A ONE DIMENSIONAL MODEL FOR A NON-LINEAR MESON FIELD

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

September 1978

MASTER OF SCIENCE (1978) (Physics)

McMASTER UNIVERSITY Hamilton, Ontario

TITLE: A one dimensional model for a non-linear meson field

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NUMBER OF PAGES: vi, 62

ABSTRACT

A repulsive meson-meson interaction was suggested many years ago (1951 by Schiff) as a possible mechanism for nuclear saturation, but very little has been done since then. This is mainly because the meson field equation becomes nonlinear due to the meson-meson interaction. We realized that the nonlinear field equation can be analytically solvable, within classical and adiabatic approximations, if the space is reduced to a one-dimensional one. Within the above context we investigate the effect of the meson-meson interaction on nuclear forces. The approximations which Schiff used are critically examined. A variational method for determining the meson field, which Schiff suggested but did not fully investigate, is found to be a very efficient approximation. Finally, quantum corrections are briefly examined.

ACKNOWLEDGEMENTS

It is my pleasure to thank my friends in the Theory Group and especially Anup, Fernando and Kumar for many helpful discussions.

I would also like to thank Mrs. Helen Kennelly for her fast and accurate typing of this work.

Financial assistance in the form of scholarships and teaching assistantships from McMaster University is gratefully acknowledged.

Saving the best for last, I would like to thank my supervisor, Dr. Yuki Nogami for his help in this project. His knowledge and interest were essential to the progress of the work and his patience and encouragement made working with him a pleasure.

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CHAPTER I

INTRODUCTION

It has long been recognized that a non-linear meson field may be useful in describing certain properties of the nuclear interaction. In 1951, Schiff[1] introduced a term proportional to the fourth power of the meson field amplitude in the Lagrangian. His purpose was to explain two features of atomic nuclei, the first being saturation, and the second being the success of the independent-nucleon approximation used in shell-model calculations. Saturation occurs in nuclei consisting of more than about ten nucleons. For such nuclei, it is experimentally found that the binding energy per nucleon, B/A, is approximately constant. For somewhat heavier nuclei, it is found that the interior, or core, density behaves similarly. These experimental facts imply that the interaction between a pair of nucleons in the presence of other nucleons, as in nuclei, is less than that of a solitary pair of nucleons. If each nucleon simply interacts pairwise with each other nucleon, the binding energy per nucleon and the density, will always increase with the addition of each nucleon to the system, until it collapses. The shell-model, which works so well in predicting nuclear spectra and decay schemes, leads to a similar conclusion. The two-body interactions in a nucleus are suppressed in favour of the interaction of each nucleon with the average nucleon density. It is thought that the individual nucleons feel only some sort of general force due to all the other nucleons in the nucleus rather than a combination of strong, short-range, two body forces, even though it is known that the latter type of forces govern the behaviour of isolated, two-nucleon systems.

Use of non-linear meson fields is by no means the only method devised to account for the aforementioned behaviour of nuclei. Many other effects are considered, alone and in combination, to explain why heavy nuclei do not collapse. Repulsive core potentials, tensor and exchange forces, and the Pauli exclusion principle all modify the nuclear potential in the desired manner, to some extent. It is felt however that these may not be sufficient. Results of some calculations, which incorporate such effects, are given in reference 2. It is found that the value of B/A that these calculations can yield is less than the experimental value by a few MeV for nuclei. Similar results are attainable for nuclear matter where the accepted value of 16 MeV, obtained from the semiempirical mass formula, does not include Coulomb and surface effects which are important in nuclei and reduce the binding energy to about 8 MeV per nucleon. Alternatively, potentials can be adjusted such that the value of B/A is reproduced, but these lead to discrepancies in other nuclear properties, the most important being the two-nucleon scattering data. In view of this rather unsatisfactory situation it is felt that investigation of a non-linear meson field would be worthwhile particularly since very little has been done following Schiff's attempt.

Having now decided to determine the effects of a non-linear meson theory, we must obtain a model with which we can pursue our investigation. It is generally true that non-linear equations are much more difficult to deal with than linear ones and in quantum field theory we have a further complication in that the fields themselves are quantized. For this reason we restrict most of the discussion to classical field theory. To investigate the validity of such an approximation we can estimate the quantum corrections. We expect that, for large field amplitudes which in the quantum formulation result from the presence of many mesons in the system, the approximation will be best, just as the correspondence principle ensures that for large quantum numbers, quantum effects disappear. This is due to the fact that, in a system consisting of many bosons, the creation or annihilation of a single one from a given state, has a negligible effect. We further restrict ourselves to discussion of neutral scalar mesons.

Following Schiff and others[3], we consider a meson field ϕ, which interacts with the nucleon field through a Yukawa interaction. The inclusion of a $\frac{\lambda}{4}\phi^4$ term in the Lagrangian is due to consideration of a meson-meson interaction. The meson field equation is then non-linear, containing $\lambda\phi^3$. In the usual linear theory which does not include the meson-meson interaction, the problem is soluble within the classical approximation and leads to a Yukawa potential between nucleons. The non-linear situation however is much harder to analyze and to our knowledge,little progress has been made, since Schiff, who resorted to variational calculations. For this reason we further simplify our model using as a guide, knowledge of several papers[4,5,6,7] in which equations similar to ours are found to be exactly soluble. The analysis of one-dimensional (1-D) few-body atomic and molecular systems using the Hartree-Fock approximation leads to equations which are of the same form as those arising in our investigation. Solution of our equations then parallels those used for the "atomic" system. We consider in the classical approximation, a one-dimensional system of point-source nucleons interacting with neutral, scalar mesons via a Yukawa interaction with a point-contact repulsion between the latter.

Investigation of models as simple as the one proposed cannot be expected to yield results which are quantitatively comparable to those obtained from more realistic three dimensional (3-D) systems, but can be useful from a pedagogical viewpoint. Indeed, this is usually the justification given for carrying out such calculations. In our case the situation is complicated because of an important difference between the 1-D and 3-D systems. It is found that a 1-D system of nuclear matter is saturated whether or not the meson-meson interaction is taken into account whereas saturation in a 3-D system is very sensitive to the strength of this interaction. In view of this difference then, the model is proposed not as a means of explaining the saturation mechanism, but as a method of comparing the accuracy of approximations employed in realistic 3-D calculations. Since our model is simple, it is possible to carry out most calculations easily and to compare several different approximations with the "exact" result. Although suppression of the two-body interaction in the many-nucleon system is investigated, our primary achievement is that we are able to propose a rather simple accurate approximation to the meson field amplitude which can be used to calculate the two-nucleon interaction energy. The success of this approximation for the 1-D system encourages us to believe that it may be the basis of a useful method for dealing with a 3-D system as well.

We begin our investigation by considering the simpler linear formulation. In Chapter II we neglect the meson-meson interaction and find that the field equation can be solved in a very general manner for any number of nucleons and that the energy of such a system is easily obtainable. Chapter III is concerned with the non-linear theory. The meson fields with one, two and three nucleon sources are obtained and the energy for each system is found. The possibility of three-body interactions is investigated. In Chapter IV, several approximations are tested and one is found to be surprisingly accurate. In our model there are two coupling-constants, λ for the meson-meson interaction and g for the meson-nucleon interaction which is taken to be of a Yukawa type. Nuclear matter and two-nucleon bound state calculations done in Appendix A, are used to test values of the two parameters which are used in the text.

CHAPTER II

A 1-D SYSTEM OF NUCLEONS IN THE ABSENCE OF A MESON-MESON INTERACTION

The Lagrangian density L, of our system of nucleons and neutral scalar mesons is given by

$$L = i\psi^{\dagger}\dot{\psi} - \frac{1}{2M}\,\psi^{\dagger\,'}\psi' + g\psi^{\dagger}\psi\phi + \frac{1}{2}\{\dot{\phi}^{2} - \phi'^{2} - m^{2}\phi^{2}\}, \quad (2.1)$$

where M, m and g are the nucleon mass, meson mass and coupling constant between mesons and nucleons, respectively. In quantum field theory, ψ and ϕ are the nucleon and meson field operators, but unless otherwise stated, we treat them simply as classical field amplitudes. Throughout the discussion we use units such that $\hbar = c = 1$. Energy and mass then have the same dimension. For m let us take the pion mass, namely

$$m = \frac{139.6 \text{ MeV}}{197.33 \text{ MeV fm}} = 0.71 \text{ fm}^{-1} .$$

(Note: $\hbar c$ = 197.33 MeV fm in more familiar units.) The coupling constant g also has units of fm^{-1}. Actual values for g and the non-linear parameter λ, are discussed in Appendix A.

The field equations can now be derived, starting with Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L\, dt = 0 , \quad (2.2)$$

where L is the Lagrangian for the system. In field theory this becomes

$$\delta \int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} L \; dx = 0 \; . \tag{2.3}$$

This can then be expressed as

$$\int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} \left\{ \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{d}{dx}\frac{\partial L}{\partial \phi'} \right\} \delta\phi dx = 0, \tag{2.4}$$

and analagously for the nucleon field. This condition gives the usual Euler-Lagrange equation

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{d}{dx}\frac{\partial L}{\partial \phi'} = 0 \; . \tag{2.5}$$

Writing this explicity for (2.1), we have

$$\ddot{\phi} - \phi'' + m^2\phi = g\psi^*\psi \tag{2.6a}$$

and

$$i\dot{\psi} + \frac{1}{2M}\psi'' = - g\phi\psi \; . \tag{2.6b}$$

These are the meson and nucleon field equations, which must be solved simultaneously in order to obtain a complete description of the system. Here, however, we consider only a first approximation to the nucleon field and solve explicitly the resulting meson field equation, (2.6a). In the linear case only, we can show how the result for an arbitrary nucleon field can be obtained. We also limit discussion to the static situation in which $\ddot{\phi}$ is zero. It is expected that this will have a negligible effect as long as the nucleons can be considered to be moving slowly with respect to the mesons. Eq. (2.6a) then becomes

$$-\phi'' + m^2\phi = g\psi^*\psi \;, \tag{2.7}$$

and it is apparent that this can be solved by the method of Green's functions. The right hand side of (2.7) is just the source term for the meson field. Eq. (2.7) can be written as

$$\mathcal{D}\phi(x) - g\psi^*(x)\psi(x) = 0 \tag{2.8}$$

with

$$\mathcal{D} \equiv -\frac{d^2}{dx^2} + m^2 \;.$$

The boundary conditions are

$$\phi(\infty) = \phi'(\infty) = \phi(-\infty) = \phi'(-\infty) = 0 \;. \tag{2.9}$$

Consider the Green's functions $G_1(x)$, $G_2(x)$ such that

$$\mathcal{D}G_1(x) = 0 \qquad -\infty \leq x \leq x' \tag{2.10a}$$

and

$$\mathcal{D}G_2(x) = 0 \qquad x' \leq x \leq \infty. \tag{2.10b}$$

These must satisfy the relations

$$G_1(x') - G_2(x') = 0 \qquad (2.11a)$$

and

$$G_1'(x') - G_2'(x') = 1,$$

which simply mean that G must be continuous at x = x' and its derivative must be discontinuous there. Then assume

$$G(x,x') = c_1 u(x) \qquad -\infty \leq x \leq x' \qquad (2.12a)$$

and

$$G(x,x') = c_2 v(x) \qquad x' \leq x \leq \infty, \qquad (2.12b)$$

where

$$\mathcal{D}u(x) = 0 \qquad -\infty \leq x \leq x' \qquad (2.13a)$$

and

$$\mathcal{D}v(x) = 0 \qquad x' \leq x \leq \infty. \qquad (2.13b)$$

Equations (2.11) and (2.13) give

$$G(x,x') = \frac{1}{2m} e^{-m|x-x'|} . \qquad (2.14)$$

It can be shown that

$$\phi(x) = g \int_{-\infty}^{\infty} G(x,x')\psi^*(x')\psi(x')dx' \qquad (2.15)$$

satisfies (2.8), by substituting (2.14) into (2.15) and evaluating $\mathcal{D}\phi(x)$. Because of (2.13) we are assured that $\phi(x)$ is the solution to (2.7). (e.g. see Arfken[8]).

We now substitute for $\phi(x)$ in (2.6b) to obtain

$$i\dot{\psi} + \frac{1}{2M}\psi'' = -g^2 \int_{-\infty}^{\infty} G(x,x')\psi^\dagger(x')\psi(x')dx' \cdot \psi \ . \quad (2.16)$$

Recall now, that an operator in configuration space having the form

$$\sum_{i<j=1}^{N} V(x_i, x_j)$$

is represented in Fock space (i.e. in the formalism of second quantization) as

$$\int dx \int dx' \ \psi^\dagger(x')\psi^\dagger(x)V(x,x')\psi(x)\psi(x')$$

(e.g. see Schweber[9]). Then the term in the Schroedinger equation corresponding to $V(x,x')$ in Fock space, is $V(x_i,x_j)$, in configuration space. This means that the nucleon potential is given by

$$V(x_i,x_j) = -g^2 \ G(x_i,x_j) \ . \quad (2.17)$$

We now consider a situation in which the nucleons are completely localized and their wavefunctions are simply of the form $\psi^*\psi \equiv \delta(x-x_i)$ where x_i is the position of the nucleon. This is known as the adiabatic approximation. More generally consider

$$\psi^*(x)\psi(x) = \sum_{i=1}^{N} \delta(x-x_i) = \sum_{i=1}^{N} f_1(x_i) \ . \quad (2.18)$$

It is assumed that the source function $f(x_i)$ is the same for all nucleons. We see immediately from (2.15) that

$$\phi_N(x) = \frac{g}{2m} \sum_{i=1}^{N} e^{-m|x-x_i|} \quad \text{for N nucleons.} \qquad (2.19)$$

The general solution for N nucleons is simply a linear combination of the single nucleon solution. More importantly

$$\sum_{i<j=1}^{N} V(x_i-x_j) = -g^2 \sum_{i<j=1}^{N} G(x_i-x_j) \; . \qquad (2.20)$$

The potential for a system of nucleons is given by a simple sum of two body interactions. There are no N-body interactions for N>2.

Eq. (2.20) can be verified by explicitly evaluating

$$H = \int_{-\infty}^{\infty} \mathcal{H} \, dx \qquad (2.21)$$

where

$$\mathcal{H} = \frac{1}{2} \pi^2 - \mathcal{L}_m - \mathcal{L}_I \; , \qquad (2.22)$$

with π being the momentum canonically conjugate to ϕ. This is just $\dot{\phi}$, so in our static approximation

$$\mathcal{H} = - \mathcal{L}_m - \mathcal{L}_I = \frac{1}{2} \phi'^2 + \frac{1}{2} m^2\phi^2 - g \sum_{i=1}^{N} \delta(x-x_i)\phi \; . \qquad (2.23)$$

$\mathcal{L}_m$, $\mathcal{L}_I$ are the Lagrangian densities which describe separately the meson field and its interaction with the nucleon field. This gives the total energy of the system which overcounts the self-energy of each N, so we must subtract the self-energy of N(N-2) nucleons in order to arrive at (2.20).

The final results for the linear system are:

$$\phi_1(x) = \frac{g}{2m} e^{-m|x-x_i|} \quad ; \text{ the meson field with one nucleon at } x = x_i . \tag{2.24}$$

$$H_1 = -\frac{g^2}{4m} \quad ; \text{ the self energy of a single nucleon.} \tag{2.25}$$

$$H_{ij} = -\frac{g^2}{2m} e^{-mr} \quad ; \text{ the interaction energy for two nucleons separated by a distance } r. \tag{2.26}$$

Here, we note that (2.26) is the one-dimensional equivalent of a Yukawa potential which has the form

$$V = -\frac{g^2}{4\pi} \frac{e^{-mr}}{r} . \tag{2.27}$$

This follows from the 3-D form of (2.1). Note that in this case g is dimensionless.

Since we were initially guided by the solution of the "atomic" systems, some comparison is in order. Lapidus[3] considers the problem of two particles interacting via a potential given by

$$V(x) = -A\{\delta(x+a) + \delta(x-a)\} , \tag{2.28}$$

which corresponds to the source term (2.18) in the two nucleon system. The resulting Schroedinger equation for the system then differs from (2.7) only slightly. The equation is

$$-\frac{\hbar^2}{2M_e} \Psi''(x) + V(x)\Psi(x) = E\Psi(x) , \tag{2.29}$$

where M_e is the mass of the particles being considered and $\psi(x)$, the wavefunction for the system, is of the same form as (2.19) with N=2 except that an odd parity solution is allowed. The linear combination of atomic orbitals as an approximation to the actual molecular orbitals (LCAO-MO) is then investigated. The result is that the approximation is good for larger "interatomic" distances, as expected. The fact that an equation similar to (2.19) does not hold exactly, is a direct consequence of the difference between (2.29) and (2.7), as can be seen from the following:

$$\mathcal{D}\phi_1(x) - gf_1(x) = 0 \ , \tag{2.30}$$

for a single nucleon. For N nucleons this can be written as

$$\mathcal{D}\phi_N(x) - gf_N(x) = \mathcal{D} \sum_{i=1}^{N} \phi_1(x_i) - g \sum_{i=1}^{N} f_1(x_i) = 0 \ , \tag{2.31}$$

so that a linear combination of one-nucleon fields satisfies the many-nucleon equation. It is not possible to write a corresponding equation for the atomic system because of the presence of the "extra" factor of $\Psi(x)$ in (2.29).

Concluding our discussion of the linear meson theory, we note that similar general results hold for the three-dimensional system: the meson fields are superposable and the interaction energy of a system of many nucleons is simply the sum of the pair interactions. We now turn our attention to a non-linear theory, but repeatedly return to the results of this section as a test of the non-linear results which must be identical in the limit of small λ.

CHAPTER III

A NON-LINEAR MESON FIELD

As in Chapter II, we consider the Lagrangian density for the system. Here we introduce a term to account for the meson-meson interaction.

$$L(x) = i\psi^\dagger\dot{\psi} - \frac{1}{2M}\psi^{\dagger\,\prime}\psi' + g\psi^\dagger\psi\phi + \frac{1}{2}\{\dot{\phi}^2 - \phi'^2 - m^2\phi^2 - \frac{\lambda}{2}\phi^4\}. \quad (3.1)$$

The (positive) coupling constant between mesons, λ, has units of fm^{-2}. It can be seen that the last term of eq. (3.1) corresponds to a point-contact repulsion between mesons, by noting that

$$\frac{\lambda}{4}\phi^4(x) = \frac{\lambda}{4}\int \phi^2(x)\delta(x-x')\phi^2(x')dx' \ . \quad (3.2)$$

The Euler-Lagrange relation then yields the field equations:

$$\ddot{\phi} - \phi'' + m^2\phi + \lambda\phi^3 = g\psi^*\psi, \quad (3.3a)$$

and

$$i\dot{\psi} + \frac{1}{2M}\psi'' = -g\phi\psi \quad (3.3b)$$

In the static approximation the meson field equation is

$$-\phi'' + m^2\phi + \lambda\phi^3 = g\psi^*\psi \ . \quad (3.4)$$

Here, unlike in Chapter II, we cannot find the solution for an arbitrary nucleon field, and resort immediately to the adiabatic approximation.

$$-\phi_N'' + m^2\phi_N + \lambda\phi_N^3 = g \sum_{i=1}^{N} \delta(x-x_i) \quad , \tag{3.5}$$

for N nucleons, with the ith being localized at $x = x_i$.

The field equation for a single nucleon located at the origin is given by (3.5) as

$$-\phi_1'' + m^2\phi_1 + \lambda\phi_1^3 = g\delta(x) \quad . \tag{3.6}$$

The solution of this is obtained by solving the corresponding homogeneous equation,

$$-\phi_1'' + m^2\phi_1 + \lambda\phi_1^3 = 0 \quad , \tag{3.7}$$

subject to certain restrictions. The first of these requires that the field amplitude be continuous everywhere and is known as the matching condition. The second requires that the derivative be discontinuous at the position of the nucleon. This jump condition is found by integrating (3.6) over a small region about the origin, which results in

$$-\phi_1'\Big|_{-\varepsilon}^{\varepsilon} + \int_{-\varepsilon}^{\varepsilon} [m^2\phi_1 + \lambda\phi_1^3]dx = g. \tag{3.8}$$

The integral vanishes in the limit $\varepsilon \to 0$ since the field is continuous and the result, giving the relation for the discontinuity in the derivative,is

$$\phi'_{0_+} - \phi'_{0_-} = -g \ . \tag{3.9}$$

The solution to (3.7) is found by multiplying by $2\phi'(x)$ and integrating so that

$$\phi_1'^2 = m^2\phi_1^2 + \frac{\lambda}{2}\phi_1^4 + c \ . \tag{3.10}$$

The integration constant must be zero so that (3.10) is satisfied for $x \to \infty$ where ϕ and ϕ' are zero. Separating variables and integrating yields

$$\int [m^2\phi_1^2 + \frac{\lambda}{2}\phi_1^4]^{-1/2} \, d\phi_1 = x \ . \tag{3.11}$$

Making the substitution

$$u^2 = m^2 + \frac{\lambda}{2}\phi_1^2 \tag{3.12}$$

results in

$$\int \frac{du}{u^2-m^2} = x \ . \tag{3.13}$$

This can be solved to give the relation

$$\frac{1}{2m}\ \ell n\{\frac{u-m}{u+m}\} + k = x \ , \tag{3.14}$$

with k being the constant of integration. Then, replacing for u and rewriting, $\phi(x)$ is obtained as

$$\phi_1(x) = 2\left[\frac{2m^2}{\lambda}\right]^{1/2} \frac{\alpha\, e^{-m|x|}}{(1-\alpha^2 e^{-2m|x|})} , \qquad (3.15)$$

where the constant of integration is now α. The jump condition (3.9) gives the relation

$$4m^2\left[\frac{2}{\lambda}\right]^{1/2} \alpha\left\{\frac{1+\alpha^2}{(1-\alpha^2)^2}\right\} = g , \qquad (3.16)$$

which can be solved numerically to obtain α. The field amplitude $\phi_1(x)$ is then known and is shown in Fig. 1a.

The energy associated with this system is just the self-energy of a single nucleon and is evaluated using (2.21). Here

$$H = \frac{1}{2}\{\phi_1'^2 + m^2\phi_1^2 + \frac{\lambda}{2}\,\phi_1^4\} - g\delta(x)\phi_1 . \qquad (3.17)$$

Setting $\beta = \frac{1}{m}\,\ell n\alpha$ and $\gamma = \frac{2m^2}{\lambda}$ gives

$$\phi_1(x) = \gamma^{1/2}\text{csch}\; m(x-\beta) \quad \text{for } x \geq 0 \qquad (3.18a)$$

and

$$\phi_1(x) = -\gamma^{1/2}\text{csch}\; m(x+\beta) \quad \text{for } x \leq 0 . \qquad (3.18b)$$

Using the relation (3.10), the nucleon self-energy H_1, is written as

$$H_1 = \gamma\left\{\int_{-\infty}^{0} \frac{\cosh^2 n(x+\beta)\,dx}{\sinh^4 m(x+\beta)} + \int_0^{\infty} \frac{\cosh^2 m(x-\beta)\,dx}{\sinh^4 m(x-\beta)}\right\} \qquad (3.19)$$

$$-2g\gamma^{1/2}\,\frac{\alpha}{1-\alpha^2} .$$

Then

$$H_1 = -\frac{2m\gamma}{3}(1 + \coth^3 m\beta) - 2g\gamma^{1/2}\frac{\alpha}{1-\alpha^2} , \qquad (3.20)$$

and upon replacing for β and substituting for g from (3.16) this becomes

$$H_1 = \frac{4}{3} m\gamma \frac{\alpha^2(\alpha^4-6\alpha^2-3)}{(1-\alpha^2)^3} . \qquad (3.21)$$

The substitution $u^2 = \frac{\alpha^2}{\lambda}$ in eq. (3.16) gives

$$\alpha^2 = \frac{g^2\lambda}{32m^4} , \qquad (3.22)$$

for small λ, so that

$$\lim_{\lambda\to 0} H_1 = -\frac{g^2}{4m} \qquad (3.23)$$

which is identical to eq. (2.25) as expected.

For a system of two nucleons (3.5) must be solved explicitly since the fields are not superposable. With nucleons at $x = \pm a$, (3.5) becomes

$$\phi_2'' - m^2\phi_2 - \lambda\phi_2^3 = -g\{\delta(x+a) + \delta(x-a)\}. \qquad (3.24)$$

For this system the solution has two distinct forms depending upon which region is considered. In the exterior region, $|x| \geq a$, the field amplitude is immediately found to be

$$\phi_2(x) = \phi_1(x\pm a) . \qquad (3.25)$$

Note however that the constant α does not satisfy (3.16) and must be found from the matching and jump conditions. For the interior region, $|x| \leq a$, the solution is found to be quite different. Multiplying by $2\phi_2'$ and integrating gives

$$\phi_2'^2 = m^2\phi_2^2 + \frac{\lambda}{2}\,\phi_2^4 + K\ . \tag{3.26}$$

The integration constant K is non-zero since we do not expect that ϕ_2 will be zero in this region; we do expect that it will have a minimum so that $\phi_2'(0) = 0$. This leads to

$$\phi_2'^2(0) = m^2\phi_2^2(0) + \frac{\lambda}{2}\,\phi_2^4(0) + K = 0\ . \tag{3.27}$$

Then (3.26) becomes

$$\phi_2'^2 = m^2[\phi_2^2 - \phi_2^2(0)] + \frac{\lambda}{2}\,[\phi_2^4 - \phi_2^4(0)] \tag{3.28}$$

which can be rewritten as

$$\phi_2' = [(\phi_2^2 - \phi_0^2)(\frac{\lambda}{2}\,\phi_2^2 + m^2 + \frac{\lambda}{2}\,\phi_0^2)]^{1/2}\quad ;\ \phi_2(0) = \phi_0\ . \tag{3.29}$$

Separating variables and integrating yield

$$\frac{4}{\lambda^2}\int_{\phi_0}^{\phi_2} [(\phi_2^2 - \phi_0^2)(\phi_2^2 + \frac{2m^2}{\lambda} + \phi_0^2)]^{-1/2} d\phi_2 = x\ . \tag{3.30}$$

This gives*

$$nc^{-1}(\frac{\phi_2}{\phi_0}\,|\mu) = x[\lambda\phi_0^2 + m^2]^{1/2} \tag{3.31}$$

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*A description of the Jacobian elliptic functions and of details leading to (3.32) is given in Appendix B.

or

$$\phi_2(x) = \phi_0 \, nc(x[\lambda\phi_0^2 + m^2]^{1/2} | \mu) \; ; \; |x| \leq a \; . \tag{3.32}$$

$$\mu = (\phi_0^2 + \frac{2m^2}{\lambda})(2\phi_0^2 + \frac{2m^2}{\lambda})^{-1} \; . \tag{3.33}$$

Matching the field amplitudes at x = a gives

$$2\gamma^{1/2} \frac{\alpha}{1-\alpha^2} = \phi_0 nc(u_a|\mu) \; , \tag{3.34}$$

where $u_a = a[\lambda\phi_0^2 + m^2]^{1/2}$. This can be solved to yield

$$\alpha = \frac{-\gamma^{1/2} \pm [\gamma + \phi_0^2 \, nc^2(u_a|\mu)]^{1/2}}{\phi_0 nc(u_a|\mu)} \; , \tag{3.35}$$

and taking the upper sign ensures that $\alpha > 0$. Imposing the jump condition at x = a gives

$$\phi'_{a_+} - \phi'_{a_-} = -g \; , \tag{3.36}$$

which results in the relation

$$2m\gamma^{1/2} \frac{\alpha(1+\alpha^2)}{(1-\alpha^2)^2} + \frac{\phi_0(\lambda\phi_0^2+m^2)^{1/2} \cdot sn(u_a|\mu) dn(u_a|\mu)}{cn^2(u_a|\mu)} = g. \tag{3.37}$$

Substituting (3.35) into (3.37) gives a relation which can be solved numerically to obtain ϕ_0 and then α. Therefore $\phi_2(x)$ is determined by the four relations (3.25), (3.32), (3.35) and (3.37), and is shown in Fig. 1b.

The nucleon-nucleon interaction energy can now be evaluated using the relation

$$H_2(r) = \frac{1}{2}\int_{-\infty}^{\infty} \{\phi_2'^2 + m^2\phi_2^2 + \frac{\lambda}{2}\phi_2^4\}dx - g\{\phi_2(a)+\phi_2(-a)\} - 2H_1, \quad (3.38)$$

where r is the internucleon separation 2a. Eq. (3.28) can be used to simplify the first term. The integration is done numerically using a simple Simpson's rule formula. For maximum accuracy, the function must be evaluated at exactly the position of the cusps which occur at x = a and x = -a. The function $H_2(r)$ is shown in Fig. 2. It is seen that for $\lambda \to 0$, this reduces to (2.26). Note that for $\lambda = 0$ (3.34) and (3.37) can be solved analytically to obtain ϕ_0 and α, though this was avoided by observing (2.19).

To investigate the possibility of many-body interactions it is necessary to obtain the solution of the field equation for (at least) a three-nucleon system. For nucleons positioned at x = a, x = -b and x = 0, (3.5) becomes

$$-\phi_3'' + m^2\phi_3 + \lambda\phi_3^3 = g\{\delta(x-a) + \delta(x+b) + \delta(x)\}. \quad (3.39)$$

The corresponding homogeneous equation,

$$-\phi_3'' + m^2\phi_3 + \lambda\phi_3^3 = 0\ , \quad (3.40)$$

must then be solved for the four regions $x \le -b$, $-b \le x \le 0$, $0 \le x \le a$ and $x \ge a$. The forms of the solutions however, are identical to those obtained for the corresponding regions in the two-nucleon system. The matching and jump conditions are much more complicated though, resulting in a system of six equations which must be solved simultaneously. Matching field amplitudes at $x = -b$, $x = 0$ and $x = a$ gives

$$2\gamma^{1/2} \frac{\alpha_b}{1-\alpha_b^2} = \phi_b \, nc(u_1|\mu_b) \tag{3.41}$$

where

$$u_1 = (x_b+b)[\lambda\phi_b^2 + m^2] \quad , \quad \mu_b = \frac{[\phi_b^2 + \frac{2m^2}{\gamma}]}{[2\phi_b^2 + \frac{2m^2}{\gamma}]}$$

and ϕ_b is the value of the field amplitude at the position $x = x_b$, where $\phi_3' = 0$. Note that for sufficiently small b and a, the situation

$$x_b < -b \ , \tag{3.42}$$

is possible. The other relations are

$$\phi_b nc(u_b|\mu_b) = \phi_a nc(u_a|\mu_a) \tag{3.43}$$

and

$$\phi_a nc(u_2|\mu_b) = 2\gamma^{1/2} \frac{\alpha_a}{1-\alpha_a^2} \ , \tag{3.44}$$

where

$$u_b = - x_b[\lambda\phi_b^2 + m^2]^{1/2} \quad , \quad u_a = -x_a[\lambda\phi_a^2 + m^2]^{1/2}$$

and

$$u_2 = (a-x_a)[\lambda\phi_a^2 + m^2]^{1/2} \ .$$

The jump conditions are

$$\frac{\phi_b(\lambda\phi_b^2 + m^2)^{1/2}\mathrm{sn}(u_1|\mu_b)\mathrm{dn}(u_1|\mu_b)}{\mathrm{cn}^2(u_1|\mu_b)} - \frac{2m\gamma\alpha_b(1 + \alpha_b)}{(1-\alpha_b)^2} = -g \qquad (3.45)$$

at $x = -b$,

$$\frac{\phi_b(\lambda\phi_b^2 + m^2)^{1/2}\mathrm{sn}(u_b|\mu_b)\mathrm{dn}(u_b|\mu_b)}{\mathrm{cn}^2(u_b|\mu_b)} - \frac{\phi_a(\lambda\phi_a^2 + m^2)^{1/2}\mathrm{sn}(u_a|\mu_a)\mathrm{dn}(u_a|\mu_a)}{\mathrm{cn}^2(u_a|\mu_a)} = g \qquad (3.46)$$

at $x = 0$, and

$$\frac{2m\gamma\alpha_a(1+\alpha_a)}{(1-\alpha_a)^2} + \frac{\phi_a(\lambda\phi_a^2+m^2)^{1/2}\mathrm{sn}(u_2|\mu_a)\mathrm{dn}(u_2|\mu_a)}{\mathrm{cn}^2(u_2|\mu_a)} = g \qquad (3.47)$$

at $x = a$. This can be reduced to a system of four equations by solving for α_a and α_b as in (3.35). The field amplitudes are

$$\phi_3(x) = \frac{2\gamma^{1/2}\alpha_b e^{m(x+b)}}{[1-\alpha_b^2 e^{2m(x+b)}]} \qquad \text{for } x \leq -b , \qquad (3.48a)$$

$$\phi_3(x) = \phi_b \mathrm{nc}(u_b|\mu_b) \qquad \text{for } - \leq x \leq 0 , \qquad (3.48b)$$

$$\phi_3(x) = \phi_a \mathrm{nc}(u_a|\mu_a) \qquad \text{for } 0 \leq x \leq a \qquad (3.48c)$$

and

$$\phi_3(x) = \frac{2\gamma^{1/2}\alpha_a e^{-m(x-a)}}{[1-\alpha_a^2 e^{-2m(x-a)}]} \qquad \text{for } x \geq a . \qquad (3.48d)$$

Here $u_a = (x-x_a)[\lambda\phi_a^2 + m^2]^{1/2}$ and $u_b = (x-x_b)[\lambda\phi_b^2 + m^2]^{1/2}$. The field amplitude $\phi_3(x)$ is shown in Fig. 1c. The interaction energy of the system is then computed using the Hamiltonian density which gives

$$H_3 = \frac{1}{2}\int_{-\infty}^{\infty} \{\phi_3'^2 + m^2\phi_3^2 + \frac{\lambda}{2}\phi_3^4\}dx - g\{\phi_3(-b) + \phi_3(0) + \phi_3(a)\} - 3H_1 \; . \tag{3.49}$$

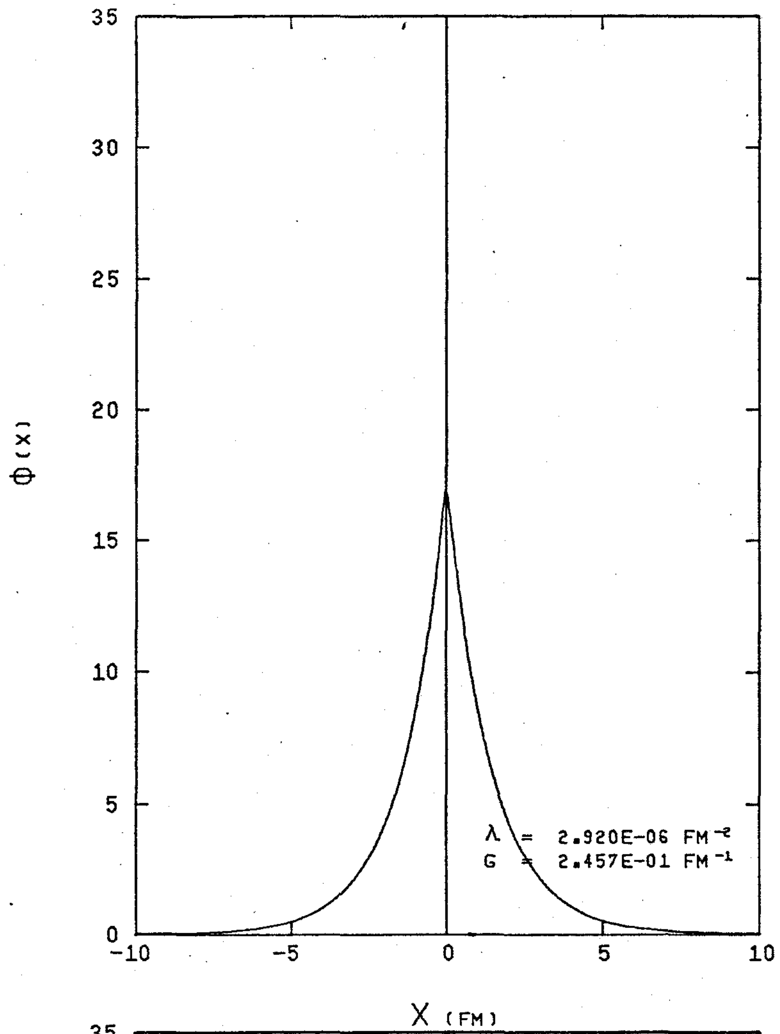
The contribution from the three-body interaction can then be determined by comparing $H_3(a,b)$ to the total contributions of the two-body interactions. This can be written as

$$V_{123}(a,b) = H_3(a,b) - \{H_2(a) + H_2(b) + H_2(a+b)\}. \tag{3.50}$$

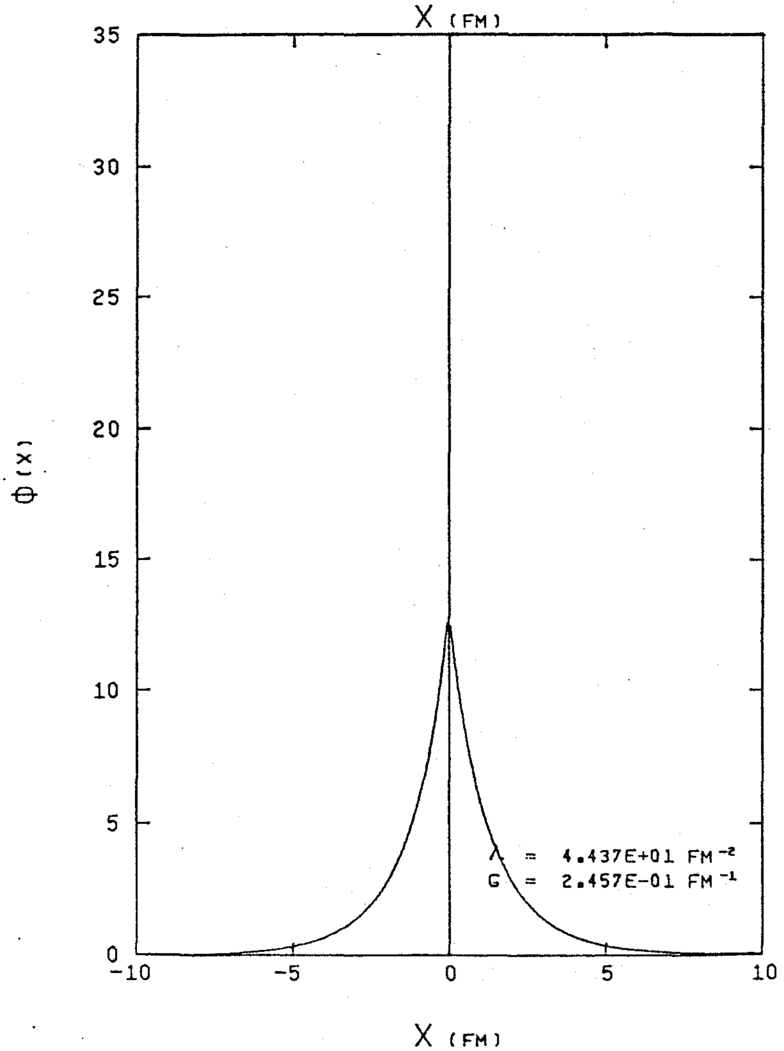
For the linear meson field ($\lambda=0$) this is always zero but, as can be seen from Fig. 3, the meson-meson repulsion gives rise to a repulsive, three-body interaction. The upper plot shows that the total interaction energy of the three-nucleon system is greater (less negative) than the sum of the three, two-body interaction energies. Note that $H_3(a,b)$ vanishes only for large a <u>and</u> b, but both plots clearly show that, as expected, V_{123} vanishes when one nucleon is removed to infinity. The lower plot also indicates that V_{123} is symmetric with respect to exchange of a and b. Figure 4 shows the behaviour of V_{123} as a function of λ.

Summarizing, we have shown how the meson field equation can be solved when any number of localized nucleon sources are involved although the system of simultaneous transcendental equations which the problem is reduced to, becomes increasingly complicated. Even though the solution for two or more nucleons can easily be found in terms of the well-known Jacobi elliptical functions, it must be evaluated numerically. The solution obtained, however, is the exact one for the model under consideration so that we are now able to compare the accuracy of several approximations to the field amplitude and to the two-nucleon interaction.

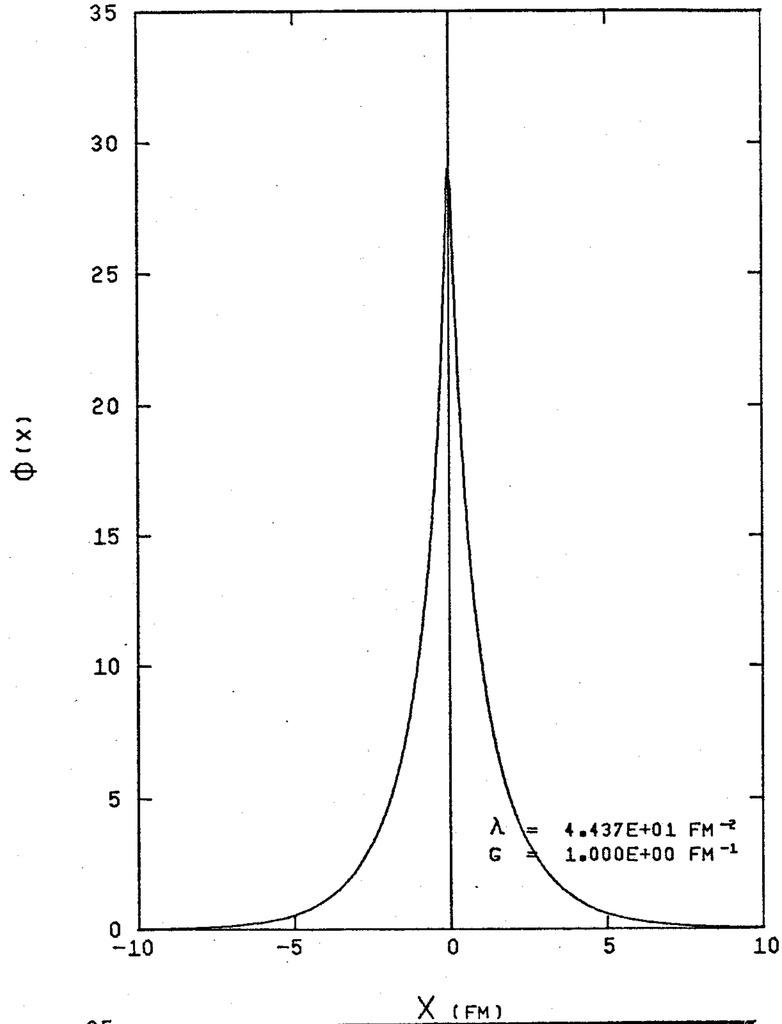
Fig. 1a The meson field amplitude for a single nucleon source. The lower plot shows the effect of the non-linear term in the field equation.



35
30
25
20
15
10
5
0
−10
−5
0
5
10
φ(x)
X (FM)
λ = 2.920E−06 FM⁻²
G = 2.457E−01 FM⁻¹



35
30
25
20
15
10
5
0
−10
−5
0
5
10
φ(x)
X (FM)
λ = 4.437E+01 FM⁻²
G = 2.457E−01 FM⁻¹


Φ(X)
X (FM)
λ = 4.437E+01 FM⁻²
G = 1.000E+00 FM⁻¹
35
30
25
20
15
10
5
0
−10
−5
0
5
10

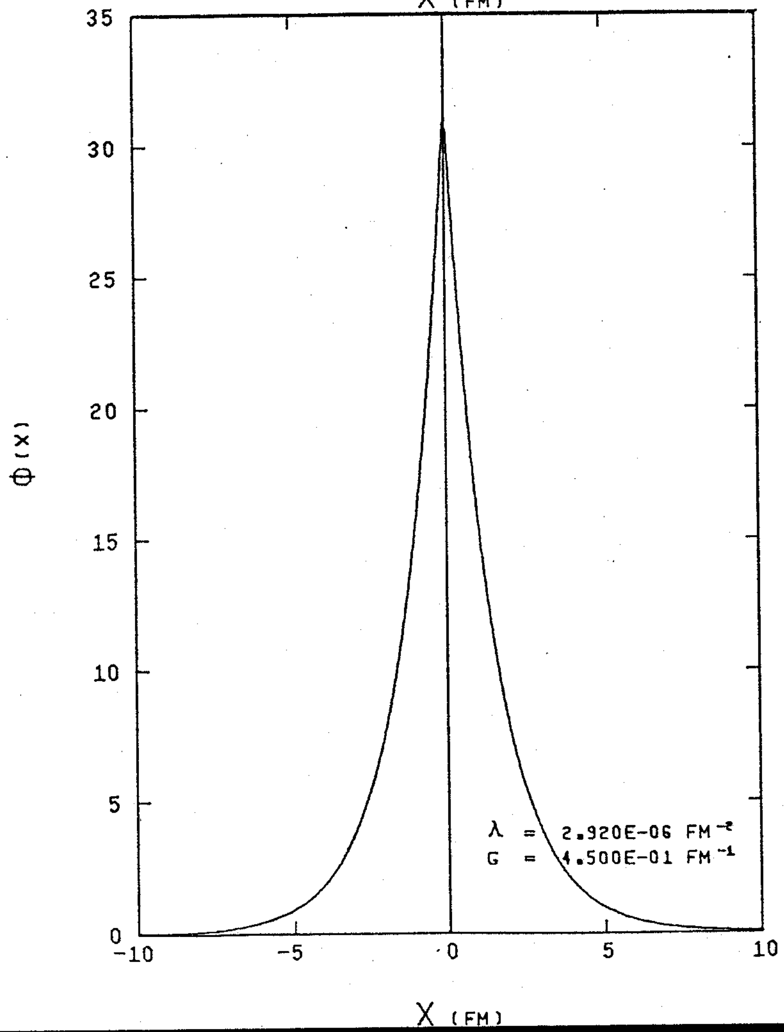
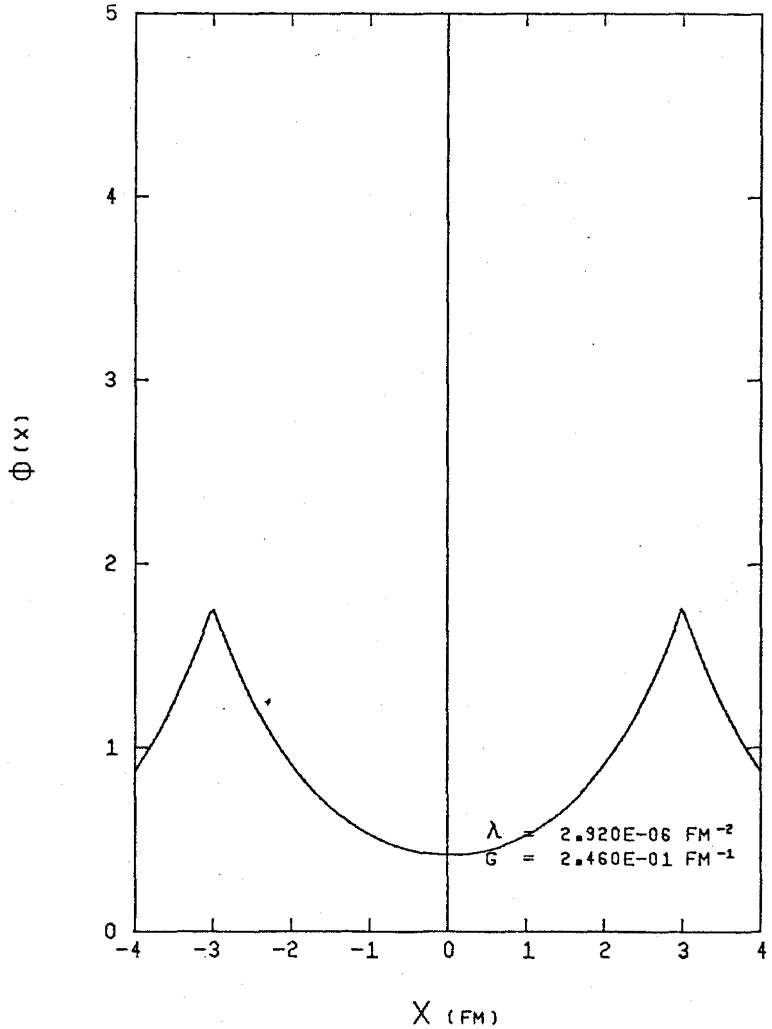
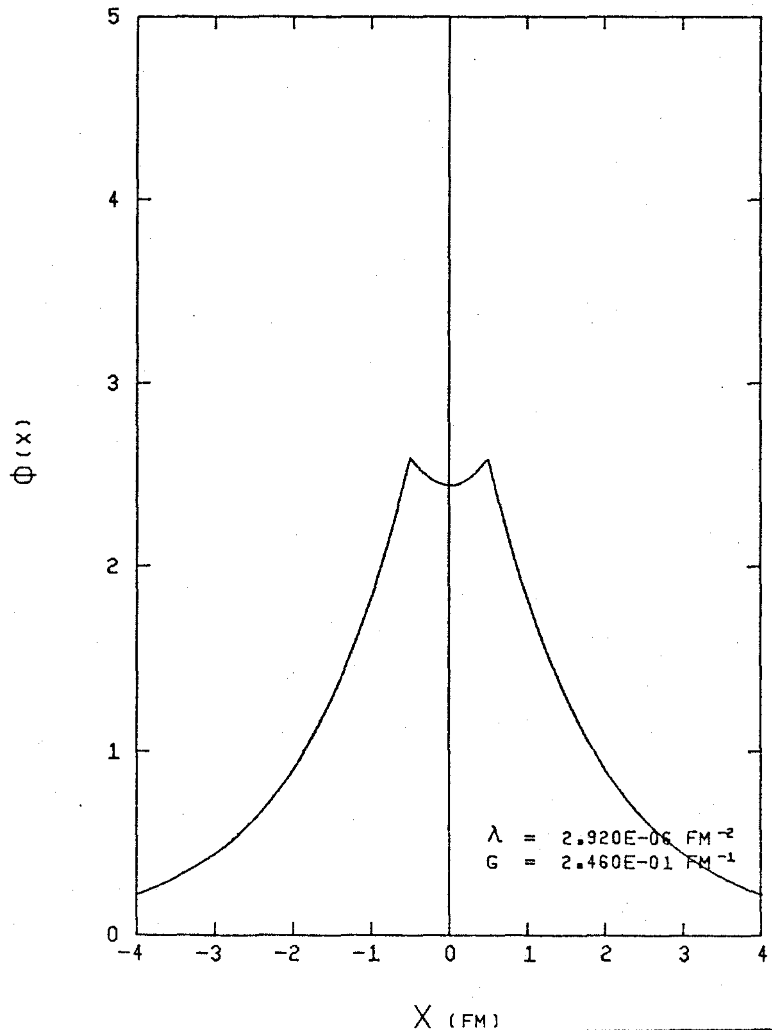

Φ(X)
X (FM)
λ = 2.920E-06 FM⁻²
G = 4.500E-01 FM⁻¹
35
30
25
20
15
10
5
0
−10
−5
0
5
10

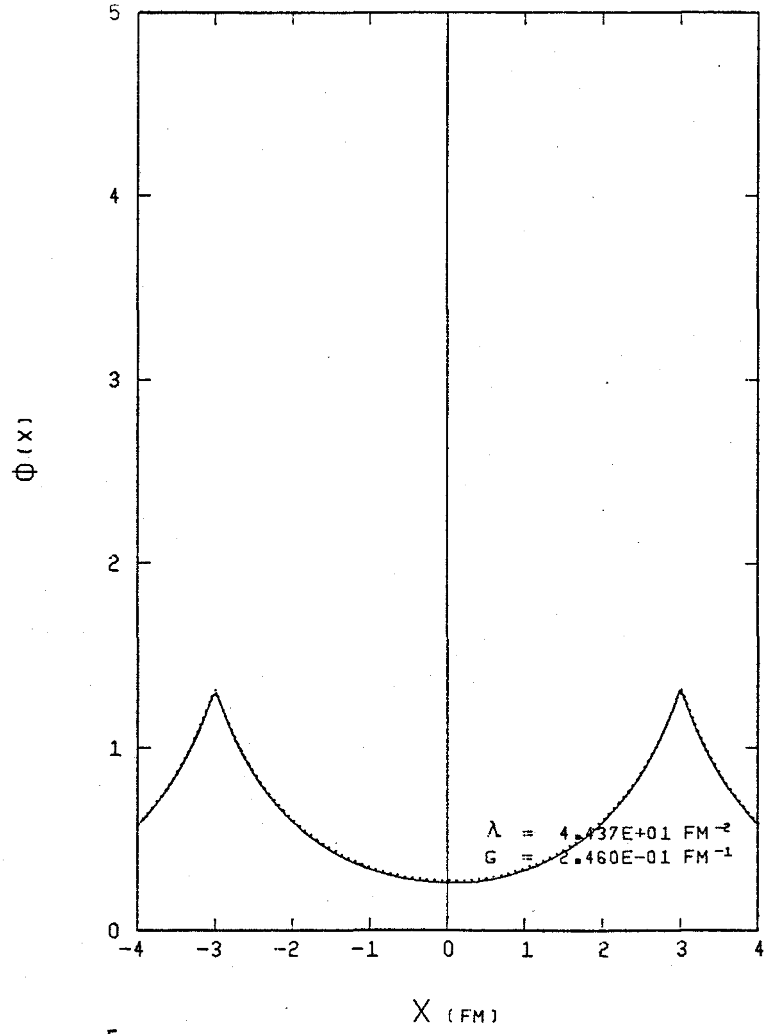
Fig. 1b The two-nucleon field for different separations. The second line, (····), shows $\phi_2^{LC}(x)$. Note that it is plotted in all six diagrams.

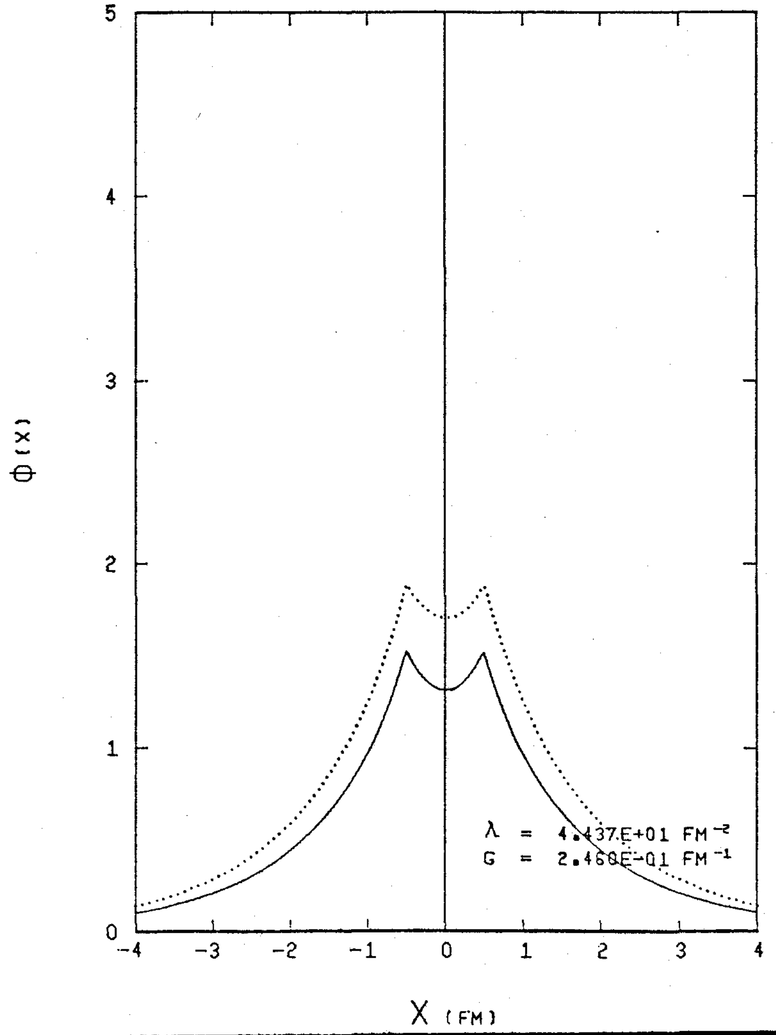


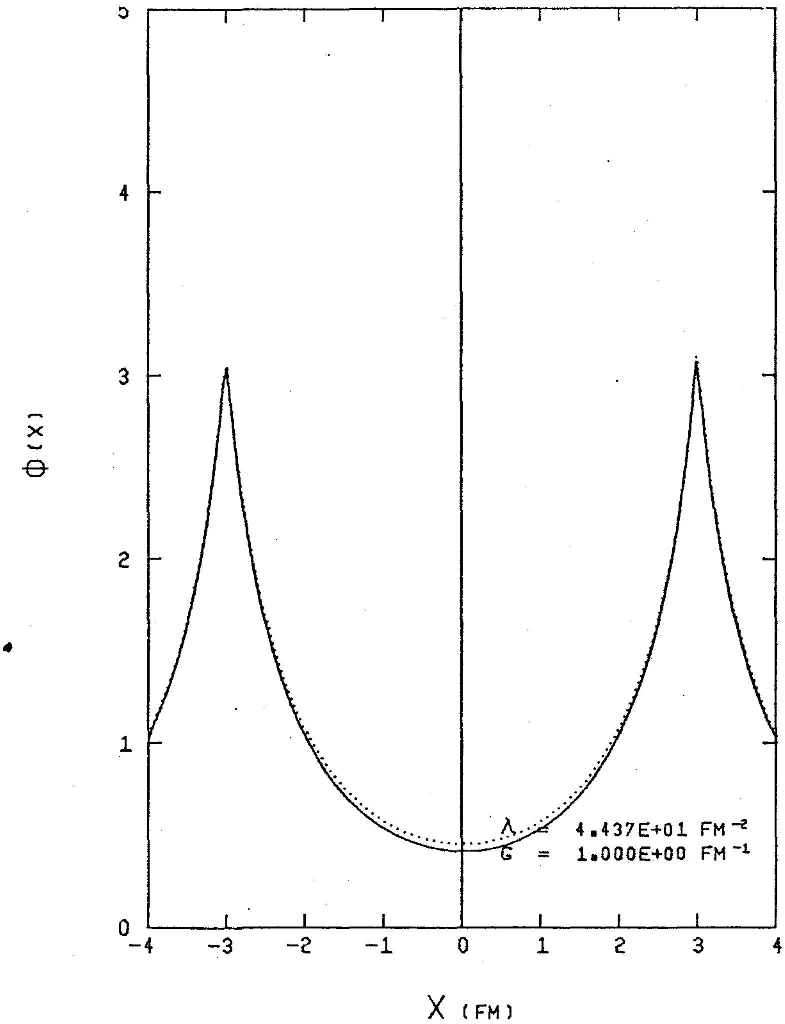
5
4
3
2
1
0
Φ(X)
-4 -3 -2 -1 0 1 2 3 4
X (FM)
λ = 2.920E-06 FM⁻²
G = 2.460E-01 FM⁻¹



5
4
3
2
1
0
Φ(X)
-4 -3 -2 -1 0 1 2 3 4
X (FM)
λ = 2.920E-06 FM⁻²
G = 2.460E-01 FM⁻¹


Φ(X)
X (FM)
λ = 4.437E+01 FM⁻²
G = 2.460E-01 FM⁻¹


Φ(X)
X (FM)
λ = 4.437E+01 FM⁻²
G = 2.460E-01 FM⁻¹


Φ(X)
X (FM)
λ = 4.437E+01 FM⁻²
G = 1.000E+00 FM⁻¹
0
1
2
3
4
5
-4
-3
-2
-1
0
1
2
3
4

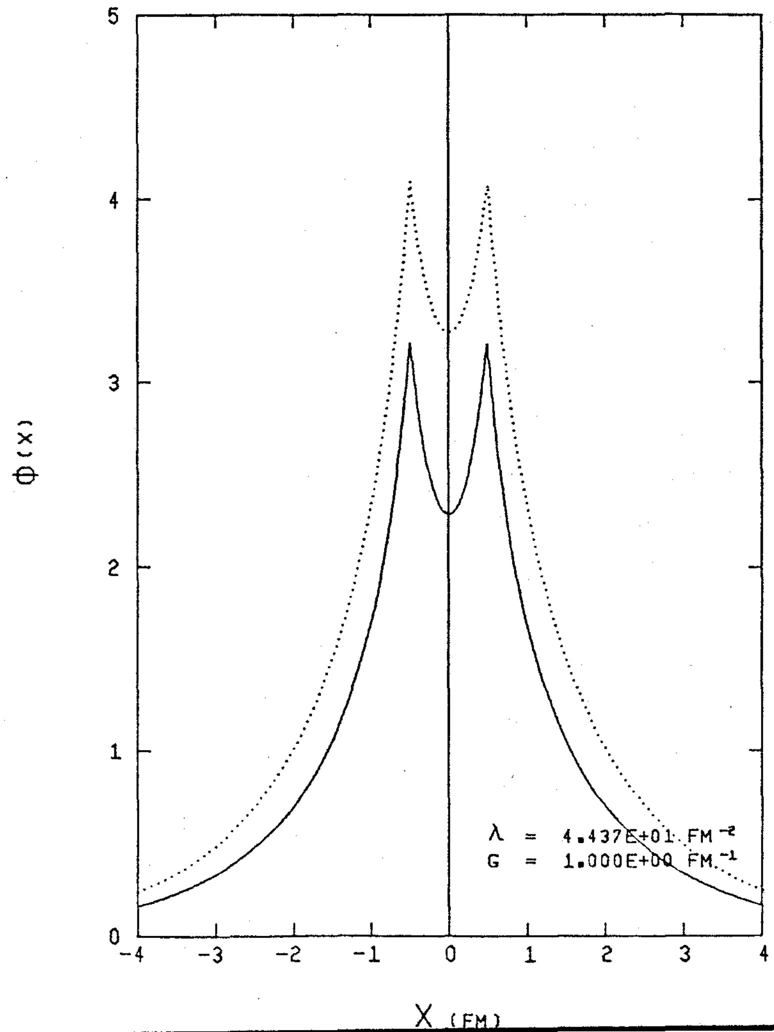
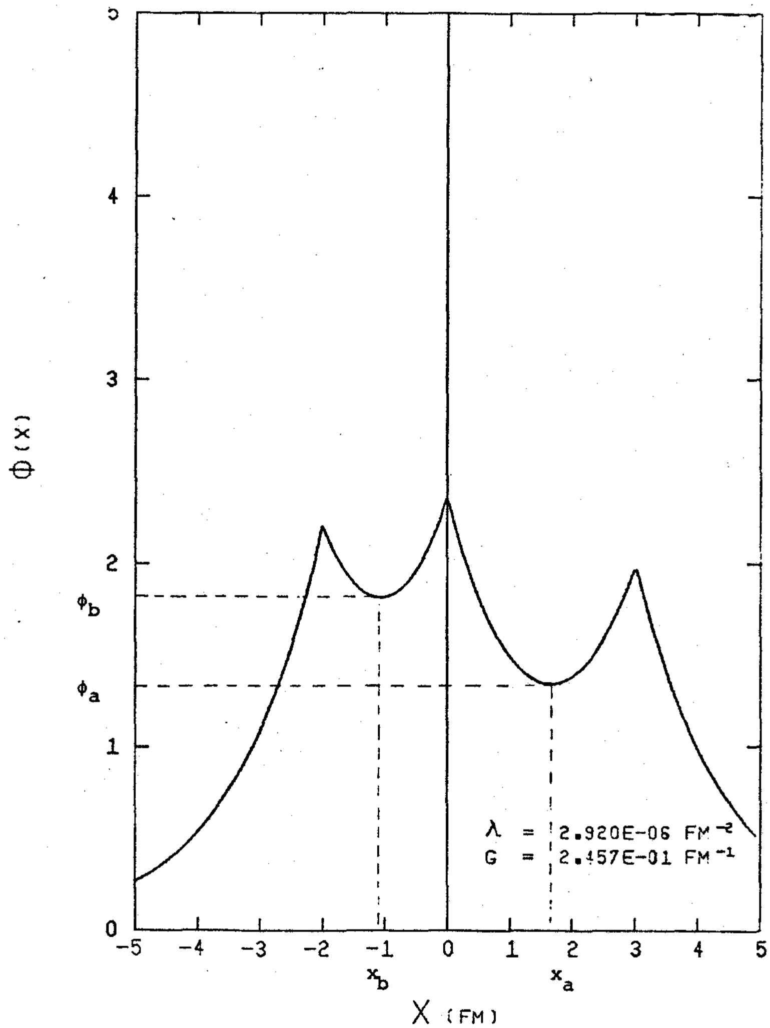
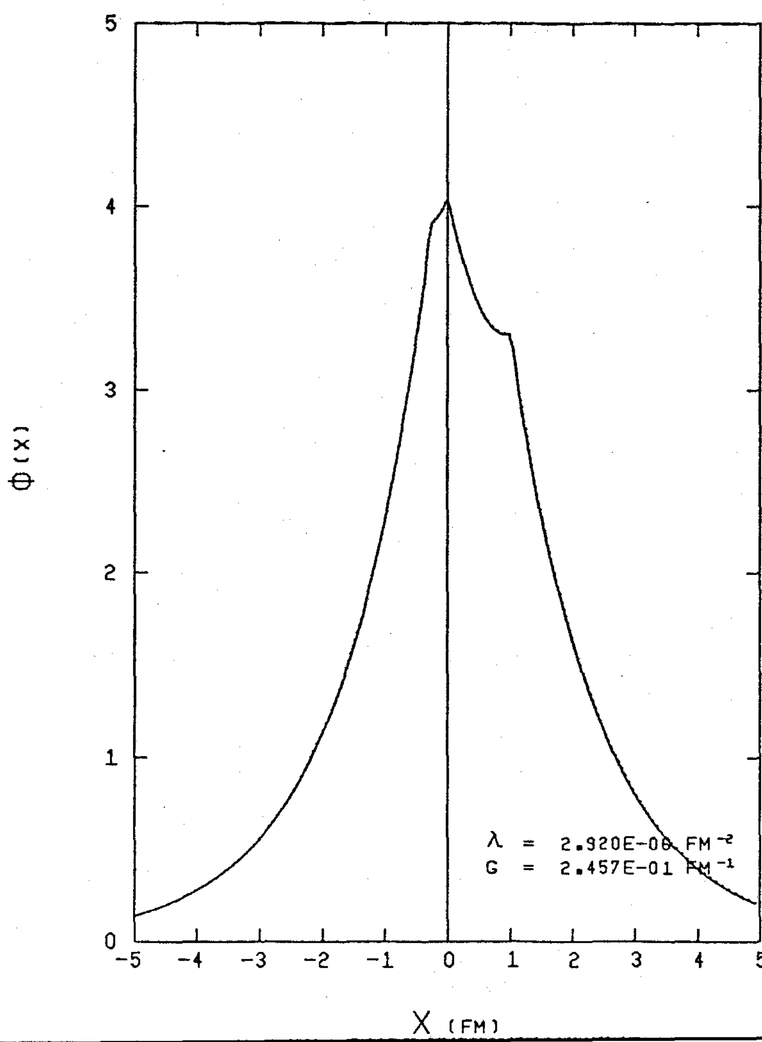

Φ(X)
X (FM)
λ = 4.437E+01 FM⁻²
G = 1.000E+00 FM⁻¹
0
1
2
3
4
5
-4
-3
-2
-1
0
1
2
3
4

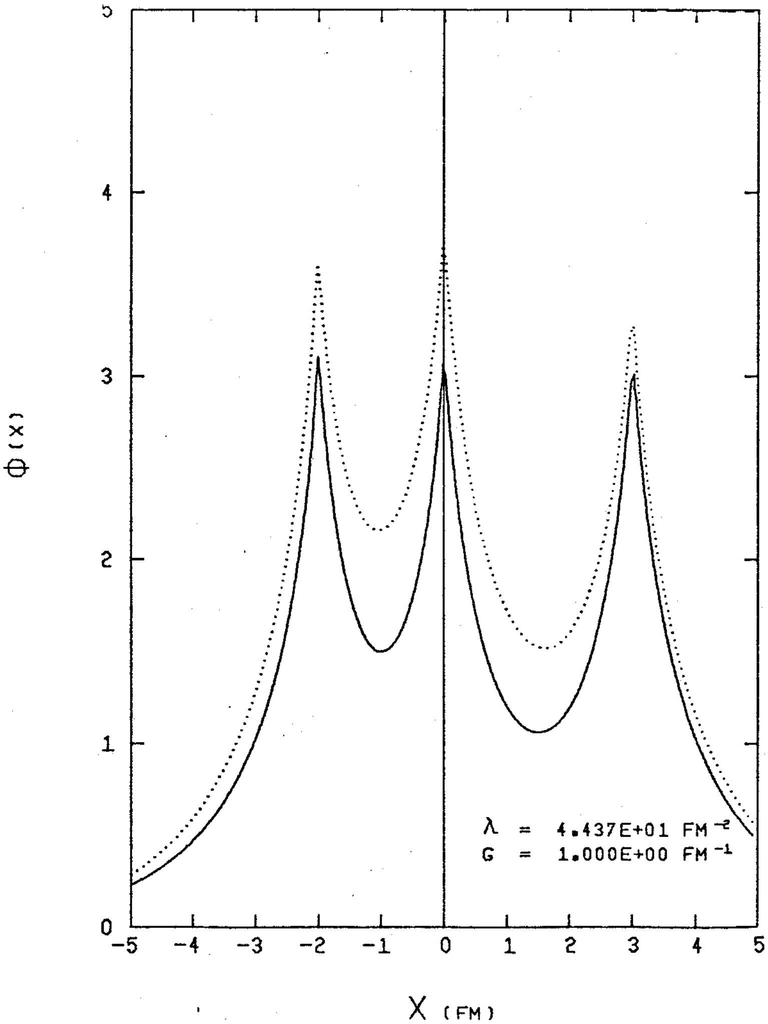
Fig. 1c The meson field with three nucleon sources. The second diagram shows the situation referred to in (3-42).



Φ(X)
X (FM)
ϕ_b
ϕ_a
x_b
x_a
λ = 2.920E-06 FM⁻²
G = 2.457E-01 FM⁻¹



Φ(X)
X (FM)
λ = 2.920E-06 FM⁻²
G = 2.457E-01 FM⁻¹


Φ(X)
X (FM)
λ = 4.437E+01 FM^-2
G = 1.000E+00 FM^-1
0
1
2
3
4
5
-5
-4
-3
-2
-1
0
1
2
3
4
5

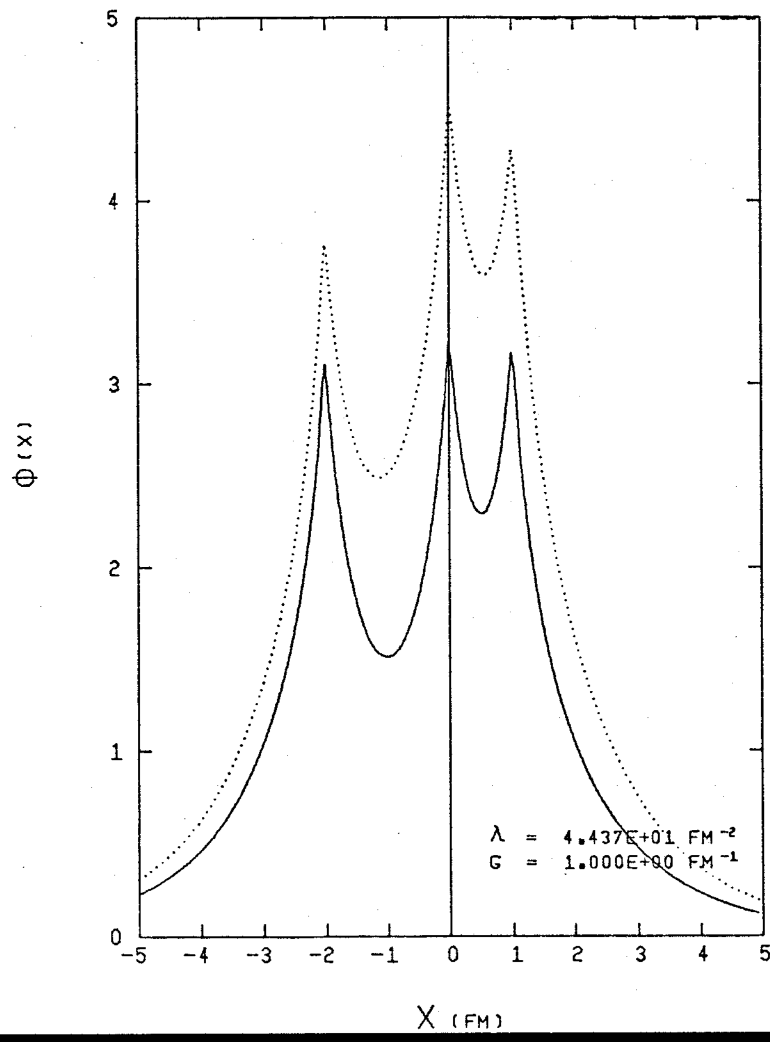
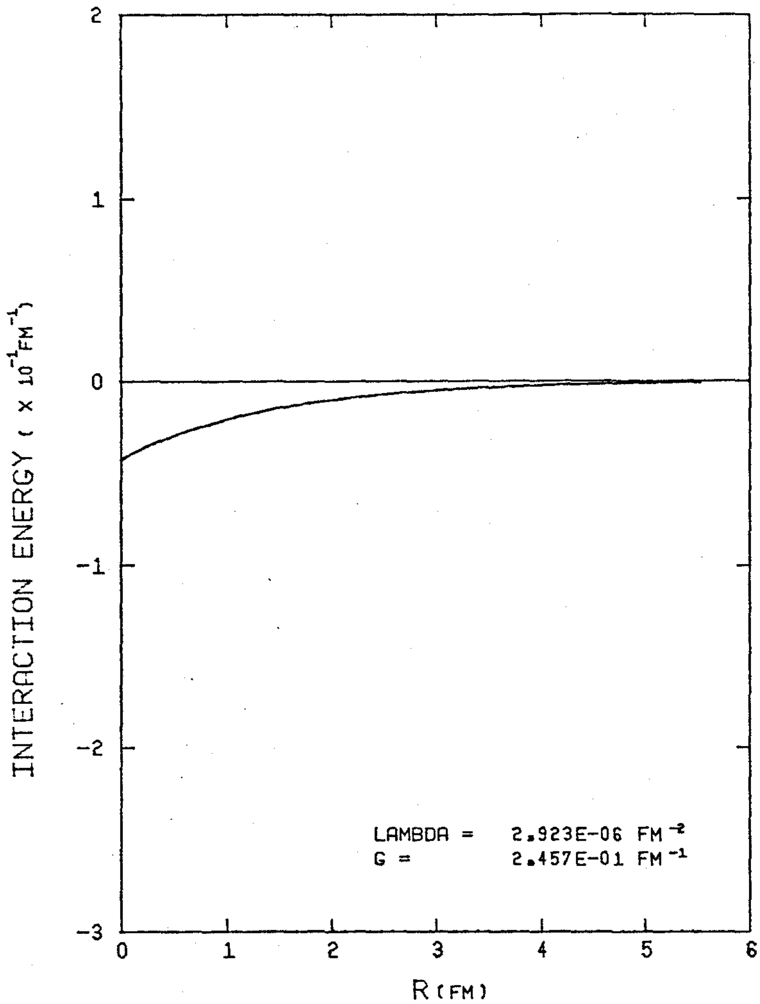

Φ(X)
X (FM)
λ = 4.437E+01 FM^-2
G = 1.000E+00 FM^-1
0
1
2
3
4
5
-5
-4
-3
-2
-1
0
1
2
3
4
5

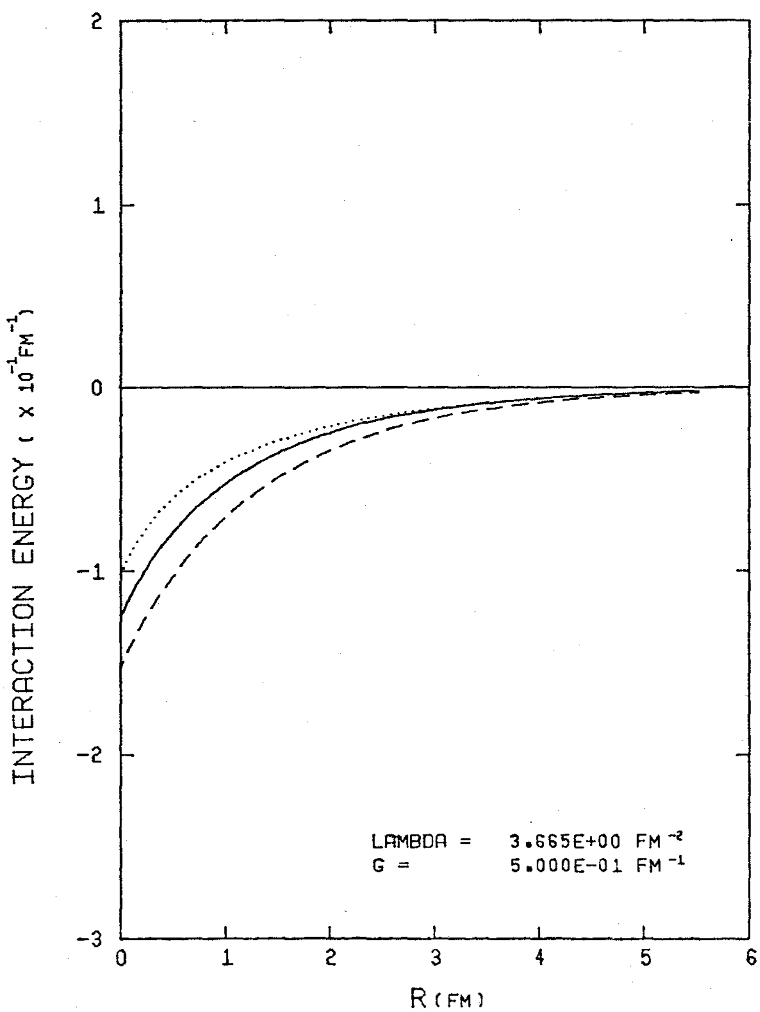
Fig. 2 The two-nucleon potential $H_2(r)$.

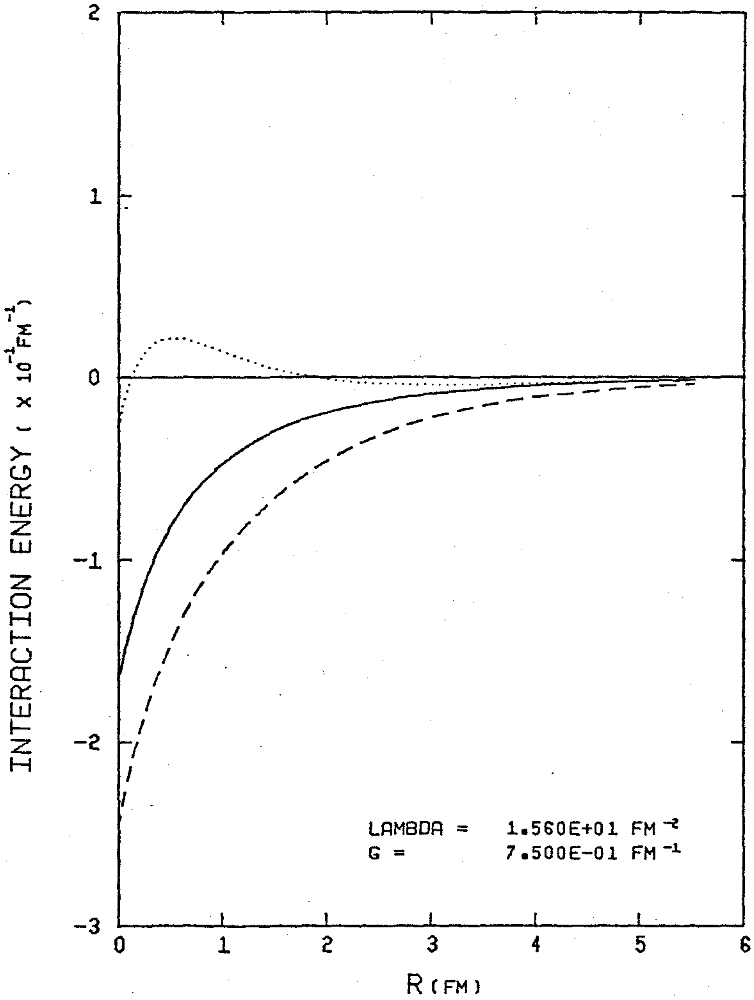
(———) exact calculation

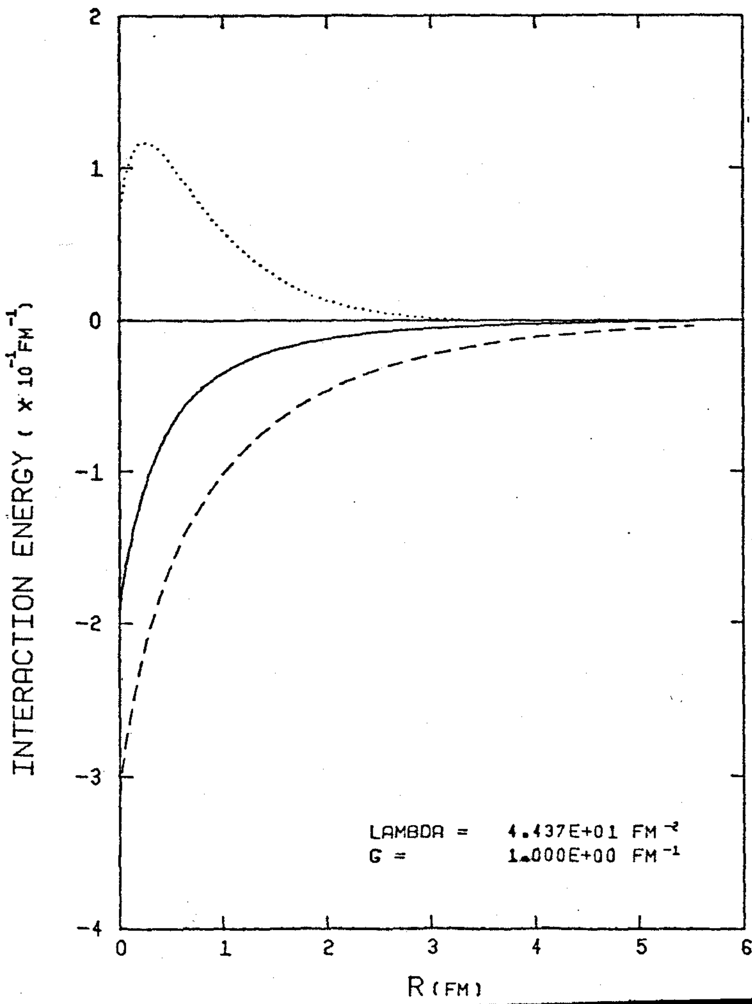
(····) using $\phi_2^{LC}(x)$

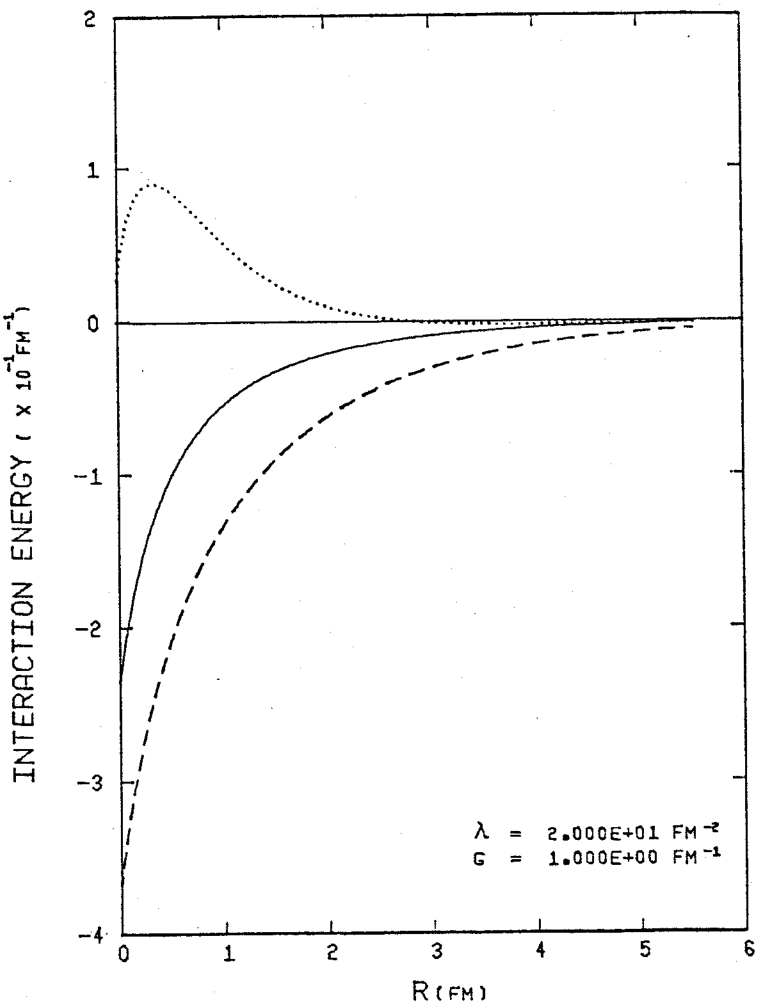
(----) $-g\phi_1(x)$.


INTERACTION ENERGY (x 10^-1 FM^-1)
R (FM)
LAMBDA = 2.923E-06 FM^-2
G = 2.457E-01 FM^-1


INTERACTION ENERGY (x 10^-1 FM^-1)
R (FM)
LAMBDA = 3.665E+00 FM^-2
G = 5.000E-01 FM^-1


INTERACTION ENERGY (x 10^{-1}FM^{-1})
R (FM)
LAMBDA = 1.560E+01 FM^{-2}
G = 7.500E-01 FM^{-1}


INTERACTION ENERGY (x 10^{-1}FM^{-1})
R (FM)
LAMBDA = 4.437E+01 FM^{-2}
G = 1.000E+00 FM^{-1}


INTERACTION ENERGY (x 10^-1 FM^-1)
R (FM)
λ = 2.000E+01 FM^-2
G = 1.000E+00 FM^-1

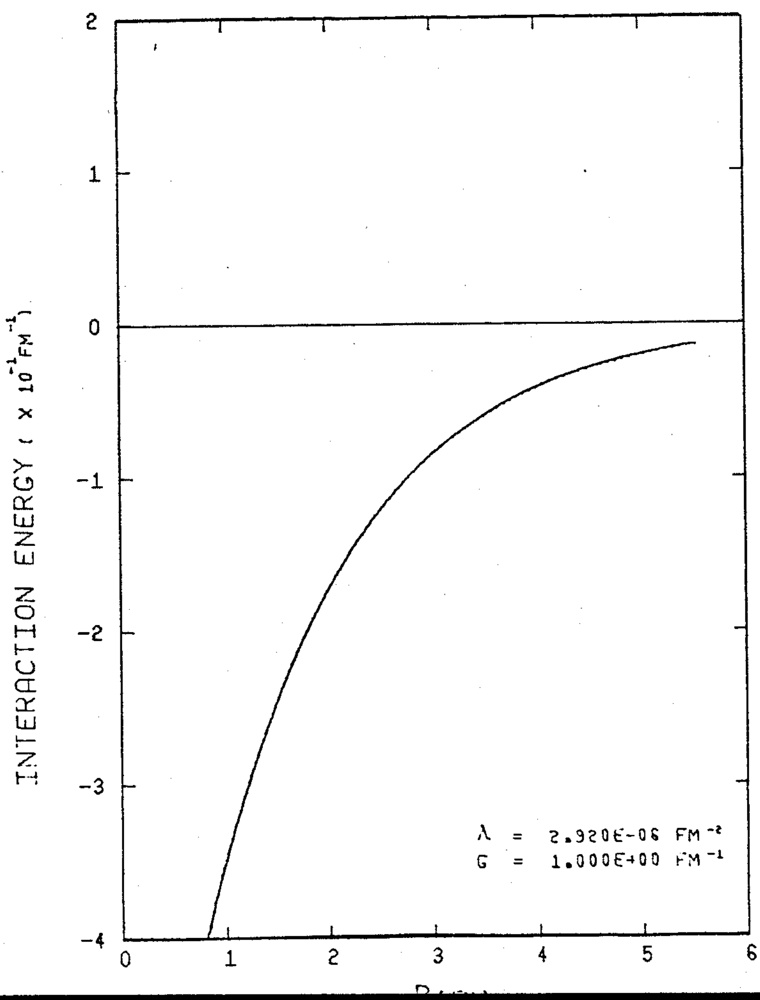
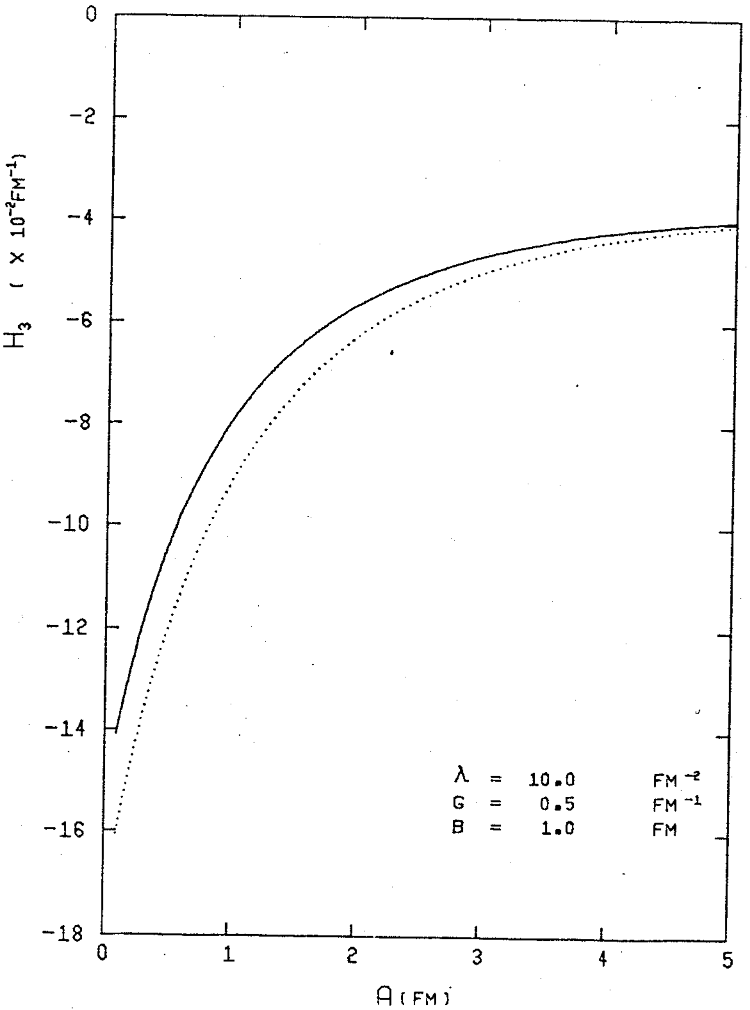

INTERACTION ENERGY (x 10^-1 FM^-1)
λ = 2.920E-06 FM^-2
G = 1.000E+00 FM^-1

Fig. 3 The three-nucleon interaction energy. The first diagram compares $H_3(a,b)$ with the sum of the three, two-body interactions:

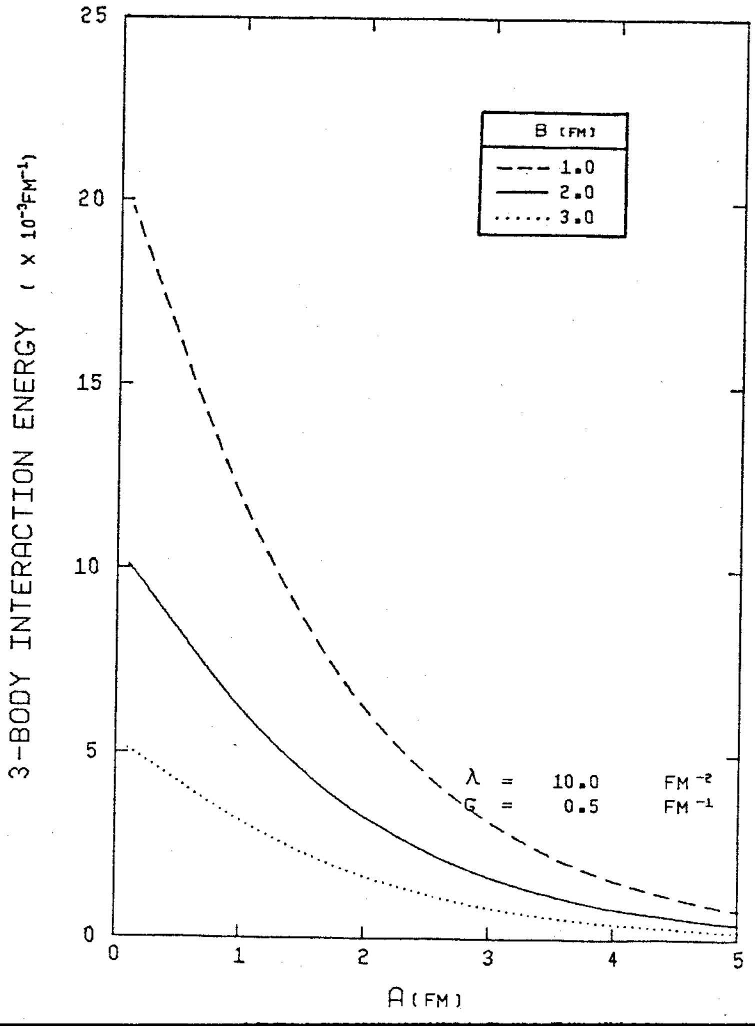
(———) $H_3(a,b)$

(••••••) $H_2(a) + H_2(b) + H_2(a+b)$.

The lower diagram shows $V_{123}(a,b)$ for various values of the internucleon separation, b.

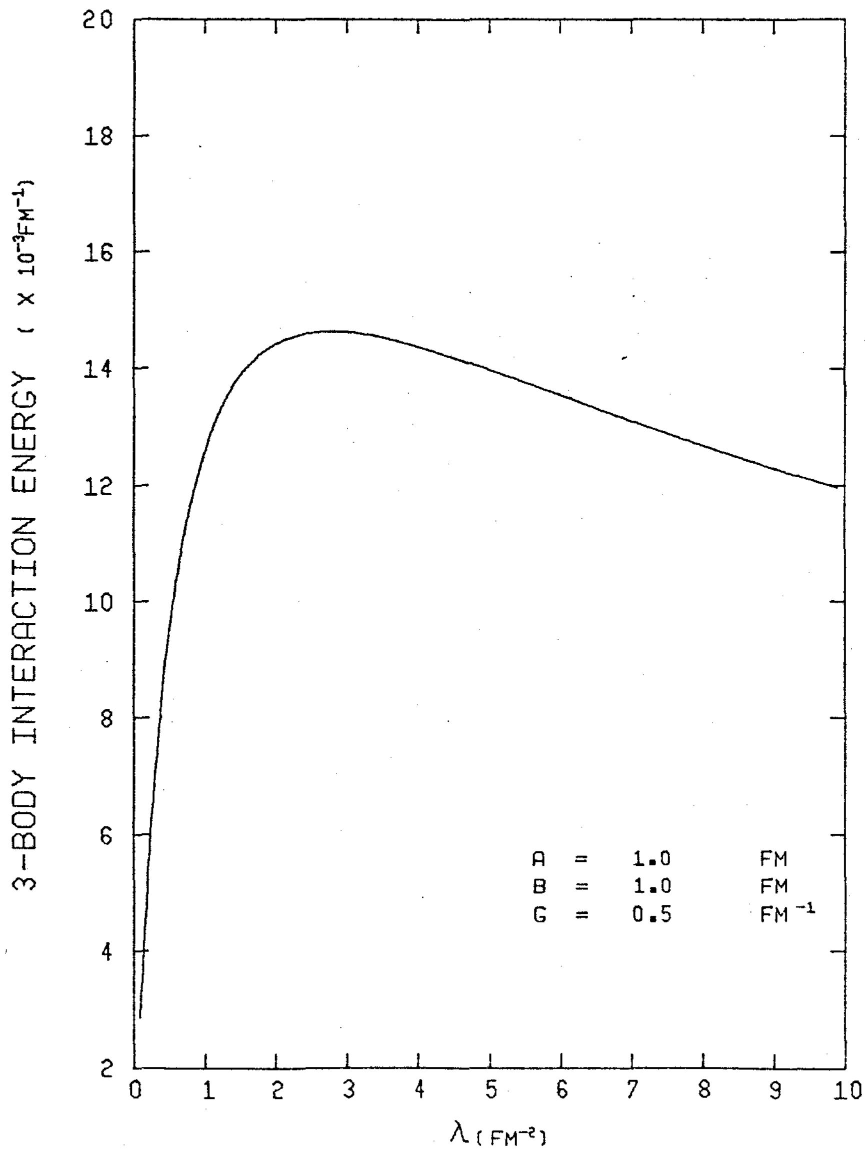


H3 (X 10^-2 FM^-1)
λ = 10.0 FM^-2
G = 0.5 FM^-1
B = 1.0 FM
A (FM)



3-BODY INTERACTION ENERGY (X 10^-3 FM^-1)
B (FM)
1.0
2.0
3.0
λ = 10.0 FM^-2
G = 0.5 FM^-1
A (FM)

Fig. 4 The three-nucleon interaction energy $V_{123}(a,b)$ as a function of the non-linear parameter λ.


3-BODY INTERACTION ENERGY (X 10⁻³FM⁻¹)
λ (FM⁻²)
A = 1.0 FM
B = 1.0 FM
G = 0.5 FM⁻¹
0 1 2 3 4 5 6 7 8 9 10
2 4 6 8 10 12 14 16 18 20

CHAPTER IV

POSSIBLE APPROXIMATION METHODS

The results of Chapter II suggest alternatives to calculating $\phi_2(x)$ and $H_2(r)$ directly when $\lambda \neq 0$. Guided by (2.19) and (2.20) we consider two approximations for the interaction energy of the two-nucleon system. The first gives $H_2(r)$ directly. In eq. (2.17), it is seen that $-g\,\phi_1(x)$ is the exact expression for the two nucleon interaction energy when the source term is a delta function. In the non-linear system this becomes

$$H(r) = -\frac{8m\gamma\alpha^2(1+\alpha^2)\,e^{-mr}}{(1-\alpha^2)^2(1-\alpha^2 e^{-2mr})}, \qquad (4.1)$$

with the constant α given by (3.16). It can be seen that for large r the expression reduces to the correct value, namely zero, and in the other limit,

$$H_2(0) = -\frac{16m^3\alpha^2(1+\alpha^2)}{\lambda(1-\alpha^2)^3}. \qquad (4.2)$$

Eq. (2.19) suggests the approximation

$$\phi_2^{LC}(x) = \phi_1(x-a) + \phi_1(x+a)\ , \qquad (4.3)$$

for the field amplitude of the two-nucleon system. $\phi_2^{LC}(x)$ is shown along with the exact result in Fig. 1. and as expected from (2.19), the agreement improves as λ is decreased.

This field can then be used in (3.38) to calculate H_2. Note that (3.28) is not satisfied. Using the example of (3.18), $H_2(r)$ may be calculated analytically. Some details of the calculation, and the result are given in Appendix C. The result, which was also computed numerically as a check, is shown in Fig. 2. As can be seen, this approximation gives poor results for large λ and small internucleon separation. For $r = 0$, $\phi_2^{LC}(x) = 2\phi_1(x)$ and

$$H_2(0) = 2H_1 + \frac{32m^2\alpha^4(3-\alpha^2)}{\lambda(1-\alpha^2)^3} . \qquad (4.4)$$

For the exact case, we cannot find a comparable expression for $H_2(0)$. The interaction energy of the two-nucleon system, in the limit of zero separation, can be found by evaluating the self-energy of a one-nucleon system with doubled source strength. For $\lambda = 0$

$$H_2(0) = -\frac{g^2}{2m} = 2H_1 \qquad (4.5)$$

but this is not generally true. The energy is given by (3.21) but the constant α now satisfies the relation

$$\phi'_{0_+} - \phi'_{0_-} = -2g , \qquad (4.6)$$

so that the limiting forms of $H_2(0)$ for the different approximations are best compared numerically.

A third approximation is suggested by comparison of the actual field amplitudes $\phi_2(x)$ and $\phi_2^{LC}(x)$ as in Fig. 1. From the plot it is seen that $\phi_2^{LC}(x)$ differs from the exact field amplitude by a factor that is approximately constant over the range shown, for a given separation r. It is in fact, this difference that is responsible for the deviation of the approximate energy from the correct value. We therefore consider an improved form of the previous approximation. For the field amplitude take

$$\phi_2^{V}(x,r) = c(r)\phi_2^{LC}(x,r) \qquad (4.7)$$

where c(r) is a constant with respect to x. Here, we explicitly include the r dependence of the fields. We may now investigate a means of obtaining c(r). The usual variational theorem states that the energy of a system is actually a minimum, which can be expressed as

$$\delta\langle H\rangle = 0. \qquad (4.8)$$

In our static approximation ($\dot{\phi} = 0$), the equivalent condition for field theory is

$$\delta \int_{\infty}^{\infty} \mathcal{H}(\phi,\phi')\,dx = 0. \qquad (4.9)$$

where $\mathcal{H}$ is the Hamiltonian density used in (3.38). This can be written in the form

$$\int_{-\infty}^{\infty} \{\frac{\partial H}{\partial \phi} - \frac{d}{dx}\frac{\partial H}{\partial \phi'}\}\delta\phi \; dx = 0 \; , \qquad (4.10)$$

which is satisfied if

$$\frac{\partial H}{\partial \phi} - \frac{d}{dx}\frac{\partial H}{\partial \phi'} = 0 \; . \qquad (4.11)$$

This is exactly eq. (3.5) which we solved to find $\phi(x)$, so we see that, in the static approximation, minimizing the energy is equivalent to solving the field equation. It is interesting to note that since

$$H = - L. \qquad (4.12)$$

eq. (4.11) is identical to (2.5) if we neglect the second term. In order to obtain the best value of c(r) or in general, any variational parameter of a trial field, we find the parameter which results in the lowest energy. We are then assured that the energy will be greater than or equal to the exact value depending on the trial field amplitude and variational parameter chosen. The accuracy of (4.7) can be seen by evaluating the two-nucleon interaction energy as done in the previous approximation. Comparison with the result of the exact calculation of eq. (3.38) shows that (4.7) is actually a very good choice for the trial field. The comparison is given numerically in Table 1 rather than graphically in Fig. 2 so that the difference is most apparent.

Schiff[1] also used eq. (4.3) as an approximation to the field amplitude. Since the exact form of the 3-D field was not known, it was not possible to determine the accuracy of the approximation, though it was thought that the potential would be attractive near the origin. The potential obtained using (4.3), however, was found to be repulsive for small r. From Fig. 2, it is apparent that, for certain values of g, the 1-D potential behaves similarly, and knowing the "exact" potential we observe that the approximation (4.3) is very poor. Schiff then suggested the use of a trial field similar to (4.7) for small r, and suggested that the potential was likely to become attractive. Our calculations indicate that the approximation is indeed very good, being able to reproduce the "exact" form of the potential. We conclude that, at least in one dimension, the variational approach using a trial field of the form (4.7), is a reasonable method of dealing with the non-linear meson field.

Table 1. Comparison of the Exact and Variational Interaction Energies

$g(fm^{-1})$	r (fm)	$-H_2(r)$ (fm^{-1})	$-H_2^V(r)$ (fm^{-1})
	0.001	0.190	0.181
	0.481	0.0725	0.0681
1.0	1.041	0.0334	0.0285
	2.561	0.00775	0.00528
	4.001	0.00265	0.00207
	0.001	0.165	0.159
	0.481	0.0837	0.0812
0.75	1.041	0.0458	0.0431
	2.561	0.0129	0.0113
	4.001	0.00464	0.00423
	0.001	0.125	0.123
	0.481	0.0859	0.0795
0.50	1.041	0.0510	0.0505
	2.561	0.0170	0.0167
	4.001	0.00631	0.00624

CHAPTER V

DISCUSSION AND CONCLUSIONS

We have found that we can solve the classical meson field equation for a system of neutral scalar mesons and any number of localized nucleon sources by reducing the problem to finding the solution of a system of simultaneous transcendental equations. The two-nucleon interaction can then be evaluated by numerically integrating the Lagrangian density. The results of most of the calculations done are best illustrated by the figures shown. First we consider the effects of the parameters g and λ. The nucleon source strength g, determines the meson field amplitude and the last four diagrams in Fig. 1b clearly show that increasing g results in an increase in the field amplitude. By comparing the first and last diagrams in Fig. 2, it is seen that this results in a stronger attraction between two nucleons as well. This is expected since we usually think of the mesons being responsible for the nucleon-nucleon attraction and an increased meson field amplitude is due to the presence of more mesons in the system. The effect of introducing a repulsive meson-meson interaction in the model is just the opposite. Increasing the strength of the repulsion, diminishes the meson field amplitude as can be seen from the first two diagrams in Fig. 1a. The last three diagrams of Fig. 2 show that this also reduces the two-nucleon interaction.

We investigate the many-body interaction by finding the energy of a three-nucleon system and subtracting the three two-body contributions. Fig. 3 shows that the three nucleon interaction is repulsive; the attraction between the three nucleons is less than that which would arise if only pair-wise interactions were present. As expected, the interaction decreases as the nucleons become separated. In Figure 4 it is seen that, for a given value of the nucleon source strength g, there is a specific value of λ for which the three-body interaction is a maximum. Like the two-nucleon interaction, V_{123} decreases (from the maximum) as λ is increased.

We have shown that a variational-like theorem holds when the static approximation is employed and that a linear combination of one-nucleon fields is a good trial field for the two-nucleon system. The two-nucleon interaction can then be evaluated quite accurately using a single variational parameter which is a function of the nuclear separation.

APPENDIX A

DETERMINATION OF THE PARAMETERS g AND λ

We consider nuclear matter in one dimension with a "volume" 2L. By comparing with a three dimensional (3-D) system, we obtain a relation between the density at nuclear saturation, and g and λ. Approximating nuclear matter as a zero-temperature gas, we have one nucleon per state in phase space, with all states occupied up to the Fermi momentum $p_f = \hbar k_f$. The total number of nucleons in the volume is

$$A = \int_{-p_f}^{p_f} \frac{2L \cdot 4dp}{2\pi\hbar} = \frac{8L\ p_f}{\pi\hbar} = \frac{8L\ k_f}{\pi}\ , \qquad (A.1)$$

where the factor 4 is due to the spin-isospin degeneracy of the nucleons. The density is

$$\rho = \frac{A}{2L} = \frac{4k_f}{\pi}\ . \qquad (A.2)$$

The kinetic energy is given by

$$T_N = \int_{-p_f}^{p_f} \frac{2L \cdot p^2}{2M} \cdot \frac{4dp}{2\pi\hbar} = \frac{4L \cdot p_f^3}{3M\pi\hbar} = \frac{\pi^2\hbar^2 L}{48M}\ \rho^3\ . \qquad (A.3)$$

In the classical approximation, the total energy of the system is

$$E = T_N + (-g\rho\Phi + \frac{m^2}{2}\ \Phi^2 + \frac{\lambda}{4}\ \Phi^4) \cdot 2L \qquad (A.4)$$

with $\Phi = \bar{\phi}(x)$; The last term is simply the product of the energy density and the volume. The energy per nucleon is then

$$\varepsilon = \frac{[\pi\hbar\rho]^2}{96M} - g\Phi + \frac{m^2\Phi^2}{2\rho} + \frac{\lambda\Phi^4}{4\rho} \quad . \tag{A.5}$$

As a first approximation, consider the simplified case, $\lambda = 0$. For the system to be stable, the energy per nucleon must be a minimum , and hence

$$\frac{\partial\varepsilon}{\partial\Phi} = -g + \frac{m^2\Phi}{\rho} = 0 \quad , \quad \text{and} \quad \Phi = \frac{g}{m^2}\rho \; . \tag{A.6}$$

Replacing this in (A.5) gives the energy of stable system with density ρ

$$\varepsilon(\rho) = \frac{[\pi\hbar\rho]^2}{96M} - \frac{g^2\rho}{2m^2} \quad . \tag{A.7}$$

The value of ρ for which this will be a minimum ρ_0 is given by

$$\frac{\partial\varepsilon(\rho)}{\partial\rho} = \frac{\pi^2\hbar^2\rho_0}{48M} - \frac{g^2}{2m^2} = 0 \; , \tag{A.8}$$

so that

$$\rho_0 = [\frac{g}{m\pi h}]^2 \; 24M \tag{A.9}$$

is the density of the saturated system. In 3-D however, there is no such ρ_0. The kinetic energy per nucleon is proportional to $\rho^{2/3}$ with the result that the second term in (A.7) dominates for large ρ and the energy of the system decreases as ρ increases so that the system collapses.

We now demand that this density correspond to that in the region of constant density of heavy nuclei. We must however make the comparison between the 1-D and 3-D systems. For a spherical Fermi surface the density is given by

$$\frac{A}{v} = \int_0^{p_f} \frac{4\pi p^2 dp}{(2\pi h)^3} = \frac{4\pi}{3} \frac{p_f^3}{(2\pi h)^3} = \frac{k_{fs}^3}{6\pi^2} , \qquad \text{(A.10)}$$

in 3 dimensions. A cubic Fermi surface results in

$$\frac{A}{v} = \int_{-p_f}^{p_f} \frac{d^3p}{(2\pi h)^3} = \frac{p_f^3}{(\pi h)^3} = \frac{k_{fc}^3}{\pi^3} , \qquad \text{(A.11)}$$

and for a given density

$$k_{fc} = (\tfrac{\pi}{6})^{1/3} k_{fs} . \qquad \text{(A.12)}$$

Using the density

$$\rho_o = 0.170 \text{ nucleons fm}^{-3} \qquad \text{(A.13)}$$

obtained from heavy nuclei, and remembering the degeneracy,

$$\rho_0 = \frac{2}{3} k_{fs}^3 \qquad \text{(A.14)}$$

so that $k_{fs} = 1.36 \text{ fm}^{-1}$. We now take the cubic Fermi surface as being more appropriate for comparison to a 1-D system so that $k_{fc} = 1.10 \text{ fm}^{-1}$ and $\rho_0 = 1.40$ nucleons fm^{-1}. Then using (A.9),

$$g_0 = 0.246 \text{ fm}^{-1}.$$

Using these values,

$$\varepsilon = -8.70 \text{ Mev or } -0.044 \text{ fm}^{-1} \text{ with } \hbar = c = 1.$$

This is not the same as the 15.68 Mev per particle predicted by the semiempirical mass formula which is not surprising since we do not expect a 1-D calculation to reproduce the energy of a 3-D system.

With $\lambda \neq 0$, the situation is more complicated. Again, minimize (A.5) with respect to Φ to obtain

$$\frac{\partial\varepsilon}{\partial\Phi} = -g + \frac{m^2\Phi}{\rho} + \frac{\lambda\Phi^3}{\rho} = 0 . \qquad (A.15)$$

Using the general result for the roots of a cubic equation, we find only one real root: (eg. see Abramowitz and Stegun[10])

$$\Phi = \{[1+[\frac{4m^6}{27g^2\rho^2\lambda} + 1]^{1/2}]^{1/3} + [1-[\frac{4m^6}{27g^2\rho^2\lambda} + 1]^{1/2}]^{1/3}\}\cdot(\frac{g\rho}{2\lambda})^{1/3} . \qquad (A.16)$$

We then replace this in (A.5) and differentiate with respect to ρ. Setting the result to zero and $\rho = \rho_0$ gives a relation between g and λ. Choosing a value for g, we then solve for λ.

As a method of fixing the remaining parameter g, we now compare the binding energy and "radius" of our two nucleon system to that of the deuteron. Schroedinger's equation

$$-\frac{\hbar^2}{2M}\frac{d^2}{dx^2}\psi + V(x)\psi = B_d\,\psi, \qquad (A.17)$$

with the potential V being taken as $H_2(r)$ calculated in Chapter III, is solved numerically, giving the wave function $\psi(x)$ and the binding energy B_d. The r.m.s. "radius" is then evaluated using the relation

$$R_d = \langle x^2 \rangle^{1/2} = \left[\int_{-\infty}^{\infty} \psi^*(x) x^2 \psi(x)\, dx\right]^{1/2} . \qquad \text{(A.18)}$$

Results for some values of g are given in Table 2, and Fig. 5 shows the form of $\psi(x)$. The values of the parameters given in the first two columns of Table 2 are related by the nuclear matter calculation. The energy per nucleon is minimized with respect to variation of the average field amplitude and nucleon density resulting in a single relation between the density and the two parameters. We then require that the Fermi level of our system correspond to that obtained from the Fermi gas model for heavy nuclei which fixes the nucleon density of the 1-D system at $\rho = 1.40$ nucleons fm^{-1}.[17] The nucleon source strength g, is arbitrarily chosen and λ is then determined. In several of the figures shown, values of g and λ which are not determined by the nuclear matter calculation are also used to illustrate the effect of the meson-meson interaction. From Table 2 it can be seen that there is no single value of g for which all these quantities can be fitted to the experimental values. It is clear, however, that although an alternate method of determining the model parameters could be employed to obtain better agreement, the lack of 1-D experimental values available for comparison limits the importance of actual values for the parameters.

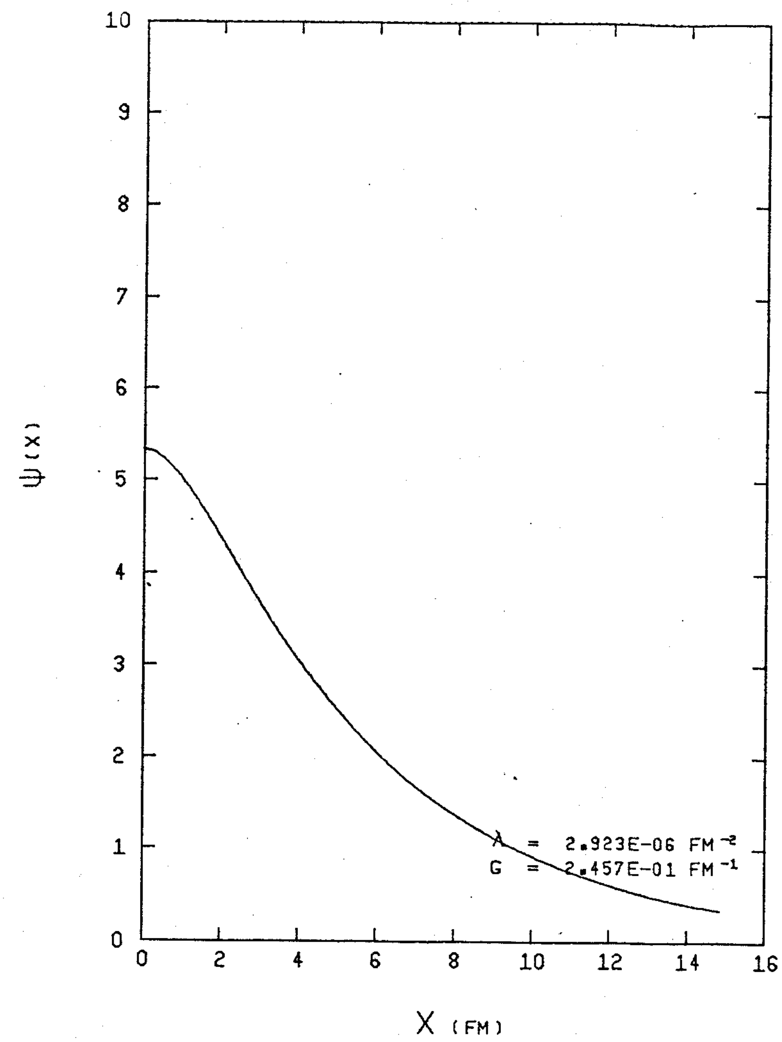
Table 2. Binding Energy and Radius of the "Deuteron"

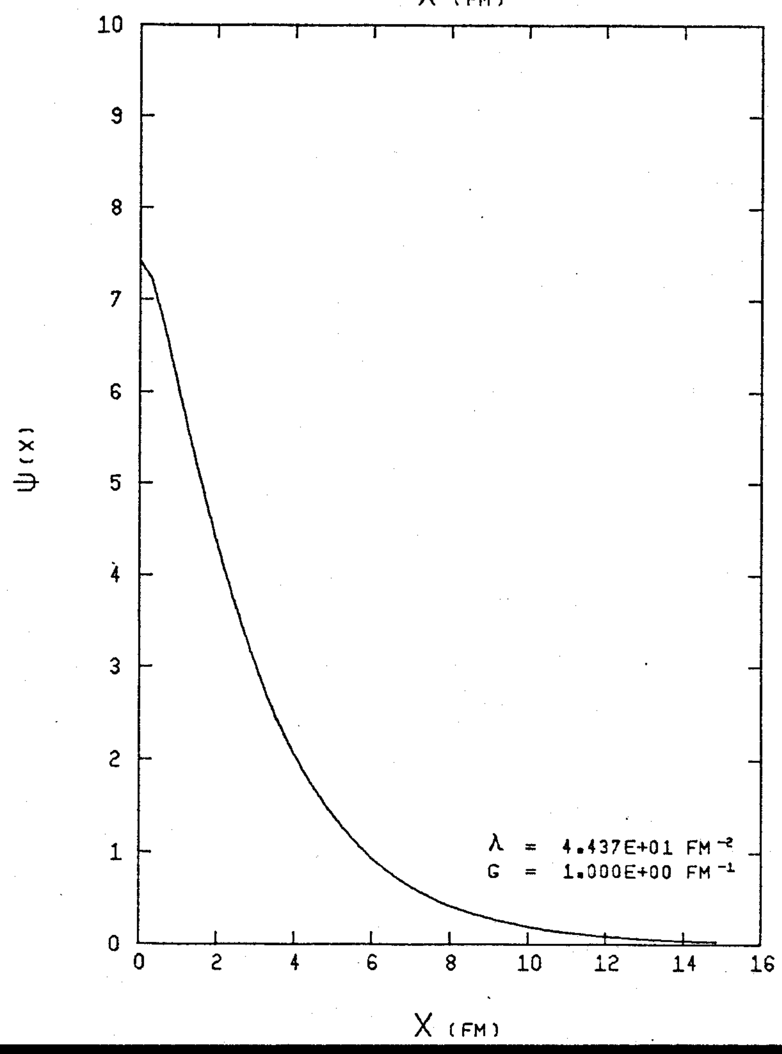
g (fm^{-1})	λ (fm^{-2})	B_d (Mev)	R_d (fm)	B/A † (Mev)
0.246	3×10^{-6}	1.73	3.64	8.31
0.30	0.39	3.02	2.91	12.81
0.40	1.58	5.57	2.26	19.32
0.50	3.67	7.24	2.02	24.09
0.75	15.60	7.64	1.90	31.33
1.00	44.37	6.50	1.97	35.02
Exp.	-	2.26	2.1*	15.68

†The last column gives the binding energy per nucleon as determined by the nuclear matter calculation.

*The experimental value taken as the radius of the deuteron depends on the definition used. It may vary by a few tenths of a Fermi, depending on whether the charge or structure radius is considered. (See e.g. Berard[16]).

Fig. 5 The wavefunction of a 1-D "deuteron"


λ = 2.923E-06 FM^-2
G = 2.457E-01 FM^-1
ψ(X)
X (FM)


λ = 4.437E+01 FM^-2
G = 1.000E+00 FM^-1
ψ(X)
X (FM)

APPENDIX B

Here, we give a short description of the Jacobian elliptic functions; these are well known and the description is intended solely for completeness. A full account of their properties may be found in reference 10.

For our purposes, we are interested in the definition given in terms of integrals[11],

$$u = \int_0^{\Phi} \frac{d\theta}{(1-m\sin^2\theta)^{1/2}} . \tag{B.1}$$

Φ is called the amplitude and

$$\Phi = \text{am } u . \tag{B.2}$$

From this define

$$\text{sn } u = \sin\Phi; \quad \text{cn } u = \cos\Phi \text{ and } \text{dn } u = (1-m\sin^2\Phi)^{1/2} . \tag{B.3}$$

The other functions are defined in terms of these; specifically

$$\text{nc } u = \frac{1}{\text{cn } u} . \tag{B.4}$$

When concerned about the parameter m, we write

$$\text{sn } u \equiv \text{sn}(u|m) . \tag{B.5}$$

The parameter m, must be between 0 and 1 to calculate the functions, but we can always transform such that this is satisfied[12]. These functions are doubly periodic with quarterperiods given by:

$$K = \int_0^{\pi/2} \frac{d\theta}{(1-m\sin^2\theta)^{1/2}} \quad ; \quad iK' = i\int_0^{\pi/2} \frac{d\theta}{(1-(1-m)\sin^2\theta)^{1/2}}$$

For m = 0 or 1, the more familiar forms

$$\begin{aligned} &sn(u|0) = \sin u; \quad cn(u|0) = \cos u \ ; \quad dn(u|0) = 1 \\ &sn(u|1) = \tanh u; \quad cn(u|1) = dn(u|1) = \text{sech } u \ , \end{aligned} \tag{B.7}$$

are obtained, as can easily be verified by evaluating (B.1). Finally[13]

$$\frac{d}{du}\,[nc(u|m)] = sc(u|m)dc(u|m) = \frac{sn(u|m)\,dn(u|m)}{cn^2(u|m)} \ . \tag{B.8}$$

To see how we arrive at (3.32), start with (B.1) then let $\theta = \sec^{-1}(\frac{t}{b})$ so that

$$\begin{aligned} u &= \int_b^{b\sec\Phi} [t(t^2-b^2)(1-m(\frac{t^2-b^2}{t^2}))]^{-1/2} dt \\ &= \int_b^{y} b[(t^2-b^2)(t^2-m(t^2-b^2))]^{1/2} dt \ ; \quad y = b\sec\Phi \\ &= b(1-m)^{-1/2}\int_b^{y} [(t^2-b^2)(t^2 + \frac{mb^2}{1-m})]^{-1/2} dt \ . \end{aligned} \tag{B.9}$$

Now make the substitution $m = a^2(a^2+b^2)^{-1}$, to get

$$u = (a^2+b^2)^{1/2}\int_b^{y} [(t^2-b^2)(t^2+a^2)]^{-1/2} dt \ . \tag{B.10}$$

Using (B.3) and (B.4)

$$nc(u|m) = \sec\Phi = \frac{y}{b} \tag{B.11}$$

or

$$u = nc^{-1}\left(\frac{y}{b}\,|m\right) . \tag{B.12}$$

This is exactly 16.4.49 of ref. 10. Eq. (3.30) can now be solved by letting

$$b^2 = \phi_0^2 \ ; \ a^2 = \frac{2m^2}{\lambda} + \phi_0^2 \quad \text{and} \quad y = \phi ,$$

so that (3.30) becomes

$$x = \frac{4}{\lambda^2}\left[2\phi_0^2 + \frac{2m^2}{\lambda}\right]^{-1/2} nc^{-1}\left(\frac{\phi}{\phi_0}\middle|\frac{\phi_0^2 + \frac{2m^2}{\lambda}}{2\phi_0^2 + \frac{2m^2}{\lambda}}\right) . \tag{B.13}$$

Here m is the meson mass; in III we write the parameter m as μ to distinguish it from the former. (B.13) gives exactly (3.32).

To calculate the Jacobian elliptic functions, we use the Arithmetic-Geometric Mean (A.G.M.)[14]. First calculate the A.G.M. scale:

$$a_0 = 1 \ ; \ b_0 = (1-m)^{1/2} \ ; \ c_0 = m^{1/2} \ ; \ m \text{ is the parameter.} \tag{B.14}$$

Then

$$a_{i+1} = \frac{1}{2}(a_i+b_i) \ ; \ b_{i+1} = (a_i b_i)^{1/2} \ ; \ c_{i+1} = \frac{1}{2}(a_i-b_i) . \tag{B.15}$$

continuing until $\frac{c_i}{a_i} < \varepsilon$ where ε is the desired accuracy; the number of iterations needed depends on m and the maximum is around

12. (For m = 1, we resort to (B.7).) Then find

$$\Phi_N = 2^N a_N u \tag{B.16}$$

and from

$$\sin(2\Phi_{N-1} - \Phi_N) = \frac{c_N}{a_N} \sin \Phi_N \tag{B.17}$$

calculate $\Phi_{N-1} \ldots \Phi_0$ whence

$$\mathrm{sn}(u|m) = \sin\Phi_0 \;, \quad \mathrm{cn}(u|m) = \cos\Phi_0$$

and (B.18)

$$\mathrm{dn}(u|m) = \cos\Phi_0/\cos(\Phi_1 - \Phi_0) \;.$$

APPENDIX C

The interaction energy is calculated using (2.21) which becomes

$$\int_{-\infty}^{\infty} [\tfrac{1}{2}\, \phi_2'^{LC^2} + \frac{m^2}{2}\, \phi_2^{LC^2} + \frac{\lambda}{4}\, \phi_2^{LC^4} - g\{(x-a) + (x+a)\}\phi_2^{LC}]dx \ . \quad (C.1)$$

First, observe that the direct terms give twice the self energy of a single nucleon field. i.e. $2H_1$; only the cross terms need to be explicitly evaluated. We are left with

$$\int_{-\infty}^{\infty} [\phi_+'\phi_-' + m^2\phi_+\phi_- + \lambda\{\phi_+^2\phi_-^2 + \tfrac{3}{2}(\phi_+^3\phi_- + \phi_+\phi_-^3)\}]-2g\phi_+(-a) \quad (C.2)$$

where $\phi_+ \equiv \phi_1(x-a)$ and $\phi_- \equiv \phi_1(x+a)$.

We now integrate over the two regions separately since:

$$\phi_+ = -\gamma^{1/2} \text{ csch } m(x-a+\beta) \ ; \quad x < a$$
$$\phi_+ = \gamma^{1/2} \text{ csch } m(x-a-\beta) \ ; \quad x > a$$
$$\phi_- = -\gamma^{1/2} \text{ csch } m(x+a+\beta) \ ; \quad x < -a$$
$$\phi_- = \gamma^{1/2} \text{ csch } m(x+a-\beta) \ ; \quad x > -a$$

with $\beta = \frac{1}{m}\, \ell n\, \alpha$.

Consider now, the two regions $|x| < a$ and $|x| > a$. In the first

$$\phi_+'\phi_-' = -m^2\gamma \frac{\cosh m(x+a-\beta)\cosh m(x-a-\beta)}{\sinh^2 m(x+a-\beta)\sinh^2 m(x-a+\beta)} \quad (C.3)$$

$$m^2\phi_+\phi_- = -\, m^2\gamma \frac{1}{\sinh m(x+a-\beta)\sinh m(x-a+\beta)} \tag{C.4}$$

$$\lambda\phi_+^2\phi_-^2 = \lambda\gamma^2 \frac{1}{\sinh^2 m(x+a-\beta)\sinh^2 m(x-a+\beta)} \tag{C.5}$$

$$\frac{3}{2}\lambda\{\phi_+^3\phi_- + \phi_+\phi_-^3\} = \frac{3}{2}\lambda\gamma^2 \frac{\{\sinh^2 m(x+a-\beta)+\sinh^2 m(x-a+\beta)\}}{\sinh^3 m(x+a-\beta)\sinh^3 m(x-a+\beta)} \tag{C.6}$$

Next, we sum (C.3,4,5) to get

$$m^2\gamma \frac{\{2-\sinh m(x+a-\beta)\sinh m(x-a+\beta)-\cosh m(x+a-\beta)\cosh m(x-a+\beta)\}}{\sinh^2 m(x+a-\beta)\sinh^2 m(x-a+\beta)}$$

which is reduced to

$$\frac{4m^2\gamma\{2-\cosh 2mx\}}{\{\cosh 2mx-\cosh 2m(a-\beta)\}^2} .$$

We integrate over $|x| < a$ using Gradshteyn and Ryzhik[15] 2.443 .1 and .3. The right hand side of equation (C.6) can be reduced to

$$\frac{12m^2\gamma\{\cosh 2mx\cdot\cosh 2m(a-\beta)-1\}}{\{\cosh 2mx - \cosh 2m(a-\beta)\}^3}$$

which can be integrated, again using Gradshteyn and Ryzhik 2.443. In a similar manner we evaluate for $|x| > a$. The final result is

$$H_2(r) = 2m\gamma\{\frac{4\sinh 2ma}{u^2} - \frac{2\sinh 2ma(3+\cosh 2m(a-\beta))}{\sinh^2 2m(a-\beta)\cdot u}$$

$$- \frac{(3\cosh 2m(a-\beta)+1)}{\sinh^2 2m(a-\beta)} \cdot I_1\} + 4m\gamma\{\frac{(3-\cosh 2ma)(1-\sinh 2m(a-\beta))}{\sinh^2 2ma\cdot u}$$

$$+ \frac{2\sinh 2m(a-\beta)}{u^2} + \frac{(1-3\cosh 2ma)}{\sinh^2 2ma} \cdot I_2\} - 2g\phi_+(-a), \qquad (C.7)$$

where $u = [\cosh 2ma - \cosh 2m(a-\beta)]$ and

$$I_1 = \int_{-2ma}^{2ma} \frac{dx}{[\cosh x - \cosh 2m(a-\beta)]}$$

$$I_2 = \int_{2m(a-\beta)}^{\infty} \frac{dx}{[\cosh x - \cosh 2ma]}$$

From Gradshteyn and Ryzhik 2.443.3

$$I_1 = \frac{2}{\sinh 2m(a-\beta)} \ell n\{\frac{T_1 + T_2}{T_1 - T_2}\}$$

and

$$I_2 = \frac{1}{\sinh 2ma} \ell n\{- \frac{T_3 + T_4}{T_3 - T_4} \cdot e^{-2ma}\} ,$$

where

$$T_1 = 1 - \cosh 2m(a-\beta)$$

$$T_2 = \sinh 2m(a-\beta) \tanh ma$$

$$T_3 = 1 - \cosh 2ma$$

$$T_4 = \sinh 2ma \tanh m(a-\beta) .$$

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