LATTICE OF TOPOLOGIES

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By

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SCOPE AND CONTENTS: This thesis deals with the lattice of all topologies which may be put on an arbitrary set E. The structure of the lattice is investigated together with its lattice properties. A chapter is included on the co-atoms of the lattice, the ultra-topologies, wherein various topological properties which do and do not hold are investigated. Various topological properties are considered as to which topologies are minimal and maximal such and also which topological properties are preserved under lattice operations and relations.

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Chapter 0

Definitions and Notations

In this thesis a topology t will be said to possess a property P rather than saying a topological space (E,t) possesses the property P since E, though arbitrary, will be kept fixed during any discussion.

Definitions:- A topology is said to be -

regular:- iff for any $x \notin A = \overline{A}$ there exist open sets M,N such that

$x \in M$, $A \subseteq N$ and $M \cap N = \emptyset$

completely regular:- iff for any neighbourhood M of any point $x \in E$

there exists a continuous function $f: E \to [0,1]$ such that $f(x) = 0$

and $f(y) = 1$ for all $y \notin M$.

normal:- iff for any two closed sets A,B with $A \cap B = \emptyset$ there exist

open sets M,N such that $A \subseteq M$, $B \subseteq N$ and $M \cap N = \emptyset$.

completely normal:- iff for any two sets A,B with $A \cap \overline{B} = \emptyset = \overline{A} \cap B$

there exist open sets M,N such that $A \subseteq M$, $B \subseteq N$ and $M \cap N = \emptyset$.

T_0:- iff for any two distinct points there exists an open set containing

one and not the other.

$T_{\frac{1}{2}}$:- iff for any $x \in E$ there exist open sets A,B such that $\{x\} = A-B$.

T_1:- iff for distinct points x and y there exist open sets O and O'

such that $x \in O$, $y \notin O$ and $y \in O'$, $x \notin O'$.

T_2:- iff for distinct points x and y there exist disjoint open sets O

and O' such that $x \in O$ and $y \in O'$.

T_3:- iff it is T_1 and regular

$T_{3\frac{1}{2}}$:- iff it is T_1 and completely regular

T_4:- iff it is T_1 and normal

T_5:- iff it is T_1 and completely normal

compact:- iff any open cover of E has a finite subcover.

paracompact: iff any open cover of E has an open locally finite refinement.

metacompact:- iff any open cover of E has a point finite refinement.

locally compact:- iff each point has at least one compact neighbourhood.

Lindelöf:- iff each open cover of E has a countable subcover.

connected:- iff for any proper non-void subset A of E at least one of A and E-A is not open.

locally connected:- iff the neighbourhood system of each point has a base consisting of connected sets.

totally disconnected:- iff only the singletons are connected.

extremally disconnected:- iff the closure of any open set is again open.

zero dimensional:- iff the neighbourhood system of each point has a base of open-closed sets.

principal:- iff the intersection of an arbitrary family of open sets is again open.

door space:- iff every subset of E is open or closed.

A topology satisfies the -

first axiom of countability:- iff the neighbourhood system of every point has a countable base.

second axiom of countability:- iff the topology has a countable base.

A topology t will be said to be <u>maximal</u> with respect to a property P iff t is not the discrete topology and any topology strictly finer than t does not possess the property P (<u>minimal</u> is defined dually).

All other terms are standard and for a definition of these the reader is referred to one of [9], [13], [18], [25], [32], [39].

Notations

t with or without subscripts or superscripts will denote a topology on a set E.

O will denote the zero of the lattice of all topologies on E, i.e. $\{\emptyset, E\}$, the trivial topology.

1 will denote the unit element of the lattice, i.e. P(E), the discrete topology.

U(y) will denote the principal ultrafilter generated by $\{y\}$, a one element subset of E.

U(A) will denote the principal filter generated by a subset A of E.

C* will denote the cofinite filter, i.e. the set of all subsets of E which have finite complements.

C' will denote the filter of all sets A such that E-A is countable.

C(x,F) will denote the topology whose open sets are precisely the sets of the filter F and the sets which do not contain x.

C(A) will denote the topology $\{\emptyset, A, E\}$ where A is a non-empty proper subset of E.

C(F) will denote the topology where the open sets are precisely the sets belonging to the filter F together with the empty set.

B(p) will denote the filterbase of all open sets containing p.

N(p) will denote the filter of all neighbourhoods of p.

$B_t(p)$ will denote the filterbase of all t-open sets containing p.

$N_t(p)$ will denote the filter of all t-neighbourhoods of p.

Chapter 1

Structure of $\mathcal{T}$

The set of all topologies on a set is a complete lattice, denoted by $\mathcal{T}$, where the order relation is that of set-theoretic inclusion. The purpose of this thesis is to study the structure of the lattice of all topologies on a given set.

The <u>meet</u> of two topologies is just the set-theoretic intersection, i.e. $t' \wedge t'' = \{O\colon O \in t' \text{ and } O \in t''\}$. The <u>join</u> of two topologies is the least topology which contains the set-theoretic union of the two topologies, i.e. the open sets of the join are arbitrary unions of finite intersections of sets from either topology. In particular if B' is a base for t' and B'' is a base for t'' then a base for the join is the set of all $B_i' \cap B_j''$ where B_i' belongs to B' and B_j'' belongs to B''.

An <u>infratopology</u> is a topology t such that the only topology strictly coarser than t is the trivial topology. Thus the infratopologies are the atoms of $\mathcal{T}$. Clearly every infratopology is of the form $C(A) = \{\emptyset, A, E\}$ where A is a non-void proper subset of E. It is equally evident that every topology is the supremum of infratopologies coarser than it.

For a point $p \in E$ and an ultrafilter different from $U(p)$ define $C(p,U) = \{X\colon p \notin X \text{ or } X \in U\}$. This topology is called an <u>ultratopology</u>.

Theorem 1.1 :- The $C(p,U)$ are the co-atoms of $\mathcal{T}$ and every topology is the meet of ultratopologies which are finer than it. (Fröhlich, [17])

Proof:- (i) Assume $C(p,U) \subset t$. Then there exists a set $X \in t$ such that $X \notin C(p,U)$. This implies $p \in X \notin U$. But $X \notin U$ implies $(E-X) \in U$ and hence $(E-X) \cup \{p\} \in U \subseteq t$. Therefore the singleton $\{p\} = ((E-X) \cup \{p\}) \cap X$ is in t. Hence all singletons are in t and therefore t is the discrete topology.

(ii) Let t be a co-atom. Then there exists a $p \in E$ such that $\{p\} \notin t$ which implies that $U(p)$ is not the neighbourhood filter of p. Now $E-\{p\}$ meets every open neighbourhood of p and $\{(E-\{p\}) \cap B: B \in B(p)\}$ is closed under finite intersection and hence is a filter basis. Let F be the filter generated by it. Since each filter is contained in an ultrafilter there exists an ultrafilter U containing F. Now if $\{p\} \in U$ then $\{p\} \cap (E-\{p\}) = \emptyset$ is also in U which is a contradiction. Therefore $U \neq U(p)$. Hence $t \subseteq \{X: p \notin X\} \cup N(p) \subseteq C(p,U) \subset P(E)$. Thus $t = C(p,U)$ since t was assumed to be a co-atom. Therefore the $C(p,U)$ are the co-atoms of $\mathcal{T}$.

(iii) Suppose $t \subset \wedge(C(p,U): C(p,U) \supseteq t) = t'$. Then there exists a set $X \in t'$ such that $X \notin t$. This implies that there exists $x \in X$ such that for all $B \in t$ with $x \in B$ we have $B \cap (E-X) \neq \emptyset$. This follows since $E-X$ is not closed in t. Let F be the filter generated by these $B \cap (E-X)$. Again, since F is proper, there exists an ultrafilter $U \supseteq F$. Hence $t \subseteq \{Y: x \in Y\} \cup F \subseteq C(x,U)$. From this it follows that $t' \subseteq C(x,U)$. But

$X \notin C(x,U)$ since the complement of X is in U and X is an element of t' which is a contradiction. Hence t equals t'.

Theorem 1.2 :- If T is a finite set of ultratopologies then $C(x,U)$ is finer than $\wedge T$ iff there exists a $y \in E$ and an ultrafilter V such that $C(y,U)$ and $C(x, V)$ are in T. (Fröhlich, [17])

Proof:- Let $T = \{C(x_i,U_i): i = 1,2,\ldots n\}$. Then $\wedge T = \bigcap_{i=1}^{n} (P(\complement x_i) \cup U_i)$. By distributivity one obtains a union of 2^n terms of these, however, the only terms of interest are $\bigcap_{i=1}^{n} U_i$ and $\bigcap_{i=1}^{n} P(\complement x_i) = P(\complement\{x_1, \ldots x_n\})$. If $\wedge T \subseteq C(x,U)$ then $P(\complement\{x_1, \ldots x_n\}) \cup \bigcap_{i=1}^{n} U_i \subseteq C(x,U) = P(\complement x) \cup U$. Because $\{x\} \notin C(x,U)$ it follows that x is one of the x_i. For this index i we have $C(x,U_i) \in T$. Since $P(\complement x) \cap U(x) = \emptyset$ we have $U(x) \cap \bigcap_{i=1}^{n} U_i \subseteq U$. Since $U(x) \notin U$ it follows that U is equal to one of the U_i. For the corresponding index i we have $C(x_i,U) \in T$. The converse is obvious.

A topology t is a principal topology iff the intersection of each subset of t is a member of t.

Lemma 1.3 :- The meet of a family of principal topologies is again principal.

Proof:- Let $(t_i: i \in I)$ be a family of principal topologies. Take a family of sets $(A_j: j \in J)$ in the meet of all t_i. This implies that $(A_j: j \in J)$ belongs to each t_i and, since each t_i is principal, $\bigcap_{j \in J} A_j \in t_i$, for all $i \in I$. Hence $\bigcap_{j \in J} A_j \in \bigwedge_{i \in I} t_i$.

Theorem 1.4 :- (i) The infratopologies are principal.

(ii) An ultratopology $C(p,U)$ is principal iff U is principal.

Proof:- (i) obvious

(ii) Assume U is principal, i.e. $U = U(q)$. Let $(A_i : i \in I)$ be an arbitrary subfamily of $C(p,U(q))$ and suppose $\bigcap_{i \in I} A_i \notin C(p,U(q))$. This implies that p is in the intersection while q is not. From this it follows that there exists a $j \in I$ with $p \in A_j$, $q \notin A_j$. But this implies $A_j \notin C(p,U(q))$ which is a contradiction. Therefore the intersection belongs to $C(p,U(q))$ and hence $C(p,U(q))$ is principal.

Conversely, assume $C(p,U)$ is principal. Let $B_p = \bigcap (B: B \in B(p))$. B_p is open since $C(p,U)$ is principal. Since $p \in B_p$ it follows that $B_p \in U$. But $E - \{p\} \in U$ since $U \neq U(p)$. Thus $B_p \cap (E - \{p\}) = B_p - \{p\}$ is an element of U. Now for any X in U, $B_p \subseteq X \cup \{p\}$ and therefore $B_p - \{p\} \subseteq X - \{p\} \subseteq X$. This implies $U = U(B_p - \{p\})$ where $B_p - \{p\}$ is a singleton.

Theorem 1.5 :- For a topology t the following are equivalent:-

(1) t is principal

(2) the t-neighbourhood system of each point has a base of one set

(3) t is the meet of principal ultratopologies containing it. (Steiner [36])

Proof:- (1)$\Rightarrow$(2) For each $p \in E$ let $B_p = \bigcap (B: B \in B(p))$. Then B_p is open since t is principal and it is the smallest open set containing p. Thus $\{B_p\}$ is a base of $N(p)$.

(2)$\Rightarrow$(3) Let $p \in E$ and define $t' = \bigwedge (C(p,U(q)) : p \in E \text{ and } q \in B_p)$. Let A be any subset of E. Now $A \in t$ iff $B_p \subseteq A$ for all $p \in A$

iff $A \in C(p,U(q))$ for all $p \in A$, for all $q \in B_p$

iff $A \in t'$

Thus $t = t'$.

(3)$\Rightarrow$(1) follows from lemma 1.3.

Theorem 1.6 :- The set π of all principal topologies of E is a meet-complete sublattice of $\mathcal{T}$. (Steiner [36])

Proof:- The meet of principal topologies is principal as proven in lemma 1.3. Consider $t = t' \vee t''$ where t' and t'' are principal. By theorem 1.5, for each $x \in E$ there exists B_x' such that $\{B_x'\}$ is a base for $N'(x)$ and B_x'' such that $\{B_x''\}$ is a base for $N''(x)$. Thus $\{B_x\} = \{B_x' \cap B_x''\}$ is a base for $N(x)$. Thus by theorem 1.5 t is principal.

The lattice of all principal topologies is not a complete sublattice of $\mathcal{T}$. This is easily seen by the following example. Take a non-principal ultrafilter U. Then, by theorem 1.4, C(p,U) is not principal. Now $C(p,U) = \vee(C(A_i)\colon i \in I)$ where each $C(A_i)$ is an infratopology coarser than C(p,U) and each infratopology is principal. Thus the join of principal topologies need not be principal.

Theorem 1.7 :- The infratopologies are not T_1-topologies if E contains at least two elements.

Proof:- Let t equal $\{\emptyset, A, E\}$. There exists an x in E such that $E-\{x\}$ is not in t which implies $\{x\}$ is not closed and hence t is not a T_1-topology.

Theorem 1.8 :- An ultratopology is a T_1-topology iff it is nonprincipal.

Proof:- Assume U is a nonprincipal ultrafilter on E. Then, for all x in E, complement of x is in U. Hence each singleton is closed and therefore the ultratopology is a T_1-topology. Take any principal ultratopology $C(x,U(y))$. Then the complement of y is not open in $C(x,U(y))$ and hence singleton y is not closed. Therefore $C(x,U(y))$ is not a T_1-topology.

Theorem 1.9 :- $C(C^*)$, the cofinite topology is T_1.

Proof:- The complement of the complement of x is singleton x and hence finite. This implies complement of x is open and hence singleton x is closed. Therefore $C(C^*)$ is a T_1-topology.

Theorem 1.10 :- The following are equivalent for a topology t:-

(1) t is a T_1-topology

(2) t is the meet of nonprincipal ultratopologies

(3) t is finer than the cofinite topology

Proof:- (1) implies (2). By theorem 1.1 every topology is the meet of ultratopologies containing it and since any topology finer than a T_1-topology is again a T_1-topology, t is a meet of nonprincipal ultratopologies.

(2) implies (3). $C(p,U)$ nonprincipal implies U nonprincipal by 1.8 hence U contains the cofinite filter and thus $C(p,U)$ contains the cofinite topology. Therefore, the cofinite topology is contained in the meet of all nonprincipal ultratopologies. Hence the cofinite topology is contained in t.

(3) implies (1). Since the cofinite topology is a T_1-topology and t is finer than it, t is also a T_1-topology. Hence the T_1-topologies form a principal filter generated by the cofinite topology which is a complete sublattice of $\mathcal{T}$. The finest T_1-topology (the unit of the sub-

lattice) is the discrete topology and the coarsest (the zero of the sublattice) is the cofinite topology.

Note that on a finite set the only T_1-topology is the discrete topology.

Theorem 1.11 :- Every topology is a meet of a principal topology and a T_1-topology.

Proof:- By 1.1 any topology t is the meet of ultratopologies t_i, $i \in I$, where each t_i is finer than t.

Let t' equal the meet over all $j \in I$ such that t_j is principal and let t'' equal the meet over all $k \in I$ such that t_k is a T_1-topology. Then $t = \wedge(t_i\colon i \in I) = t' \wedge t''$ where t' is principal by 1.3 and t'' is a T_1-topology by 1.10.

A topology on E which is neither a T_1-topology nor a principal topology is a mixed topology. A mixed topology can be represented as the meet of a T_1-topology and a principal topology, but this representation need not be unique.

The join of two mixed topologies can be a T_1-topology or a principal topology as illustrated by the following example.

$t_1 = C(x,U) \wedge C(p,U(q))$

$t_2 = C(x,U) \wedge C(q,U(p))$

$t_3 = C(y,V) \wedge C(p,U(q))$ where x is different from y and U and V are distinct nonprincipal ultrafilters.

Then $t_1 \vee t_2 = C(x,U)$ is a T_1-topology whereas

$t_1 \vee t_3 = C(p,U(q))$ is a principal topology.

The meet of two mixed topologies cannot be a T_1-topology but it can be a principal topology. Let $t = t_1 \wedge t_2$ where t_1 is a T_1-topology and t_2 is a principal topology. Since t_2 is a principal topology it is the meet of principal ultratopologies and clearly the complement of every point is not in any principal ultratopology and hence certainly not in the meet. Hence the meet of two mixed topologies is not a T_1-topology. For an example that the meet of two mixed topologies may in fact be principal consider the following. Let U and V be distinct nonprincipal ultrafilters. Hence there exists a set A in U such that the complement of A is in V. Let $t_1 = \wedge(C(x,U(q)) : q \epsilon A-\{x\})$ and $t_2 = \wedge(C(y,U(q)) : q \epsilon E-A-\{x\})$ Then $t_1 \leq C(x,U)$, $t_2 \leq C(y, V)$. Also $C(x,U) \wedge t_2$ and $C(y,V) \wedge t_1$ are mixed topologies but their meet is $t_1 \wedge t_2$ which is principal.

Chapter 2

Lattice Properties of $\mathcal{T}$

Theorem 2.1 :-

The lattice of topologies on a set E is distributive iff E has fewer than three elements. If E has three or more elements, the lattice is not even modular. (Steiner [36]).

Proof:- Obviously if E has one element or two elements $\mathcal{T}$ is a distributive lattice. Let $E \supseteq \{p,q,r\}$ and let

$$t_1 = C(p,U(q)) \wedge C(p,U(r))$$
$$t_2 = C(p,U(r))$$
$$t_3 = C(r,U(q))$$

be topologies on E.

$t_1 \leq t_2$

$(t_1 \vee t_3) \wedge t_2 = 1 \wedge t_2 = t_2$

also $t_1 \leq C(p,U(q))$ and $t_3 \wedge t_2 \leq C(p,U(q))$

Thus $t_1 \vee (t_3 \wedge t_2) \leq C(p,U(q))$. But $t_2 \nleq C(p,U(q))$ and hence

$$(t_1 \vee t_3) \wedge t_2 \neq t_1 \vee (t_3 \wedge t_2).$$

Therefore $\mathcal{T}$ is not modular.

Definition:- A lattice L is self-dual iff there exists a one-to-one mapping f of L onto intself such that $f(a \wedge b) = f(a) \vee f(b)$ and $f(a \vee b) = f(a) \wedge f(b)$.

Theorem 2.2 :-

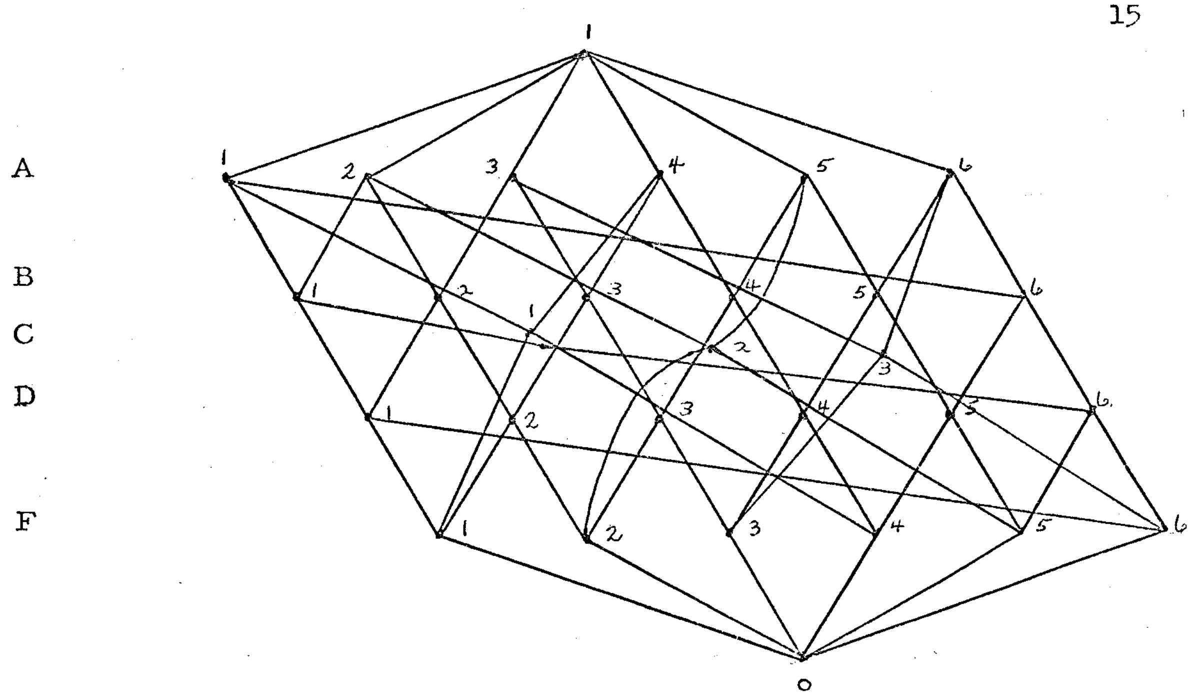
The lattice of topologies on E is self-dual iff $|E| \leq 3$. (Steiner [36]).

Proof:- If $\mathcal{T}$ is self-dual there exists a one-to-one mapping f of $\mathcal{T}$ onto itself such that if $a \leq b$ then $f(a) = f(a \wedge b) = f(a) \vee f(b)$. Hence $a \leq b$ implies $f(b) \leq f(a)$. Thus $f(0) = 1$, $f(1) = 0$, infratopologies map onto ultratopologies and ultratopologies map onto infratopologies. Therefore the number of infratopologies and untratopologies must be equal.

In the lattice of topologies on a set E, if $|E| = n < \infty$ there are $n(n-1)$ ultratopologies (all principal) and 2^n-2 infratopologies. If $|E| \geq \aleph_0$ there are $2^{|E|}$ infratopologies and $2^{2^{|E|}}$ ultratopologies on E since that is the number of ultrafilters on E (see Banaschewski, [4]). Thus the number of ultratopologies equals the number of infratopologies only when $|E| \leq 3$.

If $|E| = 1$ or $|E| = 2$ then $\mathcal{T}$ is obviously self-dual. If $|E| = 3$ there are 29 topologies on E, but it can be seen by rotating the diagram on the following page by 180° that this lattice is self-dual.

Thus the lattice $\mathcal{T}$ of all topologies on a set E is a complete, atomistic, co-atomistic, non-modular (unless $|E| < 3$), non-self-dual (unless $|E| \leq 3$), complemented lattice. It contains the sublattice of principal topologies and the complete sublattice (principal filter) of T_1-topologies. (Note - For the proof of the complementation of $\mathcal{T}$ the reader is referred to Steiner [36].)



Lattice of topologies on a three element set E = {a, b, c}

0 Trivial Topology
1 Discrete Topology

A

1.	∅	{b}	{c}	{b, c}	{c, a}	E
2.	∅	{c}	{b}	{a, b}	{b, c}	E
3.	∅	{a}	{b}	{a, b}	{b, c}	E
4.	∅	{a}	{b}	{a, b}	{a, c}	E
5.	∅	{a}	{c}	{a, b}	{a, c}	E
6.	∅	{a}	{c}	{c, a}	{c, b}	E

B

1.	∅	{b}	{c}	{b, c}	E
2.	∅	{b}	{b, a}	{b, c}	E
3.	∅	{a}	{b}	{a, b}	E
4.	∅	{a}	{a, b}	{a, c}	E
5.	∅	{a}	{c}	{a, c}	E
6.	∅	{c}	{c, a}	{c, b}	E

C

1.	∅	{a, c}	{b}	E
2.	∅	{a, b}	{c}	E
3.	∅	{a}	{b, c}	E

D

1.	∅	{b}	{b, c}	E
2.	∅	{b}	{b, a}	E
3.	∅	{a}	{a, b}	E
4.	∅	{a}	{a, c}	E
5.	∅	{c}	{c, a}	E
6.	∅	{c}	{c, b}	E

F

1.	∅	{b}	E
2.	∅	{a, b}	E
3.	∅	{a}	E
4.	∅	{a, c}	E
5.	∅	{c}	E
6.	∅	{b, c}	E

Chapter 3

Topological Properties of Ultratopologies

Theorem 3.1 :- Every ultratopology is $T_{\frac{1}{2}}$, T_0, completely normal, normal, extremally disconnected, paracompact, metacompact, a door space.

Proof:- Consider any ultratopology $C(x,U)$.

$T_{\frac{1}{2}}$ - For any $p \neq x$ we have $\{p\}=\{p\}-\emptyset$ and for x we have $\{x\} = E-\complement x$.

T_0 - $T_{\frac{1}{2}}$ implies T_0.

completely normal - Let A,B be subsets of E such that $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. If $x \notin \bar{A}$ then $\bar{A}$ is open and $E-\bar{A}$ is also open and contains B. If $x \in \bar{A}$ then $x \notin B$ and hence B is open. Now $A \cap \bar{B} = \emptyset$ implies $A \subseteq E-\bar{B} \subseteq E-B$. Thus the disjoint open sets separating A and B are $E-\bar{B}$ and B. Therefore $C(x,U)$ is completely normal.

normal - completely normal implies normal.

extremally disconnected - Let O be an open set. If $x \in O$ then $E-O \in C(x,U)$ and hence O is closed. Therefore the closure of O is O which is open. If $x \notin O$ then either $E-O \in U$ or $O \in U$. If $E-O \in U$ then O is closed and the closure of O is O which is open. If $O \in U$ then $\bar{O} = O \cup \{x\}$ which again belongs to U and hence closure of O is open.

paracompact - Any open cover of E consists of at least one set V of U such that $x \in V$. For the refinement select one of these V together with all singletons of points not in V.

metacompact - paracompact implies metacompact.

door space - For any subset A, $x \notin A$ or $x \in A$ and hence A is open or closed.

Theorem 3.2 :- No ultratopology on a set E is connected if $|E| \geq 3$. (Steiner [36])

Proof:- Let $C(x,U)$ be an ultratopology on E. If $|E| \geq 3$ then there exists a $y \in E$ such that $x \neq y$ and $U \neq U(y)$. Hence $\{y\}$ and $E - \{y\}$ are open. Thus E is the disjoint union of two open sets and therefore $C(x,U)$ is not connected.

Theorem 3.3 :- For an ultratopology $C(x,U)$ the following are equivalent -

(1) $C(x,U)$ is nonprincipal.

(2) $C(x,U)$ satisfies the separation axioms T_1 to T_5.

(3) $C(x,U)$ is totally disconnected.

(4) $C(x,U)$ is zero dimensional.

(5) $C(x,U)$ is (completely) regular.

Proof:- (1)$\Longleftrightarrow$(2) This is a consequence of the following - an ultratopology is nonprincipal iff it is T_1 (by 1.8), any ultratopology is completely normal (by 3.1) and $T_5 \Rightarrow T_4 \Longrightarrow T_{3\frac{1}{2}} \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1$.

(1)$\Longleftrightarrow$(3) Nonprincipal implies T_2(by the above) and T_2 together with extremally disconnected implies totally disconnected. If $C(x,U)$ is totally disconnected then the singletons are the maximal connected sets and hence closed which implies it is a T_1-topology and hence nonprincipal.

(1)$\Longleftrightarrow$(4) Nonprincipal implies T_3 and T_3 together with extremally disconnected implies zero dimensional. If $C(x,U)$ is principal then U is principal (by 1.4), say $U = U(y)$. Then $E-\{y\}$ is not open but $\{y\}$ is open and hence there is no base of open-closed sets for the neighbourhood system of y.

(1)$\Leftrightarrow$(5) By (2), if $C(x,U)$ is nonprincipal then it is (completely) regular. Conversely, consider any principal ultratopology $C(x,U(y))$. Then $\{x\}$ is closed and any open set containing it must also contain y. Hence there do not exist disjoint open sets separating $\{x\}$ and y. Therefore $C(x,U(y))$ is not regular.

Lemma 3.4 :- If an ultratopology $C(p,U)$ does not satisfy one of the countability axioms then for any topology t strictly coarser than $C(p,U)$ which satisfies one of the countability axioms there exists a topology t', strictly finer than t and strictly coarser than $C(p,U)$, which satisfies the axiom also.

Proof:- If t satisfies one of the countability axioms and $t < C(p,U)$ then there exists a set $A \in C(p,U)$ such that $A \notin t$. Then the infratopology $C(A)$ satisfies the axiom and clearly $t \vee C(A)$ does also.

Corollary:- If no ultratopology satisfies a countability axiom then there exist no maximal topologies satisfying that countability axiom.

Lemma 3.5 :- If U is a nonprincipal ultrafilter on E then U does not have a countable base.

Proof:- U nonprincipal implies that E is infinite. Assume U has a countable base, namely $\{B_1, \ldots\ldots, B_n, \ldots\ldots\}$, $n \in N$. Without loss of generality one can assume $B_1 \supset B_2 \supset \ldots\ldots\ldots \supset B_n \supset B_{n+1} \supset \ldots\ldots$. Further one can assume that there are at least two elements in B_n which are not in B_{n+1}. By the axiom of choice one can select a point $x_n \in B_n - B_{n+1}$. Define $M = \{x_n : n \in N\}$. M is non-empty as U was nonprincipal. Further M intersect B_n is not empty for all $n \in N$ since $x_n \in B_n \cap M$. Then $M \in U$ since if a set meets every set in an ultrafilter it is in the ultrafilter. Hence there exists a B_n such that $B_n \subseteq M$. Now there exists $y_n \in B_n - B_{n+1}$ such that $y_n \neq x_n$. Hence $y_n \in B_n - M$ and thus $B_n \not\subseteq M$ which is a contradiction. Therefore U does not have a countable base.

<u>Corollary</u>:- The nonprincipal ultratopologies do not satisfy the countability axioms.

<u>Proof</u>:- Let $C(x,U)$ be a nonprincipal ultratopology. Then, for all $y \neq x$, $\{y\}$ is a base for $N(y)$, but $N(x)$ has a countable base iff U has a countable base which it does not have as seen by the above lemma.

<u>Theorem 3.6</u> :- For an ultratopology $C(x,U)$ the following are equivalent:-

(1) $C(x,U)$ is principal.

(2) $C(x,U)$ is locally compact.

(3) $C(x,U)$ is locally connected.

(4) $C(x,U)$ satisfies the first axiom of countability.

<u>Proof</u>:- $(1) \Longleftrightarrow (2)$ Assuming $C(x,U)$ is principal then by 1.5 there exists a smallest open set B_p containing p for all $p \in E$. If $C(x,U)$ is nonprincipal then U is nonprincipal. It is sufficient to consider any open neighbourhood M of x. Thus $x \in M \in U$ and M is infinite. Now there exists a set N such that $M = N \cup (M-N)$ with both N and M-N infinite. Without any loss of generality one can assume that $N \in U$ and also that $x \in N$. Then $N \cup \bigcup_{p \in M-N} \{p\}$ is an open cover of M which has no finite subcover.

Therefore there exists no compact neighbourhood of x and hence $C(x,U)$ is not locally compact.

$(1)\Leftrightarrow(3)$ If $C(x,U)$ is principal then by 1.5 there exists a smallest open set B_p containing p for all $p \in E$. If $C(x,U)$ is nonprincipal then U is nonprincipal and for any $M \in B(x)$ we have $M \neq \{x\}$ and hence there exists a $y \in M$ with $y \neq x$. Now $E-\{y\}$ is open since $\{y\} \notin U$. Thus $M \cap \complement y$ is open. Therefore $M = (M \cap \complement y) \cup \{y\}$ and thus M is not connected. Therefore, if U is nonprincipal, $C(x,U)$ is not locally connected.

$(1)\Leftrightarrow(4)$ If $C(x,U)$ is principal then $N(p)$ has a countable basis for all $p \in E$, namely $\{B_p\}$ by 1.5. If $C(x,U)$ is nonprincipal then it does not satisfy the first countability axiom by the corollary to 3.5.

<u>Theorem 3.7</u> :- A space is a door space iff the ultratopologies in its representation have a common point or a common ultrafilter. (Steiner [36])

<u>Proof</u>:- Let $t_1 = \wedge(C(x,U_i): i \in I)$ and let $t_2 = \wedge(C(y_j,V): j \in J)$. Let A be any subset of E. Then either $x \in A$ and hence A is closed with respect to t_1 or $x \notin A$ and hence $A \in t_1$. Also $A \in V$ or $E-A \in V$ and hence A is either open or closed with respect to t_2. Therefore both t_1 and t_2 yield door spaces.

Consider the topology $t \leq C(x,U) \wedge C(y,V)$ where $x \neq y$ and $U \neq V$. Now $U \neq V$ implies that there exists a subset A of E such that $A \in U$ and $E-A \in V$. Then $B = (A \cup \{y\}) \cap \complement x$ does not belong to $C(y,V)$ for if $B \in C(y,V)$ then $B \cap (E-A) \in C(y,V)$, but $B \cap (E-A) = \{y\} \notin C(y,V)$. Also $E-B \notin C(x,U)$ for if $(E-B) \in C(x,U)$ then $A \cap B = \{x\} \in U$ which is a contradiction. Hence the set B as defined above is neither open nor closed in t. Therefore (E,t) is not a door space.

Chapter 4

Minimal and Maximal Topologies

In this chapter various properties of topological spaces are considered and the minimal and maximal topologies having these properties are investigated. A table is provided at the end of this chapter to summarize the results.

T_0, minimal:-

Definition:- Given a topology t on E define $p \leq q$ iff $N(p) \subseteq N(q)$, for $p, q \in E$. This defines a partial order on E if t is a T_0-topology. (This definition is the same as the one in Birkhoff [10], pg. 13).

Theorem 4.1:- A topology t is minimal T_0 iff $\leq$ is a total order of E and the non-void sets of t are of the form $]a, \longrightarrow]$ for $a \in E$.

Proof:- Assume $\leq$ is not a total order, that is, there exists points p and q in E, such that $p \nleq q$ and $q \nleq p$. That, by definition, is equivalent to $N(p) \nsubseteq N(q)$ and $N(q) \nsubseteq N(p)$. Consider $t' = t \wedge C(p, N(q))$. This is a topology strictly coarser than t since N(q) converges to p in t' but not in t. Claim t' is also T_0. Without loss of generality one may take x distinct from y with $x \nleq y$.

Case 1 :- $x \nleq p$ $(N_t(x) \nsubseteq N_t(p))$. Hence there exists N in B(x) with $p \notin N$ and also there exists $M \in B(x)$ with $y \notin M$. Thus $M \cap N \in B(x)$ and also $M \cap N \in C(p, N(q))$ since $p \notin M \cap N$. Therefore $M \cap N \in t'$ and $x \in M \cap N$ but $y \notin M \cap N$.

Case 2 :- $x \leq p$, $q \not\leq y$. Then there exists a set $M \in B(x)$ and a set $N \in B(q)$ such that $y \notin M$ and $y \notin N$ and hence $y \notin M \cup N \in B(q)$. Therefore $M \cup N \in t'$ and $x \in M \cup N$ but $y \notin M \cup N$.

Case 3 :- $x \leq p$, $q \leq y$. Then $N(y) \not\subseteq N(x)$ because if it were we would have $N(q) \subseteq N(y) \subseteq N(x) \subseteq N(p)$ which contradicts the assumption. Similarily $N(y) \not\subseteq N(p)$. Hence there exists $N \in B(y)$ with $p \notin N$ and there exists $M \in B(y)$ with $x \notin M$. Thus $M \cap N \in B(y)$, $M \cap N \in t'$ since $p \notin M \cap N$ and $y \in M \cap N$ while $x \notin M \cap N$.

Therefore if t is minimal T_0 then $\leq$ is a total order of E. Now any set of the form $]a,\rightarrow]$ is in t since for any $x \in]a,\rightarrow]$ we have $a < x$ and hence there exists $N \in B(x)$ with $N \subseteq]a,\rightarrow]$. Since t is minimal T_0 and the topology whose sets are of the form $]a,\rightarrow]$ is T_0 we have that the non-void sets of t are of the form $]a,\rightarrow]$ for $a \in E$.

T_0, maximal :- any ultratopology (3.1)

$T_{\frac{1}{2}}$, minimal :-

Theorem 4.2 :- A topology t is minimal $T_{\frac{1}{2}}$ iff $\leq$ is a total order of E and the non-void sets of t are of the form $[a,\rightarrow]$ and $]a,\rightarrow]$ for $a \in E$.

Proof:- Assume $\leq$ is not a total order. This implies that there exist p and q, elements of E, such that $p \not\leq q$ and $q \not\leq p$, i.e. $N(p) \not\subseteq N(q)$ and $N(q) \not\subseteq N(p)$. Consider $t' = t \wedge C(p,N(q))$. The topology t' is strictly coarser than t since N(q) converges to p in t' but not in t. Claim t' is also a $T_{\frac{1}{2}}$-topology. Let $x \in E$ and hence there exist sets $A,B \in t$ such that $\{x\} = A-B$.

Case 1 :- $x \not\leq p$. Then there exists $M \in B(x)$ with $p \notin M$. Now $A \cap M \in t'$,

$B \cap M \in t'$ and $\{x\} = (A \cap M) - (B \cap M)$.

Case 2 :- $x \leq p$. This implies $N(q) \not\subseteq N(x)$ and hence there exists a set $N \in B(q)$ such that $x \notin N$. Thus $(A \cup N) \in t'$, $(B \cup N) \in t'$ and $\{x\} = (A \cup N) - (B \cup N)$. Thus if t is minimal $T_{\frac{1}{2}}$ we have $\leq$ is a total order. It is obvious that sets of the form stated form a $T_{\frac{1}{2}}$ topology. For any $x \in]a,\longrightarrow]$ we have $a < x$ and hence there exists a set $N \in B(x)$ with $N \subseteq]a,\longrightarrow]$ and so $]a,\longrightarrow] \in t$. For any $x \in [a,\longrightarrow]$ we have $a \leq x$ and hence there exists a set $N \in B(x)$ with $N \subseteq [a,\longrightarrow]$ and so $[a,\longrightarrow]$ is open. Therefore since t is minimal $T_{\frac{1}{2}}$ we have proven that t is minimal $T_{\frac{1}{2}}$ iff $\leq$ is a total order and the sets of t are of the form $]a,\longrightarrow]$ and $[a,\longrightarrow]$ for $a \in E$.

$T_{\frac{1}{2}}$, maximal :- any ultratopology (3.1)

T_1, minimal :- Cofinite topology (1.10)

T_1, maximal :- Any nonprincipal ultratopology (1.8 and 1.10)

T_2, minimal :-

Theorem 4.3 :- A T_2-topology t is minimal T_2 iff the following condition holds:- if an open filter has a unique adherence point then it converges to this point. (See also Banaschewski [4] and Bourbaki [12]).

Proof:- Let p be the unique adherence point of an open filter F which does not converge to p in a T_2-topology t. Define $t' = t \wedge C(p,F)$. Now t' is strictly coarser than t since F converges to p in t' while it does not in t.

Claim :- t' is a T_2-topology. Take $x \neq y$. Without any loss of generality one can assume $p \neq y$. Then y is not an adherence point of F, and hence there exists an $M \in B(y)$ and an open set $N \in F$ such that $M \cap N = \emptyset$. Now since t is T_2 there exist t-open sets A,B with $x,p \in A$, $y \in B$ and $A \cap B = \emptyset$. Hence $x \in A \cup N$, $y \in B \cap M$, $A \cup N \in t'$, $B \cap M \in t'$, and $(A \cup N) \cap (B \cap M) = \emptyset$. Therefore t' is a T_2-topology strictly coarser than t and hence t is not minimal T_2.

Conversely, let t possess the given property and consider any T_2-topology t' coarser than t. Hence we have $N_{t'}(p) \subseteq N_t(p)$. Now since t' is a T_2-topology any $q \neq p$ can be separated from p by disjoint t'-open (hence t-open) sets. Hence p is the only adherence point of $N_{t'}(p)$, $N_{t'}(p)$ is an open filter relative to t and hence by the given property $N_{t'}(p)$ converges to p. Thus we have $N_t(p) \subseteq N_{t'}(p)$. Therefore $t = t'$ and t is a minimal T_2-topology.

This condition implies that compact T_2-topologies are minimal T_2. There are minimal T_2-topologies which are not compact. For an example of such the reader is referred to Berri [6].

<u>T_2, maximal</u> :- Any nonprincipal ultratopology (3.3).

<u>Lemma 4.4</u> :- Let t be a T_1-topology. Let F be a filter which has no adherence point (or, has a unique adherence point to which it does not converge). Let p be any point in E (or, the unique adherence point). Define $t' = t \wedge C(p,F)$. Then t' is a strictly coarser T_1-topology.

<u>Proof</u>:- $\{p\}$ is closed with respect to t'. If $x \neq p$ then x is not an adherence point of F and hence there exists a set $A \in F$ such that $x \notin A$. This implies that $A \subseteq \complement x$ and hence $\complement x \in F$. Therefore $\{x\}$ is closed with

respect to t'. Now t' is a strictly coarser topology since F converges to p in t' but it does not in t. Therefore t' is a strictly coarser T_1-topology.

T_3, minimal :-

Theorem 4.5 :- A T_3-topology t is minimal T_3 iff the following condition holds:- if a regular open filter has a unique adherence point then it converges to this point. (See also Banaschewski [4] and Berri [9]).

(Note - a regular open filter is a filter which has a base of open sets which is equivalent to a base of closed sets.)

Proof:- Let p be the unique adherence point of a regular open filter F which does not converge to p in a T_3-topology, t. Define $t' = t \wedge C(p,F)$. Now t' is a strictly coarser T_1-topology (4.4). Now the claim is that t' is also T_3.

Case 1 :- Consider the point p and a set A such that $p \in A \in t'$. Thus we have $A \in t$ and $A \in F$. $A \in t$ implies by the regularity of t that there exists a t-closed neighbourhood of p, B, such that $B \subseteq A$. By the regularity of F there exists a t-closed set $M \in F$ such that $M \subseteq A$. Thus $M \cup B$ is t-closed, belongs to F, contains p, is contained in A and therefore $p \in B \cup M \subseteq A$ where $B \cup M$ is a t'-closed neighbourhood of p.

Case 2 :- Consider $x \neq p$ and a t'-open set A with $x \in A$. Consider $A \cap \complement p$. By the regularity of t there exists a t-closed neighbourhood N of x such that $N \subseteq A$. Now since F is regular and x is not an adherence point there exist t-closed set B and M such that $B \in N(x)$, $M \in F$ and $B \cap M = \emptyset$. $B \cap N$ is a t-closed neighbourhood of x and $(B \cap N) \cap M = \emptyset$.

Hence $\complement(B\cap N)\supseteq M$ and thus $\complement(B\cap N)\in F$. Therefore $B\cap N$ is a t'-closed neighbourhood of x such that $B\cap N\subseteq A$.

Therefore t' is a T_3-topology and hence t is not minimal T_3.

Conversely, let t possess the given property and consider any T_3-topology t' coarser than t. Hence $N_{t'}(p)\subseteq N_t(p)$. Now since t' is T_3 and thus t_2 any $q\neq p$ can be separated from p by disjoint t'-open (hence t-open) sets. Hence p is the only adherence point of $N_{t'}(p)$, $N_{t'}(p)$ is a regular open filter relative to t and hence by the given property $N_{t'}(p)$ converges to p. Thus we have $N_t(p)\subseteq N_{t'}(p)$. Therefore t = t' and t is a minimal T_3-topology.

This condition implies that compact T_2-topologies are minimal T_3. For an example of a minimal T_3-topology which is not compact the reader is referred to Berri [9].

<u>T_3, maximal</u>:- Any nonprincipal ultratopology (3.3).

<u>T_4, minimal</u>:-

<u>Theorem 4.6</u>:- The minimal T_4-topologies are exactly the compact T_2-topologies. (See also Berri [6]).

<u>Proof</u>:- If t is compact and T_2 it is T_4 and minimal T_2 (4.3). Therefore the compact T_2-topologies are minimal T_4.

Conversely, let t be a T_4-topology which is not compact and hence there exists a closed filter F with no adherence point. Let p be any point of E and let $t' = t\wedge C(p,F)$. Then t' is a strictly coarser T_1-topology (4.4). Also t' is normal. Let X, Y be disjoint t'-closed sets. Without loss of generality let $p\notin X$ and hence

$E-X \in F$. Thus there exists a t-closed set $B \in F$ with $B \subseteq E-X$. Then X and $B \cup Y \cup \{p\}$ are disjoint t-closed sets and since t is normal there exist disjoint t-open sets U and V such that $X \subseteq U$ and $B \cup Y \cup \{p\} \subseteq V$. Since $p \notin U$ we have $U \in t'$ and since $B \subseteq V$ we have $V \in t'$. Therefore U and V are disjoint t'-open sets such that $X \subseteq U$ and $Y \subseteq V$. Therefore t' is a strictly coarser T_4-topology and hence t is not minimal T_4.

<u>T_4, maximal</u> :- Any nonprincipal ultratopology (3.3).

<u>T_5, minimal</u> :-

<u>Theorem 4.7</u> :- A T_5-topology is minimal T_5 iff it is compact. (See also [31]).

<u>Proof</u> :- If t is a T_5-topology which is not compact then we have $t' = t \wedge C(p,F)$ as in theorem 4.6. Take A and B subsets of E such that $A \cap Cl_{t'}B = \emptyset = Cl_{t'}A \cap B$. This implies $A \cap Cl_t B = \emptyset = Cl_t A \cap B$. Hence there exist t-open sets U_1 and U_2 such that $A \subseteq U_1$, $B \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$. If $x \notin A \cup B$ then $U_1 \cap \complement x$ and $U_2 \cap \complement x$ are t'-open sets with void intersection and $A \subseteq U_1 \cap \complement x$ and $B \subseteq U_2 \cap \complement x$. If $x \in A \cup B$ then without any loss of generality we may assume $x \in A$ and hence $x \notin Cl_{t'}B$. Since t' is a T_4-topology it is regular and hence there exist t'-open sets V_1 and V_2 such that $x \in V_1$, $Cl_{t'}B \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$. Thus $A \subseteq U_1 \cup V_1 \in t'$, $B \subseteq V_2 \cap U_2 \in t'$ and $(U_1 \cup V_1) \cap (V_2 \cap U_2) = \emptyset$. Thus non-compact T_5 implies non-minimal and therefore minimal T_5 implies compact.

Conversely, if t is compact T_5 then it is compact T_4 and hence minimal T_4 (4.6) and therefore minimal T_5.

<u>T_5, maximal</u> :- Any nonprincipal ultratopology (3.3).

$T_{3\frac{1}{2}}$, minimal :-

Theorem 4.8 :- The minimal $T_{3\frac{1}{2}}$-topologies are exactly the compact T_2-topologies. (See also Banaschewski [4] and Berri [6]).

Proof:- The compact T_2-topologies are minimal $T_{3\frac{1}{2}}$ (4.5). Assume (E,t) is a $T_{3\frac{1}{2}}$-space. Now (E,t) may be embedded in a T_4-space (E^*,t^*) such that E is dense in E^*. If $E = E^*$ and E is not compact then by theorem 4.6 there exists a strictly coarser T_4-topology and hence $T_{3\frac{1}{2}}$-topology. Thus it suffices to consider $E \subset E^*$ and hence there exists $q \in E^* - E$. Let $p \in E$. Define a topology $s^* = t^* \wedge C(p,N_{t^*}(q))$. Let $E' = E^* - \{q\}$. Now $E \subseteq E'$ implies $(E,s^*_E) \subseteq (E',s^*_{E'})$.

(1) $s^*_E < t^*_E = t$. Let N be a t^*-open set containing p and N' a t^*-open set containing q with $N' \cap N = \emptyset$. Then $N \cap E \in t$. Suppose there exists a t^*-open set M such that $q \in M$ and $M \cap E = N \cap E$. Then $E \cap (M \cap N') = \emptyset$ which is a contradiction since $q \in E^*$ and E is dense in E^*.

(2) $s^*_{E'}$ is T_1. Let x and y be distinct joints in E'. If $p \neq x$ then there exists a t^*-open set N with $x \in N$ and $y \notin N$ and also there exists a t^*-open set M with $x \in M$ and $p \notin M$. Thus $x \in M \cap N$ but $y \notin M \cap N$ and $M \cap N \in s^*_{E'}$. If $p = x$ then there exists a t^*-open set N with $p \in N$ and $y \notin N$ and also there exists a t^*-open set M with $q \in M$ and $y \notin M$. Thus $M \cup N \in N_{t^*}(q)$, $p \in M \cup N$ and $y \notin M \cup N$.

(3) $s^*_{E'}$ is normal. Let A, B be $s^*_{E'}$-closed sets such that $A \cap B = \emptyset$. Without loss of generality $p \notin A$ and hence $p \in E' - A \in s^*_{E'}$. Thus $(E'-A) \cup \{q\} \in N_{t^*}(q)$, but $(E'-A) \cup \{q\} = E^*-A$. Now A and $B \cup \{p,q\}$ are t^*-closed hence there exist disjoint t^*-open sets O_1 and O_2 such that

$A \subseteq O_1$ and $B \cup \{p,q\} \subseteq O_2$. Clearly O_1 and O_2 are open in s^*. Hence $O_1 \cap E'$ and $O_2 \cap E'$ are $s^*_{E'}$-open sets and they have void intersection. Thus $s^*_{E'}$ is normal.

Therefore, s^*_E is $T_{3\frac{1}{2}}$ since every subspace of a T_4-space is $T_{3\frac{1}{2}}$. (Kelly [25], pg. 118). Hence it has been shown that a strictly coarser $T_{3\frac{1}{2}}$-topology can be constructed if the topology is not compact.

$\underline{T_{3\frac{1}{2}}\text{, maximal}}$:- Any nonprincipal ultratopology (3.3).

Regular and Completely Regular, minimal:-

Lemma 4.9 :- In a regular topology, if the closure of two distinct points have non-void interesection then the closures are equal.

Proof:- Let the closure of p be denoted by $\bar{p}$ and let $p \neq q$. If $p \notin \bar{q}$ then by regularity there exists $N \in B(p)$ with $\bar{N} \cap \bar{q} \neq \emptyset$. Therefore $\bar{p} \cap \bar{q} \neq \emptyset$ implies $p \in \bar{q}$ and by symmetry $q \in \bar{p}$. Therefore $\bar{p} \cap \bar{q} \neq \emptyset$ implies $\bar{p} = \bar{q}$.

Therefore in a regular topology one can determine a partition of E by taking as elements of the partition the closures of suitably chosen single points.

Theorem 4.10 :- The minimal regular and completely regular topologies are exactly those of the form $\{\emptyset, A, E-A, E\}$ where A is a non-void proper subset of E.

Proof:- Obviously $C(A) \vee C(E-A)$ are minimal regular and minimal completely regular. Let $t \neq 0$ be a (completely) regular topology and hence from lemma 4.9 there exists a partition P of E.

(i) If P is finite then the closures of points are also open and hence

$\{\emptyset, \bar{p}, E-\bar{p}, E\} \subseteq t$.

(ii) If P is infinite then one can construct a strictly coarser topology which is (completely) regular.

<u>Regular</u> :- Select p and q in E such that $\bar{p} \neq \bar{q}$ and let $t'' = C(p,U(q)) \wedge C(q,U(p))$. Consider $t' = t \wedge t''$. Then t' is a strictly coarser topology since the closure in t' of p contains q. Now any t'-open set is a t-open set which contains both p and q or does not contain both p and q. Let $x \in E$ and $O \in B_{t'}(x)$. Hence $O \in t$ and by the regularity of t there exists $O \in t$ such that $x \in O' \subseteq Cl_{t'}O' \subseteq O$. If p and q both do not belong to O then O' is t'-open and $Cl_t O'$ is t''-closed and hence t'-closed. If p and q are both in O then by the regularity of t there exists t-open sets O_p and O_q such that $p \in O_p \subseteq Cl_t O_p \subseteq O$ and $q \in O_q \subseteq Cl_t O_q \subseteq O$. Now $x \in O' \cup O_p \cup O_q \in t'$ and $O' \cup O_p \cup O_q \subseteq Cl_t(O' \cup O_p \cup O_q) = Cl_{t'}(O' \cup O_p \cup O_q) \subseteq O$. Therefore t' is regular. Hence, if P is infinite, we can always find a strictly coarser regular topology.

<u>Completely Regular</u> :- Let t be completely regular. Then any t'-closed set contains both p and q or does not contain both p and q. Let A be a t'-closed subset of E and hence A is t-closed. Let $x \notin A$.

(i) $p,q \in A$. By the complete regularity of t there exists a function f, continuous with respect to t such that $f(x) = 0$ and $f(A) = 1$. Let M be any open set in $[0,1]$, then $f^{-1}(M) \in t$. If $1 \notin M$ then $p,q \notin f^{-1}(M)$ and hence $f^{-1}(M) \in t'$. If $1 \in M$ the $p,q \in f^{-1}(M)$ and hence $f^{-1}(M) \in t'$. Therefore f is continuous in t', $f(x) = 0$ and $f(A) = 1$.

(ii) $p,q \notin A$. By the complete regularity of t there exist f_1, f_2, f_3

continuous with respect to t such that $f_1(x) = f_2(p) = f_3(q) = 0$ and $f_i(A) = 1$, $(i = 1,2,3)$. Let f be the infimum of the f_i. Then $f(\{x,p,q\}) = 0$ and $f(A) = 1$. Let M be any open set in $[0,1]$. If $0 \in M$ then $p,q \in f^{-1}(M)$ and hence $f^{-1}(M) \in t'$. If $0 \notin M$ then $p,q \notin f^{-1}(M)$ and hence $f^{-1}(M) \in t'$. Therefore f is continuous in t'. Hence for any (completely) regular topology where the partition is infinite, we find a strictly coarser (completely) regular topology. Therefore the minimal regular and completely regular topologies are exactly those of the form $\{\emptyset, A, E-A, E\}$ where A is a non-void proper subset of E.

Regular and Completely Regular, maximal :-

Theorem 4.11 :- A topology is a maximal regular topology iff it is a nonprincipal ultratopology or it is of the form $C(x,U(y)) \wedge C(y,U(x))$ for some x,y in E.

Proof:- (i) A topology is a maximal regular T_1-topology iff it is nonprincipal ultratopology (3.3).

(ii) A principal ultratopology is not regular (3.3)

(iii) $t = C(x,U(y)) \wedge C(y,U(x))$ is a maximal regular topology (Steiner [36]). Every t-open set is also t-closed since it must contain both x and y or must not contain both x and y. Hence given any closed set A and $p \notin A$ it is possible to separate them by disjoint open sets, namely, A and E-A. Therefore t is regular. Conversely the only non-discrete topologies strictly finer than t are $C(x,U(y))$ and $C(y,U(x))$ (1.2) neither of which is regular (3.3).

(iv) Any regular non-T_1-topology is contained in a topology of the form $C(x,U(y)) \wedge C(y,U(x))$. If t is not a T_1-topology then there exists $p \in E$ such that $\bar{p}$ consists of at least two elements. Take distinct x and y in $\bar{p}$. Then the claim is that $t \leq C(x,U(y)) \wedge C(y,U(x))$. Take M any t-open set. If x and y are both in M or both not in M then $M \in C(x,U(y)) \wedge C(y,U(x))$. Therefore, without any loss of generality, we can assume $x \in M$ and $y \notin M$. Now $y \in E-M$ which is closed and hence $y \in (E-M) \cap \bar{p}$ which is closed. Also $(E-M) \cap \bar{p} \subset \bar{p}$ since $x \notin E-M$ which is a contradiction (4.9).

<u>Corollary</u>:- A topology is maximal completely regular iff it is a nonprincipal ultratopology or it is of the form $C(x,U(y)) \wedge C(y,U(x))$ for some $x,y \in E$.

<u>Proof</u>:- Nonprincipal ultratopologies are maximal completely regular (3.3) and principal ultratopologies are not completely regular (3.3). Topologies of the form $C(x,U(y)) \wedge C(y,U(x))$ are completely regular since all open sets are also closed and hence these topologies are maximal completely regular. Complete regularity implies regularity and hence one can show, using (iv) above, that any completely regular non-T_1-topology is contained in one of the form $C(x,U(y)) \wedge C(y,U(x))$.

<u>Normal and Completely Normal, minimal</u> :- infratopologies - obvious.

<u>Normal and Completely Normal, maximal</u> :- ultratopologies (3.1).

<u>Compact, minimal</u> :- infratopologies - obvious.

<u>Compact, maximal</u> :-

<u>Theorem 4.12</u> :- A space (E,t) is maximal compact (Lindelöf) iff the

compact (Lindelöf) subsets of E are identical with the closed subsets of E. (See also Smythe and Wilkins, [34]).

Proof:- Assume (E,t) is maximal compact. Since (E,t) is compact (Lindelöf) then every closed subset of E is compact (Lindelöf). Assume that there exists a subset A which is compact (Lindelöf) but is not closed. Define $t' = t \vee C(E-A)$. Then t' consists of all those subsets of E which can be expressed as $(O' \cap (E-A)) \cup O''$, where O' and O'' are t-open sets. Since $E-A \in t'$ we have $t' > t$. Let $(U_i: \ i \in I)$ be a t'-open cover of E. Then each $U_i = (O_i' \cap (E-A)) \cup O_i'')$. Hence $E = \bigcup_{i \in I} U_i = (\bigcup_{i \in I} (O_i' \cap (E-A)) \cup \bigcup_{i \in I} O_i''$ and therefore the set of all $(O_i' \cup O_i'')$ with $i \in I$ is a t-open cover of E. Since (E,t) is compact (Lindelöf) there exists a finite (countable) subcover, i.e. $E = \bigcup_{j \in J} (O_{i_j}' \cup O_{i_j}'')$ where J is finite (countable). Hence $E-A \subseteq \bigcup_{j \in J} ((O_{i_j}' \cap (E-A)) \cup O_{i_j}'')$. Now $\bigcup_{i \in I} O_i''$ is an open cover of A and, since A was assumed to be compact (Lindelöf), there exists a finite (countable) subcover, i.e. $A \subseteq \bigcup_{k \in K} O_{i_k}''$ with K finite (countable). Thus $E = \bigcup_{l \in L} ((O_{i_l}' \cap (E-A)) \cup O_{i_l}'')$ where $L = J \cup K$ which is finite (countable). Therefore (E,t') is a compact (Lindelöf) space which contradicts the maximality of (E,t).

Conversely, since E is closed (E,t) is compact (Lindelöf). Let t' be a strictly finer compact (Lindelöf) topology on E. Then there exists a subset A which is t'-closed but not t-closed. Hence A is not compact (Lindelöf) in (E,t) which is a contradiction since A is compact (Lindelöf) with respect to the space (E,t'). This implies the compact T_2-topologies are maximal compact.

Paracompact and Metacompact, minimal :- infratopologies - obvious.

Paracompact and Metacompact, maximal :- ultratopologies (3.1).

Locally Compact, minimal :- infratopologies - obvious.

Locally Compact, maximal :- principal ultratopologies (3.6). The question as to whether there are maximal locally compact T_1-topologies is open.

Lindelöf, minimal :- infratopologies - obvious.

Lindelöf, maximal :- topologies where the Lindelöf subsets are exactly the closed ones (4.12).

Connected, minimal :- infratopologies - obvious.

Connected, maximal :- Open question. It is easy to see from theorem 3.8 that $C(U)$, where U is an ultrafilter, and $C(x, \{E\})$ are maximal connected.

Locally connected, minimal :- infratopologies - obvious.

Locally connected, maximal :- Principal ultratopologies since the only maximal locally connected topologies are ultratopologies and by 3.6 only ultratopologies which are principal are locally connected. The fact that the only maximal locally connected topologies are ultratopologies is easily seen by the following :- Assume $t \neq P(E)$ is locally connected. Then there exists a point $x \in E$ such that $\{x\} \notin t$. Then $t \vee \{\emptyset, \{x\}, E\}$ is locally connected.

Totally disconnected, minimal :- open question

Totally disconnected, maximal :- nonprincipal ultratopologies (3.3).

<u>Extremally disconnected, minimal</u> :- infratopologies - obvious.

<u>Extremally disconnected, maximal</u> :- ultratopologies (3.1).

<u>Zero dimensional, minimal</u> :- $\{\emptyset, A, E-A, E\}$ where A is a non-void proper subset of E. This is easily seen by considering any non-trivial zero dimensional topology, t. Then there exists a subset B such that $\emptyset \subset B \subset E$ and let $p \in B$. Then since t is zero dimensional there exists an open-closed set A with $p \in A \subseteq B$. Hence $A \in t$, $E-A \in t$, and $\emptyset \neq A \neq E$.

<u>Zero dimensional, maximal</u> :-

<u>Theorem 4.13</u> :- The maximal zero dimensional topologies are the nonprincipal ultratopologies and topologies of the form $C(x,U(y)) \wedge C(y,U(x))$ for $x,y \in E$.

<u>Proof</u>:- A nonprincipal ultratopology is zero dimensional (3.3). A mixed topology cannot be maximal zero dimensional since it is properly contained in a nonprincipal ultratopology. In the case of principal topologies, regularity and being zero dimensional are equivalent since regularity implies that for all $p \in E$, B_p(1.5) is closed and hence open-closed. Hence the neighbourhood system of any point has a base of open-closed sets and therefore is zero dimensional. Conversely, if a topology is zero dimensional it has a base of closed sets which is equivalent to regularity (Kowalsky [26], pg. 59). Therefore the maximal zero dimensional topologies are exactly the maximal regular principal topologies which are the topologies of the form $C(x,U(y)) \wedge C(y,U(x))$ for $x,y \in E$.

<u>First Axiom of Countability, minimal</u> :- infratopologies - obvious.

<u>First Axiom of Countability, maximal</u> :- principal ultratopologies by virtue of theorem 3.6, corollary to lemma 3.5 and the corollary to lemma 3.4.

<u>Second Axiom of Countability, minimal</u> :- infratopologies - obvious.

<u>Second Axiom of Countability, maximal</u> :- none. This follows as a consequence of the corollary to lemma 3.4 since no ultratopology (in general) satisfies the second axiom of countability. (Recall also the corollary to lemma 3.5).

<u>Separable, minimal</u> :- any infratopology, $C(A)$, since if A is not itself countable take a countable subset of A.

<u>Separable, maximal</u> :-

<u>Theorem 4.14</u> :- There exist no maximal separable topologies, if E is uncountable.

<u>Proof</u> :- Let t be a topology which is separable. This implies that there exists a subset N which is countable such that $\overline{N} = E$. Now $N \subset E$ and hence there exists $p \notin N$. Thus $E-\{p\} \supseteq N$ and hence $E-\{p\}$ is not closed and so $\{p\} \notin t$. Consider $t' = t \vee C(\{p\})$ which is a strictly finer topology than t. Let $N' = N \cup \{p\}$. Consider subset A such that $N' \subseteq A \neq E$. If A is a t'-closed set then $E-A \in t'$. But $p \notin E-A$ and hence $E-A \in t$ which implies A is t-closed which is a contradiction. Hence N' is dense in t'. Therefore for any separable topology we can find a strictly finer separable topology. Also it is obvious that no ultratopology is separable. Therefore there exist no maximal separable topologies, if E is uncountable.

Principal, minimal :- infratopologies - obvious.

Principal, maximal :- principal ultratopologies - obvious.

Door, minimal :- $C(x,\{E\})$ and $C(U)$ where U is an ultrafilter. This follows directly from theorem 3.7.

Door, maximal :- ultratopologies (3.1).

Property	Minimal	Maximal
T_0	topology where $\leq$ is a total order of E and it is of the form $]a, \rightarrow]$ for a in E.	ultratopologies
$T_{\frac{1}{2}}$	topology where $\leq$ is a t.o. of E and it is of the form $[a, \rightarrow]$ and $]a, \rightarrow]$ for a in E.	ultratopologies
T_1	cofinite	nonprincipal ultratopologies
T_2	T_2-topology where if an open filter has a unique adherence point then it converges to this point	nonprincipal ultratopologies
T_3	T_3-topology where if a regular open filter has a unique adherence point then it converges to this point	nonprincipal ultratopologies
$T_{3\frac{1}{2}}$	compact T_2	nonprincipal ultratopologies
T_4	compact T_2	nonprincipal ultratopologies
T_5	compact T_5	nonprincipal ultratopologies
regular	$\{\emptyset, A, E-A, E\} = C(A) \vee C(E-A)$	nonprincipal ultratopologies or of the form $C(x, U(y) \wedge C(y, U(x))$
completely regular	same as regular	same as regular
normal	infratopologies	ultratopologies
completely normal	infratopologies	ultratopologies
compact	infratopologies	topologies where compact subsets are exactly the closed ones.
paracompact	infratopologies	ultratopologies
metacompact	infratopologies	ultratopologies
locally compact	infratopologies	principal ultratopologies and ?
Lindelöf	infratopologies	topologies where the Lindelö subsets are exactly the closed ones
connected	infratopologies	$C(U)$, $C(x, \{E\})$ and ?
locally connected	infratopologies	principal ultratopologies
totally disconnected	?	nonprincipal ultratopologies
extremally disconnected	infratopologies	ultratopologies
zero dimensional	same as regular	same as regular
1st axiom of countability	infratopologies	principal ultratopologies
2nd axiom of countability	infratopologies	none
separable	infratopologies	none
principal	infratopologies	principal ultratopologies
door	$C(x, \{E\})$ and $C(U)$	ultratopologies

Chapter 5

Preservation of Topological Properties under Lattice Operations and Relations

In this chapter the preservation of various topological properties under lattice operations and relations will be investigated. A table is provided at the end of the chapter to summarize the results. It should be noted that if a property is preserved with respect to taking coarser topologies then it is preserved under infinite meets which implies it is preserved under finite meets. This may be graphically represented as follows:-

$$\leq \;\Rightarrow\; \bigwedge \;\Rightarrow\; \wedge$$

Similarily $\quad \geq \;\Rightarrow\; \bigvee \;\Rightarrow\; \vee$

Also if a property is not preserved with respect to a lattice operation or relation we may indicate this by placing a stroke through the symbol.

Thus we have $\quad \not\leq \;\Rightarrow\; \not\bigwedge \;\Rightarrow\; \not\wedge$

$$\not\geq \;\Rightarrow\; \not\bigvee \;\Rightarrow\; \not\vee$$

Before beginning the detailed study it is useful to prove several general lemmas and theorems which will be used later.

Lemma 5.1 :- U contains N(p) with respect to a topology t iff t is coarser than C(p,U).

Proof :- If p belongs to a t-open set then the set belongs to U and hence C(p,U) and any set not containing p belongs to C(p,U). Conversely, if p is contained in a t-open set and $t \leq C(p,U)$ then the set belongs to U and hence $B(p) \subseteq U$ and so also $N(p) \subseteq U$.

Lemma 5.2 :-

(1) t is a T_2-topology iff for each ultrafilter U there exists at most one point p such that $t \leq C(p,U)$.

(2) t is compact iff for each ultrafilter U there exists at least one point p such that $t \leq C(p,U)$.

(3) t is compact T_2 iff for each ultrafilter U there exists exactly one point p such that $t \leq C(p,U)$.

Proof :- (1) t is T_2,

iff each ultrafilter U converges to at most one point p, i.e.

iff for each ultrafilter U there exists at most one point p such that

$U \supseteq N(p)$ (by the definition of convergence) i.e.

iff for each ultrafilter U there exist at most one point p such that

$t \leq C(p,U)$ (by lemma 5.1)

(2) t is compact

iff each ultrafilter U converges to at least one point p, i.e.

iff for each ultrafilter U there exists at least one point p such that

$U \supseteq N(p)$ i.e.

iff for each ultrafilter U there exists at least one point p such that

$t \leq C(p,U)$.

(3) follows from (1) and (2)

Theorem 5.3 :- The cofinite topology has the following properties:-

(1) T_0, (2) $T_{\frac{1}{2}}$, (3) T_1, (4) compact, (5) Lindelöf, (6) paracompact,

(7) metacompact, (8) locally compact, (9) connected. The cofinite topology, if E is infinte, does not have the following properties :- (10) T_2, (11) T_3, (12) $T_{3\frac{1}{2}}$, (13) T_4, (14) T_5, (15) regular, (16) completely regular, (17) normal, (18) completely normal, (19) door space, (20) principal. The cofinite topology satisfies the countability axioms iff E is countable.

<u>Proof</u> :- $C(C^*)$ is T_1 (1.9) hence $T_{\frac{1}{2}}$ and T_0. $C(C^*)$ is compact because every ultrafilter converges and hence 5,6,7,8 all hold. $C(C^*)$ is connected since any two sets in a filter have non-empty intersection. $C(C^*)$ is not T_2 since any two sets in a filter have non-empty intersection. Thus 11,12,13,14 do not hold and, since $C(C^*)$ is T_1, 15,16,17,18 also do not hold. $C(C^*)$ is not a door space since if A is infinite and has an infinite complement then it is neither open nor closed. $C(C^*)$ is not principal since it is the meet of nonprincipal ultratopologies (1.5). Now since $\cap(B: B\in C(C^*)) = \emptyset$ we have that $E = \cup(E-B)$. Now E-B is finite. Hence $C(C^*)$ has a countable base iff E is countable. Also since $\{p\} = \cap(B: p\in B\in C(C^*))$ we have $E-\{p\} = \cup(E-B)$. Again since each E-B is finite we find that the neighbourhood system of any point has a countable basis iff E is countable.

<u>Theorem 5.4</u> :- The compact T_2-topologies are incomparable. In particular, given a compact T_2-topology, then any topology strictly finer is not compact and any topology strictly coarser is not T_2.

<u>Proof</u>:- This follows directly from 4.3 and 4.12 since compact T_2-topologies are minimal T_2 and maximal compact.

Theorem 5.5 :- $C(x,C^*) = \cap(C(x,U)$: U a nonprincipal ultrafilter).

Then $C(x,C^*)$ (1) satisfies all T_i.

(2) is compact, in fact, maximal compact.

(3) is minimal T_2, T_3, $T_{3\frac{1}{2}}$, T_4, T_5.

(4) is totally disconnected.

Proof :- $C(x,C^*)$ is T_5 - Take $A,B \subseteq E$ such that $A \cap \bar{B} = \emptyset = \bar{A} \wedge B$. Without loss of generality we may assume $x \notin A$ and hence $A \in C(x,C^*)$. Now $\bar{A} \cap B = \emptyset$ implies $B \subseteq \complement\bar{A} \in C(x,C^*)$. Also $A \cap \complement\bar{A} = \emptyset$ and thus $C(x,C^*)$ is T_5. Hence $C(x,C^*)$ satisfies all T_i. $C(x,C^*)$ is compact T_2 since every ultrafilter converges to exactly one point (5.2). Thus $C(x,C^*)$ is maximal compact and minimal T_2, T_3, $T_{3\frac{1}{2}}$, T_4, T_5 (5.4). For all $y \neq x$ we have $\{y\}$ is open-closed. Also we have $\cap(E-\{y\} : y \neq x) = E-\cup\{y : y \neq x\} = \{x\}$. Each of the $E-\{y\}$ is open-closed. Now the component of a point is contained in the intersection of all open-closed sets containing the point (Kowalsky [26], 14.9). Thus for all $p \in E$, we have $\{p\}$ is the component of p. Therefore $C(x,C^*)$ is totally disconnected.

Theorem 5.6 :- Let $(t_j : j \in J)$ be an arbitrary family of (completely) regular topologies on E. Then $t = \bigvee_{j \in J} t_j$ is also (completely) regular. (Norris [29]).

Proof:- (a) regular

Let $p \in E$ and let A be a subset of E such that $p \in A \in t$. Then there exist $A_{j_i} \in t_{j_i}$ $(i=1,2,\ldots n)$ such that $p \in \bigcap_{i=1}^{n} A_{j_i} \subseteq A$. Now, since each t_{j_i} is regular, there exist $B_{j_i} \in t_{j_i}$ such that $p \in B_{j_i} \subseteq Cl_{t_{j_i}} B_{j_i} \subseteq A_{j_i}$.

Let $B = \bigcap_{i=1}^{n} B_{j_i}$. Hence $p \in B \in t$. Now $Cl_t B = Cl_t \bigcap_{i=1}^{n} B_{j_i} \subseteq \bigcap_{i=1}^{n} Cl_t B_{j_i} \subseteq \bigcap_{i=1}^{n} Cl_{t_{j_i}} B_{j_i} \subseteq \bigcap_{i=1}^{n} A_{j_i} \subseteq A$. Therefore t is regular.

(b) completely regular

Let $p \in E$ and $A \subseteq E$ such that $p \in A \in t$. Then there exist $A_{j_i} \in t_{j_i}$ $(i=1,2,\ldots n)$ such that $p \in \bigcap_{i=1}^{n} A_{j_i} \subseteq A$. Since each t_{j_i} is completely regular there exist $f_i (i=1,2,\ldots.n)$, $f_i : E \to [0,1]$, continuous with respect to t_{j_i} such that $f_i(p) = 0$ and $f_i(x) = 1$ for all $x \in \complement A_{j_i}$. Let f be the supremum of the f_i. Then $f: E \to [0,1]$, $f(p) = 0$ and $f(x) = 1$ for all $x \in \complement A$. Now, since t is finer than each t_{j_i} and the f_i are continuous with respect to t_{j_i}, f is continuous with respect to t. Therefore t is completely regular.

<u>Proposition 5.7</u> :- Consider the Euclidean plane, E, with the following topologies:-

t_1 = the product of the usual topology, s, on the reals with the topology which has as base the half-open intervals of the form $[a,b[$.

t_2 = the product of the topology which has as base the half-open intervals of the form $[a,b[$ and the usual topology, s, on the reals.

t = the product of the half-open intervals topologies of the form $[a,b[$ on the reals.

Then $t = t_1 \vee t_2$ clearly. Also t_1 and t_2 are paracompact, T_4 and T_5 but t is not T_5 nor is it metacompact.

Proof :- t is obviously T_1 and has a base of open-closed sets. Thus t is regular since if x belongs to some open set it belongs to a base set which is open-closed. Therefore t is T_3. Now t is not normal for consider the set $L = \{(x,y): x+y = 1\}$. L is closed since for all points not on the line there exists an open set which contains the point and has void intersection with L. Let M be the set of all points of L with rational co-ordinates and N the set of all points of L with irrational co-ordinates. M and N are both closed subsets of L since the induced topology on L is the discrete topology and hence M and N are t-closed. Then M and N are disjoint but obviously there exist no disjoint open sets O and O' such that $M \subseteq O$ and $N \subseteq O'$. Therefore t is not normal and hence not paracompact nor T_5. Now t is not even metacompact which may be seen by considering the following. Take as an open cover of E the left-half-plane determined by L (not including L) together with all sets $M_{(x,y)}$ where $M_{(x,y)} = \{(c,d): (x,y) \in L, x \leq c, y \leq d\}$. Now the claim is that this open cover has no point finite refinement. Assume it does. It is sufficient to confine our attention to L and the right-half-plane. For all $(x,y) \in L$ there exists a set $R_{(x,y)}$ in the refinement such that $(x,y) \in R_{(x,y)} \subseteq M_{(x,y)}$ and hence there is a "square" (a base set) $B_{(x,y)} \in t$ such that $(x,y) \in B_{(x,y)} \subseteq R_{(x,y)}$, $B_{(x,y)} = \{(p,q): p-x < b_{(x,y)}, q-y < b_{(x,y)}, b_{(x,y)}$ a real number$\}$. We form a partition P of the points of L by forming a partition P' of the $b_{(x,y)}$ by saying that (x,y) and (x',y') belong to the same class of P iff $b_{(x,y)}$ and $b_{(x',y')}$ both lie in the interval $\left]\frac{1}{n}, \frac{1}{n+1}\right]$ for some $n \in N$. Now there are uncountably many points in L and only countably many classes in P' and hence there exists a class P_i of P which contains uncountably many points of L, say that P_i corresponds to the interval $\left]\frac{1}{n-1}, \frac{1}{n}\right] = P_i'$. Hence there exists an accumulation point $(p,q) \in L$ for the class P_i. Consider the set $N' = \left\{(x,y) \in E: |(x,y) - (p,q)| < \frac{1}{2n}\right\}$. Then there exists uncountably many $B_{(x,y)}$ in N' with $b_{(x,y)} > \frac{1}{n}$ and these $B_{(x,y)}$ have non-void intersection and hence there exists a point in E which lies in infinitely many $B_{(x,y)}$ and hence in infinitely many sets of the refinement. Thus there is no point-finite refinement of the cover stated. Therefore t is not metacompact.

Now t_1 and t_2 are paracompact since each is the product of a paracompact T_3-topology and a topology which is countable at infinity (Kowalsky [26], pg. 153, 22.8). Therefore t_1 and t_2 are also T_4. It is conjectured that each is also T_5 with the proof probably being as follows: Consider t_1 and subsets A,B of E with $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Note $\overline{A}$ is the closure of A with respect to t_1 and $\widetilde{A}$ the closure of A with respect to the usual topology on E. Then we have $\widetilde{A} \supseteq \overline{A}$. By the definition of closure and since t_1 is regular we have that for all $p \in \widetilde{B} - \overline{B}$ there exists $Q_p \in t$ such that $\overline{Q}_p \cap \overline{B} = \emptyset$. Define $\mathcal{Q}' = \{Q_p: \ p \in \widetilde{B} - \overline{B}\}$. Let $\mathcal{Q}$ be a locally finite refinement of $\mathcal{Q}'$ and let $Q = \cup\mathcal{Q}$. Then $\widetilde{B} - \overline{B} \subseteq Q$ and since locally finite $\overline{Q} = \cup(\overline{X}: \ X \in \mathcal{Q})$ (Gaal, [18], pg. 153). Thus we have $\overline{Q} \cap \overline{B} = \emptyset$ and $\widetilde{B} - \overline{B} \subseteq Q \in t_1$. Similarily there exists R such that $\overline{R} \cap \overline{A} = \emptyset$ and $\widetilde{A} - \overline{A} \subseteq R \in t_1$. Let $A' = A \cap \complement Q$ and $B' = B \cap \complement R$. Now

$$\begin{aligned}
\widetilde{A}' \cap B' &= (\widetilde{A \cap \mathbb{C}Q}) \cap (B \cap \mathbb{C}R) \\
&= \widetilde{A} \cap \widetilde{\mathbb{C}Q} \cap B \cap \mathbb{C}R \\
&= (\widetilde{\mathbb{C}Q} \cap B) \cap (\widetilde{A} \cap (\mathbb{C}\widetilde{A} \cup \overline{A})) \\
&= \emptyset
\end{aligned}$$

Therefore $\widetilde{A}' \cap B' = \emptyset$ and similarily $A' \cap \widetilde{B}' = \emptyset$.

Hence there exist sets V, W which are open with respect to the usual topology and hence t-open such that $A' \subseteq V$, $B' \subseteq W$ and $V \cap W = \emptyset$.

Now $A \subseteq (V \cup Q) \cap \mathbb{C}\overline{R} \in t_1$

$B \subseteq (W \cup R) \cap \mathbb{C}\overline{Q} \in t_1$

also
$$\begin{aligned}
(V \cup Q) \cap \mathbb{C}\overline{R} \cap \mathbb{C}\overline{Q} \cap (W \cup R) &\subseteq (V \cup Q \cup R) \cap (W \cup Q \cup R) \cap \mathbb{C}(\overline{R} \cup \overline{Q}) \\
&\subseteq ((V \cup W) \cup Q \cup R) \cap \mathbb{C}(\overline{R} \cup \overline{Q}) \\
&= \emptyset
\end{aligned}$$

Therefore $(V \cup Q) \cap \mathbb{C}\overline{R}$ and $(W \cup R) \cap \mathbb{C}\overline{Q}$ are disjoint t_1-open sets and hence t_1 is a T_5-topology. Similarily t_2 is a T_5-topology. With regard to the existence of a locally finite refinement this will be the case if $t_1 \mid \widetilde{B} - \overline{B}$ is paracompact. By 13.19 of Kowalsky [26] this will be the case if $\widetilde{B} - \overline{B}$ is expressible as the countable union of closed sets. Let V be any vertical line. Then $(\widetilde{B} - \overline{B}) \cap V = \widetilde{B} \cap \mathbb{C}\overline{B} \cap V = (\widetilde{B} \cap V) \cap (\mathbb{C}\overline{B} \cap V)$. Now $\widetilde{B} \cap V$ is a closed set, say C. Also $t_1 | V = s$. Thus $\mathbb{C}\overline{B} \cap V$ is an open set and hence is the countable union of closed intervals, say $\mathbb{C}\overline{B} \cap V = \bigcup_{n \in N} Cn$. Therefore $(\widetilde{B} \cap V) \cap (\mathbb{C}\overline{B} \cap V) = C \cap \bigcup_{n \in N} Cn = \bigcup_{n \in N} C \cap Cn$. That is $(\widetilde{B} - \overline{B}) \cap V$ is the countable union of closed sets. Thus all that must be proved is that $(\widetilde{B} - \overline{B}) \cap V \neq \emptyset$ for only countably many V. This would seem to be the case but remains an open question at this time.

<u>T_0</u>:- $(\not\wedge)$ $C(x,U(y))$ and $C(y,U(x))$ are both T_0 (3.1) but there is no open set separating x and y in their meet.

$(\geqq)$ obvious.

<u>$T_{\frac{1}{2}}$</u>:- $(\not\wedge)$ $C(x,U(y)) \wedge C(y,U(x))$ is not $T_{\frac{1}{2}}$.

$(\geqq)$ obvious.

<u>T_1</u>:- $(\not\leqq)$ Coarser than T_1 need not be T_1 since the cofinite topology is the smallest T_1-topology. All other relations and operations preserve T_1 since the T_1-topologies form a complete sublattice.

<u>T_2</u>:- $(\not\wedge)$ The meet of two compact topologies (e.g. $C(x,C^*) \wedge C(y,C^*)$, $x \neq y$) is not T_2 (5.4, 5.5)

$(\geqq)$ obvious.

<u>T_3 and regularity</u>:-

$(\not\wedge)$ $C(x,C^*) \wedge C(y,C^*)$, $x \neq y$, is not T_3 but is T_1 (5.5, 1.10)

$(\vee)$ Regularity is preserved as proven in theorem 5.6 and also T_1 is preserved so T_3 is preserved.

$(\not\geqq)$ Is illustrated by the following example mentioned by Gaal [18] pg. 85 and first noticed by Hausdorff [22] pg. 264. Let t be the usual topology on the reals and $t'' = t \vee C(C')$ and let $t' = C(C')$. Then t'' is finer than t and although t is a T_3-topology t'' is not. This is easily seen by the following. Now t'' is obviously a T_1-topology. Assume t'' is regular. Let Q be the set of all rationals. Q is a t''-closed set since it is countable. The irrational number $\sqrt{2}$ is not in Q. If t'' is regular then there exist disjoint sets $M,N \in t''$ such that $\sqrt{2} \in M = A \cap A'$ where $A \in t$ and $A' \in t'$ and $Q \subseteq N = \bigcup_{i \in I}(B_i \cap B_i')$ where $B_i \in t$ and $B_i' \in t'$.

Thus $\emptyset = M \cap N = A \cap A' \cap \bigcup_{i \in I}(B_i \cap B_i') = \bigcup_{i \in I} (A \cap A' \cap B_i \cap B_i')$. Hence $A \cap A' \cap B_i \cap B_i'$ is empty for all $i \in I$ and so $A \cap B_i \subseteq \mathbb{C}(A' \cap B_i')$. Now there exists an $j \in I$ such that $A \cap B_j \neq \emptyset$ since $\sqrt{2} \in A \cap B_j$ for some $j \in I$. But then we have an uncountable set contained in a countable one which is impossible. Therefore t'' is not regular.

$\underline{T_{3\frac{1}{2}}}$ and complete regularity:- The same arguments and example apply as for T_3 and regularity.

Normal:-

$(\not\bigwedge)$ $C(x,C^*) \wedge C(y,C^*)$, $x \neq y$, is not T_4 but is T_1 (5.5, 1.10)

$(\not\bigvee)$ $C(E-\{x\}) \vee C(E-\{y\})$, $x \neq y$ is not normal since there are no disjoint open sets in the join separating the closed sets $\{x\}$ and $\{y\}$. (Note that each infratopology is normal.)

$\underline{T_4}$:-

$(\not\bigwedge)$ $C(x,C^*) \wedge C(y,C^*)$, $x \neq y$, is not T_4. (5.5)

$(\not\bigvee)$ Proposition 5.7

Completely normal :-

$(\not\bigwedge)$ $C(x,C^*) \wedge C(y,C^*)$, $x \neq y$ (5.5, 1.10)

$(\not\bigvee)$ $C(E-\{x\}) \vee C(E-\{y\})$, $x \neq y$, is not normal hence not completely normal.

$\underline{T_5}$:-

$(\not\bigwedge)$ $C(x,C^*) \wedge C(y,C^*)$, $x \neq y$, is not T_5 (5.5, 1.10)

$(\not\geqq)$ Since t, in T_3 $(\not\geqq)$ above, is also T_5.

$(\not\bigvee)$ Proposition 5.7

Compact :-

$(\leqq)$ obvious

($\not\vee$) $C(x,C^*) \vee C(y,C^*) = 1$. Each is compact (5.5) but the discrete topology is not compact if E is infinite.

Lindelöf :-

($\leq$) obvious

($\not\vee$) $C(x,C^*) \vee C(y,C^*) = 1$. Each is Lindelöf since each is compact (5.5) but the discrete topology is not Lindelöf if E is uncountable.

Paracompact :-

($\not\wedge$) This is demonstrated by the following counterexample which shows that the meet of two paracompact topologies is not necessarily even metacompact. Let R be the reals and Q the rationals. Let t_1 be the usual topology on the reals and let $t_2 = \wedge(C(q,C^*): \; q \in Q)$. Define $t = t_1 \wedge t_2$. Now t_1 is known to be paracompact and t_2 is paracompact because for any open cover of R, a finite number of sets cover Q and hence one can take as a refinement these sets together with all singletons of points not in these open sets and this refinement is locally finite.

To establish t is not metacompact it will first be shown that each infinite point finite open cover is uncountable, i.e., there exists no countable point finite open covers. Let $(M_i: \; i \in I)$ be an infinite point finite open cover of R. Now since it is infinite and point finite we have $\bigcap_{i \in I} M_i = \emptyset$ which implies $R = \bigcup_{i \in I} \complement M_i$. Now the $\complement M_i$ is countable for each $i \in I$, R is uncountable and hence $|I|$ must be uncountable. Now consider $A = \complement \left\{\sqrt{2} + n: \; n \in N\right\}$. This is an open set since it is the union of open intervals and contains all the rationals. Let O_n be the open interval about $\sqrt{2} + n$ given by $O_n =]\sqrt{2} + n - 1/3, \sqrt{2} + n + 1/3[$. Define $A_n = A \cup O_n$. Then A_n is open for all $n \in N$. It is obvious that $(A_n : n \in N)$ is an open cover of R, it is countable and hence by the previous argument it is not point finite. Clearly there is no finite subcover of this cover and it is also quite clear that there is no point finite refinement of it either. Hence t is not metacompact.

$(\vee)$ Proposition 5.7

Metacompact :-

$(\wedge)$ The counterexample in this case is the same as the one for paracompact $(\wedge)$ since both t_1 and t_2 are metacompact but t is not.

$(\vee)$ Proposition 5.7

locally compact :-

$(\wedge)$ The finite meet of locally compact topologies need not be locally compact as shown by the following counterexample. Let t be the usual topology on the reals. Consider $t' = t \wedge C(1,U(N))$ where U(N) is the principal filter generated by the natural numbers. Now t is known to be locally compact and C(1,U(N)) is locally compact since it is principal and hence for all $r \in R$ there exists a smallest open neighbourhood of N, B_r (1.5). The claim is that t' is not locally compact since there is no compact neighbourhood of 1. Assume K is a locally compact neighbourhood of 1. Then K contains an open neighbourhood of 1, $A = \bigcup_{n \in N} I_n$ where $I_n =]n-\varepsilon, n+\varepsilon[\ 0<\varepsilon\leq\frac{1}{2}$. Let $C_n = [n-\frac{\varepsilon}{2}, n+\frac{\varepsilon}{2}]$ and hence $C_n \subseteq I_n$. Let $C = \bigcup_{n \in N} C_n$. Then C is closed since it is the union of disjoint closed sets and $N \subseteq C$. Also C is obviously a closed neighbourhood. Now C is compact since it is a closed subset of a compact set. Define an open cover of C to be equal to $\{\bigcup_{n\in N}]n-\frac{\varepsilon}{2}, n+\frac{\varepsilon}{2}[\ \} \cup (]n-\varepsilon, n[: n\in N) \cup (]n, n+\varepsilon[: n\in N)$. Obviously there is no finite subcover of this cover and hence C is not compact which is a contradiction. Therefore there is no compact neighbourhood of 1 and hence t' is not locally compact.

($\vee$) The finite join of locally compact topologies need not be locally compact as shown by the following counterexample. Let $A = [-a, a]$ where a is a real number. Let t' be the usual topology of the reals restricted to A and let t'' be C(O,C*) on A. Now $t = t' \vee t''$ is not locally compact. This will be proven by showing there is no compact neighbourhood of zero. Assume there exists a compact neighbourhood K of zero. Then there exist sets $B \in C(O,C^*)$ and I, an open interval, such that $0 \in B \cap I \subseteq K$. Hence there exists an open interval J with $0 \in J \subseteq B \cap I \subseteq K$ and so there exists an open interval J' with $0 \in J' \subset J$. Take as an open cover of K the following, J' together with all $\{y\}$, $y \in K-J'$. Obviously this has no finite subcover. Therefore t is not locally compact.

connected :-

($\leq$) obvious

($\vee$) Finite join of connected topologies need not be connected as illustrated by the following example - $C(A) \vee C(E-A)$ where $\emptyset \subset A \subset E$.

locally connected :-

($\not\leq$) Since discrete topology is locally connected

($\wedge$) Lemma 5.8 :- A topology is locally connected iff the components of all open subsets are themselves open. (Kowalsky [26], pg. 108).

Proof :- Let t be a locally connected topology, M an open subset of E and K a component of M. Now a set is open in M iff it is open in E. For all $p \in K$ there exists a connected neighbourhood $O_p \subseteq M$. Since K is the largest connected subset of M containing p, $O_p \subseteq K$ for all $p \in K$ and hence $K = \bigcup_{p \in K} O_p$. Thus K is open. Conversely, take any open set O with $p \in O$. Then the component of O containing p is an open neighbourhood of p. Hence the neighbourhoods of p possess a basis of connected sets.

Lemma 5.9 :- The meet t of an arbitrary family $(t_i: i \in I)$ of locally connected topologies is locally connected. (Kowalsky [26], pg. 109).

Proof :- By lemma 5.8 it is sufficient to show all components K of M, where M is a t-open subset of E, are t-open. Now $M \in t$ implies $M \in t_i$, for all $i \in I$. Let p be any element of a component K of M. Then there exists a neighbourhood $O_i \in t_i$, with $p \in O_i \subseteq M$ and O_i connected in t_i. Hence O_i is connected in t which implies $O_i \subseteq K$. Thus, as in the last lemma, $K = \bigcup_{p \in K} O_i$, where $p \in O_i$, and hence $K \in t_i$ for all $i \in I$. Therefore $K \in t$.

($\vee$) The finite join of locally connected topologies need not be locally connected as shown by the following counterexample. Let E be the reals. Consider $t \vee C(U)$ where t is the usual topology on the reals and U is any nonprincipal ultrafilter which converges to p. Clearly both t and C(U) are locally connected. Claim that $t \vee C(U) = C(p,U)$. Now $t \vee C(U) \subseteq C(p,U)$ since for any $A \in t$, if $p \in A$ then $A \in U$ since U converges to p, and if $p \notin A$ then $A \in C(p,U)$ by definition.

To show the reverse inclusion take any $A \in C(p,U)$. If $A \in U$ then $A \in C(U)$. If $A \notin U$ then $p \notin A$. Let $q \in A$, hence $q \neq p$ and thus there exists a t-open set O such that $q \in O \notin U$ (this follows since an ultrafilter converges to only one point in a T_2-topology). Thus $\complement O \in U$ and hence $\complement O \cup \{q\} \in U$. Therefore $\{q\} = (\complement O \cup \{q\}) \cap O \in t \vee C(U)$. Thus $A \in t \vee C(U)$. Therefore $t \vee C(U) = C(p,U)$ which is not locally connected by theorem 3.6.

<u>totally disconnected</u> :-

$(\not\wedge)$ The meet of two totally disconnected topologies need not be totally disconnected as is easily seen by considering the meet of the half-open interval topologies on the real line. That is, let t_1 be the topology with a base of open sets of the form $[a,b[$ and t_2 have as base the sets of the form $]a,b]$ where a and b are real numbers. Each has a base of open-closed sets and is therefore totally disconnected but $t_1 \wedge t_2$ is the usual topology on the reals which is connected.

$(\geq)$ obvious

<u>extremally disconnected</u> :-

$(\not\wedge)$ The meet of two extremally disconnected topologies need not be extremally disconnected as shown by the following counterexample. Take $A,B \subseteq E$ such that $A \cap B = \emptyset$ and $A \cup B \subset E$. Consider $t_1 = \{\emptyset, A, B, E-A, E\}$ and $t_2 = \{\emptyset, A, B, E-B, E\}$. Obviously t_1 and t_2 are extremally disconnected since the closure of every open set is again open. Now $t = t_1 \wedge t_2 = \{\emptyset, A, B, E\}$ and t is not extremally disconnected since the closure of A is E-B which is not t-open.

($\not\vee$) The join of two extremally disconnected topologies need not be extremally disconnected as shown by the following counterexample. Let A be a non-empty proper subset of E and let $p \notin A$. The C(A) is extremally disconnected and so is $C(\complement(A\cup\{p\}))$. But, the join of these two topologies is $t = \{\emptyset, A, \complement(A\cup\{p\}), \complement\{p\}, E\}$ which is not extremally disconnected since the closure in t of A is $A\cup\{p\} \notin t$.

<u>zero dimensional</u> :-

($\not\wedge$) The counterexample in this case is the same as the one for totally disconnected ($\not\wedge$).

($\not\Rightarrow$) Obvious since the trivial topology is zero dimensional while the infratopologies are not.

($\vee$) The property is preserved under arbitrary joins since each topology in the arbitrary family has a base of open-closed sets for each point in E and a base in the join for each point consists of finite intersections of these open-closed sets and is hence open-closed.

<u>countability axioms</u> :-

($\not\wedge$) The meet of two topologies satisfying the second axiom of countability need not satisfy the first axiom of countability as illustrated by the following. Let E be the reals and t_1 the usual topology. Let t_2 be the set consisting of the empty set together with all subsets A of E such that the complement of A consists of only finitely many rational numbers. Note that t_2 is a filter topology (the Frechet filter of a sequence of all rationals) and as such has a countable base which implies t_2 satisfies the second countability axiom. Let $t = t_1 \wedge t_2$. Then for any $X \in t$ we have $\complement X$ is countable and contains no interval. Let $p \in E$ and $(B_i : i \in I)$ be a base of open neighbourhoods of p. Now t is a T_1-topology since both t_1 and t_2 are. Hence $\{p\} = \bigcap_{i \in I} B_i$ and so $E - \{p\} = \bigcup_{i \in I} \complement B_i$. But $E - \{p\}$ is uncountable and $\complement B_i$ is countable for each $i \in I$ and hence $|I|$ is uncountable. Therefore there is no countable base for the neighbourhood system and hence the first axiom of countability does not hold.

$(\not\geqq)$ since infratopologies satisfy both countability axioms.

$(\vee)$ since the base for the join is the set of all intersections of base sets from each of the topologies.

$(\not\bigvee)$ since infratopologies satisfy both countability axioms.

<u>separable</u> :-

$(\leqq)$ since closure in a coarser topology contains the closure in a finer topology.

$(\not\vee)$ Finite join does not preserve separability as illustrated by taking the join of the left-half-open and right-half-open interval topologies on the reals. This join is the discrete topology which is not separable but each of the half-open interval topologies are separable since the rationals are dense in each.

<u>principal</u> :-

$(\not\leqq)$ since the discrete topology is principal.

$(\bigwedge)$ lemma 1.3

$(\vee)$ theorem 1.6

$(\not\bigvee)$ since every infratopology is principal.

<u>door</u> :-

$(\not\wedge)$ for example $C(x,U) \wedge C(y,V)$, $x \nparallel y$, $U \nparallel V$. (3.7)

$(\dot{\geqq})$ obvious

Property	$\leq$	$\bigwedge$	$\wedge$	$\geq$	$\bigvee$	$\vee$
T_0	-	-	-	+	+	+
$T_{\frac{1}{2}}$	-	-	-	+	+	+
T_1	-	+	+	+	+	+
T_2	-	-	-	+	+	+
T_3	-	-	-	-	+	+
$T_{3\frac{1}{2}}$	-	-	-	-	+	+
T_4	-	-	-	-	-	-
T_5	-	-	-	-	-	-
regular	-	-	-	-	+	+
completely regular	-	-	-	-	+	+
normal	-	-	-	-	-	-
completely normal	-	-	-	-	-	-
compact	+	+	+	-	-	-
paracompact	-	-	-	-	-	-
metacompact	-	-	-	-	-	-
locally compact	-	-	-	-	-	-
Lindelöf	+	+	+	-	-	-
connected	+	+	+	-	-	-
locally connected	-	+	+	-	-	-
totally disconnected	-	-	-	+	+	+
extremally disconnected	-	-	-	-	-	-
zero-dimensionaly	-	-	-	-	+	+
1st axiom of countability	-	-	-	-	-	+
2nd axiom of countability	-	-	-	-	-	+
separable	+	+	+	-	-	-
principal	-	+	+	-	-	+
door	-	-	-	+	+	+

+ indicates that the property is preserved
- indicates that the property is not preserved

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