STABILITY OF CHANNEL COLUMNS

STABILITY OF OPEN THIN WALLED CHANNEL COLUMNS

By

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SCOPE AND CONTENTS:

This thesis deals with the analytical and experimental study of buckling strength, of thin walled channel struts, of different geometrical dimensions. The influence of the dimensions of the columns on the buckling strength has been studied.

The experimental work consisted of testing different channels of thin sheeting to failure. Comparison has been made with the previous work done and a comparison is made between the theoretical predicted values and the experimental results. The Appendix includes detailed mathematical procedure and matrices formulations.

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ABSTRACT

The different modes of buckling failure of a compression member in the form of an open thin-walled channel cross section are considered.

The member may lose stability in one of two ways, namely overall buckling and local buckling, depending on the length-web ratio of the member. For local buckling, the channel section is treated as a collection of plates, subjected to inplane loading. The buckling load for such a plate system is determined by the matrix method of analysis.

Depending on the flange-web ratio, two cases are considered. For the large flange-web ratios, the web and flanges of the channel are treated as a system of interconnected plates. It is assumed that their common edges remain undeflected under load. When the flange-web ratio is small, the flanges can be considered as lips to the web plate and are treated to act as elastic support for the web plate. The buckling load is determined in each case for different values of flange-web ratios and length-web ratios. A comparison is made with the previous work done.

A series of tests are carried out to verify the theoretically calculated values for different buckling conditions. Experimental results showed a satisfactory agreement with theory and followed closely the buckling behaviour predicted theoretically.

NOTATIONS

a, b, c, d	Numerical coefficients, distances
A	Cross sectional area
A_r	Cross sectional area of rib, stiffener
C	Torsional rigidity ($C = GJ$)
C_1	Warping rigidity ($C_1 = EC_w$)
C_w	Warping constant
D	Flexural rigidity of plate [$D = Et^3/12(1 - \nu^2)$]
E	Modulus of Elasticity
[F]	Field transfer matrix
G	Modulus of elasticity in shear
h	Thickness of plate, distance, height
I_x, I_y, I_z	Moment of inertia of a plane area with respect to x, y and z axes
I_o	Polar moment of inertia
J	Torsion constant
K, k	numerical factor, buckling factor [$k = \sigma_{cr}/\sigma_e$]
l	length of plate
m, n	Integers, numerical values, numbers of half waves of buckling curve.
M_η	Bending moments, couple at η $$M_\eta = -\frac{D}{b^2}\left(\frac{\partial^2 w}{\partial \eta^2} + \nu \frac{\partial^2 w}{\partial \xi^2}\right)$$
M^*_η	$= -\frac{b^2}{D} M_\eta$

p.q.	Intensity of load distribution
P_e	Euler buckling load for columns $P_e = \frac{\pi^2 EI}{\ell^2}$
P_{cr}	Critical buckling load
[P]	The product field boundaries matrix for more than one field
Q_η	Shearing force at η $Q_\eta = -\frac{D}{b^3}\left[\frac{\partial^3 w}{\partial \eta^3} + (2 - \nu)\frac{\partial^3 w}{\partial \eta \partial \xi^2}\right]$
Q^*_η	$= -\frac{b^3}{D} Q_\eta$
[R]	Rib matrix, matrix expressing transition conditions across a rib
[S]	Support condition matrix
t	Thickness
[T]	Thickness change matrix
u, v, w	Displacements in the x, y and z directions
x, y, z	Rectangular co-ordinates

α	Numerical factor [$\alpha = a/b$]
β, γ	Constant quantities
γ_B	Numerical factor [$\gamma_B = IE/b.D$]
γ_D	Numerical factor [$\gamma_D = J.G/b.D$]
γ_w	Numerical factor [$\gamma_w = C_w.E/b^3.D$]
γ_p	Numerical factor [$\gamma_p = I_p/b^3 t$]
δ	Numerical factor [$\delta = A_r/b.t$]
ξ, η, ζ	Rectangular co-ordinates refered to unit dimensions
ϕ	Equivalent spring constant for rib [$\phi = \gamma_B\theta^4 - k_r\delta\pi^2\theta^2$]
$\{\Phi\}$	State vector
ψ	Torsional spring constant [$\psi = \gamma_w\theta^4 + (\gamma_D - k_r\pi^2\gamma_p)\theta^2$]
θ	Numerical factor [$\theta = m\pi/\alpha$]
ν	Poisson's ratio
ρ	Radius of curvature
σ	Unit normal stress
σ_{cr}	Critical buckling stress
σ_e	Euler buckling stress [$\sigma_e = \pi^2 D/b^2 t$]

CHAPTER 1

INTRODUCTION

This research is mainly directed to establish the true buckling behaviour of a compression member of thin-walled channel cross section, statically loaded at the ends, considering all the possible modes of buckling failure. For simplicity, only monosymmetric channel sections are considered.

The modes of buckling considered are as follows:

1. Overall flexural buckling about the weak axis (Euler buckling) [14]*.
2. Coupled overall torsional and flexural buckling [14].
3. Local buckling
   a. of the flange and web as a connected plate system [5].
   b. of the web alone.

There is no unified analysis to predict all types of failure. All existing methods of analysis assume certain conditions on the deformation of the member, and then deduce the buckling load.

In the overall flexural buckling mode of stability

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* Numbers in square brackets refer to the list of references.

it is assumed that the cross-section is nondeformable during buckling and no torsional deformation takes place. For convenience we shall refer to this type of buckling as Euler type buckling and the buckling load as Euler buckling load. In the torsional flexural mode of buckling, it is also assumed that the cross section is nondeformable. However, rotation due to torsional deformation takes place.

For short members, the assumption of nondeformable cross section becomes unrealistic, and the instability mode changes from overall buckling to local buckling. For local buckling of the member, the buckling mode falls broadly into two catagories, depending on the flange-web ratio.

When the flange-web ratio is small, local buckling occurs at the web, with the effect of the flanges acting as lips, lending elastic supports to the web. When the flange-web ratio is large, local buckling occurs at both web and flanges as a plate system.

Therefore, the buckling mode depends on the geometry of the column, namely the length-web width ratio (α), and the flange-web ratio (η). For long columns where flexural stiffness is relatively weak, the overall buckling mode of failure will take place (Euler buckling). For short columns, local buckling criteria takes place due to deformation of the cross section.

The actual buckling load is given by the minimum of all critical buckling loads calculated by various assumed modes of failure.

The overall buckling of the column is given by the well known Euler studies [14], where he considered the flexural buckling mode of failure. The torsional overall buckling was studied [1] and [14], which allows the cross section to rotate during buckling deformations.

Kimm [5], expanded the work done in plates using the differential equation approach, to treat the channel cross section as a web plate with two overhanging flange plates, assuming that the corners of the channel section remain straight and undisplaced during the loading process. The buckling load is then claculated for different flange-web ratios.

Bleich [1], has considered Kimm's assumptions, he related the flange plate with the web plate by the proper restraining factor and obtained similar results. A state-of-art review of the local stability of such members is given by Bulson [2].

In the present analysis, the coupled torsional-flexural overall buckling is not considered due to the fact that the critical load calculated considering torsional-flexural overall buckling is generally higher than the critical load calculated based on flexural buckling [14], or regarding local buckling.

The determination of buckling load for overall flexural buckling of a compression member is well known and needs little explanation. Therefore the present work is focused on local buckling behaviour of the channel section under compression. The critical load for local buckling is determined using the matrix transfer method of analysis.

The matrix transfer method was first applied to vibrations problems by Schnell [11] and Fuhrke [3], and is also known by the name Method of Influence Coefficients [12], [13].

Marguerre [6], applied the matrix transfer method to the buckling problem and gave a detailed comparison of the mathematical development of the vibration and the buckling of beams problem.

The method has been applied to study the stability of an I shape plate girder by Scheer [10].

In this thesis the matrix transfer method is applied to study the stability of an open channel cross section. Two types of local instability are considered depending on the values of flange-web ratio of the section.

Firstly, the buckling load is determined considering the local buckling of web plate and the overhanging flange plates, as a plate system, assuming the corners of the channel section remain straight. The mode of local buckling is termed combined flange-web local buckling. The results obtained are compared with the previous works where the buckling loads are determined by different mathematical formulations.

Secondly, the buckling load is determined considering the local buckling of the web only. In this case, the corners of the channel are no longer considered to remain straight during the loading process. The flanges are considered as elastic supports to the web. Therefore, the corner line can deflect according to the stiffness of the flanges. The buckling load determined from this mode of local buckling is different from that determined from the combined flange-web local buckling. For small values of flange-web ratio, the buckling load thus determined is less than the buckling load considering the combined flange-web buckling. Therefore, for this range of flange-web ratios, the channel section will become unstable due to local buckling of the web alone.

Three sets of experiments are carried out covering a wide range of the flange-web ratios to verify the mode of buckling and the actual buckling load as calculated from theory.

CHAPTER II

APPLICATION OF MATRIX TRANSFER METHOD TO DETERMINE THE BUCKLING LOAD OF A RECTANGULAR PLATE UNDER INPLANE COMPRESSION

In this chapter, we will illustrate the use of the Matrix Transfer Method as applied to the stability study of a rectangular plate subjected to inplane loading. The method will then be used to solve the local buckling load of the channel sections under axial loading.

The analysis is based on the following assumptions:

1. Material is linear, elastic, homogenous and isotropic.

2. The web and flanges of the channel are perfectly flat and stress is applied in the mid plane of the plates.

3. The deflections in the mid plane of the plate due to the applied stress will be neglected until stress reaches the critical value.

4. At plate edges, Navier's end conditions are applied.

5. The applied stress is uniformily distributed over the thickness and span of the plate.

6. The classical plate theory is used for the analysis.

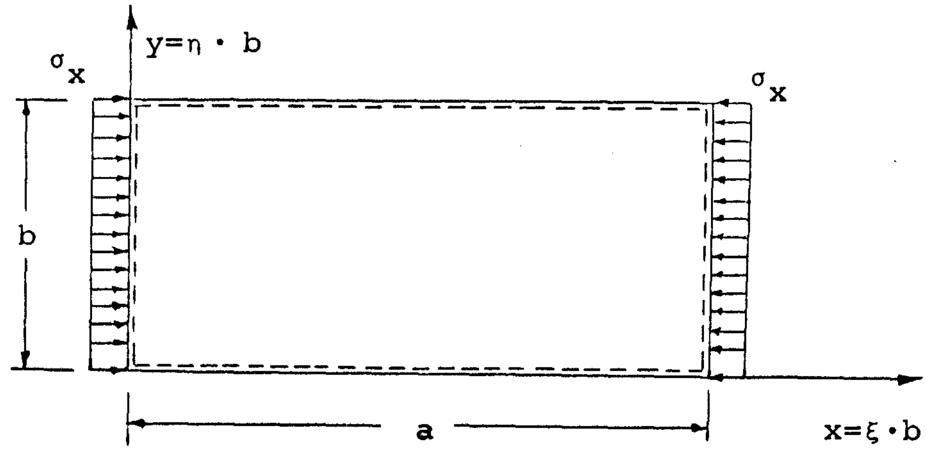


Fig. (1)

Consider an elastic rectangular plate of length a, breadth b and thickness t, with a uniformly distributed compressive stress σ_x acting in the x direction in the mid plane of the plate as shown in Fig. (1).

The governing equation of the deflection W of the plate is given by [14] as,

$$D.\left[\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}\right] + \sigma_x.t.\frac{\partial^2 W}{\partial x^2} = 0 \qquad \text{...(II-1)}$$

where D is the flexural rigidity of the plate = $Et^3/12(1 - \nu^2)$ The stress σ_x will be considered positive if the applied stess is compressive.

To nondimensionalize equation (II-1), let:

$x = \xi.b$ and

$y = \eta.b$

Expressing the differential equation in terms of the new set of axes ξ, η and introducing the buckling factor k, we can write:

$$\nabla^4 W + k\,\pi^2 \frac{\partial^2 W}{\partial \xi^2} = 0 \qquad \ldots\text{(II-2)}$$

where

$$\nabla^4 W = \frac{\partial^4 W}{\partial \xi^4} + 2\,\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 W}{\partial \eta^4}$$

$$k = \frac{\sigma_{cr}}{\sigma_e}$$

with σ_{cr} as the critical buckling stress and σ_e as the Euler buckling stress = $\pi^2 D/b^2 t$ which is the critical stress for a square plate free along the edges parallel to the applied stress and simply supported on the other edges.

Consider a plate simply supported at the edges $\xi = 0$ and $\xi = \alpha$. The boundary conditions become that both deflection and moment equal zero at both edges, that is:

$$W(0) = M(0) = W(\alpha) = M(\alpha) = 0 \qquad \ldots\text{(II-3)}$$

where

$$\alpha = a/b$$

Consider a deflection curve of the form

$$W(\xi,\ \eta) = w(\eta)\ \sin \frac{m\pi b}{a}\,\xi$$

$$= w(\eta)\ \sin \theta\xi \qquad \ldots\text{(II-4)}$$

where

$$\theta = m\pi/\alpha$$

with m as the number of half waves of the buckled curve in a direction parallel to the applied load.

Equation (II-4) satisfies the boundary conditions at $\xi = 0$ and $\xi = \alpha$.

Introducing equation (II-4) into the differential equation, we obtain an ordinary differential equation of the form:

$$\frac{d^4 w(\eta)}{d\eta^4} - 2\theta^2 \frac{d^2 w(\eta)}{d\eta^2} + (\theta^4 - k\pi^2\theta^2) w(\eta) = 0 \qquad \text{...(II-5)}$$

Seeking a solution in the form

$$w(\eta) = e^{\lambda\eta} \qquad \text{...(II-6)}$$

we obtain the characteristic equation as

$$\lambda^4 - 2\theta^2\lambda^2 + \theta^4\left(1 - \frac{\pi^2}{\theta^2} k\right) = 0 \qquad \text{...(II-7)}$$

The general solution of equation (II-5) can be written as:

$$w(\eta) = C_1 \cosh\bar{\kappa}_1\eta + C_2 \sinh\bar{\kappa}_1\eta + C_3 \cosh\bar{\kappa}_2\eta + C_4 \sinh\bar{\kappa}_2\eta \qquad \text{...(II-8)}$$

where the arbitrary constants C_1, C_2, C_3, and C_4 are to be determined by the boundary conditions at $\eta = 0$ and $\eta = 1$. $\bar{\kappa}_1$ and $\bar{\kappa}_2$ are roots of the characteristic equation (II-7) and are given by:

$$\lambda_{1,2} = \pm\sqrt{\theta(\theta + \pi^2\sqrt{k})} \equiv \pm\,\bar{\kappa}_1 \qquad \text{...(II-9)}$$

$$\lambda_{3,4} = \pm\sqrt{\theta(\theta - \pi^2\sqrt{k})} \equiv \pm\,\bar{\kappa}_2$$

Expressing the arbitrary constants in terms of the deflection, slope, moment and shear at edge $\eta = 0$, we have:

$$C_1 = -\frac{w_o(\bar{\kappa}_2^2 - \nu\theta^2) - M_o^*}{(\bar{\kappa}_1^2 - \bar{\kappa}_2^2)}$$

$$C_2 = +\frac{\frac{\partial w_o}{\partial\eta}(\bar{\kappa}_1^2 - \nu\theta^2) + Q_o^*}{\bar{\kappa}_1(\bar{\kappa}_1^2 - \bar{\kappa}_2^2)}$$

$$C_3 = +\frac{w_o(\bar{\kappa}_1^2 - \nu\theta^2) - M_o^*}{(\bar{\kappa}_1^2 - \bar{\kappa}_2^2)}$$

$$C_4 = - \frac{\frac{\partial w_o}{\partial \eta} (\bar{\kappa}_2^{\,2} - \nu\theta^2) + Q_o^*}{\bar{\kappa}_2 (\bar{\kappa}_1^{\,2} - \bar{\kappa}_2^{\,2})} \qquad \ldots \text{(II-10)}$$

where

ν is poisson's ratio

w_o is the deflection at edge $\eta = 0$

$\frac{\partial w_o}{\partial \eta}$ is the slope of the deflection curve at $\eta = 0$

M_o is the moment at edge $\eta = 0$

$$= \frac{D}{b^2} \left(\frac{\partial^2 w_o}{\partial \eta^2} + \nu \frac{\partial^2 w_o}{\partial \xi^2} \right) \qquad \ldots \text{(II-11)}$$

or $M_o^* = - \frac{b^2}{D} M_o$

Q_o is the shear at edge $\eta = 0$

$$= - \frac{D}{b^3} \left(\frac{\partial^3 w_o}{\partial \eta^3} + (2 - \nu) \frac{\partial^3 w_o}{\partial \eta \partial \xi^2} \right) \qquad \ldots \text{(II-12)}$$

or $Q_o^* = - \frac{b^3}{D} Q_o$

Using equations (II-8) and (II-10), it can be seen that the deflection of the plate can be expressed in terms of the deflection, slope, moment and shear at one edge of the plate:

$$\begin{aligned} w(\eta) = &- \frac{w_o (\bar{\kappa}_2^{\,2} - \nu\theta^2) - M_o^*}{(\bar{\kappa}_1^{\,2} - \bar{\kappa}_2^{\,2})} \cdot \cosh\bar{\kappa}_1 \eta \\ &+ \frac{\frac{\partial w_o}{\partial \eta} (\bar{\kappa}_1^{\,2} - \nu\theta^2) + Q_o^*}{\bar{\kappa}_1 (\bar{\kappa}_1^{\,2} - \bar{\kappa}_2^{\,2})} \cdot \sinh \bar{\kappa}_1 \eta \\ &+ \frac{w_o (\bar{\kappa}_1^{\,2} - \nu\theta^2) - M_o^*}{(\bar{\kappa}_1^{\,2} - \bar{\kappa}_2^{\,2})} \cdot \cosh\bar{\kappa}_2 \eta \end{aligned}$$

$$- \frac{\frac{\partial w_o}{\partial \eta}(\bar{\kappa}_2^2 - \nu\theta^2) + Q_o^*}{\bar{\kappa}_2(\bar{\kappa}^2 - \bar{\kappa}_2^2)} \cdot \sinh \bar{\kappa}_2 \eta \qquad \text{...(II-13)}$$

The general solution of the differential equation depends on the values of $\bar{\kappa}_1$ and $\bar{\kappa}_2$. There are five cases to be considered:

a) $\left(1 - \frac{\pi^2 k}{\theta^2}\right) < 0 \qquad k > (m/\alpha)^2 \qquad$ ...(II-14.a)

b) $\left(1 - \frac{\pi^2 k}{\theta^2}\right) = 0 \qquad k = (m/\alpha)^2 \qquad$ ...(II-14.b)

c) $0 < \left(1 - \frac{\pi^2 k}{\theta^2}\right) < 1 \qquad 0 < k < (m/\alpha)^2 \qquad$ ...(II-14.c)

d) $\left(1 - \frac{\pi^2 k}{\theta^2}\right) = 1 \qquad k = 0 \qquad$ ...(II-14.d)

g) $\left(1 - \frac{\pi^2 k}{\theta^2}\right) > 1 \qquad k < 0 \qquad$ ...(II-14.g)

Cases d and g where $k = 0$ (i.e. no axial stress) and $k < 0$ (i.e. tensile stress), although seemingly of no interest, they will allow any general stress applied to an assembly of plates.

For the moment we shall consider in detail the case a, where $k > (m/\alpha)^2$. The procedure for solution of the other cases follows essentially the same steps with only slight modifications.

In this case, $\bar{\kappa}_2$ becomes imaginary and the general solution takes the form:

$$w(\eta) = (A_2 \cosh \bar{\kappa}_1 \cdot \eta + A_1 \cos \bar{\kappa}_2 \eta) \cdot w_o + \left(\frac{A_1}{\bar{\kappa}_1} \sinh \bar{\kappa}_1 \cdot \eta + \frac{A_2}{\bar{\kappa}_2} \sin \bar{\kappa}_2 \eta\right) \cdot w_o'$$

$$+ \left(\cosh \bar{\kappa}_1 \cdot \eta - \cos \bar{\kappa}_2 \eta \right) \cdot M_o^*$$

$$+ \left(\frac{1}{\bar{\kappa}_1} \sinh \bar{\kappa}_1 \cdot \eta - \frac{1}{\bar{\kappa}_2} \sin \bar{\kappa}_2 \eta \right) \cdot Q_o^*$$

...(II-15)

where

$$w_o' = \frac{\partial w_o}{\partial \eta}$$

$$\bar{\kappa}_1 \equiv [\theta(\pi\sqrt{k} + \theta)]^{\frac{1}{2}}$$

$$\bar{\kappa}_2 \equiv [\theta(\pi\sqrt{k} - \theta)]^{\frac{1}{2}}$$

Equation (II-15) can be put in the form:

$$w(\eta) = \frac{1}{A_o} \left\{ (A_2 \cosh \bar{\kappa}_1 \eta + A_1 \cos \bar{\kappa}_2 \eta) \quad \left(\frac{A_1}{\bar{\kappa}_1} \sinh \bar{\kappa}_1 \eta + \frac{A_2}{\bar{\kappa}_2} \sin \bar{\kappa}_2 \eta\right) \quad (\cosh \bar{\kappa}_1 \eta - \cos \bar{\kappa}_2 \eta) \quad \left(\frac{1}{\bar{\kappa}_1} \sinh \bar{\kappa}_1 \eta - \frac{1}{\bar{\kappa}_2} \sin \bar{\kappa}_2 \eta\right) \right\} \cdot \begin{Bmatrix} w_o \\ w'_o \\ M_o^* \\ Q_o^* \end{Bmatrix}$$

...(II-16)

where

$$A_1 = \bar{\kappa}_1^2 - \nu\theta^2$$

$$A_2 = \bar{\kappa}_2^2 + \nu\theta^2$$

$$A_o = \bar{\kappa}_1^2 + \bar{\kappa}_2^2$$

The slope, moment and shear at any point inside the plate can be similarly expressed in terms of the state vector at the edge $\eta = 0$, namely,

$$\text{Slope } w'(\eta) = \frac{1}{A_o} \cdot \begin{Bmatrix} (A_2 \cosh \bar{\kappa}_1 \cdot \eta + A_1 \cos \bar{\kappa}_2 \cdot \eta) \\ (\frac{A_1}{\bar{\kappa}_1} \sinh \bar{\kappa}_1 \cdot \eta + \frac{A_2}{\bar{\kappa}_2} \sin \bar{\kappa}_2 \eta) \\ (\cosh \bar{\kappa}_1 \cdot \eta - \cos \bar{\kappa}_2 \cdot \eta) \\ (\frac{1}{\bar{\kappa}_1} \sinh \bar{\kappa}_1 \eta - \frac{1}{\bar{\kappa}_2} \sin \bar{\kappa}_2 \eta) \end{Bmatrix} \cdot \begin{Bmatrix} w_o \\ w'_o \\ M_o^* \\ Q_o^* \end{Bmatrix}$$

...(II-17)

$$\text{Moment } M^*(\eta) = \frac{1}{A_o} \cdot \begin{Bmatrix} [A_2 (\bar{\kappa}_1^{\,2} - \nu\theta^2) \cosh \bar{\kappa}_1 \eta - A_1 (\bar{\kappa}_2^{\,2} + \nu\theta^2) \cos \bar{\kappa}_2 \eta] \\ [\frac{A_1}{\bar{\kappa}_1} (\bar{\kappa}_1^{\,2} - \nu\theta^2) \sinh \bar{\kappa}_1 - \frac{A_2}{\bar{\kappa}_2} (\bar{\kappa}_2^{\,2} + \nu\theta^2) \sin \bar{\kappa}_2 \eta] \\ [(\bar{\kappa}_1^{\,2} - \nu\theta^2) \cosh \bar{\kappa}_1 \eta + (\bar{\kappa}_2^{\,2} + \nu\theta^2) \cos \bar{\kappa}_2 \eta] \\ [\frac{1}{\bar{\kappa}_1} (\bar{\kappa}_1^{\,2} - \nu\theta^2) \sinh \bar{\kappa}_1 + \frac{1}{\bar{\kappa}_2} (\bar{\kappa}_2^{\,2} + \nu\theta^2) \sin \bar{\kappa}_2] \end{Bmatrix} \cdot \begin{Bmatrix} w_o \\ w_o' \\ M_o^* \\ Q_o^* \end{Bmatrix}$$

...(II-18)

and shear $Q^*(\eta) = \frac{1}{A_o}\Big[\Big\{A_2\bar{\kappa}_1\ [\bar{\kappa}_1^{\ 2} - (2 - \nu)\theta^2]\ \sinh \bar{\kappa}_1\eta$

$+ A_1\bar{\kappa}_2\ [\bar{\kappa}_2^{\ 2} + (2 - \nu)\theta^2]\ \sinh \bar{\kappa}_2\eta\Big\}$

$\Big\{A_1[\bar{\kappa}_1^{\ 2} - (2 - \nu)\theta^2]\ \cosh \bar{\kappa}_1\ \ - A_2[\bar{\kappa}_2^{\ 2} + (2 - \nu)\theta^2]$

$\cosh \bar{\kappa}_2\eta\Big\}$

$\Big\{\bar{\kappa}_1\ [\bar{\kappa}_1^{\ 2} - (2 - \nu)\theta^2]\ \sinh \bar{\kappa}_1\eta$

$- \bar{\kappa}_2\ [\bar{\kappa}_2^{\ 2} + (2 - \nu)\theta^2]\ \sin \bar{\kappa}_2\eta\Big\}$

$\Big\{[\bar{\kappa}_1^{\ 2} - (2 - \nu)\theta^2]\ \cosh \bar{\kappa}_1\eta$

$+ [\bar{\kappa}_2^{\ 2} + (2 - \nu)\theta^2]\ \cosh \bar{\kappa}_2\eta\Big\}\Big] \cdot \begin{Bmatrix} w_o \\ w'_o \\ M_o^* \\ Q_o^* \end{Bmatrix}$

...(II-19)

To present the theory in a compact form it is convenient to use the matrix notations.

Defining the state vector $\{\Phi_\eta\}$ to be

$$\{\Phi_\eta\} = \begin{Bmatrix} w_\eta \\ w'_\eta \\ M^*_\eta \\ Q^*_\eta \end{Bmatrix} \qquad \text{for } 0 < \eta < 1$$

It can be seen that, once the state vector $\{\Phi_\eta\}$ is known for any section (η), all the information about that section will be known. From equations (II-16) to (II-19), $\{\Phi_\eta\}$ can be related to $\{\Phi_o\}$ by a matrix relation:

$$\{\Phi\}_\eta = [\ F\ ]_\eta \cdot \{\Phi_o\} \qquad \text{...(II-20)}$$

where

[F] is the field matrix which relates the state vector at any location η to the state vector at the boundary $\eta = 0$. Its terms are given by the equations from (II-16) to (II-19).

In particular, at the edge $\eta = 0$ and $\eta = 1$ of the plate, let the corresponding state vectors be $\{\Phi_o\}$ and $\{\Phi_1\}$ where:

$$\{\Phi_o\} = \begin{Bmatrix} w_o \\ w_o' \\ M_o^* \\ Q_o^* \end{Bmatrix} \qquad \{\Phi_1\} = \begin{Bmatrix} w_1 \\ w_1' \\ M_1^* \\ Q_1^* \end{Bmatrix} \qquad \text{...(II-21)}$$

They are related by the relation

$$\{\Phi_1\} = [\ F\ ]_{\eta=1} \cdot \{\Phi_o\} \qquad \text{...(II-22)}$$

where $[\ F\ ]_{\eta=1}$ denotes the matrix [F] given in equation (II-20) by replacing η by unity.

Expanding equation (II-22) we have:

$$\begin{aligned} w_1 &= f_{11}w_o + f_{12}w_o' + f_{13}M_o^* + f_{14}Q_o^* \\ w_1' &= f_{21}w_o + f_{22}w_o' + f_{23}M_o^* + f_{24}Q_o^* \\ M_1^* &= f_{31}w_o + f_{32}w_o' + f_{33}M_o^* + f_{34}Q_o^* \\ Q_1^* &= f_{41}w_o + f_{42}w_o' + f_{42}M_o^* + f_{44}Q_o^* \end{aligned} \qquad \text{...(II-23)}$$

where f_{ij} are the elements in the matrix [F].

$$f_{11} = f_{44} = A_2 \cosh \bar{\kappa}_1\eta + A_1 \cos \bar{\kappa}_2\eta$$

$$f_{12} = f_{34} = \frac{A_1}{\bar{\kappa}_1} \sinh \bar{\kappa}_1\eta + \frac{A_2}{\bar{\kappa}_2} \sin \bar{\kappa}_2\eta$$

$$f_{13} = f_{24} = \cosh \bar{\kappa}_1\eta - \cos \bar{\kappa}_2\eta$$

$$f_{14} = \frac{1}{\bar{\kappa}_1} \sinh \bar{\kappa}_1\eta - \frac{1}{\bar{\kappa}_2} \sin \bar{\kappa}_2\eta$$

$$f_{21} = f_{43} = A_2\bar{\kappa}_1 \sinh \bar{\kappa}_1\eta - A_1\bar{\kappa}_2 \sin \bar{\kappa}_2\eta \qquad \ldots\text{(II- 24)}$$

$$f_{22} = f_{33} = A_1 \cosh \bar{\kappa}_1\eta + A_2 \cos \bar{\kappa}_2\eta$$

$$f_{23} = \bar{\kappa}_1 \sinh \bar{\kappa}_1\eta + \bar{\kappa}_2 \sin \bar{\kappa}_2\eta$$

$$f_{31} = f_{42} = A_1A_2(\cosh \bar{\kappa}_1\eta-\cos \bar{\kappa}_2\eta)$$

$$f_{32} = \frac{A_1^{\,2}}{\bar{\kappa}_1} \sinh \bar{\kappa}_1\eta - \frac{A_2^{\,2}}{\bar{\kappa}_2} \sin \bar{\kappa}_2\eta$$

$$f_{41} = A_2^{\,2}\bar{\kappa}_1 \sinh \bar{\kappa}_1\eta + A_1^{\,2}\bar{\kappa}_2 \sin\bar{\kappa}_2\eta$$

The matrix $[F]_{\eta=1}$ which related the state vectors at the two boundaries $\eta = 0$ and $\eta = 1$ is known as the "field Transfer matrix" or "Field matrix" for the plate.

Field matrix [F_A] for [$k>(m/\alpha)^2$]

$$
A = \frac{1}{A_o} \cdot \begin{bmatrix}
A_2\varepsilon_2 + A_1\varepsilon_4 & \frac{A_1}{\beta_1}\varepsilon_1 + \frac{A_2}{\beta_2}\varepsilon_3 & \varepsilon_2 - \varepsilon_4 & \frac{1}{\beta_1}\varepsilon_1 - \frac{1}{\beta_2}\varepsilon_3 \\
A_2\beta_1\varepsilon_1 - A_1\beta_2\varepsilon_3 & A_1\varepsilon_2 + A_2\varepsilon_4 & \beta_1\varepsilon_1 + \beta_2\varepsilon_2 & \varepsilon_2 - \varepsilon_4 \\
A_1A_2(\varepsilon_2 - \varepsilon_4) & \frac{A_1^2}{\beta_1}\varepsilon_1 - \frac{A_2^2}{\beta_2}\varepsilon_3 & A_1\varepsilon_2 + A_2\varepsilon_4 & \frac{A_1}{\beta_1}\varepsilon_1 + \frac{A_2}{\beta_2}\varepsilon_3 \\
A_2^2\beta_1\varepsilon_1 + A_1^2\beta_2\varepsilon_3 & A_1A_2(\varepsilon_2 - \varepsilon_4) & A_2\beta_1\varepsilon_1 - A_1\beta_2\varepsilon_3 & A_2\varepsilon_2 + A_1\varepsilon_4
\end{bmatrix} \qquad \text{...(II-25)}
$$

where,

$\beta_1 = \sqrt{\theta(\pi\sqrt{k} + \theta)}$ $\qquad \varepsilon_1 = \sinh\beta_1\eta$ $\qquad A_1 = \beta_1^2 - \nu\theta^2$

$\beta_2 = \sqrt{\theta(\pi\sqrt{k} - \theta)}$ $\qquad \varepsilon_2 = \cosh\beta_1\eta$ $\qquad A_2 = \beta_2^2 + \nu\theta^2$

$\theta = m\pi/\alpha$ $\qquad \varepsilon_3 = \sin\beta_2\eta$ $\qquad A_o = A_1 + A_2 = \beta_1^2 + \beta_2^2$

$\varepsilon_4 + \cos\beta_2\eta$

Buckling Condition

Let us consider the case of the plate in Fig. (1), but the edges at $\eta = 0$ and $\eta = 1$ are free (i.e. no moment and no shear).

The boundary conditions at edge $\eta = 1$ can be expressed in terms of the boundary conditions at edge $\eta = o$ in the form:

$$\begin{Bmatrix} w_1 \\ w_1' \\ M_1^* \\ Q_1^* \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix} \cdot \begin{Bmatrix} w_o \\ w_0' \\ M_o^* \\ Q_o^* \end{Bmatrix} \qquad \text{...(II-26)}$$

For the considered example the boundary conditions are such that

at $\eta = 0$, $M_o^* = 0$ and $Q_o^* = o$

and at $\eta = 1$, $M_1^* = 0$ and $Q_1^* = 0$

Therefore, substituting the boundary conditions of the two edges in equation (II-26) we can write:

$$\begin{Bmatrix} w_1 \\ w_1' \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix} \cdot \begin{Bmatrix} w_o \\ w_o' \\ 0 \\ 0 \end{Bmatrix} \qquad \text{...(II-27)}$$

which can be expanded as

$$w_1 = f_{11} w_o + f_{12} w_o' \qquad \text{...(II-28.a)}$$

$$w_1' = f_{21} w_o + f_{22} w_o' \qquad \text{...(II-28.b)}$$

$$0 = f_{31} w_o + f_{32} w_o' \qquad \text{...(II-28.c)}$$

$$0 = f_{41} w_o + f_{42} w_o' \qquad \text{...(II-28.d)}$$

Considering equations (II-28.c) and (II-28.d), for the nontrivial solution of the values for w_o and w_o', we have the condition:

$$\begin{vmatrix} f_{31} & f_{32} \\ f_{41} & f_{42} \end{vmatrix} = 0$$

or $f_{31} \cdot f_{42} - f_{41} \cdot f_{32} = 0$ ...(II-29)

which is the buckling condition for the given case. The values of k, and hence σ_{cr}, can be determined by solving equation (II-29) by trial and error method of numerical analysis.

Consider another problem where the plate is fixed at $\eta = 0$ but free at $\eta = 1$. Equation (II-27) takes the form:

$$\begin{Bmatrix} w_1 \\ w_1' \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ M_o^* \\ Q_o^* \end{Bmatrix}$$

It can be seen that once the field matrix [F] is known, the buckling factor k, and hence the buckling stress σ_{cr} can be found for all combinations of boundary conditions at the edges $\eta = 0$ and $\eta = 1$. A complete list of combinations of boundary conditions along the two edges are shown in matrix (II-30).

In the matrix (II-30), the actual boundary conditions at both boundaries, a schematic diagram of the support

		FREE	FIXED-ROLLER	INADMISSIBLE		HINGED	FIXED
		$M^*_1 = 0$ $Q^*_1 = 0$	$w'_1 = 0$ $Q^*_1 = 0$	$w'_1 = 0$ $M^*_1 = 0$	$w_1 = 0$ $Q^*_1 = 0$	$w_1 = 0$ $M^*_1 = 0$	$w_1 = 0$ $w'_1 = 0$
FIXED	$w_0 = 0$ $w'_0 = 0$	$\begin{vmatrix} 11 & 12 \\ 21 & 22 \end{vmatrix}$	$\begin{vmatrix} 11 & 13 \\ 21 & 23 \end{vmatrix}$			$\begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix}$	$\begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix}$
HINGED	$w_0 = 0$ $M^*_0 = 0$	$\begin{vmatrix} 11 & 12 \\ 31 & 32 \end{vmatrix}$	$\begin{vmatrix} 11 & 13 \\ 31 & 33 \end{vmatrix}$			$\begin{vmatrix} 12 & 14 \\ 32 & 34 \end{vmatrix}$	$\begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix}$
INADMISSIBLE	$w_0 = 0$ $Q^*_0 = 0$						
	$w'_0 = 0$ $M^*_0 = 0$						
FIXED-ROLLER	$w'_0 = 0$ $Q^*_0 = 0$	$\begin{vmatrix} 21 & 22 \\ 41 & 42 \end{vmatrix}$	$\begin{vmatrix} 21 & 23 \\ 41 & 43 \end{vmatrix}$			$\begin{vmatrix} 22 & 24 \\ 42 & 44 \end{vmatrix}$	$\begin{vmatrix} 23 & 24 \\ 43 & 44 \end{vmatrix}$
FREE	$M^*_0 = 0$ $Q^*_0 = 0$	$\begin{vmatrix} 31 & 32 \\ 41 & 42 \end{vmatrix}$	$\begin{vmatrix} 31 & 33 \\ 41 & 43 \end{vmatrix}$			$\begin{vmatrix} 32 & 34 \\ 42 & 44 \end{vmatrix}$	$\begin{vmatrix} 33 & 34 \\ 43 & 44 \end{vmatrix}$

$$\cdot\cdot(\text{II}-30)$$

conditions and also the elements in the field matrix that gives the buckling condition are shown. For example, if both edges are free the table gives the buckling condition

$$\begin{vmatrix} 31 & 32 \\ 41 & 42 \end{vmatrix} \qquad \text{or equivalently,}$$

$$\begin{vmatrix} f_{31} & f_{32} \\ f_{41} & f_{42} \end{vmatrix} = 0$$

which is the same as given by equation (II-29).

For each combination of boundary conditions at $\eta = 0$ and $\eta = 1$, there will result an equation representing the buckling condition by equating the determinant of certain 2 x 2 matrix to zero. To facilitate computations it is convenient to define a new matrix called the Field Boundary Matrix $[\overset{\Delta}{F}_A]$. This is a 6 x 6 matrix whose elements consist of the various buckling conditions. For example, the free edges condition is found in matrix (II-30) to occupy a position in the sixth raw and the first column of the matrix. Therefore, such a buckling condition is entered as the element $(\overset{\Delta}{f}_{61})$ in the newly defined Field Boundary Matrix. Similarly, the element $(\overset{\Delta}{f}_{11})$ in the Field Boundary Matrix is the buckling condition for the free-fixed boundary condition of the plate.

This modification as will be shown later, to facilitate the calculations of the complicated cases and reduces greatly the time required for computations. The Field Boundaries Matrix $[\overset{\Delta}{F}_A]$ for the case of $k > (m/\pi)^2$ is given by matrix (II-31).

Field Boundaries Matrix $[\overset{\Delta}{F_A}][k\ (m/\alpha)^2]$

$\frac{1}{A_o^2}[2A_{12} + (A_2^2 + A_1^2)\cdot\varepsilon_4 + (\beta_4 A_1^2 - \beta_3 A_2^2)\cdot\varepsilon_1]$	$\frac{1}{A_o}(\beta_1\varepsilon_2 + \beta_2\varepsilon_3)$	$\frac{1}{A_o^2}[(A_2 - A_1)\varepsilon_5 + (\beta_3 A_2 + \beta_4 A_1)\varepsilon_1]$	$= A(1,3)$	$\frac{1}{A_o}(\frac{\varepsilon_2}{\beta_2} - \frac{\varepsilon_2}{\beta_1})$	$\frac{1}{A^2}[2\varepsilon_5 + (\beta_3 - \beta_4)\varepsilon_1]$
$\frac{1}{A_o}(\frac{A_1^2}{\beta_1}\varepsilon_2 - \frac{A_2^2}{\beta_2}\varepsilon_3)$	ε_4	$\frac{1}{A_o}(\frac{A_1}{\beta_1}\varepsilon_2 + \frac{A_2}{\beta_2}\varepsilon_3)$	$= A(2,3)$	$\frac{\varepsilon_1}{\beta_5}$	$= A(1,5)$
$\frac{1}{A_o^2}[A_{12}(A_2 - A_1)\varepsilon_5 - (\beta_3 A_2^3 + \beta_4 A_1^3)\cdot\varepsilon_1]$	$\frac{1}{A_o}(\beta_1\ {}_2\varepsilon_3 - \beta_2 A_1\varepsilon_3)$	$-\frac{1}{A_o^2}[2A_{12}\varepsilon_5 + (\beta_4 A_1^2 - \beta_3 A_2^2)\varepsilon_1]$	$= A(3,3)$	$= A(2,4)$	$= A(1,4)$
$= A(3,1)$	$= A(3,2)$	$= A(3,3)$	$= A(3,3)$	$= A(2,3)$	$= A(1,3)$
$-\frac{1}{A_o}(\beta_2 A_1^2\varepsilon_3 + \beta_1 A_2^2\varepsilon_2)$	$-\beta_5\cdot\varepsilon_1$	$= A(4,2)$	$= A(3,2)$	$= A(2,2)$	$= A(1,2)$
$\frac{1}{A_o^2}[2A_1^2 A_2^2\varepsilon_5 + (\beta_3 A_2^4 - \beta_4 A_1^4)\varepsilon_1]$	$= A(5,1)$	$= A(4,1)$	$= A(3,1)$	$= A(2,1)$	$= A(1,1)$

...(II-31)

Abbreviations in the Field Boundaries Matrix $[\overset{\Delta}{F}_A]$ $[k > (m/\alpha)^2]$

$$\beta_1 = \sqrt{\theta(\pi\sqrt{k} + \theta)}$$
$$\beta_2 = \sqrt{\theta(\pi\sqrt{k} - \theta)}$$
$$\beta_3 = \beta_1/\beta_2$$
$$\beta_4 = 1/\beta_3$$
$$\beta_5 = \beta_1\ \beta_2$$
$$A_1 = \beta_1^2 - \nu\theta^2$$
$$A_2 = \beta_2^2 + \nu\theta^2$$
$$A_o = A_1 + A_2$$
$$A_{12} = A_1 \cdot A_2$$
$$\theta = m\pi/\alpha$$
$$\varepsilon_1 = \sinh\beta_1\eta \cdot \sin\beta_2\eta$$
$$\varepsilon_2 = \sinh\beta_1\eta \cdot \cos\beta_2\eta$$
$$\varepsilon_3 = \cosh\beta_1\eta \cdot \sin\beta_2\eta$$
$$\varepsilon_4 = \cosh\beta_1\eta \cdot \cos\beta_2\eta$$
$$\varepsilon_5 = 1 - \varepsilon_4$$

It should be mentioned that all the equations from equation (II-15) to equation (II-24) are valid when the buckling factor k is such that equation (II-14.a) is satisfied, namely that $k > (m/\alpha)^2$. We can arrive at different field transfer matrices corresponding to the different values of the buckling factor k with respect to the value of $(m/\alpha)^2$ as given in equations (II-14.b) to (II-14.g).

Consequently, this will result in five Field Boundary Matrices $[\overset{\Delta}{F}_A]$, $[\overset{\Delta}{F}_B]$, $[\overset{\Delta}{F}_C]$, $[\overset{\Delta}{F}_D]$, and $[\overset{\Delta}{F}_G]$ depending on the value of k in relation to (m/α). The detailed mathematical formulation for each matrix is given in Appendix I.

To find the value of the buckling factor k that satisfies the buckling condition [e.g. equation (II-29)], a trial and error numerical method is adopted. This method is illustrated in steps as follows and by a flow chart.

1. For a certain value of α.

2. Assume a value for k.

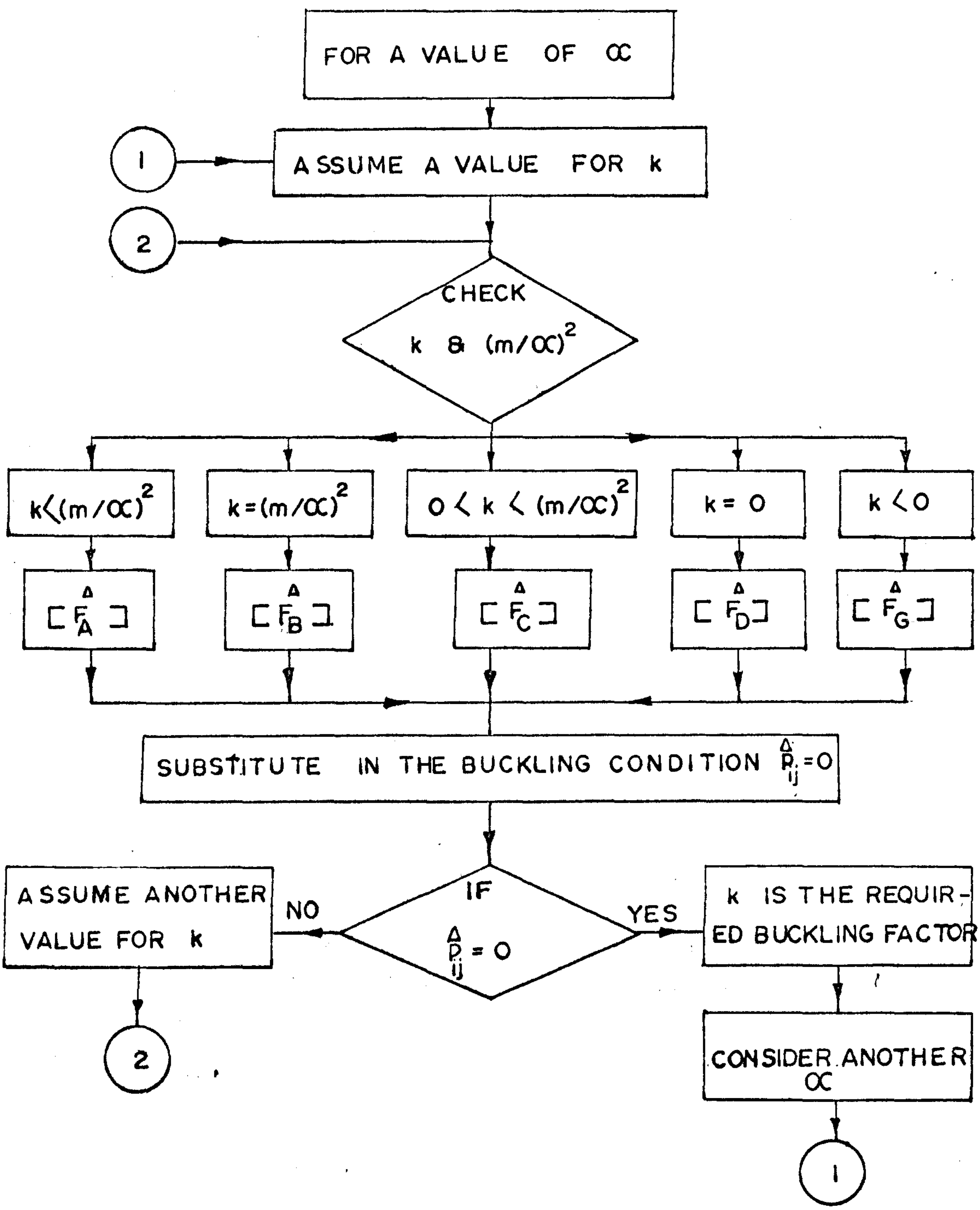
3. Applying conditions (II-14), we can determine the proper field boundary matrix to be used in forming the buckling condition.

4. Evaluate the buckling condition and if:

a) its values equal zero or a practically very small quantity then, the assumed value of k was taken as the correct value.

b) the buckling condition values differ from zero substantially, another value for the buckling factor k will be used and the whole process will be started again from step 3.

Procedure of Numerical Method Followed to Calculate the Buckling Factor k that Satisfies the Buckling Condition $\overset{\Delta}{P}_{ij}$



FLOW CHART

CHAPTER III

SPECIAL FIELD TRANSFER MATRICES

In this Chapter, we will illustrate the derivation of the special Field and Field Boundaries Matrices. These matrices relate the state vectors for special structural elements (e.g., transfer conditions across a stiffening rib, thickness change or intermediate support condition). These cases will enable us to extend the Matrix Transfer Method to be applied for continuous plates, channel cross-sections and sections of variable thickness.

A. Transfer Conditions Across a Stiffening Rib [Rib-transfer Matrix]

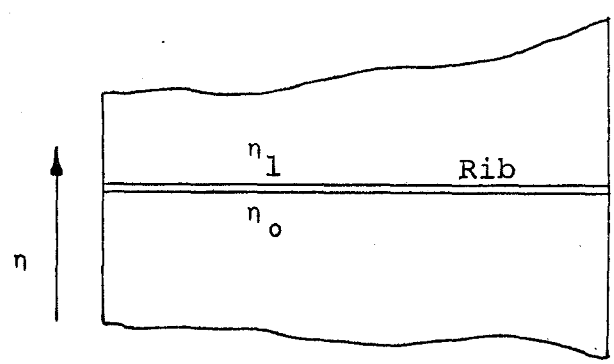


Fig. (2)

The boundary conditions before and after the stiffening rib are expressed by the state vectors, $\{\Phi_{\eta_o}\}$ and $\{\Phi_{\eta_1}\}$, to be:

$$\{\Phi_{\eta_o}\} = \begin{Bmatrix} w_{\eta_o} \\ w'_{\eta_o} \\ M^*_{\eta_o} \\ Q^*_{\eta_o} \end{Bmatrix} \quad \text{and} \quad \{\Phi_{\eta_1}\} = \begin{Bmatrix} w_{\eta_1} \\ w'_{\eta_1} \\ M^*_{\eta_1} \\ Q^*_{\eta_1} \end{Bmatrix}$$

Reconsidering equation (II-20), we can relate the two state vectors at the boundaries of the rib to follow the matrix relation:

$$\{\Phi_{\eta_1}\} = [R] \cdot \{\Phi_{\eta_o}\} \qquad \text{...(III-1)}$$

where [R] is the rib-transfer matrix which relates the two state vectors before and after the rib. The Rib Matrix satisfies the transfer conditions across a rib or a stiffener. Its terms will be calculated by the following continuation conditions:

$$w_{\eta_1} = w_{\eta_o} \qquad \text{...(III-2)}$$

$$w'_{\eta_1} = w'_{\eta_o} \qquad \text{...(III-3)}$$

$$Q_{\eta_1} = Q_{\eta_o} + EI_r \frac{\partial^4 W_{\eta_o}}{\partial \xi^4} \frac{1}{b^4} + k_r \sigma_e A_r \frac{\partial^2 W_{\eta_o}}{\partial \xi^2} \frac{1}{b^2}$$

$$\text{...(III-4)}$$

where the load carried by the rib is the load required to provide a deflection w plus taking account of the axial load $k_r\sigma_e A_r$ in the stiffener.

I_r is the moment of inertia of the rib about the centre line of the plate

A_r is the cross sectional area of rib

σ_e is Euler buckling stress

k_r is the buckling factor $[k = \frac{\sigma_{cr}}{\sigma_e}]$

To satisfy the moment compatibility we have,

$$M_{\eta_1} = M_{\eta_o} + GJ \frac{\partial^3 W_{\eta_o}}{\partial \xi^2 \partial \eta} \cdot \frac{1}{b^3} - EC_w \frac{\partial^5 W_{\eta_o}}{\partial \xi^4 \partial \eta} \cdot \frac{1}{b^5}$$

$$- k_r\sigma_e I_p \frac{\partial^3 W_{\eta_o}}{\partial \xi^2 \partial \eta} \cdot \frac{1}{b^3} \qquad \ldots(\text{III-5})$$

where the moment carried by the rib is due to the shear across the rib, its torsional rigidity and taking into consideration the effect of the axial load $k_r\sigma_e$ in the stiffener.

J is the torsion constant

C_w is the warping constant

I_p is the polar moment of inertia of the rib

Expressing the shear equation in terms of the deflection of the plate, then,

$$Q^*_{\eta_1} = Q^*_{\eta_o} - \left[\frac{EI_r}{bD} \theta^4 - k_r \frac{A_r}{bt} \theta^2 \pi^2 \right] \cdot w_{\eta_o} \qquad \ldots(\text{III-6})$$

where

$$Q^*_{\eta} = - \frac{b^3}{D} Q_{\eta}$$

$$\frac{\partial^4 W_{\eta}}{\partial \xi^4} = \theta^4 w_{\eta} \qquad \text{and} \qquad \frac{\partial^2 W_{\eta}}{\partial \xi^2} = \theta^2 w_{\eta}$$

$$\theta = m\pi/\alpha$$

To nondimensionalize the shear equation, let us introduce the following dimensionless quantities:

$$\gamma_B = \frac{EI_r}{bD} \quad \text{(relative stiffness of rib and plate)}$$

$$\delta = \frac{A_r}{bt} \quad \text{(relative area of rib and plate)}$$

Hence the shear equation reads

$$Q^*_{\eta_1} = Q^*_{\eta_o} - \phi_r w_{\eta_o} \qquad \text{...(III-7)}$$

where

ϕ_r is the equivalent spring constant of rib

$$\phi_r = \gamma_B \theta^4 - k_r \delta \pi^2 \theta^2 \qquad \text{...(III-8)}$$

to evaluate ϕ_r numerically, a nondimensional parameter (r) is introduced to be

$$\gamma = \frac{b_\omega}{t_\omega}$$

where

b_ω is the breadth of web plate

t_ω is the thickness of web plate

Nondimensionalizing the moment expression, we can write:

$$M^*_{\eta_1} = M^*_{\eta_o} + \psi_r \frac{\partial W_{\eta_o}}{\partial \eta} \qquad \text{...(III-9)}$$

where ψ_r is the torsional spring constant

$$\psi_r = \gamma_w \theta^4 + (\gamma_D - k_r \pi^2 \gamma_p)\theta^2$$

$$\gamma_D = \frac{GJ}{bD}$$

$$\gamma_w = \frac{EC_w}{b^3 D}$$

$$\gamma_p = \frac{I_p}{b^3 t}$$

$$M^*_\eta = - \frac{b^2}{D} M_\eta$$

Arranging the four boundary conditions, equations (III-2), (III-3), (III-7) and (III-9), in a matrix form, then:

$$\begin{Bmatrix} w_{\eta_1} \\ w'_{\eta_1} \\ M^*_{\eta_1} \\ Q^*_{\eta_1} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \psi_r & 1 & 0 \\ -\phi_r & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w_{\eta_o} \\ w'_{\eta_o} \\ M^*_{\eta_o} \\ Q^*_{\eta_o} \end{Bmatrix}$$

...(III-10)

Following a similar treatment to obtain the Field

Boundaries Matrix from the Field Matrix, we can simply arrive to the Rib Boundaries Matrix $\overset{\Delta}{[R]}$ to be:

$$\overset{\Delta}{[R]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \psi_r & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \phi_r & 0 & 0 & 0 & 1 & 0 \\ \psi_r\phi_r & \phi_r & 0 & 0 & \psi_r & 1 \end{bmatrix}$$

...(III-11)

B. <u>Transfer Conditions Across an Intermediate Support</u>

[Support transfer Matrix]

The case of a continuous plate over an intermediate support can be regarded as a special case of the stiffening rib. The Matrix that represents the transfer conditions across a support will be denoted as the "Support transfer Matrix", [S]. This matrix can be easily deduced from the rib transfer matrix by realizing the following facts:

a. The equivalent spring constant ϕ_s of the support is infinity, namely, no deflection is to take place.

b. The torsion spring constant ψ_s of the support is zero, namely, support does not exert rotation restraint on plate.

Introducing ψ_s=o and $\phi_s \to \infty$, the rib transfer matrix will yield the Support Boundaries Matrix giving,

$$\overset{\Delta}{[S]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{...(III-12)}$$

C. Transfer Conditions Across Thickness Variation

[Thickness Variation Matrix]

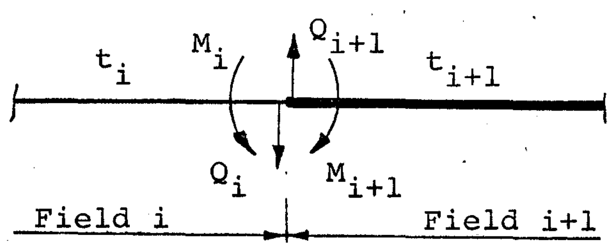


Fig. (3)

Considering a plate consists of two sections of different thickness joined together, let us denote the sections by i and i + 1 with corresponding thicknesses t_i and t_{i+1}. The continuation conditions across this sudden thickness change line require the deflection, slope moment and shear just before and after the change to satisfy the following relations:

$$w_{\eta_{i+1}} = w_{\eta_i} \qquad \ldots\text{(III-13.a)}$$

$$w'_{\eta_{i+1}} = w'_{\eta_i} \qquad \ldots\text{(III-13.b)}$$

$$M^*_{\eta_{i+1}} = M^*_{\eta_i} \frac{t_i^{\,3}}{t^3_{\,i+1}} \qquad \ldots\text{(III-13.c)}$$

$$Q^*_{\eta_{i+1}} = Q^*_{\eta_i} \frac{t_i^{\,3}}{t^3_{\,i+1}} \qquad \ldots\text{(III-13.d)}$$

where

$$M^*_{\eta_i} = -\frac{b_i^{\,2}}{D_i} M_{\eta_i} \quad \text{and}$$

$$Q^*_{\eta_i} = -\frac{b_i^{\,3}}{D_i} Q_{\eta_i}$$

The state vectors $\{\Phi_i\}$ and $\{\Phi_{i+1}\}$ will be related by the formula

$$\{\Phi_{i+1}\} = [T]\cdot\{\Phi_i\} \qquad \ldots\text{(III-14)}$$

where $[T]$ is a matrix expressing the continuation conditions across a thickness change line in the plate in a direction parallel to the applied stress.

Arranging equations (III-13) in a matrix form, we can write:

$$\begin{Bmatrix} w \\ w' \\ M^* \\ Q^* \end{Bmatrix}_{\eta_{i+1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (t_i/t_{i+1})^3 & 0 \\ 0 & 0 & 0 & (t_i/t_{i+1})^3 \end{bmatrix} \cdot \begin{Bmatrix} w \\ w' \\ M^* \\ Q^* \end{Bmatrix}_{\eta_i} \qquad \ldots\text{(III-15)}$$

Following the same procedure to obtain the Field Boundaries Matrix from the Field Matrix, we can arrive to the Thickness Change Boundaries Matrix $\overset{\Delta}{[T]}$ to be:

$$\overset{\Delta}{[T]} = (t_i/t_{i+1})^3 \cdot \begin{bmatrix} (t_{i+1}/t_i)^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & (t_i/t_{i+1})^3 \end{bmatrix} \quad \ldots\text{(III-16)}$$

CHAPTER IV

APPLICATION OF MATRIX METHOD TO STABILITY OF PLATE SYSTEM

In this chapter we are going to illustrate the application of Matrix method to a plate with any number of fields. In other words, we are going to extend the Matrix method to cover any general case of continuous plate. Also, we are going to formulate the general case of a plate with elastic supports.

Product Matrix [P] for any Number of Fields

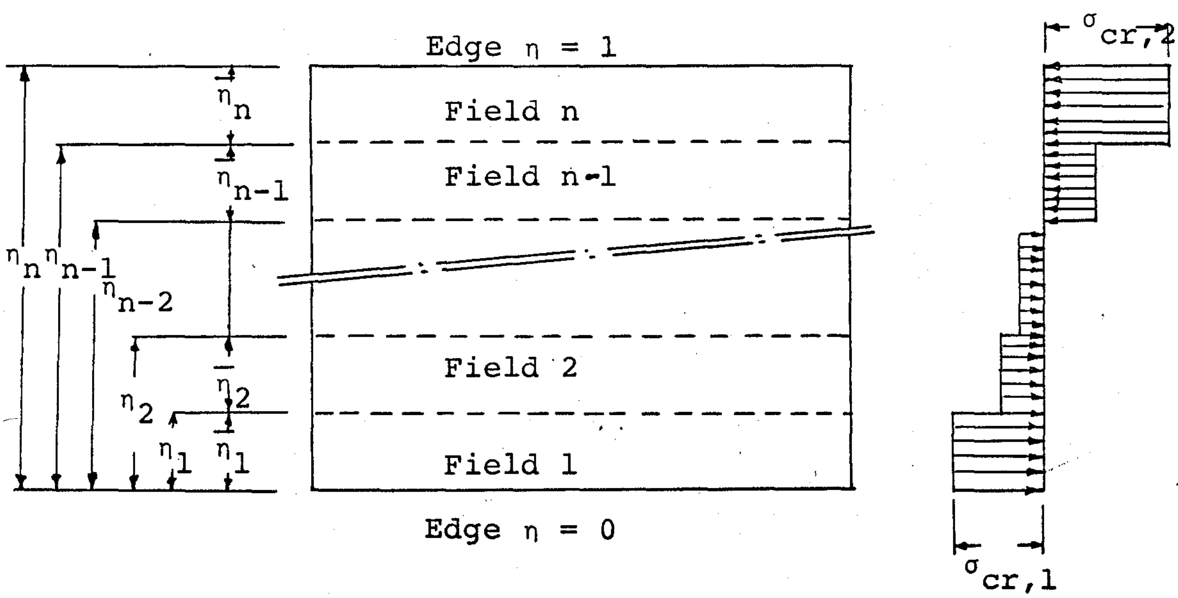


Fig. (4)

Considering a plate which is divided into a number of fields n, the ith field is subjected to a uniformly distributed stress of a value σ_i where i = 1 to n as shown in Fig. 4.

For any field i we can relate the state vectors $\{\Phi_{i-1}\}$ and $\{\Phi_i\}$ by the matrix formula:

$$\{\Phi_i\} = [F_i]\cdot\{\Phi_{i-1}\} \qquad \ldots(IV\text{-}1)$$

Equation (IV-1) can be applied to all fields of plate from i = 1 to i = n, then

$$\{\Phi_1\} = [F_1]\cdot\{\Phi_0\} \qquad \ldots(IV\text{-}2)$$

$$\{\Phi_2\} = [F_2]\cdot\{\Phi_1\}$$

--------------------------- and so on

$$\{\Phi_i\} = [F_i]\cdot\{\Phi_{i-1}\}$$

$$\{\Phi_{n-1}\} = [F_{n-1}]\cdot\{\Phi_{n-2}\}$$

$$\{\Phi_n\} = [F_n]\cdot\{\Phi_{n-1}\}$$

From the above substitutions we can express the state vector $\{\Phi_n\}$ in terms of the state vector $\{\Phi_o\}$.

$$\{\Phi_n\} = [F_n]\cdot[F_{n-1}] \,\ldots\ldots\, [F_i] \,\ldots\ldots\, [F_2]\cdot[F_1]\cdot\{\Phi_0\} \qquad \ldots(IV\text{-}3)$$

This general expression can be written in short to be:

$$\{\Phi_n\} = [P]\cdot\{\Phi_o\} \qquad \ldots(IV\text{-}4)$$

where

[P] is the Product Transfer Matrix.

This matrix can be formed by multiplying the individual Field Matrices which satisfies the intermediate boundary conditions of the fields and relating the boundary conditions, state vectors, across the whole assembly of fields.

We can arrive to the expression in equation (IV-4) through an easier computations procedure by defining $\{\bar{e}_{\eta_o}\}$ as a (6 x 1) column vector expressing the boundary conditions at edge η_o as:

Free end condition $$\{\bar{e}_1\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \qquad \text{...(IV-5.a)}$$

Fixed Roller end Condition $$\{\bar{e}_2\} = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \qquad \text{...(IV-5.b)}$$

Hinged Support $\{\bar{e}_5\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$ ...(IV-5.c)

Fixed Support $\{\bar{e}_6\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$ ...(IV-5.d)

As far as obtaining the buckling condition is concerned, in place of equation (IV-1), we can write

$$
\begin{aligned}
\{\bar{e}_{\eta_1}\} &= \overset{\Delta}{[F]}_1 \cdot \{\bar{e}_{\eta_0}\} \\
\{\bar{e}_{\eta_2}\} &= \overset{\Delta}{[F]}_2 \cdot \{\bar{e}_{\eta_1}\} \\
&\cdot \\
&\cdot \\
&\cdot \\
\{\bar{e}_{\eta_n}\} &= \overset{\Delta}{[F]}_n \cdot \{\bar{e}_{\eta_{n-1}}\}
\end{aligned}
\qquad \text{...(IV-6)}
$$

By knowing the end conditions vector $\{\bar{e}_{\eta_0}\}$as given by equations (IV-5)[1], expressions (IV-6) will result, n products of (6 x 6) matrix and a column vector (this can be reduced to n products of (5 x 5) matrix and a column vector, see

(1)

One can obtain $\{\bar{e}_{\eta_1}\}$ using the first equation in (IV-6). Similarly, knowing $\{\bar{e}_{\eta_1}\}$ enables to obtain $\{\bar{e}_{\eta_2}\}$ and so on, until we obtain $\{\bar{e}_{\eta_n}\}$ through the use of the last equation in (IV-6). Let us denote $\{\bar{e}_{\eta_n}\}$ obtained in this way as $\{\bar{e}_{\eta_n}\}_I$. However, the boundary condition at edge $\eta = 1$ is known, hence, one can write down the rector $\{\bar{e}_{\eta_n}\}$ directly. Depending on the actual boundary conditions the form of $\{\bar{e}_{\eta_n}\}$ is given by equations (IV-5). Let us denote the vector $\{\bar{e}_{\eta_n}\}$ obtained this way by $\{\bar{e}_{\eta_n}\}_{II}$. By comparing $\{\bar{e}_{\eta_n}\}_I$ which corresponds to the non zero element in $\{\bar{e}_{\eta_n}\}_I$ the buckling condition of the problem is obtained by equating this special element in $\{\bar{e}_{\eta_n}\}_I$ to zero. This method is known as the "Δ - coeeficient method". A detailed discussion of this method is given by Margeurre [18]. This method is useful for computational purposes. As can be seen, the buckling condition is obtained through the n multiplications of a (6 x 6) matrix by a column vector. If we obtain the buckling condition through the use of equation (IV-4), we need to find the product transfer matrix [P] which will involve n multiplications of (4 x 4) matrices.

Appendix) where expression (IV-3) will result, n products of (4 x 4) matrices.

Continuous plate over two spans

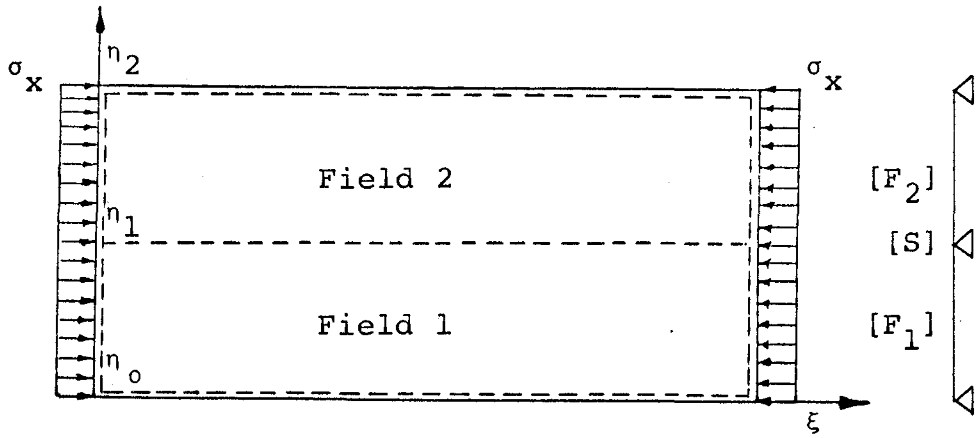


Fig. (5)

To find the buckling factor k, of a continuous plate over two spans as shown in fig. (5), we are going to relate the state vector $\{\Phi_{\eta_2}\}$ in terms of the state vector $\{\Phi_{\eta_o}\}$ by the Product matrix [P] according to the relation:

$$\{\Phi_{\eta_2}\} = [P]\cdot\{\Phi_{\eta_o}\} \qquad \text{...(IV-7)}$$

Referring to the deriviation of the Product Field Matrix [P] in equation (IV-3), we can write:

$$[P] = [F_2]\ [S]\ [F_i]$$

where $[F_i]$ is the transfer field matrix for field i (i=1,2) and [S] is the support transfer matrix as given by (III-12).

The boundary conditions at edges η_o and η_2 are assumed to be hinged , namely: deflection = 0 and moment = 0. Applying these boundary conditions to the state vectors at both ends we can establish:

$$\{\Phi_{\eta_2}\} = \begin{Bmatrix} 0 \\ w' \\ 0 \\ Q^* \end{Bmatrix}_{\eta_2} \quad \text{and} \quad \{\Phi_{\eta_o}\} = \begin{Bmatrix} 0 \\ w' \\ 0 \\ Q^* \end{Bmatrix}_{\eta_o} \qquad \ldots(\text{IV-8})$$

Expressing the formula (IV-7), by the matrix elements, we can write:

$$\begin{Bmatrix} 0 \\ w' \\ 0 \\ Q^* \end{Bmatrix}_{\eta_2} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ w' \\ 0 \\ Q^* \end{Bmatrix}_{\eta_o} \qquad \ldots(\text{IV-9})$$

Equation (IV-9) can be expanded to ~~be as~~:

$$\begin{aligned} 0 &= p_{12}\, w_o' + p_{14}\, Q_o^* \\ w_2' &= p_{22}\, w_o' + p_{24}\, Q_o^* \\ 0 &= p_{32}\, w_o' + p_{34}\, Q_o^* \\ Q_2^* &= p_{42}\, w_o' + p_{44}\, Q_o^* \end{aligned}$$

The condition for the nontrivial values of w_o' and Q_o^* will provide the condition:

$$\begin{vmatrix} p_{12} & p_{14} \\ p_{32} & p_{34} \end{vmatrix} = 0$$

or expressing the determinant gives:

$$p_{12} \cdot p_{34} - p_{14} \cdot p_{32} = 0 \qquad \ldots(\text{IV-10})$$

Equation (IV-10) forms the buckling condition for the case of continuous plate over two spans. The value of the buckling factor k that satisfies the above equation will give the buckling factor of the continuous plate.

The expression:

$$P_{12} \cdot P_{34} \underset{\Delta}{-} P_{14} \cdot P_{32} \qquad \ldots \text{(IV-11)}$$

forms the element P_{25} of the Product Field Boundaries Matrix $[\overset{\Delta}{P}]$.

Plate with Elastic Supports

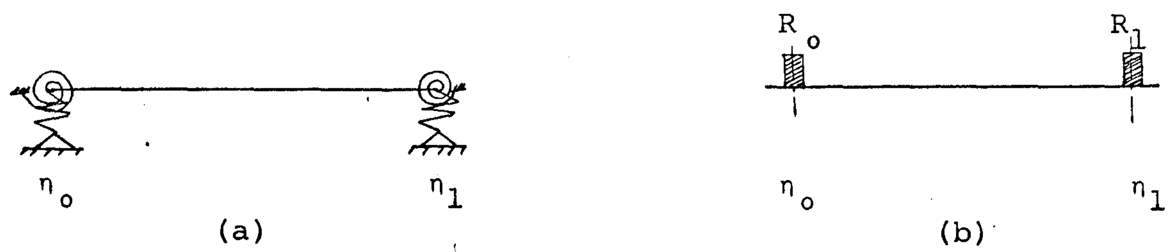


Fig. (6)

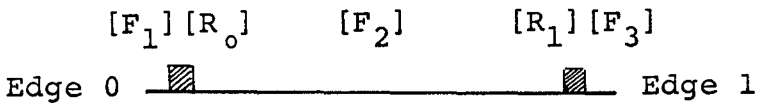
Considering the case of a plate with nonrigid supports, namely the rotation is resisted by the torsional stiffness of support and deformation of the support is proportional to the reaction in the direction of deflection. The structural model can be expressed by two sets of springs as shown in fig. (6.a)

The case of elastic supports is identical to the case of a stiffening rib fig. (6.b) where:

ϕ_r is the equivalent spring constant of the rib.

ψ_r is the torsional spring constant of the rib.

To establish the matrix formulation of this case, we are going to consider two additional fields of zero span as shown in fig. (6.b).



(Fig. 7)

The state vectors across the whole plate is related by the Product Matrix as:

$$\{\Phi_1\} = [P]\cdot\{\Phi_o\} \qquad \text{...(IV-12)}$$

where the Product Field Matrix is given by

$$[P] = [F_3]\cdot[R_1]\cdot[F_2]\cdot[R_0]\cdot[F_1]$$

where:

$[F_1]$ and $[F_3]$ are the field matrices of a plate field of zero width, (fig. 7). The value of a matrix of a plate of zero span reduces to an Identity Matrix.

$[R_1]$ and $[R_2]$ are the matrices expressing the transition conditions across the ribs R_1 and R_2.

The boundary conditions at the free edges o and 1 are:

Moment = 0 Shear Force = 0

Introducing these boundary conditions in the state vectors expression equation (IV-12), we can state:

$$\begin{Bmatrix} w_1 \\ w_1' \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \cdot \begin{Bmatrix} w_o \\ w_o' \\ 0 \\ 0 \end{Bmatrix} \quad \ldots \text{(IV-13)}$$

This matrix formulation (IV-13) can be written in equation forms to be:

$$w_1 = p_{11}\, w_o + p_{12}\, w_o'$$

$$w_1' = p_{21}\, w_o + p_{22}\, w_o'$$

$$0 = p_{31}\, w_o + p_{32}\, w_o'$$

$$0 = p_{41}\, w_o + p_{42}\, w_o'$$

The condition for the nontrivial values of the parameters w_o and w_o' is,

$$\begin{vmatrix} p_{31} & p_{32} \\ p_{41} & p_{42} \end{vmatrix} = 0 \quad \ldots \text{(IV-14)}$$

or expanding the determinant in equation (IV-14), gives:

$$p_{31}\, p_{42} - p_{41}\, p_{32} = 0 \quad \ldots \text{(IV-15)}$$

The right hand side of expression (IV-15) gives the term $\overset{\Delta}{p}_{61}$ of the Product Field Boundaries Matrix $\overset{\Delta}{[P]}$. The value of k, that satisfies the equation (IV-15) gives the buckling factor for the contineous plate over elastic support conditions.

CHAPTER V

STABILITY OF CHANNEL COLUMNS

INTRODUCTION

In this chapter we are going to apply the Matrix Transfer method to calculate the buckling load of a strut of channel cross section.

We shall only consider the local buckling of a channel cross section treated as a system of plates. The modes of buckling considered are as follows:

a) local buckling of the flanges and web and

b) local buckling of the web.

when the flange-web ratio is large. The flange, taken as as the outstanding leg of the channel, is flexible and hence local buckling occurs at the flanges and the web taken as an assembly of plates.

When the flange-web ratio is small, the flange, taken as a stiffening rib for the web plate, acts as an elastic support for the web plate, and hence local buckling occurs at the web only.

a) Local Buckling of Flanges and Web

For the case of relatively large flange-web ratio the channel section is treated as composed of three fields of the web plate with two over hanging flange plates. The structural model is shown in fig. (8).

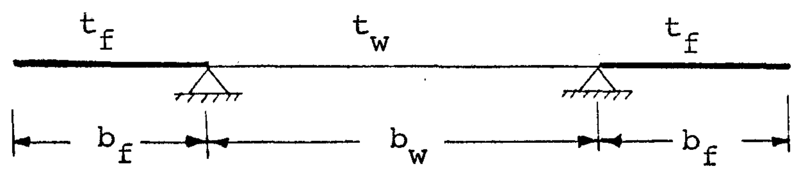


Fig. (8)

Considering a channel section of web width b_w and thickness t_w, the flanges are of thickness t_f and width b_f. The corners are assumed to remain straight which are expressed as simple supports.

The state vectors at the extreme boundaries are related by the Product Field Boundaries Matrix by the equation as:

$$\{\Phi_1\} = [P] \cdot \{\Phi_o\}$$

where

$$[P] = [F]_f \cdot [S] \cdot [T] \cdot [F]_w \cdot [S] \cdot [T] \cdot [F]_f$$

$$\overset{\Delta}{[P]} = \overset{\Delta}{[F]}_f \cdot \overset{\Delta}{[S]} \cdot \overset{\Delta}{[T]} \cdot \overset{\Delta}{[F]}_w \cdot \overset{\Delta}{[S]} \cdot \overset{\Delta}{[T]} \cdot \overset{\Delta}{[F]}_f \qquad \ldots (V\text{-}1)$$

with

$[F]_f$ as the Field Transfer Matrix of the flange plate

$[F]_w$ as the Field Transfer Matrix of the web plate

$[S]$ is the Support Transfer Matrix

$[T]$ is the Thickness Variation Matrix

The extreme ends of the plate assembly is free edges. The boundary conditions are well known to be:

$M^* = 0 \qquad\qquad Q^* = 0$

Refering to the illustrative matrix (I-30), it is clear that the term $\overset{\Delta}{p}_{61}$ of the Field Boundaries Matrix represents the free end conditions of the plate.

The expression $\overset{\Delta}{p}_{61}$ of the Product Field Boundaries Matrix given by equation (V-1), when equated to zero, will form the buckling condition of the structural model considered.

The value of the buckling factor k, which satisfies the buckling condition gives the buckling factor of the plate assembly or in other words is the buckling factor governing the local buckling of flanges and web of the channel section.

To form the Product Field Boundaries Matrix we follow simple Matrix multiplication for the equation (V-1) which gives:

$$[\overset{\Delta}{F}]_f = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix}_{\text{flange}} \quad \ldots (V-2)$$

$$\overset{\Delta}{[S]}\cdot\overset{\Delta}{[T]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\frac{t_f}{t_w})^3 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{...(V-3)}$$

$$\overset{\Delta}{[F]}_f\cdot\overset{\Delta}{[S]}\cdot\overset{\Delta}{[T]} = \begin{bmatrix} f_{15} & f_{16}\cdot t & 0 & 0 & 0 & 0 \\ f_{25} & f_{26}\cdot t & 0 & 0 & 0 & 0 \\ f_{35} & f_{36}\cdot t & 0 & 0 & 0 & 0 \\ f_{45} & f_{46}\cdot t & 0 & 0 & 0 & 0 \\ f_{55} & f_{56}\cdot t & 0 & 0 & 0 & 0 \\ f_{65} & f_{66}\cdot t & 0 & 0 & 0 & 0 \end{bmatrix}_{\text{flange}} \quad \text{...(V-4)}$$

where $t = (\frac{t_f}{t_w})^3$

$$\overset{\Delta}{[F]}_w = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix}_{\text{web}} \quad \text{...(V-5)}$$

Continuing the multiplication of matrices in equation (V-1), this will give the Product Field Boundaries Matrix $\overset{\Delta}{[P]}$, the term $\overset{\Delta}{p}_{61}$ will express the proper boundary conditions of the free flange paltes:

$$\overset{\Delta}{p}_{61} = (\overset{\Delta}{f}_{11})_f^2 \cdot (\overset{\Delta}{f}_{25})_w + 2t(\overset{\Delta}{f}_{11})_f \cdot (\overset{\Delta}{f}_{21})_f \cdot (\overset{\Delta}{f}_{15})_w + (\overset{\Delta}{f}_{21})_f^2 \cdot (\overset{\Delta}{f}_{16})_w \qquad \text{...(V-6)}$$

Equating the right hand side of equation (V-6) to zero, we have:

$$(\overset{\Delta}{f}_{11})_f^2 \cdot (\overset{\Delta}{f}_{25})_w + 2t(\overset{\Delta}{f}_{11})_f \cdot (\overset{\Delta}{f}_{21})_f \cdot (\overset{\Delta}{f}_{15})_w + (\overset{\Delta}{f}_{21})_f^2 \cdot (\overset{\Delta}{f}_{16})_w = 0 \qquad \text{...(V-7)}$$

The solution of the buckling condition is carried out numerically by trial and error by the help of the electronic computer I. B. M. 7040 to give the value of the buckling factor k, that satisfies the buckling condition. Calculations are done for different values of flange-web ratios (η) from $\eta = 0$ to $\eta = 2$ with an interval of 0·05. Results are given in Table (1) compared to the work done by:

1. Kimm, G. [5]
2. Müller-Magyari [7]
3. Kroll, W. D. [17]
4. Ritz method, two approximation terms
5. Ritz method, three approximation terms
6. Bleich, F. [1]
7. Matrix transfer method

η	1	2	3	4	5	6	7
0·0			4·000			4·000	4·000
0·1			4·440			4·450	4·450
0·2	4·259		4·520			4·650	4·585
0·3			4·400			4·260	4·377
0·4			3·660			3·300	3·755
0·5	2·328	2·908	2·840	2·948	2·948	2·780	2·910
0·75	1·352		1·480			1·356	1·497
1·00	0·80	0·884	0·864	0·904	0·904	0·847	0·888
1·25	0·531		0·600			0·584	0·588
1·50	0.375	0.415	0.408	0.424	0.420	0.426	0.419

TABLE 1

The above Table 1 shows that the values obtained by the Matrix Transfer Method agrees well with the previously obtained results. This simple comparison also serves as a check to the Matrix transfer method program.

A plot of the flange-web ratio verses the buckling factor k is given by fig. (9), comparing the values resulting of different methods.

It should be noted that for the case of a channel cross section of a zero flange-web ratio, the corresponding value of the buckling factor k equals 4·00, which is the well known case of the buckling factor of a simply supported plate [14]. This is due to the fact that in the analysis, it is assumed that there is no deflection at the corner of the channel section. It is obvious that as the flange-web ratio decreases, this assumption will not be valid and as the flange-web ratio approaches zero, such assumption presents an incorrect result.

For small flange-web ratios, the flanges essentially act as lips to the web plate. Thus, the local buckling of the channel section occurs at the web plate. The effect of flanges is to give the web plate elastic support, both transversely and torsionally. The analysis of the buckling load is given in the next section.

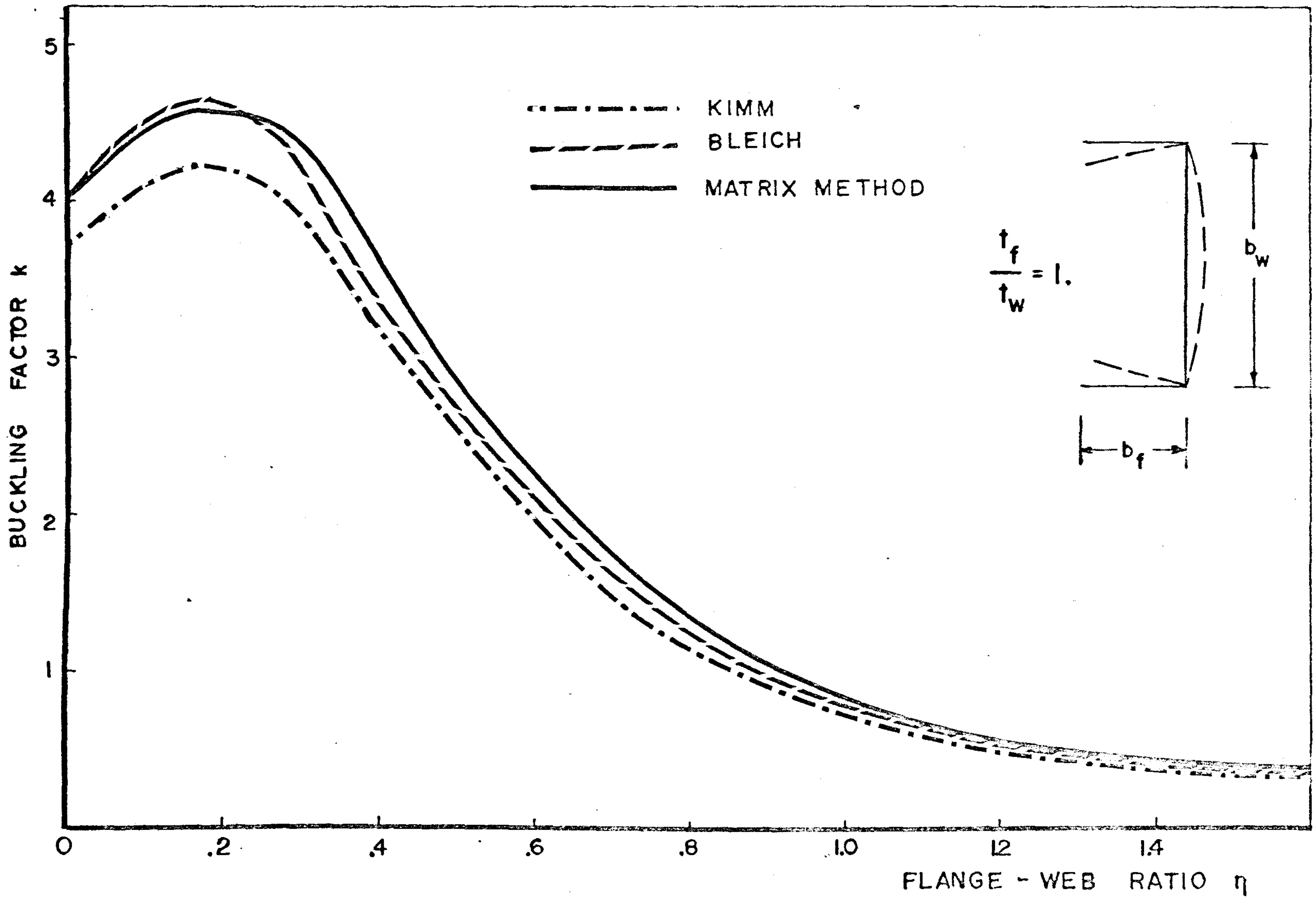
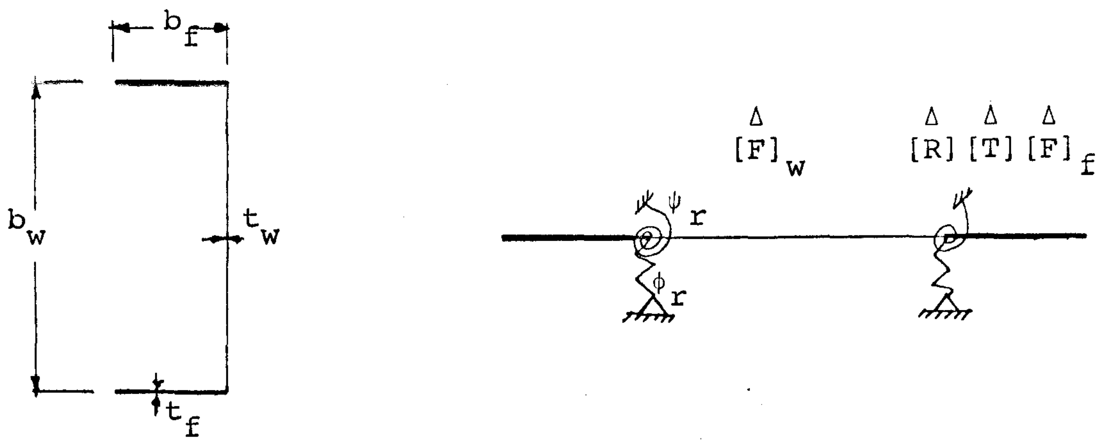

KIMM
BLEICH
MATRIX METHOD
$\frac{t_f}{t_w} = 1.$
b_w
b_f
5
4
3
2
1
0
.2
.4
.6
.8
1.0
1.2
1.4
BUCKLING FACTOR k
FLANGE - WEB RATIO η

FIG.9

b) <u>Local Buckling of Web Plate</u>



(a) (b)

Fig. (10)

For the general case of a channel of a small value of flange-web ratio, the assumption that the corners remain straight will not be realistic. A structural model represents the influence of the flange plates as providing the edges of the web plate by torsional and deflection springs is illustrated by fig. (10.b). This is the same effect as if the flange plates are considered stiffening ribs for the web plate.

In this case, the Product Field Boundaries Matrix consists of as follows:

$$\overset{\Delta}{[P]} = \overset{\Delta}{[F]}_f \cdot \overset{\Delta}{[R]} \cdot \overset{\Delta}{[T]} \cdot \overset{\Delta}{[F]}_w \cdot \overset{\Delta}{[T]} \cdot \overset{\Delta}{[R]} \cdot \overset{\Delta}{[F]}_f \qquad \text{...(V-8)}$$

where

$\overset{\Delta}{[F]}_f$ is the Field Boundaries Matrix expressing the transfer conditions across the flange plate as given in equation (II-31)

$[\overset{\Delta}{F}]_w$ is the Field Boundaries Matrix expressing the transfer conditions across the web plate.

$[\overset{\Delta}{T}]$ is the Field Boundaries Matrix expressing the transfer conditions across the thickness change line, as given in equation (III-16).

$[\overset{\Delta}{R}]$ is the Field Boundaries Matrix expressing the transfer conditions across the stiffening rib. Its elements are formed by:

ϕ_r the equivalent spring constant of the rib

$$\phi_r = \gamma_B \theta^4 - k_r \delta \pi^2 \theta^2$$

ψ_r the torsional spring constant of the rib

$$\psi_r = \gamma_w \theta^4 + (\gamma_D - k_r \pi^2 \gamma_p) \theta^2$$

The extreme edges of the structural model have the free boundary conditions. In a similar argument as before, the corresponding term of the Product Field Boundaries Matrix giving the buckling condition is:

$$\overset{\Delta}{p}_{61} = 0$$

The value of k, which satisfied the buckling condition, equation (V-9), represents the buckling factor of the channel column.

A numerical trial and error method by the help of computer, is adopted. The results are represented as a plot of the web width - column length ratio α verses the buckling factor k, for a set of flange-web ratios, fig. (11).

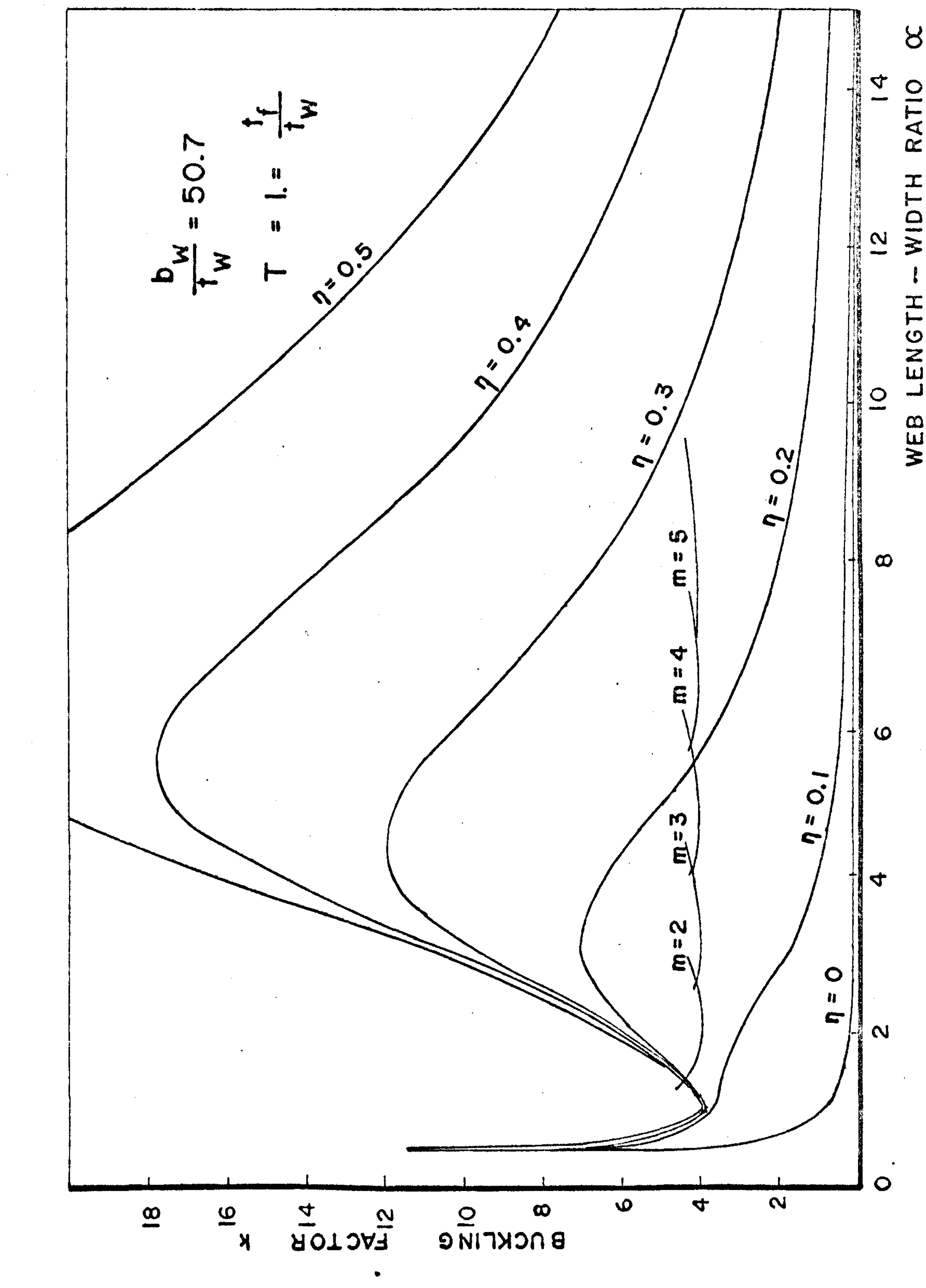
These results are imposed on the previously obtained results for the case of local buckling of the web and flange plates. A set of combined curves are obtained for a constant width of web to thickness of web plate ratio of fifty.

This set of curves shown in fig. (12), gives a clear limit between the buckling behaviour of the channel as the local buckling of web plate only and the web and flange plates. This limit is obtained by getting the same value of k, considering both forms of behaviour.

Similar curves can be obtained for different values of the parameter γ and the column height-web width ratio α.

The limit between the two modes of buckling considered, namely local buckling of web and flanges and local buckling of web only, is given in fig. (13). The dotted line shows the limit after which the buckling formula of Euler gives satisfactory results. (10%).

It is important to note that the buckling factor k, calculated by Euler's column Formula, and from the local buckling of web only, as mentioned before, will never have the same result. The Euler buckling factor is always less, due to the fact that he did not consider the energy required to deform the cross section during buckling.



$\frac{b_w}{t_w} = 50.7$
$T = 1. = \frac{t_f}{t_w}$
$\eta = 0.5$
$\eta = 0.4$
$\eta = 0.3$
$\eta = 0.2$
$\eta = 0.1$
$\eta = 0$
m=2
m=3
m=4
m=5
BUCKLING FACTOR k
2
4
6
8
10
12
14
16
18
0
2
4
6
8
10
12
14
WEB LENGTH -- WIDTH RATIO α

FIG. 11

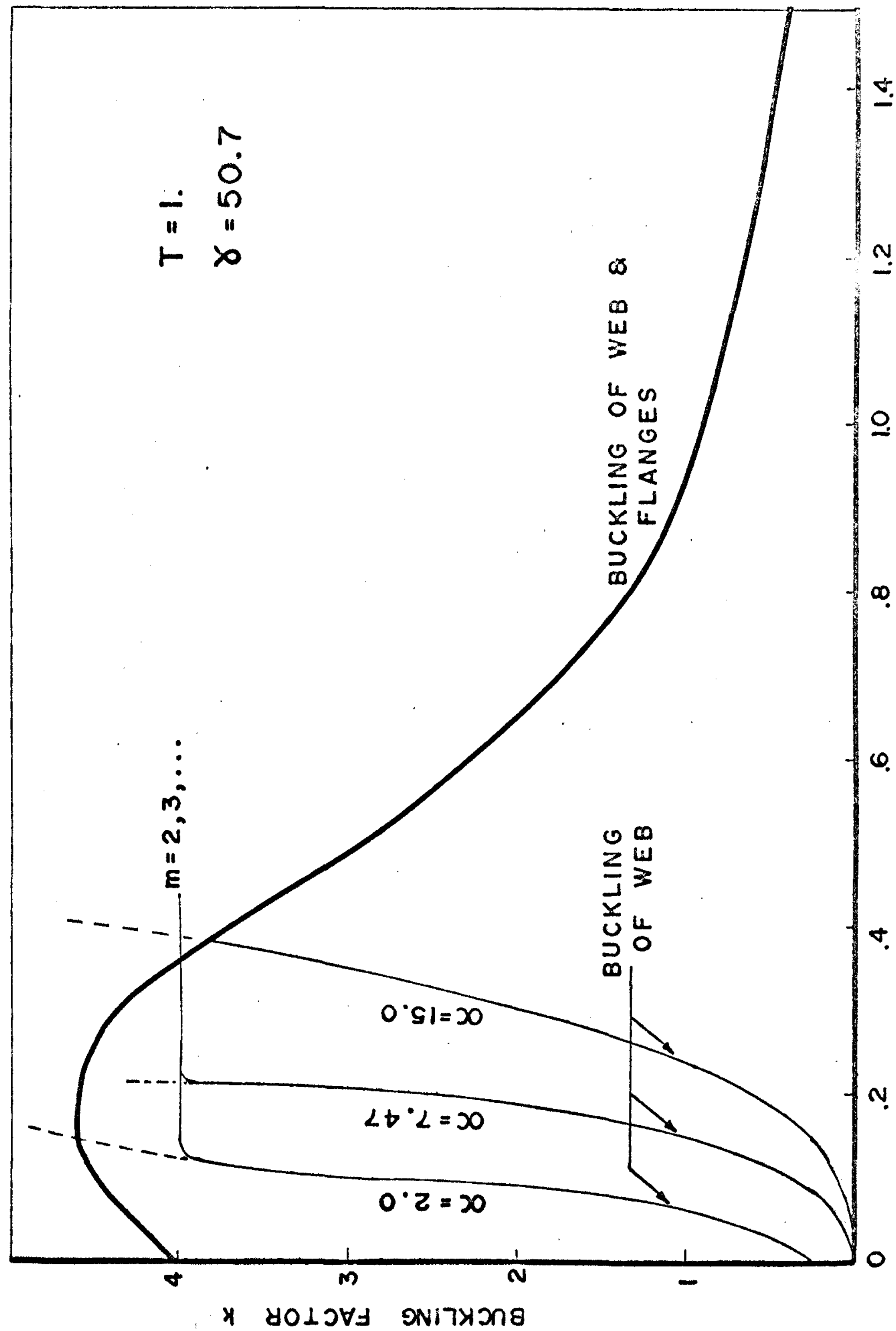

T = 1.
γ = 50.7
m=2,3,...
α = 2.0
α = 7.47
α = 15.0
BUCKLING OF WEB
BUCKLING OF WEB & FLANGES
BUCKLING FACTOR k
FLANGE — WEB RATIO η
0
.2
.4
.6
.8
1.0
1.2
1.4
1
2
3
4

FIG. 12

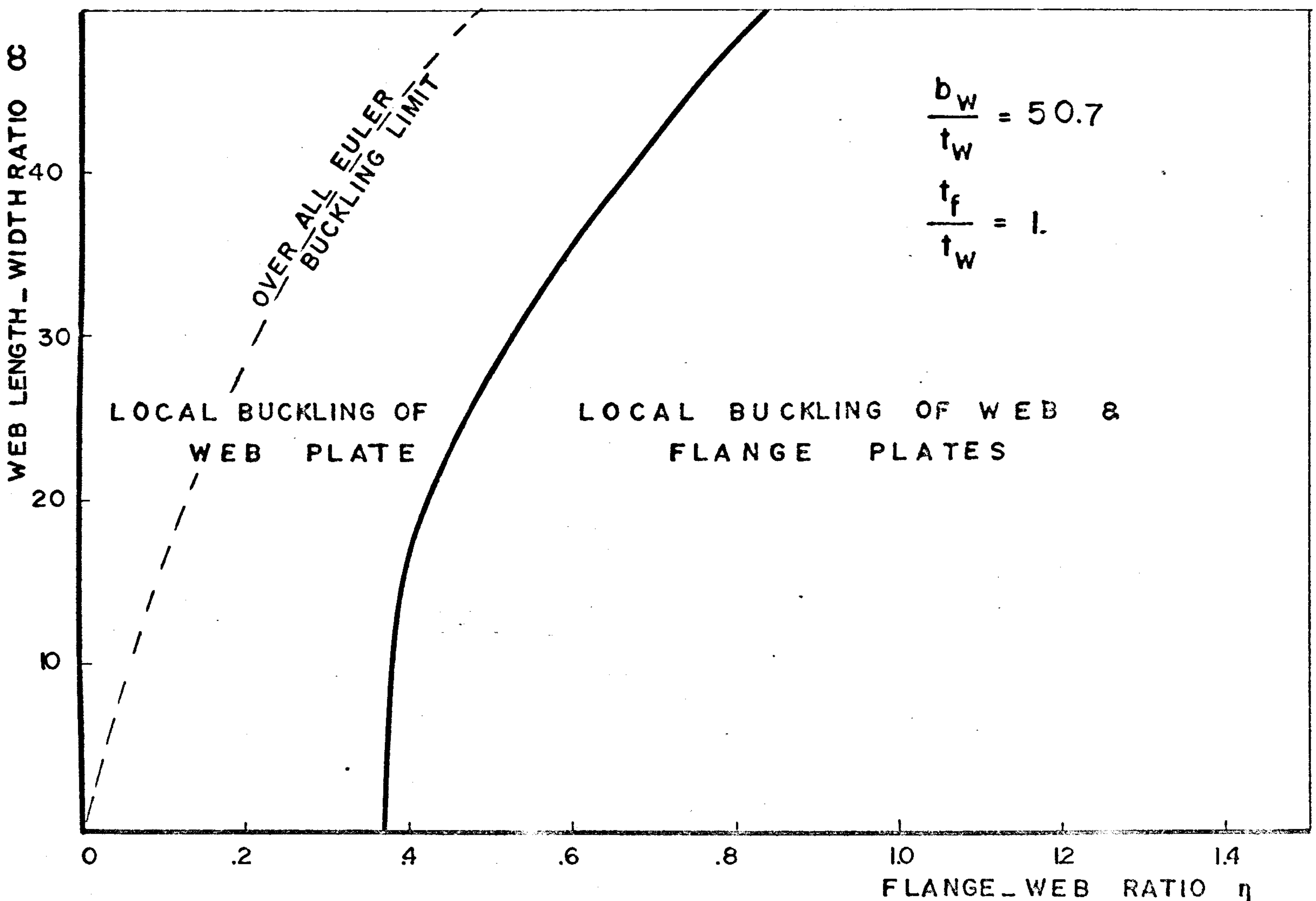

WEB LENGTH_WIDTH RATIO α
OVER ALL EULER
BUCKLING LIMIT
$\frac{b_w}{t_w} = 50.7$
$\frac{t_f}{t_w} = 1.$
LOCAL BUCKLING OF
WEB PLATE
LOCAL BUCKLING OF WEB &
FLANGE PLATES
10
20
30
40
0
.2
.4
.6
.8
1.0
1.2
1.4
FLANGE_WEB RATIO η

FIG.13

CHAPTER VI

EXPERIMENTAL WORK

INTRODUCTION

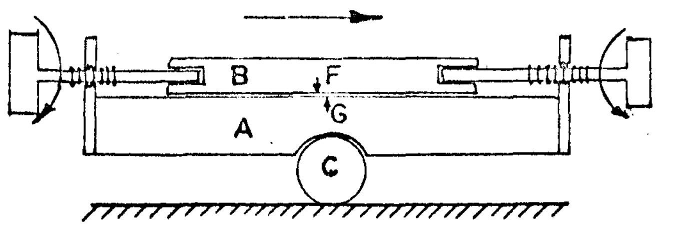
A set of experiments is carried out to study the local buckling modes of failure of channel columns. Test results are compared with the theoretically predicted behaviours.

Illustration of the test device, specimens experiments and results will be presented in this chapter.

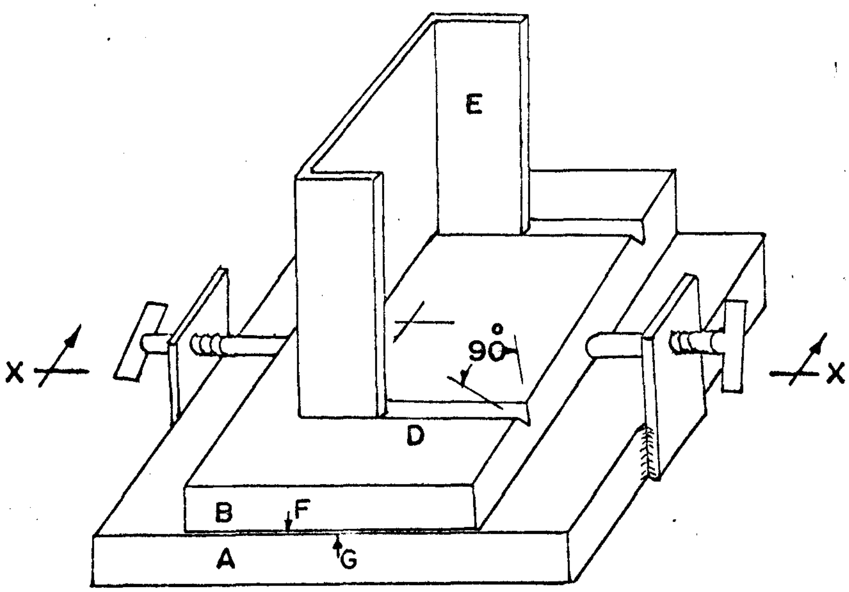
Apparatus:

The Uniform Compressive stress is applied by 120,000 lb. capacity tension-compression testing machine with an accuracy of 10. lb. within the used range of loading. The loading head is provided with a ball bearing joint.

In order to satisfy the theoretically assumed hinged supports for the web and flange plates, and to allow for all possibilities of different buckling modes of failure, a special end fixture is constructed as shown in fig. (14) and (15). The fixture consists of two plates A and B. A 90° V-groove is made on plate B, so that the channel section fits in the groove. Plate A rests on a steel ball bearing to allow rotation. By the manual controlled motion of plate B relative to plate A, the ball bearing point can be al igned with the centre of gravity of the channel cross section. The bearing plates A and B are taken thick enough to ensure a uniform stress distribution on the web and flange plates.



CROSS SEC.X-X



A_ Pl. 5 x 5 x 0.5

B_ Pl. 5 x 4 x 0.5

C_ BEARING BALL D = 1"

D_ 90° V_ GROOVE

E_ TEST SPECIMEN

F_ C.G. OF CHANNEL

G_ C.L OF BEARING BALL.

FIG.14

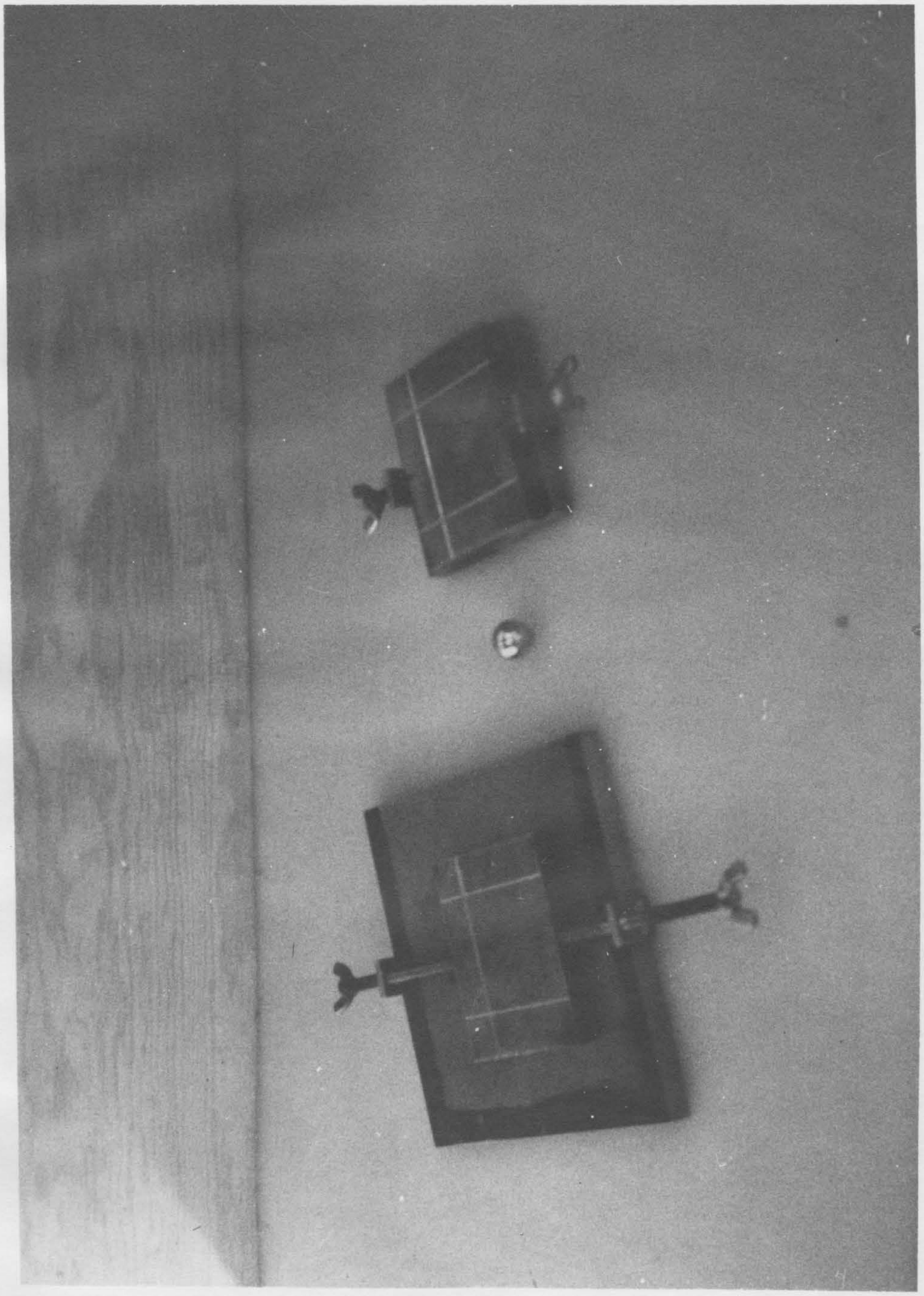


Fig. (15)

Test Specimens

Three groups of test specimens were made from Galvanized steel sheets. The steel sheets were cut and bent into channel sections of various sizes and lengths.

All channel sections have a constant thickness of 0·0589 inches.

Web width is taken a constant value of 3·00 inches, namely the web width-thickness ratio (r) is kept a constant value of 50·7.

The three test goups were of the same cross sectional dimensions but varying in column height. This enables to consider a wide range of the height-web width ratio (α) as follows:

Group	Column Height (ins.)	(α)
A	6·0	2·0
B	22·4	7·47
C	45·0	15·0

A wide range of flange-web ratio is tested covering the possibilities of different buckling modes of failure to take place, as given by table (2).

The mechanical properties of the used material was found through a simple tension test, by measuring the applied stress and the corresponding longitudinal and lateral strain, to be:

Modulous of Elasticity (E) = $3{\cdot}190 \times 10^{6}$ psi

Poisson's Ratio (γ) = 0·359

Table (2)

Group A			Group B			Group C		
Symbol	$b_{f\,(in.)}$	η	Symbol	$b_{f\,(in.)}$	η	Symbol	$b_{f\,(in)}$	η
A_1	0·00	0·00	B_1	0·00	0·00	C_1	0·00	0·00
A_2	0·15	0·05	B_2	0·15	0·05	C_2	0·15	0·05
A_3	0·30	0·10	B_3	0·30	0·10	C_3	0·30	0·10
A_4	0·33	0·11	B_4	--	--	C_4	--	--
A_5	0·36	0·12	B_5	--	--	C_5	--	--
A_6	0·39	0·13	B_6	--	--	C_6	--	--
A_7	--	--	B_7	0·51	0·17	C_7	--	--
A_8	0·60	0·20	B_8	0·60	0·20	C_8	0·60	0·20
A_9	--	--	B_9	0·72	0·24	C_9	--	--
A_{10}	--	--	B_{10}	--	--	C_{10}	0·81	0·27
A_{11}	0·90	0·30	B_{11}	0·90	0·30	C_{11}	0·90	0·30
A_{12}	--	--	B_{12}	--	--	C_{12}	1·11	0·37
A_{13}	1·20	0·40	B_{13}	1·20	0·40	C_{13}	1·20	0·40
A_{14}	1·50	0·50	B_{14}	1·50	0·50	C_{14}	1·50	0·50
A_{15}	1·80	0·60	B_{15}	1·80	0·60	C_{15}	1·80	0·60
A_{16}	2·10	0·70	B_{16}	2·10	0·70	C_{16}	2·10	0·70

Test Procedure

The channel column is fitted in the groove of the end bearing plates. The bearing ball is alligned with the centre of gravity of the channel cross section. An experimental set-up is shown in fig. (16) and fig. (17). The compressive stress is applied by the loading head of the machine moving at speed of 0.005 inch/min. This low rate of stress application is adopted to avoid the disturbance that may accompany the higher rates of load application.

A set of dial gages were set up along the column to measure the deflections at the mid-point and the quarter-point of column height. The dial readings give an indication of the loss of stability when the column starts to buckle. Comparing the readings of the gages at the same cross section will enable us to establish the deformations of the cross section during loading and buckling processes.

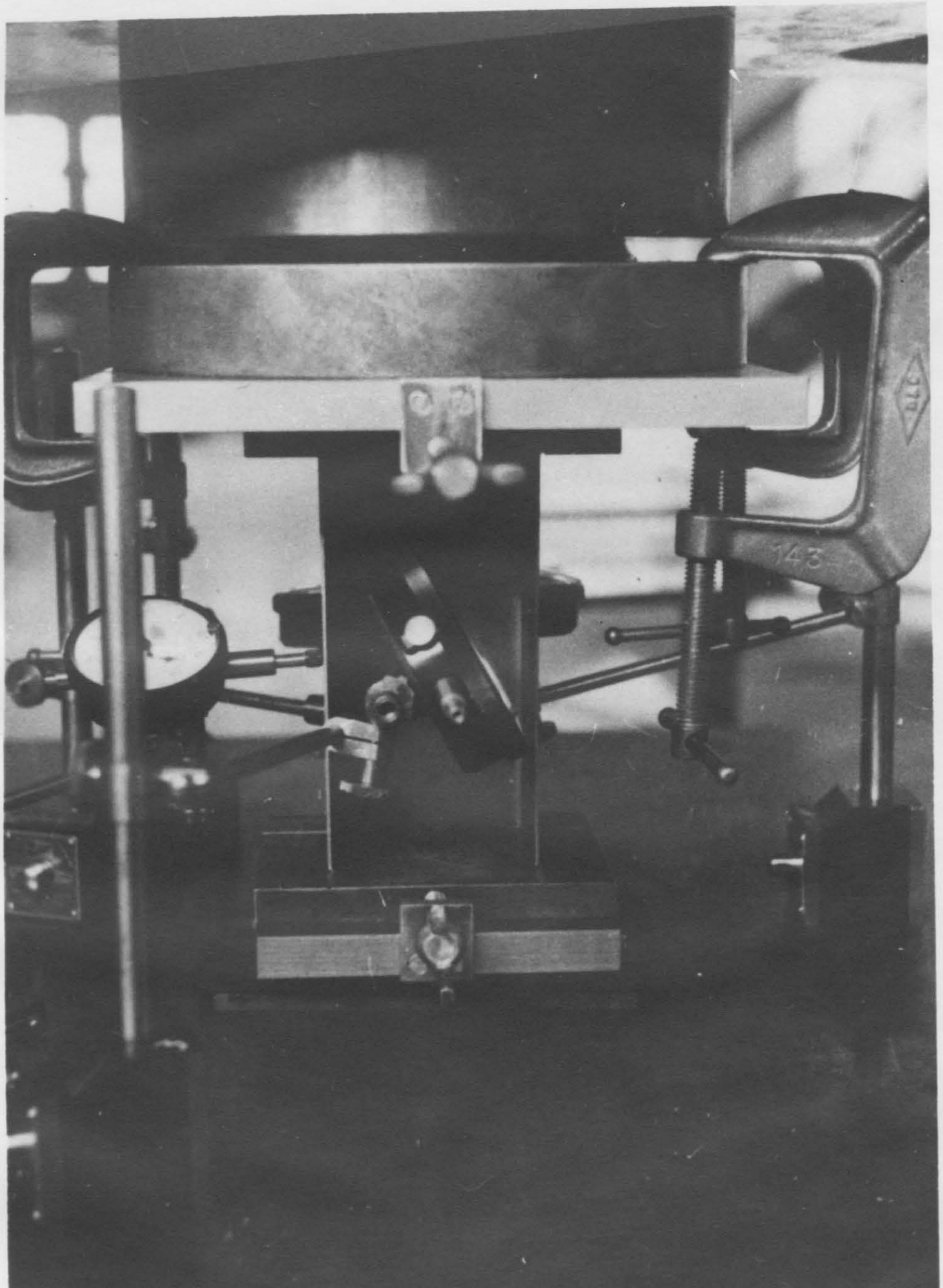


Fig. (16)

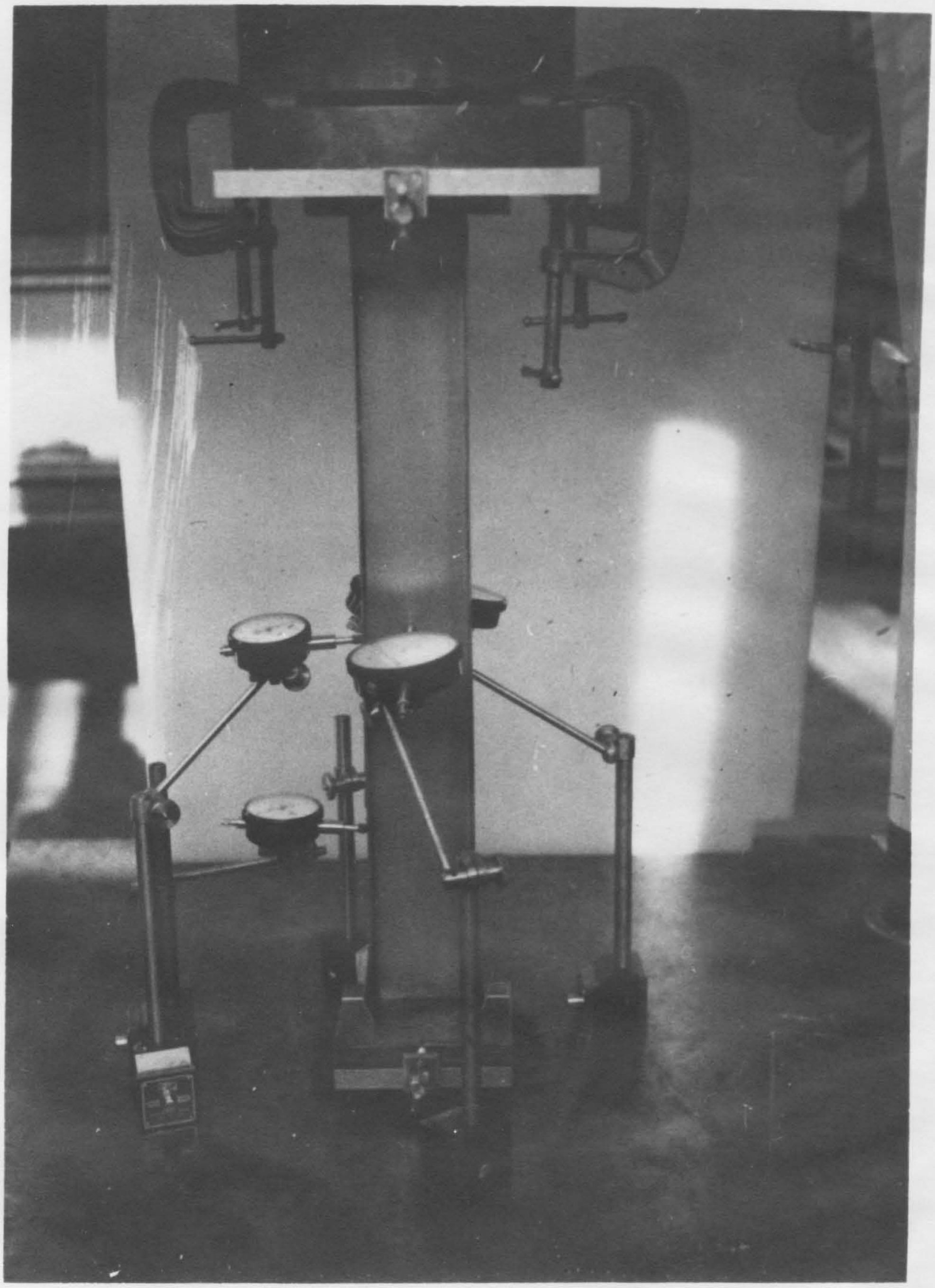


Fig. (17)

Test Results

The buckling stress is divided by the Euler stress to give the buckling factor k, which is plotted in fig. (22), (23) and (24) with the theoretically predicted values.

The following remarks are observed during loading and buckling of the channel columns.

1. No twisting or torsional failure took place. This is due to the fact that the energy required for flexural torsional mode of failure is higher than the energy required to carry the failure through flexural mode of instability.

2. Some of the test specimens failed due to the local buckling of flanges and web plates where others failed due to the buckling of the web plate only. In the first case, the channel corners are observed to remain straight while the latter case the corners follow a deflected curve fig. (18). A test specimen C_{13} followed both mentioned buckling modes of failures, fig. (19) and fig. (20).

3. Some specimens failed following a one half-wave buckling curve and some others followed a two half-waves buckling curve, depending on the dimensions of the specimens. An example is shown by fig. (21).

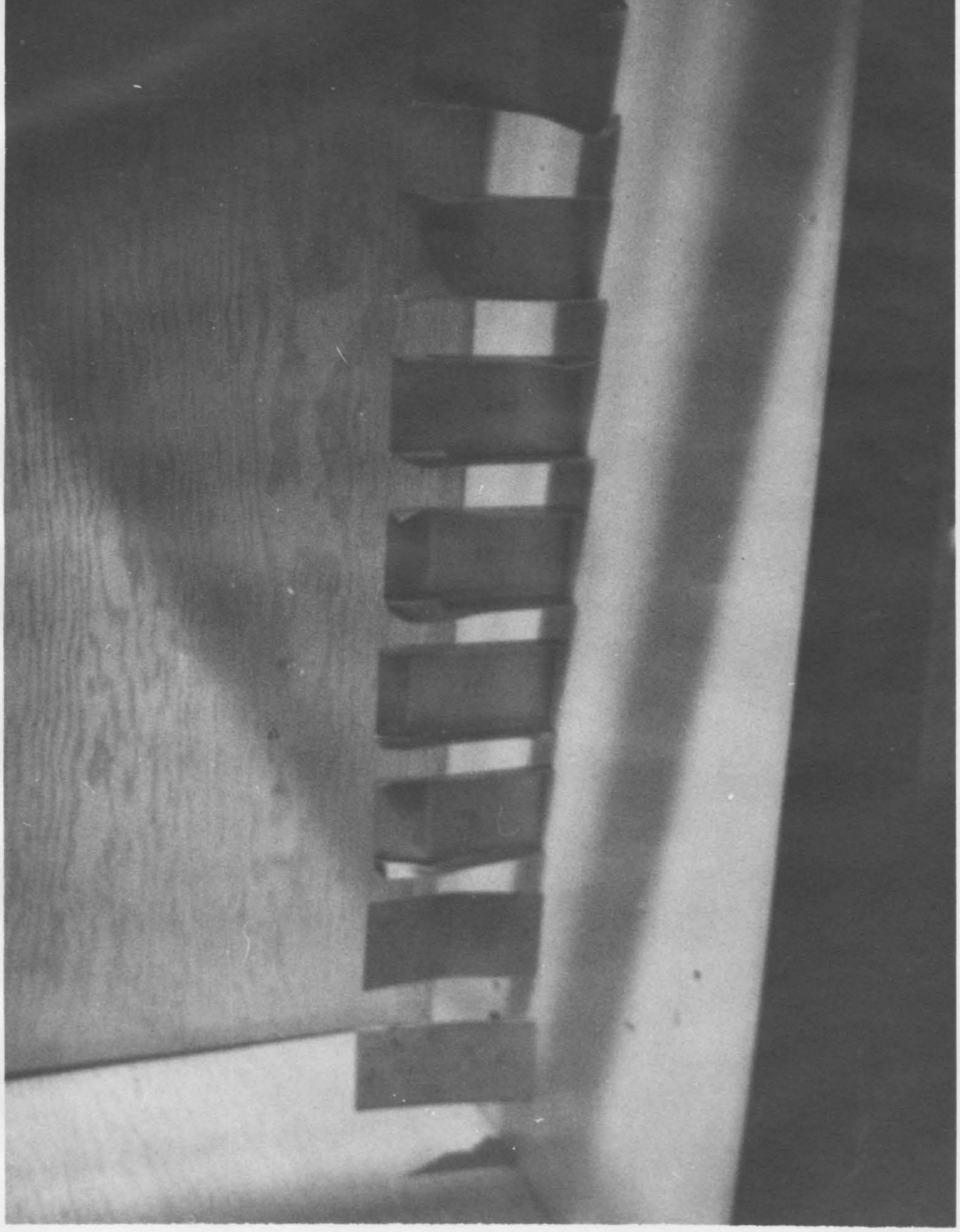


Fig. (18)

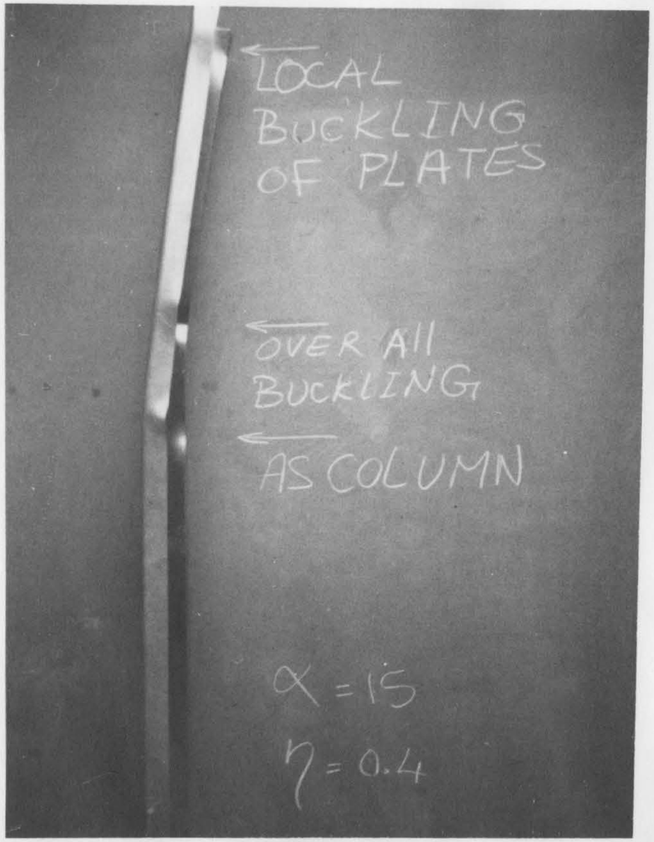

LOCAL
BUCKLING
OF PLATES
OVER All
BUCKLING
AS COLUMN
α = 15
η = 0.4

Fig. (19)

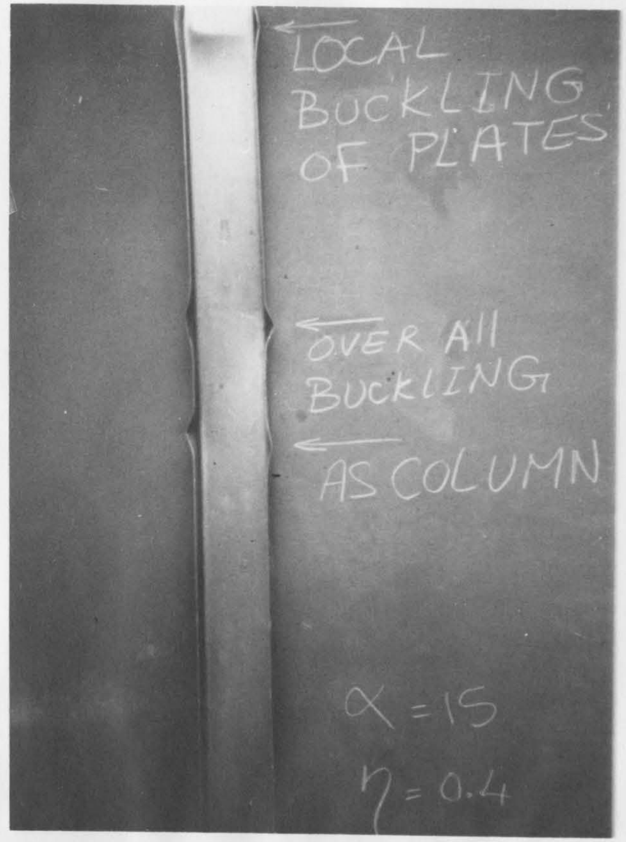

LOCAL
BUCKLING
OF PLATES
OVER All
BUCKLING
AS COLUMN
α = 15
η = 0.4

Fig. (20)

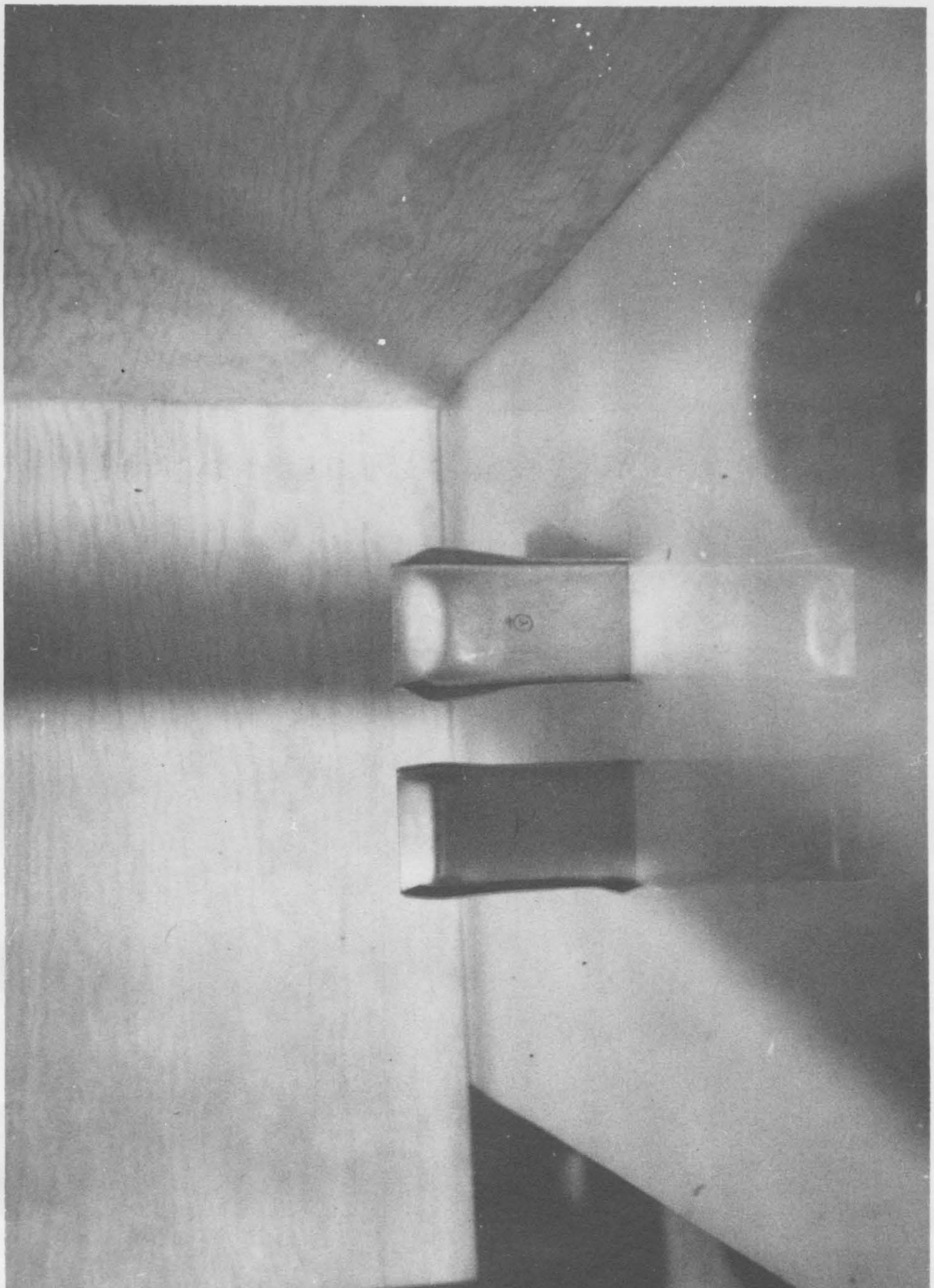


Fig. (21)

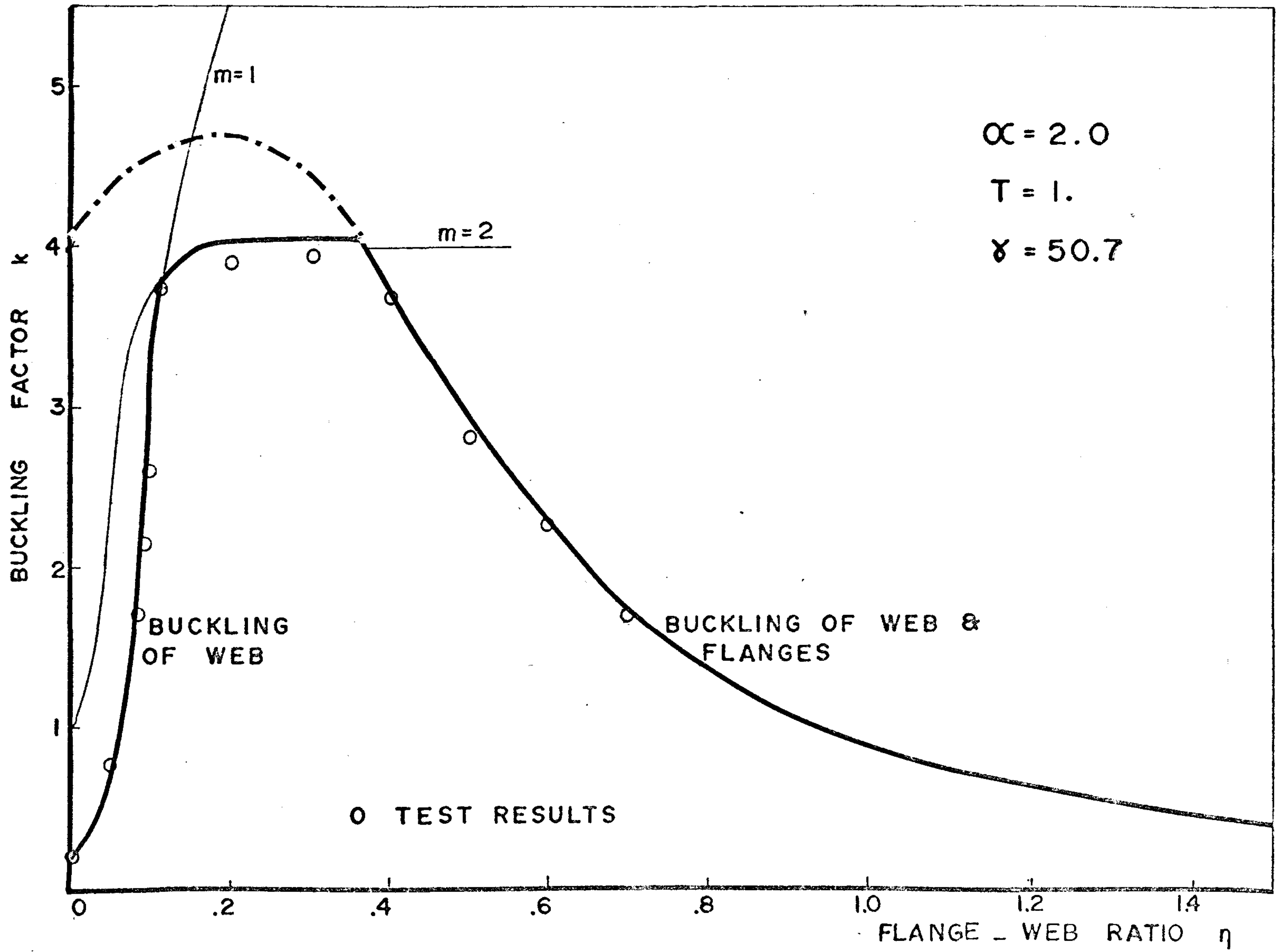

m=1
m=2
α = 2.0
T = 1.
γ = 50.7
BUCKLING FACTOR k
BUCKLING OF WEB
BUCKLING OF WEB & FLANGES
O TEST RESULTS
0
1
2
3
4
5
0
.2
.4
.6
.8
1.0
1.2
1.4
FLANGE _ WEB RATIO η

FIG. 22

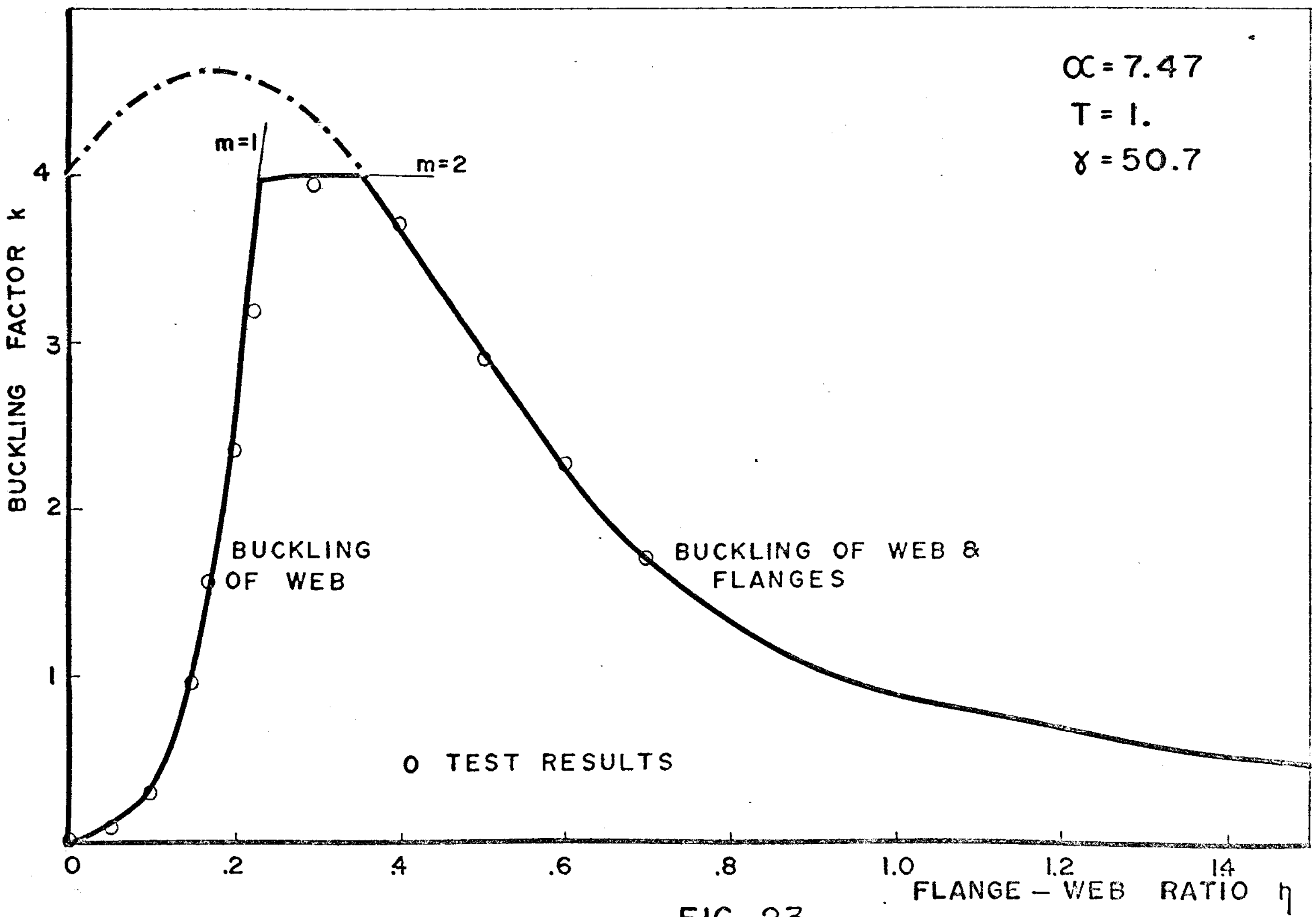

α = 7.47
T = 1.
γ = 50.7
m=1
m=2
4
3
2
1
BUCKLING FACTOR k
BUCKLING
OF WEB
BUCKLING OF WEB &
FLANGES
o TEST RESULTS
0
.2
.4
.6
.8
1.0
1.2
14
FLANGE – WEB RATIO η

FIG. 23

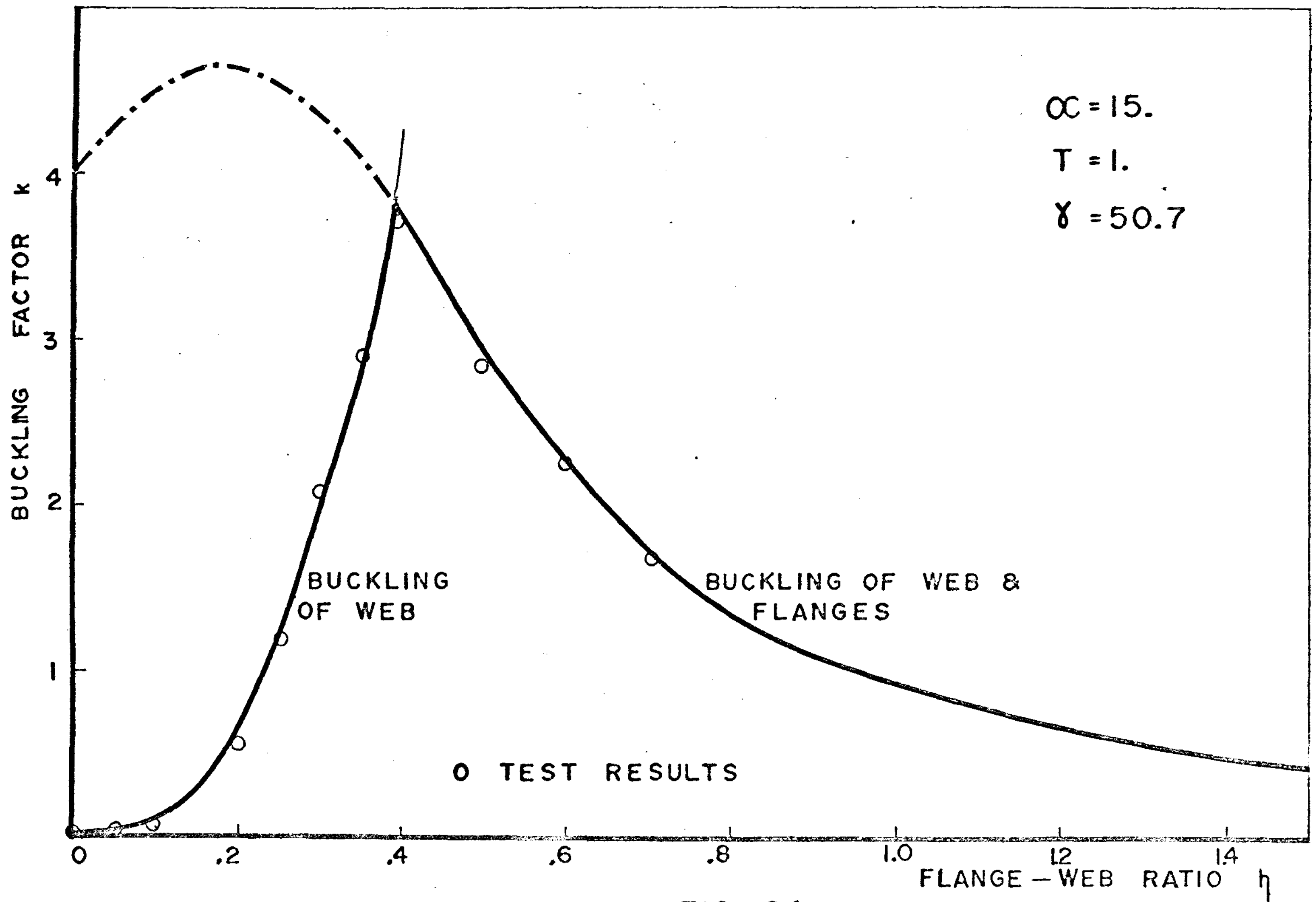

$\alpha = 15.$
$T = 1.$
$\gamma = 50.7$
BUCKLING FACTOR k
4
3
2
1
0
.2
.4
.6
.8
1.0
1.2
1.4
BUCKLING
OF WEB
BUCKLING OF WEB &
FLANGES
O TEST RESULTS
FLANGE — WEB RATIO η

FIG. 24

CHAPTER VII

STABILITY OF STANDARD ROLLED CHANNEL SECTIONS

In this chapter, we are going to present the values of the buckling factor obtained by the Matrix transfer method applied to the standard Rolled Channel sections.

The local buckling study is applied considering the local buckling of web and flange plates for the large flange-web ratios and the local buckling of web plate with the flanges acting as stiffening rib for the web plate. The instability criteria will follow the lower buckling factor resulting from either cases of buckling modes of failure.

The buckling factor (k) is plotted verses the column height-web width ratio (α), for a constant value of:

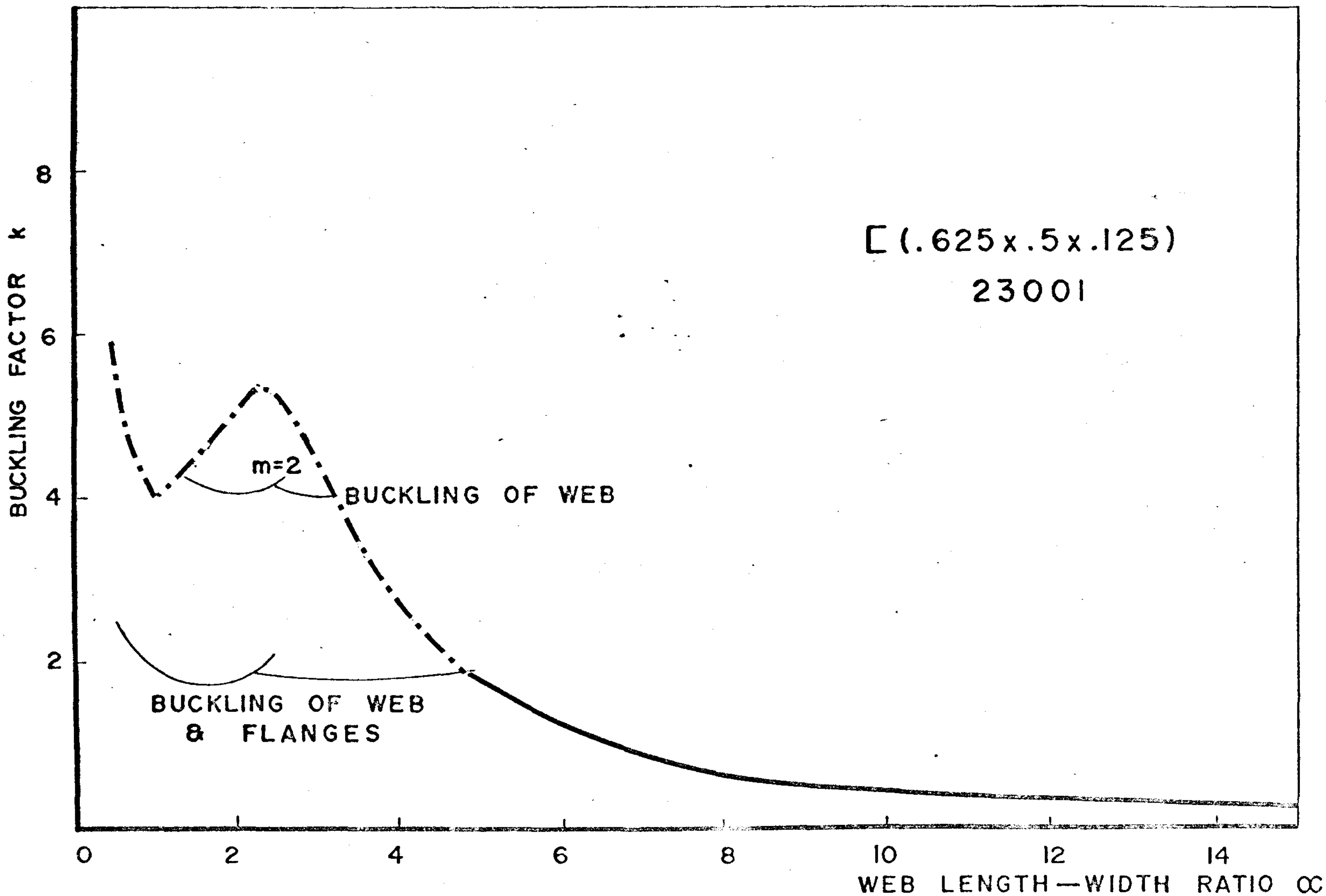
flange-web ratio	(η)
web width-web thickness ratio	(γ)
flange thickness-web thickness ratio	(t)

Results for different standard Rolled Channel sections are given by curves shown in fig. (25) to fig. (34).

The sectional property used in the calculations are taken from the Extruded Shapes manufactured by the Aluminium Company of Canada Ltd. (Alcan).

It can be seen that for a given section a long channel, (α large), fails under compression due to web buckling alone. For intermediate length channels, mode of buckling depends on the flange-web ratio. For large flange-web ratios, $\eta = 0.4$ say, local buckling of web and flanges takes place as is shown in the curves covering most of the sections chosen except sections Alcan 23032 (fig. 32). Particularly, it can be seen that for section Alcan 23003 (fig. 27) where the flange to web ratio is over 1, local buckling of web and flange is the mode of failure over a wide range of length of the channel. Section Alcan 23032 has a small flange to web ratio and it is seen from (fig. 32) that the mode of failure is the buckling of the web alone for all length of the channel.

Numbers under the channel dimensions in figures 25 to 34 refer to Alcan Series.



BUCKLING FACTOR k
8
6
4
2
0
2
4
6
8
10
12
14
WEB LENGTH—WIDTH RATIO ∝
[(.625 x .5 x .125)
23001
m=2
BUCKLING OF WEB
BUCKLING OF WEB
& FLANGES

FIG. 25

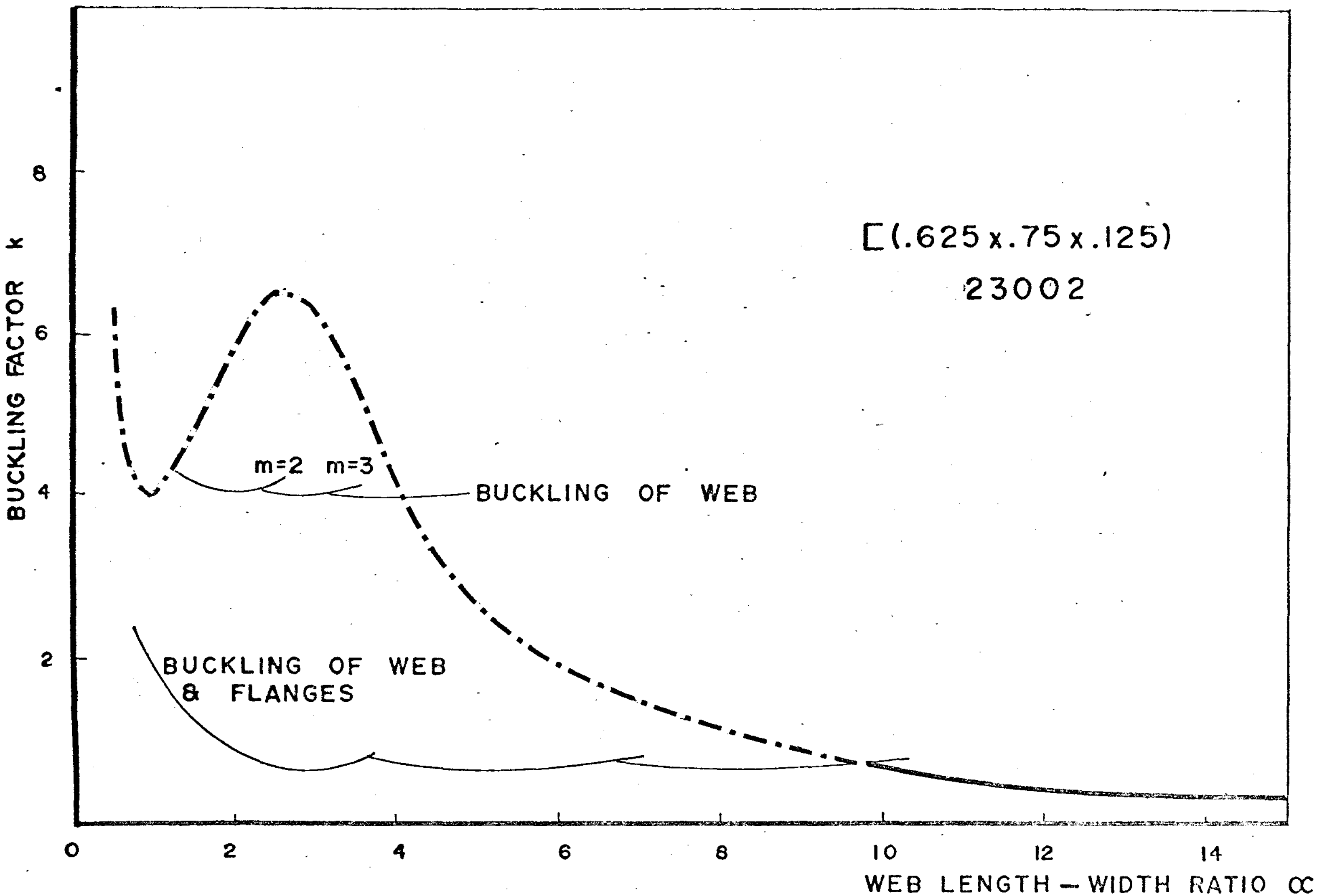

BUCKLING FACTOR k
8
6
4
2
0
2
4
6
8
10
12
14
WEB LENGTH — WIDTH RATIO α
[(.625 x .75 x .125)
23002
m=2
m=3
BUCKLING OF WEB
BUCKLING OF WEB
& FLANGES

FIG. 26

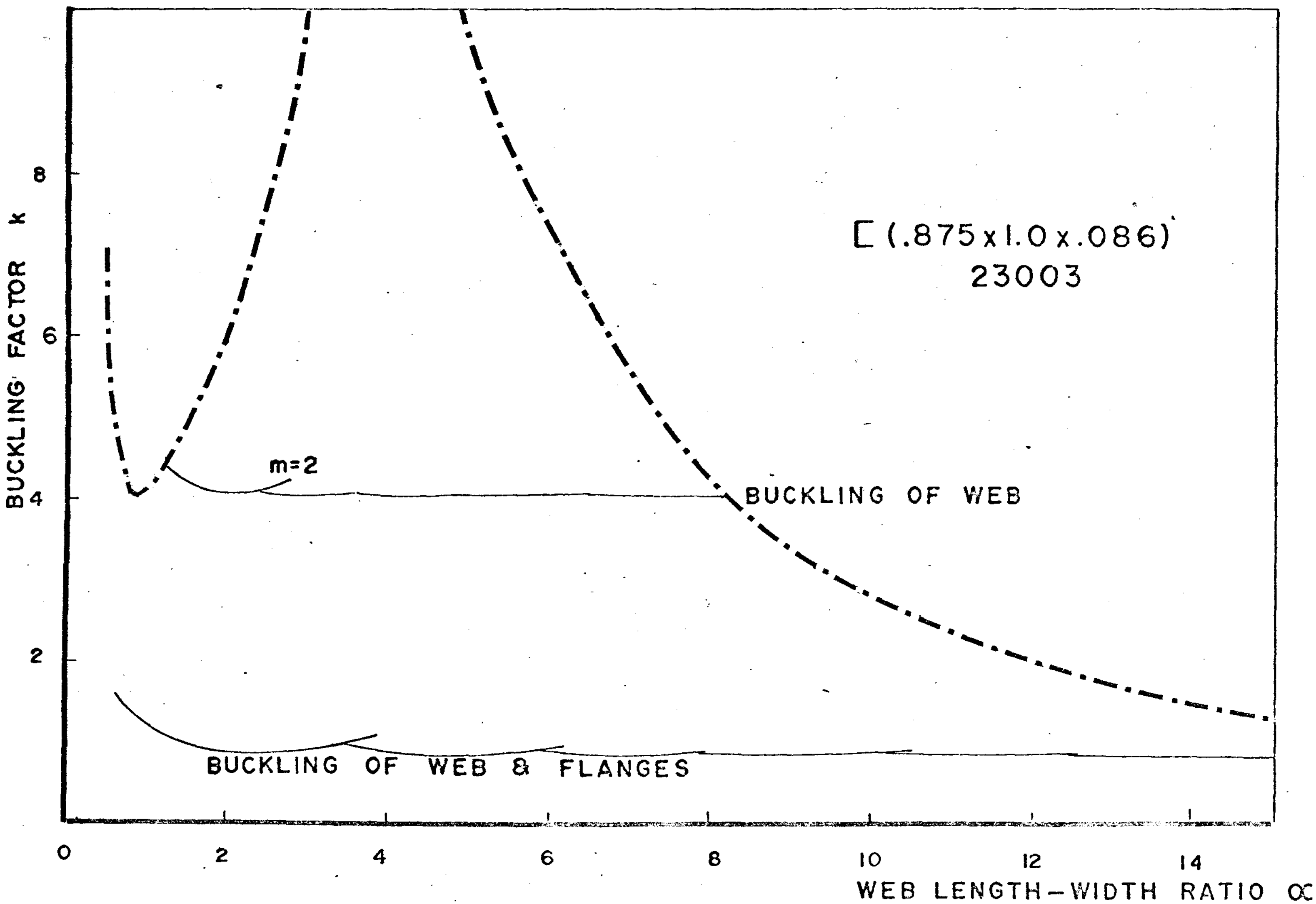

BUCKLING FACTOR k
8
6
4
2
[(.875 x 1.0 x .086)
23003
m=2
BUCKLING OF WEB
BUCKLING OF WEB & FLANGES
0
2
4
6
8
10
12
14
WEB LENGTH-WIDTH RATIO α

FIG. 27

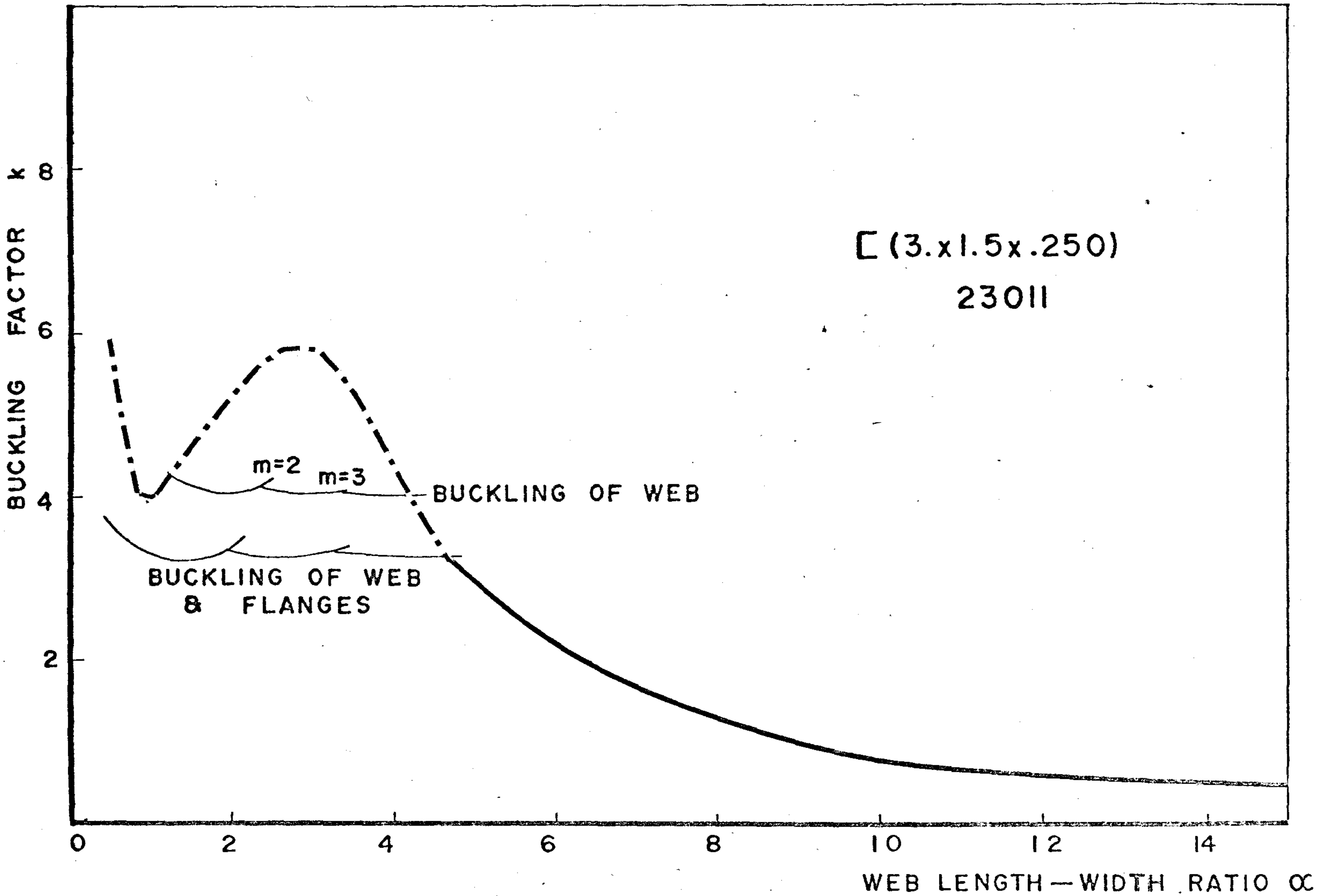

BUCKLING FACTOR k
WEB LENGTH — WIDTH RATIO α
[(3.x1.5x.250)
23011
m=2
m=3
BUCKLING OF WEB
BUCKLING OF WEB
& FLANGES
0
2
4
6
8
10
12
14

FIG. 28

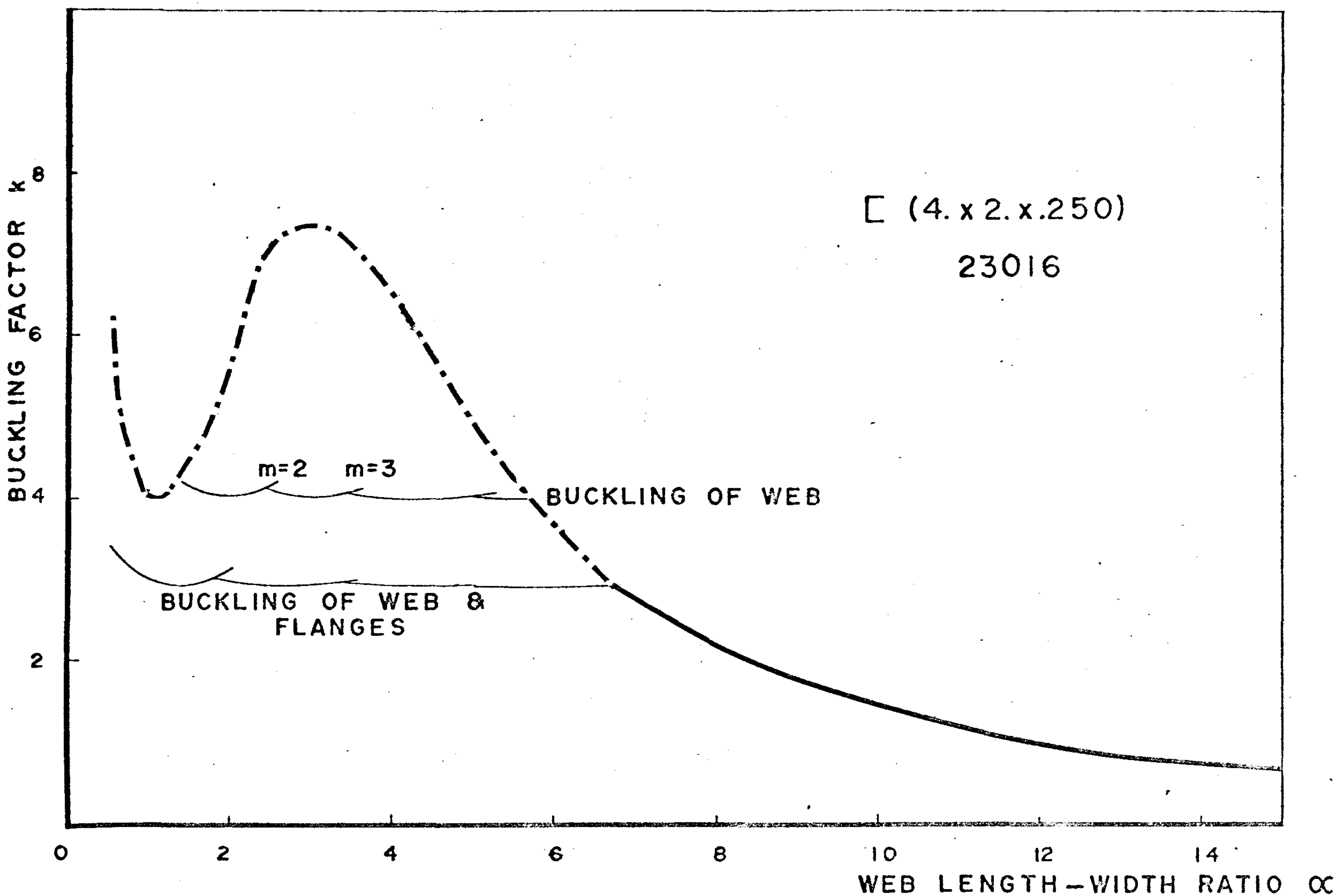

BUCKLING FACTOR k
WEB LENGTH—WIDTH RATIO α
[(4. x 2. x .250)
23016
m=2
m=3
BUCKLING OF WEB
BUCKLING OF WEB &
FLANGES
0
2
4
6
8
10
12
14

FIG. 29

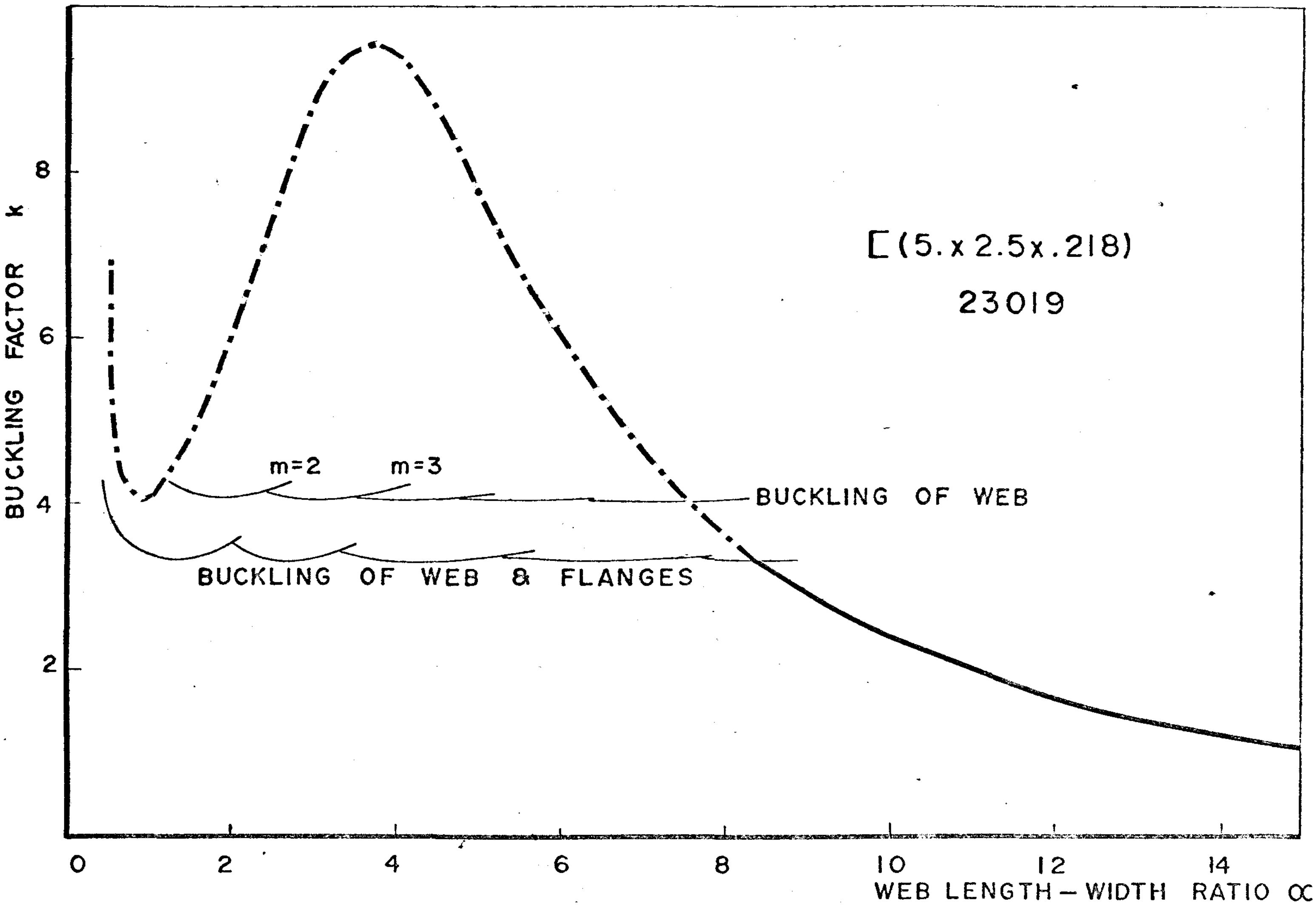

BUCKLING FACTOR k
8
6
4
2
0
2
4
6
8
10
12
14
WEB LENGTH – WIDTH RATIO ∝
[(5.x 2.5x.218)
23019
m=2
m=3
BUCKLING OF WEB
BUCKLING OF WEB & FLANGES

FIG. 30

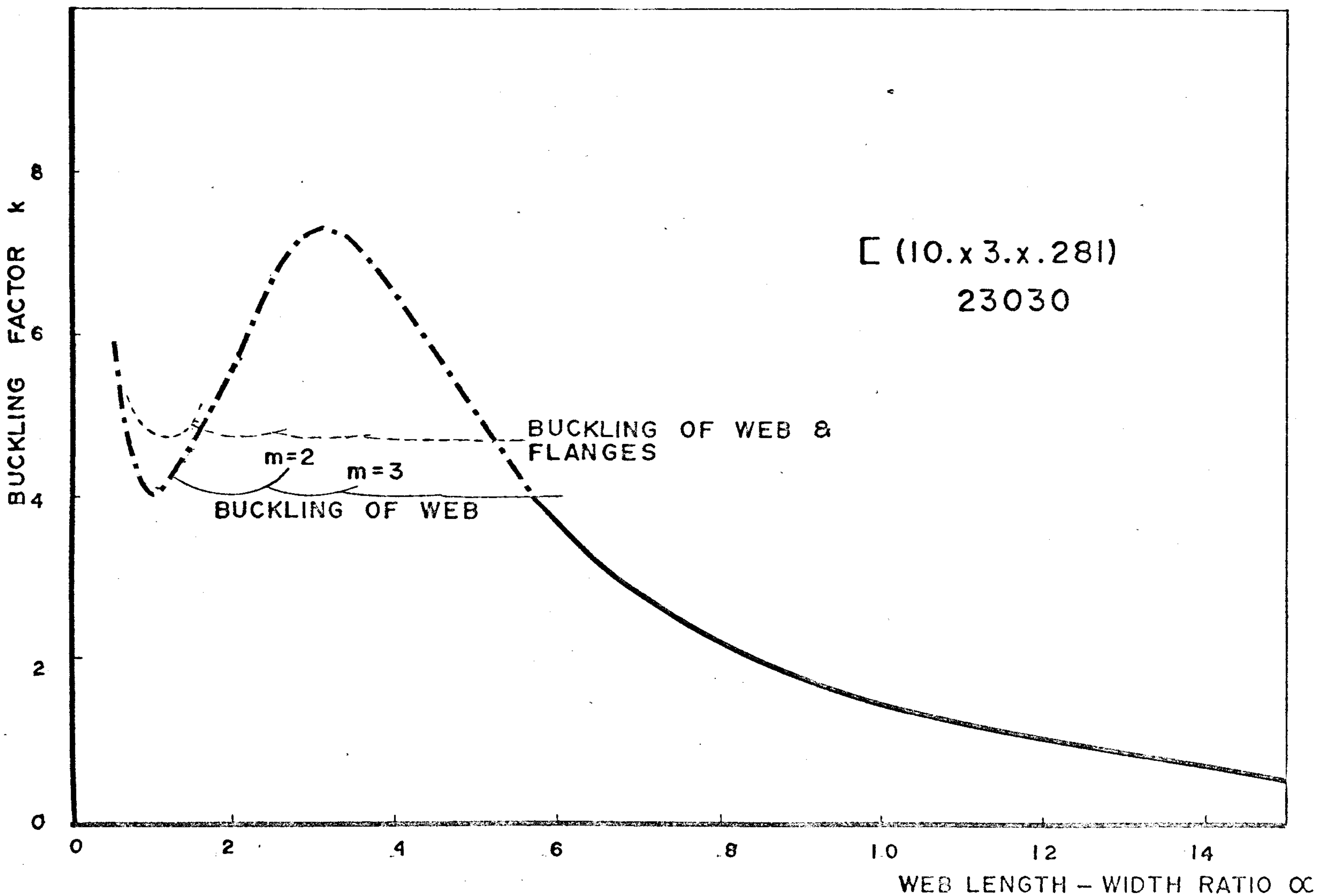
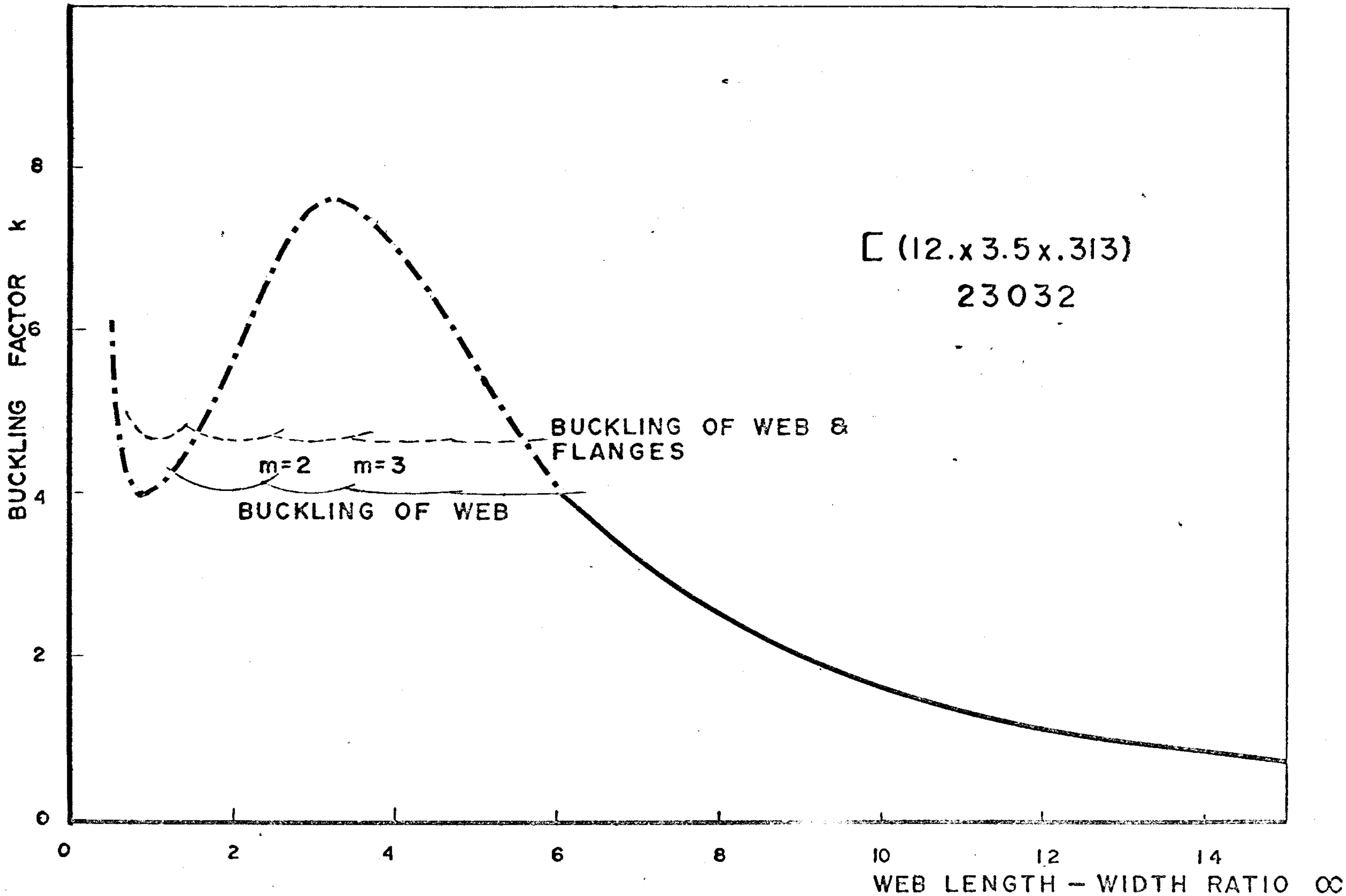

BUCKLING FACTOR k
8
6
4
2
0
0
.2
.4
.6
.8
1.0
1.2
1.4
WEB LENGTH – WIDTH RATIO ∝
[(10.x 3.x .281)
23030
BUCKLING OF WEB &
FLANGES
m=2
m=3
BUCKLING OF WEB

FIG. 31



BUCKLING FACTOR k
8
6
4
2
0
0
2
4
6
8
10
12
14
WEB LENGTH – WIDTH RATIO ∝
[(12.x 3.5 x .313)
23032
BUCKLING OF WEB & FLANGES
m=2
m=3
BUCKLING OF WEB

FIG. 32

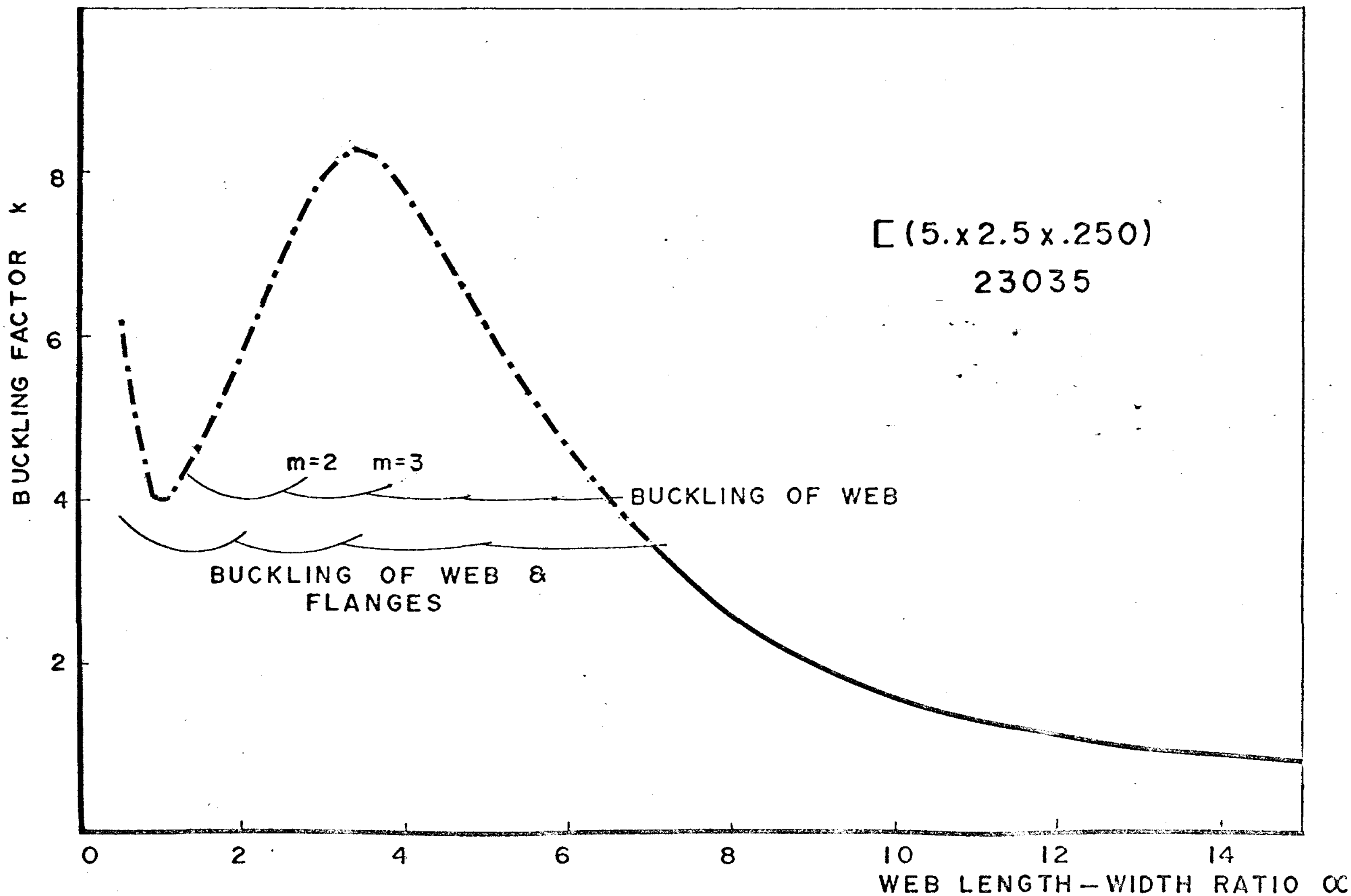

BUCKLING FACTOR k
8
6
4
2
0
2
4
6
8
10
12
14
WEB LENGTH – WIDTH RATIO α
[(5.x 2.5 x .250)
23035
m=2
m=3
BUCKLING OF WEB
BUCKLING OF WEB & FLANGES

FIG. 33

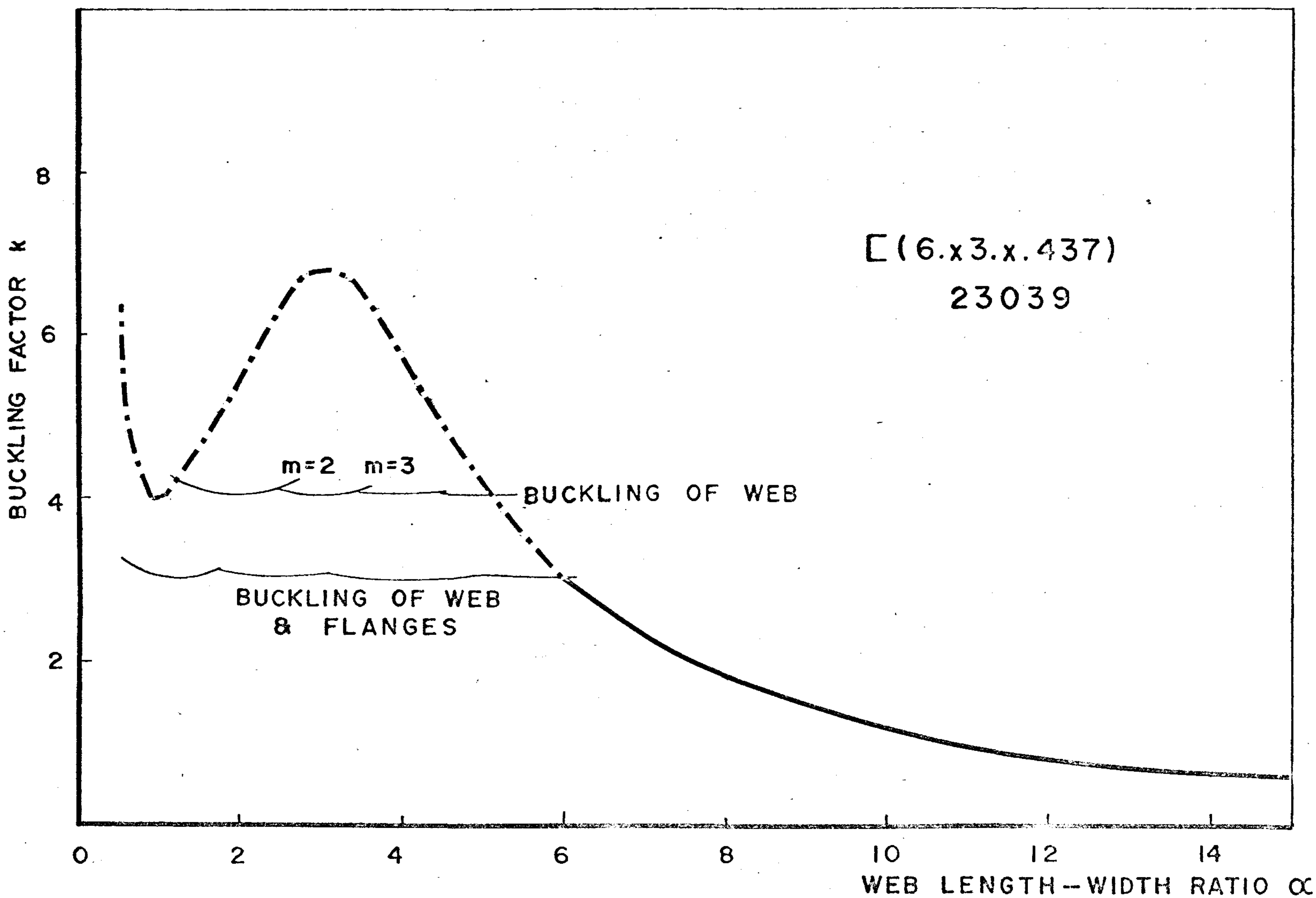

BUCKLING FACTOR k
WEB LENGTH--WIDTH RATIO α
[(6.x3.x.437)
23039
m=2
m=3
BUCKLING OF WEB
BUCKLING OF WEB & FLANGES
0
2
4
6
8
10
12
14

FIG. 34

CHAPTER VIII

CONCLUSION AND SUGGESTIONS FOR FURTHER RESEARCH

Conclusions

The introduced Matrix Transfer Method of Stability analysis proved to be a flexible, powerful and exact numerical method. The mathematical procedure can be easily programmed to be tackled by the recent large capacity computers in a considerably small interval of time.

The Matrix Transfer Method enables one to attack the more complicated problems, where the classical methods fail to solve due to the mathematical difficulties involved, and where the numerical methods fail to meet the required accuracy.

As a conclusion from applying the Matrix method to the stability of channel columns we can state that:

1. Considering the results obtained for the local buckling of web and flange plates of the channel column, and by comparing these results with the previous work done, the Matrix method is found to give comparable results.

2. Euler's buckling load for the channel strut as a column, is higher than the calculated buckling load, considering the channel as a system of plates. This agrees with the results presented by [14]. However, the experimental

results showed a buckling load higher than Euler's buckling load. This can be theoretically arrived at, by considering a part of the web plate to be acting with the flange plates. This interaction results in an increase in the flexural rigidity of the flange as a stiffener [1].

3. For a column of channel cross section of small flange-web ratios, failure occurs following the local buckling of web with the flanges acting as stiffening ribs.

4. For the case of large flange-web ratios, the instability criteria follows the local buckling of flanges and web plates which agrees with the classical curve.

5. Test results agree to a satisfactorily limit with the theoretically predicted behaviours.

6. Test results showed that flexural torsional mode of buckling does not take place. This agrees with the results arrived at by Divakaran, [15].

7. The limit between the two established local buckling modes is the intersection point of the two curves. A channel column is tested with dimensions giving theorectically a buckling factor at the intersection point. Test results showed that both local buckling modes participate in the buckling process as shown in fig. (19) and fig. (20).

8. It was observed that there is a substantial gain in buckling strenth of the channel section by increasing

the flange-web ratio η in the range $0 < \eta < 0.4$, when the length of the channel is not too long ($\alpha < 20$), this fact can be utilized for design purposes to obtain a more efficient compression member of thin walled channel sections.

Suggestions for Further Research

1. Large scale computer calculations can be expanded to obtain similar results for different loading distributions by increasing the number of plate fields considered. Variable end conditions, and different column and cross section's dimensions can be studied. Curves can be obtained giving the buckling factor and the buckling behaviour for the practical range of dimensions, end conditions and loading distribution.
2. Similar stability studies can be expanded to cover various shapes of cross sections (e.g. I and Z sections).
3. Large field of experimental work is open to establish data verifying the true buckling modes of failure for different cross sections of columns.
4. The Matrix Transfer Method can be expanded to cover the case of a plate with applied stresses in two perpendicular directions. This will be a step for a widely required generalization of the method.
5. The study of the Matrix Transfer conditions at the meeting line of more than two plates.
6. Matrix formulation approach can be extended to the new field of interest of the dynamic stability problems.

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APPENDIX

DERIVIATION OF FIELD MATRICES AND FIELD BOUNDARIES MATRICES

Field Matrix $[F_A]$ for $[k>(m/\alpha)^2]$

The solution of the characteristic equation (II-7) will be given by equation (II-8) where:

$$\lambda_{1,2} = \pm\sqrt{\theta(\pi\sqrt{k} + \theta)} = \pm\ \kappa_1 \qquad \text{...(A-1.a)}$$

$$\lambda_{3,4} = \pm\sqrt{\theta(\pi\sqrt{k} - \theta)} = \pm\ i\kappa_2 \qquad \text{...(A-1.b)}$$

Rewriting the general solution of the differential equation in the exponential form, it reads:

$$w_{(\eta)} = C_1 e^{\kappa_1\eta} + C_2 e^{-\kappa_1\eta} + C_3 e^{i\kappa_2\eta} + C_4 e^{-i\kappa_2\eta} \qquad \text{...(A-2)}$$

At the edge $\eta = 0$, the state vector $\{\Phi_o\}$ will be expressed by the deflection, slope, moment and shear components as:

$$\{\Phi_o\} \equiv \begin{Bmatrix} w \\ w' \\ M^* \\ Q^* \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1 & -\kappa_1 & \kappa_2 & -\kappa_2 \\ A_1 & A_1 & -A_2 & -A_2 \\ \kappa_1 A_2 & -\kappa_1 A_2 & -i\kappa_2 A_1 & i\kappa_2 A_1 \end{bmatrix} \cdot \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \qquad \text{...(A-3)}$$

where

$$A_1 = \kappa_1^2 - \nu\theta^2$$

$$A_2 = \kappa_2^2 - \nu\theta^2$$

$$A_o = A_1 + A_2$$

$$\theta = m\pi/\alpha$$

Expressing the values of C_1, C_2, C_3, and C_4 in terms of the end conditions parameters, then,

$$C_1 = \frac{1}{2A_o}\left(A_2 w_o + \frac{A_1}{\kappa_1} w_o' + M_o^* + \frac{1}{\kappa_1} Q_o^* \right)$$

$$C_2 = \frac{1}{2A_o}\left(A_2 w_o - \frac{A_1}{\kappa_1} w_o' + M_o^* - \frac{1}{\kappa_1} Q_o^* \right)$$

$$C_3 = \frac{1}{2A_o}\left(A_1 w_o + \frac{A_2}{i\kappa_2} w_o' - M_o^* - \frac{1}{i\kappa_2} Q_o^* \right)$$

$$C_4 = \frac{1}{2A_o}\left(A_1 w_o - \frac{A_2}{i\kappa_2} w_o' - M_o^* + \frac{1}{i\kappa_2} Q_o^* \right)$$

Introducing the values of C_1, C_2, C_3 and C_4 in the general solution (A-2), we can express the end parameters at η in terms of end parameters at η_o to be:

$$\begin{aligned} A_o w_\eta = {} & (A_2 \cdot \cosh \kappa_1 \eta + A_1 \cdot \cos \kappa_2 \eta) \cdot w_o \\ & + \left(\frac{A_1}{\kappa_1} \cdot \sinh \kappa_1 \eta + \frac{A_2}{\kappa_2} \cdot \sin \kappa_2 \eta \right) \cdot w_o' \\ & + (\cosh \kappa_1 \eta - \cos \kappa_2 \eta) \cdot M_o^* \\ & + \left(\frac{1}{\kappa_1} \cdot \sinh \kappa_1 \eta - \frac{1}{\kappa_2} \cdot \sin \kappa_2 \eta \right) \cdot Q_o^* \end{aligned}$$

...(A-4)

$$\begin{aligned} A_o w_\eta' = {} & (A_2 \kappa_1 \sinh \kappa_1 \eta - A_1 \kappa_2 \sin \kappa_2 \eta) \cdot w_o \\ & + (A_1 \cosh \kappa_1 \eta + A_2 \cos \kappa_2 \eta) \cdot w_o' \\ & + (\kappa_1 \sinh \kappa_1 \eta + \kappa_2 \sin \kappa_2 \eta) \cdot M_o^* \\ & + (\cosh \kappa_1 \eta - \cos \kappa_2 \eta) \cdot Q_o^* \end{aligned}$$

...(A-5)

$$A_oM_\eta{}^* = [\, A_2(\kappa_1{}^2 - \nu\theta^2)\ \cosh\ \kappa_1\eta - A_1(\kappa_2{}^2 + \nu\theta^2)\cos\kappa_2\eta]\cdot w_o$$
$$+ [\frac{A_1}{\kappa_1}(\kappa_1{}^2 - \nu\theta^2)\ \sinh\ \kappa_1\eta - \frac{A_2}{\kappa_2}(\kappa_2{}^2 + \nu\theta^2)\sin\ \kappa_2\eta]\cdot w_o{}'$$
$$+ [\quad (\kappa_1{}^2 - \nu\theta^2)\cosh\ \kappa_1\eta + \quad (\kappa_2{}^2 + \nu\theta^2)\cos\ \kappa_2\eta]\cdot M_o{}^*$$
$$+ [\frac{1}{\kappa_1}(\kappa_1{}^2 - \nu\theta^2)\sinh\ \kappa_1\eta \ + \frac{1}{\kappa_2}(\kappa_2{}^2 + \nu\theta^2)\sin\ \kappa_2\eta]\cdot Q_o{}^*$$

...(A-6)

$$A_oQ_\eta{}^* = \Big\{ A_2\kappa_1[\kappa_1{}^2 - (2-\nu)\theta^2]\ \sinh\ \kappa_1\eta + A_1\kappa_2[\kappa_2{}^2 + (2-\nu)\theta^2]\cdot$$
$$\sin\ \kappa_2\eta\Big\}\cdot w_o$$
$$+\Big\{A_1[\kappa_1{}^2 - (2-\nu)\theta^2]\ \cosh\kappa_1\eta - A_2[\kappa_2{}^2 + (2-\nu)\theta^2]\cdot$$
$$\cos\ \kappa_2\eta\Big\}\cdot w_o{}'$$
$$+\Big\{\kappa_1[\kappa_1{}^2 - (2-\nu)\theta^2]\ \sinh\kappa_1\eta - \kappa_2[\kappa_2{}^2 + (2-\nu)\theta^2]\cdot$$
$$\sin\ \kappa_2\eta\Big\}\cdot M_o{}^*$$
$$+\Big\{\ [\kappa_1{}^2 - (2-\nu)\theta^2]\ \cosh\ \kappa_1\eta + [\kappa_2{}^2 + (2-\nu)\theta^2]\cdot$$
$$\cos\kappa_2\eta\ \Big\}\cdot Q_o{}^*$$

...(A-7)

Refering to equation (II-20) relating the state vectors at both edges of the field,

$$\{\Phi_\eta\} = [F]\ \{\Phi_o\} \qquad ...(A-8)$$

Arranging the obtained equations for the boundary parameters at η in terms of the boundary parameters at η_o in a matrix form we get:

Field Matrix $[F_A]$ for $[k > (m/\alpha)^2]$

$$
A = \frac{1}{A_o} \cdot \begin{bmatrix}
A_2\varepsilon_2 + A_1\varepsilon_4 & \frac{A_1}{\beta_1}\varepsilon_1 + \frac{A_2}{\beta_2}\varepsilon_3 & \varepsilon_2 - \varepsilon_4 & \frac{1}{\beta_1}\varepsilon_1 - \frac{1}{\beta 2}\varepsilon_3 \\
A_2\beta_1\varepsilon_1 - A_1\beta_2\varepsilon_3 & A_1\varepsilon_2 + A_2\varepsilon_4 & \beta_1\varepsilon_1 + \beta_2\varepsilon_2 & \varepsilon_2 - \varepsilon_4 \\
A_1A_2(\varepsilon_2 - \varepsilon_4) & \frac{A_1^{\,2}}{\beta_1}\varepsilon_1 - \frac{A_2^{\,2}}{\beta_2}\varepsilon_3 & A_1\varepsilon_2 + A_2\varepsilon_4 & \frac{A_1}{\beta_1}\varepsilon_1 + \frac{A_2}{\beta_2}\varepsilon_3 \\
A_2^{\,2}\beta_1\varepsilon_1 + A_1^{\,2}\beta_2\varepsilon_3 & A_1A_2(\varepsilon_2 - \varepsilon_4) & A_2\beta_1\varepsilon_1 - A_1\beta_2\varepsilon_3 & A_2\varepsilon_2 + A_1\varepsilon_4
\end{bmatrix} \quad \ldots\text{(A-9)}
$$

where:

$$\beta_1 = \sqrt{\theta(\pi\sqrt{k} + \theta)} \qquad \varepsilon_1 = \sinh\beta_1\eta \qquad A_1 = \beta_1^{\,2} - \nu\theta^2$$

$$\beta = \sqrt{\theta(\pi\sqrt{k} - \theta)} \qquad \varepsilon_2 = \cosh\beta 1\eta \qquad A_2 = \beta_2^{\,2} + \nu\theta^2$$

$$\theta = m\pi/\alpha \qquad \varepsilon_3 = \sin\beta_2\eta \qquad A_o = A_1 + A_2 = \beta_1^{\,2} + \beta_2^{\,2}$$

$$\varepsilon_4 = \cos\beta_2\eta$$

Field Matrix $[F_B]$ for $[k = (m/\alpha)^2]$

The solution of the characteristic equation (II-7) will be given the equation (II-8) where:

$$\lambda_{1,2} = \pm\sqrt{\theta(\theta+\theta)} = \pm\sqrt{2\theta} = \pm\kappa \qquad \text{...(A-10.a)}$$

$$\lambda_{3,4} = 0 \qquad \text{...(A-10.b)}$$

where $\theta = m\pi/\alpha$

The general solution of the differential equation is rewritten as:

$$w_{(\eta)} = C_1 e^{\kappa\eta} + C_2 e^{-\kappa\eta} + C_3 + C_4 \qquad \text{...(A-11)}$$

Expressing the state vector $\{\Phi_o\}$ in terms of the derivatives of the deflection formula, then,

$$\begin{Bmatrix} w_o \\ w_o' \\ M_o^* \\ Q_o^* \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa\theta & -\kappa\theta & 0 & 1 \\ \theta^2(2-\nu) & \theta^2(2-\nu) & -\theta^2\nu & 0 \\ \kappa\cdot\theta^2\cdot\nu & -\kappa\theta^2\nu & 0 & -\theta^2(2-\nu) \end{bmatrix} \cdot \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \qquad \text{...(A-13)}$$

Proceeding same as in case of matrix $[F_A]$, the terms forming the Field Matrix $[F_B]$ are given by:

Field Matrix $[F_B]$ for $[k = (m/\alpha)^2]$

$$
B = \frac{1}{2} \cdot \begin{bmatrix} 2 + \nu(\varepsilon_2 - 1) & \eta\nu + \frac{2-\nu}{\beta}\varepsilon_1 & \frac{1}{\theta^2}(\varepsilon_2 - 1) & \frac{1}{\theta^2\beta}(\varepsilon_1 - \beta\eta) \\ \beta\cdot\nu\cdot\varepsilon_1 & (2-\nu)\varepsilon_2 + \nu & \frac{\beta}{\theta^2}\varepsilon_1 & \frac{1}{\theta^2}(\varepsilon_2 - 1) \\ \nu\theta^2(2-\nu)(\varepsilon_2 - 1) & \theta^2[\frac{(2-\nu)^2}{\beta}\varepsilon_1 - \nu^2\eta] & (2-\nu)\varepsilon_2 + \nu & \eta\nu + \frac{2-\nu}{\beta}\varepsilon_1 \\ \nu^2\theta^2\beta\varepsilon_1 & \nu\theta^2(2-\nu)(\varepsilon_2 - 1) & \beta\nu\varepsilon_1 & 2 + \nu(\varepsilon_2 - 1) \end{bmatrix} \quad \ldots\text{(A-14)}
$$

where, $\beta = \theta\sqrt{2}$ $\qquad \varepsilon_1 = \sinh \beta\eta$

$\theta = m\pi/\alpha$ $\qquad \varepsilon_2 = \cosh \beta\eta$

Field Matrix $[F_C]$ for $[0<k<(m/_\alpha)^2]$

The solution of the characteristic equation (II-7) will be given by:

$$\lambda_{1,2} = \pm\sqrt{\theta(\theta + \pi\sqrt{k})} = \pm\kappa_1 \qquad \text{...(A-15.a)}$$

$$\lambda_{3,4} = \pm\sqrt{\theta(\theta - \pi\sqrt{k})} = \pm\kappa_2 \qquad \text{...(A-15.b)}$$

The general solution of the differential equation is rewritten as:

$$w_{(\eta)} = C_1 e^{\kappa_1\eta} + C_2 e^{-\kappa_1\eta} + C_3 e^{\kappa_2\eta} + C_4 e^{-\kappa_2\eta} \qquad \text{...(A-16)}$$

Expressing the state vector $\{\Phi_o\}$ in terms of the derivatives of the deflection formula, then:

$$\begin{Bmatrix} w \\ w_o' \\ M_o^* \\ Q_o^* \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1 & -\kappa_1 & \kappa_2 & -\kappa_2 \\ C_1^* & C_1^* & C_2^* & C_2^* \\ -\kappa_1 C_2 & \kappa_1 C_2 & -\kappa_2 C_1 & \kappa_2 C_1 \end{bmatrix} \cdot \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

...(A-17)

where

$$C_1^* = \kappa_1^2 - \nu\theta^2 = -\kappa_2^2 + (2-\nu)\theta^2$$

$$C_2^* = \kappa_2^2 - \nu\theta^2 = -\kappa_1^2 + (2-\nu)\theta^2$$

$$C_o = C_1^* - C_2^* = \kappa_1^2 - \kappa_2^2 = 2\theta\pi\sqrt{k}$$

Proceeding same as in case of Matrix $[F_A]$, the terms forming the Field Matrix $[F_D]$ are given by:

Field Matrix $[F_C]$ for $[0<k<(m/\alpha)^2]$

$$C = \frac{1}{C_o}\cdot\begin{bmatrix} -(C_2\varepsilon_2 - C_1\varepsilon_4) & \frac{C_1}{\beta_1}\varepsilon_1 - \frac{C_2}{\beta_2}\varepsilon_3 & \varepsilon_2 - \varepsilon_4 & \frac{1}{\beta_1}\varepsilon_1 - \frac{1}{\beta_2}\varepsilon_3 \\ -(\beta_1 C_2\varepsilon_1 - \beta_2 C_1\varepsilon_3) & C_1\varepsilon_2 - C_2\varepsilon_4 & \beta_1\varepsilon_1 - \beta_2\varepsilon_3 & \varepsilon_2 - \varepsilon_4 \\ -C_1C_2(\varepsilon_2 - \varepsilon_4) & \frac{C_1^2}{\beta_1}\varepsilon_1 - \frac{C_2^2}{\beta_2}\varepsilon_3 & C_1\varepsilon_2 - C_2\varepsilon_4 & \frac{C_1}{\beta_1}\varepsilon_1 - \frac{C_2}{\beta_2}\varepsilon_3 \\ \beta_1 C_2^2\varepsilon_1 - \beta_2 C_1^2\varepsilon_3 & -C_1C_2(\varepsilon_2 - \varepsilon_4) & -(\beta_1 C_2\varepsilon_1 - \beta_2 C_1\varepsilon_3) & -(C_2\varepsilon_2 - C_1\varepsilon_4) \end{bmatrix} \quad \ldots\text{(A-18)}$$

where

$\beta_1 = \sqrt{\theta(\theta + \pi\sqrt{k})}$ $\qquad \varepsilon_1 = \sinh \beta_1\eta$ $\qquad C_1 = \beta_1^2 - \nu\theta^2$

$\beta_2 = \sqrt{\theta(\theta - \sqrt{\pi^2 k})}$ $\qquad \varepsilon_2 = \cosh \beta_1\eta$ $\qquad C_2 = \beta_2^2 - \nu\theta^2$

$\theta = m\pi/\alpha$ $\qquad \varepsilon_3 = \sinh \beta_2\eta$ $\qquad C_o = C_1 - C_2 = \beta_1^2 - \beta_2^2$

$\varepsilon_4 = \cosh \beta_2\eta$

Field Matrix $[F_D]$ for $[k=0]$

The solution of the characteristic equation (II-7) will be given by the equation (II-8) where:

$$\lambda_{1,2} = +\theta \qquad \text{...(A-19.a)}$$

$$\lambda_{3,4} = -\theta \qquad \text{...(A-19.b)}$$

The general solution of the differential equation (II-2) is rewritten as,

$$w_{(\eta)} = C_1 e^{\theta\eta} + C_2 e^{\theta\eta} + C_3 e^{-\theta\eta} + C_4 e^{-\theta\eta} \qquad \text{...(A-20)}$$

Expressing the state vector $\{\Phi_o\}$ in terms of the derivatives of the deflection formula, then:

$$\begin{Bmatrix} w_o \\ w_o' \\ M_o^* \\ Q_o^* \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \theta & -\theta & 1 & 1 \\ \theta^2(1-\nu) & \theta^3(1-\nu) & 2\theta^2 & -2\theta^2 \\ -\theta^3(1-\nu) & \theta^3(1-\nu) & \theta^2(1-\nu) & \theta^2(1+\nu) \end{bmatrix} \cdot \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \qquad \text{...(A-21)}$$

Proceeding same as in case of matrix $[F_A]$, the terms forming the Field Matrix $[F_D]$ are given by:

Field Matrix $[F_D]$ for $[k = 0]$

$$
D = \frac{1}{2} \cdot \begin{bmatrix}
2\varepsilon_2 - \theta\eta\nu_1\varepsilon_1 & \frac{1}{\theta}(\theta\eta\nu_1\varepsilon_2 + \nu_2\theta_1) & \frac{\eta}{\theta}\varepsilon_1 & \frac{1}{\theta^3}(\theta\eta\varepsilon_2 - \varepsilon_1) \\
-\theta(\theta\eta\nu_1\varepsilon_2 - \nu_2\varepsilon_1) & 2\varepsilon_2 + \theta\eta\nu_1\varepsilon_1 & \frac{1}{\theta}(\theta\eta\varepsilon_2 + \varepsilon_1) & \frac{\eta}{\theta}\varepsilon_1 \\
-\theta^3\eta\nu_1{}^2\varepsilon_1 & \theta[\theta\eta\nu_1{}^2\varepsilon_2 + \nu_1(2+\nu_2)\varepsilon_1] & 2\varepsilon_2 + \theta\eta\nu_1\varepsilon_1 & \frac{1}{\theta}(\theta\eta\nu_1\varepsilon_2 + \nu_2\varepsilon_1) \\
\theta^3[\theta\eta\nu_1{}^2\varepsilon_2 - \nu_1(2+\nu_2)\varepsilon_1] & -\theta^3\eta\nu_1{}^2\varepsilon_1 & -\theta(\theta\eta\nu_1\varepsilon_2 - \nu_2\varepsilon_1) & 2\varepsilon_2 - \theta\eta\nu_1\varepsilon_1
\end{bmatrix}
\qquad \ldots\text{(A-22)}
$$

where $\theta = m\pi/\alpha$ $\qquad \varepsilon_1 = \sinh\theta\eta$

$\nu_1 = 1 - \nu$ $\qquad \varepsilon_2 = \cosh\theta\eta$

$\nu_2 = 1 + \nu$

Field Matrix $[F_G]$ for $[k<0]$

The solution of the characteristic equation (II-7) will be given by the equation (II-8), where,

$$\lambda_{1,2} = \pm \beta \pm i\varepsilon \qquad \text{...(A-23.a)}$$

$$\lambda_{3,4} = \pm \beta \mp i\varepsilon \qquad \text{...(A-23.b)}$$

where,

$$\beta = \sqrt{\tfrac{1}{2}\theta\left(\sqrt{\theta^2 + \pi^2 k} + \theta\right)}$$

$$\varepsilon = \sqrt{\tfrac{1}{2}\theta\left(\sqrt{\theta^2 + \pi^2 k} - \theta\right)}$$

The general solution of the differential equation (II-2) is rewritten as:

$$w_{(\eta)} = C_1 e^{(\beta+i\varepsilon)\eta} + C_2 e^{-(\beta+i\varepsilon)\eta} + C_3 e^{(\beta-i\varepsilon)\eta} + C_4 e^{-(\beta-i\varepsilon)\eta} \qquad \text{...(A-24)}$$

Expressing the state vector $\{\Phi_o\}$ in terms of the derivatives of the deflection formula, then,

$$\begin{Bmatrix} w_o \\ w_o' \\ M_o^* \\ Q_o^* \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta+i\varepsilon & -(\beta+i\varepsilon) & \beta-i\varepsilon & -(\beta-i\varepsilon) \\ N_2 & N_2 & -N_1 & -N_1 \\ (\beta+i\varepsilon)\cdot N_1 & -(\beta+i\varepsilon)\cdot N_1 & -(\beta-i\varepsilon)\cdot N_2 & (\beta-i\varepsilon)\cdot N_2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

where, ...(A-25)

$$N_1 = 2i\beta\varepsilon - \theta^2(1-\nu)$$

$$N_2 = 2i\beta\varepsilon + \theta^2(1-\nu)$$

Proceeding same as in case of matrix $[F_A]$, the terms forming the Field Matrix $[F_G]$ are given by:

Field Matrix $[F_G]$ for $[(k<0)]$

$$
G = \frac{1}{2\beta_1\beta_2}\cdot
\begin{bmatrix}
2\beta_1\beta_2\varepsilon_4 - \theta^2\nu_1\varepsilon_1 & \frac{2}{G_o}(\beta_1 G_2\varepsilon_3 + \beta_2 G_1\varepsilon_2) & \varepsilon_1 & \frac{2}{G_o}(\beta_1\varepsilon_3 - \beta_2\varepsilon_2 \\
\beta_2 G_1\varepsilon_2 - \beta_1 G_2\varepsilon_3 & 2\beta_1\beta_2\varepsilon_4 + \lambda^2\nu_1\varepsilon_1 & \beta_1\varepsilon_3 + \beta_2\varepsilon_2 & \varepsilon_1 \\
-(4\beta_1^2\beta_2^2 + \theta^4\nu_1^2)\varepsilon_1 & \frac{2}{G_o}[\beta_2(G_1\theta^2\nu_1 + 2G_1\beta_1^2)\varepsilon_2 + \beta_1(G_2\theta^2\nu_1 - 2G_1\beta_2^2)\varepsilon_3] & 2\beta_1\beta_2\varepsilon_4 + \theta^2\nu_1\varepsilon_1 & \frac{2}{G_o}(\beta_1 G_2\varepsilon_3 + \beta_2 G_1\varepsilon_2) \\
-[\beta_2(G_1\theta^2\nu_1 + 2G_2\beta_1^2)\varepsilon_2 - \beta_1(G_2\theta^2\nu_1 - 2G_1\beta_2^2)\varepsilon_3 & -(4\beta_1^2\beta_2^2 + \theta^4\nu_1^2)\varepsilon_1 & \beta_2 G_1\varepsilon_2 - \beta_1 G_2\varepsilon_3 & 2\beta_1\beta_2\varepsilon_4 - \theta^2\nu_1\varepsilon_1
\end{bmatrix}
\quad \text{...(A-26)}
$$

where

$\beta_1 = \sqrt{\tfrac{1}{2}\theta(\sqrt{\theta^2 + \pi^2 k} + \theta)}$ $\quad \varepsilon_1 = \sinh\beta_1\eta\cdot\sin\beta_2\eta$ $\quad G_1 = 2\beta_1^2 - \theta^2\nu_1$

$\beta_2 = \sqrt{\tfrac{1}{2}\theta(\sqrt{\theta^2 + \pi^2 k} - \theta)}$ $\quad \varepsilon_2 = \sinh\beta_1\eta\cdot\cos\beta_2\eta$ $\quad G_2 = 2\beta_2^2 + \theta^2\nu_1$

$\theta = m\pi/\alpha$ $\quad \varepsilon_3 = \cosh\beta_1\eta\cdot\sin\beta_2\eta$ $\quad G_o = G_1 + G_2 = 2(\beta_1^2 + \beta_2^2)$

$\nu_1 = 1 - \nu$ $\quad \varepsilon_4 = \cosh\beta_1\eta\cdot\cos\beta_2\eta$

Field Boundaries Matrices

The Field Boundaries Matrix is deduced by simple analysis from the Field Matrix. This analysis is repeated for every term of the Field Boundaries Matrix and for the five Field Boundaries Matrices corresponding to the five cases of the buckling factor k as given by (II-14).

The Field Boundaries Construction given by Matrix (II-30), shows that if we consider the term $\overset{\Delta}{f}_{66}$ of the Field Boundaries Matrix $[\overset{\Delta}{F}]$, this term ~~is corresponding~~ corresponds to the case of a plate of free and clamped edges.

Boundary conditions are:

$$\left.\begin{array}{lll} w_o = 0 & \text{and } w_o' = 0 & \eta = 0 \\ M_1^* = 0 & \text{and } Q_1^* = 0 & \eta = 1 \end{array}\right|$$

Introducing these boundary conditions in the state vectors formula we can write:

$$\begin{Bmatrix} w_1 \\ w_1' \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ M_o^* \\ Q_o^* \end{Bmatrix} \quad \ldots \text{(A-27)}$$

The Matrix equation given by (A-27), is written in the equation forms as:

$$w_1 = f_{13}M_o^* + f_{14}Q_o^*$$

$$w_1' = f_{23}M_o^* + f_{24}Q_o^*$$

$$0 = f_{33}M_o^* + f_{34}Q_o^*$$

$$0 = f_{43}M^* + f_{44}Q^*$$

The nontrivial values of M * and Q * are established by the condition:

$$\begin{vmatrix} f_{33} & f_{34} \\ f_{43} & f_{44} \end{vmatrix} = 0 \qquad \text{...(A-28)}$$

or, $f_{33}\, f_{44} - f_{43}f_{34} = 0$ ...(A-29)

Equation (A-29) gives the stability condition for the free-clamped edges of the plate case considered.

The stability conditions gives the term f_{66} of the Field Boundaries Matrix.

The same procedure of deriviation is repeated for the different combinations of end condtions and for the five Field Matrices corresponding to the five cases of the value of k given by equation (II-14).

The Field Boundary Matrix is ~~arrived at to be~~ a (6 x 6) Matrix, symmetrical about the secondary diagonal. A simple treatment is illustrated by (A-30) and was first applied by Nassar, [8], [9] reduces the Matrices to (5 x 5) Field Boundary Matrix.

This analysis is the simple addition of columns 3 and 4 giving column 3 instead and the mean value of rows 3 and 4 giving the row 3 instead. This is due to the equivalence of columns 3 and 4, and rows 3 and 4.

$$\begin{bmatrix} \cdot & \cdot & 2\overset{\Delta}{f}_{13} & \cdot & \cdot \\ \cdot & \cdot & 2\overset{\Delta}{f}_{23} & \cdot & \cdot \\ \overset{\Delta}{f}_{31} & \overset{\Delta}{f}_{32} & 2\overset{\Delta}{f}_{33+1} & \overset{\Delta}{f}_{35} & \overset{\Delta}{f}_{36} \\ \cdot & \cdot & 2\overset{\Delta}{f}_{53} & \cdot & \cdot \\ \cdot & \cdot & 2\overset{\Delta}{f}_{63} & \cdot & \cdot \end{bmatrix} \qquad \ldots\text{(A-30)}$$

The Field Boundaries Matrices are given according to the values of k in sequence as in equations (II-14).

Abreviations in the Field Boundaries Matrix $[\overset{\Delta}{F}_A][k>(m/\alpha)^2]$

$$\beta_1 = \sqrt{\theta(\pi\sqrt{k} + \theta)}$$

$$\beta_2 = \sqrt{\theta(\pi\sqrt{k} - \theta)}$$

$$\beta_3 = \beta_1/\beta_2$$

$$\beta_4 = 1/\beta_3$$

$$\beta_5 = \beta_1 \cdot \beta_2$$

$$A_1 = \beta_1{}^2 - \nu\theta^2$$

$$A_2 = \beta_2{}^2 + \nu\theta^2$$

$$A_o = A_1 + A_2$$

$$A_{12} = A_1 \cdot A_2$$

$$\theta = m\pi/\alpha$$

$$\varepsilon_1 = \sinh \beta_1\eta \sin \beta_2\eta$$

$$\varepsilon_2 = \sinh \beta_1\eta \cos \beta_2\eta$$

$$\varepsilon_3 = \cosh \beta_1\eta \sin \beta_2\eta$$

$$\varepsilon_4 = \cosh \beta_1\eta \cos \beta_2\eta$$

$$\varepsilon_5 = 1 - \varepsilon_4$$

Field Boundaries Matrix $[\overset{\Delta}{F}_A][k_>(m/_\alpha)^2]$

$\frac{1}{A_o^2}[2A_{12} + (A_2^2 + A_1^2)\cdot\varepsilon_4$ $+ (\beta_4 A_1^2\ {}_2 - \beta_3 A_2^2)\ \varepsilon_1$	$\frac{1}{A_o}(\beta_1\varepsilon_2 + \beta_2\varepsilon_3)$	$\frac{2}{A^2}[(A_2 - A_1)\cdot\varepsilon_5$ $+ (\beta_3 A_2 + \beta_4 A_1)\cdot\varepsilon_1]$	$\frac{1}{A_o}(\frac{\varepsilon_3}{\beta_2} - \frac{\varepsilon_2}{\beta_1})$	$\frac{1}{A_o^2}[2\cdot\varepsilon_5$ $+(\beta_3 - \beta_4)\cdot\varepsilon_1]$
$\frac{1}{A_o}(\frac{A_1^2}{\beta_1}\cdot\varepsilon_2 - \frac{A_2^2}{\beta_2}\cdot\varepsilon_3)$	ε_4	$\frac{2}{A_o}(\frac{A_1}{\beta_1}\cdot\varepsilon_2 + \frac{A_2}{\beta_2}\cdot\varepsilon_3)$	$\frac{\varepsilon_1}{\beta_5}$	$= A(1,4)$
$\frac{1}{A_o^2}[A_{12}(A_2 - A_1)\cdot\varepsilon_5$ $- (\beta_3\ {}^3_2 + \beta_4\ {}^3_1)\cdot\varepsilon_1]$	$\frac{1}{A_o}(\beta_1 A_2\varepsilon_3 - \beta_2 A_1\varepsilon_3)$	$1 - \frac{1}{A_o^2}[2A_{12}\cdot\varepsilon_5$ $+(\beta_4 A_1^2 - \beta_3 A_2^2)\cdot\varepsilon_1]$	$= 0\cdot5A(2,3)$	$= 0\cdot5A(1,3)$
$-\frac{1}{A_o}(\beta_2 A_1^2\varepsilon_3 + \beta_1 A_2^2\varepsilon_2)$	$-\beta_5\cdot\varepsilon_1$	$= 2A(3,2)$	$= A(2,2)$	$= A(1,2)$
$\frac{1}{A_o^2}[2A_1^2A_2^2\varepsilon_5 + (\beta_3 A_2^4 - \beta_4 A_1^4)\cdot\varepsilon_1]$	$= A(4,1)$	$= 2A(3,1)$	$= A(2,1)$	$= A(1,1)$

$$\ldots(A\text{-}31)$$

Field Boundaries Matrix $[\overset{\Delta}{F_B}][k = (m/\lambda)^2]$

$c - \frac{\nu}{2}(\nu_2\gamma + \beta_1)$	ϕ	$\frac{2}{\beta^2}(\beta_1 + \nu_1\gamma)$	$\frac{1}{\beta^2}(\eta\cdot c - \phi)$	$\frac{1}{\beta^2\theta^2}(-\gamma + \frac{1}{\nu})$
$\frac{\theta^2}{2}(\nu_2{}^2\cdot\phi - \eta\cdot\nu^2 c)$	c	$\eta\cdot\nu c + \nu_2\cdot\phi$	$\eta\cdot\phi$	$= B(1,4)$
$\frac{\theta^2\nu}{2}(\nu_1\nu_2\cdot\gamma - \beta_1\cdot\nu)$	$\frac{\beta_1}{\eta}$	$1 + \nu(\nu_2\cdot\gamma + \beta_1)$	$= 0\cdot 5\ B(2,3)$	$= 0\cdot 5\ B(1,3)$
$-\frac{\theta^3\cdot\nu^2}{2}\cdot S$	0	$= 2\cdot B(3,2)$	$= B(2,2)$	$= B(1,2)$
$\frac{\theta^4\nu^2}{2}(\beta_1\cdot\nu - \nu_2{}^2\cdot\gamma)$	$= B(4,1)$	$2\cdot B(3,1)$	$= B(2,1)$	$= B(1,1)$

...(A-32)

Abreviations in the Field Boundaries Matrix $[\overset{\Delta}{F}_B]$ $[k = (m/\alpha)^2]$

$$\beta = 2\sqrt{\theta}$$

$$\theta = m\pi/\alpha$$

$$\delta = \theta \cdot \eta\nu/\sqrt{2}$$

$$\nu_1 = (1-\nu)$$

$$\nu_2 = (2-\nu)$$

$$c = \cosh \beta\eta$$

$$s = \sinh \beta\eta$$

$$\beta_1 = \delta \cdot s$$

$$\gamma = c - 1$$

$$\phi = s/\beta$$

Field Boundaries Matrix $[\overset{\Delta}{F}_C][0<k<(m/\alpha)^2]$

$\frac{1}{C_o^2}[-2C_{12} + (C_1^2 + C_2^2)\varepsilon_4 - C_4\cdot\varepsilon_1]$	$\frac{1}{C_o}(\beta_1\cdot\varepsilon_2 - \beta_2\cdot\varepsilon_3)$	$\frac{-2}{C_o^2}[C_3\cdot\varepsilon_5 + (\beta_3 C_2 + \beta_4 C_1)\cdot\varepsilon_1]$	$\frac{1}{C_o^2}(\frac{\varepsilon_3}{\beta_1} - \frac{\varepsilon_2}{\beta_1})$	$\frac{1}{C_o^2}[2\cdot\varepsilon_5 + (\beta_3 + \beta_4)\varepsilon_1$
$\frac{1}{G_o}(\frac{C_1^2}{\beta_1}\cdot\varepsilon_2 - \frac{C_2^2}{\beta_2}\cdot\varepsilon_3)$	ε_4	$\frac{2}{C_o}(\frac{C_1}{\beta_1}\cdot\varepsilon_2 - \frac{C_2}{\beta_2}\cdot\varepsilon_3)$	$\frac{\varepsilon_1}{\beta_5}$	$= C(1,4)$
$\frac{1}{C^2}[(C_{12}\cdot C_3\cdot\varepsilon_5) + (\beta_3 C_2^3 + \beta_4 C_1^3)\varepsilon_1]$	$\frac{1}{C}(\beta_2 C_1\varepsilon_3 - \beta_1 C_2\varepsilon_3)$	$1 + \frac{2}{G_o^2}[2C_{12}\cdot\varepsilon_5 + C_4\cdot\varepsilon_1]$	$= 0\cdot 5\ C(2,3)$	$= 0\cdot 5C(1,3)$
$\frac{1}{C_o}(\beta_2 C_1^2\varepsilon_3 - \beta_1 C_2^2\varepsilon_2)$	$\beta_5\cdot\varepsilon_1$	$= 2C(3,2)$	$= C(2,2)$	$= C(1,2)$
$\frac{1}{C_o^2}[2\cdot C_1^2\varepsilon C_2^2 \quad 5 + (\beta_3 C_2^4 + \beta_4 C_1^4)\varepsilon_1]$	$= C(4,1)$	$= 2C(3,1)$	$= C(2,1)$	$= C(1,1)$

Abbreviations in the Field Boundaries Matrix $[\overset{\Delta}{F}_C][0<k<(m/\alpha)^2]$

$$\beta_1 = \sqrt{\theta(\theta + \pi\sqrt{k})}$$

$$\beta_2 = \sqrt{\theta(\theta - \pi\sqrt{k})}$$

$$\beta_3 = \beta_1/\beta_2$$

$$\beta_4 = 1/\beta_3$$

$$\beta_5 = \beta_1 \cdot \beta_2$$

$$C = C_1 - C_2$$

$$C_1 = \beta_1{}^2 - \nu\theta^2$$

$$C_2 = \beta_2{}^2 - \nu\theta^2$$

$$C_3 = C_1 + C_2$$

$$C_4 = \beta_3\ C_2{}^2 + \beta_4 \cdot C_1{}^2$$

$$C_{12} = C_1\ C_2$$

$$\varepsilon_1 = \sinh \beta_1\eta \cdot \sinh \beta_2\eta$$

$$\varepsilon_2 = \sinh \beta_1\eta \cdot \cosh \beta_2\eta$$

$$\varepsilon_3 = \cosh \beta_1\eta \cdot \sinh \beta_2\eta$$

$$\varepsilon_4 = \cosh \beta_1\eta \cdot \cosh \beta_2\eta$$

$$\varepsilon_5 = 1 - \varepsilon_4$$

$$\theta = m\pi/\alpha$$

Field Boundaries Matrix $[\overset{\Delta}{F}_D][k = 0]$

$1+\frac{\nu_1^2}{4}\cdot\varepsilon_5 + \frac{\nu_1\nu_3}{4}\cdot\varepsilon_1$	$\frac{1}{2}(\eta + \frac{\varepsilon_3}{\theta})$	$\frac{\eta^2}{2}(\nu_1 + \nu_2\cdot\frac{\varepsilon_1}{\varepsilon_5})$	$\frac{m}{2\theta^2}(\frac{\varepsilon_3}{\varepsilon_4} - 1)$	$\frac{m^2}{4\theta^2}(\frac{\varepsilon_1}{\varepsilon_5} - 1)$
$\frac{\nu_1\theta}{2}(\varepsilon_6 + \nu_3\varepsilon_3)$	ε_2	$\frac{1}{\theta}(\varepsilon_6 + \nu_2\cdot\varepsilon_3)$	$\frac{\varepsilon_1}{\theta^2}$	= D(1,4)
$\frac{\theta^2}{4}(\nu_1\nu_2\nu_3\varepsilon_1 - \nu_1^3\varepsilon_5)$	$\frac{\theta}{2}(\nu_2\theta_3 - \nu_1\varepsilon_4)$	$1 - \frac{1}{2}(\varepsilon_5\nu_1^2 - \nu_2^2\varepsilon_1)$	= 0 5 D(2,3)	= 0 5 D(1,3)
$\frac{\theta^3}{2}\nu_1(\nu_3\varepsilon_3 - \varepsilon_6)$	$\theta^2\cdot\varepsilon_1$	= 2 D(3,2)	= D (2,2)	= D(1,2)
$\frac{\theta^4}{4}\nu_1^2(\nu_3^2\theta_1 - \nu_1^2\varepsilon_5)$	= D(4,1)	= 2 D(3,1)	= D(2,1)	= D(1,1)

...(A-34)

Abbreviations in the Field Boundaries Matrix $[\overset{\Delta}{F}_D][k = 0]$

$$\theta = m\pi/\alpha$$

$$\nu_1 = 1 - \nu$$

$$\nu_2 = 1 + \nu$$

$$\nu_3 = 3 + \nu$$

$$\varepsilon_1 = (\sinh \varepsilon_4)^2$$

$$\varepsilon_2 = (\cosh \varepsilon_4)^2$$

$$\varepsilon_3 = \sqrt{\varepsilon_1 \cdot \varepsilon_2}$$

$$\varepsilon_4 = \theta\eta$$

$$\varepsilon_5 = \theta^2\eta^2$$

$$\varepsilon_6 = \nu_1 \cdot \varepsilon_4 = \nu_1 \cdot \theta \cdot \eta$$

Field Boundaries Matrix $[\overset{\Delta}{F}_G]$ [k<0]

$\frac{1}{G_o}[G_1\varepsilon_5 + G_2\varepsilon_2 + \frac{\nu_1\lambda^2}{2}(\frac{G_2}{\beta_2^2}\cdot\varepsilon_4 + \frac{G_1}{\beta_1^2}\cdot\varepsilon_1)]$	$\frac{1}{2}(\frac{\varepsilon_6}{\beta_2} + \frac{\varepsilon_3}{\beta_1})$	$\frac{1}{G_o}(\frac{G_2}{\beta_2^2}\varepsilon_4 + \frac{G_1}{\beta_1^2}\varepsilon_1)$	$\frac{1}{G_3}(\beta_2\varepsilon_3 - \beta_1\varepsilon_6)$	$\frac{1}{2G_o}(\frac{1}{\beta_1^2} - \frac{4}{\beta_2^2})$
$\frac{1}{G_3}(\beta_2\varepsilon_7\varepsilon_3 - \beta_1\varepsilon_8\varepsilon_6)$	$\varepsilon_1 + \varepsilon_5$	$\frac{2}{G_3}(\beta_1G_2\varepsilon_6 + \beta_2G_1\varepsilon_3)$	$\frac{2}{G_o}(\varepsilon_1 + \varepsilon_4)$	= G(1,4)
$\frac{1}{2G_o}(\frac{G_1}{\beta_1^2}\varepsilon_7\varepsilon_1 + \frac{2}{\beta_2^2}\varepsilon_8\varepsilon_4)$	$\frac{1}{2}(\frac{G_1}{\beta_1}\varepsilon_3 - \frac{G_1}{\beta_1}\varepsilon_6)$	$1 + \frac{1}{G_o}(\frac{G_1^2}{\beta_1^2}\varepsilon_1 - \frac{G_2^2}{\beta_2^2}\cdot\varepsilon_4)$	= 0 5 G(2,3)	= 0 5 G(1,3)
$\frac{1}{2}(\frac{\varepsilon_7}{\beta_1}\varepsilon_3 + \frac{\varepsilon_8}{\beta_2}\varepsilon_6)$	$\frac{G_o}{2}(\varepsilon_1 + \varepsilon_4)$	= 2G(3,2)	= G(2,2)	= G(1,2)
$\frac{1}{2G_o}(\frac{\varepsilon_7^2}{\beta_1^2}\cdot\varepsilon_1 - \frac{\varepsilon_8^2}{\beta_2^2}\cdot\varepsilon_4)$	= G(4,1)	= 2G(3,1)	= G(2,1)	= G(1,1)

Abbreviations in the Field Boundaries Matrix $[\overset{\Delta}{F}_G][k<0]$

$$\beta_1 = \sqrt{\frac{\theta}{2}\left(\sqrt{\theta^2 + \pi^2 k} + \theta\right)}$$

$$\beta_2 = \sqrt{\frac{\theta}{2}\left(\sqrt{\theta^2 + \pi^2 k} - \theta\right)}$$

$$\beta_{12} = \beta_1 \cdot \beta_2$$

$$\theta = m\pi/\alpha$$

$$\nu_1 = 1 - \nu$$

$$G_1 = 2\beta_1^2 - \theta^2(1-\nu)$$

$$G_2 = 2\beta_1^2 + \theta^2(1-\nu)$$

$$G_o = G_1 + G_2$$

$$G_3 = G_o \cdot \beta_{12}$$

$$\varepsilon_1 = (\sinh \beta_1 \eta)^2$$

$$\varepsilon_2 = (\cosh \beta_1 \eta)^2$$

$$\varepsilon_3 = \sinh\beta_1\eta \cdot \cosh\beta_1\eta$$

$$\varepsilon_4 = (\sin\beta_2\eta)^2$$

$$\varepsilon_5 = (\cos\beta_2\eta)^2$$

$$\varepsilon_6 = \sin\beta_2\eta \cdot \cos\beta_2\eta$$

$$\varepsilon_7 = 2\beta_1^2\eta G_2 + \nu_1\theta^2 G_1$$

$$\varepsilon_8 = 2\beta_2^2\eta G_1 - \nu_1\theta^2 G_2$$

APPENDIX B

Table (3)

Buckling Load on Test Specimens

Group A		Group B		Group C	
Symbol	Load lb.	Symbol	Load lb.	Symbol	Load lb.
A_1	442	B_1	30	C_1	0.0
A_2	1850	B_2	160	C_2	25
A_3	6120	B_3	488	C_3	90
A_4	6180	B_4	----	C_4	----
A_5	7035	B_5	----	C_5	----
A_6	9700	B_6	----	C_6	----
A_7	----	B_7	4350	C_7	----
A_8	11150	B_8	4850	C_8	4080
A_9	----	B_9	6840	C_9	----
A_{10}	----	B_{10}	----	C_{10}	5500
A_{11}	12 800	B_{11}	8400	C_{11}	7500
A_{12}	----	B_{12}	----	C_{12}	9210
A_{13}	13350	B_{13}	13650	C_{13}	13500
A_{14}	11000	B_{14}	11 600	C_{14}	10 800
A_{15}	9900	B_{15}	10 400	C_{15}	10 100
A_{16}	7900	B_{16}	8.500	C_{16}	8300