REGULAR SETS, SCALAR MULTIPLICATIONS

AND

ABSTRACTIONS OF DISTANCE SPACES

REGULAR SETS, SCALAR MULTIPLICATIONS AND ABSTRACTIONS OF DISTANCE SPACES

By

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SCOPE AND CONTENTS: This thesis is both classically and abstractly oriented in a geometrical sense. The discussion is centred around the notion distance.

In the first chapter, the concept of a regular set is defined and discussed. The idea of a regular set is a natural generalization of equilateral triangles and regular tetrahedra in Euclidean spaces.

In chapter two, two kinds of scalar multiplication associated with metric spaces are studied.

In chapter three, the concept of distance is abstracted to a level where it loses most of its structure. This abstraction is then examined.

In chapter four, generalized metric spaces are examined. These are specializations of the abstract spaces of chapter three.

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PROLOGUE

The aim of this thesis is to define as nearly as possible standard geometrical objects and operations in metric spaces. Later, the possibility of stripping most of the structure from metric spaces yet still preserving the concept of "distance" is considered.

As far as the aim is concerned a type of regular set in metric spaces and two related types of "scalar multiplication" connected with metric spaces are defined.

The regular sets are modelled on the Euclidean geometric objects known as equilateral triangles and regular tetrahedra.

Two two scalar multiplications are in essence similar to normal scalar multiplications except that there is no addition associated with them as in vector spaces.

The work regarding metric spaces stripped of most of their structure was motivated by a graduate course in "Universal Algebra" given by Professor G. Bruns of McMaster University. Metric spaces can be considered as special cases of these abstract spaces. Proceeding along this line more structure is put back into these abstract "distance" spaces. This leads to a mathematical system which resembles a metric space in many ways but which retains certain properties which the abstract spaces possess and metric spaces do not.

NOTATION

The following is a description of notational usage within this thesis.

The capital letter R is used throughout to denote the real numbers although in certain sections it is used differently. In these situations, however, the usage is specifically indicated.

In some situations, the notation $|S|$ is used to denote the cardinality of the set S which is effectively the number of elements which S has. In other situations, the notation $|x|$ is used with regard to real numbers to denote the absolute value of x. In all cases, the usage will be clear from the context.

With regard to equivalence relations, we will denote the equivalence class of an element x in M with respect to the equivalence relation θ by $[x]_\theta$. The set of all such equivalence classes will be denoted by M/θ.

If S is a set of ordered pairs, then S^{-1} is the set of all ordered pairs (x,y) such that

$$(y,x) \in S.$$

If S and T are sets of ordered pairs, then $S \circ T$ is the set of all ordered pairs (x,y) such that there exists some z with

$$(x,z) \in S \quad \text{and} \quad (z,y) \in T.$$

Throughout this work, the symbol "iff" will be used as an abbreviation of the phrase "if and only if".

TABLE OF CONTENTS

CHAPTER ONE

- REGULAR SETS -

Introduction. This Chapter is devoted to the discussion of "regular" sets in a metric space. These sets, which are called "tets", are generalizations of the normal Euclidean objects known as equilateral triangles and regular tetrhedra.

A concept of differentiating real valued functions on a metric space is defined. Also a notion of a "localizable" direction at a point is developed. In addition, a uniform topology on the tets is defined, examined and a uniformly continuous function on a subset of tets is exhibited. In conclusion, a certain type of space based on the idea of tets is examined. It is found that a metric space generated a space of this type and each space of this type generates a metric space.

SECTION ONE

- METRIC SPACES -

1.1. Preliminary Notions.

Let M be a set and let R be the set of real numbers.

Definition: (M,d) is a metric space iff

(1) $d: M \times M \to R$

(2) for any $p,q \in M$,
$d(p,q) = 0$ iff $p=q$

(3) for any $p,q \in M$,
$d(p,q) = d(q,p)$

(4) for any $p,q,r \in M$,
$d(p,q) \leq d(p,r) + d(r,q)$

Condition (4) is commonly known as the triangle inequality.

Let (M,d) be a metric space.

Definition: If $p \in M$ and $e \in R$ and $e > 0$ we define the following subsets of M:

$$N(p; e) = \{q \mid d(p,q) < e\}$$

which is called the <u>open disk</u> of radius e at p and

$$E(p;\ e) = \{q \mid d(p,q) > e\}$$

which is called the <u>exterior</u> of the open disk of radius e at p.

<u>Definition</u>: Define $]0, \rightarrow]$ to be the set of all x such that

$$x \in R \quad \text{and} \quad x > 0.$$

<u>Definition</u>: Define $D(M,d)$ to be the set of all

$$\bigcup_{i \in I} N(p_i;\ e_i)$$

where $(p_i;\ e_i)_{i \in I}$ is a family in $M \times]0, \rightarrow]$.

<u>Theorem</u>: $D(M,d)$ <u>is a topology on</u> M <u>called the topology associated with</u> d.

<u>Proof</u>. Not given ([1], page 119).

<u>Theorem</u>: <u>If</u> $p \in M$ <u>and</u> $e \in R$ <u>and</u> $e > 0$, <u>then</u>

(1) $N(p; e)$ <u>is open</u>

(2) $E(p; e)$ <u>is open</u>.

<u>Proof of (1)</u>. It is immediate that $N(p; e)$ is a union of the form

$$\bigcup_{i \in I} N(p_i; e_i)$$

consisting of a single term: namely, itself.

<u>Proof of (2)</u>. Take $q \in E(p; e)$. Consider

$$f(q) = \tfrac{1}{2}(d(p,q) - e).$$

Then it is readily seen that

$$N(q; f(q)) \subseteq E(p; e).$$

Thus we have immediately that

$$\bigcup_{q \in E(p;e)} N(q; f(q)) = E(p; e)$$

since

$$q \in N(q;\ f(q)).$$

<u>Theorem</u>: <u>For any</u> p, q, r $\in$ M,

$$d(p,q) \geq |d(p,r) - d(r,q)|.$$

<u>Proof</u>. We have from the triangle inequality

$$d(p,r) \leq d(p,q) + d(r,q).$$

This implies

$$d(p,q) \geq d(p,r) - d(r,q).$$

Also from the triangle inequality

$$d(r,q) \leq d(r,p) + d(p,q).$$

Hence we get

$$\begin{aligned} d(p,q) &\geq d(r,q) - d(r,p) \\ &= d(r,q) - d(p,r) \end{aligned}$$

since $d(p,r) = d(r,p)$.

Theorem: For any $p,q,r,s \in M$

$$|d(p,q) - d(r,s)| \leq d(p,r) + d(q,s).$$

Proof.

$$\begin{aligned} &|d(p,q) - d(r,s)| \\ &= |(d(p,q) - d(q,r)) + (d(q,r) - d(r,s))| \\ &\leq |d(p,q) - d(q,r)| + |d(q,r) - d(r,s)| \\ &\leq d(p,r) + d(q,s) \end{aligned}$$

since we have by the preceding theorem that

$$\begin{aligned} d(p,r) &\geq |d(p,q) - d(q,r)| \\ d(q,s) &\geq |d(q,r) - d(r,s)|. \end{aligned}$$

SECTION TWO

- "TETRAHEDRA" -

1.2. Regular Sets.

Let (M,d) be a metric space and let $e \in R$.

Definition: $K \subseteq M$ is an e-tet iff

for any $x,y \in K$,

if $x \neq y$, then $d(x,y) = e$.

Note. $\emptyset$ is an e-tet for any $e \in R$. If $p \in M$, then $\{p\}$ is an e-tet for any $e \in R$.

Definition: $K \subseteq M$ is a tet iff there exists $e \in R$ such that K is an e-tet.

Definition: Tet(e) is the set of all e-tets in M.

The term tet is intended as a contraction of the word tetrahedron in order to indicate structures which bear a resemblance to the equilateral triangle of the plane or the tetrahedron of space.

Note. We may partially order Tet(e) by inclusion.

Theorem: If K is an e-tet in M, then there exists a maximal e-tet, $\overline{K}$, in M (maximal with respect to inclusion) such that $K \subseteq \overline{K}$.

Proof. Let C be a non-empty chain in Tet(e) such that

$$L \in C \text{ implies } K \subseteq L.$$

Define

$$C^* = \bigcup_{L \in C} L.$$

Trivially, we have

$$K \subseteq C^*$$

since C is non-empty.

We now show that $C^* \in \text{Tet}(e)$.

Take $x, y \in C^*$ with $x \neq y$. Then there exists $L', L'' \in C$ with

$$x \in L' \text{ and } y \in L''.$$

The fact that C is a chain gives

$$L' \subseteq L'' \quad \text{or} \quad L'' \subseteq L'.$$

We assume, without loss of generality, that

$$L' \subseteq L'' .$$

Hence $x,y \in L''$. Now L'' is an e-tet. Thus $x,y \in L''$ and $x \neq y$ implies that

$$d(x,y) = e.$$

Hence C^* is an upper bound of C in the set of all $L \in \mathrm{Tet}(e)$ with $K \subseteq L$. Also C was an arbitrary non-empty chain. Hence, by Zorn's Lemma, there exists a maximal $\overline{K} \in \mathrm{Tet}(e)$ with the property that $K \subseteq \overline{K}$.

<u>Definition</u>: Let K be an e-tet in M. Define

$$\mathrm{int}(K) = \{q \mid q \in M \text{ and } d(x,q) < e \text{ for all } x \in K\},$$

$$\mathrm{ext}(K) = \{q \mid q \in M \text{ and } d(x,q) > e \text{ for some } x \in K\},$$

$$\mathrm{bnd}(K) = \{q \mid q \in M \text{ and } d(x,q) \leq e \text{ for all } x \in K \text{ and there exists } x' \in K \text{ with } d(x',q) = e\}.$$

In this definition, ext(K) is read "the exterior of K", int(K) is read "the interior of K" and bnd(K) is read "the boundary of K".

<u>Theorem</u>: <u>If</u> K <u>is an e-tet in</u> M, <u>then</u>

(1) $\text{ext}(K) \cap \text{bnd}(K) = \emptyset$

(2) $\text{int}(K) \cap \text{bnd}(K) = \emptyset$

(3) $\text{ext}(K) \cap \text{int}(K) = \emptyset$

(4) $\text{ext}(K) \cup \text{bnd}(K) \cup \text{int}(K) = M.$

<u>In addition, it is also true that</u>

(5) $\text{ext}(K) = \bigcup_{x \in K} E(x;\ e)$

(6) $\text{int}(K) = \bigcap_{x \in K} N(x;\ e)$

(7) $\text{bnd}(K) = \bigcup_{x \in K} K(x)$

<u>where for any</u> $x' \in K$

$$K(x') = \{q \mid q \in M \text{ and } d(x,q) \leq e \text{ for all } x \in K \text{ and } d(x',q) = e\}.$$

Also it is true that

(8) *for any* $x \in K$,
$K(x)$ *is closed*.

Hence, since $E(x; e)$ is open, we have that ext(K) is open. Since $K(x)$ is closed, we have that bnd(K) is the union of closed sets. Also int(K) is the intersection of open sets. If, in particular, K is a finite set, then bnd(K) is closed and int(K) is open.

Proof of (3). Assume there exists an x such that

$$x \in \text{ext}(K) \cap \text{int}(K).$$

This implies that there exists a $y \in K$ such that

$$d(y,x) < e \quad \text{and} \quad d(y,x) > e.$$

This is impossible.

Proof of (1) and (2). The proofs of these are analogous to the proof of (3) and are left to the reader.

Proof of (4). Certainly we have

$$\text{ext}(K) \cup \text{bnd}(K) \cup \text{int}(K) \subseteq M.$$

We now show that the reverse inclusion holds.

Take $q \in M$. Then q satisfies one of the two following statements:

(a) there exists some $x \in K$ with $d(x,q) > e$

(b) for any $x \in K$, $d(x,q) \leq e$.

In case (a), x is immediately seen to belong to $ext(K)$. Case (b) may be subdivided into the following cases:

(c) for any $x \in K$, $d(x,q) \neq e$

(d) for some $x \in K$, $d(x,q) = e$.

In case (c), x is seen to be an element of $int(K)$. In case (d), x is seen to be in $bnd(K)$.

Proof of (5).

$q \in ext(K)$

iff there exists $x \in K$ with $d(x,q) > e$

iff there exists $x \in K$ with $q \in E(x; e)$

iff $q \in \bigcup_{x \in K} E(x; e)$.

Proof of (6) and (7). The proofs of these are analogous to the proof of (5) and are left to the reader.

<u>Proof of (8)</u>. Let $(x(n))$ be a sequence in $K(x')$ with $\lim(x(n)) = x$. It will suffice to show that $x \in K(x')$.

Now for all $q \in K$,

$$d(x(n), q) \leq e.$$

Hence it is seen that

$$\lim(d(x(n), q)) \leq e.$$

But ([1], pages 9 and 15)

$$\lim(d(x(n), q)) = d(\lim(x(n)), q).$$

Hence $d(x,q) \leq e$.

Now for any $n \in N$,

$$d(x', x(n)) = e.$$

Thus we have

$$\lim(d(x', x(n))) = e.$$

But ([1], pages 9 and 15)

$$\lim(d(x', x(n))) = d(x', \lim(x(n))).$$

Finally we get $d(x',x) = e$.

Theorem: If K is an e-tet in M, then K is closed and K does not have a cluster point.

Proof. Let $(p(n))$ be a sequence in K with $\lim(p(n)) = p$. It will suffice to show that $p \in K$.

Since $(p(n))$ has a limit, it is a Cauchy sequence. Thus there exists $n^* \in N$ such that if $k', k'' \geq n^*$ then

$$d(p(k'), p(k'')) < \tfrac{1}{2}e.$$

Hence, since $p(k')$ and $p(k'') \in K$, we have that $p(k')$ and $p(k'')$ are equal if $k', k'' \geq n^*$. Thus $p(n) = p(n^*)$ if $n \geq n^*$. This implies that

$$\lim(p(n)) = p(n^*).$$

Immediately we have that

$$p = p(n^*).$$

Hence $p \in K$ since $p(n^*) \in K$. Thus K is closed.

Take $p \in K$. Then it is immediate that

$$N(p; \tfrac{1}{2}e) \cap K = \{p\}.$$

Thus p is not a cluster point. p was arbitrary. Hence K does not have a cluster point.

Corollary. *If* K *is infinite, then* K *is not compact*.

Proof. Trivial.

SECTION THREE

- BISECTORS and DERIVATIVES -

1.3.1. The Bisector of a Tet.

Definition: Define $Tet(p; e)$ to be the set of all maximal e-tets K such that $p \in K$.

Definition: If $K \in Tet(p; e)$, define $b(p,K)$ to be the set of all $q \in M$ such that

$$d(q,r) = d(q,s)$$

for all $r,s \in K$ with $r \neq p$ and $s \neq p$.

$b(p,K)$ is called the bisector of K which passes through p.

Theorem: If $K \in Tet(p; e)$, then $b(p,K)$ is closed.

Proof. Let $(q(n))$ be a sequence in $b(p,K)$ with $\lim(q(n)) = q$. It will suffice to show that $q \in b(p,K)$.

Take arbitrary $r,s \in K$ with $r \neq p$ and $s \neq p$. Then for any $n \in N$,

$$d(q(n), r) = d(q(n), s).$$

Hence it is seen that

$$\lim(d(q(n), r)) = \lim(d(q(n), s)).$$

But

$$\lim(d(q(n), r)) = d(\lim(q(n)), r)$$
$$\lim(d(q(n), s)) = d(\lim(q(n)), s).$$

Hence

$$d(q,r) = d(q,s).$$

Definition: If $K \varepsilon \mathrm{Tet}(p; e)$, define $b^{+}(p,K)$ to be the set of all $q \varepsilon b(p,K)$ with $d(r,q) < e$ for all $r \varepsilon K$ with $r \neq p$.

Definition: If $K \varepsilon \mathrm{Tet}(p; e)$, define $b^{-}(p,K)$ to be the set of all $q \varepsilon b(p,K)$ with $d(r,q) > e$ for some $r \varepsilon K$ with $r \neq p$

Note. It is clear that

$$b^{+}(p,K) \cap b^{-}(p,K) = \emptyset.$$

Intuitively, in the Euclidean plane, $b^{+}(p,K)$ and $b^{-}(p,K)$

give rise to a "positive" and "negative" direction on $b(p,K)$ close to p.

1.3.2. Derivatives of Real-Valued Functions.

Let (M,d) be a metric space. Let $p \in M$. Let

$$f\colon M \to R.$$

Let $K \in Tet(p; e)$.

Definition: f is *positively* (e,p,K)-*differentiable* iff

$$(1) \qquad \lim_{\substack{y \to p \\ y \in b^{+}(p,K)}} \frac{f(y) - f(p)}{d(y,p)}$$

exists.

Definition: f *is negatively* (e,p,K)-*differentiable* iff

$$(2) \qquad \lim_{\substack{y \to p \\ y \in b^{-}(p,K)}} \frac{f(y) - f(p)}{-d(y,p)}$$

exists.

Definition: f is (e,p,K)-differentiable iff f is positively (e,p,K)-differentiable and f is negatively (e,p,K)-differentiable and

$$f^{+}_{(e,K)}(p) = f^{-}_{(e,K)}(p)$$

where $f^{+}_{(e,K)}(p)$ denotes the limit (1) and $f^{-}_{(e,K)}(p)$ denotes the limit (2).

SECTION FOUR

- DIRECTIONS -

1.4.1. Types of Maps.

Let (M,d) and (M',d') be metric spaces.

Definition: f is an isometry of (M,d) onto (M',d') iff

(1) $f\colon M \to M'$

(2) f is one-to-one and onto

(3) for any $x,y \in M$,
$d'(f(x), f(y)) = d(x,y)$.

Definition: f is a local isometry of (M,d) into (M',d') iff

(1) $f\colon M \to M'$

(2) for any $p \in M$, there exists a neighbourhood U of p and a neighbourhood V of $f(p)$ such that $f|U$ is an isometry of $(U, d|(U \times U))$ onto $(V, d'|(V \times V))$.

1.4.2. The Notion of Directions.

Let (M,d) be a metric space.

Definition: If $p \in M$, define

$$\mathrm{Tet}(p) = \bigcup_{e \in R} \mathrm{Tet}(p;\ e).$$

Definition: If K is a tet, define

$$\mathrm{char}(K) = e \quad \text{iff} \quad K \text{ is an } e\text{-tet}.$$

Note. In what follows use is made of $b^+(p,K)$ in the definitions. However, a similar discussion can be carried out using $b^-(p,K)$.

Definition: K determines a direction at p iff

(1) $K \in \mathrm{Tet}(p)$ and $p \in M$

(2) p is an accumulation point of $b^+(p,K)$.

Definition: If $p \in M$, define

$$\mathrm{Tet}^*(p) = \{K \mid K \text{ determines a direction at } p\}.$$

Definition: If $K', K'' \in \mathrm{Tet}^*(p)$, define $K' \sim K''$

iff there exists $e \in R$ with $e > 0$ such that

$$(1) \quad b^{+}(p,K') \cap N(p;\ e) \subseteq b^{+}(p,K'')$$

$$(2) \quad b^{+}(p,K'') \cap N(p;\ e) \subseteq b^{+}(p,K').$$

Claim. $\sim$ is an equivalence relation on Tet*(p).

Proof. The reflexivity of $\sim$ is immediate from the properties of containment. The symmetry is inherent in the definition. The transitivity follows from the transitivity of containment.

Definition: A regular direction at p is an equivalence class in Tet*(p) with respect to $\sim$.

Notation. If $K \in$ Tet*(p), then $[K]_{\sim}$ denotes the equivalence class of K in Tet*(p) with respect to $\sim$. Thus $[K]_{\sim}$ is called a regular direction at p.

Definition: A regular direction $[K]_{\sim}$, at p is refinable iff for any $e > 0$, there exists $f > 0$ with $f \le e$ such that there exists $L \in [K]_{\sim}$ with

$$\text{char}(L) = f.$$

<u>Claim</u>. <u>The association</u>

$$[K]_{\sim} \rightsquigarrow [K]_{\sim} \cap \wp(N(p;\, e))$$

defines a one-to-one and onto mapping of the set of refinable regular directions at p in M to the set of refinable regular directions at p in $N(p;\, e)$.

<u>Proof</u>. Clearly, since $[K]_{\sim}$ is refinable, we have that

$$[K]_{\sim} \cap \wp(N(p;\, e)) \neq \emptyset.$$

From this, it is immediate that

$$[K]_{\sim} \cap \wp(N(p;\, e))$$

is a refinable regular direction at p in $N(p;\, e)$.

Assume that

$$[K]_{\sim} \cap \wp(N(p;\, e)) = [L]_{\sim} \cap \wp(N(p;\, e)).$$

Since neither of these sets is empty, we may take J to be an element of both. Then we have that if $I \varepsilon [K]_{\sim}$ then

$$J \sim I$$

and that if $I \varepsilon [L]_{\sim}$ then

$$J \sim I.$$

Hence it is seen that

$$[J]_{\sim} = [K]_{\sim}$$

and

$$[J]_{\sim} = [L]_{\sim} .$$

Thus

$$[K]_{\sim} = [L]_{\sim} .$$

This proves that the association is one-to-one.

Let L be a refinable regular direction at p in $N(p; e)$. Let $K \varepsilon L$. Then it is certainly true that

$$L = [K]_{\sim} \cap \wp(N(p; e)).$$

Thus the association is onto.

The refinable directions seem to be of importance since they are "preserved" no matter how close one comes to the point p in question (i.e., there is a tet giving them).

1.4.3. Preservation of Refinable Regular Directions.

Let (M,d) and (M',d') be metric spaces. Let f be a local isometry of (M,d) into (M',d'). Let $p \in M$. Let U be a neighbourhood of p and let V be a neighbourhood of $f(p)$ such that $f|U$ is an isometry of $(U, d|(U \times U))$ onto $(V, d'|(V \times V))$.

Since f is a local isometry, we are guaranteed the existence of a U and a V with these properties. Choose e' and e'' with

$$N(p;\, e') \subseteq U \quad \text{and} \quad N(f(p);\, e'') \subseteq V.$$

Set $e = \min\{e',e''\}$. Then it is clear that $f|N(p;\, e)$ is an isometry of $N(p;\, e)$ onto $N(f(p);\, e)$.

Moreover under an isometry of a metric space onto a metric space refinable regular directions are preserved since such an isometry preserves distances.

Hence if we take a refinable regular direction $[K]_\sim$ at p in (M,d) and then assign to it

$$[K]_\sim \cap \rho(N(p; e))$$

and then assign to this

$$f([K]_\sim \cap \rho(N(p; e))),$$

we have associated a refinable regular direction at f(p) in N(f(p); e) with $[K]_\sim$. Now assign to this regular direction at f(p) in N(f(p); e) the regular direction at p in (M',d') guaranteed by the last claim in the preceding sub-section. Thus we now have an association of $[K]_\sim$ with a refinable regular direction at f(p) in (M',d'). Moreover, from the construction it is immediate that this assignment is one-to-one and onto between the refinable regular directions at p in (M,d) and the refinable regular directions at f(p) in (M',d').

Thus in a certain sense the local isometry, f, preserves refinable regular directions.

SECTION FIVE

- A TOPOLOGY -

1.5.1. Nearness of Tets.

Let (M,d) be a metric space.

Definition: $\mathrm{Tet}(M) = \bigcup_{e \varepsilon R} \mathrm{Tet}(e)$.

Definition: If $e \varepsilon R$ and $e > 0$, define $N(e)$ to be the set of all ordered pairs (K',K'') such that

$$K',K'' \varepsilon \mathrm{Tet}(M),$$

and there exists a function

$$h:\ K' \to K''$$

such that h is one-to-one and onto and such that for any $x \varepsilon K'$

$$d(x,h(x)) < e.$$

Definition: $U = \{N(e) | e \varepsilon R \text{ and } e > 0\}$.

Claim. U *is a filter base on* $\mathrm{Tet}(M) \times \mathrm{Tet}(M)$ *and the filter that it generates is a uniform structure on*

Tet(M) ([2], page 177; [3], page 21).

Proof. (1). First we show that U is a filter base on Tet(M) × Tet(M). Let

$$N(h'), N(h'') \in U.$$

Consider

$$h = \min\{h', h''\}.$$

Then

$$N(h) \subseteq N(h') \quad \text{and} \quad N(h) \subseteq N(h'').$$

This implies that

$$N(h) \subseteq N(h') \cap N(h'').$$

Note. If R is a set of ordered pairs then R^{-1} is the set of all ordered pairs (x,y) such that

$$(y,x) \in R.$$

If R and S are sets of ordered pairs then $R \circ S$ is the set of all ordered pairs (x,y) such that there exists some z with

$$(x,z) \in R \text{ and } (z,y) \in S.$$

(2) Now we show that U generates a uniform structure. It is sufficient to show that

(a) for every $e \in R$ with $e > 0$,
$\{(K,K) | K \in Tet(M)\} \subseteq N(e)$

(b) for every $e \in R$ with $e > 0$,
$N(e) = (N(e))^{-1}$

(c) for every $e \in R$ with $e > 0$,
there exists $h \in R$ with $h > 0$ such that
$N(h) \circ N(h) \subseteq N(e)$.

(a) Let $K \in Tet(M)$. Consider the identity map on K; denote it by $I(K)$. Now $I(K)$ is one-to-one and onto and for any $x \in K$ and any $e \in R$ with $e > 0$ we have

$$\begin{aligned} & d(x, I(K)\ (x)) \\ & = d(x,x) \\ & = 0 \\ & < e. \end{aligned}$$

Hence $(K,K) \varepsilon N(e)$ for all $e \varepsilon R$ with $e > 0$.

(b) Let $e \varepsilon R$ with $e > 0$. Let

$$(K',K'') \varepsilon N(e).$$

This implies that there exists

$$h: \quad K' \to K''$$

which is one-to-one and onto and is such that for any $x \varepsilon K'$

$$d(x,h(x)) < e.$$

Consider

$$h^{-1}: \quad K'' \to K'$$

which is one-to-one and onto. We note that for any $y \varepsilon K''$

$$\begin{aligned} &d(y,h^{-1}(y)) \\ &= d(h^{-1}(y),y) \\ &= d(h^{-1}(y), h(h^{-1}(y))) \\ &< e \end{aligned}$$

by the properties of h. Hence $(K'', K') \in N(e)$.

This implies that

$$N(e) = (N(e))^{-1}.$$

(c) Let $e \in R$ with $e > 0$. Choose

$$h = \tfrac{1}{2}e.$$

Let

$$(K', K'') \in N(h) \circ N(h).$$

This implies that there exists $K^* \in \mathrm{Tet}(M)$ with

$$(K', K^*) \in N(h) \quad \text{and} \quad (K^*, K'') \in N(h).$$

Hence there exists

$$h'\colon\ K' \to K^*$$

which is one-to-one and onto and has the property that for any $x \in K'$

$$d(x,h'(x)) < h.$$

Also there exists

$$h'': K^* \to K''$$

which is one-to-one and onto and has the property that for any $y \in K^*$

$$d(y,h''(y)) < h.$$

Consider

$$h'' \circ h': K' \to K''.$$

Then $h'' \circ h'$ is one-to-one and onto. We now see that for any $x \in K'$

$$d(x,(h'' \circ h')(x))$$
$$\leq d(x,h'(x)) + d(h'(x), h''(h'(x)))$$
$$< h + h = e$$

by the triangle inequality. Hence

$$(K',K'') \varepsilon N(e).$$

Thus

$$N(h) \circ N(h) \subseteq N(e).$$

Definition: Define $D^*(M,d)$ to be the topology induced on Tet(M) by the filter generated by U.

1.5.2. Finitary Metric Spaces.

Definition: (M,d) is *finitary* iff $K \varepsilon \text{Tet}(M)$ implies that K is finite.

Claim. *If (M,d) is finitary, then $D^*(M,d)$ is Hausdorff.*

Proof. Take $K',K'' \varepsilon \text{Tet}(M)$ with $K' \neq K''$. This implies, without loss of generality, that there exists a k such that

$$k \varepsilon K' \text{ and } k \not\varepsilon K''.$$

Since K'' is finite, we have that

$$e = \min_{x \varepsilon K''} d(k,x) > 0.$$

Assume that there exists K^* with

$$K^* \varepsilon N(K'; \tfrac{1}{2}e) \cap N(K''; \tfrac{1}{2}e).$$

where

$$N(K; h) = \{L | (K,L) \varepsilon N(h)\}$$

and is thus a neighbourhood of K in $Tet(M)$ with respect to $D^*(M,d)$. This gives that there exists

$$h': K' \to K^*$$

which is one-to-one and onto and is such that for any $x \varepsilon K'$

$$d(x,h'(x)) < \tfrac{1}{2}e.$$

Also there exists

$$h'': K^* \to K''$$

which is one-to-one and onto and is such that for any $y \in K^*$

$$d(y,h''(y)) < \tfrac{1}{2}e.$$

Hence

$$\begin{aligned} &d(k,h''(h'(k))) \\ &\leq d(k,h'(k)) + d(h'(k),h''(h'(k))) \\ &< \tfrac{1}{2}e + \tfrac{1}{2}e = e. \end{aligned}$$

This is a contradiction since

$$h''(h'(k)) \in K''.$$

<u>1.5.3</u>. <u>A Property of Arcs</u>.

<u>Claim</u>. <u>If</u> f <u>is a function with</u>

$$f: \quad I \to \mathrm{Tet}(M)$$

<u>where</u>

$$I = \{x \mid x \in R \text{ and } 0 \leq x \leq 1\}$$

<u>and if</u> f <u>is continuous with respect to the usual topology on</u> I <u>and</u> $D^*(M,d)$, <u>then</u>

$$|f(0)| = |f(1)|.$$

<u>Proof</u>. For any $x \in I$, there exists $h(x) \in R$ with $h(x) > 0$ such that for any $y \in I$ if

$$|x - y| < h(x)$$

then

$$f(y) \in N(f(x); 1)$$

by the continuity of f. In such a situation we can immediately conclude that

(1) $$|f(y)| = |f(x)|$$

by the definition of $D^*(M,d)$.

The set of all $N(x; h(x))$ is an open cover of I. Thus there exists a finite subcover

$$(N(x_i; h(x_i)))_{i=1}^{n}$$

such that

(1) for $i = 1,\ldots,n-1$,

$x_i < x_{i+1}$

(2) for $i = 1,\ldots,n-1$,

$N(x_i; h(x_i)) \cap N(x_{i+1}, h(x_{i+1})) \neq \emptyset$

(3) $0 \in N(x_1, h(x_1))$

(4) $1 \in N(x_n, h(x_n))$

since I is compact and connected. We will show that for $i = 1,\ldots,n-1$,

$$|f(x_i)| = |f(x_{i+1})|.$$

Now since for $i = 1,\ldots,n-1$,

$$N(x_i; h(x_i)) \cap N(x_{i+1}, h(x_{i+1})) \neq \emptyset$$

there exists y_i for $i=1,\ldots,n-1$ such that

$$y_i \in N(x_i; h(x_i)) \cap N(x_{i+1}, h(x_{i+1})).$$

This and (1) implies that $i = 1,\ldots,n-1$,

$$|f(x_i)| = |f(y_i)| \text{ and } |f(y_i)| = |f(x_{i+1})|.$$

Hence we get that for $i = 1,\ldots,n-1$

$$|f(x_i)| = |f(x_{i+1})|.$$

Also since $0 \in N(x_1; h(x_1))$ we have that

$$|f(0)| = |f(x_1)|,$$

and since $1 \in N(x_n; h(x_n))$ we have that

$$|f(1)| = |f(x_n)|.$$

Hence

$$|f(0)| = |f(x_1)| = \ldots = |f(x_n)| = |f(1)|.$$

SECTION SIX

- THE CHARACTERISTIC -

1.6.1. A Uniformly Continuous Function.

Let (M,d) be a metric space. We recall that if K is a tet then

$$\text{char}(K) = e \quad \text{iff} \quad K \text{ is an } e\text{-tet}.$$

Note. Let $K \in \text{Tet}(M)$. Then we have the following results:

(a) if $|K| \leq 1$, then
for any $r \in R$
$\text{char}(K) = r$

(b) if $|K| \geq 2$, then
$\text{char}(K)$ is unique.

Proof of (a). Since $|K| \leq 1$, it follows that $K = \{p\}$ for some $p \in M$ or $K = \emptyset$. In either case, K is an r-tet for any $r \in R$.

Hence $\text{char}(K) = r$ for any $r \in R$.

Proof of (b). Since $|K| \geq 2$, it follows that there exists $p,q \in K$ with $p \neq q$. Assume

$$\text{char}(K) = r' \text{ and } \text{char}(K) = r''.$$

then

$$d(p,q) = r' \text{ and } d(p,q) = r''.$$

this implies that $r' = r''$. Hence char(K) is unique.

Definition: Define Tet*(M) to be the set of all $K \in$ Tet(M) with $|K| \geq 2$.

Definition: Define D**(M,d) to be the restriction of the topology D*(M,d) to Tet*(M).

Definition: Define

$$\text{CHAR: } \text{Tet}^*(M) \to R$$

by setting

$$\text{CHAR}(K) = \text{char}(K).$$

Then CHAR is well-defined by the observations in the preceding note.

<u>Claim.</u> CHAR <u>is uniformly continuous with respect to the usual topology on</u> R <u>and</u> D**(M,d).

<u>Proof</u>. Take $e > 0$. Choose $h = \frac{1}{2}e$. Then it is sufficient to show that

$$\text{if } (K,L) \in N(h),$$
$$\text{then } |\text{char}(K) - \text{char}(L)| < e.$$

Take $(K,L) \in N(h)$. Hence there exists

$$f\colon K \to L$$

which is one-to-one and onto and is such that for any $x \in K$

$$d(x,f(x)) < h.$$

Now, since $|K| \geq 2$, there exist $p,q \in K$ with

$$p \neq q.$$

Thus

$$f(p) \neq f(q)$$

since f is one-to-one. Also we have

$$d(p,f(p)) < h \quad \text{and} \quad d(q,f(q)) < h.$$

By an inequality proved in <u>1.1</u> we find that

$$\begin{aligned} &|d(p,q) - d(f(p), f(q))| \\ &\leq d(p,f(p)) + d(q,f(q)) \\ &< h + h = e. \end{aligned}$$

But we also have that

$$d(p,q) = \text{char}(K)$$

since $p \neq q$ and that

$$d(f(p), f(q)) = \text{char}(L)$$

since $f(p) \neq f(q)$. Hence we see that

$$|\text{char}(K) - \text{char}(L)| < e.$$

1.6.2. A Function Property.

Let (M,d) be a metric space. Let

$$K \in \mathrm{Tet}(M).$$

Claim. *If* f *is a function with*

$$f\colon\ K \to M$$

such that for any $x \in K$

$$d(x,f(x)) < \tfrac{1}{2}\,\mathrm{char}(K)$$

then f *is one-to-one.*

Proof. Assume that $x,y \in K$ and $f(x) = f(y)$. Then we must show that $x = y$. Now we know that

$$d(x,f(x)) < \tfrac{1}{2}\,\mathrm{char}(K)$$

and that

$$d(y,f(y)) < \tfrac{1}{2}\,\mathrm{char}(K).$$

Computing we find that

$$\begin{aligned} & d(x,y) \\ & \leq d(x,f(x)) + d(f(x), y) \\ & = d(x,f(x)) + d(y,f(y)) \\ & < \tfrac{1}{2}\,\mathrm{char}(K) + \tfrac{1}{2}\,\mathrm{char}(K) \\ & = \mathrm{char}(K) \end{aligned}$$

since $f(x) = f(y)$. But this gives us that $x = y$ since K is a tet.

SECTION SEVEN

- STRUCTURE THEOREMS -

1.7.1. Composite Tet-Spaces.

Definition: (T,d) is a tet-space iff

(1) (T,d) is a metric space

(2) there exists $e \in R$ with $e > 0$ such that for any $x,y \in T$ with $x \neq y$: we have $d(x,y) = e$.

Definition: $((T_i,d_i)_{i\in I}, R)$ is a composite tet-space iff

(1) for any $i \in I$, (T_i,d_i) is a tet-space

(2) R is an equivalence relation on $\bigcup_{i\in I} T_i$

(3) if $x,z \in T_i$ and $y,w \in T_j$ and $x \mathrel{R} y$ and $z \mathrel{R} w$, then $d_i(x,z) = d_j(y,w)$

(4) for any $x,y \in \bigcup_{i\in I} T_i$ if it is not the case that $x \mathrel{R} y$ then there exists $j \in I$ such that $x,y \in T_j$

(5) if $x,y \in T_i$ and $y,z \in T_j$ and $x,z \in T_k$, then $d_i(x,y) + d_j(y,z) \geq d_k(x,z)$.

Let $T = ((T_i,d_i)_{i \in I}, R)$ be a composite tet-space.

<u>Definition</u>: Define $M(T)$ to be the set of all equivalence classes in $\bigcup_{i \in I} T_i$ with respect to R.

<u>Notation</u>. If $x \in \bigcup_{i \in I} T_i$, then the equivalence class of x in $\bigcup_{i \in I} T_i$ with respect to R will be denoted by $[x]_R$.

<u>Definition</u>: Define a function $d(T)$ with

$$d(T)\colon\ M(T) \times M(T) \to R$$

by setting

$$(d(T))([x]_R,[y]_R) = d_i(x,y)$$

if we have that

$$[x]_R \neq [y]_R$$

and where $i \in I$ is such that $x, y \in T_i$ and by setting

$$(d(T))([x]_R, [y]_R) = 0$$

if we have that

$$[x]_R = [y]_R .$$

<u>Claim</u>. $d(T)$ <u>is well-defined</u>.

<u>Proof</u>. In the case

$$[x]_R = [y]_R$$

$d(T)$ is certainly well-defined. Let us now consider the case

$$[x]_R \neq [y]_R.$$

Assume that

$$[z]_R = [x]_R \text{ and } [w]_R = [y]_R.$$

Then let $i, j \in I$ be such that $x, y \in T_i$ and $z, w \in T_j$.

We can do this by property (4) of T. Hence we have that

$$x \; R \; z \quad \text{and} \quad y \; R \; w$$

and that

$$x,y \in T_i \quad \text{and} \quad z,w \in T_j.$$

Thus, by property (3) of T, we see that

$$d_i(x,y) = d_j(z,w).$$

Claim. $d(T)$ is a metric on $M(T)$.

Proof. (a). Assume that

$$(d(T))([x]_R, [y]_R) = 0.$$

Assume, in addition, that

$$[x]_R \neq [y]_R.$$

Then, by the definition of $d(T)$, there exists $i \in I$ with

$$d_i(x,y) = 0 \quad \text{and} \quad x,y \in T_i.$$

Thus $x = y$ since d_i is a metric and, consequently,

$$[x]_R = [y]_R .$$

This, however, contradicts our assumption.

In the case

$$[x]_R = [y]_R$$

we are guaranteed by the definition of $d(T)$ that

$$(d(T))([x]_R, [y]_R) = 0.$$

(b). If we have that

$$[x]_R = [y]_R$$

then it is immediate from the definition of $d(T)$ that

$$(d(T))([x]_R, [y]_R) = (d(T))([y]_R, [x]_R)$$

since both sides are zero.

If we have that

$$[x]_R \neq [y]_R$$

then we have for some $i \in I$ that

$$\begin{aligned}
&(d(T)([x]_R, [y]_R) \\
&= d_i(x,y) \\
&= d_i(y,x) \\
&= (d(T))([y]_R, [x]_R)
\end{aligned}$$

since d_i is a metric on T_i.

(c). We now consider the triangle inequality. We verify only the case that

$$[x]_R \neq [y]_R \text{ and } [y]_R \neq [z]_R \text{ and } [x]_R \neq [z]_R$$

leaving the other cases since they are similar and very easy to prove. We have that for some $i,j,k \in I$

$$\begin{aligned}
&(d(T))([x]_R, [y]_R) + (d(T))([y]_R, [z]_R) \\
&= d_i(x,y) + d_j(y,z) \\
&\geq d_k(x,z) \\
&= (d(T))([x]_R, [z]_R)
\end{aligned}$$

by property (5) of T.

1.7.2. Composite Tet-Spaces of (M,d).

Let (M,d) be a metric space.

Note. If $p,q \in M$, then the set whose elements are only p and q is a d(p,q)-tet in M. This tet is also contained in a maximal d(p,q)-tet.

Definition: Define Nat to be the ordered pair consisting of the family of

$$(K, d|(K \times K))$$

where K is a tet in (M,d) and of the equivalence relation

$$\equiv|M.$$

Definition: Define Max to be the ordered pair consisting of the family of

$$(K, d|(K \times K))$$

where K is a maximal tet in (M,d) and of the equivalence relation

$$\equiv|M.$$

Claim. Nat *is a composite tet-space.*

Claim. Max *is a composite tet-space.*

Proofs. The proof of property (4) of 1.6.1 is immediate from the note at the beginning of the section. The proofs of the other properties are immediate translations of the properties of (M,d) as a metric space and of equality as an equivalence relation.

Claim. $(M(Nat), d(Nat))$ *is congruent to* (M,d) *by means of the assignment*

$$x \rightsquigarrow \{x\}.$$

Proof. It is simple to show that $M(Nat)$ is the set of all $\{x\}$ such that $x \in M$. It is also easy to show that

$$(d(Nat))(\{x\},\{y\}) = d(x,y).$$

These follow from the fact that the equivalence class of x in a set with respect to equality is just $\{x\}$.

Claim. $(M(Max), d(Max))$ *is congruent to* (M,d) *by means of the assignment*

$$x \rightsquigarrow \{x\}.$$

Proof. The proof is the same as the proof of the preceding claim .

CHAPTER TWO

- "SCALAR MULTIPLICATIONS" -

Introduction. In this Chapter, two "scalar multiplications" are defined. In the first case a special scalar multiplication space is developed and a topological embedding theorem is proved for finitely compact metric spaces with a finite basis. In the second case the scalar multiplication takes ordered pairs consisting of a real number and an element of a given metric spaces into subsets of the metric space. As regards this second scalar multiplication a relation between it and normal scalar multiplication on a real inner product space is exhibited.

SECTION ONE

- THE SPACE $V(p,S)$ -

2.1.1. The Construction of $V(p,S)$.

Let (M,d) be a metric space. In addition, let $S \subseteq M$ and $p \in S$.

Definition: Let $V(p,S)$ be the set of all functions e such that

$$e: \quad S \to R$$

and such that $e(q) \geq 0$ for any $q \in S$ and such that for any $q \in S$ with $q \neq p$ we have that

$$(1) \quad |e(p) - d(p,q)| \leq e(q)$$
$$(2) \quad e(q) \leq e(p) + d(p,q).$$

Thus $e(p)$, $e(q)$ and $d(p,q)$ satisfy the triangle inequality for all $q \in S$ with $q \neq p$ and $V(p,S)$ is a subset of all the mappings from S into the set of non-negative real numbers. It might be said that in a certain sense $V(p,S)$ is "quasi-metric at p".

If $X \in V(p,S)$, then X is a function from S into the set of non-negative real numbers. Define $X(q)$ to be the value of X at $q \in S$. We may specify a function by the set of its values as in sequence notation. Thus we write

$$X = (X(q))_{q \in S}.$$

Definition: Consider $\ell \in R$ and $X \in V(p,S)$. Then we define ℓX to be the mapping from S into the complex numbers which sends q to $(\ell X)(q)$ for any $q \in S$ by putting

$$(\ell X)(q) = \begin{cases} |\ell| X(p) & \text{if } q = p \\ \sqrt{(\ell^2-\ell)(X(p))^2+(1-\ell)(d(p,q))^2+\ell(X(q))^2} \\ \text{if } q \neq p. \end{cases}$$

Thus we have

$$\ell X = ((\ell X)(p))_p \cup ((\ell X)(q))_{q \in S, q \neq p}$$

where the terms on the right are considered as functions.

Our goal is the following theorem.

Theorem. For any $\ell \in R$ and $X \in V(p,S)$,

$$\ell X \in V(p,S).$$

Proof. The proof is given in the remainder of this subsection.

Take $\ell \in R$ and $X \in V(p,S)$.

Claim. If $q \in S$ with $q \neq p$, then

$$((\ell X)(q))^2 \geq 0$$

where we have that

$$((\ell X)(q))^2 = (\ell^2 - \ell)(X(p))^2 + (1-\ell)(d(p,q))^2 + \ell(X(q))^2. \tag{1}$$

Proof. Consider first the case where

$$X(p) \neq 0.$$

In this case, we define

$$C(q) = \frac{(X(p))^2+(d(p,q))^2-(X(q))^2}{2X(p)\ d(p,q)} . \tag{2}$$

We note that

$$\begin{aligned} &\ell^2(X(p))^2+(d(p,q))^2-2\ell X(p)\ d(p,q)C(q) \\ &= (\ell^2-\ell)(X(p))^2+(1-\ell)(d(p,q))^2+\ell(X(q))^2. \end{aligned} \tag{3}$$

This is readily verified by expansion of the top half of the equation using (2).

Now we have that

$$|X(p) - d(p,q)| \le X(q).$$

This implies that

$$(X(p))^2+(d(p,q))^2-(X(q))^2 \le 2X(p)\ d(p,q).$$

Hence

$$C(q) \le 1.$$

We also have that

$$X(q) \le X(p) + d(p,q).$$

Thus it is seen that

$$-2X(p)\ d(p,q) \le (X(p))^2+(d(p,q))^2-(X(q))^2.$$

Hence

$$C(q) \geq -1.$$

It follows that

(4) $$|C(q)| \leq 1.$$

Next, if $\ell \geq 0$, then from (1), (3) and (4) we get

$$((\ell X)(q))^2 \geq \ell^2(X(p))^2+(d(p,q))^2-2\ell X(p)\ d(p,q)$$
$$= (\ell X(p)-d(p,q))^2$$
$$\geq 0.$$

On the other hand, if $\ell \leq 0$, we get from (1), (3) and (4) that

$$((\ell X)(q))^2 \geq \ell^2(X(p))^2+(d(p,q))^2+2\ell X(p)\ d(p,q)$$
$$= (\ell X(p)+d(p,q))^2$$
$$\geq 0.$$

Consider now the case where

$$X(p) = 0.$$

Then for all $q \in S$ with $q \neq p$ we have

$$|X(p) - d(p,q)| \leq X(q)$$

which gives us that

$$d(p,q) \leq X(q).$$

We also have that for all $q \in S$ with $q \neq p$

$$X(q) \leq X(p) + d(p,q)$$

which implies that

$$X(q) \leq d(p,q).$$

Hence if $q \in S$ and $q \neq p$, then

(5) $$X(q) = d(p,q).$$

Thus we have

$$\begin{aligned} ((\ell X)(q))^2 &= (1-\ell)(d(p,q))^2+\ell(d(p,q))^2 \\ &= (d(p,q))^2 \\ &\geq 0. \end{aligned}$$

Finally, we see that if $q \in S$ and $q \neq p$, then $(\ell X)(q)$ is real and greater than or equal to zero.

<u>Claim</u>. <u>We can now complete our proof that</u>

$$(\ell X) \in V(p,S).$$

<u>Proof</u>. It is sufficient to show that for any $q \in S$ with $q \neq p$ we have that

$$|(\ell X)(p) - d(p,q)| \leq (\ell X)(q)$$

and that

$$(\ell X)(q) \leq (\ell X)(p) + d(p,q).$$

We will now consider the problem in two cases. First consider the case

$$X(p) = 0.$$

By reference to (5) of the preceding claim we see that for any $q \in S$ with $q \neq p$

$$X(q) = d(p,q).$$

Also we have that

$$(\ell X)(p) = |\ell| X(p) = 0.$$

The inequalities required reduce to

$$(1) \quad d(p,q) \leq (\ell X)(q)$$
$$(2) \quad (\ell X)(q) \leq d(p,q)$$

for any $q \in S$ with $q \neq p$. But these are valid since if $q \in S$ and $q \neq p$ then

$$\begin{aligned}(\ell X)(q) &= \sqrt{(\ell^2-\ell)0 + (1-\ell)(d(p,q))^2 + \ell(X(q))^2} \\ &= \sqrt{(1-\ell)(d(p,q))^2 + \ell(d(p,q))^2} \\ &= d(p,q).\end{aligned}$$

Now we consider the case

$$X(p) \neq 0.$$

Making reference to (3) and (4) of the preceding claim we see that

$$(a) \quad |C(q)| \leq 1$$
$$(b) \quad ((\ell X)(q))^2 = \ell^2(X(p))^2 + (d(p,q))^2 - 2\ell X(p)\, d(p,q)\, C(q).$$

Next, if $\ell \geq 0$, then from (a) and (b) we see that

$$
\begin{aligned}
((\ell X)(q))^2 &\geq \ell^2(X(p))^2+(d(p,q))^2-2\ell X(p)\, d(p,q) \\
&= (\ell X(p)-d(p,q))^2 \\
&= (|\ell|\, X(p)-d(p,q))^2 \\
&= ((\ell X)(p)-d(p,q))^2.
\end{aligned}
$$

Thus

$$(\ell X)(q) \geq |(\ell X)(p) - d(p,q)|.$$

Also, if $\ell \geq 0$, then from (a) and (b) we see that

$$
\begin{aligned}
((\ell X)(q))^2 &\leq \ell^2(X(p))^2+(d(p,q))^2+2\ell X(p)\, d(p,q) \\
&= (\ell X(p)+d(p,q))^2 \\
&= (|\ell|X(p)+d(p,q))^2 \\
&= ((\ell X)(p)+d(p,q))^2.
\end{aligned}
$$

This implies that

$$(\ell X)(q) \leq (\ell X)(p) + d(p,q).$$

Now, if $\ell \leq 0$, then from (a) and (b) we have

$$
\begin{aligned}
((\ell X)(q))^2 &\leq \ell^2(X(p))^2+(d(p,q))^2-2\ell X(p)\ d(p,q) \\
&= (\ell X(p)-d(p,q))^2 \\
&= (|\ell|X(p)+d(p,q))^2 \\
&= ((\ell X)(p)+d(p,q))^2.
\end{aligned}
$$

From this, it is immediate that

$$(\ell X(q) \leq (\ell X)(p) + d(p,q).$$

Moreover, if $\ell \leq 0$, we obtain from (a) and (b) that

$$
\begin{aligned}
((\ell X)(q))^2 &\geq \ell^2(X(p))^2+(d(p,q))^2+2\ell X(p)\ d(p,q) \\
&= (\ell X(p)+d(p,q))^2 \\
&= (-|\ell|X(p)+d(p,q))^2 \\
&= ((\ell X)(p)-d(p,q))^2.
\end{aligned}
$$

Hence

$$(\ell X)(q) \geq |(\ell X)(p) - d(p,q)|.$$

Thus in either instance of the sign of ℓ we have the required inequalities. Thus

$$\ell X \in V(p,S).$$

2.1.2. The Scalar Multiplication on V(p,S).

Definition: Define a function $O(p,S)$ with

$$O(p,S): \ R \times V(p,S) \to V(p,S)$$

by putting

$$(O(p,S))(\ell, X) = \ell X$$

for any $\ell \in R$ and $X \in V(p,S)$.

Notation: In what follows, we will use the notation

$$\ell \circ X = (O(p,S))(\ell, X).$$

Claim. If ℓ, $m \in R$ and $X \in V(p,S)$, then

$$\ell \circ (m \circ X) = (\ell m) \circ X.$$

Proof. It is sufficient to show that

$$\ell(mX) = (\ell m)X.$$

First consider the point $p \in S$. Then from the definitions we get

$$\begin{aligned}(\ell(mX))(p) &= |\ell|((mX)(p)) \\ &= |\ell|\,|m|X(p) \\ &= |\ell m|X(p)\end{aligned}$$

and

$$((\ell m)X)(p) = |\ell m|X(p).$$

Hence

$$(\ell(mX))(p) = ((\ell m)X)(p).$$

In addition if $q \in S$ with $q \neq p$ then we get

$$\begin{aligned}&(\ell(mX))(q) \\ &= \sqrt{(\ell^2-\ell)((mX)(p))^2+(1-\ell)(d(p,q))^2+\ell((mX)(q))^2}\end{aligned}$$

$$= \sqrt{(\ell^2-\ell)|m|^2\ (X(p))^2+(1-\ell)(d(p,q))^2 +\ell((m^2-m)(X(p))^2+(1-m)(d(p,q))^2+m(X(q))^2)}$$

$$= \sqrt{\ell^2 m^2(X(p))^2+(d(p,q))^2-\ell m(X(p))^2 -\ell m(d(p,q))^2+\ell m(X(q))^2}$$

$$= \sqrt{((\ell m)^2-\ell m)(X(p))^2+(1-\ell m)(d(p,q))^2+\ell m(X(q))^2}$$

$$= \ ((\ell m)X)(q).$$

Hence we conclude that

$$\ell(mX) = (\ell m)\ X.$$

Definition. We put

$$p^* = (d(p,q))_{q\varepsilon S}.$$

Note. Since the triangle inequality holds in M,

$$p^* \ \varepsilon\ V(p,S).$$

We also have

(1) $p^*(p) = 0$

(2) for any $q \ \varepsilon\ S$ with $q \neq p$

$p^*(q) = d(p,q)$.

Claim. For any $\ell \in R$

$$\ell p^* = p^*.$$

Proof. First consider the point p. Then we have that

$$\begin{aligned}(\ell p^*)(p) &= |\ell|(p^*(p)) \\ &= 0 \\ &= p^*(p).\end{aligned}$$

Now consider $q \in S$ with $q \neq p$. Then

$$\begin{aligned}(\ell p^*)(q) &= \sqrt{(\ell^2-\ell)(p^*(p))^2+(1-\ell)(d(p,q))^2+\ell(p^*(q))^2} \\ &= \sqrt{(1-\ell)(d(p,q))^2+\ell(d(p,q))^2} \\ &= d(p,q) \\ &= p^*(q).\end{aligned}$$

Hence we have shown what was claimed.

Claim. If $X \in V(p,S)$, then

$$0\,X = p^*.$$

<u>Proof</u>. First consider $p \in S$. Then we see that

$$\begin{aligned}(0\ X)(p) &= |0|X(p) \\ &= 0 \\ &= p^*(p).\end{aligned}$$

Now consider $q \in S$ with $q \neq p$. Then we have

$$\begin{aligned}(0\ X)(q) &= \sqrt{(0^2-0)(X(p))^2+(1-0)(d(p,q))^2+0(X(q))^2} \\ &= d(p,q) \\ &= p^*(q).\end{aligned}$$

Hence

$$0\ X = p^*.$$

<u>Claim</u>. <u>If</u> $X \in V(p,S)$, <u>then</u>

$$1\ X = X.$$

<u>Proof</u>. First consider the point $p \in S$. Then we get that

$$\begin{aligned}(1\ X)(p) &= |1|X(p) \\ &= X(p).\end{aligned}$$

Now consider $q \in S$ with $q \neq p$. Then

$$(1\ X)(q) = \sqrt{(1^2-1)(X(p))^2+(1-1)(d(p,q))^2+1(X(q))^2}$$
$$= X(q).$$

Hence it is seen that

$$1\ X = X.$$

SECTION TWO

- THE MAPPING C(p,S) -

2.2.1. Basic Properties of C(p,S).

Let (M,d) be a metric space.

Definition: S is a **basis** of M iff

(1) $S \subseteq M$

(2) for any $x,y \in M$
if we have for every $q \in S$
$d(x,q) = d(y,q)$
then $x=y$.

Thus if S is a basis of M, then if $x,y \in M$ and $x \neq y$, then there exists $q \in S$ such that

$$d(x,q) \neq d(y,q).$$

Note. According to our definition, M is a basis of M. For if we take $x,y \in M$ with

$$d(x,z) = d(y,z)$$

for any $z \in M$, we may consider the special case where $x = z$. Then we have

$$0 = d(y,x).$$

Thus we obtain that $x = y$. Thus we have seen that every metric space has at least one basis.

Let $S \subsetneq M$ and $p \in M$.

<u>Definition</u>: Construct a function $C(p,S)$ with

$$C(p,S): \quad M \to V(p,S)$$

by setting

$$(C(p,S))(x) = (d(x,q))_{q \in S}.$$

<u>Note</u>. Since M is a metric space we have that if $x \in M$ then

$$(d(x,q))_{q \in S} \in V(p,S).$$

Also we see immediately that

$$(C(p,S))(p) = p^*.$$

Claim. *If* S *is a basis of* M, *then* C(p,S) *is one-to-one.*

Proof. Assume that

$$(C(p,S))(x) = (C(p,S))(y).$$

This gives us, from the definition, that

$$(d(x,q))_{q\varepsilon S} = (d(y,q))_{q\varepsilon S}.$$

This says that for any $q \varepsilon S$

$$d(x,q) = d(y,q).$$

Hence we see that

$$x = y$$

since S is a basis of M.

2.2.2. *Dense Bases.*

Let (M,d) be a metric space

Claim. *If* S *is dense in* M, *then* S *is a basis of* M.

Proof. Assume that S is not a basis of M. Hence there exists $x,y \in M$ with

$$d(s,x) = d(s,y)$$

for every $s \in S$ and such that

$$x \neq y.$$

From this we get

$$\begin{aligned} d(x,y) &\leq d(x,s) + d(s,y) \\ &= 2d(s,x) \end{aligned}$$

for all $s \in S$ by the triangle inequality. Since S is dense in M we have that for any $e > 0$ there exists $s \in S$ with

$$d(s,x) \leq \tfrac{1}{2}e.$$

Thus for any $e > 0$, we have

$$d(x,y) \leq e.$$

Hence

$$d(x,y) = 0.$$

Hence $x = y$. This is a contradiction. This implies that S is a basis of M.

SECTION THREE

- THE TOPOLOGY D(p,S) -

2.3.1. Preliminaries.

Let (M,d) be a metric space. Let $S \subseteq M$ and $p \in S$. Assume S is a finite set.

Definition: If $X \in V(p,S)$ and $e \in R$ with $e > 0$, then define N(X; e) to be the set of all $Y \in V(p,S)$ such that for any $q \in S$

$$|X(q) - Y(q)| < e.$$

Definition: Let D(p,S) be the usual product topology on R^S restricted to V(p,S).

Hence D(p,S) is a topology on V(p,S) and for any $X \in V(p,S)$ and any $e \in R$ with $e > 0$ we know that N(X; e) is an open set of V(p,S).

Claim. O(p,S) is a continuous mapping.

Proof. It is sufficient to show that O(p,S) is a continuous map from $R \times V(p,S)$ into R^S since the image of O(p,S) is contained in V(p,S) which is contained in R^S. To show this it is sufficient to show that the mapping

$$(\ell, X) \rightarrow (\ell X)(q)$$

is continuous for every $q \in S$ since a map into R^S is continuous iff each of its coordinate maps is continuous.

We consider first the case where $q = p$. Now we know that the map

$$\ell \rightsquigarrow |\ell|$$

is continuous on R and that the map

$$X \rightsquigarrow X(p)$$

is continuous on $V(p,S)$ since it is continuous on R^S. Hence the map

$$(\ell, X) \to (|\ell|, X(p))$$

is continuous on $R \times V(p,S)$. But we know that multiplication on R is continuous. Thus we see that

$$(\ell, X) \to |\ell| X(p)$$

is continuous on $R \times V(p,S)$.

Now we consider the case where $q \in S$ and $q \neq p$.

We know that the maps

$$X \rightsquigarrow (X(p))^2,$$
$$X \rightsquigarrow (X(q))^2$$

and the constant map

$$X \rightsquigarrow (d(p,q))^2$$

are all continuous on $V(p,S)$ since they are continuous on R^S. Similarly we know that the maps

$$\ell \rightsquigarrow (\ell^2 - \ell),$$
$$\ell \rightsquigarrow (1 - \ell)$$

and

$$\ell \rightsquigarrow \ell$$

are continuous on R. Hence we see that the maps

$$(\ell, X) \rightsquigarrow ((\ell^2 - \ell), (X(p))^2)$$

and

$$(\ell, X) \rightsquigarrow (1 - \ell, (d(p,q))^2)$$

and

$$(\ell, X) \rightsquigarrow (\ell, (X(q))^2)$$

are all continuous. But multiplication on R is continuous and the sum of continuous functions is continuous. Thus the map

$$(\ell, X) \rightsquigarrow (\ell^2 - \ell)\ (X(p))^2 + (1 - \ell)\ (d(p,q))^2 + \ell(X(q))^2$$

is continuous on $R \times V(p,S)$. But since $X \varepsilon V(p,S)$ the image of (ℓ, X) under this map is always greater than or equal to zero. Hence this map is into the non-negative reals. But taking square roots on the non-negative reals is continuous and the composition of continuous functions is continuous. Hence

$$(\ell, X) \rightsquigarrow (\ell X)(q)$$

is continuous on $R \times V(p,S)$.

Claim. C(p,S) is continuous.

Proof. Let $x \varepsilon M$. Let A be a neighbourhood of

$(C(p,S))(x)$. Then there exists $e \in R$ with $e > 0$ such that

$$N((C(p,S))(x);\ e) \subseteq A.$$

We now consider $(C(p,S))(N(x;\ e))$. Take $y \in N(x;\ e)$. Now for all $q \in S$ we have that

$$\begin{aligned} |d(y,q) - d(x,q)| &\leq d(x,y) \\ &< e \end{aligned}$$

by a claim in <u>1.1</u>. But we also know that

$$d(y,q) = ((C(p,S))(y))(q)$$

and that

$$d(x,q) = ((C(p,S))(x))(q)$$

for any $q \in S$. Hence for any $q \in S$

$$|((C(p,S))(y))(q) - ((C(p,S))(x))(q)| < e.$$

This implies that

$$((C(p,S))(y) \in N((C(p,S))(x);\ e).$$

Thus we see that

$$(C(p,S))(N(x;\ e)) \subseteq N((C(p,S))(x);\ e).$$

Hence $C(p,S)$ is continuous.

2.3.2 An Embedding Theorem.

Theorem. *If* (M,d) *is finitely compact* ([4], page 6) *and* S *is a basis of* M, *then* $C(p,S)$ *is a homeomorphism of* M *onto its image*.

Proof. By previous results we have seen that $C(p,S)$ is continuous and one-to-one in this situation (cf. 2.2.1 and 2.3.1).

It is sufficient to show that if $x \in M$ and $(C(p,S))(x) = X$, then

for any $e \in R$ with $e > 0$
there exists $h > 0$ such that
for any $Y \in V(p,S)$
if $Y \in N(x;h) \cap (C(p,S))(M)$
then $d(x,(C(p,S))^{-1}(Y)) < e$.

for if this is true we consider the following argument. Take $e > 0$ and consider $(C(p,S))(N(x;\ e))$. Then by the above we are guaranteed the existence of $h > 0$ with the property (1). Take

$$Y \in N(X; h) \cap (C(p,S))(M).$$

Thus we obtain

$$d(x,(C(p,S))^{-1}(Y)) < e.$$

Now we see that

$$(C(p,S))^{-1}(Y) \in N(x; e).$$

Hence

$$(C(p,S))^{-1}(N(X; h) \cap (C(p,S))(M)) \subseteq N(x; e)$$

which implies that $(C(p,S))^{-1}$ is continuous. From this it is immediate that $C(p,S)$ is a homeomorphism of M onto its image.

What follows is a proof of our sufficiency hypothesis. Assume this hypothesis is false. Then for some $x^* \in M$ with $(C(p,S))(x^*) = X^*$ there exists $e^* \in R$ with $e^* > 0$ such that for any $h \in R$ if $h > 0$ then there exists Y

such that

$$Y \varepsilon N(X^*; h) \cap (C(p,S))(M)$$

and

$$d(x^*; (C(p,S))^{-1}(Y)) \geq e^*.$$

Take an x^* and e^* which satisfy this condition. Hence for any $h > 0$, there exists

$$Y(h) \varepsilon N(X^*; h) \cap (C(p,S))(M)$$

such that

$$d(x^*; (C(p,S))^{-1}(Y(h))) \geq e^*.$$

Thus for any $n \varepsilon N$, there exists

$$Y(n) \varepsilon N(X^*; 1/n) \cap (C(p,S))(M)$$

such that

(1) $$d(x^*, (C(p,S))^{-1}(Y(n))) \geq e^*.$$

We put

$$y(n) = (C(p,S))^{-1}(Y(n)).$$

Now consider the sequence $(y(n))$. Now for all $q \in S$ and all $n \in N$ we have

$$(2) \qquad |d(y(n),q) - d(x^*,q)| < 1/n$$

by the definition of $(y(n))$ and since

$$Y(n) \in N(X^*;\ 1/n).$$

From this it is seen that for all $n \in N$

$$d(y(n),p) < 1/n + d(x^*,p).$$

Now take $m,m' \in M$ and consider $d(y(m),y(m'))$. By the triangle inequality we get

$$\begin{aligned} d(y(m),y(m')) &\leq d(y(m),p) + d(y(m'),p) \\ &< 1/m + 2/m' + 2d(x^*,p) \\ &\leq 2 + 2d(x^*,p). \end{aligned}$$

Thus $(y(n))$ is a bounded sequence. But M is finitely compact. This implies that there exists $y \in M$ such that y is a cluster point of $(y(n))$. We now get that there exists a subsequence $(y(n(i)))$ of $(y(n))$ with

$$\lim_{i \to \infty} y(n(i)) = y.$$

By (2) of the above, we see that

$$|d(y(n(i)), q) - d(x^*, q)| < 1/n(i)$$

for all $q \in S$ and all $i \in N$. Taking limits as i approaches infinity, we obtain that for all $q \in S$

$$|d(y,q) - d(x^*, q)| = 0.$$

This implies that for all $q \in S$

$$d(y, q) = d(x^*, q).$$

Hence

$$x^* = y$$

since S is a basis of M. Thus we have

$$d(x^*, y) = 0.$$

Now we consider (1) of the above and obtain

$$d(x^*, y(n(i))) \geq e^*$$

for all $i \in N$. Taking limits as i approaches infinite, we get

$$\begin{aligned} d(x^*, y) &\geq e^* \\ &> 0. \end{aligned}$$

This gives us a contradiction. Hence our sufficiency hypothesis is valid and this proves the theorem.

SECTION FOUR

- THE COSINE -

2.4. Basic Properties.

Let (M,d) be a metric space.

Definition: If $p,q,r \in M$ and $p \neq q$ and $p \neq r$, then define

$$\cos(pq,pr) = \frac{(d(p,q))^2+(d(p,r))^2-(d(q,r))^2}{2d(p,q)\,d(p,r)} .$$

Claim: If $p,q,r \in M$ and $p \neq q$ and $p \neq r$, then we have

$$|\cos(pq,pr)| \leq 1.$$

Proof. First we show that

$$-1 \leq \cos(pq,\ pr).$$

By the triangle inequality we have

$$d(q,r) \leq d(p,q) + d(p,r).$$

By squaring this and rearranging terms we get

$$-2d(p,q)d(p,r) \leq (d(p,q))^2+(d(p,r))^2-(d(q,r))^2.$$

Dividing both sides by $2d(p,q)d(p,r)$ gives the desired result.

Now we show that

$$\cos(pq,pr) \leq 1.$$

This is true iff

$$(d(p,q))^2+(d(p,r))^2-(d(q,r))^2 \leq 2d(p,q)d(p,r).$$

This is equivalent to

$$(d(p,q)-d(p,r))^2 \leq (d(q,r))^2$$

which is in turn equivalent to

$$|d(p,q)-d(p,r)| \leq d(q,r).$$

But this is true by a claim of <u>1.1</u>.

SECTION FIVE

- THE MAPPING P(p,S) -

2.5. Scalar Multiplication in (M,d).

Let (M,d) be a metric space. Let $S \subseteq M$ and let $p \in S$.

Definition: If $x \in M$ and $e \in R$ with $e \geq 0$, then we define $S(x; e)$ to be the set of all $y \in M$ such that

$$d(x,y) = e.$$

Definition: Let us define a function $P(p,S)$ with

$$P(p,S): V(p,S) \to \mathcal{P}(M)$$

by putting

$$(P(p,S))(X) = \bigcap_{q \in S} S(q; X(q))$$

for any $X \in V(p,S)$.

Now consider $x \in M$ and $\ell \in R$. Then it is readily seen that

$$(P(p,S))(\ell((C(p,S))(x)))$$
$$= S(p;|\ell|\ d(p,x))$$
$$\cap \bigcap_{\substack{q \in S \\ q \neq p}} S(q;\ \sqrt{(\ell^2-\ell)(d(p,x))^2+(1-\ell)(d(p,q))^2+\ell(d(q,x))^2}$$

<u>Definition</u>: We put

$$(\ell x) = (P(p,S))(\ell((C(p,S))(x)))$$

for any $\ell \in R$ and any $x \in M$. We also define a function $(p,S)^*$ with

$$(p,S)^*\colon R \times M \to \wp(M)$$

by putting

$$(p,S)^*(\ell,x) = (\ell x)$$

for any $\ell \in R$ and any $x \in M$.

<u>Claim</u>. <u>For any</u> $x \in M$,

$$(0\,x) = \{p\}.$$

<u>Proof</u>. We clearly have that

$$(\Theta x) \subseteq S(p;\ \Theta\, d(p,x))$$
$$= \{p\}.$$

In addition we notice that for any $q \in S$ with $q \neq p$

$$S(q;\ \sqrt{(\Theta^2-\Theta)(d(p,x))^2+(1-\Theta)(d(p,q))^2+\Theta(d(q,x))^2}\ )$$
$$= S(q;\ d(p,q)).$$

This implies that

$$(\Theta x) = \{p\} \cap \bigcap_{\substack{q \in S \\ q \neq p}} S(q;d(p,q)).$$

Now for all $q \in S$ with $q \neq p$ we note that

$$p \in S(q;\ d(p,q)).$$

Thus we have that

$$p \in (\Theta x).$$

Hence we have proved that

$$\{p\} = (\Theta x).$$

<u>Claim</u>. <u>For any</u> $x \varepsilon M$

$$x \varepsilon (1x).$$

<u>Proof</u>. First we consider $p \varepsilon S$. Then it is immediate that

$$S(p; |1| d(p,x)) = S(p; d(p,x)).$$

This gives us that

$$x \varepsilon S(p; |1| d(p,x)).$$

Now let us consider all $q \varepsilon S$ with $q \neq p$. In this case we observe that

$$S(q; \sqrt{(1^2-1)(d(p,x))^2+(1-1)(d(p,q))^2+1(d(q,x))^2})$$
$$= S(q; d(q,x)).$$

Moreover we know that

$$x \varepsilon S(q; d(q,x)).$$

Hence we have proved that

$$x \varepsilon (1x).$$

<u>Note</u>. For any $x \in M$ with $x \neq p$ and for any $q \in S$ with $q \neq p$ we have that

$$\ell^2(d(p,x))^2+(d(p,q))^2-2\ell d(p,x)d(p,q)\cos(px,pq)$$
$$= (\ell^2-\ell)(d(p,x))^2+(1-\ell)(d(p,q))^2+\ell(d(q,x))^2.$$

This equation is readily proved by expanding the first term with the aid of the definition of cos(px,pq).

<u>Claim</u>. <u>For any</u> $\ell, m \in R$ <u>with</u>

$$|\ell| \neq |m|$$

<u>and for any</u> $x \in M$ <u>with</u> $x \neq p$ <u>we can prove that</u>

$$(\ell x) \cap (mx) = \emptyset .$$

<u>Proof</u>. From the definitions of (ℓx) and (mx) we obtain that

$$(\ell x) \cap (mx)$$
$$\subseteq S(p; |\ell| d(p,x)) \cap S(p; |m| d(p,x))$$
$$= \emptyset$$

since $|\ell| \neq |m|$ and $d(p,x) \neq 0$.

<u>Claim</u>. <u>For any</u> $\ell, m \in R$ <u>with</u>

$$\ell \neq m \text{ and } |\ell| = |m|$$

<u>and for any</u> $x \neq p$ <u>we have that</u>

<u>if there exists</u> $q \in S$ <u>with</u> $q \neq p$
<u>such that</u> $\cos(px, pq) \neq 0$
<u>then</u> $(\ell x) \cap (mx) = \emptyset$.

<u>Proof</u>. Since we have that

$$\ell \neq m \text{ and } |\ell| = |m|$$

we see that

$$\ell = -m \text{ and } \ell \neq 0.$$

Assume that

$$(\ell x) \cap (mx) \neq \emptyset.$$

Then we have that for any $q \in S$ with $q \neq p$

$$\ell^2(d(p,x))^2+(d(p,q))^2-2\ell d(p,x)d(p,q)\cos(px,pq)$$
$$= m^2(d(p,x))^2+(d(p,q))^2-2md(p,x)d(p,q)\cos(px,pq).$$

This implies that for any $q \in S$ with $q \neq p$

$$\ell \cos(px,pq) = m \cos(px,pq)$$

since our hypothesis guarantees that

$$|\ell| = |m|.$$

Hence for any $q \in S$ with $q \neq p$ we see that

$$\cos(px,pq) = 0$$

since $\ell = -m$ and $\ell \neq 0$. Thus if there exists $q \in S$ with $q \neq p$ such that

$$\cos(px,pq) \neq 0$$

then it is immediate that

$$(\ell x) \cap (mx) = \emptyset.$$

Claim. For any $x \in M$ with $x \neq p$ and for any $r \in M$ with

$$r \in (\ell x)$$

where $\ell \neq 0$ and $\ell \in R$ we have that for any $q \in S$ with $q \neq p$

$$\cos(pr,pq) = (\ell/|\ell|)\cos(px,pq).$$

Proof. Since $r \in (\ell x)$ we can conclude that

$$d(p,r) = |\ell| d(p,x)$$

and also that for any $q \in S$ with $q \neq p$

$$d(r,q) = \sqrt{\ell^2(d(p,x))^2+(d(p,q))^2-2\ell d(p,x)d(p,q)\cos(px,pq)}.$$

Hence for any $q \in S$ with $q \neq p$

$$\begin{aligned}
&\cos(pr,pq)\\
&= \frac{(d(p,r))^2+(d(p,q))^2-(d(r,q))^2}{2d(p,r)d(p,q)}\\
&\quad \frac{\ell^2(d(p,x))^2+(d(p,q))^2-\ell^2(d(p,x))^2-(d(p,q))^2+2\ell d(p,x)d(p,q)\cos(px,pq)}{2|\ell|d(p,x)d(p,q)}\\
&= (\ell/|\ell|)\cos(px,pq).
\end{aligned}$$

<u>Claim.</u> <u>For any</u> $x \in M$ <u>with</u> $x \neq p$ <u>and for any</u> $r \in M$ <u>with</u>

$$r \in (\ell x)$$

<u>where</u> $\ell \in R$ <u>with</u> $\ell \neq 0$ <u>we have that for any</u> $m \in R$

$$(mr) \subseteq ((m\ell)x).$$

<u>Proof.</u> We consider $y \in M$ such that

$$y \in (mr).$$

Then we see that

$$\begin{aligned} d(p,y) &= |m|d(p,r) \\ &= |m|\,|\ell|\ d(p,x) \\ &= |m\ell|d(p,x). \end{aligned}$$

Moreover for any $q \in S$ with $q \neq p$ we see that

$$\begin{aligned} &d(q,y) \\ &= \sqrt{m^2(d(p,r))^2+(d(p,q))^2-2md(p,r)d(p,q)\cos(pr,pq)} \\ &= \sqrt{m^2\,|\ell|^2(d(p,x))^2+d(p,q))^2 -2m\,|\ell|d(p,x)d(p,q)(\ell/\,|\ell|)\cos(px,pq)} \\ &= \sqrt{(m\ell)^2(d(p,x))^2+(d(p,q))^2-2(m\ell)d(p,x)d(p,q)\cos(px,pq)} \end{aligned}$$

Hence it is immediate that

$$y \in ((m\ell)x).$$

SECTION SIX

- A RESULT FOR INNER PRODUCT SPACES -

2.6.

Let (H,i) be a real inner product space. Let $S \subseteq H$ and let $o \in S$.

Claim. For any $\ell \in R$ and any $x \in H$

$$\ell x \in (\ell x),$$

where

$$\ell x$$

is the scalar product of and x in H and

$$(\ell x)$$

is defined with respect to $S \subseteq H$ and $o \in S$ as in 2.5 and H has the usual metric derived from i associated with it and $|| \ ||$ denotes the associated norm on H.

<u>Proof</u>. First we consider $d(\ell x,o)$. Then we have

$$\begin{aligned} d(\ell x,o) &= ||\ell x-o|| \\ &= |\ell|\,||x-o|| \\ &= |\ell|\,d(x,o). \end{aligned}$$

Now we consider $d(\ell x,q)$ for any $q \in S$ with $q \neq o$.

It is sufficient to show that

$$\begin{aligned} &(d(\ell x,q))^2 \\ &= (\ell^2 - \ell)(d(o,x))^2+(1-\ell)(d(o,q))^2+ (d(q,x))^2. \end{aligned}$$

This is equivalent to showing that

$$\begin{aligned} &||x-q||^2 \\ &= (\ell^2 - \ell)||x||^2+(1-\ell)||q||^2 + \ell||q-x||^2 \end{aligned}$$

which in turn is equivalent to

$$\begin{aligned} &i(\ell x-q,\ell x-q) \\ &= (\ell^2-\ell)i(x,x)+(1-\ell)i(q,q)+ i(x-q,x-q). \end{aligned}$$

The right member of this equation is equal to

$$(\ell^2-\ell)i(x,x)+(1-\ell)i(q,q)$$
$$+\ell i(x,x)+\ell i(q,q)-\ell i(q,x)-\ell i(x,q)$$

which equals

$$\ell^2 i(x,x)-\ell i(q,x)-\ell i(x,q)+i(q,q).$$

The left member of this equation is equal to

$$i(\ell x,\ell x)-\ell i(q,x)-\ell i(x,q)+i(q,q).$$

This in turn is equal to

$$\ell^2 i(x,x)-\ell i(q,x)-\ell i(x,q)+i(q,q).$$

Thus both halves of the equation equal the same thing. Hence the equation is true. Thus for any $\ell \varepsilon R$ and $x \varepsilon H$

$$\ell x \;\varepsilon\; (\ell x).$$

CHAPTER THREE

- ABSTRACT DISTANCE SPACES -

Introduction. This chapter discusses abstractions of metric spaces or distance spaces as they are sometimes called. In this chapter, the terminology "distance spaces" is used for these abstract spaces. For these spaces the basic notion of assigning a "distance" to pairs of points is retained but all other structure is deleted. Notions of homomorphisms, quotient spaces and product spaces are defined and examined.

SECTION ONE

- INTRODUCTION -

3.1.1. Basic Definitions.

Let M and F be given sets.

Definition: (M,d,F) is a distance space iff

$$d: M \times M \to F.$$

Definition: (M,d,F) is a symmetric distance space iff

(1) $d: M \times M \to F$

(2) for any $x,y \in M$

$d(x,y) = d(y,x)$.

Notation. The notation $\mathfrak{M} = (M,d,F)$ and occasionally $\mathfrak{N} = (N,e,G)$ with or without subscripts will be used throughout this chapter to denote distance spaces. If symmetry is used then it will be noted at the time.

3.1.2. Bases.

Let $\mathfrak{M} = (M,d,F)$ be a distance space.

Definition: S is a right basis of $\mathfrak{M}$ iff

(1) $S \subseteq M$

(2) for any $x,y \in M$

if for any $s \in S$, $d(x,s) = d(y,s)$

then $x = y$.

Definition: S is a left basis of $\mathfrak{M}$ iff

(1) $S \subseteq M$

(2) for any $x,y \in M$

if for any $s \in S$, $d(s,x) = d(s,y)$

then $x = y$.

If $\mathfrak{M}$ is a symmetric distance space then every right basis of $\mathfrak{M}$ is a left basis of $\mathfrak{M}$ and conversely. In this case we refer to a basis of $\mathfrak{M}$.

3.1.3. Homomorphisms.

Let $\mathfrak{M}_1 = (M_1,d_1,F_1)$ and $\mathfrak{M}_2 = (M_2,d_2,F_2)$ be distance spaces.

Definition: (g_1,g_2) is a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ if

(1) $g_1\colon\ M_1 \to M_2$

(2) $g_2\colon\ F_1 \to F_2$

(3) for any $x,y \in M_1$

$(g_2 \circ d_1)(x,y) = d_2(g_1(x),g_1(y))$.

Definition: (g_1,g_2) is an isomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ iff

(1) (g_1,g_2) is a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$

(2) g_1 and g_2 are both one-to-one and onto.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are symmetric distance spaces, then homomorphism and isomorphisms are defined in exactly the same way.

Notation. The fact that (g_1,g_2) is a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ will be denoted by

$$(g_1,g_2)\colon \mathcal{M}_1 \to \mathcal{M}_2.$$

If it is stated that g is a homomorphism then it is assumed that there exists g_1 and g_2 such that $g = (g_1,g_2)$

and (g_1, g_2) is a homomorphism.

If $g = (g_1, g_2)$ and $h = (h_1, h_2)$ are homomorphisms, then, as a notational device, we put

$$h \circ g = (h_1 \circ g_1, h_2 \circ g_2).$$

SECTION TWO

- HOMOMORPHISMS -

3.2.1. Isomorphisms.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be distance spaces.

Claim. If (g_1, g_2) is an isomorphism from $\mathfrak{M}_1$ into $\mathfrak{M}_2$, then (g_1^{-1}, g_2^{-1}) is an isomorphism from $\mathfrak{M}_2$ into $\mathfrak{M}_1$.

Proof. For any $x,y \in M_2$

$$(g_2 \circ d_1)(g_1^{-1}(x), g_1^{-1}(y))$$
$$= d_2(g_1(g_1^{-1}(x)), g_1(g_1^{-1}(y)))$$
$$= d_2(x,y)$$

since g is a homomorphism and g_1 is one-to-one and onto. Hence

$$d_1(g_1^{-1}(x), g_1^{-1}(y)) = g_2^{-1}(d_2(x,y))$$

since g_2 is one-to-one and onto.

3.2.2. Composition of Homomorphisms.

Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be distance spaces. Let

$$(g_1, g_2): \mathcal{M}_1 \to \mathcal{M}_2$$
$$(h_1, h_2): \mathcal{M}_2 \to \mathcal{M}_3.$$

Claim. $(h_1 \circ g_1, h_2 \circ g_2): \mathcal{M}_1 \to \mathcal{M}_3.$

Proof.

$$\begin{aligned}
&(h_2 \circ g_2)(d_1(x,y)) \\
&= h_2(g_2(d_1(x,y))) \\
&= h_2(d_2(g_1(x), g_1(y))) \\
&= d_3(h_1(g_1(x)), h_1(g_1(y))) \\
&= d_3((h_1 \circ g_1)(x), (h_1 \circ g_1)(y)).
\end{aligned}$$

SECTION THREE

- CONGRUENCE RELATIONS -

3.3. Quotient Spaces.

Note. If θ is an equivalence relation on M, then $[x]_\theta$ will denote the equivalence class of x in M with respect to θ.

Let $\mathfrak{M}$ be a distance space.

Definition: $\theta = (\theta_M, \theta_F)$ is a congruence relation on $\mathfrak{M}$ iff

(1) θ_M is an equivalence relation on M

(2) θ_F is an equivalence relation on F

(3) for any $x,y,z,w \in M$
if $z\theta_M x$ and $w\theta_M y$,
then $d(z,w)\theta_F\, d(x,y)$.

Let θ be a congruence relation on $\mathfrak{M}$.

Definition: $\mathfrak{M}/\theta = (M/\theta_M,\ d/\theta,\ F/\theta_F)$ where M/θ_M is the set of equivalence classes in M with respect to θ_M and similarly for F/θ_F and where

$$(d/\theta)([x]_{\theta_M}, [y]_{\theta_M})\ [d(x,y)]_{\theta_F}\ .$$

Remark. $\mathfrak{M}/\theta$ is called the quotient of $\mathfrak{M}$ by θ and is referred to as a quotient space.

Claim. d/θ is well-defined.

Proof. If

$$[z]_{\theta_M} = [x]_{\theta_M} \text{ and } [w]_{\theta_M} = [y]_{\theta_M}$$

then

$$z\theta_M x \quad \text{and} \quad w\theta_M y.$$

Hence

$$d(z,w)\theta_F\, d(x,y)$$

since θ is a congruence relation. Thus

$$[d(z,w)]_{\theta_F} = [d(x,y)]_{\theta_F}.$$

As a result of this claim, $\mathfrak{M}/\theta$ is a distance space. It is symmetric if $\mathfrak{M}$ is symmetric.

Definition: Define

$$\kappa_\theta = (\kappa_{\theta_M}, \kappa_{\theta_F})$$

by

$$\kappa_{\theta_M}(x) = [x]_{\theta_M} \quad \text{for} \quad x \in M$$

$$\kappa_{\theta_F}(a) = [a]_{\theta_F} \quad \text{for} \quad a \in F$$

Claim.

$$\kappa_\theta \colon \mathfrak{M} \to \mathfrak{M}/\theta.$$

Proof.

$$\kappa_{\theta_F}(d(x,y)) = d(x,y)_{\theta_F}$$

and

$$(d/\theta)(\kappa_{\theta_M}(x), \kappa_{\theta_M}(y))$$

$$= (d/\theta)([x]_{\theta_M}, [y]_{\theta_M})$$

$$= [d(x,y)]_{\theta_F}.$$

Note. The notation here of $\mathfrak{M}/\theta$ and κ_θ exemplifies the notation used elsewhere in this chapter for these objects.

SECTION FOUR

- THE CONGRUENCE RELATION INDUCED By a HOMOMORPHISM -

3.4. Equality Transformed.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be distance spaces. Let $g = (g_1, g_2)$ be a homomorphism from $\mathfrak{M}_1$ into $\mathfrak{M}_2$.

Definition: Define θ_M by

$$(x,y) \in \theta_M \text{ iff } g_1(x) = g_1(y).$$

Define θ_F by

$$(a,b) \in \theta_F \text{ iff } g_2(a) = g_2(b).$$

Claim. $\theta = (\theta_M, \theta_F)$ is a congruence relation on $\mathfrak{M}_1$.

Proof. It is easily seen that θ_M and θ_F are equivalence relations from the properties of equality.

Consider

$$z\theta_M x \text{ and } w\theta_M y.$$

Hence

$$g_1(z) = g_1(x) \quad \text{and} \quad g_1(w) = g_1(y).$$

Thus

$$d_2(g_1(z),\ g_1(w)) = d_2(g_1(x),\ g_1(y)).$$

This implies

$$g_2(d_1(z,w)) = g_2(d_1(x,y))$$

since (g_1, g_2) is a homomorphism.

Hence

$$d_1(z,w)\,\theta_F\, d_1(x,y).$$

SECTION FIVE

- SUBSPACES -

3.5.1. Definition.

Let M be a distance space.

Definition: (N,G) is a subspace of $\mathcal{M}$ iff

(1) $N \subseteq M$

(2) $G \subseteq F$

(3) $d(N \times N) \subseteq G$.

3.5.2. Subspaces and Homomorphisms.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be distance spaces. Let $g = (g_1, g_2)$ be a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$.

Claim. If (N, G) is a subspace of $\mathcal{M}_1$, then $(g_1(N), g_2(G))$ is a subspace of $\mathcal{M}_2$.

Proof. Take $z, w \in g_1(N)$. This implies that there exist $x, y \in N$ with

$$g_1(x) = z \quad \text{and} \quad g_1(y) = w.$$

$$d_2(z,w) = d_2(g_1(x), g_1(y))$$
$$= (g_2 \circ d_1)(x, y)$$
$$= g_2(d_1(x, y)).$$

But $x,y \in N$ implies

$$d_1(x,y) \in G.$$

Hence

$$g_2(d_1(x,y)) \in g_2(G).$$

Thus

$$d_2(z,w) \in g_2(G).$$

<u>Claim.</u> <u>If</u> (N, G) <u>is a subspace of</u> $\mathfrak{M}_2$, <u>then</u> $(g_1^{-1}(N), g_2^{-1}(G))$ <u>is a subspace of</u> $\mathfrak{M}_1$.

<u>Proof.</u> Take $x,y \in g_1^{-1}(N)$. This implies that there exist $z,w \in N$ with

$$g_1(x) = z \quad \text{and} \quad g_1(y) = w.$$

$$g_2(d_1(x,y)) = d_2(g_1(x), g_1(y))$$
$$= d_2(z, w).$$

But $z, w \in N$ implies

$$d_2(z, w) \in G.$$

Hence

$$d_1(x,y) \in g_2^{-1}(G).$$

3.5.3. Families of Subspaces.

Let $\mathcal{M}$ be a distance space. Let $((N_i, G_i))_{i \in I}$ be a family of subspaces of $\mathcal{M}$.

Claim. (N, G) is a subspace of $\mathcal{M}$ where

$$N = \bigcap_{i \in I} N_i \quad \text{and} \quad G = \bigcap_{i \in I} G_i.$$

Proof. Take $x, y \in N$. Hence

$$x, y \in N_i \quad \text{for all} \quad i \in I.$$

Thus

$$d(x,y) \in G_i \quad \text{for all} \quad i \in I.$$

That is

$$d(x,y) \in G.$$

SECTION SIX

- PRODUCTS -

3.6.1. Definition.

Let $(\mathcal{M}_i)_{i\varepsilon I}$ be a family of distance spaces.

Definition: Define

$$\prod_{i\varepsilon I} \mathcal{M}_i = \left(\prod_{i\varepsilon I} M_i, \prod_{i\varepsilon I} d_i, \prod_{i\varepsilon I} F_i \right)$$

where

$$\left(\prod_{i\varepsilon I} d_i \right)\left((x_i)_{i\varepsilon I}, (y_i)_{i\varepsilon I} \right) = (d_i(x_i,y_i))_{i\varepsilon I} .$$

It is obvious that $\prod_{i\varepsilon I} \mathcal{M}_i$ is a distance space which is symmetric if and only if for every $i \varepsilon I$, $\mathcal{M}_i$ is symmetric.

$\prod_{i\varepsilon I} \mathcal{M}_i$ is called the <u>product</u> of the family $(\mathcal{M}_i)_{i\varepsilon I}$. In the rest of this section $(\mathcal{M}_i)_{i\varepsilon I}$ is considered fixed and the notation $\prod \mathcal{M} = (\prod M, \prod d, \prod F)$ is used for the product $\prod_{i\varepsilon I} \mathcal{M}_i$.

3.6.2. The Projection Maps.

Let

$$pr_j^{\prod M}: \quad \prod M \to M_j$$

be defined by

$$pr_j^{\prod M}((x_i)_{i \in I}) = x_j.$$

The mapping $pr_j^{\prod M}$ is called the jth projection of $\prod M$.

Let

$$pr_j^{\prod F}: \quad \prod F \to F_j$$

be defined by

$$pr_j^{\prod F}((a_i)_{i \in I}) = a_j.$$

The mapping $pr_j^{\prod F}$ is called the jth projection of $\prod F$.

<u>Definition</u>: For any $j \varepsilon I$, define

$$pr_j^{\Pi \mathfrak{M}} = (pr_j^{\Pi M}, pr_j^{\Pi F}).$$

<u>Claim</u>. <u>For any</u> $j \varepsilon I$,

$$pr_j^{\Pi \mathfrak{M}} : \Pi \mathfrak{M} \to \mathfrak{M}_j$$

<u>Proof</u>.

$$(pr_j^{\Pi F} \circ \Pi d)((x_i)_{i \varepsilon I}, (y_i)_{i \varepsilon I})$$

$$= pr_j^{\Pi F}(\Pi d((x_i)_{i \varepsilon I}, (y_i)_{i \varepsilon I})$$

$$= pr_j^{\Pi F}((d_i(x_i, y_i))_{i \varepsilon I})$$

$$= d_j(x_j, y_j)$$

and

$$d_j(pr_j^{\Pi M}((x_i)_{i \varepsilon I}), pr_j^{\Pi M}((y_i)_{i \varepsilon I}))$$

$$= d_j(x_j, y_j).$$

3.6.3. The Extension Property.

Definition: A triple

$$((\mathcal{M}_i)_{i\varepsilon I}, \mathcal{M}, (\phi_i)_{i\varepsilon I})$$

where

(1) all $\mathcal{M}_i$ and $\mathcal{M}$ are distance spaces

(2) for any $i \varepsilon I$, $\phi_i : \mathcal{M} \to \mathcal{M}_i$

is said to have the extension property iff

for any distance space $\mathcal{N}$

and $(\psi_i)_{i\varepsilon I}$ with $\psi_i : \mathcal{N} \to \mathcal{M}_i$

there exists exactly one ψ with

$\psi : \mathcal{N} \to \mathcal{M}$

such that for any $i \varepsilon I$,

$\phi_i \circ \psi = \psi_i$.

Claim.

$$((\mathcal{M}_i)_{i\varepsilon I}, \Pi\mathcal{M}, (pr_i^{\Pi\mathcal{M}})_{i\varepsilon I})$$

has the extension property.

<u>Proof</u>. Take

$$\mathfrak{N} = (N,e,G) \quad \text{and} \quad (\psi_i)_{i\varepsilon I}$$

with

$$\psi_i = (\psi_i^N, \psi_i^G) \quad \text{and} \quad \psi_i : \mathfrak{N} \to \mathfrak{M}_i .$$

Define

$$\psi_N : N \to \prod M$$

by

$$\psi_N(n) = (\psi_i^N(n))_{i\varepsilon I} .$$

Define

$$\psi_G : G \to \prod F$$

by

$$\psi_G(g) = (\psi_i^G(g))_{i\varepsilon I} .$$

Put

$$\psi = (\psi_N, \psi_G).$$

Then

$$pr_i^{\Pi\mathfrak{M}} \circ \psi = \psi_i.$$

That is,

$$pr_i^{\Pi M} \circ \psi_N = \psi_i^N$$

$$pr_i^{\Pi F} \circ \psi_G = \psi_i^G.$$

For instance,

$$pr_i^{\Pi M}((\psi_i^N(n))_{i\varepsilon I}) = \psi_i^N(n).$$

It will now be shown that ψ is a homomorphism from $\mathfrak{N}$ into $\Pi\mathfrak{M}$.

$$(\psi_G \circ e)(n_1,n_2)$$

$$= (\psi_i^G(e(n_1,n_2)))_{i\varepsilon I}$$

$$= (d_i(\psi_i^N(n_1), \ \psi_i^N(n_2)))_{i\varepsilon I}$$

$$= (\Pi d)((\psi_i^N(n_1))_{i\varepsilon I}, (\psi_i^N(n_2))_{i\varepsilon I})$$

$$= (\Pi d)(\psi_N(n_1), \psi_N(n_2)).$$

The mapping ψ is unique by the application of the projections. If $\phi = (\phi_N, \phi_G)$ is another such map, then

$$pr_i^{\Pi\mathfrak{M}} \circ \psi = \psi_i \quad \text{and} \quad pr_i^{\Pi\mathfrak{M}} \circ \phi = \psi_i.$$

This implies, for instance, that

$$pr_i^{\Pi M}(\psi_N(n)) = \psi_i^N(n) = pr_i^{\Pi M}(\phi_N(n))$$

and so

$$\psi_N = \phi_N.$$

<u>Claim.</u> <u>If</u> $((\mathfrak{M}_i)_{i\varepsilon I}, \mathfrak{N}, (\phi_i)_{i\varepsilon I})$ <u>has the extension property then there exists exactly one isomorphism</u>

$$\phi : \mathfrak{N} \to \Pi\mathfrak{M}$$

<u>such that</u>

$$pr_i^{\Pi\mathfrak{M}} \circ \phi = \phi_i.$$

<u>Proof</u>. If such a ϕ exists it is unique by application of the projections as in the preceding claim.

Using the fact that

$$((\mathfrak{M}_i)_{i\varepsilon I}, \Pi\mathfrak{M}, (pr_i^{\Pi\mathfrak{M}})_{i\varepsilon I})$$

has the extension property there exists exactly one

$$\phi : \mathfrak{N} \to \Pi\mathfrak{M}$$

with

$$pr_i^{\Pi\mathfrak{M}} \circ \phi = \phi_i .$$

Using the fact that

$$((\mathfrak{M}_i)_{i\varepsilon I}, \mathfrak{N}, (\phi_i)_{i\varepsilon I})$$

has the extension property there exists exactly one

$$\psi : \Pi\mathfrak{M} \to \mathfrak{N}$$

with

$$\phi_i \circ \psi = pr_i^{\Pi\mathfrak{M}} .$$

Hence

$$\begin{aligned} & pr_i^{\Pi\mathfrak{M}} \circ (\phi \circ \psi) \\ &= (pr_i^{\Pi\mathfrak{M}} \circ \phi) \circ \psi \\ &= \phi_i \circ \psi \\ &= pr_i^{\Pi\mathfrak{M}} \\ &= pr_i^{\Pi\mathfrak{M}} \circ I_{\Pi\mathfrak{M}}. \end{aligned}$$

Similarly

$$\begin{aligned} & \phi_i \circ (\psi \circ \phi) \\ &= \phi_i \\ &= \phi_i \circ I_{\mathfrak{N}}. \end{aligned}$$

By the extension property applied to the triple

$$((\mathfrak{M}_i)_{i\varepsilon I}, \Pi\mathfrak{M}, (pr_i^{\Pi\mathfrak{M}})_{i\varepsilon I})$$

with respect to $\Pi\mathfrak{M}$ and $(pr_i^{\Pi\mathfrak{M}})_{i\varepsilon I}$ $\phi \circ \psi$ is unique. Similarly $\psi \circ \phi$ is unique. Hence

$$\phi \circ \psi = I_{\Pi\mathfrak{M}} \text{ and } \psi \circ \phi = I_{\mathfrak{N}}.$$

And so

$$\phi^{-1} = \psi.$$

Thus

$$\phi : \mathfrak{N} \to \Pi\mathfrak{M} \text{ is an isomorphism.}$$

Note. In the above claim,

$$I_{\mathfrak{N}} = (I_N, I_G)$$

where I_N and I_G are the identity maps on N and G. Similarly

$$I_{\Pi\mathfrak{M}} = (I_{\Pi M}, I_{\Pi F}).$$

SECTION SEVEN

- BASES IN QUOTIENT SPACES -

3.7. Distances from Relations.

Let $\mathfrak{M}$ be a distance space. Let $\theta = (\theta_M, \theta_F)$ be a congruence relation on $\mathfrak{M}$. Let $S \subseteq M$.

Claim. $\kappa_\theta(S)$ is a right basis of $\mathfrak{M}$ iff

for any $x, y \in M$,
if for any $s \in S$, $d(x,s)\theta_F\, d(y,s)$
then $x \theta_M y$.

Proof. $\kappa_\theta(S)$ is a right basis of iff

for any $x, y \in M$
if for any $s \in S$,
$(d/\theta)([x]_{\theta_M}, [s]_{\theta_M}) = (d/\theta)([y]_{\theta_M}, [s]_{\theta_M})$
then $[x]_{\theta_M} = [y]_{\theta_M}$

iff

$$\text{for any } x,y \in M$$
$$\text{if for any } s \in S,$$
$$[d(x,s)]_{\theta_F} = [d(y,s)]_{\theta_F}$$
$$\text{then } x\theta_M y$$

iff

$$\text{for any } x,y \in M,$$
$$\text{if for any } s \in S,$$
$$d(x,s)\theta_F\, d(y,s)$$
$$\text{then } x\theta_M y.$$

Note. A similar result holds for left bases.

SECTION EIGHT

- BASES IN PRODUCTS -

3.8.1. Separate Bases.

Let $(\mathfrak{M}_i)_{i\varepsilon I}$ be a family of distance spaces. Let $(S_i)_{i\varepsilon I}$ be a family of sets such that for any $i \varepsilon I$, $S_i \neq \emptyset$ and S_i is a right basis of $\mathfrak{M}_i$

Claim. $\prod_{i\varepsilon I} S_i$ is a right basis of $\prod_{i\varepsilon I} \mathfrak{M}_i$.

Proof. Take $(x_i)_{i\varepsilon I}$ and $(y_i)_{i\varepsilon I}$ in $\prod_{i\varepsilon I} M_i$. Assume that for any $(s_i)_{i\varepsilon I} \varepsilon \prod_{i\varepsilon I} S_i$

$$(\prod_{i\varepsilon I} d_i)((x_i)_{i\varepsilon I}, (s_i)_{i\varepsilon I})$$
$$= (\prod_{i\varepsilon I} d_i)((y_i)_{i\varepsilon I}, (s_i)_{i\varepsilon I}).$$

Take $i_0 \varepsilon I$ arbitrary. Take $s \varepsilon S_{i_0}$ arbitrary. Construct

$$(r_i)_{i\varepsilon I} \text{ in } \prod_{i\varepsilon I} S_i \text{ with } r_{i_0} = s.$$

Applying the above to $(r_i)_{i \varepsilon I}$ and equating the i_0 th coordinates, it is found that

$$d_{i_0}(x_{i_0}, s) = d_{i_0}(y_{i_0}, s).$$

But $s \varepsilon S_{i_0}$ was arbitrary. Hence

$$x_{i_0} = y_{i_0}.$$

But $i_0 \varepsilon I$ was arbitrary. Hence

$$(x)_{i \varepsilon I} = (y_i)_{i \varepsilon I}.$$

Note. A similar result holds for left bases.

3.8.2. Common Base.

Let $(\mathfrak{M}_i)_{i \varepsilon I}$ be a family of distance spaces. Let S be a set such that for any $i \varepsilon I$ it is true that S is a right basis of $\mathfrak{M}_i$.

Definition: $\overline{S}$ is the set of all families $(s_i)_{i \varepsilon I}$ in $\prod_{i \varepsilon I} S$ such that there exists $s \varepsilon S$ with $s_i = s$ for any $i \varepsilon I$.

<u>Claim</u>. $\overline{S}$ <u>is a right basis of</u> $\prod_{i \in I} \mathfrak{M}_i$.

<u>Proof</u>. Take $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ in $\prod_{i \in I} \mathfrak{M}_i$.
Assume that for $(s_i)_{i \in I}$ in $\overline{S}$

$$(\prod_{i \in I} d_i)((x_i)_{i \in I}, (s_i)_{i \in I})$$
$$= (\prod_{i \in I} d_i)((y_i)_{i \in I}, (s_i)_{i \in I}).$$

Take $i_0 \in I$ arbitrary. Take $s \in S$ arbitrary.

Construct $(s_i)_{i \in I}$ in $\overline{S}$ by $s_i = s$ for all $i \in I$.
Applying the above to this $(s_i)_{i \in I}$ and equating the
i_0 th coordinates it is seen that

$$d_{i_0}(x_{i_0}, s) = d_{i_0}(y_{i_0}, s).$$

But $s \in S$ was arbitrary. Hence

$$x_{i_0} = y_{i_0}.$$

But $i_0 \in I$ was aribtrary. Hence

$$(x_i)_{i \in I} = (y_i)_{i \in I}.$$

<u>Note</u>. A similar result holds for left bases.

SECTION NINE

- PRESERVATION OF BASES UNDER ISOMORPHISM -

3.9. Invariance of Bases.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be distance spaces. Let S be a right basis of $\mathfrak{M}_1$. Let $g = (g_1, g_2)$ be an isomorphism from $\mathfrak{M}_1$ into $\mathfrak{M}_2$.

Claim. $g_1(S)$ is a right basis of $\mathfrak{M}_2$.

Proof. Take $z, w \in M_2$. Assume that for any $r \in g_1(S)$

$$d_2(z,r) = d_2(w,r).$$

Let

$$x = g_1^{-1}(z) \quad \text{and} \quad y = g_1^{-1}(w).$$

Take $s \in S$. Then $g_1(s) \in g_1(S)$. Thus for any $s \in S$

$$d_2(g_1(x), g_1(s)) = d_2(g_1(y), g_1(s)).$$

Hence

$$g_2(d_1(x,s)) = g_2(d_1(y,s)).$$

But g_2 is one-to-one. Hence for any $s \in S$,

$$d_1(x,s) = d_1(y,s).$$

Thus

$$x = y$$

and finally,

$$g = g_1(x) = g_1(y) = w.$$

SECTION TEN

- A HOMOMORPHISM THEOREM -

3.10. The Congruence Relation Induced by a Homomorphism.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be distance spaces. Let $\phi = (\phi_{M_1}, \phi_{F_1})$ be a homomorphism from $\mathfrak{M}_1$ into $\mathfrak{M}_2$. Let θ be the congruence relation induced on $\mathfrak{M}_1$ by the homomorphism ϕ (cf. 3.4).

Claim. There exists exactly one homomorphism

$$\psi\colon \mathfrak{M}_1/\theta \to \mathfrak{M}_2$$

such that

(1) $\psi \circ \kappa_\theta = \phi$

(2) ψ is one-to-one.

Moreover,

$$\psi\colon \mathfrak{M}_1/\theta \to \phi(\mathfrak{M}_1)$$

is an isomorphism.

<u>Proof</u>. Define ψ_{M_1} by

$$\psi_{M_1}([x]_{\theta_{M_1}}) = \phi_{M_1}(x).$$

Define ψ_{F_1} by

$$\psi_{F_1}([a]_{\theta_{F_1}}) = \phi_{F_1}(a).$$

Put

$$\psi = (\psi_{M_1}, \psi_{F_1}).$$

Now

$$x\theta_{M_1}y \quad \text{implies} \quad \phi_{M_1}(x) = \phi_{M_1}(y)$$

and

$$a\theta_{F_1}b \quad \text{implies} \quad \phi_{F_1}(a) = \phi_{F_1}(b).$$

Hence ψ is well-defined.

Since

$$\phi_{M_1}(x) = \phi_{M_1}(y) \text{ implies } [x]_{\theta_{M_1}} = [y]_{\theta_{M_1}}$$

and

$$\phi_{F_1}(a) = \phi_{F_1}(b) \text{ implies } [a]_{\theta_{F_1}} = [b]_{\theta_{F_1}}$$

it is seen that ψ is one-to-one.

Trivially $\psi \bullet \kappa_\theta = \phi$.

The mapping ψ is seen to be unique through the application of this equation.

Now

$$\begin{aligned}
&(\psi_{F_1} \bullet d_1/\theta)([x]_{\theta_{M_1}}, [y]_{\theta_{M_1}}) \\
&= \psi_{F_1}((d_1/\theta)([x]_{\theta_{M_1}}, [y]_{\theta_{M_1}})) \\
&= \psi_{F_1}([d_1(x,y)]_{\theta_{F_1}}) \\
&= \phi_{F_1}(d_1(x,y)).
\end{aligned}$$

Also

$$d_2(\psi_{M_1}([x]_{\theta_{M_1}}), \ \psi_{M_1}([y]_{\theta_{M_1}}))$$

$$= d_2(\phi_{M_1}(x), \phi_{M_1}(y)).$$

Thus ψ is a homomorphism since ϕ is.

If $z \in \phi_{M_1}(M_1)$, then there exists $x \in M_1$ with

$$\phi_{M_1}(x) = z.$$

Hence

$$\psi_{M_1}([x]_{\theta_{M_1}}) = z.$$

If $c \in \phi_{F_1}(F_1)$, then there exists $a \in F_1$ with

$$\phi_{F_1}(a) = c.$$

Hence

$$\psi_{F_1}([a]_{\theta_{F_1}}) = c.$$

Thus ψ maps $\mathfrak{M}_1/\theta$ onto $\phi(\mathfrak{M}_1)$.

SECTION ELEVEN

- FIRST ISOMORPHISM THEOREM -

3.11. Subspace Quotients.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be distance spaces. Let $\mathfrak{N}$ be a subspace of $\mathfrak{M}$; that is, $N \subseteq M$ and $G \subseteq F$ and $e = d|(N \times N)$.

Definition: If θ is a congruence relation on $\mathfrak{M}$, define

$$\theta_{\mathfrak{N}} = (\theta_N, \theta_G)$$

by

$$\theta_N = \theta_M \cap (N \times N)$$
$$\theta_G = \theta_F \cap (G \times G).$$

It is clear that $\theta_{\mathfrak{N}}$ is a congruence relation on $\mathfrak{N}$.

Let θ be a congruence relation on $\mathfrak{M}$.

Claim. There exists exactly one homomorphism

$$\psi : \mathfrak{N}/\theta_{\mathfrak{N}} \to \mathfrak{M}/\theta$$

<u>such that</u>

(1) ψ is one-to-one

(2) $\psi \circ \kappa_{\theta_{\mathfrak{N}}} = \kappa_\theta \circ \mathrm{inj}$

<u>where</u>

(a) $\mathrm{inj} = (\mathrm{inj}_N, \mathrm{inj}_G)$

(b) $\mathrm{inj}_N(x) = x$ <u>for</u> $x \in N$

(c) $\mathrm{inj}_G(a) = a$ <u>for</u> $a \in G$.

Moreover, ψ is an isomorphism from $\mathfrak{N}/\theta_{\mathfrak{N}}$ onto $\kappa_\theta(\mathfrak{N})$.

<u>Note</u>. $\mathrm{inj} = (\mathrm{inj}_N, \mathrm{inj}_G)$ is a homomorphism from $\mathfrak{N}$ into $\mathfrak{M}$.

<u>Proof</u>. By condition (2) on ψ, it is unique if it exists. Define

$$\psi_N([x]_{\theta_N}) = \kappa_{\theta_M}(x).$$

Define

$$\psi_G([a]_{\theta_G}) = \kappa_{\theta_F}(a)$$

Now

$$x\theta_N y \quad \text{implies} \quad x\theta_M y \quad \text{implies} \quad [x]_{\theta_M} = [y]_{\theta_M}.$$

Hence

$$\psi_N : N/\theta_N \to \kappa_{\theta_M}(N)$$

is well-defined.

Similarly

$$\psi_G : G/\theta_G \to \kappa_{\theta_F}(G)$$

is well-defined.

Put

$$\psi = (\psi_N, \psi_G).$$

Now

$$\kappa_{\theta_M}(x) = (\kappa_{\theta_M} \circ \text{inj}_N)(x) \quad \text{if} \quad x \in N$$

and

$$\kappa_{\theta_F}(a) = (\kappa_{\theta_F} \circ \mathrm{inj}_G)(a) \quad \text{if} \quad a \in G.$$

Hence ψ satisfies condition (2).

Moreover,

$$(\psi_G \circ e/\theta_{\mathcal{N}})([x]_{\theta_N}, [y]_{\theta_N})$$

$$= \psi_G((e/\theta_{\mathcal{N}})([x]_{\theta_N}, [y]_{\theta_N})$$

$$= \psi_G([e(x,y)]_{\theta_G})$$

$$= \psi_G([d(x,y)]_{\theta_G})$$

$$= \kappa_{\theta_F}(d(x,y))$$

$$= [d(x,y)]_{\theta_F}.$$

and,

$$(d/\theta)(\psi_N([x]_{\theta_N}), \psi_N([y]_{\theta_N}))$$

$$= (d/\theta)(\kappa_{\theta_M}(x), \kappa_{\theta_M}(y))$$

$$= (d/\theta)([x]_{\theta_M}, [y]_{\theta_M})$$

$$= [d(x,y)]_{\theta_F}.$$

Next if $x,y \in N$, then

$$\kappa_{\theta_M}(x) = \kappa_{\theta_M}(y)$$

implies

$$x\theta_M y,$$

which implies

$$x\theta_N y$$

since $x,y \in N$ and hence

$$[x]_{\theta_N} = [y]_{\theta_N}.$$

Hence ψ_N is one-to-one.

Similarly ψ_G is one-to-one.

Finally,

$$[x]_{\theta_M} \in \kappa_{\theta_M}(N)$$

implies that there exists $x_N \in N$ with

$$x_N\theta_M x.$$

Hence

$$\psi_N([x_N]_{\theta_N}) = [x_N]_{\theta_M} = [x]_{\theta_M}.$$

Hence ψ_N is onto $\kappa_{\theta_M}(N)$.

Similarly ψ_G is onto $\kappa_{\theta_F}(G)$.

Hence ψ is an isomorphism from $\mathcal{N}/\theta_{\mathcal{N}}$ onto $\kappa_\theta(\mathcal{N})$.

SECTION TWELVE

- SECOND ISOMORPHISM THEOREM -

3.12. Quotient Spaces of Quotient Spaces.

Let $\mathcal{M}$ be a distance space. Let θ and ψ be congruence relations on $\mathcal{M}$ with $\theta \subseteq \psi$.

Definition: Define ψ/θ by

$$[x]_{\theta_M} (\psi/\theta)_{M/\theta_M} [y]_{\theta_M} \quad \text{iff} \quad x \; \psi_M \; y,$$

$$[a]_{\theta_F} (\psi/\theta)_{F/\theta_F} [b]_{\theta_F} \quad \text{iff} \quad a \; \psi_F \; b.$$

Claim. ψ/θ is a congruence relation on $\mathcal{M}/\theta$.

Proof. $(\psi/\theta)_{M/\theta_M}$ and $(\psi/\theta)_{F/\theta_F}$ are equivalence relations since ψ_M and ψ_F are equivalence relations.

Assume

$$[z]_{\theta_M} (\psi/\theta)_{M/\theta_M} [x]_{\theta_M}$$

and

$$[w]_{\theta_M} (\psi/\theta)_{M/\theta_M} [y]_{\theta_M}.$$

Then

$$z \ \psi_M \ x \quad \text{and} \quad w \ \psi_M \ y.$$

Hence

$$d(z,w)\psi_F \ d(x,y).$$

Thus

$$[d(z,w)]_{\theta_F} \ (\psi/\theta)_{F/\theta_F} \ [d(x,y)]_{\theta_F}.$$

Finally

$$(d/\theta)([z]_{\theta_M}, [w]_{\theta_M})(\psi/\theta)_{F/\theta_F}(d/\theta)([x]_{\theta_M}, [y]_{\theta_M}).$$

Claim. There exists exactly one isomorphism

$$\chi: \mathfrak{M}/\psi \rightarrow (\mathfrak{M}/\theta)/(\psi/\theta)$$

<u>such that</u>

$$\chi \circ \kappa_{\psi} = \kappa_{(\psi/\theta)} \circ \kappa_{\theta} .$$

<u>Proof</u>. If χ exists, it is unique by the condition on it. Define χ_M by

$$\chi_M([x]_{\psi_M}) = [[x]_{\theta_M}]_{(\psi/\theta)_{M/\theta_M}}.$$

Define χ_F by

$$\chi_F([a]_{\psi_F}) = [[a]_{\theta_F}]_{(\psi/\theta)_{F/\theta_F}}.$$

Put

$$\chi = (\chi_M, \chi_F).$$

Then

$$
\begin{aligned}
& x \ \psi_M y \\
\text{iff } & [x]_{\theta_M} (\psi/\theta)_{M/\theta_M} [y]_{\theta_M} \\
\text{iff } & [[x]_{\theta_M}]_{(\psi/\theta)_{M/\theta_M}} = [[y]_{\theta_M}]_{(\psi/\theta)_{M/\theta_M}} \\
\text{iff } & \chi_M([x]_{\psi_M}) = \chi_M([y]_{\psi_M}).
\end{aligned}
$$

Hence χ_M is well-defined and one-to-one.

Similarly χ_F is well-defined and one-to-one.

From the definition of χ,

$$\chi \circ \kappa_\psi = \kappa_{(\psi/\theta)} \circ \kappa_\theta \,.$$

Also we see that

$$\begin{aligned}
&(\chi_F \circ d/\psi)([x]_{\psi_M}, [y]_{\psi_M}) \\
&= \chi_F([d(x,y)]_{\psi_F}) \\
&= [[d(x,y)]_{\theta_F}]_{(\psi/\theta)_{F/\theta_F}}
\end{aligned}$$

and we obtain that

$$\begin{aligned}
&((d/\theta)/(\psi/\theta))(\chi_M([x]_{\psi_M}), \chi_M([y]_{\psi_M})) \\
&= ((d/\theta)/(\psi/\theta))([[x]_{\theta_M}]_{(\psi/\theta)_{M/\theta_M}}, [[y]_{\theta_M}]_{(\psi/\theta)_{M/\theta_M}}) \\
&= [(d/\theta)([x]_{\theta_M}, [y]_{\theta_M})]_{(\psi/\theta)_{F/\theta_F}} \\
&= [[d(x,y)]_{\theta_F}]_{(\psi/\theta)_{F/\theta_F}} \,.
\end{aligned}$$

Finally, for

$$[[x]_{\theta_M}]_{(\psi/\theta)_{M/\theta_M}} \quad \text{in} \quad (M/\theta_M)/(\psi/\theta)_{M/\theta_M}$$

it is seen by inspection that

$$\chi_M([x]_{\psi_M}) = [[x]_{\theta_M}]_{(\psi/\theta)_{M/\theta_M}} .$$

Hence χ_M is onto.

Similarly χ_F is onto.

SECTION THIRTEEN

- AN ALTERNATIVE WAY OF DEFINING A BASIS -

3.13. A Definition by Functions.

Let $\mathcal{M}$ be a distance space. Let $S \subseteq M$.

Definition: Define

$$f_S\colon \ M \to F^S$$

by

$$f_S(x) = (d(s,x))_{s \in S}.$$

Claim. S is a left basis of M iff

f_S is one-to-one.

Proof. Let S be a left basis of M. Assume

$$f_S(x) = f_S(y).$$

Hence

$$(d(s,x))_{s \varepsilon S} = (d(s,y))_{s \varepsilon S}.$$

Thus

$$d(s,x) = d(s,y) \quad \text{for any} \quad s \varepsilon S.$$

Thus $x = y$ since S is a basis of M. Hence f_S is one-to-one.

Let f_S be one-to-one. Assume

$$d(s,x) = d(s,y) \quad \text{for any} \quad s \varepsilon S.$$

This implies

$$(d(s,x))_{s \varepsilon S} = (d(s,y))_{s \varepsilon S}.$$

Hence

$$f_S(x) = f_S(y).$$

Thus $x = y$ since f_S is one-to-one. Hence S is a left basis of M.

<u>Note</u>. A similar result holds for right bases.

CHAPTER FOUR

- GENERALIZED METRIC SPACES -

Introduction. In this Chapter, a new type of abstract space is considered and is called a generalized metric space. These spaces are specializations of the spaces of Chapter Three. Homomorphisms, quotient spaces and product spaces are developed in this context. These spaces are close to metric spaces in that all of the conditions on a metric space are preserved in a formal sense. In fact, every metric space is a generalized metric space.

SECTION ONE

- HOMOMORPHISMS -

4.1.1. Definitions.

Definition. (F, +, 0, $\leq$) is a lattice-ordered group iff

(1) (F, +, 0) is an abelian group

(2) (F, $\leq$) is a lattice

(3) for any a, b, c ε F
if a $\leq$ b
then a + c $\leq$ b + c.

Definition. (M, d, (F, +, 0, $\leq$)) is a generalized metric space iff

(1) (M, d, F) is a distance space

(2) (F, +, 0, $\leq$) is a lattice-ordered group

(3) for any x,y ε M
d(x,y) = 0 iff x = y

(4) for any x,y ε M
d(x,y) = d(y,x)

(5) for any x,y,z ε M
d(x,y) + d(y,z) $\geq$ d(x,z).

Note. The notational device

$$\mathcal{M} = (M, d, (F, +, 0, \leq))$$

with or without subscripts will be used to denote a generalized metric space throughout the rest of this chapter.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two generalized metric spaces.

Definition. $g = (g_1, g_2)$ is a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ iff

(1) g is a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ considered as distance spaces

(2) for any $a, b \in F_1$

$$g_2(a +_1 b) = g_2(a) +_2 g_2(b)$$

(3) for any $a, b \in F_1$

(a) $g_2(a \wedge_1 b) = g_2(a) \wedge_2 g_2(b)$

(b) $g_2(a \vee_1 b) = g_2(a) \vee_2 g_2(b)$.

If, in addition, g_1 and g_2 are one-to-one and onto, then g is an isomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$.

Notation. The fact that g is a homomorphism from $\mathfrak{M}_1$ into $\mathfrak{M}_2$ is written symbolically as

$$g: \mathfrak{M}_1 \to \mathfrak{M}_2.$$

4.1.2. Composition of Homomorphisms.

Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ be generalized metric spaces. Let

$$g: \mathfrak{M}_1 \to \mathfrak{M}_2 \quad \text{and} \quad h: \mathfrak{M}_2 \to \mathfrak{M}_3.$$

Claim. In this situation we have that

$$h \circ g: \mathfrak{M}_1 \to \mathfrak{M}_3.$$

Proof. It is sufficient to show that properties (2) and (3) of the definition of a homomorphism hold for $h \circ g$(cf. 3.2). Consider $a, b \in F_1$. Then as regards property (2) we see that

$$\begin{aligned}(h_2 \bullet g_2)(a +_1 b) &= h_2(g_2(a +_2 b)) \\ &= h_2(g_2(a) +_2 g_2(b)) \\ &= h_2(g_2(a)) +_3 h_2(g_2(b)) \\ &= (h_2 \bullet g_2)(a) +_3 (h_2 \bullet g_2)(b).\end{aligned}$$

In regarding property (3) we follow a procedure parallel to what was used to prove property (2).

4.1.3. The Identity Map.

Let $\mathfrak{M}$ be a generalized metric space.

Claim. The ordered pair

$$I(\mathfrak{M}) = (I(M), I(F))$$

is an isomorphism from $\mathfrak{M}$ into $\mathfrak{M}$ where $I(M)$ is the identity map on M and $I(F)$ is the identity map on F.

Proof. The proof is obvious and is left to the reader.

4.1.4. Inverses of Isomorphisms.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be generalized metric spaces. Let $g = (g_1, g_2)$ be an isomorphism of $\mathfrak{M}_1$ into $\mathfrak{M}_2$.

Claim. (g_1^{-1}, g_2^{-1}) is a homomorphism (and hence an isomorphism) from $\mathfrak{M}_2$ into $\mathfrak{M}_1$.

Proof. It is sufficient to show properties (2) and (3) of the definition of a homomorphism (cf. 3.2.1). Let $c, d \in F_2$. Let

$$a = g_2^{-1}(c) \quad \text{and} \quad b = g_2^{-1}(d).$$

Then to demonstrate property (2) we see that

$$\begin{aligned} g_2^{-1}(c +_2 d) &= g_2^{-1}(g_2(a) +_2 g_2(b)) \\ &= g_2^{-1}(g_2(a +_1 b)) \\ &= a +_1 b \\ &= g_2^{-1}(c) + g_2^{-1}(d). \end{aligned}$$

In order to demonstrate property (3), we follow a procedure parallel to what was used to prove property (2).

SECTION TWO

- QUOTIENT SPACES -

4.2. Congruence Relations.

Let $\mathfrak{M} = (M, d, (F, +, 0, \leq))$ be a generalized metric space. For any $a, b \in F$ we let

$$a \vee b = \sup\{a,b\}$$

denote the join of a and b in F considered as a lattice and we let

$$a \wedge b = \inf\{a,b\}$$

denote the meet of a and b in F considered as a lattice.

Definition. $\theta = (\theta_M, \theta_F)$ is a congruence relation on $\mathfrak{M}$ iff

(1) θ is a congruence relation on $\mathfrak{M}$ considered as a distance space

(2) for any $a, b, a', b' \in F$
if $a \ \theta_F \ a'$ and $b \ \theta_F \ b'$
then $a \wedge b \ \theta_F \ a' \wedge b'$
and $a \vee b \ \theta_F \ a' \vee b'$
and $a + b \ \theta_F \ a' + b'$

(3) for any $x, y \in M$

if $d(x, y)\ \theta_F\ 0$, then $x\ \theta_M\ y$

Note. We will use the notation previously used in this chapter for equivalence classes and for the set of these equivalence classes.

Let $\theta = (\theta_M, \theta_F)$ be a congruence relation on $\mathcal{M}$.

Definition. If we have that

$$[a]_{\theta_F},\ [b]_{\theta_F} \in F/\theta_F$$

then we define

$$[a]_{\theta_F}\ (\wedge/\theta)\ [b]_{\theta_F} = [a \wedge b]_{\theta_F}$$

and we define

$$[a]_{\theta_F}\ (\vee/\theta)\ [b]_{\theta_F} = [a \vee b]_{\theta_F}.$$

Note. By property (2) of the definition of a congruence relation we immediately see that

$$(\wedge/\theta) \quad \text{and} \quad (\vee/\theta)$$

are well-defined operations on F/θ_F.

Claim 1. For any $[a]_{\theta_F}, [b]_{\theta_F}, [c]_{\theta_F} \in F/\theta_F$ we have that

$$\text{(a)} \quad [a]_{\theta_F} = [a]_{\theta_F} (\wedge/\theta) [b]_{\theta_F}$$

$$\text{iff } [b]_{\theta_F} = [b]_{\theta_F} (\vee/\theta) [a]_{\theta_F}$$

$$(b_1) \quad [a]_{\theta_F} = [a]_{\theta_F} (\wedge/\theta) [a]_{\theta_F}$$

$$(b_2) \quad [a]_{\theta_F} = [a]_{\theta_F} (\vee/\theta) [a]_{\theta_F}$$

$$(c_1) \quad [a]_{\theta_F} (\wedge/\theta) ([b]_{\theta_F} (\wedge/\theta) [c]_{\theta_F})$$

$$= ([a]_{\theta_F} (\wedge/\theta) [b]_{\theta_F}) (\wedge/\theta) [c]_{\theta_F}$$

$$(c_2) \quad [a]_{\theta_F} (\vee/\theta) ([b]_{\theta_F} (\vee/\theta) [c]_{\theta_F})$$

$$= ([a]_{\theta_F} (\vee/\theta) [b]_{\theta_F}) (\vee/\theta) [c]_{\theta_F}$$

$$(d_1) \quad [a]_{\theta_F} (\wedge/\theta) [b]_{\theta_F} = [b]_{\theta_F} (\wedge/\theta) [a]_{\theta_F}$$

$$(d_2) \quad [a]_{\theta_F} (\vee/\theta) [b]_{\theta_F} = [b]_{\theta_F} (\vee/\theta) [a]_{\theta_F}.$$

Proof. The proof of (b_1), (b_2), (c_1), (c_2), (d_1) and (d_2) are straightforward and are left to the reader.

Proof of (a). Assume that we have

$$[a]_{\theta_F} = [a]_{\theta_F} (\wedge/\theta) [b]_{\theta_F}.$$

This gives us from our definitions that

$$[a]_{\theta_F} = [a \wedge b]_{\theta_F}.$$

Hence there exists $a' \in [a]_{\theta_F}$ with

$$a' = a \wedge b.$$

By taking meets of both sides with a we see that

$$a \wedge a' = a \wedge b.$$

By taking meets of both sides with a' we see that

$$a \wedge a' = (a \wedge a') \wedge b.$$

Since $(F, \leq)$ is a lattice, we are guaranteed as a consequence that

$$a \wedge a' \leq b$$

and

$$b = b \vee (a \wedge a').$$

Hence we see that

$$[b]_{\theta_F} = [b]_{\theta_F} \; (\vee/\theta) \; [a \wedge a']_{\theta_F}.$$

Now consider property (2) in the definition of a congruence relation with regard to the points a, a, a', a of F. We know that

$$a\theta_F \; a' \quad \text{and} \quad a\theta_F \; a.$$

Hence we obtain that

$$a\theta_F \; (a \wedge a').$$

But this translates into

$$[a]_{\theta_F} = [a \wedge a']_{\theta_F}.$$

Thus we see that

$$[b]_{\theta_F} = [b]_{\theta_F} \; (\vee/\theta) \; [a]_{\theta_F}.$$

The proof of the reverse implication parallels the above proof. Thus we have proved what we claimed.

Definition: If $[a]_{\theta_F}$, $[b]_{\theta_F} \varepsilon F/\theta_F$ we define

$$[a]_{\theta_F} \ (\leq/\theta) \ [b]_{\theta_F} \text{ iff } [a]_{\theta_F} = [a]_{\theta_F} \ (\Lambda/\theta) \ [b]_{\theta_F}.$$

Claim 2. (F, $(\leq/\theta)$) is a lattice whose joins and meets are given by the operations (V/θ) and (Λ/θ) respectively.

Proof. The proof that $\leq/\theta$ is a partial order on F/θ_F is simple and left to the reader. Now we consider $[c]_{\theta_F} \varepsilon F/\theta_F$ with

$$[c]_{\theta_F} \ (\leq/\theta) \ [a]_{\theta_F} \text{ and } [c]_{\theta_F} \ (\leq/\theta) \ [b]_{\theta_F}.$$

Then this implies that

$$[c]_{\theta_F} = [c]_{\theta_F} \ (\ /\theta) \ [a]_{\theta_F} \text{ and } [c]_{\theta_F} = [c]_{\theta_F} (\ /\theta) \ [b]_{\theta_F}.$$

This gives us that

$$\begin{aligned} [c]_{\theta_F} &= [c]_{\theta_F} \ (\Lambda/\theta) \ [c]_{\theta_F} \\ &= [c]_{\theta_F} \ (\Lambda/\theta)(\ [a]_{\theta_F} \ (\Lambda/\theta) \ [b]_{\theta_F}). \end{aligned}$$

Thus we see that

$$[c]_{\theta_F} \leq [a]_{\theta_F} \ (\Lambda/\theta) \ [b]_{\theta_F}.$$

Thus $[a]_{\theta_F}$ $(\wedge/\theta)$ $[b]_{\theta_F}$ is an upper bound for all lower bounds of $[a]_{\theta_F}$ and $[b]_{\theta_F}$. We now show that

$$[a]_{\theta_F} (\wedge/\theta)\ [b]_{\theta_F}\ (\leq/\theta)\ [a]_{\theta_F}.$$

This equivalent to showing that

$$[a]_{\theta_F}\ (\wedge/\theta)\ [b]_{\theta_F} = ([a]_{\theta_F}\ (\wedge/\theta)[b]_{\theta_F})\ (\wedge/\theta)\ [a]_{\theta_F}.$$

But this is obviously true. Hence $[a]_{\theta_F}$ $(\wedge/\theta)$ $[b]_{\theta_F}$ is the greatest lower bound of $[a]_{\theta_F}$ and $[b]_{\theta_F}$.

A similar proof where meets are replaced with joins and property (a) of the preceding claim is used to enable one to define $\leq/\theta$ in terms of joins gives one that $[a]_{\theta_F}$ $(\vee/\theta)$ $[b]_{\theta_F}$ is the least upper bound of $[a]_{\theta_F}$ and $[b]_{\theta_F}$. This complete the proof of this claim.

Definition: If $[a]_{\theta_F}$, $[b]_{\theta_F}$ ε F/θ_F, then we define

$$[a]_{\theta_F}(+/\theta)\ [b]_{\theta_F} = [a + b]_{\theta_F}.$$

Note. By property (2) of the definition of a congruence relation in 4.2 we see immediately that

$(+/\theta)$

is a well-defined operation on F/θ_F.

Claim 3. $(F/\theta_F, (+/\theta))$ is a group with unit $[0]_{\theta_F}$.

Proof. This is obvious and left to the reader.

Claim 4. $(F/\theta_F, (+/\theta), [0]_{\theta_F}, (\leq/\theta))$ is a lattice-ordered group.

Proof. It is sufficient to demonstrate property (3) of a lattice-ordered group. Take $[a]_{\theta_F}, [b]_{\theta_F} \varepsilon F/\theta_F$ such that

$$[a]_{\theta_F} (\leq/\theta) [b]_{\theta_F}.$$

Let $[c]_{\theta_F} \varepsilon F/\theta_F$. Now we have that

$$[a]_{\theta_F} = [a]_{\theta_F} (\wedge/\theta) [b]_{\theta_F}.$$

Hence we see that there exists $a' \varepsilon [a]_{\theta_F}$ with

$$a' = a \wedge b.$$

By considering the proof of the Claim 1 we see that

$$a \wedge a' = (a \wedge a') \wedge b.$$

But this means that

$$a \wedge a' \leq b.$$

Hence we obtain that

$$(a \wedge a') + c \leq b + c$$

since $(F, +, 0, \leq)$ is a lattice-ordered group. This implies that

$$(a \wedge a') + c = ((a \wedge a') + c) \wedge (b + c)$$

which gives us that

$$[(a \wedge a') + c]_{\theta_F} \leq [b + c]_{\theta_F}.$$

But this translates immediately into

$$[a \wedge a']_{\theta_F} \, (+/\theta) \, [c]_{\theta_F} \leq [b]_{\theta_F} \, (+/\theta) \, [c]_{\theta_F}.$$

We also know from the proof of Claim 1 that

$$[a]_{\theta_F} = [a \wedge a']_{\theta_F}.$$

Hence we have that

$$[a]_{\theta_F} (+/\theta)\ [c]_{\theta_F} \leq [b]_{\theta_F}\ (+/\theta)\ [c]_{\theta_F}.$$

Claim 5. $(M/\theta_M,\ d/\theta,\ (F/\theta_F,\ (+/\theta),\ [0]_{\theta_F},\ (\leq/\theta))$ is a generalized metric space where d/θ is defined as it was for distance spaces in 3.3.

Proof. It is sufficient to verify properties (3), (4) and (5) of a generalized metric space. Property (3) is a direct consequence of property (3) of the definition of a congruence relation. Property (4) is a direct consequence of property (4) for $\mathcal{M}$ and the definition of d/θ. In order to prove property (5) let $[x]_{\theta_M},\ [y]_{\theta_M},\ [z]_{\theta_M} \in M/\theta_M$.

Hence we have that

$$\begin{aligned}&(d/\theta)([x]_{\theta_M},[y]_{\theta_M})\ (+/\theta)\ (d/\theta)([y]_{\theta_M},[z]_{\theta_M})\\ &= [d(x,y)]_{\theta_F}\ (+/\theta)\ [d(y,z)]_{\theta_F}\\ &= [d(x,y) + d(y,z)]_{\theta_F}.\end{aligned}$$

But we know that

$$d(x,y) + d(y,z) \geq d(x,z).$$

Hence we obtain that

$$(d(x,y) + d(y,z)) \wedge d(x,z) = d(x,z).$$

This implies that

$$[d(x,y) + d(y,z)]_{\theta_F} \wedge [d(x,z)]_{\theta_F} = [d(x,z)]_{\theta_F}$$

which gives us that

$$[d(x,y) + d(y,z)]_{\theta_F} \geq [d(x,z)]_{\theta_F}.$$

Hence we have that

$$(d/\theta)([x]_{\theta_M},[y]_{\theta_M}) \ (+/\theta) \ (d/\theta) \ ([y]_{\theta_M}, [z]_{\theta_M})$$

$$\geq (d/\theta)([x]_{\theta_M},[z]_{\theta_M}).$$

<u>Claim 6.</u> <u>The ordered pair of maps</u> $\kappa = (\kappa_M,\kappa_F)$ <u>where</u> κ_M <u>is defined by</u>

$$\kappa_M(x) = [x]_{\theta_M}$$

<u>and</u> κ_F <u>is defined by</u>

$$\kappa_F(a) = [a]_{\theta_F}$$

<u>is a homomorphism from</u> $\mathfrak{M}$ <u>into</u> $\mathfrak{M}/\theta$ <u>where we put</u>

$$\mathfrak{M}/\theta = (M/\theta_M, d/\theta, (F/\theta_F, (+/\theta), [0]_{\theta_F}, (\leq/\theta)).$$

<u>Proof</u>. The proof is an obvious extension of a similar result for distance spaces proved in <u>3.3</u> and is left to the reader.

<u>Note</u>. It is possible in this context to prove results similar to those proved for quotients of distance spaces by merely extending the proofs of these results to give the validity of properties (2) and (3) of the definition of homomorphism for generalized metric spaces.

SECTION THREE

- PRODUCTS -

4.3. Products.

Let $(\mathfrak{M}_i)_{i \varepsilon I}$ be a family of generalized metric spaces.

Definition. We define $\prod_{i \varepsilon I} \mathfrak{M}_i$ to be

$$\left(\prod_{i \varepsilon I} M_i,\ \prod_{i \varepsilon I} d_i,\ \left(\prod_{i \varepsilon I} F_i,\ \prod_{i \varepsilon I} +_i,\ (0_i)_{i \varepsilon I},\ \prod_{i \varepsilon I} \leq_i\right)\right)$$

where $\prod_{i \varepsilon I} d_i$ is as defined previously for distance spaces (see 3.6) and where

$$\left(\prod_{i \varepsilon I} +_i\right)\left((a_i)_{i \varepsilon I},\ (b_i)_{i \varepsilon I}\right) = (a_i +_i b_i)_{i \varepsilon I}$$

and where

$$(a_i)_{i \varepsilon I} \prod_{i \varepsilon I} \leq_i (b_i)_{i \varepsilon I}$$

iff for any $i \varepsilon I$, $a_i \leq_i b_i$.

Notation. For convenience in the rest of this sub-section we write $\prod_{i \varepsilon I} \mathfrak{M}_i$ as

$$(M,\ d,\ (F,\ +,\ 0,\ \leq)).$$

We call $\prod_{i\varepsilon I} \mathfrak{M}_i$ the "<u>product</u> of the $\mathfrak{M}_i$".

<u>Claim 1</u>. $(F, +, 0, \leq)$ <u>is a lattice-ordered group</u>.

<u>Proof</u>. It is easily seen that

$$(F, +, 0)$$

is a group and that

$$(F, \leq)$$

is a lattice. We now demonstrate property (3) of a lattice-ordered group. Let

$$(a_i)_{i\varepsilon I},\ (b_i)_{i\varepsilon I},\ (c_i)_{i\varepsilon I}\ \varepsilon\ F.$$

Assume that

$$(a_i)_{i\varepsilon I} \leq (b_i)_{i\varepsilon I}.$$

This implies that for all $i \varepsilon I$

$$a_i \leq_i b_i.$$

Hence we have that for all $i \in I$

$$a_i +_i c_i \leq b_i +_i c_i.$$

Thus we see that

$$(a_i +_i c_i)_{i\varepsilon I} \leq (b_i +_i c_i)_{i\varepsilon I}.$$

From the deifnition of $+$ we see that

$$(a_i)_{i\varepsilon I} + (c_i)_{i\varepsilon I} \leq (b_i)_{i\varepsilon I} + (c_i)_{i\varepsilon I}.$$

Claim 2. $(M, d, (F, +, 0, \leq))$ *is a generalized metric space.*

Proof. Property (1) of the definition of a generalized metric space is satisfied which is seen from our work on distance spaces (cf. 3.6.1). Property (2) is satisfied by the result of the preceding claim. It remains only to verify properties (3), (4) and (5).

First consider property (3). Assume that

$$(x_i)_{i\varepsilon I}, (y_i)_{i\varepsilon I} \varepsilon M$$

are such that

$$(0_i)_{i\varepsilon I} = d((x_i)_{i\varepsilon I}, (y_i)_{i\varepsilon I}).$$

This is true iff

$$(0_i)_{i\varepsilon I} = (d_i(x_i, y_i))_{i\varepsilon I}$$

which is true iff for all $i \varepsilon I$

$$0_i = d_i(x_i, y_i).$$

This is equivalent to the statement that for all $i \varepsilon I$

$$x_i = y_i$$

which is true iff

$$(x_i)_{i\varepsilon I} = (y_i)_{i\varepsilon I}.$$

Next consider property (4). Then we have that

$$\begin{aligned} & d((x_i)_{i\varepsilon I}, (y_i)_{i\varepsilon I}) \\ &= (d_i(x_i, y_i))_{i\varepsilon I} \\ &= (d_i(y_i, x_i))_{i\varepsilon I} \\ &= d((y_i)_{i\varepsilon I}, (x_i)_{i\varepsilon I}). \end{aligned}$$

Finally consider property (5). We observe that

$$d((x_i)_{i\varepsilon I}, (y_i)_{i\varepsilon I}) + d((y_i)_{i\varepsilon I}, (z_i)_{i\varepsilon I})$$

$$= (d_i(x_i, y_i) + d_i(y_i, z_i))_{i\varepsilon I}$$

$$\geq (d_i(x_i, z_i))_{i\varepsilon I}$$

$$= d((x_i)_{i\varepsilon I}, (z_i)_{i\varepsilon I})$$

since for each $i \varepsilon I$, $\mathfrak{M}_i$ is a generalized metric space.

We recall from Section <u>3.6.2</u> the definition of the functions

$$pr_j^M \quad \text{and} \quad pr_j^F$$

which are the j-th projections of M and F respectively and we define

$$pr_j^{\mathfrak{M}} = (pr_j^M, pr_j^F).$$

<u>Claim 3</u>. <u>For any</u> $j \varepsilon I$, $pr_j^{\mathfrak{M}}$ <u>is a homomorphism from</u> $\mathfrak{M}$ <u>into</u> $\mathfrak{M}_j$.

Proof. By referring to 3.6.2, we see that it is sufficient to verify properties (2) and (3) of the definition of a homomorphism given in 4.1.1.

As regards property (2), we see that

$$\begin{aligned} & pr_j^F((a_i)_{i\in I} + (b_i)_{i\in I}) \\ &= pr_j^F((a_i +_i b_i)_{i\in I}) \\ &= a_j +_j b_j \\ &= pr_j^F((a_i)_{i\in I}) + pr_j^F((b_i)_{i\in I}). \end{aligned}$$

In order to demonstrate property (3) we follow a procedure parallel to what was used to prove property (2).

Note. It is possible to define the notion of extension property for generalized metric spaces in exactly the same way as was done for distance spaces in 3.6. Then it is readily seen that

$$\mathfrak{M} = (M, d)F, +, 0, \leq))$$

has the extension property and that any triple with the extension property is essentially the same as $\mathfrak{M}$. The proofs are the same as those for distance spaces except that in certain instances one must verify properties (2) and (3) of the definition of a generalized metric space homomorphism given in 4.1.1.

SECTION FOUR

- A METHOD OF CONSTRUCTION OF UNIFORM STRUCTURES ON GENERALIZED METRIC SPACES -

4.4. The Construction.

Let $\mathcal{M} = (M, d, (F, +, 0, \leq))$ be a generalized metric space. Let F^* be a filter on F such that

(1) for any U, if $U \varepsilon F^*$
then $0 \varepsilon U$

(2) for any V, if $V \varepsilon F^*$
then there exists $W \varepsilon F^*$
such that $W + W \subseteq V$

(3) for any V, if $V \varepsilon F^*$
then there exists $W \varepsilon F^*$
such that $W \subseteq V$ and W is convex.

Note. We say that $W \subseteq F$ is convex iff for any $a, b \varepsilon W$ and for any $x \varepsilon F$

if $a \leq x \leq b$
then $x \varepsilon W$.

<u>Definition</u>: For any $V \in F^*$, define $V(M)$ to be the set of all ordered pairs

$$(x,y)$$

such that $x,y \in M$ and

$$d(x,y) \in V.$$

<u>Claim</u>. <u>For any</u> $V \in F^*$,

$$V(M) = (V(M))^{-1}.$$

<u>Proof</u>. This is automatic due to property (4) of the definition of a generalized metric space.

<u>Claim</u>. <u>For any</u> $V \in F^*$,

$$\{(x,x) \mid x \in M\} \subseteq V(M).$$

<u>Proof</u>. Take $x \in M$. Then we have that

$$d(x,x) = 0.$$

Hence since $0 \in V$ we have that

$$d(x,x) \in V.$$

Thus we obtain that

$$(x,x) \in V(M).$$

<u>Note</u>. <u>For any</u> $x,y \in M$,

$$d(x,y) \geq 0.$$

<u>Proof</u>. Assume that

$$d(x,y) < 0.$$

Then it is immediate that

$$d(x,y) + d(x,y) < 0.$$

But by the triangle inequality

$$d(x,x) \leq d(x,y) + d(y,x).$$

This implies that

$$0 \leq d(x,y) + d(x,y)$$

which is a contradiction.

<u>Claim</u>. <u>For any</u> $V \in F^*$, <u>there exists</u> $W \in F^*$ <u>such that</u>

$$W(M) \circ W(M) \subseteq V(M).$$

Proof. Take $T \subseteq V$ such that

$$T \in F^* \text{ and } T \text{ is convex.}$$

Take $W \in F^*$ such that

$$W + W \subseteq T.$$

Take $x,y \in M$ such that

$$(x,y) \in W(M) \circ W(M).$$

This implies that there exists $z \in M$ such that

$$(x,z) \in W(M) \text{ and } (z,y) \in W(M).$$

Hence we have that

$$d(x,z) \in W \text{ and } d(z,y) \in W.$$

Thus we see that

$$d(x,z) + d(z,y) \in T.$$

Now we know that

$$0 \leq d(x,y) \leq d(x,z) + d(z,y).$$

Using the facts that $0 \in T$ and

$$d(x,z) + d(z,y) \in T$$

and that T is convex, we conclude that

$$d(x,y) \in T.$$

This implies that

$$d(x,y) \in V.$$

Hence we obtain that

$$(x,y) \in V(M).$$

Now consider F'' to be the filter on $M \times M$ generated by the set of all $V(M)$ where $V \in F^*$. Then F'' is a uniform structure on M as a consequence of what we have proved ([2], page 177; [3], page 21).

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