5 provides the value of λ for various values of $E(x)/\sigma$, where $E(x)$ is the first moment or the mean and σ is the standard deviation for the Weibull distribution. The mean is given by

$$E(x) = x_o^{\lambda}\,\Gamma(\lambda + 1) \qquad (4\text{-}36)$$

So x_o can be obtained as

$$x_o = \left[\frac{E(x)}{\Gamma(\lambda+1)}\right]^{1/\lambda} \qquad (4\text{-}37)$$

Fig. 4-4 can also be used to obtain the value of λ, if the quotient $E(x)/\sigma$ is known. If the experimental data x_i $(i = 1,2,\ldots,n)$ is known, the arithmetic mean $\bar{x}$ is calculated as

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

and the standard deviation is computed as

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}}$$

The quotient $\frac{\bar{x}}{s}$ is computed and used instead of population value $\frac{E(x)}{\sigma}$

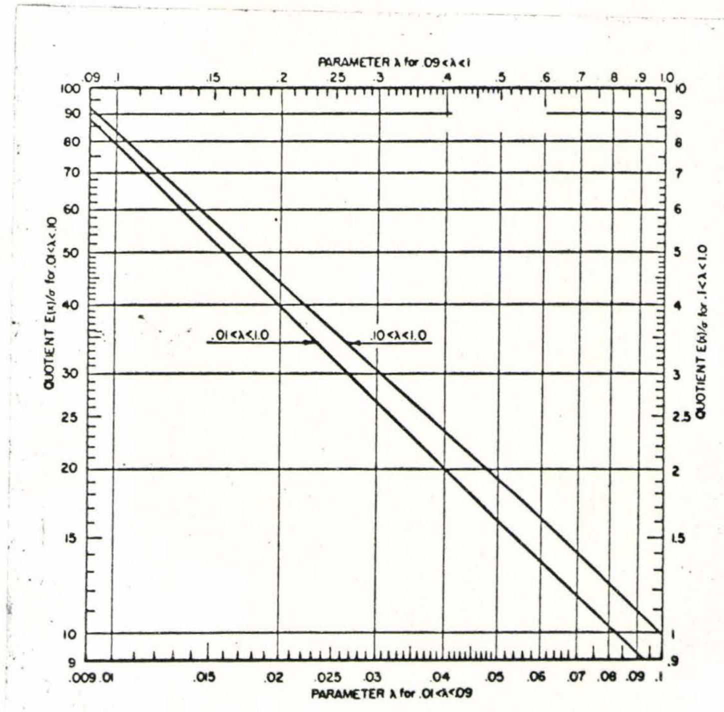


Fig. 4-4 Graph for Estimating λ

and the estimate $\hat{\lambda}$ of the parameter λ is obtained from Fig. 4-4 or from Table 4.5 by interpolation. To illustrate the precedure, we will use the same example as used in Sec. 4.3.2. Here the test was not truncated and all 20 relays were tested to failure. Relay 18 failed at 13.42×10^5 actuations, relay 19 failed at 18.07×10^5 actuations and relay 20 failed at 20.63×10^5 actuations. The data has been plotted on extremal probability paper and is shown in Fig. 4-5. A good fit has been obtained to a straight line.

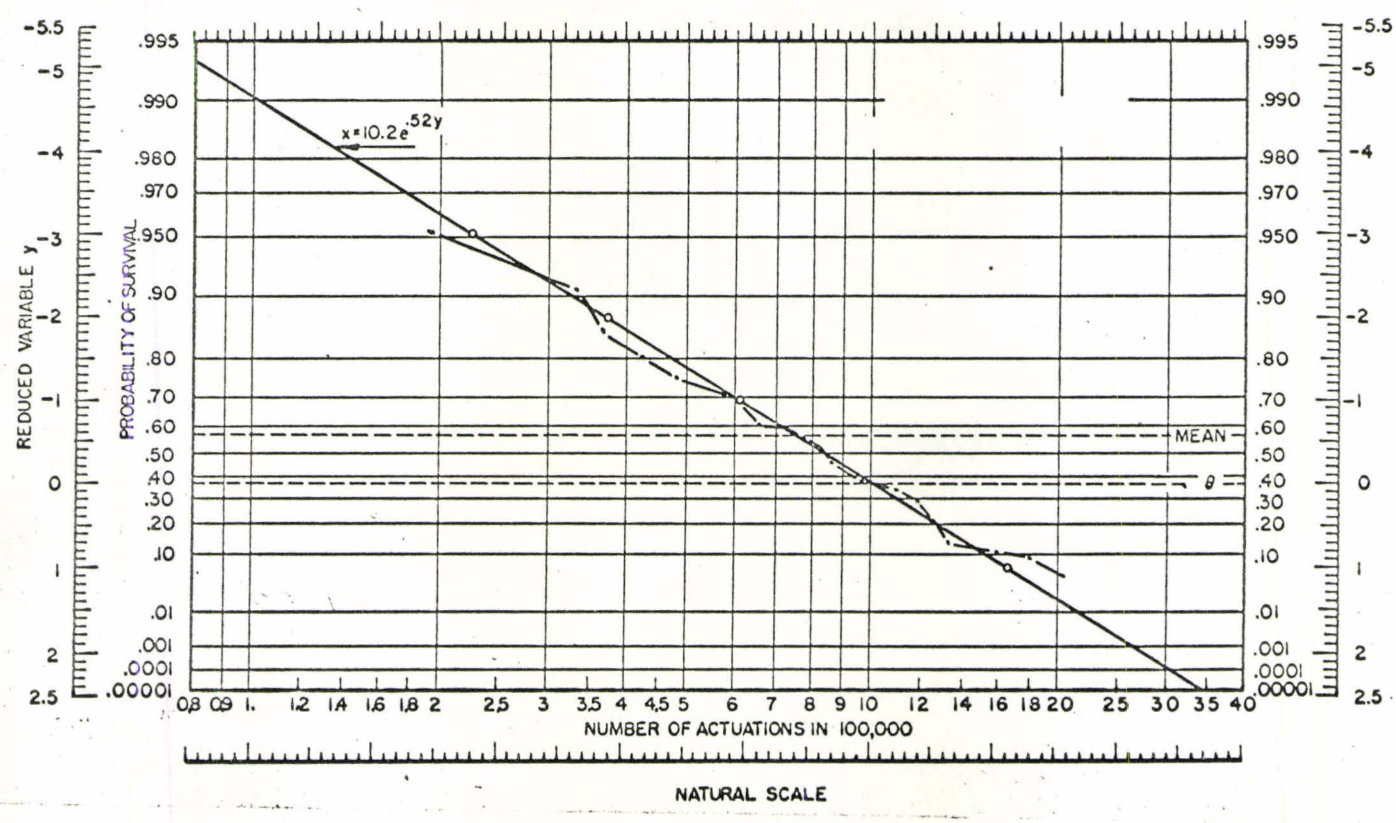


Fig. 4-5 Probability of Survival of a Relay Submitted to Actuation

The arithmetic mean of the sample is

$$\bar{x} = \frac{\sum_{i=1}^{20} x_i}{n} = \frac{180 \cdot 36 \times 10^5}{20} = 9 \cdot 018 \times 10^5 \text{ actuations.}$$

and the standard deviation of the sample is

$$s = \sqrt{\frac{\sum_{i=1}^{20} (x_i - \bar{x})^2}{19}} = 4 \cdot 900 \times 10^5 \text{ actuations.}$$

Hence

$$\frac{\bar{x}}{s} = 1.840$$

From Table 4.5 or from Fig. 4-4, we obtain

$$\hat{\lambda} = 0.52$$

and hence the estimate of scale parameter $m = \frac{1}{.52} = 1.923$

To obtain x_o, we use the values obtained from the sample in eqn. (4-37).

We get

$$x_o = \left[\frac{\bar{x}}{\Gamma(\hat{\lambda}+1)} \right]^{1/\hat{\lambda}}$$

$$= \left[\frac{9 \cdot 018 \times 10^5}{\Gamma(1 \cdot 52)} \right]^{1 \cdot 923}$$

$$= 0 \cdot 3581 \times 10^{12} \text{ actuations.}$$

The estimate of x_o differs from the value calculated earlier because here the test was not truncated and also due to approximations used by Gumbel.

4.3.7 The Method of Order Statistics for Ungrouped Data

This method has been suggested by Lieblein and Zelen (Ref. G.2.18) and will be discussed in Section 4.6.2.

4.4 Different Measures of Life Quality

We shall now derive the expressions for various measures of life quality of a product. It is advisable to use the notation suggested by Kao, since it is now widely used in reliability work. In Kao's notation, the Weibull c.d.f. becomes

$$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}} \tag{4-38}$$

$$\eta > 0,\ \beta > 0,\ t > 0$$

The Weibull p.d.f. becomes

$$f(t) = \left(\frac{\beta}{\eta}\right)\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{t}{\eta}\right)^{\beta}} \tag{4-39}$$

The expression for reliability is then

$$R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}} \tag{4-40}$$

The expression for hazard rate is

$$Z(t) = \left(\frac{\beta}{\eta}\right)^{\beta-1} \tag{4-41}$$

β is referred to as the Weibull slope and η is called the characteristic life. An expression for mean life μ (or first moment) can now be derived.

$$\mu = \int_0^\infty t\, f(t)\, dt$$

$$= \int_0^\infty t \left(\frac{\beta}{\eta}\right)\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{t}{\eta}\right)^{\beta}} dt$$

$$= \frac{\eta}{\beta}\Gamma\left(\frac{1}{\beta}\right)$$

$$= \eta\, \Gamma\left(\frac{1}{\beta}+1\right) \tag{4-42}$$

where the gamma function is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx \quad ; \quad n > 0$$

The variance (or the second moment) can also be similarly derived as

$$\sigma^2 = \eta^2 \left[\Gamma\left(\frac{2}{\beta} + 1\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \right] \tag{4-43}$$

The standard deviation is

$$\sigma = \eta \left[\Gamma\left(\frac{2}{\beta} + 1\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \right]^{1/2} \tag{4-44}$$

If we define the quantile of order p as ζ_p, then

$$p = \int_0^{\zeta_p} f(t) \, dt$$

We then have

$$\zeta_p = \eta \left(\ln \frac{1}{1-p} \right)^{1/\beta} \tag{4-45}$$

An expression for median life can now be obtained by setting $p = \frac{1}{2}$.

Median $\xi = \eta\ (\ln 2)^{1/\beta}$ (4-46)

The expressions derived here will be used in the later sections.

4.5 Sampling Plans Based on Weibull Distribution

Sampling plans based on Weibull distribution have been developed in recent years. Depending on the measure of life quality which is of importance, namely either the mean life or the hazard rate, two different types of sampling plans are available.

4.5.1 Acceptance Sampling Plan Based On Mean Life Criterion

This approach is taken when the mean life of the components is deemed more important. The plan is developed as follows (Ref. C.2.26). The c.d.f. for the Weibull distribution is

$$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

If we substitute

$$b = \frac{1}{\beta} \qquad (4\text{-}47)$$

we get

$$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{1/b}} \qquad (4\text{-}48)$$

Eqn. (4-42) for mean life then becomes

$$\mu = \eta\,\Gamma(b + 1) \qquad (4\text{-}49)$$

Let $þ'$ = the probability of an item failing before the end of test time t

Then

$$þ' = 1 - e^{-\left(\frac{t}{\eta}\right)^{1/b}} \qquad (4\text{-}50)$$

This can be rewritten as

$$\frac{1}{1 - þ'} = e^{\left(\frac{t}{\eta}\right)^{1/b}}$$

Taking natural logarithms,

$$-\ln\,(1-þ') = \left(\frac{t}{\eta}\right)^{1/b}$$

and hence

$$\left[-\ln\,(1-þ')\right]^{b} = \frac{t}{\eta}$$

which yields

$$\eta = \frac{t}{\left[-\ln\,(1-p')\right]^{b}} \tag{4-51}$$

From eqn. (4-49), we get

$$\eta = \frac{\mu}{\Gamma(b + 1)} \tag{4.52}$$

Equating equations (4-51) and (4-52), we get

$$\frac{\mu}{\Gamma(b+1)} = \frac{t}{[-\ln(1-p')]^{b}}$$

$$\text{or} \quad \frac{t}{\mu} = \frac{[-\ln(1-p')]^{b}}{\Gamma(b+1)} \tag{4-53}$$

$$\text{or} \quad \frac{t}{\mu} = \frac{\exp\{\ln[-\ln(1-p')]^{b}\}}{\Gamma(b+1)}$$

$$\text{or} \quad \frac{t}{\mu} = \frac{\exp\{b\ln[-\ln(1-p')]\}}{\Gamma(b+1)} \tag{4-54}$$

Values of the expression $-\ln\left[-\ln(1-p')\right]$ can be obtained from the table of the inverse of the cummulative probability function of extremes (Ref. F.3.11, Table 2, pp. 19-25). Again from equation (4-53), we get

$$\left(\frac{t}{\mu}\right)^{\beta} = -\frac{\ln(1-p')}{\left[\Gamma(b+1)\right]^{\beta}}$$

$$\text{or} \quad p' = 1 - \exp\left\{-\left(\frac{t}{\mu}\right)^{\beta}\left[\Gamma(b+1)\right]^{\beta}\right\} \qquad (4\text{-}55)$$

Using eqns. (4-54) and (4-55), Kao (Ref.C.2.26) has tabulated values of $\frac{t}{\mu}$ for different values of p' and values of p' for different values of $\frac{t}{\mu}$. The inspection is done on an attribute basis and the number of failed items y at the end of test time t is counted. If the lot size N is large compared to the sample size n, the cumulative binomial distribution may be used to obtain the probability of acceptance P (A).

$$P(A) = P(y \leqslant c) = \sum_{y=0}^{c} \binom{n}{y} p'^{\,y} (1-p')^{n-y} \qquad (4\text{-}56)$$

If the sample size n is relatively large compared to the lot size N, the cumulative Poisson distribution must be used.

$$P(A) = P(y \leq C) = \sum_{y=0}^{C} \frac{(np')^{y}}{y!} e^{-np'} \qquad (4\text{-}57)$$

To set up a sampling plan, a convenient testing time t is first selected. Then the ratio $\frac{t}{\mu}$ is computed for mean life μ_1 corresponding to ARL (Acceptable Reliability Level) and mean life μ_2 corresponding to LTFR (Lot Tolerance Failure Rate). Then from tables of $\frac{t}{\mu}$ and p', two values of p' are obtained. Then from the tables of cumulative binomial distribution (Ref. F.3.6 or Ref. F.3.8) or cumulative Poisson distribution (if Poisson distribution is used), the sample size n and acceptance number C can be obtained for the two specified values of the probability that the sample will accept the lot, P (A). These two values of P (A) are the producer's risk α and consumer's risk β. An O.C. Curve may also be plotted. Kao (Ref. C.2.26) has provided tables for designing such sampling plans.

In summary, the plan works as follows -

(i) Select a random sample of n items from the lot.

(ii) Put the sample items on life test for some preassigned period t.

(iii) Denote by y the number of failures observed prior to time t.

(iv) Accept the lot if $y \leq C$, the acceptance number and if $y > C$, reject the lot.

Double and multiple sampling plans can also be designed.

4.5.2 Acceptance Sampling Plan Based On Hazard Rate Criterion

This type of sampling plan is used when the hazard rate during any part of the lifetime of the component is deemed to be the most important life quality. The plan is developed as follows (Ref. C.2.22, C.2.23 and C.2.24). In eqn. (4-41), the hazard rate for Weibull distribution was given as

$$Z(t) = \left(\frac{\beta}{\eta}\right)\left(\frac{t}{\eta}\right)^{\beta-1}$$

Multiplying both sides by $\frac{t}{\beta}$ gives

$$\frac{t\,Z(t)}{\beta} = \left(\frac{t}{\eta}\right)^{\beta} \qquad (4\text{-}58)$$

The probability of an item failing before the end of test time t is given by

$$p' = F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

Substituting from eqn. (4-58), we get

$$p' = 1 - \exp\left[-\frac{t\,Z(t)}{\beta}\right] \tag{4-59}$$

and therefore

$$t\,Z(t) = -\,\beta \ln\,(1-p') \tag{4-60}$$

or

$$t\,Z(t) = \beta \exp\left\{\ln\left[-\ln(1-p')\right]\right\}$$

Values of the expression $-\ln\left[-\ln(1-p')\right]$ are obtained from the table of the inverse of the cumulative probability function of extremes (Ref. F.3.11, Table 2, pp. 12-25).

Here the lot quality of interest is the hazard rate at a life of t hours. Acceptable hazard rate $Z_1(t)$ and rejectable hazard rate $Z_2(t)$ are specified. Using eqn. (4-59), the two values of p' at the two specified hazard levels are computed. For specified consumer's risk α and producer's risk β, the sample size n and acceptance number C can be obtained from eqns. (4-56) or (4-57). Kao (Ref. C.2.24) has provided tables for obtaining p' for different values of $t\,Z(t)$ and for obtaining t Z(t) for different values of β'. He has also provided tables of sampling plans for a number of values of β.

4.6 Use of Weibull Distribution On Fatigue Studies of Mechanical Elements

4.6.1 Theoretical Considerations

The Weibull distribution has been extensively used as a model for the fatigue life of mechanical elements. Weibull in his original paper (Ref. F.1.67) applied this distribution to the analysis of fatigue data. Failures in most mechanical elements exposed to dynamic loading are caused due to fatigue. The failure of rolling contact bearings, when operated under proper conditions (sufficient lubrication, absence of dust and foreign material etc.) is due to fatigue and is manifested as flaking of the raceways, cracks and fractures etc. (Ref. G.1.11). The failure of gears are also mostly due to fatigue. Bending fatigue failures are caused by gear tooth being stressed many times. Pitting and spalling of the gear flanks occur due to fatigue caused by repetition of compressive loading. Failures of shafts, machine structural elements etc. are also frequently caused by fatigue. Due to the importance of predicting fatigue life, various methods have been proposed for representing fatigue reliability data. The Weibull distribution has been found to be a very good mathematical model for approximating fatigue phenomena. The theoretical justification for using the Weibull distribution for fatigue studies has been given by Freudenthal and Gumbel (Ref. G.2.7) on the basis that fatigue is an extreme-value phenomenon, related in some manner to the strength at the weakest point in the material under stress.

This leads to a distribution of the type suggested by Weibull. Lieblein and Zelen (Ref. G.2.18) have stated that this explanation has not received universal acceptance. Nevertheless, various investigators have shown that the Weibull distribution represents experimental fatigue data quite satisfactorily and hence there are practical reasons favouring the use of this distribution. Gumbel (F.1.28) has stated that the probability theory developed by Fisher and Tippet (F.1.22) and Gnedenko postulates the following conditions for the existance of a Weibull distribution:

A Number of independent occurances will asymptotically approach a Weibull distribution (generally with finite lower bound) if

(i) Each of these occurances is the earliest one of a very large parent population of mutually independent, actual or potential occurances. In fatigue testing, each observed failure must be the earliest one of a very large population of mutually independent potential (or actual but subcritical)failures.

(ii) The possible range of values assumed by the variate characterizing the occurances has a lower bound that may be finite or zero. In fatigue testing the possible range of fatigue lives has a lower bound which is either a finite minimum life or zero.

(iii) The parent distribution G(L) of potential occurances from which the observed earliest occurances are derived must satisfy the following necessary and sufficient condition

$$\lim_{L \to L_b} \frac{G[C(L-L_b)]}{G[L-L_b]} = C^e$$

Here L is the endurance life of a potential (subcritical failure), L_b is the lower bound of L, C is an arbitarily selected fixed quantity and e is a positive constant.

Gumbel (Ref. F.1.28) has shown that, if the above conditions are satisfied, the following Weibull distribution is asymptotically approached:

$$S = \exp\left[-\left(\frac{L-L_b}{L_o-L_b}\right)^e\right]$$

where L_o is the scale parameter.

The first two conditions intuitively fit the customary model of a rolling contact which fails in fatigue, such as flaking of rolling contact bearing raceways and pitting of gear tooth flanks. It is generally assumed that such a contact will fail as a result of subcritical damage accumulating at numerous mutually independent weak points within the metal, which are stressed in turn as the rolling contact traverses over them. The weakest of these weak points will precipitate macroscopic failure of the specimen. Because of the small stressed volume surrounding a rolling contact, the assumption of independence of failure probabilities of different weak points is plausible.

4.6.2. Analysis of Rolling Contact Bearing Life Using Weibull Distribution

Statistical methods have been used in analysis of rolling contact bearing failure data for a long time and the Weibull distribution has been used by a number of investigators. It is advisable to use the notations used in the bearing industry. The Weibull cumulative distribution function is stated as

$$F = 1 - \exp\left[-\left(\frac{L}{L_o}\right)^e\right] \; ; \; L \geq 0 \qquad (4\text{-}62)$$

$$S = \exp\left[-\left(\frac{L}{L_o}\right)^e\right] \; ; \; L \geq 0 \qquad (4\text{-}63)$$

The notations used are as follows -

L = life in stress cycles or shaft revolutions

L_o = Scale parameter

e = Weibull slope

F = cumulative probability of failure

S = cumulative probability of survival or reliability

L_b = location parameter, which is generally assumed to be zero.

L_{50} = Median life, which is the life when 50% of the bearings have failed

and

L_{10} = design life or rating life, when 10% of the bearings have failed.

Lieblein and Zelen (Ref. G.2.18) analyzed the failure data of about 5000 deep-groove ball bearings manufactured by a number of manufacturers. The Weibull parameters were estimated by the method of order statistics. The results obtained were as follows -

Mean value of Weibull slope e = 1.51

Median value of Weibull slope e = 1.43

$$\text{Ratio } \frac{L_{50}}{L_{10}} = 4.08$$

But ASA standard B 3.11 states that

$$\frac{L_{50}}{L_{10}} \approx 5.0$$

and this value is widely used. The rating life L_{10} is obtained for any bearing using the relationship

$$L_{10} = \left(\frac{C}{P}\right)^n \text{ million revolutions} \qquad (4\text{-}64)$$

where C = Basic Dynamic Load Rating (lbs.)

= the load which will give a rating life of 1 million revolutions

P = constant bearing load (lbs.)

The exponent n is given as

$n = 3$ for ball bearings

and

$n = 10/3$ for roller bearings.

Values of C has been listed by the bearing manufactures for all types of bearings. For most purposes, the rating life is used. But it must be remembered that the probability of failure at rating life is 10% or in other words, the reliability is 90%. For calculating the life for higher reliability, the following method may be used -

From eqn. (4-63),

$$S = \exp\left[-\left(\frac{L}{L_o}\right)^e\right]$$

For 90% probability of survival

$$0.90 = \exp\left[-\left(\frac{L_{10}}{L_o}\right)^e\right] \tag{4-65}$$

Since precise estimates of Weibull slope e are not available, the relationship $L_{50}/L_{10} = 4.08$ may be used.

Then we can write

$$0.50 = \exp\left[-\left(\frac{4.08\ L_{10}}{L_o}\right)^e\right] \tag{4-66}$$

Equations (4-65) and (4-66) have been solved by Shube (Ref. G.2.25) to yield

$$e = 1.34$$

and $L_o = 5.35\ L_{10}$.

Hence eqn. (4-63) can be written as

$$S = \exp\left[-\left(\frac{L}{5.35\ L_{10}}\right)^{1.34}\right] \tag{4-67}$$

So for a particular application, where any specified reliability (or probability of survival) S_1 is warranted, L_{10} life is obtained from eqn. (4-64) and solving equation (4-67), the required life can be obtained. If the ratio $L_{50}/L_{10} = 5$ is used, then we get

$$e = 1.17$$

and $L_o = 6.84\ L_{10}$

Eqn. (4-63) then becomes

$$S = \exp\left[-\left(\frac{L}{6.84\ L_{10}}\right)^{1.17}\right] \tag{4-68}$$

Equations (4-67) and (4-68) can be expressed in the alternative form

$$\frac{L}{L_{10}} = (9.49 \ln S)^{0.746}$$

and

$$\frac{L}{L_{10}} = (9.49 \ln S)^{0.854}$$

Shube (Ref. G.2.25) has plotted the ratio $L/_{L_{10}}$ against F and the graph is shown in Fig. 4-5. The life L for a given reliability requirement S can be obtained from the graph, knowing the value of L_{10} from eqn. (4-64).

We had assumed that the bearing failure data follows Weibull distribution at any part of the lifetime. But Tallian (Ref. G.2.27) has shown that in the early failure part of the bearing life, the experimental data does not fit the Weibull line (on Weibull probability paper) closely. The Weibull line yields a much lower value of failure life than is observed by actual experimentation. This discrepancy is shown in Fig. 4-6 and Fig. 4-7. In these figures, $\ln \ln\left(\frac{1}{s}\right)$ is plotted on the ordinate and ln L on the abscissa. Tallian has suggested the following mathematical causes of deviations from Weibull distribution -

(i) The specific assumption of location parameter $L_b = 0$ may be incorrect.

(ii) Independent potential or subcritical failures are not sufficiently numerous to warrant use of the asympotic Weibull form.

(iii) Macroscopic failure is not instantaneous.

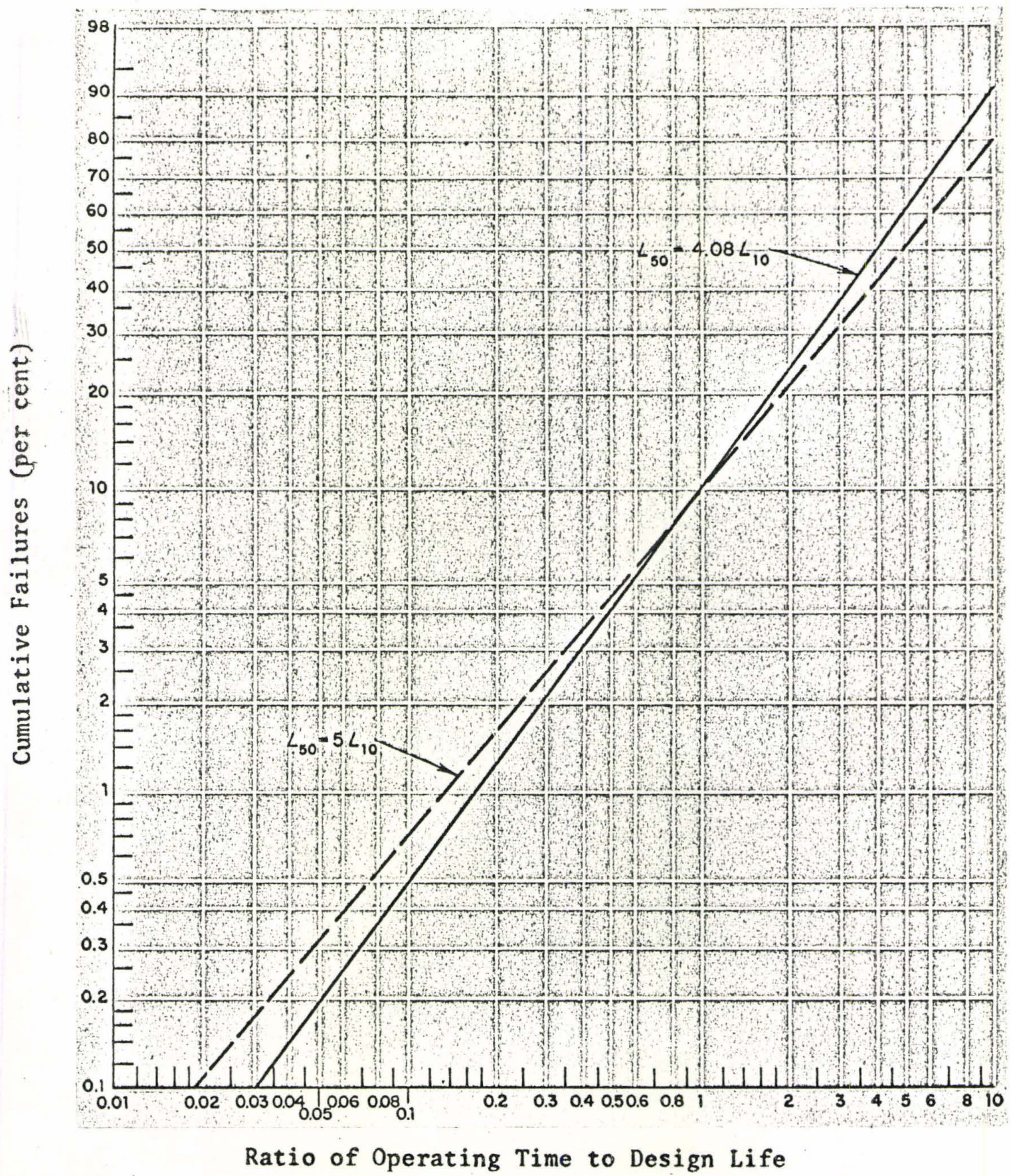


Fig. 4-5 Rolling Contact Bearing Failure Characteristics Plotted on Weibull Probability Paper

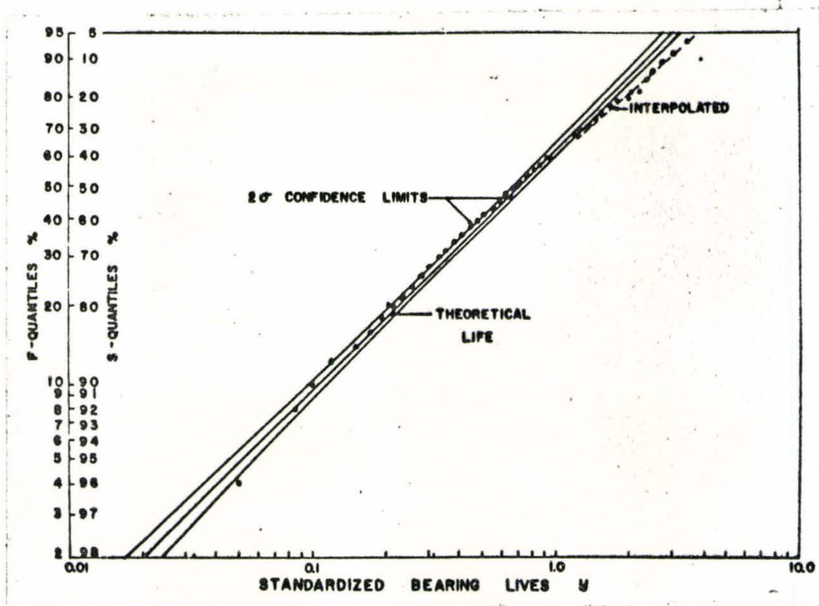


Fig. 4-6 Life Distribution of Rolling Contact Bearings - General

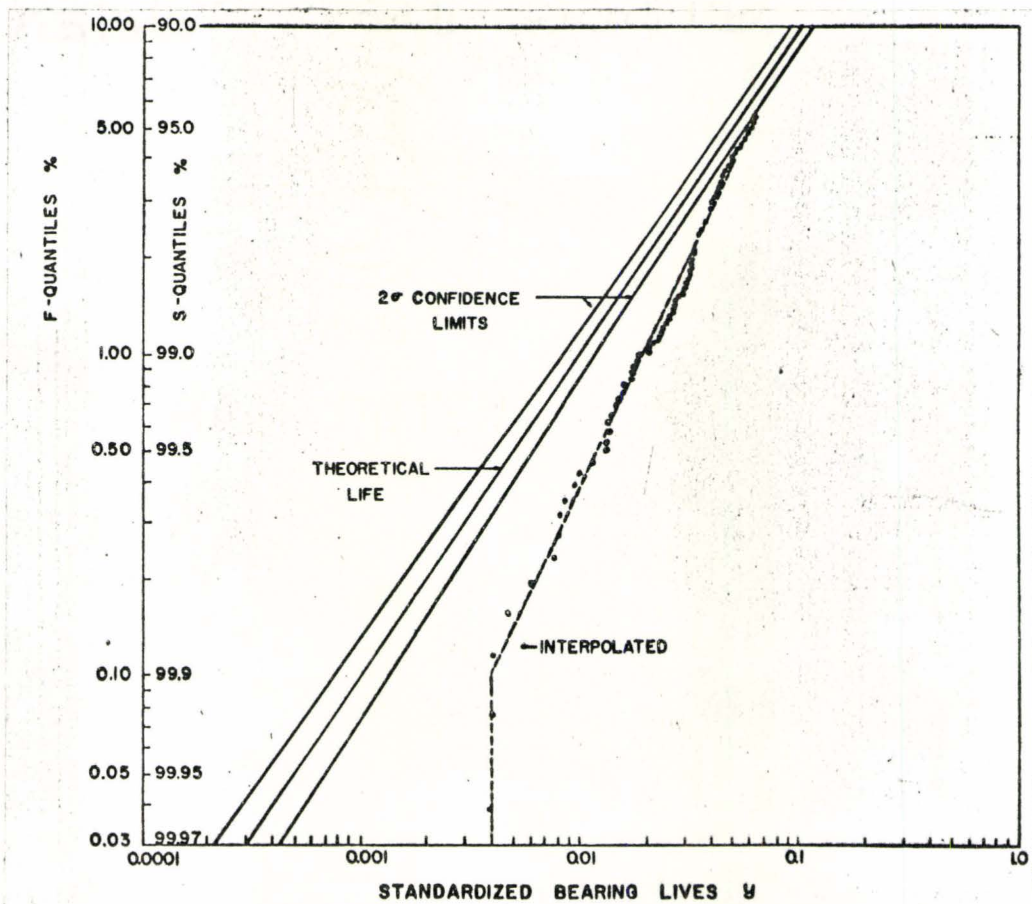


Fig. 4-7 Life Distribution of Rolling Contact Bearings in the Early Failure Region

The mechanism of fatigue in metals has been explained by some investigations (Ref. G.2.27, pp. 192) as a succession of phases of structural transformation as follows -

(1) A relatively short phase of work hardening occurs, which in itself does not necessarily lead to fatigue failure.

(2) Concurrently with work hardening, and after its termination, a crack in excess of critical size forms.

(3) The crack propagates until macroscopic failure occurs.

Based on this mechanism, Tallian (Ref. G.2.27) hypothesized that Phases (1) and (2) are concurrent and life up to the end of Phase (2) is Weibull distributed with zero lower bound. Phase (3) requires a finite length of time since a crack in rolling contact fatigue initiates at a substantial depth beneath the surface and must therefore traverse a finite distance in the metal before it can reach the surface and produce fatigue spalling. Phase (3) life will therefore have a finite lower bound.

On the above hypothesis, the experimental results in the region of early failures can be explained by assuming that in this region, the excess experimental life y_E is Phase (3) life and as such it is dependent on Phase (2) life y_T. The reliability equation can then be modified as

$$\frac{L}{L_{10}} = \left[\frac{\ln\left(\frac{1}{S}\right) + y_E}{\ln\left(\frac{1}{0.9}\right)} \right]^{1/e}$$

Tallian (Ref. G.2.27) has indicated some methods of estimating y_E.

5. CONCLUDING REMARKS

We have briefly reviewed some methods for determination of reliability of components and also sampling plans to assess the reliability of components purchased by outside suppliers.

In designing for high reliability, reliability should be considered right from the preliminary stage and a reliability control program should be effective during the entire design stage and extend beyond it to prototype approval and production. A number of different approaches should be tried and a tentative design should be made. Necessary testing for determining the reliability of the components should be done, if such data is not available, and the reliability of the entire machine or system should be calculated. Reliability analysis of alternative designs using different arrangement of components or using different components themselves should be made. Finally, a design best satisfying the requirements of reliability, economy etc. should be selected. Then the detailed design work can be done and a precise reliability analysis of the final design can be made. Finally the prototype should be subjected to life tests and the test results should be compared with the reliability requirements and modifications and redesign should be made as necessary. Only then should the design be released for production.

Space vehicles are designed with strict reliability supervision and control. Most aircraft manufacturers also base their design on reliability theory. Many automobile manufacturing firms are paying increasing attention to reliability and some of them have done substantial amount of reliability testing. Some of them have remarked about the high cost of reliability testing due to frequent change in automobile models. But it has also been remarked that if a large volume of test data on older models are available, testing for subsequent models need not be so rigorous and testing expense may be reduced. Reliability studies on rolling contact bearings have been made for a long time and the life of a bearing associated with a certain probability of survival can be easily obtained from the information supplied by the manufacturers.

A similar approach has also been taken by the manufacturers some electrical components, such as electric bulbs and vacuum tubes. More and more designers and producers are getting conscious of reliability and are setting up reliability programs. There is a huge potential for use of reliability theory for mass produced items and this would be beneficial both for the producer in making his product more competitive and for the user in getting some precise idea of the product quality during its lifetime. One difficulty with specifying reliability is the wide variety of environmental conditions a product is exposed to. In order to overcome this, a certain product may be tested under some stardardized environment and load. The automobile tire of a certain size may be tested in this manner and life specified. The consumer can then obtain a more precise idea of the quality of the particular tire and may also compare the reliability data given by different manufacturers.

The Weibull distribution seems to be the most powerful and most modern mathematical model for mechanical components. It is a safe practice to assume the experimental data to follow the Weibull distribution and obtain estimates of the parameters. However, up to the present time, the Weibull distribution is not as well developed theoretically as the exponential or normal distribution. There are some aspects, such as plotting position and confidence intervals, which are still controversial. Hence, if, on analyzing the experimental data, the estimate of the shape parameter is found to be close to unity, the exponential distribution may be used; and if the estimate of the shape parameter is found to be nearly 4, the normal distribution may be used, thus permitting use of the better developed statistical theory for these distributions.

In conclusion, it may be stated that a better understanding and acceptance of the principles of reliability will be beneficial both for the industry and the consumer.

APPENDIX A-1

The FORTRAN program for estimating the parameters of the Weibull distribution by the method of least squares is given below

```
C     ESTIMATION OF SCALE AND SHAPE PARAMETERS OF WEIBULL DISTRIBUTION
C     LOCATION PARAMETER ASSUMED TO BE ZERO
      DIMENSION X(100),FX(100), ABSCA(100), ORD(100), YEST(100), DEVX(10
     10), DEVY(100), DATA(100)
      READ 11, N,M
      READ 12, (X(I),I=1,N)
      READ 13, (DATA(I),I=1,N)
   11 FORMAT(2I4)
   12 FORMAT(5(F12.3,4X))
   13 FORMAT(5(F9.3,7X))
      DO 15 I=1,N
   15 FX(I)=DATA(I)/(FLOAT(N)+FLOAT(M)+1.0)
C     CALCULATE TRANSFORMED COORDINATES
      DO 21 I=1,N
      ABSCA(I)=ALOG(X(I))
   21 ORD(I)=ALOG(ALOG(1.0/(1.0-FX(I))))
C     FIT STRAIGHT LINE BY LEAST SQUARES METHOD
      SUMX1=0.0
      SUMX2=0.0
      SUMY1=0.0
      SUMY2=0.0
      SUMXY=0.0
```

```
      DO 25 I=1,N
      SUMX1=SUMX1+ABSCA(I)
      SUMX2=SUMX2+((ABSCA(I))*(ABSCA(I)))
      SUMY1=SUMY1+ORD(I)
      SUMY2=SUMY2+((ORD(I))*(ORD(I)))
      SUMXY=SUMXY+((ABSCA(I))*(ORD(I)))
   25 CONTINUE
      A=(((FLOAT(N))*SUMXY)-(SUMX1*SUMY1))/(((FLOAT(N))*SUMX2)-(SUMX1*SU
     1MX1))
      B=((SUMX2*SUMY1)-(SUMX1*SUMXY))/(((FLOAT(N))*SUMX2)-(SUMX1*SUMX1))
      XMEAN=SUMX1/FLOAT(N)
      YMEAN=SUMY1/FLOAT(N)
      SUM5=0.0
      SUM6=0.0
      SUM7=0.0
      DO 31 I=1,N
      YEST(I)=(A*ABSCA(I))+B
      DEVX(I)=ABSCA(I)-XMEAN
      DEVY(I)=ORD(I)-YMEAN
      SUM5=SUM5+(DEVX(I)*DEVY(I))
      SUM6=SUM6+(DEVX(I)*DEVX(I))
      SUM7=SUM7+(DEVY(I)*DEVY(I))
   31 CONTINUE
      CORR=SUM5/((SUM6*SUM7)**0.5)
      PRINT 33,
   33 FORMAT(73H NO.          X          LN(X)        F(X)    LN(LN(1/(1-F(X)))
     1    EXPTD. VALUE   )
      DO 35 I=1,N
   35 PRINT 36, I, X(I), ABSCA(I), FX(I), ORD(I),YEST(I)
```

```
   36 FORMAT(I4,2X,F12.3,2X,F9.5,2X,F7.4,2X,F12.5, 9X,F7.4)
      PRINT 41, CORR
   41 FORMAT(35H THE COEFFICIENT OF CORRELATION IS ,F9.5)
      SCALE=EXP(-B)
      PRINT 51, SCALE
   51 FORMAT(24H THE SCALE PARAMETER IS ,E14.5)
      PRINT 53,A
   53 FORMAT(24H THE SHAPE PARAMETER IS ,F12.5)
      STOP
      END
$ENTRY
  17   3
  190000.0       334000.0       365000.0       420000.0       472000.0
  589000.0       610000.0       662000.0       792000.0       840000.0
  850000.0       900000.0       960000.0      1102000.0      1195000.0
 1240000.0      1303000.0
       1.0            2.0            3.0            4.0            5.0
       6.0            7.0            8.0            9.0           10.0
      11.0           12.0           13.0           14.0           15.0
      16.0           17.0
```

APPENDIX A-2

The FORTRAN program for estimating the parameters of the Weibull Distribution by the method of maximum likelihood for ungrouped data is given below

```
C      MAXIMUM LIKELIHOOD ESTIMATES OF WEIBULL PARAMETERS
       DIMENSION X(100)
       READ 11,N,K
       READ 12, (X(I),I=1,N)
    11 FORMAT(2I4)
    12 FORMAT(5(F12.3,4X))
       PRINT 16,
    16 FORMAT(78H ITERATION NO.     PARAMETER M     X(0) FROM EQN.(4-24)
      1 X(0) FROM EQN.(4-25) )
       NNN=1
       SLOPE=1.834
    21 SUM1=0.0
       SUM2=0.0
       SUM3=0.0
       DO 23 I=1,N
       SUM1=SUM1+((X(I))**(SLOPE))
       SUM2=SUM2+(((X(I))**(SLOPE))*(ALOG(X(I))))
    23 SUM3=SUM3+ALOG(X(I))
       KKK=K-N
       DE=FLOAT(KKK)*((X(N))**(SLOPE))
       DF=((DE)*ALOG(X(N)))
       DG=(FLOAT(N))/SLOPE
```

```
      GH=(1.0/FLOAT(N))*(SUM1+DE)
      GI=(SUM2+DF)
      GJ=(DG+SUM3)
      GK=((GI)/(GJ))
      PRINT 26,NNN,SLOPE,GH,GK
   26 FORMAT(7X,I5,8X,F8.4,10X,E14.5,12X,E14.5)
      GL=ABS(GH-GK)
      IF (GL-0.001)61,61,32
   32 IF (NNN-50)34,34,61
   34 NNN=NNN+1
      SLOPE=SLOPE+0.0001
      GO TO 21
   61 STOP
      END
$ENTRY
  17   20
  190000.0        334000.0        365000.0        420000.0        472000.0
  589000.0        610000.0        662000.0        792000.0        840000.0
  850000.0        900000.0        960000.0       1102000.0       1195000.0
 1240000.0       1303000.0
```

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