EQUATIONAL CLASSES OF IDEMPOTENT SEMIGROUPS

THE LATTICE

OF

EQUATIONAL CLASSES OF IDEMPOTENT SEMIGROUPS

By

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SCOPE AND CONTENTS: The lattice of equational classes of idempotent semigroups is completely described. It is shown that every equational class of idempotent semigroups is determined by a single equation (in addition to the associative and idempotent equations). A method is presented for finding which class a given equation determines, and when the class determined by one equation is contained in the class determined by a second equation.

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INTRODUCTION

The concept of an equational class of algebras was introduced by Birkhoff in 1935, and has since been discussed by several authors (see for example Tarski, 1954). If one ignores the foundation problems, which in any case are easily circumvented, the equational classes of algebras of a given type form a lattice under class inclusion. It is of interest, then, to investigate the properties of this lattice for a given type of algebra, and if possible to describe the lattice.

The former problem has been dealt with more successfully than the latter. For example, distributivity of the lattice has been characterized in special cases by Jónsson (in press). Up to the present, the only lattice to be described is the lattice of equational classes of algebras with one unary operation (Jacobs and Schwabauer, 1964). The next type to consider would seem to be the lattice of equational classes of algebras with one binary operation. Here the problem is much more difficult. Kalicki (1955) showed that there are uncountably many atoms in the lattice of equational classes of groupoids. The lattice of all equational classes of semigroups (associative groupoids) forms a sublattice of the lattice of equational classes of groupoids. This sublattice is uncountable (Evans, 1967), and has been investigated by Kalicki and Scott (1955), who listed its countably many atoms.

The lattice of equational classes of commutative idempotent semigroups, i.e. of semilattices, consists of only two elements. Non-trivial sublattices of the lattice of equational classes of semigroups may be obtained by removing one of these restrictions. Partial results have been obtained for the lattice of equational classes of commutative semigroups (Schwabauer, 1966; Nelson, 1967). In the case of idempotent semigroups, it is relatively easy to show that the lattice of equational classes has three atoms, and in fact the sublattice generated by the atoms has been shown by Tamura (1966) to be the eight-element Boolean lattice. Kimura (1958-IV) has described all equations on idempotent semigroups in three variables.

In this thesis, a complete description is given of the lattice of equational classes of idempotent semigroups. An outline of the thesis, by chapter, follows.

Chapter I: The solution of the word problem for free idempotent semigroups given by Green and Rees (1952) is described. Those equations on semigroups which are no restriction on idempotent semigroups are characterized. Several invariants are introduced for use in succeeding chapters.

Chapter II: The relation $\sim_n$ (for every natural number n) is introduced and characterized. The special characterization given in proposition 2.7 is of central importance, and leads to the definition of certain other relations which are used in later chapters to classify equations.

Chapter III: In this chapter, the equations (f = g) in n variables which satisfy $f \underset{n}{\sim} g$ are singled out for special attention. For fixed n, we define the n-skeleton to be that subposet of the lattice of equational classes whose elements are the classes determined by a single equation of this kind. The n-skeleton is completely described, using the relations defined in chapter II.

Chapter IV: The skeleton of the lattice is formed from the union of the n-skeletons by finding the inclusions between elements of successive n-skeletons. The relations defined in chapter II are used together with relations introduced in this chapter, to show that every equation determines an equational class which either is a member of the skeleton or is equal to the meet of 2 skeletal elements. Finally from properties of equations, and of the skeleton of the lattice, it is shown that every equational class of idempotent semigroups is determined by a single equation. A description of the lattice is therefore complete.

CHAPTER I

INVARIANTS AND FREE IDEMPOTENT SEMIGROUPS

In this chapter, we present the solution of the word problem for free idempotent semigroups given by Green and Rees in 1952. We then introduce complete sets of invariants, and use them to further describe the algorithm for deciding when two words represent the same element of the free idempotent semigroup. Various additional invariants are introduced and their properties discussed. These invariants will prove to be useful in the succeeding chapters.

1. The word problem.

Throughout the paper X will be a fixed countable set, and F(X) the free semigroup generated by X. Every semigroup generated by X may be thought of as consisting of all products $x_1x_2\ldots x_n$ of finite non-empty sequences in X. The free semigroup F(X) is characterized by the property that two products $x_1x_2\ldots x_n$, $y_1y_2\ldots y_m$ are equal iff $n = m$ and $x_i = y_i$, $(i = 1,2,\ldots,n)$. We also refer to the elements of F(X) as <u>words</u> or <u>terms</u>.

A semigroup equation is a pair (f,g) of elements $f,g \in F(X)$. The equation (f,g) is said to <u>hold</u>, to be <u>valid</u>, or to be <u>satisfied</u> in a semigroup S iff for every homomorphism $\varphi: F(X) \to S$, $\varphi(f) = \varphi(g)$. In order to conform with the usual notation we also say in this case that the equation $(f = g)$ holds in S.

We will be concerned here with idempotent semigroups, i.e. semigroups which satisfy the equation $(x = x^2)$, $(x \in X)$. Let $FI(X)$ be the free idempotent semigroup generated by X. Since every homomorphism of $F(X)$ into an idempotent semigroup factors through $FI(X)$ in such a way that X is mapped identically, the semigroup $FI(X)$ plays an important role. Let $\kappa: F(X) \longrightarrow FI(X)$ be the homomorphism which maps X identically. An equation $(f = g)$ holds in every idempotent semigroup iff $\kappa(f) = \kappa(g)$. If $\kappa(f) = \kappa(g)$, we also write $f \sim g$.

For technical reasons we also introduce $F'(X)$, the free monoid generated by X. $F'(X)$ contains $F(X)$ as a subsemigroup and has one additional element e, which may be thought of as the product of the empty sequence, and which satisfies $ae = ea = a$ for all $a \in F'(X)$. The definition of $\sim$ may be extended to $F'(X)$ by setting $e \sim e$ and asserting that $e \not\sim a$ for all $a \neq e$.

It is well known (Cohn, 1965, p156) that the relation $\sim$ on $F(X)$ can be described as follows:

(1.1) $f \sim g$ iff there exists $h_0, h_1, \ldots, h_n \in F(X)$ and $p_i, q_i, r_i \in F'(X)$, $(i = 0,1,\ldots,n-1)$, such that $h_0 = f$, $h_n = g$, and for every $i = 0,1,\ldots,n-1$, either $h_i = p_i q_i r_i$ and $h_{i+1} = p_i q_i^2 r_i$, or $h_i = p_i q_i^2 r_i$ and $h_{i+1} = p_i q_i r_i$.

Using this description, the relation $\sim$ will be characterized in a more convenient way. The following notation will be used.

The <u>dual</u> $(S^*, *)$ of a semigroup $(S, \cdot)$ is defined by $S^* = S$, and $a*b = b.a$ for all $a,b \in S$. Since every term f determines uniquely the sequence in X of which it is the product, the dual f^* of $f = x_1 x_2 \ldots x_n$

may also be introduced by defining $f^* = x_1 * x_2 * \dots * x_n = x_n x_{n-1} \dots x_1$. Clearly $f^{**} = f$, and $f \sim g$ iff $f^* \sim g^*$.

If $f = x_1 x_2 \dots x_n \in F(X)$, let $L(f) = n$ be the length of f, and $E(f) = \{x_1, x_2, \dots, x_n\}$, the set of variables occurring in f. Let $L(e) = 0$, and $E(e) = \emptyset$.

Since $E(pqr) = E(pq^2r)$, it follows from the above description of $\sim$ that

(1.2) If $f \sim g$ then $E(f) = E(g)$.

For $f = x_1 x_2 \dots x_n \in F(X)$, define $\overline{f}(0) = x_j$ where j is determined by the properties

(i) $i < j$ implies $x_i \neq x_j$

(ii) $E(x_1 \dots x_j) = E(f)$

Define $f(0) = x_1 \dots x_{j-1}$. Note that $f(0) = e$ if $|E(f)| = 1$. By duality define $\overline{f}(1) = \overline{f^*}(0)$ and $f(1) = (f^*(0))^*$. Define $e(i) = e$, and $\overline{e}(i) = e$, $(i = 0,1)$.

It follows easily from these definitions that

(1.3) $E(f) = E(f(0)\overline{f}(0)) = E(\overline{f}(1)f(1))$.

(1.4) If $f \neq e$, $|E(f)| = |E(f(0))| + 1 = |E(f(1))| + 1$.

By a <u>substitution</u> we mean here a homomorphism $\varphi : F(X) \to F(X)$ which maps X into itself. Every substitution may be extended to equations by defining $\varphi(f,g) = (\varphi(f), \varphi(g))$. φ is said to be a <u>substitution in $(f = g)$ by n variables</u> iff $|E(\varphi(f)) \cup E(\varphi(g))| = n$. If the equation $(f = g)$ holds in a semigroup S, then every equation obtained from $(f = g)$ by substitution holds in S. In particular, if $f \sim g$ then $\varphi(f) \sim \varphi(g)$. Clearly for every substitution φ we have:

(1.5) $\varphi(f^*) = (\varphi(f))^*$.

(1.6) If $E(f) = E(g)$, then $E(\varphi(f)) = E(\varphi(g))$.

Moreover:

(1.7) If $E(f) \neq E(g)$, then there exists a substitution φ in $(f = g)$ by two variables with $E(\varphi(f)) \neq E(\varphi(g))$, and hence, by (1.2), with $\varphi(f) \not\sim \varphi(g)$.

<u>Proof</u>: W.l.o.g. there exists $b \in E(f) - E(g)$. With an arbitrary $a \in X$, $a \neq b$, define φ by

$$\varphi(x) = \begin{cases} b \text{ if } x = b. \\ a \text{ if } x \in X - \{b\}. \end{cases}$$

Clearly $E(\varphi(g)) = \{a\}$, and $b \in E(\varphi(f))$. Therefore $E(\varphi(f)) \neq E(\varphi(g))$.

<u>Lemma 1.1</u>: (Brown, 1964). If $f, g \in F'(X)$ and $E(g) \subseteq E(f)$, then $f \sim fgf$.

<u>Proof</u>: (Brown, 1964). We first show that if $f \sim pgr$, $(p, r \in F'(X))$, then $f \sim fgf$. If $f \sim pgr$, then $fgf \sim (pgr)g(pgr)$ $\sim (pg)(rg)(pgr) \sim (pg)(pg)(rg)(pgr) \sim p(gpgr)(gpgr) \sim pgr \sim f$.

In order to prove the lemma it is therefore enough to show that if $E(g) \subseteq E(f)$, then $f \sim pgr$ for some $p, r \in F'(X)$. We show, by induction on $L(g)$, that in fact $f \sim fgr$ for some r. If $L(g) = 0$, take $r = f$. If $L(g) \geq 1$, then $g = hx$, $x \in X$, $h \in F'(X)$. By inductive hypothesis, $f \sim fhr$. Since $x \in E(g) \subseteq E(f)$ we have $f = f'xf''$ for some $f', f'' \in F'(X)$. It follows that $f \sim (f'xf'')hr = f'(xf''h)r$ $\sim f'(xf''n)(xf''h)r = fg(f''hr)$.

<u>Lemma 1.2</u>: (McLean, 1954). If $f,g,h \in F'(X)$ and $E(h) \subseteq E(f)$ $= E(g)$ then $fg \sim fhg$.

<u>Proof</u>: By lemma 1.1, $fhg \sim (fhg)(fg)(fhg)$ $= (fhgf)(gfhg) \sim fg$.

<u>Corollary 1.3</u>: (Green and Rees, 1952). Assume $f,g \in F(X)$. If $f(0) \sim g(0)$, $f(1) \sim g(1)$, $\overline{f}(0) = \overline{g}(0)$ and $\overline{f}(1) = \overline{g}(1)$, then $f \sim g$. In particular, $f \sim f(0)\overline{f}(0)\overline{f}(1)f(1)$ for all $f \in F(X)$.

<u>Proof</u>: It is enough to show that $f \sim f(0)\overline{f}(0)\overline{f}(1)f(1)$. From lemma 1.2 it follows immediately that if $L(f) \geq L(f(0)\overline{f}(0)\overline{f}(1)f(1))$ then $f \sim f(0)\overline{f}(0)\overline{f}(1)f(1)$. If $L(f) < L(f(0)\overline{f}(0)\overline{f}(1)f(1))$, then there exist $p, q, r \in F'(X)$ such that $f = pqr$, $pq = f(0)\overline{f}(0)$ and $qr = \overline{f}(1)f(1)$. From this it follows that $f \sim pq^2r = f(0)\overline{f}(0)\overline{f}(1)f(1)$.

<u>Lemma 1.4</u>: (Green and Rees, 1952). If $f,g \in F(X)$ and $f \sim g$, then $\overline{f}(0) = \overline{g}(0)$, $\overline{f}(1) = \overline{g}(1)$, $f(0) \sim g(0)$, and $f(1) \sim g(1)$.

<u>Proof</u>: (Green and Rees, 1952). By duality and the description of $\sim$, (1.1), it is enough to show, for $p,q,r \in F'(X)$ with $pqr \in F(X)$, that $\overline{pqr}(0) = \overline{pqqr}(0)$ and $(pqr)(0) \sim (pq^2r)(0)$. The first of these statements is obvious. If $\overline{pqr}(0) \in E(p) \cup E(q)$, then the second statement is also obvious. Otherwise there exists $r' \in F'(X)$ with $(pqr)(0)$ $= pqr' \sim pq^2r' = (pq^2r)(0)$.

Corollary 1.3 and lemma 1.4 give the following:

<u>Theorem 1.5</u>: If $f,g \in F(X)$, then $f \sim g$ iff

(i) $\bar{f}(0) = \bar{g}(0)$, $\bar{f}(1) = \bar{g}(1)$

(ii) $f(0) \sim g(0)$, $f(1) \sim g(1)$.

A mapping π of $F(X)$ into an arbitrary set will be called an <u>invariant</u> iff for all $f,g \in F(X)$, if $f \sim g$ then $\pi(f) = \pi(g)$. A set M of such mappings will be called a <u>complete set of invariants</u> iff for all $f,g \in F(X)$, $f \sim g$ iff $\pi(f) = \pi(g)$ for all $\pi \in M$. The set $\{K\}$, where $K\colon F(X) \to FI(X)$ is the canonical homomorphism, is a trivial complete set of invariants. Theorem 1.5 states that the mappings $f \rightsquigarrow \bar{f}(0)$, $f \rightsquigarrow \bar{f}(1)$, $f \rightsquigarrow K(f(0))$, $f \rightsquigarrow K(f(1))$ constitute a complete set of invariants. In order to obtain a more refined set of invariants, it is necessary to introduce some more notation.

Let $F(2)$ be the free semigroup generated by the set $\{0,1\}$. $F(2)$ consists of all products of finite non-empty sequences in $\{0,1\}$ where two products are equal iff the sequences are the same. The length $L(\alpha)$ of $\alpha \in F(2)$ is defined as before. For $f \in F'(X)$, $\alpha \in F(2)$, we define objects $f(\alpha)$ and $\bar{f}(\alpha)$ by induction on $L(\alpha)$. If $L(\alpha) = 1$, then $f(\alpha)$ and $\bar{f}(\alpha)$ have already been defined. If $L(\alpha) \geq 2$, say $\alpha = \beta i$, $(i = 0,1)$, put

$$f(\alpha) = (f(\beta))(i) \quad \text{and} \quad \bar{f}(\alpha) = \overline{f(\beta)}(i)$$

By induction on $L(\alpha) + L(\beta)$ it is easy to see that for all $\alpha, \beta \in F(2)$,

$$f(\alpha\beta) = (f(\alpha))(\beta). \tag{1.8}$$

$$\bar{f}(\alpha\beta) = \overline{f(\alpha)}(\beta). \tag{1.9}$$

Furthermore, if for $\alpha = i_1 i_2 \ldots i_k$, ($i_j \in \{0,1\}$), we put $\alpha' = i'_1 i'_2 \ldots i'_k$ where $i'_k = 1 - i_k$, then

(1.10) $f(\alpha') = (f^*(\alpha))^*$ and

(1.11) $\bar{f}(\alpha') = \overline{f^*}(\alpha)$.

(1.12) If $L(\alpha) \leq |E(f)|$, then $|E(f)| = |E(f(\alpha))| + L(\alpha)$.

By induction it now follows from Theorem 1.5

Corollary 1.6: The set $\{f \rightsquigarrow \bar{f}(\alpha) \mid \alpha \in F(2)\}$ is a complete set of invariants.

2. Some additional invariants.

From corollary 1.6 it follows that in particular the mappings $f \rightsquigarrow \bar{f}(0^k)$ and $f \rightsquigarrow \bar{f}(1^k)$ are invariants for any k. By (1.2), if $f \sim g$ then $E(f) = E(g)$. Therefore the mappings H and H*, defined by

$$H(f) = \bar{f}(0^{|E(f)|})$$

$$H^*(f) = H(f^*) = \bar{f}(1^{|E(f)|}),$$

are invariants. H(f) is simply the first variable, and H*(f) the last variable which occurs in f. For any $n \geq 0$, define

$$H^{*^n} = \begin{cases} H \text{ if } n \text{ is even} \\ H^* \text{ if } n \text{ is odd} \end{cases}$$

For all $n \geq 0$, $(H^{*^n})^* = H^{*^{(n+1)}}$.

Furthermore, the mappings I and F, defined by

$$I(f) = \bar{f}(0^{|E(f)|})\bar{f}(0^{|E(f)|-1})\ldots\bar{f}(0) \quad \text{and}$$

$$F(f) = (I(f^*))^* = \bar{f}(1)\bar{f}(1^2)\ldots f(1^{|E(f)|}),$$

(where the products are taken in F(X)), are invariants. I(f) indicates the order of first occurrence of the variables of f. For any $n \geq 0$, define

$$I^n = \begin{cases} I \text{ if } n \text{ is even} \\ F \text{ if } n \text{ is odd} \end{cases}$$

The mappings $\bar{I}$ and $\bar{F}$, defined by

$$\bar{I}(f) = f(0)\ \bar{f}(0) \quad \text{and}$$

$$\bar{F}(f) = (\bar{I}(f^*))^* = \bar{f}(1)f(1),$$

(where products are taken in $F(X)$), are not themselves invariants. However, it follows from theorem 1.5 that the composed mappings $\mathcal{K}\circ\bar{I}$ and $\mathcal{K}\circ\bar{F}$ are invariants. For any $n \geq 0$, define

$$\bar{I}^n = \begin{cases} \bar{I} \text{ if } n \text{ is even.} \\ \bar{F} \text{ if } n \text{ is odd.} \end{cases}$$

H, I, E, F were introduced by Tamura (1966).

If φ is any substitution, it is clear that $H^{*n}(\varphi(f)) = \varphi(H^{*n}(f))$ for all $n \geq 0$. In particular, for all $n \geq 0$,

(1.13) If $H^{*n}(f) = H^{*n}(g)$, then $H^{*n}(\varphi(f)) = H^{*n}(\varphi(g))$

Furthermore, we have, for all $n \geq 0$,

(1.14) If $H^{*n}(f) \neq H^{*n}(g)$, then there exists a substitution φ in $(f = g)$ by two variables such that $H^{*n}(\varphi(f)) \neq H^{*n}(\varphi(g))$ and in particular such that $\varphi(f) \not\sim \varphi(g)$.

(1.15) Let k be the largest number r such that $\varphi(\bar{f}(0^r)) = \overline{\varphi(f)}(0)$. Then $\bar{I}(\varphi(f)) = \varphi(f(0^k))\,\varphi(\bar{f}(0^k))$.

<u>Proof</u>: By definition $\overline{\varphi(f)}(0) = \varphi(\bar{f}(0^k))$. Moreover since $E(f(0^k)) = \left\{ \bar{f}(0^i) \mid k + 1 \leq i \leq |E(f)| \right\}$, it follows that

$\overline{\varphi(f)}(0) \notin E(\varphi(f(0^k)))$. Therefore by the definition of $\bar{I}$, $\bar{I}(\varphi(f)) = \varphi(f(0^k))\,\varphi(\bar{f}(0^k))$.

(1.16) Let k be the largest number r such that $\varphi(\bar{f}(1^r)) = \overline{\varphi(f)}(1)$. Then $\bar{F}(\varphi(f)) = \varphi(\bar{f}(1^k))\varphi(f(1^k))$

(1.17) $I^n(\varphi(f)) = I^n(\varphi(I^n(f)))$.

Proof: We will show by induction on $|E(\varphi(f))|$ that $I(\varphi(f)) = I(\varphi(I(f)))$. Then, since $F(f) = (I(f^*))^*$, it follows easily that $F(\varphi(f)) = F(\varphi(F(f)))$.

If $|E(\varphi(f))| = 1$, the result is trivial. Assume $|E(\varphi(f))| > 1$. From (1.15) it follows that $\bar{I}(\varphi(f)) = \varphi(f(0^k))\varphi(\bar{f}(0^k))$ for k the largest number r such that $\varphi(\bar{f}(0^r)) = \overline{\varphi(f)}(0)$. By inductive hypothesis, $I(\varphi(f(0^k))) = I(\varphi(I(f(0^k))))$. Moreover, it is an immediate consequence of the definitions that $\overline{\varphi(I(f))}(0) = \overline{\varphi(f)}(0) = \varphi(\bar{f}(0^k))$. Therefore $I(\varphi(f)) = I(\varphi(f(0^k)))\varphi(\bar{f}(0^k)) = I(\varphi(I(f(0^k))))\overline{\varphi(I(f))}(0) = I(\varphi(I(f)))$, (from the definition of k).

It follows immediately that

(1.18) If $I^n(f) = I^n(g)$, then $I^n(\varphi(f)) = I^n(\varphi(g))$.

(1.19) If $I^n(f) \neq I^n(g)$, then there exists a substitution φ in (f=g) by at most three variables such that $I^n(\varphi(f)) \neq I^n(\varphi(g))$ and hence such that $\varphi(f) \neq \varphi(g)$.

Proof: By (1.7) and (1.14) we can assume that $E(f) = E(g)$, and $H(f) = H(g)$. Since $I(f) \neq I(g)$ it follows that $I(f) = f_1 x f_2 y f_3$, and

$I(g) = g_1 y g_2 x g_3$, with $x, y \in X$, $f_i, g_i \in F'(X)$, $(i = 1,2,3)$. Setting $x = b$, $y = c$, and $z = a$ for $z \in E(f) - \{x,y\}$ gives a substitution φ with $\varphi(f) = abf'(a,b,c)$, $\varphi(g) = acg'(a,b,c)$. It follows that $\overline{\varphi(f)}(0) = c \neq b = \overline{\varphi(g)}(0)$, and therefore that $I(\varphi(f)) \neq I(\varphi(g))$.

Let the equational class of idempotent semigroups determined by the equation $(f = g)$ be $[f = g]$. Similarly if $(f_\alpha = g_\alpha)_{\alpha \in I}$ is a family of equations, let $[(f_\alpha = g_\alpha)_{\alpha \in I}]$ be the equational class of idempotent semigroups determined by the family. Then $[p = q] \subseteq [f = g]$ iff $(p = q) \Rightarrow (f = g)$.

Lemma 1.7: If $E(f) = E(g)$, then
$[f = g] = [(f = \overline{I}(f)\overline{F}(g)), (g = \overline{I}(f)\overline{F}(g))]$.

Proof: It is clear that the right hand side is contained in the left. Conversely, if $(f = g)$ holds in an idempotent semigroup S, then in S, $f = \overline{I}(f)f = \overline{I}(f)\overline{I}(g)\overline{F}(g) = \overline{I}(f)\overline{F}(g)$, by lemma 1.2. By symmetry the second equation also holds in S.

The dual $\mathcal{O}^*$ of a class $\mathcal{O}$ of semigroups is the class defined by $S \in \mathcal{O}^*$ iff $S^* \in \mathcal{O}$. Then

(1.20) $\quad [f = g]^* = [f^* = g^*]$ and

(1.21) $\quad [f = g] \subseteq [p = q]$ iff $[f = g]^* \subseteq [p = q]^*$.

CHAPTER II

SOME RELATIONS

In this chapter, the existence of certain substitutions will be used to define several relations in $F(X)$. These relations will be used in succeeding chapters to classify equations.

1. The relation $\sim_n$.

Definition 2.1: For $f,\ g \in F(X)$, $f \sim_n g$ iff for every substitution φ in $(f = g)$ by less than n variables, $\varphi(f) \sim \varphi(g)$.

That the relation $\sim_n$ is an equivalence relation will be proved later in this chapter (statement 2.4).

Since $f \sim g$ iff $\varphi(f) \sim \varphi(g)$ for all substitutions φ in $(f = g)$, the relation $\sim$ could be included in definition 2.1 by defining $f \sim_\infty g$ iff $f \sim g$.

Since $\varphi(f) \sim \varphi(g)$ if $|E(\varphi(f)) \cup E(\varphi(g))| = 1$, it follows that

(2.1) $\qquad f \sim_2 g$, for all $f,\ g \in F(X)$.

Moreover:

(2.2) $\qquad$ If $f \sim_n g$ then $f \sim_k g$ for all $k \leq n$, ($n = \infty$ included),

(2.3) $\qquad$ If $|E(f) \cup E(g)| < n$ then $f \sim_n g$ iff $f \sim g$.

2. Characterization of $\sim_n$ for $n \geq 3$.

Proposition 2.2: $f \sim_3 g$ iff $E(f) = E(g)$, $H(f) = H(g)$, and $H^*(f) = H^*(g)$.

Proof: Assume first that $E(f) = E(g)$, $H(f) = H(g)$, and $H^*(f) = H^*(g)$, and let φ be any substitution in $(f = g)$ by two variables. From (1.6) and (1.13), it follows that $E(\varphi(f)) = E(\varphi(g))$, $H(\varphi(f)) = H(\varphi(g))$, and $H^*(\varphi(f)) = H^*(\varphi(g))$. Since $|E(\varphi(f))| = 2$, (and $E(\varphi(f)) = E(\varphi(g))$), it follows that $\overline{\varphi(f)}(0) = E(f) - \{H(\varphi(f))\}$ $= E(g) - \{H(\varphi(g))\} = \overline{\varphi(g)}(0)$, and similarly that $\overline{\varphi(f)}(1)$ $= \overline{\varphi(g)}(1)$. Theorem 1.5 then gives

$$\varphi(f) \sim H(\varphi(f))\overline{\varphi(f)}(0)\overline{\varphi(f)}(1)H^*(\varphi(f))$$
$$= H(\varphi(g))\overline{\varphi(g)}(0)\overline{\varphi(g)}(1)H^*(\varphi(g)) \sim \varphi(g),$$ and hence $f \sim_3 g$.

The inverse implication follows immediately from (1.7) and (1.14).

Proposition 2.3: For $n \geq 4$, $f \sim_n g$ iff $I(f) = (I(g)$, $F(f) = F(g)$, and for all $r \geq 1$, $f(0^r) \sim_{n-1} g(0^r)$ and $f(1^r) \sim_{n-1} g(1^r)$.

Proof: Assume first that $I(f) = I(g)$, and that for all $r \geq 1$, $f(0^r) \sim_{n-1} g(0^r)$. Let φ be any substitution in $(f = g)$ by less than n variables. From (1.18) it follows that $I(\varphi(f)) = I(\varphi(g))$ and in particular that $\overline{\varphi(f)}(0) = \overline{\varphi(g)}(0)$. Moreover, by (1.15), $(\varphi(f))(0)$ $= \varphi(f(0^k))$, where k is the largest r with $\varphi(\bar{f}(0^r)) = \overline{\varphi(f)}(0)$. Since k depends only on $I(f)$ and $\overline{\varphi(f)}(0)$, it follows that k is also the largest r with $\varphi(\bar{g}(0^r)) = \overline{\varphi(g)}(0)$ and that $(\varphi(g))(0) = \varphi(g(0^k))$. By construction, φ is a substitution in $(f(0^k) = g(0^k))$ by less than $n - 1$ variables. But $f(0^k) \sim_{n-1} g(0^k)$, and therefore $\varphi(f(0^k)) \sim \varphi(g(0^k))$. By (1.15) it now follows that $\bar{I}(\varphi(f))$ $= \varphi(f(0^k))\varphi(\bar{f}(0^k)) \sim \varphi(g(0^k))\varphi(\bar{g}(0^k)) = \bar{I}(\varphi(g))$.

By duality, (using (1.10) and the definition of F and $\overline{F}$), we obtain $\overline{F}(\varphi(f)) = \overline{F}(\varphi(g))$, under the conditions $F(f) = F(g)$ and, for all $r \geq 1$, $f(1^r) \sim_{n-1} g(1^r)$. It follows therefore from the conditions of the proposition that $\varphi(f) \sim \overline{I}(\varphi(f))\overline{F}(\varphi(f)) \sim \overline{I}(\varphi(g))\overline{F}(\varphi(g)) \sim \varphi(g)$, and therefore that $f \sim_n g$.

Conversely, assume $f \sim_n g$. Since $n \geq 4$, we have $f \sim_4 g$, and therefore by (1.19), $I(f) = I(g)$ and $F(f) = F(g)$. Assume $f(0^r) \not\sim_{n-1} g(0^r)$ for some $r \geq 1$. Then there exists a substitution φ_o in $(f(0^r) = g(0^r))$ by less than $n - 1$ variables such that $\varphi_o(f(0^r)) \not\sim \varphi_o(g(0^r))$. Extend φ_o to a substitution φ in $(f = g)$ by less than n variables, by setting $\varphi(x) = a \in X - E(\varphi(f(0^r)))$ for all $x \in X - E(f(0^r))$. Then $(\varphi(f))(0) = \varphi(f(0^r)) \not\sim \varphi(g(0^r)) = (\varphi(g))(0)$, and therefore $\varphi(f) \not\sim \varphi(g)$, which contradicts $f \sim_n g$. This proves $f(0^r) \sim_{n-1} g(0^r)$ for all $r \geq 1$. Dually we can show that $f(1^r) \sim_{n-1} g(1^r)$ for all $r \geq 1$, completing the proof.

Every $\alpha \in F(2)$ can be written as $\alpha = i_1^{n_1} i_2^{n_2} \ldots i_k^{n_k}$ where $i_j \neq i_{j+1}$ for $1 \leq j < k$ and $n_j \geq 1$, $(1 \leq j \leq k)$. Define $\overline{\alpha} = i_1 i_2 \ldots i_k$. The following corollary is obtained from proposition 2.3 by induction.

<u>Corollary 2.4</u>: If $n \geq 4$, $f \sim_n g$ and $n - L(\overline{\alpha}) = 3$, then $f(\alpha) \sim_{n-L(\overline{\alpha})} g(\alpha)$.

From propositions 2.2 and 2.3, it follows that for all $n \geq 3$, if $f \sim_n g$ then $E(f) = E(g)$. With this remark it is easy to prove the transitivity of $\sim_n$. The remainder of the proof of the following statement is trivial.

(2.4) $\sim_n$ is an equivalence relation for all $n \geq 2$.

3. Characterization of $\sim n$ for (p,q) in the special case $E(p) = E(g)$, $|E(p)| = n \geq 3$.

(i) Notation.

For any natural number n, define

$$d(n) = \frac{1 + (-1)^{n+1}}{2} = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

For $n \geq 3$ define $(02)_n$ and $(12)_n \in F(2)$ as follows:

$$(02)_n = (01)^{\frac{n-d(n)-2}{2}} 0^{d(n)}$$

$$(12)_n = ((02)_n)' = (10)^{\frac{n-d(n)-2}{2}} 1^{d(n)}$$

A straightforward calculation shows

(2.5) $$1(02)_n = (12)_{n+1}$$

$$0(12)_n = (02)_{n+1}$$

If $f \in F'(X)$, let $f^0 = e$, and $f^1 = f$.

For $p \in F(X)$ define

$$p(0,n) = (\bar{p}((02)_n))^{d(n+1)} p((02)_n)(\bar{p}((02)_n))^{d(n)}$$

$$p(1,n) = (\bar{p}((12)_n))^{d(n)} p((12)_n)(\bar{p}((12)_n))^{d(n+1)}$$

(2.6) If $|E(p)| = n \geq 3$ then $|E(p(0,n))| = |E(p(1,n))| = 3$

(2.7) $$(p(0))(1,n) = p(0,n+1)$$

$$(p(1))(0,n) = p(1,n+1)$$

Proof: From the definitions and (2.5), it follows that:

$$(p(0))(1,n) = (\overline{p(0)}((12)_n))^{d(n)}(p(0))((12)_n)(\overline{p(0)}((12)_n))^{d(n+1)}$$
$$= (\bar{p}((02)_{n+1}))^{d(n+2)}p((02)_{n+1})(\bar{p}((02)_{n+1}))^{d(n+1)}$$
$$= p(0,n+1).$$

(2.8) $p(1,n) = (p^*(0,n))^*$

$p(0,n) = (p^*(1,n))^*$

Proof: From (1.10), (1.11), and the definitions, it follows that:

$$(p^*(0,n))^* = ((\overline{p^*}((02)_n))^{d(n+1)}p^*((02)_n)(\overline{p^*}((02)_n))^{d(n)})^*$$
$$= (\overline{p^*}((02)_n))^{d(n)}(p^*((02)_n))^*(\overline{p^*}((02)_n))^{d(n+1)}$$
$$= (\bar{p}((12)_n))^{d(n)}p((12)_n)(\bar{p}((12)_n))^{d(n+1)}$$
$$= p(1,n).$$

(ii) Expansions and the standard expansion.

A product P in F(X) of terms $p(\alpha)$ and $\bar{p}(\alpha)$, $\alpha \in F(2)$, is said to be an expansion for p iff $p \sim P$. Thus corollary 1.3 shows that $p(0)\bar{p}(0)\bar{p}(1)p(1)$ is an expansion for p.

If $f_i \in F'(X)$, $0 \leq i \leq r$, let $\prod_{i=0}^{r}{}^* f_i = f_r f_{r-1} \cdots f_0$.
In order to simplify the notation, if $\alpha \in F(2)$, let
$\underline{p\bar{p}}(\alpha) = p(\alpha)\bar{p}(\alpha)$ and $\underline{\bar{p}p}(\alpha) = \bar{p}(\alpha)p(\alpha)$.

Lemma 2.5: For each $r \geq 0$, the following is an expansion for f:

$$\left(\prod_{i=0}^{r}(\underline{f\bar{f}}((01)^i 00)\bar{f}((01)^{i+1}))\right)f((01)^{r+1})\prod_{j=0}^{r}{}^*(\bar{f}((01)^j 0)\underline{\bar{f}f}((01)^j 1)).$$

Proof: The lemma is proved by induction on r. For r = 0 it is easy to see that $f \sim \underline{f\bar{f}}(00)\bar{f}(01)f(01)\bar{f}(0)\underline{\bar{f}f}(1)$. Assume, by the inductive hypothesis, that the expression is an expansion for some $r = k \geq 0$. Now $f(01)^{k+1} \sim \underline{f\bar{f}}((01)^{k+1}00)\underline{\bar{f}f}((01)^{k+2})\bar{f}((01)^{k}0)\underline{\bar{f}f}((01)^{k+1}1)$ and therefore, from the inductive hypothesis, it follows that

$$f \sim \Big(\prod_{i=0}^{k+1}\underline{f\bar{f}}((01)^{i}00)\bar{f}((01)^{i+1})\Big)f((01)^{k+2})\Big(\prod_{j=0}^{k+1}{}^{*}\,\bar{f}((01)^{j}0)\underline{\bar{f}f}((01)^{j}1)\Big).$$

The expression is therefore an expansion for each $r \geq 0$.

For each $n \geq 3$ and $p \in F(X)$, we define below an expansion $A_n(p)$, called the <u>standard expansion for p of order n.</u> This expansion will be of particular interest, since in general it is the simplest expansion in which $p(0,n)$ and $p(1,n)$ occur.

For all $n \geq 3$, define

$$A_n(p) = (\hat{A}_n(p))(\hat{A}_n(p^*))^*$$

where, for the case $n = 3$,

$$\hat{A}_3(p) = p(03)$$

and, for the case $n \geq 4$,

$$\hat{A}_n(p) = \Big(\prod_{i=0}^{\frac{n+d(n)}{2}-3}\big(\underline{p\bar{p}}((01)^{i}00)\bar{p}((01)^{i+1})\big)\Big)$$

$$\big(\underline{p\bar{p}}((01)^{\frac{n+d(n)}{2}-2}00)\big)^{d(n+1)}p(0,n)\big(\underline{\bar{p}p}((01)^{\frac{n+d(n)}{2}-2}1)\big)^{d(n)}$$

$$\Big(\prod_{j=1}^{\frac{n+d(n)}{2}-(2+d(n))}{}^{*}\big(\bar{p}((01)^{j}0)\underline{\bar{p}p}((01)^{j}1)\big)\Big)\bar{p}(0).$$

An explicit description of $(\hat{A}_n(p^*))^*$ can be given in terms of p by applying (1.10), (1.11), and (2.8). Thus, for $n \geq 4$,

$$(\hat{A}_n(p^*))^* = \bar{p}(1)\Big(\prod_{j=1}^{\frac{n+d(n)}{2}-(2+d(n))} (\underline{p}\bar{p}((10)^j 0)\bar{p}((10)^j 1))\Big)$$

$$(\underline{p}\bar{p}((10)^{\frac{n+d(n)}{2}-2}0))^{d(n)}p(1,n)(\bar{p}\underline{p}((10)^{\frac{n+d(n)}{2}-2}11))^{d(n+1)}$$

$$\Big(\prod_{i=0}^{\frac{n+d(n)}{2}-3}\Big)^* (\bar{p}((10)^{i+1})\underline{p}\bar{p}((10)^i 11))).$$

Lemma 2.6: For all $p \in F(X)$ and $n \geq 3$,

(1) $\bar{I}(p) \sim \bar{I}(A_n(p)) = \hat{A}_n(p)$

(2) $\bar{F}(p) \sim \bar{F}(A_n(p)) = (\hat{A}_n(p^*))^*$

In particular, for all $n \geq 3$, $A_n(p)$ is an expansion for p.

Proof: We show first, by induction on $n \geq 3$, that $\bar{I}(p) \sim \hat{A}_n(p)$.

If $n = 3$, $\bar{I}(p) = p(0)\bar{p}(0) = p(0,3) = \hat{A}_3(p)$.

Assume $k \geq 3$. Then

$$p((02)_k) \sim (p((02)_k))(0)\overline{p((02)_k)}(0)\overline{p((02)_k)}(1)(p((02)_k))(1)$$

$$= (\underline{p}\bar{p}((01)^{\frac{k-d(k)-2}{2}}00))^{d(k)}p(0,k+1)(\bar{p}\underline{p}((01)^{\frac{k-d(k)-2}{2}}1))^{d(k+1)}$$

By replacing $p((02)_k)$ in $\hat{A}_k(p)$ by the above expression, we obtain, for $k = 3$,

$$\hat{A}_3(p) \sim \underline{p}\bar{p}(00)p(0,4)\bar{p}(0) = \hat{A}_4(p),$$

and for $k \geq 4$,

$$\hat{A}_k(p) \sim \Big(\prod_{i=0}^{\frac{k+d(k)}{2}-3} (\underline{p}\bar{p}((01)^i 00)\bar{p}((01)^{i+1}))\Big)$$

$$(\underline{p}\bar{p}((01)^{\frac{k+d(k)}{2}-2}00)\bar{p}((01)^{\frac{k-d(k)-2}{2}}0^{d(k)}))^{d(k+1)}$$

$$(\underline{p\bar{p}}((01)^{\frac{k-d(k)-2}{2}}00))^{d(k+2)}p(0,k+1)(\underline{\bar{p}p}((01)^{\frac{k-d(k)-2}{2}}1))^{d(k+1)}$$

$$(\bar{p}((01)^{\frac{k-d(k)-2}{2}}0^{d(k)})\underline{\bar{p}p}((01)^{\frac{k+d(k)}{2}-2}1))^{d(k)}$$

$$\Big(\prod_{j=1}^{\frac{k+d(k)}{2}\,(2+d(k))}{}^{*}\;(\bar{p}((01)^{j}0)\underline{\bar{p}p}((01)^{j}1)))\bar{p}(0) = \hat{A}_{k+1}(p).$$

It follows therefore by induction that $\bar{I}(p) \sim \hat{A}_n(p)$ for all $n \geq 3$. Since $E(\hat{A}_n(p)) = E((\hat{A}_n(p^*))^*)$, and $E(\hat{A}_n(p)) - \{\bar{p}(0)\} \neq E(\hat{A}_n(p))$, it follows that in fact $\bar{I}(A_n(p)) = \hat{A}_n(p)$, and therefore part (1) has been established.

Part (2) follows from part (1) by duality. In fact, replacing p by p^* in (1) gives that $\bar{F}(p) = (\bar{I}(p^*))^* = (\hat{A}_n(p^*))^*$.

<u>(iii) The characterization.</u>

<u>Proposition 2.7</u>: Assume $p,q \in F(X)$, $E(p) = E(q)$, $|E(p)| = n \geq 3$. Then $p \sim_n q$ iff (for $\alpha \in F(2)$)

(1) $p(\alpha) \sim q(\alpha)$ for all $p(\alpha)$ occurring in $A_n(p)$, $q(\alpha)$ occurring in $A_n(q)$, and $\alpha \neq (02)_n$, $\alpha \neq (12)_n$;

(2) $\bar{p}(\alpha) = \bar{q}(\alpha)$ for all $\bar{p}(\alpha)$ occurring in $A_n(p)$, $q(\alpha)$ occurring in $A_n(q)$, and $\alpha \neq (02)_n$, $\alpha \neq (12)_n$;

(3) $H^{*n+1}(p(0,n)) = H^{*n+1}(q(0,n))$, $H^{*n}(p(1,n)) = H^{*n}(q(1,n))$.

<u>Proof</u>: We show first by induction on n that conditions (1), (2), and (3) imply $p \sim_n q$.

If $n = 3$, (3) implies $H(p(3)) = H(p(0,3)) = H(q(0,3)) = H(q(3))$ and $H^*(p(3)) = H^*(p(1,3)) = H^*(q(1,3)) = H^*(q(3))$. By proposition 2.2 it follows that $A_3(p) \sim_3 A_3(q)$, and therefore, by lemma 2.6, $p \sim_3 q$.

If $n = k + 1 \geq 4$, it is sufficient, by proposition 2.3, to prove that $I(p) = I(q)$, $F(p) = F(q)$, and $p(0^r) \sim_k q(0^r)$, $p(1^r) \sim_k q(1^r)$, $(r \geq 1)$.

The expressions $\bar{p}(0)$, $\bar{p}(00)$, $p(00)$ occur in $p(k+1)$, and therefore by (1) and (2) it follows that $\bar{p}(0) = \bar{q}(0)$, $\bar{p}(00) = \bar{q}(00)$, and $p(00) \sim q(00)$. By theorem 1.5, therefore, $\bar{p}(0^r) = \bar{q}(0^r)$ for all $r \geq 1$ and hence $I(p) = I(q)$. Moreover, it follows that $p(0^r) \sim q(0^r)$ for all $r \geq 2$. We must show that $p(0) \sim_k q(0)$.

By inductive hypothesis it is enough to show that (1), (2), and (3) hold for $p(0)$ and $q(0)$. If $H(\alpha) = 0$, then $\alpha = 0\beta$, and it follows from $p(00) \sim q(00)$ that $(p(0))(\alpha) = (p(00))(\beta) \sim (q(00))(\beta) = (q(0))(\alpha)$ and $\overline{p(0)}(\alpha) = \overline{(p(00))}(\beta) = \overline{(q(00))}(\beta) = \overline{q(0)}(\alpha)$. Therefore we need only verify conditions (1) and (2) in case $H(\alpha) = 1$.

If $\overline{p(0)}(\alpha)$ or $(p(0))(\alpha)$ occur in $A_k(p(0))$; if $H(\alpha) = 1$, and if $\alpha \neq (12)_k$, then a straightforward calculation to check the many cases shows that $p(0\alpha)$ and $\bar{p}(0\alpha)$ occur in $A_{k+1}(p)$, that $q(0\alpha)$ and $\bar{q}(0\alpha)$ occur in $A_{k+1}(q)$ and that $0\alpha \neq (02)_{k+1}$. Therefore $p(0\alpha) \sim q(0\alpha)$, and $\bar{p}(0\alpha) = \bar{q}(0\alpha)$. Hence $(p(0))(\alpha) \sim (q(0))(\alpha)$ and $\overline{p(0)}(\alpha) = \overline{q(0)}(\alpha)$ for all these α, and therefore (1) and (2) hold for $p(0)$ and $q(0)$.

From (2.8), $(p(0))(1,k) = p(0,k+1)$. Moreover, by (3), for p and q, $H_*^{k+2}(p(0,k+1)) = H_*^{k+2}(q(0,k+1))$ and therefore $H_*^k((p(0))(1,k)) = H_*^{k+2}(p(0,k+1))) = H_*^k((q(0))(1,k))$. Since $p(00) \sim q(00)$, it follows by theorem 1.5 that $H_*^{k+1}((p(0))(0,k)) = H_*^{k+1}((q(0))(0,k))$. Therefore (3) also holds for $p(0)$ and $q(0)$. By inductive hypothesis we can therefore conclude that $p(0) \sim_k q(0)$.

By duality, using the definition of F and (1.10), we can now show that $F(p) = F(q)$ and that $p(1^r) \sim_k q(1^r)$ for all $r \geq 1$. The proof that (1), (2) and (3) imply $p \sim_n q$ is therefore complete.

Conversely, we assume $p \sim_n q$, and show (1), (2), and (3). If $n = 3$, (1) and (2) are vacuously true, and (3) follows from proposition 2.2, since $p \sim p(0,3)p(1,3)$.

Assume $n \geq 4$. If $p(\alpha)$ occurs in $A_n(p)$, $\alpha \neq (02)_n, \alpha \neq (12)_n$, then by corollary 2.4, $p(\alpha) \sim_{n-L(\bar{\alpha})} q(\alpha)$. Moreover it can be shown by direct calculation that $n - L(\bar{\alpha}) \geq 3$, and therefore by proposition 2.2 that $E(p(\alpha)) = E(q(\alpha))$. Since, for these α, $L(\alpha) > L(\bar{\alpha})$, it follows that $|E(p(\alpha)) \cup E(q(\alpha))| < n - L(\bar{\alpha})$, by (1.12). Then by (2.3), $p(\alpha) \sim q(\alpha)$.

If $\bar{p}(\alpha)$ occurs in $A_n(p)$, $\alpha \neq (02)_n, \alpha \neq (12)_n$, then it can be shown by corollary 2.4 that if $\alpha = \alpha_0 i^r$, some $r \geq 1$, $i \in \{0,1\}$, then $p(\alpha_0) \sim_4 q(\alpha_0)$. By proposition 2.3 it follows that $\bar{p}(\alpha) = \bar{q}(\alpha)$, and hence (2) is proved.

We establish (3) by induction on $n \geq 3$, using (2.7). It has already been shown that (3) holds if $n = 3$. Assume that (3) holds if $n = k \geq 3$. Then:

$$H_*^{k+2}(p(0,k+1)) = H_*^{k}((p(0))(1,k)) = H_*^{k}((q(0))(1,k)) = H_*^{k+2}(q(0,k+1))$$

and similarly $\quad H_*^{k+1}(p(1,k+1)) = H_*^{k+1}(q(1,k+1))$.

Therefore by induction (3) holds for all n.

This completes the proof of proposition 2.7.

<u>Corollary 2.8</u>: Assume $p,q \in F(X)$ with $|E(p) \cup E(q)| = n \geq 3$ and $p \sim_n q$. Then $\bar{I}(p) \sim \bar{I}(q)$ iff $p(0,n) \sim q(0,n)$, and $\bar{F}(p) \sim \bar{F}(q)$

iff $p(1,n) \sim q(1,n)$. In particular, $p \sim q$ iff $p(0,n) \sim q(0,n)$ and $p(1,n) \sim q(1,n)$.

<u>Proof</u>: Proposition 2.7 and the definition of $A_n(p)$ show that if $p(0,n) \sim q(0,n)$, then $\overline{I}(p) \sim \overline{I}(q)$, and if $p(1,n) \sim q(1,n)$ then $\overline{F}(p) \sim \overline{F}(q)$. The converse is trivial, since $p \rightsquigarrow p(0,n)$ and $p \rightsquigarrow p(1,n)$ are invariants.

4. The relations Θ_n, Θ_n^*, $\overline{\Theta}_n$ and $\overline{\Theta}_n^*$.

Corollary 2.8 shows that the significant parts of p and q with $p \sim_n q$ and $|E(p) \cup E(q)| = n \geq 3$, are $p(0,n)$, $p(1,n)$, $q(0,n)$, and $q(1,n)$. This suggests that the following relations be defined.

<u>Definition 2.9</u>: Assume $p,q \in F(X)$, $p \sim_n q$, $n \geq 3$. Then, for every substitution φ in $(p = q)$ by n variables,

(1) $p\ \Theta_n q$ iff $I^{n+1}((\varphi(p))(0,n)) = I^{n+1}((\varphi(q))(0,n))$,

(2) $p\ \Theta_n^* q$ iff $p^* \Theta_n q^*$, the dual of Θ_n,

(3) $p\ \overline{\Theta}_n q$ iff $(\varphi(p))(0,n) \sim (\varphi(q))(0,n)$,

(4) $p\ \overline{\Theta}_n^* q$ iff $p^* \overline{\Theta}_n q^*$, the dual of $\overline{\Theta}_n$.

<u>Lemma 2.10</u>: Assume $p,q \in F(X)$, $p \sim_n q$, $n \geq 3$. Then for every substitution φ in $(p = q)$ by n variables,

(1) $p\ \Theta_n^* q$ iff $I^n((\varphi(p))(1,n)) = I^n((\varphi(q))(1,n))$,

(2) $p\ \overline{\Theta}_n^* q$ iff $(\varphi(p))(1,n) \sim (\varphi(q))(1,n)$.

<u>Lemma 2.11</u>: If $f \sim_{n+1} g$, then $f\ \overline{\Theta}_n g$, and $f\ \overline{\Theta}_n^* g$.

<u>Proof</u>: If $f \sim_{n+1} g$, then for every substitution φ in $(f = g)$

by n variables it follows that $\varphi(f) \sim \varphi(g)$. In particular, $((\varphi(f))(0,n) \sim ((\varphi(g))(0,n)$, and $((\varphi(f))(1,n) \sim ((\varphi(g))(1,n)$, and therefore $f\ \overline{\theta}_n g$ and $f\ \overline{\theta}^*_n g$.

The following statement is an immediate consequence of definition 2.9:

(2.9) If $p\ \overline{\theta}_n q$, then $p\ \theta_n q$ for all $n \geq 3$.

If $p\ \overline{\theta}^*_n q$, then $p\ \theta^*_n q$ for all $n \geq 3$.

Moreover, from definition 2.9 and corollary 2.8,

(2.10) If $|E(p)| = n \geq 3$, then $p\ \overline{\theta}_n q$ and $p\ \overline{\theta}^*_n q$ imply $p \sim q$.

CHAPTER III

THE n-SKELETON

An equation $(p = q)$, with $|E(p) \cup E(q)| = n$, $p \sim_n q$, $p \not\sim_{n+1} q$, will be called an equation in <u>n essential variables</u>. If $(p = q)$ is such an equation, then there is an idempotent semigroup in which $(p = q)$ does not hold, but for every substitution φ in $(p = q)$ by less than n variables $\varphi(p) \sim \varphi(q)$.

For fixed n, we define the <u>n-skeleton</u> to be that subposet of the lattice of equational classes whose elements are the classes determined by a single equation in n essential variables. In this chapter we will show that for all n, the n-skeleton is in fact a meet subsemilattice of the lattice of equational classes. The n-skeleton will be completely described, and shown to consist of seven elements if $n = 2$, and eight elements if $n \geq 3$.

1. The n-skeleton, $n \geq 3$.

For the proof of the following lemma, it is useful to observe that

(3.1) If $|E(p)| = n$, then

$$E(p) = \left\{ \bar{p}((01)^i 0^j) \mid i \geq 0,\ j \in \{0,1\},\ 1 \leq L((01)^i 0^j) \leq n \right\}.$$

This follows from the fact that the $\bar{p}((01)^i 0^j)$ form a set of n mutually distinct elements.

For $f \in F(X)$, and $n \geq 0$, define

$$f^{*^n} = \begin{cases} f \text{ if } n \text{ is even} \\ f^* \text{ if } n \text{ is odd} \end{cases}$$

From this definition, and the fact that $f^{**} = f$, it follows that $(f^{*^n})^* = f^{*^{n+1}}$ for all $n \geq 0$.

Lemma 3.1: For $f, p \in F(X)$, assume $|E(p)| = n \geq 3$, and let $f(0,n) = (y_1 y_2 \cdots y_n)^{*^{n+1}}$, $y_i \in X$. Let ψ: $F(X) \to F(X)$ be any homomorphism which satisfies the following four properties:

(1) $\psi(\bar{p}((01)^i)) = \bar{f}((01)^i)f\bar{f}((01)^i 00), 1 \leq i \leq \frac{n+d(n)}{2} - (2+d(n))$

(2) $\psi(\bar{p}((01)^j 0)) = \bar{f}f((01)^{j+1}1)\bar{f}((01)^j 0), 0 \leq j \leq \frac{n+d(n)}{2} - 3$

(3) $\psi(\bar{p}((01)^{\frac{n+d(n)}{2}-2} 0^{d(n+1)})) = (y_u \cdots y_1)^{d(n+1)} \bar{f}((01)^{\frac{n+d(n)}{2}-2} 0^{d(n+1)}) (y_1 \cdots y_u)^{d(n)}$ for some $0 \leq u \leq r$.

(4) $E(\psi(p(0,n))) = E(f(0,n))$.

Then the following is an expansion for f:

$f\bar{f}(00)$

$$\psi\left(\prod_{i=0}^{\frac{n+d(n)}{2}-3} (p\bar{p}((01)^i 00)\bar{p}((01)^{i+1}))(p\bar{p}((01)^{\frac{n+d(n)}{2}} 00)p(0,n))^{d(n+1)}\right)$$

$f(0,n)$

$$\psi((p(0,n)\bar{p}p((01)^{\frac{n+d(n)}{2}-2} 1))^{d(n)} \prod_{j=1}^{\frac{n+d(n)}{2}-(2+d(n))^*} (\bar{p}((01)^j 0)\bar{p}p((01)^j 1))\bar{p}(0))$$

$\bar{F}(f)$.

Proof: We show first that

$E(\psi(p((01)^i 0))) = E(f((01)^i 0)), 0 \leq i \leq \frac{n+d(n)}{2} - 3$, and

$$E(\psi(p((01)^j))) = E(f((01)^j)), 1 \leq j \leq \frac{n+d(n)}{2} - (2+d(n)).$$

$$E(\psi(p((01)^i0))) = \bigcup_{i+1 \leq k} (E(\psi(\bar{p}((01)^k))) \cup E(\psi(\bar{p}((01)^k0))))$$

$$= \bigcup_{i+1 \leq k = \frac{n+d(n)}{2} - (2+d(n))} E(\bar{f}((01)^k)\underline{f\bar{f}}((01)^k00)) \cup E(\bar{f}((01)^{\frac{n+d(n)}{2}-2}0^{d(n+1)}))$$

$$\cup E(f(0,n)) \cup \{y_1, \ldots, y_u\} \cup \bigcup_{i+1 \leq k \leq \frac{n+d(n)}{2} - 3} E(\underline{\bar{f}f}((01)^{k+1}1)\bar{f}((01)^k0))$$

$$= \left\{\bar{f}((01)^k0^v) \mid i+1 \leq k,\ v \in \{0,1\},\ L((01)^k0^v) \leq n\right\}.$$

$$= E(f((01)^i0)).$$

Similarly, it can be shown that $E(\psi(p((01)^j))) = E(f((01)^j))$.

By lemma 1.2, it follows that

$$\underline{f\bar{f}}((01)^i00)\underline{\bar{f}f}((01)^{i+1}) \sim \underline{f\bar{f}}((01)^i00)\psi(\underline{p\bar{p}}((01)^i00))\underline{\bar{f}f}((01)^{i+1}),$$

for $0 \leq i \leq \frac{n+d(n)}{2} - 3$, and

$$\underline{f\bar{f}}((01)^j0)\underline{\bar{f}f}((01)^j1) \sim \underline{f\bar{f}}((01)^j0)\psi(\underline{\bar{p}p}((01)^j1))\underline{\bar{f}f}((01)^j1)$$

for $1 \leq j \leq \frac{n+d(n)}{2} - (2+d(n))$.

Moreover, since $E(f(0,n)) = E(\psi(p(0,n)))$,

$$(\underline{f\bar{f}}((01)^{\frac{n+d(n)}{2}-2}00))^{d(n+1)}f(0,n)(\underline{\bar{f}f}((01)^{\frac{n+d(n)}{2}-2}1))^{d(n)}$$

$$\sim (\underline{f\bar{f}}((01)^{\frac{n+d(n)}{2}-2}00)\psi(\underline{p\bar{p}}((01)^{\frac{n+d(n)}{2}-2}00)p(0,n)))^{d(n+1)}f(0,n)$$

$$(\psi(p(0,n)\bar{p}p((01)^{\frac{n+d(n)}{2}-2}1))\underline{f\bar{f}}((01)^{\frac{n+d(n)}{2}-2}1))^{d(n)}.$$

The proof of the lemma is now completed by replacing those terms which occur in $A_n(f)$ by the terms which have been established to be in the relation $\sim$ to them.

<u>Proposition 3.2</u>: Let $p, q, f, g \in F(X)$ satisfy $|E(f)| = n \geq 3$,

$f \sim_n g$, $p \sim_n q$, $p \not\sim_n q$. Then $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

<u>Proof:</u> Since $p \not\sim_n q$, there exists a substitution φ in $(p = q)$ by n variables such that $\varphi(p)\not\sim_n\varphi(q)$. Since $[p = q] \subseteq [\varphi(p) = \varphi(q)]$, we may assume w.l.o.g. that $|E(p)| = n$.

Since $p \not\sim_n q$ and $|E(p)| = n$, $I^{n+1}(p(0,n)) \neq I^{n+1}(q(0,n))$. Since $p \sim_n q$, it follows that $H^{*n+1}(p(0,n)) = H^{*n+1}(q(0,n))$, by proposition 2.7, and therefore, w.l.o.g.,

$p(0,n) = (cb)^{d(n)}p_3(bc)^{d(n+1)}$ and $q(0,n) = (ca)^{d(n)}q_3(ac)^{d(n+1)}$,

where $a,b,c \in X$, and $E(p_3) \cup E(q_3) \subseteq \{a,b,c\}$.

Let $f(0,n) = (y_1 \ldots y_r)^{*n+1}$, and

$g(0,n) = (z_1 \ldots z_s)^{*n+1}$, $y_i, z_i \in X$.

Since $f \sim_n g$, $|E(f)| = n$, it follows from proposition 2.7 that $y_1 = H^{*n+1}(f(0,n)) = H^{*n+1}(g(0,n)) = z_1$. If $f(0,n) \neq g(0,n)$, there exists t such that $y_j = z_j$, $1 \leq j < t$, and $y_t \neq z_t$. (Since $\bar{I}^{n+1}(f(0,n)) = f(0,n)$).

Let ψ be the mapping referred to in lemma 3.1 with the additional conditions that in property (3), $u = 0$, and in property (4),

$\psi(c) = (y_1 \ldots y_{t-1})^{*n+1} = (z_1 \ldots z_{t-1})^{*n+1}$,

$\psi(b) = (y_t \ldots y_r)^{*n+1}$, and

$\psi(a) = z_t$.

Then

$(\psi(p(0,n)))^{d(n+1)}f(0,n)(\psi(p(0,n)))^{d(n)}$

$= \psi(((cb)^{d(n)}p_3(bc)^{d(n+1)})^{d(n+1)}(cb)^{*n+1}((cb)^{d(n)}p_3(bc)^{d(n+1)})^{d(n)})$ [1]

$= \psi((p_3bc)^{d(n+1)}(cb)^{*n+1}(cbp_3)^{d(n)}) \sim \psi(p_3{}^{d(n+1)}(cb)^{*n+1}p_3{}^{d(n)})$

$= \psi(p(0,n))$.

[1] Note: We use the convention $(f^*g^*)^* = g^*f^*$ and not gf, for $f,g \in F(X)$.

Combining this result with lemma 3.1, we obtain $f \sim \underline{f\overline{f}}(00)\psi(A_n(p))\overline{F}(f)$. Therefore $[p = q] \subseteq [\psi(A_n(p)) = \psi(A_n(q))]$ $\subseteq [f = \underline{f\overline{f}}(00)\psi(A_n(q))\overline{F}(f)]$.

Define f_1 by replacing $f(0,n)$ with $\psi(q(0,n))$ in $A_n(f)$. It can be shown that $f_1(\alpha) = f(\alpha)$, and $\overline{f}_1(\alpha) = \overline{f}(\alpha)$, for all $f(\alpha)$, $\overline{f}(\alpha)$ occurring in $A_n(f)$, $\alpha \neq (02)_n$. From lemma 3.1, $f_1 \sim \underline{f\overline{f}}(00)\psi(A_n(q))\overline{F}(f)$, and therefore $[p = q] \subseteq [f = f_1]$. Moreover, $f_1(0,n) = \psi(q(0,n)) = \psi((ca)^{d(n)}q_3(ac)^{d(n+1)})$
$= ((z_1 \ldots z_{t-1})^{*n+1}z_t)^{d(n)}\psi(q_3)(z_t(z_1 \ldots z_{t-1})^{*n+1})^{d(n+1)}$
$= (z_1 \ldots z_t)^{d(n)}\psi(q_3)(z_t \ldots z_1)^{d(n+1)}$. Therefore if $f(0,n) = (y'_1 \ldots y'_{r'})^{*n+1}$, it follows that $y'_j = z_j$ for $0 \leq j \leq t$.

Thus we have found an f_1 from f such that $[p = q] \subseteq [f = f_1]$. Moreover, we have shown that if $f(0,n), f_1(0,n)$, and $g(0,n)$ are written as above, then $y_j = z_j (0 \leq j < t)$, $y_t \neq z_t$, and $y'_j = z_j (0 \leq j \leq t)$.

The method just described can be repeated at most $(s-t)+1$ times to find $f_2, f_3, \ldots f_v$, where $[p = q] \subseteq [f_i = f_{i+1}]$ for $1 \leq i \leq v-1$, and where $f_v(0,n), (=I^{n+1}(f_v(0,n))) = g(0,n)$. Moreover, $f_v(\alpha) = f(\alpha)$ and $\overline{f}_v(\alpha) = \overline{f}(\alpha)$ for all $f(\alpha)$ and $\overline{f}(\alpha)$ occurring in $A_n(f)$, $(\alpha \neq (02)_n)$. Since by proposition 2.7 $f(\alpha) \sim g(\alpha)$ and $\overline{f}(\alpha) = \overline{g}(\alpha)$ for all $f(\alpha)$ and $\overline{f}(\alpha)$ occurring in $A_n(f)$, $(\alpha \neq (02)_n$, $\alpha \neq (12)_n)$, it follows by corollary 2.8 that $f_v \sim \overline{I}(g)\overline{F}(f)$. Hence $[p = q] \subseteq [f = \overline{I}(g)\overline{F}(f)]$.

The dual of proposition 3.2 states that if $p^*, q^*, f^*, g^* \in F(X)$ satisfy $|E(f^*)| = n \geq 3$, and $f^* \sim_n g^*$, $p^* \sim_n q^*$, $p^* \not\sim_n q^*$, then $[p^* = q^*] \subseteq [f^* = \overline{I}(g^*)\overline{F}(f^*)]$. This is equivalent to the statement that if $p,q,f,g \in F(X)$ satisfy $|E(f)| = n \geq 3$, and $f \sim_n g$, $p \sim_n g$, $p \not\theta^*_n q$, then $\left[(p^{**} = q^{**}\right] \subseteq \left[f^{**} = (\bar{I}(g^*)\bar{F}(f^*))^*\right]$. Since $(\bar{F}(f^*))^*(\bar{I}(g^*))^* =$ $= \bar{I}(f)\bar{F}(g)$, we can state the dual of proposition 3.2 as

<u>(Proposition 3.2)*</u>: Let $p,q,f,g \in F(X)$ satisfy $|E(f)| = n \geq 3$ and $f \sim_n g$, $p \sim_n q$, $p \not\theta^*_n q$. Then $\left[p = q\right] \subseteq \left[f = \bar{I}(f)\bar{F}(g)\right]$.

<u>Proposition 3.3</u>: Let $p,q,f,g \in F(X)$ satisfy $|E(f)| = n \geq 3$ and $f \sim_n g$, $f\ \theta_n g$, $p \sim_n q$, $p \not\theta_n q$. Then $\left[p = q\right] \subseteq \left[f = \bar{I}(g)\bar{F}(f)\right]$.

<u>Proof</u>: We may assume w.l.o.g. that $|E(p)| = n$, since there exists a substitution φ in $(p = q)$ by n variables with $\varphi(p) \not\theta_n \varphi(q)$. Moreover we may assume that $p\ \theta_n q$ by proposition 3.2. Therefore $I^{n+1}(p(0,n)) = I^{n+1}(q(0,n))$ and $p(0,n) \sim q(0,n)$, and so w.l.o.g. $p(0,n) = (cbca)^{d(n)}(acbc)^{d(n+1)} = (acbc)_*{}^n$, and $q(0,n) = (abc)_*{}^n$, where $a,b,c \in X$.

Let $f(0,n) = (y_1y_2\ldots y_r)_*{}^{n+1}$, and assume there exist t and j such that $y_t = y_j$ for some $j < t - 1$. Let ψ be the mapping referred to in lemma 3.1, with the additional conditions that in property (3), $u = j - 1$, and in property (4),

$$\psi(c) = y_j = y_t,$$
$$\psi(b) = (y_{j+1}\ldots y_{t-1})_*{}^{n+1}, \text{ and}$$
$$\psi(a) = (y_{t+1}\ldots y_r)_*{}^{n+1}.$$

Then

$$\psi((\bar{p}((01)^{\frac{n+d(n)}{2}-2}))^{d(n)}(\underline{p\bar{p}}((01)^{\frac{n+d(n)}{2}-2}00)p(0,n))^{d(n+1)})$$

$$f(0,n)\psi((p(0,n)\underline{\bar{p}p}((01)^{\frac{n+d(n)}{2}-2}1))^{d(n)}(\bar{p}((01)^{\frac{n+d(n)}{2}-2}0))^{d(n+1)})$$

$$\sim(\bar{f}((01)^{\frac{n+d(n)}{2}-2})y_1\ldots y_{j-1})^{d(n)}(\psi(\underline{p\bar{p}}((01)^{\frac{n+d(n)}{2}-2}00)(acbc)_*{}^n))^{d(n+1)}$$

$$((y_1\cdots y_j)_*^{n+1} y_j (y_{j+1}\cdots y_{t-1})_*^{n+1} y_t (y_{t+1}\cdots y_r)_*^{n+1})_*^{n+1}$$

$$(\psi((acbc)_*^n \underline{\bar{p}p}((01)^{\frac{n+d(n)}{2}-2}1)))^{d(n)}(y_{j-1}\cdots y_1\ \bar{f}((01)^{\frac{n+d(n)}{2}-2}0))^{d(n+1)}$$

$$\sim(\bar{f}((01)^{\frac{n+d(n)}{2}-2}))^{d(n)}(\psi(\underline{p\bar{p}}((01)^{\frac{n+d(n)}{2}-2}00)))^{d(n+1)}$$

$$((y_1\cdots y_j)_*^{n+1} y_j (y_{j+1}\cdots y_{t-1})_*^{n+1} y_t (y_t\cdots y_r)_*^{n+1})_*^{n+1}$$

$$(\psi(\underline{\bar{p}p}((01)^{\frac{n+d(n)}{2}-2}1)))^{d(n)}(\bar{f}((01)^{\frac{n+d(n)}{2}-2}0))^{d(n+1)}$$

$$=\psi((\bar{p}((01)^{\frac{n+d(n)}{2}-2}))^{d(n)}(\underline{p\bar{p}}((01)^{\frac{n+d(n)}{2}-2}00))^{d(n+1)}$$

$$p(0,n)(\underline{\bar{p}p}((01)^{\frac{n+d(n)}{2}-2}1))^{d(n)}(\bar{p}((01)^{\frac{n+d(n)}{2}-2}0))^{d(n+1)}).$$

It follows from lemma 3.1 that $f\sim \underline{f\bar{f}}(00)\psi(A_n(0))\bar{F}(f)$ and therefore $\left[p = q\right]\subseteq\left[\psi(f) = \psi(q)\right] \subseteq \left[f = \underline{f\bar{f}}(00)\psi(A_n(q))\bar{F}(f)\right]$.

Define f_1 by replacing $f(0,n)$ with $(y_1\cdots y_{j-1})^{d(n)}\ (q(0,n))(y_{j-1}\cdots y_1)^{d(n+1)}$ in $A_n(f)$. Then $f_1(\alpha) = f(\alpha)$ and $\bar{f}_1(\alpha) = \bar{f}(\alpha)$ for $f(\alpha)$ and $\bar{f}(\alpha)$ occurring in $A_n(f)$, $(\alpha \neq (02)_n)$. By lemma 3.1, $f_1\sim\underline{f\bar{f}}(00)\psi(A_n(q))\bar{F}(f)$, and therefore $\left[p = q\right]\subseteq\left[f = f_1\right]$. Moreover, $f_1(0,n) = (y_1\cdots y_{j-1})^{d(n)}((y_{t+1}\cdots y_r)_*^{n+1}(y_{j+1}\cdots y_{t-1})_*^{n+1}y_j)_*^n$

$$(y_{j-1}\cdots y_1)^{d(n+1)} = (y_1\cdots y_{j-1})^{d(n)}(y_j y_{j+1}\cdots y_{t-1}y_{t+1}\cdots y_r)_*^{n+1}$$

$$(y_{j-1}\cdots y_1)^{d(n+1)}.$$

We have now found an f_1 from f such that $\left[p = q\right]\subseteq\left[f = f_1\right]$, $I^{n+1}(f(0,n)) = I^{n+1}(f_1(0,n))$, and $L(f_1(0,n)) < L(f(0,n))$. By repeating the process finitely many times, we can finally find an h such that $\left[p = q\right]\subseteq\left[f = h\right]$ and $h(0,n) = I^{n+1}(f(0,n))$.

Similarly, we can find h_o from g such that $\left[p = q\right]\subseteq\left[g = h_o\right]$

and $h_o(0,n) = I^{n+1}(g(0,n)) = I^{n+1}(f(0,n)) = h(0,n)$.

Now $f(\alpha) = h(\alpha)$ and $\overline{f}(\alpha) = \overline{h}(\alpha)$ for all $f(\alpha)$ and $\overline{f}(\alpha)$ occurring in $A_n(f)$, $(\alpha \neq (02)_n)$, and $g(\alpha) = h_o(\alpha), \overline{g}(\alpha) = \overline{h}_o(\alpha)$, for all $g(\alpha)$ and $\overline{g}(\alpha)$ occurring in $A_n(g)$, $(\alpha \neq (02)_n)$. Since $f \sim_n g$, we can conclude by proposition 2.7 and corollary 2.8 that $h \sim \overline{I}(h_o)\overline{F}(h)$. Therefore $[p = q] \subseteq [f = \overline{I}(g)\overline{F}(f)]$.

Using a method similar to that used in determining an equivalent statement for the dual of proposition 3.2, the dual of proposition 3.3 can be shown to be equivalent to

<u>(Proposition 3.3)*</u>: Let $p,q,f,g \in F(X)$ satisfy $|E(f)| = n \geq 3$ and $f \sim_n g$, $f\ \theta_n^* g$, $p \sim_n q$, $p\ \overline{\emptyset}_n^*\ q$. Then $[p = q] \subseteq [f = \overline{I}(f)\overline{F}(g)]$.

<u>Theorem 3.4</u>: Assume $p,q,f,g \in F(X)$ such that $|E(f)| = n \geq 3$, $p \sim_n q$, $f \sim_n g$. Then any of the following eight conditions is sufficient for $[p = q] \subseteq [f = g]$.

(1) $p\ \emptyset_n\ q$, $p\ \emptyset_n^*\ q$

(2) $p\ \overline{\emptyset}_n\ q$, $p\ \emptyset_n^*\ q$, $f\ \theta_n\ g$

(3) $p\ \overline{\emptyset}_n^*\ q$, $p\ \emptyset_n\ q$, $f\ \theta_n^*\ g$

(4) $p\ \emptyset_n^*\ q$, $f\ \overline{\theta}_n\ g$

(5) $p\ \emptyset_n\ q$, $f\ \overline{\theta}_n^*\ g$

(6) $p\ \overline{\emptyset}_n\ q$, $p\ \overline{\emptyset}_n^*\ q$, $f\ \theta_n\ g$, $f\ \theta_n^*\ g$

(7) $p\ \overline{\emptyset}_n^*\ q$, $f\ \overline{\theta}_n\ g$, $f\ \theta_n^*\ g$

(8) $p\ \overline{\emptyset}_n\ q$, $f\ \overline{\theta}_n^*\ g$, $f\ \theta_n\ g$

<u>Proof</u>: By lemma 1.7, it is sufficient to show in each case that the conditions given imply that $(f = \overline{I}(f)\overline{F}(g))$ and $(g = \overline{I}(f)\overline{F}(g))$

hold. This can be done by applying corollary 2.8, proposition 3.2, (proposition 3.2)*, proposition 3.3, and (proposition 3.3)* as needed. We will prove (1) and (4) in detail.

If $p \not\phi_n q$ then, by proposition 3.2, $[p = q] \subseteq [g = \bar{I}(f)\bar{F}(g)]$. If $p \not\phi^*_n q$ then, by (proposition 3.2)*, $[p = q] \subseteq [f = \bar{I}(f)\bar{F}(g)]$. Therefore (1) is proved.

Again, if $p \not\phi^*_n q$ then, by (proposition 3.2)*, $[p = q] \subseteq [f = \bar{I}(f)\bar{F}(g)]$. If $f \bar{\theta}_n g$, it follows by corollary 2.8 (and the definition of $\bar{\theta}_n$) that $\bar{I}(f) \sim \bar{I}(g)$. It is therefore trivial in this case that $(g = \bar{I}(f)\bar{F}(g))$ holds. Hence (4) is proved.

By (2.10), if $f \bar{\theta}_n g$ and $f \bar{\theta}^*_n g$ then $[f = g]$ is the class of all idempotent semigroups. By (2.9), if $f \bar{\theta}_n g$ then $f \theta_n g$, and if $f \bar{\theta}^*_n g$ then $f \theta^*_n g$. Therefore

<u>Corollary 3.5</u>: For $n \geq 3$, there are at most eight distinct equational classes determined by a single equation in n essential variables.

The information contained in theorem 3.4 is summarized in figure 1. In order to show that this represents the n-skeleton, for $n \geq 3$, it is sufficient to show that all the non-trivial inclusions of the classes determined by equations in n essential variables are given by theorem 3.4. This will show in particular that the eight elements of figure 1 are distinct elements of the lattice of equational classes.

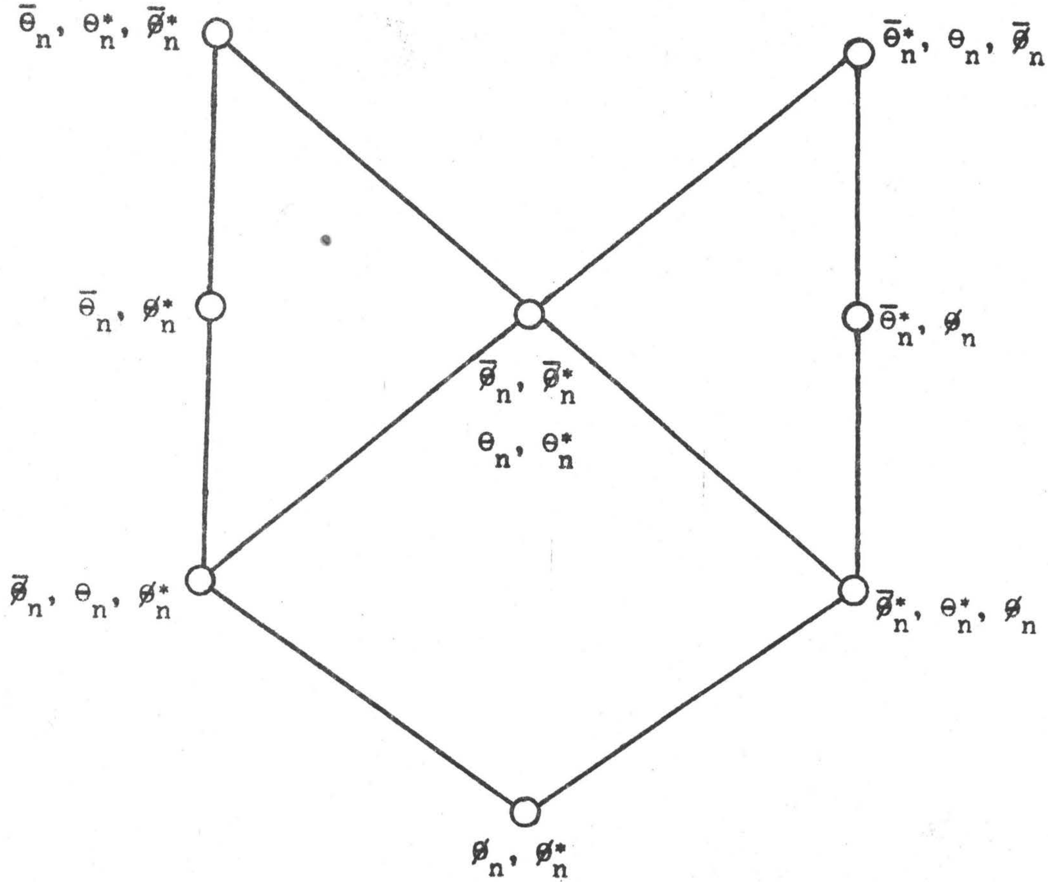

Figure 1: The n-skeleton, $n \geq 3$.

Lemma 3.6: Assume $p,q \in F(X)$ with $p \sim_n q$, and let $\psi: F(X) \longrightarrow F(X)$ be any homomorphism. Then $\psi(p) \sim_n \psi(q)$. In particular, if $|E(\psi(p)) \cup E(\psi(q))| < n$, then $\psi(p) \sim \psi(q)$.

Proof: Since we have to show that for any substitution φ in $(\psi(p) = \psi(q))$ by less than n variables, $\varphi(\psi(p)) \sim \varphi(\psi(q))$, it is enough to show that if $|E(\psi(p)) \cup E(\psi(q))| < n$, then $\psi(p) \sim \psi(q)$.

We will show that $\bar{I}(\psi(p)) \sim \bar{I}(\psi(q))$. The remainder of the proof is dual. If $n = 2$, the result is trivial. If $n = 3$, then w.l.o.g. $E(\psi(p)) = E(\psi(q)) = \{x,y\}$, say. Since $p \sim_3 q$, $H(p) = H(q)$, and therefore $H(\psi(p)) = H(\psi(q)) = x$, say. It follows that $\bar{I}(\psi(p)) \sim xy \sim \bar{I}(\psi(g))$.

We proceed by induction on n. Assume $n \geq 4$. Then there are two cases to consider. If $E(\psi(p)) - \{\psi(\bar{p}(0))\} = E(\psi(p))$, then $E(\psi(p)) = E(\psi(p\bar{p}(00)))$, and therefore since $n \geq 4$, $\bar{I}(\psi(p))w = \psi(p\bar{p}(00)) \sim \psi(q\bar{q}(00)) = \bar{I}(\psi(q))w$, and hence $\bar{I}(\psi(p)) \sim \bar{I}(\psi(q))$. If $E(\psi(p)) - \{\psi(\bar{p}(0))\} \neq E(\psi(p))$, then $E(\psi(p(0))) < n-1$. Moreover by proposition 2.3, $\bar{p}(0) \sim_{n-1} q(0)$, and it follows by induction that $\psi(p(0)) \sim \psi(q(0))$. Also by proposition 2.3, $\bar{p}(0) = \bar{q}(0)$, and therefore $\bar{I}(\psi(p)) \sim \psi(p(0))\psi(\bar{p}(0)) \sim \bar{I}(\psi(q))$.

Proposition 3.7: Let $f,g,p,q \in F(X)$ satisfy $|E(p)| = |E(f)| = n \geq 3$, $f \sim_n g$, $p \sim_n q$, $p\, \theta_n\, q$, $f\, \not\theta_n\, g$. Then $[p = q] \not\subseteq [f = g]$.

Proof: $[p = q] \subseteq [f = g]$ iff there exists a finite sequence $h_0, h_1, \ldots, h_n \in F(X)$ such that $h_0 = f$, $h_n = g$, and such that for each $0 \leq i < n$, there exists u_i, $v_i \in F'(X)$ and homomorphism $\Psi_i: F(X) \to F(X)$ with $h_i = u_i(\Psi_i(p))v_i$ and $h_{i+1} = u_i(\Psi_i(q))v_i$, or $h_i = u_i(\Psi_i(q))v_i$ and $h_{i+1} = u_i(\Psi(p))v_i$. Since $f \bar{\theta}\!\!\!/_n\, g$, it follows that $\bar{I}(f) \not\sim \bar{I}(g)$. It is therefore enough to show that if $f = u(\Psi(p))v$, $f_1 = u(\Psi(q))v$, for some $u, v \in F'(X)$ and homomorphism $\Psi: F(X) \longrightarrow F(X)$, then $\bar{I}(f) \sim \bar{I}(f_1)$.

There are two cases. If $|E(\Psi(p))| < n$, then by lemma 3.6, $\bar{I}(f) = u(\Psi(p))v_1 \sim u(\Psi(q))v_1 = \bar{I}(g)$, for some $v_1 \in F'(X)$. If $|E(\Psi(p))| = n$, then since $p\, \bar{\theta}_n\, q$, $\bar{I}(\Psi(p)) \sim \bar{I}(\Psi(q))$, and therefore $\bar{I}(f)w = u\bar{I}(\Psi(p)) \sim u\bar{I}(\Psi(q)) = \bar{I}(f_1)w_1$ for some $w, w_1 \in F'(X)$. Hence $\bar{I}(f) \sim \bar{I}(f_1)$.

Proposition 3.8: Let $f, g, p, q \in F(X)$ satisfy $|E(f)| = |E(p)| = n \geq 3$, $f \sim_n g$, $p \sim_n q$, $p\, \theta_n\, q$, $f\, \theta\!\!\!/_n\, g$. Then $[p = q] \not\subseteq [f = g]$.

Proof: Since $f\, \theta\!\!\!/_n\, g$, it follows that $I^{n+1}(f(0,n)) \not\sim I^{n+1}(g(0,n))$. Since $|E(f(0,n))| = 3$, $E(f(0,n)) = E(g(0,n))$, and $H^{*n+1}(f(0,n)) = H^{*n+1}(g(0,n))$, it follows that $H^{*n}(f(0,n)) \not\sim H^{*n}(g(0,n))$. As in the proof of proposition 3.7, it is therefore enough to show that if $f = u(\Psi(p))v$ and $g = u(\Psi(q))v$, for $u, v \in F'(X)$ and homomorphism $\Psi: F(X) \to F(X)$, then $H^{*n}(f(0,n)) = H^{*n}(g(0,n))$.

Since $p \sim_n q$, and $|E(p)| = n \geq 3$, we can assume w.l.o.g., by proposition 2.7, that $p(\alpha) = q(\alpha)$ and $\bar{p}(\alpha) = \bar{q}(\alpha)$ for all $p(\alpha)$ and $\bar{p}(\alpha)$ occurring in $A_n(p)$, $\alpha \neq (02)_n$. Moreover since $p\, \theta_n\, q$ it follows that $I^{n+1}(p(0,n)) = I^{n+1}(q(0,n))$. Let $u(\Psi(p)) = (x_1 x_2 \ldots x_r)^{*n}$ and $u(\Psi(q)) = (y_1 y_2 \ldots y_s)^{*n}$. Then, if $H^{*n}((\Psi(p))(0,n)) = x_j$, then w.l.o.g. $x_k = y_k$ for all $k \leq j$.

Let $H_*^n((u(\Psi(p)))(0,n)) = x_i$. Then $i \leq j$, and $H_*^n((u(\Psi(p)))(0,n)) = x_i = x_j = H_*^n((u(\Psi(q)))(0,n))$. Therefore $H_*^n(f(0,n)) = H_*^n((u(\Psi(p)))(0,n)) = H_*^n(g(0,n))$.

Theorem 3.9: The elements of figure 1 are distinct elements of the lattice of equational classes of idempotent semigroups. The order indicated in figure 1 is the restriction of the lattice order. In particular, figure 1 represents the n-skeleton for $n \geq 3$.

Proof: The theorem is an immediate consequence of propositions 3.7 and 3.8, and their duals.

Theorem 3.10: For $n \geq 3$, the n-skeleton is a meet subsemilattice of the lattice of equational classes.

Proof: Since the poset of figure 1 is a meet semilattice and a subposet of the lattice of equational classes, it is sufficient to show that the meets in figure 1 are actually meets in the lattice. The non-trivial inclusions can be established in each case by appropriate application of propositions 3.2 and 3.3, their duals, and lemma 1.7.

For example let $p_i, q_i \in F(X)$, $i = 1,2,3$, and assume $|E(p_i)| = n \geq 3$, $p_i \sim_n q_i$, $i = 1,2,3$, and that $p_1 \,\overline{\theta}_n\, q_1$, $p_1 \,\emptyset^*_n\, q_1$, $p_2 \,\overline{\emptyset}_n\, q_2$, $p_2 \,\overline{\emptyset}^*_n\, q_2$, $p_2 \,\theta_n\, q_2$, $p_2 \,\theta^*_n\, q_2$, $p_3 \,\overline{\emptyset}_n\, q_3$, $p_3 \,\theta_n\, q_3$, $p_3 \,\emptyset^*_n\, q_3$. We must show that $[p_1 = q_1] \cap [p_2 = q_2] \subseteq [p_3 = q_3]$. Since $p_1 \,\emptyset^*_n\, q_1$, it follows by (proposition 3.2)*, that $[p_1 = q_1] \subseteq [p_3 = \overline{I}(p_3)\overline{F}(q_3)]$. Moreover since $p_2 \,\overline{\emptyset}_n\, q_2$ and $p_3 \,\theta_n\, q_3$, then by proposition 3.3 $[p_2 = q_2] \subseteq [p_3 = \overline{I}(q_3)\overline{F}(p_3)]$. Therefore by lemma 1.7,

$$[p_1 = q_1] \cap [p_2 = q_2] \subseteq [p_3 = q_3] .$$

2. The 2-skeleton.

The 2-skeleton has essentially been described by Tamura (1966) and Kimura (1958,IV). Kimura also listed all equations in three variables (without proofs), and has therefore described the 3-skeleton.

Proposition 3.10: (Tamura,1966). Let $(f = g)$ be any equation in 2 essential variables, and let $a, b \in X$. Then

(1) If $E(f) \neq E(g)$, $H(f) \neq H(g)$, $H^*(f) \neq H^*(g)$,

then $[f = g] = [a = b]$,

(2) If $E(f) \neq E(g)$, $H(f) = H(g)$, $H^*(f) \neq H^*(g)$,,

then $[f = g] = [a = ab]$,

(3) If $E(f) \neq E(g)$, $H(f) \neq H(g)$, $H^*(f) = H^*(g)$,

then $[f = g] = [a = ba]$,

(4) If $E(f) \neq E(g)$, $H(f) = H(g)$, $H^*(f) = H^*(g)$,

then $[f = g] = [a = aba]$,

(5) If $E(f) = E(g)$, $H(f) \neq H(g)$, $H^*(f) \neq H^*(g)$,

then $[f = g] = [ab = ba]$,

(6) If $E(f) = E(g)$, $H(f) = H(g)$, $H^*(f) \neq H^*(g)$,

then $[f = g] = [ab = aba]$,

(7) If $E(f) = E(g)$, $H(f) \neq H(g)$, $H^*(f) = H^*(g)$,

then $[f = g] = [ab = bab]$.

Moreover, these seven equational classes are distinct, and are all the classes determined by an equation in 2 essential variables.

Proof: It follows from the idempotent law that there are only nine equations in 2 essential variables. These are the seven given in the statement of proposition 3.11, together with (aba = b) and (aba = bab). However it is easy to check that

$\left[aba = b\right] = \left[a = b\right]$ and $\left[aba = bab\right] = \left[ab = ba\right]$.

It is also easy to check that the seven classes listed are distinct. For example, if $H(f) = H(g)$, and $H(f_1) \neq H(g_1)$, then $\left[f = g\right] \nsubseteq \left[f_1 = g_1\right]$.

The results of this proposition are summarized in figure 2.

Theorem 3.12: The poset given in figure 2 is a meet subsemilattice of the lattice of equational classes, and is in fact the 2-skeleton.

Proof: From proposition 3.11, it is only necessary to show that the meets in figure 2 are meets in the lattice of equational classes. It is essentially trivial to check each case.

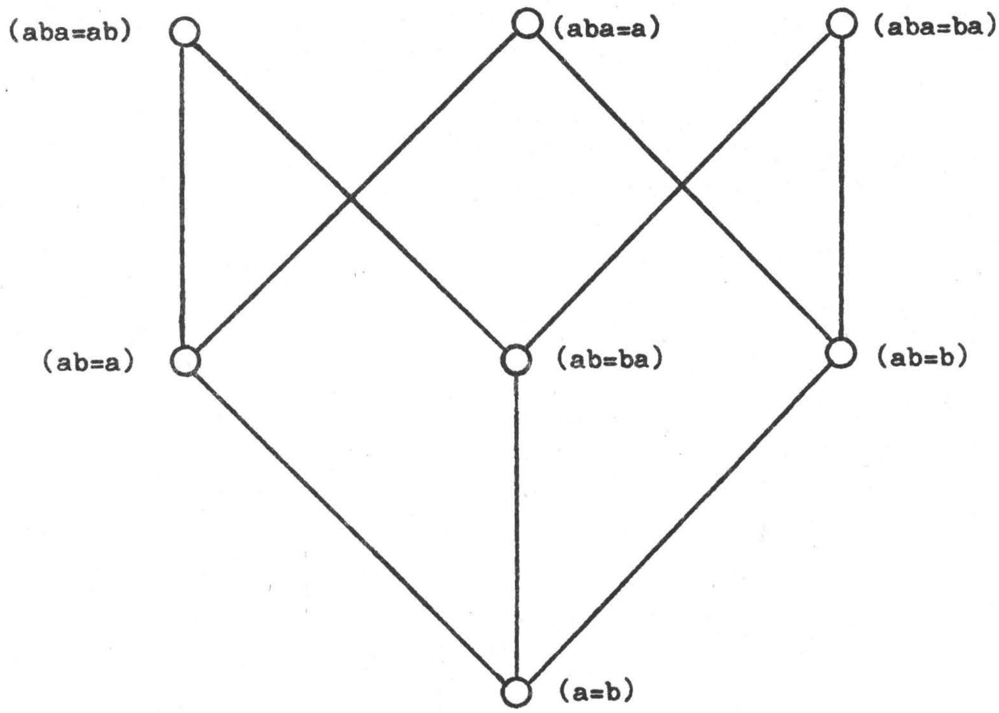

Figure 2: The 2-skeleton

CHAPTER IV

THE LATTICE

The union of the n-skeletons, for all $n \geq 2$, together with the restriction of the order in the lattice of equational classes of idempotent semigroups, will be called the skeleton of the lattice. Since we have previously described the n-skeletons for each $n \geq 2$, the description of the skeleton is completed by finding the inclusions which hold among the elements of n-skeletons for different n. It will be shown that in a certain sense the (n+1)-skeleton covers the n-skeleton for all $n \geq 2$. A description of the classes of idempotent semigroups which may be described by a single equation will then be completed by considering equations $(f = g)$ for which there exists a substitution φ by less than $|E(f) \cup E(g)|$ variables such that $\varphi(f) \not\sim \varphi(g)$, and relating these equations to the elements of the skeleton. Finally, it is shown that every class determined by finitely many equations is determined by one equation, and from this and the description of the skeleton it is concluded that every class may be described by a single equation.

1. Extension of propositions 3.2 and 3.3.

Proposition 4.1: (Extension of proposition 3.2). For $n \geq 3$ let $p,q,f,g \in F(X)$ satisfy $p \sim_n q$, $p \not\sim_n q$, $f \sim_n g$. Then $\left[p = q\right] \subseteq \left[f = \bar{I}(g)\bar{F}(f)\right]$.

Proof: Let $E(f) = n + k$ where w.l.o.g. $k \geq 0$. The proof is by induction on $k \geq 0$.

Define f_1 by replacing $f(0,n)$ with $g(0,n)$ in $A_n(f)$. From the proof of proposition 3.2, we can conclude that $\left[p = q\right] \subseteq \left[f = f_1\right]$ (since the cardinality restriction on $E(f)$ in proposition 3.2 was not used to prove this statement).

It is therefore enough to show that

$$\left[p = q\right] \subseteq \left[f = \bar{I}(g_1)\bar{F}(f)\right] \tag{4.1}$$

where g_1 is defined by replacing $g(0,n)$ with $f(0,n)$ in $A_n(g)$.

As was shown in the proof of proposition 3.2, (4.1) is trivial if $k = 0$, since in that case $g(\alpha) \sim f(\alpha)$ and $\bar{g}(\alpha) = \bar{f}(\alpha)$ for all $g(\alpha)$ and $\bar{g}(\alpha)$ occurring in $A_n(g)$, ($\alpha \neq (02)_n$, $\alpha \neq (12)_n$).

Assume by induction that (4.1) is true for $k-1$ (for some $k \geq 1$). Let r be an arbitrary but fixed integer with $0 \leq r \leq \frac{n+d(n)}{2} - (2+d(n))$. Let φ be a substitution which satisfies the properties

$$\varphi(\bar{f}((01)^r0)) = \bar{f}((01)^r00)$$

$$\varphi(x) = x \text{ for all } x \in E(f) - \left\{\bar{f}((01)^r0)\right\} .$$

Then φ is a substitution in $(f = g)$ by $(n+k-1)$ variables.

By lemma 2.5,

$$\varphi(f) \sim \varphi\Big(\prod_{i=0}^{r}(\underline{f\bar{f}}((01)^i00)\bar{f}((01)^{i+1}))f((01)^{r+1})\prod_{j=0}^{r}{}^{*}(\bar{f}((01)^j0)\underline{\bar{f}f}((01)^j1))\Big).$$

Now $\varphi(\bar{f}((01)^r0)) = \bar{f}((01)^r00) \in E(f((01)^r0))$, and therefore

$$E(\varphi(\underline{\bar{f}f}((01)^{r+1})\bar{f}((01)^r))) = E(\varphi(\underline{f\bar{f}}((01)^r00))) = E(\varphi(\underline{\bar{f}f}((01)^r1))).$$

It follows by lemma 1.1 that

$$\varphi(\underline{f\bar{f}}((01)^r00)\underline{\bar{f}f}((01)^{r+1})\bar{f}((01)^r0)\underline{\bar{f}f}((01)^r1)) \sim \varphi(\underline{f\bar{f}}((01)^r00)\underline{\bar{f}f}((01)^r1)).$$

Moreover $\varphi(\underline{f\bar{f}}((01)^r00)) = \underline{f\bar{f}}((01)^r00)$. From these results it follows that

$$\varphi(f)\sim\varphi\Big(\prod_{i=0}^{r-1}\big(\underline{f\bar{f}}((01)^{i}00)\bar{f}((01)^{i+1})\big)\Big)\underline{f\bar{f}}((01)^{r}00)$$
$$\varphi\Big(\underline{\bar{f}f}((01)^{r}1)\prod_{j=0}^{r-1}{}^{*}\big(\bar{f}((01)^{j}0)\underline{\bar{f}f}((01)^{j}1)\big)\Big).$$

Since $\varphi(f)((01)^{r})\sim\underline{\varphi(f)\overline{\varphi(f)}}((01)^{r}0)\underline{\overline{\varphi(f)}\varphi(f)}((01)^{r}1)$,

it follows from lemma 2.5 that

$$\varphi(f)\sim\prod_{i=0}^{r-1}\big(\underline{\varphi(f)\overline{\varphi(f)}}((01)^{i}00)\overline{\varphi(f)}((01)^{i+1})\big)\underline{\varphi(f)\overline{\varphi(f)}}((01)^{r}0)$$
$$\underline{\overline{\varphi(f)}\varphi(f)}((01)^{r}1)\prod_{j=0}^{r-1}{}^{*}\big(\overline{\varphi(f)}((01)^{j}0)\underline{\overline{\varphi(f)}\varphi(f)}((01)^{j}1)\big).$$

By comparing these two expansions for $\varphi(f)$ we can conclude that

$$(4.2)\qquad (\varphi(f))((01)^{r}0)\sim f((01)^{r}00).$$

Consider a homomorphism $\psi: F(X)\longrightarrow F(X)$ which satisfies the following properties:

$$(4.3)\qquad \begin{aligned}
&\psi(x) = x \text{ for all } x\in E((\varphi(f))((01)^{r}0))\\
&\psi(\overline{\varphi(f)}((01)^{i})) = \bar{f}((01)^{i})\underline{f\bar{f}}((01)^{i}00),\ 1\le i\le r-1\\
&\psi(\overline{\varphi(f)}((01)^{r})) = \bar{f}((01)^{r})\\
&\psi(\overline{\varphi(f)}((01)^{r}0)) = \bar{f}((01)^{r}00)\underline{f\bar{f}}((01)^{r+1})\bar{f}((01)^{r}0)\\
&\psi(\overline{\varphi(f)}((01)^{j}0)) = \underline{\bar{f}f}((01)^{j+1}1)\bar{f}((01)^{j}0),\ 0\le j\le r-1
\end{aligned}$$

From (4.2), it follows that

$$(4.4)\qquad \psi((\varphi(f))((01)^{r}0))\sim f((01)^{r}00)$$

Using (4.3) and (3.1), we can conclude that

$E(\psi(\underline{\varphi(f)\overline{\varphi(f)}}((01)^{i}00))) = E(\underline{f\bar{f}}((01)^{i}00))$, $0\le i\le r-1$, and that

$E(\psi(\underline{\overline{\varphi(f)}\varphi(f)}((01)^j1))) = E(\underline{\bar{f}f}((01)^j1)),\ 1 \leq j \leq r.$

Therefore, for $0 \leq i \leq r-1$,

$$(4.5)\qquad \underline{f\bar{f}}((01)^i00)\underline{\bar{f}f}((01)^{i+1})$$
$$\sim \underline{f\bar{f}}((01)^i00)\psi(\underline{\varphi(f)\overline{\varphi(f)}}((01)^i00))\underline{\bar{f}f}((01)^{i+1}),$$

and for $0 \leq j \leq r$

$$(4.6)\qquad \underline{\bar{f}f}((01)^j0)\underline{\bar{f}f}((01)^j1)$$
$$\sim \underline{\bar{f}f}((01)^j0)\psi(\underline{\overline{\varphi(f)}\varphi(f)}((01)^j1))\underline{\bar{f}f}((01)^j1).$$

Since $f(01)^r \sim \underline{f\bar{f}}((01)^r00)\underline{\bar{f}f}((01)^{r+1})\bar{f}((01)^r0)\underline{\bar{f}f}((01)^r1)$,

it follows from lemma 2.5 that

$$(4.7)\qquad f \sim \prod_{i=0}^{r-1}(\underline{f\bar{f}}((01)^i00)\bar{f}((01)^{i+1}))f((01)^r00)\bar{f}((01)^r00)$$
$$\underline{\bar{f}f}((01)^{r+1})\prod_{j=0}^{r}{}^{*}(\bar{f}((01)^j0)\underline{\bar{f}f}((01)^j1)).$$

By (4.5) and (4.6), it follows from (4.7) that

$$f \sim \underline{f\bar{f}}(00)\prod_{i=0}^{r-1}(\psi(\underline{\varphi(f)\overline{\varphi(f)}}((01)^i00))\bar{f}((01)^{i+1})\underline{f\bar{f}}((01)^{i+1}00))$$
$$\underline{\bar{f}f}((01)^{r+1})\bar{f}((01)^r0)\psi(\underline{\overline{\varphi(f)}\varphi(f)}((01)^r1))$$
$$\prod_{j=1}^{r}{}^{*}(\underline{\bar{f}f}((01)^j1)\bar{f}((01)^{j-1}0)\psi(\underline{\overline{\varphi(f)}\varphi(f)}((01)^{j-1}1)))\bar{F}(f)$$

and therefore, by (4.3) and (4.4), that

$$(4.8)\qquad f \sim \underline{f\bar{f}}(00)\psi(\varphi(f))\bar{F}(f).$$

Since $E(\varphi(f)) = n+k-1$, it follows by induction that

$$[p = q] \subseteq [\psi(\varphi(f)) = \bar{I}(\psi(\varphi(g)))\bar{F}(\psi(\varphi(f)))]$$

Since $\bar{I}(\psi(\varphi(g)))\bar{F}(\psi(\varphi(f)))\bar{F}(f) \sim \psi(\varphi(g))\bar{F}(f)$, it therefore follows from (4.8) that $[p = q] \subseteq [f = \underline{f\bar{f}}(00)\psi(\varphi(g))F(f)]$.

Since $\varphi(f) \sim_n \varphi(g)$, and $i \leq r$ where $r \leq \frac{n+d(n)}{2} - (2+d(n))$, it follows by corollary 2.4 that $\varphi(f)((01)^{i-1}0) \sim_4 \varphi(g)((01)^{i-1}0)$ and therefore by proposition 2.3 that

$$(4.9) \qquad \overline{\varphi(f)}((01)^i) = \overline{\varphi(g)}((01)^i) \text{ for } 1 \leq i \leq r.$$

Similarly, since $j \leq r$, it follows that $\varphi(f)((01)^j) \sim_4 \varphi(g)((01)^j)$, and therefore that

$$(4.10) \qquad \overline{\varphi(f)}((01)^j0) = \overline{\varphi(g)}((01)^j0) \text{ for } 0 \leq j \leq r.$$

Moreover $\varphi(f)((01)^r0) \sim_3 \varphi(g)((01)^r0)$ and therefore $E(\varphi(f)((01)^r0)) = E(\varphi(g)((01)^r0))$. Hence (4.3) holds with f replaced by g. It follows that

$$(4.11) \qquad \underline{f\bar{f}}(00)\psi(\varphi(g))\bar{F}(f)$$

$$= \underline{f\bar{f}}(00)\prod_{i=0}^{r-1}(\psi(\underline{\varphi(g)\overline{\varphi(g)}}((01)^i00))\bar{f}((01)^{i+1})\underline{f\bar{f}}((01)^{i+1}00))\underline{\bar{f}f}((01)^{r+1})$$

$$\bar{f}((01)^r0)\psi(\underline{\overline{\varphi(g)}\varphi(g)}((01)^r1))$$

$$\prod_{j=1}^{r}{}^{*}(\underline{\bar{f}f}((01)^j1)\bar{f}((01)^{j-1}0)\psi(\underline{\overline{\varphi(g)}\varphi(g)}((01)^{j-1}1)))\bar{F}(f).$$

Moreover, with the method used to establish (4.5) and (4.6), we can prove the statements obtained from (4.5) and (4.6) by replacing $\varphi(f)$ with $\varphi(g)$ and $\overline{\varphi(f)}$ with $\overline{\varphi(g)}$. We can also show $\psi(\varphi(g)((01)^r0)) \sim g((01)^r00)$ by a method similar to that used in establishing (4.4).

From these results, and (4.11), we can conclude

(4.12) For any $0 \le r \le \frac{n+d(n)}{2} - (2+d(n))$, $[p = q]$

$$\subseteq \Big[f = \prod_{i=0}^{r-1} (f\bar{f}((01)^i 00)\bar{f}((01)^{i+1}))g((01)^r 00)\bar{f}((01)^r 00)\bar{f}f((01)^{r+1})$$

$$\prod_{j=0}^{r}{}^{*} (\bar{f}((01)^j 0)\bar{f}f((01)^j 1)) \Big] .$$

Following the pattern of proof leading to statement (4.12), we now let φ be a substitution with the properties

$$\varphi(\bar{f}((01)^s)) = \bar{f}((01)^s 1)$$

$$\varphi(x) = x \text{ for all } x \in E(f) - \{\bar{f}((01)^s)\}$$

where s is an arbitrary but fixed integer with $1 \le s \le \frac{n+d(n)}{2} - 2$. As we proved statement (4.2), we can now establish

(4.13) $\varphi(f)((01)^s) \sim f((01)^s 1)$.

We consider a homomorphism $\psi : F(X) \longrightarrow F(X)$ which satisfies the following properties:

$$\psi(x) = x \text{ for all } x \in E(\varphi(f)((01)^s))$$

$$\psi(\overline{\varphi(f)}((01)^i)) = \bar{f}((01)^i)f\bar{f}((01)^i 00),\ 1 \le i \le s-1$$

(4.14) $$\psi(\overline{\varphi(f)}((01)^{s-1}0)) = \bar{f}((01)^{s-1}0)$$

$$\psi(\overline{\varphi(f)}((01)^s)) = \bar{f}((01)^s)f\bar{f}((01)^s 0)\bar{f}((01)^s 1)$$

$$\psi(\overline{\varphi(f)}((01)^j 0)) = \bar{f}f((01)^{j+1}1)\bar{f}((01)^j 0),\ 0 \le j \le s-2$$

Using (4.13) and (4.14), and the pattern of proof of (4.12), we can finally arrive at a statement which corresponds to statement (4.12),

(4.15) For any $1 \le s \le \frac{n+d(n)}{2} - 2$, $[p = q]$

$$\subseteq \Big[f = \prod_{i=0}^{s-1} (\underline{f\bar{f}}((01)^i 00)\bar{f}((01)^{i+1}))\underline{f\bar{f}}((01)^s 0)\bar{f}((01)^s 1)$$

$$g((01)^s 1) \prod_{j=0}^{s-1}{}^{*} (\bar{f}((01)^j 0)\underline{\bar{f}f}((01)^j 1))\Big] .$$

Since $f \sim_n g$, it follows from corollary 2.4 that $f(01)^t \sim_4 g(01)^t$ for $0 \leq t \leq \frac{n+d(n)}{2} - (2+d(n))$, and $f((01)^{u-1}0) \sim_4 g((01)^{u-1}0)$ for $1 \leq u \leq \frac{n+d(n)}{2} - 2$, and therefore, by proposition 2.3, that

$$\left.\begin{array}{ll} (4.16) & \bar{f}((01)^t 0) = \bar{g}((01)^t 0), \\ (4.17) & \bar{f}((01)^t 00) = \bar{g}((01)^t 00), \\ (4.18) & \bar{f}((01)^t 1) = \bar{g}((01)^t 1), \text{ and} \end{array}\right\} \; 0 \leq t \leq \frac{n+d(n)}{2} - (2+d(n))$$

$$(4.19) \qquad \bar{f}((01)^u) = \bar{g}((01)^u),\ 1 \leq u \leq \frac{n+d(n)}{2} - 2.$$

The statements (4.12) and (4.15), together with (4.16), (4.17), (4.18), and (4.19), can now be used to establish (4.1) as follows:

By (4.12) and (4.17), with $r = t = 0$, it follows that $\big[p = q\big] \subseteq \big[f = \underline{g\bar{g}}(00)\underline{f\bar{f}}(01)\bar{f}(0)\bar{F}(f)\big]$. By (4.19), with $u = 1$, $\bar{f}(01) = \bar{g}(01)$, and therefore $\big[p = q\big] \subseteq \big[f = \underline{g\bar{g}}(00)\bar{g}(01)f(01)\bar{f}(0)\bar{F}(f)\big]$.

Let $\hat{f}_{r-1} = \prod_{i=0}^{r-1}(\underline{g\bar{g}}((01)^i 00)\bar{g}((01)^{i+1}))f((01)^r 00)\bar{f}((01)^r 00)\underline{f\bar{f}}((01)^{r+1})$ $\prod_{j=0}^{r}{}^{*}(\bar{f}((01)^j 0)\underline{\bar{f}f}((01)^j 1))$, and assume by induction that $\big[p = q\big] \subseteq \big[f = \hat{f}_{r-1}\big]$. By (4.12) and (4.19), with f replaced by $\hat{f}_{r-1}$, it follows that

$$\big[p = q\big] \subseteq \Big[\hat{f}_{r-1} = \prod_{i=0}^{r-1}(\underline{g\bar{g}}((01)^i 00)\bar{g}((01)^{i+1}))\underline{g\bar{g}}((01)^r 00)\underline{\bar{f}f}((01)^{r+1})$$

$$\prod_{j=0}^{r}{}^{*}(\bar{f}((01)^j 0)\underline{\bar{f}f}((01)^j 1))\Big]$$, and therefore if $r+1 \leq \frac{n+d(n)}{2} - 2$, then

$\left[p = q\right] \subseteq \left[\hat{f}_{r-1} = \hat{f}_r\right]$.

By induction it therefore follows that if

$$g_2 = \prod_{i=0}^{\frac{n+d(n)}{2}-3} (g\bar{g}((01)^i 00)\bar{g}((01)^{i+1}))(g\bar{g}((01)^{\frac{n+d(n)}{2}-2} 00))^{d(n+1)}$$

$$f(0,n)(\bar{f}f((01)^{\frac{n+d(n)}{2}-2} 1))^{d(n)} \prod_{j=0}^{\frac{n+d(n)}{2}-(2+d(n))^*} (\bar{g}((01)^j 0)\bar{g}g((01)^j 1)),$$

then $\left[p = q\right] \subseteq \left[f = \bar{I}(g_2)\bar{F}(f)\right]$.

In a similar way, we can show, by induction on s, that $\left[p = q\right] \subseteq \left[g_2 = \bar{I}(g_1)\bar{F}(g)\right]$, and therefore that $\left[p = q\right] \subseteq \left[f = \bar{I}(g_1)\bar{F}(f)\right]$. This concludes the proof of statement (4.1), and hence of the proposition.

Lemma 4.2: Let $f, g \in F(x)$ satisfy $f \sim_n g$ and $f\ \Theta_n\ g$. Then $I^{n+1}(f(0,n)) = I^{n+1}(g(0,n))$.

Proof: Assume $I^{n+1}(f(0,n)) \neq I^{n+1}(g(0,n))$ for some $f, g \in F(X)$ with $f\ \Theta_n\ g$. By (1.19), it follows that there exists a substitution φ in $(f(0,n) = g(0,n))$ by 3 variables such that $I^{n+1}(\varphi(f(0,n))) \neq I^{n+1}(\varphi(g(0,n)))$. We can assume that $\varphi(x) = x$ for all $x \in X - E(f(0,n))$. Then φ is a substitution in $(f = g)$ by n variables. We will show that $(\varphi(f))(0,n) \sim \varphi(f(0,n))$.

Since $\bar{f}(0) \notin E(f(0,n))$, it follows that $\varphi(\bar{f}(01)) = \bar{f}(0)$, and $\varphi(x) \neq \bar{f}(0)$ if $x \in \bar{f}(0)$. Therefore $\varphi(f(0)) \sim (\varphi(f))(0)$. Assume by inductive hypothesis that $\varphi(f((01)^i)) \sim (\varphi(f))((01)^i)$ for some i, and that $\bar{f}((01)^i 0) \notin E(f(0,n))$. Then $\varphi(f((01)^i 0)) \sim (\varphi(f))((01)^i 0)$. Similarly if by inductive hypothesis $\varphi(f((01)^j 0)) \sim (\varphi(f))((01)^j 0)$, and if $\bar{f}((01)^{j+1}) \notin E(f(0,n))$, then $\varphi(f((01)^{j+1})) \sim (\varphi(f))((01)^{j+1})$.

It follows by induction that

$$\varphi(f((01)^{\frac{n-d(n)-2}{2}-d(n+1)}0^{d(n+1)}))\sim(\varphi(f))((01)^{\frac{n-d(n)-2}{2}-d(n+1)}0^{d(n+1)}).$$

Therefore $(\varphi(f))(0,n) = \overline{I}^{n+1}((\varphi(f))((01)^{\frac{n-d(n)-2}{2}-d(n+1)}0^{d(n+1)}))$

$$\sim\overline{I}^{n+1}(\varphi(f((01)^{\frac{n-d(n)-2}{2}-d(n+1)}0^{d(n+1)})))$$

$$\sim\overline{I}^{n+1}(\varphi(\overline{I}^{n+1}(f((01)^{\frac{n-d(n)-2}{2}-d(n+1)}0^{d(n+1)}))))$$

$$= \overline{I}^{n+1}(\varphi(f(0,n))) = \varphi(f(0,n)).$$

By definition of θ_n, it follows that $\varphi(f)\ \theta_n\ \varphi(g)$, and therefore $I^{n+1}((\varphi(f))(0,n)) = I^{n+1}((\varphi(g))(0,n))$. Then $I^{n+1}(\varphi(f(0,n)))$ $= I^{n+1}((\varphi(f))(0,n)) = I^{n+1}(\varphi(g(0,n)))$, which is a contradiction. Therefore $I^{n+1}(f(0,n)) = I^{n+1}(g(0,n))$.

<u>Proposition 4.3</u>: (Extension of proposition 3.3). For $n \geq 3$ let $p,q,f,g \in F(X)$ satisfy $p \sim_n q$, $p\ \overline{\theta}_n\ q$, $f \sim_n g$, and $f\ \theta_n\ g$. Then $\left[p = q\right] \subseteq \left[f = \overline{I}(g)\overline{F}(f)\right]$.

<u>Proof</u>: (Cf. proof of proposition 4.1). We must first show $\left[p = q\right] \subseteq \left[f = f_1\right]$ (f_1 defined as in the proof of proposition 4.1). This statement was proved in proposition 3.3 with no cardinality restriction on $E(f)$, except to show $I^{n+1}(f(0,n)) = I^{n+1}(g(0,n))$. Lemma 4.2 removes the cardinality restriction on this statement. Therefore, using lemma 4.2 and proposition 3.3 we can show that $\left[p = q\right] \subseteq \left[f = f_1\right]$ under the conditions of proposition 4.3.

It remains to establish statement (4.1) under the assumptions of proposition 4.3.

If $f\ \theta_n\ g$, then $\varphi(f)\ \theta_n\ \varphi(g)$ for any substitution by $|E(f)|-1$ variables. Statement (4.1) can therefore be proved from proposition 3.3 and the definition of θ_n, by the same method as was used in proposition 4.1 .

2. <u>The skeleton of the lattice.</u>

We are now in a position to prove a series of propositions which will describe the inclusions among the elements of different n-skeletons, thus giving a description of the skeleton of the lattice.

<u>Proposition 4.4</u>: Let $p,q,f,g \in F(X)$ satisfy $|E(p)| = n \geq 3$, $p \sim_n q$, $p\ \bar{\varnothing}_n\ q$, $|E(f)| = n+1$, $f \sim_{n+1} g$, $f\ \theta^*_{n+1}\ g$. Then $\left[p = q\right] \subseteq \left[f = g\right]$.

<u>Proof</u>: Since $f \sim_{n+1} g$, and $|E(f)| = n+1 \geq 4$, it follows by proposition 2.7 that $f(11) \sim g(11)$. Therefore $(f(1))(1,n)$

$= (\overline{f(1)}((12)_n))^{d(n)}(f(1))((12)_n)(\overline{f(1)}((12)_n))^{d(n+1)}$

$= (\bar{f}(1(12)_n))^{d(n)} f(1(12)_n)(\bar{f}(1(12)_n))^{d(n+1)}$

$\sim (\bar{g}(1(12)_n))^{d(n)} g(1(12)_n)(\bar{g}(1(12)_n))^{d(n+1)}$

$= (g(1))(1,n)$. Also since $f \sim_{n+1} g$, and $|E(f)| = n+1 \geq 4$, it follows by proposition 2.3 that $f(1) \sim_n g(1)$. Therefore, by corollary 2.8, $\bar{F}(f(1)) \sim \bar{F}(g(1))$.

By (2.7), $(f(1))(0,n) = f(1,n+1)$, and therefore, since $f\ \theta^*_{n+1}\ g$, it follows that $f(1)\ \theta_n\ g(1)$. Applying proposition 3.3 we have that $\left[p = q\right] \subseteq \left[f(1) = \bar{I}(g(1))\bar{F}(f(1))\right]$, and since $\bar{F}(f(1)) \sim \bar{F}(g(1))$ and (by proposition 2.3) $\bar{f}(1) = \bar{g}(1)$, it follows that $\left[p = q\right] \subseteq \left[f = \bar{I}(f)\bar{F}(g)\right]$.

Since $f \sim_{n+1} g$, it follows by lemma 2.11 that $f\ \theta_n\ g$. By proposition 4.3, $\left[p = q\right] \subseteq \left[g = \bar{I}(f)\bar{F}(g)\right]$, and therefore $\left[p = q\right] \subseteq \left[f = g\right]$.

Proposition 4.5: Let $p,q,f,g \in F(X)$ satisfy $p \sim_n q$, $|E(p)| = n \geq 3$, $p \not\!\phi_n q$, $f \sim_{n+1} g$, $|E(f)| = n+1$. Then $[p = q] \subseteq [f = g]$.

Proof: It follows immediately by proposition 4.1 that $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$ and since $f(1) \sim_n g(1)$, that $[p = q] \subseteq [f(1) = \bar{I}(g(1))\bar{F}(f(1))]$. As in the proof of proposition 4.4, $\bar{F}(f(1)) \sim \bar{F}(g(1))$ and $\bar{f}(1) = \bar{g}(1)$, and therefore $[p = q] \subseteq [f = \bar{I}(f)\bar{F}(g)]$. It follows that $[p = q] \subseteq [f = g]$.

Proposition 4.6: Let $p,q,f,g \in F(X)$ satisfy $p \sim_n q$, $|E(p)| = n \geq 3$, $p \bar{\phi}_n q$, $p \bar{\phi}^*_n q$, $f \sim_{n+1} g$, $|E(f)| = n+1$. Then $[p = q] \subseteq [f = g]$.

Proof: Let f_1, g_1, f_2, $g_2 \in F(X)$ satisfy $f_i \sim_{n+1} g_i$, $|E(f_i)| = n+1$ $(i = 1,2)$, $f_1 \; \theta^*_{n+1} \; g_1$, $f_2 \theta_{n+1} g_2$, $f_1 \; \bar{\phi}^*_{n+1} \; g_1$, $f_1 \phi_{n+1} g_1$, $f_2 \; \bar{\phi}_{n+1} \; g_2$, $f_2 \; \phi^*_{n+1} \; g_2$. By proposition 4.4 and its dual, $[p = q] \subseteq [f_1 = g_1] \wedge [f_2 = g_2]$.

Let f_3 and $g_3 \in F(X)$ satisfy $f_3 \sim_{n+1} g_3$, $|E(f_3)| = n+1$, $f_3 \; \phi_{n+1} \; g_3$, $f_3 \phi^*_{n+1} g_3$. Then by theorem 3.10, $[f_1 = g_1] \wedge [f_2 = g_2] = [f_3 = g_3]$. By proposition 3.2 and its dual, $[f_3 = g_3] \subseteq [f = g]$, and therefore $[p = q] \subseteq [f = g]$.

Proposition 4.7: Let $p,q,f,g \in F(X)$ satisfy $|E(p)| = n \geq 3$, $p \sim_n q$, $p \; \bar{\theta}_n \; q$, $p \; \theta^*_n \; q$, $|E(f)| = n+1$, $f \sim_{n+1} g$, $f \; \phi_{n+1} \; g$. Then $[p = q] \not\subseteq [f = g]$.

Proof: As was noted in the proof of proposition 3.7, it is enough to show that if $f = u(\psi(p))v$ and $f_1 = u(\psi(q))v$, for some

$u, v \in F'(X)$ and homomorphism $\Psi: F(X) \longrightarrow F(X)$, then $f\ \theta_{n+1}\ f_1$.

There are two cases to consider:— either $\bar{I}(f)w = u\bar{I}(\Psi(p))$ and $\bar{I}(f_1)w_1 = u\bar{I}(\Psi(q))$ for some $w, w_1 \in F'(X)$, or $f(0) = u(\Psi(p))v_1$ and $f_1(0) = u(\Psi(q))v_1$ for some $v_1 \in F'(X)$.

If $\bar{I}(f)w = u\bar{I}(p)$ and $\bar{I}(f_1)w_1 = u\bar{I}(q)$, then, since $p\ \bar{\theta}_n\ q$ implies $\bar{I}(\Psi(p)) \sim \bar{I}(\Psi(q))$, it follows that $\bar{I}(f)w \sim \bar{I}(f_1)w_1$, and therefore (by the definition of $\bar{I}$) that $\bar{I}(f) \sim \bar{I}(f_1)$. In particular, it follows that $f\ \theta_{n+1}\ f_1$.

Assume $f(0) = u(\Psi(p))v_1$ and $f_1(0) = u(\Psi(q))v_1$. Let φ be any substitution in $(f(0) = f_1(0))$ by less than $n = |E(f(0))|$ variables. Then $|E(\varphi(\Psi(p)))| < n$, and therefore, by lemma 3.6 $\varphi(\Psi(p)) \sim \varphi(\Psi(q))$. It follows that $\varphi(f(0)) \sim \varphi(f_1(0))$, and therefore that $f(0) \sim_n f_1(0)$. By definition of $f(0)$ and $f_1(0)$, it is trivial that $[p = q] \subseteq [f(0) = f_1(0)]$. Therefore from the dual of proposition 3.8 it follows that $f(0)\ \theta_n^*\ f_1(0)$. Since $f(0, n+1) = (f(0))(1, n)$, it follows from the definitions of θ_{n+1} and θ_n^* that $f\ \theta_{n+1}\ f_1$.

<u>Corollary 4.8</u>: Let $p_0, q_0, p_1, q_1, p_2, q_2 \in F(X)$ satisfy $p_0 \sim_n q_0$, $p_0\ \bar{\theta}_n q_0$, $p_0\ \not\theta_n^*\ q_0$, $p_1 \sim_n q_1$, $p_1\ \bar{\not\theta}_n\ q_1$, $p_1\ \bar{\not\theta}_n^*\ q_1$, $p_1\ \theta_n\ q_1$, $p_1\ \theta_n^*\ q_1$, $p_2 \sim_n q_2$, $p_2\ \bar{\theta}_n\ q_2$, $p_2\ \theta_n^*\ q_2$, $p_2\ \bar{\not\theta}_n^*\ q_2$, $|E(p_0)| = |E(p_1)| = |E(p_2)| = n \geq 3$. Then $[p_0 = q_0] \vee [p_1 = q_1] \subsetneqq [p_2 = q_2]$.

<u>Proof</u>: From the description of the n-skeleton, it is clear that $[p_0 = q_0] \vee [p_1 = q_1] \subseteq [p_2 = q_2]$. Consider $f, g \in F(X)$ with $f \sim_{n+1} g$, $|E(f)| = n+1$, $f\ \not\theta_{n+1}\ g$, $f\ \not\theta_{n+1}^*\ g$. From the dual of proposition 4.5, and propositions 4.6 and 4.7, it follows that

$\left[p_0 = q_0\right] \vee \left[p_1 = q_1\right] \subseteq \left[f = g\right]$, and $\left[p_2 = q_2\right] \not\subseteq \left[f = g\right]$.

In order to complete the description of the skeleton, it is necessary to find the inclusions which exist between elements of the 2-skeleton and the 3-skeleton. This will be done in proposition 4.9 and corollary 4.10.

Proposition 4.9: For $a,b,c \in X$, consider ($abc = abac$) and ($abca = acba$). Then $abc \sim_3 abac$, $abc\ \bar{\emptyset}_3\ abac$, $abc\ \theta_3\ abac$, $abc\ \emptyset^*\ abac$; and $abca \sim_3 acba$, $abca\ \emptyset_3\ acba$, $abca\ \emptyset^*_3\ acba$. Also $\left[aba = ab\right] \subseteq \left[abc = abac\right]$, $\left[ab = a\right] \subseteq \left[abca = acba\right]$, $\left[ab = ba\right] \subseteq \left[abca = acba\right]$, and $\left[aba = ab\right] \not\subseteq \left[abca = acba\right]$.

Proof: The only non-trivial statement is $\left[aba = ab\right] \not\subseteq \left[abca = acba\right]$. It is clear that if $p = aba$ and $q = ab$, then $I(u(\psi(p))v) = I(u(\psi(q))v)$ for any $u,v \in F'(X)$ and homomorphism $\psi: F(X) \longrightarrow F(X)$. Therefore since $I(abca) \neq I(acba)$, $\left[aba = ab\right] \not\subseteq \left[abca = acba\right]$.

Corollary 4.10: $\left[ab = a\right] \vee \left[ab = ba\right] \subsetneq \left[aba = ab\right]$.

The above results and their duals are summarized in figure 3.

3. Equational classes determined by one equation.

In this section we show that all equational classes determined by one equation are equal to an element of the skeleton of the lattice, or to the meet of two elements of the skeleton.

Proposition 4.11: Let $p,q,f,g \in F(X)$ satisfy $|E(p)| = n \geq 3$,

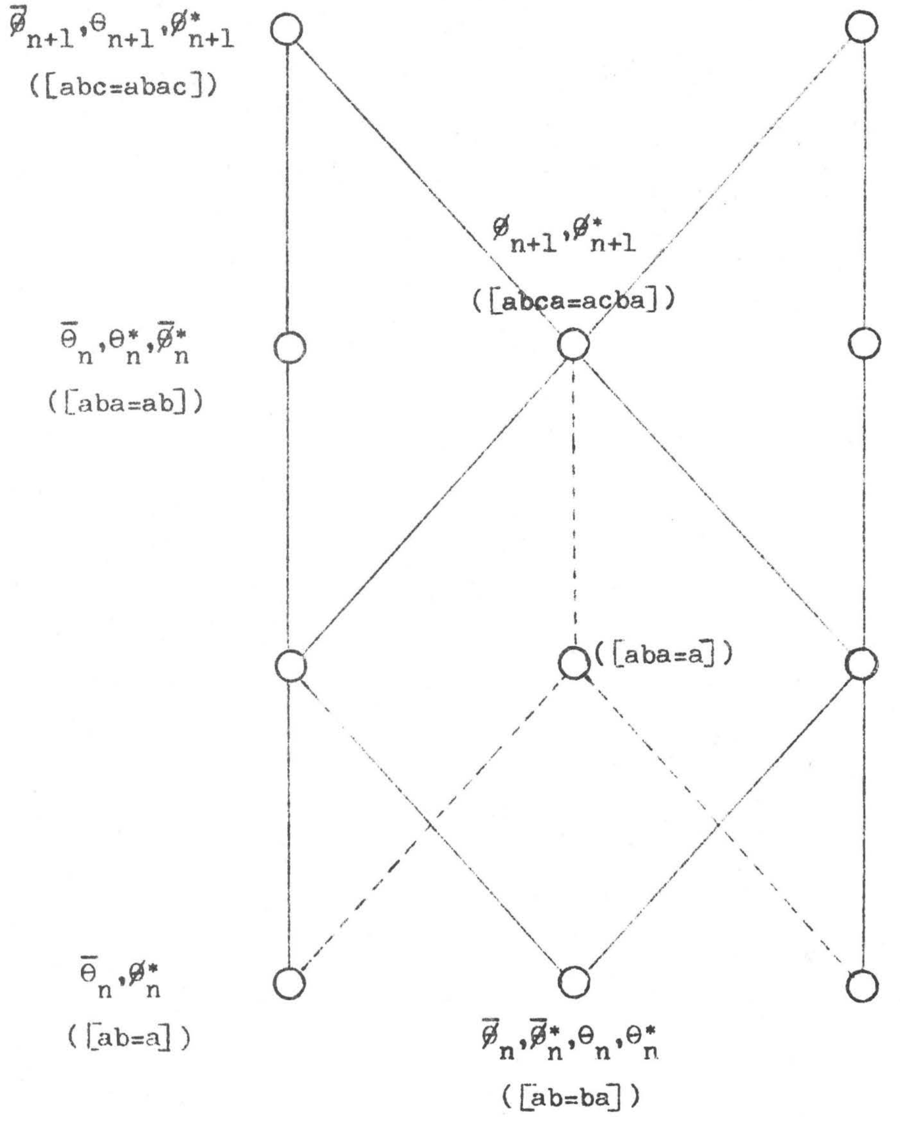

Figure 3: The inclusions between elements of the n-skeleton and the (n+1)-skeleton. The broken lines, and classes in brackets, refer to n=2 only.

$p \sim_n q$, $p \not\!\phi^*_n q$, $f \sim_n g$, $f\ \overline{\theta}_n\ g$. Then $\left[p = q\right] \subseteq \left[f = \overline{I}(g)\overline{F}(f)\right]$.

Proof: Since $f\ \overline{\theta}_n\ g$, it follows that for every substitution φ in $(f = g)$ by n variables, $\overline{I}(\varphi(f)) \sim \overline{I}(\varphi(g))$. Therefore $f \sim_{n+1} \overline{I}(g)\overline{F}(f)$.

If p_0 and $q_0 \in F(X)$ satisfy $|E(p_0)| = n+1$, $p_0 \sim_{n+1} q_0$ and $p_0\ \phi_{n+1}\ q_0$, then by the dual of proposition 4.5, and by proposition 4.1, it follows that $\left[p = q\right] \subseteq \left[p_0 = q_0\right] \subseteq \left[f = \overline{I}(g)\overline{F}(f)\right]$.

Definition 4.12: Let $f, g \in F(X)$ satisfy $f \sim_n g$, $n \geq 3$. Then

(1) $f\ \hat{\theta}_n\ g$ iff $f\ \theta_{n+1}\ \overline{I}(g)\overline{F}(f)$, and

(2) $f\ \hat{\theta}^*_n\ g$ iff $f^*\ \theta_n\ g^*$.

It should be noted that if $f\ \hat{\theta}_n\ g$, it follows that $f \sim_{n+1} \overline{I}(g)\overline{F}(f)$, but not necessarily that $f \sim_{n+1} g$. From this remark it follows easily

(4.20) If $p\ \hat{\theta}_n\ q$, then $p\ \overline{\theta}_n\ q$ for all $n \geq 3$.

If $p\ \hat{\theta}^*_n\ q$, then $p\ \overline{\theta}^*_n\ q$ for all $n \geq 3$.

Moreover:

(4.21) If $p\ \hat{\theta}_n\ q$ and $p\ \hat{\theta}^*_n\ q$ then $p \sim_{n+1} q$.

Proposition 4.13: Let $p, q, f, g \in F(X)$ satisfy $|E(p)| = n \geq 3$, $p \sim_n q$, $p \not\sim_{n+1} q$, $f \sim_n g$, $f\ \hat{\theta}_n\ g$. Then $\left[p = q\right] \subseteq \left[f = \overline{I}(g)\overline{F}(f)\right]$.

Proof: Let $p_0, q_0 \in F(X)$ satisfy $|E(p_0)| = n+1$, $p_0\ \overline{\phi}_{n+1}\ q_0$, $p_0\ \theta_{n+1}\ q_0$, $p_0\ \overline{\theta}^*_{n+1}\ q_0$. Since $f\ \theta_{n+1}\ \overline{I}(g)\overline{F}(f)$, and $p_0\ \overline{\phi}_{n+1}\ q_0$, it follows by proposition 4.3 that $\left[p_0 = q_0\right] \subseteq \left[f = \overline{I}(g)\overline{F}(f)\right]$. Since $p \not\sim_{n+1} q$, and $|E(p)| = n$, either $p\ \overline{\phi}_n\ q$ or $p\ \overline{\phi}^*_n\ q$. Since $p_0\ \theta_{n+1}\ q_0$ and $p_0\ \theta^*_{n+1}\ q_0$, it follows by proposition 4.4, or its dual, that

$\left[p = q\right] \subseteq \left[p_0 = q_0\right]$. Therefore $\left[p = q\right] \subseteq \left[f = \bar{I}(g)\bar{F}(f)\right]$.

Proposition 4.14: Let $p,q,f,g \in F(X)$ satisfy $f \sim_n g$, $f\ \hat{\phi}_n\ g$, $f\ \bar{\theta}_n\ g$, $|E(p)| = n+1$, $p \sim_{n+1} q$. Then $\left[f = g\right] \subseteq \left[p = \bar{I}(q)\bar{F}(p)\right]$.

Proof: Since $f\ \bar{\theta}_n\ g$, it follows that for every substitution φ in $(f = \bar{I}(g)\bar{F}(f))$ by n variables, $\varphi(f) \sim \varphi(\bar{I}(g)\bar{F}(f))$, and hence that $f \sim_{n+1} \bar{I}(g)\bar{F}(f)$. Moreover, since $f\ \hat{\phi}_n\ g$, it follows that $f\ \phi_{n+1} \bar{I}(g)\bar{F}(f)$. By proposition 4.1, it follows that $\left[f = \bar{I}(g)\bar{F}(f)\right] \subseteq \left[p = I(q)F(p)\right]$, and therefore that $\left[f = g\right] \subseteq \left[p = \bar{I}(q)\bar{F}(p)\right]$.

Proposition 4.15: Let $p,q,f,g \in F(X)$ satisfy $|E(p)| = n \geq 3$, $p \sim_n q$, $f \sim_n g$. Then any of the following conditions is sufficient for $\left[p = q\right] = \left[f = g\right]$.

(1) $p\ \phi_n\ q$, $p\ \phi^*_n\ q$, $f\ \phi_n\ g$, $f\ \phi^*_n\ g$.

(2) $p\ \theta_n\ q$, $p\ \bar{\phi}_n\ q$, $p\ \phi^*_n\ q$, $f\ \theta_n\ g$, $f\ \bar{\phi}_n\ g$, $f\ \phi^*_n\ g$.

(3) $p\ \theta^*_n\ q$, $p\ \bar{\phi}^*_n\ q$, $p\ \phi_n\ q$, $f\ \theta^*_n\ g$, $f\ \bar{\phi}^*_n\ g$, $f\ \phi_n\ g$.

(4) $p\ \bar{\theta}_n\ q$, $p\ \phi^*_n\ q$, $f\ \bar{\theta}_n\ g$, $f\ \phi^*_n\ g$.

(5) $p\ \bar{\theta}^*_n\ q$, $p\ \phi_n\ q$, $f\ \bar{\theta}^*_n\ g$, $f\ \phi_n\ g$.

(6) $p\ \bar{\phi}_n\ q$, $p\ \bar{\phi}^*_n\ q$, $p\ \theta_n\ q$, $p\ \theta^*_n\ q$, $f\ \bar{\phi}_n\ g$, $f\ \bar{\phi}^*_n\ g$, $f\ \theta_n\ g$, $f\ \theta^*_n\ g$.

(7) $p\ \bar{\theta}_n\ q$, $p\ \theta^*_n\ q$, $p\ \bar{\phi}^*_n\ q$, $f\ \hat{\theta}_n\ g$, $f\ \theta^*_n\ g$, $f\ \bar{\phi}^*_n\ g$.

(8) $p\ \bar{\theta}^*_n\ q$, $p\ \theta_n\ q$, $p\ \bar{\phi}_n\ q$, $f\ \hat{\theta}^*_n\ g$, $f\ \theta_n\ g$, $f\ \bar{\phi}_n\ g$.

Moreover, if $p_1, q_1 \in F(X)$ satisfy $|E(p_1)| = n+1$, $p_1 \sim_{n+1} q_1$, then either of the following conditions implies that $\left[f = g\right] = \left[p = q\right] \wedge \left[p_1 = q_1\right]$.

(9) $p\ \theta^*_n\ q$, $p\ \bar{\phi}^*_n\ q$, $p\ \bar{\theta}_n\ q$, $p_1\ \phi_{n+1}\ q_1$, $p_1\ \bar{\theta}^*_{n+1}\ q_1$, $f\ \hat{\phi}_n\ g$, $f\ \bar{\theta}_n\ g$, $f\ \theta^*_n\ g$, $f\ \bar{\phi}^*_n\ g$.

(10) $p\ \theta_n\ q$, $p\ \bar{\phi}_n\ q$, $p\ \bar{\theta}^*_n\ q$, $p_1\ \phi^*_{n+1}\ q_1$, $p_1\ \bar{\theta}_{n+1}\ q_1$, $f\ \hat{\phi}^*_n\ g$, $f\ \bar{\theta}^*_n\ g$, $f\ \theta_n\ g$, $f\ \bar{\phi}_n\ g$.

Proof: The proof consists for the most part of listing those propositions already proved which can be applied.

(1) propositions 4.1, (4.1)*.

(2) propositions (4.1)*, 4.3.

(3) dual of (2).

(4) propositions (4.1)*, 4.11.

(5) dual of (4).

(6) propositions 4.3, (4.3)*.

(7) propositions 4.13, (4.3)*.

(8) dual of (7)

(9) Since $p\ \bar{\theta}_n\ q$ and $p_1\ \bar{\theta}^*_{n+1}\ q_1$, it follows from propositions (4.3)* and 4.14 that $[f = g] \subseteq [p = q] \wedge [p_1 = q_1]$. Since $f \sim_{n+1} \bar{I}(g)\bar{F}(f)$, it follows from proposition 4.1 that $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$. By proposition (4.3)*, $[p = q] \subseteq [g = \bar{I}(g)\bar{F}(f)]$, and therefore $[p_1 = q_1] \wedge [p = q] \subseteq [f = g]$.

(10) dual of (9).

The following is the corresponding proposition for $n = 2$.

Proposition 4.16: (Tamura, 1966; lemma 13). Let $f,g \in F(X)$ satisfy $f \sim_2 g$, and let $a,b,c \in X$. Then

(1) If $E(f) \neq E(g)$, $H(f) \neq H(g)$, $H^*(f) \neq H^*(g)$, then $[f = g] = [a = b]$.

(2) If $E(f) \neq E(g)$, $H(f) = H(g)$, $H^*(f) \neq H^*(g)$, then $[f = g] = [ab = a]$.

(3) If $E(f) \neq E(g)$, $H(f) \neq H(g)$, $H^*(g) = H^*(g)$, then $[f = g] = [ab = b]$.

(4) If $E(f) \neq E(g)$, $H(f) = H(g)$, $H^*(f) = H^*(g)$, then $[f = g] = [aba = a]$.

(5) If $E(f) = E(g)$, $H(f) \neq H(g)$, $H^*(f) \neq H^*(g)$, then $[f = g] = [ab = ba]$.

(6) If $E(f) = E(g)$, $H(f) = H(g)$, $H^*(f) \neq H^*(g)$, $I(f) = I(g)$, then $[f = g] = [aba = ab]$.

(7) If $E(f) = E(g)$, $H(f) \neq H(g)$, $H^*(f) = H^*(g)$, $F(f) = F(g)$, then $[f = g] \quad [aba = ba]$.

(8) If $E(f) = E(g)$, $H(f) = H(g)$, $H^*(f) \neq H^*(g)$, $I(f) \neq I(g)$, then $[f = g] = [abc = acb] = [aba = ab] \wedge [abca = acba]$.

(9) If $E(f) = E(g)$, $H(f) \neq H(g)$, $H^*(f) = H^*(g)$, $F(f) \neq F(g)$, then $[f = g] = [abc = bac] = [aba = ba] \wedge [abca = acba]$.

<u>Proof</u>: From (1.7) and (1.14) we can conclude that if $E(f) \neq E(g)$ then $[f = g] \subseteq [aba = a]$; if $H(f) \neq H(g)$ then $[f = g] \subseteq [aba = ba]$; and if $H^*(f) \neq H^*(g)$, then $[f = g] \subseteq [aba = ab]$. It is trivial that if $E(f) = E(g)$, then $[ab = ba] \subseteq [f = g]$; if $H(f) = H(g)$, then $[ab = a] \quad [f = g]$; and if $H^*(f) = H^*(g)$, then $[ab = b] \subseteq [f = g]$.

Thus if $E(f) \neq E(g)$, $H(f) \neq H(g)$, and $H^*(f) \neq H^*(g)$, then $[f = g] \subseteq [aba = ab] \wedge [aba = a] \wedge [aba = ba] = [a = b]$, and (1) is proved.

If $E(f) \neq E(g)$, $H(f) = H(g)$, and $H^*(f) \neq H^*(g)$, then then $[f = g] \subseteq [aba = a] \wedge [aba = ab] = [ab = a] \subseteq [f = g]$, and (2) is proved. The proof of (3) is dual to the proof of (2).

If $E(f) \neq E(g)$, $H(f) = H(g)$, and $H^*(f) = H^*(g)$, then trivially $[abc = ac] \subseteq [f = g]$. But if $(aba = a)$ holds, then $abc = abcacabc = ac$, and therefore $[aba = a] \subseteq [abc = ac]$, and hence $[aba = a] = [abc = ac]$. Therefore $[aba = a] \subseteq [f = g] \subseteq [aba = a]$, and (4) is proved.

If $E(f) = E(g)$, $H(f) \neq H(g)$, and $H^*(f) \neq H^*(g)$, then $[f = g] \subseteq [aba = ab] \wedge [aba = ba] = [ab = ba] \subseteq [f = g]$, proving (5).

If $I(f) = I(g)$, then $[aba = ab] \subseteq [f = g]$, since if $(aba = ab)$ holds, then $(f = I(f))$ holds. Therefore if $I(f) = I(g)$, (which of course implies $E(f) = E(g)$, and $H(f) = H(g)$), and if $H^*(f) \neq H^*(g)$, then $[f = g] = [aba = ab]$, and (6) is proved. The proof of (7) is dual to that of (6).

If $E(f) = E(g)$ and $H(f) = H(g)$, it is trivial that $[abc = acb] \subseteq [f = g]$. If $H^*(f) \neq H^*(g)$, then $[f = g] \subseteq [aba = ab]$. If $E(f) = E(g)$, $H(f) = H(g)$, and $I(f) \neq I(g)$, then $[f = g] \subseteq [abf_1(a,b,c) = acg_1(a,b,c)]$ by the obvious substitution, and if in addition $H^*(f) \neq H^*(g)$, then since $[f = g] \quad [aba = ab]$, $[f = g] \subseteq [abc = acb]$. Therefore (8) is proved. The proof of (9) is dual to that of (8).

<u>Theorem 4.17</u>: For each $n \geq 3$, there are exactly ten equational classes determined by equations $(f = g)$ with $f \sim_n g$, $f \nsim_{n+1} g$. There are exactly nine equational classes determined by equations $(f = g)$ with $f \sim_3 g$. In particular, figures 1, 2, and 3 depict that subposet of the lattice of equational classes of idempotent semigroups which consists of the classes determined by a single equation.

Proof: The description of the n-skeleton, corollaries 4.8 and 4.10, and theorems 4.15 ((9) and (10)) and 4.16 ((8) and (9)), show that for $n \geq 3$, there exist at least ten classes determined by an equation $(f=g)$ with $f \sim_n g$, $f \not\sim_{n+1} g$, and that there exist at least nine classes determined by an equation $(f = g)$ with $f \sim_3 g$. But in fact for each n, all equations of this type have been shown to be equal to one of ten equations for $n \geq 3$, or one of nine, for $n = 2$. Therefore there are at most ten or nine such equations, respectively.

Thus, since all classes determined by a single equation have been given in figures 1, 2, and 3, these figures do in fact depict the subposet of the lattice of equational classes determined by one equation.

4. Equational classes determined by arbitrarily many equations.

In order to show that the whole lattice has been described in figures 1, 2, and 3, we will show that every equational class is determined by a single equation.

Proposition 4.18: Every equational class determined by finitely many equations is determined by one equation.

Proof: We will show that the meet of two classes, each of which is determined by a single equation, is itself determined by a single equation.

Let $(f_1 = g_1)$ and $(f_2 = g_2)$, be equations. If $f_1 \sim_3 g_1$, and $f_2 \sim_3 g_2$, then $E(f_i) = E(g_i)$, $H(f_i) = H(g_i)$, and $H^*(f_i) = H^*(g_i)$, $(i = 1,2)$. Take $E(f_1) \cap E(f_2) = \emptyset$. It is trivial that

$$\left[f_1 = g_1\right] \wedge \left[f_2 = g_2\right] \subseteq \left[f_1 f_2 = g_1 g_2\right].$$

Moreover, if φ_1 is a substitution which satisfies $\varphi_1(x) = H^*(f_1)$ for all $x \in E(f_2)$, $\varphi_1(y) = y$ for all $y \in E(f_1)$, then $\varphi_1(f_1 f_2) = f_1$ and $\varphi_1(g_1 g_2) = g_1$. Therefore $[f_1 f_2 = g_1 g_2] \subseteq [f_1 = g_1]$. Similarly, if φ_2 satisfies $\varphi_2(x) = H(f_2)$, for all $x \in E(f_1)$, $\varphi_2(y) = y$ for all $y \in E(f_2)$, then it follows that $[f_1 f_2 = g_1 g_2] \subseteq [f_2 = g_2]$.

If $f_1 \not\sim_3 g_1$, or $f_2 \not\sim_3 g_2$, then $[f_1 = g_1] \wedge [f_2 = g_2]$ $= [p = q] \wedge [p_1 = q_1]$, where either both $[p = q]$ and $[p_1 = q_1]$ are equational classes given in proposition 4.16, or $[p_1 = q_1]$ is one of these classes, and p and q satisfy $p \sim_3 q$ and one of the following:

(i) $p \, \bar{\phi}^*_3 \, q$, $p \, \theta_3 \, q$, $p \, \phi^*_3 \, q$,

(ii) $p \, \bar{\phi}^*_3 \, q$, $p \, \theta^*_3 \, q$, $p \, \phi_3 \, q$,

(iii) $p \, \phi_3 \, q$, $p \, \phi^*_3 \, q$.

It is straightforward to check that all such meets are classes generated by a single equation.

<u>Theorem 4.19</u>: Every equational class of idempotent semigroups is determined by one equation.

<u>Proof</u>: The poset of equational classes which are determined by one equation is a lattice which satisfies the descending chain condition. Moreover, every set of incomparable elements in this lattice is finite. It follows that every meet in the lattice of equational classes is a finite meet in the lattice of classes determined by one equation, and therefore determined by one equation (by theorem 4.18).

From this theorem we can now conclude that the lattice of equational classes of idempotent semigroups has been completely described. Figure 4 depicts the complete lattice.

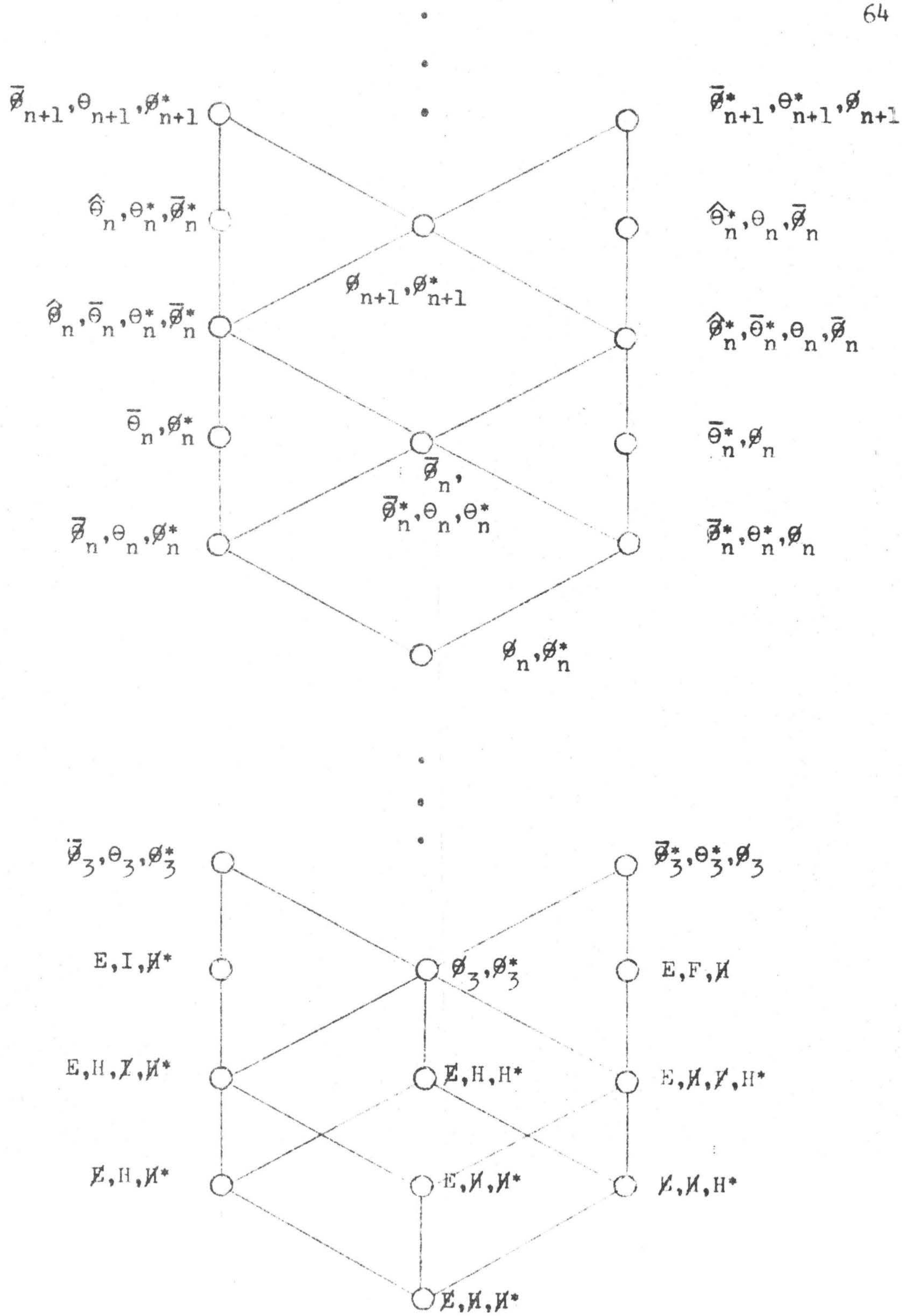

Figure 4: The lattice.

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