KÖTHE RINGS

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By

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INTRODUCTION

The problems considered in this thesis have their basis in a paper by Köthe [15]. Köthe considered those rings R for which each right and each left R-module is a direct sum of cyclic modules. Such rings are called Köthe rings. In the commutative case the class of Köthe rings has been determined. By combining results of Köthe [15] and Cohen and Kaplansky [4] one obtains that a commutative ring R is Köthe if and only if R is a principal ideal artinian ring. It has been shown by Nakayama [18] that in the non-commutative case the class of serial rings (which properly contains the principal ideal artinian rings) are Köthe rings. However, Nakayama [19] also showed that in the non-commutative case the class of serial rings does not coincide with the class of Köthe rings.

One can generalize the original Köthe problem by considering those rings R such that each right and each left R-module is a direct sum of finitely generated modules. The name generalized Köthe has been suggested for this class of rings. In the commutative case the classes of Köthe rings and of generalized Köthe rings coincide (i.e. they are precisely the principal ideal artinian rings by Griffith [11, Thm. 4.3]). In the non-commutative case not much is known about the generalized Köthe rings except they must be artinian (Faith and Walker [9, Thm. 3.1]).

It is possible to further generalize the original Köthe problem. For instance given an infinite cardinal d consider those rings R such that each left R-module is a direct sum of modules each generated by at most d elements. In Chapter 1 we state some results concerning this problem and similar generalizations of the original Köthe problem.

In Chapters 2 and 3 we study results due largely to Eisenbud and Griffith [5, 6, 11] concerning two restricted versions of the original Köthe problem. We characterize those rings R for which every left R-module is a direct sum of left R-modules with a unique composition series. We show these rings are exactly the serial rings. Also we characterize those rings for which every left R-module is a direct sum of torsionless left R-modules with a unique composition series. These are shown to be the serial quasi-Frobenius rings.

In Chapter 4 we consider commutative rings. We show that the Köthe problem, the generalized Köthe problem and the two restricted versions studied in Chapter 3 all give rise to the same class of commutative rings, namely the principal ideal artinian rings. These results are largely due to Griffith [11]. We also show that if R is a commutative ring and there exists a cardinal number n such that every R-module is a summand of a direct sum of R-modules with at most n generators, then R is a principal ideal artinian ring. This result is due to Warfield [26, Thm. 2].

Throughout this paper all rings have identities and all modules will be unital.

CHAPTER I

In this chapter we state without proof some general results which will be used in the succeeding chapters.

Lemma 1: If M is a module which is a direct sum of modules each generated by at most c elements where c is an infinite cardinal number, then each direct summand of M is a direct sum of modules each generated by at most c elements.

This result is a generalization of a well known theorem of Kaplansky's [14, Thm. 1] in which the c above is taken to be countable. The proof of Lemma 1, which is similar to the original proof of Kaplansky's, can be found in Walker [21, Thm. 4.3].

Lemma 2: A ring R is left noetherian if and only if there exists a cardinal number c such that each left R-module is contained in a direct sum of modules each generated by at most c elements.

Lemma 3: If each left R-module is contained in a direct sum of finitely generated modules then R is left artinian.

Lemmas 2 and 3 are due to Faith and Walker [9, Thms. 3.3, 3.1].

Lemma 4: If d is a fixed cardinal number and if each left R-module is a direct sum of modules each generated by at most

d elements then R is left artinian.

Lemma 4 is due to Griffith [11, Thm. 2.2]. We give a sketch of Griffith's proof. By Lemma 2 R is left noetherian. The following remarks show R is left perfect which combined with left noetherian implies R is left artinian. In [12, Thm. 2.2] Griffith proves a ring R is left-perfect if and only if each $\aleph_1$-separable left R-module is a direct sum of countably generated modules (A left R-module M is called $\aleph_1$-separable if M is flat, torsionless and if each countably generated submodule of M is contained in a countably generated direct summand of M.) Therefore if each flat left R-module is a direct sum of countably generated modules, R is left perfect. The techniques of Griffith in [12] can be extended to prove that if there exists an infinite cardinal number d such that each flat left R-module is a direct sum of modules each generated by at most d elements, then R is left perfect.

CHAPTER II

For a given ring R Proposition 1 gives a sufficient condition for every left R-module to be written as the direct sum of modules from a given class of finitely presented modules. Proposition 2 considers the same type of problem for an artinian ring; that is sufficient conditions are given for every R-module to be written as the direct sum of cyclic modules from a given class.

We briefly state the concepts needed for Proposition 1 and the Lemmas leading up to it.

A module M is finitely presented if $M \cong \frac{F}{K}$ with F and K finitely generated and F projective.

A left module M is cyclically presented if $M \cong \frac{R}{Ra}$ for some $a \in R$.

A submodule A of a left R-module B is a pure (relatively divisible) submodule of B if for any finitely presented (cyclically presented) module F the natural homomorphism $\mathrm{Hom}(F,B) \to \mathrm{Hom}(F, \frac{B}{A})$ is surjective.

A module P is pure projective (relatively divisible projective) if for any module B and pure (relatively divisible) submodule A, the natural homomorphism $\mathrm{Hom}(P,A) \to \mathrm{Hom}(P, \frac{B}{A})$ is surjective.

We state without proof two Lemmas of elementary results on purity and pure projectivity. Lemma 5 can be found in Warfield [22, Cor. 1 and 3] and Lemma 6 in Griffith [11, Lemma 3.1].

Lemma 5: A left R-module is pure projective (relatively divisible projective) if and only if it is a direct summand of a direct sum of finitely presented (cyclically presented) modules.

Lemma 6: Let R be any ring and A a left R-module.

(a) If B is a submodule of A and C a pure (relatively divisible) submodule of A such that $C \subseteq B \subseteq A$ and such that $\frac{B}{C}$ is pure (relatively divisible) in $\frac{A}{C}$, then B is a pure (relatively divisible) submodule of A.

(b) If $\{B_\lambda\}_{\lambda\varepsilon\Lambda}$ is an ascending chain of pure (relatively divisible) submodules of A then $B = \bigcup_{\lambda\varepsilon\Lambda} B_\lambda$ is a pure (relatively divisible) submodule of A.

Proposition 1: Let R be any ring and let $\mathcal{F}$ be a class of finitely presented left R-modules. If each non-zero left R-module contains a copy of a non-zero module of $\mathcal{F}$ as a pure submodule then each left R-module is a direct sum of copies of modules in $\mathcal{F}$.

Proof: Let M be a non-zero left R-module. By assumption there exists at least one independent family $\mathcal{G}$ of non-zero submodules from (i.e. isomorphic to members of) $\mathcal{F}$ such that $G = \sum_{\lambda\varepsilon\Lambda} \oplus G_\lambda$ is pure in M. We show the existence of a maximal family of this type.

We consider a chain $\mathfrak{K}$, ordered by set-theoretic inclusion, consisting of such families. We define $\mathcal{L} = \bigcup_{K \varepsilon \mathfrak{K}} K$, $K' = \sum_{k \varepsilon K} \oplus\, k$ and $L = \sum_{\ell \varepsilon \mathcal{L}} \oplus\, \ell$. It can easily be shown by a set theoretic inclusion argument that $L = \bigcup_{K \varepsilon \mathfrak{K}} K'$. By assumption each K' is pure in M and since the K' form a chain, $\bigcup_{K \varepsilon \mathfrak{K}} K'$ is pure in M by Lemma 6 (b). Therefore L is pure in M and there exists a maximal family $\mathfrak{G}$ by Zorn's Lemma. If $G = M$ we are done. Otherwise we consider the left R-module $\frac{M}{G}$. By assumption there exists a submodule A of M such that $\frac{A}{G} \subsetneq \frac{M}{G}$ and $\frac{A}{G} \cong B$ in $\mathfrak{F}$. We have $G \subsetneq A \subsetneq M$ with G pure in M and $\frac{A}{G}$ pure in $\frac{M}{G}$. By Lemma 6 (a) A is pure in M. It is trivial that G is pure in A. Consider the following diagram

$$\begin{tikzcd}
 & B \arrow[dl, "r"'] \arrow[d, tail, "j"] \\
A \arrow[r, two heads, "\pi"'] & \frac{A}{G}
\end{tikzcd}$$

where j is the isomorphism between B and $\frac{A}{G}$, π the natural projection map and r exists since G is pure in A and B is finitely presented (hence pure projective). πr is an isomorphism and therefore $A \cong \operatorname{im} r \oplus \ker \pi \cong B' \oplus G$ where $B' \cong B$. A is pure in M and hence $[B'] \cup \{\mathfrak{G}\}$ is an independent family of submodules from $\mathfrak{F}$ whose direct sum is a pure submodule of M. This is a contradiction to the maximality of the family $\mathfrak{G}$ and hence $G = M$.

A module B is an essential extension of a module A if there exists a monomorphism α: $A \rightarrowtail B$ such that if $0 \neq X \subseteq B$ then $\alpha(A) \cap X \neq 0$. If B is injective then B is called the injective hull of A, denoted E(A). It can be shown that any essential extension of A is contained in E(A) up to isomorphism over A.

A module C is a coessential extension of a module D if there exists an epimorphism β: $C \twoheadrightarrow D$ such that $\ker \beta + X = C$ implies that $X = C$ for all submodules $X \subseteq C$. If C is projective then C is called the projective cover of D, denoted P(D). It can be shown that any coessential extension of D is an epimorphic image of P(D).

Proposition 2 is essentially a result of Griffith's (Corollary 1) strengthened slightly by a suggestion of B. Banaschewski (oral commun-ication).

<u>Proposition 2</u>: Let R be a left artinian ring and $\mathcal{F}$ a class of cyclic left R-modules containing the simple left R-modules. If $\mathcal{F}$ is closed under essential and coessential extensions then each left R-module is a direct sum of modules in $\mathcal{F}$.

<u>Proof</u>: Let $A \neq 0$ be a left R-module. Since R is left artinian soc $A \neq 0$ and thus by assumption A contains a non-zero submodule B from $\mathcal{F}$. Since R is left artinian the length of the composition series of each cyclic left R-module is finite and is

bounded by the length of the compositon series of ${}_RR$. Therefore we can choose B as the module from $\mathcal{F}$ with longest length which is isomorphic to a submodule of A. By Zorn's Lemma there exists $M \subseteq A$ such that M is maximal with respect to $B \cap M = 0$. We consider the map $A \overset{\pi}{\twoheadrightarrow} \frac{A}{M}$ where π is the natural projection. $\pi|_B$ is a monomorphism since $M \cap B = 0$. Also $\pi|_B$ is essential since, if $\pi(B) \cap L' = 0$ for $L' = \frac{L}{M}$ a submodule of $\frac{A}{M}$, then $B \cap L = 0$ which implies $L \subseteqq M$ and thus $L' = 0$. Since $\frac{A}{M}$ is an essential extension of B we have that $\frac{A}{M} \in \mathcal{F}$. If $\pi|_B = \frac{A}{M}$ then $\pi|_B$ would be an isomorphism which implies that B is a direct summand of A. In that case B would be pure in A and we could apply Proposition 1 to achieve the desired result. So we assume that $\pi(B) \neq \frac{A}{M}$. Since $\frac{A}{M} \in \mathcal{F}$, $\frac{A}{M}$ is cyclic and therefore we can find a cyclic submodule $Ra \subseteq A$ such that $\pi(Ra) = \frac{A}{M}$. For the same reasons as given above we can choose D in A such that D has the smallest compositon length with respect to $\pi(D) = \frac{A}{M}$. Also if $\ker(\pi|_D) + E = D$ then $\pi(E) = \frac{A}{M}$ and our choice of D gives us that $E = D$. Therefore D is a coessential extension of $\frac{A}{M}$ and since $\frac{A}{M}$ is in $\mathcal{F}$, D is in $\mathcal{F}$. Comparing composition lengths we have that $\ell(D) \geq \ell(\frac{A}{M}) > \ell(B)$. This is a contradiction to the original choice of B. Therefore $\pi|_B$ is an isomorphism and as indicated above the result follows by Proposition 1.

<u>Corollary 1</u>: Let R be a left artinian ring and $\mathcal{F}$ a class of cyclic left R-modules containing the simple left R-modules.

If $\mathcal{F}$ is closed under the operations of taking submodules, homomorphic images, projective covers and injective hulls then each left R-module is a direct sum of modules in $\mathcal{F}$.

<u>Proof</u>: By the remarks before Proposition 2 $\mathcal{F}$ is closed under essential and coessential extensions.

Corollary 1 appears in Griffith [11, Thm. 3.3] and the proof of Proposition 2 is similar to Griffith's proof of Corollary 1. Whether the conditions of Corollary 1 are actually stronger than those of Proposition 2 is not known.

It is clear that two such classes as mentioned in Corollary 1 (and Proposition 2) both contain all indecomposable modules. However, such classes need not be unique. For instance consider R a principal ideal artinian ring which is not local. We will show in Theorem 1 that the class of left R-modules with a unique compositon series satisfies the conditions of Corollary 1 (and of Proposition 2). But we show that the class of cyclic left R-modules (which properly contains the above class since R is not local) also satisfies the conditions of Corollary 1. Let A be a cyclic left R-module. By Faith [8, Thm. 2] E(A) is cyclic. Also, since A is cyclic, there exists the natural map of R onto A. Therefore it follows that P(A) is a direct summand of R and since R is principal ideal artinian, P(A) is cyclic. Obviously the class of cyclic left R-modules is closed with respect to homomorphic images and it is closed with respect to submodules since R is principal ideal.

If R is a left artinian ring a finitely generated left R-module is essential over its socle. Therefore, if given a left artinian ring R and a class $\mathcal{F}$ as described in Proposition 2 we have that the injective hull of a finitely generated left R-module is finitely generated (this is also true if R is a generalized Köthe ring). This condition for left artinian rings has been studied in [16] and [20].

CHAPTER III

3.1 If R is a ring J will denote its Jacobson radical.

If M is a R-module then soc M denotes the sum of all simple submodules of M and $\ell(M)$ the length of a compositon series for M (if one exists).

M is called a uniserial module if it has a unique composition series of finite length. If M is a uniserial left R-module it can be shown that its composition series is $M \supseteq JM \supseteq J^2M \ldots \supseteq J^nM = 0$. ${}_R\mathcal{U}$ will denote the class of uniserial left R-modules.

A ring R is a left serial ring if R is left artinian and for each indecomposable idempotent e, Re is uniserial. R is serial if it is both left serial and right serial (serial rings are sometimes referred to as generalized uniserial rings).

If R is artinian then the left ideal Re, e an indecomposable idempotent, is called a dominant left summand of R if $J^ke = 0$ implies that $J^k = 0$.

A module N has the exchange property if whenever $M = N \oplus X = \bigoplus_{i \in I} M_i$ there exists $M_i' \subseteq M_i$ such that $M = N \oplus (\bigoplus_{i \in I} M_i')$. One can show that in this case there exist

$M_i'' \subseteq M_i$ such that $M_i = M_i' \oplus M_i''$ for all $i \in I$ and $N \cong \bigoplus_{i+I} M_i''$.

In Theorem 1 we show that, among other ways, serial rings may be characterized by saying that each of their left modules is a direct sum of uniserial modules. Thus serial rings are Köthe rings but the containment is proper. (Nakayama [19]). Nakayama [18, Thm. 21] showed that serial rings possess this property while Fuller [10, Thm. 5.4] showed that it characterizes serial rings. The proof here is due to Eisenbud and Griffith [6].

Theorem 1: The following are equivalent for any ring R.

(1) Every left R-module is a direct sum of modules in ${}_R\mathcal{U}$

(2) R is left artinian and ${}_R\mathcal{U}$ is closed under the operations of taking essential and coessential extensions.

(3) R is a left serial ring and for each simple left R-module S, E(S) is in ${}_R\mathcal{U}$.

(4) Every left R-module is relatively divisible projective and every indecomposable cyclic left R-module is in ${}_R\mathcal{U}$.

(5) R is left artinian and the dominant left summands of $\frac{R}{J^k}$ are $\frac{R}{J^k}$ - injective for each k.

(6) R is serial.

(7) The left-right symmetry of (1)-(5).

Proof:

3.2 We first prove the equivalence of (1), (2) and (3).

These results are due to Griffith [6, Thm. 4.1]. We note that the proof of (3) —> (2) shows that for A in ${}_R\mathcal{U}$, $P(A) \cong Re$, e an indecomposable idempotent. Therefore ${}_R\mathcal{U}$ consists of the R-modules $\frac{Re}{J^k e}$, e an indecomposable idempotent.

(1) —> (3) It follows from Lemma 4 that R is left artinian. By assumption we have that, for each indecomposable idempotent e, Re is in ${}_R\mathcal{U}$. Therefore R is a left serial ring. If S is simple then $E(S)$ is indecomposable and thus by assumption is in ${}_R\mathcal{U}$.

(3) —> (2) Since ${}_R\mathcal{U}$ is clearly closed under submodules and homomorphic images, it's enough to show that for A in ${}_R\mathcal{U}$, $E(A)$ and $P(A)$ are in ${}_R\mathcal{U}$.

A in ${}_R\mathcal{U}$ implies that soc A is simple (due to the unique composition series of A). Since A is essential over its socle $E(A) = E(\text{soc } A)$. By assumption $E(\text{soc } A)$ and thus $E(A)$ is in ${}_R\mathcal{U}$.

To show that $P(A)$ is in ${}_R\mathcal{U}$ we note that since A has a unique composition series it has a unique maximal submodule M. Since R is left artinian $\frac{A}{M} \cong \frac{Re}{Je}$, e an indecomposable idempotent.

We consider the diagram

$$\begin{tikzcd}
& Re \arrow[dl, "\varphi"'] \arrow[d, two heads, "\pi_1"] \\
A \arrow[r, two heads, "\pi_2"'] & \frac{A}{M} \cong \frac{Re}{Je}
\end{tikzcd}$$

where φ exists by the projectivity of Re and π_1 and π_2 are the natural projections. M contains all proper submodules of A and thus by the commutativity of the above diagram φ is an epimorphism. Also by the commutativity of the diagram $\ker\varphi$ is contained in Je and thus by Nakayama's Lemma $\ker\varphi$ is small in Re. Therefore $Re \cong P(A)$ and since R is left serial, P(A) is in ${}_R\mathcal{U}$.

(2) —> (1) Trivially ${}_R\mathcal{U}$ contains the class of simple left R-modules. Therefore (1) follows by Proposition 2.

<u>3.3</u> In this section we prove the equivalence of (1), (4) and (5) of Theorem 1. (1) and (4) are due to Griffith [11, Thm. 4.1] and (5) to Eisenbud and Griffith [5, Prop. 1.1]. The conditions in (4) have been slightly changed from those originally stated by Griffith in [11]. Whereas in (4) we have that each indecomposable cyclic left R-module is in ${}_R\mathcal{U}$, Griffith has that each indecomposable cyclically presented left R-module is in ${}_R\mathcal{U}$. The reason we changed the conditions is that there is a mistake in Griffith's proof that (1) implies (4). In that proof he shows that a left R-module A in ${}_R\mathcal{U}$ is isomorphic to $\frac{Re}{Rre}$, r in R and e an indecomposable idempotent (this follows from $P(A) \cong Re$ and since Re is in ${}_R\mathcal{U}$, all of its submodules are cyclic). He then states that $\frac{Re}{Rre} \cong \frac{R}{Rx}$ where $x = re + (1-e)$, which is not always true. We consider the serial ring R (any serial ring obviously satisfies the conditions of Theorem 1) consisting of all 2×2 upper triangular matrices over a given field. Choosing $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

and $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ it is possible to show that $\frac{Re}{Rre}$ is a one-dimensional vector space over R and $\frac{R}{Rx}$ is a two-dimensional space over R. We don't know if Griffith's original statement of (4) is actually equivalent to the rest of Theorem 1.

(1) –> (4) By (1) R is a left serial ring. Let A be in $_R\mathcal{U}$. By the remark just above $A \cong \frac{Re}{Rre}$ for some $r \in R$. We now show that $\frac{Re}{Rre}$ is relatively divisible projective.

We wish to complete the diagram

$$\begin{tikzcd} & \frac{Re}{Rre} \arrow[d, "\varphi"] \\ X \arrow[r, two heads, "\pi"'] & \frac{X}{Y} \end{tikzcd}$$

where π is the natural map and Y is a relatively divisible submodule of X. An equivalent characterization of relatively divisible (see [22, Prop. 2]) is that $Y \cap sX = sY$ for all $s \in R$. We select $x \in X$ such that $\pi(x) = \varphi(\bar{e})$. We have $\pi(rex) = \varphi(\overline{ree}) = \varphi(\overline{re}) = 0$ and hence rex is in Y. Since Y is relatively divisible in X there exists $y \in Y$ such that $rex = rey$. We define ψ : Re –> X by $\psi(ae) = ae(x-y)$ for $a \in R$. Clearly ψ is well-defined and since $\psi(Rre) = Rre(x-y) = 0$ we have the map $\overline{\psi} : \frac{Re}{Rre} \to X$ given by $\overline{\psi}(\overline{ae}) = \psi(ae) = ae(x-y)$. The map $\overline{\psi}$ completes the above diagram since $\pi\overline{\psi}(\overline{ae}) = \pi(ae(x-y)) = ae(\pi(x) - \pi(y)) = ae\varphi(\bar{e}) = \varphi(\overline{ae})$. The assumption gives us that every left R-module is a direct sum of modules of the form $\frac{Re}{Rre}$, e an indecomposable idempotent. Since the $\frac{Re}{Rre}$ are relatively divisible projective every left R-module is relatively divisible projective. Trivially each indecomposable left R-module is in ${}_R\mathcal{U}$.

(4) —> (5) It follows from Lemma 3 that R is left artinian. By Lemma 5 we have that each left R-module is a direct summand of a direct sum of cyclically presented left R-modules. It's clear that (4) is true for any homomorphic image of R and thus for $\frac{R}{J^k}$. Therefore it is enough to prove every dominant left summand of R is injective. Suppose Re is a dominant left summand of R. Since R is left artinian a module of the form $\frac{R}{Rx}$ can be written as the direct sum of indecomposable cyclic left R-modules. Therefore the indecomposable injective E(Re) is a direct summand of cyclic indecomposable left R-modules and since injectives have the exchange property (Warfield [23, Lemma 2]) E(Re) is a cyclic indecomposable left R-module. Thus by assumption E(Re) is in ${}_R\mathcal{U}$. Now $\ell(Re) \leq \ell(E(Re))$ but since Re is a dominant left summand of R, Re $\cong$ E(Re). Therefore Re is R-injective.

(5) —> (1) To show R is left serial it is enough to show that, for each indecomposable idempotent e, $\frac{J^{k-1}e}{J^k e}$ is simple or zero for each k. If $\frac{J^{k-1}e}{J^k e} \neq 0$ then $\frac{Re}{J^k e}$ is a dominant left summand of $\frac{R}{J^k}$ and thus by assumption is an indecomposable $\frac{R}{J^k}$ - injective. This

implies that $\operatorname{soc}\left(\frac{Re}{J^k e}\right)$ is simple and since $\frac{J^{k-1}e}{J^k e} \subseteq \operatorname{soc}\frac{Re}{J^k e}$,
this gives us that $\frac{J^{k-1}e}{J^k e}$ is simple.

Next we show that every non-zero left R module has a uniserial summand and thus (1) is true by Proposition 1. Let M be a left R-module. Clearly M is generated by its cyclic submodules and if Ra is a cyclic submodule of M then $Ra = \sum_{i=1}^{n} Re_i a$ e_i indecomposable idempotents. $Re_i a$ is a homomorphic image of Re_i and since R is left serial $Re_i a$ is uniserial. Therefore M is generated by its uniserial submodules. Since R is left artinian we can choose $X \subseteq M$ to be a uniserial submodule of maximal length, say length k. Since M is the sum of its uniserial submodules and by the way X was chosen, it follows that $J^k M = 0$. Thus the embedding of X in M can be considered as a $\frac{R}{J^k}$ monomorphism. As in the proof of (3) $\to$ (2) there exists an indecomposable idempotent e such that Re is the projective cover of X. But since $\ell(X) = k$, X is isomorphic to $\frac{Re}{J^k e}$. $\frac{Re}{J^k e}$ is a dominant left summand of $\frac{R}{J^k}$ and thus by assumption is $\frac{R}{J^k}$ - injective. Therefore X is $\frac{R}{J^k}$ - injective and is a direct summand of M.

<u>3.4</u> Nakayama proved in [18, Thm. 21] that over a serial ring every finitely generated module is a direct sum of uniserial modules. The proof here is essentially that of Eisenbud and Griffith [5, Prop. 1.1]. We note that Lemma 8 gives a condition for a projective module over a serial ring to be injective.

(6) $\rightarrow$ (1) We state without proof the following result of Auslander's [2, Prop. 10].

<u>Lemma 7</u>: Let R be an artinian ring and X an R-module. Suppose $\mathrm{Ext}^1_R(S,X) = 0$ for every simple module S. Then X is injective.

<u>Lemma 8</u>: Let R be a serial ring, e an indecomposable idempotent. Re is injective iff for every indecomposable idempotent f, $Re \not\cong Jf$.

<u>Proof</u>:

($\Longrightarrow$) If Re is injective then $Re \not\cong Jf$ since f is assumed to be an indecomposable idempotent.

($\Longleftarrow$) On the other hand to show that Re is injective it is enough by Lemma 7 to show that for every primitive idempotent f, $\mathrm{Ext}^1_R(\frac{Rf}{Jf}, Re) = 0$. This is equivalent to showing that every map φ: $Jf \rightarrow Re$ extends to a map $Rf \rightarrow Re$. We consider the diagram

$$\begin{tikzcd}
Jf \arrow[r, tail, "i"] \arrow[dr, "\varphi"'] & Rf \arrow[r, tail, shift right, "j_1"'] & R \arrow[l, two heads, shift right, "\pi_1"'] \\
& Re \arrow[r, tail, shift right, "j_2"'] & R \arrow[l, two heads, shift right, "\pi_2"']
\end{tikzcd}$$

where i, j_1, j_2 are the natural inclusions and π_1, π_2 are the natural projections. We wish to construct a map α: $Rf \rightarrow Re$ such that the left hand triangle of the diagram commutes.

Since R is a serial ring, Jf is uniserial. Therefore there exists a primitive idempotent g of R such that $Rg \cong P(Jf)$. The epimorphism $Rg \twoheadrightarrow Jf$ induces a monomorphism

$$\mathrm{Hom}_R(Jf, R) \rightarrowtail \mathrm{Hom}_R(Rg, R) \cong gR$$

Now $j_1 i$ and $j_2 \varphi$ are in $\mathrm{Hom}_R(Jf, R)$ and hence by the above monomorphism can be viewed as elements of gR. Since R is serial, one of these elements is a multiple of the other in gR.

Suppose $j_2\varphi$ is a multiple of $j_1 i$. Then there exists an $a\colon R \to R$ such that $j_2\varphi = a\, j_1 i$. Let $\alpha = \pi_2\, a j_1$. We obtain the following diagram

$$\begin{tikzcd}
Jf \arrow[r, tail, "i"] \arrow[dr, "\varphi"'] & Rf \arrow[r, tail, shift right, "j_1"'] \arrow[d, dashed, "\alpha"] & R \arrow[l, two heads, shift right, "\pi_1"'] \arrow[d, "a"] \\
& Re \arrow[r, tail, shift right, "j_2"'] & R \arrow[l, two heads, shift right, "\pi_2"']
\end{tikzcd}$$

where $\alpha i = \pi_2\, a j_1 i = \pi_2\, j_2 \varphi = \varphi$.

If on the other hand $j_1 i = b j_2 \varphi$ we set $\beta = \pi_1\, b j_2$ and consider the following diagram

$$\begin{tikzcd}
Jf \arrow[r, tail, "i"] \arrow[dr, "\varphi"'] & Rf \arrow[r, tail, shift right, "j_1"'] & R \arrow[l, two heads, shift right, "\pi_1"'] \\
& Re \arrow[u, dashed, "\beta"] \arrow[r, tail, shift right, "j_2"'] & R \arrow[l, two heads, shift right, "\pi_2"'] \arrow[u, "b"']
\end{tikzcd}$$

with $\beta\varphi = \pi_1 bj_2\varphi = \pi_1 j_1 i = i$. We show that β is an isomorphism. By the commutativity of the diagram $Jf \subseteq \mathrm{Im}(\beta)$. If $\mathrm{Im}(\beta) = Jf$ then $1_{Jf} = \beta\varphi$ and so Jf is a summand of Re. Since Re is indecomposable this implies that $Jf \cong Re$ which contradicts the hypothesis. Therefore $Jf \subsetneq \mathrm{Im}(\beta)$ and β is an epimorphism since Jf is the unique maximal submodule of Rf. Since e is indecomposable and Rf is projective this implies that β is an isomorphism. Therefore β^{-1} will also complete the diagram.

Proof of (6) —> (1)

We have already shown that (1) is equivalent to (5). Since $\frac{R}{J^k}$ is serial for every k, it sufficies by (5) to prove that the dominant left summands of Re are injective. If Re is a dominant left summand of R, $Re \ncong Jf$ for any indecomposable idempotent f because of the maximl length of Re. By Lemma (8) Re is injective.

<u>3.5</u> In this section we prove that if every left R-module is a direct sum of uniserial modules then R is serial. Thus by the above every right R-module is a direct sum of uniserial modules. Since we have already proved the equivalence of (1) through (5) we obtain the left-right symmetry of (7). This proof is due to Eisenbud and Griffith [6, Thm. 1.3] although the result was first obtained by Fuller [10, Prop. 5.4].

To establish a duality between the category of left R-modules and the category of right R-modules we use the stable duality functor of Auslander and Bridger [3]. We give a brief account of this functor. Let M be a finitely presented module over any ring R and let $P \xrightarrow{\varphi} Q \twoheadrightarrow M$ be exact with P and Q finitely generated projectives. If we apply the functor $\_\_^* = \mathrm{Hom}_R(\_\_, R)$ to this sequence we obtain the exact sequence $M^* \rightarrowtail Q^* \xrightarrow{\varphi^*} P^*$ and we define $D(M)$ to be the module which makes the sequence $Q^* \longrightarrow P^* \longrightarrow D(M) \longrightarrow 0$ exact (i.e. $D(M) = \mathrm{Coker}(Q^* \xrightarrow{\varphi^*} P^*)$). For any other exact sequence $P_1 \xrightarrow{\varphi_1} Q_1 \longrightarrow M$ with finitely generated projectives P_1 and Q_1 one can show that there exist finitely generated projectives F and G such that $F \oplus D(M) \cong G \oplus D_1(M)$ where $D_1(M) = \mathrm{Coker}\ \varphi_1^*$. We say that $D(M)$ is unique up to stable equivalence. (For the rest of this chapter Projective will denote a finitely generated projective). If M is a left module then $D(M)$ is a right module. Therefore $D(D(M))$ is a left module and one can show that M is stably isomorphic to $D(D(M))$ (i.e. $M \oplus$ Projective $\cong D(D(M)) \oplus$ Projective). We note the following properties of the functor D.

(1) If M is a finitely generated projective then $D(M)$ is also a finitely generated projective. To see this consider the exact sequence $0 \longrightarrow M \overset{i}{\rightarrowtail\!\!\!\twoheadrightarrow} M$ with i the identity map and also the fact that $\mathrm{Hom}_R(\_\_, R)$ preserves finitely generated projectives.

(2) If M is a finitely presented module with a non-projective summand then $D(M)$ also has a non-projective summand. If $D(M)$ is a projective, $M \oplus$ Projective $\cong D(D(M)) \oplus$ Projective would imply that $D(D(M))$ and thus M are projectives.

(3) D preserves finite direct sums up to stable equivalence. This follows from the fact that the two sequences $P_1 \longrightarrow P_0 \longrightarrow\!\!\!\!\!\rightarrow M$ and $Q_1 \longrightarrow Q_0 \longrightarrow\!\!\!\!\!\rightarrow N$ induce the sequence $P_1 \oplus Q_1 \longrightarrow P_0 \oplus Q_0 \longrightarrow\!\!\!\!\!\rightarrow M \oplus N$.

Proof:

Lemma 9: If $A \oplus B = A_1 \oplus C$ where $A \cong A_1$ and $\mathrm{End}_R(A)$ is local then $B \cong C$.

Proof: From Warfield [21, Prop. 1] we have that if N is indecomposable then N has the exchange property iff N has local endomorphism ring. Therefore A has the exchange property and there exist $A_1', A_1'' \subseteq A_1$, $C', C'' \subseteq C$ such that $A_1 = A_1' \oplus A''$, $C = C' \oplus C''$, $A \cong A_1'' \oplus C''$ and $A \oplus B = A \oplus A_1' \oplus C'$. Since $\mathrm{Endo}_R(A)$ is local, A is indecomposable and therefore either $A_1'' = 0$ or $C'' = 0$. If $C'' = 0$ this implies that $C' = C$ and $A_1'' = A$. A_1 is also indecomposable and $A \cong A_1''$ gives that $A_1'' = A_1$ and $A_1' = 0$. Since $C' = C$ and $A_1' = 0$, $A \oplus B = A \oplus A_1' \oplus C'$ implies that $A \oplus B = A \oplus C$ and therefore $B \cong C$. On the other hand if $A_1'' = 0$ then $A_1' = A_1$ and $A \cong C''$. This gives $A \oplus B = A \oplus A_1' \oplus C'$ and therefore $B \cong A_1 \oplus C'$. Since $A_1 \cong A \cong C''$, we have $B \cong C \oplus C'' = C$.

Note that the result holds if $A \oplus B \cong A_1 \oplus C$.

A ring R is a semi-primary ring if J is nilpotent and $\frac{R}{J}$ is a semi-simple ring.

Lemma 10: Let R be a semi-primary ring, e an indecomposable idempotent. Let T be a submodule of eR such that T has finite length. Then $\mathrm{End}_R\left(\frac{eR}{T}\right)$ is a local ring.

Proof: The result follows trivially if $T = 0$ since $\mathrm{End}_R(eR) \cong eRe$ which is a local ring. Since eJ is the unique maximal submodule of eR we may assume that $T \subseteqq eJ$. It is clear that for any R-endomorphism ε of $\frac{eR}{T}$ that $\varepsilon\left(\frac{eJ}{T}\right) \subseteqq \frac{eJ}{T}$. It then follows that the set $A = \{ \varphi \in \mathrm{End}_R\left(\frac{eR}{T}\right) \mid \mathrm{Im}(\varphi) \subseteqq \frac{eJ}{T} \}$ is an ideal of $\mathrm{End}_R\left(\frac{eR}{T}\right)$. We show that A is the unique maximal ideal of $\mathrm{End}_R\left(\frac{eR}{T}\right)$. Consider $\emptyset \in \mathrm{End}_R\left(\frac{eR}{T}\right)$ such that $\emptyset \notin A$ (i.e. $\mathrm{Im}\,\emptyset \not\subseteqq \frac{eJ}{T}$ and thus $\emptyset$ is onto). We show that $\emptyset$ is an isomorphism and therefore that the non-units form the unique maximal ideal A. We have two projective extensions for the finitely presented module $\frac{eR}{T}$ namely $T \xrightarrow{i} eR \xrightarrow{\pi} \frac{eR}{T}$ and $\ker \emptyset\pi \rightarrowtail eR \xrightarrow{\pi\emptyset} \frac{eR}{T}$ where i is the inclusion map and π the projection map. By Schanuel's Lemma it follows that $T \oplus eR \cong \mathrm{Ker}\,\emptyset\pi \oplus eR$. Since the endomorphism ring of eR is local it follows by Lemma 9 that $T \cong \mathrm{Ker}\,\emptyset\pi$. It follows that $\mathrm{Ker}\,\emptyset\pi$ has the same finite length as T and since $T \subseteqq \mathrm{Ker}\,\emptyset\pi$, $T = \mathrm{Ker}\,\emptyset\pi$. Therefore $\emptyset$ is a monomorphism and thus an isomorphism.

Lemma 11: Suppose R is a left artinian ring with only finitely many nonisomorphic finitely generated indecomposable left modules. Then this statement holds when "left" is replaced by "right".

Proof: We first show that R is right artinian making essential use of the stable duality theory of Auslander and Bridger [3]. We assume R is not right artinian. R is at least semi-perfect and thus can be written as the direct sum of principal ideals generated by indecomposable idempotents. Our assumption implies that R is not right noetherian since R being right perfect plus right noetherian would give R right artinian. Thus for some indecomposable idempotent e, eR is not noetherian and therefore does not have a finite composition series. Looking at the finite chain of modules $eR \supseteq eJ \supseteq \dots \supseteq eJ^n = 0$ it must be that the composition length of $\frac{eJ^k}{eJ^{k+1}}$ is infinite for at least one k, $0 \le k \le n$. We select the largest such k. Since $\frac{eJ^k}{eJ^{k+1}}$ is an $\frac{R}{J}$ module it can be written as the direct sum of simple $\frac{R}{J}$ modules (and thus simple R-modules) i.e. $\frac{eJ^k}{eJ^{k+1}} = \sum_{i \in I} \oplus \frac{A_i}{eJ^{k+1}}$ where I is an infinite set. We can construct an (infinite) composition series for $\frac{eJ^k}{eJ^{k+1}}$ where the n+1 th term is $\sum_{i=1}^{n} \oplus \frac{A_i}{eJ^{k+1}}$. We construct a chain from eJ^{k+1} to eJ^k with the n+1 th term of the chain being $\sum_{i=1}^{n} A_i$. The modules $\sum_{i=1}^{n} A_i$ have finite length by the choice of k. This can be seen from the isomorphisms

$$\sum_{i=1}^{n} \oplus \frac{A_i}{eJ^{k+1}} \cong \frac{\sum_{i=1}^{n} A_i}{eJ^{k+1}} \quad \text{and} \quad \frac{\sum_{i=1}^{n} A_i}{\sum_{i=1}^{n-1} A_i} \cong \frac{\dfrac{\sum_{i=1}^{n} A_i}{eJ^{k+1}}}{\dfrac{\sum_{i=1}^{n-1} A_i}{eJ^{k+1}}} \cong \frac{\sum_{i=1}^{n} \oplus \dfrac{A_i}{eJ^{k+1}}}{\sum_{i=1}^{n-1} \oplus \dfrac{A_i}{eJ^{k+1}}}$$

which is simple. We have constructed an infinite chain $S_1 \subsetneqq S_2 \subsetneqq S_3 \dots \subseteq eR$ where each S_i has finite length. Also we have that $\frac{eR}{S_i} \not\cong \frac{eR}{S_j}$ for $i \neq j$ since otherwise we could apply Schanuel's Lemma to the two short exact sequences

$$S_i \rightarrowtail eR \twoheadrightarrow \frac{eR}{S_i} \quad \text{and} \quad S_j \rightarrowtail eR \twoheadrightarrow \frac{eR}{S_j}$$

to obtain $S_i \oplus eR \cong S_j \oplus eR$. But since eR has a local endomorphism ring $S_i \cong S_j$ by Lemma 9. Since S_i and S_j have finite lengths and one contains the other, $S_i \cong S_j$ implies that $S_i = S_j$.

We denote the representatives (finitely many by assumption) of the finitely generated indecomposable non-projective left R-modules by $U_1, U_2, \dots U_n$. We note that $\frac{eR}{S_i}$ is non-projective for each i and therefore $D(\frac{eR}{S_i}) = V_i \oplus$ Projective where V_i is a direct sum of certain U_j's, say $V_i = \sum_{i=1}^{n'} \oplus U_j$. This follows from property (2) of D mentioned at the beginning. From property (3) of D we have that $D(V_i)$ is stably isomorphic to the direct sum of the $D(U_j)$'s. By applying D again we get (*) $\frac{eR}{S_i} \oplus$ Projective $\cong \bigoplus_{j=1}^{n'} D(U_j) \oplus$ Projective.

Since $\frac{eR}{S_i}$ has a local endomorphism ring it has the exchange property. But $\frac{eR}{S_i}$ is indecomposable and thus must be isomorphic to a direct summand of a module on the right hand side of (*). Since $\frac{eR}{S_i}$ is non-projective the module must necessarily be $D(U_j)$ for some j, $0 \leq j \leq n'$.

Therefore for every $i \in I$ there is an index $j = j(i)$ such that U_j is a summand of V_i and $\frac{eR}{S_i}$ is a summand of $D(U_j)$. By Lemma 9 we have that the complement of $\frac{eR}{S_i}$ on the right hand side of ⊛ is a finitely generated projective. Therefore we can write $\frac{eR}{S_i} \oplus \text{Projective} \cong D(U_{j(i)}) \oplus \text{Projective}$ for every $i \in I$. Since there are finitely many U_j's and infinitely many S_i's, there are indices i, i' such that $i \neq i'$ but $j(i) = j(i')$ i.e. $D(U_{j(i)}) \cong D(U_{j(i')})$. Then we would have $\frac{eR}{S_i} \oplus \text{Projective} \cong \frac{eR}{S_j} \oplus \text{Projective}$. Since R is semi-perfect, Projective can be written uniquely (up to isomorphism) as the direct sum of principal ideals generated by indecomposable idempotents. Therefore both sides of $\frac{eR}{S_i} \oplus \text{Projective} \cong \frac{eR}{S_j} \oplus \text{Projective}$ are sums of modules with local endomorphism rings and so by the Krull-Schmidt theorem $\frac{eR}{S_i} \cong \frac{eR}{S_j}$ which is a contradiction. Therefore R is right artinian.

Since R is right artinian the Krull-Schmidt theorem holds in the category of finitely generated right R-modules. We use this to show that for any finitely generated (left or right) R-module M, M and $D(M)$ have the same number of non-projective indecomposable summands. For suppose M is a finitely generated (left or right) R-module then M can be decomposed into the direct sum of indecomposable non-projective and indecomposable projective summands. Since R is both left artinian and right artinian the Krull-Schmidt theorem gives that this decomposition is unique up to isomorphism. Let $M = \bigoplus_{i=1}^{n} A_i \oplus \bigoplus_{j=1}^{m} B_j$ where the A_i's are the indecomposable non-projective summands and the B_j's are the indecomposable projective summands. Then we can also decompose $D(M)$ in the same way i.e. $\bigoplus_{i=1}^{n'} A_i' \oplus \bigoplus_{j=1}^{m'} B_j' \cong D(M) \cong \bigoplus_{i=1}^{n} D(A_i) \oplus \text{Projective}$

where the A_i''s and B_j''s have the same properties as the A_i's and B_j's. By (2) of the properties of D mentioned at the beginning each $D(A_i)$ has a non-projective indecomposable summand and in fact it only has one. For suppose $D(A_i) = C_1 \oplus C_2 \oplus \text{Projective}$ where C_1, C_2 are non-projective indecomposable summands. Then $A_i \oplus \text{Projective} \cong D(C_1) \oplus D(C_2) \oplus \text{Projective}$ and each of $D(C_1)$, $D(C_2)$ must have a non-projective indecomposable summand, say $D(C_1)'$ and $D(C_2)'$ respectively. Since A_i has local endomorphism ring, it has the exchange property and therefore we may assume without loss of generality that $A_i \cong D(C_1)'$. Lemma 9 gives us that $D(C_2)'$ is a projective. This is a contradiction. Therefore for $1 \leq i \leq n$, $D(A_i)$ has one non-projective indecomposable direct summand which must be isomorphic to some A_i'. By the Krull-Schmidt theorem $n = n'$.

Also we have that two finitely generated R-modules without projective summands are isomorphic if and only if they are stably isomorphic. That isomorphic implies stably isomorphic is true in general. On the other hand suppose we have A and B finitely generated R-modules such that A and B have no projective summands and $D(A) \oplus \text{Projective} \cong D(B) \oplus \text{Projective}$. This implies that $A \oplus \text{Projective} \cong B \oplus \text{Projective}$. We can decompose both sides of this equation into the direct sum of finitely generated indecomposables and since neither A nor B have projective summands, $A \cong B$ by the Krull-Schmidt theorem.

These last two remarks give us the desired result. For if B is a finitely generated non-projective indecomposable right module then by the above $D(B)$ has only one non-projective indecomposable summand. Then $B \oplus \text{Projective} \cong D(U_i) \oplus \text{Projective}$ and by the last remark $B \cong$ the unique (up to isomorphism) non-projective direct summand of $D(U_i)$. Thus R has the same number of non-projective finitely generated indecomposable modules on the right as on the left and since R is semi-perfect, the same can be said for the left and right finitely generated indecomposable projective modules

Proof of (1) $\longrightarrow$ (6)

R is left artinian by Lemma 4. Therefore if U is a uniserial left module we can show (as in the proof of (3) $\longrightarrow$ (2)), by considering the projective cover for U, that $U \cong \dfrac{Re}{J^k e}$, e an indecomposable idempotent. Since $J^k e$ is uniserial there exists an indecomposable idempotent e' such that Re' is the projective cover for $J^k e$ (and therefore $J^k e \cong \dfrac{Re'}{J^{k'} e'}$ for some k')

Then we have the following exact sequence for the finitely presented module U $Re' \xrightarrow{\pi_1} J^k e \xrightarrow{i} Re \xrightarrow{\pi_2} U$ where π_1, π_2 are the maps resulting from Re' and Re being projective covers and i the natural inclusion. Therefore $D(U) = \mathrm{Cok}((Re)^* \xrightarrow{(i\pi_1)^*} (Re')^*)$ is a homomorphic image of $(Re')^*$, a principal indecomposable right ideal.

By Lemma 11 R is right artinian. For if N is a finitely generated indecomposable left R-module, then $N \cong \frac{Re}{J^k e}$ for some k and some indecomposable idempotent e. Therefore there are only a finite number of non-isomorphic finitely generated indecomposable left R-modules.

Thus if we are given M any finitely generated right module then D(M) is a direct sum of uniserial left modules, say $D(M) = \bigoplus_{i=1}^{n} U_i$. Then $M \oplus \text{Projective} \cong \bigoplus_{i=1}^{n} D(U_i) \oplus \text{Projective}$ where by the above the $D(U_i)$ are homomorphic images of principal indecomposable right ideals. Since R is right artinian we have by the Krull-Schmidt theorem that M is a direct sum of homomorphic images of principal indecomposable right R-modules. By Nakayama [19, Thm. 3] R is a serial ring.

3.6

If X is a subset of a ring R, set $(X:0) = \{ a \in R \mid Xa = 0 \}$ and $(0:X) = \{ a \in R \mid aX = 0 \}$. Any right (left) ideal of R of the form (X:0) ((0:X)) is a right (left) annulet.

A ring R is quasi-Frobenius in case

(1) each right ideal is a right annulet.

(2) each left ideal is a left annulet

and (3) R is right (or left) artinian.

Faith and Walker [9, Thm. 5.3] showed that R is quasi-Frobenius if and only if each injective right (left) R-module is projective (this characterization is still valid when the substitution injective <——> projective is made (Faith [7, Thm. (A)])).

R is called a right S-ring if for any left ideal I of R $(I:0) \neq 0$.

A left R-module M is called torsionless provided M can be embedded (as a left R-module) into a direct product of copies of R.

${}_R\mathcal{U}^*$ will denote the class of torsionless modules in ${}_R\mathcal{U}$.

Theorem 2 gives several equivalent characterizations of serial quasi-Frobenius rings. The statements of Theorem 2 are similar to those of Theorem 1 with the main difference being that the class ${}_R\mathcal{U}$ of uniserial left R-modules in Theorem 1 is replaced by the class ${}_R\mathcal{U}^*$ of torsionless uniserial left R-modules.

<u>Theorem 2</u>: The following are equivalent for any ring R.

(1) Each left R-module is a direct sum of modules in ${}_R\mathcal{U}^*$.

(2) R is a left artinian, right S- ring and ${}_R\mathcal{U}^*$ is closed under the operations of taking essential and coessential extensions.

(3) Every left R-module is relatively divisible projective and each indecomposable cyclic left R-module is in ${}_R\mathcal{U}^*$.

(4) R is a left serial quasi-Frobenius ring.

(5) $\frac{R}{J^k}$ is quasi-Frobenius for each k.

(6) The left-right symmetry of (1)-(5).

Proof:

(1) <--> (3) This is true by Theorem 1.

(2) --> (1) We show that ${}_R\mathcal{U}^*$ contains the simple R-modules and therefore the result follows by Proposition 2. If A is a simple left R-module then $A \cong \frac{R}{M}$ where M is a maximal left ideal. Since R is a right S-ring there exists $x \neq 0 \in (M:0)$. We define a map

$\varphi: \frac{R}{M} \longrightarrow Rx$ by $\varphi(\bar{r}) = rx$. φ is clearly an isomorphism and therefore $A \cong Rx$ which is torsionless.

(4) --> (2) Since R is left serial it is left artinian. From the proof of Theorem 5.3 in Faith and Walker [9] one obtains that if R is a quasi-Frobenius ring then every R-module is torsionless. Therefore for R quasi-Frobenius ${}_R\mathcal{U} = {}_R\mathcal{U}^*$ and as in the proof of (3) --> (2) of Theorem 1 it is enough to show that ${}_R\mathcal{U}$ is closed with respect to projective covers and injective hulls. If A is in ${}_R\mathcal{U}$ then as before we can show that $P(A) \cong Re$, e an indecomposable idempotent. Since R is left serial, Re is in ${}_R\mathcal{U}$. Also we have that $E(A) = E(\text{soc } A)$ and since A is in ${}_R\mathcal{U}$, soc A is simple. $E(\text{soc } A)$ is torsionless and therefore there exists a monomorphism $f\colon E(\text{soc } A) \rightarrowtail \prod_{i\in I} (R)_i$. We consider the maps $\pi_i f$ where the π_i are the natural projections. Suppose that $\pi_i f$ is not injective for all i. This implies that $\ker \pi_i f \neq 0$ for all i and therefore that $\text{soc } A \subseteq \ker \pi_i f$ for all i. But this contradicts that f is a monomorphism and thus for some i, $\pi_i f$ is a monomorphism. This implies that $E(A)$ is isomorphic to a direct summand of R which is

necessarily indecomposable. Since R is a left serial ring, $E(A)$ is in ${}_R\mathcal{U}$. R is a right S-ring since R quasi-Frobenius implies that every left ideal is a left annulet.

(1) $\to$ (4) We have that R is serial from Theorem 1. If Q is an indecomposable injective left R-module then by assumption Q is in ${}_R\mathcal{U}^*$. Since Q is an indecomposable injective and R is left artinian we can consider Q as being the injective hull of some simple module. Therefore, as in the proof of (4) $\to$ (2) above, we obtain that Q is a direct summand of R and is thus also projective. Since R is left artinian, each injective is the direct sum of indecomposable injectives and hence each injective is projective. By the result of Faith and Walker quoted at the beginning R is quasi-Frobenius.

(5) $\to$ (4) To prove that R is left serial it is enough to show that, for e an indecomposable idempotent, $\dfrac{J^k e}{J^{k+1} e}$ is simple or zero for all k. For e an indecomposable idempotent $\bar{R}\,e$ is $\bar{R}(=\frac{R}{J})$ projective. By assumption $\bar{R}$ is quasi-Frobenius and therefore by the result of Faith's mentioned at the beginning $\bar{R}\,e$ is $\bar{R}$ injective. Since $\bar{R}\,e$ is an indecomposable injective, soc $\bar{R}\,e$ is simple. But $\dfrac{J^{k-1}e}{J^k e} \subseteqq \text{soc}\,\bar{R}\,e$ and therefore $\dfrac{J^{k-1}e}{J^k e}$ is simple or zero for all k.

(4) $\to$ (5) We denote $\dfrac{R}{J^k}$ by $\bar{R}$. Since R is serial $\bar{R}\,e = \dfrac{Re}{J^k e}$, e an indecomposable idempotent. We let $\bar{E}$ denote the $\bar{R}$-injective hull of $\overline{Re}$. We have already proved the equivalence of (1) through (4)

and thus can assume (1). By (1) every left R-module can be written as the direct sum of modules in ${}_R\mathcal{U}$. This is true for any homomorphic image of R and in particular for $\overline{R}$. Therefore $\overline{E}$ is in ${}_{\overline{R}}\mathcal{U}$. The projective cover of $\overline{E}$ is $\overline{Rf}$, f an indecomposable idempotent, and therefore $\overline{E} \cong \frac{Rf}{If}$ where $J^k f \subseteq If$. We show that $\overline{Re}$ is $\overline{R}$ injective by comparing $\ell(\overline{E})$ and $\ell(\overline{Re})$. If $\ell(\overline{Re}) = \ell(\overline{E})$ this implies that $\overline{Re} \cong \overline{E}$ and thus $\overline{Re}$ is $\overline{R}$-injective. If $\ell(\overline{Re}) < \ell(\overline{E})$ then since $\ell(\overline{Re}) \leqq \ell(\overline{Rf}) \leqq k$ we have $\ell(\overline{Re}) < \ell(\overline{Rf}) \leqq k$. Therefore $J^{k-1}e = J^k e$ and by Nakayama's Lemma $J^k e = 0$. Thus $\frac{Re}{J^k e} = Re$ and $\overline{E}$ is the $\overline{R}$-injective hull of Re. Considered as an R-module $\overline{E}$ is indecomposable and since R is quasi-Frobenius, Re is R-injective. Therefore $Re = \overline{E}$. This implies that Re and thus $\overline{Re}$ is $\overline{R}$-injective. By Faith's result mentioned before $\overline{R}$ is quasi-Frobenius.

Remark: Our condition (2) is slightly different from the corresponding condition of Griffith [11, Thm. 4.2]. His condition is that R is left artinian and ${}_R\mathcal{U}^*$ is closed under the operations of taking injective hulls and projective covers. He then quotes Corollary 1 to obtain (1). However, one of the conditions of Corollary 1 is that the class of cyclic modules under consideration contains the simple R-modules. Griffith's assumption does not guarantee this. For instance consider the serial ring consisting of all 2 x 2 upper triangular matrices over a given field. It is easy to show that over this ring ${}_R\mathcal{U}^*$ is closed with respect to injective hulls and projective covers. However, this ring is not quasi-Frobenius. The condition that R is right S-ring was added to insure that the simple R-modules are torsionless.

For an example of a ring that satisfies Theorem 1 but does not satisfy Theorem 2 consider the ring of all 2 x 2 matrices over a given field with the usual matrix multiplication except that the product of off-diagonal entries is zero. This ring is serial but is not quasi-Frobenius (see Mueller [17]).

CHAPTER IV

4.1 In this chapter we give several characterizations of commutative Köthe rings. Köthe [15] showed that if R is a principal ideal artinian ring (not necessarily commutative) then every R-module can be written as the direct sum of cyclic modules. The main result of this chapter is that if R is a commutative ring and there exists a cardinal number n such that every R-module is a summand of a direct sum of R-modules with at most n generators, then R is a principal ideal artinian ring. This result, due to Warfield [26, Thm. 2], generalized results of Cohen and Kaplansky [4] in which $n = 1$ and of Griffith [11, Thm. 4.3] in which n is finite. It follows from this that in the commutative case the class of Köthe rings coincides with the class of generalized Köthe rings.

The other characterizations which appear in Theorem 3 are due mainly to Griffith [11, Thm. 4.3]. It is shown that in the commutative case the rings considered in Theorems 1 and 2, namely the class of serial rings and the class of serial quasi-Frobenius rings coincide with each other and with the class of Köthe rings.

<u>Theorem 3</u>: The following are equivalent for any commutative ring R.

(1) R is a principal ideal artinian ring.

(2) R is a serial quasi-Frobenius ring.

(3) Every R-module is a direct sum of cyclic modules.

(4) For some cardinal number n every R-module is a summand of a direct sum of modules each generated by at most n elements.

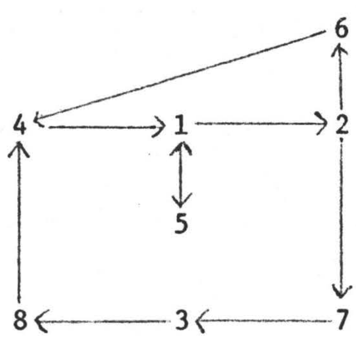
(5) Every R-module is a direct sum of indecomposable modules.

(6) Every R-module is isomorphic to a direct sum of ideals of R.

(7) Every R-module is relatively divisible projective.

(8) Every R-module is pure projective.

The following is a diagram of the proof.



<u>4.2</u> In this section we prove the implications (4) $\rightarrow$ (1) and (5) $\rightarrow$ (1). The results and proof are due to Warfield [26, Thms. 2,3]. We note that (4) implies R is noetherian by Lemma 2 and (5) implies R is noetherian by Faith and Walker [12, Cor. 1.3]. Thus we consider

commutative noetherian rings which are not principal ideal artinian. We construct arbitrarily large modules over such rings. Although these modules are not necessarily indecomposable they can only have finite direct sum decompositions. Lemma 13 is basic to the construction of such modules and the techniques used there are similar to those of Griffith [11] to prove that if every R-module is a direct sum of finitely generated R-modules then R is principal ideal artinian.

For m a maximal ideal of the commutative ring R, R_m will denote the localization of R by m. If L is a $\frac{R}{m}$ module then $\dim_{\frac{R}{m}}(L)$ will denote the dimension of L as a $\frac{R}{m}$ vector space.

R is called a special PIR if it is a commutative local ring with identity whose maximal ideal m is principal and nilpotent (see Zariski-Samuel [27, p. 245]). If R is also noetherian it follows from [1, Prop. 8.6] that R is a local artinian ring. If in this case $\dim_{\frac{R}{m}}\left(\frac{m}{m^2}\right) \leq 1$, then R is a principal ideal artinian ring by [1, Prop. 8.8].

<u>Lemma 12</u>: Let R be a commutative noetherian ring which is not a principal ideal artinian ring. Then either

(i) R has a maximal ideal m such that R_m is a discrete valuation ring, or

(ii) R has a factor ring $S(= \frac{R}{L}$ for some ideal $L)$ which is a local ring with maximal ideal m such that $m = aS \oplus bS$, aS and bS simple.

<u>Proof</u>: If for all maximal ideals m of R, $\dim_{\frac{R}{m}}\left(\frac{m}{m^2}\right) \leq 1$ then by Warfield [25, Thm. 4] this is equivalent to R_m a discrete valuation ring or a special PIR for every maximal ideal m. In [25, Thm. 4] Warfield also shows that another equivalent condition is that R is the direct sum of Dedekind domains and special PIR's. From Warfield's proof of the equivalence of these conditions it follows that if R_m is a special PIR for every maximal ideal m, then R is a direct sum of PIR's. By the remark before Lemma 12 these PIR's are necessarily principal ideal artinian and therefore R is principal ideal artinian. This is a contradiction and therefore R_m is a discrete valuation ring for some maximal ideal m.

If for some maximal ideal m, $\dim_{\frac{R}{m}}\left(\frac{m}{m^2}\right) = n > 1$ we consider the ring $\frac{R}{m^2}$. We note that $\frac{R}{m^2}$ is a local ring with maximal ideal $\frac{m}{m^2}$ (For if I is an ideal such that $m^2 \subseteq I$ and $I \not\subseteq m$ then $I + m = R$. Multiplication of both sides by m gives us $Im + m^2 = m$ and therefore $m^2 + mI + I = R$. However, since $m^2 \subseteq I$ and $mI \subseteq I$ the last equation implies that $I = R$). We consider a subspace of $\frac{m}{m^2}$ of dimension $n-2$. This necessarily has the form $\frac{L}{m^2}$, L an ideal contained in m. By the above $S = \frac{R}{L}$ is a local ring with maximal ideal $\frac{m}{L}$. The $\frac{R}{m}$ dimension of $\frac{m}{L}$ $\left(\cong \frac{\frac{m}{m^2}}{\frac{L}{m^2}}\right)$ is 2. Therefore

$\frac{m}{L} = a\frac{R}{L} \oplus b\frac{R}{L}$, $a\frac{R}{L}$ and $b\frac{R}{L}$ simple and the ring S is of the form of (ii).

We now give the construction which is basic in obtaining the large modules mentioned in the introduction. We consider a ring S as in Lemma 12(ii). We let F be the Cartesian product of a countable number of copies of S indexed by the non-negative integers. If x is an element of F, the co-ordinates of x are denoted by $(x_0, x_1, x_2, \ldots)$. We define a mapping σ on F, co-ordinate-wise, by $\sigma(x)_0 = 0$ and $\sigma(x)_{i+1} = x_i$. We let K be the set of elements of F of the form $ax - b\sigma(x)$. K can easily be shown to be a submodule and we define A to be the S-module $\frac{F}{K}$.

Lemma 13: Let S be a ring satisfying (ii) of Lemma 12, $k = \frac{S}{m}$, $k[t]$ the polynomial ring in one variable over k, $k[[t]]$ the power series ring in one variable over k, and S the A-module defined above. Then

(1) $\frac{A}{mA} \cong k[[t]]$ as $k[t]$ modules and $\frac{\mathrm{End}_S(A)}{J} \cong k[[t]]$ as rings where J is the ideal of $\mathrm{End}_S(A)$ consisting of those endomorphisms f such that $f(A) \subseteq mA$.

(2) A is an indecomposable S-module which is not countably generated.

Proof: We first set down the basic notation that will be used throughout the proof of this lemma. $\overline{x}$ will denote an element of $\frac{A}{mA}$ where $x \varepsilon A$. If $x \varepsilon A(=\frac{F}{K})$ then $x = f + K$ where $f \varepsilon F$.

Since $F = \prod_{i=0}^{\infty} (S)_i$ we can also write $f = (s_i)$, $s_i \varepsilon S$.

We also have the following isomorphisms. Since $K \subseteq mF$ and $A = \frac{F}{K}$ we have the isomorphisms $mA \cong \frac{mF}{K}$ given for $y \varepsilon mA$, $y = r(f' + K)$, $r \varepsilon m$ and $f' \varepsilon F$ by $y \longrightarrow rf' + K$ and

$\frac{A}{mA} = \frac{\frac{F}{K}}{\frac{mF}{K}} \cong \frac{F}{mF}$ given by $x + mA \longrightarrow f + mF$ (x and f as assigned above). We first make $\frac{A}{mA}$ into a $k[t]$ module. We define maps α, β: $\frac{A}{mA} \longrightarrow mA$ by $\alpha(\bar{x}) = ax$ and $\beta(\bar{x}) = bx$. These maps are well defined since $m^2 = 0$ and we show that they are also injective. To prove α is injective we must show that $ax = 0$ implies that $x \varepsilon mA$. By the above isomorphisms this is equivalent to showing $af \varepsilon K$ implies $f \varepsilon mF$. Now $af \varepsilon K$ implies there exists an element $y = (y_i)$ of F such that $af = ay - b\sigma(y)$. Written co-ordinate wise this gives us $(as_0, as_1, \ldots, as_i, \ldots) = (ay_0, ay_1 - by_0, \ldots, ay_i - by_i, \ldots)$. Starting from the second co-ordinate we have $by_i = a(s_{i+1} - y_{i+1})$ for all i. Since $bS \cap aS = 0$, $by_i = 0$ for all i. This gives us $y_i \varepsilon m$ for all i since otherwise S a local ring implies that y_i is a unit and thus $b = 0$. Since $y_i \varepsilon m$ for all i and $m^2 = 0$, $ay - b\sigma(y) = 0$ and thus $af = 0$. Again, since S is a local ring, $af = 0$ gives us that $s_i \varepsilon m$ for all i and thus $f \varepsilon mF$. A similar proof shows β is injective. Also, from the definition of K $af + K = b\sigma(f) + K$ and hence $aA \subseteq bA$ and $mA = bA$. Therefore β is surjective and thus is an isomorphism.

We define a map $\emptyset$ on $\frac{A}{mA}$ by $\emptyset = \beta\alpha^{-1}$. We make $\frac{A}{mA}$ into a module over $k[t]$ by defining $ty = \emptyset(y)$. We showed at the beginning of this proof that $\frac{A}{mA} \cong \frac{F}{mF}$. But $\frac{F}{mF} = \frac{\Pi(S)_i}{m\Pi(S)_i} = \frac{\Pi(S)_i}{\Pi(m)_i}$ and $\frac{\Pi(S)_i}{\Pi(m)_i} \cong \Pi(\frac{S}{m})_i$ under the map $(s_i) + \Pi m_i \longrightarrow ((s_i + m)_i)$. It is trivial that $\prod_{i=0}^{\infty} (\frac{S}{m})_i = \prod_{i=0}^{\infty} (k)_i = k[[t]]$. Therefore we have $\frac{A}{mA} \cong k[[t]]$ where the isomorphism is given by $\bar{x} \longrightarrow \sum_{i=0}^{\infty} \overline{s_i}\, t^i$ $(\overline{s_i} = s_i + m)$. We show this is a $k[t]$ isomorphism. The computation that follows is used repeatedly in the rest of the proof. We show $\emptyset(\bar{x})$ can be described explicitly in terms of $\sigma(f)$. We have $\emptyset(\bar{x}) = \beta^{-1}\alpha(\bar{x}) = \beta^{-1}(ax)$ and as above $ax = b\sigma(f) + K$. Therefore $\beta^{-1}(ax) = \beta^{-1}(b\sigma(f) + K) = \overline{\overline{\sigma(f)}}$ where $\overline{\overline{\sigma(f)}} = (\sigma(f) + K) + \frac{mF}{K}$. Therefore $\emptyset^i(\bar{x}) = \overline{\overline{\sigma^i(f)}}$ where $\overline{\overline{}}$ has the same meaning as above. Thus for $k(t) = \sum_{i-0}^{n} k_i t^i$, $k(t)\,\bar{x} = k_0\,\bar{x} + \ldots + k_n\, \emptyset^n(\bar{x})$ maps onto $k_0 \sum_{i=0}^{\infty} \overline{s_i}\, t^i + k_1 \sum_{i=0}^{\infty} \overline{s_i}\, t^{i+1} + \ldots + k_n \sum_{i=0}^{\infty} \overline{s_i}\, t^{i+n}$ which is equal to $\sum_{i=0}^{n} k_i t^i (\sum_{i=0}^{\infty} \overline{s_i}\, t^i)$. Therefore $\frac{A}{mA} \cong k[[t]]$ as $k[t]$ modules.

Next we show $\frac{\mathrm{End}_S(A)}{J} \cong \mathrm{End}_{k[[t]]}(k[[t]])$ as rings and since $\mathrm{End}_{k[[t]]}(k[[t]]) \cong k[[t]]$ this would complete the proof of (1). Suppose we are given $\varphi \in \frac{\mathrm{Endo}_S(A)}{J}$ i.e. φ is an S-endomorphism of

A such that $\varphi(A) \not\subseteq mA$. Therefore φ induces a non-zero S-endomorphism $\overline{\varphi}$ of $\frac{A}{mA}$ given by $\overline{\varphi}(\overline{x}) = \overline{\varphi(x)}$ ($\overline{\varphi(x)} = \varphi(x) + mA$). $\overline{\varphi}$ is in fact a k-endomorphsim of $\frac{A}{mA}$ since if $k = s + m$ then $\overline{\varphi}(k\overline{x}) = \overline{\varphi}(\overline{sx}) = \overline{\varphi(sx)}$ $= \overline{s\,\varphi(x)} = k\overline{\varphi(x)} = k\overline{\varphi}(\overline{x})$. We now prove $\overline{\varphi}$ is a k[t] endomorphism of $\frac{A}{mA}$. We define $k(t) = \sum_{i=0}^{n} k_i t^i$ and we must show $\overline{\varphi}(k(t)\overline{x}) = k(t)\overline{\varphi}(\overline{x})$.

Since $\overline{\varphi}$ is a k-endomorphism

$$\overline{\varphi}(k(t)\overline{x}) = k_0\overline{\varphi}(\overline{x}) + k_1\overline{\varphi}(\emptyset(\overline{x})) + \ldots + k_n\overline{\varphi}((\emptyset)^n(\overline{x}))$$

and we also have

$$k(t)\overline{\varphi}(\overline{x}) = k_0\overline{\varphi}(\overline{x}) + k_1\emptyset(\overline{\varphi}(\overline{x})) + \ldots + k_n\,\emptyset^n(\overline{\varphi}(\overline{x}))$$

To prove these are equal it is enough to show $\overline{\varphi}(\emptyset(\overline{x})) = \emptyset\overline{\varphi}(\overline{x})$. From above we have $\overline{\varphi}(\emptyset(\overline{x})) = \overline{\varphi}(\overline{\overline{\sigma(f)}})$ where $\overline{\overline{\sigma(f)}} = (\sigma(f) + K) + \frac{mF}{K}$.

On the other hand $\emptyset\overline{\varphi}(\overline{x}) = (\beta^{-1}\alpha)\ (\overline{\varphi(x)}) = \beta^{-1}(a\varphi(x))$.

But φ is an S-endomorphism and hence $\beta^{-1}(a\varphi(x)) = \beta^{-1}(\varphi(ax))$.

As before $ax = b\sigma(f) + K$ and since φ is an S-endomorphism $\beta^{-1}(\varphi(ax)) = \beta^{-1}(\varphi(b\sigma(f) + K)) = \beta^{-1}(b\varphi(\sigma(f) + K)) = \overline{\varphi(\sigma(f) + K)} =$ $\overline{\varphi}(\overline{\overline{\sigma(f)}})$ where $\overline{\overline{}}$ has the same meaning as above. Therefore $\overline{\varphi}(\emptyset(\overline{x})) =$ $\emptyset\overline{\varphi}(\overline{x})$ and hence $\overline{\varphi}$ is a k[t] endomorphism of $\frac{A}{mA}$ and thus of k[[t]].

We now show any k[t] endomorphism $\overline{\varphi}$ of k[[t]] is actually a k[[t]] endomorphism of k[[t]]. We wish to show $\overline{\varphi}(\pi) = \pi\overline{\varphi}(1)$ for $\pi \in k[[t]]$. We write $\pi = \pi_n + t^n \pi^n$ where π_n is the polynomial consisting of the first n terms of π and $t^n \pi^n$ the remainder. The following equalities are true since $\overline{\varphi}$ is a k[t] endomorphism of k[[t]]. $\overline{\varphi}(\pi) = \overline{\varphi}(\pi_n + t^n \pi^n) = \varphi(\pi_n) + \overline{\varphi}(t^n \pi^n)$. Substituting $\pi - t^n \pi^n$ for π_n we have $\overline{\varphi}(\pi) - \pi\overline{\varphi}(1) = t^n(\overline{\varphi}(\pi) - \pi^n\overline{\varphi}(1))$ for all n. We let (t^n) denote the ideal generated by t^n and obviously $\bigcap_{n=0}^{\infty} (t^n) = 0$. Therefore, $\overline{\varphi}(\pi) = \pi\overline{\varphi}(1)$ and thus $\overline{\varphi}$ is a k[[t]] endomorphism of k[[t]].

The map of $\dfrac{\mathrm{End}_S(A)}{J}$ into k[[t]] described above is obviously one to one and a ring homomorphsim. To complete the proof of (1) we need to show any k[[t]] endomorphism f of k[[t]] is induced by an S-endomorphism of A. First we note any endomorphism f of k[[t]] is given by multiplication of an element of k[[t]], say $\tau(t) = \sum_{i=0}^{\infty} \overline{y_i}\, t^i$ $(\overline{y_i} = y_i + m,\ y_i \in S)$. We consider $s(t) = \sum_{i=0}^{\infty} y_i\, t^i$ in S[[t]]. s(t) can be made to induce an endomorphism on F by defining $t^n f = \sigma^n(f)$ for $f \in F$. The endomorphism induced by s(t) maps $f = (s_0, s_1, \ldots, s_n, \ldots)$ onto $(y_0 s_0,\ y_0 s_1 + y_1 s_0,\ \ldots,\ \sum_{i+j=n} y_i s_j, \ldots)$.

For $k = (ax_0,\ ax_1 - bx_0, \ldots,\ ax_{i+1} - bx_i, \ldots) \in K$ the endomorphism induced by s(t) maps k into the element $(a(y_0x_0),\ a(y_0x_1 + y_1x_0) - by_0x_0, \ldots$ $\ldots, a(y_0x_{i+1} + \ldots + y_{i+1}x_0) - b(y_0x_{i-1} + \ldots + y_{i-1}x_0) \ldots)$ which also is in K. If we call the endomorphism induced by s(t) θ, then since $\theta(K) \subseteq K$, θ induces an endomorphism $\overline{\theta}$ on A given by $\overline{\theta}(x) = \theta(f) + K$.

It is clear that if $s(t) \neq 0$, $\overline{\theta}$ is in $\frac{\mathrm{End}_S(A)}{J}$. It is also clear that $\overline{\theta}\left(\frac{mF}{K}\right) \subseteq \frac{mF}{K}$ and thus $\overline{\theta}$ induces an endomorphism $\overline{\overline{\theta}}$ on $\frac{A}{mA}$. The action of $\overline{\overline{\theta}}$ on $\overline{x} \in \frac{A}{mA}$ is given by $\overline{\overline{\theta}}\,(\overline{x}) = \overline{\theta(x)} + mA =$ $(\theta(f) + \frac{F}{K}) + \frac{mF}{K} = ((y_0s_0,\ y_0s_1 + y_1s_0, \ldots,\ y_0s_n + \ldots + y_ns_0,\ \ldots) + \frac{F}{K}) + \frac{mF}{K}$.

By the isomorphism between $\frac{A}{mA}$ and $k[[t]]$ given at the beginning this element maps into $\left(\sum_{i=0}^{\infty} \overline{y_i}\, t^i\right)\left(\sum_{i=0}^{\infty} \overline{s_i}\, t^i\right)$ in $k[[t]]$. But this is just $f\left(\sum_{i=0}^{\infty} \overline{s_i}\, t^i\right)$ and thus we obtain our given endomorphism of $k[[t]]$.

To prove (2) we use the fact that $\frac{A}{mA} \cong \prod_{i=0}^{\infty} (k)_i$. $\prod_{i=0}^{\infty} (k_i) \cong$ $\mathrm{Hom}_k\left(\bigoplus_{i=0}^{\infty} (k)_i,\ k\right)$ under the map $\sum_{i=0}^{\infty} k_i t^i \longrightarrow f_k \in \mathrm{Hom}\left(\bigoplus_{i=0}^{\infty} (k)_i,\ k\right)$ where $f_k\left(\bigoplus_{j=1}^{n} \ell_{ij}\right) = \sum_{j=1}^{n} \ell_{ij}\, k_{ij}$. The dimension of $\bigoplus_{i=0}^{\infty} k_i$ as a k vector space is card $N = \omega$ and by Jacobson [13, Thm. 1, p. 86], $\dim_k\left(\prod_{i=0}^{\infty} k_i\right) = (\text{card } k)^{\omega} \geq 2^{\omega}$. Therefore the k-dimension of $\frac{A}{mA}$ is uncountable and thus A is not countably generated.

If A has a direct sum decomposition we can assume without loss of generality it decomposes into two summands, say $A = M \oplus N$. This induces a decomposition of mA, namely $mA = m\,M \oplus m\,N$. We note the projection maps of A, p_M and p_N, are not in the ideal J.

For if $p_M(A) \subseteq m\,A$ then M is necessarily contained in $m\,M$ and thus $M = m\,M$. But since $m^2 = 0$ this can't be so. Therefore we have the non-trivial idempotents p_M and p_N in the ring $\frac{\mathrm{Endo}_S(A)}{J}$ and thus by (1) we have two non-trivial idempotents in the ring $k[[t]]$. But $k[[t]]$ is a local ring and hence does not have any non-trivial idempotents. Therefore A is indecomposable.

Lemma 14: Let R be a noetherian ring which is not a principal ideal artinian ring. Then for any cardinal number n there exists a module M which cannot be generated by n or fewer elements such that any direct sum decomposition of M has a finite number of summands.

Proof: By Lemma 12 either

(i) R_η is a discrete valuation ring for some maximal ideal η of R, or

(ii) R has a factor ring S satisfying (ii) of Lemma 12.

In case (i) we can assume R is a discrete valuation ring with maximal ideal m. We show if X is an R_η-module with the required properties of the lemma it also has the same properties as an R-module. For if the R_η-module X has the decomposition $X = \sum_{i\in I} \oplus X_i$ as an R-module then by tensoring both sides with R_η we obtain

$X_{R_\eta} \cong X \underset{R}{\otimes} R_\eta \cong \sum_{i\in I} \oplus (X_i \underset{R}{\otimes} R_\eta)$ (where X_{R_η} denotes X as an R_η-module). This is since direct sums commute with tensor products and $X_{R_\eta} \cong X \underset{R}{\otimes} R_\eta$ under the map $x \otimes \frac{r}{s} \longrightarrow x\,\frac{r}{s}$, $x \in X$, $\frac{r}{s} \in R_\eta$.

Since this map is injective $X_i \bigotimes_R R_\eta \neq 0$ for any i. Also if X cannot be generated by n or fewer elements as an R_η-module, $X_{R_\eta} \cong X \bigotimes_R R_\eta$ gives us that X cannot be generated by n or fewer elements as an R-module.

We define R^* to be the completion of R in the m-adic topology i.e. the basic neighbourhoods of R are the ideals m^k. For a fixed cardinal number n we define M to be the Cartesian product of n copies of R^*. Clearly M is a complete Hausdorff R-module in the product topology. We show M is also a complete module in the m-adic topology on M. The m-adic topology on M has, as its basic neighbourhoods the modules $m^k M$; since m^k is finitely generated for each k, one can easily show that $m^k M = \prod_{i \varepsilon I} (m^k R^*)_i$, I an index set of cardinality n. But $m^k R^* = (m^*)^k$ where m^* is the completion of m([1, Prop. 10.15]).

We let (x_α) be a Cauchy sequence of elements of M in the m-adic topology. Since the m-adic topology is finer than the product topology, (x_α) is also a Cauchy sequence in the product topology. But M is a complete module in the product topology and thus (x_α) converges to some element $x \varepsilon M$ which we assume without loss of generality is 0, (x_α) a Cauchy sequence in the m-adic topology implies that for every k there exists $\varepsilon(k)$, such that for all $\gamma,\delta \geq \varepsilon(k)$, $x_\gamma - x_\delta \varepsilon \Pi(m^*)^k_i$. But if we assume (x_α) does not converge to zero in the m-adic topology then there exists a k such that for all α there exists $\beta > \alpha$ such that $x_\beta \notin \Pi(m^*)^k_i$. $x_\beta \notin \Pi(m^*)^k_i$

implies there exists a co-ordinate j such that $X_{\beta j} \notin (m^*)^k$. If we choose $\alpha = \varepsilon(k)$ then by the Cauchy condition $x_{\gamma j} - x_{\beta j} \in (m^*)^k$ for all $\gamma \geq \varepsilon(k)$. Since $x_{\beta j} \notin (m^*)^k$, $x_{\gamma j} \notin (m^*)^k$ for all $\gamma > \alpha$. This is a contradiction since (x_α) converging to zero in the product topology implies $x_{\alpha j}$ converges to zero. Thus M is a complete module in the m-adic topology.

To show M cannot be generated by n or fewer elements is similar to the proof in Lemma 13 that A was not countably generated. As in Lemma 13 we have the following isomorphisms $\frac{M}{mM} = \frac{\Pi(R^*)_i}{m\Pi(R^*)_i} = \frac{\Pi(R^*)_i}{\Pi(m^*)_i} \cong \Pi(\frac{R^*}{m^*})_i$. As in Lemma 13, $\dim_{\frac{R}{m}} (\Pi(\frac{R^*}{m^*})_i) \geq 2^n$ and therefore M cannot be generated by n or fewer elements.

Now we suppose M has an infinite direct sum decomposition. Then there exists a countably infinite direct sum decomposition $M = \sum_{i=0}^{\infty} \oplus N_i$ $(N_i \neq 0)$. We show any direct summand of M is closed in the m-adic topology. For suppose $M = N \oplus N'$ and that we have a Cauchy sequence $(m_\alpha) \in N$ such that (m_α) converges to $n' \in N'$. We consider the sequence $m_\alpha - n'$ which converges to zero in the m-adic topology. Thus for every k there exists a β such that $\alpha > \beta$ implies that $m_\alpha - n' \in m^k M = m^k N \oplus m^k N'$. $m_\alpha \in N$ for each α implies $(-n') \in m^k N'$ for all k which is true only if $n' = 0$. This implies N is complete and thus closed.

We define $N^r = \sum_{i=0}^{r} N_i$. By the above the N^r are all closed and their union is M. Therefore by the Baire Category Theorem some submodule N^r contains an open subset. Then N^r contains a neighbourhood of zero which implies for some integer j, $m^j M \subseteqq N^r$. But $M = N^r \oplus \sum_{i=r+1}^{\infty} N_i$ and therefore $m^j M \subseteqq N^r$ implies $m^j (\sum_{i=r+1}^{\infty} N_i) = 0$. Since R is a discrete valuation ring it is an integral domain and thus so is R^*. Since M is the direct product of R^* it is torsion free. Thus $m^j (\sum_{i=r+1}^{\infty} \oplus N_i) = 0$ gives a contradiction and there does not exist an infinite direct sum decomposition of M. This completes the proof of case (i).

For case (ii) we let M be the Cartesian product of n copies of the module A considered in Lemma 13. As above the $\frac{S}{m}$ dimension of $\frac{M}{mM}$ is $\geq 2^n$ and thus M cannot be generated by n or fewer elements. As in Lemma 13 we have the following isomorphisms.

$$\frac{M}{mM} = \frac{\Pi(A)_i}{m\Pi(A)_i} = \frac{\Pi(A)_i}{\Pi(mA)_i} \cong \Pi(\frac{A}{mA})_i \cong \Pi(\frac{F}{mF})_i$$

Similar to Lemma 13 we show any S-endomorphism of M induces a $k[[t]]$-module endomorphism of $\frac{M}{mM}$. We let $x \in M$ where $x = (x_i)$, $x_i \in A$ and $x_i = f_i + K$ where $f_i \in F$. We define $\bar{x} = x + mM$ and the above isomorphism maps $\bar{x} \in \frac{M}{mM}$ onto $(x_i + mA)_i \in \Pi(\frac{A}{mA})_i$. Let φ be an S-endomorphism of M. Then φ induces an S-endomorphism $\bar{\varphi}$ of $\frac{M}{mM}$ given by $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)}$ $(\overline{\varphi(x)} = \varphi(x) + mM)$. By the proof of

Lemma 13 the k[t] endomorphisms of k[[t]] coincide with the k[[t]] endomorphisms of k[[t]] and thus it suffices to prove $k(t)\overline{\varphi}(\bar{x}) = \overline{\varphi}(k(t)\,\bar{x})$ where $k(t) = \sum_{j=0}^{m} k_j\, t^j$. Now $\overline{\varphi}(\bar{x}) = \Pi(p_i\overline{\varphi}(\bar{x}))_i$ where p_i is the natural projection onto $(\frac{A}{mA})_i$. Thus it is enough to show $k(t)\, p_i\overline{\varphi}(\bar{x}) = p_i\overline{\varphi}(k(t)\,\bar{x})$ and in fact it is enough to show $k_1\, t\, p_i\overline{\varphi}(\bar{x}) = p_i\overline{\varphi}(k_1\, t\,\bar{x})$. We have $k_1\, t\, p_i\overline{\varphi}(\bar{x}) = k_1\, t(p_i\varphi(x) + mA) = k_1(\beta^{-1}\alpha)\,(p_i\varphi(x) + mA) = k_1\,\beta^{-1}(ap_i\varphi(x)) = k_1\,\beta^{-1}(p_i\varphi\,(ax))$. The last steps are true since p_i and φ are S-endomorphisms. Due to the fact that $ax = (ax_i) = (b\,\sigma(f_i) + K)_i$ and φ and p_i are S-endomorphisms we have the following equalities. $k_1\,\beta^{-1}(p_i\varphi(ax)) = k_1\,\beta^{-1}\, p_i\varphi((b\,\sigma(f_i) + K)_i) = k_1\,\beta^{-1}(b\, p_i\,\varphi((\sigma(f_i) + K)_i)) = k_1(p_i\varphi((\sigma(f_i) + K)_i) + mA) = k_1\, p_i\overline{\varphi}((\sigma(f_i) + K)_i + mA) = k_1\, p_i\overline{\varphi}((\beta^{-1}\alpha)\,((x_i + mA)_i)) = k_1\, p_i\overline{\varphi}(tx)$. The above establishes a ring homomorphism of $End_S(M)$ into $End_{k[[t]]}(\frac{M}{mM})$ and thus $End_{k[[t]]}(\Pi(k[[t]])_i)$. Therefore any S-decomposition of M into direct summands gives α non-trivial orthogonal idempotents in the ring $End_{k[[t]]}(\Pi(k[[t]])_i)$ and thus a decomposition of $\Pi(k[[t]])_i$ into α parts (as in Lemma 13 we use the fact that if N is an S-module and $N = mN$ then $N = 0$). Since k[[t]] is a complete discrete valuation ring α must be finite by the first part of the proof of this lemma. Therefore M has the required properties.

<u>Proof</u> of (4) $\longrightarrow$ (1).

We assume R is not principal ideal artinian. First we consider the case when n is finite. By Lemma 3, R is artinian. Since R is a commutative artinian ring, R is the direct sum of local rings and thus we can consider R a local ring with maximal ideal m. $\frac{R}{m^2}$ is not a principal ideal ring since if $\frac{m}{m^2}$ is principal then by [1, Prop. 8.8] R is a principal ideal ring. Therefore by the proof of Lemma 12 there exists a factor ring S of R of the form of (ii) of Lemma 12. The module A considered in Lemma 13 is an indecomposable S-module which is not countably generated. The existence of such a module contradicts the hypothesis.

Thus we can assume n is an infinite cardinal. By Lemma 2 R is noetherian. Since we have assumed R is not a principal ideal artinian ring there exists by Lemma 14 a module M which cannot be generated by n or fewer elements such that any direct sum decomposition of M has a finite number of summands. Since the number of summands is finite and n is an infinite cardinal, at least one of the summands cannot be generated by n or fewer elements. By Lemma 1 the assumption implies every R-module is a direct sum of modules each generated by at most n elements. This is a contradiction and thus R must be principal ideal artinian.

(5) $\longrightarrow$ (1) Corollary 1.3 in Faith and Walker [9] states that if each injective is a direct sum of indecomposables then R is noetherian. We assume R is a noetherian ring which is not principal ideal artinian. By Lemma 12 either (1) R_η is a discrete valuation ring for

some maximal ideal η of R or (2) R has a factor ring S of the type described in Lemma 12, (ii). In both cases we consider the modules constructed in Lemma 14, i.e. in the first case we let M be an infinite product of copies of the completion R_η^* of R_η and in the second case we let M be an infinite product of copies of the S-module A described in Lemma 13.

We show the rings $\mathrm{End}_R(R_\eta^*)$ and $\mathrm{End}_S(A)$ are local. To show $\mathrm{End}_R(R_\eta^*)$ is local it suffices to prove $\mathrm{End}_R(R_\eta^*) = \mathrm{End}_{R_\eta^*}(R_\eta^*)$ since $\mathrm{End}_{R_\eta^*}(R_\eta^*)$ is isomorphic to the local ring R_η^*. Obviously any R_η^* endomorphism of R_η^* is an R-endomorphism of R_η^*. On the other hand suppose φ is an R-endomorphism of R_η^*. Let x be in R_η^*. x is the limit of a Cauchy sequence (x_α) in R_η . We wish to show $\varphi(x) = x\varphi(1)$. Since R_η is a discrete valuation ring the ideals of R_η are exactly powers of the maximal ideal m of R_η and in fact it can be shown there exists an element $p \in R_\eta$ such that $m^k = (p^k)$ ([1, Prop. 9.2]). Since (x_α) is a Cauchy sequence, for each k there exists an α_k such that $x - x_{\alpha_k} \in (p^k)$, i.e. $x - x_{\alpha_k} = p^k r_k$, $r_k \in R_\eta$. Thus we have $\varphi(x) = \varphi(x_{\alpha_k} + p^k r_k) = \varphi(x_{\alpha_k}) + \varphi(p^k r_k)$. But it is easy to show φ is an R_η-endomorphism. Thus $\varphi(x) = \varphi(x_{\alpha_k}) + \varphi(p^k r_k) = x_{\alpha_k}\varphi(1) + p^k\varphi(r_k)$ for each k. Since R is noetherian $\bigcap(p^k) = 0$. Therefore $\varphi(x) = x\varphi(1)$ and φ is a R_η^* endomorphism of R_η^* .

To show $End_S(A)$ is local we consider the ideal J defined in Lemma 13. Since $m^2 = 0$, $J^2 = 0$ and $J^2 = 0$ implies J is contained in any maximal left ideal of $End_S(A)$. For if L is a maximal left ideal of $End_S(A)$ such that $J \not\subseteq L$ then $J + L = End_S(A)$ and thus $JL = J(End_S(A)) = J$. This gives us $JL + L = End_S(A)$. Since $JL \subseteq L$ this leads to a contradiction. By (1) of Lemma 13 $\frac{End_S(A)}{J}$ is a commutative local ring and by the above there is a one-to-one correspondence between the maximal left ideals of $End_S(A)$ and the maximal ideals of $\frac{End_S(A)}{J}$. Therefore $End_S(A)$ is a local ring.

We will prove the module M (for both cases) is not a direct sum of indecomposable modules. From Lemma 14 any direct sum decomposition of M (in both cases) has only a finite number of summands. From now on we just consider the module M constructed for case 2 but the fact that $End_{\hat{R}}(R^*_\eta)$ is local makes the proof for case 1 identical. Since any direct sum decomposition of M is finite we can choose the decomposition $M = B_1 \oplus \ldots \oplus B_r$ such that r is minimal and the B_i are indecomposable and non-zero. Since M is the infinite product of copies of A, $M = A_1 \oplus N$ where $A_1 \cong A$ and $N \cong M$. A has a local endomorphism ring and thus has the exchange property by Warfield [24, Prop. 1]. Therefore there exist submodules $B_i' \subseteq B_i$ such that B_i' is a summand of B_i and $M = A_1 \oplus B_1' \oplus \ldots \oplus B_r'$. Since B_i' is a summand of B_i and B_i is indecomposable, either $B_i' = B_i$ or $B_i' = 0$ for each i. For at least one i $B_i' = 0$, for otherwise $M = A_1 \oplus M$. Since A has local endomorphism ring, $N \cong B_1' \oplus \ldots \oplus B_r'$ by Lemma 9, where $B_i' = 0$ for some i. Since $N \cong M$ this is a contradiction to the original choice of r and thus R must be principal ideal artinian.

4.3 We complete the proof of Theorem 3 in this section. As mentioned before these results are due mainly to Griffith although they contain results originally due to Köthe [15] and to Cohen and Kaplansky [4].

(1) $\longrightarrow$ (2) To show R is serial it is enough to show $\frac{J^{k-1}e}{J^k e}$ is simple or zero for each k and each indecomposable idempotent e. We consider the ideal $\mathrm{soc}\left(\frac{R}{J^k}\right)$ in $\frac{R}{J^k}$. Since R is a principal ideal ring the preimage of $\mathrm{soc}\left(\frac{R}{J^k}\right)$ in R is a principal ideal and thus $\mathrm{soc}\left(\frac{R}{J^k}\right)$ is a principal ideal. However, $\mathrm{soc}\left(\frac{R}{J^k}\right)$ is also an $\frac{R}{J}$ - module and is also principal as an $\frac{R}{J}$ - module. This implies that $\ell\left(\mathrm{soc}\left(\frac{R}{J^k}\right)\right) \leq \ell\left(\frac{R}{J}\right)$. Since R is artinian, $\frac{R}{J} = \sum_{i=1}^{n} \oplus \frac{Re_i}{Je_i}$ where the $\frac{Re_i}{Je_i}$ are simple. From this decomposition it follows that $\frac{R}{J^k} = \sum_{i=1}^{n} \oplus \frac{Re_i}{J^k e_i}$ and thus $\mathrm{soc}\left(\frac{R}{J^k}\right) = \sum_{i=1}^{n} \oplus \mathrm{soc}\left(\frac{Re_i}{J^k e_i}\right)$. Since $\frac{Re_i}{Je_i}$ is simple for all i, $\ell\left(\mathrm{soc}\left(\frac{R}{J^k}\right)\right) \leq \ell\left(\frac{R}{J}\right)$ implies $\mathrm{soc}\left(\frac{Re_i}{J^k e_i}\right)$

is simple for all i (soc $\left(\frac{Re_i}{J^k e_i}\right) \neq 0$ since R is artinian). Therefore the result follows from $\frac{J^{k-1}e_i}{J^k e_i} \subseteq$ soc $\left(\frac{Re_i}{J^k e_i}\right)$.

R is a commutative artinian ring implies it is the direct sum of local artinian rings (necessarily serial). Therefore we can assume R is a local serial ring. In this case $R \supsetneq J \supsetneq J^2 \supsetneq \ldots \supsetneq J^n = 0$ is the unique composition series for R and thus J^{n-i} is the annihilator ideal for J^i. Therefore every ideal is an annulet and R is quasi-Frobenius.

We note that this proof actually shows commutative serial rings are quasi-Frobenius and therefore in the commutative case the class of serial rings is equal to the class of serial quasi-Frobenius rings.

(2) $\longrightarrow$ (7) This is true by Theorem 2.

(7) $\longrightarrow$ (3) By Lemma 5 every R-module is a direct summand of a direct sum of cyclically presented modules. By Lemma 3 R is artinian and therefore every cyclically presented module can be decomposed into the direct sum of cyclic indecomposable modules (which necessarily have local endomorphism rings). It follows that every R-module is a direct summand of modules with local endomorphism rings and by a result of Warfield [24, Thm. 1] every R-module is a direct sum of such modules.

(3) $\longrightarrow$ (8)   By Lemma 3 R is artinian. Over an artinian ring every finitely generated module is finitely presented and thus (9) follows by Lemma 5.

(8) $\longrightarrow$ (4)   This is obvious by Lemma 5.

(1) $\longrightarrow$ (5)   By the above (1) is equivalent to (2) and the result follows by Theorem 2.

(2) $\longrightarrow$ (7)   By Theorem 2 each R-module is a direct sum of modules in ${}_R\mathcal{U}^*$. Each module in ${}_R\mathcal{U}^*$ is necessarily an ideal in R and the conclusion follows.

(6) $\longrightarrow$ (4)   By the assumption we can take n in (5) to be the cardinality of R.

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