STABILITY OF A STRUCTURAL SYSTEM
UNDER CIRCULATORY LOADING AND
PARAMETRIC EXCITATION
STABILITY OF A STRUCTURAL SYSTEM UNDER CIRCULATORY LOADING
AND PARAMETRIC EXCITATION

By

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SCOPE AND CONTENTS:

This thesis describes the analytical study of the stability of a structural system under circulatory loading and/or parametric excitation. The model is a double pendulum, composed of two rigid weightless bars of equal length and two concentrated masses at the ends of each bar, on an oscillating base. The vertical oscillation of the base produces parametric excitation to the system. A circulatory force is applied at the free end. At the joints the restoring moments are produced by spring and damping. The damping coefficients are taken as positive, and the gravitational effects are included.

The combined effect of the circulatory loading and parametric excitation on stability of the system is investigated. The problem is so formulated that the stability of the system is represented by coupled Mathieu equations. The effect of damping on the boundary of stability is also determined.
ACKNOWLEDGEMENT

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B Behavior of the Solutions of the Differential Equation

\[ \ddot{f}(t) + \epsilon \dot{f}(t) + (\delta^2 + \mu \phi^2 \cos \theta t)f(t) = 0 \]

REFERENCES:
NOTATIONS

a  Lag parameter
b, b_k  Damping coefficients
c, c_k  Spring constant coefficients
D  Dissipation function
f(t)  Vertical movement of the base
g  Acceleration of gravity
h(t)  Horizontal movement of the base
\(\ell\)  Length of the bar
k  Indices
m, m_k  Masses
N  Amplitude of the vertical vibration of the base
P  Circulatory end force
P*  Critical value of the circulatory force P for the undamped system without base motion
P**  Critical value of the circulatory force P (with the lag parameter
  \(a = 0.811\)) for the undamped system without base excitation. At this value, the system could lose stability by divergence.
P*  Critical value of the circulatory force P for the damped system without base excitation.
P**  Critical value of the circulatory force P (with the lag parameter
  \(a = 0.811\)) for the damped system with base excitation. At this value, the system could lose stability by divergence.
Generalized forces

$t$  Time

$T$  Kinetic energy

$V$  Potential energy

$x$  $x$ axis

$y$  $y$ axis

$z_k$  Angular coordinates

$\alpha$  Parametric amplitude, $\alpha = \frac{N}{\kappa}$

$\beta$  Loading parameter, $\beta = \frac{p}{mg}$

$\beta_*$  Critical value of the loading parameter $\beta$ for the undamped system without base excitation.

$\beta_{**}$  Critical value of the loading parameter $\beta$ (with the lag parameter $a = 0.811$) for the undamped system without base excitation. At this value, the system could lose stability by divergence.

$\beta_*$  Critical value of the loading parameter $\beta$ for the damped system without base excitation.

$\beta_{**}$  Critical value of the loading parameter $\beta$ (with the lag parameter $a = 0.811$) for the damped system without base excitation. At this value, the system could lose stability by divergence.

$\gamma$  Spring constant parameter, $\gamma = \frac{c}{\kappa mg}$

$\eta_k$  Angular coordinates

$\phi_k$  Angular coordinates

$\xi$  Damping parameter, $\xi = \frac{b^2}{\kappa^3 m^2 g}$
Frequency of the vertical vibration of the base

\[ w_0 \] Natural frequency of a simple pendulum, \( w_0 = \sqrt{\frac{g}{l}} \)

\[ w_1 \] Natural frequency of the first mode for the double pendulum,

\[ w_1 = \sqrt{\gamma(3 - 2\sqrt{2}) - (2 - \sqrt{2})} \]

\[ w_2 \] Natural frequency of the second mode for the double pendulum,

\[ w_2 = \sqrt{\gamma(3 + 2\sqrt{2}) - (2 + \sqrt{2})} \]

\[ \lambda \] \( \lambda = \left(\frac{w}{\omega}\right)^2 = \frac{(g/l)}{\theta^2} \)
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CHAPTER 1

INTRODUCTION

It is well known that Euler is the first man who discussed the stability concept in 1744\(^{(3)}\). He determined the "critical load" for a compressed column and defined this load as the smallest force under which the column could be in equilibrium not only in its original straight configuration, but also in an infinitely close (adjacent) curved configuration (bifurcation of equilibrium). If the load is smaller than the critical value, the column remains straight and this straight configuration of equilibrium is said to be stable. As the load equals to the critical value, branching (bifurcation) of the forms of equilibrium occurs. For forces larger than the critical value, if the column is pushed slightly, then the deflection increases indefinitely and the vertical equilibrium configuration is said to be unstable.

But the Euler's "critical load" concept is not universal, and there are a number of cases in which loss of stability is not expressed as a transition to proximate forms of equilibrium, but to some completely different form. It was later discovered that the Euler's "critical load" concept is applicable if the external forces are conservative forces (i.e. if they have a potential, or we call them noncirculatory) and if the forces belong to some special range of non-conservative forces (The limitation to use Euler's method in studying non-conservative problems has been mentioned in many articles; for instance, Panovko and Gurbanova's book\(^{(22)}\). In general, the Euler's "critical load" concept is not
applicable if the forces are non-conservative and do not belong to such range (we can call non-conservative forces as circulatory).

The concept of a loss of stability for the general non-conservative problems can be understood clearly, if we borrow an expression from Aeroelasticity - "flutter" (oscillations with increasing amplitude). For instance, consider a non-conservative loading is applied on a system and the system is disturbed. By increasing the loading, the behavior of the system starts to flutter when a certain load value is exceeded. The critical load is defined as the smallest load under which the system flutters. Obviously this value of critical load can be different from Euler's "critical load". It is necessary to use the dynamical method, which is based on the investigation of the oscillations of the system close to its position of equilibrium, to find the critical value of the load. Thus a structural system may lose stability by flutter or by a transition to proximate forms of equilibrium (which we shall denote as divergence).

The non-conservative problems may occur under two different types of loading. One is instationary load, which means the magnitude of the load is dependent on the time. The other is stationary load, which is the load with constant magnitude. In the first case the system could lose stability by flutter only, but in the second case the system could lose stability either by flutter or by divergence, depending on the loading parameter.

In the 1950's the non-conservative problems of the stability of elastic systems were discussed in works by Ziegler, Bolotin and some other Russian and European scientists. Ziegler's discovery of the destabilizing effect of linear viscous damping in a non-conservative elastic
system provided an impetus for further studies of this remarkable phenomenon. In particular, Bolotin\(^{(1)}\) found that the destabilizing effect in an elastic system with two degrees of freedom is highly dependent on the ratio of the damping coefficients and that it could be eliminated for a certain particular value of this ratio. Herrmann and Bungay (in 1964) published a paper\(^{(7)}\) in which they proposed that two different types of structural instability could be designated as divergence and flutter. Since that time, those two terms were widely used in the theory of structural stability. The destabilizing effect of damping in non-conservative systems was further explored by Herrmann and Jong\(^{(10)}\). Nemat-Nasser and Herrmann (1966)\(^{(18)}\) proved that the critical load of a system with \(N\) degrees of freedom without damping, subjected to non-conservative forces which are linear functions of the generalized coordinates is an upper bound for the critical load of the same system with some sufficiently small velocity dependent forces. From 1960 to 1967, besides Herrmann's group\(^{(9,11,19,20,21)}\), there were a lot of non-conservative problems of the theory of elastic stability solved by Leipholz and other German scientists. A thorough survey of the stability of elastic systems subjected to non-conservative forces can be found in an article by Herrmann (1967)\(^{(6)}\).

Among the problems of the dynamic stability of structures, probably the best known subclass is constituted by the problems of parametric excitation, or parametric resonance. Their differential equations of motion in this class of problems are characterized by time-dependent coefficients, more specially by periodic coefficients which we call...
Mathieu-Hill equations. According to Bolotin\(^{(2)}\), the first solution of this kind of problem was given by Beliaev in 1924. This was followed by analysis by some other Russian scientists. In the United States, Lubkin, a student of Stoker, solved the problem in a doctoral dissertation submitted to New York University in 1939. An article by Lubkin and Stoker\(^{(15)}\), published in 1943, can be considered to be the most original and important work about this topic in the United States. The results of a theoretical and experimental investigation of the subject were published by Utida and Sezawa\(^{(26)}\) in 1940. Bolotin's book\(^{(2)}\), published in 1956 in Russia, could be considered as the most important reference about this topic. The problem of a restricted class of coupled Hill's equations was treated by Hsu in 1961\(^{(12)}\). In this article, he discussed how to plot the stability chart for two uncoupled Mathieu-Hill's equations. Later a paper on the parametric excitation of a dynamic system having multiple degrees of freedom was published by Hsu\(^{(13)}\). In an article, published in 1965, a general nonlinear problem about parametric excitation was solved by Tso and Caughey\(^{(24)}\). More recently, parametric torsional stability of a bar under axial excitation was discussed by Tso\(^{(25)}\). A more complete literature survey about the parametric response of structures is given in an article by Evan-Iwanowski\(^{(4)}\).

The purpose of this thesis is to study what is the combined effect (interaction) of non-conservative (circulatory) force and parametric excitation (or parametric loading) in the stability of structural system. This means the problem treated in this thesis is related to those in the theory of parametric excitation and the stability of non-conservative
elastic systems.

Here we can imagine a pipe conveying a jet of fluid and the base of the nozzle has vibration due to the engine of the pump or some other factors. In such a case the jet produces non-conservative (circulatory) loading and the vibration of the base produces parametric excitation to the pipe. Then the effect of the vibration on the stability of the jet nozzle will be of practical interest. Hence, the study of the combined effect of circulatory loading and parametric excitation becomes necessary.

To simplify the problem, we restrict ourselves to consider a two degree of freedom system. The reason is that a two degree of freedom system contains the essential features of a continuous system and without its mathematical complexity the mechanics can be understood more clearly. This thesis mainly deals with two degree of freedom system, but some special features of three, four and multiple degree of freedom systems will be treated in a later chapter.
Consider a simple model consisting of a two degree of freedom system under parametric base excitation only. The effect of parametric base excitation on the stability of such a system will be studied.

Figure 2.1 shows a double pendulum composed of two weightless bars of equal length \( l \), which carry concentrated masses \( m_1 \) and \( m_2 \), on an oscillating base. \( f(t) \) and \( h(t) \) describe the vertical and the horizontal motion of the base. The generalized coordinates \( \phi_1 \) and \( \phi_2 \) are taken to be small in the sense such that \( \sin \phi \sim \phi \). When the system is displaced from its equilibrium position, the restoring moments of magnitude \( c_1 \dot{\phi}_1 + b_1 \phi_1 \) and \( c_2(\phi_2 - \phi_1) + b_2(\dot{\phi}_2 - \dot{\phi}_1) \) are induced at the hinges. The damping coefficients \( b_1 \) and \( b_2 \) are taken as positive, and the gravitational effects will also be included. \( c_1 \) and \( c_2 \) represent the spring constants of the torsional springs at the hinges. First of all, the differential equations of motion will be derived. Then from those equations, we could investigate the regions of instability in a parameter space with the amplitude and the frequency of the vibration of the base. The effect of ground (base) motion on stability for the system without spring and damping will be treated as special cases. Finally, the effect of spring constant on stability for the system without ground motion will be considered. That is the case when the amplitude of the vibration of the ground is zero.
2.1 DERIVATION OF DIFFERENTIAL EQUATION OF MOTION

According to Figure 2.1 and using rectangular cartesian \( x \) and \( y \) coordinates lead the position vector of mass \( m_1 \) as

\[
\vec{r}_1 = \vec{r}_0 + \vec{p}_{01}
\]

where \( \vec{r}_0 \) is the position vector of point \( O \), and \( \vec{p}_{01} \) is the position vector of point \( 1 \) referring to the moving point \( O \). Hence

\[
\vec{r}_1 = (h(t) + \ell \sin \phi_1)\hat{i} + (f(t) + \ell \cos \phi_1)\hat{j} \tag{2.1}
\]

Similarly, the position vector of mass \( m_2 \) can be written as

\[
\vec{r}_2 = \vec{r}_1 + \vec{p}_{12}
\]

where \( \vec{r}_1 \) is the position vector of point \( 1 \), and \( \vec{p}_{12} \) is the position vector of point \( 2 \) referring to the moving point \( 1 \). Hence

\[
\vec{r}_2 = (h(t) + \ell \sin \phi_1 + \ell \sin \phi_2)\hat{i} + (f(t) + \ell \cos \phi_1 + \ell \cos \phi_2)\hat{j} \tag{2.2}
\]

At the time equals to zero, i.e. \( t = 0 \), it is assumed that the whole system stays at the original equilibrium position (see Figure 2.2) i.e.

\[
\phi_1 = 0, \quad \phi_2 = 0, \quad f(t) = f_0, \quad h(t) = h_0
\]

Hence the position vectors of masses \( m_1 \) and \( m_2 \) at \( t = 0 \) are

\[
\vec{r}_{1(t=0)} = h_0\hat{i} + (f_0 + \ell)\hat{j}
\]

\[
\vec{r}_{2(t=0)} = h_0\hat{i} + (f_0 + 2\ell)\hat{j}
\]

The changes of the position vectors of masses \( m_1 \) and \( m_2 \) in vertical component between \( t = 0 \) and arbitrary \( t \) are given by

\[
(\Delta \vec{r}_1)_y = (\vec{r}_1)_{0y} - (\vec{r}_1)_y = [(f_0 - f(t)) + \ell(1 - \cos \phi_1) + \ell(1 - \cos \phi_2)]\hat{j} \tag{2.3}
\]

and

\[
(\Delta \vec{r}_2)_y = (\vec{r}_2)_{0y} - (\vec{r}_2)_y = [(f_0 - f(t)) + \ell(1 - \cos \phi_1) + \ell(1 - \cos \phi_2)]\hat{j} \tag{2.4}
\]
Figure 2.1 A Double Pendulum on Oscillating Base
Figure 2.2: A Double Pendulum in the Original Equilibrium Position
The potential energy $V$ can be expressed as

$$V = -m_1 g |(Δ\vec{r}_1)_{\text{v}}| - m_2 g |(Δ\vec{r}_2)_{\text{v}}| + \frac{1}{2} c_1 \phi_1^2 + \frac{1}{2} c_2 (\phi_2 - \phi_1)^2$$

where $| |$ means absolute value. If equations 2.3 and 2.4 are substituted into the above equation, there is obtained

$$V = -m_1 g ((f_o - f(t)) + \xi (1 - \cos \phi_1)) - m_2 g ((f_o - f(t)) + \xi (1 - \cos \phi_1)$$

$$+ \xi (1 - \cos \phi_2)) + \frac{1}{2} c_1 \phi_1^2 + \frac{1}{2} c_2 (\phi_2 - \phi_1)^2$$

$$- - - - (2.5)$$

Let equations 2.1 and 2.2 be differentiated with respect to $t$. Then there are

$$\vec{r}_1 = (\vec{h}(t) + \xi \vec{\phi}_1 \cos \phi_1) \vec{i} + (\vec{f}(t) - \xi \vec{\phi}_1 \sin \phi_1) \vec{j}$$

$$\vec{r}_2 = (\vec{h}(t) + \xi \vec{\phi}_1 \cos \phi_1 + \xi \vec{\phi}_2 \cos \phi_2) \vec{i} + (\vec{f}(t) - \xi \vec{\phi}_1 \sin \phi_1 - \xi \vec{\phi}_2 \sin \phi_2) \vec{j}$$

The kinetic energy $T$ can be written as

$$T = \frac{1}{2} m_1 |\vec{r}_1|^2 + \frac{1}{2} m_2 |\vec{r}_2|^2$$

where $|\vec{r}_2|$ means absolute magnitude of $\vec{r}$.

Thus

$$T = \frac{1}{2} m_1 \{\vec{h}(t)^2 + \vec{f}(t)^2 + \xi^2 \vec{\phi}_1^2 + 2 \xi \vec{\phi}_1 (-\vec{f}(t) \sin \phi_1 + \vec{h}(t) \cos \phi_1)$$

$$+ \frac{1}{2} m_2 \{\vec{h}(t)^2 + \vec{f}(t)^2 + \xi^2 \vec{\phi}_2^2 + \xi^2 \vec{\phi}_2^2 + 2 \xi \vec{\phi}_2 (\vec{h}(t) \cos \phi_1 - \vec{f}(t) \sin \phi_1)$$

$$+ 2 \xi^2 \vec{\phi}_1 \vec{\phi}_2 \cos(\phi_1 - \phi_2) + 2 \xi \vec{\phi}_2 (\vec{h}(t) \cos \phi_2 - \vec{f}(t) \sin \phi_2)\}$$

Because $\phi_1$ and $\phi_2$ are considered as small quantities the approximation

$$\cos (\phi_1 - \phi_2) \approx 1$$

are used. Hence the kinetic energy $T$ can be rewritten as
\[ T = \frac{1}{2} m_1 (\ddot{h}(t))^2 + \dot{f}(t)^2 + \omega_1^2 \phi_1^2 + 2 \omega_1 \dot{\phi}_1 (\ddot{h}(t) \cos \phi_1 - \dot{f}(t) \sin \phi_1) \]
\[ + \frac{1}{2} m_2 (\ddot{h}(t))^2 + \dot{f}(t)^2 + \omega_2^2 \phi_2^2 + 2 \omega_2 \dot{\phi}_2 (\ddot{h}(t) \cos \phi_2 - \dot{f}(t) \sin \phi_2) \]
\[ + 2 \omega_1 \omega_2 \dot{\phi}_2 + 2 \omega_2 \ddot{\phi}_2 (\ddot{h}(t) \cos \phi_2 - \dot{f}(t) \sin \phi_2) \]  
\[ - - - - (2.6) \]

Next the dissipation function \( D \) can be written as
\[ D = \frac{1}{2} b_1 \dot{\phi}_1^2 + \frac{1}{2} b_2 (\dot{\phi}_2 - \dot{\phi}_1)^2 \]  
\[ - - - - (2.7) \]

Equations 2.5, 2.6 and 2.7 and Lagrange's equations in the form
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}_k} \right) - \frac{\partial T}{\partial \phi_k} + \frac{\partial V}{\partial \phi_k} + \frac{\partial D}{\partial \dot{\phi}_k} = 0 \quad (k = 1, 2) \]

are employed to establish the equation of motion.

\[
\begin{bmatrix}
\left( m_1 + m_2 \right) & m_2 \\
m_2 & m_2
\end{bmatrix}
\begin{bmatrix}
\ddot{\phi}_1 \\
\ddot{\phi}_2
\end{bmatrix}
+ \begin{bmatrix}
(b_1 + b_2) & -b_2 \\
-b_2 & b_2
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2
\end{bmatrix}
+ \begin{bmatrix}
(c_1 + c_2) & -c_2 \\
-c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
\]
\[
\begin{bmatrix}
\omega_1^2 (m_1 + m_2) (g + \dddot{f}(t)) & 0 \\
0 & \omega_1^2 m_2 (g + \dddot{f}(t))
\end{bmatrix}
\begin{bmatrix}
\sin \phi_1 \\
\sin \phi_2
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\omega_2 (m_1 + m_2) \ddot{h}(t) & 0 \\
0 & \omega_2 m_2 \ddot{h}(t)
\end{bmatrix}
\begin{bmatrix}
\cos \phi_1 \\
\cos \phi_2
\end{bmatrix}
\]
Linearizing the equation of motion by noting that
\[
\sin \phi_k \sim \phi_k, \quad \cos \phi_k \sim 1 \quad (k = 1, 2)
\]
there is obtained
\[
\ddot{\mathbf{z}} = \left[ \begin{array}{cc} (m_1 + m_2) & m_2 \\ m_2 & m_2 \end{array} \right] \left[ \begin{array}{c} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{array} \right] + \left[ \begin{array}{cc} (b_1 + b_2) & -b_2 \\ -b_2 & b_2 \end{array} \right] \left[ \begin{array}{c} \dot{\phi}_1 \\ \dot{\phi}_2 \end{array} \right] + \left[ \begin{array}{cc} (c_1 + c_2) - 1(m_1 + m_2)(g + \ddot{f}(t)) & -c_2 \\ -c_2 & c_2 - \ddot{h}_2(m_2(g + \ddot{f}(t)) \end{array} \right] \left[ \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right]
\]
\[
= - \left[ \begin{array}{c} (m_1 + m_2) \\ m_2 \end{array} \right] \ddot{h}(t)
\]

where \(\ddot{f}(t)\) denotes the acceleration of the vertical ground motion and \(\ddot{h}(t)\) expresses the acceleration of the horizontal ground motion.

Hence, it could be concluded that the vertical ground motion contributes to parametric excitation, but the horizontal ground motion induces forced oscillation. Because we are interested mainly in parametric excitation, only the case of vertical ground (or base) motion will be considered in subsequent analysis. Thus \(h(t) = 0\), (i.e. \(\ddot{h}(t) = 0\)) is assumed. Using the transformation
\[
\left[ \begin{array}{c} z_1 \\ z_2 \end{array} \right] = \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] \left[ \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right]
\]
leads the equation of motion
To be specific, let us consider the case

\[ c_1 = c_2 = c, \quad m_1 = m_2 = m, \quad b_1 = b_2 = b \]

Then the equation of motion will be written as

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
z''_1 \\
z''_2 \\
\end{bmatrix}
+ \begin{bmatrix}
5 & -2 \\
-2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
z'_1 \\
z'_2 \\
\end{bmatrix}
+ \begin{bmatrix}
5 & -2 \\
-2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
\end{bmatrix}
+ \ell(g + f(t) )
\begin{bmatrix}
-3 & 1 \\
1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
\end{bmatrix}
= 0
\]

Using similarity transformation and setting

\[ \{z\} = [\phi] \{\eta\} \]

where

\[ [\phi] =
\begin{bmatrix}
1 & 1 \\
(1 + \sqrt{2}) & (1 - \sqrt{2}) \\
\end{bmatrix} \]
\{z\} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \{\eta\} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}

lead the following equation of motion

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{\eta}_1 \\
\ddot{\eta}_2
\end{bmatrix}
+ b
\begin{bmatrix}
(3-2\sqrt{2}) & 0 \\
0 & (3+2\sqrt{2})
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{bmatrix}
+ c
\begin{bmatrix}
(3-2\sqrt{2}) & 0 \\
0 & (3+2\sqrt{2})
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix}
+ \left(g + \ddot{f}(t)\right) \ell m
\begin{bmatrix}
-(2 - \sqrt{2}) & 0 \\
0 & -(2 + \sqrt{2})
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix}
= 0
\]

Assume the vertical vibration of the base is periodic; then this vibration can be denoted by

\[f(t) = N \cos \theta t\]

where \(N\) is the amplitude of the vibration and \(\theta\) is the frequency. The acceleration of the vibration can be also written as

\[\ddot{f}(t) = -N \theta^2 \cos \theta t\]

Therefore, the equation of motion becomes

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{\eta}_1 \\
\ddot{\eta}_2
\end{bmatrix}
+ \frac{b}{\ell^2 m}
\begin{bmatrix}
(3-2\sqrt{2}) & 0 \\
0 & (3+2\sqrt{2})
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{bmatrix}
+ \frac{c}{\ell^2 m}
\begin{bmatrix}
(3-2\sqrt{2}) & 0 \\
0 & (3+2\sqrt{2})
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix}
= 0
\]

(Equation continued on next page)
\[
- \left( \frac{g}{\xi} - \frac{N G^2}{\xi} \cos \theta t \right) \begin{bmatrix}
(2 - \sqrt{2}) & 0 \\
0 & (2 + \sqrt{2})
\end{bmatrix} \begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} = 0
\]

It should be noted that the equations for \( \eta_1 \) and \( \eta_2 \) are uncoupled. With the substitution \( \tau = \theta t \), those two equations can be put in the standard form

\[
\frac{d^2 \eta_k}{d\tau^2} + \frac{\varepsilon_k}{\theta} \frac{d\eta_k}{d\tau} + (\delta_k + \varepsilon_k \cos \tau) \eta_k = 0 \quad (k = 1, 2) \quad (2.8)
\]

where

\[
\delta_1 = \left( \frac{w_1}{\theta} \right)^2 = \frac{1}{\theta^2} \left[ \frac{c}{\xi^2 m} (3-2\sqrt{2}) - \frac{g}{\xi} (2-\sqrt{2}) \right] = \frac{\xi}{\theta^2} \left[ \gamma (3-2\sqrt{2}) - (2-\sqrt{2}) \right]
\]

\[
\delta_2 = \left( \frac{w_2}{\theta} \right)^2 = \frac{1}{\theta^2} \left[ \frac{c}{\xi^2 m} (3+2\sqrt{2}) - \frac{g}{\xi} (2+\sqrt{2}) \right] = \frac{\xi}{\theta^2} \left[ \gamma (3+2\sqrt{2}) - (2+\sqrt{2}) \right]
\]

\[
\varepsilon_1 = \frac{N}{\xi} (2 - \sqrt{2}) = \alpha (2 - \sqrt{2})
\]

\[
\varepsilon_2 = \frac{N}{\xi} (2 + \sqrt{2}) = \alpha (2 + \sqrt{2})
\]

\[
\varepsilon_1 = \frac{b}{\xi^2 m} (3 - 2\sqrt{2})
\]

---

\( \gamma = (2 - \sqrt{2}) \)
\[ e_2 = \frac{b}{k^2_m} (3 + 2\sqrt{2}) \]

and

\[ \gamma = \frac{c}{k^2 mg} \quad \alpha = \frac{N}{k} \quad w_o = \sqrt{\frac{k}{\lambda}} \quad \lambda = \left( \frac{b}{w_o} \right)^2 \]

\[ w_1 = w_o \sqrt{\gamma (3 - 2\sqrt{2}) - (2 - \sqrt{2})} \]

\[ w_2 = w_o \sqrt{\gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2})} \]

2.2 DETERMINATION OF THE REGIONS OF DYNAMIC INSTABILITY

In this section, the construction of the stability chart for Equation 2.8 will be discussed. First, let us consider the case that there is no damping, i.e. \( b_k = 0 \) \( (k = 1, 2) \). Hence Equations 2.8 reduce to the two uncoupled Mathieu equations

\[ \frac{d^2 \eta_k}{d\tau^2} + (\delta_k + e_k \cos \tau) \eta_k = 0 \quad (k = 1, 2) \quad (2.10) \]

where the notations remain the same as in Equations 2.9. The stability chart for those two uncoupled Mathieu equations could be constructed easily (see Reference [12]). The stable and unstable regions can be given in a parameter space with \( \frac{w_2^2}{b^2} \) and \( \alpha \) as shown in Figure 2.3. (In this figure we use \( \gamma = \frac{c}{k^2 mg} = \frac{11}{2} \)). The regions in which the solutions of the equations are unbounded (unstable solutions) are noncrosshatched. Information concerning the stability of the system can then be obtained from the stability
Figure 2.3: The Stability Chart for the System on the Oscillating Base
chart. If the parametric excitation parameters correspond to a point in the crosshatched region, it means that the initial straight form of the system is dynamically stable. However, if the same point is found in the noncrosshatched region, then any initial deviation from the straight form of the system will be dynamically unstable.

For small value of \( \alpha \), i.e. small value of \( N \) (parametric amplitude), the boundary of two principal regions of instability can be expressed by the equations

\[
\delta_k \pm \frac{e_k}{2} - \frac{1}{4} = 0 \quad \text{(k = 1, 2)} \quad - - - - (2.11)
\]

or in the forms of

\[
\left(\frac{W_0}{\theta}\right)^2 \left[\gamma(3 - 2\sqrt{2}) - (2 - \sqrt{2})\right] \pm \frac{(2 - \sqrt{2})}{2} \alpha - \frac{1}{4} = 0
\]

and

\[
\left(\frac{W_0}{\theta}\right)^2 \left[\gamma(3 + 2\sqrt{2}) - (2 + \sqrt{2})\right] \pm \frac{(2 + \sqrt{2})}{2} \alpha - \frac{1}{4} = 0
\]

Thus when the parametric frequency \( \theta \) satisfies the following conditions

\[
\frac{1}{4} \leq \delta_k \leq \frac{1}{4} + \frac{e_k}{2} \quad \text{(k = 1, 2)} \quad - - - - (2.12)
\]

or in the forms of

\[
\frac{1}{4} - \frac{(2 - \sqrt{2})}{2} \alpha \leq \left(\frac{W_0}{\theta}\right)^2 \left[\gamma(3 - 2\sqrt{2}) - (2 - \sqrt{2})\right] \leq \frac{1}{4} + \frac{(2 - \sqrt{2})}{2} \alpha
\]

and

\[
\frac{1}{4} - \frac{(2 + \sqrt{2})}{2} \alpha \leq \left(\frac{W_0}{\theta}\right)^2 \left[\gamma(3 + 2\sqrt{2}) - (2 + \sqrt{2})\right] \leq \frac{1}{4} + \frac{(2 + \sqrt{2})}{2} \alpha
\]

- - - - (2.13)
then the system will become unstable. It should be noted that the above inequalities are only applicable for small values of $\alpha$. Inequalities 2.13 can be rewritten in the following forms

\[
\frac{w_o^2(y(3 - 2\sqrt{2}) - (2 - \sqrt{2}) \alpha)}{1 + \frac{(2 - \sqrt{2}) \alpha}{2}} \geq \theta^2 \geq \frac{w_o^2(y(3 - 2\sqrt{2}) - (2 - \sqrt{2}) \alpha)}{1 + \frac{(2 - \sqrt{2}) \alpha}{2}}
\]

and

\[
\frac{w_o^2(y(3 + 2\sqrt{2}) - (2 + \sqrt{2}) \alpha)}{1 + \frac{(2 + \sqrt{2}) \alpha}{2}} \geq \theta^2 \geq \frac{w_o^2(y(3 + 2\sqrt{2}) - (2 + \sqrt{2}) \alpha)}{1 + \frac{(2 + \sqrt{2}) \alpha}{2}}
\]

which denote the two principal regions of instability.

Next, the case with damping will be considered, (i.e. $b_k \neq 0$ ($k = 1, 2$)). Equations 2.8 could be rewritten in the forms of

\[
\ddot{\eta}_k + \varepsilon_k \dot{\eta}_k + (\delta_k + \mu_k \theta^2 \cos \theta t)\eta_k = 0 \quad (k = 1, 2)
\]

where \(\ddot{\eta}_k = \frac{d^2\eta_k}{dt^2}\), \(\dot{\eta}_k = \frac{d\eta_k}{dt}\)

\[
\delta_k = \delta_k \theta^2
\]

\[
\mu_k = \mu_k
\]

The Equations 2.14 are two uncoupled periodic differential equations with damping terms. The stability of the solution of this kind of differential equation will be discussed in the Appendix. Here the results from the Appendix will be used to plot the stability chart for the Equations 2.14.
The boundaries of the regions of instability are shown in Figure 2.3 by dotted lines. In this figure the spring constant parameter, \( \gamma = \frac{c}{\ell mg} = \frac{11}{2} \), is used. The damping parameter

\[
\xi = \frac{b^2}{\ell^3 m^2 g}
\]

is assumed to be 0.05, which corresponds to a fractional critical damping value for the first mode of 0.032.

2.3 EFFECT OF GROUND MOTION ON STABILITY FOR THE SYSTEM WITHOUT SPRING AND DAMPING

Here we consider a special case that the double pendulum is just under parametric base excitation, but there are no restoring moments at the hinges, (i.e. \( c_k = 0, b_k = 0 \) \( k = 1, 2 \)). In this case the system will be governed by the equations of motion,

\[
\frac{d^2 \eta_k}{d\tau^2} + (\delta'_k + e'_k \cos \tau) \eta_k = 0 \quad (k = 1, 2) \quad - - - - (2.16)
\]

where

\[
\delta'_1 = -(2 - \sqrt{2}) \frac{g}{\ell \theta^2}
\]

\[
\delta'_2 = -(2 + \sqrt{2}) \frac{g}{\ell \theta^2}
\]

\[
e'_1 = \frac{N}{\ell} (2 - \sqrt{2}) = \alpha (2 - \sqrt{2})
\]

\[
e'_2 = \frac{N}{\ell} (2 + \sqrt{2}) = \alpha (2 + \sqrt{2})
\]
The stability chart for those equations is shown in Figure 2.4. This chart has been plotted by Hsu\(^{(12)}\) in his analysis of the stability of a double inverted pendulum. Theoretically one can stabilize a basically unstable system (inverted pendulum) by base vibration. But since the region of stability is so small, it will not be easy to achieve it in practice.

### 2.4 EFFECT OF SPRING CONSTANT ON STABILITY FOR THE SYSTEM WITHOUT GROUND (BASE) MOTION

Let us neglect the ground motion, which is the special case that the amplitude of the ground vibration is zero, the system will be governed by the equations of motion

\[
\ddot{\eta}_k + \varepsilon_k \dot{\eta}_k + (\delta_k)\eta_k = 0 \quad (k = 1, 2) \tag{2.18}
\]

The notations remain the same as in Equations 2.9

First let us study the case that the damping is neglected, i.e. \(b_k = 0\) \((k = 1, 2)\). The equations of motion for this system become

\[
\ddot{\eta}_k + \delta_k \eta_k = 0 \quad (k = 1, 2) \tag{2.19}
\]

But from Equations 2.15 and 2.19, the following relationship is obtained

\[
\delta_k = w_k^2 \quad (k = 1, 2)
\]

Hence, the stability conditions for Equations 2.19 can be presented as

\[
\delta_k = w_k^2 \geq 0 \quad (k = 1, 2)
\]

The quantities \(w_1\) and \(w_2\) are the natural frequencies for the first and the second modes of the system, which are known as
Figure 2.4: The Stability Chart for the System Without Spring and Damping
Thus the critical value of the parameter \( \gamma \) is the larger one of

\[
\gamma = \frac{(2 - \sqrt{2})}{(3 - 2\sqrt{2})} = (2 + \sqrt{2})
\]

\[
\gamma = \frac{(2 + \sqrt{2})}{(3 + 2\sqrt{2})} = (2 - \sqrt{2})
\]

And so the spring constant \( c \) must satisfy the condition such that

\[
c \geq (2 + \sqrt{2})mg
\]

--- (2.20)

otherwise the system will be unstable.

For the case that the damping is taken into account, the governing equations are

\[
\ddot{\eta}_k + \varepsilon_k \dot{\eta}_k + \tilde{\delta}_k \eta_k = 0 \quad (k = 1, 2) \quad - - - - (2.21)
\]

If the conditions

\[
\tilde{\delta}_k \geq 0 \quad (k = 1, 2) \quad - - - -
\]

are satisfied, the solutions of the Equations 2.21 will be bounded. Therefore the critical value of the spring constant parameter \( \gamma \) will still be the larger one of

\[
\gamma = \frac{(2 - \sqrt{2})}{(3 - 2\sqrt{2})} = (2 + \sqrt{2})
\]

and

\[
\gamma = \frac{(2 + \sqrt{2})}{(3 + 2\sqrt{2})} = (2 - \sqrt{2})
\]
The spring constant $c$ must satisfy the same stability condition

$$
c \geq (2 + \sqrt{2})\lambda mg
$$

as in the case damping is neglected. A conclusion can be drawn that the damping does not have any destabilizing or stabilizing effect in this system. The reason can be understood easily, if we notice the system is conservative and may lose stability by divergence only. So damping should have no effect on stability at all. Concerning to this phenomenon, further information would be obtained in the next chapter.
CHAPTER 3

TWO DEGREE OF FREEDOM SYSTEM SUBJECTED TO NONCONSERVATIVE (CIRCULATORY) END FORCE WITH THE LAG PARAMETER "a" (FIGURE 3.1)

In this chapter, the problem of two degree of freedom system subjected to nonconservative (circulatory) end force will be considered. Similar problems have been discussed by Ziegler(29), Herrmann and associates(7,8,9,10). The main purpose of this chapter is to seek the effect of nonconservative force on stability of the two degree of freedom system that we are studying. First, the influence of the lag parameter "a" on the critical value of the force will be investigated. Then the further features about the stability of the system for several specific cases of the lag parameter "a" will be discussed.

3.1 DERIVATION OF THE EQUATION OF MOTION

The model under investigation is the same as in Chapter 2. A double pendulum, Figure 3.1, composed of two rigid weightless bars of equal length l, with two concentrated masses m₁ and m₂ at the ends of each bar. A circulatory force is applied at the free end such that it always makes an angle "aφ₂" with the vertical. The parameter "a" is known as the lag parameter. If a = 0, we have a vertical force; and when a = 1, the force becomes a follower force. At the hinges the restoring moments are \( c₁φ₁ + b₁\dot{φ₁} \) and \( c₂(φ₂ - φ₁) + b₂(\dot{φ₂} - \dot{φ₁}) \). The damping coefficients \( b₁ \) and \( b₂ \) are still taken as positive, and the gravitational effects are considered.

In this case, there is no ground motion. Hence the potential energy \( V \), the kinetic energy \( T \), and the dissipation function \( D \), can be reduced
Figure 3.1: A Double Pendulum Subjected to Circulatory End Loading
from Equations 2.5, 2.6, and 2.7 in Chapter 2, namely,

\[ V = -m_1 g \ell (1 - \cos \phi_1) - m_2 g \ell [(1 - \cos \phi_1) + (1 - \cos \phi_2)] \]

\[ + \frac{1}{2} c_1 \dot{\phi}_1^2 + \frac{1}{2} c_2 (\dot{\phi}_2 - \dot{\phi}_1)^2 \]  
- - - (3.1)

\[ T = \frac{1}{2} m_1 \ell^2 \dot{\phi}_1 + \frac{1}{2} m_2 (\ell^2 \dot{\phi}_1^2 + \ell^2 \dot{\phi}_2^2 + 2 \ell \dot{\phi}_1 \dot{\phi}_2) \]  
- - - (3.2)

\[ D = \frac{1}{2} b_1 \dot{\phi}_1^2 + \frac{1}{2} b_2 (\dot{\phi}_2 - \dot{\phi}_1)^2 \]  
- - - (3.3)

The generalized forces \( Q_1 \) and \( Q_2 \) are

\[ Q_1 = P \ell (\phi_1 - a \phi_2) \]  
- - - (3.4)

\[ Q_2 = P \ell (1 - a) \phi_2 \]

Equations 3.1, 3.2, 3.3, 3.4 and Lagrange's equations in the form

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}_k} \right) - \frac{\partial T}{\partial \phi_k} + \frac{\partial V}{\partial \phi_k} + \frac{\partial D}{\partial \ddot{\phi}_k} = Q_k \]  
(\( k = 1, 2 \))

are employed to establish the linear equation of motion

\[
\begin{bmatrix}
(m_1 + m_2) & m_2 \\
m_2 & m_2
\end{bmatrix}
\begin{bmatrix}
\ddot{\phi}_1 \\
\ddot{\phi}_2
\end{bmatrix}
+ \begin{bmatrix}
(b_1 + b_2) & -b_2 \\
\dot{\phi}_2 & b_2
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2
\end{bmatrix}
+ \begin{bmatrix}
(c_1 + c_2) - \ell (m_1 + m_2)g & -c_2 \\
-c_2 & c_2 - \ell m_2 g
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
\]

\[
+ P \ell
\begin{bmatrix}
-1 & a \\
0 & -(1 - a)
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
= 0
\]
The transformation

\[
\begin{bmatrix}
    z_1 \\
    z_2
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 \\
    1 & 1
\end{bmatrix}
\begin{bmatrix}
    \phi_1 \\
    \phi_2
\end{bmatrix}
\]

is used again. Then the equation of motion becomes

\[
\begin{bmatrix}
    m_1 & 0 \\
    0 & m_2
\end{bmatrix}
\begin{bmatrix}
    \ddot{z}_1 \\
    \ddot{z}_2
\end{bmatrix} + \begin{bmatrix}
    (b_1 + 4b_2) & -2b_2 \\
    -2b_2 & b_2
\end{bmatrix}
\begin{bmatrix}
    \dot{z}_1 \\
    \dot{z}_2
\end{bmatrix} + \begin{bmatrix}
    (c_1 + 4c_2) & -2c_2 \\
    -2c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
    z_1 \\
    z_2
\end{bmatrix}
\]

\[
+ g \begin{bmatrix}
    -(m_1 + 2m_2) & m_2 \\
    m_2 & -m_2
\end{bmatrix}
\begin{bmatrix}
    z_1 \\
    z_2
\end{bmatrix} + \begin{bmatrix}
    -(1+a) & a \\
    (1-a) & -(1-a)
\end{bmatrix}
\begin{bmatrix}
    z_1 \\
    z_2
\end{bmatrix} = 0
\]

Let us consider the specific case

\[c_1 = c_2 = c, \quad m_1 = m_2 = m, \quad b_1 = b_2 = b\]

The equation of motion will be rewritten as

\[
\begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    \dddot{z}_1 \\
    \dddot{z}_2
\end{bmatrix} + \begin{bmatrix}
    5 & -2 \\
    -2 & 1
\end{bmatrix}
\begin{bmatrix}
    \ddot{z}_1 \\
    \ddot{z}_2
\end{bmatrix} + \begin{bmatrix}
    5 & -2 \\
    -2 & 1
\end{bmatrix}
\begin{bmatrix}
    \dot{z}_1 \\
    \dot{z}_2
\end{bmatrix} + \begin{bmatrix}
    -3 & 1 \\
    1 & -1
\end{bmatrix}
\begin{bmatrix}
    z_1 \\
    z_2
\end{bmatrix} + \xi_m g
\]

(Equation continued on next page)
\[
\begin{pmatrix}
-(1 + a) & a \\
(1 - a) & -(1 - a)
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} + p^2 
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} = 0 
\]

It is interesting to note that when \( a = 1/2 \), the last matrix becomes symmetric and has the same form as the matrix in the fourth term.

Using similarity transformation and setting

\[
\{z\} = [\phi] \{n\}
\]

where

\[
[\phi] = \begin{pmatrix}
1 & 1 \\
(1 + \sqrt{2}) & (1 - \sqrt{2})
\end{pmatrix}, \quad \{z\} = \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}, \quad \{n\} = \begin{pmatrix}
\dot{n}_1 \\
\dot{n}_2
\end{pmatrix}
\]

lead the equation of motion to the form of

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\ddot{n}_1 \\
\ddot{n}_2
\end{pmatrix} + b
\begin{pmatrix}
0 \\
(3 + 2\sqrt{2})
\end{pmatrix}
\begin{pmatrix}
\dot{n}_1 \\
\dot{n}_2
\end{pmatrix} + c
\begin{pmatrix}
0 \\
(3 + 2\sqrt{2})
\end{pmatrix}
\begin{pmatrix}
\dot{n}_1 \\
\dot{n}_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
(2 - \sqrt{2}) & 0 \\
0 & (2 + \sqrt{2})
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix} - \frac{p^2}{2\sqrt{2}}
\begin{pmatrix}
(2\sqrt{2} - 1 - 2a) & (2a - 1) \\
-(2a - 1) & (2\sqrt{2} + 1 + 2a)
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix} = 0
\]

or in the matrix notation
\[
[M] \{\ddot{\eta}_n\} + [B] \{\dot{\eta}_n\} + ([E] - g[A]) \{\eta_n\} = 0 \quad \ldots \quad (3.6)
\]

where

\[
[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
[B] = \begin{bmatrix} (3 - 2\sqrt{2}) & 0 \\ 0 & (3 + 2\sqrt{2}) \end{bmatrix}
\]

\[
[E] = \begin{bmatrix} (3 - \sqrt{2}) & 0 \\ 0 & (3 + \sqrt{2}) \end{bmatrix} - \frac{p_0}{2\sqrt{2}} \begin{bmatrix} (2\sqrt{2} - 1 - a)(2a - 1) \\ -(2a - 1)(2\sqrt{2} + 1 + a) \end{bmatrix}
\]

\[
[A] = \begin{bmatrix} (2 - \sqrt{2}) & 0 \\ 0 & (2 + \sqrt{2}) \end{bmatrix}
\quad \ldots \quad (3.7)
\]

3.2 EFFECT OF THE NONCONSERVATIVE FORCE WITH THE LAG PARAMETER "a" ON STABILITY FOR THE TWO DEGREE OF FREEDOM SYSTEM

Seeking a solution of the form

\[
\eta_k = A_k e^{\lambda t} \quad \text{(k = 1, 2)}
\]

to Equation 3.6 yields the characteristic equation

\[
p_0^4 + p_1^3 + p_2^2 + p_3 + p_4 = 0 \quad \ldots \quad (3.8)
\]
with the coefficients

\[ p_0 = 1 \]

\[ p_1 = 6\sqrt{\xi} \]

\[ p_2 = \xi + 2(3\gamma - 2 - \beta) \]

\[ p_3 = \sqrt{\xi}(2\gamma + 4(a - 1)\beta - 4) \]

\[ p_4 = \gamma^2 + 4(a - 1)\beta + \beta^2(1 - a) + \beta(3 - 2a) - 4\gamma + 2 \]

and the quantities

\[ \Omega = \frac{D}{w} \]

\[ \beta = \frac{p}{mg} \]

\[ w = \sqrt{\frac{g}{\xi}} \]

\[ \gamma = \frac{c}{kmg} \]

\[ \xi = \frac{b^2}{L^3m^2g} \]

For a given spring constant, say, \( \gamma = \frac{c}{kmg} = \frac{11}{2} \), the variation of the critical value of loading parameter "\( \beta \)" with respect to the lag parameter "\( a \)" will be studied next. Setting \( \gamma = \frac{11}{2} \) and applying the Routh-Hurwitz criteria lead the stability conditions

\[ \bar{p}_1 = 6\sqrt{\xi} \geq 0 \]

\[ \bar{p}_1\bar{p}_2 - \bar{p}_0\bar{p}_3 = \sqrt{\xi}(6\xi + 167 - 4(2 + a)\beta) \geq 0 \]

\[ \bar{p}_1\bar{p}_2\bar{p}_3 - \bar{p}_0\bar{p}_3\bar{p}_4 = 4(a-1)(1-4a)\beta^2 + (24\xi(a-1) - 40(2a+1)\beta \]

\[ + (800 + 42\xi) \geq 0 \]
\[ p_4 = \beta^2(1 - a) + \beta(20a - 19) + \frac{41}{4} \geq 0 \]

The damping coefficients \( b_k \) are taken as positive; hence the above four inequalities can be reduced to the forms of

\[ 6\xi + 167 - 4(2 + a)\beta \geq 0 \]

\[ 4(a-1)(1-4a)\beta^2 + [24\xi(a-1) - 40(2a+1)] + (800 + 42\xi) \geq 0 \]

\[ \beta^2(1 - a) + \beta(20a - 19) + \frac{41}{4} \geq 0 \quad - - - - (3.10) \]

According to the inequalities of Equation 3.10 a chart showing the critical value of nonconservative loading parameter "\( \beta \)" as the function of the lag parameter "\( a \)" is plotted (see Figure 3.2). In this figure the damping parameter, \( \xi = 0.05 \) is used. At \( a = 0.811^* \) there is a jump discontinuity of the curve. Checking the equation reveals that in the region \( a < 0.811 \) the system may lose stability by divergence (static instability) and in the domain \( a \geq 0.811 \) the system loses stability by flutter (dynamic instability).

If the damping is neglected, \( (\xi = 0) \), then the characteristic equation becomes

\[ -A\Omega^4 + \bar{B}\Omega^2 + \bar{C} = 0 \]

where

\[ A = 1 \]

\[ \bar{B} = 29 - 2\beta \]

\[ \bar{C} = \beta^2(1 - a) + \beta(20a - 19) + \frac{41}{4} \]

* The more accurate value is 0.8107125.
Both damped and undamped

Spring Constant Parameter

Figure 3.2: The Critical Value of the Loading Parameter "β"
the stability conditions become

(a) \( \bar{B} - 4\bar{A}\bar{C} = 4(\beta - 20)(\beta a - 10) \geq 0 \)

(b) \( \bar{B} = 29 - 2\beta \geq 0 \)

(c) \( \bar{C}\bar{A} = \beta^2(1 - a) + \beta(20a - 19) + \frac{41}{4} \geq 0 \)

According to these inequalities a similar chart in the parameter space of \( \beta \) and \( a \) can be drawn. This stability chart is also shown in the Figure 3.2 labelled "undamped". It should be noted that the same jump discontinuity at \( a = 0.811 \) exists. In the domain \( a \leq 0.811 \) loss of stability through divergence as in the damped case may occur, and when \( a \geq 0.811 \) flutter happens if loading is larger than the critical value. A careful examination of Figure 3.2 shows that in the region \( a \leq 0.811 \), the critical value of \( \beta \) is not influenced by damping and in the range \( a \geq 0.811 \) the destabilizing effect of damping becomes obvious. Since damping force is a velocity-dependent force, the effect of damping will be significant in the case of loss of stability by flutter (dynamical instability). But when the system loses stability by divergence (static instability), damping force will have no effect at all.

Figure 3.2 shows there are two critical values of loading parameter \( \beta \) at \( a = 0.811 \) for both the damped and undamped cases. For the undamped case, we call the large one as \( \beta^* \) corresponding to \( P^* \), and the small one as \( \beta^{**} \) corresponding to \( P^{**} \). We can express \( \beta^* \), \( \beta^{**} \), \( P^* \), and \( P^{**} \) in the following equations

\[ \beta^* = 12.3348 \]

\[ \beta^{**} = 7.359 \] - - - - (3.11)
\[ P_\star = 12.3348 \text{ mg} \]
\[ P_{\star\star} = 7.359 \text{ mg} \]  
**(3.11)**

For the damped case the quantities for those parameters are

\[ \tilde{\beta}_\star = 8.9182 \]
\[ \tilde{\beta}_{\star\star} = 7.359 \]  
**(3.12)**

and

\[ \tilde{P}_\star = 8.9182 \text{ mg} \]
\[ \tilde{P}_{\star\star} = 7.359 \text{ mg} \]

### 3.3 EFFECT OF THE NONCONSERVATIVE (CIRCULATORY) FORCE WITH THE LAG PARAMETER "a" = 1 ON STABILITY FOR THE TWO DEGREE OF FREEDOM SYSTEM

Let us study further the case where the lag parameter \( a = 1 \).

This case corresponds to a tangential nonconservative end force (see Figure 3.3). The problem is very similar to the solved problem which was discussed by Herrmann and associates\(^{(8,10)}\). The main difference is that in our problem the gravitational effects are included and the concentrated masses \( m_1 \) and \( m_2 \) are assumed equal. In Herrmann's study, they used a different mass distribution and neglected the gravitational effects.

From the Equation 3.8 the characteristic equation for the specific case \( a = 1 \) reduces to

\[ a^{\Omega^4}_0 + a^{\Omega^3}_1 + a^{\Omega^2}_2 + a^{\Omega}_3 + a = 0 \]  
**(3.13)**
Figure 3.3: A Double Pendulum Subjected to Tangential End Force
with the coefficients

\[ a_0 = 1 \]
\[ a_1 = 6\sqrt{\xi} \]
\[ a_2 = \xi + 2(3\gamma - 2 - \beta) \]
\[ a_3 = \sqrt{\xi}(2\gamma - 4) \]
\[ a_4 = (\gamma^2 + 2 - 4\gamma + \beta) \]

and the quantities

\[ \Omega = \frac{D}{(w^2)} \]
\[ \beta = \frac{p}{mg} \]
\[ w_0 = \sqrt{\frac{g}{\xi}} \]
\[ \gamma = \frac{c}{\xi mg} \]
\[ \xi = \frac{b^2}{\xi^3 m^2 g} \]

Considering the undamped case first, (\( \xi = 0 \)), the characteristic Equation 3.13 can be reduced to the form

\[ \bar{A}\Omega^4 + \bar{B}\Omega^2 + \bar{C} = 0 \]

where

\[ \bar{A} = 1 \]
\[ \bar{B} = 2(3\gamma - 2 - \beta) \]
\[ \bar{C} = (\gamma^2 + 2 - 4\gamma + \beta) \]

Hence the stability conditions for the undamped system are

(a) \( \bar{B} - 4\bar{A}\bar{C} = 4(\beta - 4\gamma + 2)(\beta - 2\gamma + 1) \geq 0 \)
\[ B = (1 - \frac{3}{2\gamma} + \frac{\beta}{2}) \leq 0 \]  - - - - (3.17)

(c) \[ CA = (\gamma^2 - 4\gamma + 2 + \beta) \geq 0 \]

According to inequalities (Equation 3.17), a stability chart is plotted as shown in Figure 3.4. For \( \beta = 0 \), which means there is no loading, the critical value of the spring constant parameter \( \gamma \) is \( (2 + \sqrt{2}) \). This value has been obtained in inequality (2.20) already. It is known that for the system without loading when the spring constant \( c \) is less than \( (2 + \sqrt{2})mg \), the system will become unstable. However, addition of certain values of tangential force will stabilize the system as shown in Figure 3.4. Therefore, suitable amount of tangential force have stabilizing effect. This effect exists when the parameter \( \gamma \) is equal to or larger than 1. But when \( \gamma \) is less than 1, there is no value of loading which can put the system to become stable. In Figure 3.4 it can be seen that

\[ \beta^* = 10 \quad \text{if} \quad \gamma = \frac{11}{2} \]

where \( \beta^* \) denotes the critical value of \( \beta \). Hence the critical value of the nonconservative force \( P \) is

\[ P^* = 10 \text{ mg} \]

when the spring constant parameter of the system is

\[ \gamma = \frac{11}{2} \]

Going back to the damped case, the characteristic equation is given in Equation 3.13. Applying the Routh-Hurwitz criteria, we obtain the stability conditions
Figure 3.4: The Critical Value of "β" for the Undamped System with "α" = 1
\[ a = 6\sqrt{\xi} > 0 \]
\[ a - a_0 = \sqrt{\xi}(6\xi + 34\gamma - 20 - 12\beta) > 0 \]
\[ a_{12} = a_{12} a_{03} = a_{12} a_{03} a_{14} = \xi[12(\gamma - 2)\xi + (32\gamma^2 - 32\gamma + 8 + 12\beta - 24\gamma\beta)] > 0 \]
\[ a = \gamma^2 + 2 - 4\gamma + \beta > 0 \]

With the damping coefficients \( b \) taken as positive, the above four inequalities can be reduced to the forms of

\[ 6\xi + 34\gamma - 20 - 12\beta > 0 \]
\[ 12(\gamma - 2)\xi + (32\gamma^2 - 32\gamma + 8 + 12\beta - 24\gamma\beta) > 0 \]
\[ \gamma^2 + 2 - 4\gamma + \beta > 0 \]

According to inequalities (3.18), another stability chart is plotted as shown in Figure 3.5. In this figure the value of damping is taken to be \( \xi = 0.05 \). A comparison of this figure with Figure 3.5 shows that damping has destabilizing effect. This effect has been well known since Ziegler(27) discovered this remarkable phenomenon. For instance, with a given spring constant parameter, \( \gamma = \frac{11}{2} \), the system without damping gives the critical value of loading parameter \( \beta \) equal to 10; but the system includes damping gives the critical value of \( \beta \) to be 6.68.

Next, let us investigate the case when the spring constant parameter \( \gamma \) is less than \((2 + \sqrt{2})\), which means that the system is unstable originally. From Figure 3.5, we find in the range
Figure 3.5: The Critical Value of "β" for the Damped System with "a" = 1
application of certain values of tangential loading could stabilize the system. Hence we know that in this range the tangential force has stabilizing effect. But when

\[ \gamma \leq 2 \]

there is no value of the tangential force which can stabilize the system. This is in contrast to the case when damping is neglected. It is useful to mention here that in Figure 3.5 the boundary of the stable region always passes through the point \((\gamma = 2, \beta = 2)\) for any value of damping parameter \(\xi\), and even the parameter \(\xi\) is very small and close to zero.

### 3.4 EFFECT OF THE NONCONSERVATIVE FORCE WITH THE LAG PARAMETER \(a = 1/2\) ON STABILITY OF THE TWO DEGREE OF FREEDOM SYSTEM

If the value \(a = \frac{1}{2}\) is put into Equation 3.6, this equation will be reduced to two uncoupled differential equations with the forms of

\[
\ddot{\eta}_k + \epsilon_k \dot{\eta}_k + \delta_k \eta_k = 0 \quad (k = 1, 2) \quad - - - (3.19)
\]

where

\[
\epsilon_1 = \frac{b}{\xi^2 m} (3 - 2\sqrt{2})
\]

\[
\epsilon_2 = \frac{b}{\xi^2 m} (3 + 2\sqrt{2})
\]

\[
\delta_1 = \dot{w}^2 \frac{c}{\xi^2 m} (3 - 2\sqrt{2}) - \frac{g}{\xi} (2 - \sqrt{2}) - \frac{p}{\xi m} (\sqrt{2})
\]

\[
= \left( \frac{g}{\xi} \right) [\gamma(3 - 2\sqrt{2}) - (2 - \sqrt{2}) - \beta(\frac{2 - \sqrt{2}}{2})]
\]

(Equation continued on next page)
\[
\delta_k = \frac{w^2}{2} = \frac{c}{\ell^2 m} (3 + 2\sqrt{2}) - \frac{p}{\ell} (2 + \sqrt{2}) - \frac{p}{\ell m} \left( \frac{2 + \sqrt{2}}{2} \right)
\]

\[
\delta_k = \frac{g}{\ell} \left[ \gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right) \right]
\]

\[
\delta_k = \frac{w^2 o}{2} \left[ \gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right) \right]
\]

and

\[
\gamma = \frac{c}{\ell m g}, \quad \beta = \frac{p}{mg}, \quad w_o = \sqrt{\frac{p}{\ell}}
\]

\[
\dot{w}_1 = w_o \sqrt{\gamma (3 - 2\sqrt{2}) - (2 - \sqrt{2}) - \beta \left( \frac{2 - \sqrt{2}}{2} \right)}
\]

\[
\dot{w}_2 = w_o \sqrt{\gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right)}
\]

In this case, the mathematical analysis becomes relatively simple. The damping coefficients \( \delta_k \) (\( k = 1, 2 \)) are taken as positive. The conditions, such that the solutions of Equation 3.19 will be bounded, are

\[
\delta_k > 0 \quad (k = 1, 2) \quad - - - (3.21)
\]

By neglecting damping, the governing equations of motion become

\[
\ddot{\eta}_k + \delta_k \dot{\eta}_k = 0 \quad (k = 1, 2) \quad - - - (3.22)
\]

where the notations remain the same as in Equations 3.20. Hence the stability conditions are still the same as the damped case, namely,

\[
\delta_k > 0 \quad (k = 1, 2) \quad - - - (3.23)
\]
Thus a conclusion can be made that for the case "a" = $\frac{1}{2}$, the damping has no destabilizing effect. This phenomenon has been mentioned in Section 3.2. The reason is that at "a" = $\frac{1}{2}$, the system loses stability by divergence. Inequalities (3.21) can be rewritten in the following forms

$$\frac{g}{k} [\gamma (3 - 2\sqrt{2}) - (2 - \sqrt{2}) - \beta (\frac{2 - \sqrt{2}}{2}) ] \geq 0$$

and

$$\frac{g}{k} [\gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta (\frac{2 + \sqrt{2}}{2}) ] \geq 0$$

Using the above two inequalities leads a stability as shown in Figure 3.6. We mention here once more that this figure is applicable for both the damped and the undamped cases.
Figure 3.6: The Critical Value of \( \rho \) for \( a = 1/2 \)
In Chapter 2 the stability of a two degree of freedom system under parametric base excitation is studied, and in Chapter 3 the stability of the same system subjected to nonconservative (circulatory) end force is investigated. Now it will be of interest to study the combined effect (interaction) of the parametric base excitation and the nonconservative end force on the stability of the two degree of freedom system. The main purpose of this chapter is to seek for this combined effect. At first the problem of two degree of freedom system under parametric base excitation and nonconservative end force with the lag parameter "a" is formulated. Then the special cases $a = 1$, $a = \frac{1}{2}$, $a = 0.811^*$ will be further studied and the principal regions of instability for those cases are determined. $a = 1$ is the case the system is subjected to tangential end force (follower force), and $a = \frac{1}{2}$ is the case that the classical modes will exist. $a = 0.811$ is of interest, because it is at the boundary of the regions separating the behaviour of the system without base excitation from loss of stability through divergence and flutter (see Figure 3.2).

4.1 DERIVATION OF THE EQUATION OF MOTION

Consider a double pendulum (Figure 4.1) composed of two rigid weightless bars of equal length $l$, with two concentrated masses $m_1$ and $m_2$.

* The more accurate value is 0.8107125
Figure 4.1: Two Degree of Freedom System Under Base Motion and Circulatory End Force
at the ends of each bar. A circulatory force is applied at the free end at an angle "a_2" to the vertical. At the joints the restoring moments are \( c_1 \dot{\phi}_1 + b_1 \dot{\phi}_1 \) and \( c_2 (\phi_2 - \phi_1) + b_2 (\dot{\phi}_2 - \dot{\phi}_1) \). The damping coefficients \( b_1 \) and \( b_2 \) are taken as positive, and the gravitational effects are included. The vertical and the horizontal vibrations of the base are denoted by \( f(t) \), and \( h(t) \).

The potential energy \( V \), the kinetic energy \( T \), the dissipation function \( D \), and the generalized forces \( Q_1 \) and \( Q_2 \) are known in Equations 2.5, 2.6, 2.7 in Chapter 2, and Equations 3.4 in Chapter 3, as

\[
V = -m_1 g \{ (f_o - f(t)) + \ell (1 - \cos \phi_1) \}
\]

\[
- m_2 g \{ (f_o - f(t)) + \ell (1 - \cos \phi_1) + \ell (1 - \cos \phi_2) \}
\]

\[
+ \frac{1}{2} c_1 \dot{\phi}_1^2 + \frac{1}{2} c_2 (\dot{\phi}_2 - \dot{\phi}_1)^2
\]

\[
T = \frac{1}{2} m_1 (\dot{h}(t)^2 + \dot{f}(t)^2 + \ell^2 \dot{\phi}_1^2 + 2 \ell \dot{\phi}_1 (\dot{h}(t) \cos \phi_1 - \dot{f}(t) \sin \phi_1))
\]

\[
+ \frac{1}{2} m_2 (\dot{h}(t)^2 + \dot{f}(t)^2 + \ell^2 \dot{\phi}_2^2 + \ell^2 \dot{\phi}_1^2 + 2 \ell \dot{\phi}_1 (\dot{h}(t) \cos \phi_1 - \dot{f}(t) \sin \phi_1))
\]

\[
+ 2 \ell^2 \dot{\phi}_2 (\dot{h}(t) \cos \phi_2 - \dot{f}(t) \sin \phi_2)
\]

\[
D = \frac{1}{2} b_1 \dot{\phi}_1^2 + \frac{1}{2} b_2 (\dot{\phi}_2 - \dot{\phi}_1)^2
\]

\[
Q_1 = P \ell (\phi_1 - a \phi_2)
\]

\[
Q_2 = P \ell (1 - a) \phi_2
\]

Lagrange's equation in the form

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}_k} - \frac{\partial T}{\partial \phi_k} + \frac{\partial V}{\partial \phi_k} + \frac{\partial D}{\partial \dot{\phi}_k} \right) = Q_k \quad (k = 1, 2)
\]
are employed to establish the linear equation of motion

\[
\varepsilon^2 \begin{bmatrix}
(m_1 + m_2) & m_2 \\
\quad & m_2
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2
\end{bmatrix}
+ \begin{bmatrix}
b_1 + b_2 & -b_2 \\
- b_2 & b_2
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2
\end{bmatrix}
+ \begin{bmatrix}
(c_1 + c_2) & -c_2 \\
- c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
+ \varepsilon (g + f(t))
\]

\[
\begin{bmatrix}
-m_1 + m_2 & 0 \\
0 & -m_2
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
+ \varepsilon P \dot{\phi}
\begin{bmatrix}
-1 & a \\
0 & -(1-a)
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
= -\begin{bmatrix}
(m_1 + m_2) \varepsilon \\
m_2 \varepsilon
\end{bmatrix} \ddot{h}(t)
\]

As mentioned in Chapter 2, we are mainly interested in parametric excitation. Hence the case of no horizontal base (or ground) motion will be considered in subsequent analysis. Thus, \( h(t) = 0 \) (i.e., \( \ddot{h}(t) = 0 \)) is assumed. The transformation

\[
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2
\end{bmatrix}
\]

is used. The equation of motion becomes

\[
\varepsilon^2 \begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\begin{bmatrix}
\ddot{z}_1 \\
\ddot{z}_2
\end{bmatrix}
+ \begin{bmatrix}
(b_1 + 4b_2) & -2b_2 \\
-2b_2 & b_2
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix}
+ \begin{bmatrix}
(c_1 + 4c_2) & -2c_2 \\
-2c_1 & c_2
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
+ \varepsilon (g + f(t))
\]

\[
\begin{bmatrix}
-(m_1 + 2m_2) & m_2 \\
m_2 & -m_2
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
+ \varepsilon P \dot{\phi}
\begin{bmatrix}
-1+a & a \\
1-a & -(1-a)
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
= 0
\]
Consider the specific case
\[ c_1 = c_2 = c, \quad b_1 = b_2 = b, \quad m_1 = m_2 = m \]

The equation of motion is reduced to the form of

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix}
+ b
\begin{bmatrix}
5 & -2 \\
-2 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{z}_1 \\
\ddot{z}_2
\end{bmatrix}
+ c
\begin{bmatrix}
5 & -2 \\
-2 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix}
\]

\[
+ \ell(g+f(t))m
\begin{bmatrix}
-3 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
+ P \ell
\begin{bmatrix}
-(1+a) & a \\
(1-a) & -(1-a)
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
= 0
\]

Using similarity transformation and setting
\[
\{z\} = [\phi]\{\eta\}
\]

where
\[
[\phi] = \begin{bmatrix}
1 & 1 \\
(1+\sqrt{2}) & (1-\sqrt{2})
\end{bmatrix}
\]

\[
\{z\} = \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
\quad \{\eta\} = \begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix}
\]

lead the equation of motion as

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{bmatrix}
+ b
\begin{bmatrix}
(3-2\sqrt{2}) & 0 \\
0 & (3+2\sqrt{2})
\end{bmatrix}
\begin{bmatrix}
\ddot{\eta}_1 \\
\ddot{\eta}_2
\end{bmatrix}
+ c
\begin{bmatrix}
(3-2\sqrt{2}) & 0 \\
0 & (3+2\sqrt{2})
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{bmatrix}
\]

(Equation continued on next page)
where \( f(t) \) denotes the acceleration of the vertical vibration of the base (ground). If we assume the vertical vibration of the base is periodic, this vibration can be expressed by

\[
f(t) = N \cos \theta t
\]

where \( N \) is the amplitude and \( \theta \) is the frequency of the vibration. The acceleration of the vibration will be

\[
\ddot{f}(t) = -N \theta^2 \cos \theta t
\]

Therefore the equation of motion becomes

\[
\begin{pmatrix}
1 & 0 & \eta_1 \\
0 & 1 & \eta_2
\end{pmatrix}
+ \begin{pmatrix}
(3-2\sqrt{2}) & 0 & \eta_1 \\
0 & (3+2\sqrt{2}) & \eta_2
\end{pmatrix}
+ \begin{pmatrix}
(3-2\sqrt{2}) & 0 & \eta_1 \\
0 & (3+2\sqrt{2}) & \eta_2
\end{pmatrix}
+ (g - N\theta^2 \cos \theta t) \lambda m
\begin{pmatrix}
-(2-\sqrt{2}) & 0 & \eta_1 \\
0 & -(2+\sqrt{2}) & \eta_2
\end{pmatrix}
= 0
\]

\[
- \frac{p_l}{2\sqrt{2}} \begin{pmatrix}
(2\sqrt{2} - 1 - 2a) & (2a - 1) & \eta_1 \\
-(2a - 1) & (2\sqrt{2} + 1 + 2a) & \eta_2
\end{pmatrix}
= 0
\]

or in the matrix notation
\[ [M] \{\ddot{\eta}\} + [B] \{\dot{\eta}\} + ([E] - (g - \pi^2 \cos \theta t) [A])\{\eta\} = 0 \quad (4.1) \]

where

\[ [M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ [B] = \begin{bmatrix} (3 - 2\sqrt{2}) & 0 \\ 0 & (3 + 2\sqrt{2}) \end{bmatrix} \]

\[ [E] = \begin{bmatrix} (3-2\sqrt{2}) & 0 \\ 0 & (3+2\sqrt{2}) \end{bmatrix} - \frac{2\gamma}{2\sqrt{2}} \begin{bmatrix} (2\sqrt{2}-1-2a) & (2a-1) \\ -(2a-1) & (2\sqrt{2}+1+2a) \end{bmatrix} \]

\[ [A] = \begin{bmatrix} (2 - \sqrt{2}) & 0 \\ 0 & (2 + \sqrt{2}) \end{bmatrix} \]

\[ \{\eta\} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \]

It should be noted that the matrix \([E]\) is not symmetric if "a" is not equal to \(\frac{1}{2}\).

To investigate exactly the stability of the solutions for Equation 4.1 is very complicated. Hence the principal regions of instability are considered only.

4.2 EQUATIONS FOR THE BOUNDARIES OF THE PRINCIPAL INSTABILITY REGIONS

Let us study the undamped case (i.e. \(b = 0\)) first. Then Equation 4.1 reduces to the form of
\[ [M] \dot{\{\eta\}} + ([E] - (g - N\theta^2 \cos \theta t) [A]) \{\eta\} = 0 \quad \text{(4.3)} \]

where notations remain the same as in Equations 4.2. Equation 4.3 represents two coupled Mathieu equations. The boundaries of the principal regions of instability for the coupled Mathieu equations can be expressed as a first approximation (see Reference 2) by:

\[ |[E] - (g \pm \frac{N\theta^2}{2})[A]| - \frac{1}{4} \theta^2[M] = 0 \quad \text{(4.4)} \]

Determinant (4.4) can be rewritten in the form of

\[ \det \begin{pmatrix} -(2a-1) \frac{p}{2\sqrt{2}} & (2\sqrt{2} - 1 - 2a) + c(3 - 2\sqrt{2}) - (g \pm \frac{N\theta^2}{2}) \lambda m(2 - \sqrt{2}) - \frac{\theta^2 \lambda^2 m}{4} \frac{\theta^2 \lambda^2 m}{4} \\ \frac{2a-1}{2\sqrt{2}} \frac{p}{2\sqrt{2}} & (2\sqrt{2} + 1 + 2a) + c(3 + 2\sqrt{2}) - (g \pm \frac{N\theta^2}{2}) \lambda m(2 + \sqrt{2}) - \frac{\theta^2 \lambda^2 m}{4} \frac{\theta^2 \lambda^2 m}{4} \end{pmatrix} = 0 \]

Expanding the above determinant and setting

\[ \frac{p}{mg} = \beta, \quad \frac{c}{\lambda mg} = \gamma, \quad \frac{N}{\lambda} = \alpha, \quad \lambda = \frac{\theta^2}{(g/\ell)} \quad \text{(4.5)} \]

lead the equations of the boundaries of the principal instability regions for the undamped case as

\[ a_0 \lambda^2 + a_1 \lambda + a_2 = 0 \quad \text{(4.6)} \]

with the coefficients

\[ a_0 = \frac{1}{16} + \frac{\alpha}{2} + \frac{\alpha^2}{2} \]

\[ a_1 = 1 \mp 2\alpha \gamma + 2\alpha - \frac{3}{2} \gamma + \frac{\beta}{2} + \frac{3\beta \alpha}{2} \pm \beta \alpha a \quad \text{(4.7)} \]

\[ a_2 = \gamma^2 - 4\gamma + 2 + 3\beta - 2\alpha \beta - 4\beta \gamma (1 - a) + \beta^2 (1 - a) \]
If damping is taken into account, the equation of motion for this case is shown in Equation 4.1. The boundaries of the principal regions of instability for this differential equation can be expressed as a first approximation by:

\[
\begin{align*}
[E] - g[A] - \frac{N_0^2}{2} [A] - \frac{1}{4} \theta^2 [C] - \frac{1}{2} \theta [B] &= 0 \\
\frac{1}{2} \theta [B] & \quad [E] - g[A] + \frac{N_0^2}{2} [A] - \frac{1}{4} \theta^2 [C]
\end{align*}
\]

or in the form of

\[
\begin{pmatrix}
\begin{array}{cccc}
1 & -(2a-1) & \frac{1}{2} \theta b & 0 \\
\frac{(2a-1)}{2\sqrt{2}} p & u_2 & 0 & -\frac{\theta b}{2} (3+2\sqrt{2}) \\
\frac{\theta b}{2} (3-2\sqrt{2}) & 0 & u_3 & -(2a-1) \frac{p}{2\sqrt{2}} \\
0 & \frac{\theta b}{2} (3-2\sqrt{2}) & \frac{(2a-1)}{2\sqrt{2}} p & u_4
\end{array}
\end{pmatrix}
= 0
\]

where

\[
\begin{align*}
u_1 &= c(3-2\sqrt{2}) - (g + \frac{N_0^2}{2}) \frac{m(2-\sqrt{2})}{2} - \frac{\theta^2 \rho^2 m}{4} - \frac{2(\sqrt{2} - 1 - 2a)}{2\sqrt{2}} p \\
u_2 &= c(3+2\sqrt{2}) - (g + \frac{N_0^2}{2}) \frac{m(2+\sqrt{2})}{2} - \frac{\theta^2 \rho^2 m}{4} - \frac{2(\sqrt{2} + 1 + 2a)}{2\sqrt{2}} p \\
u_3 &= c(3-2\sqrt{2}) - (g - \frac{N_0^2}{2}) \frac{m(2-\sqrt{2})}{2} - \frac{\theta^2 \rho^2 m}{4} - \frac{2(\sqrt{2} - 1 - 2a)}{2\sqrt{2}} p \\
u_4 &= c(3+2\sqrt{2}) - (g - \frac{N_0^2}{2}) \frac{m(2+\sqrt{2})}{2} - \frac{\theta^2 \rho^2 m}{4} - \frac{2(\sqrt{2} + 1 + 2a)}{2\sqrt{2}} p
\end{align*}
\]
Setting

\[ \lambda = \frac{6^2}{(g/k)} \quad \alpha = \frac{N}{k} \quad \beta = \frac{P}{mg} \quad \gamma = \frac{c}{km} \quad \xi = \frac{b^2}{m^2gk^3} \]

leads the following determinant

\[
\begin{vmatrix}
\tilde{u}_1 & -\frac{(2a-1)}{2\sqrt{2}} \beta & -\frac{(3-2\sqrt{2})}{2} \sqrt{\lambda \xi} & 0 \\
\frac{(2a-1)}{2\sqrt{2}} \beta & \tilde{u}_2 & 0 & -\frac{(3+2\sqrt{2})}{2} \sqrt{\lambda \xi} \\
\frac{(3-2\sqrt{2})}{2} \sqrt{\lambda \xi} & 0 & \tilde{u}_3 & -\frac{(2a-1)}{2\sqrt{2}} \beta \\
0 & \frac{(3+2\sqrt{2})}{2} \sqrt{\lambda \xi} & \frac{(2a-1)}{2\sqrt{2}} \beta & \tilde{u}_4
\end{vmatrix}
= 0
\]

where

\[
\tilde{u}_1 = \gamma(3-2\sqrt{2}) - (1+\frac{a\lambda}{2})(2-\sqrt{2}) - \lambda \frac{4}{4} - \frac{(2\sqrt{2} - 1 - 2a)}{2\sqrt{2}} \beta
\]

\[
\tilde{u}_2 = \gamma(3+2\sqrt{2}) - (1+\frac{a\lambda}{2})(2+\sqrt{2}) - \lambda \frac{4}{4} - \frac{(2\sqrt{2} + 1 + 2a)}{2\sqrt{2}} \beta
\]

\[
\tilde{u}_3 = \gamma(3-2\sqrt{2}) - (1-\frac{a\lambda}{2})(2-\sqrt{2}) - \lambda \frac{4}{4} - \frac{(2\sqrt{2} - 1 - 2a)}{2\sqrt{2}} \beta
\]

\[
\tilde{u}_4 = \gamma(3+2\sqrt{2}) - (1-\frac{a\lambda}{2})(2+\sqrt{2}) - \lambda \frac{4}{4} - \frac{(2\sqrt{2} + 1 + 2a)}{2\sqrt{2}} \beta
\]

After the determinant is expanded, the equation of the boundaries of the principal regions of instability for the damped case becomes:

\[
a_0 \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0 \quad - - - - (4.8)
\]

with the coefficients
\[ a_0 = \left(\frac{1}{16}\right)^2 - \frac{3}{16} a^2 + \frac{a^4}{4} \]

\[ a_1 = a^2(\gamma - 1 + \beta(a-1)) + \left(\frac{8}{16} - \frac{3\gamma}{16} + \frac{1}{8}\right) + \xi\left(\frac{17}{32} - \frac{3a^2}{4}\right) \]

\[ a_2 = a^2(4\gamma - \beta(3-2a) - 2 + 2\gamma\beta - 3\gamma^2 - \beta^2(a^2 - 2a + \frac{5}{4}) \]

\[ + \left(\frac{19}{8}\gamma^2 + \frac{(a-3)\beta^2}{8} + \left(\frac{a}{2} - 2\right)\gamma\beta + \beta\left(\frac{11-2a}{8}\right) - \frac{7}{2}\gamma + \frac{5}{4}\right) \]

\[ a_3 = (2 - 3\gamma + \beta)(\gamma^2 + \beta^2(1-a) + 4(a-1)\gamma\beta + \beta(3-2a) - 4\gamma + 2) \]

\[ + \frac{\xi}{4}(2\gamma^2 + \left(\frac{57}{4} - 31a + 17\beta^2\right)\beta^2 + 12 + 2(13-14a)\beta - 8\gamma + 8(a-1)\gamma\beta) \]

\[ - \frac{\xi}{16}(2a - 17)^2 \beta^2 \]

\[ a_4 = [\gamma^2 + 4(a - 1)\gamma\beta + \beta^2(1-a) + \beta(3 - 2a) - 4\gamma + 2]^2 \]

It should be noted that Equations 4.6 and 4.8 denote the boundaries of the principal instability regions only. For the higher instability regions those equations are not useful. It should also be pointed out that Equations 4.6 and 4.8 are only applicable for the small value of \(a\) (i.e., for the small amplitude of the vertical vibration of the base).

4.3 **DETERMINATION OF THE REGIONS OF DYNAMICAL INSTABILITY FOR THE CASE**

**THE LAG PARAMETER "a" = 1**

The lag parameter \(a = 1\) is the case that the system is subjected to tangential end force (see Figure 4.2). In this case the equations of the boundaries of principal instability regions for the undamped system
Figure 4.2: Two Degree of Freedom System Under Base Motion and Tangential End Force
(b = 0) become (see Equations 4.6)

\[ a_0 \lambda^2 + a_1 \lambda + a_2 = 0 \]  \hspace{1cm} - - - - (4.10)

with the coefficients

\[ a_0 = \frac{1}{16} + \frac{\alpha}{2} + \frac{\alpha^2}{2} \]

\[ a_1 = 1 \pm 2\alpha \pm 2\alpha \gamma - \frac{3}{2} \alpha + \frac{\beta}{2} + \frac{\beta \alpha}{2} \]

\[ a_2 = \gamma^2 + 2 + \beta - 4\gamma \]  \hspace{1cm} - - - - (4.11)

According to Equation 4.10 and by using a computer, stability charts are plotted as shown in a series of figures (from 4.3 to 4.12). In these figures the value \( \gamma = \frac{11}{2} \) is used and only the principal regions are concerned.

By examining those charts, it is seen that when the tangential loading is increased, the principal region of instability for the first mode (in the figure the region caused by the first mode has a higher value of \( \left( \frac{W_2}{\theta} \right)^2 \)), will become narrow; but the principal region of instability for the second mode (the left region in the figure), will become large. The positions of those two principal regions of instability will change also. When the tangential end force increases to the critical value \( P_* \) (see Figure 3.2 in Chapter 3); which is the critical value of the nonconservative end force for the system without base (ground) motion, those two regions of instability merge into one. In those figures (4.3 to 4.12) a parameter space with \( \left( \frac{W_2}{\theta} \right)^2 \) and \( \alpha \) is used, where \( W_2 \) is the natural frequency of the second mode for the system without any end loading. The quantity \( W_2 \) has been defined in Equation 2.9, and \( \alpha \) and \( \theta \)
Figure 4.3: Principal Regions of Instability for "a" = 1 and p = 0
Figure 4.4: Principal Regions of Instability for \( a^* = 1 \) and \( p = 0.1P^* \)
Figure 4.5: Principal Regions of Instability for "a" = 1 and p = 0.3p*
Figure 4.6: Principal Regions of Instability for "a" = 1 and p = 0.6P*
Figure 4.7: Principal Regions of Instability for \( a = 1 \) and \( p = 0.668 P_* \).
Figure 4.8: Principal Regions of Instability for "a" = 1 and p = 0.7P*

\[ \alpha = 1. \]
\[ \frac{P}{P_*} = 0.7 \]
\[ \gamma = \frac{c}{lg} = \frac{11}{2} \]
\[ P_* = 10mg \]
Figure 4.9: Principal Regions of Instability for "a" = 1 and p = 0.8P

\[
a = 1 \\
\frac{p}{P_*} = 0.8 \\
\gamma = \frac{c}{\ell mg} = \frac{11}{2} \\
P_* = 10mg
\]
Figure 4.10: Principal Regions of Instability for "a" = 1 and p = 0.9P*
Figure 4.11: Principal Regions of Instability for \( a = 1 \) and \( p = 0.95P* \)
Figure 4.12: Principal Regions of Instability for "a" = 1 and p = P*

\[ a = 1 \]
\[ \frac{P}{P^*} = 1 \]
\[ \gamma = \frac{c}{\lambda mg} = \frac{11}{2} \]
\[ P^* = 10 \text{ mg} \]
are the parameters which denote the amplitude and the frequency of the vibration of the base.

Equation 4.10 is also plotted in a three dimensional diagram (see Figure 4.13 and 4.14). The parameters plotted are \( \alpha, \beta, \) and \( \lambda, \) with fixed \( \gamma = \frac{11}{2} \). Figure 4.14 is a part of Figure 4.13. Observe in Figure 4.14 that there is a special line. Along this line the instability region vanishes. This means that for certain combinations of \( \alpha, \beta, \) and \( \lambda, \) the principal region of instability for the first mode can be eliminated.

If damping is taken into account, the equations of the boundaries of the principal instability regions for this case \( (a = 1) \) will be (see Equation 4.8):

\[
a_0 \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0 \tag{4.12}
\]

where

\[
a_0 = \left( \frac{1}{16} \right)^2 - \frac{3}{16} \alpha^2 + \frac{\alpha^4}{4}
\]

\[
a_1 = \alpha^2 \left( \frac{\gamma}{2} - 1 \right) + \left( \frac{\beta}{16} - \frac{3\gamma}{16} + \frac{1}{8} \right) + \xi \left( \frac{17}{32} - \frac{3\alpha^2}{4} \right)
\]

\[
a_2 = \alpha^2 \left( 4\gamma - \beta - 2 + 2\beta \gamma - 3\gamma^2 - \frac{\beta^2}{4} \right) + \left( \frac{19\gamma^2}{8} + \frac{\beta^2}{4} + \frac{5}{4} \right)
\]

\[
- \frac{3\gamma \beta}{2} + \frac{9\beta}{8} - \frac{7\gamma}{2} \right) + \xi \left( \frac{5}{2} - \frac{3\gamma}{4} - \frac{\beta}{4} \right) + \frac{\xi^2}{16}
\]

\[
a_3 = (2-3\gamma+\beta) \left( \gamma^2 + 2 + \beta - 2\gamma \right) + \frac{\xi}{4} \left( 2\gamma^2 + \frac{\beta^2}{4} + 12 - 2\beta - 8\gamma \right) - \frac{\xi^2 \beta^2}{16}
\]

\[
a_4 = (\gamma^2 + 2 + \beta - 4\gamma)^2 \tag{4.13}
\]
Figure 4.13: Principal Regions of Instability for \( a = 1 \) (Three Dimensional Diagram)
Figure 4.14: A Part of Figure 4.13.

\[ a = 1 \]
\[ \gamma = \frac{c}{f \cdot mg} = \frac{11}{2} \]
\[ P_x = 10 \text{ mg} \]

Along this line instability region vanishes

Loading Parameter
According to Equation 4.12 and by using computer, stability charts are plotted in the parameter space with \( \left( \frac{w^2}{\theta} \right)^2 \) and \( \alpha \) as shown in Figures 4.3 to 4.12. The dotted lines indicate the boundaries of instability for the system including damping. It can be seen that as the loading increases, the boundary of the principal region of instability for the second mode will move up little by little, but the boundary of the instability region for the first mode will move down rapidly until the loading is increased to the critical value \( P = \bar{P}_* = 0.668 P_* \) (\( P_* \) is the critical value of the nonconservative (circulatory) end force for the damped system without base excitation). At this value of loading, the boundary of instability region touches with the \( \left( \frac{w^2}{\theta} \right)^2 \) axis. After the loading becomes larger than \( \bar{P}_* \), the boundary of the instability region for the first mode will move up very fast. When the load \( P \) increases to \( 0.9 P_* \), the minimum point of the boundary (for the first mode) is above \( \alpha = 0.12 \). The second interesting result is that when \( P \) is quite close to the critical value \( \bar{P}_* \), the shifting of the boundary of instability region for the first mode will become significant. This means that when \( P \) is close to \( \bar{P}_* \), the instability region for the first mode will be slightly shifted (in the direction of decreasing frequencies) with respect to the region of instability for the system without damping. But the instability region for the second mode does not be shifted. The shifting effect of damping for the instability region has been mentioned by Bolotin\(^{(2)}\). He stated that unless the damping was sufficiently large, such a phenomenon would not exist. For the present problem, the damping is quite small, which can be considered as an exception to Bolotin's statement. The third result from the figures is that when the force is
equal to the critical value $\bar{P}_*$, the boundary of the principal region of instability for the first mode will touch with the horizontal axis. This means that when $P$ equals to $\bar{P}_*$, the critical value of the excitation parameter $\alpha$ for the first mode is zero.

4.4 DETERMINATION OF THE REGIONS OF DYNAMICAL INSTABILITY FOR THE CASE

THE LAG PARAMETER $\alpha = 1/2$

If $\alpha = 1/2$ is assumed, Equation 4.1 will reduce to two uncoupled differential equations. With the substitution $\tau = \theta t$, these equations can be put in the standard forms

$$\frac{d^2 \eta_k}{d\tau^2} + \frac{\varepsilon_k}{\theta} \frac{d\eta_k}{d\tau} + (\delta_k + \varepsilon_k \cos \tau) \eta_k = 0 \quad (k = 1, 2) \quad \cdots \quad (4.14)$$

where

$$\varepsilon_1 = \frac{b}{\ell_2^m} (3 - 2\sqrt{2})$$

$$\varepsilon_2 = \frac{b}{\ell_2^m} (3 + 2\sqrt{2})$$

$$\delta_1 = \left( \frac{\omega_v}{\theta} \right) = \frac{1}{\theta^2} \left[ \frac{c}{\ell_2^m} (3 - 2\sqrt{2}) - \frac{g}{\ell} (2 - \sqrt{2}) - \frac{p}{\dot{\xi}_m} \left( \frac{2 - \sqrt{2}}{2} \right) \right]$$

$$= \frac{(g/\theta)}{\theta^2} \left[ \gamma (3 - 2\sqrt{2}) - (2 - \sqrt{2}) - \beta \left( \frac{2 - \sqrt{2}}{2} \right) \right]$$

$$= \left( \frac{\omega_v}{\theta} \right)^2 \left[ \gamma (3 - 2\sqrt{2}) - (2 - \sqrt{2}) - \beta \left( \frac{2 - \sqrt{2}}{2} \right) \right]$$

$$= \left( \frac{\omega_v}{\theta} \right)^2 \left[ \gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right) \right]$$

$$= \frac{(g/\theta)}{\theta^2} \left[ \gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right) \right]$$

$$= \left( \frac{\omega_v}{\theta} \right)^2 \left[ \gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right) \right]$$

$$= \frac{(g/\theta)}{\theta^2} \left[ \gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right) \right]$$

$$= \left( \frac{\omega_v}{\theta} \right)^2 \left[ \gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right) \right]$$

$$= \frac{(g/\theta)}{\theta^2} \left[ \gamma (3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta \left( \frac{2 + \sqrt{2}}{2} \right) \right]$$
\[ e_1 = \frac{N}{k} (2 - \sqrt{2}) = \alpha(2 - \sqrt{2}) \]

\[ e_2 = \frac{N}{k} (2 + \sqrt{2}) = \alpha(2 + \sqrt{2}) \]

and

\[ \gamma = \frac{c}{\xi mg}, \quad \alpha = \frac{N}{\xi}, \quad \beta = \frac{P}{mg}, \quad \omega_o = \sqrt{\frac{k}{\xi}} \]

\[ w_1 = \omega_o \sqrt{\frac{\gamma(3 - 2\sqrt{2}) - (2 - \sqrt{2}) - \beta\left(\frac{2 - \sqrt{2}}{2}\right)}{2}} \]

\[ w_2 = \omega_o \sqrt{\frac{\gamma(3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta\left(\frac{2 + \sqrt{2}}{2}\right)}{2}} \]

\[ \frac{d^2 \eta_k}{dt^2} + (\delta_k + \epsilon_k \cos \tau) \eta_k = 0 \quad (k = 1, 2) \quad (4.16) \]

where notations remain the same as in Equations 4.5. For small value of \( \alpha \) (i.e. small value of parametric amplitude N), the boundaries of the two principal regions of instability can be expressed by

\[ \delta_k + \frac{\epsilon_k}{\sqrt{2}} - \frac{1}{4} = 0 \quad (k = 1, 2) \quad (4.17) \]

or in the forms of

\[ \frac{1}{\lambda} \left[ \gamma(3 - 2\sqrt{2}) - (2 - \sqrt{2}) - \beta\left(\frac{2 - \sqrt{2}}{2}\right) \right] - \alpha\left(\frac{2 - \sqrt{2}}{2}\right) - \frac{1}{4} = 0 \]

and

\[ \frac{1}{\lambda} \left[ \gamma(3 + 2\sqrt{2}) - (2 + \sqrt{2}) - \beta\left(\frac{2 + \sqrt{2}}{2}\right) \right] - \alpha\left(\frac{2 + \sqrt{2}}{2}\right) - \frac{1}{4} = 0 \]

From the above two equations, the equations for the boundaries of the two principal instability regions for the undamped differential equations
Figure 4.15: Principal Regions of Instability for "a" = 1 and p = 0.1P_∗.
Parameter Frequency

Figure 4.16: Principal Regions of Instability for $a' = 1/2$ and $p = 0.3P_*$
Figure 4.17: Principal Regions of Instability for \( a = \frac{1}{2} \) and \( p = 0.6p_\ast \)

\[
\frac{p}{p_\ast} = 0.6 \\
\gamma = \frac{c}{\ell mg} = \frac{11}{2} \\
p = 0.7331 \text{ mg} \\
p_\ast = 1.222 \text{ mg}
\]
Figure 4.18: Principal Regions of Instability for "a" = 1/2 and p = 0.7Pₚ
Figure 4.19: Principal Regions of Instability for \( a = \frac{1}{2} \) and \( p = 0.8P^* \)
Figure 4.20: Principal Regions of Instability for $a = 1/2$ and $p = 0.9P_*$
Figure 4.21: Principal Regions of Instability for "a" = 1/2 and p = P_*
(4.16) can be written as

$$a_0 \lambda^2 + a_1 \lambda + a_2 = 0$$  \hspace{1cm} (4.18)

with the coefficients

$$a_0 = \frac{1}{16} \alpha + \frac{a^2}{2}$$

$$a_1 = 1 \pm 2 \alpha \gamma \pm 2 \alpha - \frac{3}{2} \gamma + \frac{\beta}{2} + \frac{3 \beta \alpha}{2} \pm \frac{1}{2} \beta \alpha$$

$$a_2 = \gamma^2 - 4 \gamma + 2 + 2 \gamma^2 + 2 \beta - 2 \beta \gamma + \frac{\beta^2}{2}$$  \hspace{1cm} (4.19)

Equation 4.18 can be obtained directly by substituting $a = 1/2$ into Equation 4.6. According to Equations 4.18 and by using a computer, the stability charts could be plotted in the parameter space with $(\frac{\omega_2}{\theta})^2$ and $\alpha$ as shown in a series of figures (from 4.15 to 4.21). In those figures we have taken $\gamma = \frac{11}{2}$. If those figures are examined, it could be seen that when the nonconservative (circulatory) end force increases, the instability regions for both the first and the second modes will be shifted towards the direction decreasing frequencies. But the instability region for the first mode is shifted much faster than the instability region for the second mode. And when the circulatory end force increases to the critical value $P_*$ (see Figure 3.2 in Chapter 3, which is in respect to the system without base excitation), the position of the instability region for the first mode will move to infinity. This corresponds to the fact that as the end load approaches the critical value, the natural frequency of the first mode tends to zero. For principal region of instability, the parametric frequency that induces instability is in the
n eighborhood of twice the natural frequency. Since the natural frequency tends to zero, the parametric frequency to cause principle region of instability will tend to zero also. But the boundaries of the instability region must intersect with the vertical $\alpha$ axis at fixed points for any value of the loading parameter $\beta$. Hence when $P$ increases to $P_*$, the boundary of the principal regions of instability for the first mode becomes a horizontal line passing through that specified point and parallel to the horizontal $\left(\frac{w_2}{\beta}\right)^2$ axis (see Appendix 1). Thus in Figure 4.21 there is only one principal region of instability. Another interesting phenomenon is that the larger the circulatory end force, the bigger the instability regions for both the first and the second modes.

Let us go back to Equation 4.16. The stability chart with larger value of $\alpha$ for those two uncoupled Mathieu equations is easy to construct (12). In different cases of nonconservative force $P$ the stability charts are shown in Figures 4.22 to 4.25. It should be noted that when $p = 0$, the stability chart would be the same as we showed in Figure 2.3. From Figure 4.25 it will be known that even the nonconservative end force $P$ is larger than the critical value $P_*$ (which respects to the critical value for the system without base excitation), there still exists stable regions.

If damping is taken into account, the equations of the boundaries of the principal instability regions for this case ($a = 1/2$) will be (see Equation 4.8):

$$a_0 \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0$$

(4.20)

where
Figure 4.22: The Stability Chart for $a = \frac{1}{2}$ and $p = 0.1P_*$
Figure 4.23: The Stability Chart for \( a'' = 1/2 \) and \( p = 0.9p_* \).
Figure 4.24: The Stability Chart for "a" = 1/2 and p = P*
Figure 4.25: The Stability Chart for "a" = 1/2 and p = 2P*
According to Equation 4.20 and using a computer, we plot the stability charts in the parameter space with \( \left( \frac{w_2}{\theta} \right)^2 \) and \( \alpha \) as shown in Figures 4.15 to 4.21. The dotted lines indicate the boundaries of instability regions for the system including damping. In those figures we use the damping parameter \( \xi = 0.05 \). An examination of these boundaries shows that as the nonconservative end force increases, the boundary of the principal instability regions for the second mode will move up very slowly, but the boundary of the instability region for the first mode will move up more rapidly. When the force \( P \) increases to 0.7 \( P_\ast \) (\( P_\ast \) is the critical value of the force \( P \) for the system without the base excitation), the region of instability for the first mode will be settled above the range \( \alpha \leq 0.12 \). Hence in our figures there are no boundaries of the instability regions.
for the first mode after \( P \) is equal to or larger than 0.7 \( P \).

4.5 DETERMINATION OF THE REGIONS OF DYNAMICAL INSTABILITY FOR THE CASE

THE LAG PARAMETER \( a = 0.811^* \)

A nonconservative and loading with the lag parameter \( a = 0.811 \) is of particular interest. Without base excitation this case separates the two types of behaviour of the system when the loading becomes critical. When \( a < 0.811 \), instability may occur by divergence and when \( a > 0.811 \), instability may occur by flutter. Hence as the magnitude of the nonconservative (circulatory) end force changes in this case (\( a = 0.811 \) and the base excitation is included), the variation of the principal regions of instability should have the features of both the cases \( a = 1 \) and \( a = 1/2 \).

Let us study the undamped system first. According to Equation 4.6 and by substituting \( a = 0.8107125 \), \( \gamma = \frac{11}{2} \) into this equation, the principal instability regions could be plotted in the parameter space with \( \left( \frac{w_2}{\rho} \right)^2 \) and \( \alpha \) as shown in Figures 4.26 to 4.35. If those charts are investigated it will be seen that when the nonconservative (circulatory) end force is increased, the principal region of instability for the first mode will grow very rapidly and the position of the instability region will shift towards the direction of decreasing frequencies. When the nonconservative force increases to \( P^{**} \), (see Section 3.2 in Chapter 3, that is the critical value of the force under which the system without base excitation will lose stability by divergence), the vertex of the instability region for the first mode will move to infinity. But

* The more accurate value is 0.8107125
Equation 4.6 shows that the boundaries of the two principal regions of instability must intersect with the vertical $\alpha$ axis at the fixed points for any value of the loading parameter $\beta$. Hence when $P$ increases to $P^{**}$, the boundary of instability region for the first mode will become a horizontal line passing through that specified point and parallel to horizontal $\left(\frac{w_2}{\delta}\right)^2$ axis. This is similar to the stability chart for the case $\alpha = 1/2$ with the critical end loading and the base excitation (see Figure 4.24). (If the system without base excitation is held by some external means so that it does not lose its stability by divergence at $P^{**}$, it becomes possible to load it beyond the critical value $P^{**}$).

After the circulatory end force becomes larger than $P^{**}$, the instability region for the first mode will become narrow rapidly and the position of this instability region will move back from the infinity. The variation of the instability region for the second mode is simple. As the circulatory end force increases, this instability region will become larger slowly.

When the nonconservative (circulatory) end loading continues to increase beyond $P^{**}$ and up to $P^*$ (see Section 3.2 in Chapter 3: this is the critical value of the loading $P$ over which the system without base excitation will lose instability by flutter), the two principal regions of instability will merge into one as shown in Figure 4.35.

Next the damped system is considered. According to Equation 4.8 and by setting $a = 0.8107125$, $\gamma = \frac{11}{2}$, into this equation, the stability charts are drawn as shown in Figures 4.26 to 4.35 in the parameter space $\left(\frac{w_2}{\delta}\right)^2$ and $\alpha$. The dotted lines indicate the boundaries of instability regions for the system including damping. It is seen that when the
circulatory end force increases, the boundary of the instability region for the second mode will move up little by little as in the case \( a'' = 1 \), but the boundary of the instability region for the first mode will move up very fast. When the end force \( P \) increases to 0.4 \( P_\ast \), the principal region of instability for the first mode will be settled above \( \alpha = 0.12 \):

hence in the figures there are no boundaries of the principal instability region for the first mode after \( P \) is equal to, or larger than 0.4 \( P_\ast \).

But when the circulatory end force increases and becomes larger than the critical value \( P_{**} \) the principal region of instability for the first mode will move down rapidly. When \( P \) equals to \( \bar{P}_\ast \), (see Section 3.2 in Chapter 3) the boundary of the instability region for the first mode will touch with the horizontal axis \( \left( \frac{\omega_2}{\omega_0} \right)^2 \) and shift (in the direction of decreasing frequencies) with respect to the region of instability for the system without damping. Hence we have the same argument as in the case \( a = 1 \).

After \( P \) becomes larger than \( \bar{P}_\ast \), the principal region of instability for the first mode will move up again. When \( P \) increases to 11.1013 mg (i.e. \( \frac{p}{p_\ast} = 0.9 \)), this region of instability will be settled above \( \alpha = 0.12 \): thus in our figures, there is no boundary of the instability region for the first mode after \( P \) is equal to, or larger than, 0.9 \( P_\ast \).

\[ \bar{p}_\ast \] is the critical value of the force \( P \) for the damped system without base excitation. If \( P \) is over this value, the system will lose stability by flutter.
**Figure 4.26:** Principal Regions of Instability for $a = 0.811$ and $p = 0.1$.

- $a = 0.811$
- $\frac{p}{p_x} = 0.1$
- $r = \frac{c}{\lambda m g} = \frac{11}{2}$
- $p = 1.2235 \text{ mg}$
- $p_x = 12.3348 \text{ mg}$

**Variables:**
- $\xi$
- $\zeta = 0.05$
- $\zeta = 0$
Figure 4.28: Principal Regions of Instability for "a" = 0.811 and p = 0.4P.*
Figure 4.29: Principal Regions of Instability for \(a = 0.811\) and \(p = 0.5p_*\)
Figure 4.30: Principal Regions of Instability for $\alpha = 0.811$ and $p = 0.6p_*$.
Figure 4.31: Principal Regions of Instability for "a" = 0.811 and \( p = 0.7p_\ast \).
Figure 4.32: Principal Regions of Instability for "a" = 0.811 and $p = 0.723P_\ast = P_\ast$. 
Figure 4.33: Principal Regions of Instability for \(a = 0.811\) and \(p = 0.8p^*\).
Figure 4.34: Principal Regions of Instability for \( a = 0.811 \) and \( p = 0.9P_* \).
Figure 4.35: Principal Regions of Instability for "a" = 0.811 and p = P*
CHAPTER 5
MULTIPLE DEGREE OF FREEDOM SYSTEM

In the previous chapters the problems about a two degree of
freedom system were studied. From Chapter 2, we know for a two degree
of freedom system (without circulatory end force, but under the base
excitation), that when the spring constants $c_k$, the damping coefficients
$b_k$ and the concentrated masses $m_k$ ($k = 1, 2$) have the following
relationship:

$$c_1 = c_2 = c, \quad b_1 = b_2 = b, \quad m_1 = m_2 = m$$

the classical normal modes exist. In this chapter we seek the conditions
for a $N$ degree of freedom system of those $c_k$, $b_k$ and $m_k$ ($k = 1, 2, 3, \ldots n$),
such that the classical modes can be obtained.

First a three degree of freedom system will be investigated:
then we expand the analysis to four, five .... and $N$ degree of freedom
systems.

5.1 THREE DEGREE OF FREEDOM SYSTEM

The model is a triple pendulum, as shown in Figure 5.1, com­
oposed of three rigid weightless bars of equal length $l$, with three con­
centrated masses $m_1$, $m_2$ and $m_3$ at the ends of each bar. At the joints
the restoring moments are $c_1\dot{\phi}_1 + b_1\dot{\phi}_1$, $c_2(\dot{\phi}_2 - \dot{\phi}_1) + b_2(\ddot{\phi}_2 - \ddot{\phi}_1)$ and
$c_3(\dot{\phi}_3 - \dot{\phi}_2) + b_3(\ddot{\phi}_3 - \ddot{\phi}_2)$. The damping coefficients $b_1$ and $b_2$ and $b_3$ are
taken as positive, and the gravitational effects are included. The
vertical and the horizontal vibrations of the base are denoted by $f(t)$
Figure 5.1: A Triple Pendulum on Oscillating Base
and \( h(t) \).

The potential energy \( V \), the kinetic energy \( T \), and the dissipation function \( D \) are

\[
V = m_1 g \left\{ (f_0 - f(t)) + \ell(1 - \cos \phi_1) \right\}
\]

\[-m_2 g \left\{ (f_0 - f(t)) + \ell(1 - \cos \phi_1) + \ell(1 - \cos \phi_2) \right\}.
\]

\[-m_3 g \left\{ (f_0 - f(t)) + \ell(1 - \cos \phi_1) + \ell(1 - \cos \phi_2) + \ell(1 - \cos \phi_3) \right\}
\]

\[+ \frac{1}{2} c_1 \phi_1^2 + \frac{1}{2} c_2 (\phi_2 - \phi_1)^2 + \frac{1}{2} c_3 (\phi_3 - \phi_2)^2 \]

\[
T = \frac{1}{2} m_1 \left\{ (\dot{h}(t) + \ell \dot{\phi}_1 \cos \phi_1)^2 + (\ddot{h}(t) - \ell \dot{\phi}_1 \sin \phi_1)^2 \right\}
\]

\[+ \frac{1}{2} m_2 \left\{ (\dot{h}(t) + \ell \dot{\phi}_1 \cos \phi_1 + \ell \dot{\phi}_2 \cos \phi_2)^2 + (\ddot{h}(t) - \ell \dot{\phi}_1 \sin \phi_1 - \ell \dot{\phi}_2 \sin \phi_2)^2 \right\}
\]

\[+ \frac{1}{2} m_3 \left\{ (\dot{h}(t) + \ell \dot{\phi}_1 \cos \phi_1 + \ell \dot{\phi}_2 \cos \phi_2 + \ell \dot{\phi}_3 \cos \phi_3)^2 \right\}
\]

\[+ (f(t) - \ell \dot{\phi}_1 \sin \phi_1 - \ell \dot{\phi}_2 \sin \phi_2 - \ell \dot{\phi}_3 \sin \phi_3)^2 \]

\[
D = \frac{1}{2} b_1 \dot{\phi}_1^2 + \frac{1}{2} b_2 (\dot{\phi}_2 - \dot{\phi}_1)^2 + \frac{1}{2} b_3 (\dot{\phi}_3 - \dot{\phi}_2)^2
\]

Lagrange's equations in the form

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}_k} \right) - \frac{\partial T}{\partial \phi_k} + \frac{\partial V}{\partial \phi_k} + \frac{\partial D}{\partial \dot{\phi}_k} = 0 \quad (k = 1, 2)
\]

are employed to establish the linear equation of motion.
\[
\ddot{z}^2 = \begin{pmatrix}
    (m_1 + m_2 + m_3) & (m_2 + m_3) & m_3 \\
    (m_2 + m_3) & (m_2 + m_3) & m_3 \\
    m_3 & m_3 & m_3
\end{pmatrix}
\begin{pmatrix}
    \phi_1 \\
    \phi_2 \\
    \phi_3
\end{pmatrix}
+ \begin{pmatrix}
    (b_1 + b_2) & -b_2 & 0 \\
    -b_2 & (b_2 + b_3) & -b_3 \\
    0 & -b_3 & b_3
\end{pmatrix}
\begin{pmatrix}
    \phi_1 \\
    \phi_2 \\
    \phi_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
    (c_1 + c_2) & -c_2 & 0 \\
    -c_2 & (c_2 + c_3) & -c_3 \\
    0 & -c_3 & c_3
\end{pmatrix}
\begin{pmatrix}
    \phi_1 \\
    \phi_2 \\
    \phi_3
\end{pmatrix}
- \begin{pmatrix}
    (m_1 + m_2 + m_3) & 0 & 0 \\
    0 & (m_2 + m_3) & 0 \\
    0 & 0 & m_3
\end{pmatrix}
\begin{pmatrix}
    \phi_1 \\
    \phi_2 \\
    \phi_3
\end{pmatrix}
\]

\[
= - \begin{pmatrix}
    (m_2 + m_3) \\
    (m_2 + m_3) \\
    m_3
\end{pmatrix} h(t)
\]

Interested primarily in parametric excitation (resonance) we will consider the case where there is no horizontal base motion. Thus \( h(t) = 0 \) (i.e. \( \ddot{h}(t) = 0 \)) is assumed. And the transformation

\[
\begin{pmatrix}
    z_1 \\
    z_2 \\
    z_3
\end{pmatrix} = \begin{pmatrix}
    1 & 0 & 0 \\
    1 & 1 & 0 \\
    1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
    \phi_1 \\
    \phi_2 \\
    \phi_3
\end{pmatrix}
\]

is used. The equation of motion becomes
We consider the specific case

\[ c_1 = 3c \quad b_1 = 3b \quad m_1 = m \]
\[ c_2 = 6c \quad b_2 = 6b \quad m_2 = m \]
\[ c_3 = 2c \quad b_3 = 2b \quad m_3 = m \]

(Under such conditions the coefficient matrices of the Equation 5.1 commute).

Then the equation of motion is reduced to the form of

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{z}_1 \\
\ddot{z}_2 \\
\ddot{z}_3 \\
\end{bmatrix}
= b
\begin{bmatrix}
-16 & 14 & -4 \\
29 & -16 & 2 \\
2 & -4 & 2 \\
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\end{bmatrix}
+ c
\begin{bmatrix}
-16 & 14 & -4 \\
29 & -16 & 2 \\
2 & -4 & 2 \\
\end{bmatrix}
\begin{bmatrix}
\dddot{z}_1 \\
\dddot{z}_2 \\
\dddot{z}_3 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
-5 & 2 & 0 \\
2 & -3 & 1 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix} + (g + \dot{f}(t)) \mathbf{I}_m \begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix} = 0
\]

Using similarity transformation and setting \(\{z\} = [\phi] \{\eta\}\)

where
\[
[\phi] = \begin{bmatrix}
8.2018 & -0.9567 & 0.2549 \\
-5.2899 & -1.2943 & 0.5842 \\
1 & 1 & 1
\end{bmatrix}
\]

We obtain the equation of motion
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2 \\
\dot{\eta}_3
\end{bmatrix} + b
\begin{bmatrix}
39.5634 & 0 & 0 \\
0 & 5.2637 & 0 \\
0 & 0 & 0.1729
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix} + c
\begin{bmatrix}
39.5634 & 0 & 0 \\
0 & 5.2637 & 0 \\
0 & 0 & 0.1729
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix} - (g + \ddot{f}(t)) \mathbf{I}_m
\begin{bmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2 \\
\dot{\eta}_3
\end{bmatrix} = 0
\]
where \( f(t) \) denotes the acceleration of the vertical vibrations of the base (ground). If the vertical vibration of the base is assumed to be periodic, this vibration can be expressed by

\[
f(t) = N \cos \theta t
\]

where \( N \) is the amplitude and \( \theta \) is the frequency of the vibration. The acceleration of the vibration will be

\[
\ddot{f}(t) = -N\theta^2 \cos \theta t
\]

The equation of motion becomes

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{\eta}_1 \\
\ddot{\eta}_2 \\
\ddot{\eta}_3
\end{bmatrix}
+ b
\begin{bmatrix}
39.5634 & 0 & 0 \\
0 & 5.2637 & 0 \\
0 & 0 & 0.1729
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix}
+ c
\begin{bmatrix}
39.5634 & 0 & 0 \\
0 & 5.2637 & 0 \\
0 & 0 & 0.1729
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix}
\begin{bmatrix}
6.2899 & 0 & 0 \\
0 & 2.2943 & 0 \\
0 & 0 & 0.4158
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix}
\begin{bmatrix}
-\theta^2 \cos \theta t
\end{bmatrix}
\]

Hence we have three uncoupled differential equations. By setting \( \tau = \theta t \), those equations can be put in the standard forms.
\[
\frac{d^2 \eta_k}{d\tau^2} + \frac{\varepsilon_k}{\varepsilon} \frac{d \eta_k}{d\tau} + (\delta_k + \varepsilon_k \cos \tau) \eta_k = 0 \quad (k = 1, 2, 3)
\]  
--- (5.2)

where

\[\varepsilon_1 = 39.5634 \frac{b}{\xi^2 m}\]
\[\varepsilon_2 = 5.2637 \frac{b}{\xi^2 m}\]
\[\varepsilon_3 = 0.1729 \frac{b}{\xi^2 m}\]

\[\delta_1 = \left( \frac{w_1}{\theta} \right)^2 = \frac{1}{\theta^2} \left[ 39.5634 \frac{c}{\xi^2 m} - 6.2899 \frac{g}{\xi} \right]
\]
\[= \frac{g/\xi}{\theta^2} \left[ 39.5634 \gamma - 6.2899 \right] = \left( \frac{w_0}{\theta} \right)^2 \left[ 39.5634 \gamma - 6.2899 \right]
\]

\[\delta_2 = \left( \frac{w_2}{\theta} \right)^2 = \frac{1}{\theta^2} \left[ 5.2637 \frac{c}{\xi^2 m} - 2.2943 \frac{g}{\xi} \right]
\]
\[= \frac{g/\xi}{\theta^2} \left[ 5.2637 \gamma - 2.2943 \right] = \left( \frac{w_0}{\theta} \right)^2 \left[ 5.2637 \gamma - 2.2943 \right]
\]

\[\delta_3 = \left( \frac{w_3}{\theta} \right)^2 = \frac{1}{\theta^2} \left[ 0.1729 \frac{c}{\xi^2 m} - 0.4158 \frac{g}{\xi} \right]
\]
\[= \frac{g/\xi}{\theta^2} \left[ 0.1729 \gamma - 0.4158 \right] = \left( \frac{w_0}{\theta} \right)^2 \left[ 0.1729 \gamma - 0.4158 \right]
\]

\[e_1 = 6.2899 \frac{N}{\xi} = 6.2899 \alpha\]
\[e_2 = 2.2943 \frac{N}{\xi} = 2.2943 \alpha\]
\[e_3 = 0.4188 \frac{N}{\xi} = 0.4158 \alpha\]
Neglecting the damping (i.e. \( b = 0 \)), the equations of motion will be

\[
\frac{d^2 \eta_k}{d\tau^2} + (\delta_k + e_k \cos \tau) \eta_k = 0 \quad (k = 1, 2, 3) \quad - - - - (5.4)
\]

where \( \delta_k \) and \( e_k \) are the same quantities as was defined in Equation 5.3.

The stability chart for those equations is easy to construct (see Reference 12), and is given in Figure 5.2. In this chart only the principal regions of instability are concerned, and the spring constant parameter \( \gamma \) is used as \( \frac{11}{2} \).

If the damping is taken into account, the system will be
governed by Equation 5.2. The three principal regions of instability for those equations are also shown in Figure 5.2. The dotted lines denote the boundaries of the principal instability regions for the system including damping. The damping parameter \( \xi (= \frac{b^2}{\ell^2 m^2 g}) \) is used as 0.05 which corresponds to the case the damping factor \( \xi (= \frac{c_3}{w^{3}}) \) for the lowest mode (the first mode) is equal to 0.026.

It should be noted that in the damped case the principal instability region for the first mode is settled much above the range.
Figure 5.2: Principal Regions of Instability

\[ \gamma = \frac{c}{\lambda m g} = \frac{11}{2} \]
Figure 5.3: Four Degree of Freedom System on Oscillating Base
\( a < 0.1 \). Hence in the figure there is no boundary of the instability region for the first mode including damping.

5.2 FOUR DEGREE OF FREEDOM SYSTEM

For a four degree of freedom system (see Figure 5.3) consider at the hinges, the restoring moments are

\[
\begin{align*}
&c_1 \dot{q}_1 + b_1 \ddot{q}_1, \\
&c_2 (\dot{q}_2 - \dot{q}_1) + b_2 (\ddot{q}_2 - \ddot{q}_1), \\
&c_3 (\dot{q}_3 - \dot{q}_2) + b_3 (\ddot{q}_3 - \ddot{q}_2), \\
&c_4 (\dot{q}_4 - \dot{q}_3) + b_4 (\ddot{q}_4 - \ddot{q}_3).
\end{align*}
\]

Using Lagrange's equations

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial U}{\partial q_k} \right) + \frac{\partial D}{\partial q_k} = 0 \quad (k = 1, 2, 3, 4)
\]

and following the same procedure as we did before, we have

\[
\begin{bmatrix}
(m_1 + m_2 + m_3 + m_4) & (m_2 + m_3 + m_4) & (m_3 + m_4) & m_4 \\
(m_2 + m_3 + m_4) & (m_2 + m_3 + m_4) & (m_3 + m_4) & m_4 \\
(m_3 + m_4) & (m_3 + m_4) & (m_3 + m_4) & m_4 \\
4 & 4 & 4 & 4
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2 \\
\dot{\phi}_3 \\
\dot{\phi}_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
(b_1 + b_2) & -b_2 & 0 & 0 \\
-b_2 & (b_2 + b_3) & -b_3 & 0 \\
0 & -b_3 & (b_3 + b_4) & -b_4 \\
0 & 0 & -b_4 & b_4
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
(c_1 + c_2) & -c_2 & 0 & 0 \\
-c_2 & (c_2 + c_3) & -c_3 & 0 \\
0 & -c_3 & (c_3 + c_4) & -c_4 \\
0 & 0 & -c_4 & c_4
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{bmatrix}
\]

(Equation continued on next page)
\[-(g + f(t)) \mathbf{z} = \begin{bmatrix} (m_1 + m_2 + m_3 + m_4) & 0 & 0 & 0 \\ 0 & (m_2 + m_3 + m_4) & 0 & 0 \\ 0 & 0 & (m_3 + m_4) & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = - \begin{bmatrix} (m_1 + m_2 + m_3 + m_4) \\ (m_2 + m_3 + m_4) \\ (m_3 + m_4) \\ m_4 \end{bmatrix} \ddot{h}(t)\]

\[h(t) = 0 \text{ (i.e. } \ddot{h}(t) = 0) \text{ is assumed as in the case of three degree of freedom system. And the transformation}\]

\[
\begin{bmatrix}
    z_1 \\
    z_2 \\
    z_3 \\
    z_4
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    1 & 1 & 0 & 0 \\
    1 & 1 & 1 & 0 \\
    1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
    \eta_1 \\
    \eta_2 \\
    \eta_3 \\
    \eta_4
\end{bmatrix}
\]

is used. The equation of motion becomes

\[
\begin{bmatrix}
    m_1 & 0 & 0 & 0 \\
    0 & m_2 & 0 & 0 \\
    0 & 0 & m_3 & 0 \\
    0 & 0 & 0 & m_4
\end{bmatrix}
\begin{bmatrix}
    \ddot{z}_1 \\
    \ddot{z}_2 \\
    \ddot{z}_3 \\
    \ddot{z}_4
\end{bmatrix} +
\begin{bmatrix}
    (b_1 + 4b_2 + b_3) & -2(b_2 + b_3) & b_3 & 0 \\
    -2(b_2 + b_3) & (b_2 + 4b_3 + b_4) & -2(b_3 + b_4) & b_4 \\
    b_3 & -2(b_3 + b_4) & (b_3 + 4b_4) & -2b_4 \\
    0 & b_4 & -2b_4 & b_4
\end{bmatrix}
\begin{bmatrix}
    z_1 \\
    z_2 \\
    z_3 \\
    z_4
\end{bmatrix} = 0
\]

(Equation continued on next page)
Consider the case

\[ \begin{align*}
    b_1 &= 2b_2 \\
    & \quad b_2 = 6b_2 \\
    & \quad b_3 = 3b_3 \\
    & \quad b_4 = b_4 \\
    c_1 &= 2c_2 \\
    & \quad c_2 = 6c_2 \\
    & \quad c_3 = 3c_3 \\
    & \quad c_4 = c_4 \\
    m_1 &= m_2 = m_3 \\
    & \quad m_4 = m_4
\end{align*} \]

Thus, the equation of motion becomes

\[ \begin{bmatrix}
    1 & 0 & 0 & 0 & \frac{z_1}{z_1} \\
    0 & 1 & 0 & 0 & \frac{z_2}{z_2} \\
    0 & 0 & 1 & 0 & \frac{z_3}{z_3} \\
    0 & 0 & 0 & 1 & \frac{z_4}{z_4}
\end{bmatrix}
\begin{bmatrix}
    29 & -18 & 3 & 0 & \frac{\dot{z}_1}{z_1} \\
    -18 & 19 & -8 & 1 & \frac{\dot{z}_2}{z_2} \\
    3 & -8 & 7 & -2 & \frac{\dot{z}_3}{z_3} \\
    0 & 1 & -2 & 1 & \frac{\dot{z}_4}{z_4}
\end{bmatrix} + \begin{bmatrix}
    b \\
    3 \\
    -8 \\
    0
\end{bmatrix} = 0 \]
The above equation of motion can be rewritten in the form of

\[ \ddot{z} + c \begin{pmatrix} 29 & -18 & 3 & 0 \\ -18 & 19 & -8 & 1 \\ 3 & -8 & 7 & -2 \\ 0 & 1 & -2 & 1 \end{pmatrix} z + (g + f(t)) \lambda_m \begin{pmatrix} -7 & 3 & 0 & 0 \\ 3 & -5 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} z = 0 \]

The above equation of motion can be rewritten in the form of

\[ \ddot{z} + b [B] \{z\} + (c[k] + (g + f(t)) \lambda_m [A]) \{z\} = 0 \]

where \([M]\) is unit matrix

\[ [B] = [K] = \begin{pmatrix} 29 & -18 & 3 & 0 \\ -18 & 19 & -8 & 1 \\ 3 & -8 & 7 & -2 \\ 0 & 1 & -2 & 1 \end{pmatrix} \]

\[ [A] = \begin{pmatrix} -7 & 3 & 0 & 0 \\ 3 & -5 & 2 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \]

and matrices \([k]\) and \([A]\) commute: i.e.

\[ [k] [A] = [A] [k] \]

Hence we know those matrices can be simultaneously transformed into
Figure 5.4: Five Degree of Freedom System on Oscillating Base
diagonal form. And so in this case, the classical modes will exist. The
reason is that $b_k$, $c_k$ and $m_k$ ($k = 1, 2, 3, 4$) are taken to have the
relationship as expressed in Equations 5.5. If $b_k$, $c_k$ and $m_k$ do not
satisfy such conditions, the classical modes will not exist.

### 5.3 FIVE DEGREE OF FREEDOM SYSTEM

Referring to Section 5.2, we know that the equation of motion
for the five degree of freedom system (see Figure 5.4) is

$$\begin{align*}
\kappa^2 \left[ \begin{array}{cccccc}
(m_1 + m_2 + m_3 + m_4 + m_5) & (m_2 + m_3 + m_4 + m_5) & (m_3 + m_4 + m_5) & (m_4 + m_5) & m_5 \\
(m_2 + m_3 + m_4 + m_5) & (m_3 + m_4 + m_5) & (m_4 + m_5) & (m_5) & m_5 \\
(m_3 + m_4 + m_5) & (m_4 + m_5) & (m_4 + m_5) & (m_5) & m_5 \\
(m_4 + m_5) & (m_4 + m_5) & (m_4 + m_5) & (m_5) & m_5 \\
m_5 & m_5 & m_5 & m_5 & m_5
\end{array} \right] \left[ \begin{array}{c}
\dot{\phi}_1 \\
\dot{\phi}_2 \\
\dot{\phi}_3 \\
\dot{\phi}_4 \\
\dot{\phi}_5
\end{array} \right] \\
+ \left[ \begin{array}{cccccc}
(b_1 + b_2) & -b_2 & 0 & 0 & 0 \\
-b_2 & (b_2 + b_3) & -b_3 & 0 & 0 \\
0 & -b_3 & (b_3 + b_4) & -b_4 & 0 \\
0 & 0 & -b_4 & (b_4 + b_5) & -b_5 \\
0 & 0 & 0 & -b_5 & b_5
\end{array} \right] \left[ \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\phi_5
\end{array} \right]
\end{align*}$$

(Equation continued on following page)
\[
\begin{bmatrix}
(c_1 + c_2) & -c_2 & 0 & 0 & 0 \\
-c_2 & (c_3 + c_4) & -c_3 & 0 & 0 \\
0 & -c_3 & (c_4 + c_5) & -c_4 & 0 \\
0 & 0 & -c_4 & (c_4 + c_5) & -c_5 \\
0 & 0 & 0 & -c_5 & -c_5
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\phi_5
\end{bmatrix}
= (g + \ddot{f}(t))k
\]

\[
\begin{bmatrix}
(m_1 + m_2 + m_3 + m_4 + m_5) & 0 & 0 & 0 & 0 \\
0 & (m_2 + m_3 + m_4 + m_5) & 0 & 0 & 0 \\
0 & 0 & (m_3 + m_4 + m_5) & 0 & 0 \\
0 & 0 & 0 & (m_4 + m_5) & 0 \\
0 & 0 & 0 & 0 & m_5
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\phi_5
\end{bmatrix}
= h(t)
\]

\[h(t) = 0: \quad \text{(i.e. } h(t) = 0) \text{ will be assumed as before, and the transformation}
\]

\[
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\phi_5
\end{bmatrix}
\]
is used. The equation of motion becomes

\[
\begin{bmatrix}
m_1 & 0 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 & 0 \\
0 & 0 & m_3 & 0 & 0 \\
0 & 0 & 0 & m_4 & 0 \\
0 & 0 & 0 & 0 & m_5
\end{bmatrix}
\begin{bmatrix}
\ddot{z}_1 \\
\ddot{z}_2 \\
\ddot{z}_3 \\
\ddot{z}_4 \\
\ddot{z}_5
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix}
(b_1 + 4b_2 + b_3) & -2(b_2 + b_3) & b_3 & 0 & 0 \\
-2(b_2 + b_3) & (b_2 + 4b_3 + b_4) & -2(b_3 + b_4) & b_4 & 0 \\
0 & -2(b_3 + b_4) & (b_3 + 4b_4 + b_5) & -2(b_4 + b_5) & b_5 \\
0 & 0 & b_4 & -2(b_4 + b_5) & (b_4 + 4b_5) & -2b_5 \\
0 & 0 & 0 & b_5 & -2b_5 & b_5
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
(c + c_3) & -2(c_2 + c_3) & c_3 & 0 & 0 \\
-2(c_1 + c_2) & (c_1 + 4c_2 + c_3) & -2(c_2 + c_3) & c_4 & 0 \\
0 & -2(c_3 + c_4) & (c_3 + 4c_4 + c_5) & -2(c_4 + c_5) & c_5 \\
0 & 0 & -2(c_4 + c_5) & (c_4 + 4c_5) & -2c_5 \\
0 & 0 & 0 & c_5 & -2c_5 & c_5
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
(g + f(t)) x
\end{bmatrix}
\]

\[
\begin{bmatrix}
-(m_1 + 2m_2 + 2m_3 + 2m_4 + 2m_5) & (m + m + m + m) \\
(m + m + m + m) & -(m_2 + 2m_3 + 2m_4 + 2m_5) & (m + m + m + m) & 0 & 0 \\
0 & (m + m + m + m) & -(m_3 + 2m_4 + m_5) & (m + m + m + m) & 0 \\
0 & 0 & (m + m + m + m) & -(m + 2m + m + m) & m \\
0 & 0 & 0 & m & -m
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5
\end{bmatrix}
= 0

- - - - - (5.6)
Let us consider the case

\[
\begin{align*}
\frac{b_1}{1} &= 2.5b \\
\frac{b_2}{2} &= 10b \\
\frac{b_3}{3} &= 6b \\
\frac{b_4}{4} &= 3b \\
\frac{b_5}{5} &= b \\
\frac{c_1}{1} &= 2.5c \\
\frac{c_2}{2} &= 10c \\
\frac{c_3}{3} &= 6c \\
\frac{c_4}{4} &= 3c \\
\frac{c_5}{5} &= c \\
\frac{m_1}{2} &= m \\
\frac{m_2}{3} &= m \\
\frac{m_3}{4} &= m \\
\frac{m_4}{5} &= m \\
\end{align*}
\]

- - - (5.7)

Then Equation 5.6 will reduce to be

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\varepsilon^2 m & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\frac{z_1}{z} \\
\frac{z_2}{z} \\
\frac{z_3}{z} \\
\frac{z_4}{z} \\
\frac{z_5}{z} \\
\end{pmatrix}
= 
\begin{pmatrix}
48.5 & -32 & 6 & 0 & 0 \\
-32 & 37 & -18 & 3 & 0 \\
6 & -18 & 19 & -8 & 1 \\
0 & 3 & -8 & 7 & -2 \\
0 & 0 & 1 & -2 & 1 \\
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5 \\
\end{pmatrix}
+ (g + f(t)) \varepsilon m
\]

(Equation continued on next page)
We can express the above equation of motion in the following form:

\[ \ddot{z} m [M] \{z\} + b[B] \{z\} + (c[k] + (g + \dot{f}(t)) \text{Im}[A]) \{z\} = 0 \]

(5.8)

where \([M]\) is unit matrix

\[
[M] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
[B] = [k] = \begin{bmatrix}
48.5 & -32 & 6 & 0 & 0 \\
-32 & 37 & -18 & 3 & 0 \\
6 & -18 & 19 & -8 & 1 \\
0 & 3 & -8 & 7 & -2 \\
0 & 0 & 1 & -2 & 1
\end{bmatrix}
\]

\[
[A] = \begin{bmatrix}
-9 & 4 & 0 & 0 & 0 \\
4 & -7 & 3 & 0 & 0 \\
0 & 3 & -5 & 2 & 0 \\
0 & 0 & 2 & -3 & 1 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}
\]

and matrices \([k]\) and \([A]\) commute; i.e.

\([k][A] = [A][k]\)
Hence we know those matrices can be simultaneously transformed into diagonal form. Meanwhile, the classical modes exist.

5.4 N-DEGREE OF FREEDOM SYSTEM

By referring to Sections 5.1, 5.2, and 5.3 it can be concluded that the equation of motion for the N degree of freedom system must be
No horizontal base movement is assumed (i.e. $h(t) = 0$) as before, and the transformation

$$
\begin{bmatrix}
z_1 \\
z_2 \\
z_n
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_{n-1} \\
\phi_n
\end{bmatrix}
$$
is used. The equation of motion will be

\[
\begin{bmatrix}
\begin{array}{cccc}
  m_1 & 0 & 0 & \vdots \\
  0 & m_2 & 0 & \vdots \\
  & \ddots & \ddots & \vdots \\
  0 & 0 & m_{n-1} & 0 \\
  0 & 0 & 0 & m_n
\end{array}
\end{bmatrix}

\begin{bmatrix}
  z_1 \\
  \vdots \\
  z_{n-2} \\
  z_{n-1} \\
  z_n
\end{bmatrix}

+ \begin{bmatrix}
  0 \\
  \vdots \\
  0 \\
  0 \\
  0
\end{bmatrix}

\begin{bmatrix}
  (b_1 + 4b_2 + b_3) & -2(b_2 + b_3) & b_3 & 0 & 0 & \vdots \\
  -2(b_2 + b_3) & (b_2 + 4b_3 + b_4) & -2(b_3 + b_4) & b_4 & 0 & \vdots \\
  b_3 & -2(b_3 + b_4) & (b_3 + 4b_4 + b_5) & \ddots & \vdots \\
  0 & b_4 & \ddots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & b_n & -2b_n \\
  0 & 0 & 0 & -2b_n & b_n \\
\end{bmatrix}
\]
Combining the results in Sections 5.1, 5.2, and 5.3 the necessary conditions for the existence of the classical modes for a two degree of freedom system are
\[ b_1 = b_2 = b \]
\[ c_1 = c_2 = c \]
\[ m_1 = m_2 = m \]

For a three degree of freedom are:
\[ b_1 = 1.5b \quad b_2 = 3b \quad b_3 = b \quad \text{(or } b_1 = 3b', \, b_2 = 6b', \, b_3 = 2b') \]
\[ c_1 = 1.5c \quad c_2 = 3c \quad c_3 = c \]
\[ m_1 = m_2 = m_3 = m \]

For a four degree of freedom system are:
\[ b_1 = 2b \quad b_2 = 6b \quad b_3 = 3b \quad b_4 = b \]
\[ c_1 = 2c \quad c_2 = 6c \quad c_3 = 3c \quad c_4 = c \]
\[ m_1 = m_2 = m_3 = m_4 = m \]

For a five degree of freedom system are:
\[ b_1 = 2.5b \quad b_2 = 10b \quad b_3 = 6b \quad b_4 = 3b \quad b_5 = b \]
\[ c_1 = 2.5c \quad c_2 = 10c \quad c_3 = 6c \quad c_4 = 3c \quad c_5 = c \]
\[ m_1 = m_2 = m_3 = m_4 = m_5 = m \]

Hence, for a \( N \) degree of freedom system we can infer that when
\[ b_1 = \frac{n}{2} b \quad b_2 = \frac{n(n-1)}{2} b \quad b_3 = \frac{(n-1)(n-2)}{2} b \quad b_4 = \frac{(n-2)(n-3)}{2} b \quad b_5 = \frac{2.1}{n} b \]
Then the equation of motion is reduced to \( n \) uncoupled equations. Therefore the classical mode will exist. It should be pointed out that the above argument does not constitute a proof, but an extrapolations of the previous consideration of two, three, four and five degrees of freedom systems.
6.1 CONCLUSIONS

From the previous five chapters, the following conclusions can be made:

(1) If a structural system is under base oscillation, the component of the base oscillation in the direction of the axis of the structure will contribute to parametric excitation of the system, but the component of the base oscillation perpendicular to the direction of the axis of the structure will induce ordinary force vibration only.

(2) If an undamped structural system subjected to nonconservative (circulatory) end force lost stability by divergence, then the same system including damping would lose its stability by divergence also. On the other hand, if the undamped structural system subjected to circulatory end force lost stability by flutter, then the same system including damping would still lose its stability by flutter. This means that the damping does not change the nature of the system as far as loss of stability is concerned.

(3) If an undamped structural system subjected to circulatory end force loses stability by divergence, then for the same system including damping, the damping has no destabilizing or stabilizing effect. But, if the undamped system subjected to circulatory end force loses stability by flutter, then for the same system including damping, the damping has destabilizing effect.
(4) For an original unstable two degree of freedom system, we can apply a proper circulatory end force to make the system stable. But if the system includes damping, the stabilizing effect of the nonconservative (circulatory) end force will be diminished. This is shown clearly at Figures 3.4 and 3.5. When the system is without damping, in the domain $1 \leq \gamma \leq 2$ there exists some stable region; but when the same system includes damping, this stable region will be eliminated for any value of damping, even if the damping is very small.

(5) If a two degree of freedom system subjected to nonconservative (circulatory) end force loses stability by flutter, then for the same system under both parametric base excitation and circulatory end force, the principal instability region of the first mode will become smaller as the circulatory force increases and the principal instability region for the second mode will enlarge as the circulatory force increases. Finally, those two principal instability regions will merge into one as the circulatory force increases to the critical value (this critical value is for the same system without base excitation). But if the two degree of freedom system subjected to circulatory force loses stability by divergence, then for the same system under both parametric base excitation and circulatory end force, the principal instability regions of both the first and the second modes will enlarge as the circulatory force increases.

(6) For a two degree of freedom system with damping subjected to circulatory force and under parametric base excitation, when the circulatory force is close to the critical value (which is the critical load for the same system without base excitation), the
shifting effect of damping will become obvious even for small value of damping. This means that when the circulatory force is close to the critical value, the principal instability region for the first mode will be slightly shifted (in the direction of decreasing frequencies) with respect to the principal region of instability for the same system without damping.

(7) For a two degree of freedom system, if certain mass, damping and spring constant distributions are given, it can be expected that the classical normal modes exist. Even for the same system subjected to a circulatory end force with a proper lag parameter "a", the classical normal modes can also be expected.

6.2 SUGGESTIONS

So far, the two degree of freedom system is analyzed by assuming small displacement, i.e. \( \sin \phi \approx \phi \). Such approximation is adequate for the onset of stability and calculation of critical parameters. However, as soon as the motion starts to grow, the geometric nonlinear effects become important. In particular, in order to study the steady state behavior after instability sets in, some nonlinear terms need to be included. Hence, if one more term is taken and

\[
\sin \phi_k = \phi_k - \frac{\phi_k^3}{3!} \quad (k = 1, 2)
\]

are assumed, then the equation of motion becomes

\[
[M] \{\ddot{\phi}_k\} + [\phi(t)] \{\dot{\phi}_k\} + [T(t)]\{\phi_k^3\} = 0
\]
where

\[ [M] = \begin{bmatrix} (m_1 + m_2) & m_2 \\ m_2 & m_2 \end{bmatrix}, \quad \{\phi_k\} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \]

\[ [\phi(t)] = \begin{bmatrix} (c_1 + c_2) - \xi (m_1 + m_2)(g + f'(t)) & -c_2 \\ -c_2 & c_2 - m_2 \xi (g + f'(t)) \end{bmatrix} \]

\[ [T(t)] = \frac{1}{3!} \xi (g + f'(t)) \begin{bmatrix} (m_1 + m_2) & 0 \\ 0 & m_2 \end{bmatrix} \]

In this equation, damping and horizontal vibration of the foundation are not considered; i.e. \( b = 0, h(t) = 0 \). If \( f(t) \) is a periodic function of time, then two coupled nonlinear differential equations with periodic coefficients are obtained. It is suggested that the study of such a system of equation is worthwhile since it will give some insight to the post critical behavior of the system.
APPENDIX

SOME PROPERTIES OF THE DIFFERENTIAL EQUATIONS
WITH PERIODIC COEFFICIENTS

In this thesis we restrict ourselves to consideration of the
differential equation with periodic coefficients, so called Mathieu-Hill
equation (2). The stability of the solution of this kind of differential
equation is well-known (3), (23). But some special cases of Mathieu-
Hill equations, which are widely used in this thesis, have their own
peculiar properties. In this appendix those special cases of Mathieu-
Hill equation will be discussed.

A. BEHAVIOR OF THE SOLUTIONS OF THE DIFFERENTIAL EQUATION

\[ f''(t) + (\mu \theta^2 \cos \theta t) f(t) = 0 \quad - - - - (A.1) \]

Floquet theory for the linear differential equations with
periodic coefficients is applicable for the Equation A.1. Hence the
well-known method (2) for analyzing those differential equations, could
be employed here. Let us seek the periodic solution in the form

\[ f(t) = \sum_{n}^{\infty} \left( a_n \sin \frac{n \theta t}{2} + b_n \cos \frac{n \theta t}{2} \right) \quad - - - - (A.2) \]

where

\[ n = 1, 3, 5 \ldots \ldots \]

if the periodic solution has a period 2T, and

\[ n = 0, 2, 4, 6 \ldots \ldots \]

if this solution has a period T.

Substituting the series (A.2) into Equation A.1 and simplifying
the equation leads the following equation:

\[ \sum_{n} \sin \frac{n \omega t}{2} \left[ a_n \left( -\frac{n^2 \theta^2}{4} \right) + (a_{n-2} + a_{n+2}) \frac{\mu \theta^2}{2} \right] + \cos \frac{n \omega t}{2} \]

\[ [b_n \left( -\frac{n^2 \theta^2}{2} \right) + (b_{n-2} + b_{n+2}) \frac{\mu \theta^2}{2} ] = 0 \]

Let us seek the periodic solution with a period 2T first. Equating the coefficients of identical \( \sin \left( \frac{n \omega t}{2} \right) \) and \( \cos \left( \frac{n \omega t}{2} \right) \) leads to the following system of linear homogeneous algebraic equations in terms of \( a_n \) and \( b_n \):

\[ -a_1 \left( \frac{\mu \theta^2}{2} + \frac{\theta^2}{4} \right) + a_3 \left( \frac{\mu \theta^2}{2} \right) = 0 \]

\[ a_n \left( -\frac{n^2 \theta^2}{4} \right) + (a_{n-2} + a_{n+2}) \frac{\mu \theta^2}{2} = 0 \quad (n = 3, 5, 7 \ldots) \]

and

\[ b_1 \left( -\frac{\mu \theta^2}{4} - \frac{\theta^2}{4} \right) + b_3 \left( \frac{\mu \theta^2}{2} \right) = 0 \]

\[ b_n \left( -\frac{n^2 \theta^2}{4} \right) + (b_{n-2} + b_{n+2}) \frac{\mu \theta^2}{2} = 0 \quad (n = 3, 5, 7 \ldots) \]

where the first system contains only \( a \) coefficients, and the second \( b \) contains only \( b \) coefficients. Next we seek the periodic solution with a period 2T. The following system of algebraic equations will be obtained:

\[ -a_2(\theta^2) + a_4 \left( \frac{\mu \theta^2}{2} \right) = 0 \]

\[ a_n \left( -\frac{n \theta^2}{4} \right) + (a_{n-2} + a_{n+2}) \frac{\mu \theta^2}{2} = 0 \quad (n = 4, 6, 8 \ldots) \]

and

\[ b_2 \left( \frac{\theta^2}{2} \right) = 0 \]

\[ b_0 \left( \frac{\mu \theta^2}{2} \right) - b_2(\theta^2) + b_4 \left( \frac{\mu \theta^2}{2} \right) = 0 \]
The algebraic Equations A.3 and A.4 can be expressed in the following form:

(a) In the case period = 2T

\[
\begin{bmatrix}
-(\frac{\mu}{2} + \frac{1}{4}) & \frac{\mu}{2} & 0 & 0 & 0 & \cdots \\
\frac{\mu}{2} & \frac{9}{4} & -\frac{\mu}{2} & 0 & 0 & \cdots \\
0 & \frac{\mu}{2} & \frac{25}{4} & \frac{\mu}{2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_3 \\
a_5 \\
\vdots
\end{bmatrix} = 0
\]

and

\[
\begin{bmatrix}
\frac{\mu}{2} & \frac{1}{4} & \frac{\mu}{2} & 0 & 0 & 0 & \cdots \\
\frac{\mu}{2} & \frac{9}{4} & -\frac{\mu}{2} & 0 & 0 & \cdots \\
0 & \frac{\mu}{2} & \frac{25}{4} & \frac{\mu}{2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_3 \\
b_5 \\
\vdots
\end{bmatrix} = 0
\]

(b) In the case period = T

\[
\begin{bmatrix}
-1 & \frac{\mu}{2} & 0 & 0 & 0 & \cdots \\
\frac{\mu}{2} & -4 & \frac{\mu}{2} & 0 & 0 & \cdots \\
0 & \frac{\mu}{2} & -16 & \frac{\mu}{2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
a_2 \\
a_4 \\
a_6 \\
\vdots
\end{bmatrix} = 0
\]
and

\[ \begin{vmatrix}
0 & \mu & 0 & 0 & 0 & \ldots & b_0 \\
\frac{\mu}{2} & -1 & \frac{\mu}{2} & 0 & 0 & \ldots & b_2 \\
0 & \frac{\mu}{2} & -4 & \frac{\mu}{2} & 0 & \ldots & b_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{vmatrix} = 0 \]

If the periodic solution exists, the above equations are true, only when the obtained determinants of the homogeneous systems be equal to zero, i.e.

(a) In the case period = \(2T\)

\[ \begin{vmatrix}
-\left(\frac{\mu}{2} + \frac{1}{4}\right) & \frac{\mu}{2} & 0 & 0 & 0 & \ldots \\
\frac{\mu}{2} & -\frac{9}{4} & \frac{\mu}{2} & 0 & 0 & \ldots \\
0 & \frac{\mu}{2} & -\frac{25}{4} & \frac{\mu}{2} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{vmatrix} = 0 \]

\[ \begin{vmatrix}
\frac{\mu}{2} - \frac{1}{4} & \frac{\mu}{2} & 0 & 0 & 0 & \ldots \\
\frac{\mu}{2} & -\frac{9}{4} & \frac{\mu}{2} & 0 & 0 & \ldots \\
0 & \frac{\mu}{2} & -\frac{25}{4} & \frac{\mu}{2} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{vmatrix} = 0 \]
(b) In the case period = T

\[
\begin{vmatrix}
-1 & \mu/2 & 0 & 0 & 0 & \cdots \\
\mu/2 & -4 & \mu/2 & 0 & 0 & \cdots \\
0 & \mu/2 & -16 & \mu/2 & 0 & \cdots \\
& & & & & \ddots
\end{vmatrix} = 0
\]

and

\[
\begin{vmatrix}
0 & \mu & 0 & 0 & 0 & \cdots \\
\mu/2 & -1 & \mu/2 & 0 & 0 & \cdots \\
0 & \mu/2 & -4 & \mu/2 & 0 & \cdots \\
& & & & & \ddots
\end{vmatrix} = 0
\]

Note that the four determinants are independent of θ. It means existence of periodic solution is dependent on certain values of \( \mu \) only. The boundaries of instability regions can be determined from the above four determinants. But from the Floquet theory it is known that two solutions of identical periods bound the region of instability, and two solutions of different periods bound the region of stability. Hence the stability chart can be constructed as shown in Figure A.1.

B. BEHAVIOR OF THE SOLUTION OF DIFFERENTIAL EQUATION

\[
\ddot{f}(t) + \epsilon \dot{f}(t) + (\delta^2 + \mu \delta^2 \cos \theta t) f(t) = 0 \quad - - - (A.5)
\]

(i) This equation is the standard form of Mathieu-Hill equation with damping term; except in the second part of the coefficient of \( f(t) \)
Figure A.1: The Stability Chart for $\ddot{f}(t) + (\mu_0^2 \cos \theta t) f(t) = 0$
we have parameters $\mu \theta^2$ where $\theta$ is in the same quantity as parametric frequency. Seeking the periodic solution and following the same procedure as we did in Section A, we will find the system of linear homogeneous algebraic equations in terms of $a$ and $b$:

$$\left(\frac{\delta^2}{\theta^2} - \frac{\mu}{2} - \frac{1}{4}\right) a_1 - \frac{\epsilon}{2\theta} b_1 + \frac{\mu}{2} a_3 = 0$$

$$\left(\frac{\delta^2}{\theta^2} - \frac{n^2}{4}\right) a_n - \epsilon \left(\frac{\theta}{2}\right) b_n + \frac{\mu}{2} \left(a_{n-2} + a_{n+2}\right) = 0$$

$$\quad (n = 3, 5, \ldots) \quad - - - \quad (A.6)$$

and

$$\left(\frac{\delta^2}{\theta^2} + \frac{\mu}{2} - \frac{1}{4}\right) b_1 + \frac{\epsilon}{2\theta} a_1 + \frac{\mu}{2} b_3 = 0$$

$$\left(\frac{\delta^2}{\theta^2} - \frac{n^2}{4}\right) b_n + \epsilon \left(\frac{\theta}{2}\right) a_n + \frac{\mu}{2} \left(b_{n-2} + b_{n+2}\right) = 0$$

$$\quad (n = 3, 5, \ldots)$$

(b) In the case period = $T$:

$$\left(\frac{\delta^2}{\theta^2} - 1\right) a_2 - \frac{\epsilon}{2} b_2 + \frac{\mu}{2} a_4 = 0$$

$$\left(\frac{\delta^2}{\theta^2} - \frac{n^2}{4}\right) a_n - \epsilon \left(\frac{\theta}{2}\right) b_n + \frac{\mu}{2} \left(a_{n-2} + a_{n+2}\right) = 0$$

$$\quad (n = 4, 6, 8, \ldots)$$

$$\left(\frac{\delta^2}{\theta^2}\right) b_0 + \mu b_2 = 0$$

$$\left(\frac{\delta^2}{\theta^2} - 1\right) b_2 + \frac{\epsilon}{2} a_2 + \frac{\mu}{2} \left(b_0 + b_4\right) = 0$$
\[
\left( \frac{\delta^2}{\theta^2} - \frac{n^2}{4} \right) b_n + \epsilon \left( \frac{n}{2\theta} \right) a_n + \frac{\mu}{2} \left( b_{n-2} + b_{n+2} \right) = 0
\]

\[(n = 4, 6, 8, \ldots)\]

It is convenient to express the above equations in another form. First in the case period = 2T:

\[
\begin{pmatrix}
\left( \frac{\delta^2}{\theta^2} - \frac{9}{4} \right) & \mu & 0 & \frac{3\epsilon}{2\theta} \\
\frac{\mu}{2} & \left( \frac{\delta^2}{\theta^2} - \frac{1}{4} \right) & -\frac{\epsilon}{2\theta} & 0 \\
0 & \frac{\epsilon}{2\theta} & \left( \frac{\delta^2}{\theta^2} + \frac{1}{4} \right) & \frac{\mu}{2} \\
\frac{3\epsilon}{2\theta} & 0 & \frac{\mu}{2} & \left( \frac{\delta^2}{\theta^2} - \frac{9}{4} \right)
\end{pmatrix}
\begin{pmatrix}
a_3 \\
a_1 \\
b_1 \\
b_3
\end{pmatrix}
= 0 \quad (A.7)
\]

Next in the case period = T:

\[
\begin{pmatrix}
\left( \frac{\delta^2}{\theta^2} - 4 \right) & \mu & 0 & 0 & -\frac{2\epsilon}{\theta} \\
\frac{\mu}{2} & \left( \frac{\delta^2}{\theta^2} - 1 \right) & 0 & \frac{\epsilon}{\theta} & 0 \\
0 & 0 & \frac{\delta^2}{\theta^2} & \mu & 0 \\
0 & \frac{\epsilon}{\theta} & \frac{\mu}{2} & \left( \frac{\delta^2}{\theta^2} - 1 \right) & \frac{\mu}{2} \\
\frac{2\epsilon}{\theta} & 0 & 0 & \frac{\mu}{2} & \left( \frac{\delta^2}{\theta^2} - 4 \right)
\end{pmatrix}
\begin{pmatrix}
a_4 \\
a_2 \\
b_0 \\
b_2 \\
b_4
\end{pmatrix}
= 0 \quad (A.8)
\]

By using the same argument as before, the equations of the boundary frequencies must be:
(a) In the case period = 2T:

\[
\begin{vmatrix}
\left(\frac{\delta^2}{\theta^2} - \frac{9}{4}\right) & \frac{\mu}{2} & 0 & -\frac{3\varepsilon}{2\theta} \\
\frac{\mu}{2} & \left(\frac{\delta^2}{\theta^2} - \frac{\mu}{2} - \frac{1}{4}\right) & -\frac{\varepsilon}{2\theta} & 0 \\
0 & \frac{\varepsilon}{2\theta} & \left(\frac{\delta^2}{\theta^2} + \frac{\mu}{2} - \frac{1}{4}\right) & \frac{\mu}{2} \\
\frac{3\varepsilon}{2} & 0 & \frac{\mu}{2} & \left(\frac{\delta^2}{\theta^2} - \frac{9}{4}\right)
\end{vmatrix} = 0 \quad - - - \quad (A.9)
\]

(b) In the case period = T:

\[
\begin{vmatrix}
\left(\frac{\delta^2}{\theta^2} - 4\right) & \frac{\mu}{2} & 0 & 0 & -\frac{2\varepsilon}{\theta} \\
\frac{\mu}{2} & \left(\frac{\delta^2}{\theta^2} - 1\right) & 0 & -\frac{\varepsilon}{\theta} & 0 \\
0 & 0 & \frac{\delta^2}{\theta^2} & \mu & 0 \\
0 & \frac{\varepsilon}{\theta} & \frac{\mu}{2} & \left(\frac{\delta^2}{\theta^2} - 1\right) & \frac{\mu}{2} \\
\frac{2\varepsilon}{\theta} & 0 & 0 & \frac{\mu}{2} & \left(\frac{\delta^2}{\theta^2} - 4\right)
\end{vmatrix} = 0 \quad - - - \quad (A.16)
\]

From the above two determinants, the boundary of stable and unstable regions can be determined. To determine the critical values of the excitation parameter below which no parameter resonance is possible, we will begin with the principal region of instability by retaining the central elements in the determinant (Equation A.9).
Solving the Equation A.11 with respect to the exciting frequency, we have

\[
\frac{\delta^2}{\theta^2} = \frac{-\left(\frac{\epsilon^2}{4\delta^2} - \frac{1}{2}\right) \pm \sqrt{\mu^2 - \frac{\epsilon^2}{4\delta^2} + \frac{\epsilon^4}{16\delta^4}}}{2}
\]

Since \(\frac{\epsilon}{\delta}\) is small, this formula can be simplified by neglecting the terms containing higher powers of \(\frac{\epsilon}{\delta}\):

\[
\frac{\delta^2}{\theta^2} = \frac{1}{2} \pm \sqrt{\mu^2 - \frac{\epsilon^2}{4\delta^2}}
\]

Equation A.12 is plotted in the following diagram (Figure A.2).

The limiting case is when the expression under the square root of Equation A.12 becomes zero, namely,

\[
\mu^2 - \frac{\epsilon^2}{4\delta^2} = 0
\]

This defines the minimum value of \(\mu\) such that below this value the system always remains stable. Thus, the critical value of the parameter \(\mu\) is

\[
\mu^* = \frac{\epsilon}{2\delta}
\]

or we can write in the form of

\[
\frac{\mu^*}{2} = \left(\frac{1}{2}\right)^2 \frac{\epsilon}{\delta}
\]
Figure A.2: Principal Regions of Instability for $\ddot{f}(t) + \epsilon \dot{f}(t) + (\delta^2 + \mu \delta^2 \cos \delta t) f(t) = 0$
where \( 1! \) means 1 factorial.

(ii) In order to determine the boundaries of the second region of instability, the central elements in Equation A.10 are retained.

\[
\begin{vmatrix}
\frac{\delta^2}{\theta^2} - 1 & 0 & -\frac{\epsilon}{\theta} \\
0 & \frac{\delta^2}{\theta^2} & \mu \\
\frac{\epsilon}{\theta} & \frac{\mu}{2} & (\frac{\delta^2}{\theta^2} - 1)
\end{vmatrix} = 0
\]

(A.14)

It is difficult to obtain an exact analytical solution of such an equation. Therefore we will substitute the approximate value of the boundary frequency \( \theta_\infty = \delta \) in all the elements. This only slightly influences the final result in all but the upper and lower diagonal elements \(^{(1)}\). Equation A.14 can be rewritten then in the form:

\[
\begin{vmatrix}
q & 0 & -\frac{\epsilon}{\delta} \\
0 & 1 & \mu \\
\frac{\epsilon}{\delta} & \frac{\mu}{2} & q
\end{vmatrix} = 0
\]

where

\[
q = \frac{\delta^2}{\theta^2} - 1
\]

Expanding the determinant leads the following equation

\[
q^2 - \frac{\mu^2}{2} + \frac{\epsilon^2}{\delta^2} = 0
\]
Hence
\[ q = \frac{\mu}{2} \pm \sqrt{\left(\frac{\mu}{2}\right)^2 - \frac{4\varepsilon^2}{\delta^2}} \]

The boundary frequency will be calculated by
\[ \frac{\delta^2}{\theta^2} = 1 + \frac{\mu}{2} \pm \sqrt{\left(\frac{\mu}{2}\right)^2 - \frac{4\varepsilon^2}{\delta^2}} \]

- - - (A.15)

The critical case in Equation A.15 is
\[ \left(\frac{\mu}{2}\right) - \frac{4\varepsilon^2}{\delta^2} = 0 \]

Thus
\[ \frac{\mu}{2} = \sqrt{\frac{\varepsilon}{\delta}} \]

or in the form of
\[ \frac{\mu}{2} = \sqrt{\frac{2!}{2}} \frac{\varepsilon}{\delta} \]

- - - (A.16)

(iii) For determining the boundaries of the third region of instability, we return to the determinant (Equation A.9) and use the same approximation method as before, replacing all \( \theta \) by \( 2/3 \varepsilon \) (i.e. \( \frac{\delta^2}{\theta^2} = \frac{9}{4} \)) except the upper and lower diagonal elements. The equation of the boundaries of the third region of instability becomes:
\[ q \begin{pmatrix} \frac{\mu}{2} & 0 & -\frac{9}{4} \frac{\epsilon}{\delta} \\ \frac{\mu}{2} & (2 - \frac{\mu}{2}) & -\frac{3}{4} \frac{\epsilon}{\delta} & 0 \\ 0 & \frac{3}{4} \frac{\epsilon}{\delta} & (2 + \frac{\mu}{2}) & \frac{\mu}{2} \\ \frac{9}{4} \frac{\epsilon}{\delta} & 0 & \frac{\mu}{2} & q \end{pmatrix} = 0 \]

where

\[ q = \frac{\delta^2}{\epsilon^2} - \frac{9}{4} \]

Expanding the determinant leads the following equation

\[ q^2 \left( 4 - \frac{\mu^2}{4} + \frac{9}{16} \frac{\epsilon^2}{\delta^2} \right) - \mu^2 q + \left( \frac{\mu^6}{16} + 36 \left[ \frac{9}{16} \frac{\epsilon^2}{\delta^2} \right] - 3 \left( \frac{\mu^2}{4} \left[ \frac{9}{16} \frac{\epsilon^2}{\delta^2} \right] \right) \right) = 0 \]

Neglecting quantities of the higher order of \( \epsilon^h \) leads to

\[ q = \frac{\mu^2 \pm \sqrt{\frac{\mu^6}{16} - 4 \left( \frac{9}{16} \frac{\epsilon^2}{\delta^2} \right) \left( 12 - \frac{\mu^2}{2} \right)^2}}{2 \left( 4 - \frac{\mu^2}{4} + \frac{9}{16} \frac{\epsilon^2}{\delta^2} \right)} \]

The boundary frequency will be calculated by the equation

\[ \frac{\delta^2}{\epsilon^2} = \frac{9}{4} + \frac{\mu^2 \pm \sqrt{\frac{\mu^6}{16} - 4 \left( \frac{9}{16} \frac{\epsilon^2}{\delta^2} \right) \left( 12 - \frac{\mu^2}{2} \right)^2}}{2 \left( 4 - \frac{\mu^2}{4} + \frac{9}{16} \frac{\epsilon^2}{\delta^2} \right)} \]

- - - (A.17)

In this case the critical condition is
\[
\frac{\mu^6}{16} - 4\left(\frac{\delta^2}{16}\right) \frac{\varepsilon^2}{\delta^2} (12 - \frac{\mu^2}{2})^2 = 0
\]

Simplifying the equation leads to
\[
\frac{\mu^3}{4} - 24\left(\frac{3\varepsilon}{4}\right) \left(1 - \frac{\mu^2}{24}\right) = 0
\]

Since \(\mu\) is a small quantity, the above equation gives, approximately,
\[
\frac{\mu^3}{4} \approx 24\left(\frac{3\varepsilon}{4}\right) \left(1 - \frac{\mu^2}{24}\right)
\]

Thus,
\[
\frac{\mu^3}{4} = \frac{\varepsilon}{\delta}
\]

or we can write in the form of
\[
\frac{\mu^3}{4} = \frac{\varepsilon}{\delta} = \sqrt{\frac{3\varepsilon}{2}}
\]

(iv) Now we are going to determine the boundaries of the fourth region of instability. Returning to the determinant (Equation A.10) and using the same approximation method as before, replace all \(\theta_*\) by \(\frac{1}{2}\) \(\delta\) (i.e. \(\frac{\delta^2}{\theta^2} = 4\)) except the upper and lower diagonal elements, the equation of the boundaries of the fourth region of instability is obtained as:

\[
\frac{\mu^3}{4} = \frac{\varepsilon}{\delta} = \sqrt{\frac{3\varepsilon}{2}}
\]  

---  (A.18)
where \( q = \frac{\delta^2}{\theta^2} - 4 \)

Expanding the determinant,

\[
q^2(36 - \frac{3}{2} \mu^2 + 16 \frac{\varepsilon^2}{\delta^2}) - 3q(2 - \frac{\mu^2}{24})\mu^2 + (\frac{\mu^4}{4} - 8 \frac{\varepsilon^2}{\delta} \mu^2 + 36 \frac{16\varepsilon^2}{\delta^2}) = 0
\]

Solving the equation and neglecting quantity of the order of \( \varepsilon^4 \) leads the following equation

\[
q = \frac{3\mu^2(2 - \frac{\mu^2}{24}) \pm \sqrt{\frac{9}{24^2} \mu^8 - [(36 \times 8)^2 + 32 \times 36 \times 4\mu^2 - 64\mu^4]} \frac{\varepsilon^2}{\delta^2}}{2(36 - \frac{3}{2} \mu^2 + 16 \frac{\varepsilon^2}{\delta^2})}
\]

Put \( q = \frac{\delta^2}{\theta^2} - 4 \) into the above equation,

\[
\frac{\delta^2}{\theta^2} = 4 + \frac{3\mu^2(2 - \frac{\mu^2}{24}) \pm \sqrt{\frac{9}{24^2} \mu^8 - [(36 \times 8)^2 + 32 \times 36 \times 4\mu^2 - 64\mu^4]} \frac{\varepsilon^2}{\delta^2}}{2(36 - \frac{3}{2} \mu^2 + 16 \frac{\varepsilon^2}{\delta^2})}
\]  

--- (A.19)
From the equation A.19 it can be seen that the critical case is

\[ \frac{9}{24^2} \mu^8 - [36 \times 8]^2 \left( \frac{32 \times 36 \times 4 \mu^2 - 64 \mu^4}{(36 \times 8)^2} \right) \frac{\varepsilon^2}{\delta^2} = 0 \]

or

\[ \frac{9}{24^2} \mu^8 - (36 \times 8)^2 \left( 1 + \frac{32 \times 36 \times 4}{(36 \times 8)^2} \mu^2 - \frac{64 \mu^4}{(36 \times 8)^2} \right) \frac{\varepsilon^2}{\delta^2} = 0 \]

Since \( \mu \) is small quantity, the above equation can be expressed approximately by:

\[ \frac{9}{24^2} \mu^8 \sim (36 \times 8)^2 \frac{\varepsilon^2}{\delta^2} \]

Thus

\[ \frac{\mu_{*1}}{2} = \frac{4}{\sqrt{\frac{16 \times 9}{\delta} \frac{\varepsilon}{\delta}}} \]

or we can write in the form of

\[ \frac{\mu_{*4}}{2} = \frac{4}{\sqrt{\frac{(41)}{2} \frac{\varepsilon}{\delta}}} \]

(v) From Equations A.13, A.16, A.18 and A.20, it is possible to deduce that in every region of instability the critical value of the parameter \( \mu \) has the following relationship:

\[ \frac{\mu_{*1}}{2} = \frac{11}{2} \frac{\varepsilon}{\delta} \]

\[ \frac{\mu_{*2}}{2} = \sqrt{\frac{21}{2} \frac{\varepsilon}{\delta}} \]

\[ \frac{\mu_{*5}}{2} = \frac{3}{\sqrt{\frac{31}{2} \frac{\varepsilon}{\delta}}} \]
\[
\frac{\mu_4}{2} = 4 \sqrt{\left( \frac{4l}{2} \right)^2 \frac{\varepsilon}{\delta}}
\]

\[
\frac{\mu_k}{2} = \kappa \sqrt{\left( \frac{kl}{2} \right)^2 \frac{\varepsilon}{\delta}}
\]
REFERENCES


