NONLINEAR BEHAVIOUR OF OPEN THIN-WALLED ELASTIC BEAMS

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by

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TITLE: NONLINEAR BEHAVIOUR OF OPEN THIN-WALLED ELASTIC BEAMS

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SCOPE AND CONTENTS:

A general,consistent,nonlinear theory for open thin-walled elastic beams is presented. The theory takes into account geometric nonlinearities caused by large rotation of the cross section of the beam. The nonlinear differential equations of deformation and response are derived by means of application of Hamilton's principle. It is found that the set of equations reduces to the results obtained by Cullimore and Gregory in the special cases of large uniform torsion of thin-walled members. A solution of a thin-walled beam, subjected to large non-uniform torsional deformation due to application of torques at the ends, is obtained. Comparison is made on the torque - rotation characteristics of a thin-walled beam subjected to large uniform torsion and large non-uniform torsion to show the effect of end constraint from warping.

A set of nonlinear equations to study the stability of a thin-walled beam of open cross section, under axial loading (spatial stability) and lateral loading (lateral stability), is presented. Using the derived equations, the dynamic stability of thin-walled beams of symmetrical and monosymmetrical cross sections subjected to axial loads,is investigated. The regions of parametric instability, the steady state amplitudes of oscillations, once parametric instability takes place, and the non-steady state solutions, to show the growth of the parametric oscillations, are carried out.

The effect of viscous damping on the steady state amplitude and the growth behaviour of the parametrically excited oscillations is shown. The dynamic stability of a thin-walled beam of symmetrical I section and a monosymmetrical split ring section are worked out in detail as examples.

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This work is dedicated to my parents in recognition of the sacrifices and support so that this goal might be achieved.

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NOTATIONS

a	arbitrary point on the profile line of the cross section
a_o	distance from center of circle to centroid of the split ring section
$\{a_i\},\{b_i\}$	vectors of constant coefficients
A	cross sectional area
$[A],[B]$	matrices of coefficients of coupled Mathieu's equations
b	depth of a narrow rectangular section
B	width of the flanges of an I section
c	thickness of the thin-walled section
c_d	fractional critical damping
$c_i,\bar{c}_i$	constant quantities
c_f,c_w	thickness of flange and web plates of an I section, respectively
c_x,c_y	co-ordinates of shear center relative to centroid in the X, Y, directions, respectively
C	the shear center
$[C_d]$	the damping matrix
$[D],[E]$	matrices of coefficients of coupled Mathieu's equations
$e(e_x,e_y)$	point of application of the resultants of axial loads

$\bar{e}(\bar{e}_x,\bar{e}_y)$	point of application of lateral loads
E	Young's modulus
E^*	$E/(1 - \nu^2)$
$\{f\}$	vector of variables
$g_i,\bar{g}_i$	functions of the amplitude components
G	shearing modulus of elasticity
h	perpendicular distance from shear center to the tangent of profile line at point a
H	height of web plate of an I section
i,j	integer index
I_d	St. Venant's torsional constant
I_{xx},I_{yy}	principal moments of inertia
I_{pc}	half the polar moment of inertia relative to shear center
I_R	fourth moment of inertia about shear center
I_{Rx},I_{Ry}	third moments of inertia
$I_{\omega\omega}$	warping torsional constant
$[I]$	the identity matrix
k	nondimensionalized quantity $= [0.5(1 - \nu)\ell^2 I_d/I_{\omega\omega}]^{1/2}$
$K_{ij},\bar{K}_{ij}$	constant coefficients
K_i,K_i^*	constants of integration
ℓ	length of the thin-walled beam
m_t	twisting moment distributed over the length of the thin-walled beam

M	nondimensionalized moment quantity
M_x, M_y	total bending moments along the $\bar{X}$ and $\bar{Y}$ directions due to axial and lateral loads
$\bar{M}_x, \bar{M}_y$	bending moments along the $\bar{X}$ and $\bar{Y}$ directions due to axial loads
M_x^*, M_y^*	bending moments due to lateral forces along the $\bar{X}$ and $\bar{Y}$ axis, respectively
M_t	moment of twist applied at the end of the thin-walled beam
M_ω	bimoment
n	perpendicular distance from shear center to the normal of the profile line at point a
N	resultant of axial loads
N_o, N_t	nondimensionalized axial load parameters
O	centroid of the cross section
O'	center of the circle of the split ring section
OX, OY, OZ	principal axes through centroid and rotate with the cross section
$O\bar{X}$, $O\bar{Y}$, $O\bar{Z}$	principal axes through centroid in the undeformed state
p	distributed axial end load
$\bar{p}$	distributed axial load
P	resultant of axial end loads
P_o, P_t	static and dynamic components of axial load

$P_\zeta, P_\eta, P_\theta$	buckling loads for uncoupled flexural and torsional modes of buckling
$P_{1,2}$	buckling loads for coupled flexural torsional modes of buckling
q	distributed lateral load
q_x, q_y	lateral load components along the $\bar{X}$ and $\bar{Y}$ directions, respectively
Q_x, Q_y	shearing forces along the $\bar{X}$ and $\bar{Y}$ directions, respectively
r	$= (h^2 + n^2)^{1/2}$
$\bar{r}$	radius of the split ring section
R_s	radius of curvature in the s direction
s	co-ordinate along the contour of the cross section
t	time variable
T	change in the kinetic energy
$\bar{T}$, T_o	total kinetic energy and energy before buckling, respectively
u, v, w	displacements along z, tangent and normal to the tangent of profile line of cross section, respectively
U	strain energy density of thin shell
U_ε	strain energy density due to stretching
U_κ	strain energy density due to bending
V	change in the strain energy

$\bar{V}$, V_o	total strain energy and energy before buckling, respectively
W	change in the work done by external loads
$\bar{W}$, W_o	total work done and work done before buckling, respectively
W_A, W_L	change in the work done by axial and lateral loads, respectively
W_E	change in work done by axial end loads
W_T	work done due to end twisting moment
x, y, z	variables in the X, Y and Z directions, respectively
z	co-ordinates in the $\bar{Z}$ direction
$\bar{z}$	z/ℓ
$\alpha, \bar{\alpha}, \alpha_{ij}$	constant quantities
β	nondimensionalized buckling load
γ	a small parameter $\gamma << 1$
γ_{zs}	the shear strain in the sz plane
δ	variational operator
$\varepsilon_z, \varepsilon_s$	strain in the z and s directions, respectively
η_o, $\tilde{\eta}$	displacements from undeformed to deformed state and from deformed to buckled state, respectively
η_c, η_s	amplitude components of steady state solution
θ	rotational displacement of cross section about shear center

$\theta_1, \theta_2, \theta_3$	components of perturbation solution
θ_o	rotation of cross section prior to buckling
$\tilde{\theta}$	rotation of cross section during buckling
θ_c, θ_s	amplitude components of the steady state solution
θ_n	amplitude of torsional oscillations $= [\theta_c^2 + \theta_s^2]^{1/2}$
κ_z, κ_s	change of curvature in the z and s directions, respectively
λ	frequency of forcing functions
ν	Poisson's ratio (taken to be 0.33 in all calculations)
ζ, ξ, η	translational displacements of shear center in the Z, X and Y directions, respectively
$\zeta_a, \zeta_c, \zeta_s$	amplitude components of the steady state solution
ζ_o	longitudinal displacement prior to buckling
$\tilde{\zeta}$	longitudinal displacement during buckling
$\zeta_t, \eta_t, \xi_t, \theta_t$	time dependent part of the solution
$\bar{\xi}, \bar{\eta}$	translational displacements of shear center in the $\bar{X}$ and $\bar{Y}$ directions, respectively

ξ_0, $\tilde{\xi}$	displacements from the undeformed to deformed state and from deformed to buckled state, respectively
ξ_c, ξ_s	amplitude components of the staady state solution
ρ	mass per unit volume of the member
τ	twist in the zs plane
ϕ	angle measured from the axis of symmetry of the split ring section
Φ	semi central angle (radians) of the split ring section
ψ	angle between the tangent of the cross section at point a and the X axis
Ψ	nonlinear functions
ω	sectorial area
$\Omega_\zeta, \Omega_\xi, \Omega_\eta, \Omega_\theta$	frequency of free vibrations under constant axial load of the uncoupled longitudinal, flexural and torsional type of vibrations
$\Omega_{1,2}$	frequency of free vibration under constant axial load for the coupled flexural-torsional type of vibrations
$( )^{\cdot}$	dots above a function indicate derivatives and the order of the derivative with respect to time t.
$( )'$	primes indicate derivatives and the order of the derivative of the associated quantity with respect to z.

CHAPTER I

INTRODUCTION

1.1. Preamble

Thin-walled structures are classified as bodies that have the form of a long prismatic shell. These bodies are characterized in that their three dimensions are all of different order of magnitude. The thickness of the shell is small compared with other characteristic dimensions of the cross section. The cross sectional dimensions are small compared with the length of the shell. Thin-walled elements are classified further if they are of open or closed cross section.

Thin-walled elements are widely used in trusses, bridges, aerospace structures and many other areas of engineering design. An open thin-walled section, in general, has large flexural rigidity which makes it an efficient structural element to resist bending. However, it has relatively low torsional rigidity. Unless the applied loads pass through the line of shear centers of the beam, both flexural and torsional deformation occur. Due to its inherent low torsional rigidity, large rotations of the cross sections of the beam may occur. It is likely that thin-walled beams will be used more often as the improved fabrication techniques in metals, made it possible to form thin-walled members of various shapes of cross section such as channel, I, Z and thin-walled aerofoil sections. Thus, a more detailed examination of the behaviour of such structural elements, taking into account the geometrical nonlinearity caused by large rotations of the cross section, is desirable.

1.2. Literature Review

There is a large volume of literature on the behaviour of open thin-walled elastic beams. Most of the literature is concerned with small deflections and rotations of the member. For clarification, it is convenient to sub-divide the literature on thin-walled beams into the following categories:

(a) Linear Deformational Analysis Under Static Loads

Under the action of static load, a thin-walled beam may deflect and rotate. Such deformations are accompanied by a state of strain and a state of stress. Linear analysis is concerned with small deflections and rotations of a thin-walled beam subjected to static loads. It predicts deformation, strains and the state of stress in the member. Thin-walled beams subjected to static loads may respond in different ways depending on the character of the applied load and geometrical configurations of the beam. The response of the member takes the form of flexural deformations, if the applied load passes through the line of shear centers of the beam. However, when the load does not pass through the line of shear centers, both flexural and torsional deformations occur. As torsional deformations take place, plane sections do not remain plane and warping of the section may develop. If warping of the thin-walled beam is prevented, additional axial strains are created and the beam is said to be in the state of "non-uniform torsion". The concept of non-uniform torsion was introduced by Timoshenko [49]*. He considered warping of the cross section of a symmetrical I beam subjected to a twisting moment. Theories of bending and torsion were studied by Goodier [21], Timoshenko and Gere [51]. Extensive study of the behaviour of thin-walled members was condensed in book form, in Russian in 1958, by Vlasov and translated into English in 1961 [54].

(b) Linear Stability Analysis Under Static Loads

Thin-walled beams subjected to static loads may lose stability in different ways. Under the action of axial loads, local buckling or overall buckling of the member may take place. The overall buckling may take the form of flexural, torsional or coupled flexural torsional type of buckling. Under the action of lateral loads, the member may lose stability due to lateral buckling. Lateral buckling, again may take the form of flexural, torsional or coupled flexural - torsional type of buckling. The type of buckling depends on the point of application, direction of the load and the geometrical configurations of the thin-walled cross

* Numbers in parentheses refer to entries in the Bibliography.

section. Linear stability analysis predicts the critical load corresponding to buckling of the member. The elastic stability of thin-walled beams was investigated by Timonshenko [49,51], Pflüger [46], Bleich [4] and Vlasov [54]. The effect of axial stress on the torsional buckling of thin-walled beams was studied by Goodier [21] and Biot [2]. It was shown that the torsional rigidity of the member is increased if the beam is under tensile stresses. On the other hand, compressive stresses reduce the torsional rigidity of a thin-walled beam.

(c) Nonlinear Deformational Analysis Under Static Loads

Due to the fact that thin-walled members are weak in torsion, studies have been carried out to take account of possible large rotation of the cross section of the beam. In particular, the behaviour of a thin-walled beam subjected to static twisting moments applied at the end cross sections, has been studied by Cullimore [11] and Gregory [23,24]. Cullimore considered the case of uniform torsion of a beam of narrow rectangular section and I section, subjected to static end moments. Gregory extended the study of uniform torsion to cover a beam of monosymmetrical angle sections. They showed that if large torsional deformation was considered, the torsional and longitudinal deformations are coupled together. Such coupling is known as the "shortening effect". The name was derived from the fact that when a thin-walled beam is subjected to equal and opposite torques applied at its ends, the end sections tend to approach each other due to the torsional and longitudnial coupling. This coupling is nonlinear in nature and is usually neglected in the linear analysis of the problem.

(d) Linear Response Analysis Under Dynamic Loads

Thin-walled beams subjected to dynamic loads may respond in different ways depending on the character of the applied load and the geometrical configuration of the thin-walled beam. The response of the member may take the form of flexural, torsional or coupled flexural - torsional type of oscillations. The coupled frequencies of free vibrations for thin-walled beams of monosymmetrical cross section, were determined by Gere and Lin [19,20] for various end conditions. Vibrations of thin-walled beams when subjected to dynamic loads were discussed by Vlasov [54]. The effect of shear strain due to bending and warping of the beam was included by Tso [53]. He considered torsional deformations in addition to bending and shearing deformations. This approach can be considered as a "Timoshenko beam theory" analogy for thin-walled elements.

(e) Linear Stability Analysis Under Dynamic (Periodic) Loads

Under dynamic, periodic, axial loading a thin-walled beam may lose stability at load values much smaller than the static critical load, depending on the frequency of the loading. It is known that there exists a number of frequency ranges in which the beam can be excited into lateral oscillations. This type of loss of stability is commonly known as parametric instability or dynamic instability of the beam.

The ranges of frequency, under which parametric instability can take place, depends on the natural frequencies of the beam and also on the amplitude of the dynamic load. Therefore, in parametric stability studies of systems, it is convenient to refer to conditions under which parametric instability occur to plots in the applied load amplitude and applied load frequency space. The regions in the load amplitude - load frequency space that corresponds to possible parametric instability of the system are referred to as unstable regions. The unstable region, located in the neighbourhood of the applied load frequency equals twice the natural frequency of the system, is referred to as the"principal unstable region". It is in this region that the system is most susceptible to parametric instability. The study of linear parametric stability analysis consists of the determination of the locations of the unstable regions, and in particular, the "principal unstable region".

For a system which has more than one single natural frequency, such as thin-walled beams, there is a number of "principal unstable regions". For each natural frequency, there exists a principal unstable region that corresponds to it.

Literature on the paremetric stability of mechanical and structural systems were reviewed by Evan-Iwanowski [18]. In particular, the parametric stability of thin-walled beams was given by Bolotin [6]. Using Vlasov's thin-walled beam equations, Bolotin constructed the boundaries of the principal regions of parametric instability for a thin-walled beam of symmetrical and monosymmetrical cross sections.

It can be seen that most work on open thin-walled beams are based on the linear theory of thin-walled beams. The need for a nonlinear theory of thin-walled beams arises in many aspects of study. Due to the low torsional rigidity of open thin-walled beam, rotations of the beam cross section cannot be considered as small in general. The effect of large twist on the warping stress need to be studied. Such a nonlinear theory would generalize the work of Gregory [25] which has been verified experimentally. Nonlinear behaviour of the response of a thin-walled beam during coupled flexural - torsional type of vibration was observed experimentally by Tso [52]. Theoretical explanation of the nonlinear behaviour is yet to be investigated. In the static stability analysis, the linear theory of Vlasov becomes inadequate in the study of post buckling behaviour of the member. In the parametric stability analysis, it is necessary to include nonlinear terms in order to predict bounded response and to establish the steady state amplitudes of vibrations once the system is excited into the parametric resonance. The need for nonlinear terms arise because a linear theory predicts unbounded response once parametric instability sets in. Unlike in forced vibration studies inclusion of viscous damping terms in the linear analysis does not predict bounded responses.

Recent attempts have been made [3, 15, 17, 47] to develop a nonlinear theory of open thin-walled beams. However, the geometrical nonlinearities under consideration were not introduced in a systematic manner in the analysis. Therefore, the resulting theories are either too complicated to use or they are of inconsistent order of approximation. Therefore, there is a need for a nonlinear theory of thin-walled beams of open cross section which is consistent and relatively simple to apply.

1.3. Statement of the Problem

This thesis is concerned with the derivation of a general,consistent,nonlinear theory of thin-walled beams of open cross section subjected to time dependent loads. Due to the low torsional rigidity of open thin-walled beams, rotations of the cross section cannot be considered as small in general. In other words, if the torsional deformations are taken to be of the order ε , where $\varepsilon < 1$, then it is reasonable to consider the flexural deformations to be of the order ε^2. This assumption was found to be justifiable due to the fact that the flexural rigidity of a thin-walled member is large as compared to torsional rigidity. Treating torsional deformational quantities as of order ε , where $\varepsilon < 1$, and flexural deformational quantities as of order ε^2, terms retained in the formulation are such that the resulting equations of motion (or equilibrium) contain terms up to an order of ε^3. Thus, products of torsional deformations and products of torsional and flexural deformations are retained in the final expressions. However, products of flexural deformations are neglected, being treated as terms of order ε^4 or higher. Only a consistent third order approximation was maintained throughout the derivation. Therefore, the nonlinear theory is accurate for large rotations, but essentially linear in the longitudinal and flexural deformations. The present theory is different from previous formulations in that the effect of large rotations and coupling between axial shortening and torsional deformation is taken into account. The present approach starts from the nonlinear shell theory of elastic shells. By means of special assumptions, the theory can be reduced to a theory to represent the behaviour of thin-walled beams. The energy formulation is adopted in the derivation. Such an approach has two advantages.

Firstly, energy expressions obtained during formulation are useful for the direct application of approximate variational methods, such as the Rayleigh-Ritz method of solution. Secondly, a set of boundary conditions, of consistent order of approximation, is obtained. The energy approach consists of formulating the strain energy, work done by external forces and kinetic energy. Application of Hamilton's principle leads to the governing set of nonlinear differential equations and the associated boundary conditions. The nonlinear deformation equations in the simplest form reduce to Gregory's formulations. The nonlinear deformation theory was then applied to the case of non-uniform torsion of thin-walled beams. An example of a cantilever thin-walled beam of symmetrical I section, subjected to end twisting moment is worked out in detail.

The second part of this research is the formulation of a general,consistent,nonlinear stability theory. The problems considered are "bifurcation" type of stability problems in which other forms of equilibrium, different from the precritical shape, become possible. The stability study is further divided into cases of axial stability and lateral stability. Axial stability problems refer to the class of problems where the structure losses stability under axial loading. Lateral stability problems refer to the loss of stability of the beam under lateral loading.

The energy approach is used again to formulate the nonlinear axial and lateral stability theories. The nonlinear stability theory is then applied to study the parametric stability of thin-walled beams subjected to axial dynamic (periodic) loads. The case of a simply supported thin-walled beam of symmetrical I section and the case of a thin-walled beam, of built-in ends, of monosymmetrical split ring section, are considered. In each case, the principle unstable regions are determined. In addition, the steady state amplitudes of parametric oscillations are found and the nonsteady state solutions are worked out which show the growth of the amplitudes of vibrations. The influence of viscous damping on the steady state amplitude and the growth history of the amplitude of vibration is also given.

CHAPTER II

GENERAL CONSISTENT NONLINEAR DEFORMATION THEORY OF OPEN THIN-WALLED ELASTIC BEAMS

2.1. Introduction

In this chapter, a nonlinear theory of thin-walled beams of open sections is formulated. Some attempts have been made to introduce nonlinearities into the theory of thin-walled beams of open sections [3, 15, 17, 47]. The nonlinearities, however, were not introduced in a systematic manner. Therefore, the resulting theories are either too complicated to use or they are of inconsistent order of approximations.

In order to develop a nonlinear theory which is sufficiently accurate to describe the behaviour of the open thin-walled beam and yet relatively simple to apply, it is necessary to examine more closely the basic characteristics of the thin-walled beam. Fundamentally, a thin-walled beam of open section possesses large flexural stiffness, but small torsional stiffness. A simple example will illustrate this. Consider a beam of standard channel section of 10" x 2 5/8" x 15.3 lb/ft. The length of the beam is taken to be 125 inches. The beam is assumed to be built in at one end while the other end is free. A concentrated load P is applied at the free end in the plane of the major axis of the channel passing through the centroid of the section. Calculations of deformations based on Vlasov's theory of thin-walled beams [54], show that the angle of twist at the free end is 61.87 10^{-6} P, while the maximum slope of the deflected curve is 3.89 10^{-6} P. Thus, the angle of twist is in general much larger than the slope of the beam caused by bending action. One can generalize the argument to state that the torsional deformations are large in comparison with flexural deformations. In other words, if the torsional deformations are of the order ε, where $\varepsilon < 1$, then it is reasonable to consider the flexural deformations to be of the order ε^2.

It is with this frame of mind that the nonlinear theory of thin-walled beams was derived. Instead of using the direct approach of equating internal and applied forces and moments to obtain the equations of motion (or equilibrium), the energy formulation was chosen. The indirect approach offers two distinct advantages. First of all, a set of boundary conditions was obtained. These conditions are of consistent accuracy with the derived equations of motion (or equilibrium). Secondly, with the energy expression readily available, it is possible to apply Rayleigh-Ritz type of technique to obtain approximate solution to the problems at hand.

Starting from the nonlinear theory of elastic prismatic shells, the nonlinear thin-walled beam theory is formulated by making special assumptions and neglecting terms of high order of smallness. Treating torsional deformational quantities as of order ε, where $\varepsilon < 1$, and flexural deformational quantities as of order ε^2, terms are retained in the energy expressions such that the resulting equations of motion (or equilibrium) contain terms up to an order of ε^3. Thus, products of torsional deformations and products of torsional deformation and flexural deformation are retained in the final expressions. However, products of flexural deformations are neglected, being treated as terms of order ε^4 or higher. Therefore, the derived theory can be considered as a nonlinear theory of thin-walled members of open cross sections for large angle of twist. It takes into account the nonlinear nature of torsional deformations and the coupling between torsional and flexural deformations. However, the theory is "Linear" in bending deformations in the sense that it does not take into account large deformation of the beam due to flexure.

2.2. Strain Energy Expression

A thin-walled beam of open section can be treated as a special case of a thin-walled prismatic shell. Consider a prismatic shell and let the origin of the co-ordinate axes be placed at the centroid of the cross section. Axes $O\bar{X}$ and $O\bar{Y}$ are taken to be in the direction of the principal axes of the cross section while $O\bar{Z}$ coincides with the line passing through the centroids of the sections. The shell's generator is then parallel to the $O\bar{Z}$ axis. The lines of principal curvature of the shell are orthogonal lines parallel and perpendicular to the generator, denoted by z and s respectively as shown in figure [1].

According to Love's first approximation, the strain energy density U of a thin elastic shell is given by [30];

$$U = U_\varepsilon + U_\kappa \tag{2.1}$$

$$= \frac{Ec}{2(1 - \nu^2)} [(\varepsilon_z + \varepsilon_s)^2 - 2(1 - \nu)(\varepsilon_z\varepsilon_s - \frac{1}{4}\gamma_{zs}^2)]$$

$$+ \frac{Ec^3}{24(1 - \nu^2)} [(\kappa_z + \kappa_s)^2 - 2(1 - \nu)(\kappa_z\kappa_s - \tau^2)] \tag{2.2}$$

where

U_ε is the strain energy density due to stretching

U_κ is the strain energy density due to bending

ε_z is the strain in the z direction

ε_s is the strain in the s direction

γ_{zs} is the shear strain in the zs plane

κ_z is the change of curvature in the z direction

κ_s is the change of curvature in the s direction

τ is the twist in the zs plane

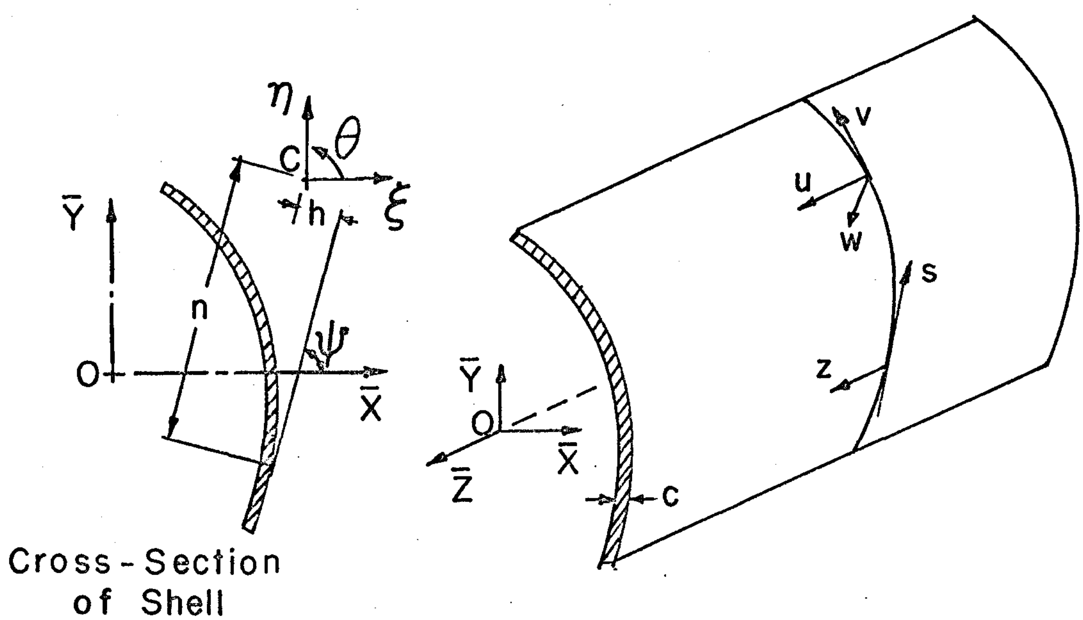

FIG.(1) OPEN THIN-WALLED PRISMATIC SHELL

E represents the modulus of elasticity

ν denotes Poisson's ratio

c is the thickness of the thin-walled section

In the strain energy density expression, the state of stress is assumed to be approximately plane. In other words, the effect of the transverse shear stress and normal stress acting on surfaces parallel to the middle surface of the shell are considered to be negligible.

From the nonlinear formulation of thin shell theory, the strains are related to the displacements u, v and w by the expressions [42],

$$\varepsilon_z = \frac{\partial u}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \tag{2.3}$$

$$\gamma_{zs} = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial z} + \frac{\partial v}{\partial z}\left(\frac{\partial v}{\partial s} + \frac{w}{R_s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial w}{\partial s} - \frac{v}{R_s}\right) \tag{2.4}$$

$$\tau = \frac{\partial^2 w}{\partial s \partial z} + \frac{\partial}{\partial z}\left(\frac{v}{R_s}\right) \tag{2.5}$$

where

u(z,s,t) is the longitudinal displacement along the z direction

v(z,s,t) is the transverse displacement along the tangent of the profile line of the cross section

w(z,s,t) is the transverse displacement along the normal to the tangent of the profile line of the cross section

R_s is the principal radius of curvature of the shell

The fundamental assumption in the theory of thin-walled beams is that the contour of the cross section of the prismatic shell is not deformable in its own plane. Thus, the displacements of the contour in its own plane consist of rigid body displacements only. The shape of the cross section is unchanged after deformation. Thus, the displacements of the cross section in its plane can be described by orthogonal displacements ξ and η of any point C in the plane of the cross section and a rotation θ about the point C. The displacements ξ and η represent the lateral displacements of the cross section in the principal directions while θ represents the rotation of the section. The displacements ξ, η and rotation θ are independent of the co-ordinate s along the contour of the cross section. They are a function of z only.

In the formulation of nonlinear theory due to large rotation, it is necessary to distinguish the deformed and the undeformed state. Two sets of co-ordinate axes are defined. The co-ordinate axes $O\bar{X}$ and $O\bar{Y}$, as shown in figure [2] , are along the principal directions of the cross section in the undeformed state. The $O\bar{Z}$ axis is in the z direction. The displacements of the cross section in the $O\bar{X}$ and $O\bar{Y}$ directions are denoted by $\bar{\xi}$ and $\bar{\eta}$ respectively. The second set of co-ordinate axes OX, OY and OZ is attached to the cross section and thus rotate with the principal directions. The two sets of axes thus coincide in the undeformed state. During deformation, however, $O\bar{X}$, $O\bar{Y}$ and $O\bar{Z}$ axes retain their direction in space while axes OX, OY and OZ will rotate with the cross section. The displacements of the cross section in the OX and OY directions are denoted by ξ and η respectively. In the present formulation bending deformations are considered small, namely rotations about the OX and OY axis are small. Therefore, in relating deformations the cosine of the angle between $O\bar{Z}$ and OZ axes is taken unity.

The displacements along the undeformed axes are related to the displacements in the direction of the deformed axes by co-ordinate transformations using the direction cosines.

The direction cosines for large rotation as shown in figure [3] can be written as:

	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
X	$\cos\theta$	$\sin\theta$	0
Y	$-\sin\theta$	$\cos\theta$	0
Z	0	0	1

(2.6)

The transformation given by (2.6) accounts only for large rotation θ, while slopes due to flexural

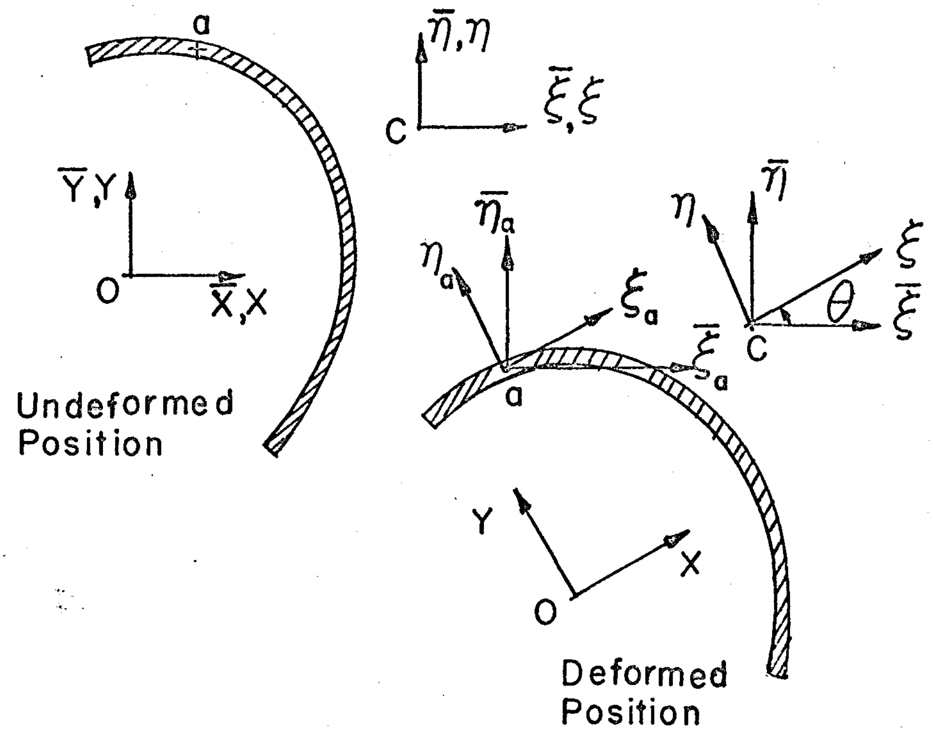

FIG (2) RIGID BODY MOTION OF THE CROSS SECTION

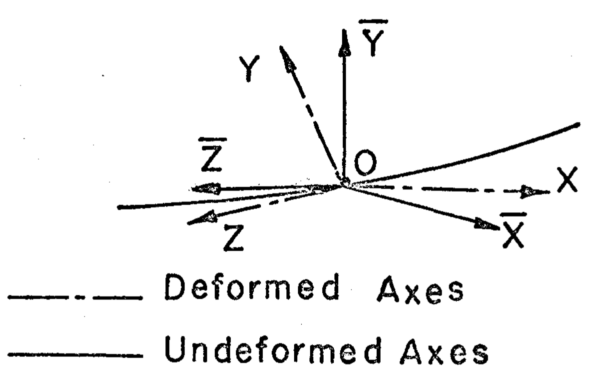

FIG. (3) LARGE ROTATION OF CO-ORDINATE AXES deformations are considered small. Therefore, the displacements along the $O\bar{Z}$ and OZ axes are taken to be the same. The curvatures in the $\bar{X}$ and $\bar{Y}$ directions can be related to the curvatures in the X and Y directions by the relations

$$\xi'' = \bar{\xi}'' \cos\theta + \bar{\eta}'' \sin\theta \qquad (2.7)$$

$$\eta'' = \bar{\eta}'' \cos\theta - \bar{\xi}'' \sin\theta \qquad (2.8)$$

where $(\ )' = \frac{\partial}{\partial z}$

Consider a point a on the profile line of the cross section. The displacements of point a in the $O\bar{X}$ and $O\bar{Y}$ directions $\bar{\xi}_a$ and $\bar{\eta}_a$ can be related to the displacements of point C in the $O\bar{X}$ and $O\bar{Y}$ directions by the expressions

$$\bar{\xi}_a = \bar{\xi} - (y - c_y)\sin\theta + (x - c_x)(\cos\theta - 1) \qquad (2.9)$$

$$\bar{\eta}_a = \bar{\eta} + (x - c_x)\sin\theta + (y - c_y)(\cos\theta - 1) \qquad (2.10)$$

The displacements v(s,z,t) and w(s,z,t) of point a(x,y) represent displacements in the plane of the cross secion. Therefore, they can be expressed in terms of $\bar{\xi}$, $\bar{\eta}$ and θ by the relationships,

$$v(s,z,t) = \bar{\xi}(z,t)\cos\psi(s) + \bar{\eta}(z,t)\sin\psi(s) + n(s)[\cos\theta(z) - 1] + h(s)\sin\theta(s) \qquad (2.11)$$

$$w(s,z,t) = -\bar{\xi}(z,t)\sin\psi(s) + \bar{\eta}(z,t)\cos\psi(s) - h(s)[\cos\theta(z) - 1] + n(s)\sin\theta(z) \qquad (2.12)$$

where ψ is the angle between the tangent of the cross section at point a and the OX axis.

$$h(s) = (x - c_x) \sin \psi - (y - c_y) \cos \psi \quad (2.13)$$

perpendicular distance from point $C(c_x, c_y)$ to the tangent of the profile line at point a(x,y).

$$n(s) = (x - c_x) \cos \psi - (y - c_y) \sin \psi \quad (2.14)$$

perpendicular distance from point $C(c_x, c_y)$ to the normal of the profile line at point a(x,y).

Since the rotation θ of the section is considered to be large (say θ = 30°), it is necessary to use the approximations:

$$\sin \theta \approx \theta - \frac{\theta^3}{3!} \quad (2.15)$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2!} \quad (2.16)$$

The displacement u(s,z,t) can be expressed in terms of ξ, η and θ by assuming the shear strain γ_{zs} to be zero. This assumption was found to be acceptable when the beam is long; thus,

$$\gamma_{zs} = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial z} + \frac{\partial v}{\partial z}\left(\frac{\partial v}{\partial s} + \frac{w}{R_s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial w}{\partial s} - \frac{v}{R_s}\right) \quad (2.17)$$

substituting expressions (2.11) and (2.12) into equation (2.17) and using the approximations (2.15) and (2.16), there is obtained;

$$u(s,z,t) = \zeta(z,t) - [\bar{\xi}'(z,t) + \bar{\eta}'(z,t)\theta(z,t]x(s)$$

$$- [\bar{\eta}'(z,t) - \bar{\xi}'(z,t)\theta]y(s) - \theta'(z,t)\omega(s)$$

$$+ \theta^2(z,t)\theta'(z,t)\ \Omega_R(s) \qquad (2.18)$$

$\zeta(z,t)$ represents the overall longitudinal displacement of the cross section. Ω_R and ω are geometric properties of the cross section defined by

$$\Omega_R(s) = \int_0^s (n^2 + h^2)\ \frac{d\psi}{ds}\ d\bar{s} \qquad (2.19)$$

$$\omega(s) = \int_0^s h(\bar{s})\ d\bar{s} \qquad (2.20)$$

$\omega(s)$ is commonly referred to as the sectorial area in thin-walled beam theory.

Since the contour of the cross section of a thin-walled beam is assumed to be non-deformable in its own plane, the strain and the change in curvature ε_s and κ_s vanish. Also, in the theory of thin-walled beams the stretching effect is more important than the bending effect, it follows that the effect of the change of curvature κ_z in the z direction is neglected in comparison with the effect of the longitudinal strain ε_z. Neglecting κ_z is equivalent to the assumption that the distribution of the normal stresses over the wall thickness is constant. For thin-walled beams, where the wall thickness c is small, this assumption is reasonable.

The strain energy density from equation (2.2) can be written as:

$$U = \frac{Ec}{2(1 - \nu^2)} \varepsilon_z^2 + \frac{Ec^3}{12(1 + \nu)} \tau^2 \qquad (2.21)$$

where the strain in the z direction is derived from expression (2.3), as

$$\varepsilon_z = \zeta' - (\bar{\xi}'' + \bar{\eta}''\theta)x - (\bar{\eta}'' - \bar{\xi}''\theta)y - \theta''\omega$$
$$+ \frac{1}{2}(\eta^2 + h^2)\theta'^2 + \Omega_R(\theta^2\theta'' + 2\theta\theta'^2) \qquad (2.22)$$

$$\tau = \theta' - \frac{1}{2}\theta^2\theta' \qquad (2.23)$$

The nonlinear term in the twist expression (2.23) is multiplied by c^3 where the thickness c is small. Therefore, this term will be neglected due to the fact that it results in smaller terms than the adopted consistent third order approximation.

The total strain energy for the thin-walled beam V, is then given as:

$$V = \int_0^{\ell}\int_s \{ \frac{Ec}{2(1 - \nu^2)} [\zeta' - (\bar{\xi}'' + \bar{\eta}''\theta)x$$
$$- (\bar{\eta}'' - \bar{\xi}''\theta)y - \theta''\omega + \frac{1}{2}(\eta^2 + h^2)\theta'^2$$
$$+ \Omega_R(\theta^2\theta'' + 2\theta\theta'^2)]^2 + \frac{Ec^3}{12(1 + \nu)}\theta'^2\} \, ds \, dz \qquad (2.24)$$

2.3. Work Done Expression

Two types of external loads will be considered. Firstly, there are axial loads applied at the ends of the beam. Let the stress distribution over the end cross section of the open thin-walled beam in the z direction be denoted by p(s,t). Then one can define the longitudinal end load, end bending moments and bimoments by the following relationships:

$$P(t) = \int_s c\ p(s,t)\ ds \tag{2.25}$$

$$M_y(t) = -\int_s c\ \bar{x}(s)\ p(s,t)\ ds \tag{2.26}$$

$$M_x(t) = \int_s c\ \bar{y}(s)\ p(s,t)\ ds \tag{2.27}$$

$$M_\omega(t) = \int_s c\ \omega(s)\ p(s,t)\ ds \tag{2.28}$$

M_x and M_y denote the bending moments along the $\bar{X}$ and $\bar{Y}$ axes respectively. The bimoment is represented by M_ω.

The work done by the longitudinal stresses applied at the end can be written as:

$$W_E = \int_s c\ p(s,t)\ u(s,z,t)\Big|_{z=0}^{z=\ell} ds$$

$$- \frac{1}{2}\int_s c\ p(s,t) \int_0^\ell (\bar{\xi}_a'^2 + \bar{\eta}_a'^2)\ dzds \tag{2.29}$$

The displacements $\bar{\xi}_a$ and $\bar{\eta}_a$ can be related to the displacements of the point C, in the $\bar{X}$ and $\bar{Y}$ directions by using equations (2.9) and (2.10). Also, substituting equation (2.18) into (2.29), the work done can be written as:

$$W_E = \int_s c\ p\ [\zeta - (\bar{\xi}' + \bar{\eta}'\theta)x - (\bar{\eta}' - \bar{\xi}'\theta)y - \omega\theta' + \Omega_R\ \theta^2\theta']\Big|_{z=0}^{z=\ell} ds$$

$$- \frac{1}{2}\int_s c\ p\int_0^{\ell}\{(\bar{\xi}' - (y - c_y)\theta' - (x - c_x)\theta\theta')^2 + (\bar{\eta}' + (x - c_x)\theta' - (y - c_y)\theta\theta')^2\}\ dzds \quad (2.30)$$

In addition, one can apply twisting moments M_t at the ends also. The work done by the end torque is:

$$W_T = M_t\ \theta\Big|_{z=0}^{z=\ell} \quad (2.31)$$

The second type of loading considered are loads applied along the length of the beam. Only lateral loads will be considered. Let $q(z,t)$ be a laterally applied distributed load along the length of the beam. This load can be always resolved into an equivalent load acting through point C, plus a couple $m_t(z,t)$ about the longitudinal axis through the point C. Further, this equivalent load can be resolved into two components $q_x(z,t)$ and $q_y(z,t)$ in the $O\bar{X}$ and $O\bar{Y}$ directions, respectively.

The work done by the lateral force components can be written as:

$$W_L = \int_0^{\ell}\{q_x(z,t)\bar{\xi} + q_y(z,t)\bar{\eta} + m_t(z,t)\theta\}\ dz \quad (2.32)$$

Therefore, the total work done by the longitudinal end stresses, twisting moment and lateral force components is,

$$W = W_E + W_T + W_L \tag{2.33}$$

2.4. Kinetic Energy Expression

The kinetic energy of the beam is given as:

$$T = \frac{1}{2}\int_s\int_0^{\ell} \rho c \; (\dot{u}^2 + \dot{\xi}_a^2 + \dot{\bar{\eta}}_a^2) \; dzds \tag{2.34}$$

where

ρ is the mass density of the material of the beam

$\dot{u}$ is the velocity component in the z direction

$\dot{\bar{\xi}}_a$ and $\dot{\bar{\eta}}_a$ are the velocity components in the $\bar{X}$ and $\bar{Y}$ directions. They can be related to the velocity components at the point C by equations (2.9) and (2.10).

Dots are used to denote time derivatives.

The kinetic energy expression (2.34) can be expressed in terms of displacements refer to axis $\bar{X}$ and $\bar{Y}$.

$$\begin{aligned} T = \frac{1}{2}\int_0^{\ell}\int_s \rho c \; \{ & [\dot{\zeta} - (\bar{\xi}' + \bar{\eta}'\theta)^{\bullet} x \\ & - (\bar{\eta}' - \bar{\xi}'\theta)^{\bullet} y - \dot{\theta}'\omega + \Omega_R(\theta^2\theta')^{\bullet}]^2 \\ & + [\dot{\bar{\xi}} - (y - c_y)\dot{\theta} - (x - c_x)\theta\dot{\theta}]^2 \\ & + [\dot{\bar{\eta}} + (x - c_x)\dot{\theta} - (y - c_y)\theta\dot{\theta}]^2\} \; dsdz \end{aligned} \tag{2.35}$$

2.5. Differential Equations and Boundary Conditions

The governing nonlinear differential equations of thin-walled beam are derived by variational procedure from Hamilton's principle, which is written as:

$$\delta \int_{t_1}^{t_2} (T - V + W)dt = 0 \tag{2.36}$$

Introducing the expressions for the strain energy (2.24), work done (2.33) and kinetic energy (2.35) into Hamilton's principle (2.36), and carrying out the variation, a set of four coupled nonlinear differential equations were obtained.

To simplify the algebra, it is convenient to require the following relations to be true.

$$\int_s xc \; ds = \int_s yc \; ds = \int_s yxc \; ds = 0 \tag{2.37}$$

$$\int_s x\omega c \; ds = \int_s y\omega c \; ds = 0 \tag{2.38}$$

$$\int_s \omega c \; ds = \tag{2.39}$$

The three integrals given by (2.37) will be zero if the origin of the co-ordinate system is at the centroid of the cross section and the OX and OY are principle axes.

Integrals given by (2.38) are satisfied by selecting the arbitrary point C to be the shear center. The integral (2.39) is considered as a mathematical definition for the origin of co-ordinate s on the contour of the cross section.

The nonlinear differential equations resulting from the variation process in Hamilton's principle can thus be written as:

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')' - \rho A\ddot{\zeta} = 0 \qquad (2.40)$$

$$E^*I_{yy}\xi^{iv} - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')'' - \rho I_{yy}\ddot{\xi}''$$
$$+ \rho A[\ddot{\xi} + c_y\ddot{\theta} + c_x(\theta\ddot{\theta} + \dot{\theta}^2)]$$
$$- P\{\xi'' - [(e_y - c_y) + (e_x - c_x)\theta]\theta''$$
$$- (e_x - c_x)\theta'^2\} - q_x - q_y\theta = 0 \qquad (2.41)$$

$$E^*I_{xx}\eta^{iv} - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')'' - \rho I_{xx}\ddot{\eta}''$$
$$+ \rho A[\ddot{\eta} - c_x\ddot{\theta} + c_y(\theta\ddot{\theta} + \dot{\theta}^2)]$$
$$- P\{\eta'' + [(e_x - c_x) - (e_y - c_y)\theta]\theta''$$
$$- (e_y - c_y)\theta'^2\} - q_y + q_x\theta = 0 \qquad (2.42)$$

$$E^*I_{\omega\omega}\theta^{iv} - 6E(I_R\theta'^2\theta'' - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)''$$
$$- 2E^*[\theta'(I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'')]'$$
$$- GI_d\theta'' - \rho I_{\omega\omega}\ddot{\theta}'' + 2\rho I_{pc}\ddot{\theta} + \rho A[c_y(\ddot{\bar{\xi}} + \ddot{\bar{\eta}}\theta)$$
$$- c_x(\ddot{\bar{\eta}} - \ddot{\bar{\xi}}\theta)] + P[(e_y - c_y)(\bar{\xi}'' + \bar{\eta}''\theta)$$
$$- (e_x - c_x)(\bar{\eta}'' - \bar{\xi}''\theta)] - m_t + q_x\eta - q_y\xi = 0 \qquad (2.43)$$

The associated boundary conditions are:

$$\{E^*A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta') - P\}\delta\zeta\Big|_0^{\ell} = 0 \qquad (2.44)$$

$$\{E^*I_{yy}\xi'' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta') - M_\eta\}\delta\xi'\Big|_0^{\ell} = 0 \qquad (2.45)$$

$$\{E^*I_{yy}\xi''' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')'$$

$$- P\{\xi' - [(e_y - c_y) + (e_x - c_x)\theta]\theta'\}\delta\xi\Big|_0^{\ell} = 0 \qquad (2.46)$$

$$\{E^*I_{xx}\eta'' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta') + M_\xi\}\delta\eta'\Big|_0^{\ell} = 0 \quad (2.47)$$

$$\{E^*I_{xx}\eta''' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')'$$

$$- P\{\eta' + [(e_x - c_x) - (e_y - c_y)\theta]\theta'\}\delta\eta\Big|_0^{\ell} = 0 \qquad (2.48)$$

$$\{E^*I_{\omega\omega}\theta'' - E^*[I_{R\omega}\theta'^2 + I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)]$$

$$+ M_\omega - M_\Omega\theta^2\}\delta\theta'\Big|_0^{\ell} = 0 \qquad (2.49)$$

$$\{E^*I_{\omega\omega}\theta''' - 2E^*I_R\theta'^3 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)'$$

$$- 2E^*\theta'(I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'')$$

$$- GI_d\theta' + 2M_\Omega\theta\theta' + P[(e_y - c_y)(\bar{\xi}' + \bar{\eta}'\theta)$$

$$- (e_x - c_x)(\bar{\eta}' - \bar{\xi}'\theta)] + M_t\}\delta\theta\Big|_0^{\ell} = 0 \qquad (2.50)$$

where $\bar{\xi}$ and $\bar{\eta}$ can be related to ξ and η by equations (2.7), (2.8) and transformation (2.6)

$$E^* = E/(1 - \nu^2) \tag{2.51}$$

G is the shear modulus of elasticity

A is the cross sectional area

$$A = \int_s c \, ds \tag{2.52}$$

$$I_{xx} = \int_s c \, y^2 \, ds \tag{2.53}$$

$$I_{yy} = \int_s c \, x^2 \, ds \tag{2.54}$$

$$I_d = \int_s \frac{c^3}{3} \, ds \tag{2.55}$$

$$I_{pc} = \int_s \frac{1}{2} c \, (h^2 + n^2) \, ds \tag{2.56}$$

$$I_{\omega\omega} = \int_s c \, \omega^2 \, ds \tag{2.57}$$

$$I_{Rx} = \int_s \frac{1}{2} cx \, (h^2 + n^2) \, ds \tag{2.58}$$

$$I_{Ry} = \int_s \frac{1}{2} cy \, (h^2 + n^2) \, ds \tag{2.59}$$

$$I_{R\omega} = \int_s \frac{1}{2} c\omega \, (h^2 + n^2) \, ds \tag{2.60}$$

$$I_R = \int_s c \, [\tfrac{1}{2} (h^2 + n^2)]^2 \, ds \tag{2.61}$$

$$I_{\Omega c} = \int_s c \, \Omega_R \, ds \tag{2.62}$$

$$I_{\Omega x} = \int_s cx \, \Omega_R \, ds \tag{2.63}$$

$$I_{\Omega y} = \int_s cy \, \Omega_R \, ds \tag{2.64}$$

$$I_{\Omega\omega} = \int_s c\omega \, \Omega_R \, ds \tag{2.65}$$

$$M_\Omega = \int_s c \, p\Omega_R \, ds \tag{2.66}$$

The differential equations given by (2.40) to (2.43) are the general nonlinear differential equations governing the behaviour of thin-walled beam of an open section under applied loads. The beam is subjected to applied longitudinal stresses and twisting moments at the ends as well as lateral loads along the beam. The equations are generally coupled. If the applied loads are static loads, the coupling occurs with the nonlinear terms only. For the case of dynamic loads, the coupling between the equations will involve both linear and nonlinear terms.

2.6. Reducing the General Theory to Elementary Theory

The linear differential equations governing the behaviour of thin-walled beam of an open cross section can be deducted from the nonlinear theory. Neglecting the nonlinear terms in the equations (2.40) to (2.43) and allowing the application of lateral loads only, Vlasov's equations [54] are obtained:

$$E\,A\zeta'' - \rho A\ddot{\zeta} = 0 \tag{2.67}$$

$$E\,I_{yy}\xi^{iv} - \rho I_{yy}\ddot{\xi}'' + \rho A\ddot{\xi} + \rho A\,c_y\ddot{\theta} - q_\xi = 0 \tag{2.68}$$

$$E\,I_{xx}\eta^{iv} - \rho I_{xx}\ddot{\eta}'' + \rho A\ddot{\eta} - \rho A\,c_x\ddot{\theta} - q_\eta = 0 \tag{2.69}$$

$$E\ I_{\omega\omega}\theta^{iv} - \rho I_{\omega\omega}\ddot{\theta}'' - GI_d\theta'' + 2\ \rho I_{pc}\ddot{\theta}$$

$$+ \rho A\ c_y\ \ddot{\xi} - \rho A\ c_x\ddot{\eta} - m_t = 0 \qquad (2.70)$$

The associated boundary conditions are

$$E\ A\zeta'\ \ \delta\zeta\Big|_0^\ell = 0 \qquad (2.71)$$

$$E\ I_{yy}\xi''\ \ \delta\xi'\Big|_0^\ell = 0 \qquad (2.72)$$

$$E\ I_{yy}\xi'''\ \ \delta\xi\Big|_0^\ell = 0 \qquad (2.73)$$

$$E\ I_{xx}\eta''\ \ \delta\eta'\Big|_0^\ell = 0 \qquad (2.74)$$

$$E\ I_{xx}\eta'''\ \ \delta\eta\Big|_0^\ell = 0 \qquad (2.75)$$

$$E\ I_{\omega\omega}\theta''\ \ \delta\theta'\Big|_0^\ell = 0 \qquad (2.76)$$

$$(E\ I_{\omega\omega}\theta''' - GI_d\theta')\delta\theta\Big|_0^\ell = 0 \qquad (2.77)$$

CHAPTER III

STATIC ANALYSIS OF OPEN THIN-WALLED BEAMS SUBJECTED TO END TORQUES

3.1. Introduction

To investigate the behaviour of an open thin-walled elastic beam subjected to static laods, the terms derived from the kinetic energy expression need not be included; namely, terms containing time derivatives from equation (2.35) can be omitted.

In particular, the case of a thin-walled beam subjected to twisting moments applied at the ends is studied in this chapter. It is shown that the set of nonlinear ordinary differential equations admit a simple solution in the case of the uniform torsion of the structural member. The equations can be solved and results obtained reduce to the results given by Cullimore [11] and Gregory [23,24,25] for a beam with the cross section shape of an angle, I section and a beam in the form of a rectangular strip.

The effect of non-uniform twist is considered next. In particular, the case of having one end of the beam fixed with a torque applied at the other free end is studied. Due to the fixed end condition, warping is prevented and this causes additional axial strain. The perturbation technique is used to obtain an approximate solution of the problem. The results are then compared to the non-uniform torsion calculated according to the linear theory of Vlasov.

3.2. Thin-Walled Beam Subjected to Uniform Torsion

Consider a thin-walled beam subjected to end twisting moments M_t. The nonlinear differential equations and boundary conditions as given by equations (2.40) to (2.43), will reduce in this case to:

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')' = 0 \tag{3.1}$$

$$E^*I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta)'' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')'' = 0 \tag{3.2}$$

$$E^*I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta)'' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')'' = 0 \tag{3.3}$$

$$E^*I_{\omega\omega}\theta^{iv} - 6\,E^*I_R\theta'^2\theta'' - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)''$$

$$- 2\,E^*\{\theta'[I_{pc}\zeta' - I_{Rx}(\bar{\xi}'' + \bar{\eta}''\theta)$$

$$- I_{Ry}(\bar{\eta}'' - \bar{\xi}''\theta)]\} - GI_d\theta'' = 0 \tag{3.4}$$

$$[E^*A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')]\delta\zeta\Big|_0^\ell = 0 \tag{3.5}$$

$$[E^*I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta) - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')]\delta\xi'\Big|_0^\ell = 0 \tag{3.6}$$

$$[E^*I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta)' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')']\delta\xi\Big|_0^\ell = 0 \tag{3.7}$$

$$[E^*I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta) - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')]\delta\eta'\Big|_0^{\ell} = 0 \tag{3.8}$$

$$[E^*I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta)' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')']\delta\eta\Big|_0^{\ell} = 0 \tag{3.9}$$

$$[E^*I_{\omega\omega}\theta'' - E^*I_{R\omega}\theta'^2 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)]\delta\theta'\Big|_0^{\ell} = 0 \tag{3.10}$$

$$[E^*I_{\omega\omega}\theta''' - 2\,E^*I_R\theta'^3 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)'$$

$$- 2\,E^*\theta'\{I_{pc}\zeta' - I_{Rx}(\bar{\xi}'' + \bar{\eta}''\theta)$$

$$- I_{Ry}(\bar{\eta}'' - \bar{\xi}''\theta)\} - GI_d\theta' + M_t]\delta\theta\Big|_0^{\ell} = 0 \tag{3.11}$$

By definition of uniform torsion, the variation of the angle of twist in the longitudinal direction of the beam is linear. Accordingly, derivatives of the angle of twist higher than the first derivative will vanish in the differential equations and the boundary conditions. Applying the above simplifications to equations (3.1) to (3.11) for the case of uniform torsion of a thin-walled beam, there is obtained

$$A\zeta'' + 2\,I_{\Omega c}\theta\theta'^2 = 0 \tag{3.12}$$

$$I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta)'' - 2\,I_{\Omega x}\theta'^3 = 0 \tag{3.13}$$

$$I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta)'' - 2\, I_{\Omega y}\theta'^{3} = 0 \tag{3.14}$$

$$\{\theta'[I_{pc}\zeta' - I_{Rx}(\bar{\xi}'' + \bar{\eta}''\theta) - I_{Ry}(\bar{\eta}'' - \bar{\xi}''\theta)]\}' = 0 \tag{3.15}$$

The boundary conditions are,

$$\{A\zeta' + I_{pc}\theta'^{2} + I_{\Omega c}\theta^{2}\theta'\}\delta\zeta\Big|_0^{\ell} = 0 \tag{3.16}$$

$$\{I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta) - I_{Rx}\theta'^{2} - I_{\Omega x}\theta^{2}\theta'\}\delta\xi'\Big|_0^{\ell} = 0 \tag{3.17}$$

$$\{I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta)' - 2\, I_{\Omega x}\theta\theta'^{2}\}\delta\xi\Big|_0^{\ell} = 0 \tag{3.18}$$

$$\{I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta) - I_{Ry}\theta'^{2} - I_{\Omega x}\theta^{2}\theta'\}\delta\eta'\Big|_0^{\ell} = 0 \tag{3.19}$$

$$\{I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta)' - 2\, I_{\Omega y}\theta\theta'^{2}\}\delta\eta\Big|_0^{\ell} = 0 \tag{3.20}$$

$$\{I_{R\omega}\theta'^{2} + I_{\Omega\omega}\theta\theta'^{2}\}\delta\theta'\Big|_0^{\ell} = 0$$

$$\{E^*I_{\Omega\omega}\theta'^{3} + 2\, E^*I_{R}\theta'^{3} + 2\, E^*\theta'[I_{pc}\zeta' - I_{Rx}(\bar{\xi}'' + \bar{\eta}''\theta) - I_{Ry}(\bar{\eta}'' - \bar{\xi}''\theta)] + GI_{d}\theta' - M_{t}\}\delta\theta\Big|_0^{\ell} = 0 \tag{3.21}$$

Integrating equations (3.12) and (3.15) once and integrating equation (3.13) and (3.14) twice, there is obtained

$$A\zeta' + I_{\Omega c}\theta^2\theta' + K_1 = 0 \tag{3.22}$$

$$I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta) - I_{\Omega x}\theta^2\theta' + K_2 z + K_3 = 0 \tag{3.23}$$

$$I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta) - I_{\Omega t}\theta^2\theta' + K_y z + K_5 = 0 \tag{3.24}$$

$$\theta'[I_{pc}\zeta' - I_{Rx}(\bar{\xi}'' + \bar{\eta}''\theta) - I_{Ry}(\bar{\eta}'' - \bar{\xi}''\theta)] + K_6 = 0 \tag{3.25}$$

where K_1 to K_6 are the constants of integrations. Using the boundary conditions (3.16) to (3.21), $K_i(i = 1,6)$ can be determined to yield:

$$A\zeta' + I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta' = 0 \tag{3.26}$$

$$I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta) - I_{Rx}\theta'^2 - I_{\Omega x}\theta^2\theta' = 0 \tag{3.27}$$

$$I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta) - I_{Ry}\theta'^2 - I_{\Omega y}\theta^2\theta' = 0 \tag{3.28}$$

$$E^*I_{\Omega\omega}\theta'^3 + 2\,E^*I_R\theta'^3 + 2\,E^*\theta'[I_{pc}\zeta' - I_{Rx}(\bar{\xi}'' + \bar{\eta}''\theta) - I_{Ry}(\bar{\eta}'' - \bar{\xi}''\theta)] + GI_d\theta' - M_t = 0 \tag{3.29}$$

The first derivative of the angle of twist θ', is a constant value for uniform torsion.

Eliminating ζ', $\bar{\xi}''$ and $\bar{\eta}''$ from equation (3.29) a first order nonlinear differential equation is obtained; namely,

$$GI_d\theta' + 2\,E^*\theta'^3[I_R - \frac{(I_{pc})^2}{A} - \frac{(I_{Rx})^2}{I_{yy}} - \frac{(I_{Ry})^2}{I_{xx}} + \frac{1}{2}\,I_{\Omega\omega}] - M_t = 0 \qquad (3.30)$$

This leads to the nonlinear moment-rotation relationship:

$$M_t = GI_d\theta'\{1 + \frac{2E^*}{GI_d}\;[I_R - \frac{(I_{pc})^2}{A} - \frac{(I_{Rx})^2}{I_{yy}} - \frac{(I_{Ry})^2}{I_{xx}} + \frac{1}{2}\,I_{\Omega\omega}]\theta'^2\} \qquad (3.31)$$

The longitudinal strain due to shortening effect of nonlinear twist can be found using equation (2.22), which is written as:

$$\varepsilon_z = \zeta' - (\bar{\xi}'' + \bar{\eta}''\theta)x - (\bar{\eta}'' - \bar{\xi}''\theta)y + \frac{1}{2}\,(h^2 + n^2)\theta'^2 \qquad (3.32)$$

Introducing the derivatives ζ', $\bar{\xi}''$ and $\bar{\eta}''$ given by equations (3.26) to (3.28) into the strain equation (3.32), there is obtained

$$\varepsilon_z = [\,-\,I_{pc}/A - (I_{Rx}/I_{yy})x - (I_{Ry}/I_{xx})y + \frac{1}{2}\,(h^2 + n^2)]\theta'^2 + 2\Omega_R\theta\theta'^2 \qquad (3.33)$$

The moment-rotation relationship and the strain-rotation expressions as given by equations (3.31) and (3.33), respectively, are general formulas for a thin-walled beam of unsymmetrical cross section under uniform torsion. Uniform torsion is possible only if there are applied torques at the end of the beam and also the end conditions as given by (3.16) to (3.21) are satisfied.

3.3. Thin-Walled Beam of Monosymmetrical Section Subjected to Uniform Torsion.

Consider a thin-walled beam of monosymmetrical cross section, subjected to end twisting moment M_t. The axis OX is considered to be the axis of symmetry. It can be easily shown, that for a monosymmetrical section, I_{Ry} and $I_{\Omega\omega} = 0$. Introducing this result into equations (3.31) and (3.33). The moment-rotation and the strain-rotation expressions for a thin-walled beam of monosymmetrical cross section become,

$$M_t = GI_d\theta' \{1 + \frac{2E^*}{GI_d} [I_R - \frac{(I_{pc})^2}{A} - \frac{(I_{Rx})^2}{I_{yy}}]\theta'^2\} \tag{3.34}$$

$$\varepsilon_z = [- I_{pc}/A - (I_{Rx}/I_{yy})x + \frac{1}{2}(h^2 + n^2) + 2\Omega_R\theta]\theta'^2 \tag{3.35}$$

The (twisting moment - angle of twist) and the (axial strain - rotation) relationships for a few commonly used cross sections of thin-walled beam, are calculated in the subsequent sections.

EXAMPLE 1: Narrow Rectangular Cross Section

A beam of narrow rectangular cross section is the simplest shape, thin-walled beam. The cross section is symmetric and also there is no warping of the cross section under any end conditions.

Consider a cantilever beam of rectangular cross section of thickness c and height b as shown in figure [4a]. Assuming c << b, the properties of the cross section can then be calculated as follows:

$$I_{pc} = \frac{1}{2} \int_s c\ (h^2 + n^2)\ ds = \frac{cb^3}{24} \tag{3.36}$$

$$I_R = \frac{1}{4} \int_s c\ (h^2 + n^2)^2\ ds = \frac{cb^5}{320} \tag{3.37}$$

$$I_d = \int_s \frac{c^3}{3}\ ds = \frac{c^3 b}{3} \tag{3.38}$$

$$\Omega_R = 0 \tag{3.39}$$

$$I_{Rx} = \frac{1}{2} \int_s c\ (h^2 + n^2)x\ ds = 0 \tag{3.40}$$

Substituting the values of the above constants into the moment and the strain expressions as given by

(3.34) and (3.35), respectively, gives:

$$M_t = \frac{1}{3} Gbc^3\theta'[1 + \frac{Eb^4}{120Gc^2}\theta'^2] \quad (3.41)$$

$$\varepsilon_z = [\frac{1}{2} y^2 - \frac{1}{24} b^2]\theta'^2 \quad (3.42)$$

The moment-rotation expression as given by (3.41) is identical to that given by Cullimore [11]. The strain expression (3.42) is the same as that given by Gregpry [23].

The strain at the centroid can be obtained from expression (3.42), at y = 0, thus

$$\varepsilon_z = - \frac{1}{24} b^2\theta'^2 \quad (3.43)$$

The strain distribution across the cross section is parabolic as shown in figure [4b].

A numerical example for a narrow strip is worked out to compare between the linear and nonlinear analyses. Consider a beam of rectangular cross section. The length of the beam ℓ is taken as 10 times the depth b of the rectangular strip section. The ratio of the depth b to the thickness c is taken as 50. The nonlinear moment of twist - angle of rotation relationship calculated from equation (3.41) is plotted in figure [5]. Neglecting the nonlinear terms in equation (3.41), the linear relationship is obtained. The results are also shown in

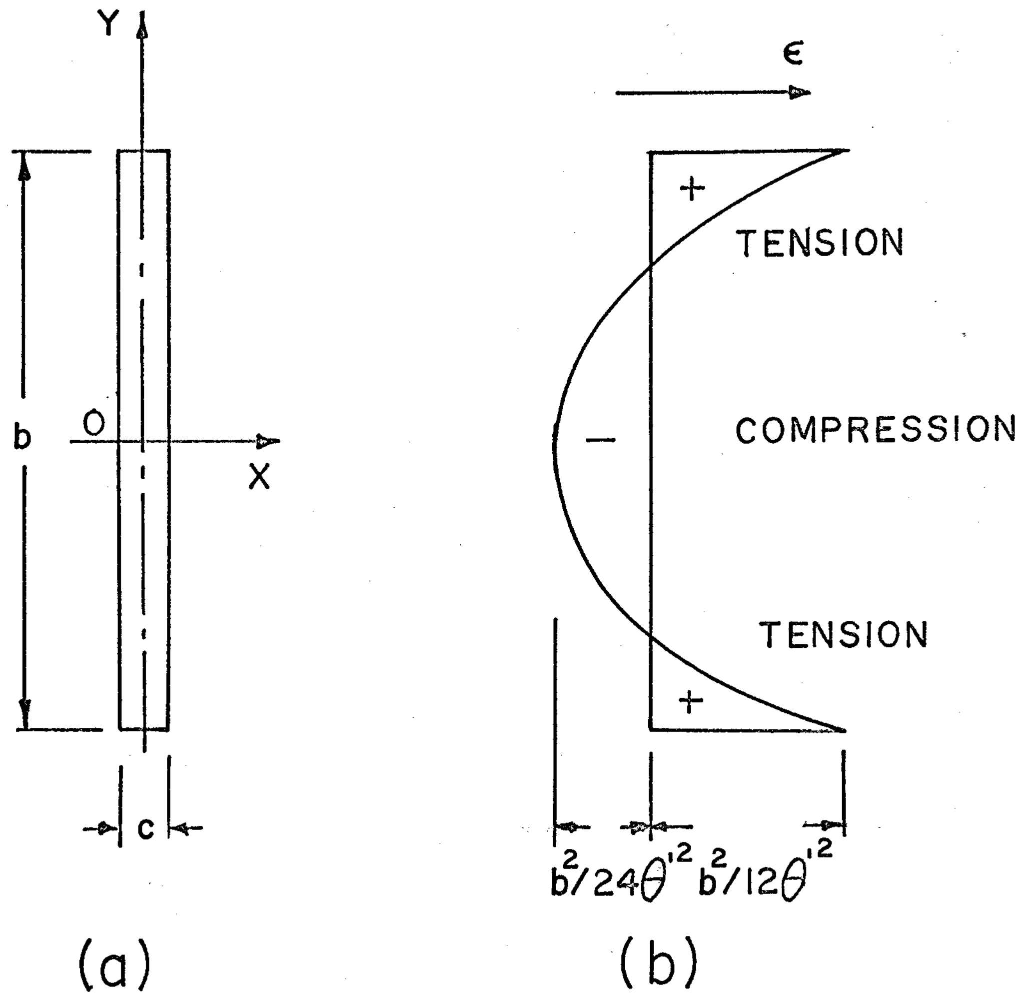

FIG. (4) AXIAL STRAIN DISTRIBUTION AT THE FREE END OF A BEAM OF NARROW RECTANGULAR STRIP SECTION

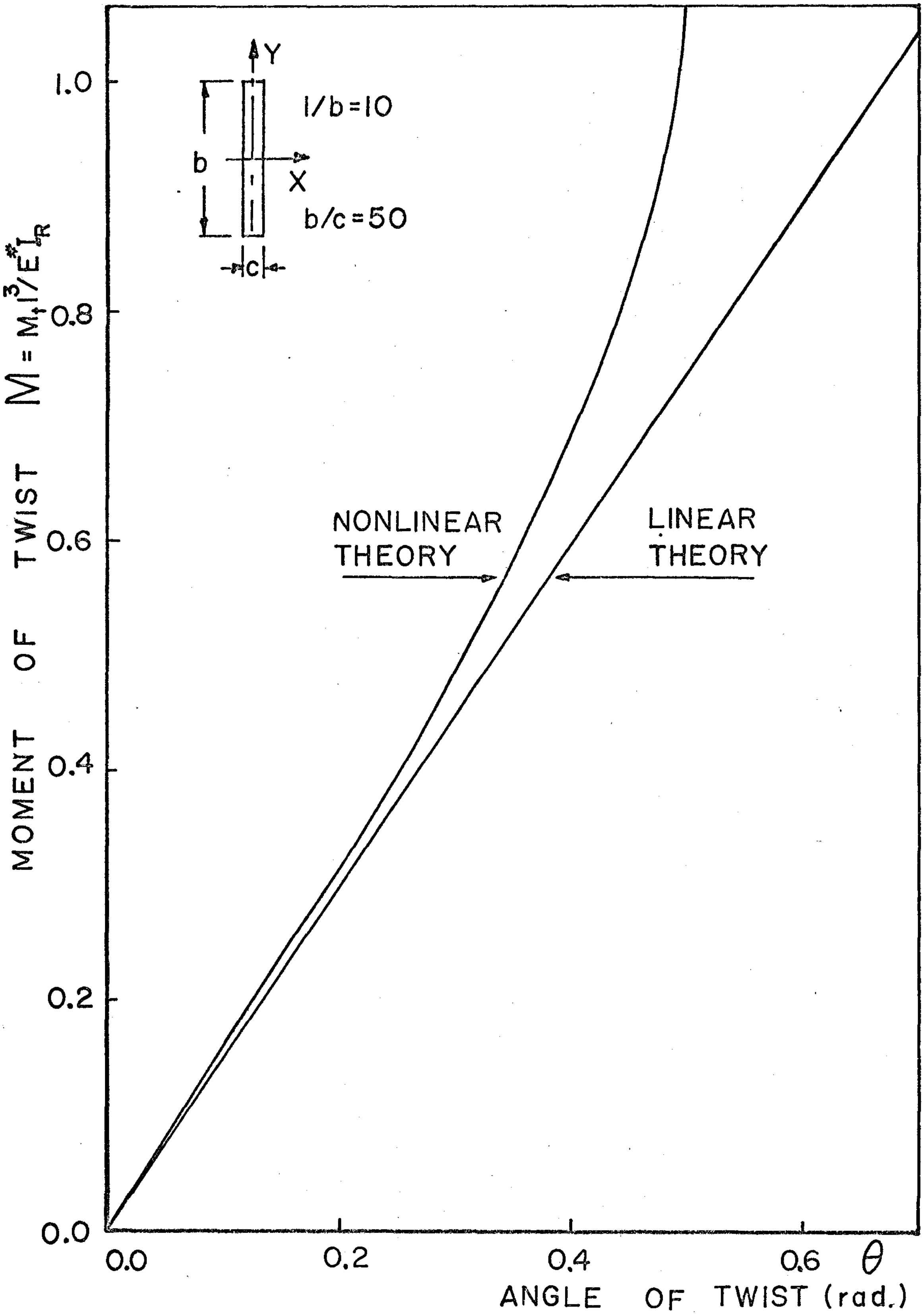

FIG. (5) TWISTING MOMENT - ANGLE OF ROTATION RELATIONSHIP FOR A BEAM OF NARROW RECTANGULAR STRIP SECTION figure [5] for comparison.

From the plots in figure [5], it is shown that 10% error exists in the value of the moment required to cause rotation of 0.35 radians.

EXAMPLE 2: Thin-Walled Beam of I Section

A thin-walled beam of I section may be a beam of symmetrical section, but unlike the narrow rectangular cross section, the section tends to warp under torsion.

Consider a symmetrical I section of flange and web thickness c_f and c_w, respectively, of height H, flange width B as shown in figure [6]. The flange thickness is taken to be small compared with the height H, i.e. $c_f << H$.

The geometrical properties of the cross section are evaluated as follows:

$$A = H\, c_w + 2\, B\, c_f \qquad (3.44)$$

$$I_{pc} = \frac{1}{24}\left[H^3 c_w + 2\, B^3\, c_f + 6\, H^2\, B\, c_f\right] \qquad (3.45)$$

$$I_{Rx} = \Omega_R = 0 \qquad (3.46)$$

$$I_d = 2\,\frac{c_f^{\,3}}{3}\, B + \frac{c_w^{\,3}}{3}\, H \qquad (3.47)$$

$$r^2 = n^2 + h^2 \qquad (3.48)$$

$$I_R = \frac{c_w}{320}\, H^5 + \frac{c_f}{32}\, B\, H^4 + \frac{c_f}{160}\, B^5 + \frac{c_f}{48}\, H^2 B^3 \qquad (3.49)$$

The moment - rotation expression is obtained by substituting the values of the above geometrical constants into equation (3.34), thus

$$M_t = \left(\frac{2c_f^3}{3} B + \frac{c_w^3}{3} H\right) G\theta' \{1 + \frac{6E^*}{G(2c_f^3 B + c_w^3 H)}$$

$$\left[\frac{c_w H^5}{320} + \frac{c_f}{32} BH^4 + \frac{c_f}{160} B^5 + \frac{c_f}{48} H^2 B^3\right.$$

$$\left. - \frac{(H^3 c_w + 2B^3 c_f + 6H^2 B c_f)^2}{576\,(Hc_w + 2Bc_f)}\right] \theta'^2 \} \tag{3.50}$$

The strain expression can be obtained by substutiting equations (3.44) to (3.49) into equation (3.35), thus:

$$\varepsilon_z = \frac{1}{24}\left[12\, r^2 - \frac{H^3 c_w + 6H^2 B c_f + 2B^2 c_f}{Hc_w + 2Bc_f}\right] \theta'^2 \tag{3.51}$$

The axial strain - rotation expression for a thin-walled beam of I section,as given by equation (3.51), agrees with Cullimore's [11] formula which was verified experimentally.

EXAMPLE 3: Equal Angle Section

A thin-walled beam of an angle section, represents an example of a thin-walled beam with monosymmetrical section.

Consider a monosymmetrical angle section of thickness c and side width b as shown in figure [7].

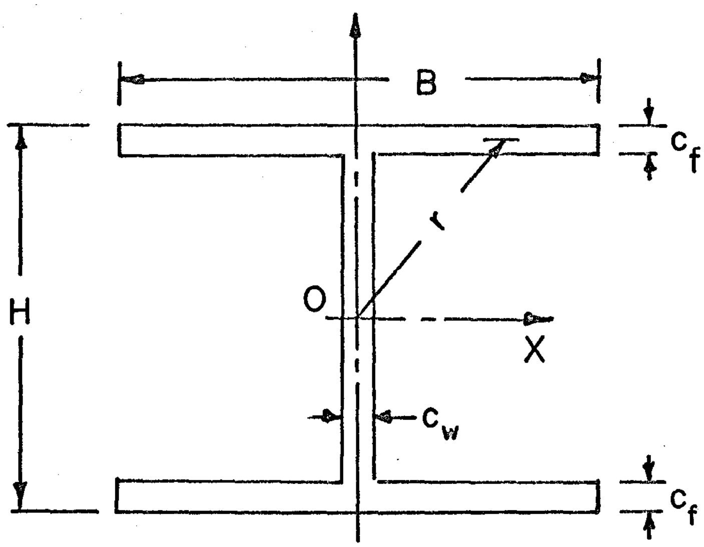

FIG. (6) SYMMETRICAL I SECTION

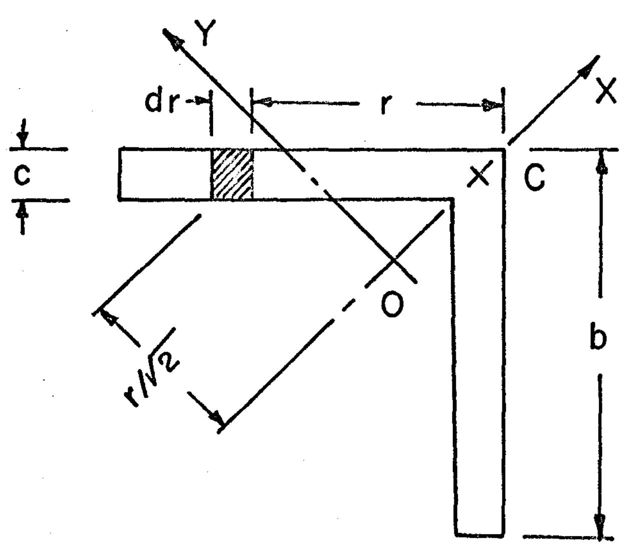

FIG. (7) EQUAL ANGLE SECTION

The geometrical properties can be found to be:

$$A = 2\ cb \tag{3.52}$$

$$n = r \tag{3.53}$$

$$x = \frac{1}{\sqrt{2}}(r - b/2) \tag{3.54}$$

$$I_{pc} = \frac{1}{3}\ cb^3 \tag{3.55}$$

$$I_{Rx} = \frac{5}{12\ \sqrt{2}}\ cb^4 \tag{3.56}$$

$$I_{yy} = \frac{1}{12}\ cb^3 \tag{3.57}$$

$$I_R = \frac{1}{10}\ cb^5 \tag{3.58}$$

$$I_d = \frac{2}{3}\ c^3 b \tag{3.59}$$

substituting the values of the constants as given by equations (3.52) to (3.59) into the moment and strain expressions as given by (3.34) and (3.35), then

$$M_t = \frac{2}{3}\ Gc^3 b\theta'(1 - \frac{359}{120}\ \frac{E^* b^4}{Gc^2}\ \theta'^2) \tag{3.60}$$

and

$$\varepsilon_z = (\frac{1}{2}\ r^2 + \frac{1}{2}\ b^2 - \frac{1}{2}\ br)\theta'^2 \tag{3.61}$$

For uniform torsion, θ' is constant along the beam and can be written as

$$\theta' = \theta/\ell \tag{3.62}$$

then the strain expression is identical to Gregory's [24] formula.

The M_t - θ relationship (3.60) has not been given before. It should be noted that the works of Cullimore and Gregory are supported by experimental verification. The fact that the set of equations (3.1) to (3.4) reduces identically to their results in the case of uniform torsion, gives indication that the equations are capable of predicting accurately the behaviour of the beam under more complicated loadings and support conditions.

3.4. Thin-Walled Beam Subjected to Non-uniform Torsion

Consider a thin-walled beam subjected to a twisting moment M_t at one end and built-in at the other. The non-uniform torsion is different from the uniform torsion in the fact that when warping is prevented, additional axial strain is created in the beam. Accordingly, the variation of the angle of twist in the longitudinal direction of the beam is no longer linear. The governing equations are given by equations (3.1) to (3.4) and the boundary conditions are given by equations (3.5) to (3.11). Since the equations are nonlinear, it is necessary to use approximate methods to obtain the solution. In what follows, the perturbation technique is used to obtain an approximate solution of the problem.

Integrating equations (3.1) and (3.4) once and integrating equations (3.2) and (3.3) twice, there is obtained

$$E^*A\zeta' + E^*(I_{pc}\theta' + I_{\Omega c}\theta^2\theta') + K_1^* = 0 \tag{3.63}$$

$$E^*I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta) - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta') + K_2^*z + K_3 = 0 \tag{3.64}$$

$$E^*I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta) - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta') + K_4^*z + K_5 = 0 \tag{3.65}$$

$$E^*I_{\omega\omega}\theta''' - 2E^*I_R\theta'^3 - E^*I_{\Omega w}(\theta^2\theta'' + \theta\theta'^2)' - 2E^*\theta'[I_{pc}\zeta' - I_{Rx}(\bar{\xi}'' + \bar{\eta}''\theta) - I_{Ry}(\bar{\eta}'' - \bar{\xi}''\theta)] - GI_d\theta' + K_6^* = 0 \tag{3.66}$$

where K_i^* (i = 1,6) are constants of integration to be determined using the boundary conditions (3.5) to (3.11). Considering the case of a cantilever beam; namely, the end $z = 0$ is fixed while the end $z = \ell$ is free, the boundary conditions can be written as:

at $z = 0$

$$\zeta = \xi = \bar{\xi}' = \eta = \eta' = \theta = \theta' = 0 \tag{3.67}$$

at $z = \ell$

$$E^*A\zeta' + E^*I_{pc}(\theta'^2 + I_{\Omega c}\theta^2\theta') = 0 \quad (3.68)$$

$$E^*I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta) - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta') = 0 \quad (3.69)$$

$$E^*I_{yy}(\bar{\xi}'' + \bar{\eta}''\theta)' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta') = 0 \quad (3.70)$$

$$E^*I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta) - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta') = 0 \quad (3.71)$$

$$E^*I_{xx}(\bar{\eta}'' - \bar{\xi}''\theta)' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')' = 0 \quad (3.72)$$

$$E^*I_{\omega\omega}\theta'' - E^*I_{R\omega}\theta'^2 - E^*I_{\Omega\omega}\theta^2\theta'' + \theta\theta'^2) = 0 \quad (3.73)$$

$$E^*I_{\omega\omega}\theta''' - 2E^*I_R\theta'^3 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)'$$
$$- 2E^*\theta'[I_{pc}\zeta' - I_{Rx}(\bar{\xi}'' + \bar{\eta}''\theta)$$
$$- I_{Ry}(\bar{\eta}'' - \bar{\xi}''\theta)] - GI_d\theta' + M_t = 0 \quad (3.74)$$

Using the boundary conditions given by equations (3.67) to (3.74), the constants of integration can be determined to be:

$$K_i^* = 0 \qquad i = 1,5 \quad (3.75)$$

$$K_6^* = M_t \quad (3.76)$$

Eliminating ζ', $\bar{\xi}''$ and $\bar{\eta}''$ from equation (3.66) by substituting from equations (3.63), (3.64), (3.65), (3.75) and (3.76), a third order nonlinear differential equation is obtained, namely

$$E^*I_{\omega\omega}\theta''' - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)' - GI_d\theta' - 2E^*\theta'^3[I_R - \frac{(I_{pc})^2}{A} - \frac{(I_{Rx})^2}{I_{yy}} - \frac{(I_{Ry})^2}{I_{xx}}] + M_t = 0 \tag{3.77}$$

The boundary conditions associated with equation (3.77) are

at $z = 0$

$$\theta = \theta' = 0 \tag{3.78}$$

at $z = \ell$

$$E^*I_{\omega\omega}\theta'' - E^*I_{R\omega}\theta'^2 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2) = 0 \tag{3.79}$$

To reduce the volume of algebra involved in the solution of the nonlinear differential equation (3.77), further analysis will be restricted to thin-walled cross section that posses the geometrical property giving $I_{R\omega}$ and $I_{\Omega\omega} = 0$. This is true for all symmetrical and monosymmetrical sections. Furthermore, the quantities $I_{R\omega}$ and $I_{\Omega\omega}$ are small for beams with cross sections of small curvature, such as sections containing piece-wise linear segments.

It is convenient to normalize the differential equation by letting

$$\bar{z} = z/\ell \tag{3.80}$$

Thus, equation (3.77) can be written in the form:

$$\frac{d^3\theta}{d\bar{z}^3} - k^2 \frac{d\theta}{d\bar{z}} - 2\gamma \left(\frac{d\theta}{d\bar{z}}\right)^3 + M = 0 \qquad (3.81)$$

where the non-dimensional constants are

$$k^2 = 0.5\ (1 - \nu) I_d \ell^2 / I_{\omega\omega} \qquad (3.82)$$

$$M = M_t I_R / (E^* A \ell I_{\omega\omega}) \qquad (3.83)$$

$$\gamma = [I_R - (I_{pc})^2/A - (I_{Rx})^2/I_{yy} - (I_{Ry})^2/I_{xx}]/I_{\omega\omega} << 1 \qquad (3.84)$$

The boundary conditions associated with equation (3.81) become

at $\bar{z} = 0$

$$\theta = \frac{d\theta}{d\bar{z}} = 0 \qquad (3.85)$$

at $\bar{z} = 1$

$$\frac{d^2\theta}{d\bar{z}^2} = 0 \qquad (3.86)$$

A solution for the differential equation (3.81) will be sought in the form:

$$\theta = \theta_1 + \gamma\theta_2 + \gamma^2\theta_3 + \ldots. \qquad (3.87)$$

For small values of γ, first order approximation of the solution, can be written as

$$\theta = \theta_1 + \gamma\theta_2 \qquad (3.88)$$

Substituting the assumed solution given by equation (3.88) into the differential equation (3.81) and equating the coefficients of the powers of γ, there is obtained

$$\frac{d^3\theta_1}{d\bar{z}^3} - k^2 \frac{d\theta_1}{d\bar{z}} + M = 0 \tag{3.89}$$

$$\frac{d^3\theta_2}{d\bar{z}^3} - k^2 \frac{d\theta_2}{d\bar{z}} - 2 \left(\frac{d\theta_1}{d\bar{z}}\right)^3 = 0 \tag{3.90}$$

The associated boundary conditions are

$$\theta_1(0) = \frac{d\theta_1}{d\bar{z}}(0) = \frac{d^2\theta_1}{d\bar{z}^2}(1) = 0 \tag{3.91}$$

$$\theta_2(0) = \frac{d\theta_2}{d\bar{z}}(0) = \frac{d^2\theta_2}{d\bar{z}^2}(1) = 0 \tag{3.92}$$

The solution of the linear equation (3.89), which satisfies the boundary conditions (3.91), is

$$\theta_1 = \frac{M}{k^3} [-\sinh k\bar{z} + \tanh k(\cosh k\bar{z} - 1) + k\bar{z}] \tag{3.93}$$

The solution of the differential equation (3.90), that satisfies the boundary conditions (3.92), can be written as

$$\theta_2 = \frac{c_2}{24k^3} \sinh 3k\bar{z} + \frac{c_1}{24k^3} \cosh 3k\bar{z} + \frac{c_4}{6k^3} \sinh 2k\bar{z} + \frac{c_3}{6k^3} \cosh 2k\bar{z} + \frac{c_5}{2k^2} \bar{z} \sinh k\bar{z} + \frac{c_6}{2k^2} \bar{z} \cosh k\bar{z}$$

$$+ \bar{c}_1 \sinh k\bar{z} + \bar{c}_2 \cosh k\bar{z}$$

$$- \frac{c_7}{k^2} \bar{z} + \bar{c}_3 \tag{3.94}$$

where:

$$c_1 = \frac{1}{2} (\frac{M}{k^2})^3 \tanh k (\tanh^2 k + 3) \tag{3.95}$$

$$c_2 = - \frac{1}{2} (\frac{M}{k^2})^3 (3 \tanh^2 k + 1) \tag{3.96}$$

$$c_3 = - 6 (\frac{M}{k^2})^3 \tanh k \tag{3.97}$$

$$c_4 = 3 (\frac{M}{k^2})^3 (\tanh^2 k + 1) \tag{3.98}$$

$$c_5 = - \frac{3}{2} (\frac{M}{k^2})^3 \tanh k (\tanh^2 k - 5) \tag{3.99}$$

$$c_6 = \frac{3}{2} (\frac{M}{k^2})^3 (\tanh^2 k - 5) \tag{3.100}$$

$$c_7 = - 3 (\frac{M}{k^2})^3 (\tanh^2 k - 5/3) \tag{3.101}$$

$$\bar{c}_1 = - \frac{1}{k^3} (\frac{c_2}{8} + \frac{c_4}{3} + \frac{c_6}{2} - c_7) \tag{3.102}$$

$$\bar{c}_2 = - [\bar{c}_1 \tanh k + \frac{3}{8k^3} (c_2 \frac{\sinh 3k}{\cosh k} + c_1 \frac{\cosh 3k}{\cosh k})$$

$$+ \frac{2}{3 k^3 \cosh k} (c_4 \sinh 2k + c_3 \cosh 2k)$$

$$+ \frac{c_5}{2k^2} (\tanh k + \frac{2}{k}) + \frac{c_6}{2k^2} (1 + \frac{2}{k} \tanh k)] \tag{3.103}$$

$$\bar{c}_3 = - (\bar{c}_2 + \frac{c_1}{24k^3} + \frac{c_3}{6k^3}) \tag{3.104}$$

The solution given by equation (3.88) can be expressed in terms of the twisting moment by substituting equations (3.93) and (3.94) into (3.88). The angle of twist can be expressed in terms of the twisting moment in the form:

$$\theta = \Gamma_1(\bar{z}, M, M^3) \tag{3.105}$$

To establish the axial strain - rotation expression, strain formula given by equation (2.22) is used. Eliminating ζ', $\bar{\xi}''$ and $\bar{\eta}''$ from equation (2.22) by substituting from equations (3.63), (3.64) and (3.65), the strain expression can be written as

$$\varepsilon_z = [\frac{1}{2}(\bar{h}^2 + \bar{n}^2) - \bar{I}_{pc} - \bar{I}_{Rx}\,\bar{x} - \bar{I}_{Ry}\,\bar{y}]$$

$$(\frac{d\theta}{d\bar{z}})^2 - \bar{\omega}\,\frac{d^2\theta}{d\bar{z}^2} \tag{3.106}$$

where the values of nondimensional constants are

$$\bar{h} = h/\ell \qquad \text{and} \qquad \bar{n} = n/\ell \tag{3.107}$$

$$\bar{I}_{pc} = I_{pc}/A\ell^2, \quad \bar{I}_{Rx} = I_{Rx}/I_{yy}\ell$$

$$\text{and } \bar{I}_{Ry} = I_{Ry}/I_{xx}\ell \tag{3.108}$$

$$\bar{x} = x/\ell \qquad \text{and} \qquad \bar{y} = y/\ell \tag{3.109}$$

$$\bar{\omega} = \omega/\ell^2 \tag{3.110}$$

Once the solution (3.105) is obtained, the axial strain expression is defined. It is convenient to write it in the form

$$\varepsilon_z = \Gamma_2(\bar{x}, \bar{y}, \bar{\omega}, \theta'^2, \theta'') \tag{3.111}$$

An example of the non-uniform torsion of a thin-walled beam of an I section will be discussed. Results are compared with the non-uniform torsion calculated according to the linear theory.

3.5. Non-uniform Torsion of Thin-walled Beam of I Section

As an example, consider the non-uniform torsion of a thin-walled beam of I section. Due to the symmetry of the cross section $I_{Rx} = I_{Ry} = 0$ and no bending of the beam under the action of twisting moment will occur.

As a numerical example, let the dimensions of the I section be:

$$\ell/H = 10 \tag{3.112}$$

$$B/H = 0.5 \tag{3.113}$$

$$H/c_w = 30 \tag{3.114}$$

$$c_w/c_f = 1 \tag{3.115}$$

where

ℓ is the length of the thin-walled beam

H is the height of the cross section

B is the width of the flanges

c_w is the thickness of the web plate

c_f is the thickness of the flanges

Given a value for the twisting moment M_t, the corresponding angle of twist at the free end can be calculated using equation (3.105). Results are plotted in figure [8].

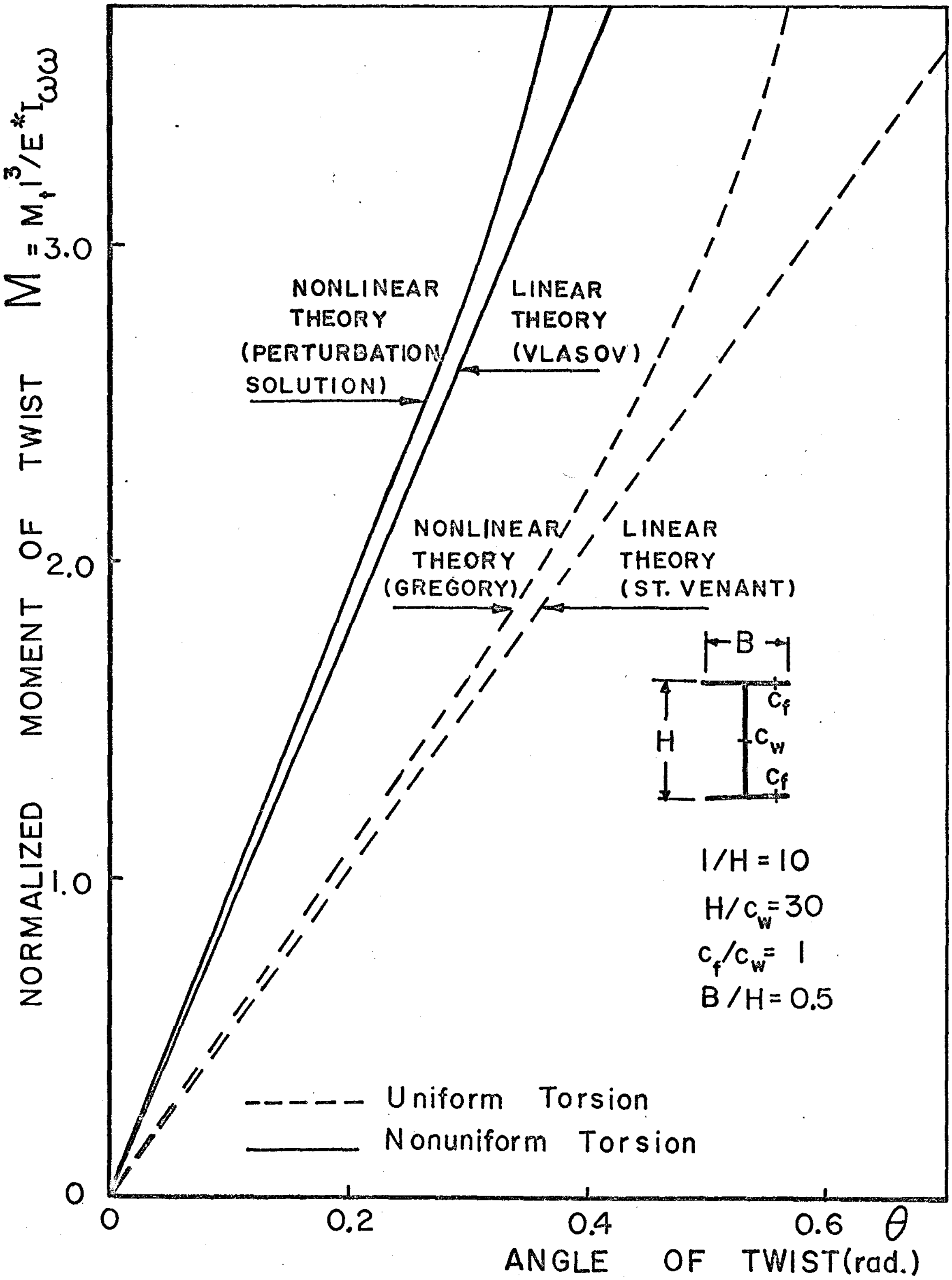

FIG. (8) UNIFORM AND NON UNIFORM TORSION OF I SECTION

To compare the results with the linear theory, equation (3.93) is used to give the moment - rotation relationship according to Vlasov's theory of thin-walled beam.

The twisting moment - rotation relationship in the linear analysis is plotted in figure [8]. A comparison between the linear and nonlinear analyses for the uniform torsion of thin-walled beam of I section,as given by equation (3.50), is also shown in figure [8].

From the plots in figure [8] it is shown that if the angle of twist is 0.35 radians, a difference of 10% exists between the linear and non-linear theory. It is interesting to note that the effect of the nonlinear terms is a "hardening" effect. In other words, the larger the twisting moment,the smaller the rate of increase of the angle of rotation.

The axial strain - rotation relationship is given by equation (3.111). The variation of the strain at a point on the cross section taken at the corners of the flanges is investigated in detail. Flange corners are where the maximum axial strain is predicted.

For a given value of the applied twisting moment, the angle of twist and its derivatives can be found using equation (3.105). The axial strain at the corners of the flanges of the I section can then be calculated from equation (3.111). Results are plotted in figure [9].

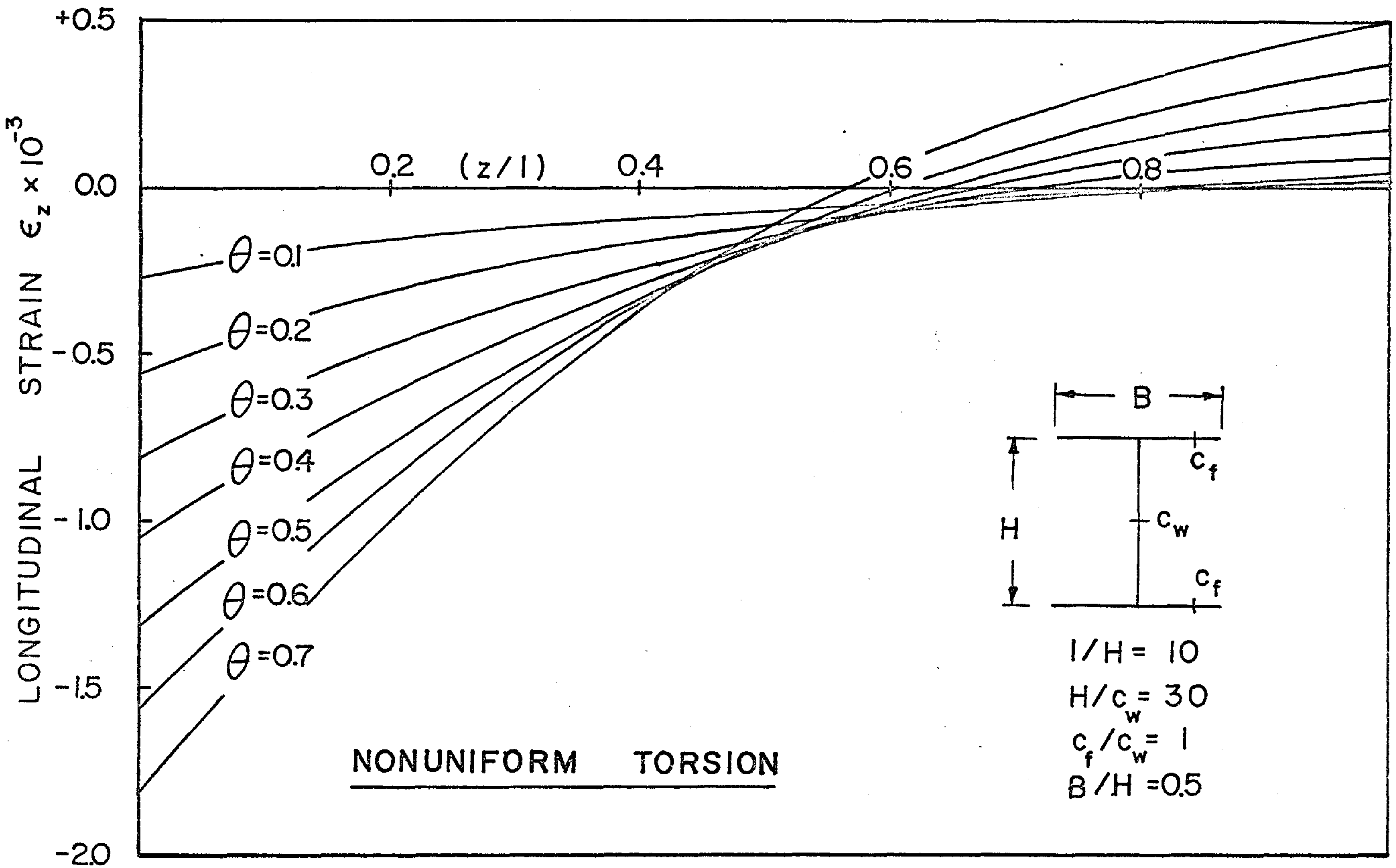

FIG. (9) AXIAL STRAIN VARIATION ALONG THE LENGTH OF AN I BEAM

Comparison between the axial strain variation as calculated by linear and non-linear analyses are given by figure [10]. For a given moment of twist, the angle of twist and its derivatives can be found by the linear analysis using equation (3.93). The axial strain according to Vlasov's theory can be calculated using the linear terms of equation (3.111). From the comparision shown in figure [10] it is clear that the linear theory is inadequate for predicting the axial strain variation. This is due to the fact that the coupling between the axial shortening, bending and the twisting is a nonlinear coupling. At the fixed boundary, however, both analyses give identical results as both solutions are required to satisfy the same boundary condition $\theta'(0) = 0$. The maximum strain is found to take place at the fixed end $\bar{z} = 0$. At the free end $\bar{z} = 1$, the nonlinear theory predicst positive strain. This observation constitutes a criteria for a simple experimental verification for the nonlinear theory.

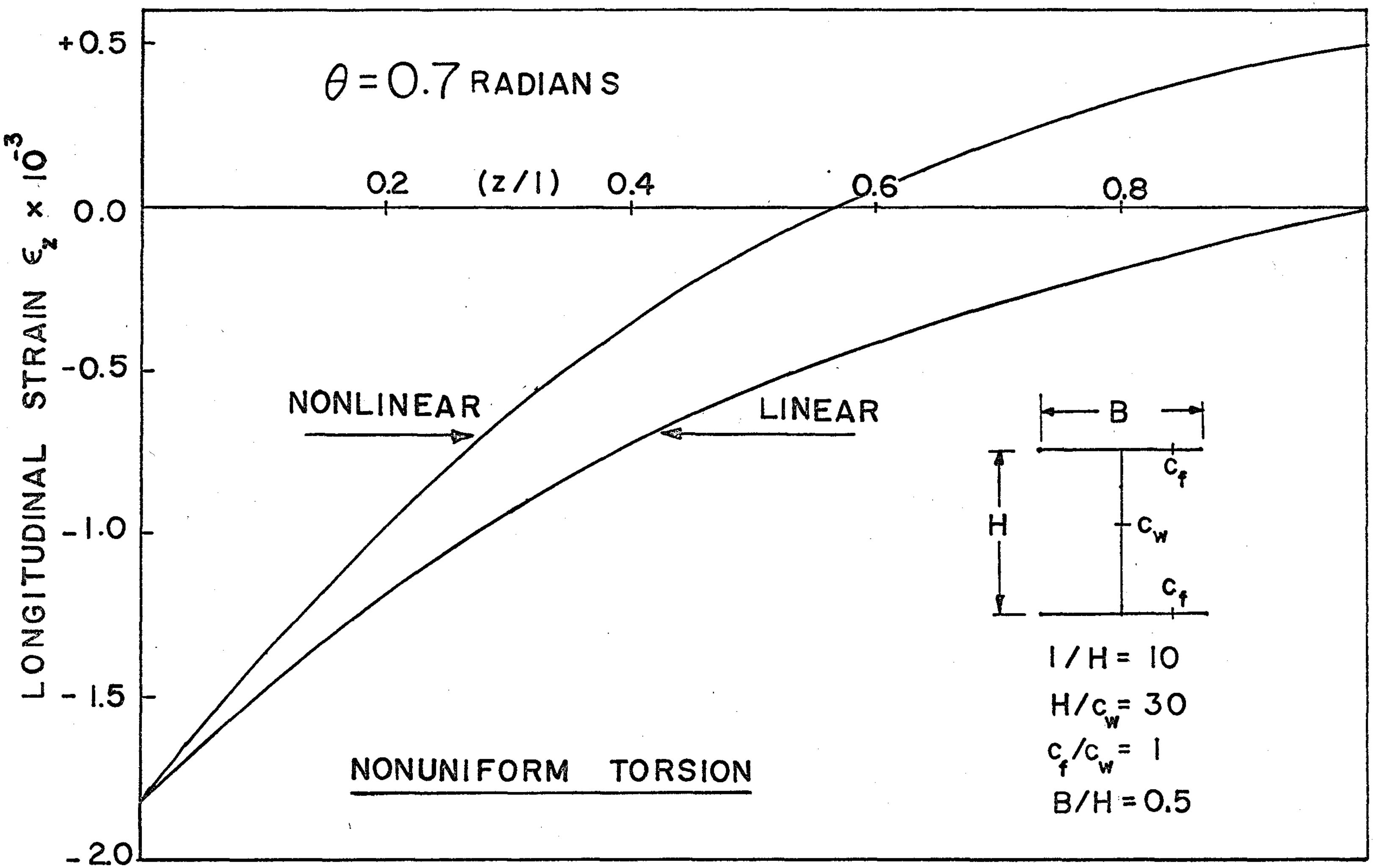

FIG. (10) LINEAR AND NONLINEAR AXIAL STRAIN VARIATION

CHAPTER IV

GENERAL CONSISTENT NONLINEAR STABILITY THEORY OF THIN-WALLED ELASTIC BEAMS

4.1. Introduction

In the study of stability of an elastic system in its critical state, there are two kinds of stability problems depending on the character of the applied load. Stability problems of the first kind are bifurcation problems in which other forms of equilibrium, different from the precritical shape, become possible. Stability problems of the second kind are problems in which the deformation of the system increases with the increase of the external load. At buckling, the deformation of the equilibrium shape, shows only quantitative and not qualitative variations from the precritical shape. In this chapter, the first kind of stability problems (bifurcation), will be considered.

A thin-walled beam may lose stability in the form of overall buckling, local buckling or a combination of both types of buckling. In this thesis only overall buckling of thin-walled members is considered. A beam of symmetrical cross section subjected to an axial end load, passing through the centroid of the end section, may result in either flexural buckling or torsional buckling. However, a thin-walled beam of unsymmetrical cross section, under axial loading, results in combined flexural - torsional mode of buckling. This is due to the fact that the shear center does not coincide with the centroid of the section. Accordingly, twisting of the beam is accompanied by bending. Coupled flexural and torsional buckling may also occur when the axial load is eccentrically applied to the beam.

In addition, a thin-walled beam may lose stability due to loads applied laterally. A lateral load acting in a plane parallel to the plane of largest ridigity and passing through the line of shear centers, may cause lateral instability of the system. At critical conditions, a neighbouring equilibrium configuration takes the form of combined flexural and torsional deformation of the beam. This phenomena is also known in literature as "plane form of bending".

Linear stability theory of thin-walled elastic beams was studied by Vlasov [54]. In the static analysis the linear theory is capable of predicting the buckling load for different modes of overall buckling configurations. In dynamic stability analysis, it is capable of predicting the natural frequencies as well as establishing the boundaries of parametric instability regions [6]. The need for a nonlinear stability theory of thin-walled beam, arises in two areas. In the static stability analysis, the linear stability formulation of Vlasov becomes inadequate in the study of the post buckling behaviour of the member.

In the parametric stability analysis, it is necessary to include nonlinear terms in the analysis in order to predict bounded response and establish the steady state amplitudes of vibrations once the system is excited into the parametric resonance.

In this chapter a general nonlinear stability theory is formulated using the energy approach. Stability of thin-walled members subjected to both axial end loads and lateral loads are considered.

4.2. Stability of Open Thin-Walled Beam Subjected to Axial Loads

In formulating the nonlinear stability theory of thin-walled beams under axial loads it is necessary to distinguish the behaviour of the beam before buckling. Three cases are needed to be considered: (a) before buckling, there is neither flexural nor torsional deformations in the beam. An example of such behaviour is a thin-walled beam of symmetrical cross section subjected to uniformly applied axial stress at its ends. The buckling condition is then characterized by the existence of an adjacent equilibrium state involving flexural or torsional deformations of the member. (b) before buckling, there is only flexural deformation in one principal direction of the beam. The buckling condition is then characterized by the existence of an adjacent equilibrium state involving additional flexural deformation in the other principal direction and rotation of the section. (c) before buckling, the applied end stress causes the beam to have flexural deflections in both the principal directions. In this case, the loss of stability is characterized by torsional deformations of the beam.

In each case, the governing stability equations can be obtained by calculating the additional energy (potential, kinetic and work done) involved when the beam is transferred from deformed state to the buckled state.

Let V, T and W denote the additional potential energy, kinetic energy and work done during buckling, respectively. Then the stability equations can be obtained by the application of Hamilton's Principle, namely

$$\delta \int_{t_1}^{t_2} (T - V + W)\, dt = 0 \tag{4.1}$$

CASE 1: Loss of Stability in the Form of Coupled Torsional Buckling and Flexural Buckling in Two Principal Directions.

Consider a thin-walled beam subjected to a uniformly distributed axial stress p(t), applied at the end sections. The distribution of the applied stress is such that the resultant P(t) is acting through the centroid of the cross section, namely

$$P(t) = \int_s c\ p(t)\ ds \tag{4.2}$$

$$M_y(t) = -\int_s c\ \bar{x}(s)\ p(t)\ ds = 0 \tag{4.3}$$

$$M_x(t) = \int_s c\ \bar{y}(s)\ p(t)\ ds = 0 \tag{4.4}$$

$$M_\omega(t) = \int_s c\ \omega(s)\ p(t)\ ds = 0 \tag{4.5}$$

M_x and M_y denote the bending moments along the $\bar{X}$ and $\bar{Y}$ axes, respectively, M_ω denotes the bimoment. Due to the applied end load the thin-walled beam will deform in the longitudinal direction while no flexural or torsional deformations will take place. As the magnitude of the applied load reaches the critical value, flexural buckling or torsional buckling may occur and the beam will show flexural deflection in the principal directions as well as a rotation θ of its cross section.

The strain energy of deformation $\bar{V}$, from the undeformed state to the buckled configuration, is given by equation (2.24), namely

$$\bar{V} = \frac{1}{2}\int_0^\ell\int_s \{E^*c[\zeta' - (\bar{\xi}'' + \bar{\eta}''\theta)x - (\bar{\eta}'' - \bar{\xi}''\theta)y - \omega\theta'' + \frac{1}{2}(h^2 + n^2)\theta'^2 + \Omega_R(\theta^2\theta'' + 2\theta\theta'^2)]^2 + G\ c^3/3\ \theta'^2\}\ ds\ dz \tag{4.6}$$

It is convenient to divide the deformation variables into two parts, as follows

$$\zeta = \zeta_0 + \tilde{\zeta} \tag{4.7}$$

$$\xi = \xi_0 + \tilde{\xi} \tag{4.8}$$

$$\eta = \eta_0 + \tilde{\eta} \tag{4.9}$$

$$\theta = \theta_0 + \tilde{\theta} \tag{4.10}$$

The quantities ζ_0, ξ_0, η_0 and θ_0, represent the deformations from the undeformed state to the deformed state. In our present case ξ_0, η_0 and θ_0 are zero. The variables $\tilde{\zeta}$, $\tilde{\xi}$, $\tilde{\eta}$ and $\tilde{\theta}$ represent deformations from the deformed state to the buckled state.

The strain energy of deformation V_0, from the undeformed state to the deformed state, (i.e. prior to buckling) is given by

$$V_0 = \frac{1}{2} \int_0^{\ell} \int_s E^* \, c \, \zeta_0'^2 \, ds \, dz \tag{4.11}$$

Therefore, the strain energy of deformation V, from the deformed state to the buckled state can be written as

$$V = \bar{V} - V_0 \tag{4.12}$$

Subtracting equation (4.11) from equation (4.6), the strain energy due to buckling is given by

$$V = \frac{1}{2} \int_0^{\ell}\int_s \{E^* c[\zeta' - (\bar{\xi}'' + \bar{\eta}''\theta)x$$

$$- (\bar{\eta}'' - \bar{\xi}''\theta)y - \omega\theta''$$

$$+ \frac{1}{2}(n^2 + h^2)\theta'^2 + \Omega_R(\theta^2\theta'' + 2\theta\theta'^2)]^2$$

$$- \frac{1}{2} E^* c\, \zeta_0'^2 + G\, c^3/3\, \theta'^2\}\, ds\, dz \qquad (4.13)$$

The work done $\bar{W}$ by the applied forces, from the undeformed state to the buckled configurations, is given by equation (2.30), namely

$$\bar{W} = \int_s c\, p[\zeta - (\bar{\xi}' + \bar{\eta}'\theta)x$$

$$- (\bar{\eta}' - \bar{\xi}'\theta)y - \omega\theta'$$

$$+ \Omega_R\theta^2\theta']\Big|_0^{\ell}\, ds$$

$$- \frac{1}{2}\int_s c\, p \int_0^{\ell} \{[\bar{\xi}' - (y - c_y)\theta'$$

$$- (x - c_x)\theta\theta']^2 + [\bar{\eta}' + (x - c_x)\theta'$$

$$- (y - c_y)\theta\theta']^2\}\, dz\, ds \qquad (4.14)$$

The work done W_0, to deform the beam from the undeformed state to the deformed state, can be written as

$$W_0 = \int_s c\, p\, \zeta_0\Big|_0^{\ell}\, ds \qquad (4.15)$$

The work done W during buckling is therefore given by the formula

$$W = \bar{W} - W_0 \tag{4.16}$$

Substituting the expressions for $\bar{W}$ and W_0 from equations (4.14) and (4.15) into equation (4.16), there is obtained

$$\begin{aligned} W = \int_s c\, p[\tilde{\zeta} - (\bar{\xi}' + \bar{\eta}'\theta)x - (\bar{\eta}' - \bar{\xi}'\theta)y \\ - \omega\theta' + \Omega_R\theta^2\theta']\Big|_0^{\ell} ds \\ - \frac{1}{2}\int_s c\, p \int_0^{\ell} \{[\bar{\xi}' - (y - c_y)\theta' \\ - (x - c_x)\theta\theta']^2 + [\bar{\eta}' + (x - c_x)\theta' \\ - (y - c_y)\theta\theta']^2\}\, dz\, ds \end{aligned} \tag{4.17}$$

Following a similar procedure, the kinetic energy of the beam $\bar{T}$, after buckling takes place, is given by equation (2.35) and can be written as

$$\begin{aligned} \bar{T} = \frac{1}{2}\int_0^{\ell}\int_s \rho c\{[\dot{\zeta} - (\bar{\xi}' + \bar{\eta}'\theta)^{\cdot}\, x - (\bar{\eta}' - \bar{\xi}'\theta)^{\cdot}\, y \\ + \Omega_R(\theta^2\theta')^{\cdot}\,]^2 + [\dot{\bar{\xi}} - (y - c_y)\dot{\theta} \\ - (x - c_x)\theta\dot{\theta}]^2 + [\dot{\bar{\eta}} + (x - c_x)\dot{\theta} \\ - (y - c_y)\theta\dot{\theta}]^2\}\, ds\, dz \end{aligned} \tag{4.18}$$

The kinetic energy T_0, before buckling occurs, is

$$T_0 = \frac{1}{2} \int_0^{\ell}\int_s \rho \; c \; \dot{\zeta}_0^2 \;\; ds \; dz \tag{4.19}$$

The change of kinetic energy of the beam T, during buckling, is thus

$$T = \bar{T} - T_0 \tag{4.20}$$

Substituting the expressions for $\bar{T}$ and T_0 as given by equation (4.18) and (4.19) into equation (4.20) there is obtained

$$\begin{aligned} T = \frac{1}{2} \int_0^{\ell}\int_s \rho \; c\{[\dot{\zeta} - (\bar{\xi}' + \bar{\eta}'\theta)^{\bullet} x - (\bar{\eta}' - \bar{\xi}'\theta)^{\bullet} y \\ - \omega\dot{\theta}' + \Omega_R(\theta^2\theta')^{\bullet}]^2 + [\dot{\bar{\xi}} - (y - c_y)\dot{\theta} \\ - (x - c_x)\theta\dot{\theta}]^2 + [\dot{\bar{\eta}} + (x - c_x)\dot{\theta} \\ - (y - c_y)\theta\dot{\theta}]^2 - \dot{\zeta}_0^2\} \; ds \; dz \end{aligned} \tag{4.21}$$

Introducing the strain energy (4.13), work done (4.17) and kinetic energy (4.21) into Hamilton's principle (4.1) and carrying out the variation procedure, the following three nonlinear stability differential equations are obtained

$$\begin{aligned} E^*I_{yy}\xi^{iv} - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')'' \\ - P[\xi'' + c_y\theta'' + c_x(\theta\theta'' + \theta'^2) - \rho \, I_{yy}\ddot{\xi}'' \\ + \rho \, A[\ddot{\xi} + c_y\ddot{\theta} + c_x(\theta\ddot{\theta} + \dot{\theta}^2)] = 0 \end{aligned} \tag{4.22}$$

$$E^*I_{xx}\eta^{iv} - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')''$$

$$- P\,[\eta'' - c_x\theta'' + c_y(\theta\theta'' + \theta'^2)] - \rho\, I_{yy}\ddot{\eta}''$$

$$+ \rho A\,[\ddot{\eta} - c_x\ddot{\theta} + c_y(\theta\ddot{\theta} + \dot{\theta}^2)] = 0 \qquad (4.23)$$

$$E^*I_{\omega\omega}\theta^{iv} - 6\,E^*I_R\theta'^2\theta'' - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)''$$

$$- 2\,E^*\,\{\theta'[I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'']\}'$$

$$- GI_d\theta'' - P\,[c_y(\bar{\xi}'' + \bar{\eta}''\theta) - c_x(\bar{\eta}'' - \bar{\xi}''\theta)]$$

$$- \rho I_{\omega\omega}\ddot{\theta}'' + 2\,\rho I_{pc}\ddot{\theta} + \rho A[c_y(\ddot{\bar{\xi}} + \ddot{\bar{\eta}}\theta)$$

$$- c_x(\ddot{\bar{\eta}} - \ddot{\bar{\xi}}\theta)] = 0 \qquad (4.24)$$

The associated boundary conditions are:

$$\{E^*I_{yy}\xi'' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')\}\ \delta\xi'\Big|_0^{\ell} = 0 \qquad (4.25)$$

$$\{E^*I_{yy}\xi''' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')'$$

$$- P(\xi' + c_y\theta' + c_x\theta\theta')\}\ \delta\xi\Big|_0^{\ell} = 0 \qquad (4.26)$$

$$\{E^*I_{xx}\eta'' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')\}\ \delta\eta'\Big|_0^{\ell} = 0 \qquad (4.27)$$

$$\{E^*I_{xx}\eta''' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')'$$

$$- P(\eta' - c_x\theta' + c_y\theta\theta')\}\ \delta\eta\Big|_0^{\ell} = 0 \qquad (4.28)$$

$$\{E^*I_{\omega\omega}\theta'' - E^*[I_{R\omega}\theta'^2 + I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)]$$

$$- M_\Omega\theta^2\}\ \delta\theta'\Big|_0^\ell = 0 \qquad (4.29)$$

$$\{E^*I_{\omega\omega}\theta''' - 2\,E^*I_R\theta'^3 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)'$$

$$- 2\,E^*\theta'(I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'')$$

$$- GI_d\theta' + 2\,M_\Omega\theta\theta' - P[c_y(\bar{\xi}' + \bar{\eta}'\theta)$$

$$- c_x(\bar{\eta}' - \bar{\xi}'\theta)]\}\ \delta\theta\Big|_0^\ell = 0 \qquad (4.30)$$

Equations (4.22) to (4.24) are stability equations. The deformations in the longitudinal ζ direction are governed by the deformation equation

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')' - \rho A\ddot{\zeta} = 0 \qquad (4.31)$$

The associated non-homogeneous boundary conditions are

$$\{E^*A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')$$

$$- P\}\ \delta\zeta\Big|_0^{\tilde{\ell}} = 0 \qquad (4.32)$$

The linear differential equations, governing the stability of thin-walled beams of unsymmetrical cross section subjected to axial loads applied at the ends, can be obtained from the nonlinear differential equations (4.22) to (4.24).

Neglecting the nonlinear terms, Vlasov's equations are obtained

$$EI_{yy}\xi^{iv} - P\xi'' - P\,c_y\theta'' + \rho A\ddot{\xi} + \rho A\,c_y\ddot{\theta} = 0 \quad (4.33)$$

$$EI_{xx}\eta^{iv} - P\eta'' + P\,c_x\theta'' + \rho A\ddot{\eta} - \rho A\,c_x\ddot{\theta} = 0 \quad (4.34)$$

$$EI_{\omega\omega}\theta^{iv} - GI_d\theta'' - P\,c_y\xi'' + P\,c_x\eta''$$

$$- 2\,PI_{pc}\theta''/A + 2\,\rho I_{pc}\ddot{\theta} + \rho A\,c_y\ddot{\xi}$$

$$- \rho A\,c_x\ddot{\eta} = 0 \quad (4.35)$$

The differential equations (4.33) to (4.35) are coupled. These equations represent the coupled flexural - torsional type of vibrations. Neglecting the time dependent terms, equations (4.33) to (4.35) will then represent the case of coupled flexural - torsional buckling. The linear differential equations can be uncoupled for the case of thin-walled beams of symmetrical cross section. For a symmetrical section the shear center coincides with the centroid, thus

$$c_x = c_y = 0 \quad (4.36)$$

Introducing equation (4.36) into the linear differential equations (4.33) to (4.35), the uncoupled equations can be written as

$$EI_{yy}\xi^{iv} - P\xi'' + \rho A\ddot{\xi} = 0 \quad (4.37)$$

$$EI_{xx}\eta^{iv} - P\eta'' + \rho A\ddot{\eta} = 0 \tag{4.38}$$

$$EI_{\omega\omega}\theta^{iv} - GI_d\theta'' - 2\,P\,I_{pc}\theta''/A$$

$$+ 2\,\rho\,I_{pc}\ddot{\theta} = 0 \tag{4.39}$$

The linear differential equations (4.37) to (4.39) are the familiar Vlasov's equations. The uncoupled equations represent flexural and torsional vibrations. Equations (4.37) and (4.38) are the linear equations of flexural vibrations under axial load. Neglecting the time dependent terms the static stability equations are obtained. The uncoupled equations will then represent the flexural buckling and the torsional buckling of a thin-walled beam.

A detailed study of dynamic stability of a thin-walled beam, of symmetrical and monosymmetrical cross section, is presented in Chapter V and Chapter VI.

CASE 2: Loss of Stability in the Form of Coupled Torsional Buckling and Flexural Buckling in One Principal Direction.

Consider a thin-walled beam subjected to a distribution of axial stress p(s,t) applied at the end sections. The distribution of the load over the end section is such that the resultant P is acting in the OYZ plane. The point of application of the resultant P is taken to be at a point with co-ordinates $(0, e_y)$, namely

$$P(t) = \int_s c\, p(s,t)\, ds \qquad (4.40)$$

$$M_y(t) = -\,P(t)\, e_x = -\int_s c\, \bar{x}(s)\, p(s,t)\, ds = 0 \qquad (4.41)$$

$$M_x(t) = P(t)\, e_y = \int_s c\, \bar{y}(s)\, p(s,t)\, ds \qquad (4.42)$$

$$M_\omega(t) = \int_s c\, \omega(s)\, p(s,t)\, ds = 0 \qquad (4.43)$$

Due to the applied load and moment the thin-walled beam will deform in the longitudinal ζ direction as well as in the η direction, while no deflection will take place in the ξ direction. Also, there is no rotation θ. As the magnitude of the applied stress reaches the critical value, coupled flexural - torsional mode of buckling will occur and the beam will show deflection in the ξ direction as well as a rotation θ of its cross section. If deformation variables are expressed in the form given by equations (4.7) to (4.10) the quantities ξ_0 and θ_0 are zero in this case.

Following derivations similar to those done in CASE I, the strain energy V, work done W and kinetic energy T of the beam,from the deformed state to the buckled state, can be written as

$$V = \frac{1}{2} \int_0^{\ell}\int_s \{E^*c[\zeta' - (\bar{\xi}'' + \bar{\eta}''\theta)x - (\bar{\eta}'' - \bar{\xi}''\theta)y$$

$$- \omega\theta'' + \frac{1}{2}(h^2 + n^2)\theta'^2 + \Omega_R(\theta^2\theta'' + 2\theta\theta'^2)]^2$$

$$- \frac{1}{2} E^* c[\zeta_0' - \eta_0''y]^2$$

$$+ Gc^3/3\theta'^2\} \, ds \, dz \qquad (4.44)$$

$$W = \int_s c \; p[\tilde{\zeta} - (\bar{\xi}' + \bar{\eta}'\theta)x - (\bar{\eta}' - \bar{\xi}'\theta)y$$

$$- \omega\theta' + \Omega_R\theta^2\theta'] \Big|_0^{\ell} \, ds$$

$$- \frac{1}{2}\int_s c \; p \int_0^{\ell} \{[\bar{\xi}' - (y - c_y)\theta' - (x - c_x)\theta\theta']^2$$

$$+ [\bar{\eta}' + (x - c_x)\theta' - (y - c_y)\theta\theta']^2$$

$$- \eta_0'^2\} \, dz \, ds \qquad (4.45)$$

$$T = \frac{1}{2}\int_0^{\ell}\int_s \rho c \; \{[\dot{\zeta} - (\bar{\xi}' + \bar{\eta}'\theta)^{\cdot} x - (\bar{\eta}' - \bar{\xi}'\theta)^{\cdot} y$$

$$- \omega\dot{\theta}' + \Omega_R(\theta^2\theta')^{\cdot}]^2 + [\dot{\bar{\xi}} - (y - c_y)\dot{\theta}$$

$$- (x - c_x)\theta\dot{\theta}]^2 + [\dot{\bar{\eta}} + (x - c_x)\dot{\theta}$$

$$- (y - c_y)\theta\dot{\theta}]^2 - [\dot{\zeta}_0 - \dot{\eta}_0'y]^2$$

$$- \dot{\eta}_0^2\} \, ds \, dz \qquad (4.46)$$

Introducing the strain energy (4.44), work done (4.45) and kinetic energy (4.46) into Hamilton's principle (4.1) and carrying out the variation procedure, the following two nonlinear stability differential equations are obtained

$$E^*I_{yy}\xi^{iv} - E^*(I_{\Omega x}\theta^2\theta')'' - P[\xi'' - (e_y - c_y)\theta''$$

$$+ c_x(\theta\theta'' + \theta'^2)] - \rho I_{yy}\ddot{\xi}'' - E^*(I_{Rx}\theta'^2)''$$

$$+ \rho A[\ddot{\xi} + c_y\ddot{\theta} + c_x(\theta\ddot{\theta} + \dot{\theta}^2)] = 0 \qquad (4.47)$$

$$E^*I_{\omega\omega}\theta^{iv} - 6\, E^*I_R\theta'^2\theta''$$

$$- E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)''$$

$$- 2\, E^*\{\theta'[I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'']\}'$$

$$- GI_d\theta'' + P[(e_y - c_y)(\bar{\xi}'' + \bar{\eta}''\theta)$$

$$+ c_x(\bar{\eta}'' - \bar{\xi}''\theta)]$$

$$- \rho I_{\omega\omega}\ddot{\theta}'' + 2\,\rho I_{pc}\ddot{\theta} + \rho A[c_y(\ddot{\bar{\xi}} + \ddot{\bar{\eta}}\theta)$$

$$- c_x(\ddot{\bar{\eta}} - \ddot{\bar{\xi}}\theta)] = 0 \qquad (4.48)$$

The associated boundary conditions are:

$$\{E^*I_{yy}\xi'' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta') - M_\eta\}\ \delta\xi'\Big|_0^\ell = 0 \qquad (4.49)$$

$$\{E^*I_{yy}\xi''' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')'$$

$$- P[\xi' - (e_y - c_y)\theta' + c_x\theta\theta']\}\ \delta\xi\Big|_0^\ell = 0 \qquad (4.50)$$

$$\{E^*I_{\omega\omega}\theta'' - E^*[I_{R\omega}\theta'^2 + I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)]$$

$$- M_\Omega\theta^2\}\ \delta\theta'\Big|_0^\ell = 0 \qquad (4.51)$$

$$\{E^*I_{\omega\omega}\theta''' - 2\ E^*I_R\theta'^3 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)'$$

$$- 2\ E^*\theta'[I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'']$$

$$- GI_d\theta' - 2\ M_\Omega\theta\theta' + P[(e_y - c_y)(\bar{\xi}' + \bar{\eta}'\theta)$$

$$+ c_x(\bar{\eta}' - \bar{\xi}'\theta)]\}\ \delta\theta\Big|_0^\ell = 0 \qquad (4.52)$$

The nonlinear deformation equations, in the longitudinal and η directions, are

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')' - \rho A\ddot{\zeta} = 0 \qquad (4.53)$$

$$E^*I_{xx}\eta^{iv} - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')''$$

$$- \rho I_{xx}\ddot{\eta}'' + \rho A[\ddot{\eta} + c_x\ddot{\theta} + c_y(\theta\ddot{\theta} + \dot{\theta}^2)]$$

$$- P\{\eta'' - [c_x + (e_y - c_y)\theta]\theta''$$

$$- (e_y - c_y)\theta'^2\} = 0 \qquad (4.54)$$

The associated non-homogeneous boundary conditions are

$$\{E^*A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta') - P\}\ \delta\tilde{\zeta}\Big|_0^\ell = 0 \qquad (4.55)$$

$$\{E^*I_{xx}\eta'' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')$$

$$+ M_\xi\}\ \delta\tilde{\eta}'\Big|_0^\ell = 0 \qquad (4.56)$$

$$\{E^*I_{xx}\eta''' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')'$$

$$- P[\eta' - c_x\theta' - (e_y - c_y)\theta\theta']\} \left.\delta\eta\right|_0^{\ell} = 0 \qquad (4.57)$$

where

$$\bar{\xi}'' = [\tilde{\xi}'' - (\eta_0 + \tilde{\eta})''\tilde{\theta}] \qquad (4.58)$$

$$\bar{\eta}'' = [(\eta_0 + \tilde{\eta})'' + \tilde{\xi}''\tilde{\theta}] \qquad (4.59)$$

$$M_\eta = - M_x \sin\theta \qquad (4.60)$$

$$M_\xi = M_x \cos\theta \qquad (4.61)$$

The stability equations (4.47) and (4.48) describe the bifurcation type of stability in the coupled flextural - torsional mode. The deformation equations in the longitudinal ζ and coupled flexural η direction are given by equations (4.53) and (4.54). For the case of a thin-walled beam of symmetrical cross section, where $c_x = c_y = I_{Rx} = I_{Ry} = 0$, also $I_{\Omega x} = I_{\Omega y} = I_{\Omega\omega} = 0$, subjected to an axial static load, the stability equations reduce to

$$E^*I_{yy}\xi^{iv} - P\xi'' + P\,e_y\theta'' = 0 \qquad (4.62)$$

$$E^*I_{\omega\omega}\theta^{iv} - 6\,E^*I_R\theta'^2\theta'' + 2\,E^*I_{pc}(\theta'\zeta')'$$

$$- GI_d\theta'' + P\,e_y(\bar{\xi}'' + \bar{\eta}''\theta) = 0 \qquad (4.63)$$

The deformation equations become:

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')' = 0 \qquad (4.64)$$

$$E^*I_{xx}\eta^{iv} - P(\eta'' - e_y\theta\theta'' - e_y\theta'^2) = 0 \qquad (4.65)$$

To obtain a solution describing the behaviour of the thin-walled beam in the immediate post-buckling zone of coupled flexural - torsional type of buckling, the four coupled differential equations (4.62) to (4.65) should be solved simultaneously.

A solution of the coupled equations is rather difficult, thus an approximate approach should be adopted. Before buckling there are no flexural deflections in the ξ directions and no rotation θ. However, the longitudinal deformations as well as the flexural deflection in the η directions can be expressed by the linear equations

$$E^*A\zeta'' = 0 \qquad (4.66)$$

$$E^*I_{xx}\eta^{iv} - P\eta'' = 0 \qquad (4.67)$$

Solutions for equations (4.66) and (4.67) can be easily obtained. These solutions represent the longitudinal and flexural deformations prior to buckling. Introducing the solutions of equations (4.66) and (4.67) into the nonlinear stability equations (4.62) and (4.63), two coupled nonlinear equations are obtained. These two stability equations, represent coupled flexural - torsional type of buckling. An approximate solution will describe the behaviour of the thin-walled beam in the immediate post buckling zone. It is to be noted that the obtained solution is unable to accurately predict the behaviour beyond the immediate post buckling zone due to the fact that no large flexural deformations are accounted for in the present nonlinear stability theory.

CASE 3: Loss of Stability in the Form of Torsional Buckling

Consider a thin-walled beam subjected to axial end stresses. The point of application of the resultant of the applied stress has co-ordinates (e_x, e_y). The end force and moments can be written as

$$P(t) = \int_s c\, p(s,t)\, ds \tag{4.68}$$

$$M_y(t) = -P(t)e_x = -\int_s c\, \bar{x}(s)\, p(s,t)\, ds \tag{4.69}$$

$$M_x(t) = P(t)e_y = \int_s c\, \bar{y}(s)\, p(s,t)\, ds \tag{4.70}$$

$$M_\omega(t) = \int_s c\, \omega(s)\, p(s,t)\, ds = 0 \tag{4.71}$$

Due to the applied load and moments, the thin-walled beam will deform in the longitudinal ζ direction as well as in the principal directions. No rotation θ will take place prior to buckling. As the magnitude of the applied stress reaches the critical value, buckling will occur and the cross section will rotate. To express the deformation variables, in this case, as given by equations (4. 7) to

(4.10), the quantity θ_0 is zero.

The strain energy V, work done W and kinetic energy T, from the deformed state to the buckled state, as given by equations (4.12), (4.16) and (4.20) respectively, can then be written as

$$V = \frac{1}{2} \int_0^{\ell}\int_s \{E^*c[\zeta' - (\bar{\xi}'' + \bar{\eta}''\theta)x - (\bar{\eta}'' - \bar{\xi}''\theta)y$$

$$- \omega\theta'' + \frac{1}{2}(h^2 + n^2)\theta'^2 + \Omega_R(\theta^2\theta'' + 2\theta\theta'^2)]^2$$

$$- E^*c[\zeta_0' - \xi_0''x - \eta_0''y]^2$$

$$+ G\, c^3/3\ \theta'^2\}\ ds\ dz \qquad (4.72)$$

$$W = \int_s c\ p[\tilde{\zeta} - (\tilde{\bar{\xi}}' + \bar{\eta}'\theta)x - (\tilde{\bar{\eta}}' - \bar{\xi}'\theta)y$$

$$- \omega\theta' + \Omega_R\theta^2\theta']\Big|_0^{\ell}\ ds$$

$$- \frac{1}{2}\int_s c\ p\int_0^{\ell}\{[\bar{\xi}' - (y - c_y)\theta'$$

$$- (x - c_x)\theta\theta']^2 + [\bar{\eta}' + (x - c_x)\theta'$$

$$- (y - c_y)\theta\theta']^2 - [\xi_0'^2 + \eta_0'^2]\}\ dz\ ds \qquad (4.73)$$

$$T = \frac{1}{2}\int_0^{\ell}\int_s \rho c\{[\dot{\zeta} - (\bar{\xi}' + \bar{\eta}'\theta)^{\cdot}\,x - (\bar{\eta}' - \bar{\xi}'\theta)^{\cdot}\,y$$

$$- \omega\dot{\theta}' + \Omega_R(\theta^2\theta')^{\cdot}\,]^2$$

$$+ [\dot{\bar{\xi}} - (y - c_y)\dot{\theta} - (x - c_x)\theta\dot{\theta}]^2$$

$$+ [\dot{\bar{\eta}} + (x - c_x)\dot{\theta} - (y - c_y)\theta\dot{\theta}]^2$$

$$- [\dot{\zeta}_0 - \dot{\xi}'x - \dot{\eta}'y]^2 - (\dot{\xi}_0^2 + \dot{\eta}_0^2)\}\ ds\ dz \qquad (4.74)$$

Introducing the strain energy (4.72), work done (4.73) and kinetic energy (4.74) into Hamilton's principle (4.1) and carrying out the variation procedure, the following nonlinear stability differential equation is obtained

$$E^*I_{\omega\omega}\theta^{iv} - 6\,E^*I_R\theta'^2\theta'' - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)''$$
$$- 2\,E^*[\theta'\,(I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'')]'$$
$$- GI_d\theta'' + P[(e_y - c_y)(\bar{\xi}'' + \bar{\eta}''\theta)$$
$$- (e_x - c_x)(\bar{\eta}'' - \bar{\xi}''\theta)] - \rho I_{\omega\omega}\ddot{\theta}''$$
$$+ 2\,\rho I_{pc}\ddot{\theta} + \rho A[c_y(\ddot{\bar{\xi}} + \ddot{\bar{\eta}}\theta) - c_x(\ddot{\bar{\eta}} - \ddot{\bar{\xi}}\theta)] = 0 \tag{4.75}$$

The associated boundary conditions are

$$\{E^*I_{\omega\omega}\theta'' - E^*[I_{R\omega}\theta'^2 + I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)$$
$$- M_\Omega\theta^2\}\;\delta\theta'\Big|_0^\ell = 0 \tag{4.76}$$

$$\{E^*I_{\omega\omega}\theta''' - 2\,E^*I_R\theta'^3 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)'$$
$$- GI_d\theta' - 2\,E^*\theta'(I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'')$$
$$+ 2\,M_\Omega\theta\theta' + P[(e_y - c_y)(\bar{\xi}' + \bar{\eta}'\theta)$$
$$- (e_x - c_x)(\bar{\eta}' - \bar{\xi}'\theta)]\}\;\delta\theta\Big|_0^\ell = 0 \tag{4.77}$$

The nonlinear deformation equations, in the ζ, ξ and η directions, are

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')' - \rho A\ddot{\zeta} = 0 \quad (4.78)$$

$$E^*I_{yy}\xi^{iv} - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')'' - \rho I_{yy}\ddot{\xi}''$$
$$+ \rho A[\ddot{\xi} + c_y\ddot{\theta} + c_x(\theta\ddot{\theta} + \dot{\theta}^2)]$$
$$- P[\xi'' - (e_y - c_y)\theta'' - (e_x - c_x)$$
$$(\theta\theta'' + \theta'^2)] = 0 \quad (4.79)$$

$$E^*I_{xx}\eta^{iv} - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')'' - \rho I_{xx}\ddot{\eta}''$$
$$+ \rho A[\ddot{\eta} - c_x\ddot{\theta} + c_y(\theta\ddot{\theta} + \dot{\theta}^2)]$$
$$- P[\eta'' + (e_x - c_x)\theta'' - (e_y - c_y)$$
$$(\theta\theta'' + \theta'^2)] = 0 \quad (4.80)$$

The associated boundary conditions are

$$\{E^*A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta') - P\}\ \delta\tilde{\zeta}\Big|_0^{\ell} = 0 \quad (4.81)$$

$$\{E^*I_{yy}\xi'' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')$$
$$- M_\eta\}\ \delta\tilde{\xi}'\Big|_0^{\ell} = 0 \quad (4.82)$$

$$\{E^*I_{yy}\xi''' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')'$$
$$- P\{\xi' - [(e_y - c_y) + (e_x - c_x)\theta]$$
$$\theta'\}\ \delta\tilde{\xi}\Big|_0^{\ell} = 0 \quad (4.83)$$

$$\{E^*I_{xx}\eta'' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta') + \tilde{M}_\xi\} \; \delta\eta' \Big|_0^\ell = 0 \quad (4.84)$$

$$\{E^*I_{xx}\eta''' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')' - P\{\bar{\eta}' + [(e_x - c_x) - (e_y - c_y)\theta]\theta'\} \; \delta\eta \Big|_0^{\tilde{\ell}} = 0 \quad (4.85)$$

where

$$\bar{\xi}'' = (\xi_0 + \tilde{\xi})'' - (\eta_0 + \tilde{\eta})''\tilde{\theta} \quad (4.86)$$

$$\bar{\eta}'' = (\eta_0 + \tilde{\eta})'' + (\xi_0 + \tilde{\xi})''\tilde{\theta} \quad (4.87)$$

$$M_\xi = M_x \cos\theta + M_y \sin\theta \quad (4.88)$$

$$M_\eta = M_y \cos\theta - M_x \sin\theta \quad (4.89)$$

The linear differential equation governing the torsional stability of thin-walled beam of an open cross section subjected to axial static loads applied at the ends, can be obtained from the general nonlinear differential equation (4.75). Neglecting the nonlinear terms as well as the time dependent terms, Vlasov's equation is obtained

$$EI_{\omega\omega}\theta^{iv} - GI_d\theta'' + P(e_y - c_y)\xi'' - P(e_x - c_x)\eta'' - 2(PI_{pc}/A + M_xI_{Ry}/I_{xx} - M_yI_{Rx}/I_{yy})\theta'' = 0 \quad (4.90)$$

where ξ and η are expressed by the linear terms of deformation equations (4.79) and (4.80).

4.3. Lateral Stability of Open Thin-Walled Beam

In formulating the nonlinear stability theory of thin-walled beam under axial and lateral loads, it is necessary to distinguish the behaviour of the beam before buckling. Two cases need to be considered: (a) before buckling, there is only flexural deformation in one principle direction of the beam. As the magnitude of the lateral loads increases to the critical value, buckling of the member is characterized by flexural deformation in the other principal direction and rotation of the section. (b) before buckling, the applied loads causes the beam to have flexural deflections in both the principal directions. In this case, as the magnitude of the lateral loads reaches the critical value, loss of stability is characterized by torsional deformations (rotation of the cross sections) of the beam.

In each case, the governing stability equations can be obtained by calculating the additional energy involved when the beam is transferred from deformed state to buckled state and applying Hamilton's principle.

CASE 1: Lateral Flexural - Torsional Stability

Consider a thin-walled beam subjected to axial as well as lateral loads. The axial loads are distributed along the member. However, the resultant of the axial loads is restricted so that before buckling there exists flexural deflection in one principal direction only, and no rotation of the section will take place.

$$N(z,t) = \int_s c\, \bar{p}(s,z,t)\, ds \tag{4.91}$$

$$\bar{M}_y(z,t) = -\int_s c\, \bar{x}(s)\, \bar{p}(s,z,t,)\, ds = 0 \tag{4.92}$$

$$\bar{M}_x(z,t) = \int_s c\, \bar{y}(s)\, \bar{p}(s,z,t)\, ds \tag{4.93}$$

$$\bar{M}_\omega(z,t) = \int_s c\, \omega(s)\, \bar{p}(s,z,t)\, ds = 0 \tag{4.94}$$

N, $\bar{M}_x$ and $\bar{M}_y$ denote normal force and bending moments along the OZ, $O\bar{X}$ and $O\bar{Y}$ axes, respectively. The bimoment is denoted by $\bar{M}_\omega$.

Lateral loads are applied along the length of the beam. It is assumed that the applied lateral loads pass through the line of shear centers and are acting in a plane parallel to the $O\bar{Y}Z$ plane. This restriction insures that there is no lateral deformation in the ξ direction and also no rotational deformation of the beam would occur before buckling.

Let $q_y(z,t)$ denote the resultant of laterally applied distributed loads at any given cross section of the beam. The point of application of $q_y(z,t)$ is taken to have co-ordinates $(\bar{e}_x, \bar{e}_y)$ as shown in figure [11].

Dividing the deformation variables in the form as given by equations (4.7) to (4.10), the quantities ξ_0 and

θ_0 denoting the flexural deflection and rotation of the cross section before buckling are zero.

The change of strain energy and kinetic energy during buckling will be the same as those given by equations (4.44) and (4.46), respectively. The change in the work done W, during buckling, will be the sum of the change in the work done by axial loads W_A and that of the lateral loads W_L as

$$W = W_A + W_L \tag{4.95}$$

Using equation (4.45) the change in the work done by axial loads can be written as

$$\begin{aligned} W_A = \int_0^{\ell}\int_s c\ \bar{p}[\tilde{\zeta}' - (\bar{\xi}' + \bar{\eta}'\theta)'x - (\tilde{\bar{\eta}}' - \bar{\xi}'\theta)'y \\ - \omega\theta'' + \Omega_R(\theta^2\theta')']ds\ dz \\ - \frac{1}{2}\int_0^{\ell}\int_s c\ \bar{p}\{[\bar{\xi}' - (y - c_y)\theta' \\ - (x - c_x)\theta\theta']^2 + [\bar{\eta}' + (x - c_x)\theta' \\ - (y - c_y)\theta\theta']^2 - \eta_0'^2\}\ ds\ dz \end{aligned} \tag{4.96}$$

The work done by lateral forces $\bar{W}_L$, from the undeformed state to the buckled state, will take the form

$$\bar{W}_L = \int_0^{\ell} q_y(z,t)\ \bar{\eta}_e\ dz \tag{4.97}$$

$\bar{\eta}_e$ donotes the displacement of point of application $\bar{e}$ of the lateral load in the OY direction, as shown in figure [11]. The displacement $\bar{\eta}_e$ of point $\bar{e}$ can be related to the displacement of the shear center $\bar{\eta}$ by the relation

$$\bar{\eta}_e = \bar{\eta} + (\bar{e}_x - c_x)\sin\theta + (\bar{e}_y - c_y)(\cos\theta - 1) \tag{4.98}$$

Introducing equation (4.98) into equation (4.97) there is obtained

$$\bar{W}_L = \int_0^{\ell} [q_y\bar{\eta} + q_y(\bar{e}_y - c_y)(\cos\theta - 1) + q_y(\bar{e}_x - c_x)\sin\theta\,]\,dz \tag{4.99}$$

Since $q_y(z,t)$ was originally assumed to pass through the line of shear centers, thus

$$q_y(\bar{e}_x - c_x) = 0 \tag{4.100}$$

Introducing (4.100) into equation (4.99) the work done is thus

$$\bar{W}_L = \int_0^{\ell} [q_y\bar{\eta} + q_y(\bar{e}_y - c_y)(\cos\theta - 1)]dz \tag{4.101}$$

To express the displacement $\bar{\eta}$ in terms of ξ, η and θ, the first term of the integral in equation (4.74) can be rewritten as follows

$$\int_0^{\ell} q_y\bar{\eta}\ \ dz = -\ Q_y\bar{\eta}\Big|_0^{\ell} + M_x^*\bar{\eta}'\Big|_0^{\ell} - \int_0^{\ell} M_x^*\bar{\eta}''\ dz \tag{4.102}$$

where

$$-M_x^{*''} = -\ Q_y' = q_y \tag{4.103}$$

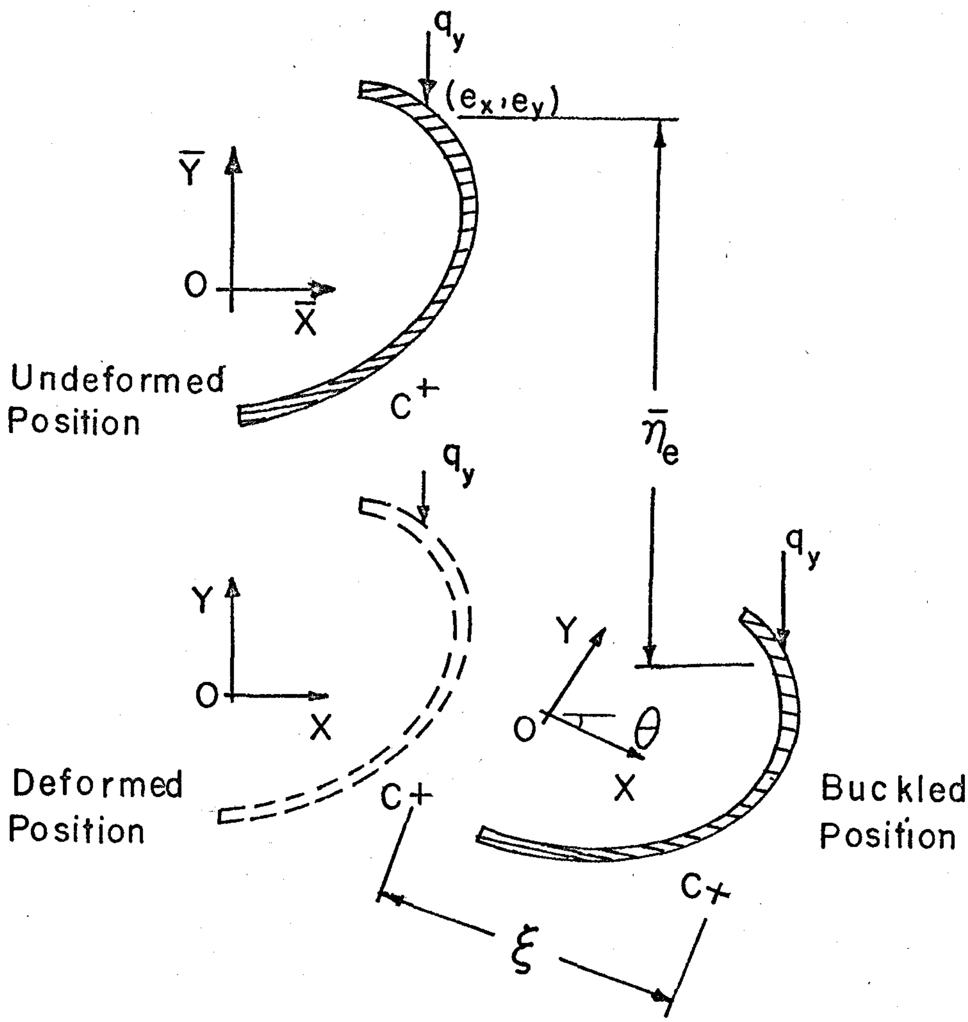

FIG. (11) RIGID BODY MOTION OF THE CROSS SECTION TO BUCKLED CONFIGURATION

M_x^* and Q_y' denote bending moment in the $\overline{OX}$ direction and shearing force in the $\overline{OY}$ direction, resulting from the applied lateral loads only.

The second derivative of $\bar{\eta}$ appearing in equation (4.102) denotes the curvature on the $\overline{OXZ}$ plane. This can be related to curvatures in the ξ and η directions by using equations (2.7) and (2.8). Using equations (2.7),(2.8) and (4.102) the work done expression (4.101) can be written as

$$\bar{W}_L = - Q_y\bar{\eta}\Big|_0^\ell + M_x^*\bar{\eta}'\Big|_0^\ell - \int_0^\ell [M_x^*(\ \xi'' \sin\theta + \eta'' \cos\theta)$$

$$+ q_y(\bar{e}_y - c_y)(\cos\theta - 1)]\ dz \qquad (4.104)$$

Prior to buckling there would be no deformation in the ξ direction, as well as no rotation θ. The work done before buckling W_{L_0}, can thus be written as

$$W_{L_0} = - Q_y\eta_0\Big|_0^\ell + M_x^*\eta_0'\Big|_0^\ell - \int_0^\ell [M_x^*\eta_0''] \ dz \qquad (4.105)$$

The change in work done by lateral loads during buckling is

$$W_L = \bar{W}_L - W_{L_0} \qquad (4.106)$$

Substituting the expressions for $\bar{W}_L$ and W_{L_0} as given by equations (4.104) and (4.105) into equation (4.106), then

$$W_L = - Q_y \tilde{\eta} \Big|_0^{\ell} + M_x^* \tilde{\eta}' \Big|_0^{\ell} - \int_0^{\ell} \{ M_x^* [\xi'' \sin\theta$$

$$+ \eta_0'' (\cos\theta - 1) + \tilde{\eta}'' \cos\theta$$

$$+ q_y(\bar{e}_y - c_y)(\cos\theta - 1)]\} \; dz \qquad (4.107)$$

The approximations (2.15) and (2.16) for the sin θ and cos θ will be used.

Introducing the strain energy (4.44), kinetic energy (4.46) and the work done (4.95) into Hamilton's principle (4.1) and carrying out the variation procedure, two nonlinear stability equations are obtained.

$$E^* I_{yy} \xi^{iv} - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')''$$

$$+ [M_x(\theta - \theta^3/_6)]''$$

$$- \{N[\bar{\xi}' + c_y\theta'(1 - \theta^2/_2) + c_x\theta\theta']\}'$$

$$- \rho I_{yy}\ddot{\bar{\xi}}'' + \rho A[\ddot{\bar{\xi}} + c_y\ddot{\theta} + c_x(\theta\ddot{\theta} + \dot{\theta}^2)] = 0 \quad (4.108)$$

$$E^* I_{\omega\omega}\theta^{iv} - 6\, E^* I_R \theta'^2\theta'' - E^* I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)''$$

$$- 2\, E^*[\theta'(I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'')]'$$

$$- GI_d\theta'' - \rho I_{\omega\omega}\ddot{\theta}'' + M_x(\xi'' - \eta''\theta)$$

$$+ c_x[N(\bar{\eta}' - \bar{\xi}'\theta)]' - c_y[N(\bar{\xi}' + \bar{\eta}'\theta)]'$$

$$+ 2\, \rho I_{pc}\ddot{\theta} + \rho A[c_y(\ddot{\bar{\xi}} + \ddot{\bar{\eta}}\theta) - c_x(\ddot{\bar{\eta}} - \ddot{\bar{\xi}}\theta)]$$

$$+ q_y(\bar{e}_y - c_y)(\theta - \theta^3/_6) = 0 \qquad (4.109)$$

where M_x is the total moment due to axial and lateral loads

$$M_x = \bar{M}_x + M_x^* \quad (4.110)$$

The associated boundary conditions are

$$\{E^*I_{yy}\xi'' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta') - \tilde{M}_\eta\} \, \delta\xi' \Big|_0^\ell = 0 \quad (4.111)$$

$$\{E^*I_{yy}\xi''' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')' + M_x\,\theta' - N[\xi' + c_y\theta'(1 - \theta^2/_2) + c_x\theta\theta']\} \, \delta\xi \Big|_0^{\tilde{\ell}} = 0 \quad (4.112)$$

$$\{E^*I_{\omega\omega}\theta'' - E^*[I_{R\omega}\theta'^2 + I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)] - M_\Omega\theta^2\} \, \delta\theta' \Big|_0^\ell = 0 \quad (4.113)$$

$$\{E^*I_{\omega\omega}\theta''' - 2\,E^*I_R\theta'^3 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2) - 2\,E^*\theta'(I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'') - GI_d\theta' + 2\,M_\Omega\theta\theta' + M_x(\xi' - \eta'\theta) + c_xN(\bar{\eta}' - \bar{\xi}'\theta) - c_yN(\bar{\xi}' + \bar{\eta}'\theta)\} \, \delta\theta \Big|_0^\ell = 0 \quad (4.114)$$

The nonlinear deformation equations are

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega y}\theta^2\theta')'' - \rho A\zeta - N' = 0 \quad (4.115)$$

$$E^*I_{xx}\eta^{iv} - E^*(I_{Ry}\theta'^2 + I_{\ y}\theta^2\theta')'' + [M_x(1 - \theta^2/_2)]'' - N[\eta' + c_y\theta\theta' - c_x\theta'(1 - \theta^2/_2)]\} - \rho I_{xx}\ddot{\eta}'' + \rho A[\ddot{\eta} - c_x\ddot{\theta} + c_y(\theta\ddot{\theta} + \dot{\theta}^2)] = 0 \quad (4.116)$$

The associated boundary conditions are

$$\{E^*A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta') - N\}\ \delta\tilde{\zeta}\Big|_0^{\ell} = 0 \quad (4.117)$$

$$\{E^*I_{xx}\eta'' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta') + M_{\xi}\}\ \delta\tilde{\eta}'\Big|_0^{\ell} = 0 \quad (4.118)$$

$$\{E^*I_{xx}\eta''' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')' - N[\eta' + c_y\theta\theta' - c_x(1 - \theta^2/2)\theta']\}\ \delta\tilde{\eta}\Big|_0^{\ell} = 0 \quad (4.119)$$

<u>CASE 2: Lateral Torsional Stability</u>

Consider a thin-walled beam subjected to axial as well as lateral loads. In this case, the point of application of the resultant of the applied axial loads is of co-ordinates (e_x, e_y), namely

$$N(z,t) = \int_s c\ \bar{p}(s,z,t)\ ds \quad (4.120)$$

$$\bar{M}_y(z,t) = -\int_s c\ \bar{x}(s)\ \bar{p}(s,z,t)\ ds \quad (4.121)$$

$$\bar{M}_x(z,t) = \int_s c\ \bar{y}(s)\ \bar{p}(s,z,t)\ ds \quad (4.122)$$

$$\bar{M}_\omega(z,t) = \int_s c\ \omega(s)\ \bar{p}(s,z,t)\ ds = 0 \quad (4.123)$$

Lateral loads are applied along the length of the beam. It is assumed that the applied lateral loads $q(z,t)$ pass through the line of shear centers before buckling occurs.

The point of application of the applied lateral load in the plane of any cross section is denoted by $\bar{e}$, of co-ordinates $(\bar{e}_x, \bar{e}_y)$. The restriction that the lateral load pass through the line of shear centers, during deformation prior to buckling, is to ensure that there is no twisting moment that could arise during deformation and consequently no torsional deformation of the beam would occur before buckling. At buckling, however, the beam will experience torsional deformations. It should be pointed out that a conservative applied load, such as gravity loads, does not satisfy this condition in general.

This lateral load q(z,t) can be always resolved into two components $q_x(z,t)$ and $q_y(z,t)$ in the $\bar{X}$ and $\bar{Y}$ directions, respectively.

Expressing the variables ζ, ξ, η and θ in the form given by equations (4.7) to (4.10), the quantity θ_0 is zero in the present case.

The change of the strain energy and the kinetic energy during buckling will be the same as given by equations (4.72) and (4.74), respectively. The change in the work done by axial loads during buckling, can be written using equation (4.73), as

$$\begin{aligned} W_A = \int_0^{\ell}\int_s c\ \bar{p}[\tilde{\zeta}' - (\tilde{\bar{\xi}}' + \tilde{\bar{\eta}}'\theta)'x - (\tilde{\bar{\eta}}' - \bar{\xi}'\theta)'y \\ - \omega\theta'' + \Omega_R(\theta^2\theta')'] \, ds dz - \frac{1}{2}\int_0^{\ell}\int_s c\ \bar{p} \\ \{[\bar{\xi}' - (y - c_y)\theta' - (x - c_x)\theta\theta']^2 \\ + [\bar{\eta}' + (x - c_x)\theta' - (y - c_y)\theta\theta']^2 \\ - [\xi_0'^2 + \eta_0'^2]\} \, ds\, dz \end{aligned} \qquad (4.124)$$

The work done by lateral forces $\bar{W}_L$, from the undeformed state to the buckled state, will take the form

$$\bar{W}_L = \int_0^{\ell} (q_x \bar{\xi}_e + q_y \bar{\eta}_e)\, dz \qquad (4.125)$$

$\bar{\xi}_e$ and $\bar{\eta}_e$ are the displacement components of point $\bar{e}$ in the $O\bar{X}$ and $O\bar{Y}$ directions.

The change in the work done by lateral forces W_L can be obtained following similar derivations as given in CASE 1, to be

$$W_L = - Q_y \tilde{\bar{\eta}}\Big|_0^{\ell} + M_x^* \tilde{\bar{\eta}}'\Big|_0^{\ell} - \int_0^{\ell} \{M_x^*[\, \xi'' \sin\theta - \eta_0'' (\cos\theta - 1)$$

$$- \tilde{\eta}'' \cos\theta] + q_y (e_x - c_x)(\cos\theta - 1)\}\, dz$$

$$- Q_x \tilde{\bar{\xi}}\Big|_0^{\ell} - M_y^* \tilde{\bar{\xi}}'\Big|_0^{\ell} + \int^{\ell} \{M_y^*[\eta'' \sin\theta + \xi_0''(\cos\theta - 1)$$

$$+ \tilde{\xi}'' \cos\theta] + q_x (\bar{e}_y - c_y)(\cos\theta - 1)\}\, dz \qquad (4.126)$$

where

$$-M_x^{*''} = - Q_y' = q_y \qquad (4.127)$$

$$-M_y^{*''} = Q_x' = - q_x \qquad (4.128)$$

The change in the total work done W, by axial and lateral loads, can be obtained by introducing equations (4.124) and (4.126) into the equation

$$W = W_A + W_L \qquad (4.129)$$

Introducing the strain energy (4.72), kinetic energy (4.74) and the work done (4.129) into Hamilton's principle (4.1) and carrying out the variation procedure, a nonlinear stability equation is obtained

$$E^*I_{\omega\omega}\theta^{iv} - 6\ E^*I_R\theta'^2\theta'' - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)''$$

$$- 2\ E^*[\theta'(I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'')]'$$

$$- GI_d\theta'' - \rho I_{\omega\omega}\ddot{\theta}'' + 2\ \rho I_{pc}\ddot{\theta}$$

$$+ \rho A[c_y(\ddot{\bar{\xi}} + \ddot{\bar{\eta}}\theta) - c_x(\ddot{\bar{\eta}} - \ddot{\bar{\xi}}\theta)]$$

$$+ M_x(\xi'' - \eta''\theta) + M_y(\eta'' + \xi''\theta)$$

$$+ c_x[N(\bar{\eta}' - \bar{\xi}'\theta)]' - c_y[N(\bar{\xi}' - \bar{\eta}'\theta)]'$$

$$+ [q_x(\bar{e}_x - c_x) + q_y(\bar{e}_y - c_y)](\theta - \theta^3/6) = 0 \quad (4.130)$$

The associated boundary conditions are

$$\{E^*I_{\omega\omega}\theta'' - E^*[I_{R\omega}\theta'^2 + I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)]$$

$$- M_\Omega\theta^2\}\ \delta\theta'\Big|_0^\ell = 0 \quad (4.131)$$

$$\{E^*I_{\omega\omega}\theta''' - 2\ E^*I_R\theta'^3 - E^*I_{\Omega\omega}(\theta^2\theta'' + \theta\theta'^2)$$

$$- 2\ E^*\theta'[I_{pc}\zeta' - I_{Rx}\xi'' - I_{Ry}\eta'']$$

$$- GI_d\theta' + M_x(\xi' - \eta'\theta) + M_y(\eta' + \xi'\theta)$$

$$+ c_x\ N(\bar{\eta}' - \bar{\xi}'\theta) - c_y\ N(\bar{\xi}' + \bar{\eta}'\theta)$$

$$+ 2\ M_\Omega\theta\theta'\}\ \delta\theta\Big|_0^\ell = 0 \quad (4.132)$$

where

$$M_x = \bar{M}_x + M_x^* \quad (4.133)$$

$$M_y = \bar{M}_y + M_y^* \tag{4.134}$$

The nonlinear deformation equations are

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')'$$
$$- \rho A\ddot{\zeta} - N' = 0 \tag{4.135}$$

$$E^*I_{yy}\xi^{iv} - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')''$$
$$- [M_y(1 - \theta^2/_2)]'' + [M_x(\theta - \theta^3/_6)]''$$
$$- \{N[\xi' + c_y(1 - \theta^2/_2)\theta' + c_x\theta\theta']\}'$$
$$- \rho I_{yy}\ddot{\xi}''$$
$$+ \rho A[\ddot{\xi} + c_y\ddot{\theta} + c_x(\theta\ddot{\theta} + \dot{\theta}^2)] = 0 \tag{4.136}$$

$$E^*I_{xx}\eta^{iv} - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')''$$
$$+ [M_x(1 - \theta^2/_2)]'' + [M_y(\theta - \theta^3/_6)]''$$
$$- \{N[\eta' + c_y\theta\theta' - c_x(1 - \theta^2/_2)\theta']\}' - \rho I_{xx}\ddot{\eta}''$$
$$+ \rho A[\ddot{\eta} - c_x\ddot{\theta} + c_y(\theta\ddot{\theta} + \dot{\theta}^2)] = 0 \tag{4.137}$$

The associated nonhomogeneous boundary conditions are

$$\{E^*A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta') - N\}\ \delta\zeta\Big|_0^{\ell} = 0 \tag{4.138}$$

$$\{E^*I_{yy}\xi'' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta') - M_\eta\}\ \delta\tilde{\xi}'\Big|_0^{\ell} = 0$$

(4.139)

$$\{E^*I_{yy}\xi''' - E^*(I_{Rx}\theta'^2 + I_{\Omega x}\theta^2\theta')' + M_x\theta'$$
$$- N[\xi' + c_y(1 - \theta^2/_2)\theta' + c_x\theta\theta']\}\ \delta\tilde{\xi}\Big|_0^{\ell} = 0$$

(4.140)

$$\{E^*I_{xx}\eta'' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta') + M_\xi\}\ \delta\tilde{\eta}'\Big|_0^{\ell} = 0$$

(4.141)

$$\{E^*I_{xx}\eta''' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')$$
$$+ M_y\theta' - N[\eta' + c_y\theta\theta' - c_x(1 - \theta^2/_2)\theta']\}\ \delta\tilde{\eta}\Big|_0^{\ell} = 0$$

(4.142)

The linear differential equations governing the lateral stability of thin-walled beams, of an open cross section subjected to lateral as well as axial static loads, can be obtained from the general nonlinear differential equations (4.108) and (4.130). Neglecting the nonlinear terms and the time dependent terms Vlasov's equations are obtained

$$EI_{yy}\xi^{iv} + (M_x\theta)'' - [N(\xi' + c_y\theta')]' = 0 \qquad (4.143)$$

$$EI_{xx}\eta^{iv} + (M_y\theta)'' - [N(\eta' - c_x\theta')]' = 0 \qquad (4.144)$$

$$EI_{\omega\omega}\theta^{iv} - GI_d\theta'' + M_x\xi'' + M_y\eta'' - 2[(NI_{pc}/A$$
$$+ M_xI_{Ry}/I_{xx} - M_yI_{Rx}/I_{yy})\theta']' + c_x(N\eta')'$$
$$- c_y(N\xi')' + [q_x(\bar{e}_x - c_x) + q_y(\bar{e}_y - c_y)]\theta = 0 \qquad (4.145)$$

CHAPTER V

DYNAMIC STABILITY OF THIN-WALLED BEAM OF SYMMETRICAL CROSS SECTION

5.1. Introduction

As examples to illustrate the use of the stability equations derived in Chapter IV, the remaining portion of the thesis concentrates on studying the problem of parametric stability of thin-walled elastic beams of symmetrical and monosymmetrical sections, subjected to axial periodic end loads.

Under the action of periodic end loading, a state of longitudinal vibration is set up in the beam as a direct consequence of the applied load. Such behaviour is termed as "forced response". However, it is known that at certain frequency ranges of applied load, flexural and/or torsional vibrations are set up. This phenomenon is termed as "parametric instability". The parametric stability study of thin-walled beams consists of two parts. The first part is to determine the range of frequencies that such instability can take place. Since the range of "unstable" frequencies depends on the parametric amplitude, (i.e. magnitude of time varying part of the end load), the critical ranges of frequency are most conveniently expressed in terms of unstable regions in the parametric amplitude - parametric frequency space. Thus, the first part of the parametric stability study is to determine such regions of instability. To achieve this, only the linear theory is necessary. Such analysis has been done by Bolotin [6] for the case of a symmetrical I section. The technique used in the present analysis will be similar to that of Bolotin.

While it has been verified that the linear stability theory predicts accurately the stability regions, it also predicts that the flexural and/or torsional response grows without bound once the applied load is in the unstable frequency range. This is known as "parametric resonance". However, experimental observations show that while the amplitude of the oscillations initially grows exponentially as predicted by the linear theory, it soon reaches a steady state amplitude. Thus, the second part of the parametric stability study in this thesis consists of finding the steady state amplitude and also the transient growth of the parametric response.

Two examples of studying parametric stability of thin-walled beams are given. In this chapter, the case of a simply supported, thin-walled beam of symmetrical I section is studied. In Chapter VI the case of thin-walled beams, of built-in ends, of monosymmetrical split ring section will be discussed.

5.2. Differential Equations

Consider a thin-walled beam of symmetrical cross section, subjected to dynamic axial forces at the ends of the beam. It is assumed that the resultant of the applied forces passes through the centroid of the cross section. Due to the applied end load, the thin-walled beam will deform in the longitudinal direction. In the unstable regions, the beam will show additional flexural deflection in the principal directions of the cross section or rotational deformations. The stability equations for such a case was discussed in Chapter IV, and the governing partial differential equations were formulated. The geometrical properties of a symmetrical cross section will simplify the form of the differential equations because for a symmetrical section the shear center coincides with the centroid, i.e.

$$c_x = c_y = 0 \qquad (5.1)$$

and

$$I_{Rx} = I_{Ry} = I_{R\omega} = I_{\Omega x} = I_{\Omega y} = I_{\Omega\omega} = 0 \qquad (5.2)$$

The longitudinal deformation in the ζ direction is given by equation (4.29)

$$E^* A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')' - \rho A\ddot{\zeta} = 0 \qquad (5.3)$$

The associated non-homogeneous boundary conditions are

$$\{E^* A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta') - (P_0 + P_t \cos \lambda t)\}\ \delta\zeta\Big|_0^{\ell} = 0 \qquad (5.4)$$

where the resultant of the applied forces is considered a periodic function which can be expressed in the form

$$P(t) = P_0 + P_t \cos \lambda t \tag{5.5}$$

Using equations (4.22) to (4.24), the flexural and torsional stability equations for symmetric sections are written as

$$E^*I_{yy}\xi^{iv} - (P_0 + P_t \cos \lambda t)\xi'' + \rho A\ddot{\xi} = 0 \tag{5.6}$$

$$E^*I_{xx}\eta^{iv} - (P_0 + P_t \cos \lambda t)\eta'' + \rho A\ddot{\eta} = 0 \tag{5.7}$$

$$E^*I_{\omega\omega}\theta^{iv} - 6\ E^*I_R\theta'^2\theta'' - 2\ E^*I_{pc}(\theta'\zeta')'$$

$$- GI_d\theta'' + 2\ \rho I_{pc}\ddot{\theta} = 0 \tag{5.8}$$

The associated boundary conditions are

$$\xi''\ \delta\ \xi'\Big|_0^{\ell} = 0 \tag{5.9}$$

$$\{E^*I_{yy}\xi'' - (P_0 + P_t \cos \lambda t)\xi'\}\ \delta\xi\Big|_0^{\ell} = 0 \tag{5.10}$$

$$\eta''\ \delta\ \eta'\Big|_0^{\ell} = 0 \tag{5.11}$$

$$\{E^*I_{xx}\eta'' - (P_0 + P_t \cos \lambda t)\eta'\}\ \delta\eta\Big|_0^{\ell} = 0 \tag{5.12}$$

$$\{E^*I_{\omega\omega}\theta'' - M_\Omega\theta^2\}\ \delta\theta'\Big|_0^{\ell} = 0 \tag{5.13}$$

$$\{E^*I_{\omega\omega}\theta''' - 2\ E^*I_R\theta'^3 - 2\ E^*I_{pc}\theta'\zeta'$$

$$+ 2\ M_\Omega\theta\theta' - GI_d\theta'\}\ \delta\theta\Big|_0^{\ell} = 0 \tag{5.14}$$

It is to be noted that the boundary conditions of the ζ equation, as given by (5.4), are time dependent. It is convenient to modify the differential equations so that a new variable ζ_1 is introduced into the equation. ζ_1 is then required to satisfy homogeneous boundary conditions. This can be done by a technique that was suggested by Mindlin and Goodman [36].

Let a new variable be defined as

$$\zeta_1(z,t) = \zeta(z,t) - (P_0 + P_t \cos \lambda t)z/E^*A \quad (5.15)$$

where

$$\zeta_1(0,t) = 0 \quad (5.16)$$

$$\zeta_1'(\ell,t) = 0 \quad (5.17)$$

Introducing equation (5.15) into equations (5.3) and (5.8), thus the differential equations of the system can be written as

$$E^*A\zeta_1'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')'$$

$$- \rho A\ddot{\zeta}_1 + \rho A z\lambda^2 P_t \cos \lambda t/E^*A = 0 \quad (5.18)$$

$$E^*I_{yy}\xi^{iv} - (P_0 + P_t \cos \lambda t)\xi'' + \rho A\ddot{\xi} = 0 \quad (5.19)$$

$$E^*I_{xx}\eta^{iv} - (P_0 + P_t \cos \lambda t)\eta'' + \rho A\ddot{\eta} = 0 \quad (5.20)$$

$$E^*I_{\omega\omega}\theta^{iv} - 6\ E^*I_R\theta'^2\theta'' - 2\ E^*I_{pc}[\theta'\zeta_1''$$

$$+ \theta''(E^*A\zeta_1' + P_0 + P_t \cos \lambda t)/E^*A]$$

$$- GI_d\theta'' + 2\ \rho I_{pc}\ddot{\theta} = 0 \quad (5.21)$$

The boundary conditions of the thin-walled beam are such that

at $z = 0$

$$\zeta_1 = \xi = \xi'' = \eta = \eta'' = \theta = \theta'' = 0 \tag{5.22}$$

at $z = \ell$

$$\zeta_1' = \xi = \xi'' = \eta = \eta'' = \theta' = \theta''' = 0 \tag{5.23}$$

The end conditions expressed by equations (5.22) and (5.23) satisfy the boundary conditions given by equations (5.16), (5.17) and (5.9) to (5.14).

It is convenient at this stage to nondimensionalize the constants and variables. The applied force is best expressed as a factor less than or equal to 1, multiplied by the smallest static buckling load. Frequencies are best normalized in terms of the frequencies of free vibrations.

The static buckling loads in the uncoupled flexural and torsional modes of buckling are

$$P_\xi = \pi^2 E^* I_{yy}/\ell^2 \tag{5.24}$$

$$P_\eta = \pi^2 E^* I_{xx}/\ell^2 \tag{5.25}$$

$$P_\theta = A/(E^* I_{\omega\omega}\pi^2/4\ell^2 + GI_d)/(2I_{pc}) \tag{5.26}$$

where P_ξ, P_η and P_θ denote the buckling load in the two principal directions ξ and η, due to flexure and torsional buckling, respectively.

The applied loads will be expressed in the normalized form

$$E^*A\ \beta(N_0 + N_t \cos \lambda t) = P_0 + P_t \cos \lambda t \qquad (5.27)$$

The axial load parameters N_0 and N_t are nondimensionalized with respect to the smallest buckling stress ($E^*\beta$) in the weaker mode. $N_0 = -1$ corresponds to an applied axial compressive load which will cause buckling in the weaker direction.

The frequency of the first mode of free vibration, under a constant applied axial load P_0 can be written as

$$\Omega_\zeta^2 = \pi^2 E^*/(4\rho \ell^2) \qquad (5.28)$$

$$\Omega_\xi^2 = (1 + P_0/P_{cr})\ \pi^4 E^* I_{yy}/(\rho A \ell^4) \qquad (5.29)$$

$$\Omega_\eta^2 = (1 + P_0/P_{cr})\ \pi^4 E^* I_{xx}/(\rho A \ell^4) \qquad (5.30)$$

$$\Omega_\theta^2 = (1 + P_0/P_{cr})\ \pi^2 (E^* I_{\omega\omega} \pi^2/4\ell^2 + GI_d)/ (8\ell^2 I_{pc} \rho A) \qquad (5.31)$$

where Ω_ζ is the first natural frequency in the longitudinal direction. Ω_ξ and Ω_η are the frequencies of free vibration in the two principal directions ξ and η. Ω_θ is the frequency of free torsional type of vibrations.

Displacement variables are nondimensionalized with respect to the length, in the form

$$\zeta^* = \zeta_1/\ell \tag{5.32}$$

$$\xi_1 = \xi/\ell \tag{5.33}$$

$$\eta_1 = \eta/\ell \tag{5.34}$$

$$\bar{z} = z/\ell \tag{5.35}$$

The partial differential equations given by equations (5.18) to (5.21) can be transformed to ordinary differential equations by Galerkin's averaging technique. This is a first step towards attempting to solve the equations. To achieve this, seek solutions that satisfy the boundary conditions (5.22) and (5.23) in the form

$$\zeta^*(\bar{z},t) = \zeta_t(t) \sin \pi\bar{z}/2 \tag{5.36}$$

$$\xi_1(\bar{z},t) = \xi_t(t) \sin \pi\bar{z} \tag{5.37}$$

$$\eta_1(\bar{z},t) = \eta_t(t) \sin \pi\bar{z} \tag{5.38}$$

$$\theta(\bar{z},t) = \theta_t(t) \sin \pi\bar{z}/2 \tag{5.39}$$

where ζ_t, ξ_t, η_t and θ_t are functions of time only.

Solutions in the form of (5.36) to (5.39) represent the first approximation of longitudinal, flexural and torsional motion. Substituting the solutions (5.36) to (5.39) into the partial differential equations (5.18) to (5.21) and applying Galerkin's averaging technique, a set of ordinary differential equations are obtained

$$\ddot{\zeta}_t + \Omega_\zeta^2 \zeta_t - (8\beta\lambda^2 N_t \cos \lambda t)/\pi^2 + E^* I_{pc} \pi^2 \theta_t^2/(3\rho A \ell^4)$$

$$+ E^* I_{\Omega c} \pi^2 \theta_t^3/(16\rho A \ell^3) = 0 \qquad (5.40)$$

$$\ddot{\xi}_t + \Omega_\xi^2 \xi_t + \pi^2 E^* \beta N_t \cos \lambda t \; \xi_t/(\rho \ell^2) = 0 \qquad (5.41)$$

$$\ddot{\eta}_t + \Omega_\eta^2 \eta_t + \pi^2 E^* \beta N_t \cos \lambda t \; \eta_t/(\rho \ell^2) = 0 \qquad (5.42)$$

$$\ddot{\theta}_t + \Omega_\theta^2 \theta_t + \pi^2 E^* \beta N_t \cos \lambda t \; \theta_t/(4\rho \ell^2)$$

$$+ \pi^2 E^* \; \theta_t \zeta_t/(3\rho \ell^2) - 3 \; \pi^4 E^* I_R \theta_t^3/(64\rho I_{pc} \ell^4) = 0 \qquad (5.43)$$

It is convenient to express the system of ordinary differential equations in matrix notations as

$$[D]\{\ddot{f}\} + ([E] - \beta N_t \cos \lambda t [B])\{f\} + \{\Psi\} = 0 \qquad (5.44)$$

where [B], [D], and [E] are matrices of constant coefficients.

{f} is a vector of variables

{Ψ} is a vector consisting of nonlinear terms

At this stage, it is also convenient to introduce damping effects. The present investigation will be restricted to the effect of linear viscous damping. This effect is represented by a coefficient c_d known as "fractional critical damping" times the fundamental frequency of the mode of vibrations. Generally speaking, there can be different values for the fractional critical damping for longitudinal, flexural and torsional vibrations. In the present analysis, the value of the fractional critical damping will be assumed to remain the same for all three types of vibrations.

The presence of viscous damping will reduce the frequencies of free vibrations; however, this effect can be neglected as $c_d << 1$.

Introducing linear damping terms in the system of ordinary differential equations (5.44), there is obtained

$$[D]\{\ddot{f}\} + [C_d]\{\dot{f}\} + ([E] - \beta N_t \cos \lambda t[B])\{f\} + \{\Psi\} = 0 \qquad (5.45)$$

where $[C_d]$ is the damping matrix which contains diagonal terms only. Damping matrix can be established experimentally.

The linear terms of the ordinary differential equation (5.40), including linear damping terms, are

$$\ddot{\zeta}_t + 2\, c_d\, \Omega_\zeta \dot{\zeta}_t + \Omega_\zeta^2 \zeta_t = (8\beta\lambda^2\, N_t \cos \lambda t)/\pi^2 \qquad (5.46)$$

Equation (5.46) is in the standard form of forced vibration equations.

Neglecting the nonlinear terms of the parametric stability equations (5.41) to (5.43), a set of linear equations is obtained

$$\ddot{\xi}_t + 2\, c_d \Omega_\xi \dot{\xi}_t + \Omega_\xi^2 \xi_t + \pi^2 E^* \beta N_t \cos \lambda t \xi_t/(\rho \ell^2) = 0 \qquad (5.47)$$

$$\ddot{\eta}_t + 2\, c_d \Omega_\eta \dot{\eta}_t + \Omega_\eta^2 \eta_t + \pi^2 E^* \beta N_t \cos \lambda t \eta_t / (\rho \ell^2) = 0 \qquad (5.48)$$

$$\ddot{\theta}_t + 2\, c_d \Omega_\theta \dot{\theta}_t + \Omega_\theta^2 \theta_t + \pi^2 E^* \beta N_t \cos \lambda t \theta_t / (4 \rho \ell^2) = 0 \qquad (5.49)$$

Equations (5.47) to (5.49) have variable coefficients. They are of the Mathieu-Hill type equations, a mathematical characteristic of the parametric stability problems.

5.3. Boundaries of the Principal Regions of Parametric Instability

Consider a thin-walled beam of symmetrical I section of the gemoetrical dimensions

$$c_f = c_w = c \qquad (5.50)$$

$$H/c = 50 \qquad (5.51)$$

$$H = B \qquad (5.52)$$

$$\ell/H = 10 \qquad (5.53)$$

where c_f and c_w are thicknesses of the flange and web plates, respectively.

H is the height of the web plate,

B is the width of the flanges

ℓ is the length of the thin-walled beam

The four ordinary differential equations (5.46) to (5.49) are uncoupled due to the fact that the resultant of the applied loads passes through the centroid of a symmetrical cross section. The parametric stability equations (5.47) to (5.49) are in the form of uncoupled Mathieu equations as

$$[I]\{\ddot{f}\} + [C_d]\{\dot{f}\} + ([E] - \beta N_t \cos \lambda t\, [B])\{f\} = 0 \tag{5.54}$$

where [I] is the Identity matrix.

$$\{f\} = \begin{Bmatrix} \xi \\ \eta \\ \theta \end{Bmatrix} \tag{5.55}$$

$$[C_d] = 2\, c_d \begin{bmatrix} \Omega_\xi & 0 & 0 \\ 0 & \Omega_\eta & 0 \\ 0 & 0 & \Omega_\theta \end{bmatrix} \tag{5.56}$$

$$[E] = \begin{bmatrix} \Omega_\xi^2 & 0 & 0 \\ 0 & \Omega_\eta^2 & 0 \\ 0 & 0 & \Omega_\theta^2 \end{bmatrix} \tag{5.57}$$

$$[B] = -\pi^2 E^*/(\rho \ell^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \tag{5.58}$$

Equations (5.47) to (5.49) as were expressed in the form (5.54) are uncoupled parametric excitation equations. Extensive study of the uncoupled Mathieu's equations is available [32]. The condition for instability is when a bounded harmonic solution does not exist. This condition defines a region in the parametric amplitude - parametric frequency plane which is a region of instability.

To establish the boundaries of instability regions corresponding to different types of vibrations, seek a solution for equation (5.54) in the form of a Fourier expansion, with terms of the period $4\pi/\lambda$. It takes the form

$$\{f(t)\} = \sum_{i=1,3,5} (\{a_i\} \sin i\lambda t/2 + \{b_i\} \cos i\lambda t/2) \tag{5.59}$$

where the vectors $\{a_i\}$ and $\{b_i\}$ are of constant coefficients.

Substituting the Fourier solution (5.59) into the system of linear equations (5.54), and applying harmonic balance technique, a system of algebraic equations were obtained in the order of two equations for each value of the parameter i in the periodic solution (5.59). The condition for the existence of the solution with period of $4\pi/\lambda$ is when the determinant of coefficients vanishes.

If the interest is only in the principal region of instability, which is the case corresponding to parameter i = 1, the condition for the existence of the solution can be written as

$$\begin{vmatrix} [E] + N_t\beta[B]/2 - [I]\lambda^2/4 & -[G_d]\lambda/2 \\ [G_d]\lambda/2 & [E] - N_t\beta[B]/2 - [I]\lambda^2/4 \end{vmatrix} = 0 \tag{5.60}$$

This condition (5.60) is a first approximation for the infinite determinant. The approximation is equivalent to the assumption that the periodic solutions on the boundaries are the harmonic functions:

$$\{f(t)\} = \{a\} \sin \lambda t/2 + \{b\} \cos \lambda t/2 \tag{5.61}$$

For a relatively small amplitude of exciting force, this approximation gives reasonably good results.

The boundaries of the principal regions of instability given by condition (5.60), can be written as

$$[\Omega_\xi^2 - \pi^2 E^* \beta N_t/(2\rho \ell^2) - \lambda^2/4]$$
$$[\Omega_\xi^2 + \pi^2 E^* \beta N_t/(2\rho \ell^2) - \lambda^2/4]$$
$$+ (\lambda c_d \Omega_\xi)^2 = 0 \tag{5.62}$$

$$[\Omega_\eta^2 - \pi^2 E^* \beta N_t/(2\rho \ell^2) - \lambda^2/4]$$
$$[\Omega_\eta^2 + \pi^2 E^* \beta N_t/(2\rho \ell^2) - \lambda^2/4]$$
$$+ (\lambda c_d \Omega_\eta)^2 = 0 \tag{5.63}$$

$$[\Omega_\theta^2 - \pi^2 E^* \beta N_t/(8\rho \ell^2) - \lambda^2/4]$$
$$[\Omega_\theta^2 + \pi^2 E^* \beta N_t/(8\rho \ell^2) - \lambda^2/4]$$
$$+ (\lambda c_d \Omega_\theta)^2 = 0 \tag{5.64}$$

where λ is the frequency of the parametric load. Ω_ξ, Ω_η and Ω_θ are frequencies of free flexural vibrations in the ξ and η directions and torsional vibrations under a constant applied load P_0.

Equations (5.62) to (5.64) are plotted in figure [12] and figure [13] for the case of a thin-walled beam of symmetrical I section. The geometrical dimensions of the I section are as given by equations (5.50) to (5.53). Figure [12] represents the case when damping terms are absent. The vertical axis represents the parametric load, i.e. the amplitude of the exciting force. The horizontal axis, with different scales, represents the parametric frequency, i.e. the normalized frequency of the exciting force with respect to the frequency of free vibrations. The effect of different values of the parameter N_0 on the boundaries of the principal regions of parametric instability is shown. The three instability regions are the principal regions associated with flexural and torsional instability.

Figure [13] represents the effect of viscous damping on the principal region of torsional instability when $N_0 = 0$. Different values of the damping coefficients were considered. Similar graphs can be easily plotted for the other two principal regions of flexural instability.

It can be easily concluded from figure [12] that the principal region of torsional instability takes place before the principal regions of flexural instability. This is due to the fact that the lowest frequency of free vibrations is the frequency of first mode of free torsional oscillation.

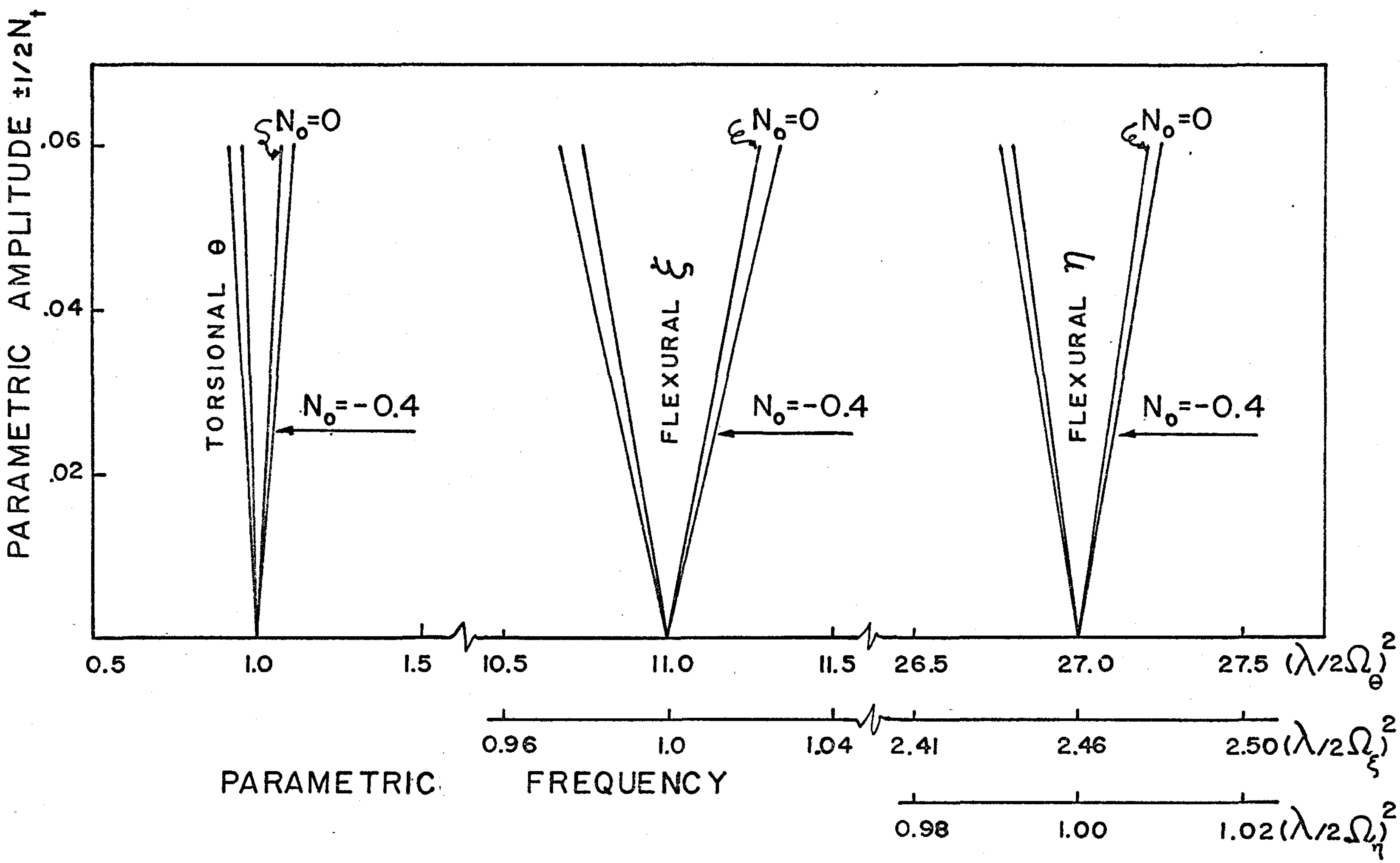

FIG. (12) PRINCIPAL INSTABILITY REGIONS FOR SYMMETRICAL I BEAM

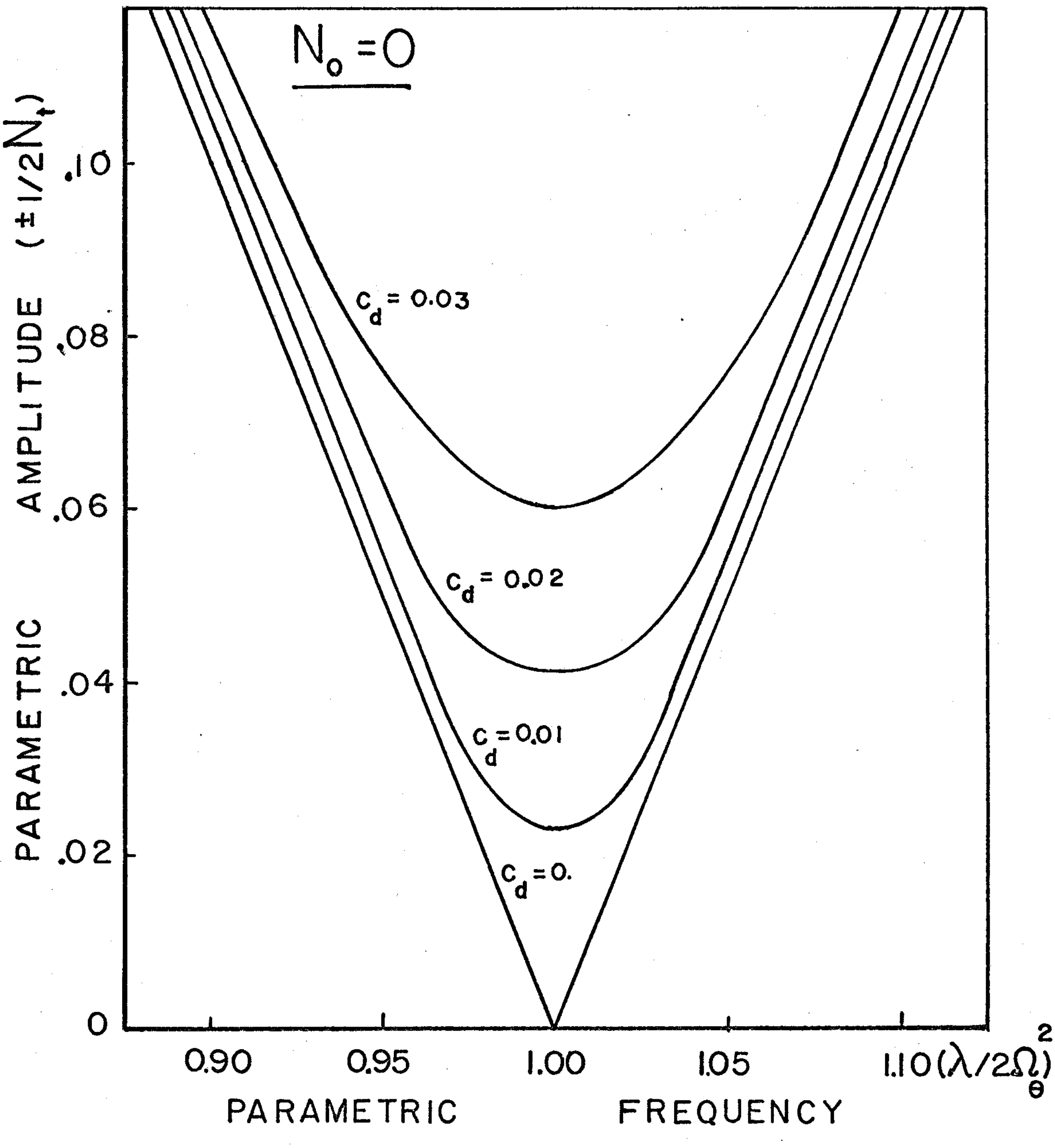

FIG. (13) PRINCIPAL REGION OF TORSIONAL INSTABILITY FOR SYMMETRICAL I BEAM

5.4. Steady State Amplitudes of Vibrations

The linear theory of stability was used to predict the boundaries of the principal regions of instability. It also predicted that the flexural or torsional response grows without bound once the applied load is in the unstable frequency range. However, experimental observations show that while the amplitude of oscillations originally grows exponentially as predicted by the linear theory, it soon reaches a steady state amplitude. To find the steady state amplitudes near the principal region of parametric instability, nonlinear theory should be used. The four ordinary nonlinear differential equations (5.40) to (5.43) are recalled. The steady state amplitude of oscillations near the principal region of instability can be obtained by seeking solutions of equations (5.40) to (5.43) of the approximate harmonic form:

$$\zeta_t = \zeta_a + \zeta_s \sin \lambda t + \zeta_c \cos \lambda t \tag{5.65}$$

$$\xi_t = \xi_s \sin \lambda t/2 + \xi_c \cos \lambda t/2 \tag{5.66}$$

$$\eta_t = \eta_s \sin \lambda t/2 + \eta_c \cos \lambda t/2 \tag{5.67}$$

$$\theta_t = \theta_s \sin \lambda t/2 + \theta_c \cos \lambda t/2 \tag{5.68}$$

The assumed solution of the response in the longitudinal direction (5.65) is a forced oscillation case where the response frequency λ equals the exciting frequency.

The parametric flexural response in the ξ and η directions and the parametric torsional response θ are of response frequency $\lambda/2$ which is half the exciting frequency. Since the frequency of parametric response is taken $\lambda/2$, solutions (5.66) to (5.68) apply only to the principal regions of parametric flexural and torsional instability.

The steady state amplitude of the uncoupled flexural type of parametric vibrations in the ξ and η directions cannot be obtained using the approximate solutions (5.66) and (5.67). Higher order theory has to be used. However, the present order of approximations adopted is able to predict the amplitudes of the coupled longitudinal and torsional type of vibrations. Therefore, the interest will be focused on the coupled longitudinal and torsional responses.

Introducing the adopted solutions (5.65) and (5.68) into the differential equations (5.40) and (5.43) and applying harmonic balance technique, a set of algebraic equations is obtained. The components of the amplitude of longitudinal oscillations are given by:

$$\zeta_a = -\alpha(\theta_c^2 + \theta_s^2) \tag{5.69}$$

$$\zeta_c = \{(\Omega_\zeta^2 - \lambda^2)[8\lambda^2\beta N_t/\pi^2 - \alpha\Omega_\zeta^2(\theta_c^2 - \theta_s^2)] + 4\alpha\, c_d\,\lambda\,\Omega_\zeta^3\theta_c\theta_s\}/D_0 \tag{5.70}$$

$$\zeta_s = \{- 2\alpha \Omega_\zeta^2(\Omega_\zeta^2 - \lambda^2)\theta_c\theta_s$$

$$+ 2 c_d\Omega_\zeta\lambda[8\lambda^2\beta N_t/\pi^2 - \alpha\Omega_\zeta^2(\theta_c^2 - \theta_s^2)\}/D_0 \quad (5.71)$$

where

$$\alpha = 2 I_{pc}/(3A\ell^2) \quad (5.72)$$

$$D_0 = (\Omega_\zeta^2 - \lambda^2)^2 + (2 c_d \Omega_\zeta\lambda)^2 \quad (5.73)$$

The amplitude of parametric torsional vibrations near the principal region of parametric instability is given by

$$\theta_c = \theta_s = 0 \quad (5.74)$$

or

$$\theta_n^2 = \{- (\alpha_1 + \alpha_3) \pm [(\alpha_1 + \alpha_3)^2$$

$$- 4(1 + \alpha_4^2)(\alpha_1\alpha_3 + \alpha_2^2)]^{1/2}\}$$

$$/[2D_1(1 + \alpha_4^2)] \quad (5.75)$$

where

$$\theta_n^2 = \theta_c^2 + \theta_s^2 \quad (5.76)$$

The values of the quantities α_i, $i = 1,4$ are

$$\alpha_1 = \Omega_\theta^2 - \lambda^2/4 + 0.5\, \Omega_\zeta^2\beta N_t$$

$$+ [16\, \Omega_\zeta^2(\Omega_\zeta^2 - \lambda^2)\lambda^2\beta N_t/(3\pi^2)]/D_0 \quad (5.77)$$

$$\alpha_2 = c_d\Omega_\theta\lambda + 32\, c_d\, \Omega_\zeta^3\, \lambda^3\beta N_t/(3\pi^2) \quad (5.78)$$

$$\alpha_3 = \Omega_\theta^2 - \lambda^2/4 - 0.5\ \Omega_\zeta^2 \beta N_t$$

$$- [16\ \Omega_\zeta^2(\Omega_\zeta^2 - \lambda^2)\lambda^2 \beta N_t/(3\pi^2)]/D_0 \qquad (5.79)$$

$$\alpha_4 = 4\ \Omega_\zeta^4\ \alpha\ c_d\ \lambda/(3D_0D_1) \qquad (5.80)$$

$$D_1 = 9\ \pi^2\Omega_\zeta^2\ I_R/(64\ I_{pc}\ell^2)$$

$$- 2\ \Omega_\zeta^2\alpha[2 + (\Omega_\zeta^2 - \lambda^2)\Omega_\zeta^2/D_0]/3 \qquad (5.81)$$

Further discussions will involve the behaviour of the amplitude of parametric torsional vibrations. This is due to the fact that the first principal region of parametric instability corresponds to the first mode of the torsional type of vibrations. The amplitude of torsional oscillations is of larger magnitude compared to the amplitude of flexural type of vibrations. This is consistent with the order of approximations in the present theory for large rotation.

The steady state amplitude of parametric torsional oscillations, given by equation (5.74), represents a trivial solution. It will be shown in Chapter VI that the trival solution for the amplitude of torsional vibration near the principal region of parametric instability is an unstable solution. The non-trivial solution for the amplitude of steady state torsional vibrations is given by equation (5.75) where the quantities α_i, i = 1,4 are given by equations (5.77) to (5.80). For a certain value of the parametric frequency $(\lambda/2\Omega_\theta)^2$ near the principal region of torsional instability, equation (5.75) gives two solutions for the amplitude of torsional vibrations θ_n. Applying the criteria governing the stability of the two non-trivial solutions,as will be discussed in detail in Chapter VI, one of the solutions was found to be stable while the other is unstable.

The steady state amplitude of parametric torsional oscillations, as given by equation (5.75), is calculated for the case of thin-walled beams of symmetrical I section. The geometrical properties of the section are as given by equations (5.50) to (5.53). Solutions of equation (5.75) are plotted in figure [14]. For $N_0 = 0$ and a constant value of the parametric load, $N_t/2 = 0.05$, the variation of the amplitude of steady state torsional vibration θ_n is shown versus the parametric frequency $(\lambda/2\Omega_\theta)^2$. Solutions for different values of damping coefficients were carried out to illustrate the effect of damping on the steady state amplitude of vibration. Solid lines in figure [14] represent the stable branches of the solution while the unstable branches are represented by dotted lines.

To discuss the behaviour of the structure near the principal region of parametric instability as shown in

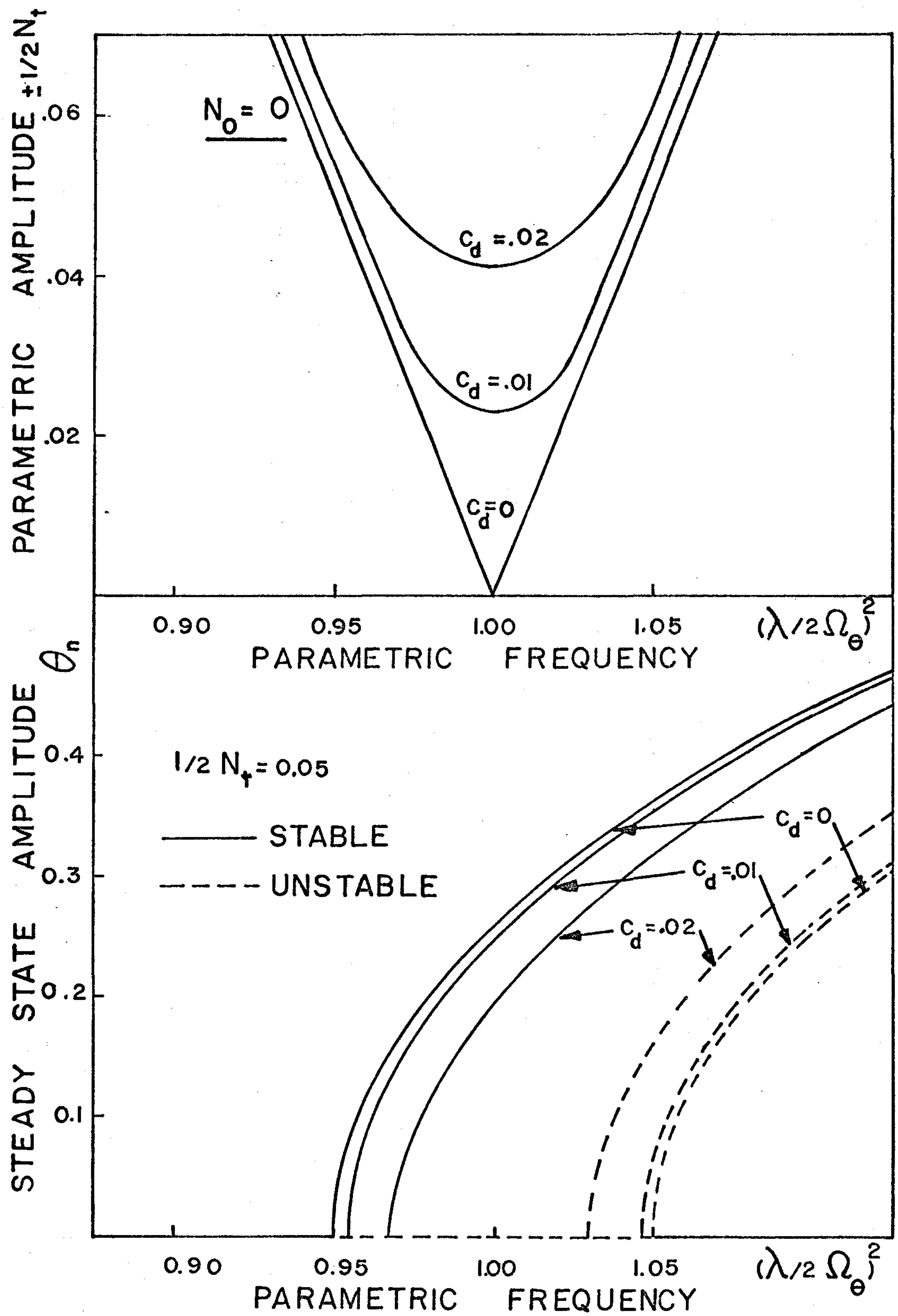

FIG. (14) STEADY STATE AMPLITUDE OF TORSIONAL VIBRATIONS FOR AN I BEAM figure [14], two cases should be considered depending on whether the structure is approaching the instability region with increasing or decreasing parametric frequencies. For the case of an increasing parametric frequency, the steady state amplitude of torsional vibration follows the trival solution. In other words, no torsional response takes place. As the parametric frequency enters the unstable range, the amplitude of torsional oscillations will grow. Continuous growth of the amplitude of vibrations takes place as the parametric frequency increases. As the paremetric frequency represents a point on the second boundary of the unstable region,a sudden drop of the amplitude takes place and the trivial solution of zero amplitude becomes stable.

For the case of approaching the unstable region by a decreasing frequency, a sudden jump of the amplitude will take place at the right boundary of the unstable region. As the parametric frequency decreases further, the steady state amplitude of torsional oscillation decreases gradually following the solid curve in figure [14]. When the parametric frequency represents a point on the left boundary of the unstable region, the trival solution (5.74), i.e. no torsional response, becomes stable.

5.5. Nonsteady State Solution

Nonsteady state solutions represent the time history of the amplitude growth near the principal region of instability. The transient response of a thin-walled beam of symmetrical cross section is carried out by taking the form of the solutions as

$$\zeta_t = \zeta_a + \zeta_s(t) \sin \lambda t + \zeta_c(t) \cos \lambda t \tag{5.82}$$

$$\xi_t = \xi_s(t) \sin \lambda t/2 + \xi_c(t) \cos \lambda t/2 \tag{5.83}$$

$$\eta_t = \eta_s(t) \sin \lambda t/2 + \eta_c(t) \cos \lambda t/2 \tag{5.84}$$

$$\theta_t = \theta_s(t) \sin \lambda t/2 + \theta_c(t) \cos \lambda t/2 \tag{5.85}$$

These solutions are of time dependent amplitudes of oscillations. It is assumed that the time dependent coefficients are slowly varying parameters; namely , the variation of the amplitude over a complete cycle is very small compared to the amplitude itself. Also, the variation of the first derivative of the amplitude over a complete cycle is very small compared to the derivative of the amplitude with respect to time.

At this stage the interest will be limited to the transient response of the parametric torsional vibration near the principal region of parametric torsional instability. The differential equations (5.40) and (5.43) represent the coupled longitudinal and torsional responses. The solutions (5.82) and (5.85) are introduced into equations (5.40) and (5.43) and applying harmonic balance technique, a set of first order,coupled,nonlinear, differential equations are obtained. The components of the amplitude of longitudinal oscillations are given by

$$\zeta_a = -\alpha(\theta_c^2 + \theta_s^2) \tag{5.86}$$

$$\dot{\zeta}_c = (D_3 + c_d\,\Omega_\zeta D_4/\lambda)/D_2 \tag{5.87}$$

$$\dot{\zeta}_s = (D_4 - c_d\,\Omega_\zeta\,D_3/\lambda)/D_2 \tag{5.88}$$

where

$$\alpha = 2\,I_{pc}/(3A\ell^2) \tag{5.89}$$

$$D_2 = 1 + (c_d\Omega_\zeta/\lambda)^2 \tag{5.90}$$

$$D_3 = 0.5\,[\zeta_s(\Omega_\zeta^2 - \lambda^2) + \alpha\Omega_\zeta^2\theta_c\theta_s]/\lambda - c_d\,\Omega_\zeta\zeta_c \tag{5.91}$$

$$D_4 = -\,0.5\,[\zeta_c(\Omega_\zeta^2 - \lambda^2) - 8\,\lambda^2\beta N_t/\pi^2 + 0.5\alpha\Omega_\zeta^2(\theta_c^2 - \theta_s^2)]/\lambda - c_d\,\Omega_\zeta\zeta_s \tag{5.92}$$

The components of the amplitude of parametric torsional vibrations near the principal region of parametric torsional instability are given by

$$\dot{\theta}_c = [c_d\Omega_\theta \, (D_6 - c_d \, \Omega_\theta \, \theta_s)/\lambda + (D_7 - c_d \, \Omega_\theta \, \theta_c)]/D_5 \qquad (5.93)$$

$$\dot{\theta}_s = [- \, c_d\Omega_\theta(D_7 - c_d \, \Omega_\theta \, \theta_c)/\lambda + (D_6 - c_d \, \Omega_\theta \, \theta_s)]/D_5 \qquad (5.94)$$

where

$$D_5 = [1 + (2 \, c_d\Omega_\theta/\lambda)^2] \qquad (5.95)$$

$$D_6 = - \, [\theta_c(\Omega_\theta^2 - \lambda^2/4) + 0.5 \, \Omega_\zeta^2\beta N_t\theta_c + 9 \, \pi^2\Omega_\zeta^2 \, I_R \, (\theta_c^3 + \theta_s^2\theta_c)/(I_{pc}\ell^2) + 2 \, \Omega_\zeta^2(2\zeta_a\theta_c + \zeta_c\theta_c + \zeta_s\theta_s)/3]/\lambda \qquad (5.96)$$

$$D_7 = [\theta_s(\Omega_\theta^2 - \lambda^2/4) - 0.5 \, \Omega_\zeta^2\beta N_t\theta_s + 9 \, \pi^2\Omega_\zeta^2 \, I_R \, (\theta_s^3 + \theta_c^2\theta_s)/(I_{pc}\ell^2) + 2 \, \Omega_\zeta^2(2\zeta_a\theta_s - \zeta_c\theta_s - \zeta_s\theta_c)/3]/\lambda \qquad (5.97)$$

Analytical solutions of equations (5.86) to (5.88), (5.93) and (5.94) are difficult to obtain. Therefore, numerical solutions are adopted. Numerical integration procedure was carried out for the time interval (π/λ) for the case of a thin-walled beam of symmetrical I section. The geometrical properties of the member are as given by equations (5.50) to (5.53). Arbitrary initial conditions were assumed, $\theta_c = 0$, $\theta_s = 0.02$. Constant values of the load parameter $N_0 = 0$, the parametric amplitude $N_t/2 = 0.05$ and the parametric frequency $(\lambda/2\Omega_\theta)^2 = 1$, are considered.

Solutions of equations (5.86) to (5.88), (5.93) and (5.94) for different values of damping coefficients, were carried out. The time history of the behaviour of the transient amplitude of parametric torsional oscillation θ_n is plotted in figure [15]. In the absence of damping, if the parametric frequency is within the unstable range, the amplitude of parametric torsional oscillations grows almost exponentially from an initial value to oscillate about the value of the steady state amplitude. This behaviour is shown in figure [15] by the curve denoted by $c_d = 0$. The presence of damping coefficients decreases the rate of initial growth of the amplitude. Three cases can be discussed for different values of the damping coefficient. In the presence of relatively small damping coefficients, $c_d = 0.01$, the amplitude of oscillations initially grows, then damped vibrations take place. The damped oscillations converge to the amplitude of steady state oscillations. For a larger value of the damping coefficient, $c_d = 0.02$ in this case, the amplitude of torsional oscillations grows to take the value of the steady state solution. However, for a relatively large

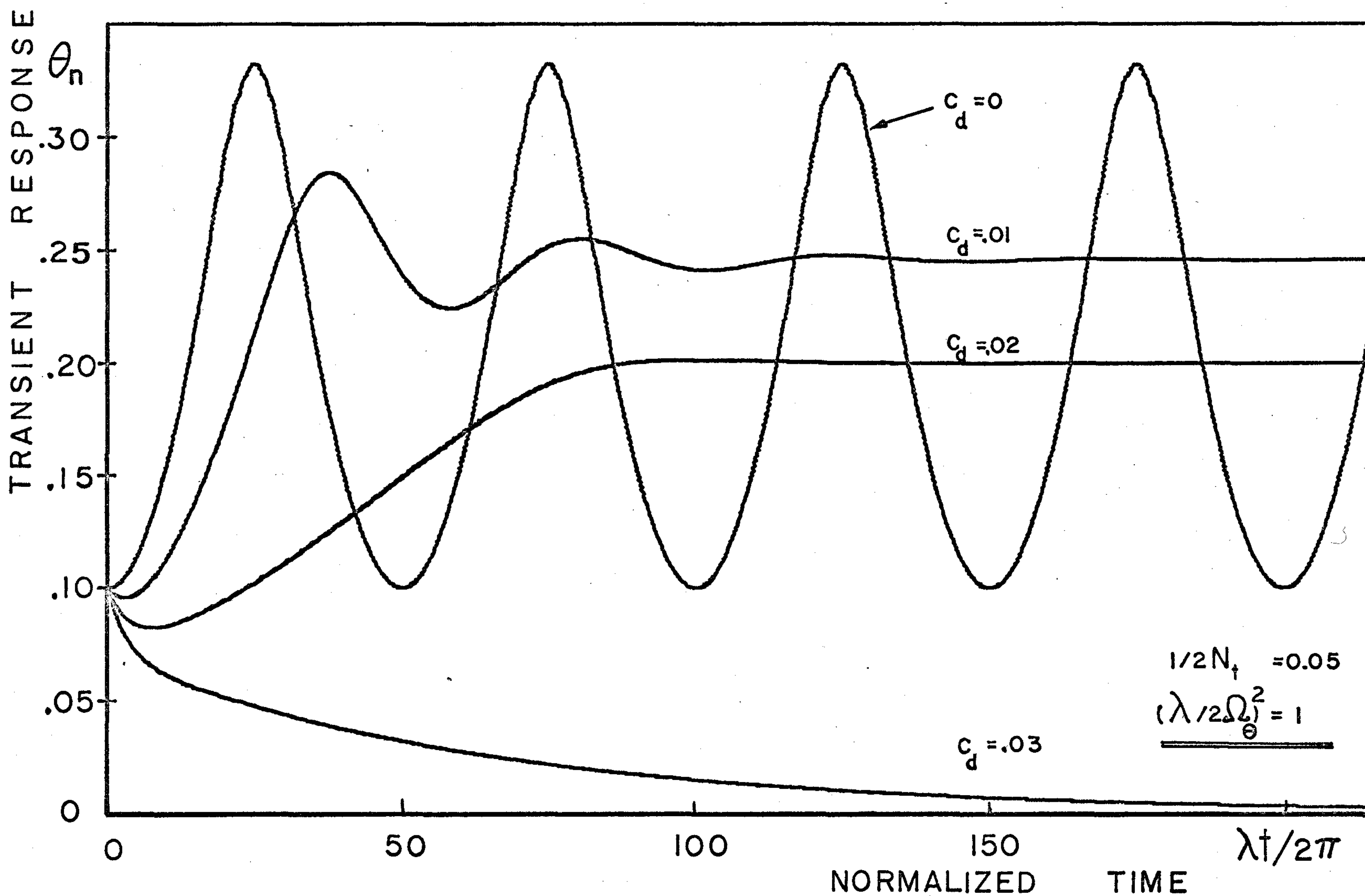

FIG. (15) TRANSIENT TORSIONAL RESPONSE OF SYMMETRICAL I BEAM viscous damping, $c_d = 0.03$, the same value of the parametric frequency $(\lambda/2\Omega_\theta)^2 = 1$, and parametric amplitude $N_t/2 = 0.05$, represent a point outside the unstable region, refer to figures [13] and [14]. The transient solution for this case decreases continuously from an arbitrary initial value to become asymptotic to the stable trivial solution $\theta_n = 0$.

To summarize the parametric response of a thin-walled beam of symmetric I section and to correlate between solutions for the boundaries of parametric instability regions, steady state and transient responses, figure [16] is presented. Figure [16] represents the parametric torsional response for a fixed value of the damping coefficient. For the constant load parameter $N_0 = 0$ the boundaries of the principal region of instability are plotted in the upper left corner of the figure. For a parametric amplitude of $N_t/2 = 0.05$, the steady state amplitude of parametric torsional oscillations are shown in the lower left corner of the graph. Solid lines represent the stable solution while the unstable solution is plotted in dotted lines. The nonsteady state response, for arbitrary initial values, is shown on the right side of figure [16] for parametric frequency of $(\lambda/2\Omega_\theta)^2 = 1$. To follow the parametric response, the parametric amplitude of 0.05 and

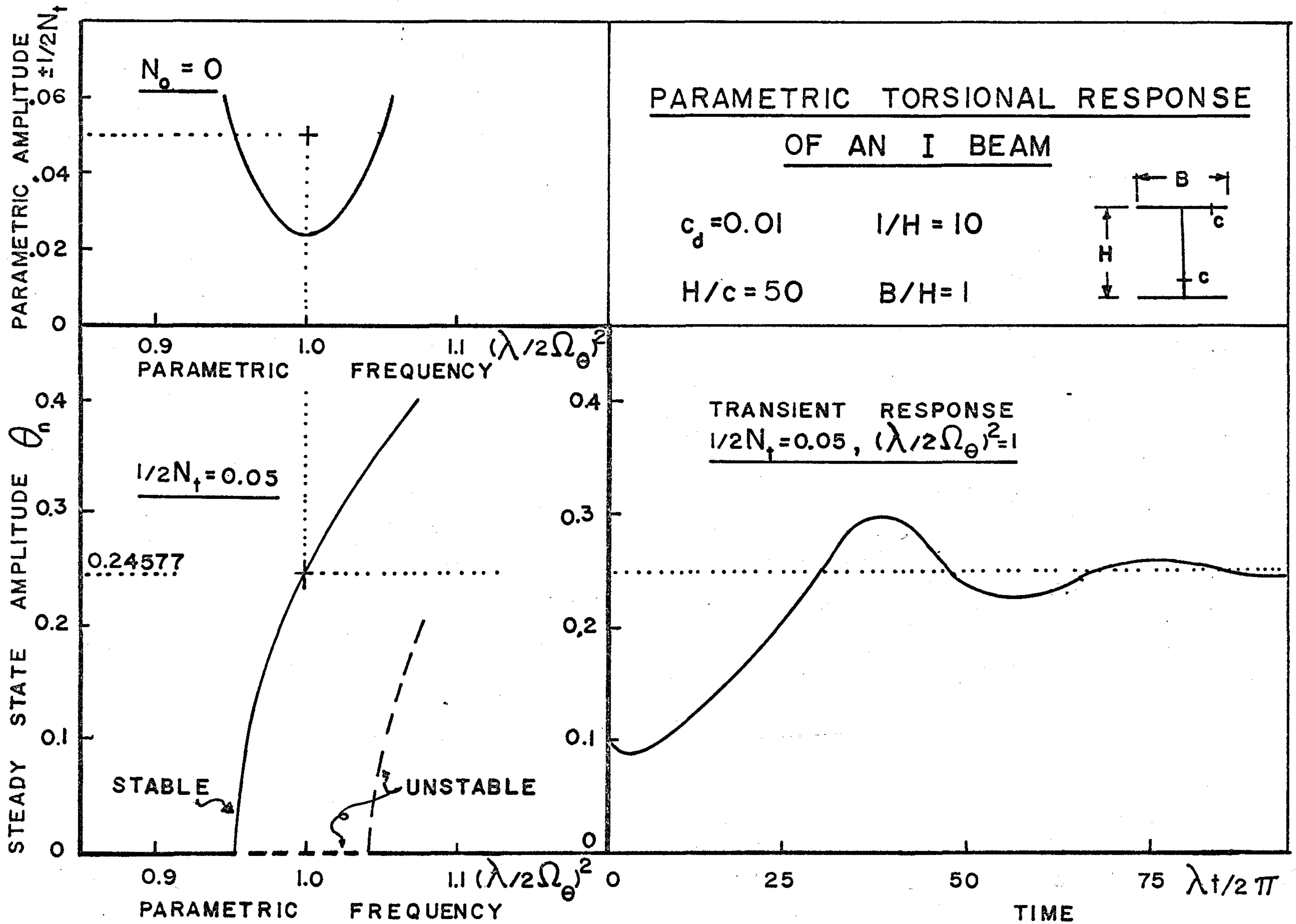

FIG. (16) PARAMETRIC RESPONSE AT THE PRINCIPAL REGION OF TORSIONAL INSTABILITY FOR AN I BEAM.

parametric frequency of 1, represent a point inside the region of parametric torsional instability, thus, the system is unstable. The steady state amplitude of torsional oscillations is determined by projecting the point (1, 0.05) from the stability chart to meet the stable branch of the solution at the magnitude of the steady state amplitude (0.24577). The transient response is shown to grow from an initial value, then damped oscillations take place. The oscillations converge to the steady state amplitude of parametric torsional oscillations. Quantitiative results can be easily obtained, then plotted in graphs similar to figure [16] for different values of N_0, parametric amplitude $(N_t/2)$, parametric frequency $(\lambda/2\Omega_\theta)^2$ and damping coefficient c_d.

CHAPTER VI

DYNAMIC STABILITY OF THIN-WALLED BEAM OF MONOSYMMETRICAL CROSS SECTION

6.1. Introduction

The dynamic stability of thin-walled beams of monosymmetrical cross section is presented in this chapter. As an example of monosymmetrical section, a split ring section is considered. Under the action of periodic end loading a state of longitudinal vibration is set up in the beam as a direct consequence of the applied load. In general, there is no flexural or torsional deformation. However, at certain frequency ranges of the applied load flexural and/or torsional vibrations are set up. Under such conditions, the beam is said to be "parametrically unstable". The present example differs from the symmetrical I section case, which was presented in Chapter V, in two aspects. Firstly, the case of a built-in boundary condtion is adopted. Secondly, coupled flexural-torsional type of vibrations will take place. This fundamental difference in behaviour is due to the fact that for a monosymmetrical section the shear center does not coincide with the centroid. Therefore, coupled flexural-torsional type of vibrations will result, once parametric instability sets in. The parametric stability study of thin-walled beam, in this chapter, consists of two parts. The first part is to determine the range of frequencies at which such instability can take place. This can be expressed as unstable regions in the parametric amplitude-parametric frequency space. The second part of the parametric stability study consists of establishing the steady state amplitudes of vibration and investigating the transient growth of the parametric response.

6.2. Differential Equations

Consider a thin-walled beam of monosymmetrical cross section where the axis of symmetry is taken to be the OY axis. The beam is subjected to dynamic axial forces at the ends of the beam. It is assumed that the resultant of the applied forces passes through the centroid of the cross section. Due to the applied end load the thin-walled beam will deform in the longitudinal direction while no flexural or torsional deformation will take place. However, if the magnitude and frequency of the end loading corresponds to a point in the unstable regions, the beam will show flexural deformation in the principal direction of symmetry or a combined flexural deformation in the other principal direction and torsional deformation. The geometrical properties of monosymmetrical cross section will simplify the differential equations, as

$$c_x = 0 \qquad (6.1)$$

and

$$I_{Rx} = I_{R\omega} = I_{\Omega x} = I_{\Omega\omega} \tag{6.2}$$

The longitudinal deformation in the ζ direction is given by equation (4.31)

$$E^*A\zeta'' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta')' - \rho A\ddot{\zeta} = 0 \tag{6.3}$$

The associated non-homogeneous boundary conditions are

$$\{E^*A\zeta' + E^*(I_{pc}\theta'^2 + I_{\Omega c}\theta^2\theta') - (P_0 + P_t \cos \lambda t)\} \delta\zeta\Big|_0^{\ell} = 0 \tag{6.4}$$

where the resultant of the applied forces is considered to be a periodic function which can be expressed in the form

$$P(t) = P_0 + P_t \cos \lambda t \tag{6.5}$$

Using equations (4.22) to (4.24) the flexural and torsional stability equations for a monosymmetrical section are written as

$$E^*I_{yy}\xi^{iv} - (P_0 + P_t \cos \lambda t)(\xi'' + c_y\theta'') + \rho A\ddot{\xi} + \rho A c_y\ddot{\theta} = 0 \tag{6.6}$$

$$E^*I_{xx}\eta^{iv} - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')'' - (P_0 + P_t \cos \lambda t)[\eta'' + c_y(\theta\theta'' + \theta'^2)] + \rho A[\ddot{\eta} + c_y(\theta\ddot{\theta} + \dot{\theta}^2)] = 0 \tag{6.7}$$

$$E^*I_{\omega\omega}\theta^{iv} - 6\,E^*I_R\theta'^2\theta'' - 2\,E^*[\theta'(I_{pc}\zeta' - I_{Ry}\eta'')]'$$

$$- GI_d\theta'' + 2\,\rho I_{pc}\ddot{\theta}$$

$$- (P_0 + P_t \cos \lambda t)c_y(\bar{\xi}'' + \bar{\eta}''\theta)$$

$$+ \rho A c_y(\ddot{\bar{\xi}} + \ddot{\bar{\eta}}\theta) = 0 \qquad (6.8)$$

The associated boundary conditions are

$$\xi''\delta\xi'\Big|_0^\ell = 0 \qquad (6.9)$$

$$\{E^*I_{yy}\xi''' - (P_0 + P_t \cos \lambda t)(\xi' + c_y\theta')\}\ \delta\xi\Big|_0^\ell = 0$$

$$(6.10)$$

$$\{E^*I_{xx}\eta'' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')\}\ \delta\eta'\Big|_0^\ell = 0 \quad (6.11)$$

$$\{E^*I_{xx}\eta''' - E^*(I_{Ry}\theta'^2 + I_{\Omega y}\theta^2\theta')'$$

$$- (P_0 + P_t \cos \lambda t)(\eta' + c_y\theta\theta')\}\ \delta\eta\Big|_0^\ell = 0 \quad (6.12)$$

$$\{E^*I_{\omega\omega}\theta'' - M_\Omega\theta^2\}\ \delta\theta'\Big|_0^\ell = 0 \qquad (6.13)$$

$$\{E^*I_{\omega\omega}\theta''' - 2\,E^*I_R\theta'^3 - 2\,E^*\theta'(I_{pc}\zeta' - I_{Ry}\eta'')$$

$$- GI_d\theta' - 2\,M_\Omega\theta\theta'$$

$$- (P_0 + P_t \cos \lambda t)c_y(\bar{\xi}' + \bar{\eta}'\theta)\}\ \delta\theta\Big|_0^\ell = 0 \quad (6.14)$$

The boundary conditions of the ζ equation as given by (6.4) are time dependent. It is convenient to modify the differential equations so that a new variable ζ_1 is required to satisfy homogeneous boundary conditions only. This can be done by the Mindlin and Goodman's technique.

Let a new variable ζ_1 be defined as

$$\zeta_1(z,t) = \zeta(z,t) - (P_0 + P_t \cos \lambda t)z/E^*A \quad (6.15)$$

where

$$\zeta_1(0,t) = 0 \quad (6.16)$$

$$\zeta_1'(\ell,t) = 0 \quad (6.17)$$

Introducing equation (6.15) into equation (6.3) and (6.8) a modified set of differential equations is obtained.

For fixed end conditions, the boundary conditions of the thin-walled beam can be expressed mathematically as

at $z = 0$

$$\zeta_1 = \xi = \xi' = \eta = \eta' = \theta = \theta' = 0 \quad (6.18)$$

at $z = \ell$

$$\zeta_1' = \xi = \xi' = \eta = \eta' = \theta = \theta' = 0 \quad (6.19)$$

It is convenient at this stage to nondimensionalize the problem. The applied force is expressed as a factor less than or equal to 1, multiplied by the smallest static buckling load. The frequencies are best normalized in terms of the frequencies of free vibrations.

Let the static flexural buckling load in the uncoupled η direction be P_η and the coupled flexural torsional buckling loads be denoted as P_1 and P_2. Mathematically they can be expressed as

$$P_\eta = 4\,\pi^2\, E^* I_{xx}/\ell^2 \qquad (6.20)$$

$$P_{1,2} = [-\bar{D}_2 \pm (\bar{D}_2 - 4\,\bar{D}_1\bar{D}_3)^{1/2}]/(2\bar{D}_1) \qquad (6.21)$$

where the constants $\bar{D}_1$, $\bar{D}_2$ and $\bar{D}_3$ are

$$\bar{D}_1 = c_y^2\, A/(2I_{pc}) - 1 \qquad (6.22)$$

$$\bar{D}_2 = 4\,\pi^2\, E^* I_{yy}/\ell^2 + A(4\,\pi^2\, E^* I_{\omega\omega}/\ell^2 + GI_d)/ (2I_{pc}\ell^2) \qquad (6.23)$$

$$\bar{D}_3 = 4\,\pi^2\, E^* A I_{yy}(4\,\pi^2 E^* I_{\omega\omega}/\ell^2 + GI_d)/ (2I_{pc}\ell^2) \qquad (6.24)$$

In general, P_1 is the critical load that corresponds to the buckling mode whose torsional deformation is predominant. This is called "predominant torsional" mode. P_2 is the critical load that corresponds to the buckling mode whose flexural deformation is predominant. This is called "predominant flexural" mode.

The applied loads will then be expressed in the form

$$E^* A\beta(N_0 + N_t \cos \lambda t) = P_0 + P_t \cos \lambda t \qquad (6.25)$$

where

$$\beta = P_{cr}/(E^*A) \qquad (6.26)$$

The axial load parameters N_0 and N_t are nondimensionalized with respect to the smallest buckling load P_{cr} in the weaker mode. $N_0 = -1$ corresponds to an applied axial compressive load which will cause buckling in the weaker mode.

The frequency of free vibrations, under a constant applied axial load P_0, in the uncoupled longitudinal and flexural type of vibrations can be written as

$$\Omega_\zeta^2 = \pi^2 E^*/(4\ell^2\rho) \qquad (6.27)$$

$$\Omega_\eta^2 = (1 + P_0/P_{cr})\, 16\, \pi^4 E^* I_{xx}/(3\rho A\ell^4) \qquad (6.28)$$

The frequency of free vibration, under a constant applied axial load P_0, of the coupled flexural-torsional types of oscillations, can be written as

$$\Omega_{1,2} = (-D_2^* \pm D_3^*)/(2\bar{D}_1) \qquad (6.29)$$

where the constant $\bar{D}_1$ is given by equation (6.22), D_2^* and D_3^* are given as

$$D_2^* = d_1 + d_0 + 2\, d_0\, \bar{D}_1 \qquad (6.30)$$

$$D_3^* = (D_2^{*2} - 4\, \bar{D}_1 D_4^*)^{1/2} \qquad (6.31)$$

$$D_4^* = -(d_1 - d_0)(d_2 - d_0) + c_y^2\, A\, d_o^2/(2I_{pc}) \qquad (6.32)$$

$$d_0 = 4\,\pi^2 P_0/(3\rho A\ell^2) \qquad (6.33)$$

$$d_1 = 16\,\pi^4 E^* I_{yy}/(3\rho A\ell^4) \qquad (6.34)$$

$$d_2 = 2\,\pi^2(4\,\pi^2 E^* I_{\omega\omega}/\ell^2 + GI_d)/(3\rho\ell^2 I_{pc}) \qquad (6.35)$$

Ω_1 and Ω_2 are frequencies that correspond to a coupled flexural-torsional type of vibration. The mode shape corresponding to one of the frequencies, say Ω_1, is characterized by predominant torsional deformations while Ω_2 corresponds to a mode shape of predominant flexural characteristic. In general, for a thin-walled beam of open cross section, the torsional rigidity is small compared with the flexural rigidity, thus $\Omega_1 < \Omega_2$.

Consider the nondimensionalized variables to be

$$\zeta^* = \zeta_1/\ell \qquad (6.36)$$

$$\xi_1 = \xi/\ell \qquad (6.37)$$

$$\eta_1 = \eta/\ell \qquad (6.38)$$

$$\bar{z} = z/\ell \qquad (6.39)$$

The partial differential equations given by equations (6.3) and (6.6) to (6.8) can be changed to ordinary differential equations by applying Galerkin's averaging technique to the equations.

As our interest is directed to the first mode of oscillations in each type of deformation, approximate mode shapes are used. The approximate first mode shape functions are taken to be

$$\zeta^*(\bar{z},t) = \zeta_t(t) \sin \pi\bar{z}/2 \tag{6.40}$$

$$\xi_1(\bar{z},t) = \xi_t(t) (1 - \cos 2\pi\bar{z}) \tag{6.41}$$

$$\eta_1(\bar{z},t) = \eta_t(t) (1 - \cos 2\pi\bar{z}) \tag{6.42}$$

$$\theta(\bar{z},t) = \theta_t(t) (1 - \cos 2\pi\bar{z}) \tag{6.43}$$

where ζ_t, ξ_t, η_t and θ_t are functions of time only.

The first natural frequency of different types of vibrations, as calculated using the approximate mode shape functions given by equations (6.40) to (6.43) are compared to the frequency values calculated using a four term "beam type" mode function. The difference was found to be of the order of 0.4%. The approximations in mode shapes as expressed in (6.40) to (6.43) are, therefore, regarded as reasonable.

Substituting the solutions (6.40) to (6.43) into the partial differential equations (6.3) and (6.6) to (6.8) and applying Galerkin's averaging technique, a set of ordinary differential equations is obtained

$$\ddot{\zeta}_t + 2c_d\Omega_\zeta\dot{\zeta}_t + \Omega_\zeta^2\zeta_t - (8\beta N_t\lambda^2 \cos \lambda t)/\pi^2$$
$$+ 1024\ I_{pc}\Omega_\zeta^2\theta_t^2/(63A\ell^2) - 11\ I_{\Omega c}\Omega_\zeta^2\theta_t^3/(3\pi A\ell^2) = 0 \tag{6.44}$$

$$\ddot{\xi}_t + c_y\ddot{\theta}_t/\ell + 2c_d\Omega_2\dot{\xi}_t + d_1\xi_t$$
$$+ 4\pi^2E^*\beta(N_0 + N_t \cos \lambda t)(\xi_t + c_y\theta_t/\ell)/(3\rho\ell^2) = 0 \tag{6.45}$$

$$\ddot{\eta}_t + 2c_d\Omega_\eta\dot{\eta}_t + \Omega_\eta^2\eta_t$$
$$+ 4\pi^2E^*\beta(N_0 + N_t \cos \lambda t)(\eta_t + c_y\theta_t^2/\ell)/(3\rho\ell^2)$$
$$- 5c_y[\theta_t\ddot{\theta}_t + \dot{\theta}_t^2]/(3\ell) = 0 \tag{6.46}$$

$$\ddot{\theta}_t + c_yA\ell\ddot{\xi}_t/(2I_{pc}) + 2c_d\Omega_1\dot{\theta}_t + d_2\theta_t$$
$$+ 4\pi^2E^*\beta(N_0 + N_t \cos \lambda t)[\theta_t + c_yA\ell\xi_t/(2I_{pc})$$
$$]/(3\rho\ell^2) + 4\pi^4E^*I_R\theta_t^3/(\rho I_{pc}\ell^4)$$
$$+ 256\pi^2E^*\theta_t\zeta_t/(189\rho\ell^2) = 0 \tag{6.47}$$

where c_d is the damping coefficient known as the fractional critical damping. The damping terms were introduced in the ordinary differential equations (6.44) to (6.47) after Galerkin's technique was applied. The effect on damping on the frequency of free vibration can be neglected as $c_d << 1$.

The linear terms of the ordinary differential equation (6.44) including damping terms, are written as

$$\ddot{\zeta}_t + 2\ c_d\ \Omega_\zeta\ \dot{\zeta}_t + \Omega_\zeta^2\ \zeta_t = (8\beta N_t \lambda^2 \cos \lambda t)/\pi^2 \qquad (6.48)$$

Equation (6.46) is in the standard form of the forced vibration equations. The linear terms of the parametric stability equations are

$$\ddot{\xi}_t + c_y \ddot{\theta}_t/\ell + 2\ c_d\ \Omega_2\ \dot{\xi}_t + d_1\ \xi_t$$

$$+ 4\pi^2 E^* \beta (N_0 + N_t \cos \lambda t)(\xi_t + c_y \theta_t/\ell)/(3\rho\ell^2) = 0$$

$$(6.49)$$

$$\ddot{\eta}_t + 2\ c_d\ \Omega_\eta\ \dot{\eta}_t + \Omega_\eta^2\ \eta_t$$

$$+ 4\pi^2 E^* \beta N_t \cos \lambda t\ \eta_t/(3\rho\ell^2) = 0 \qquad (6.50)$$

$$\ddot{\theta}_t + c_y A\ell\ddot{\xi}_t/(2I_{pc}) + 2\ c_d\ \Omega_1\ \dot{\theta}_t + d_2\theta_t$$

$$+ 4\pi^2 E^* \beta (N_0 + N_t \cos \lambda t)[\theta_t + c_y A\ell\xi_t/(2I_{pc})]/$$

$$(3\rho\ell^2) = 0 \qquad (6.51)$$

Equations (6.49) to (6.51) are of variable coefficients which is a mathematical characteristic of the parametric stability problems.

6.3. Boundaries of the Principal Regions of Parametric Instability

For a thin-walled beam of monosymmetrical cross section, there exists three types of oscillations. Namely, the uncoupled flexural, the coupled flexural-torsional type of vibrations of predominant torsional characteristics and the coupled flexural-torsional type of oscillations of predominant flexural characteristics. For each type of vibration, there exists an infinite number of frequencies of free vibration corresponding to an infinite number of mode shapes. There is a region of principal parametric instability corresponding to each frequency of free vibrations. If only the first mode of each type of vibration is considered, there exists three principal regions of parametric instability for a thin-walled beam of monosymmetrical cross section. The three principal regions will be distinguished by the type of vibration they correspond to. Namely, flexural, coupled flexural-torsional of predominant torsional or flexural characteristics. In addition, there exists regions of parametric instability corresponding to combinations of frequencies of free vibrations of the system. However, this type of "combination parametric instability" will not be considered in this thesis. The present analysis will be limited to the study of the principal regions of parametric instability.

To establish the boundaries of the principal regions of parametric instability, only the linear equations of stability are used.

The equations governing the parametric stability of thin-walled beams of monosymmetrical cross section (6.49) to (6.51) can be written in the form

$$[D]\{\ddot{f}\} + [C_d]\{\dot{f}\} + ([E] - \beta(N_0 + N_t \cos \lambda t)[B])\{f\} = 0 \tag{6.52}$$

$$\{f\} = \begin{Bmatrix} \xi_t \\ \eta_t \\ \theta_t \end{Bmatrix} \tag{6.53}$$

$$[C_d] = 2\, c_d \begin{bmatrix} \Omega_2 & 0 & 0 \\ 0 & \Omega_n & 0 \\ 0 & 0 & \Omega_1 \end{bmatrix} \tag{6.54}$$

$$[D] = \begin{bmatrix} 1 & 0 & c_y/\ell \\ 0 & 1 & 0 \\ c_y A\ell/(2I_{pc}) & 0 & 1 \end{bmatrix} \tag{6.55}$$

$$[E] = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & 16\pi^4 E^* I_{xx}/(3\rho A\ell^4) & 0 \\ 0 & 0 & d_2 \end{bmatrix} \tag{6.56}$$

$$[B] = - 4\pi^2 E^*[D]/(3\rho\ell^2) \tag{6.57}$$

Equation (6.50) is an uncoupled differential equation representing parametric stability of the uncoupled flexural type of vibrations. The boundaries of the principal region of parametric flexural instability can be written as

$$[\Omega_\eta^2 - 2\pi^2 E^*\beta N_t/(3\rho\ell^2) - \lambda^2/4][\Omega_\eta^2 + 2\pi^2 E^*\beta N_t/(3\rho\ell^2) - \lambda^2/4] + (c_d\lambda\Omega_\eta)^2 = 0 \tag{6.58}$$

Using the method suggested by Bolotin [6], the boundaries of principal regions of the coupled flexural-torsional types of parametric oscillations can be written using conditions(5.60) as

$$\begin{vmatrix} K_1 & c_y\mu/\ell & -c_d\Omega_2\lambda & 0 \\ c_yA\ell\mu/(2I_{pc}) & K_2 & 0 & -c_d\Omega_1\lambda \\ c_d\Omega_2\lambda & 0 & \bar{K}_1 & c_y\bar{\mu}/\ell \\ 0 & c_d\Omega_1\lambda & c_yA\ell\bar{\mu}/(2I_{pc}) & \bar{K}_2 \end{vmatrix} = 0 \tag{6.59}$$

where

$$K_1 = \mu + 16\pi^4 E^* I_{yy}/(3\rho A\ell^4) \tag{6.60}$$

$$\bar{K}_1 = K_1 - \mu + \bar{\mu} \tag{6.61}$$

$$\mu = 4\pi^2 E^*(N_0 + N_t/2)/(3\rho\ell^2) - \lambda^2/4 \tag{6.62}$$

$$\bar{\mu} = 4\pi^2 E^*(N_0 - N_t/2)/(3\rho\ell^2) - \lambda^2/4 \qquad (6.63)$$

$$K_2 = \mu + 2\pi^2(4\pi^2 E^* I_{\omega\omega}/\ell^2 + GI_d)/(3\rho\ell^2 I_{pc})$$

$$\bar{K}_2 = K_2 - \mu + \bar{\mu} \qquad (6.65)$$

The condition (6.59) for the boundaries of the principal regions of coupled flexural-torsional type of parametric oscillations, can be solved numerically by trial and error method of analysis. The boundaries of the principal regions can thus be constructed in the parametric amplitude-parametric frequency space. It should be pointed out that there is NO GUARANTEE that the boundaries, determined by Bolotin's method (6.59) are the actual boundaries of unstable regions for coupled Mathieu equations (6.52). It remains to show that in the present example, the boundaries obtained by condition (6.59) are in fact the boundaries of an unstable region. This can be accomplished by a more detailed study of the behaviour of the solutions inside the determined region.

To be specific, consider a thin-walled beam of monosymmetrical split ring section as shown in figure [17]

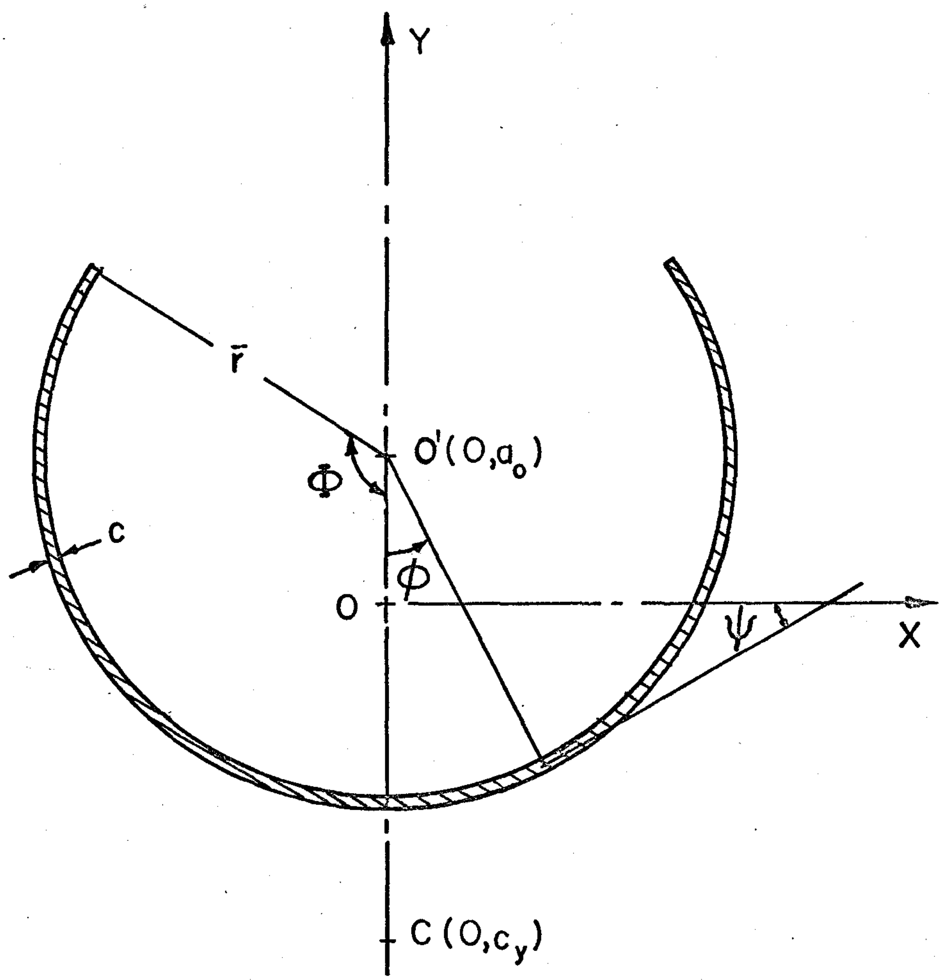

FIG. (17) SPLIT RING SECTION

of geometrical dimensions

$$c/\bar{r} = 0.065 \tag{6.66}$$

$$\ell/\bar{r} = 36 \tag{6.67}$$

$$\Phi = \pi/2 \tag{6.68}$$

where c is the thickness of the thin wall

ℓ is the length of the beam

$\bar{r}$ is the radius of the ring

Φ is the semi-central angle in radians

The geometrical properties of the cross section are calculated using the formulas given in Appendix A.

Numerical substitution in conditions (6.58) and (6.59) will give the boundaries of the three principal regions. However, the interest will be directed towards the principal region of parametric instability of the coupled flexural torsional type of vibrations of the predominant torsional characteristics. This is due to the fact that the lowest frequency of free vibrations is that of the coupled flexural-torsional type of vibrations of the predominant torsional characteristics. Also, the amplitudes of oscillations within this unstable region are expected to be larger than the amplitudes of oscillations within the uncoupled flexural and coupled predominant flexural instability regions.

Limiting the discussion to the principal region of parametric instability of the predominant torsional characteristics is regarded as sufficient since no qualitative change in the behaviour is expected by studying the principal regions of flexural and predominant flexural parametric instability.

The solution of condition (6.59) is plotted as shown in figures [18] and [19]. The plots are in the parametric amplitude-parametric frequency space. Figure [18] represents the case when damping terms are absent $c_d = 0$, for different values of the parameter N_0. Figure [19] represents the effect of viscous damping on the principal region of parametric stability of the coupled flexural-torsional type of oscillations of predominant torsional characteristics, when $N_0 = 0$.

6.4. Steady State Amplitudes of Vibrations

Since the stability equations for ξ and θ are coupled Mathieu equations, the unstable region as determined by Bolotin's method need further verification. This can be achieved by studying the steady state amplitudes of vibrations near the principal regions. An unstable region will correspond to a nontrivial steady state solution near this region. To study the steady state equations, the nonlinear theory should be used. The four ordinary nonlinear differential equations (6.44) to (6.47) are recalled. The steady state amplitudes of oscillations can be obtained by

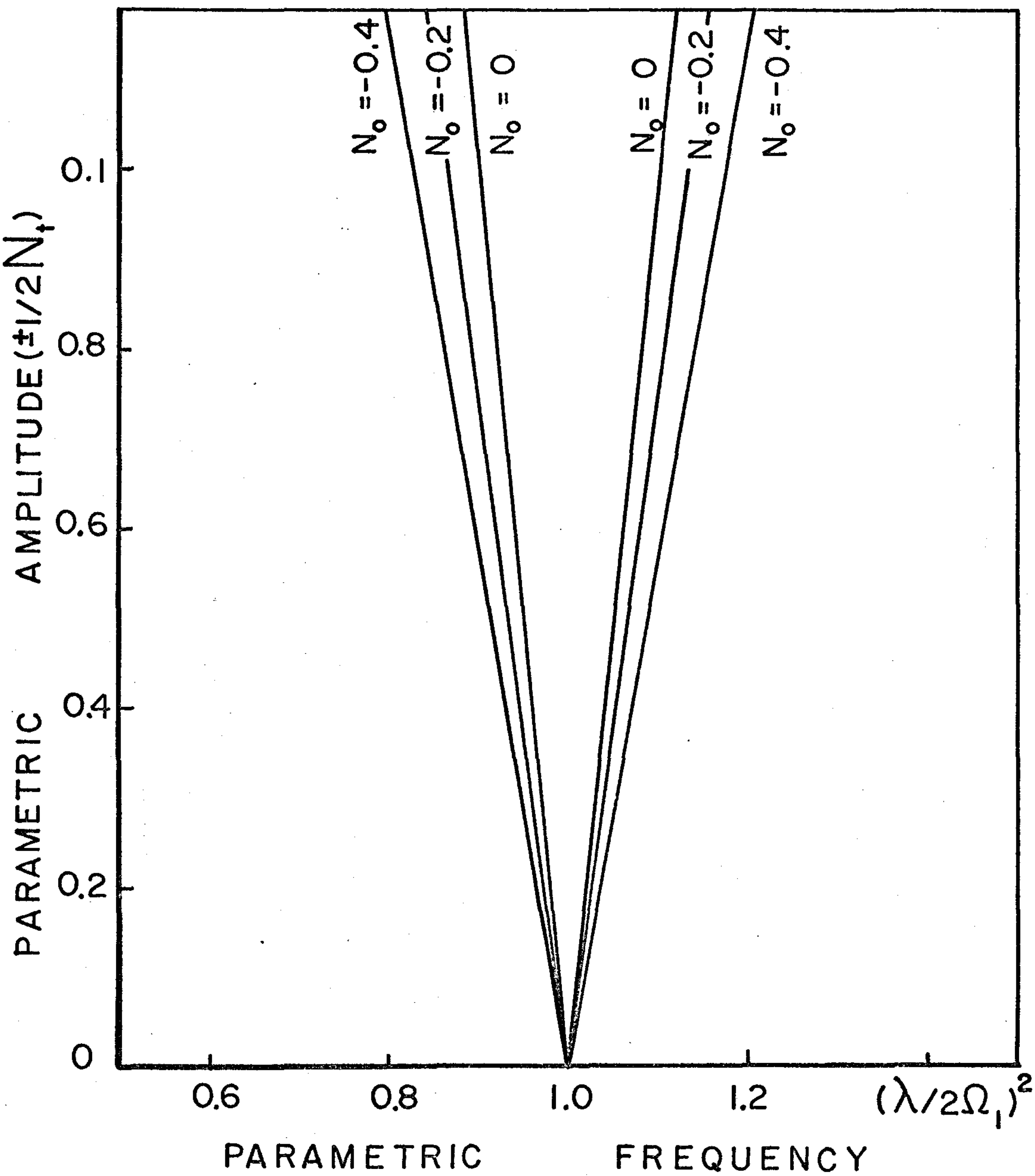

FIG. (18) PRINCIPAL INSTABILITY REGION FOR SPLIT RING SECTION (PREDOMINANT TORSION)

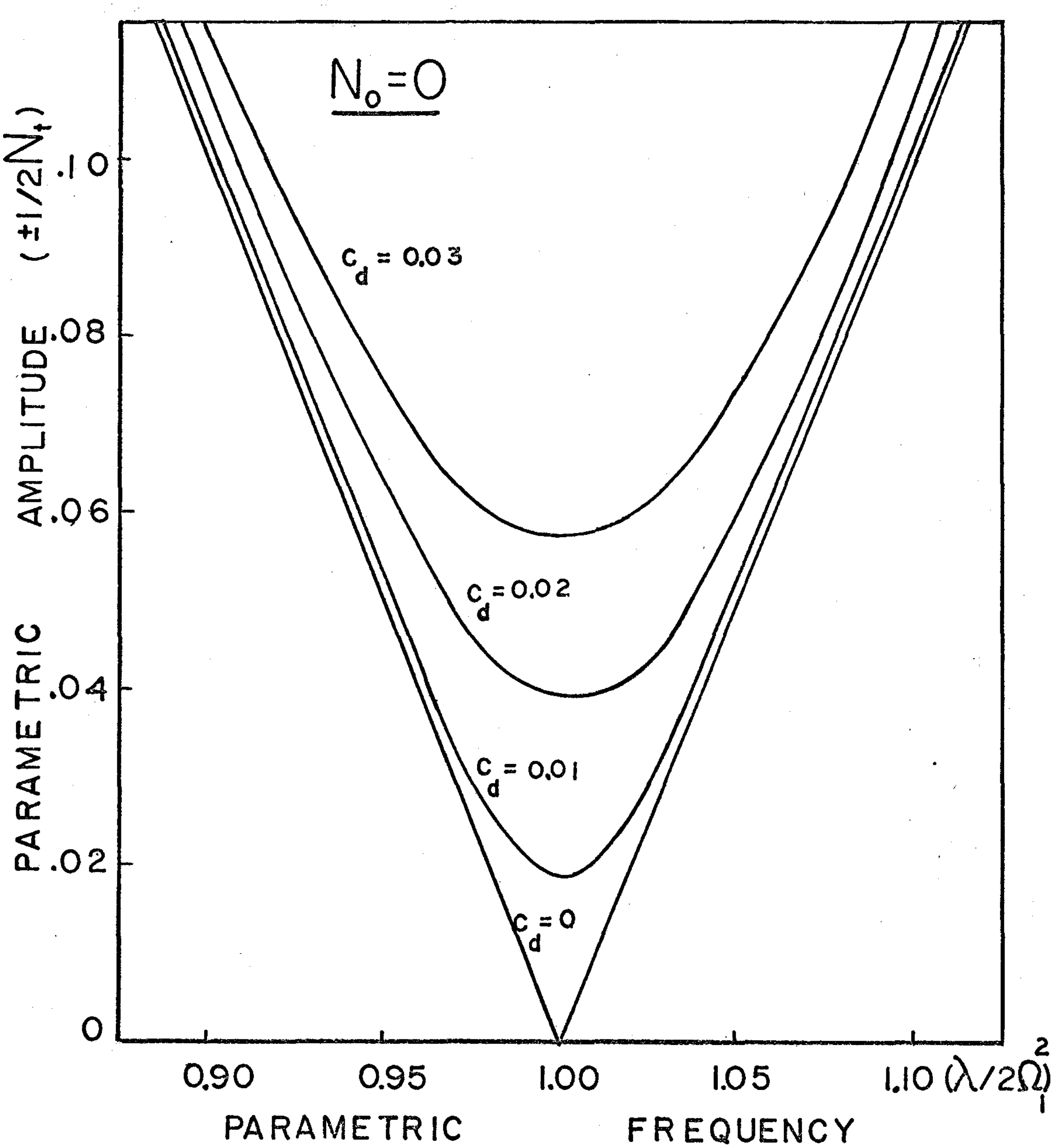

FIG. (19) EFFECT OF DAMPING ON THE PRINCIPAL INSTABILITY REGION FOR SPLIT RING SECTION (PREDOMINANT TORSION) seeking solutions of the approximate harmonic form (5.65) to (5.68).

Introducing the adopted solutions into the differential equations (6.44) to (6.47) and applying the harmonic balance technique, a set of nonlinear algebraic equations is obtained

$$\zeta_a = \bar{\alpha}_0 \theta_n^2 \tag{6.69}$$

$$\zeta_c = \bar{\alpha}_1 \theta_c \theta_s + \bar{\alpha}_2 + \bar{\alpha}_3(\theta_c^2 - \theta_s^2) \tag{6.70}$$

$$\zeta_s = \bar{\alpha}_4 \theta_c \theta_s + \bar{\alpha}_5 + \bar{\alpha}_6(\theta_c^2 - \theta_s^2) \tag{6.71}$$

$$\xi_c = \bar{\alpha}_7 \theta_c + \bar{\alpha}_8 \theta_s \tag{6.72}$$

$$\xi_s = \bar{\alpha}_9 \theta_c + \bar{\alpha}_{10} \theta_s \tag{6.73}$$

$$\theta_c = \theta_s = 0 \tag{6.74}$$

or

$$\theta_n^2 = [-\bar{\alpha}_{12} \pm (\bar{\alpha}_{12}^2 - 4\,\bar{\alpha}_{11}\bar{\alpha}_{13})^{1/2}]/(2\bar{\alpha}_{11}) \tag{6.75}$$

where the coefficients $\bar{\alpha}_0$, $\bar{\alpha}_i$, $i = 1,11$ are given in Appendix B, and

$$\theta_n^2 = \theta_c^2 + \theta_s^2 \tag{6.76}$$

The steady state amplitudes of the uncoupled flexural type of parametric vibrations in the η direction cannot be obtained using the approximate solutions (5.61). Higher order theory should be used. However, the order of approximations adopted can predict the amplitude of coupled longitudinal-torsional and flexural-torsional type of parametric oscillations.

Further discussions will be focused on the behaviour of the amplitude of coupled flexural-torsional vibrations. This is due to the fact that the first principal unstable region is that corresponding to the coupled flexural-torsional type of vibrations of the predominant torsional characteristics.

A trivial solution for θ as given by expression (6.74) implies trivial solutions for flexural deflections ξ and η also. A nontrivial solution for the amplitude of torsional oscillations as given by equation (6.75) implies nontrivial solutions for ξ and η as given by equations (6.72) and (6.73) due to the coupling between θ and flexural deflection terms.

The nontrivial solution for the amplitude of torsional oscillations θ_n(6.75) has two values. This means that for each value of the parametric frequency near the principal region there exist two solutions for the amplitude of torsional vibrations. However, studying the stability of the two solutions shows that one solution is stable while the other is unstable. The stability analysis of the steady state solutions will be discussed later. The existence of a stable solution for the steady state amplitudes of flexural-torsional types of vibrations in the principal regions shows that this region is in fact a principal region of parametric instability.

The numerical solutions of equation (6.75) are shown in figure [20]. The steady state amplitude of torsional vibrations near the principal region of predominant torsional instability is plotted versus the parametric frequency for a certain value of the parametric load ($N_t/2 = 0.05$). An open thin-walled split ring section of the geometrical dimensions (6.66) to (6.68) were used. The nontrivial solutions for the steady state amplitude, are plotted in the lower part of figure [20]. The stable branches of the solution are shown in solid lines while the unstable solutions are shown dotted.

The effect of damping on the steady state amplitude of oscillations is illustrated in figure [20]. Different values of the damping coefficients c_d are used. The presence of small viscous damping (c_d = .01, .02) reduces the magnitude of the steady state amplitude of oscillations. For large values of damping coefficients (e.g. c_d = .03) the system is not parametrically excited. Therefore, no steady state amplitude of oscillations is expected and the trivial solution (6.74) will hold.

To discuss the behaviour of the steady state amplitude for varying parametric frequency it is necessary to state

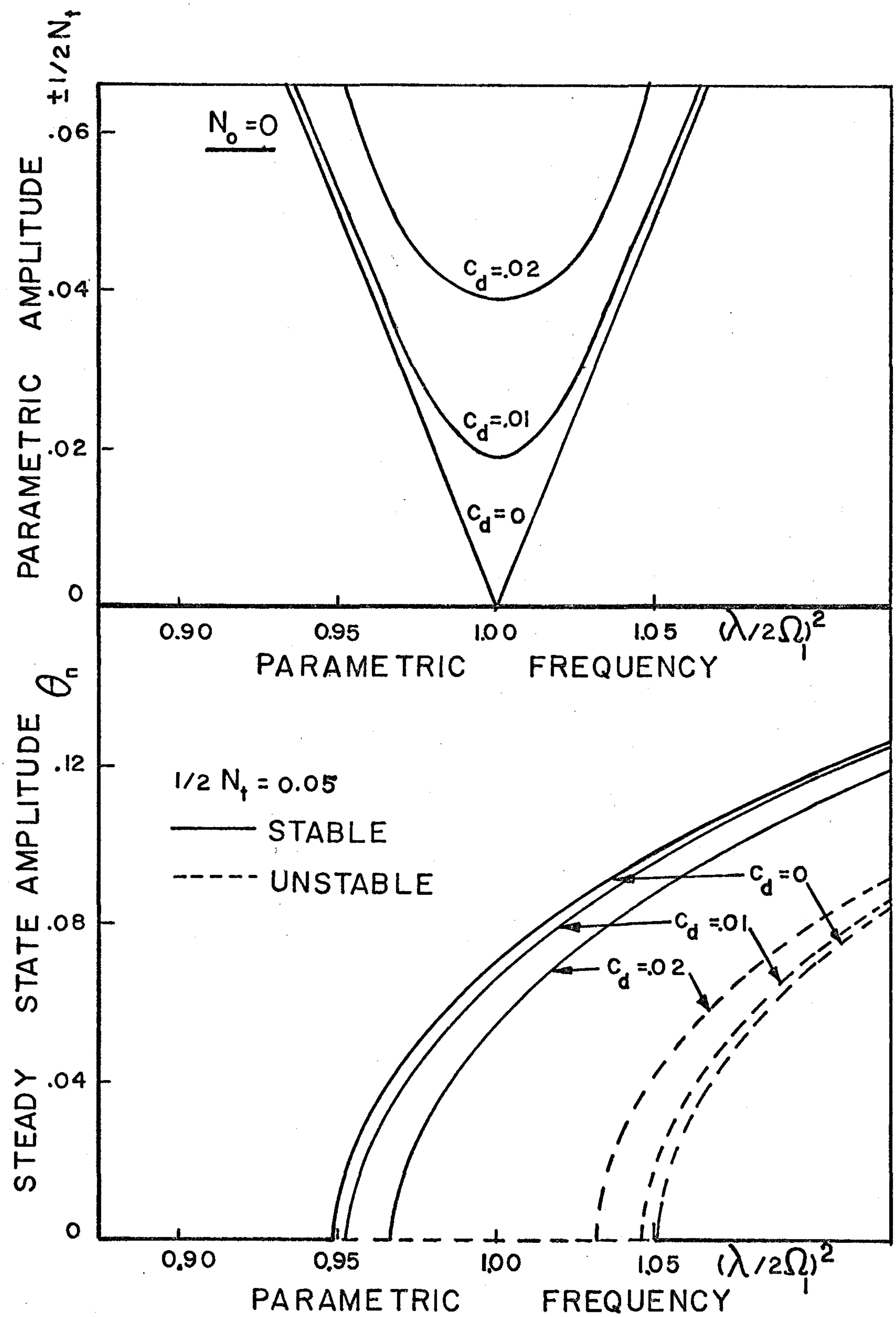

FIG. (20) STEADY STATE AMPLITUDE OF PREDOMINANT TORSIONAL VIBRATIONS FOR A SPLIT RING SECTION whether the unstable region is approached by increasing or decreasing parametric frequency for a constant value of the parametric amplitude. Starting from a frequency below the critical range the system is stable and the steady state amplitude will correspond to the trivial solution $\theta_n = 0$. As the parametric frequency enters the unstable range, the amplitude of torsional oscillations increases gradually. When the parametric frequency exits from the unstable range there is a sudden drop of the steady state amplitude to zero.

On the other hand, if the unstable region is approached by a decreasing parametric frequency different behaviour is predicted. For a parametric frequency above the critical range the trivial solution will hold indicating no amplitudes of vibrations. This shows that the system is parametrically stable. As the parametric frequency decreases until it reaches a point on the right boundary of the unstable region a sudden jump of the steady state amplitude will take place. As the parametric frequency decreases further the steady state amplitude of parametric torsional oscillations decreases gradually. Finally,as the parametric frequency exits from the unstable range, the trivial solution becomes a stable one.

6.5. Nonsteady State Solution

The time history of the amplitude growth at the principal region of instability is studied in this section. The growth of the amplitude of oscillations, from an initial value to the value of the steady state amplitude, is referred to as Nonsteady state or Transient solution. Transient solution can be obtained by allowing the amplitude of the assumed solutions to vary with time. This can be achieved by considering the solutions of the form as given in equations (5.82) to (5.85). In these solutions the amplitude components are taken as time dependent. It is assumed that the time dependent coefficients are slowly varying. In other words, the variation of the amplitude component over a complete cycle is considered to be small compared to the amplitude itself. Also, the variation of the first derivative of the amplitude over a complete cycle is considered to be small compared to the derivative of the amplitude component with respect to time.

Solutions (5.82) to (5.85) are introduced into the system of differential equations (6.44) to (6.47) and applying harmonic balance technique a set of first order nonlinear differential equations is obtained

$$\zeta_a = \bar{g}_1(\theta_c, \theta_s) \qquad (6.77)$$

$$\dot{\zeta}_c = \bar{g}_2(\zeta_c, \zeta_s, \theta_c, \theta_s) \tag{6.78}$$

$$\dot{\zeta}_s = \bar{g}_3(\zeta_c, \zeta_s, \theta_c, \theta_s) \tag{6.79}$$

$$\{\dot{f}\} = [M]^{-1} \{\bar{g}\} \tag{6.80}$$

where $\{\dot{f}\}$ is a vector of the first derivatives of the amplitude components

$$\{\dot{f}\} = \begin{Bmatrix} \dot{\xi}_c \\ \dot{\xi}_s \\ \dot{\theta}_c \\ \dot{\theta}_s \end{Bmatrix} \tag{6.81}$$

$$\{\bar{g}\} = \begin{Bmatrix} \bar{g}_4 \\ \bar{g}_5 \\ \bar{g}_6 \\ \bar{g}_7 \end{Bmatrix} \tag{6.82}$$

The functions $\bar{g}_i$, i = 1,7 are nonlinear functions of the amplitude components given in Appendix B.

$[M]^{-1}$ is the inverse of matrix $[M]$ where

$$[M] = \begin{bmatrix} 1 & -2c_d\Omega_2/\lambda & c_y/\ell & 0 \\ 2c_d\Omega_2/\lambda & 1 & 0 & c_y/\ell \\ c_yA\ell/(2I_{pc}) & 0 & 1 & -2c_d\Omega_1/\lambda \\ 0 & c_yA\ell/(2I_{pc}) & 2c_d\Omega_1/\lambda & 1 \end{bmatrix} \quad (6.83)$$

An analytical solution of equations (6.77) to (6.80) is rather difficult to attempt and, therefore, numerical solution is used. It should be noted, however, that if the first derivatives of the amplitude components (the left hand side of equations (6.77) to (6.80)) are taken to be zero, the steady state equations (6.69) to (6.75) are obtained. As an example to study the behaviour of the transient solution, consider a thin walled beam of dimensions monosymmetrical split ring section. The geometrical properties of the section are as given by equations (6.66) to (6.68). The geometrical properties of the cross section are as given by Appendix A.

The transient solution as given by equations (6.77) to (6.80) can be established for a given value of the constant component of the load N_0, parametric amplitude $N_t/2$ and parametric frequency $(\lambda/2\Omega_1)^2$. In addition, initial values of the nonsteady state amplitude components should be specified initially, $\theta_c = 0$, $\theta_s = 0.03$. The values of the parametric amplitude and frequency represent a point in the parametric amplitude - parametric frequency space. If this point is located outside the boundaries of the unstable regions the transient solution will decay rapidly and become asymptotic to the stable trivial solution. On the other hand, if the parametric amplitude and frequency represent a point in the unstable region, the transient solution shows the growth of the amplitude from an initial value to a value corresponding to the steady state amplitude of oscillation. The manner with which the amplitude approaches the steady state value depends on the damping of the system. Further discussions are limited to the behaviour of the transient solution of the torsional amplitude near the first principal region of parametric instability of the predominant torsional characteristics.

The behaviour of the nonsteady state amplitude of torsional oscillations is presented in figure [21]. Numerical solutions of the system of equations (6.77) to (6.80) are carried out for the constant load component $N_0 = 0$, parametric amplitude $N_t/2 = 0.05$ and parametric frequency $(\lambda/2\Omega_1)^2 = 1$. These values represent a point inside the principal region of parametric instability of predominant

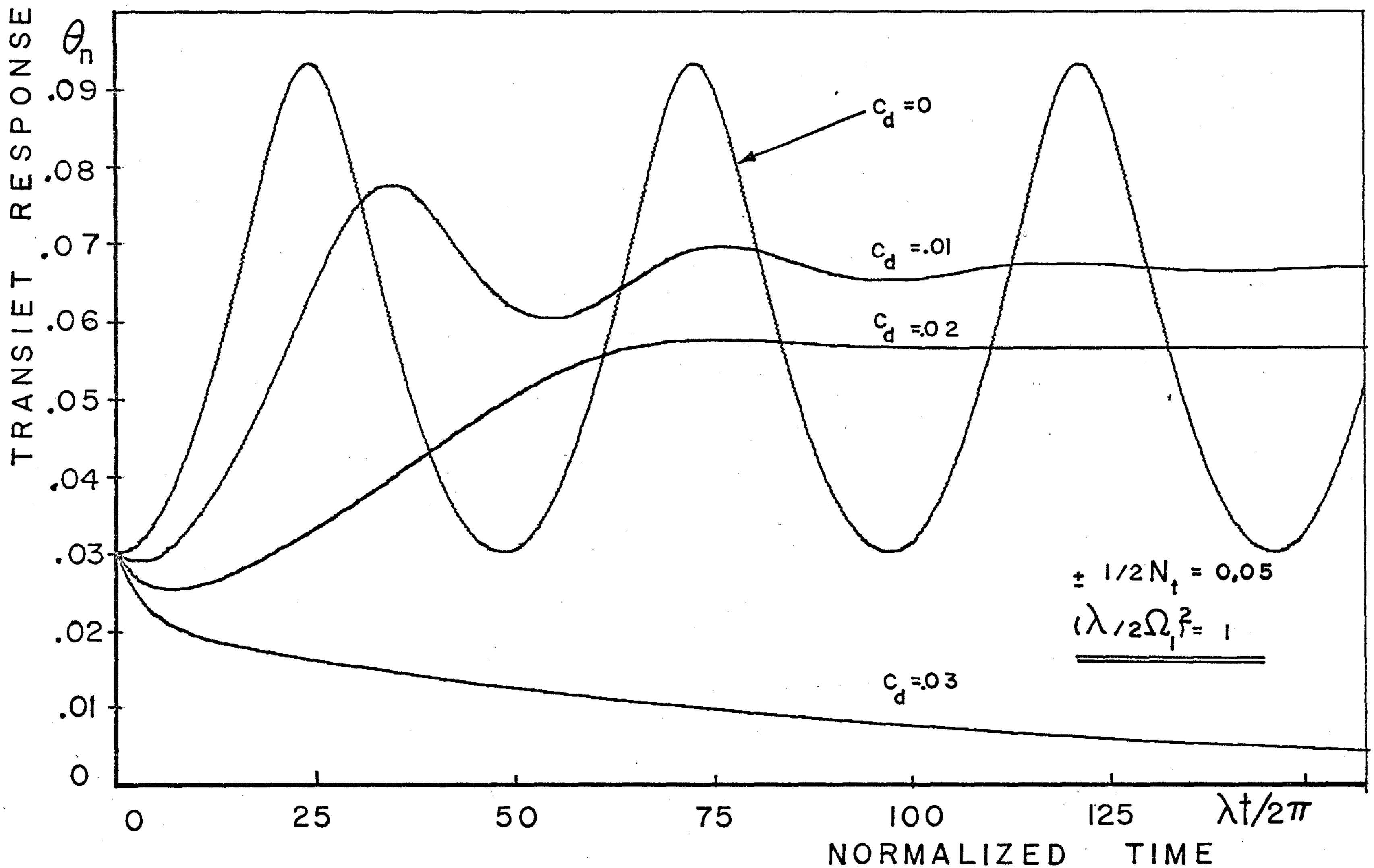

FIG. (21) TRANSIENT RESPONSE OF PREDOMINANT TORSIONAL VIBRATIONS FOR A SPLIT RING SECTION torsional characteristics (refer to figure [19] and figure [20]). Arbitrary initial values of the amplitude of torsional vibrations is considered. Computations were repeated for different values of the damping coefficient c_d. Results plotted in figure [21] show the time history of the nonsteady state amplitude of torsional vibrations for different values of the damping coefficient.

In the absence of damping $c_d = 0$ the amplitude of parametric torsional oscillations increases initially and then oscillates about the value predicted by the steady state solution. When relatively small damping is present (e.g. $c_d = .01$) the oscillations of the amplitude are damped down causing the transient solution to converge to the value of the steady state amplitude. For the case of larger damping coefficient $c_d = .03$ the considered parametric amplitude and parametric frequency represent a point that lies outside the unstable region corresponding to $c_d = .03$, figure [19]. The behaviour of the transient solution in this case is characterised by a rapidly decreasing amplitude ot become asymptotic to the trivial solution.

6.6. Stability of Steady State Solutions

In the study of the steady state amplitudes of parametric oscillations,in particular the steady state amplitude of torsional oscillations, it was shown that more than one solution exists. The trivial solution is given by equation (6.74) while equation (6.75) gives two nontrivial solutions. When the parametric frequency is outside the unstable range only the trivial solution exists. If the parametric frequency is within the unstable range, the trivial solution is still valid in addition to one or two nontrivial solutions from equation (6.74). Therefore, it is important to study the stability of the steady state solutions.

To check the stability of the steady state solutions, a technique given by Bolotin is used. The steady state amplitudes of parametric oscillations are disturbed. If the small disturbance considered remains bounded, the steady state amplitude value is considered stable. If the disturbances grow the steady state solution is considered unstable. This stability check can be achieved by using the nonsteady state solutions. To be specific, an example to study the stability of the steady state solution is worked out for the case of a thin-walled beam of monosymmetrical section.

The nonsteady state solutions given by equations (6.77) to (6.80) are recalled. This system of equations contain six first order, coupled, differential equations whose coefficients do not explicitly contain time. Let one of the steady state solutions which satisfy equations (6.69) to (6.75) be written as

$$\zeta_c = \zeta_{c0} \tag{6.84}$$

$$\zeta_s = \zeta_{s0} \tag{6.85}$$

$$\xi_c = \xi_{c0} \tag{6.86}$$

$$\xi_s = \xi_{s0} \tag{6.87}$$

$$\theta_c = \theta_{c0} \tag{6.88}$$

$$\theta_s = \theta_{s0} \tag{6.89}$$

To investigate the stability of the solutions given by equations (6.84) to (6.89) small disturbances of the amplitude values are considered in the form

$$\zeta_c = \zeta_{c0} + \gamma_1 \tag{6.90}$$

$$\zeta_s = \zeta_{s0} + \gamma_2 \tag{6.91}$$

$$\xi_c = \xi_{c0} + \gamma_3 \tag{6.92}$$

$$\xi_s = \xi_{s0} + \gamma_4 \tag{6.93}$$

$$\theta_c = \theta_{c0} + \gamma_5 \tag{6.94}$$

$$\theta_s = \theta_{s0} + \gamma_6 \tag{6.95}$$

where γ_i, $i = 1,6$ is a small time dependent disturbance of the amplitude components.

Substituting equations (6.90) to (6.95) into equations (6.78) to (6.80) the equations of the disturbed motion are obtained in the form

$$\frac{d\gamma_1}{dt} = \sum_{i=1,6} g_{1i}\,\gamma_i \tag{6.96}$$

$$\frac{d\gamma_2}{dt} = \sum_{i=1,6} g_{2i}\,\gamma_i \tag{6.97}$$

$$\{\bar{\gamma}\} = [M]^{-1}\,[A]\,\{\gamma\} \tag{6.98}$$

where g_{ji} ($j = 1,2$ and $i = 1,6$) are coefficients given in Appendix B.

$\{\bar{\gamma}\}$ is a vector representing the time derivatives of the disturbance quantities, in the form

$$\{\bar{\gamma}\} = \begin{Bmatrix} \frac{d\gamma_3}{dt} \\ \frac{d\gamma_4}{dt} \\ \frac{d\gamma_5}{dt} \\ \frac{d\gamma_6}{dt} \end{Bmatrix} \tag{6.99}$$

$[M]^{-1}$ represents the inverse of the 4 x 4 matrix [M] given by equation (6.83).

Matrix [A] is a 4 x 6 matrix whose elements are a_{ji} , j = 1,4 and i = 1,6. The coefficients a_{ji} are given in Appendix B.

$\{\gamma\}$ is a column vector of six components representing the disturbance quantities γ_i, i = 1,6.

It is convenient at this stage to express equations (6.96) to (6.98) in the matrix form

$$\{\dot{\gamma}\} = [G]\ \{\gamma\} \qquad (6.100)$$

where $\{\gamma\}$ is a column vector of six components γ_i, i = 1,6. [G] represents a matrix of coefficients whose elements can be obtained from equations (6.96) to (6.98). The first two rows are g_{1i} and g_{2i}; i = 1,6 while the rows 3 to 6 are b_{ji}; j = 1,4 and i = 1,6 where

$$[B] = [M]^{-1}\ [A] \qquad (6.101)$$

The solution for equation (6.100) is expressed in the exponential form as

$$\{\gamma\} = \{\gamma_0\}\ e^{(\bar{\lambda} t/\lambda)} \qquad (6.102)$$

where $\{\gamma_0\}$ is a column vector of six constant components and λ is the frequency of the exciting force while $\bar{\lambda}$ represents the eigenvalues of [G].

The disturbances $\{\gamma\}$ decay with time only if all roots $\bar{\lambda}$ have negative real parts.

Analytical quantities and coefficients are given for equations (6.96) to (6.98) for the case of a thin-walled beam of monosymmetrical cross section. However, further analytical formulation of the matrix [G] is rather difficult as it involves the inversion of the matrix [M]. The elements of the matrix [G] can thus be constructed numerically for each specific case.

Numerical computations were carried out to check the stability of the steady state solutions near the principal regions of parametric instabilities by evaluating the eigenvalues of the matrix [G].

The cases considered included the trivial and nontrivial steady state solutions of a thin-walled beam of symmetrical I section. Results are applied to the case of zero damping as shown in figures [14] and [16]. The stable solutions of the steady state are in solid lines while the unstable solutions are dotted.

Stability of the solutions of the steady state amplitudes of parametric torsional vibrations is studied for a thin-walled beam of monosymmetrical split ring section near the principal region of parametric stability of the predominant torsional characteristics. Results are applied to figure [20] as solid lines for the stable solutions while the unstable solutions are dotted.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

7.1. Conclusions

From the present investigation, the following conclusions are arrived at:

1. The nonlinear deformation theory of thin-walled beams of open cross sections, as presented, accounts for large rotations of the cross section. The nonlinear differential equations were checked in two ways. Firstly, it was shown to reduce to the well established linear theory of Vlasov. Secondly, the nonlinear deformation equations in the simplest form reduce to Gregory's formulations. Both the linear theory of Vlasov and Gregory's formulations were supported by experimental work.

2. The present nonlinear theory has four main advantages over the formulations available in the literature, namely: (a) the deformations in the longitudinal direction of the thin-walled beam were accounted for. (b) a consistent third order approximation was maintained throughout the formulation. The approximations were based on the large rotation assumption where the angle of rotation of the cross section is treated as first order while the slope of the deflection curve was considered as a second order quantity.

(c) the nonlinear differential equations are accompanied by the appropriate boundary conditions of a consistent order of approximation. (d) the present theory is relatively simple to apply. This was illustrated by the different examples considered.

3. The solutions of a thin-walled beam, of symmetrical and monosymmetrical cross sections subjected to end twisting moments, were shown to agree with existing solutions for the case of uniform torsion, as given by Cullimore and Gregory. Numerical solutions for the case of a cantilever thin-walled beam, of narrow, rectangular, cross section as well as symmetrical I and angle sections, were carried out. Comparison between the linear and nonlinear solutions for the moment of twist - angle of rotation behaviour showed substantial difference for large angles of rotation. For the specific cases considered, there was a difference of over 10% in the values of the moment of twist, when the angle of rotation was 0.35 radians.

4. The nonlinear solution of a cantilever thin-walled beam, subjected to end torque, was carried out by means of perturbation analysis. This solution, to the author's knowledge, has not been done before. The twisting moment - angle of rotation relationship for a thin-walled beam of symmetrical I section was compared with the linear solution of non-uniform torsion.

The variation of the axial strain component along the longitudinal axis of the beam, as predicted by linear and nonlinear theories, were compared for the case of non-uniform torsion of a thin-walled beam of a symmetrical I section. A fundamental characteristic difference in the axial strain was shown to take place near the free end of the beam. Although the linear theory predicts no axial strain at the free end of the cantilever beam, the nonlinear theory shows that a positive strain component exists due to the "shortening effect" of large torsion.

5. A general consistent nonlinear stability theory of thin-walled beams was presented. The nonlinear differential equations governing the stability under axial loads and lateral stability were formulated. The nonlinear stability theory was shown to reduce to the linear equations of Vlasov.

6. The study of the parametric stability of thin-walled beams of symmetrical I section, subjected to axial periodic load passing through the centroid of the end section, was carried out. In the case of the specific example considered, the first principal region of parametric instability is that of the torsional type of parametric oscillations. The steady state amplitude of vibrations near the principal region of parametric torsional instability was shown to behave in two different manners depending on whether the unstable region is approached by increasing or

decreasing parametric frequencies. As the parametric frequency enters the unstable range from below, the steady state amplitude of vibrations grows from the trivial solution. Continuous growth of the amplitude takes place as the parametric frequency increases. As the parametric frequency leaves the unstable range a sudden drop of the steady state amplitude to the trivial solution will take place. On the other hand, approaching the unstable region from above the unstable range, a sudden jump of the value of the amplitude occurs at the right boundary of the unstable region. As the parametric frequency decreases further, the steady state amplitude of oscillations decreases gradually until the trivial solution becomes valid at the other boundary of the unstable region.

7. The study of the transient solution near the principal region of parametric instability shows growth of the amplitude of oscillation. From the initial value, the amplitude grows monotonically at first, then oscillates about the corresponding magnitude as predicted by the steady state analysis.

8. The presence of viscous damping will affect not only the regions of parametric instability as was established by the linear analysis, but also it affects the steady state and the nonsteady state amplitudes of oscillations. The steady state amplitude of vibrations of a damped system are

smaller than those of an undamped system. The rate of growth and duration at which the steady state amplitude is reached varies considerably with damping. The presence of viscous damping was found to decrease the rate of initial growth of the amplitude. Three cases can be discussed for different values of damping coefficients. A relatively small damping causes the amplitude of the transient vibrations to grow and over shoot the steady state value. Then the amplitude approaches the steady state value in a decreasing oscillation manner. For a larger value of the damping coefficient, the amplitude of nonsteady state response grows with a smaller rate until it reaches the steady state value. For a constant parametric amplitude, the system with large damping will not be parametrically excited. Thus, the transient solution for this case decreases continuously from an arbitrary initial value to become asymptotic to the trivial solution of zero magnitude.

9. The study of the parametric stability of thin-walled beams of monosymmetrical cross section, subjected to axial periodic load passing through the centroid of the end section, was found to differ from the case of a thin-walled beam of symmetrical cross section, in one aspect only. In the case of a thin-walled beam of monosymmetrical sections, coupled flexural - torsional type of parametric oscillations can be excited. The first principal region of parametric

instability for a thin-walled beam of split ring section was found to be that of the predominant torsional characteristics.

The effect of viscous damping on the unstable regions, steady state amplitudes and the transient solutions for the case of thin-walled beams of monosymmetrical cross section was studied. However, when compared with the effect of damping on the parametric response of a thin-walled beam of symmetrical cross sections, no qualitative difference was observed.

7.2. Suggestion for Further Research

The nonlinear theory of thin-walled elastic beams, as presented in this work, was checked in two ways. Firstly, it reduces to Vlasov's linear theory. Secondly, it reduces in its simplest form to Gregory's nonlinear formulations which were verified experimentally. However, further experimental verification of the nonlinear thin-walled theory is mandatory. Experiments can be conducted to establish the behaviour of thin-walled beams when subjected to static and dynamic loading. Deflections and rotations, of a thin-walled beam subjected to non-uniform torsion under the action of static twisting moment applied at its ends, can be measured to be compared with theoretical predictions. The response of thin-walled beams subjected to dynamic loads can be conducted experimentally in different ways. Lateral dynamic (periodic) loads will create a "Forced Vibration" type of response.

The magnitude of steady state amplitudes of vibrations, for an exciting frequency close to the frequency of free vibrations, can be measured experimentally to be compared with theoretical calculations. Also, measuring the longitudinal strain component at resonance conditions may be done experimentally to verify theoretical strains. Applying periodic loads in the axial direction of a thin-walled beam at its ends will give rise to parametric stability problems. The amplitudes of parametric oscillations near the principal region of parametric instability may be established experimentally. Experimental values of steady state amplitudes of oscillations can be compared to the steady state amplitudes of parameteric oscillations as predicted by the nonlinear theory.

Further theoretical investigations may be directed to study the phenomenon of combination resonance of variable orders. The existence of sub-harmonic and super-harmonic oscillations may be investigated. Interaction of different instability regions may take place if two frequencies of free vibrations of different modes or types approach each other. This will cause the unstable regions corresponding to these frequencies to interact. The effect of interaction of different instability regions on the steady state and the growth of the amplitude may be studied using the nonlinear theory.

Lateral dynamic loads applied to a thin-walled beam will create a parametric stability problem. This aspect of stability may be a wide field of investigation.

Problems including nonlinear, plastic, viso-elastic and hysteretic material properties, approach a more realistic representation of practical problems.

Deformation and stability theories of thin-walled structures are one of the important branches of the mechanics of deformable solids. Although much has been done recently to clarify many problems, a wide field of investigation remains.

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APPENDIX A

GEOMETRICAL PROPERTIES OF SPLIT RING SECTION

To define a circular split ring, three independent quantities should be stated, namely:

c is the thickness of the ring's wall

$\bar{r}$ is the radius to the center line of the wall

Φ is the semi-central angle in radians

The origin of the axes OX and OY is the centroid of the cross section. Axis OY is taken as the axis of symmetry as shown in figure [17]. The center of the circle is of co-ordinates $(0, a_0)$, ϕ is an angle measured from the axis of symmetry. The origin of the co-ordinate s for this monosymmetrical section is at the intersection of the section's profile with the axis of symmetry.

From the geometry of the split ring section,

$$s = \bar{r}\,\phi \tag{A.1}$$

$$x = \bar{r}\sin\phi \tag{A.2}$$

$$y = a_0 - \bar{r}\cos\phi \tag{A.3}$$

$$\psi = \phi \tag{A.4}$$

$$A = 2\,\bar{r}\,c\,\Phi \tag{A.5}$$

The value of a_0 is found by using the formula

$$\int_s y\,c\,ds = 0 \tag{A.6}$$

thus

$$a_o = \bar{r} \sin \Phi/\Phi \qquad (A.7)$$

$$I_{yy} = \int_s c\, x^2\, ds$$

$$= c\, \bar{r}^3 (\Phi - \sin \Phi \cos \Phi) \qquad (A.8)$$

$$I_{xx} = \int_s c\, y^2\, ds$$

$$= c\, \bar{r}^3 (\Phi \sin \Phi \cos \Phi - 2 \sin^2 \Phi/\Phi) \qquad (A.9)$$

To determine the co-ordinates of the shear center $C(0, c_y)$

$$c_y = a_0 - \frac{1}{I_{yy}} \int_s x\, c\, \omega_B\, ds \qquad (A.10)$$

in which $\omega_B = \bar{r}^2\phi$, twice the area swept by the radius moving along the section starting at s = 0, therefore

$$c_y = \frac{-\bar{r}(\Phi \sin \Phi + \sin^2 \Phi \cos \Phi - 2\, \Phi^2 \cos \Phi)}{\Phi(\Phi - \sin \Phi \cos \Phi)} \qquad (A.11)$$

To determine the expressions for h(s) and n(s)

$$h(s) = (x - c_x) \sin \psi - (y - c_y) \cos \psi \qquad (A.12)$$

$$n(s) = (x - c_x) \cos \psi + (y - c_y) \sin \psi \qquad (A.13)$$

$$h(s) = \bar{r} - \frac{2\bar{r}(\sin \Phi - \Phi \cos \Phi)}{\Phi - \sin \Phi \cos \Phi} \cos \phi \qquad (A.14)$$

$$n(s) = \frac{2\bar{r}(\sin \Phi - \Phi \cos \Phi)}{\Phi - \sin \Phi \cos \Phi} \sin \phi \qquad (A.15)$$

$$\omega(s) = \int_0^s h(\bar{s})\, d\bar{s}$$

$$= \bar{r}^2 [\phi - \frac{2(\sin \Phi - \Phi \cos \Phi)}{\Phi - \sin \Phi \cos \Phi} \sin \phi] \qquad (A.16)$$

$$I_d = \int_s \frac{c^3}{3}\, ds$$

$$= \frac{2}{3}\, \bar{r}\, c^3\, \phi \qquad (A.17)$$

$$I_{pc} = \frac{1}{2} \int_s c\, (h^2 + n^2)\, ds$$

$$= c\, \bar{r}^3 \left[\phi + \frac{4\,\phi(\sin\phi - \phi\cos\phi)^2}{(\phi - \sin\phi\cos\phi)^2}\right.$$

$$\left. - \frac{4\sin\phi(\sin\phi - \phi\cos\phi)}{\phi - \sin\phi\cos\phi}\right] \qquad (A.18)$$

$$I_{\omega\omega} = \int_s c\, \omega^2\, ds$$

$$= \frac{2}{3}\, c\, \bar{r}^5 \left[\phi^3 - \frac{6\sin\phi - \phi\cos\phi)^2}{\phi - \sin\phi\cos\phi}\right] \qquad (A.19)$$

$$I_{Ry} = \int_s \frac{1}{2}\, c\, y(h^2 + n^2)\, ds$$

$$= 2\, c\, \bar{r}^4 \left[\frac{\phi(\phi + \sin\phi\cos\phi - 2\sin^2\phi)(\sin\phi - \phi\cos\phi)}{\phi - \sin\phi\cos\phi}\right] \qquad (A.20)$$

$$I_R = \int_s \frac{1}{4}\, c\, (h^2 + n^2)^2\, ds$$

$$= \frac{1}{2}\, c\, \bar{r}^5 \left[\phi - \frac{8\sin\phi(\sin\phi - \phi\cos\phi)}{\phi - \sin\phi\cos\phi}\right.$$

$$+ 8\,(2\phi + \sin\phi\cos\phi)\left(\frac{\sin\phi - \phi\cos\phi}{\phi - \sin\phi\cos\phi}\right)^2$$

$$- 32\sin\phi\left(\frac{\sin\phi - \phi\cos\phi}{\phi - \sin\phi\cos\phi}\right)^3$$

$$\left. + 16\,\phi\left(\frac{\sin\phi - \phi\cos\phi}{\phi - \sin\phi\cos\phi}\right)^4\right] \qquad (A.21)$$

APPENDIX B

CONSTANTS AND FUNCTIONS

Coefficient of equation (6.69)

$$\bar{\alpha}_0 = - 512\ I_{pc}/(63\ A\ell^2) \tag{B.1}$$

Coefficients of equation (6.70)

$$\bar{\alpha}_1 = - 4\ \bar{\alpha}_0\ c_d\lambda\ \Omega_\zeta^3/D_0 \tag{B.2}$$

$$\bar{\alpha}_2 = 8(\Omega_\zeta^2 - \lambda^2)\lambda^2\beta\ N_t/(\pi^2 D_0) \tag{B.3}$$

$$\bar{\alpha}_3 = (\Omega_\zeta^2 - \lambda^2)\bar{\alpha}_0\ \Omega_\zeta^2/D_0 \tag{B.4}$$

where $\bar{\alpha}_0$ is given by equation (B.1)

$$D_0 = (\Omega_\zeta^2 - \lambda^2)^2 + (2\ c_d\ \Omega_\zeta\lambda)^2 \tag{B.5}$$

Coefficients of equation (6.71)

$$\bar{\alpha}_4 = 2\ \bar{\alpha}_0\ \Omega_\zeta^2\ (\Omega_\zeta^2 - \lambda^2)/D_0 \tag{B.6}$$

$$\bar{\alpha}_5 = 16\ c_d\ \Omega_\zeta\lambda^3\ \beta\ N_t/(\pi^2 D_0) \tag{B.7}$$

$$\bar{\alpha}_6 = 2\ c_d\ \Omega_\zeta^3\ \lambda\ \bar{\alpha}_0/D_0 \tag{B.8}$$

where $\bar{\alpha}_0$ and D_0 are given by equations (B.1) and (B.5), respectively.

Coefficients of equation (6.72)

$$\bar{\alpha}_7 = \bar{K}_1 \; c_y \; \mu/(\bar{D}_0 \ell) \qquad (B.9)$$

$$\bar{\alpha}_8 = - \; c_d \; \Omega_2 \, \lambda \, c_y \; \bar{\mu}/(\bar{D}_0 \ell) \qquad (B.10)$$

where

$$\bar{D}_0 = K_1 \; \bar{K}_1 - (c_d \Omega_2 \lambda)^2 \qquad (B.11)$$

$$\mu = 4 \; \pi^2 E^*(N_0 + 0.5N_t)/(3\rho\ell^2) - \lambda^2/4 \qquad (B.12)$$

$$\bar{\mu} = 4 \; \pi^2 E^*(N_0 - 0.5N_t)/(3\rho\ell^2) - \lambda^2/4 \qquad (B.13)$$

$$K_1 = \mu + 16 \; \pi^4 E^* I_{yy}/(3\rho A \ell^4) \qquad (B.14)$$

$$\bar{K}_1 = \bar{\mu} + 16 \; \pi^4 E^* I_{yy}/(3\rho A \ell^4) \qquad (B.15)$$

Coefficients of equation (6.73)

$$\bar{\alpha}_9 = c_d \; \Omega_2 \, \lambda \, c_y \; \mu/(\bar{D}_0 \ell) \qquad (B.16)$$

$$\bar{\alpha}_{10} = K_1 \; c_y \; \bar{\mu}/(\bar{D}_0 \ell) \qquad (B.17)$$

where D_o, μ, $\bar{\mu}$ and K_1 are given by equations (B.11) to (B.14)

Coefficients of equation (6.75)

$$\bar{\alpha}_{11} = B_1^2 + B_2^2 \qquad (B.18)$$

$$\bar{\alpha}_{12} = B_1(A_{11} + A_{22}) + B_2(A_{12} - A_{21}) \qquad (B.19)$$

$$\bar{\alpha}_{13} = A_{11}A_{22} - A_{12}A_{21}$$

where

$$A_{11} = K_2 + c_y^2 A \; \mu^2 \; \bar{K}_1/(2I_{pc}\bar{D}_0) + 4K_3 \; \bar{\alpha}_2 \qquad (B.20)$$

$$A_{12} = c_d \; \Omega_1 \lambda - c_y^2 A \; \mu\bar{\mu} \; c_d \; \Omega_2 \lambda/(2I_{pc}\bar{D}_0) + 2K_3 \; \bar{\alpha}_5 \qquad (B.21)$$

$$A_{21} = - A_{12} + 4\ K_3\ \bar{\alpha}_5 \tag{B.22}$$

$$A_{22} = \bar{K}_2 + c_y^2\ A\ \mu^2 K_1/(2I_{pc}\bar{D}_0) - 4\ K_3\ \bar{\alpha}_2 \tag{B.23}$$

$$B_1 = 3\ \pi^4\ E^*I_R/(\rho\ell^4 I_{pc}) + 8\ K_1(\bar{\alpha}_0 - 0.5\ \bar{\alpha}_3) \tag{B.24}$$

$$B_2 = K_1\ \bar{\alpha}_1 \tag{B.25}$$

where $D_0, \mu, \bar{\mu}, K_1, \bar{K}_1$ are given by equations (B.11) to (B.15). $\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2$ and $\bar{\alpha}_3$ are given by equations (B.1) to (B.4), while $\bar{\alpha}_5$ is given by (B.7).

$$K_2 = \mu + 2\ \pi^2(4\ \pi^2\ E^*I_{\omega\omega}/\ell^2 + GI_d)/(3\rho\ell^2 I_{pc}) \tag{B.26}$$

$$\bar{K}_2 = K_2 - \mu + \bar{\mu} \tag{B.27}$$

$$K_3 = 32\ \pi^2\ E^*/(189\rho\ell^2) \tag{B.28}$$

<u>Function $\bar{g}_1$ in equation (6.77)</u>

$$\bar{g}_1 = - \bar{\alpha}_0(\theta_c^2 + \theta_s^2) \tag{B.29}$$

where $\bar{\alpha}_0$ is given by equation (B.1)

<u>Function $\bar{g}_2$ in equation (6.78)</u>

$$\bar{g}_2 = (\bar{D}_3 + c_d\,\Omega_\zeta\ \bar{D}_4/\lambda)/D_2 \tag{B.30}$$

where

$$D_2 = 1 + (c_d\ \Omega_\zeta/\lambda)^2 \tag{B.31}$$

$$\bar{D}_3 = 0.5\ [\zeta_s(\Omega_\zeta^2 - \lambda^2) + \bar{\alpha}_0\ \Omega_\zeta^2\theta_c\theta_s]/\lambda$$

$$- c_d\ \Omega_\zeta\ \zeta_c \quad \text{(B.32)}$$

$$\bar{D}_4 = -\ 0.5\ [\zeta_c(\Omega_\zeta^2 - \lambda^2) - 8\ \lambda^2\beta N_t/\pi^2$$

$$+ 0.5\ \bar{\alpha}_0\ \Omega_\zeta^2(\theta_c^2 - \theta_s^2)]/\lambda - c_d\ \Omega_\zeta\ \zeta_s$$

(B.33)

where $\bar{\alpha}_0$ is given by (B.1)

Function $\bar{g}_3$ in equation (6.79)

$$\bar{g}_3 = (\bar{D}_4 - c_d\,\Omega_\zeta\ \bar{D}_3/\lambda)/D_2 \quad \text{(B.34)}$$

where D_2, $\bar{D}_3$ and $\bar{D}_4$ are given by equations (B.31) to (B.33)

Functions in equation (6.82)

$$\bar{g}_4 = (\xi_s\ \bar{K}_1 + c_y\ \bar{\mu}\theta_s/\ell)/\lambda\ - c_d\ \Omega_2\ \xi_c \quad \text{(B.35)}$$

$$\bar{g}_5 = -\ (\xi_c\ K_1 + c_y\ \mu\theta_c/\ell)/\lambda - c_d\ \Omega_2\ \xi_s \quad \text{(B.36)}$$

$$\bar{g}_6 = [\theta_s\ \bar{K}_2 + c_y\ A\ \ell\ \bar{\mu}\ \xi_s/(2I_{pc})$$

$$+ 3\ \pi^4\ E^*I_R(\theta_s^3 + \theta_c^2\theta_s)/(\rho\ell^4 I_{pc})$$

$$+ 4\ K_3(2\zeta_a\theta_s - \zeta_c\theta_s + \zeta_s\theta_c)]/\lambda$$

$$- c_d\ \Omega_1\ \theta_c \quad \text{(B.37)}$$

$$\bar{g}_7 = -[\theta_c K_2 + c_y A \ell \mu \xi_c/(2I_{pc}) + 3\pi^4 E^* I_R(\theta_c^3 + \theta_s^2\theta_c)/(\rho\ell^4 I_{pc}) + 4 K_3 (2\zeta_a\theta_c + \zeta_c\theta_c + \zeta_s\theta_s)]/\lambda - c_d \Omega_1 \theta_s \quad \text{(B.38)}$$

where $\mu, \bar{\mu}$, K_1 and $\bar{K}_1$ are given by equations (B.12) to (B.15). K_2, $\bar{K}_2$ and K_3 are given by equations (B.26) to (B.28).

Coefficients of equation (6.96)

$$g_{11} = -c_d \Omega_\zeta[1 + 0.5(\Omega_\zeta^2 - \lambda^2)/\lambda^2]/D_2 \quad \text{(B.39)}$$

$$g_{12} = [0.5(\Omega_\zeta^2 - \lambda^2)/\lambda - c_d^2 \Omega_\zeta^2/\lambda]/D_2 \quad \text{(B.40)}$$

$$g_{13} = g_{14} = 0 \quad \text{(B.41)}$$

$$g_{15} = -0.5\, \bar{\alpha}_0 \Omega_\zeta^2[\theta_{s0}/\lambda - c_d \Omega_\zeta \theta_{c0}/\lambda^2]/D_2 \quad \text{(B.42)}$$

$$g_{16} = -0.5\, \bar{\alpha}_0 \Omega_\zeta^2[\theta_{c0}/\lambda + c_d \Omega_\zeta \theta_{s0}/\lambda^2]/D_2 \quad \text{(B.43)}$$

where $\bar{\alpha}_0$ and D_2 are given by equations (B.1) and (B.31), respectively.

Coefficients of equation (6.97)

$$g_{21} = -g_{12} \quad \text{(B.44)}$$

$$g_{22} = g_{11} \quad \text{(B.45)}$$

$$g_{23} = g_{24} = 0 \quad \text{(B.46)}$$

$$g_{25} = -g_{16} \quad \text{(B.47)}$$

$$g_{26} = g_{15} \tag{B.48}$$

where g_{11} to g_{16} are given by equations (B.39) to (B.43).

Elements of the matrix [A] in equation (6.98)

$$\begin{bmatrix} 0 & 0 & -c_d\Omega_2 & \bar{K}_1/\lambda & 0 & c_y\bar{\mu}/(\lambda\ell) \\ 0 & 0 & -K_1/\lambda & -c_d\Omega_2 & -c_y\mu/(\lambda\ell) & 0 \\ -4K_3\theta_{s0}/\lambda & 4K_3\theta_{c0}/\lambda & 0 & a_{34} & a_{35} & a_{36} \\ -4K_3\theta_{c0}/\lambda & -4K_3\theta_{s0}/\lambda & a_{43} & 0 & a_{45} & a_{46} \end{bmatrix}$$

where

$$a_{34} = c_y\ A\ \ell\ \bar{\mu}/(2I_{pc}\lambda) \tag{B.50}$$

$$a_{35} = [4K_3\zeta_{s0} + 6\ \pi^4 E^* I_R \theta_{c0}\theta_{s0}/(\rho\ell^4 I_{pc})]/\lambda - c_d\Omega_1 \tag{B.51}$$

$$a_{36} = [\bar{K}_2 + 3\ \pi^4 E^* I_R(3\theta_{s0}^2 + \theta_{c0}^2)/(\rho\ell^4 I_{pc}) + 4K_3(2\zeta_a - \zeta_{c0})]/\lambda \tag{B.52}$$

$$a_{43} = -\ c_y\ A\ \ell\ \mu/(2I_{pc}\lambda) \tag{B.53}$$

$$a_{45} = -\ [K_2 + 3\pi^4 E^* I_R(3\theta_{c0}^2 + \theta_{s0}^2)/(\rho\ell^4 I_{pc}) + 4K_3(2\zeta_a + \zeta_{c0})]/\lambda \tag{B.54}$$

$$a_{46} = [-4K_3\zeta_{s0} + 6\ \pi^4 E^* I_R \theta_{c0}\theta_{s0}/(\rho\ell^4 I_{pc})]/\lambda - c_d\Omega_1 \tag{B.55}$$

where $\mu, \bar{\mu}$, K_1 and $\bar{K}_1$ are given by equations (B.12) to (B.15). $K_2, \bar{K}_2$ and K_3 are given by equations (B.26) to (B.28).