GENERALIZED FIBRE SPACES

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SCOPE AND CONTENTS: This thesis deals with generalized fibre spaces. It improves upon existing definitions and introduces new ones. It establishes the category of pairs and the category of g.f.s. The relationship between classical fibre spaces and generalized fibre spaces is examined. The induced g.f.s. is defined as well as the concept of section and it is established that the lifting of a fully regular continuous g-function is equivalent to the existence of a section in the induced g.f.s. Finally, the lifting theorem for g.f.s. is stated and proved.

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CHAPTER I

Category of Pairs and Category of Generalized Fibre Spaces

It is the intention in this manuscript to improve existing definitions and theorems on generalized fibre spaces and to develop new concepts required to prove the homotopy lifting theorem for generalized fibre spaces.

R. G. Lintz, in his paper "On Generalized Fibre Spaces" [5], introduced a definition of generalized fibre space. This definition has proven difficult to work with for numerous reasons. Among them are the following. The concept of inverse limit in the definition was difficult to use, with many problems arising when considering other properties, and also it unnecessarily restricted one to compact metric spaces. Group action on the fibres was not permitted under the old definition and hence, for example, the Moebius strip was not a generalized fibre space. Many of the subsequent considerations became trivial under the old definition. In addition, with the new definition, extensions of more of the classical results are possible.

It is with these considerations in mind that we begin anew the study of generalized fibre spaces. Because serious difficulties arise in the more general situation, we shall assume in this manuscript that

all spaces are compact Hausdorff, although it is possible that most of the theorems might be true in a somewhat more general situation such as paracompact Hausdorff.

<u>Definition</u>: Given a topological space X, then a <u>pair</u> $(M, \tilde{\mathcal{V}})$ in X is a topological subspace $M \subseteq X$ and a family $\tilde{\mathcal{V}}$ of collections $\tilde{\delta}$ of open sets of X such that if $V \in \tilde{\delta} \in \tilde{\mathcal{V}}$ then $V \cap M \neq \emptyset$.

When we say simply that $(X, \mathcal{V})$ is a pair we mean that it is a pair in X itself.

<u>Definition</u>: Given a pair $(X, \mathcal{V})$ in a space Y then a <u>subpair</u> $(M, \tilde{\mathcal{V}})$ of $(X, \mathcal{V})$ is a subspace M of X with $\tilde{\mathcal{V}} = \{\tilde{\delta} \mid \delta \in \mathcal{V}\}$ where $\tilde{\delta} = \{A \mid A \in \delta, A \cap M \neq \emptyset\}$.

<u>Definition</u>: A <u>g-function</u> is a mapping $f: (M, \tilde{\mathcal{V}}) \longrightarrow (M', \tilde{\mathcal{V}}')$ where M' is a subspace of a topological space X' and M is a subspace of a topological space X as above. More specifically f is a family of mappings $f_{\tilde{\mathcal{V}}}$ and $(f_{\tilde{\delta}})_{\tilde{\delta} \in \tilde{\mathcal{V}}}$ where $f_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \longrightarrow \tilde{\mathcal{V}}'$ and $f_{\tilde{\delta}}: \tilde{\delta} \longrightarrow \tilde{\delta}' = f_{\tilde{\mathcal{V}}}(\tilde{\delta})$ where $\tilde{\delta} \in \tilde{\mathcal{V}}$ and $\tilde{\delta}' \in \tilde{\mathcal{V}}'$.

<u>Definition</u>: A g-function is said to be <u>one-to-one (1-1)</u> iff $f_{\tilde{\mathcal{V}}}$ and $f_{\tilde{\delta}}$, for all $\tilde{\delta} \in \tilde{\mathcal{V}}$, are 1-1.

<u>Definition</u>: A g-function is said to be <u>onto</u> iff $f_{\tilde{\mathcal{V}}}$ and $f_{\tilde{\delta}}$, for all $\tilde{\delta} \in \tilde{\mathcal{V}}$, are onto.

<u>Definition</u>: A g-function is <u>continuous</u> if for all $\tilde{\delta}$, $\tilde{\lambda} \in \tilde{\mathcal{V}}$ such that $\tilde{\lambda}$ refines $\tilde{\delta}$ $(\tilde{\delta} \leq \tilde{\lambda})$ and for all $A \in \tilde{\delta}$, $B \in \tilde{\lambda}$ with $B \subseteq A$ we have $f_{\tilde{\lambda}}(B) \subseteq f_{\tilde{\delta}}(A)$.

In this work we will be dealing mainly with topological spaces and coverings. Thus a pair, for the most part, will be a topological space X together with a family of coverings $\mathcal{V}$. The definition of continuity is as above in the particular case that $M = X$ and $\tilde{\mathcal{V}} = \mathcal{V}$. If we have a continuous g-function $f\colon (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$ we will have need in the future to talk about the restriction of f to a subspace M of X.

<u>Definition</u>: Given a g-function $f\colon (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$, we define f^M to be the <u>restriction of f</u> to the subspace M of X where

$$f^M\colon (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}') \text{ given by}$$

$$f^M_{\mathcal{V}}(\delta) = f_{\mathcal{V}}(\delta)$$

$$f^M_{\delta}(A) = \begin{cases} f_{\delta}(A) & \text{if } A \cap M \neq \emptyset \\ \emptyset & \text{otherwise .} \end{cases}$$

From the above it is clear that one tacitly assumes every $\delta \in \mathcal{V}$, for any $\mathcal{V}$, possesses the empty set as a member.

If $\mathcal{V}$ is a family of coverings then $\tilde{\mathcal{V}} = \{\tilde{\delta} \mid \delta \in \mathcal{V}\}$, where $\tilde{\delta} = \{A \mid A \in \delta \in \mathcal{V},\ A \cap M \neq \emptyset\}$, is again a family of coverings in the sense that $M \subseteq \bigcup \tilde{\delta}$ for any $\tilde{\delta} \in \tilde{\mathcal{V}}$. It is obvious that the restriction of a continuous g-function is again continuous.

<u>Definition</u>: Given a pair $(X, \mathcal{V})$, a subpair $(M, \tilde{\mathcal{V}})$ of $(X, \mathcal{V})$, a continuous g-function $\tilde{f}$: $(M, \tilde{\mathcal{V}}) \longrightarrow (Y, \mathcal{V}')$ and an arbitrary covering α of Y, then we say that $\tilde{f}$ has an <u>α-extension</u> f: $(X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$ iff

1) for all $\delta \in \mathcal{V}$ and for all $A \in \delta$ such that $A \cap M \neq \emptyset$ we have $\tilde{f}_{\tilde{\delta}}(A) = f_{\delta}(A)$.

2) there exists $\delta \in \mathcal{V}$ such that $\tilde{f}_{\tilde{\delta}}(\tilde{\delta}) \geq \alpha$ and then $f_{\delta}(\delta) \geq \alpha$.

If only condition 1) is true, we say that f is an <u>extension</u> of $\tilde{f}$.

We can define a category of pairs in which the objects are all pairs $(M, \tilde{\mathcal{V}})$ in X for all possible topological spaces X and the morphisms are continuous g-functions f: $(M, \tilde{\mathcal{V}}) \longrightarrow (N, \tilde{\mathcal{V}}')$ such that f_{δ} is onto for all $\delta \in \tilde{\mathcal{V}}$. The requirement that each f_{δ} is onto is necessary to obtain the result that the composition of two morphisms is again a morphism.

<u>Definition</u>: We will say that two pairs $(M, \tilde{\mathcal{V}})$ and $(M', \tilde{\mathcal{V}}')$ are isomorphic iff there exist continuous g-functions f: $(M, \tilde{\mathcal{V}}) \longrightarrow (M', \tilde{\mathcal{V}}')$ and g: $(M', \tilde{\mathcal{V}}') \longrightarrow (M, \tilde{\mathcal{V}})$ such that $g \circ f$ is the identity on $(M, \tilde{\mathcal{V}})$ and $f \circ g$ is the identity on $(M', \tilde{\mathcal{V}}')$.

<u>Notation</u>: $(M, \tilde{\mathcal{V}}) \approx (M', \tilde{\mathcal{V}}')$ iff $(M, \tilde{\mathcal{V}})$ and $(M', \tilde{\mathcal{V}}')$ are isomorphic.

<u>Lemma 1.1</u>: $(M, \tilde{\mathcal{V}})$ and $(M', \tilde{\mathcal{V}}')$ are isomorphic iff there exists a continuous g-function h: $(M, \tilde{\mathcal{V}}) \longrightarrow (M', \tilde{\mathcal{V}}')$ such that h is 1-1, onto and h^{-1} is also continuous.

<u>Proof</u>: Given such an h, define maps $f = h$ and $f' = h^{-1}$. Then it is clear that $f \circ f' = id_{(M', \tilde{\mathcal{V}}')}$ and $f' \circ f = id_{(M, \tilde{\mathcal{V}})}$. Conversely, assuming they are isomorphic we have

$g_{\tilde{\mathcal{V}}'} \circ f_{\tilde{\mathcal{V}}}(\tilde{\delta}) = \tilde{\delta}$ which implies $f_{\tilde{\mathcal{V}}}$ is 1-1 and similarily $f_{\tilde{\delta}}$ is 1-1 for all $\tilde{\delta} \in \tilde{\mathcal{V}}$

$f_{\tilde{\mathcal{V}}} \circ g_{\tilde{\mathcal{V}}'}(\tilde{\delta}') = \tilde{\delta}'$ which implies $g_{\tilde{\mathcal{V}}'}$ is 1-1 and similarily $g_{\tilde{\delta}'}$ is 1-1 for all $\tilde{\delta}' \in \tilde{\mathcal{V}}'$.

It is obvious that f and g are onto maps and hence $f^{-1} = g$ and the proof of the lemma is complete.

<u>Definition</u>: A continuous g-function $f: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$ is <u>cofinal</u> iff $f_{\mathcal{V}}(\mathcal{V})$ is cofinal in $\mathcal{V}'$, i.e., given any $\delta' \in \mathcal{V}'$ there exists a $\lambda' \in f_{\mathcal{V}}(\mathcal{V})$ such that $\delta' \leq \lambda'$.

<u>Definition</u>: A g-function $f: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$ is <u>regular</u> iff for all $A, B \in \delta \in \mathcal{V}$ such that $A \cap B \neq \emptyset$ then $f_\delta(A) \cap f_\delta(B) \neq \emptyset$.

<u>Definition</u>: A g-function $f: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$ is <u>fully regular</u> iff for all $A \in \delta \in \mathcal{V}$, $B \in \lambda \in \mathcal{V}$ such that $A \cap B \neq \emptyset$ then $f_\delta(A) \cap f_\lambda(B) \neq \emptyset$.

<u>Definition</u>: A g-function $f: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$ <u>generates</u> the function $\varphi : X \longrightarrow Y$ iff for any $x \in X$ and any neighbourhood N of $\varphi(x)$ there exists $\delta \in \mathcal{V}$, $A \in \delta$ such that $x \in A$, $\varphi(x) \in \overline{f_\delta(A)}$ and $f_\delta(A) \subseteq N$.

It has been shown by A. Jansen [3] that there is a relation between usual continuous functions and continuous g-functions. Specifically, he has shown the following two results.

1) If X is regular and X' is compact T_2, $f: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$, $\mathcal{V}$ and $\mathcal{V}'$ are cofinal in the set of all coverings of X and X' respectively, f is continuous and cofinal then f generates a continuous

function $\varphi : X \longrightarrow X'$.

2) If X and X' are topological spaces and $\varphi : X \longrightarrow X'$ a continuous function onto X' then there exists a g-function $f: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$ which generates φ.

A <u>generalized arc</u>, abbreviated g.a., is a connected, locally connected space π, irreducibly connected between two points $a, b \in \pi$ called its extremeties. Such a space can be totally ordered such that the order topology is the same as before (Wilder [8]) and so we can use the notation (π, a, b). Immediately, for any $x \in \pi$, we have $a \leq x \leq b$. For example, the interval $[0, 1]$ is a g.a. as well as the space $[0, \Omega]$ where Ω is the first uncountable ordinal. The space $[0, \Omega]$ is obtained in the following manner. Consider the set A of all ordinals from 0 to Ω and let $X = (A - \Omega) \times [0, 1]$. Identify two elements (a, x) and (a', x') of X if $a' = a + 1$ and $x = 1$, $x' = 0$ obtaining a space $\tilde{X}$ and then $[0, \Omega]$ is the one-point compactification of $\tilde{X}$.

A <u>regular subdivision</u> of a g.a. π (also called a regular covering of π for the interval $[0, 1]$) is an open covering of π, with finitely many open sets of the form $c < x < d$ for pairs of points $c, d \in \pi$, of order two. An open covering of π of order two is an open covering such that each point in π lies in at most two of the open sets.

<u>Notation</u>: Given a g-function $f: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$, for any $\delta' \in \mathcal{V}'$ and for all $A' \in \delta'$ $\underline{\delta(A')} = f_{\delta}^{-1}(A')$ which is contained in $\delta \in \mathcal{V}$.

<u>Definition</u>: A <u>g-fibre space</u> (g.f.s.) is a 5-tuple $\mathcal{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ where

1) Γ' is a finite open covering of X' and any covering in $\mathcal{V}'$ refines Γ'

2) $\mathcal{V}_F$ is cofinal, $\mathcal{V}'$ is directed by refinements and there is a function $r: \mathcal{V}' \longrightarrow \mathcal{V}_F$ onto such that for all $\delta', \lambda' \in \mathcal{V}'$ we have $\delta' \leq \lambda'$ iff $r(\delta') \leq r(\lambda')$.

3) p is a continuous g-function, $p: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$, such that $p_{\mathcal{V}}$ is 1-1, onto and p_δ is onto for all $\delta \in \mathcal{V}$.

4) for all $A' \in \delta'$, $A' \subseteq \mathcal{U}' \in \Gamma'$ there exists an isomorphism $i^{\mathcal{U}'}_{\delta(A')} : N_{\delta(A')} \longrightarrow N_{r(\delta')}$ where $N_{\delta(A')}$ is the nerve of $\delta(A')$.

5) for all $A' \in \delta'$, $B' \in \lambda'$, $\delta', \lambda' \in \mathcal{V}'$, $\delta' \leq \lambda'$, $B' \subseteq A' \subseteq \mathcal{U}'$ we have $\delta(A') \leq \lambda(B')$ and for any other $\tilde{A}' \in \delta'$, $\tilde{B}' \in \lambda'$ with $\tilde{B}' \subseteq \tilde{A}' \subseteq \mathcal{U}'$ there exist projections $p^{\lambda(B')}_{\delta(A')}$ and $p^{\lambda(\tilde{B}')}_{\delta(\tilde{A}')}$ such that the following diagram is commutative

$$\begin{tikzcd}
N_{\delta(A')} \arrow[r, "i^{\mathcal{U}'}_{\delta(A')\delta(\tilde{A}')}"] & N_{\delta(\tilde{A}')} \\
N_{\lambda(B')} \arrow[u, "p^{\lambda(B')}_{\delta(A')}"] \arrow[r, "i^{\mathcal{U}'}_{\lambda(B')\lambda(\tilde{B}')}"'] & N_{\lambda(\tilde{B}')} \arrow[u, "p^{\lambda(\tilde{B}')}_{\delta(\tilde{A}')}"']
\end{tikzcd}$$

where $i^{\mathcal{U}'}_{\delta(A')\delta(\tilde{A}')} = \left(i^{\mathcal{U}'}_{\delta(\tilde{A}')}\right)^{-1} \circ i^{\mathcal{U}'}_{\delta(A')}$

$(X, \mathcal{V})$ is called the total pair.

$(X', \mathcal{V}')$ is called the base pair.

$(F, \mathcal{V}_F)$ is called the g-fibre

p is called the projection.

<u>Definition</u>: A <u>locally trivial</u> g-fibre space is a g-fibre space $\mathfrak{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ such that for all $\mathcal{U}' \in \Gamma'$ there exists a subset $X_{\mathcal{U}'} \subseteq X$ such that $(X_{\mathcal{U}'}, \tilde{\mathcal{V}}) \approx (\mathcal{U}' \times F, \tilde{\mathcal{V}}' \times \mathcal{V}_F)$.

Before considering locally trivial g-fibre spaces and properties relating thereto let us consider the category of g-fibre spaces.

<u>Category of g-fibre spaces</u>: The objects in this category are the g.f.s. and the morphisms are pairs of continuous g-functions $(f, \bar{f}): \mathfrak{X} \longrightarrow \mathcal{Y}$ such that $\bar{f}_\delta$ and f_δ are onto and the following diagram is commutative.

$$\begin{tikzcd}
(X, \mathcal{V}) \arrow[r, "f"] \arrow[d, "p"'] & (Y, \mathcal{W}) \arrow[d, "q"] \\
(X', \mathcal{V}') \arrow[r, "\bar{f}"] & (Y', \mathcal{W}')
\end{tikzcd}$$

This is a "fibre preserving" map.

$\mathfrak{X}$ and $\mathcal{Y}$ are said to be <u>isomorphic</u> iff there exist morphisms $(f, \bar{f})$ and $(g, \bar{g})$ such that

$$(f, \bar{f}): \mathfrak{X} \longrightarrow \mathcal{Y} \qquad (g, \bar{g}): \mathcal{Y} \longrightarrow \mathfrak{X} \text{ and}$$

$$(g, \bar{g}) \circ (f, \bar{f}) = id_{\mathfrak{X}} \qquad (f, \bar{f}) \circ (g, \bar{g}) = id_{\mathcal{Y}} .$$

For the sake of simplicity the morphisms will often be referred to as f or g instead of the pairs $(f, \bar{f})$ and $(g, \bar{g})$ when there is no possibility of confusion. That g.f.s. and morphisms so defined form a category is easily seen.

<u>Lemma 1.2</u>: The isomorphisms in the category of g.f.s. are precisely the pairs of continuous g-functions $(f, \bar{f})$ where $f_{\mathcal{V}}$, $\bar{f}_{\mathcal{V}'}$, f_δ, $\bar{f}_{\delta'}$ are all 1-1, onto and f^{-1} and $\bar{f}^{-1}$ are continuous.

Proof: The proof of this lemma is an immediate consequence of lemma 1.1.

We now turn to a necessary condition that a g.f.s. be locally trivial. It is this condition which Lintz used to define local triviality in [5].

Theorem 1.3: Given a g.f.s. $\mathcal{X}$ such that for all $\mathcal{U}' \in \Gamma'$ there exists a closed subset $X_{\mathcal{U}'} \subseteq X$ together with a family of coverings $\bar{\mathcal{V}} = X_{\mathcal{U}'} \cap \mathcal{V} = \{\bar{\delta} \mid \delta \in \mathcal{V}\}$ where $\bar{\delta} = \{A \cap X_{\mathcal{U}'} \mid A \in \delta\}$. such that

1) there exists a restriction of p to $(X_{\mathcal{U}'}, \bar{\mathcal{V}})$ and this restriction agrees on intersections, i.e.

$$p \mid (X_{\mathcal{U}'}, \bar{\mathcal{V}}) = \bar{p}: (X_{\mathcal{U}'}, \bar{\mathcal{V}}) \longrightarrow (\bar{\mathcal{U}}', \bar{\mathcal{V}}')$$

and if $\mathcal{U}'$ and $\mathcal{U}_0' \in \Gamma'$ and $\bar{p}$ and $\bar{q}$ are restrictions then if $A \subseteq X_{\mathcal{U}'} \cap X_{\mathcal{U}_0'}$ and $A \in \bar{\delta} \cap \bar{\delta}_0$, $\bar{\delta} \in \bar{\mathcal{V}}$, $\bar{\delta}_0 \in \bar{\mathcal{V}}_0$ we have $\bar{p}_{\bar{\delta}}(A) = \bar{q}_{\bar{\delta}_0}(A)$.

2) $A' \in \delta' \in \mathcal{V}'$ with $\emptyset \neq \bar{A}' = \bar{\mathcal{U}}' \cap A \in \bar{\delta}' \in \bar{\mathcal{V}}'$ implies that $\delta(\bar{A}') = X_{\mathcal{U}'} \cap \delta(A')$ is equal to $\bar{p}_{\bar{\delta}}^{-1}(\bar{A}')$ and its nerve is isomorphic to $N_{\delta(A')}$ by inclusion.

i.e. $A \rightsquigarrow X_{\mathcal{U}'} \cap A$ $\quad A \in \delta(A')$ induces an isomorphism

$N_{\delta(A')}$ onto $N_{\delta(\bar{A}')}$

then, $\mathcal{X}$ is locally trivial.

Remark: Due to 2) above we will identify $N_{\delta(A')}$ and $N_{\delta(\bar{A}')}$.

Proof: It possibly should be noted here that for all $\delta \in \mathcal{V}$ there is associated a unique $\delta' \in \mathcal{V}'$ and a unique $\delta_F \in \mathcal{V}_F$.

For any $\bar{A} \in \bar{\delta} \in \bar{\mathcal{V}}$ there exists $A \in \delta \in \mathcal{V}$ such that $A \cap X_{\mathcal{U}'} = \bar{A}$. Then for $\tilde{\delta} = \{A \mid A \in \delta, A \cap X_{\mathcal{U}'} \neq \emptyset\}$ we have $\tilde{\delta} \cap X_{\mathcal{U}'} = \bar{\delta}$. Similarily $\tilde{\delta}' \cap \bar{\mathcal{U}}_1 = \bar{\delta}'$. Define $\mathcal{W} = \{\tilde{\delta}' \times \delta_F \mid \tilde{\delta}' \in \tilde{\mathcal{V}}'\}$, $\tilde{\delta}' = \bar{\mathcal{U}}' \cap \delta'$ for some $\mathcal{U}' \in \Gamma'$.

Define $f: (X_{\mathcal{U}'}, \tilde{\mathcal{V}}) \longrightarrow (\bar{\mathcal{U}}' \times F, \mathcal{W})$ by

$f_{\tilde{\mathcal{V}}}(\tilde{\delta}) = \tilde{\delta}' \times \delta_F$

$f_{\tilde{\delta}}: \tilde{\delta} \longrightarrow \tilde{\delta}' \times \delta_F$ by $f_{\tilde{\delta}}(\tilde{A}) = \tilde{A}' \times i^{\mathcal{U}'}_{\bar{\delta}(\bar{A}')}(\bar{A})$ for all $\tilde{A} \in \tilde{\mathcal{V}}$

where $\tilde{A}' \cap \bar{\mathcal{U}}' = \bar{p}_{\bar{\delta}}(\bar{A})$.

f is continuous:

$\tilde{\delta}, \tilde{\lambda} \in \tilde{\mathcal{V}}$ such that $\tilde{\delta} \leq \tilde{\lambda}$ and $\tilde{A} \in \tilde{\delta}$, $\tilde{B} \in \tilde{\lambda}$ such that $\tilde{B} \subseteq \tilde{A}$

$$\bar{p}_{\bar{\lambda}}(\bar{B}) \times i^{\mathcal{U}'}_{\bar{\lambda}(\bar{B}')} \subseteq \bar{p}_{\bar{\delta}}(\bar{A}) \times i^{\mathcal{U}'}_{\bar{\delta}(\bar{A}')}(\bar{A})$$

and hence $f_{\tilde{\lambda}}(\tilde{B}) \subseteq f_{\tilde{\delta}}(\tilde{A})$

f is 1-1:

$f_{\tilde{\mathcal{V}}}$ is obviously 1-1. Take any $\tilde{\delta} \in \tilde{\mathcal{V}}$, $\tilde{A}, \tilde{B} \in \tilde{\delta}$, $\tilde{A} \neq \tilde{B}$. Now if $\bar{p}_{\bar{\delta}}(\bar{A}) \neq \bar{p}_{\bar{\delta}}(\bar{B})$ then $f_{\tilde{\delta}}(\tilde{A}) \neq f_{\tilde{\delta}}(\tilde{B})$. If $\bar{p}_{\bar{\delta}}(\bar{A}) = \bar{p}_{\bar{\delta}}(\bar{B})$ then since $\tilde{A} \neq \tilde{B}$ we have $i^{\mathcal{U}'}_{\bar{\delta}(\bar{A}')}(\bar{A}) \neq i^{\mathcal{U}'}_{\bar{\delta}(\bar{B}')}(\bar{B})$ and hence $f_{\tilde{\delta}}(\tilde{A}) \neq f_{\tilde{\delta}}(\tilde{B})$.

f is onto:

Take any $\delta_w \in \mathcal{W}$. This implies $\delta_w = \tilde{\delta}' \times \delta_F$. Since $\bar{p}$ is onto we have that there exists a $\bar{\delta} \in \tilde{\mathcal{V}}$ such that $\bar{p}_{\bar{\delta}}(\bar{\delta}) = \bar{\delta}'$ and hence $f_{\tilde{\mathcal{V}}}$ is onto. For $A_w \in \delta_w$ we have $A_w = \tilde{A}' \times A_F$. Now there exists $\bar{A} \in \bar{\delta}$ such that $\bar{p}_{\bar{\delta}}(\bar{A}) = \bar{A}'$. Now by the definition of δ_w and A_w the choice of δ_F and A_F are uniquely defined and hence $f_{\tilde{\delta}}(\tilde{A}) = A_w$ and thus $f_{\tilde{\delta}}$ is onto. Therefore f is onto.

f^{-1} is continuous:

Consider $\omega_1, \omega_2 \in \mathcal{W}$ such that $\omega_1 \leq \omega_2$ and $W_1 \in \omega_1$, $W_2 \in \omega_2$ such that $W_2 \subseteq W_1$

$$\omega_1 = \tilde{\delta}_1' \times \delta_{1_F}$$

$$\omega_2 = \tilde{\delta}_2' \times \delta_{2_F}$$

Now $\omega_1 \leq \omega_2$ implies $\tilde{\delta}_1' \leq \tilde{\delta}_2'$ and $\delta_{1_F} \leq \delta_{2_F}$. Also W_1 is the image of some set $\tilde{A}$ under f and similarily W_2 is the image of some set $\tilde{B}$.

$W_2 \subseteq W_1$ gives $\bar{p}_{\bar{\delta}_1}(\bar{B}) \times i^{U'}_{\bar{\delta}(\bar{B}')}(\bar{B}) \subseteq \bar{p}_{\bar{\delta}_2}(\bar{A}) \times i^{U'}_{\bar{\delta}(\bar{A}')}(\bar{A})$ which yields $\tilde{B} \subseteq \tilde{A}$.

Given a locally trivial g.f.s $\mathcal{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ and a fully regular continuous onto g-function $\bar{f}: (Y', \mathcal{W}') \longrightarrow (X', \mathcal{V}')$ we will now define a g-fibre space over $(Y', \mathcal{W}')$ which we will call the induced g.f.s.

<u>Definition</u>: Given a locally trivial g.f.s. $\mathcal{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ and a fully regular continuous onto g-function $\bar{f}: (Y', \mathcal{W}') \longrightarrow (X', \mathcal{V}')$ we define the <u>induced fibre space</u> (with respect to $\bar{f}$ and $\mathcal{X}$) to be $\mathcal{Y} = ((Y, \mathcal{W}), (Y', \mathcal{W}'), q, (F, \mathcal{V}_F), \Delta')$ where

$$D' = \bigcup_{\omega' \in \mathcal{W}'} \bigcup_{\substack{W' \in \omega' \\ \bar{f}(W') \subseteq U'}} W' \quad \text{for some } U' \in \Gamma',$$

$$\Delta' = \{D' \mid U' \in \Gamma'\},$$

$$Y = \bigcup_{D' \in \Delta'} D' \times F.$$

For each $D' \varepsilon \Delta'$ let $\mathcal{W}'_{D'} = \{ \omega'_{D'} \mid \omega' \varepsilon \mathcal{W}' \}$ where

$$\omega'_{D'} = \{ W' \mid W' \varepsilon \omega', W' \cap D' \neq \emptyset \}.$$

Define $\omega = \bigcup_{D' \varepsilon \Delta'} \omega'_{D'} \times \delta_F$ where $r \circ \bar{f}(\omega') = \delta_F$.

Let $\mathcal{W}$ be the set of all such ω. Define

$q: (Y, \mathcal{W}) \longrightarrow (Y', \mathcal{W}')$ by $q_{\mathcal{W}}(\omega) = \omega'$ and

$q_\omega(W) = q_\omega(W' \times A_F) = W'$.

To justify this definition we will prove that $\mathcal{Y} = ((Y, \mathcal{W}), (Y', \mathcal{W}'), q, (F, \mathcal{V}_F), \Delta')$ is indeed a g.f.s., in fact even locally trivial. We shall use the notation $\mathcal{Y} = f^{-1}(\mathcal{X})$.

<u>Theorem 1.4</u>: Given a locally trivial g.f.s. $\mathcal{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ and a fully regular continuous onto g-function $\bar{f}: (Y', \mathcal{W}') \longrightarrow (X', \mathcal{V}')$ then $\mathcal{Y} = f^{-1}(\mathcal{X})$ is a locally trivial g.f.s.

<u>Proof</u>: It is obvious that $q_{\mathcal{W}}$ is 1-1 and that q is continuous and onto. It is clear that $\Delta' \leq \omega'$ for all $\omega' \varepsilon \mathcal{W}'$. That Δ' is a cover follows from the fact that $\Gamma' \leq \delta'$ for all $\delta' \varepsilon \mathcal{V}'$.

It should be noted here that for any $W' \varepsilon \omega' \varepsilon \mathcal{W}'$, if $W' \cap D' \neq \emptyset$ then $\bar{f}_{\omega'}(W') \cap U' \neq \emptyset$. This follows from the definition of D' and the fact that $\bar{f}$ is fully regular. $W' \cap D' \neq \emptyset$ implies that there exists $W'_0 \varepsilon \omega'_0$ such that $W' \cap W'_0 \neq \emptyset$. But this implies that $\bar{f}_{\omega'}(W') \cap \bar{f}_{\omega'_0}(W'_0) \neq \emptyset$ since $\bar{f}$ is fully regular. Now $\bar{f}_{\omega'_0}(W'_0) \subseteq U'$ and hence $W' \cap D' \neq \emptyset$ implies $\bar{f}_{\omega'}(W') \cap U' \neq \emptyset$.

It might be worthwhile to point out that one tacitly assumes in the above statement that D' and U' correspond to one another as per the definition of D'.

Define f by $f_{\mathcal{W}}(\omega) = p^{-1}_{\mathcal{V}'} \circ \bar{f}_{\mathcal{W}'} \circ q_{\mathcal{W}}(\omega)$

$$f_\omega(W) = f_\omega(W' \times A_F) = \mathrm{iso}^{U'}_{\delta' \times \delta_F}(\bar{f}_{\omega'}(W') \times A_F)$$

where $\bar{f}_{\omega'}(W') \subseteq U' \in \Gamma'$

and $\mathrm{iso}^{U'}$ is the isomorphism one has since $\mathcal{X}$ is locally trivial, i.e.,

$$\mathrm{iso}^{U'}: (X_{U'}, \tilde{\mathcal{V}}) \longrightarrow (U' \times F, \tilde{\mathcal{V}}' \times \mathcal{V}_F).$$

It is obvious that f so defined is continuous and hence by the construction of f we have that the following diagram is commutative.

$$\begin{tikzcd}
(Y, \mathcal{W}) \arrow[r, "f"] \arrow[d, "q"'] & (X, \mathcal{V}) \arrow[d, "p"] \\
(Y', \mathcal{W}') \arrow[r, "\bar{f}"'] & (X', \mathcal{V}')
\end{tikzcd}$$

Hence f is a "fibre preserving" map.

Define $r': \mathcal{W}' \longrightarrow \mathcal{V}_F$ by $r' = r \circ \bar{f}_{\mathcal{W}'}$

$$q_\omega^{-1}(W') = \{ W' \times A_F \mid A_F \in \delta_F \}.$$

It is clear then that $N_{\omega(W')}$ is isomorphic in a natural way to $N_{r'(\omega')} = N_{\delta_F}$ and it is this natural isomorphism which we shall use.

Consider $\omega_1', \omega_2' \in \mathcal{W}'$ such that $\omega_1' \leq \omega_2'$ and sets $W_2' \in \omega_2'$ and $W_1' \in \omega_1'$ such that $W_2' \subseteq W_1'$. Then there exists a set $D' \in \Delta'$ such that $W_1' \subseteq D'$ and hence $W_2' \subseteq D'$.

$$\omega_1(W_1') = W_1' \times \delta_F \quad \text{where} \quad r'(\omega_1') = \delta_F$$

$$\omega_2(W'_2) = W'_2 \times \lambda_F \quad \text{where} \quad r'(\omega'_2) = \lambda_F$$

Now $\omega'_1 \leq \omega'_2$ implies that $\delta_F \leq \lambda_F$ and hence clearly we have $\omega_1(W'_1) \leq \omega_2(W'_2)$.

Thus we have that $\mathcal{Y} = ((Y, \mathcal{W}), (Y', \mathcal{W}'), q, (F, \mathcal{V}_F), \Delta')$ is a g.f.s.

We now will show that $\mathcal{Y}$ is also locally trivial.

Define $Y_{D'} = Y - \bigcup_{\omega \varepsilon \mathcal{W}} \bigcup_{\substack{W \varepsilon \omega \\ q_\omega(W) \subseteq Y'-D'}} W$.

Now $W \varepsilon \omega$ with $W \cap Y_{D'} \neq \emptyset$ implies that $q_\omega(W) \cap D' \neq \emptyset$. The proof is obvious since if one assumes $q_\omega(W) \cap D' = \emptyset$ then one has $q_\omega(W) \subseteq Y' - D'$ which yields $W \subseteq Y - Y_{D'}$ and hence $W \cap Y_{D'} = \emptyset$ which contradicts the assumption. Therefore, it is obvious that $(Y_{D'}, \tilde{\mathcal{W}}) \approx (D' \times F, \tilde{\mathcal{W}}' \times \mathcal{V}_F)$ with the isomorphism being the natural isomorphism.

It is natural that $\mathcal{Y}$ should be locally trivial as this was indicated by its construction.

Remark: It is quite probable that the definition of induced fibre space can be stated without assuming that $\mathcal{X}$ is locally trivial or that $\bar{f}$ is fully regular but at the present time the technical details of such a definition have not been worked out. When we will be using the induced fibre space in this paper we will have that $\mathcal{X}$ is locally trivial and that $\bar{f}$ is fully regular.

CHAPTER II

Relation Between Classical and Generalized Fibre Spaces

We will now consider the position of fibre-spaces in the usual sense among the generalized fibre spaces. The definition of locally trivial fibre space we use is the following: -

$(X, X', \varphi, F, \Gamma')$ is a locally trivial fibre space where X, X', F are topological spaces such that

(i) φ is a continuous function from X onto X' and $\varphi^{-1}(x')$ is homeomorphic to F.

(ii) Γ' is an open covering of Y such that to each $U' \varepsilon \Gamma'$ there exists a homeomorphism $\psi : U' \times F \longrightarrow \varphi^{-1}(U')$ with $\psi(x', a) \varepsilon \varphi^{-1}(x')$ if $x' \varepsilon U'$, $a \varepsilon F$. (Hilton [2], pg. 46)

Before we begin, it might be useful to make the following observation.

Proposition 2.1: Given a cofinal family $\mathcal{V}$ of open coverings of a topological space then for any $\alpha \varepsilon \mathcal{V}$ the family $\mathcal{V}'$ of all $\delta \varepsilon \mathcal{V}$ such that $\delta \geq \alpha$ is cofinal.

Proof: Let $(\mathcal{H}, \leq)$ be the family of all open coverings of a given topological space. Then $(\mathcal{H}, \leq)$ is directed by refinements. Let $\mathcal{V}' = \{\delta \mid \delta \varepsilon \mathcal{V}, \alpha \leq \delta\}$. Let β be any element of $\mathcal{H}$. $\mathcal{V}$ cofinal implies that there exists $\lambda \varepsilon \mathcal{V}$ such that $\beta \leq \lambda$. Now

$\mathcal{W}$ directed by refinements implies that there exists $\mu \in \mathcal{W}$ such that $\alpha \leq \mu$, $\lambda \leq \mu$. $\mathcal{V}$ cofinal implies that there exists $\omega \in \mathcal{V}$ such that $\mu \leq \omega$ which implies $\alpha \leq \omega$ which gives $\omega \in \mathcal{V}'$,

$\beta \leq \lambda \leq \mu \leq \omega$ hence $\beta \leq \omega$.

Therefore $\mathcal{V}'$ is cofinal.

In addition to the classical definition of fibre-space we need a general condition relating coverings of F with coverings of X'.

<u>Definition</u>: A pair of spaces (X, Y) is well related relative to coverings, abbreviated w.r.c., iff there exists two families of coverings $\mathcal{V}_X$ and $\mathcal{V}_Y$, cofinal in X and Y respectively, and a one-to-one function

$$r: \mathcal{V}_X \xrightarrow{\text{onto}} \mathcal{V}_Y \text{ preserving refinements,}$$

i.e., for all $\delta_X \in \mathcal{V}_X$ and $\lambda_X \in \mathcal{V}_X$, $\delta_X \leq \lambda_X$ iff $r(\delta_X) \leq r(\lambda_X)$.

The following lemma shows that important classes of spaces may be w.r.c.

<u>Lemma 2.2</u>: If X and Y are compact metric spaces, then the pair (X, Y) is w.r.c. (Lintz [5]).

<u>Proof</u>: Take $\mathcal{V}_X$ as a cofinal family of finite coverings and we can suppose the coverings of $\mathcal{V}_X$ form a sequence $(\delta_n^X)_{n \in N}$ such that $i \leq j$ implies $\delta_i^X \leq \delta_j^X$. We can also suppose the same for $\mathcal{V}_Y$. Define $r: \mathcal{V}_X \longrightarrow \mathcal{V}_Y$ by $r(\delta_i^X) = \delta_i^Y$. This r satisfies the above conditions and hence (X, Y) is w.r.c.

<u>Theorem 2.3</u>: If $\mathcal{F} = (X, X', \varphi, F, \Gamma')$ is a locally trivial classical fibre space with projection $\varphi : X \longrightarrow X'$ and

fibre F then there exists a locally trivial g.f.s. $\mathcal{X} = ((X,\mathcal{V}),(X',\mathcal{V}'),$ $p, (F, \mathcal{V}_F), \Gamma')$ with $\mathcal{V}$ and $\mathcal{V}'$ cofinal in the set of all coverings of X and X' respectively such that for all $A \in \delta \in \mathcal{V}$ we have $p_\delta(A) = \varphi(A)$.

<u>Proof</u>: Let Γ' be the covering of X' as given in the definition of the classical fibre space $\mathcal{F}$. Take cofinal families $\mathcal{V}_F$ and $\mathcal{V}'$ for F and X' respectively and as (F, X') are w.r.c. there exists $r: \mathcal{V}_F \longrightarrow \mathcal{V}'$ 1-1, onto and refinement preserving. We may also assume, since $\mathcal{V}'$ is cofinal, that any set $A' \in \delta' \in \mathcal{V}'$ is contained in some $U' \in \Gamma'$ for all $\delta' \in \mathcal{V}'$. Let $X_{U'}$ be the set given by $\varphi^{-1}(U')$. By the definition of $\mathcal{F}$ there exists a homeomorphism $h: U' \times F \longrightarrow \varphi^{-1}(U')$.

For any $\delta_F \in \mathcal{V}_F$ we have $r(\delta_F) = \delta' \in \mathcal{V}'$ and thus we can cover U' by $\bar{\delta}'$ given by $\bar{\delta}' = U' \cap \delta'$. Let $\bar{\mathcal{V}}' = \{\bar{\delta}' \mid \delta' \in \mathcal{V}'\}$ which is cofinal in the set of all coverings of U' by the definition of $\mathcal{V}_F$ and $\mathcal{V}'$. Let $\mathcal{W} = \{\delta_W = \bar{\delta}' \times \delta_F \mid \delta_F \in \mathcal{V}_F\}$. Then $\mathcal{W}$ is cofinal in the set of all coverings of $U' \times F$. It is clear that if $U'_0 \in \Gamma'$ is such that $U'_0 \cap U' \neq \emptyset$ then $\bar{\mathcal{V}}'$ and $\bar{\mathcal{V}}'_0$ agree on intersections, i.e. $(U'_0 \cap U') \cap \bar{\mathcal{V}}' = (U'_0 \cap U') \cap \bar{\mathcal{V}}'_0$.

Let $\bar{\mathcal{V}} = h(\mathcal{W})$ which is a family of coverings in $\varphi^{-1}(U')$. Let $\mathcal{V} = \{\delta \mid \delta = \bigcup_{U' \in \Gamma'} \bar{\delta}$ determined by a fixed $\delta_F \in \mathcal{V}_F\}$. It is easy to see that $\mathcal{V}$ is cofinal in X.

It might be pointed out here that every $A \in \delta$ is the image under some homeomorphism of a set $A' \times A_F$, $A' \in \delta'$, $A_F \in \delta_F$. Now for all $\delta \in \mathcal{V}$ we have that δ is determined by some $\delta_F \in \mathcal{V}_F$

and by virtue of $r: \mathcal{V}_F \longrightarrow \mathcal{V}'$ we have $r(\delta_F) = \delta'$. Hence we can define a g-function $p: (X, \mathcal{V}) \longrightarrow (X', \mathcal{V}')$ as follows:

$$p_{\mathcal{V}}(\delta) = \delta'$$

$$p_\delta(A) = A' \quad \text{where } A' \in \bar{\delta}' \in \bar{\mathcal{V}} \text{ for some } U' \in \Gamma'.$$

Now p is continuous and $p_\delta(A) = \varphi(A)$ for all $A \in \delta \in \mathcal{V}$ is easily seen since φ is fibre preserving.

We now wish to show $\mathfrak{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ is a locally trivial g.f.s. That $\mathcal{V}'$ is directed by refinements and that p is onto and continuous are trivial since these follow from the construction. For all $A' \in \delta'$, $A' \subseteq U' \in \Gamma'$ there exists isomorphism onto

$$i^{U'}_{\delta(A')} : N_{\delta(A')} \longrightarrow N_{\delta_F}$$

since $p_\delta^{-1}(A') = \{ h_{\delta_W}(A' \times S) \mid S \in \delta_F \}$.

Therefore $N_{\delta(A')} = p_\delta^{-1}(A') \approx N_{\delta_F}$

Now for all $A' \in \delta'$, $B' \in \lambda'$; $\delta', \lambda', \in \mathcal{V}'$, $\delta' \leq \lambda'$, $B' \subseteq A' \subseteq U'$ we have $T \in \lambda(B') = B' \times \lambda_F$ implies $T = B' \times B_F$, $B_F \in \lambda_F$. Any $T \in \lambda(B')$, since $T = B' \times B_F$, is contained in $A' \times B_F$. Now since $\delta' \leq \lambda'$ we have $\delta_F \leq \lambda_F$ and thus there exists an $A_F \in \delta_F$ such that $B_F \subseteq A_F$. Hence $A' \times B_F \subseteq A' \times A_F \in \delta(A')$. Therefore $\delta(A') \leq \lambda(B')$.

Now $N_{\delta(A')} = A' \times \delta_F$ and $N_{\lambda(B')} = B' \times \lambda_F$ and there exists the natural projection from λ_F to δ_F which gives the natural projection $N_{\lambda(B')}$ to $N_{\delta(A')}$. The remainder of condition 5 in the

definition is easy to see. That $\mathcal{X}$ is locally trivial is evident from its construction. This completes the proof of this theorem.

As a consequence of this theorem we can "identify" a usual fibre space $\mathcal{F}$ with a g.f.s. $\mathcal{X}$ obtained from it as above, in the class of compact metric spaces. In this sense the g.f.s. are generalizations of the usual fibre spaces. Also it can be proved that if we go from $\mathcal{F}$ to $\mathcal{X}$, as above, actually $\varphi: X \longrightarrow Y$ is the function generated by p in the sense as defined earlier.

We shall now turn to the converse problem, i.e., under what conditions can we obtain a usual fibre space $\mathcal{F}$ from a given g.f.s. $\mathcal{X}$.

<u>Theorem 2.4</u>: Given a locally trivial g.f.s. $\mathcal{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ where X, X', F are compact metric spaces and $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{V}_F$ are cofinal, then there exists a classical fibre space $\mathcal{F} = (X, X', \varphi, F, \Gamma')$ where φ is generated by p.

<u>Proof</u>: Since $\mathcal{V}$ and $\mathcal{V}'$ are cofinal and p is a cofinal g-function because it is onto then p generates a continuous function $\varphi: X \longrightarrow X'$.

(a) <u>φ is onto</u>: Let y be any point in X'. Let $\{V_i(y), i = 1, 2, 3, \ldots\}$ be a sequence of open balls containing y with radius 1/i. Now since $\mathcal{V}'$ is cofinal, for all $V_i(y)$ there exists $\delta' \varepsilon \mathcal{V}'$ (denoted by δ_i') such that there exists open sets A_i' with $y \varepsilon A_i' \subseteq V_i(y)$. Now consider the corresponding coverings $\delta_i \varepsilon \mathcal{V}$ and the sets $p^{-1}_{\delta_i}(A_i')$.

We have $i \leq j$ implies $\delta(A'_i) \leq \delta(A'_j)$ because $V_j(y) \subseteq V_i(y)$ and we can choose $A'_j \subseteq A'_i$. We even have $\bar{A}'_j \subseteq A'_i$ since X' is a regular space. Therefore for $A_j \in \delta(A'_j)$ there exists $A_i \in \delta(A'_i)$ with $A_j \subseteq A_i$ and again $\bar{A}_j \subseteq A_i$ even. Thus we can define by induction a sequence $(A_k)_{k \in N}$ such that $\bigcap_{k=1}^{\infty} A_k \neq \emptyset$ and also for the corresponding A'_k, $\bigcap_{k=1}^{\infty} A'_k = \{y\}$. Therefore for any $x \in \bigcap_{k=1}^{\infty} A_k$ by the definition of φ we have $\varphi(x) = y$ and thus φ is onto.

(b) As $\mathfrak{X}$ is a locally trivial g.f.s. there exists for all $U' \in \Gamma'$ an isomorphism $h: (X_{U'}, \tilde{\mathcal{V}}) \longrightarrow (U' \times F, \mathcal{W})$. Now $\tilde{\mathcal{V}}$, $\mathcal{W}$ and h are cofinal and hence there exists continuous function ψ generated by h, $\psi: X_{U'} \longrightarrow U' \times F$.

(c) Consider $x, y \in X_{U'}$, $x \neq y$. Then since X is T_2 we have open sets $V(x)$ and $V(y)$ such that $V(x) \cap V(y) = \emptyset$. $\tilde{\mathcal{V}}$ cofinal implies that there exists $\tilde{\delta} \in \tilde{\mathcal{V}}$ such that there are sets $A, B \in \tilde{\delta}$ with $x \in A \subseteq V(x)$ and $y \in B \subseteq V(y)$. Since h is an isomorphism $h_{\tilde{\delta}}(A) \cap h_{\tilde{\delta}}(B) = \emptyset$ and hence $\psi(x) \neq \psi(y)$.

(d) Let y be any point of $U' \times F$. Then there exists a sequence of neighbourhoods of y, $\{V_i(y)\}$, each belonging to some $\delta^i_w \in \mathcal{W}$ such that $\bigcap_{i=1}^{\infty} V_i(y) = \{y\}$ and $V_i(y) \supseteq \overline{V_{i+1}(y)}$. Consider the open sets $A_i = h^{-1}_{\delta^i_w}[V_i(y)] \in \tilde{\delta}_i \in \tilde{\mathcal{V}}$. Now $i < j$ implies that $A_i \supset A_j$ and since compact metric $A = \bigcap_{i=1}^{\infty} A_i \neq \emptyset$ which yields the existence of $x \in A$ and $\psi(x) = y$.

Therefore, since ψ is continuous, 1-1, onto we have ψ is a homeomorphism onto. It remains to show that $\varphi^{-1}(y)$ is homeomorphic with F. The proof here is similar to the one for ψ. Noting $y \in A' \in \delta' \in \mathcal{V}'$ gives $\delta(A') = p_\delta^{-1}(A')$. We note also that $\varphi^{-1}(y) = \bigcap \delta(A')$ and hence we can define a cofinal family of coverings $\mathcal{V}_y$ for $\varphi^{-1}(y)$ and an isomorphism $g:(\varphi^{-1}(y), \mathcal{V}_y) \longrightarrow (F, \mathcal{V}_F)$. Let $\lambda: \varphi^{-1}(y) \longrightarrow F$ be the function generated by g and as above we find that λ is a homeomorphism onto. Therefore we have that ψ is fibre preserving and thus $\mathcal{F} = (X, X', \varphi, F, \Gamma')$ is a locally trivial classical fibre space.

CHAPTER III

Lifting Theorem for Generalized Fibre Spaces

In this chapter we intend to establish the lifting theorem for generalized fibre spaces.

Definition: A section $s: (Y', \mathcal{W}') \longrightarrow (Y, \mathcal{W})$ is a continuous g-function such that $q \circ s = id_{\mathcal{W}'}$.

Now it is always true that $q_{\mathcal{W}} \circ s_{\mathcal{W}'}$ = identity on $\mathcal{W}'$ since q is one-to-one and onto but the problem arises when one tries to define $s_{\omega'}(W')$ so that $q_{\omega} \circ s_{\omega'}(W') = W'$. Hence we might have the case that a section need not exist.

Theorem 3.1: Given a g.f.s. $\mathcal{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ and a continuous fully regular g-fn $\bar{f}: (Y', \mathcal{W}') \xrightarrow{onto} (X', \mathcal{V}')$ then $\bar{f}$ can be lifted to a continuous map $\tilde{f}: (Y', \mathcal{W}') \longrightarrow (X, \mathcal{V})$ iff there exists a section $s: (Y', \mathcal{W}') \longrightarrow (Y, \mathcal{W})$ where $\mathcal{Y} = ((Y, \mathcal{W}), (Y', \mathcal{W}'), q, (F, \mathcal{V}_F), \Delta')$ is the generalized fibre space induced by $\bar{f}$.

Proof: If one has a section $s: (Y', \mathcal{W}') \longrightarrow (Y, \mathcal{W})$ then one can define $\tilde{f}$ by $\tilde{f} = f \circ s$.

Conversely, given $\tilde{f}: (Y', \mathcal{W}') \longrightarrow (X, \mathcal{V})$ we can define s by $s_{\mathcal{W}'} = q_{\mathcal{W}}^{-1}$ and

$$s_{\omega'}(W') = W' \times pr_F\,(iso_{\delta}^{U'} \circ \tilde{f}_{\omega'}(W')) \quad \text{where } \bar{f}_{\omega'}(W') \subseteq U'$$

Consider $\omega', \omega'' \in \mathcal{W}'$ such that $\omega' \leq \omega''$ and sets $W' \in \omega'$ and $W'' \in \omega''$ with $W'' \subseteq W'$. Then

$$s_{\omega''}(W'') = W'' \times pr_F(iso_\lambda^{U'} \circ \tilde{f}_{\omega''}(W''))$$

$$s_{\omega'}(W') = W' \times pr_F(iso_\delta^{U'} \circ \tilde{f}_{\omega'}(W'))$$

and hence, clearly, s is continuous.

Also, $q_{\mathcal{W}} \circ s_{\mathcal{W}'}(\omega') = q_{\mathcal{W}} \circ q_{\mathcal{W}'}^{-1}(\omega') = \omega'$

$$q_\omega \circ s_{\omega'}(W') = q_\omega(W' \times pr_F(iso_\delta^{U'} \circ \tilde{f}_{\omega'}(W'))) = W'.$$

Therefore $q \circ s$ is the identity on $(Y', \mathcal{W}')$ and hence s is a section.

We now wish to define what we mean by homotopy of g-functions [6]. Let Y' and X' be topological spaces and X_0' and X_1' be subspaces of X'. Let $\mathcal{W}_0'$ and $\mathcal{W}_1'$ be families of coverings of Y' and $\mathcal{V}_0'$ and $\mathcal{V}_1'$ families of coverings of X_0' and X_1' respectively. Let

$\bar{f}$: $(Y', \mathcal{W}_0') \longrightarrow (X_0', \mathcal{V}_0')$

$\bar{g}$: $(Y', \mathcal{W}_1') \longrightarrow (X_1', \mathcal{V}_1')$ be two continuous g-functions.

Let α be a covering of X'.

<u>Definition</u>: We define $\bar{f}$ and $\bar{g}$ to be <u>α-homotopic</u> in X' if there exists a generalized arc (π, a, b) and a regular g-function $\bar{H}$: $(Y' \times \pi, \mathcal{M}') \longrightarrow (X', \mathcal{V}')$ such that

1) $\mathcal{W}_0' = \mathcal{M}' \cap Y_0'$ where $Y_0' = Y' \times \{a\}$

$\mathcal{W}_1' = \mathcal{M}' \cap Y_1'$ where $Y_1' = Y' \times \{b\}$

and $\mathcal{W}_0'$ and $\mathcal{W}_1'$ are regarded, in the obvious way, as families of coverings of Y_0' and Y_1'.

2) For any $\mu' \in \mathcal{M}'$ such that $\overline{H}_{\mu'}(\mu') \geq \alpha$ and $M' \in \mu'$ with $M' \cap Y_0' \neq \emptyset$ $[M' \cap Y_1' \neq \emptyset]$ we have $\overline{H}_{\mu'}(M') \cap X_0' = \overline{f}_{\mu' \cap Y_0'}(M' \cap Y_0')$ $[H_{\mu'}(M') \cap X_1' = \overline{g}_{\mu' \cap Y_1'}(M' \cap Y_1')]$.

3) There exists $\mu' \in \mathcal{M}'$ such that $\overline{H}_{\mu'}(\mu') \geq \alpha$.

<u>Notation</u>: $\overline{f} \underset{Y,\alpha}{\sim} \overline{g}$

<u>Definition</u>: We say $\overline{f}$ and $\overline{g}$ are homotopic iff they are α-homotopic for all coverings α of X'. In this case we write $f \underset{Y}{\sim} g$.

It might be useful to point out that condition 3) is essential to guarantee that for some $\mu' \in \mathcal{M}'$ the collection of open sets $\overline{f}_{\mu' \cap Y_0'}(\mu' \cap Y_0')$ and $\overline{g}_{\mu' \cap Y_1'}(\mu' \cap Y_1')$ can be "deformed continuously" into one another through "arbitrarily small" open sets of $\overline{H}_{\mu'}(\mu')$. Without this condition the reader can easily show that any two g-functions would be always homotopic.

In this paper we shall be considering only the special case where $\overline{f}$ and $\overline{g}$ are homotopic and $\mathcal{W}_0' = \mathcal{W}_1'$, $X_0' = X_1' = X'$ and $\mathcal{V}_0' = \mathcal{V}_1'$. That is,

$$\overline{f}, \overline{g} : (Y', \mathcal{W}') \longrightarrow (X', \mathcal{V}')$$

$$\overline{H} : (Y' \times \pi, \mathcal{M}') \longrightarrow (X', \mathcal{V}') \text{ such that}$$

$$\mathcal{M}' \cap (Y' \times \{a\}) = \mathcal{W}' = \mathcal{M}' \cap (Y' \times \{b\}), \text{ etc.}$$

At this point it is necessary to have some preliminary considerations before proving the main theorem.

<u>Definition</u>: An α-homotopy $H: (X \times \pi, \mathcal{W}) \longrightarrow (Y, \mathcal{W}')$ between two continuous g-functions $f, g: (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$ is called a cartesian α-homotopy iff $\mathcal{W} \subseteq \mathcal{V} \times \mathcal{V}_\pi$ where $\mathcal{V}_\pi$ is a family of regular coverings of π. [7]

<u>Definition</u>: Given $(X, \mathcal{V})$ and M a subspace of X, and continuous g-functions $f, g: (M, \mathcal{V}_M) \longrightarrow (Y, \mathcal{V}')$. Assume H is a cartesian α-homotopy between f and g. Suppose f has an extension $\tilde{f}$ to $(X, \mathcal{V})$ and that $M \subseteq V \subseteq X$. We say that H has a <u>V-extension compatible with $\tilde{f}$</u> iff there exists a regular g-function $\tilde{H}: (X \times \pi, \tilde{\mathcal{W}}) \longrightarrow (Y, \mathcal{V}')$ such that

1) for any $\tilde{\delta} \in \tilde{\mathcal{W}}$ and any $\tilde{A} \in \tilde{\delta}$ such that $\tilde{A} \cap (M \times \pi) \neq \emptyset$ we have $\tilde{H}_{\tilde{\mathcal{W}}}(\tilde{\delta}) = H_{\mathcal{W}}(\delta_{\mathcal{W}}) \in \mathcal{V}'$

$$\tilde{H}_{\tilde{\delta}}(\tilde{A}) = H_{\delta_{\mathcal{W}}}(\tilde{A} \cap (M \times \pi))$$

where $\delta_{\mathcal{W}} = \tilde{\delta} \cap (M \times \pi) \in \mathcal{W}$.

2) for any $\tilde{\delta} \in \tilde{\mathcal{W}}$ such that $\tilde{H}_{\tilde{\delta}}(\tilde{\delta}) \geq \alpha$ and any $\tilde{A} \in \tilde{\delta}$ such that $\tilde{A} \cap (X \times \{a\}) \neq \emptyset$ we have $\tilde{H}_{\tilde{\delta}}(\tilde{A}) = f_\delta(A)$ where $A = \tilde{A} \cap (X \times \{a\})$

3) for any $\delta_M \in \mathcal{V}_M$ such that $H_{\delta_{\mathcal{W}}}(\delta_{\mathcal{W}}) \geq \alpha$ and any $\tilde{\delta}$ such that $\delta_{\mathcal{W}} = \tilde{\delta} \cap (M \times \pi) = \delta_M \times \delta_\pi$ we have $\tilde{H}_{\tilde{\delta}}(\tilde{\delta}) \geq \alpha$.

In the above $\tilde{\mathcal{W}}$ is the family of all $\tilde{\delta}$ where $\tilde{\delta}$ is the collection of all sets of $\delta \times \delta_\pi$ which have non-void intersection with $V \times \pi$. If $V = X$ then we speak only of an extension of H. [7]

<u>Lemma 3.2</u>: Given $(X, \mathcal{V})$ and $(Y, \mathcal{V}')$ where X is a normal, connected T_1 space and $\mathcal{V}$ is cofinal in X, let

$f, g: (M, \mathcal{V}_M) \longrightarrow (Y, \mathcal{V}')$ be two continuous g-functions where M is a closed subspace of X. Let α be a given covering of Y and suppose f has an α-extension $\tilde{f}$ to $(X, \mathcal{V})$ and that $H: (M \times \pi, \mathcal{W}) \longrightarrow (Y, \mathcal{W}')$ is a cartesian α-homotopy between them satisfying the conditions

1) H has a V-extension $\tilde{H}$ compatible with $\tilde{f}$, where V is an open subset of X containing M.

2) $\mathcal{W} \subseteq \mathcal{V}_M \times \mathcal{V}_\pi$, where $\mathcal{V}_\pi$ is a family of regular coverings δ_π of π.

3) the sub-family $\hat{\mathcal{V}}_\pi$ of $\mathcal{V}_\pi$ of all δ_π such that $H_{\delta_M \times \delta_\pi}(\delta_M \times \delta_\pi) \geq \alpha$, $\delta_M \in \mathcal{V}_M$ is countable and well ordered by refinement.

Then there exists an extension $\tilde{g}$ of g to $(X, \tilde{\mathcal{V}})$ where $\tilde{\mathcal{V}}$ is a countable sub-family of $\mathcal{V}$ and besides that $\tilde{f} \mid (X, \tilde{\mathcal{V}})$ and $\tilde{g}$ are α-homotopic under an α-homotopy which is an extension of H.

We do not intend to reproduce here the proof of this theorem as given by R. G. Lintz [7].

<u>Theorem 3.3</u>: (Lifting Theorem for g.f.s.)

<u>Conditions A</u>: $\mathcal{X} = ((X, \mathcal{V}), (X', \mathcal{V}'), p, (F, \mathcal{V}_F), \Gamma')$ a locally trivial g.f.s., continuous g-functions $\bar{f}, \bar{g}: (Y', \mathcal{W}') \longrightarrow (X', \mathcal{V}')$ onto, $\mathcal{V}'$ and $\mathcal{W}'$ directed by refinements, $\bar{H}: (Y' \times \pi, \mathcal{M}') \longrightarrow (X', \mathcal{V}')$ a homotopy between $\bar{f}$ and $\bar{g}$, $(\pi, \mathcal{V}_\pi)$ a generalized arc with extremities a and b.

<u>Conditions B:</u> Y' is connected, $\mathcal{W}'$ is cofinal, $\mathcal{V}_\pi$ is a countable family of regular coverings of π well ordered by refinement.

If $\bar{f}$ can be lifted so also can $\bar{g}$ and the lifting of $\bar{H}$ is a homotopy between them.

<u>Proof:</u> Let $\mathcal{E} = ((E, \mathcal{M}), (Y' \times \pi, \mathcal{M}'), q, (F, \mathcal{V}_F), \Delta') = \bar{H}^{-1}(\mathcal{X})$ be the generalized fibre space induced by $\bar{H}$ over $Y' \times \pi$. Now by 1.4 we have that $\mathcal{E}$ is locally trivial. By 3.1 we know that a lifting exists iff $\mathcal{E}$ has a section. Hence we must only show that there exists a section $s: (Y' \times \pi, \mathcal{M}') \longrightarrow (E, \mathcal{M})$. We know that there exists a section s_a over $Y' \times \{a\}$ since $\bar{f}$ can be lifted.

First we will show that for any $y' \varepsilon Y'$ there exists a neighbourhood $U_{y'}$ such that E is trivial over $U_{y'} \times \pi$. Since E is locally trivial we know that for all points (y', p), $p \varepsilon \pi$, there exists a D' such that $(y', p) \varepsilon D'$ and E is trivial over D', i.e., $(E_{D'}, \tilde{\mathcal{M}}) \approx (D' \times F, \tilde{\mathcal{M}}' \times \mathcal{V}_F)$.

The union of all such D' certainly covers $\{(y', p) \mid p \varepsilon \pi\}$. and only a finite number of D' are required. For each D' one can find an interval I of π and an open neighbourhood U of y' such that $U \times I$ is contained in D'. Hence we have a finite number of intervals $I_1, \ldots, I_k$ in π and open neighbourhoods $U_1, \ldots, U_k$ of y such that E is trivial over $U_i \times I_i$. Therefore by induction E is trivial over $\bigcup_{i=1}^{k} U_i \times I_i$. Let $U_y = \bigcap_{i=1}^{k} U_i$ then E is trivial over $U_{y'} \times \pi$. Since any open cover has a finite subcover

because of compactness let $\{ V_i \times \pi \mid i = 1, 2, \ldots, n \}$ be such a finite subcover of $\{ U_{y'} \times \pi \mid y' \in Y' \}$.

Let us consider T any subspace of $Y' \times \pi$ and assume there is a section t defined on $(T, \tilde{\mathcal{M}}')$ into $(E_T, \tilde{\mathcal{M}})$. Further assume E is trivial over T, i.e., $(E_T, \tilde{\mathcal{M}}) \approx (T \times F, \tilde{\mathcal{M}}' \times \mathcal{V}_F)$. Then there exists a continuous g-function $\varphi : (T, \tilde{\mathcal{M}}') \longrightarrow (F, \mathcal{V}_F)$ defined by $\varphi = pr_F \circ t$.

Now, we have a section $s_a : (Y' \times \{a\}, \tilde{\mathcal{M}}') \longrightarrow (E, \mathcal{M})$ and hence there exists a g-function $\varphi : (Y' \times \{a\}, \tilde{\mathcal{M}}') \longrightarrow (F, \mathcal{V}_F)$. We can easily, therefore, extend s_a to a section $s' : (V_1 \times \pi, \tilde{\mathcal{M}}') \longrightarrow (E, \mathcal{M})$ by defining $s'_{\mu'}(M') = i_\mu^{-1}(M' \times \varphi_{\mu'} \circ pr_{Y'}(M'))$ where i is the isomorphism between $(E_{V_1 \times \pi}, \tilde{\mathcal{M}})$ and $((V_1 \times \pi) \times F, \tilde{\mathcal{M}}' \times \mathcal{V}_F)$.

One can assume V_2 intersects V_1. Now since $Y' \times \pi$ is compact T_2 there exists an open set V_{11} such that $V_{11} \subseteq \overline{V}_{11} \subseteq V_1$. Let $C_1 = V_2 \cap \overline{V}_{11}$. Then C_1 is a closed subset of V_2.

The section s_a gives a g-function $\varphi : (Y' \times \{a\}, \tilde{\mathcal{M}}') \longrightarrow (F, \mathcal{V}_F)$ and the section s' gives us the g-function $\psi : (V_1 \times \pi, \tilde{\mathcal{M}}') \longrightarrow (F, \mathcal{V}_F)$ and φ and ψ agree on $(V_1 \times \{a\}, \tilde{\mathcal{M}}')$.

Now let us consider what we have in the light of 3.2. We have φ_{C_1} which is the restriction of φ to C_1 and $\psi_{C_1 \times \pi}$ which is the restriction of ψ to $C_1 \times \pi$. The map $\varphi_{V_1 \cap V_2}$ extends φ_{C_1} to the open set $V_1 \cap V_2$ which contains C_1 and

the map $\psi_{(V_1 \cap V_2) \times \pi}$ is the extension of the homotopy $\psi_{C_1 \times \pi}$ to $(V_1 \cap V_2) \times \pi$. Therefore we may conclude by 3.2 that $\psi_{C_1 \times \pi}$ is extendable to $V_2 \times \pi$ and hence there exists a section s^2 on $V_2 \times \pi$ which agrees with s' on $(V_1 \cap V_2) \times \pi$. One proceeds inductively to obtain a section $s: (Y' \times \pi, \mathcal{M}') \longrightarrow (E, \mathcal{M})$.

Before concluding, one should point out that conditions B are required because of lemma 3.2. If this lemma can be improved then these restrictions can be removed here.

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