SOME CONTRIBUTIONS TO INFERENTIAL ISSUES OF CENSORED EXPONENTIAL FAILURE DATA

By

DONGHOON HAN, B.Sc., M.Sc.

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

June 2008

DOCTOR OF PHILOSOPHY (2008)	McMaster University
(Mathematics)	Hamilton, Ontario

TITLE:	Some Contributions to Inferential Issues of Censored Exponential Failure Data
AUTHOR:	Donghoon Han, B.Sc., M.Sc. (McMaster University)
SUPERVISOR:	Professor N. Balakrishnan
NUMBER OF PAGES:	xx, 185

Acknowledgements

First of all, I would like to express my sincere thanks to my supervisor, Dr. N. Balakrishnan without whose supervision, guidance, patience, and encouragement, this thesis would not have been possible. As well, I would like to thank my family whose constant love and invaluable support have helped me to continue my studies.

In addition to my supervisor, I would also like to extend my appreciation to Dr. R. Viveros-Aguilera, Dr. A. Childs and Dr. I. Akhundov for serving on my supervisory committee. Lastly, I would like to thank the Department of Mathematics and Statistics at McMaster University for providing a gracious learning environment and computing facilities as well as *Journal of Statistical Planning and Inference* for granting permission to reproduce the materials from the published articles as a part of this thesis.

Abstract

In this thesis, we investigate several inferential issues regarding the lifetime data from exponential distribution under different censoring schemes. For reasons of time constraint and cost reduction, censored sampling is commonly employed in practice, especially in reliability engineering. Among various censoring schemes, progressive Type-I censoring provides not only the practical advantage of known termination time but also greater flexibility to the experimenter in the design stage by allowing for the removal of test units at non-terminal time points. Hence, we first consider the inference for a progressively Type-I censored life-testing experiment with k uniformly spaced intervals. For small to moderate sample sizes, a practical modification is proposed to the censoring scheme in order to guarantee a feasible life-test under progressive Type-I censoring. Under this setup, we obtain the maximum likelihood estimator (MLE) of the unknown mean parameter and derive the exact sampling distribution of the MLE through the use of conditional moment generating function under the condition that the existence of the MLE is ensured. Using the exact distribution of the MLE as well as its asymptotic distribution and the parametric bootstrap method, we discuss the construction of confidence intervals for the mean parameter and their performance is then assessed through Monte Carlo simulations.

Next, we consider a special class of accelerated life tests, known as step-stress tests in reliability testing. In a step-stress test, the stress levels increase discretely at pre-fixed time points and this allows the experimenter to obtain information on the parameters of the lifetime distributions more quickly than under normal operating conditions. Here, we consider a k-step-stress accelerated life testing experiment with an equal step duration τ. In particular, the case of progressively Type-I censored data with a single stress variable is investigated. For small to moderate sample sizes, we introduce another practical modification to the model for a feasible k-step-stress test under progressive censoring, and the optimal τ is searched using the modified model. Next, we seek the optimal τ under the condition that the step-stress test proceeds to the k-th stress level, and the efficiency of this conditional inference is compared to the preceding models. In all cases, censoring is allowed at each change stress point $i\tau$, $i = 1, 2, \ldots, k$, and the problem of selecting the optimal τ is discussed using C-optimality, D-optimality, and A-optimality criteria.

Moreover, when a test unit fails, there are often more than one fatal cause for the failure, such as mechanical or electrical. Thus, we also consider the simple step-stress models under Type-I and Type-II censoring situations when the lifetime distributions corresponding to the different risk factors are independently exponentially distributed. Under this setup, we derive the MLEs of the unknown mean parameters of the different causes under the assumption of a cumulative exposure model. The exact distributions of the MLEs of the parameters are then derived through the use of conditional moment generating functions. Using these exact distributions as well as the asymptotic distributions and the parametric bootstrap method, we discuss the construction of confidence intervals for the parameters and then assess their performance through Monte Carlo simulations.

KEY WORDS: **A-optimality; accelerated life-testing; C-optimality; change-point; competing risks; conditional inference; conditional moment generating function; confidence interval; cumulative exposure model; D-optimality; exponential distribution; maximum likelihood estimation; order statistics; parametric bootstrap method; progressive Type-I censoring; step-stress model; tail probability; Type-I censoring; Type-II censoring**

List of Publications from the Thesis

- Balakrishnan, N., Han, D., 2008. Optimal step-stress testing for progressively Type-I censored data from exponential distribution. J. Statist. Plann. Inference, to appear.

- Balakrishnan, N., Han, D., 2008. Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring. J. Statist. Plann. Inference, to appear.

- Balakrishnan, N., Han, D., 2008. Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint. Comput. Statist. Data Analysis, revised.

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Chapter 1

Introduction

1.1 Accelerated life-test

In industrial and system engineering, reliability theory plays an important role. As the customers and users are constantly looking for products and systems with high quality and longer life, the reliability assessment or life-testing has become an essential and integral part of the manufacturing process. In order to guarantee the service life and performance of a product, or even to compare alternative manufacturing designs, life-testing under normal operating conditions is obviously most reliable. However, due to continual improvement in manufacturing design and technology, one often experiences difficulty in obtaining sufficient information about the failure time distribution of the products. As the products become highly reliable with substantially long life-spans, time-consuming and expensive tests are often required to collect enough failure data, which are necessary to draw inference about the relationship of lifetime with external stress variables (*i.e.*, covariates) such as temperature, pressure, vibration, and cycling rate. In such situations, standard life-testing methods are not suitable, especially when developing prototypes of potential new products. This difficulty is overcome by accelerated life-test (ALT) wherein the units are subjected to higher stress levels than normal in order to cause rapid and more failures in a short period of time. ALT is popular in industrial quality assurance and reliability engineering, and it allows the experimenter to apply more severe stresses to obtain information on the parameters of the lifetime distributions more quickly than would be possible under normal operating conditions. Some key references in the area of accelerated life-testing include Nelson and Meeker (1978), Nelson (1990), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2002).

There are three major types of stress loadings in ALT. They are:

- Constant-stress ALT: the selected high stress level is maintained until the end of the experiment;
- Step-stress ALT: the stress level is discretely increased on the surviving test units of an experiment at some pre-fixed time points;
- Progressive-stress ALT (also known as Linearly increasing stress ALT): the stress level is continuously (linearly) increased until the designed threshold level of the stress is reached,

and one can also create other possible variations by combining any of the modes described above. In this thesis, our focus is mainly on addressing some of the inferential issues related to the step-stress failure data, and so the next subsections are devoted

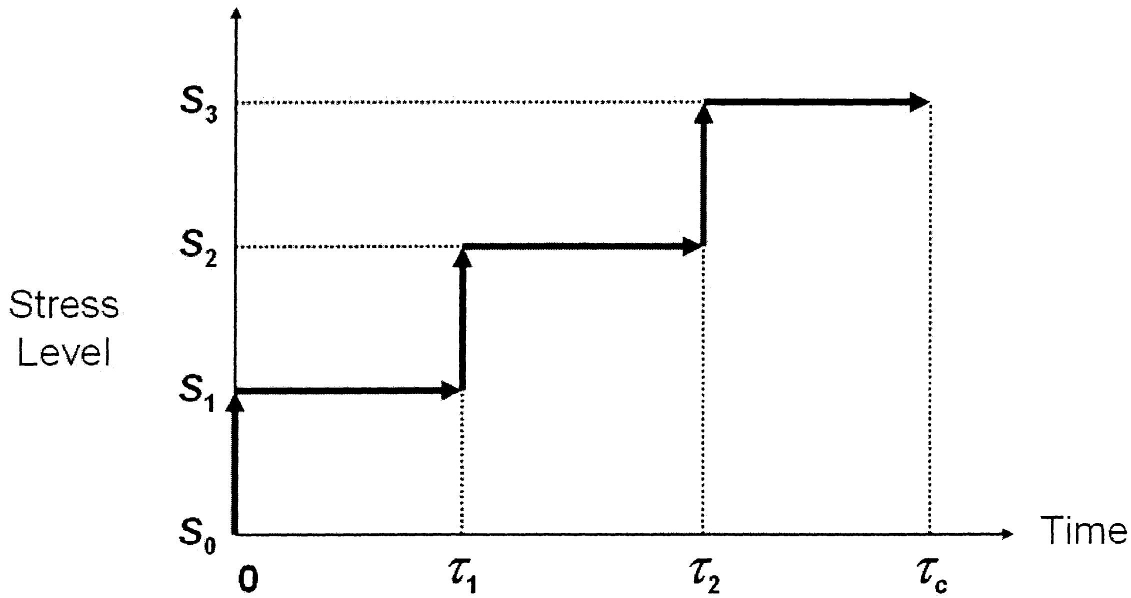

Figure 1.1: Stress Loading Scheme of a
3-level Step-stress Accelerated Life-test

to the current statistical modeling approach of the step-stress ALT in more detail.

1.1.1 Step-stress life-test

As mentioned briefly earlier, among various stress loading schemes, the step-stress ALT allows the experimenter to gradually increase the stress levels at some fixed time points during the experiment. We have graphically illustrated how the step-stress ALT proceeds in Figure 1.1. Here, s_0 is the stress level at the normal operating condition while s_1, s_2, s_3 are the design stress levels in order. The pre-determined stress change time points are denoted by τ_1 and τ_2, while τ_c denotes the termination (censoring) time point of this step-stress test. Under this setup, the life-test initially starts at the level s_1 until time τ_1 at which point the stress level changes to s_2. Then, the stress is maintained at that level until time τ_2 at which the stress level increases to s_3. Then, this final stress level is kept constant until the experiment is terminated at time τ_c.

Once the failure data are collected from an ALT, it is then necessary to come up with a physically motivated and reasonable statistical model in order to estimate the underlying distribution of failure times under normal condition, the reliability or some other characteristics about the product under study. Since it is usually required to extrapolate the data from higher stress levels to normal (lower) working stress, the model needs to explain the effect of rising stress levels on the remaining life-time of products. That is, it should relate the stress levels to the progress of the acceleration of failure. For analyzing the step-stress failure data, there are three models studied in the literature. They are:

- Tampered random variable model, proposed by DeGroot and Goel (1979);
- Cumulative exposure model, proposed by Sedyakin (1966) and further discussed and generalized by Bagdonavicius (1978) and Nelson (1980);
- Tampered failure rate model, proposed by Bhattacharyya and Zanzawi (1989) and later generalized by Madi (1993).

Rao (1992) has shown that for the step-stress ALT, if the life-time distributions under the two stress levels belong to the same scale-parameter family, then the cumulative exposure model is equivalent to the tampered random variable model. Also, if the life-time distribution belongs to a class of life distributions having the property of setting the clock back to zero, then the tampered failure rate model is equivalent to the tampered random variable model. For a concise review of step-stress ALT, one may refer to Gouno and Balakrishnan (2001).

1.1.2 Cumulative exposure model

Among the three models suggested for the step-stress ALT in the preceding subsection, our interest specifically lies in the cumulative exposure model. It assumes that the residual lifetime of units depends only on the current cumulative fraction failed and the current stress. That is, if the surviving units are held at the current stress, they will fail according to the cumulative distribution function (CDF) for that stress level but starting at the previously accumulated fraction failed. Hence, there is no memory of how the fraction has accumulated at the stress change time point. Since the key (implicit) assumption in analyzing the failure data from an ALT is that there is an effect of cumulative damage history due to the rising stress levels, this is well reflected in the cumulative exposure model.

Let us now formulate the distribution function of the life-time variable T of a test unit under the simple step-stress ALT (*i.e.*, two stress levels) with the single stress change point τ. Given F_i, the life-time distribution of a test unit when it is held at the constant stress level s_i for $i = 1, 2$, the CDF of T is derived according to the cumulative exposure assumption and it is given by

$$F(t) = \begin{cases} F_1(t) & \text{if } 0 < t < \tau \\ F_2(\xi + t - \tau) & \text{if } \tau \leq t < \infty \end{cases},$$

where ξ is the solution of $F_2(\xi) = F_1(\tau)$ $\big(viz., \xi = F_2^{-1}(F_1(\tau))\big)$. This implies that

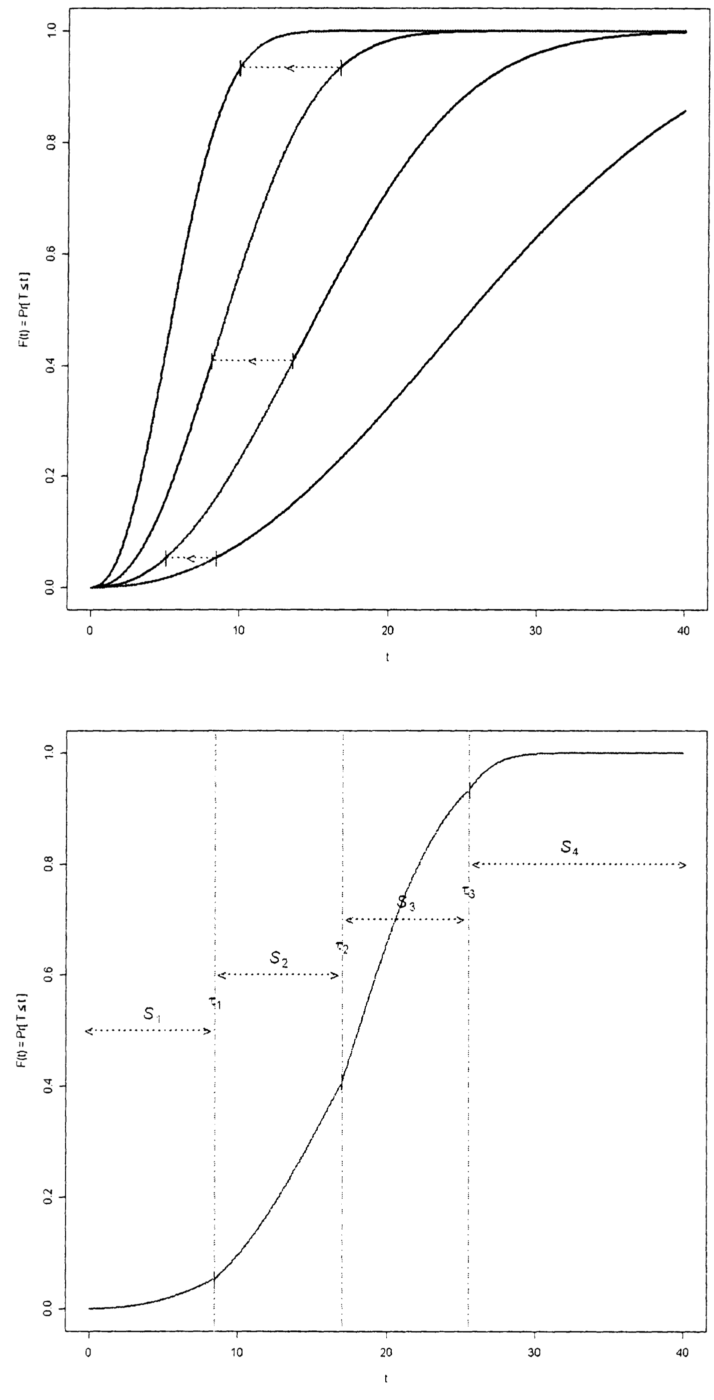

Figure 1.2: Life-time Distribution under the Cumulative Exposure Model for an Arbitrary 4-level Step-stress Test

when the stress level increases, the distribution is switched in such a way that the distribution at the higher level starts at the previously accumulated fraction. Thus, the continuity of the overall distribution is effectively preserved at each stress change time point.

We have graphically illustrated this aspect in Figure 1.2. It describes the cumulative distribution of the life-time of a test unit under an arbitrary 4-level step-stress ALT when the cumulative exposure model holds. In the upper plot of Figure 1.2, the four black solid lines correspond to the continuous CDFs of the life-time of a unit at constant stress levels. The steeper the distribution is, the higher the corresponding stress level is. The red line then depicts how the CDF under the step-stress condition is obtained according to the cumulative exposure model. Starting from the CDF at the lowest stress level, the distribution shifts horizontally to the CDF at the higher level whenever the stress change occurs. By joining these four segments, a continuous life-time distribution based on the cumulative exposure model is produced, which is presented in the lower plot of Figure 1.2.

1.2 Censored data

For any statistical analyses, a complete collection of data is the most favorable scenario prior to the actual analysis step as the inference made is considered relatively resistant to the uncertainty. In reality, however, statistical analysts and practitioners frequently encounter situations where the data are not all observable. Then, it is questionable whether comparable inference based on the incomplete sample can be developed analogous to the complete sample case. One type of such incomplete data which arises commonly in practice is censored data. Censored data arise when the experiments involving lifetimes of test units (*e.g.*, machines, products, patients, etc.) have to be terminated before collecting complete observations (*e.g.*, time to failure or death). For many pragmatic reasons such as cost reduction and time constraint, intentional censored sampling is unavoidable and it is a typical feature associated with the data especially from reliability testing and survival analysis. Ever since the necessity and importance of censoring have been recognized, properties of order statistics associated with various censoring schemes and inferential procedures based on censored samples have been studied extensively in the literature.

1.2.1 Type-I and Type-II censoring situations

Among various types of censoring, the two fundamental modes of (right) censoring which have been studied extensively in the literature are Type-I and Type-II censoring situations. Within these, right censoring is the most predominant and natural form. That is, failure observations are missing beyond the censoring time of the test (*i.e.*, to the right). Type-I right censoring occurs when the experiment is terminated at a prefixed time τ, independent of the failure times. Hence, no failures would be observed beyond this time point τ. This known termination time point makes Type-I censoring feasible for actual implementation and it provides a practical advantage when one designs a life-test. Prefixing the time of termination, however, makes the number of failures random and this may result in ineffective inference with high variability when the failure observations are too few or insufficient.

While Type-I censoring restricts the duration of the test, conventional Type-II censoring restricts the number of failures to be observed. As such, in Type-II right censoring, there would be a prefixed number r so that the experiment is terminated at the time of the r-th failure and all the remaining units are removed from the experiment. As one can see, in contrast to Type-I censoring, the number of failures is the pre-specified quantity for Type-II censoring while the time of termination is now random. This censoring scheme guarantees r failure observations and thus, it provides enormous help when one is planning a test. The principal disadvantage is, however, that the experimenter can not know in advance exactly how long it will take to complete the test since the test termination time is unknown for Type-II censoring. Hence, from a management point of view, Type-II censoring is a bit impractical and as a result, its application is less common than Type-I censoring in practice. For additional details and references regarding Type-I and Type-II censoring, one may refer to the early work of Epstein and Sobel (1953), Mann, Schafer and Singpurwalla (1974), Lawless (1982), Cohen and Whitten (1988), and Balakrishnan and Cohen (1991).

1.2.2 Progressive censoring situations

More recently, a generalized form of censoring called progressive censoring (PC) has been discussed in the literature. The concept of PC was first introduced by Herd (1956) in his Ph.D. thesis entitled *Estimation of the parameters of a population from a multi-censored sample*. The subject was further developed by Cohen (1963) and it has attracted considerable attention since then. The importance of PC lies in its efficient exploitation of the available resources compared to the traditional sampling. For an elaborate overview of various developments on PC, interested readers may refer to Cohen (1991), Balakrishnan and Aggarwala (2000), Viveros and Balakrishnan (1994), and the recent discussion paper by Balakrishnan (2007).

PC can also be either Type-I or Type-II, and in fact, it includes both the conventional Type-I and Type-II censoring situations as special cases. Progressively Type-I right censored samples are observed when a pre-specified number (or proportion) of unfailed units are continuously removed during the experiment at each predetermined time point until the time of termination is reached. On the other hand, progressive Type-II right censoring corresponds to the situation where a pre-specified number of surviving units are continuously withdrawn from the experiment at each failure time observed until the pre-determined number of units have failed from the life-test. Both censoring schemes provide greater flexibility to the experimenter in the design stage by allowing for the removal of operating test units at non-terminal time points of the test. Those withdrawn unfailed test units are typically used in other experiments in the same or at a different facility. If no intermediate censoring takes place but the censoring is allowed only at the terminal time point of an experiment, these PC schemes simply reduce to the conventional Type-I and Type-II censoring situations, respectively.

1.3 Scope of the thesis

In this thesis, we discuss some of the inferential issues related to life-time data from exponential distribution under several censoring schemes described in the last section. Each chapter has been composed in such a way that it is self-containing as much as possible and it stands alone with its own literature review whenever deemed appropriate.

We begin Chapter 2 with the exact inference for a progressively Type-I censored life-test with equi-spaced censoring points. For small to moderate sample sizes, we propose a simple modification to the censoring scheme for a feasible life-test under Type-I PC. Under this setup, the MLE of the unknown mean parameter is obtained and its exact sampling distribution is derived through the use of conditional moment generating function under the condition to ensure the existence of the MLE. Using the exact distribution of the MLE as well as its asymptotic distribution and the parametric bootstrap method, we discuss the construction of confidence intervals for the mean parameter and their performance is then assessed through Monte Carlo simulations. An example is also presented to illustrate all the methods of inference developed here.

In Chapter 3, we discuss the optimal progressive Type-I censoring scheme in the context of a k-level step-stress ALT with an equal step duration under the assumption of cumulative exposure model described earlier in Section 1.1. That is, censoring is allowed only at each stress change time point. For small to moderate sample sizes, we propose another suitable modification to the model previously considered by Gouno, Sen and Balakrishnan (2004), and the optimal step duration is searched for this model under C-optimality, D-optimality, and A-optimality criteria. Next, we discuss the determination of optimal step duration under the condition that the step-stress test proceeds to the k-th stress level, and the efficiency of this conditional inference is compared to that of the previous case.

When a test unit fails, there are often two or more fatal causes governing the failure mechanism of the unit. These are known as competing risks. In Chapter 4, we consider the simple step-stress model under Type-II censoring when these different risk factors are independently exponentially distributed, while the same situation is considered under Type-I censoring in Chapter 5. Applying the techniques developed in Chapter 2, we obtain the MLEs of the unknown mean parameters of the different risk factors in both cases. The exact distributions of the MLEs are then derived through the use of conditional moment generating functions. Subsequently, using these exact distributions along with the asymptotic distributions and the parametric bootstrap method, we discuss different ways to construct confidence intervals for the unknown parameters and then assess their performance through Monte Carlo simulations. The methods of inference discussed here are also illustrated with suitable examples.

Finally, in Chapter 6, we describe some interesting problems currently being investigated and worth considering for future research in the area of step-stress ALT under censoring.

Chapter 2

Exact Inference for Progressively Type-I Censored Exponential Failure Data

2.1 Introduction and motivation

We have discussed the general idea of progressive censoring situations in Section 1.2.2. Even though both Type-I and Type-II PC schemes have their own virtues and shortcomings, Type-I PC in comparison to Type-II PC provides a significant advantage of the known termination time point for a life-test, which makes Type-I PC quite appealing for actual implementation. However, despite such a practical benefit, most of the inferential work carried out in the literature of PC have mainly focused on Type-II rather than Type-I situation. This is because Type-I PC poses some difficulties in developing exact inference as well as in studying the theoretical properties of ordered failure times arising from such a censoring scheme, while Type-II PC possesses more tractable mathematical properties. In fact, this analytical predicament originates from the random nature of the failures occurring within each time interval, giving rise to a possibility that the life-test under Type-I PC may terminate before reaching the planned terminal stage, without yielding any failure observations, or both. Consequently, the inferential analysis for progressively Type-I censored data is approximation-based and numerical in nature; see, for example, Cohen (1963, 1966, 1975, 1976, 1991), Ringer and Sprinkle (1972), Wingo (1973, 1993), Cohen and Norgaard (1977), Nelson (1982), Gibbons and Vance (1983), Cohen and Whitten (1988), Balakrishnan and Cohen (1991), and Wong (1993). Gajjar and Khatri (1969) considered the Type-I PC situation in which at each censoring time the population parameters change, and discussed the corresponding inference for log-normal and logistic distributions. Sampford (1952) and London (1988) have studied the suitability of Type-I PC model for the case wherein patients randomly withdraw from a study before its termination. Chatterjee and Sen (1973) and Majumdar and Sen (1978) discussed nonparametric tests under Type-I PC, while Sinha and Sen (1982) considered clinical trials with staggered entry times and random withdrawals under the model of Type-I PC. Recently, Gouno, Sen and Balakrishnan (2004) and Han *et al.* (2006) have considered the model for a multiple step-stress test with exponential lifetimes under equi-spaced Type-I PC and discussed the problem of determining the optimal interval duration using several optimality criteria based on the Fisher information matrix.

In this chapter, our objective is to devise the method for exact inference regarding a life-test from which the available data are progressively Type-I censored. Here, we consider equi-spaced k intervals with τ denoting the uniform duration between the consecutive censoring time points. Under the assumption that the lifetime of each test unit is independently exponentially distributed, we obtain an explicit expression for the MLE of the unknown mean parameter in Section 2.2. For small to moderate sample sizes, a practical modification is suggested to the Type-I PC scheme in Section 2.3 in order to guarantee a feasible life-test under Type-I PC with an arbitrary number of censoring time points. In Section 2.4, we then derive the exact sampling distribution of the MLE through the use of conditional moment generating function under the condition that the existence of the MLE is ensured. Using the exact distribution of the MLE as well as its asymptotic distribution and the parametric bootstrap method, Section 2.5 discusses the construction of confidence intervals for the unknown mean parameter and their performance is then assessed through Monte Carlo simulations in Section 2.6. An example is presented in Section 2.7 to illustrate all the methods of inference discussed here. A brief concluding remark is made finally in Section 2.8.

2.2 Model description and MLE

In order to describe a life-testing procedure involving Type-I PC, we must first choose k ordered time points for censoring: $\tau_1 < \tau_2 < \cdots < \tau_k$. Now, for $i = 1, 2, \ldots, k$, let us denote n_i for the number of units failed in time interval $[\tau_{i-1}, \tau_i)$ and $y_{i,l}$ to be the l-th ordered failure time of n_i units during the i-th time interval, $l = 1, 2, \ldots, n_i$, while c_i denotes the number of units randomly removed or censored

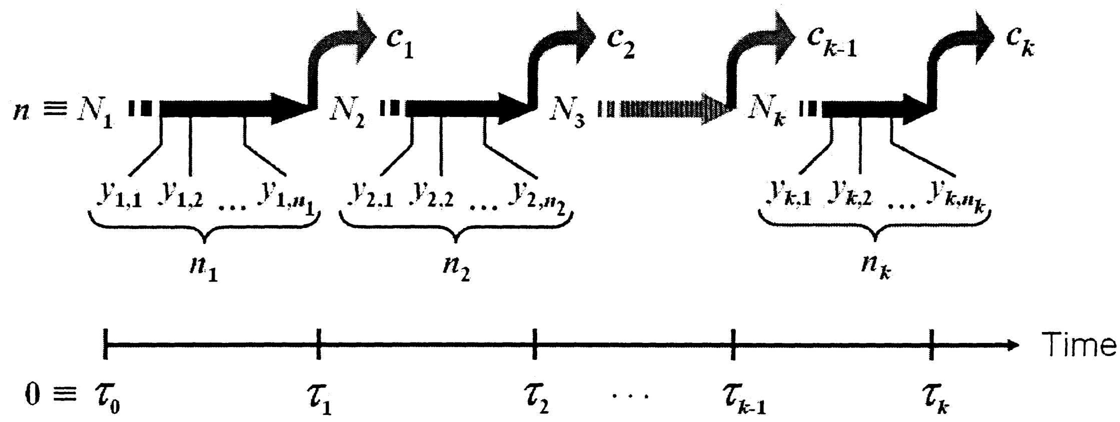

Figure 2.1: Schematic Representation of a Progressively Type-I Censored Life-test with k Censoring Time Points

at time τ_i. Furthermore, let N_i denote the number of units operating and remaining on test at the start of the i-th time interval $\left(viz., N_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} c_j\right)$.

Under this setup, a progressively Type-I censored life-testing experiment proceeds as follows (see Figure 2.1 for a diagrammatic illustration). A total of $N_1 \equiv n$ test units is initially placed at time $\tau_0 \equiv 0$ and tested until time τ_1 at which point c_1 live items are randomly withdrawn from the test. In this first time interval, a random number of n_1 failure times is also collected and the test is continued on $N_2 = n - n_1 - c_1$ units until time τ_2, at which point c_2 items are randomly withdrawn from the test, and so on. Finally, at time τ_k, all the surviving items are removed, thereby terminating the life-test. Note that since $n \equiv \sum_{i=1}^{k}(n_i + c_i)$, the number of surviving items at time τ_k is $c_k = n - \sum_{i=1}^{k} n_i - \sum_{i=1}^{k-1} c_i = N_k - n_k$. Obviously, when there is no intermediate censoring ($viz.$, $c_1 = c_2 = \cdots = c_{k-1} = 0$), this situation corresponds to a life-test under the conventional Type-I right censoring as a special case.

Under the assumption that the lifetime of a test unit follows an exponential distribution, the probability density function (PDF) and the cumulative distribution function (CDF) of the failure time of a test unit are given by

$$\begin{aligned} f(t) &= \frac{1}{\theta}\exp\left(-\frac{t}{\theta}\right), & t > 0, \quad (2.2.1)\\ F(t) &= 1 - S(t) = 1 - \exp\left(-\frac{t}{\theta}\right), & t > 0, \quad (2.2.2)\end{aligned}$$

respectively for $\theta > 0$. For convenience, no notational distinction will be made in this chapter between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$. Then, the joint probability density function (JPDF) of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k)$ with $\mathbf{y}_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is obtained as

$$f_J(\mathbf{y}, \mathbf{n}) = \left[\prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!}\right] \theta^{-D} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{k} U_i\right), \quad (2.2.3)$$

where

$$\begin{aligned} D &= \sum_{i=1}^{k} n_i, & (2.2.4)\\ U_i &= \sum_{l=1}^{n_i} (y_{i,l} - \tau_{i-1}) + (N_i - n_i)\Delta_i, & (2.2.5)\\ \Delta_i &= \tau_i - \tau_{i-1}, \qquad i = 1, 2, \ldots, k. \end{aligned}$$

Note that U_i in (2.2.5) is precisely the *Total Time on Test* statistic at the i-th time interval, while D in (2.2.4) is the total number of failure observations until τ_k. Now, using (2.2.3), the log-likelihood function of θ can be written as

$$l(\theta) = -D\log\theta - \frac{1}{\theta}\sum_{i=1}^{k} U_i$$

from which the MLE of θ is readily obtained as

$$\hat{\theta} = \frac{1}{D}\sum_{i=1}^{k} U_i. \quad (2.2.6)$$

We can easily see that the MLE of θ does not exist if $D = 0$. That is, at least one failure must be observed from the life-test in order to be able to estimate θ. This imposes the condition $D \geq 1$ in order to guarantee the existence of the MLE of θ. Since it is apparent from Corollary A.2 in Appendix A that

$$\lim_{\substack{n\to\infty \\ \tau_k\to\infty}} Pr\Big[D \geq 1\Big] = 1,$$

increase in the sample size asymptotically ensures the existence of $\hat{\theta}$. However, for a small sample size, this is not the case unless the experimenter is prepared with extremely long test duration, which is definitely not practical. Therefore, for a small sample size, which is common in reliability experiments, the analysis of the lifetime data under Type-I PC has to be a conditional one based on the condition that $D \geq 1$.

2.3 Progressively Type-I censored life-testing with small samples

Fully implementing the pre-determined PC scheme $\mathbf{c} = (c_1, c_2, \ldots, c_{k-1})$ in the model bears an inherent mathematical lapse since there is a positive probability that all the test units cease before reaching the planned k-th interval, resulting in an early termination of the life-test. For this reason, the assumption of a large sample size is required in order to guard enough surviving items to be censored at the end of each time interval and the analysis of progressively Type-I censored data has been approximately done under this assumption.

As mentioned in the previous section, however, in a reliability experiment, the sample size is usually small and there might be severe censoring due to various reasons such as budgetary constraints and facility requirements. Under such circumstances, the assumption of a large sample size will be unreasonable and therefore, a modification is necessary in order to guarantee a feasible PC scheme. One simple and natural modification is first to decide on a sequence of fixed numbers of unfailed items to be removed at the end of each time interval. Then, if, at any censoring time point during the test, the number of surviving items is less than or equal to the pre-determined number of items to be censored at that point, all the remaining items will be removed and the test is terminated. Since the number of live units at the end of each time interval before censoring takes place is random, the proposed change essentially makes the number of progressively censored units also random.

In order to revise the model according to the proposed modification, we first define a vector of non-negative integers

$$\mathbf{c}^* = (c_1^*, c_2^*, \ldots, c_{k-1}^*)$$

such that $\sum_{i=1}^{k-1} c_i^* < n$. Note that $\mathbf{c}^*$ is composed of fixed constants defining the (desired) number of surviving items to be removed at each censoring time point. Then, the actual number of censored items at the end of the i-th time interval is

$$c_i = \min\left\{c_i^*, N_i - n_i\right\} \qquad \text{for } i = 1, 2, \ldots, k-1.$$

Since all the remaining items are withdrawn from the test at τ_k, one could also state $c_k^* = n - \sum_{i=1}^{k-1} c_i^*$ so that $c_k = N_k - n_k$. When $c_i = N_i - n_i \geq 0$, the life-test terminates at the end of the i^*-th time interval, where i^* is the minimum of such i's satisfying $c_i = N_i - n_i$. Consequently, this results in $N_{i^*+1} = N_{i^*+2} = \cdots = N_k = 0$, $n_{i^*+1} = n_{i^*+2} = \cdots = n_k = 0$, and $c_{i^*+1} = c_{i^*+2} = \cdots = c_k = 0$ since

$N_{i+1} = N_i - n_i - c_i$. Hence, the proposed modification effectively allows the life-test to terminate earlier than scheduled whenever there are insufficient items remaining on the test. We should also point out that $\mathbf{c} = (c_1, c_2, \ldots, c_{k-1})$ is random under this setup. When $\mathbf{c}^* = (0, 0, \ldots, 0) = \mathbf{0}_{k-1}$, we also have $\mathbf{c} = \mathbf{0}_{k-1}$ and it is clear that this case corresponds to a life-test under the conventional Type-I right censoring. In addition, if $c_k > 0$ or $n_k > 0$ (equivalently, $N_k = n_k + c_k > 0$), it implies that the life-test has proceeded onto the last k-th time interval.

2.4 Conditional distribution of MLE

From here on, let us assume that the k time intervals of the life-test under Type-I PC are uniformly spaced (*viz.*, $\Delta_i = \tau > 0$ for $i = 1, 2, \ldots, k$). Then, in order to find the exact distribution of $\hat{\theta}$ under the condition $D \geq 1$, we first derive the conditional moment generating function (CMGF) of $\hat{\theta}$ denoted by $M_c(t)$. Using a simple conditioning argument, it can be expressed as

$$\begin{aligned} M_c(t) &= E\left[e^{t\hat{\theta}}\middle|D \geq 1\right] \\ &= \sum_{d=1}^{n} E\left[e^{t\hat{\theta}}\middle|D = d\right] \times Pr\left[D = d\middle|D \geq 1\right] \end{aligned} \tag{2.4.1}$$

for some t in the neighborhood of zero. The explicit expression of $M_c(t)$ is obtained in (A.7) using the lemmas and corollaries presented in Appendix A. Subsequently, by inverting $M_c(t)$, we can establish the following theorem regarding the conditional distribution of $\hat{\theta}$, the proof of which is also presented in Appendix A.

Theorem 2.4.1. *The conditional PDF of $\hat{\theta}$, given $D \geq 1$, is*

$$f_{\hat{\theta}}(x) = f_{\hat{\theta}}\big(x \big| D \geq 1\big) = \sum_{d=1}^{n} \sum_{\{\mathbf{n}:D=d\}} \sum_{j=0}^{d} C_{\mathbf{n},j}^{[\theta]} \ \gamma\Big(x - \tau_{\mathbf{n},j} \ ; \ d, \frac{d}{\theta}\Big), \tag{2.4.2}$$

where

$$\tau_{\mathbf{n},j} = \frac{\tau}{d}\Big(\sum_{i=1}^{k}(N_i - n_i) + j\Big), \tag{2.4.3}$$

$$C_{\mathbf{n},j}^{[\theta]} = \frac{(-1)^j}{1 - Pr\big[D = 0\big]}\Big[\prod_{i=1}^{k}\binom{N_i}{n_i}\Big]\binom{d}{j}\exp\Big\{-\frac{d}{\theta}\tau_{\mathbf{n},j}\Big\}, \tag{2.4.4}$$

$$\gamma(y \ ; \ \alpha, \lambda) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} & , \ y > 0 \\ 0 & , \ otherwise \end{cases} \qquad for \quad \alpha, \lambda > 0. \tag{2.4.5}$$

Corollary 2.4.1. *The conditional mean and variance of $\hat{\theta}$ are*

$$E\big[\hat{\theta}\big] = \theta + B_\theta(\hat{\theta})$$

and

$$Var\big[\hat{\theta}\big] = E\big[\hat{\theta}^2\big] - E^2\big[\hat{\theta}\big] = MSE_\theta(\hat{\theta}) - B_\theta^2(\hat{\theta}), \tag{2.4.6}$$

where the terms of bias and mean squared error are given by

$$B_\theta(\hat{\theta}) = \sum_{d=1}^{n} \sum_{\{\mathbf{n}:D=d\}} \sum_{j=0}^{d} C_{\mathbf{n},j}^{[\theta]} \ \tau_{\mathbf{n},j} \tag{2.4.7}$$

and

$$MSE_\theta(\hat{\theta}) = E\big[(\hat{\theta} - \theta)^2\big] = \sum_{d=1}^{n} \sum_{\{\mathbf{n}:D=d\}} \sum_{j=0}^{d} C_{\mathbf{n},j}^{[\theta]} \ \Big(\frac{\theta^2}{d} + \tau_{\mathbf{n},j}^2\Big), \tag{2.4.8}$$

respectively.

Proof. These expressions follow readily from (2.4.2). □

As presented above, the conditional distribution of $\hat{\theta}$, given $D \geq 1$, is a generalized mixture of gamma distributions. The expression for its mean clearly reveals that $\hat{\theta}$ is a biased estimator of θ. The expressions for the moments given in Corollary 2.4.1 can be used to calculate the standard error of the estimate. We can also derive an expression for the tail probability by integrating the conditional PDF of $\hat{\theta}$ given above. This expression, presented in the following corollary, is used to construct the exact confidence interval for θ later in Section 2.5.

Corollary 2.4.2. *The tail probability of $\hat{\theta}$ is given by*

$$Pr\Big[\hat{\theta} > \xi\Big] = \sum_{d=1}^{n} \sum_{\{\mathbf{n}:D=d\}} \sum_{j=0}^{d} C_{\mathbf{n},j}^{[\theta]} \, \Gamma\Big(\frac{d}{\theta}\langle \xi - \tau_{\mathbf{n},j}\rangle \; ; \; d\Big), \qquad (2.4.9)$$

where

$$\begin{aligned} \langle\, \epsilon \,\rangle &= \max\{0, \epsilon\}, \\ \Gamma(\epsilon \; ; \; \alpha) &= \begin{cases} \displaystyle\int_{\epsilon}^{\infty} \gamma(y \; ; \; \alpha, 1)\, dy = \int_{\epsilon}^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy & , \; \epsilon > 0 \\ 1 & , \; \epsilon \leq 0 \end{cases} . \end{aligned}$$

2.5 Confidence intervals

In this section, we discuss different methods of constructing confidence intervals (CIs) for the unknown parameter θ. Based on the exact conditional distribution of the MLE from Theorem 2.4.1, we can construct the exact CI for θ. Since the exact conditional distribution of $\hat{\theta}$ is quite complicated, we also present the approximate CI for θ based on the asymptotic distribution of the estimator for a large sample size. Finally, the parametric bootstrap method is used to construct a CI for θ.

2.5.1 Exact confidence interval

In order to guarantee the invertibility of the pivotal quantity for the parameter θ, we assume that the tail probability of $\hat{\theta}$ presented in Corollary 2.4.2 is a monotonically increasing function of θ. Several authors including Chen and Bhattacharya (1988), Gupta and Kundu (1998), Kundu and Basu (2000), Childs *et al.* (2003), Balakrishnan *et al.* (2007), Balakrishnan and Xie (2007a,b), and Balakrishnan *et al.* (2008) have all used this approach for constructing exact CIs in different contexts. Like these authors, we are unable to establish the required monotonicity in an analytical way due to the complex structure of the pivotal quantity in (2.4.9). However, the extensive numerical computations seem to support this monotonicity assumption (see Figure 2.2).

Let us now construct the exact $100(1-\alpha)\%$ CI for θ. We first denote θ_L and θ_U to be the lower and upper bounds of the two-sided CI for θ, respectively. Then, by the monotonicity assumption, they are the unique solutions of the equations

$$Pr\left[\hat{\theta} > \tilde{\theta}^{obs}\right] = \frac{\alpha}{2}$$

and

$$Pr\left[\hat{\theta} > \tilde{\theta}^{obs}\right] = 1 - \frac{\alpha}{2},$$

respectively, where $\tilde{\theta}^{obs}$ is the bias-corrected observed value of the MLE of θ. That is, $\tilde{\theta}^{obs} = \hat{\theta}^{obs} - \hat{B}^{obs}_{\theta}(\hat{\theta})$, where $\hat{\theta}^{obs}$ is simply the observed value of the MLE of θ and $\hat{B}^{obs}_{\theta}(\hat{\theta})$ is the observed value of the bias $B_{\theta}(\hat{\theta})$ given in (2.4.7) at $\theta = \hat{\theta}^{obs}$. One remark to be made is that if one wants to be mathematically strict, $\hat{\theta}^{obs}$ should be used rather than $\tilde{\theta}^{obs}$ in the above two equations for obtaining θ_L and θ_U. In many sampling situations, however, extremely slow increase of $Pr\left[\hat{\theta} > \hat{\theta}^{obs}\right]$ with respect to θ does not enable us to find θ_U in reasonable time and range. Hence, because of this empirical reason, the use of $\tilde{\theta}^{obs}$ is recommended in order to obtain the two-sided CI for θ as this adjustment enhances the steepness of the probability function without deteriorating the performance of the exact CI.

Since θ_L and θ_U can not be expressed in an explicit closed form, they are numerically obtained by solving the following two non-linear equations using some iterative techniques such as the bisection method, Newton-Raphson method or Brent's method:

$$\frac{\alpha}{2} = \sum_{d=1}^{n} \sum_{\{\mathbf{n}:D=d\}} \sum_{j=0}^{d} C^{[\theta_L]}_{\mathbf{n},j} \, \Gamma\left(\frac{d}{\theta_L} \langle \tilde{\theta}^{obs} - \tau_{\mathbf{n},j} \rangle \; ; \; d \right), \tag{2.5.1}$$

$$1 - \frac{\alpha}{2} = \sum_{d=1}^{n} \sum_{\{\mathbf{n}:D=d\}} \sum_{j=0}^{d} C^{[\theta_U]}_{\mathbf{n},j} \, \Gamma\left(\frac{d}{\theta_U} \langle \tilde{\theta}^{obs} - \tau_{\mathbf{n},j} \rangle \; ; \; d \right), \tag{2.5.2}$$

where $\tau_{\mathbf{n},j}$, $C^{[\theta]}_{\mathbf{n},j}$ and $\Gamma(\cdot \; ; \alpha)$ are as defined earlier. Note that the coefficients $C^{[\theta]}_{\mathbf{n},j}$ in the above two equations are functions of θ. Hence, before solving for the confidence limits, we replace θ in $C^{[\theta]}_{\mathbf{n},j}$ in an appropriate manner. That is, θ_L is substituted for θ in $C^{[\theta]}_{\mathbf{n},j}$ of (2.5.1) and likewise θ_U for θ in $C^{[\theta]}_{\mathbf{n},j}$ of (2.5.2).

2.5.2 Approximate confidence interval

As the sample size grows, the MLE exhibits some special characteristics which are asymptotically optimal. First of all, under certain regularity conditions, the MLE is asymptotically unbiased and efficient. That is, its bias tends to zero and its variance achieves the Cramer-Rao lower bound as the sample size increases to infinity. Furthermore, its distribution approaches that of a normal with the variance given by the inverse of Fisher information; see Silvey (1975), and Casella and Berger (2002) for details. Thus, inference about the unknown parameter can be based on the asymptotic normality of the MLE. In this subsection, we present an approximate method to construct the CI for θ using these properties of the MLE for large sample sizes. Although the exact method described in the preceding subsection is preferable, its computation encounters some difficulties for large samples. On the other hand, the approximate method provides not only the computational ease but also reasonable probability coverage (close to the nominal level) when the sample size is large. This finding is further discussed in Section 2.6.

First, the observed Fisher information of θ is given by

$$\begin{aligned}\mathbf{I}_O(\theta) &= -\left.\frac{d^2 l(\theta)}{d\theta^2}\right|_{\theta=\hat{\theta}} \\ &= \left.\left[-\frac{D}{\theta^2}+\frac{2}{\theta^3}\sum_{i=1}^{k} U_i\right]\right|_{\theta=\hat{\theta}} = \frac{D}{\hat{\theta}^2}\end{aligned}$$

with U_i as defined in (2.2.5). Upon inverting, we obtain the asymptotic variance of $\hat{\theta}$ as $V_O = \mathbf{I}_O^{-1}(\theta) = \hat{\theta}^2/D$. Since $\hat{\theta}$ is asymptotically unbiased for θ, we can then use $(\hat{\theta}-\theta)/\sqrt{V_O}$ as the pivotal quantity for θ to construct two-sided $100(1-\alpha)\%$

approximate CI for θ, which is given by

$$\left(\max\left\{0, \hat{\theta} - z_{\alpha/2}\sqrt{V_O}\right\},\ \hat{\theta} + z_{\alpha/2}\sqrt{V_O} \right),$$

where $z_{\alpha/2}$ is the $(1-\alpha/2)$-th quantile of a standard normal distribution.

2.5.3 Bootstrap confidence interval

In this subsection, we construct the CI for θ using a parametric bootstrap method, *viz.*, the bias-corrected and accelerated (BCa) percentile bootstrap method; see Efron (1987), Hall (1988), and Efron and Tibshirani (1993) for details. Compared to the ordinary percentile bootstrap intervals or the Studentized-t bootstrap intervals, the BCa percentile bootstrap intervals are known to perform better. Before we obtain the BCa percentile bootstrap CI for θ, the following algorithm is implemented to generate the bootstrap sample of size B based on the original progressively Type-I censored sample of size D:

Step 1 Given the initial sample size n, the k censoring time points (*i.e.*, $i\tau$ for $i = 1, 2, \ldots, k$), the desired censoring scheme $\mathbf{c}^* = (c_1^*, c_2^*, \ldots, c_{k-1}^*)$ with $c_k^* = n - \sum_{i=1}^{k-1} c_i^*$, and the original progressively Type-I censored sample of size D, calculate $\hat{\theta}$, the MLE of θ, from (2.2.6).

Step 2 Generate a simple random sample of size n from exponential distribution with mean parameter $\hat{\theta}$ obtained from Step 1, and sort them in an ascending order.

Step 3 Initialize $i = 1$ and $N_1^* = n$ with $N_2^* = N_3^* = \cdots = N_k^* = 0$ and $n_1^* = n_2^* = \cdots = n_k^* = 0$.

Step 4 Count the number of the ordered sample points in the interval $[(i-1)\tau,\ i\tau)$ and store it as n_i^*. Let $\mathbf{y}^*{}_i = (y_{i,1}^*, y_{i,2}^*, \ldots, y_{i,n_i^*}^*)$ be those collected sample points in the given interval.

Step 5 After taking out n_i^* selected sample points in Step 4, further reduce the pool of the sample points by randomly choosing and removing c_i^* points from the remaining $N_i^* - n_i^*$ points if $N_i^* - n_i^* > c_i^*$. Otherwise, terminate this loop and go to Step 7.

Step 6 Assuming $N_i^* - n_i^* > c_i^*$ in the previous step, the reduced sample now contains $N_{i+1}^* = N_i^* - n_i^* - c_i^*$ ordered points. Unless $i = k$, increment i by 1 (*i.e.*, $i = i+1$) and repeat the procedure from Step 4.

Step 7 Based on the simulated progressively Type-I censored observations $\mathbf{y}^* = (\mathbf{y}^*{}_1, \mathbf{y}^*{}_2, \ldots, \mathbf{y}^*{}_k)$, calculate the new MLE of θ, denoted by $\hat{\theta}^*$, from (2.2.6).

Step 8 Repeat Steps 2-7 B times. Then, arrange all the values of $\hat{\theta}^*$ in an ascending order to obtain the bootstrap sample of

$$\left\{\hat{\theta}^{*[1]} < \hat{\theta}^{*[2]} < \cdots < \hat{\theta}^{*[B]}\right\}.$$

With the bootstrap sample generated as above, we now obtain the two-sided $100(1-\alpha)\%$ BCa percentile bootstrap CI for θ as

$$\left(\hat{\theta}^{*[\alpha B]},\ \hat{\theta}^{*[\beta B]}\right),$$

where

$$\alpha = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 - z_{\alpha/2}}{1 - \hat{a}(\hat{z}_0 - z_{\alpha/2})}\right)$$

and

$$\beta = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{\alpha/2}}{1 - \hat{a}(\hat{z}_0 + z_{\alpha/2})}\right);$$

see Efron and Tibshirani (1993). Here, $\Phi(\cdot)$ denotes the CDF of the standard normal distribution and the value of the bias-correction $\hat{z}_0$ is given by

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\sum_{j=1}^{B} J\left(\hat{\theta}^{*[j]} < \hat{\theta}\right)}{B}\right),$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal CDF and $J(\cdot)$ is an indicator function that takes on the value 1 if the argument is true and 0 otherwise. A good estimate of the acceleration factor a is suggested to be

$$\hat{a} = \frac{\sum_{j=1}^{D}\left(\hat{\theta}^{(\cdot)} - \hat{\theta}^{(j)}\right)^3}{6\left\{\sum_{j=1}^{D}\left(\hat{\theta}^{(\cdot)} - \hat{\theta}^{(j)}\right)^2\right\}^{3/2}},$$

where $\hat{\theta}^{(j)}$ is the MLE of θ based on the original progressively Type-I censored sample with the j-th observation deleted (*i.e.*, the jackknife estimate) for $j = 1, 2, \ldots, D$ and

$$\hat{\theta}^{(\cdot)} = \frac{1}{D}\sum_{j=1}^{D}\hat{\theta}^{(j)}.$$

2.6 Numerical study

In order to evaluate the performance of all the different methods of constructing CIs discussed in Section 2.5, a Monte Carlo simulation study was carried out and the results are detailed in this section. In particular, the study is to examine a scenario in which a practitioner has to deal with small sample sizes and high censoring proportions, bringing the life-test to termination earlier than scheduled with a high probability. Hence, the value of the mean parameter was chosen to be $\theta = 6.0$ with the initial sample size n being 10 and 20. For the purpose of illustration, we also considered several different choices for the uniform interval duration τ with the number of the planned censoring points k ranging from 2 to 4. For fixed n and k, the pre-fixed Type-I PC scheme was determined by a given censoring proportion $0 < \pi^* < 1$ so that $c_i^* = n\pi^*$ for $i = 1, \ldots, k-1$. Based on 1000 Monte Carlo simulations with $B = 1000$ bootstrap replications, the true coverage probabilities of the 90%, 95% and 99% CIs for θ were then determined. The results are presented in Tables 2.1-2.5 along with the estimated mean widths and bounds of the CIs from this simulation.

Although the crude approximate method based on the asymptotic normality of the MLE is quick and easy, one major problem associated with it is that it does not necessarily take the parameter space into account when constructing the CI. There is no built-in procedure to prevent this and as a result, the lower bounds of the approximate CIs frequently hit below zero for small sample sizes or for high levels of confidence even though the parameter θ can take only a positive value in this setting. In order to turn such intervals into sensible ones, the negative lower bounds were all replaced by zero in Tables 2.3-2.5.

From Table 2.1, we clearly see that the exact method performs very well as its CIs attain the actual coverage probabilities close to the nominal levels. Similar behavior is also observed for the CIs based on the BCa bootstrap method. However, the performance of the approximate CIs is unsatisfactory for a small sample size as their actual coverage probabilities are well below the specified nominal levels in most cases. A possible explanation for this may rely on the high degree of skewness for the exact distribution of $\hat{\theta}$ and hence, a much larger sample size is required to justify the use

Table 2.1: Estimated Coverage Probabilities (in %)

based on 1000 Simulations with $\theta = 6.0$ and $B = 1000$

Nominal CL				90%			95%			99%		
n	k	π^*	τ	Exact	Approx	Boot	Exact	Approx	Boot	Exact	Approx	Boot
10	2	60%	3	90.4	85.8	90.6	95.4	87.5	95.8	99.4	94.1	98.2
			4	91.6	87.9	91.2	95.4	90.3	94.4	99.6	95.2	99.0
			5	91.2	89.2	87.8	97.2	90.8	94.6	99.8	95.1	98.2
	3	30%	3	92.6	84.7	90.6	97.4	89.1	94.8	99.6	94.8	99.2
			4	91.8	87.9	88.6	96.4	88.2	91.8	99.6	95.3	98.6
			5	92.8	86.4	88.6	96.0	91.6	95.4	99.0	94.7	98.4
	4	20%	3	90.4	87.7	89.0	97.0	89.6	96.8	98.8	94.1	98.2
			4	92.2	86.5	89.4	95.8	90.9	93.8	99.6	93.9	98.6
			5	92.0	85.5	86.6	97.0	89.7	93.8	99.4	94.6	99.2
20	2	80%	3	91.8	90.5	90.0	95.2	92.0	92.6	99.0	96.2	98.4
			4	91.2	89.4	89.2	95.8	91.0	95.0	99.6	97.1	99.2
			5	90.2	90.7	89.8	95.8	92.8	93.8	99.6	97.1	98.4
	3	40%	3	91.8	89.4	89.0	97.0	90.1	95.2	99.0	96.1	99.2
			4	91.0	90.6	91.0	95.2	93.4	94.4	99.0	94.6	98.6
			5	91.4	89.8	91.2	96.0	93.3	95.0	99.6	96.4	98.4
	4	25%	3	90.2	89.0	90.4	95.4	92.8	95.4	99.4	96.8	99.0
			4	90.4	88.8	89.0	96.4	91.9	93.8	99.2	96.5	99.0
			5	90.6	88.7	90.6	95.2	91.1	93.2	99.4	96.1	99.4

Table 2.2: Average Widths of Confidence Intervals

based on 1000 Simulations with $\theta = 6.0$ and $B = 1000$

Nominal CL				90%			95%			99%		
n	k	π^*	τ	Exact	Approx	Boot	Exact	Approx	Boot	Exact	Approx	Boot
10	2	60%	3	9.604	13.848	16.277	12.812	14.626	20.881	20.451	18.323	29.182
			4	9.813	11.202	15.555	12.832	13.217	20.281	17.895	15.746	29.913
			5	9.508	9.743	12.943	11.957	12.006	16.158	18.459	14.316	27.248
	3	30%	3	9.863	10.025	14.791	13.008	12.187	19.112	18.515	15.402	29.664
			4	9.516	9.402	12.539	12.169	11.237	16.275	18.605	14.061	27.701
			5	9.042	8.488	10.966	11.503	10.313	13.878	17.428	13.006	23.304
	4	20%	3	9.453	9.186	13.758	12.120	11.154	17.933	18.875	14.095	27.956
			4	9.112	8.712	11.092	11.737	10.308	13.828	17.826	12.844	22.848
			5	8.672	8.090	9.790	10.950	9.847	12.610	16.558	12.498	18.296
20	2	80%	3	7.685	8.694	10.730	9.520	10.533	15.556	13.567	13.108	24.835
			4	6.885	7.241	8.618	8.498	8.612	11.688	12.381	11.398	19.605
			5	6.517	6.664	7.773	8.101	7.740	10.164	11.227	10.149	15.186
	3	40%	3	7.272	7.216	8.840	8.854	8.520	11.229	12.888	10.940	18.996
			4	6.784	6.542	7.303	8.422	7.647	9.583	12.461	10.019	15.747
			5	6.313	6.171	6.875	7.823	7.147	8.451	11.023	9.585	13.300
	4	25%	3	6.489	6.515	7.520	8.185	7.726	9.450	12.517	10.231	15.775
			4	6.383	6.058	6.722	7.630	7.027	8.536	11.410	9.188	12.371
			5	5.924	5.556	6.237	7.466	6.691	7.899	10.428	8.853	10.989

Table 2.3: Average Bounds of 90% Confidence Intervals

based on 1000 Simulations with $\theta = 6.0$ and $B = 1000$

n	k	π^*	τ	Exact CI	Approximate CI	BCa Bootstrap CI
10	2	60%	3	(2.451, 12.055)	(0.878, 14.726)	(3.691, 19.968)
			4	(2.801, 12.614)	(1.333, 12.536)	(3.372, 18.927)
			5	(3.032, 12.540)	(1.708, 11.451)	(3.361, 16.304)
	3	30%	3	(2.913, 12.776)	(1.484, 11.510)	(3.492, 18.283)
			4	(3.134, 12.651)	(1.853, 11.254)	(3.407, 15.946)
			5	(3.249, 12.291)	(2.068, 10.556)	(3.463, 14.430)
	4	20%	3	(3.075, 12.528)	(1.820, 11.006)	(3.534, 17.292)
			4	(3.268, 12.380)	(2.119, 10.832)	(3.506, 14.597)
			5	(3.340, 12.012)	(2.263, 10.352)	(3.472, 13.262)
20	2	80%	3	(3.331, 11.016)	(2.397, 11.092)	(3.708, 14.438)
			4	(3.561, 10.447)	(2.829, 10.070)	(3.828, 12.445)
			5	(3.763, 10.281)	(3.111, 9.776)	(3.954, 11.727)
	3	40%	3	(3.636, 10.908)	(2.815, 10.030)	(3.824, 12.664)
			4	(3.805, 10.589)	(3.068, 9.610)	(3.857, 11.160)
			5	(3.846, 10.159)	(3.231, 9.402)	(4.022, 10.897)
	4	25%	3	(3.669, 10.157)	(3.057, 9.572)	(3.909, 11.429)
			4	(3.940, 10.323)	(3.255, 9.313)	(3.973, 10.695)
			5	(3.946, 9.870)	(3.310, 8.866)	(4.003, 10.240)

Table 2.4: Average Bounds of 95% Confidence Intervals

based on 1000 Simulations with $\theta = 6.0$ and $B = 1000$

n	k	π^*	τ	Exact CI	Approximate CI	BCa Bootstrap CI
10	2	60%	3	(2.192, 15.004)	(0.262, 14.888)	(2.844, 23.725)
			4	(2.512, 15.344)	(0.531, 13.748)	(2.995, 23.276)
			5	(2.692, 14.649)	(0.821, 12.827)	(3.021, 19.179)
	3	30%	3	(2.610, 15.618)	(0.634, 12.822)	(2.834, 21.946)
			4	(2.826, 14.995)	(0.988, 12.225)	(2.933, 19.208)
			5	(2.926, 14.429)	(1.274, 11.587)	(3.035, 16.913)
	4	20%	3	(2.782, 14.902)	(0.954, 12.108)	(3.059, 20.992)
			4	(2.971, 14.707)	(1.276, 11.583)	(3.010, 16.838)
			5	(3.028, 13.978)	(1.490, 11.336)	(3.095, 15.704)
20	2	80%	3	(2.992, 12.513)	(1.572, 12.105)	(3.564, 19.120)
			4	(3.248, 11.746)	(2.121, 10.733)	(3.582, 15.270)
			5	(3.493, 11.593)	(2.468, 10.208)	(3.707, 13.871)
	3	40%	3	(3.298, 12.152)	(2.109, 10.629)	(3.475, 14.704)
			4	(3.511, 11.933)	(2.442, 10.089)	(3.606, 13.189)
			5	(3.572, 11.395)	(2.595, 9.742)	(3.673, 12.124)
	4	25%	3	(3.422, 11.607)	(2.422, 10.148)	(3.587, 13.037)
			4	(3.570, 11.200)	(2.641, 9.668)	(3.700, 12.236)
			5	(3.739, 11.206)	(2.794, 9.484)	(3.766, 11.665)

Table 2.5: Average Bounds of 99% Confidence Intervals

based on 1000 Simulations with $\theta = 6.0$ and $B = 1000$

n	k	π^*	τ	Exact CI	Approximate CI	BCa Bootstrap CI
10	2	60%	3	(1.711, 22.162)	(0.005, 18.328)	(2.213, 31.395)
			4	(1.935, 19.830)	(0.021, 15.767)	(2.265, 32.178)
			5	(2.193, 20.652)	(0.062, 14.379)	(2.233, 29.480)
	3	30%	3	(2.019, 20.534)	(0.013, 15.415)	(2.072, 31.736)
			4	(2.289, 20.893)	(0.040, 14.101)	(2.278, 29.979)
			5	(2.400, 19.828)	(0.094, 13.100)	(2.365, 25.669)
	4	20%	3	(2.258, 21.134)	(0.034, 14.129)	(2.299, 30.255)
			4	(2.435, 20.261)	(0.102, 12.947)	(2.360, 25.208)
			5	(2.506, 19.065)	(0.191, 12.688)	(2.368, 20.664)
20	2	80%	3	(2.493, 16.061)	(0.434, 13.542)	(2.733, 27.568)
			4	(2.779, 15.160)	(0.872, 12.270)	(2.959, 22.564)
			5	(2.922, 14.149)	(1.264, 11.413)	(3.001, 18.188)
	3	40%	3	(2.777, 15.665)	(0.858, 11.798)	(2.949, 21.945)
			4	(3.047, 15.508)	(1.229, 11.248)	(3.118, 18.865)
			5	(3.045, 14.068)	(1.478, 11.063)	(3.154, 16.455)
	4	25%	3	(3.033, 15.550)	(1.223, 11.454)	(3.072, 18.846)
			4	(3.175, 14.585)	(1.538, 10.726)	(3.090, 15.461)
			5	(3.198, 13.626)	(1.750, 10.603)	(3.116, 14.105)

of the asymptotic approach to construct CIs. Moreover, from Table 2.2, we observe that the widths of the CIs obtained from the exact method are quite comparable to those from the approximate method. On the other hand, the parametric bootstrap method exhibits unduly large interval widths compared to the other methods in general. Therefore, among the different approaches we have considered for constructing CIs, the performance of the exact method is overall the best with respect to both the interval widths and probability coverages.

Nevertheless, we realize from both Tables 2.1 and 2.2 that a larger sample size eventually improves the probability coverages and widths for both the approximate and BCa bootstrap CIs. As the sample size grows, the large computational time as well as the unstable precision becomes a problematic issue for constructing CIs by the exact method. Hence, based on a more comprehensive simulation study, we recommend the use of the bootstrap approach to construct the CIs for θ when the initial sample size is considerably large since it offers computational feasibility and also performs quite well in terms of probability coverages for large sample sizes (*e.g.*, $n \geq 30$).

2.7 Illustrative example

In order to illustrate the methods of inference described in the preceding sections, a progressively Type-I censored sample was generated from the complete dataset presented in Nelson (1990, p.129). The original dataset contains $n = 19$ observations on the times (in minutes) to breakdown of insulating fluid in a test conducted at the electrical stress of 36 kilovolts. The test employed two parallel plate electrodes of a certain area and gap with the constant electrode geometry. We considered $k = 5$ equi-spaced censoring time points and the uniform interval duration of $\tau = 2$ minutes in order to generate the desired sample. The pre-fixed censoring scheme was chosen to be $\mathbf{c}^* = (3, 0, 3, 5)$ for this life-test. From the initial sample size of $n = 19$, we observed a total of $D = 8$ failure times (*i.e.*, overall 58% right censoring) and they are presented in Table 2.6 along with the realized values of other essential variables.

It is noted from Table 2.6 that although it was planned to run this life-test until the end of the 5th interval, it was actually terminated earlier than scheduled since there were no more units remaining on test after the 4th censoring took place. From the dataset given above, the observed MLE of θ is calculated from (2.2.6) and it is found to be $\hat{\theta}^{obs} = 10.861$. Using this estimate in place of θ, the observed values of the bias, standard error and mean squared error of $\hat{\theta}$ are obtained from (2.4.6)-(2.4.8) to be

$$\begin{aligned} \hat{B}_{\theta}^{obs}(\hat{\theta}) &= 0.946, \\ se(\hat{\theta}) &= 4.910, \\ \widehat{MSE}_{\theta}^{obs}(\hat{\theta}) &= 25.004, \end{aligned}$$

respectively. Hence, the bias-corrected observed value of $\hat{\theta}$ is simply $\tilde{\theta}^{obs} = \hat{\theta}^{obs} - \hat{B}_{\theta}^{obs}(\hat{\theta}) = 9.915$.

The CIs for θ are also presented in Table 2.7 using all three methods described in Section 2.5. Since the exact CIs for θ require the monotonicity of the tail probability function of $\hat{\theta}$, we provide a numerical justification of this assumption by plotting

Table 2.6: Progressively Type-I Censored Sample from $n = 19$

Breakdown Times (minutes) of Insulating Fluid at 36 kV

in Nelson (1990, p.129) with $k = 5$, $\tau = 2$ and $\mathbf{c}^* = (3, 0, 3, 5)$

i	N_i	Failure Times ($\mathbf{y_i}$)	n_i	c_i	U_i
1	19	0.19, 0.78, 0.96, 1.31	4	3	33.24
2	12	2.78	1	0	22.78
3	11	4.67, 4.85	2	3	19.52
4	6	7.35	1	5	11.35
5	0	NA	0	0	0.00
		Total	$D = 8$	11	86.89

Table 2.7: Interval Estimation for θ based on the Progressively Type-I Censored Insulating Fluid Breakdown Data in Table 2.6 with $B = 1000$

CL	Exact CI	Approximate CI	BCa Bootstrap CI
90%	(5.962, 18.379)	(4.545, 17.178)	(6.580, 22.777)
95%	(5.458, 20.983)	(3.335, 18.388)	(5.886, 29.240)
99%	(4.626, 27.592)	(0.970, 20.753)	(5.056, 42.594)

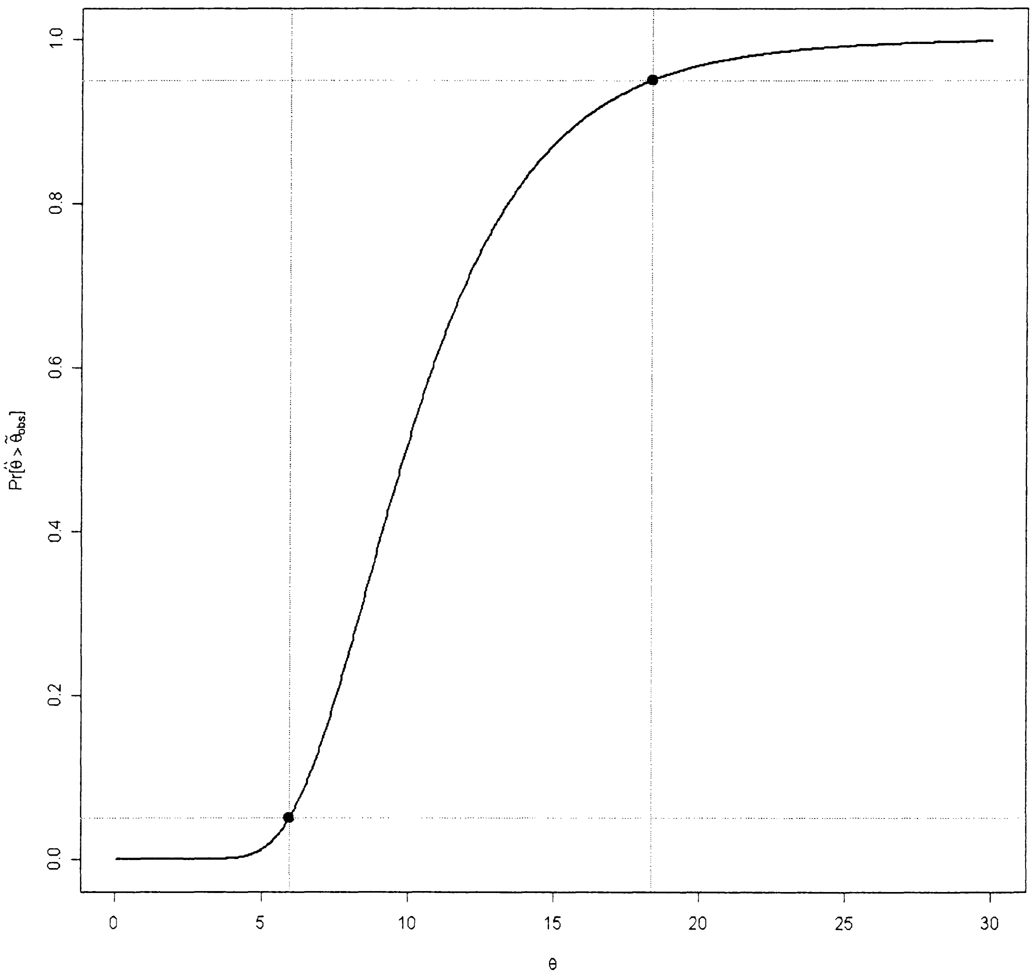

Figure 2.2: Tail Probability Plot of $\hat{\theta}$, $Pr\left[\hat{\theta} > \tilde{\theta}^{obs}\right]$, with respect to θ and Exact 90% Confidence Interval for θ from the Progressively Type-I Censored Insulating Fluid Breakdown Data in Table 2.6

the tail probability with the bias-corrected observed value of the MLE from the sample. From Figure 2.2, it is evident that the plot shows the monotonically increasing behavior of the tail probability with respect to the unknown parameter θ. In addition, the two horizontal lines corresponding to the values of $\alpha/2$ and $1 - \alpha/2$ for $\alpha = 0.10$ are overlaid in Figure 2.2 to illustrate how the exact CIs are constructed by inverting the tail probability of $\hat{\theta}$. For example, the values of θ from the two intersecting points are the unique solutions of Eqs. (2.5.1) and (2.5.2), respectively, and together they provide the exact 90% CI for θ.

From Table 2.7, we observe that the exact CIs are quite comparable to the CIs obtained by the approximate method while the BCa bootstrap CIs are much wider in all cases. This empirically justifies the superiority of the CIs by the exact method since they provide better coverage probabilities, closer to the nominal levels than those of the approximate CIs for similar interval widths (see Table 2.1). Moreover, compared to the BCa bootstrap CIs, the exact CIs have narrower interval widths giving higher specificity even though both perform equally well in terms of probability coverages (see Table 2.2).

2.8 Summary and conclusions

In this chapter, we have discussed the progressively Type-I censored life-test with uniformly spaced k intervals when the lifetimes of the test units are from exponential distribution. For small sample sizes, we have proposed a practical modification to the censoring scheme for a feasible test under Type-I PC. We have then derived the MLE of the unknown mean parameter θ and its exact conditional distribution through the use of the CMGF. We have also proposed several different procedures for constructing CIs for θ. We have then conducted a simulation study to assess the performance of all these procedures and a numerical example has been presented to illustrate all the methods of inference developed in this chapter. Based on the results of a more comprehensive simulation study, our recommendation for constructing CIs for θ is to use the exact method whenever possible, especially in the case of small sample sizes (*e.g.*, $n < 30$) since the other two methods are unsatisfactory in terms of probability coverages or widths. For larger sample sizes, however, the BCa percentile bootstrap method is more appropriate due to its computational ease as well as for its probability coverages being close to the nominal levels.

Chapter 3

Optimal Step-stress Testing for Progressively Type-I Censored Data from Exponential Distribution

3.1 Introduction and motivation

We have introduced the general idea of ALT and its special class known as the step-stress testing earlier in Section 1.1. During the past two decades, the problem of optimal scheduling of the step-stress sampling scheme has attracted great attention in the reliability literature. Miller and Nelson (1983) initiated research in this area by assuming that the lifetimes are exponentially distributed and complete failure data are available under two stress levels (*i.e.*, the case of a simple step-stress model). The basic model used was the one proposed by Sedyakin (1966), which is referred to in the literature as the cumulative exposure model. Bai, Kim and Lee (1989) extended the results of Miller and Nelson (1983) to the case of time-censored data, and the case of three stress levels was dealt by Khamis and Higgins (1996). For the general k-level, M-variable case, some numerical investigation was undertaken by Khamis (1997). Khamis and Higgins (1998) also considered the problem under a Weibull distribution for the lifetimes of units subjected to stress. Yeo and Tang (1999) investigated the optimality problem in the situation when a target acceleration factor is pre-specified. Inferential issues with the cumulative exposure model under exponentiality were discussed by Xiong (1998) and Xiong and Milliken (1999). Balakrishnan *et al.* (2007) derived the exact conditional distributions of the MLEs under the assumption of exponential failures and Type-II censoring. Recently, Gouno, Sen and Balakrishnan (2004) tackled the problem of determining the optimal stress change points for a general k-level model under the large-sample case when the available data are progressively Type-I censored; see also Han *et al.* (2006) for some related comments.

The main focus of this chapter is to build a feasible ALT model combined with PC for a small to moderate sample size, and then to investigate the choice of optimal change points of the stress levels with or without the condition that the life-test proceeds to the last stage of stress. A practical modification is suggested to the asymptotic model discussed by Gouno, Sen and Balakrishnan (2004) for a feasible step-stress analysis under a PC scheme with an arbitrary number of stress levels. Here, we consider the equi-spaced step with τ denoting the duration of each testing stage.

Since we must decide upon the length of an inspection interval, this setup for a k-step-stress test seems reasonable and pragmatic. Using three different optimality criteria (*viz.*, variance, determinant and trace), the efficiency of the conditional approach to the optimality problem is also discussed, and a comparison of the numerical results from the asymptotic and the modified models is presented as well.

3.2 Model description and MLEs

Compared to the conventional censoring, PC provides more flexibility to the experimenter in the design stage by allowing the removal of test units at non-terminal time points and is therefore highly efficient and effective in utilizing the available resources. In order to describe the step-stress testing procedure implemented with a popular form of PC, progressive Type-I censoring, let us first define $x_1 < x_2 < \ldots < x_k$ to be the ordered stress levels to be used in the test. Then, for $i = 1, 2, \ldots, k$, let n_i denote the number of units failed at stress level x_i (*i.e.*, in time interval $[(i-1)\tau, i\tau)$) and $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \ldots, n_i$, while c_i denotes the number of units censored at time $i\tau$. Furthermore, let N_i denote the number of units operating and remaining on test at the start of stress level x_i $\Big($*viz.*, $N_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} c_j\Big)$.

Under this setup, a step-stress test with an equal step duration τ proceeds as follows. A total of $N_1 \equiv n$ test units is initially placed at stress level x_1 and tested until time τ at which point the stress is changed to level x_2 and c_1 live items are randomly withdrawn from the test. The test is continued on $N_2 = n - n_1 - c_1$ units until time 2τ, when the stress is changed to level x_3 and c_2 items are withdrawn from the test, and so on. Finally, at time $k\tau$, all the surviving items are withdrawn, thereby terminating the life-test. Note that since $n \equiv \sum_{i=1}^{k}(n_i + c_i)$, the number of surviving items at time $k\tau$ is $c_k = n - \sum_{i=1}^{k} n_i - \sum_{i=1}^{k-1} c_i = N_k - n_k$. Obviously, when there is no intermediate censoring (*viz.*, $c_1 = c_2 = \cdots = c_{k-1} = 0$), this situation corresponds to the k-level step-stress testing under Type-I right censoring as a special case. Now, the following assumptions are crucial for constructing subsequent step-stress models.

Assumptions :

(i) A cumulative exposure model holds;

(ii) For any stress level, the lifetime of a test unit follows an exponential distribution;

(iii) At stress level x_i, the mean time to failure (MTTF) of a test unit, θ_i, is a log-linear function of stress given by

$$\log \theta_i = \alpha + \beta x_i, \tag{3.2.1}$$

where the regression parameters α and β are unknown and need to be estimated.

Under the assumptions (i) and (ii), the PDF of a test unit is

$$f(t) = f_i(t - (i-1)\tau) \prod_{j=1}^{i-1} S_j(\tau) \quad \text{if} \begin{cases} (i-1)\tau \le t \le i\tau & \text{for } i = 1, 2, \ldots, k-1 \\ (k-1)\tau \le t < \infty & \text{for } i = k \end{cases}, \tag{3.2.2}$$

where $f_i(t) = \dfrac{1}{\theta_i} \exp\left(-\dfrac{t}{\theta_i} \right)$. The corresponding CDF is then given by

$$F(t) = 1 - \left[\prod_{j=1}^{i-1} S_j(\tau) \right] S_i(t - (i-1)\tau) \qquad \text{if } \begin{cases} (i-1)\tau \le t \le i\tau & \text{for } i = 1, 2, \ldots, k-1 \\ (k-1)\tau \le t < \infty & \text{for } i = k \end{cases}, \tag{3.2.3}$$

where

$$F_i(t) = 1 - S_i(t) = 1 - \exp\left(-\frac{t}{\theta_i} \right).$$

As in Chapter 2, no notational distinction will be made in this chapter between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$. Then, using (3.2.2) and (3.2.3), the joint probability density function (JPDF) of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k)$ with $\mathbf{y}_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is obtained as

$$f_J(\mathbf{y}, \mathbf{n}) = \left[\prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!} \right] \left[\prod_{i=1}^{k} \theta_i^{-n_i} \right] \exp\left(-\sum_{i=1}^{k} \frac{U_i}{\theta_i} \right), \tag{3.2.4}$$

where

$$U_i = \sum_{j=1}^{n_i} (y_{i,j} - (i-1)\tau) + (N_i - n_i)\tau, \qquad i = 1, 2, \ldots, k. \tag{3.2.5}$$

Note that U_i in (3.2.5) is precisely the *Total Time on Test* statistic at stress level x_i. Now, using (3.2.4) and assumption (iii), the log-likelihood function of (α, β) can be written as

$$l(\alpha, \beta) = -\alpha \sum_{i=1}^{k} n_i - \beta \sum_{i=1}^{k} n_i x_i - \sum_{i=1}^{k} U_i \exp[-(\alpha + \beta x_i)]. \tag{3.2.6}$$

After differentiating $l(\alpha, \beta)$ in (3.2.6) with respect to α and β, we obtain the likelihood equations as

$$\begin{aligned} 0 = \frac{\partial}{\partial \alpha} l(\alpha, \beta) &= -\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} U_i \exp[-(\alpha + \beta x_i)], \\ 0 = \frac{\partial}{\partial \beta} l(\alpha, \beta) &= -\sum_{i=1}^{k} n_i x_i + \sum_{i=1}^{k} U_i x_i \exp[-(\alpha + \beta x_i)]. \end{aligned}$$

The MLEs $\hat{\alpha}$ and $\hat{\beta}$ are then obtained as simultaneous solutions to the following two equations:

$$\hat{\alpha} = \log\left(\frac{\sum_{i=1}^{k} U_i \exp(-\hat{\beta} x_i)}{\sum_{i=1}^{k} n_i}\right),$$

$$\left[\sum_{i=1}^{k} n_i\right]\left[\sum_{i=1}^{k} U_i x_i \exp(-\hat{\beta} x_i)\right] - \sum_{i=1}^{k} n_i x_i \sum_{i=1}^{k} U_i \exp(-\hat{\beta} x_i) = 0.$$

As shown above, $\hat{\alpha}$ and $\hat{\beta}$ are non-linear functions of random quantities, which make it impossible to find their exact marginal/joint distributions for exact inference. Thus, statistical inference with these MLEs are based on the asymptotic distributional result that the vector $(\hat{\alpha}, \hat{\beta})$ is approximately distributed as a bivariate normal with mean vector (α, β) and variance-covariance matrix $[\mathbf{I}_n(\alpha, \beta)]^{-1}$, where $\mathbf{I}_n(\alpha, \beta)$ is the Fisher information.

3.3 k-level step-stress test under progressive censoring with small samples

As addressed in Chapter 2, prefixing $\mathbf{c} = (c_1, c_2, \ldots, c_{k-1})$ in the model bears an inherent mathematical lapse as there is a positive probability that all the test units cease before reaching the planned k-th stress level, failing to fully apply the predetermined PC scheme $\mathbf{c}$. In Gouno, Sen and Balakrishnan (2004), a large sample size, small global censoring proportions, and a small number of stress levels were assumed in order to guard enough surviving items to be censored at the end of each stress level. As a result of these assumptions, we had to restrict our search for optimal τ in the region defined by

$$\mathcal{C}_\tau = \{\tau : A_i(\tau) > 0, \qquad i = 2, 3, \ldots, k\},$$

where

$$A_i(\tau) = \left[1 - \sum_{j=1}^{i-1} \frac{\pi_j}{G_j(\tau)}\right] G_{i-1}(\tau) F_i(\tau) \qquad \text{with} \qquad G_j(\tau) = \prod_{i=1}^{j} S_i(\tau)$$

and $\pi_j = c_j/n$ denoting the overall censoring proportion at stress level x_j, $j = 1, 2, \ldots, k-1$.

Although $\mathcal{C}_\tau$ is interpreted as a region to ensure the availability of sufficient items at the end of each stage to censor from, a careful look reveals that it only does that *on average* but not for each sample. This is an inevitable problem associated with the basic protocol of step-stress testing. Even in the case of a simple step-stress test, the assigned n test units could be all exhausted before the experiment reaches the second stress level x_2, resulting in an early termination of the life-test. Besides, in a reliability experiment, the sample size is usually small and there might be severe censoring due to various reasons such as budgetary constraints and facility requirements. Under such circumstances, the assumptions made by Gouno, Sen and Balakrishnan (2004) are violated and therefore, a modification is required to their proposed model so that a feasible PC scheme can be guaranteed. Apart from the one suggested in Chapter 2, another modification which can be entertained in practice is to decide on a fixed proportion of unfailed items to be removed at the end of each stage, rather than to decide on a global proportion over the initial sample size. Again, since the number of live units at the end of each stage before censoring takes place is random, the proposed change essentially makes the number of progressively censored units also random.

In order to revise the model according to the newly proposed modification, we first define a vector of proportions

$$\boldsymbol{\pi}^* = (\pi_1^*, \pi_2^*, \ldots, \pi_{k-1}^*),$$

where $0 \le \pi_i^* < 1$ for $i = 1, 2, \ldots, k-1$. Note that $\boldsymbol{\pi}^*$ is composed of fixed constants defining the proportion of surviving items to be censored at each stress transition. Thus, $\pi_i = c_i/n$, the overall censoring proportion at the i-th stage defined over the total number of testing units is distinguished from π_i^*. Since all the remaining items are withdrawn from the test at the end of stress level x_k, one could also state $\pi_k^* = 1$. In this setting, the number of censored items at the end of stress level x_i is

$$c_i = \Upsilon((N_i - n_i)\pi_i^*) \qquad \text{for } i = 1, 2, \ldots, k-1, \tag{3.3.1}$$

where $\Upsilon(\cdot)$ is a discretizing function of one's choice, mapping its argument to a non-negative integer. $\Upsilon(\cdot)$ could be one of $round(\cdot)$, $floor(\cdot)$, $ceiling(\cdot)$ and $trunc(\cdot)$, for example. Since $0 \le \pi_i^* < 1$, we have $0 \le c_i \le N_i - n_i$ for $i = 1, 2, \ldots, k-1$. When $c_i = N_i - n_i \ge 0$, the life-test terminates at the end of the i^*-th stage, where i^* is the minimum of such i's satisfying $c_i = N_i - n_i$. Consequently, this results in $N_{i^*+1} = N_{i^*+2} = \cdots = N_k = 0$, $n_{i^*+1} = n_{i^*+2} = \cdots = n_k = 0$, and $c_{i^*+1} = c_{i^*+2} = \cdots = c_k = 0$ since $N_{i+1} = N_i - n_i - c_i$. Hence, under the proposed modification, we allow the life-test to terminate before reaching the last stress level x_k. We should also point out that $\mathbf{c} = (c_1, c_2, \ldots, c_{k-1})$ is random as well as $\boldsymbol{\pi} = \mathbf{c}/n = (\pi_1, \pi_2, \ldots, \pi_{k-1})$ under this setup. When $\boldsymbol{\pi}^* = (0, 0, \ldots, 0) = \mathbf{0}_{k-1}$, we have $\mathbf{c} = \mathbf{0}_{k-1}$ and $\boldsymbol{\pi} = \mathbf{0}_{k-1}$, and it is clear that this case corresponds to the case of a k-level step-stress testing under Type-I right censoring. In addition, if $c_k > 0$ or $n_k > 0$ (equivalently, $N_k = n_k + c_k > 0$), it implies that the life-test has proceeded onto the last stress level x_k.

The definition of c_i in (3.3.1) nevertheless complicates the derivation of distributions of associated random quantities. For simplicity, we shall assume in all subsequent derivations that

$$c_i = (N_i - n_i)\pi_i^* \qquad \text{for } i = 1, 2, \ldots, k-1, \tag{3.3.2}$$

as $\Upsilon((N_i - n_i)\pi_i^*) \approx (N_i - n_i)\pi_i^*$. Then, by using the following properties of the counts and order statistics, we can derive the expectation of N_i and also obtain the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$. Proofs of all the subsequent lemmas and theorems are presented in Appendix B.

Properties :

(1) The random variable n_1 has a binomial distribution with parameters $(n, F_1(\tau))$. For $i = 2, 3, \ldots, k$, given $n_1, n_2, \ldots, n_{i-1}$, the random variable n_i has a binomial distribution with parameters $(N_i, F_i(\tau))$.

(2) Given $n_1, n_2, \ldots, n_i$, the random variables $(y_{i,j} - (i-1)\tau)$, $j = 1, 2, \ldots, n_i$, are distributed jointly as order statistics from a random sample of size n_i from a right-truncated exponential distribution with PDF $f_{i,\tau}(z) = \dfrac{f_i(z)}{F_i(\tau)}$ for $0 \leq z \leq \tau$ and $i = 1, 2, \ldots, k$.

Lemma 3.3.1. *For* $i = 1, 2, \ldots, k$,

$$E[N_i] = n \prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*). \tag{3.3.3}$$

Theorem 3.3.1. *Under the proposed modification, the Fisher information matrix is*

$$\mathbf{I}_n(\alpha, \beta) = n \begin{pmatrix} \sum_{i=1}^{k} A_i(\tau) & \sum_{i=1}^{k} A_i(\tau) x_i \\ \sum_{i=1}^{k} A_i(\tau) x_i & \sum_{i=1}^{k} A_i(\tau) x_i^2 \end{pmatrix}, \tag{3.3.4}$$

where

$$A_i(\tau) = F_i(\tau) \prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*). \tag{3.3.5}$$

3.4 Optimality criteria and existence of optimal stress change points

In this section, we define different optimality criteria for determining an optimal stress duration τ. These objective functions are purely based on the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$ derived in the preceding section. Unlike $A_i(\tau)$ in Gouno, Sen and Balakrishnan (2004), $A_i(\tau)$ in (3.3.5) is positive for all $\tau > 0$. This, in turn, eliminates any disconcerting anomalies and ensures a positive determinant of $\mathbf{I}_n(\alpha, \beta)$ as well as a positive variance function. Since the censoring is performed based on the number of surviving units at the end of each stage, the case of censoring beyond what is available on test is completely avoided. Therefore, there is no restriction on the search region for the optimal τ in this case (*i.e.*, $\mathcal{C}_\tau = \{\tau : \tau > 0\}$).

3.4.1 C-optimality

In an ALT experiment, researchers often wish to estimate the parameters of interest with maximum precision and minimum variability. In the step-stress setting under consideration here, such a parameter of interest is the mean lifetime of a unit at the use-condition (*viz.*, θ_0). For this purpose, we consider an objective function from (3.3.4) as

$$\begin{aligned} \phi(\tau) &= n \ \mathrm{AVar}(\log \hat{\theta}_0) = n \ \mathrm{AVar}(\hat{\alpha} + \hat{\beta} x_0) \\ &= n \ (1, x_0) \mathbf{I}_n^{-1}(\alpha, \beta) \begin{pmatrix} 1 \\ x_0 \end{pmatrix} \\ &= \frac{2 \sum_{i=1}^{k} A_i(\tau)(x_i - x_0)^2}{\sum_{i=1}^{k} \sum_{j=1}^{k} A_i(\tau) A_j(\tau)(x_i - x_j)^2}, \end{aligned} \tag{3.4.1}$$

where AVar stands for asymptotic variance and x_0 is the normal use-stress. The C-optimal τ (*viz.*, τ_C^*) is the one that minimizes $\phi(\tau)$ in (3.4.1). In the case of $k = 2$ (*i.e.*, the case of a simple step-stress test), the objective function in (3.4.1) under the C-optimality can be shown to reduce to

$$\begin{aligned} \phi(\tau) &= \frac{A_1(\tau)(x_1 - x_0)^2 + A_2(\tau)(x_2 - x_0)^2}{A_1(\tau) A_2(\tau)(x_2 - x_1)^2} \\ &= \frac{(1+\xi)^2}{A_1(\tau)} + \frac{\xi^2}{A_2(\tau)}, \end{aligned} \tag{3.4.2}$$

where $\xi = \dfrac{x_1 - x_0}{x_2 - x_1}$.

Theorem 3.4.1. *In the case of a simple step-stress test under progressive Type-I censoring, there exists an optimal step duration τ_C^* which is the unique solution of the equation $\phi'(\tau) = 0$.*

3.4.2 D-optimality

Another optimality criterion often used in planning ALT is based on the determinant of the Fisher information matrix, which is the same as the reciprocal of the determinant of the asymptotic variance-covariance matrix. Note that the overall volume of the asymptotic joint confidence region of (α, β) is proportional to $|\mathbf{I}_n^{-1}(\alpha, \beta)|^{1/2}$ at a fixed confidence level. In other words, it is inversely proportional to $|\mathbf{I}_n(\alpha, \beta)|^{1/2}$, the square root of the determinant of $\mathbf{I}_n(\alpha, \beta)$. Consequently, a larger value of $|\mathbf{I}_n(\alpha, \beta)|$ would correspond to a smaller asymptotic joint confidence ellipsoid of (α, β), and thus a higher joint precision of the estimators of α and β. Motivated by this, our second objective function is simply given by

$$\begin{aligned} \delta(\tau) &= n^{-2}|\mathbf{I}_n(\alpha, \beta)| \\ &= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} A_i(\tau)A_j(\tau)(x_i - x_j)^2. \end{aligned} \tag{3.4.3}$$

The D-optimal τ (*viz.*, τ_D^*) is obtained by maximizing (3.4.3) for the maximal joint precision of $(\hat{\alpha}, \hat{\beta})$. For $k = 2$, the objective function (3.4.3) under the D-optimality reduces to

$$\delta(\tau) = A_1(\tau)A_2(\tau)(x_2 - x_1)^2. \tag{3.4.4}$$

Theorem 3.4.2. *In the case of a simple step-stress test under progressive Type-I censoring, the D-optimal stress change point τ_D^* is the solution of $A_1'(\tau)A_2(\tau) + A_1(\tau)A_2'(\tau) = 0$.*

3.4.3 A-optimality

Another optimality criterion considered in our study is based on the sum of marginal Fisher information terms of the parameters of the model. It is identical to the sum of the diagonal elements, that is, trace of $\mathbf{I}_n(\alpha,\beta)$. Like the D-optimality, the A-optimality criterion is a general measure of the size of the Fisher information $\mathbf{I}_n(\alpha,\beta)$. The A-optimal τ (*viz.*, τ_A^*) maximizes the objective function defined by

$$\begin{aligned} a(\tau) &= \frac{1}{n} tr(\mathbf{I}_n(\alpha,\beta)) \\ &= \sum_{i=1}^{k} A_i(\tau) + \sum_{i=1}^{k} A_i(\tau) x_i^2 = \sum_{i=1}^{k} A_i(\tau)(1+x_i^2). \end{aligned} \tag{3.4.5}$$

In the case of the simple step-stress test ($k = 2$), the objective function in (3.4.5) under the A-optimality simply becomes

$$a(\tau) = A_1(\tau)(1+x_1^2) + A_2(\tau)(1+x_2^2). \tag{3.4.6}$$

Theorem 3.4.3. *For the simple step-stress test under progressive Type-I censoring, the A-optimal stress change point is*

$$\tau_A^* = \theta_2 \log\left[\left(1+\frac{\theta_1}{\theta_2}\right)(1-Q_1^A)^{-1}\right], \qquad \textit{where} \qquad Q_1^A = \frac{1+x_1^2}{(1-\pi_1^*)(1+x_2^2)},$$

and it exists when $\dfrac{x_2^2 - x_1^2}{1+x_2^2} > \pi_1^*$. *Otherwise,* τ_A^* *does not exist.*

3.5 Conditional analysis of k-step-stress test under progressive censoring

As mentioned earlier, all the distributional results in Gouno, Sen and Balakrishnan (2004) have been established under the assumption of large n, small π_i's and small k in order to ensure sufficient items left for censoring at every stress change. The assumption is actually equivalent to the life-test terminating at the very last stress level x_k. In that respect, the distributional properties and analysis carried out by these authors should be regarded as conditional ones, subject to that assumption. Conditional analysis is particularly useful as we deal with a finite sample size because the assumption of an infinite sample size is not a practical one.

In this section, we adopt the notation and intermediate results from Sections 3.2 and 3.3, and formulate the distributional results required to tackle the problem of selecting an optimal stress duration using the conditional approach. Since the probability of premature termination of a life-test with a small sample size is much greater than the one with a large sample size, the derivation of the distributional results for a finite sample case is based on the condition that the planned censoring scheme is fully applied to the test. That is, there are enough testing units for censoring at each stress change. This condition is translated into the set $\{\mathbf{n} : N_2 > 0, N_3 > 0, \ldots, N_k > 0\}$, where $\{\mathbf{n} : N_i > 0\}$ defines a set of all the possible values $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ can take on satisfying the condition $N_i > 0$ (*i.e.*, successful censoring at time $(i-1)\tau$ for $i = 2, 3, \ldots, k$). As we find that

$$\{\mathbf{n} : N_k > 0\} \subset \{\mathbf{n} : N_{k-1} > 0\} \subset \cdots \subset \{\mathbf{n} : N_1 \equiv n > 0\} = \{\mathbf{n}\},$$

the condition simply yields $\{\mathbf{n} : N_2 > 0, N_3 > 0, \ldots, N_k > 0\} = \{\mathbf{n} : N_k > 0\}$. This proves that the condition of successful censoring at every stress level is equivalent to the condition of the test proceeding to the last stress level x_k. The probability of $N_k > 0$ is then easily obtained from the following lemma.

Lemma 3.5.1. *For $i = 1, 2, \ldots, k-1$,*

$$Pr(N_k = 0|n_1, n_2, \ldots, n_{i-1}) = [H_i(\tau)]^{N_i}, \tag{3.5.1}$$

where

$$H_i(\tau) = \begin{cases} F_i(\tau) + S_i(\tau)[H_{i+1}(\tau)]^{1-\pi_i^*}, & \text{for } i = 1, 2, \ldots, k-1 \\ 0, & \text{for } i = k \end{cases} . \tag{3.5.2}$$

Corollary 3.5.1. *For k stress levels, the probability of a life-test proceeding to stress level x_k is*

$$Pr(N_k > 0) = 1 - [H_1(\tau)]^n. \tag{3.5.3}$$

Proof. Since $N_k \geq 0$, we obtain from Lemma 3.5.1 that

$$Pr(N_k > 0) = 1 - Pr(N_k = 0) = 1 - [H_1(\tau)]^{N_1} = 1 - [H_1(\tau)]^n.$$

□

With the above results, the following lemma gives an expression for the expected number of failures observed at each stress level, conditioned on $N_k > 0$. For this purpose, we denote $E_c[\ \cdot\ ] = E[\ \cdot\ |N_k > 0]$ for the conditional expectation given $N_k > 0$.

Lemma 3.5.2. *For $i = 1, 2, \ldots, k$,*

$$E\Big[N_i[H_i(\tau)]^{N_i}\Big] = n[H_1(\tau)]^n \prod_{j=1}^{i-1}(1-\pi_j^*)\left(1 - \frac{F_j(\tau)}{H_j(\tau)}\right). \tag{3.5.4}$$

Theorem 3.5.1. *For $i = 1, 2, \ldots, k$,*

$$E_c[n_i] = E[n_i|N_k > 0] = E[n_i]\frac{1 - V_i(\tau)}{1 - [H_1(\tau)]^n},$$

where

$$V_i(\tau) = \begin{cases} \dfrac{[H_1(\tau)]^{n-1}}{\prod_{j=1}^{i-1}[H_{j+1}(\tau)]^{\pi_j^*}}, & \text{for } i = 1, 2, \ldots, k-1 \\ 0, & \text{for } i = k \end{cases}$$

and

$$E[n_i] = n\left[\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*)\right] F_i(\tau).$$

We are now set to derive the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$, conditioned on $N_k > 0$, using the results presented above along with the following lemma.

Lemma 3.5.3. *For $i = 1, 2, \ldots, k$,*

$$E_c[N_i] = E[N_i|N_k > 0] = E[N_i]\left(\frac{1 - H_i(\tau)V_i(\tau)}{1 - [H_1(\tau)]^n}\right), \tag{3.5.5}$$

where $E[N_i]$ is as given in (3.3.3).

Theorem 3.5.2. *The Fisher information matrix, conditioned on $N_k > 0$, is given by*

$$\mathbf{I}_n(\alpha, \beta) = n\begin{pmatrix} \sum_{i=1}^{k} A_i(\tau) & \sum_{i=1}^{k} A_i(\tau)x_i \\ \sum_{i=1}^{k} A_i(\tau)x_i & \sum_{i=1}^{k} A_i(\tau)x_i^2 \end{pmatrix}, \tag{3.5.6}$$

where

$$
\begin{aligned}
A_i(\tau) &= \frac{E[N_i]}{n(1-[H_1(\tau)]^n)}\left[(1-V_i(\tau))F_i(\tau)+\frac{\tau}{\theta_i}(1-H_i(\tau))V_i(\tau)\right] \\
&= \frac{1}{1-[H_1(\tau)]^n}\left[\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*)\right] \\
&\quad \times\left[(1-V_i(\tau))F_i(\tau)+\tau(1-H_i(\tau))V_i(\tau)\exp(\alpha+\beta x_i)\right]. \quad (3.5.7)
\end{aligned}
$$

Before presenting numerical results, we make a remark on the asymptotic behavior of the distributional results obtained in this section. For this purpose, we first need to observe a simple property of recursive equation (3.5.2) as given below.

Theorem 3.5.3. *For $i = 1, 2, \ldots, k$, we have $0 \le H_i(\tau) < 1$.*

From this property, it is apparent that $0 \le H_1(\tau) < 1$, and so it follows immediately from (3.5.3) that

$$\lim_{n\to\infty} Pr(N_k > 0) = 1 - \lim_{n\to\infty}[H_1(\tau)]^n = 1.$$

As expected, it reveals that the probability of a k-level step-stress test terminating at level x_k converges to 1 as the sample size n increases. Based on this observation, the following limits result:

$$
\begin{aligned}
\lim_{n\to\infty} V_i(\tau) &= \frac{\lim_{n\to\infty}[H_1(\tau)]^{n-1}}{\prod_{j=1}^{i-1}[H_{j+1}(\tau)]^{\pi_j^*}} = 0, \\
\lim_{n\to\infty} E_c[n_i] &= E[n_i], \\
\lim_{n\to\infty} E_c[N_i] &= E[N_i]
\end{aligned}
$$

for $i = 1, 2, \ldots, k$. Consequently, from (3.5.7) in Theorem 3.5.2, we get

$$\lim_{n\to\infty} nA_i(\tau) = E[N_i]F_i(\tau) = E[n_i],$$

which is identical to $nA_i(\tau)$ in (3.3.5) defined earlier in Theorem 3.3.1. Thus, we observe that all the distributional results obtained in this section by conditioning on $N_k > 0$ ultimately converge to the unconditional results in Section 3.3 when the sample size n gets larger. Since the conditional information matrix of α and β in Theorem 3.5.2 eventually approaches the unconditional information matrix presented in Theorem 3.3.1, it is clear that the optimization results based on these information matrices should produce close results for large n. In other words, conditioning does not make much difference to the analysis when the initial sample size is large.

As done in Section 3.4, we can also define objective functions based on the conditional information matrix in (3.5.6) for determining optimal step duration using C-optimality, D-optimality, and A-optimality criteria. Unfortunately, the complexity of $A_i(\tau)$ in (3.5.7) makes it impossible to analytically prove the existence of the optimal τ even in the case of a simple step-stress testing. Nevertheless, the determination of optimal τ can be done numerically.

All the optimality criteria considered here, as well as some other information-based criteria, have been used extensively in the design selection process for linear designed experiments. In the context of step-stress ALT, however, C-optimality is the only criterion which has been explored and the D-optimality was considered as an alternative criterion by Gouno, Sen and Balakrishnan (2004). Here, we suggest A-optimality as another criterion to be used in the design selection process. From a practitioner's viewpoint, the choice of the optimality criterion will be certainly guided by the objective of the experiment. In cases where the planner is more interested in the precise estimation of the MTTF θ_0 at the normal use-condition, C-optimality is surely the criterion of choice. On the other hand, if one is more concerned about estimating the mean function given in assumption (i) or estimating the regression parameters α and β with high precision, a more reasonable criterion of choice should be D-optimality or A-optimality.

3.6 Numerical results

A numerical study was conducted in order to investigate the existence of the optimal stress change points and to evaluate them as a function of varying parameters (*viz.*, the sample size, MTTF, the number of stress levels, and the degree of censoring). For the purpose of illustration, we considered equi-spaced stress levels as $x_i = x_0 + id$ with the use-stress level $x_0 = 10$ and the stress increment $d = 5$. Under this setup, optimizing with respect to either the C-optimality or the D-optimality criterion is independent of the values of x_0 and d in the framework of Section 3.4. On the other hand, optimizing with respect to the A-optimality criterion is sensitive to the choice of x_0 and d. Moreover, optimization based on the conditional distribution results in Section 3.5 inherently depends on the sample size n under any optimality criterion since the sample size largely influences the probability of the test terminating at stress level x_k. We also chose the ordered MTTF as

$$\theta_{i+1} = \rho\theta_i, \qquad i = 1, 2, \ldots, k-1, \qquad 0 < \rho < 1,$$

with selected choices of θ_1 and ρ. Under this setup, therefore, a decreasing geometric sequence of MTTF is simulated with an increasing arithmetic sequence of stress levels.

Tables 3.1 and 3.2 present the values of τ_C^*, τ_D^* and τ_A^* determined from the model in Section 3.3 for a feasible PC scheme. Rather than the specific values of the

Table 3.1: Optimal Stress Change Points under the Modification of $c_i = (N_i - n_i)\pi_i^*$ with the Expected Overall PC Proportion being 10%

$\pi_i = 0.1$		$k=2$			$k=3$			$k=4$		
		τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	91.6	60.6	30.9	10.1	6.6	3.1	1.0	0.7	0.3
	$\rho = 0.3$	93.6	72.7	64.1	31.4	21.6	16.2	9.9	6.7	4.7
	$\rho = 0.5$	95.1	81.2	87.7	45.5	34.6	30.9	21.4	15.9	13.2
$\theta_1 = 300$	$\rho = 0.1$	274.9	181.7	92.8	30.4	19.9	9.2	2.9	2.1	1.0
	$\rho = 0.3$	280.7	218.0	192.4	94.2	64.7	48.7	29.6	20.0	14.1
	$\rho = 0.5$	285.4	243.5	263.0	136.6	103.8	92.8	64.1	47.7	39.5
$\theta_1 = 500$	$\rho = 0.1$	458.2	302.9	154.7	50.7	33.1	15.4	4.8	3.4	1.6
	$\rho = 0.3$	467.8	363.3	320.6	157.0	107.9	81.1	49.3	33.4	23.5
	$\rho = 0.5$	475.7	405.8	438.3	227.7	173.0	154.7	106.7	79.6	65.9

Table 3.2: Optimal Stress Change Points under the Modification of $c_i = (N_i - n_i)\pi_i^*$ with the Expected Overall PC Proportion being 20%

$\pi_i = 0.2$		$k = 2$			$k = 3$			$k = 4$		
		τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	76.3	52.3	29.5	7.2	5.1	2.8	0.6	0.5	0.3
	$\rho = 0.3$	77.9	63.1	59.1	20.8	16.3	13.9	5.0	4.2	3.6
	$\rho = 0.5$	78.4	69.3	79.0	30.0	25.3	25.4	10.8	9.4	9.4
$\theta_1 = 300$	$\rho = 0.1$	228.8	156.9	88.4	21.5	15.4	8.5	1.7	1.4	0.8
	$\rho = 0.3$	233.6	189.2	177.3	62.5	49.0	41.6	15.0	12.5	10.8
	$\rho = 0.5$	235.3	207.9	237.0	90.1	76.0	76.1	32.4	28.2	28.1
$\theta_1 = 500$	$\rho = 0.1$	381.3	261.5	147.4	35.9	25.7	14.2	2.9	2.3	1.4
	$\rho = 0.3$	389.4	315.3	295.5	104.2	81.7	69.4	25.0	20.8	17.9
	$\rho = 0.5$	392.2	346.6	395.0	150.2	126.6	126.8	54.0	47.1	46.8

optimal stress durations, the tables are intended to provide a qualitative insight into the way the optimal choice changes as a function of the relevant parameters. To be able to compare the results with those from the large-sample results of Gouno, Sen and Balakrishnan (2004), the overall PC proportion was kept uniform *on average* for all stages. That is, we let $E[c_i] = n\pi_i$ or simply $\pi_i^* = \dfrac{n\pi_i}{E[N_i]S_i(\tau)}$ for $i = 1, 2, \ldots, k-1$, where π_i is constant for all i. Tables 3.3 and 3.4 present the values of the censoring proportion π_i^* at the optima achieved by the time points presented in Tables 3.1 and 3.2, respectively.

Surprisingly, Tables 3.1 and 3.2 are identical to the ones presented in Gouno, Sen and Balakrishnan (2004) even for the newly added A-optimality criterion. With the chosen parameters, the optimal stress change points under a large sample (*i.e.*, early termination of a test disallowed) coincide with the optimal points under the modification of censoring by proportion (*i.e.*, early termination of a test allowed) as long as the number of items progressively censored at each stress level is the same *on average*. Nevertheless, the advantages of the modified model in Section 3.3 are clear when a practitioner or a test designer has to deal with a small sample size, high censoring proportions, or quite a few stress levels. Such situations prohibit us from using the protocol based on a large sample because the search region $\mathcal{C}_\tau$ for the optimal τ may not be defined at all. However, the modified model suggested here does not impose any restrictions on $\mathcal{C}_\tau$ and consequently, the optimal stress change points can be searched for any combinations of the parameter values.

We now summarize the findings from Tables 3.1 and 3.2 below:

- It is observed that $\tau_C^* > \tau_D^* > \tau_A^*$ except for the simple step-stress case with $\rho =$

0.5. This order, however, is a consequence of the specific setting chosen and does not necessarily hold for general stress levels. For the example considered here, the differences among τ_C^*, τ_D^* and τ_A^* are more pronounced for the simple step-stress case and they reduce rapidly as the number of stress levels k increases. Also, for a given k and ρ, the ratios $\frac{\tau_C^*}{\tau_D^*}$ and $\frac{\tau_D^*}{\tau_A^*}$ seem to remain constant over varying ranges of θ_1, and they form a decreasing function of the overall PC proportion.

- The optimal values in Table 3.1 dominate the corresponding values in Table 3.2. Interestingly, for a fixed k and ρ, the percentage reduction in the optimal values in Table 3.2 with respect to the corresponding ones in Table 3.1 remains reasonably constant across the choices of θ_1. For $k = 2$, for instance, the ratio $\frac{\tau^*_{C,Table3.2}}{\tau^*_{C,Table3.1}}$ is roughly stable around 83.2% with ρ fixed. As ρ increases, $\frac{\tau^*_{C,Table3.2}}{\tau^*_{C,Table3.1}}$, $\frac{\tau^*_{D,Table3.2}}{\tau^*_{D,Table3.1}}$ and $\frac{\tau^*_{A,Table3.2}}{\tau^*_{A,Table3.1}}$ decrease slightly for a given k. The dependence on ρ, however, is less noticeable for smaller values of k. These ratios also decrease steadily with increasing k.

- The behavior of the optimal τ as a function of the MTTF values is also interesting. For fixed k and ρ, as θ_1 increases, τ_C^*, τ_D^* and τ_A^* increase in a manner such that the ratios τ_C^*/θ_1, τ_D^*/θ_1 and τ_A^*/θ_1 are constant across the values of θ_1. This translates to τ_C^*, τ_D^* and τ_A^* being fixed percentiles from the stage-1 distribution, irrespective of the value of θ_1. This feature prevails in both Tables 3.1 and 3.2.

- As the shrinkage amount ρ increases with θ_1 and k fixed, τ_C^*, τ_D^* and τ_A^* all increase in such a way that the ratio of the increase is independent of the values of θ_1. Intuitively, this means that the more severe the successive stress levels are (*viz.*, smaller ρ), the more likely it is to observe failures in a short time interval. Hence, the choice of the optimal τ automatically forces the experiment to be terminated faster. The only exception is the simple step-stress case where it seems that ρ has very little effect in determining the optimal τ, especially τ_C^*.

- τ_C^*, τ_D^* and τ_A^* decrease quite rapidly as a function of k. In fact, both Tables 3.1 and 3.2 demonstrate that for $k = 4$ and small values of ρ, these optimal values are in the lower tail of the stage-1 life distribution. Consequently, it may frequently force to terminate the first stage of a life-test even before observing any failures. In that case, one practical strategy may be to continue the first-stage testing beyond τ_C^*, τ_D^* or τ_A^*.

Furthermore, the behavior of the objective functions were consistent for every optimality criterion. Figure 3.1 represents these behaviors under the modification introduced in Section 3.3. It presents the plots of $\phi(\tau)$, $\delta(\tau)$ and $a(\tau)$ for $k = 2, 3, 4$ with $\theta_1 = 100$, $\rho = 0.3$, and the expected overall PC proportion $\pi_i = 0.1$. The optimal stress change points are marked by the red dots, and the vertical dotted lines indicate the upper bounds of τ beyond which it is not possible to make the expected overall PC proportion uniform for all stress levels. For $k = 2, 3, 4$, these bounds are 230.3, 77.2, 24.5, respectively, and the bound decreases quite rapidly as k increases. As depicted in the figure, irrespective of the values of k, $\phi(\tau)$ is a convex function with a unique minimum while $\delta(\tau)$ and $a(\tau)$ are concave giving a unique maximum.

As mentioned earlier, Tables 3.3 and 3.4 list the values of π_i^* required to produce each optimal τ in Tables 3.1 and 3.2. We see that these fixed PC proportions

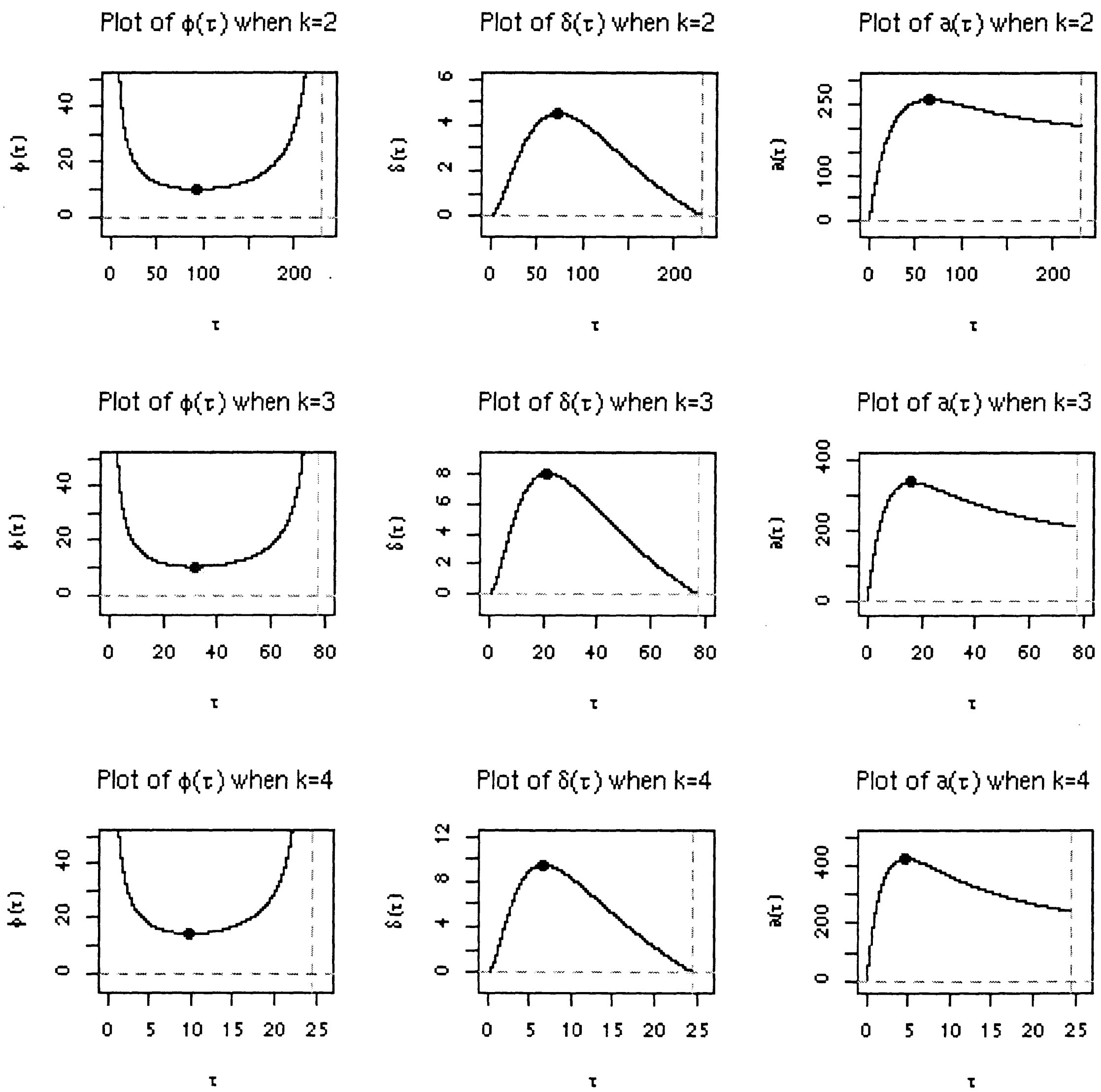

Figure 3.1: Plots of the Objective Functions for Each Optimality Criterion under the Modification of $c_i = (N_i - n_i)\pi_i^*$ with $\theta_1 = 100$, $\rho = 0.3$, and the Expected Overall PC Proportion at 10% (*viz.*, $\pi_i = 0.1$)

Table 3.3: Fixed PC Proportions under the Modification of $c_i = (N_i - n_i)\pi_i^*$

for the Expected Overall PC Proportion at 10% with $\theta_1 = 100, 300, 500$

$\pi_i = 0.1$	$k=2$			$k=3$						$k=4$								
Optimality	C	D	A	C		D		A		C			D			A		
	π_1^*	π_1^*	π_1^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*	π_3^*	π_1^*	π_2^*	π_3^*	π_1^*	π_2^*	π_3^*
$\rho = 0.1$	0.25	0.18	0.14	0.11	0.34	0.11	0.23	0.10	0.16	0.10	0.12	0.37	0.10	0.12	0.27	0.10	0.12	0.18
$\rho = 0.3$	0.25	0.21	0.19	0.14	0.45	0.12	0.29	0.12	0.23	0.11	0.17	0.62	0.11	0.15	0.37	0.10	0.14	0.27
$\rho = 0.5$	0.26	0.23	0.24	0.16	0.47	0.14	0.33	0.14	0.29	0.12	0.22	0.65	0.12	0.18	0.42	0.11	0.17	0.34

Table 3.4: Fixed PC Proportions under the Modification of $c_i = (N_i - n_i)\pi_i^*$

for the Expected Overall PC Proportion at 20% with $\theta_1 = 100, 300, 500$

$\pi_i = 0.2$	$k=2$			$k=3$						$k=4$								
Optimality	C	D	A	C		D		A		C			D			A		
	π_1^*	π_1^*	π_1^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*	π_3^*	π_1^*	π_2^*	π_3^*	π_1^*	π_2^*	π_3^*
$\rho = 0.1$	0.43	0.34	0.27	0.21	0.56	0.21	0.45	0.21	0.34	0.20	0.27	0.65	0.20	0.26	0.56	0.20	0.26	0.46
$\rho = 0.3$	0.44	0.38	0.36	0.25	0.65	0.24	0.53	0.23	0.47	0.21	0.31	0.80	0.21	0.30	0.69	0.21	0.29	0.62
$\rho = 0.5$	0.44	0.40	0.44	0.27	0.67	0.26	0.58	0.26	0.58	0.22	0.36	0.85	0.22	0.34	0.75	0.22	0.34	0.75

are at least the overall PC proportion specified, and for a fixed k and ρ, they form an increasing sequence (*i.e.*, $\pi_1^* < \pi_2^* < \cdots < \pi_{k-1}^*$) in order to keep the overall PC proportion uniform. In general, π_i^* is the highest for the C-optimality and the lowest for the A-optimality criterion. Of course, the higher the overall PC proportion is, the higher the fixed PC proportions are. One remark to make about these fixed PC proportions at the optima is that they do not depend on the value of θ_1 but slightly increase with ρ. The dependence on ρ, however, is little for the first stage of the test.

Similarly, we also constructed the objective function for each optimality criterion, using the conditional distribution results established in Section 3.5. Tables 3.5 and 3.6 present the results of this numerical study for a simple step-stress case with varying sample sizes. Again, to be able to compare the results with those from Gouno, Sen and Balakrishnan (2004) as well as the values in Tables 3.1 and 3.2, the expected overall PC proportion was kept constant by setting $E_c[c_1] = n\pi_1$ or simply $\pi_1^* = \dfrac{n\pi_1}{n - E_c[n_1]} = \dfrac{\pi_1(1-[F_1(\tau)]^n)}{S_1(\tau)}$. Tables 3.7 and 3.8 present these values of π_1^* at each optimal τ in Tables 3.5 and 3.6, respectively. From Tables 3.5 and 3.6, it is also noted that with the chosen parameters, the sample size required to produce the same optimal change points as in Tables 3.1 and 3.2 is at least 20. Intuitively, this means that the probability of a simple step-stress test terminating at the second stage is effectively 1 if the sample size is 20 or larger. Hence, we have numerically shown that the optimal τ conditioned on $N_k > 0$ converges to the unconditional one as the sample size increases.

Unfortunately, for small sample sizes, τ_A^* does not exist globally since the objective function $a(\tau)$ keeps increasing over the unrestricted range of τ. Thus, in the case of nonexistent τ_A^*, the choice of the optimal τ is completely up to the decision of

Table 3.5: Optimal Stress Change Points of the Simple Step-stress Testing ($k = 2$) under the Condition of $N_k > 0$ with the Expected Overall PC Proportion being 10%

$\pi_1 = 0.1$		$n = 5$			$n = 10$			$n \geq 20$		
		τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	119.6	71.2	(31.4)[a]	93.6	60.8	(30.9)	91.6	60.6	30.9
	$\rho = 0.3$	123.2	90.6	DNE[b]	95.7	73.3	(64.6)	93.6	72.7	64.1
	$\rho = 0.5$	130.5	113.6	DNE	97.7	82.5	(92.8)	95.1	81.2	87.7
$\theta_1 = 300$	$\rho = 0.1$	358.7	213.7	(94.2)	280.7	182.5	(92.8)	274.9	181.7	92.8
	$\rho = 0.3$	369.7	271.7	DNE	287.2	220.0	(193.7)	280.7	218.0	192.4
	$\rho = 0.5$	391.6	340.9	DNE	293.1	247.6	(278.4)	285.4	243.5	263.0
$\theta_1 = 500$	$\rho = 0.1$	597.9	356.2	(157.0)	467.9	304.1	(154.7)	458.2	302.9	154.7
	$\rho = 0.3$	616.1	452.9	DNE	478.7	366.7	(322.9)	467.8	363.3	320.6
	$\rho = 0.5$	652.6	568.1	DNE	488.5	412.7	(463.9)	475.7	405.8	438.3

[a]does not exist globally but locally exists under the constraint of $F(\tau) \leq 0.8$ or equivalently $\tau \leq \theta_1 \log 5$

[b]does not exist globally or locally

Table 3.6: Optimal Stress Change Points of the Simple Step-stress Testing ($k = 2$) under the Condition of $N_k > 0$ with the Expected Overall PC Proportion being 20%

$\pi_1 = 0.2$		$n = 5$			$n = 10$			$n \geq 20$		
		τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	87.1	56.9	(29.9)	76.8	52.4	29.5	76.3	52.3	29.5
	$\rho = 0.3$	89.8	71.2	DNE	78.5	63.3	59.4	77.9	63.1	59.1
	$\rho = 0.5$	91.9	81.8	DNE	79.1	69.8	81.3	78.4	69.3	79.0
$\theta_1 = 300$	$\rho = 0.1$	261.3	170.8	(89.6)	230.4	157.1	88.4	228.8	156.9	88.4
	$\rho = 0.3$	269.3	213.6	DNE	235.5	189.9	178.1	233.6	189.2	177.3
	$\rho = 0.5$	275.8	245.3	DNE	237.4	209.3	243.8	235.3	207.9	237.0
$\theta_1 = 500$	$\rho = 0.1$	435.5	284.6	(149.3)	384.0	261.9	147.4	381.3	261.5	147.4
	$\rho = 0.3$	448.8	356.0	DNE	392.5	316.5	296.8	389.4	315.3	295.5
	$\rho = 0.5$	459.6	408.8	DNE	395.7	348.8	406.3	392.2	346.6	395.0

a practitioner. In some cases, $a(\tau)$ exhibits a local maximum, and in order to capture this, we have imposed a constraint upon the search region for τ_A^* that the probability of observing a failure at the first stage should be at most 80%. That is, $F(\tau) \leq 0.8$ or equivalently $\tau \leq \theta_1 \log 5$. A summary of the findings from Tables 3.5 and 3.6 is as follows:

- The order of $\tau_C^* > \tau_D^* > \tau_A^*$ is again maintained except for the cases of $\rho = 0.5$, but this is merely an outcome of the specific setting chosen. For a given n and ρ, the ratios $\frac{\tau_C^*}{\tau_D^*}$ and $\frac{\tau_D^*}{\tau_A^*}$ seem to remain constant irrespective of the value of θ_1.

- The optimal values in Table 3.5 are larger than the corresponding values in Table 3.6. For a fixed n and ρ, the ratios of the optimal values in Table 3.6 to the corresponding values in Table 3.5 remain nearly constant across the choices of θ_1. For example, when $n = 5$, the ratio $\frac{\tau_{C,Table3.6}^*}{\tau_{C,Table3.5}^*}$ is roughly stable around 72.0% for fixed ρ. As ρ increases, $\frac{\tau_{C,Table3.6}^*}{\tau_{C,Table3.5}^*}$, $\frac{\tau_{D,Table3.6}^*}{\tau_{D,Table3.5}^*}$ and $\frac{\tau_{A,Table3.6}^*}{\tau_{A,Table3.5}^*}$ decrease slightly for fixed n. These ratios also form an increasing convergent sequence as n increases.

- With n and ρ fixed, τ_C^*, τ_D^* and τ_A^* increase as θ_1 increases in such a way that the ratios τ_C^*/θ_1, τ_D^*/θ_1 and τ_A^*/θ_1 are invariant no matter what the value of θ_1 is. This is interpreted as τ_C^*, τ_D^* and τ_A^* being fixed percentiles from the stage-1 distribution, irrespective of θ_1. This characteristic is persistent in both Tables 3.1 and 3.2.

- With θ_1 and n given, as ρ increases, τ_C^*, τ_D^* and τ_A^* all increase in such a way that the ratio of the increase is independent of the choice of θ_1. Nevertheless, the effect of ρ appears to be slight in determining the optimal τ, especially τ_C^*.

- As n increases, τ_C^*, τ_D^* and τ_A^* all decrease but converge to their respective unconditional ones.

Figure 3.2 describes the behavior of the objective functions under each optimality criterion. The plots of $\phi(\tau)$, $\delta(\tau)$ and $a(\tau)$ are shown for $k = 2$ with $n = 5$, $\theta_1 = 300$, and the expected overall PC proportion $\pi_1 = 0.2$. The optimal stress change points are marked by the red dots. The figure reveals that independent of the value of ρ, $\phi(\tau)$ is convex with a unique minimum and $\delta(\tau)$ yields a unique maximum with a horizontal asymptote at 0. Under the chosen setting, on the other hand, $a(\tau)$ yields a constrained local maximum only when $\rho = 0.1$ and it decreases as ρ increases.

The value of π_1^* for each optimal τ in Tables 3.5 and 3.6 are tabulated in Tables 3.7 and 3.8, respectively. Again, these fixed PC proportions are greater than the specified overall PC proportion. We also observe that π_1^* is generally the highest for the C-optimality and the lowest for the A-optimality criterion under the chosen setting. Moreover, the fixed PC proportions get higher if the overall PC proportion increases, just like in Tables 3.3 and 3.4. What is interesting about these fixed PC proportions is that they are not dependent on θ_1 but exhibit a very slight increment with ρ. As expected, they form a decreasing convergent sequence to the unconditional π_1^* as n increases.

In an attempt to assess the efficiencies of the different approaches to the optimization problem and to contrast the results obtained here, pairwise ratios of the optima under each criterion were calculated based on the optimal stress change points determined by Gouno, Sen and Balakrishnan (2004) and by the results developed here. Since Tables 3.1 and 3.2 yield not only the identical stress change points but

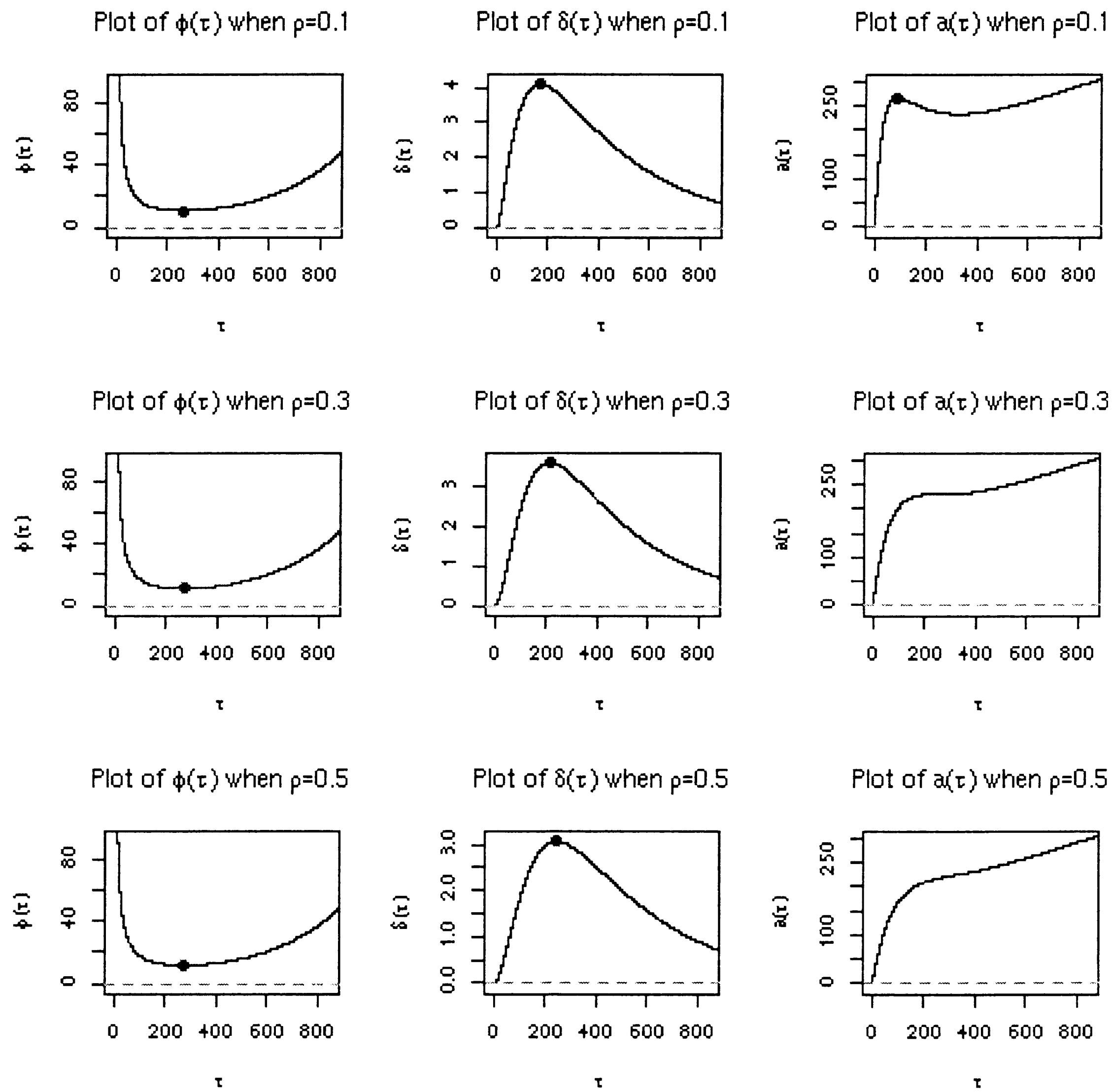

Figure 3.2: Plots of the Objective Functions for Each Optimality Criterion of the Simple Step-stress Testing ($k = 2$) under the Condition of $N_k > 0$ with $n = 5$, $\theta_1 = 300$, and the Expected Overall PC Proportion $\pi_1 = 0.2$

Table 3.7: Fixed PC Proportions π_1^* of the Simple Step-stress Testing ($k = 2$) under the Condition of $N_k > 0$ for the Expected Overall PC Proportion at 10% with $\theta_1 = 100, 300, 500$

$\pi_1 = 0.1$	$n = 5$			$n = 10$			$n \geq 20$		
Optimality	C	D	A	C	D	A	C	D	A
$\rho = 0.1$	0.28	0.20	(0.14)	0.25	0.18	(0.14)	0.25	0.18	0.14
$\rho = 0.3$	0.28	0.23	DNE	0.26	0.21	(0.19)	0.25	0.21	0.19
$\rho = 0.5$	0.29	0.27	DNE	0.26	0.23	(0.25)	0.26	0.23	0.24

Table 3.8: Fixed PC Proportions π_1^* of the Simple Step-stress Testing ($k = 2$) under the Condition of $N_k > 0$ for the Expected Overall PC Proportion at 20% with $\theta_1 = 100, 300, 500$

$\pi_1 = 0.2$	$n = 5$			$n = 10$			$n \geq 20$		
Optimality	C	D	A	C	D	A	C	D	A
$\rho = 0.1$	0.45	0.35	(0.27)	0.43	0.34	0.27	0.43	0.34	0.27
$\rho = 0.3$	0.46	0.39	DNE	0.44	0.38	0.36	0.44	0.38	0.36
$\rho = 0.5$	0.46	0.43	DNE	0.44	0.40	0.45	0.44	0.40	0.44

also exactly the same optima compared to the results in Gouno, Sen and Balakrishnan (2004), the efficiency between the modified (unconditional) model and the large sample model is not different with respect to the matched overall PC proportions. On the other hand, the efficiency of the conditional method relies upon the sample size n. Table 3.9 presents the ratios of the conditional optima to the unconditional ones for the simple step-stress case with varying sample sizes. Although these ratios are invariant across the values of θ_1, how they change with respect to other parameters is noticeable. With small n, large ρ and small π_1, we find that the efficiency of the conditional approach is higher for both C-optimality and D-optimality. For both optimality criteria, however, the differences become negligible as n gets larger since the conditional optima eventually converge to the unconditional ones obtained from the modified model. Another interesting observation is that irrespective of the sample size, the constrained τ_A^* presented in Tables 3.5 and 3.6 attains the local optimum that is identical to the global maximum attained by τ_A^* from the modified (unconditional) model. Therefore, one can always choose to increase the efficiency of the conditional approach under the A-optimality criterion by selecting an arbitrary τ which bears a higher optimum than the one achieved by τ_A^* from the unconditional model. For boosting the efficiency, however, one must be prepared to take a drastic increase in the whole test duration, too.

Table 3.9: Efficiency of the Simple Step-stress Testing ($k = 2$) under the Condition of $N_k > 0$ for the Expected Overall PC Proportion at 10% & 20% with $\theta_1 = 100, 300, 500$

		$n = 5$			$n = 10$			$n \geq 20$		
Optimality		C	D	A	C	D	A	C	D	A
$\pi_1 = 0.1$	$\rho = 0.1$	1.09	1.04	(1.00)	1.00	1.00	(1.00)	1.00	1.00	1.00
	$\rho = 0.3$	1.10	1.09	DNE	1.01	1.00	(1.00)	1.00	1.00	1.00
	$\rho = 0.5$	1.10	1.16	DNE	1.00	1.03	(1.00)	1.00	1.00	1.00
$\pi_1 = 0.2$	$\rho = 0.1$	1.05	1.02	(1.00)	1.01	1.00	1.00	1.00	1.00	1.00
	$\rho = 0.3$	1.05	1.06	DNE	1.01	1.00	1.00	1.00	1.00	1.00
	$\rho = 0.5$	1.06	1.11	DNE	1.00	1.00	1.00	1.00	1.00	1.00

Chapter 4

Exact Inference for a Simple Step-stress Model with Competing Risks for Failure from Exponential Distribution under Type-II Censoring

4.1 Introduction and motivation

As mentioned in Section 1.1, the step-stress ALT allows gradual increase of the stress levels at some pre-fixed time points during the experiment. This testing method has attracted great attention in the reliability literature. Sedyakin (1966) proposed one of the fundamental models in this area, known as the *cumulative damage* or cumulative exposure model. Recently, Balakrishnan *et al.* (2007), Balakrishnan and Xie (2007a,b), and Balakrishnan *et al.* (2008) have all discussed different inferential issues regarding the ALT under the assumption of this cumulative exposure model.

Furthermore, in reliability analysis, it is common that a failure is associated with one of several fatal risk factors the test unit is exposed to. Since it is not usually possible to study the test units with an isolated risk factor, it becomes necessary to assess each risk factor in the presence of other risk factors. In order to analyze such a competing risks model, each failure observation must come in a bivariate form composed of a failure time and the cause of failure. It is also assumed here that these competing risk factors are independent in the absence of covariates. Cox (1959), David and Moeschberger (1978), Klein and Basu (1981, 1982), and Crowder (2001) have all investigated the competing risks models and considered some specific parametric lifetime distributions for each risk factor. In addition to multiple causes of failure, censoring is also common in reliability experiments for various reasons as discussed in Section 1.2. Among different censoring schemes, the conventional Type-II right censoring corresponds to the situation when the experiment gets terminated once a pre-specified number of failures are observed.

In this chapter, we consider the simple step-stress model (*i.e.*, two stress levels) under Type-II censoring when the lifetime distributions of the different risk factors are independently exponentially distributed. In Section 4.2, we present the MLEs of the mean parameters of the different risk factors and show that these MLEs do not always exist. The conditional MLEs are therefore proposed and the exact conditional distributions of these MLEs are derived in Section 4.3. Based on the exact distributions of the MLEs, we propose exact confidence intervals for the unknown mean parameters in Section 4.4. We also present the asymptotic distributions of the MLEs and the corresponding asymptotic confidence intervals as well as the confidence intervals from a parametric bootstrap method. In Section 4.5, the performance of these confidence intervals is evaluated in terms of probability coverages via Monte Carlo simulations. In Section 4.6, we present a numerical example to illustrate all the methods of inference developed in this chapter, and some concluding remarks are finally made in Section 4.7.

4.2 Model description and MLEs

A random sample of n identical units is placed on a life test under the initial stress level s_1. The successive failure times are then recorded along with the information about which risk factor caused each failure. At a pre-fixed time τ, the stress level is increased to s_2 and the life test continues until a pre-specified r $(\leq n)$ number of failures are observed. When r is taken to be n, then a complete set of failure observations would result for this simple step-stress test (*i.e.*, no censoring). Suppose each unit fails by one of two fatal risk factors and the time-to-failure by each competing risk has an independent exponential distribution which obeys the cumulative exposure model. Let θ_{ij} be the mean time-to-failure of a test unit at the stress level s_i by the risk factor j for $i, j = 1, 2$. Then, the CDF of the lifetime T_j due to the risk factor j is given by

$$G_j(t) = G_j(t;\theta_{1j},\theta_{2j}) = \begin{cases} 1-\exp\Big\{-\frac{1}{\theta_{1j}}t\Big\} & \text{if } 0<t<\tau \\ 1-\exp\Big\{-\frac{1}{\theta_{1j}}\tau-\frac{1}{\theta_{2j}}(t-\tau)\Big\} & \text{if } \tau\le t<\infty \end{cases}$$

for $j=1,2$, and the corresponding PDF of T_j is given by

$$g_j(t) = g_j(t;\theta_{1j},\theta_{2j}) = \begin{cases} \frac{1}{\theta_{1j}}\exp\Big\{-\frac{1}{\theta_{1j}}t\Big\} & \text{if } 0<t<\tau \\ \frac{1}{\theta_{2j}}\exp\Big\{-\frac{1}{\theta_{1j}}\tau-\frac{1}{\theta_{2j}}(t-\tau)\Big\} & \text{if } \tau\le t<\infty \end{cases}$$

for $j=1,2$. Since we will observe only the smaller of T_1 and T_2, let $T=\min\{T_1,T_2\}$ denote the overall failure time of a test unit. Then, its CDF and PDF are readily obtained to be

$$\begin{aligned} F(t) &= F(t;\boldsymbol{\theta}) = 1-\big(1-G_1(t)\big)\big(1-G_2(t)\big) \\ &= \begin{cases} 1-\exp\Big\{-\Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)t\Big\} & \text{if } 0<t<\tau \\ 1-\exp\Big\{-\Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)\tau-\Big(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\Big)(t-\tau)\Big\} & \text{if } \tau\le t<\infty \end{cases}, \end{aligned} \tag{4.2.1}$$

$$\begin{aligned} f(t) &= f(t;\boldsymbol{\theta}) \\ &= \begin{cases} \Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)\exp\Big\{-\Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)t\Big\} & \text{if } 0<t<\tau \\ \Big(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\Big)\exp\Big\{-\Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)\tau-\Big(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\Big)(t-\tau)\Big\} & \text{if } \tau\le t<\infty \end{cases}, \end{aligned} \tag{4.2.2}$$

respectively, where $\boldsymbol{\theta}=(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$ with $\boldsymbol{\theta}_i=(\theta_{i1},\theta_{i2})$ for $i=1,2$. Furthermore, let C denote the indicator for the cause of failure. Then, under the assumptions specified earlier, the joint PDF of (T,C) is given by

$$\begin{aligned} f_{T,C}(t,j) &= g_j(t)\big(1-G_{j'}(t)\big) \\ &= \begin{cases} \frac{1}{\theta_{1j}}\exp\Big\{-\Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)t\Big\} & \text{if } 0<t<\tau \\ \frac{1}{\theta_{2j}}\exp\Big\{-\Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)\tau-\Big(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\Big)(t-\tau)\Big\} & \text{if } \tau\le t<\infty \end{cases} \end{aligned} \tag{4.2.3}$$

for $j, j' = 1, 2$ and $j' \neq j$. We also denote the relative risk imposed on a test unit before τ due to the risk factor j by

$$\pi_{1j} = Pr\big[C = j \big| 0 < T < \tau\big] = \frac{\theta_{1j}^{-1}}{\theta_{11}^{-1} + \theta_{12}^{-1}}, \qquad j = 1, 2. \tag{4.2.4}$$

Similarly, the relative risk after τ due to the factor j is denoted by

$$\pi_{2j} = Pr\big[C = j \big| T \geq \tau\big] = \frac{\theta_{2j}^{-1}}{\theta_{21}^{-1} + \theta_{22}^{-1}}, \qquad j = 1, 2. \tag{4.2.5}$$

They are simply the proportion of failure rates in the given time frame. One can then easily see from (4.2.3)-(4.2.5) that T and C are independent given the time frame in which a failure has occurred.

Let us now define

$$\begin{aligned} N_{1j} &= \text{the number of units that fail before } \tau \text{ due to the risk factor } j, \\ N_{2j} &= \text{the number of units that fail after } \tau \text{ due to the risk factor } j \end{aligned}$$

for $j = 1, 2$. If we let $N_{1\cdot}$ denote the total number of failures before τ and $N_{2\cdot}$ the total number of failures after τ, then according to the testing scheme, we have $N_{1\cdot} = N_{11} + N_{12}$ and $N_{2\cdot} = N_{21} + N_{22}$ with $N_{1\cdot} + N_{2\cdot} = r \leq n$. With the life-testing scheme described above, the following ordered failure times will then be observed:

$$\Big\{0 < t_{1:n} < \cdots < t_{n_{1\cdot}:n} < \tau \leq t_{n_{1\cdot}+1:n} < \cdots < t_{r:n}\Big\},$$

where $n_{1\cdot}$ denotes the observed value of $N_{1\cdot}$. For notational simplicity, let us express $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2)$ with $\mathbf{N}_i = (N_{i1}, N_{i2})$ for $i = 1, 2$, and let $\mathbf{n}$ denote the observed integer vector of $\mathbf{N}$.

Since each failure time is also accompanied by the corresponding cause of failure, let $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ be the observed sequence of the cause of failure corresponding to the observed failure times $\mathbf{t} = (t_{1:n}, t_{2:n}, \ldots, t_{r:n})$. Then, under the assumption of the cumulative exposure model, we formulate the likelihood function of $\boldsymbol{\theta}$ based on this Type-II censored data as

$$\begin{aligned} L(\boldsymbol{\theta}) &= L\big(\boldsymbol{\theta}|(\mathbf{t},\mathbf{c})\big) \\ &= \frac{n!}{(n-r)!}\left\{\prod_{i=1}^{n_{1\cdot}} f_{T,C}\big(t_{i:n},c_i\big)\right\}\left\{\prod_{i=n_{1\cdot}+1}^{r} f_{T,C}\big(t_{i:n},c_i\big)\right\}\Big\{1-F(t_{r:n})\Big\}^{n-r} \\ &= \frac{n!}{(n-r)!}\left\{\prod_{i,j=1}^{2}\theta_{ij}^{-n_{ij}}\right\}\exp\left\{-\Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)U_1-\Big(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\Big)U_2\right\} \end{aligned} \tag{4.2.6}$$

for $0 < t_{1:n} < \cdots < t_{n_{1\cdot}:n} < \tau \le t_{n_{1\cdot}+1:n} < \cdots < t_{r:n} < \infty$, where

$$r = n_{1\cdot} + n_{2\cdot} = (n_{11}+n_{12}) + (n_{21}+n_{22}),$$

$$U_1 = \sum_{i=1}^{n_{1\cdot}} t_{i:n} + (n-n_{1\cdot})\tau, \tag{4.2.7}$$

$$U_2 = \sum_{i=n_{1\cdot}+1}^{r} (t_{i:n}-\tau) + (n-r)(t_{r:n}-\tau). \tag{4.2.8}$$

Note that U_i is precisely the *Total Time on Test* statistic at the stress level s_i. From the likelihood function in (4.2.6), one can easily see that the MLE of θ_{ij} does not exist if $n_{ij} = 0$ for any $i, j = 1, 2$. That is, at least one failure caused by each risk factor must be observed at each stress level in order to estimate $\boldsymbol{\theta}$ simultaneously. This imposes the condition that $N_{ij} \ge 1$ for all $i, j = 1, 2$ and consequently, we have to ensure $4 \le r \le n$ in the planning stage of the experiment. In general, r has to be at least the product of the number of stress levels implemented and the number of fatal risk factors under consideration. Once this condition is fulfilled, the log-likelihood function of $\boldsymbol{\theta}$ is readily obtained from (4.2.6) as

$$\begin{aligned} l(\boldsymbol{\theta}) &= l\big(\boldsymbol{\theta}|(\mathbf{t},\mathbf{c})\big) = \log L(\boldsymbol{\theta}) \\ &= \log\frac{n!}{(n-r)!} - \sum_{i,j=1}^{2} n_{ij}\log\theta_{ij} - \Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)U_1 - \Big(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\Big)U_2 \end{aligned}$$

from which the MLE of θ_{ij} is readily obtained as

$$\hat{\theta}_{ij} = \frac{U_i}{n_{ij}}, \qquad i, j = 1, 2. \tag{4.2.9}$$

Remark 4.2.1. *In the model considered above, we have not assumed any relationships among the mean failure times by the two risk factors under the two stress levels. In some situations, we may know that some particular relationships hold among them; for instance, $\theta_{2j} = \rho_j \theta_{1j}$ with known ρ_j $(0 < \rho_j < 1)$ for $j = 1, 2$. In that case, the MLE of $\boldsymbol{\theta}$ exists whenever at least one failure occurs by each risk factor, and their exact distributions can be derived explicitly. One can also use the likelihood ratio test statistic to test the multiple hypotheses $H_0:\ \theta_{2j} = \rho_j \theta_{1j}$ for specified ρ_j's.*

Remark 4.2.2. *The model proposed above can be easily extended and generalized to accommodate multiple stress levels and multiple competing risks. In fact, the model under consideration is also general in the sense that it includes its marginal models as special cases. For instance, when $\theta_{11}, \theta_{21} \to \infty$ or $\theta_{12}, \theta_{22} \to \infty$, the limiting case of the above model is the simple step-stress model without the competing risk structure. If we rather let $\tau \to \infty$, then the model converges to the ordinary single stress model (i.e., one stress level only) with two competing risks.*

4.3 Conditional distributions of MLEs

To find the exact distributions of $\hat{\theta}_{ij}$, we first derive the conditional moment generating function (CMGF) of $\hat{\theta}_{ij}$, conditioned on $\left\{N_{i'j'} \geq 1 \text{ for } i', j' = 1, 2 \text{ and } \sum_{i',j'=1}^{2} N_{i'j'} = r\right\}$. Let $\mathfrak{S}$ be the set of positive integer vectors that $\mathbf{N}$ can take on satisfying this condition. That is,

$$\mathfrak{S} \equiv \left\{ \mathbf{n} \;\middle|\; n_{ij} \geq 1 \text{ for } i,j = 1,2 \text{ and } \sum_{i,j=1}^{2} n_{ij} = r \right\}.$$

Then, the given condition is equivalent to $\mathbf{N} \in \mathfrak{S}$. For notational convenience, we denote $M_{ij}(t)$ for the CMGF of $\hat{\theta}_{ij}$, $i,j = 1,2$. We can then write

$$\begin{aligned} M_{ij}(t) &= E\Big[e^{t\hat{\theta}_{ij}}\Big|\mathbf{N} \in \mathfrak{S}\Big] \\ &= \sum_{\mathbf{n}\in\mathfrak{S}} E\Big[e^{t\hat{\theta}_{ij}}\Big|\mathbf{N}_i = \mathbf{n}_i\Big] \times Pr\Big[\mathbf{N} = \mathbf{n}\Big|\mathbf{N} \in \mathfrak{S}\Big] \end{aligned} \tag{4.3.1}$$

for $i,j = 1,2$. Using the lemmas presented in Appendix C, the joint probability mass function (JPMF) of $\mathbf{N}$ is given by

$$Pr\big[\mathbf{N} = \mathbf{n}\big] = \binom{n}{n_{11}, n_{12}, n - n_{1\cdot}} \binom{n_{2\cdot}}{n_{21}} \left\{ \prod_{i,j=1}^{2} \pi_{ij}^{n_{ij}} \right\} \big\{F(\tau)\big\}^{n_{1\cdot}} \big\{1 - F(\tau)\big\}^{n-n_{1\cdot}}$$

for $\mathbf{n} = (n_{11}, n_{12}, n_{21}, n_{22}) \in \mathfrak{S}$, where π_{ij} are as defined in (4.2.4) and (4.2.5) and

$$F(\tau) = 1 - \exp\left\{ -\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\tau \right\}. \tag{4.3.2}$$

We then simply have

$$Pr\Big[\mathbf{N} = \mathbf{n}\Big|\mathbf{N} \in \mathfrak{S}\Big] = \frac{Pr\big[\mathbf{N} = \mathbf{n}\big]}{\sum_{\mathbf{m}\in\mathfrak{S}} Pr\big[\mathbf{N} = \mathbf{m}\big]}, \qquad \mathbf{n} \in \mathfrak{S}. \tag{4.3.3}$$

Subsequently, $E\Big[e^{t\hat{\theta}_{ij}}\Big|\mathbf{N}_i = \mathbf{n}_i\Big]$ (for $i,j = 1,2$) can be derived using the lemmas presented in Appendix C. Then, by inverting $M_{ij}(t)$, the CMGF of $\hat{\theta}_{ij}$, we can establish the following theorems regarding the conditional distribution of $\hat{\theta}_{ij}$, the proofs of which are presented in Appendix C.

Theorem 4.3.1. *The conditional PDF of $\hat{\theta}_{1j}$, given $\mathbf{N} \in \mathfrak{S}$, is*

$$f_{\hat{\theta}_{1j}}(x) = f_{\hat{\theta}_{1j}}\big(x\big|\mathbf{N} \in \mathfrak{S}\big) = \sum_{\mathbf{n}\in\mathfrak{S}} \sum_{k=0}^{n_{1\cdot}} C_{\mathbf{n},jk}^{[1]}\, \gamma\left(x - \tau_{1jk}\ ;\ n_{1\cdot}, n_{1j}\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\right) \tag{4.3.4}$$

for $j = 1, 2$*, where*

$$\tau_{1jk} = (n - n_{1\cdot} + k)\,\frac{\tau}{n_{1j}}, \tag{4.3.5}$$

$$C^{[1]}_{\mathbf{n},jk} = \frac{(-1)^k}{\sum_{\mathbf{m}\in\mathfrak{S}} Pr\big[\mathbf{N}=\mathbf{m}\big]}\binom{n}{n_{11}, n_{12}, n-n_{1\cdot}}\binom{n_{1\cdot}}{k}\binom{n_{2\cdot}}{n_{21}}\left\{\prod_{i',j'=1}^{2}\pi_{i'j'}^{n_{i'j'}}\right\} \times \exp\left\{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)\tau_{1jk}n_{1j}\right\}, \tag{4.3.6}$$

$$\gamma(y\ ;\ \alpha,\lambda) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1}e^{-\lambda y} & ,\ y > 0 \\ 0 & ,\ otherwise \end{cases} \qquad for\ \ \alpha,\lambda > 0. \tag{4.3.7}$$

Corollary 4.3.1. *The first two raw moments of* $\hat{\theta}_{1j}$ *are*

$$E\big[\hat{\theta}_{1j}\big] = \sum_{\mathbf{n}\in\mathfrak{S}}\sum_{k=0}^{n_{1\cdot}} C^{[1]}_{\mathbf{n},jk}\left\{\frac{n_{1\cdot}}{n_{1j}}\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)^{-1} + \tau_{1jk}\right\}$$

and

$$E\big[\hat{\theta}^2_{1j}\big] = \sum_{\mathbf{n}\in\mathfrak{S}}\sum_{k=0}^{n_{1\cdot}} C^{[1]}_{\mathbf{n},jk}\left\{\frac{n_{1\cdot}(n_{1\cdot}+1)}{n^2_{1j}}\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)^{-2} + \tau^2_{1jk} + \frac{2n_{1\cdot}}{n_{1j}}\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)^{-1}\tau_{1jk}\right\},$$

respectively, for $j = 1, 2$.

Proof. These expressions follow readily from (4.3.4). □

Theorem 4.3.2. *The conditional PDF of* $\hat{\theta}_{2j}$*, given* $\mathbf{N}\in\mathfrak{S}$*, is*

$$f_{\hat{\theta}_{2j}}(x) = f_{\hat{\theta}_{2j}}\big(x\big|\mathbf{N}\in\mathfrak{S}\big) = \sum_{\mathbf{n}\in\mathfrak{S}} C^{[2]}_{\mathbf{n}}\ \gamma\left(x\ ;\ n_{2\cdot}, n_{2j}\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)\right) \tag{4.3.8}$$

for $j = 1, 2$, where

$$\begin{aligned} C_{\mathbf{n}}^{[2]} &= Pr\Big[\mathbf{N} = \mathbf{n}\Big|\mathbf{N} \in \mathfrak{S}\Big] = \frac{Pr\big[\mathbf{N} = \mathbf{n}\big]}{\sum_{\mathbf{m}\in\mathfrak{S}} Pr\big[\mathbf{N} = \mathbf{m}\big]} \\ &= \frac{1}{\sum_{\mathbf{m}\in\mathfrak{S}} Pr\big[\mathbf{N} = \mathbf{m}\big]} \binom{n}{n_{11}, n_{12}, n - n_{1\cdot}} \binom{n_{2\cdot}}{n_{21}} \Bigg\{ \prod_{i,j=1}^{2} \pi_{ij}^{n_{ij}} \Bigg\} \\ &\quad \times \Bigg(1 - \exp\Bigg\{ - \bigg(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\bigg)\tau \Bigg\}\Bigg)^{n_{1\cdot}} \exp\Bigg\{ - \bigg(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\bigg)\tau(n - n_{1\cdot}) \Bigg\} \end{aligned} \tag{4.3.9}$$

and $\gamma(\cdot\ ;\ \alpha, \lambda)$ is as defined in (4.3.7).

Corollary 4.3.2. *The first two raw moments of $\hat{\theta}_{2j}$ are*

$$E\big[\hat{\theta}_{2j}\big] = \sum_{\mathbf{n}\in\mathfrak{S}} C_{\mathbf{n}}^{[2]} \left\{ \frac{n_{2\cdot}}{n_{2j}} \bigg(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\bigg)^{-1} \right\}$$

and

$$E\big[\hat{\theta}_{2j}^2\big] = \sum_{\mathbf{n}\in\mathfrak{S}} C_{\mathbf{n}}^{[2]} \left\{ \frac{n_{2\cdot}(n_{2\cdot}+1)}{n_{2j}^2} \bigg(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\bigg)^{-2} \right\},$$

respectively, for $j = 1, 2$.

Proof. These expressions follow readily from (4.3.8). □

For $j = 1, 2$, the distribution of $\hat{\theta}_{1j}$, conditioned on $\mathbf{N} \in \mathfrak{S}$, is a generalized mixture of gamma distributions while the conditional distribution of $\hat{\theta}_{2j}$ is a true mixture of gamma distributions since $C_{\mathbf{n}}^{[2]}$ in (4.3.9) is the conditional JPMF of $\mathbf{N}$, given $\mathbf{N} \in \mathfrak{S}$. The expressions for the expected values clearly reveal that $\hat{\theta}_{ij}$ is a biased estimator of θ_{ij} for all $i, j = 1, 2$. The expressions for the first two raw moments given in Corollaries 4.3.1 and 4.3.2 can be used to calculate the standard errors of the estimates. We can also derive the expressions for the tail probabilities by integrating the conditional PDFs of $\hat{\theta}_{ij}$ given above. These expressions, presented in the following corollary, are used to construct exact confidence intervals for θ_{ij} later in Section 4.4.

Corollary 4.3.3. *The tail probabilities of $\hat{\theta}_{ij}$ (for $i, j = 1, 2$) are given by*

$$Pr\left[\hat{\theta}_{1j} > \xi\right] = \sum_{\mathbf{n}\in\mathfrak{S}} \sum_{k=0}^{n_{1\cdot}} C^{[1]}_{\mathbf{n},jk}\, \Gamma\left(n_{1j}\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\langle \xi - \tau_{1jk} \rangle \; ; \; n_{1\cdot}\right)$$

and

$$Pr\left[\hat{\theta}_{2j} > \xi\right] = \sum_{\mathbf{n}\in\mathfrak{S}} C^{[2]}_{\mathbf{n}}\, \Gamma\left(n_{2j}\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)\xi \; ; \; n_{2\cdot}\right),$$

where

$$\begin{aligned} \langle\, \epsilon \,\rangle &= \max\{0, \epsilon\}, \\ \Gamma(\epsilon \; ; \; \alpha) &= \begin{cases} \displaystyle\int_{\epsilon}^{\infty} \gamma(y \; ; \; \alpha, 1)\,dy = \int_{\epsilon}^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\,dy & , \ \epsilon > 0 \\ 1 & , \ \epsilon \le 0 \end{cases}. \end{aligned}$$

4.4 Confidence intervals

Different methods of constructing confidence intervals (CIs) for the unknown parameters θ_{ij} are discussed in this section. Based on the exact conditional distributions of the MLEs from Theorems 4.3.1 and 4.3.2, we can construct exact CIs for θ_{ij}. Due to the complexity of the exact conditional distributions of $\hat{\theta}_{ij}$, we also present the approximate CIs for θ_{ij} based on the asymptotic distributions of the estimators for a large sample size. Finally, the parametric bootstrap method is used to construct CIs for θ_{ij}.

4.4.1 Exact confidence intervals

In order to ensure the invertibility of the pivotal quantities for the parameters θ_{ij}, we assume that the tail probability of $\hat{\theta}_{ij}$ presented in Corollary 4.3.3 is monotonically increasing with respect to θ_{ij} for each $i, j = 1, 2$. As in Section 2.5, we are unable to establish the required monotonicity in an analytical way due to the complex structure of the exact distributions of the pivotal quantities. The extensive numerical computations, however, seem to support this monotonicity assumption (see Figure 4.1).

Let us now construct the exact $100(1-\alpha)\%$ CI for θ_{1j}, $j = 1, 2$. We first denote θ_{1j}^L and θ_{1j}^U for the lower and upper bounds of the two-sided CI for θ_{1j}, respectively. Then, by the monotonicity assumption, they are the unique solutions of the equations

$$Pr\left[\hat{\theta}_{1j} > \hat{\theta}_{1j}^{obs}\right] = \frac{\alpha}{2}$$

and

$$Pr\left[\hat{\theta}_{1j} > \hat{\theta}_{1j}^{obs}\right] = 1 - \frac{\alpha}{2},$$

respectively, where $\hat{\theta}_{1j}^{obs}$ is simply the observed value of the MLE of θ_{1j}. Since θ_{1j}^L and θ_{1j}^U can not be expressed in an explicit form, they are numerically obtained by solving the following two non-linear equations using some iterative techniques such as the bisection method, Newton-Raphson method or Brent's method:

$$\frac{\alpha}{2} = \sum_{\mathbf{n}\in\mathfrak{S}}\sum_{k=0}^{n_{1\cdot}} C_{\mathbf{n},jk}^{[1]}\,\Gamma\biggl(n_{1j}\biggl(\frac{1}{\theta_{1j}^{L}}+\frac{1}{\theta_{1j'}}\biggr)\bigl\langle\hat{\theta}_{1j}^{obs}-\tau_{1jk}\bigr\rangle\ ;\ n_{1\cdot}\biggr), \quad (4.4.1)$$

$$1-\frac{\alpha}{2} = \sum_{\mathbf{n}\in\mathfrak{S}}\sum_{k=0}^{n_{1\cdot}} C_{\mathbf{n},jk}^{[1]}\,\Gamma\biggl(n_{1j}\biggl(\frac{1}{\theta_{1j}^{U}}+\frac{1}{\theta_{1j'}}\biggr)\bigl\langle\hat{\theta}_{1j}^{obs}-\tau_{1jk}\bigr\rangle\ ;\ n_{1\cdot}\biggr) \quad (4.4.2)$$

for $j = 1,2$ with $j' \neq j$, where τ_{1jk}, $C_{\mathbf{n},jk}^{[1]}$ and $\Gamma(\cdot\ ;\alpha)$ are as defined earlier. Note that the coefficients $C_{\mathbf{n},jk}^{[1]}$ in the above two equations are functions of $\boldsymbol{\theta}$. Hence, before solving for the confidence limits for fixed j, we replace θ_{1j} in $C_{\mathbf{n},jk}^{[1]}$ in an appropriate manner. That is, θ_{1j}^{L} is substituted for θ_{1j} in $C_{\mathbf{n},jk}^{[1]}$ of (4.1) and likewise θ_{1j}^{U} for θ_{1j} in $C_{\mathbf{n},jk}^{[1]}$ of (4.2). The observed values of the MLEs are also substituted for all the other unknown parameters in the expressions given above.

Using a similar argument, the two-sided $100(1-\alpha)\%$ CI for θ_{2j}, denoted by $\bigl(\theta_{2j}^{L},\ \theta_{2j}^{U}\bigr)$, can be numerically obtained as the unique solution of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{\mathbf{n}\in\mathfrak{S}} C_{\mathbf{n}}^{[2]}\,\Gamma\biggl(n_{2j}\biggl(\frac{1}{\theta_{2j}^{L}}+\frac{1}{\theta_{2j'}}\biggr)\hat{\theta}_{2j}^{obs}\ ;\ n_{2\cdot}\biggr), \quad (4.4.3)$$

$$1-\frac{\alpha}{2} = \sum_{\mathbf{n}\in\mathfrak{S}} C_{\mathbf{n}}^{[2]}\,\Gamma\biggl(n_{2j}\biggl(\frac{1}{\theta_{2j}^{U}}+\frac{1}{\theta_{2j'}}\biggr)\hat{\theta}_{2j}^{obs}\ ;\ n_{2\cdot}\biggr) \quad (4.4.4)$$

for $j = 1,2$ with $j' \neq j$ where $\hat{\theta}_{2j}^{obs}$ is an observed value of the MLE of θ_{2j}, and $C_{\mathbf{n}}^{[2]}$ and $\Gamma(\cdot\ ;\alpha)$ are as defined earlier. In the above two equations, the coefficients $C_{\mathbf{n}}^{[2]}$ are functions of $\boldsymbol{\theta}$ and thus, for fixed j, θ_{2j}^{L} is substituted for θ_{2j} in $C_{\mathbf{n}}^{[2]}$ of (4.3) and likewise θ_{2j}^{U} for θ_{2j} in $C_{\mathbf{n}}^{[2]}$ of (4.4) before solving the equations. Again, the other unknown parameters are replaced by the observed values of their respective MLEs in the above expressions.

4.4.2 Approximate confidence intervals

As discussed in Section 2.5.2, the MLEs exhibit asymptotically optimal characteristics when the sample size grows. Under certain regularity conditions, the MLEs are asymptotically unbiased and efficient. That is, their bias tends to zero and their variances achieve the Cramer-Rao lower bounds as the sample size grows to infinity. Furthermore, their distribution approaches normal with the variance-covariance matrix given by the inverse of the Fisher information matrix; see Silvey (1975), and Casella and Berger (2002) for details. Hence, inference about the unknown parameters can be based on the asymptotic normality of the MLEs. In this subsection, we present an approximate method to construct CIs for θ_{ij} using these properties of the MLEs for large sample sizes. Although the exact method described in the preceding subsection is preferable, its computation encounters some difficulties for large samples. On the other hand, the approximate method provides not only the computational ease but also a good probability coverage (close to the nominal level) when the sample size gets large. This finding is further discussed in Section 4.5.

Let us first denote the (expected) Fisher information matrix of $\boldsymbol{\theta}$ by

$$\mathbf{I}_E(\boldsymbol{\theta}) = \left[I_{ij:i'j'}\right]_{i,j,i',j'=1,2},$$

where

$$I_{ij:i'j'} = -E\left[\frac{\partial l(\boldsymbol{\theta})}{\partial\theta_{ij}\partial\theta_{i'j'}}\right] = \begin{cases} E\left[-\dfrac{N_{ij}}{\theta_{ij}^2} + \dfrac{2U_i}{\theta_{ij}^3}\right] & , \ i = i' \text{ and } j = j' \\ 0 & , \ otherwise \end{cases}$$

with U_1 and U_2 being as defined in (4.2.7) and (4.2.8), respectively. It is clear that $\mathbf{I}_E(\boldsymbol{\theta})$ is a diagonal matrix and by substituting $\hat{\theta}_{ij}$ for θ_{ij}, the observed Fisher infor-

Table 4.7: Average Widths of Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 40$, $r = 30$ and $B = 1000$

Nominal CL		90%			95%			99%		
Parameter	τ	Exact	Approx	Boot	Exact	Approx	Boot	Exact	Approx	Boot
θ_{11}	1	35.421	12.627	16.093	41.213	15.046	21.041	54.793	19.773	29.681
	2	18.864	7.380	9.718	20.447	8.794	13.187	30.406	11.558	24.267
	3	8.485	5.818	7.067	13.252	6.933	9.084	18.291	9.111	14.990
	4	6.869	5.329	6.000	9.065	6.350	7.535	15.697	8.345	11.539
	5	5.901	4.870	5.298	7.225	5.803	6.452	12.023	7.626	9.346
	6	5.234	4.649	4.968	6.417	5.539	6.094	11.776	7.280	8.650
θ_{12}	1	66.671	38.387	25.034	76.901	45.741	28.688	83.608	60.114	32.394
	2	48.090	27.734	33.713	56.240	33.047	42.926	70.153	43.431	57.971
	3	32.826	21.038	29.332	42.244	25.068	40.402	65.338	32.945	64.914
	4	21.354	17.826	23.415	28.351	21.241	32.684	44.428	27.916	56.474
	5	18.070	15.848	20.567	22.859	18.884	30.161	34.862	24.817	52.738
	6	15.948	14.360	18.568	20.045	17.111	25.531	30.218	22.488	45.642
θ_{21}	1	2.434	2.261	2.338	2.974	2.694	2.824	4.167	3.540	3.890
	2	2.864	2.597	2.786	3.537	3.095	3.408	5.095	4.068	4.819
	3	3.460	3.023	3.310	4.342	3.602	4.098	6.563	4.734	6.123
	4	4.120	3.393	4.137	5.369	4.043	5.280	9.062	5.314	8.380
	5	6.454	4.261	5.153	9.323	5.077	6.562	18.807	6.672	10.690
	6	10.662	5.299	6.446	15.384	6.314	8.435	35.594	8.298	13.424
θ_{22}	1	8.621	7.331	9.028	10.893	8.736	12.081	16.719	11.481	21.151
	2	9.999	7.978	11.340	13.005	9.506	15.346	21.553	12.493	26.297
	3	17.434	11.099	14.204	22.537	13.225	19.108	38.199	17.381	31.251
	4	28.243	13.704	17.190	36.297	16.330	22.580	60.667	21.461	34.379
	5	41.147	15.726	18.583	53.024	18.738	23.147	85.390	24.626	33.388
	6	60.692	17.684	18.222	74.906	21.072	22.897	108.851	27.694	31.464

Table 4.8: Average Bounds of 90% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 40$, $r = 30$ and $B = 1000$

Parameter	τ	Exact CI	Approximate CI	BCa Bootstrap CI
θ_{11}	1	(4.628, 40.049)	(1.109, 13.736)	(4.172, 20.265)
	2	(4.446, 23.310)	(2.923, 10.304)	(4.146, 13.864)
	3	(4.459, 12.944)	(3.409, 9.227)	(4.234, 11.302)
	4	(4.518, 11.387)	(3.695, 9.024)	(4.318, 10.318)
	5	(4.586, 10.487)	(3.815, 8.685)	(4.307, 9.605)
	6	(4.633, 9.867)	(3.920, 8.569)	(4.344, 9.312)
θ_{12}	1	(6.718, 73.389)	(0.000, 34.771)	(8.360, 33.393)
	2	(7.076, 55.166)	(1.430, 29.164)	(8.135, 41.848)
	3	(8.293, 41.119)	(3.699, 24.737)	(8.131, 37.464)
	4	(7.606, 28.960)	(4.818, 22.644)	(7.831, 31.246)
	5	(7.731, 25.801)	(5.489, 21.337)	(7.952, 28.519)
	6	(7.717, 23.665)	(5.860, 20.220)	(8.090, 26.658)
θ_{21}	1	(2.172, 4.606)	(1.933, 4.194)	(2.070, 4.409)
	2	(2.086, 4.950)	(1.782, 4.380)	(2.000, 4.786)
	3	(2.004, 5.464)	(1.617, 4.640)	(1.876, 5.187)
	4	(1.850, 5.970)	(1.378, 4.771)	(1.767, 5.904)
	5	(1.782, 8.236)	(1.116, 5.377)	(1.645, 6.798)
	6	(1.689, 12.352)	(0.752, 6.050)	(1.528, 7.974)
θ_{22}	1	(3.928, 12.549)	(2.905, 10.236)	(3.908, 12.936)
	2	(3.601, 13.600)	(2.392, 10.370)	(3.710, 15.050)
	3	(3.572, 21.006)	(1.528, 12.627)	(3.611, 17.816)
	4	(3.387, 31.630)	(0.550, 14.255)	(3.630, 20.820)
	5	(3.180, 44.327)	(0.000, 15.315)	(3.654, 22.237)
	6	(2.993, 63.685)	(0.000, 16.248)	(3.357, 21.579)

Table 4.9: Average Bounds of 95% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 40$, $r = 30$ and $B = 1000$

Parameter	τ	Exact CI	Approximate CI	BCa Bootstrap CI
θ_{11}	1	(4.278, 45.491)	(0.000, 14.946)	(3.826, 24.867)
	2	(4.354, 24.801)	(2.216, 11.011)	(3.850, 17.036)
	3	(4.467, 17.719)	(2.852, 9.785)	(3.947, 13.031)
	4	(4.573, 13.638)	(3.184, 9.534)	(4.032, 11.566)
	5	(4.599, 11.824)	(3.349, 9.151)	(4.005, 10.457)
	6	(4.664, 11.081)	(3.475, 9.014)	(4.065, 10.159)
θ_{12}	1	(5.958, 82.859)	(0.000, 38.448)	(7.437, 36.125)
	2	(6.536, 62.776)	(0.000, 31.820)	(7.422, 50.348)
	3	(7.613, 49.857)	(1.684, 26.752)	(7.500, 47.902)
	4	(6.907, 35.258)	(3.111, 24.352)	(7.238, 39.921)
	5	(7.057, 29.916)	(3.971, 22.855)	(7.559, 37.720)
	6	(7.072, 27.117)	(4.484, 21.596)	(7.488, 33.019)
θ_{21}	1	(2.038, 5.012)	(1.716, 4.410)	(1.906, 4.730)
	2	(1.940, 5.477)	(1.534, 4.629)	(1.818, 5.226)
	3	(1.844, 6.186)	(1.328, 4.930)	(1.678, 5.776)
	4	(1.683, 7.052)	(1.053, 5.096)	(1.553, 6.833)
	5	(1.592, 10.915)	(0.708, 5.785)	(1.408, 7.970)
	6	(1.479, 16.862)	(0.244, 6.558)	(1.280, 9.715)
θ_{22}	1	(3.590, 14.483)	(2.203, 10.938)	(3.572, 15.653)
	2	(3.258, 16.263)	(1.628, 11.134)	(3.360, 18.706)
	3	(3.169, 25.706)	(0.465, 13.690)	(3.223, 22.331)
	4	(2.947, 39.245)	(0.000, 15.568)	(3.171, 25.751)
	5	(2.714, 55.738)	(0.000, 16.821)	(2.988, 26.135)
	6	(2.509, 77.414)	(0.000, 17.942)	(2.773, 25.670)

Table 4.10: Average Bounds of 99% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 40$, $r = 30$ and $B = 1000$

Parameter	τ	Exact CI	Approximate CI	BCa Bootstrap CI
θ_{11}	1	(3.149, 57.942)	(0.000, 17.309)	(3.247, 32.928)
	2	(3.590, 33.996)	(0.835, 12.393)	(3.343, 27.610)
	3	(3.733, 22.024)	(1.762, 10.874)	(3.440, 18.430)
	4	(3.822, 19.519)	(2.187, 10.532)	(3.512, 15.051)
	5	(3.857, 15.880)	(2.437, 10.063)	(3.488, 12.834)
	6	(3.930, 15.706)	(2.604, 9.884)	(3.541, 12.190)
θ_{12}	1	(4.148, 87.756)	(0.000, 45.634)	(6.171, 38.565)
	2	(5.359, 75.512)	(0.000, 37.012)	(6.290, 64.261)
	3	(6.548, 71.886)	(0.000, 30.690)	(6.416, 71.330)
	4	(5.777, 50.205)	(0.000, 27.689)	(6.233, 62.707)
	5	(5.952, 40.813)	(1.004, 25.822)	(6.499, 59.237)
	6	(6.001, 36.219)	(1.796, 24.284)	(6.448, 52.090)
θ_{21}	1	(1.804, 5.970)	(1.293, 4.834)	(1.568, 5.458)
	2	(1.687, 6.782)	(1.047, 5.115)	(1.432, 6.250)
	3	(1.568, 8.131)	(0.762, 5.496)	(1.263, 7.386)
	4	(1.395, 10.457)	(0.418, 5.731)	(1.103, 9.483)
	5	(1.269, 20.075)	(0.000, 6.583)	(0.928, 11.618)
	6	(1.129, 36.723)	(0.000, 7.550)	(0.795, 14.219)
θ_{22}	1	(3.026, 19.745)	(0.830, 12.311)	(2.976, 24.127)
	2	(2.690, 24.244)	(0.134, 12.627)	(2.750, 29.047)
	3	(2.522, 40.721)	(0.000, 15.768)	(2.559, 33.810)
	4	(2.261, 62.927)	(0.000, 18.133)	(2.423, 36.801)
	5	(2.014, 87.404)	(0.000, 19.765)	(2.144, 35.532)
	6	(1.802, 110.652)	(0.000, 21.253)	(1.845, 33.309)

observation is apparent for the exact CIs and the approximate CIs. This is not surprising because when τ is small, the number of failures before τ will be fewer than those after τ and so inference for θ_{1j} will not be very precise requiring much wider intervals than θ_{2j} for a fixed level of confidence. However, as τ increases, the situation is reversed providing more information about θ_{1j} as compared to θ_{2j} and in turn, the width of the interval is shortened for θ_{1j} while it is lengthened for θ_{2j}.

As the sample size grows, the larger computational time as well as the unstable precision (especially for θ_{1j}, $j = 1, 2$) becomes a problematic issue for constructing CIs by the exact method. Hence, based on the simulation study, we recommend the use of the bootstrap approach to construct CIs for θ_{ij} when r and n are considerably large since it offers computational feasibility and also performs reasonably well in terms of probability coverages and widths for large sample sizes. But, for small sample sizes (say, $r \leq n \leq 40$), the exact method developed here is the one that is recommended.

4.6 Illustrative example

We have simulated a Type-II censored sample from a simple step-stress test with two competing risks in order to illustrate the methods of inference described in the preceding sections. The dataset was generated with the following choices of the parameters:

$$\theta_{11} = 8.96,\ \theta_{12} = 12.18,\ \theta_{21} = 4.48,\ \theta_{22} = 4.06$$

along with the stress change time point $\tau = 3$. In this setup, when the stress level increases, there is a 50% decrease in the mean time to failure caused by the risk factor

1 and a 67% decrease in the mean time to failure caused by the risk factor 2. Also, at the initial stress level, there is a 58% chance for a test unit to fail by the risk factor 1 but it drops to 47% after the increment of the stress level at τ. From the initial sample size of $n = 25$ with $r = 20$ for 20% right censoring, the observed times to failure are presented in Table 4.11 below.

From this dataset, we have $n_{11} = 7$, $n_{12} = 5$, $n_{21} = 4$, $n_{22} = 4$ and hence, the observed MLEs of θ_{ij} are found from (4.2.9) to be

$$\hat{\theta}_{11} = 7.510, \; \hat{\theta}_{12} = 10.514, \; \hat{\theta}_{21} = 4.128, \; \hat{\theta}_{22} = 4.128.$$

The CIs for θ_{ij} are also presented in Table 4.12 using all three methods described in Section 4.4. Since the exact CIs for θ_{ij} require the monotonicity of the tail probability functions of $\hat{\theta}_{ij}$, we provide the numerical justification of this assumption by plotting the tail probabilities with the observed values of the MLEs from the sample. From Figure 4.1, it is evident that all the plots show the monotonically increasing behavior of the tail probabilities with respect to the unknown parameter θ_{ij} for each $i, j = 1, 2$. In addition, the two horizontal lines corresponding to the values of $\alpha/2$ and $1 - \alpha/2$ for $\alpha = 0.10$ are overlaid in each plot of Figure 4.1 to illustrate how the exact CIs are constructed by inverting the tail probabilities of $\hat{\theta}_{ij}$. In the first plot of Figure 4.1, for example, the values of θ_{11} from the two intersecting points are the unique solutions of Eqs. (4.4.1) and (4.4.2), respectively, and together they provide the exact 90% CI for θ_{11}.

From Table 4.12, we observe that the exact CIs are always wider than the other two intervals. The approximate method provides the narrowest CIs in general while the BCa bootstrap CIs are sometimes narrower and at other times wider than the

Table 4.11: Type-II Censored Sample from $n = 25$ units on a Simple Step-stress Test with Two Competing Risks, $\tau = 3$ and $r = 20$

Stress Level 1 (before $\tau = 3$)		**Stress Level 2** (after $\tau = 3$)	
Failure Time	Failure Cause	Failure Time	Failure Cause
0.145	1	3.105	1
0.289	1	3.537	2
0.345	2	3.608	2
0.382	1	3.621	1
0.575	2	3.640	2
0.577	1	3.814	1
1.126	1	4.514	2
1.588	1	4.946	1
1.597	2		
1.772	1		
2.428	2		
2.744	2		
$n_{1\cdot} = 12$		$n_{2\cdot} = 8$	

Table 4.12: Interval Estimation based on the Type-II Censored Step-stress Data in Table 4.11 with $B = 1000$

Parameter	CL	Exact CI	Approximate CI	BCa Bootstrap CI
	90%	(4.065, 14.534)	(2.841, 12.178)	(4.253, 13.572)
$\theta_{11} = 8.96$	95%	(3.647, 16.912)	(1.947, 13.073)	(3.906, 15.606)
	99%	(3.031, 23.207)	(0.198, 14.821)	(3.375, 22.179)
	90%	(5.213, 23.442)	(2.780, 18.247)	(5.691, 20.453)
$\theta_{12} = 12.18$	95%	(4.635, 28.457)	(1.298, 19.729)	(5.216, 24.401)
	99%	(3.755, 43.314)	(0.000, 22.625)	(4.534, 34.546)
	90%	(1.902, 13.310)	(0.733, 7.524)	(1.706, 10.861)
$\theta_{21} = 4.48$	95%	(1.633, 19.177)	(0.083, 8.174)	(1.412, 14.185)
	99%	(1.162, 83.142)	(0.000, 9.445)	(1.097, 26.105)
	90%	(1.902, 13.310)	(0.733, 7.524)	(1.956, 12.303)
$\theta_{22} = 4.06$	95%	(1.633, 19.177)	(0.083, 8.174)	(1.637, 18.534)
	99%	(1.162, 83.142)	(0.000, 9.445)	(1.263, 30.015)

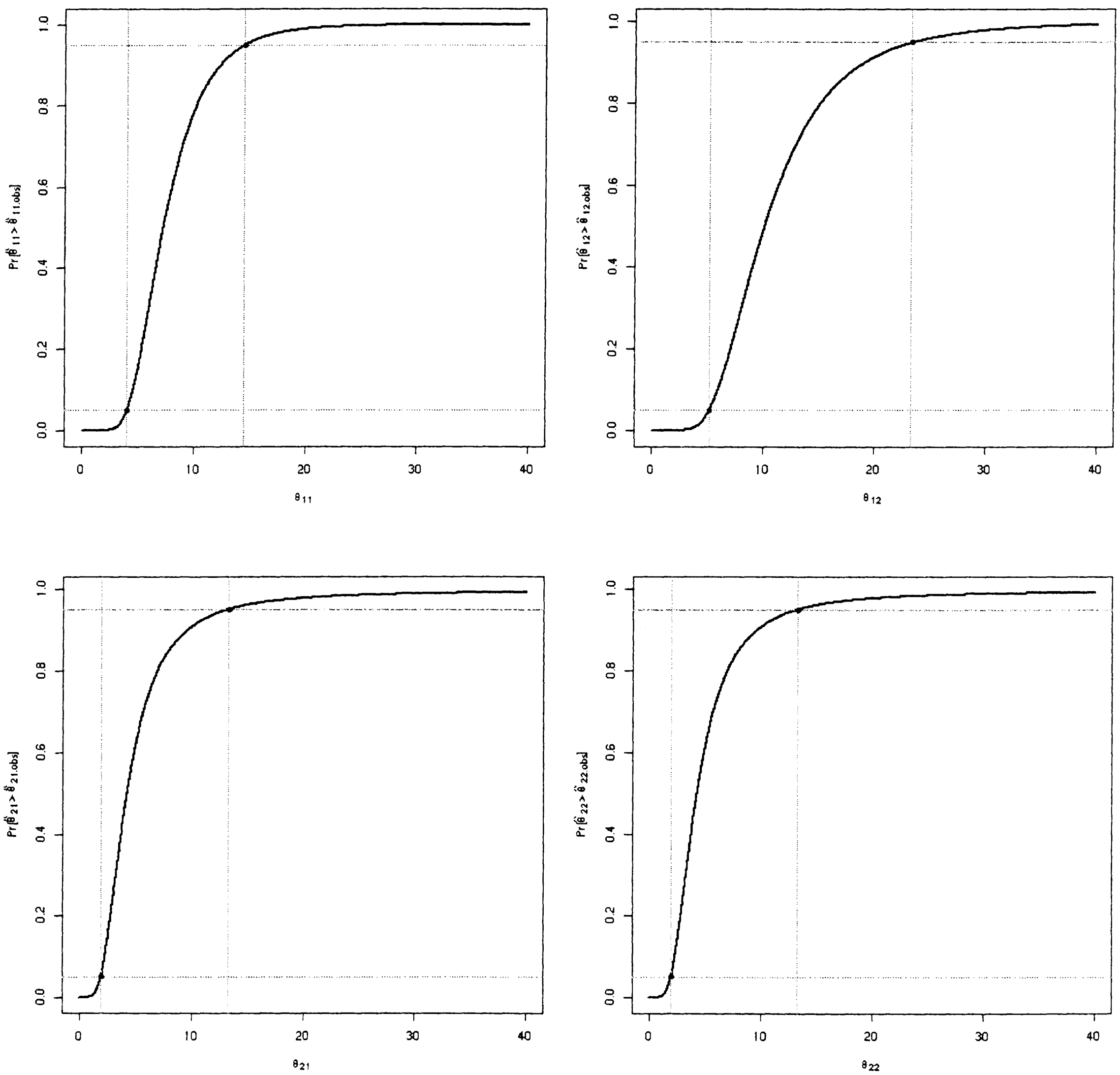

Figure 4.1: Plots of Tail Probabilities of $\hat{\theta}_{ij}$, $Pr\left[\hat{\theta}_{ij} > \hat{\theta}_{ij}^{obs}\right]$, with respect to θ_{ij} and Exact 90% Confidence Intervals for θ_{ij} from the Step-stress Data in Table 4.11

approximate ones. This is so since the coverage probabilities for the approximate CIs and the bootstrap CIs are significantly lower than the nominal levels (see Tables 4.1 and 4.6). We also note from Table 4.12 that the CIs for θ_{11} are consistently narrower than those for θ_{12}. The primary reason for this is that when θ_{11} is smaller than θ_{12}, we expect a relatively large number of failures to occur before τ by the risk factor 1 than by the risk factor 2, resulting in lower variability in the estimation of θ_{11} than θ_{12}, and vice versa. It is also interesting to observe from Table 4.12 that the exact CIs for θ_{21} and θ_{22} are identical as well as their approximate CIs. This is to be expected because the exact method and the approximate method both depend on the observed values of the MLEs and the MLEs are sensitive to the number of failures caused by each risk factor before and after τ. Since $n_{21} = n_{22} = 4$ in this example, the realized values of $\hat{\theta}_{21}$ and $\hat{\theta}_{22}$ happen to be equal and this in turn yields the same estimates for the tail probabilities of $\hat{\theta}_{21}$ and $\hat{\theta}_{22}$ (see Figure 4.1). Consequently, the inference regarding θ_{21} and θ_{22} comes out be identical in this case.

4.7 Summary and conclusions

In this chapter, we have discussed the simple step-stress model under Type-II censoring when the lifetimes corresponding to different risk factors have independent exponential distributions. We have derived the MLEs of the unknown mean parameters θ_{ij} under the assumption of a cumulative exposure model and their exact conditional distributions through the use of the CMGF. We have also proposed several different procedures for constructing CIs for θ_{ij}. We have then conducted a simulation study to assess the performance of all these procedures and a numerical example has been presented to illustrate all the methods of inference developed in this chapter. Based on the results of the simulation study, our recommendation for constructing CIs for θ_{ij} is to apply the exact method whenever possible, especially in the case of small sample sizes (*e.g.*, $r \leq n \leq 40$) since the other two methods are unsatisfactory in terms of probability coverages. For larger sample sizes, however, the BCa percentile bootstrap method is more appropriate because of its computational ease as well as for its improved probability coverages being close to the nominal levels.

Chapter 5

Exact Inference for a Simple Step-stress Model with Competing Risks for Failure from Exponential Distribution under Time Constraint

5.1 Introduction and motivation

In Chapter 4, we have developed the method of the exact inference for the simple step-stress model under Type-II censoring. In this chapter, we consider the simple step-stress model under time constraint (*i.e.*, Type-I censoring) when the lifetime distributions of the different risk factors are independently exponentially distributed. Type-I right censoring corresponds to the situation when the experiment gets terminated at a pre-fixed time point. Compared to the Type-II censoring, it has the clear advantage of the known termination time for the test, which makes it more appealing for the actual implementation in the test design. In Section 5.2, we present the MLEs of the mean parameters of the different risk factors and show that these MLEs do not always exist. The conditional MLEs are therefore proposed and the exact conditional distributions of these MLEs are derived in Section 5.3. Based on the exact distributions of the MLEs, we propose exact confidence intervals for the unknown mean parameters in Section 5.4 along with the confidence intervals from the asymptotic distributions of the MLEs and the parametric bootstrap method. In Section 5.5, the performance of these confidence intervals is evaluated in terms of probability coverages via Monte Carlo simulations. In Section 5.6, we present a numerical example to illustrate the methods of inference developed in this chapter, and some concluding remarks are finally made in Section 5.7.

5.2 Model description and MLEs

A random sample of n identical units is placed on a life test under the initial stress level s_1. The successive failure times are then recorded along with the information about which risk factor caused each failure. At a pre-fixed time τ, the stress level is increased to s_2 and the life test continues until a pre-specified censoring time τ_c ($> \tau$). When all n units fail before τ_c, then a complete set of failure observations would result for this simple step-stress test (*i.e.*, no censoring). Suppose each unit fails by one of two fatal risk factors and the time-to-failure by each competing risk has an independent exponential distribution which obeys the cumulative exposure model. Let θ_{ij} be the mean time-to-failure of a test unit at the stress level s_i by the risk factor j for $i, j = 1, 2$. Then, as in Section 4.2, the CDF of the lifetime T_j due to the risk factor j is given by

$$G_j(t) = G_j(t; \theta_{1j}, \theta_{2j}) = \begin{cases} 1 - \exp\left\{ -\frac{1}{\theta_{1j}} t \right\} & \text{if } 0 < t < \tau \\ 1 - \exp\left\{ -\frac{1}{\theta_{1j}}\tau - \frac{1}{\theta_{2j}}(t-\tau) \right\} & \text{if } \tau \le t < \infty \end{cases}$$

for $j = 1, 2$, and the corresponding PDF of T_j is given by

$$g_j(t) = g_j(t; \theta_{1j}, \theta_{2j}) = \begin{cases} \frac{1}{\theta_{1j}} \exp\left\{ -\frac{1}{\theta_{1j}} t \right\} & \text{if } 0 < t < \tau \\ \frac{1}{\theta_{2j}} \exp\left\{ -\frac{1}{\theta_{1j}}\tau - \frac{1}{\theta_{2j}}(t-\tau) \right\} & \text{if } \tau \le t < \infty \end{cases}$$

for $j = 1, 2$. Since we will observe only the smaller of T_1 and T_2, let $T = \min\{T_1, T_2\}$ denote the overall failure time of a test unit. Then, its CDF and PDF are readily obtained to be

$$\begin{aligned} F(t) &= F(t; \boldsymbol{\theta}) = 1 - \big(1 - G_1(t)\big)\big(1 - G_2(t)\big) \\ &= \begin{cases} 1 - \exp\left\{ -\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right) t \right\} & \text{if } 0 < t < \tau \\ 1 - \exp\left\{ -\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\tau - \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)(t-\tau) \right\} & \text{if } \tau \le t < \infty \end{cases}, \end{aligned} \tag{5.2.1}$$

$$\begin{aligned} f(t) &= f(t; \boldsymbol{\theta}) \\ &= \begin{cases} \left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right) \exp\left\{ -\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right) t \right\} & \text{if } 0 < t < \tau \\ \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right) \exp\left\{ -\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\tau - \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)(t-\tau) \right\} & \text{if } \tau \le t < \infty \end{cases}, \end{aligned} \tag{5.2.2}$$

respectively, where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ with $\boldsymbol{\theta}_i = (\theta_{i1}, \theta_{i2})$ for $i = 1, 2$. Furthermore, let C denote the indicator for the cause of failure. Then, the joint PDF of (T, C) is given by

$$f_{T,C}(t,j) = g_j(t)\big(1 - G_{j'}(t)\big)$$
$$= \begin{cases} \frac{1}{\theta_{1j}} \exp\Big\{ -\Big(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\Big)t\Big\} & \text{if } 0 < t < \tau \\ \frac{1}{\theta_{2j}} \exp\Big\{ -\Big(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\Big)\tau - \Big(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\Big)(t-\tau)\Big\} & \text{if } \tau \le t < \infty \end{cases} \tag{5.2.3}$$

for $j, j' = 1, 2$ and $j' \neq j$. We also denote the relative risk imposed on a test unit before τ due to the risk factor j by

$$\pi_{1j} = Pr\big[C = j \big| 0 < T < \tau\big] = \frac{\theta_{1j}^{-1}}{\theta_{11}^{-1} + \theta_{12}^{-1}}, \qquad j = 1, 2. \tag{5.2.4}$$

Similarly, the relative risk after τ due to the factor j is denoted by

$$\pi_{2j} = Pr\big[C = j \big| T \ge \tau\big] = \frac{\theta_{2j}^{-1}}{\theta_{21}^{-1} + \theta_{22}^{-1}}, \qquad j = 1, 2. \tag{5.2.5}$$

They are simply the proportion of failure rates in the given time frame. One can then easily see from (5.2.3)-(5.2.5) that T and C are independent given the time frame in which a failure has occurred.

With the life-testing scheme described above, the following ordered failure times will be observed:

$$\Big\{0 < t_{1:n} < \cdots < t_{n_{1\cdot}:n} < \tau \le t_{n_{1\cdot}+1:n} < \cdots < t_{n_{\cdot\cdot}:n} < \tau_c\Big\},$$

where $n_{1\cdot}$ denotes the observed value of $N_{1\cdot}$, the total number of failures before τ. Similarly, $n_{\cdot\cdot}$ denotes the observed value of $N_{\cdot\cdot}$, the accumulated number of failures until τ_c according to the testing scheme. If we let $N_{2\cdot}$ denote the total number of failures between τ and τ_c so that $N_{1\cdot} + N_{2\cdot} = N_{\cdot\cdot} \le n$, then we can express

N_{1j} = the number of units that fail between 0 and τ due to the risk factor j,

N_{2j} = the number of units that fail between τ and τ_c due to the risk factor j

for $j = 1, 2$ such that $N_{1\cdot} = N_{11} + N_{12}$ and $N_{2\cdot} = N_{21} + N_{22}$. Let us denote $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2)$ with $\mathbf{N}_i = (N_{i1}, N_{i2})$ for $i = 1, 2$, and let $\mathbf{n}$ denote the observed integer vector of $\mathbf{N}$.

Since each failure time is also accompanied by the corresponding cause of failure, let $\mathbf{c} = (c_1, c_2, \ldots, c_{n..})$ be the observed sequence of the cause of failure corresponding to the observed failure times $\mathbf{t} = (t_{1:n}, t_{2:n}, \ldots, t_{n..:n})$. Then, under the assumption of the cumulative exposure model, we formulate the likelihood function of $\boldsymbol{\theta}$ based on this Type-I censored data as

$$
\begin{aligned}
L(\boldsymbol{\theta}) &= L\big(\boldsymbol{\theta}|(\mathbf{t},\mathbf{c})\big) \\
&= \frac{n!}{(n-n..)!}\left\{\prod_{i=1}^{n_{1\cdot}} f_{T,C}\big(t_{i:n}, c_i\big)\right\}\left\{\prod_{i=n_{1\cdot}+1}^{n..} f_{T,C}\big(t_{i:n}, c_i\big)\right\}\Big\{1 - F(\tau_c)\Big\}^{n-n..} \\
&= \frac{n!}{(n-n..)!}\left\{\prod_{i,j=1}^{2} \theta_{ij}^{-n_{ij}}\right\}\exp\left\{-\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)U_1 - \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)U_2\right\}
\end{aligned}
\tag{5.2.6}
$$

for $0 < t_{1:n} < \cdots < t_{n_{1\cdot}:n} < \tau \leq t_{n_{1\cdot}+1:n} < \cdots < t_{n..:n} < \tau_c$, where

$$
\begin{aligned}
n.. &= n_{1\cdot} + n_{2\cdot} = (n_{11} + n_{12}) + (n_{21} + n_{22}), \\
U_1 &= \sum_{i=1}^{n_{1\cdot}} t_{i:n} + (n - n_{1\cdot})\tau, && (5.2.7) \\
U_2 &= \sum_{i=n_{1\cdot}+1}^{n..} (t_{i:n} - \tau) + (n - n..)(\tau_c - \tau). && (5.2.8)
\end{aligned}
$$

Note that U_i is the *Total Time on Test* statistic at the stress level s_i. From the likelihood function in (5.2.6), one can easily see that the MLE of θ_{ij} does not exist if $n_{ij} = 0$ for any $i, j = 1, 2$. That is, at least one failure caused by each risk factor must be observed at each stress level in order to estimate $\boldsymbol{\theta}$ simultaneously. This imposes the condition that $N_{ij} \geq 1$ for all $i, j = 1, 2$ and consequently, the acceptable sample size needs to be much larger than 4 in the planning stage of the experiment. In general, n has to be at least the product of the number of stress levels implemented and the number of fatal risk factors under consideration. Once this condition is fulfilled, the log-likelihood function of $\boldsymbol{\theta}$ is readily obtained from (5.2.6) as

$$\begin{aligned} l(\boldsymbol{\theta}) &= l\big(\boldsymbol{\theta}|(\mathbf{t},\mathbf{c})\big) = \log L(\boldsymbol{\theta}) \\ &= \log\frac{n!}{(n-n..)!} - \sum_{i,j=1}^{2} n_{ij}\log\theta_{ij} - \left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)U_1 - \left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)U_2 \end{aligned}$$

from which the MLE of θ_{ij} is readily obtained as

$$\hat{\theta}_{ij} = \frac{U_i}{n_{ij}}, \qquad i,j=1,2. \tag{5.2.9}$$

Remark 5.2.1. *Again, in the model considered above, we have not assumed any relationships among the mean failure times by the two risk factors under the two stress levels. In some situations, we may know that some particular relationships hold among them; for instance,* $\theta_{2j} = \rho_j\theta_{1j}$ *with known* ρ_j $(0 < \rho_j < 1)$ *for* $j = 1, 2$. *In that case, the MLE of* $\boldsymbol{\theta}$ *exists whenever at least one failure occurs by each risk factor, and their exact distributions can be derived explicitly. One can also use the likelihood ratio test statistic to test the multiple hypotheses* $H_0 : \theta_{2j} = \rho_j\theta_{1j}$ *for specified* ρ_j*'s.*

5.3 Conditional distributions of MLEs

To find the exact distribution of $\hat{\theta}_{ij}$, we first derive the CMGF of $\hat{\theta}_{ij}$, conditioned on $\left\{N_{i'j'} \geq 1 \text{ for } i',j' = 1,2 \text{ and } \sum_{i',j'=1}^{2} N_{i'j'} \leq n\right\}$. Let $\mathfrak{S}$ be the set of positive integer vectors that $\mathbf{N}$ can take on satisfying this condition. That is,

$$\mathfrak{S} \equiv \left\{ \mathbf{n} \;\middle|\; n_{ij} \geq 1 \text{ for } i,j = 1,2 \text{ and } \sum_{i,j=1}^{2} n_{ij} \leq n \right\}.$$

Then, the given condition is equivalent to $\mathbf{N} \in \mathfrak{S}$. For notational convenience, we denote $M_{ij}(t)$ for the CMGF of $\hat{\theta}_{ij}$, $i,j = 1,2$. We can then write

$$\begin{aligned} M_{1j}(t) &= E\left[e^{t\hat{\theta}_{1j}} \middle| \mathbf{N} \in \mathfrak{S}\right] \\ &= \sum_{\mathbf{n} \in \mathfrak{S}} E\left[e^{t\hat{\theta}_{1j}} \middle| \mathbf{N}_1 = \mathbf{n}_1\right] \times Pr\left[\mathbf{N} = \mathbf{n} \middle| \mathbf{N} \in \mathfrak{S}\right] \end{aligned} \tag{5.3.1}$$

and

$$\begin{aligned} M_{2j}(t) &= E\left[e^{t\hat{\theta}_{2j}} \middle| \mathbf{N} \in \mathfrak{S}\right] \\ &= \sum_{\mathbf{n} \in \mathfrak{S}} E\left[e^{t\hat{\theta}_{2j}} \middle| N_{1\cdot} = n_{1\cdot},\ \mathbf{N}_2 = \mathbf{n}_2\right] \times Pr\left[\mathbf{N} = \mathbf{n} \middle| \mathbf{N} \in \mathfrak{S}\right] \end{aligned} \tag{5.3.2}$$

for $j = 1,2$. From the lemmas presented in Appendix D, we find that $\mathbf{N}$ has a multinomial distribution with its JPMF as

$$\begin{aligned} Pr\left[\mathbf{N} = \mathbf{n}\right] &= \binom{n}{n_{11}, n_{12}, n_{21}, n_{22}, n - n_{\cdot\cdot}} \left\{ \prod_{i,j=1}^{2} \pi_{ij}^{n_{ij}} \right\} \\ &\quad \times \left\{F(\tau)\right\}^{n_{1\cdot}} \left\{F(\tau_c) - F(\tau)\right\}^{n_{2\cdot}} \left\{1 - F(\tau_c)\right\}^{n - n_{\cdot\cdot}} \end{aligned}$$

for $\mathbf{n} = (n_{11}, n_{12}, n_{21}, n_{22}) \in \mathfrak{S}$, where π_{ij} are as defined in (5.2.4) and (5.2.5) and $F(t)$ is as given in (5.2.1). Hence, we simply obtain

$$Pr\left[\mathbf{N} = \mathbf{n} \middle| \mathbf{N} \in \mathfrak{S}\right] = \frac{Pr\left[\mathbf{N} = \mathbf{n}\right]}{\sum_{\mathbf{m} \in \mathfrak{S}} Pr\left[\mathbf{N} = \mathbf{m}\right]}, \qquad \mathbf{n} \in \mathfrak{S}. \tag{5.3.3}$$

Subsequently, $E\left[e^{t\hat{\theta}_{1j}} \middle| \mathbf{N}_1 = \mathbf{n}_1\right]$ and $E\left[e^{t\hat{\theta}_{2j}} \middle| N_{1\cdot} = n_{1\cdot},\ \mathbf{N}_2 = \mathbf{n}_2\right]$ (for $j = 1,2$) can be derived using the lemmas presented in Appendix D. Then, by inverting $M_{ij}(t)$, the CMGF of $\hat{\theta}_{ij}$, we can establish the following theorems regarding the conditional distribution of $\hat{\theta}_{ij}$, the proofs of which are presented in Appendix D.

Theorem 5.3.1. *The conditional PDF of $\hat{\theta}_{1j}$, given $\mathbf{N} \in \mathfrak{S}$, is*

$$f_{\hat{\theta}_{1j}}(x) = f_{\hat{\theta}_{1j}}\big(x \big| \mathbf{N} \in \mathfrak{S}\big) = \sum_{\mathbf{n} \in \mathfrak{S}} \sum_{k=0}^{n_{1\cdot}} C^{[1]}_{\mathbf{n},jk}\ \gamma\bigg(x - \tau_{1jk}\ ;\ n_{1\cdot}, n_{1j}\Big(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\Big)\bigg) \quad (5.3.4)$$

for $j = 1, 2$, where

$$\begin{aligned} \tau_{1jk} &= (n - n_{1\cdot} + k)\ \frac{\tau}{n_{1j}}, & (5.3.5)\\ C^{[1]}_{\mathbf{n},jk} &= \frac{(-1)^k}{\sum_{\mathbf{m} \in \mathfrak{S}} Pr\big[\mathbf{N} = \mathbf{m}\big]} \binom{n}{n_{11}, n_{12}, n_{21}, n_{22}, n - n_{\cdot\cdot}} \binom{n_{1\cdot}}{k} \Bigg\{ \prod_{i',j'=1}^{2} \pi_{i'j'}^{n_{i'j'}} \Bigg\} \\ &\quad \times \exp\bigg\{ - \Big(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\Big) \tau_{1jk} n_{1j} - \Big(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\Big)(\tau_c - \tau)(n - n_{\cdot\cdot}) \bigg\} \\ &\quad \times \bigg(1 - \exp\bigg\{ - \Big(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\Big)(\tau_c - \tau)\bigg\}\bigg)^{n_{2\cdot}} & (5.3.6) \end{aligned}$$

and $\gamma(\cdot\ ;\ \alpha, \lambda)$ is as defined in (4.3.7).

Corollary 5.3.1. *The first two raw moments of $\hat{\theta}_{1j}$ are*

$$E\big[\hat{\theta}_{1j}\big] = \sum_{\mathbf{n} \in \mathfrak{S}} \sum_{k=0}^{n_{1\cdot}} C^{[1]}_{\mathbf{n},jk} \left\{ \frac{n_{1\cdot}}{n_{1j}} \Big(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\Big)^{-1} + \tau_{1jk} \right\}$$

and

$$\begin{aligned} E\big[\hat{\theta}_{1j}^2\big] &= \sum_{\mathbf{n} \in \mathfrak{S}} \sum_{k=0}^{n_{1\cdot}} C^{[1]}_{\mathbf{n},jk} \Bigg\{ \frac{n_{1\cdot}(n_{1\cdot} + 1)}{n_{1j}^2} \Big(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\Big)^{-2} + \tau_{1jk}^2 \\ &\qquad + \frac{2n_{1\cdot}}{n_{1j}} \Big(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\Big)^{-1} \tau_{1jk} \Bigg\}, \end{aligned}$$

respectively, for $j = 1, 2$.

Proof. These expressions follow readily from (5.3.4). ∎

Theorem 5.3.2. *The conditional PDF of $\hat{\theta}_{2j}$, given $\mathbf{N} \in \mathfrak{S}$, is*

$$f_{\hat{\theta}_{2j}}(x) = f_{\hat{\theta}_{2j}}\big(x \big| \mathbf{N} \in \mathfrak{S}\big) = \sum_{\mathbf{n}\in\mathfrak{S}} \sum_{k=0}^{n_{2\cdot}} C_{\mathbf{n},jk}^{[2]}\, \gamma\left(x - \tau_{2jk}\ ;\ n_{2\cdot}, n_{2j}\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)\right) \quad (5.3.7)$$

for $j = 1, 2$, where

$$\tau_{2jk} = (n - n_{\cdot\cdot} + k)\, \frac{(\tau_c - \tau)}{n_{2j}}, \quad (5.3.8)$$

$$\begin{aligned} C_{\mathbf{n},jk}^{[2]} &= \frac{(-1)^k}{\sum_{\mathbf{m}\in\mathfrak{S}} Pr\big[\mathbf{N} = \mathbf{m}\big]} \binom{n}{n_{11}, n_{12}, n_{21}, n_{22}, n - n_{\cdot\cdot}} \binom{n_{2\cdot}}{k} \left\{ \prod_{i',j'=1}^{2} \pi_{i'j'}^{n_{i'j'}} \right\} \\ &\quad \times \left(1 - \exp\left\{ - \left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\tau \right\}\right)^{n_{1\cdot}} \\ &\quad \times \exp\left\{ - \left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\tau(n - n_{1\cdot}) - \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)\tau_{2jk} n_{2j} \right\} \quad (5.3.9) \end{aligned}$$

and $\gamma(\cdot\ ;\ \alpha, \lambda)$ is as defined in (4.3.7).

Corollary 5.3.2. *The first two raw moments of $\hat{\theta}_{2j}$ are*

$$E\big[\hat{\theta}_{2j}\big] = \sum_{\mathbf{n}\in\mathfrak{S}} \sum_{k=0}^{n_{2\cdot}} C_{\mathbf{n},jk}^{[2]} \left\{ \frac{n_{2\cdot}}{n_{2j}} \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)^{-1} + \tau_{2jk} \right\}$$

and

$$\begin{aligned} E\big[\hat{\theta}_{2j}^2\big] &= \sum_{\mathbf{n}\in\mathfrak{S}} \sum_{k=0}^{n_{2\cdot}} C_{\mathbf{n},jk}^{[2]} \left\{ \frac{n_{2\cdot}(n_{2\cdot}+1)}{n_{2j}^2} \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)^{-2} + \tau_{2jk}^2 \right. \\ &\qquad \left. + \frac{2n_{2\cdot}}{n_{2j}} \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)^{-1} \tau_{2jk} \right\}, \end{aligned}$$

respectively, for $j = 1, 2$.

Proof. These expressions follow readily from (5.3.7). □

As presented above, for $i, j = 1, 2$, the conditional distribution of $\hat{\theta}_{ij}$, given $\mathbf{N} \in \mathfrak{S}$, is a generalized mixture of gamma distributions. The expressions for the expected values clearly reveal that $\hat{\theta}_{ij}$ is a biased estimator of θ_{ij} for all $i, j = 1, 2$. The expressions for the first two raw moments given in Corollaries 3.1 and 3.2 can be used to calculate the standard errors of the estimates. We can also derive the expressions for the tail probabilities by integrating the conditional PDFs of $\hat{\theta}_{ij}$ given above. These expressions, presented in the following corollary, are used to construct exact confidence intervals for θ_{ij} later in Section 5.4.

Corollary 5.3.3. *The tail probability of $\hat{\theta}_{ij}$ is given by*

$$Pr\left[\hat{\theta}_{ij} > \xi\right] = \sum_{\mathbf{n}\in\mathfrak{S}} \sum_{k=0}^{n_{i\cdot}} C_{\mathbf{n},jk}^{[i]} \, \Gamma\left(n_{ij}\left(\frac{1}{\theta_{i1}} + \frac{1}{\theta_{i2}}\right)\langle \xi - \tau_{ijk} \rangle \,;\, n_{i\cdot}\right)$$

for $i, j = 1, 2$, where

$$\begin{aligned} \langle\, \epsilon \,\rangle &= \max\{0, \epsilon\}, \\ \Gamma(\epsilon \,;\, \alpha) &= \begin{cases} \displaystyle\int_{\epsilon}^{\infty} \gamma(y \,;\, \alpha, 1) dy = \int_{\epsilon}^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy &, \epsilon > 0 \\ 1 &, \epsilon \leq 0 \end{cases}. \end{aligned}$$

Remark 5.3.1. *The model proposed in Section 5.2 and the distributional results obtained above can be easily extended and generalized to accommodate multiple stress levels and multiple competing risks. In fact, the model under consideration is also general in the sense that it includes its marginal models as special cases. For instance, when $\theta_{11}, \theta_{21} \to \infty$ or $\theta_{12}, \theta_{22} \to \infty$, the failure of a test unit will be caused by a single risk factor with probability* 1. *Hence, the limiting case of the proposed model is the simple step-stress model under Type-I censoring without the competing risk structure, which was considered by Balakrishnan et al. (2008). Consequently, the distributional results derived above simply reduce to those obtained by Balakrishnan et al. (2008) when $\theta_{11}, \theta_{21} \to \infty$ or $\theta_{12}, \theta_{22} \to \infty$. On the other hand, if we rather let $\tau \to \infty$, then* *the model developed here converges to the ordinary single stress model (i.e., one stress level only) with two competing risks.*

5.4 Confidence intervals

In this section, we discuss different methods of constructing CIs for the unknown parameters θ_{ij}. Based on the exact conditional distributions of the MLEs from Theorems 5.3.1 and 5.3.2, we can construct exact CIs for θ_{ij}. Since the exact conditional distributions of $\hat{\theta}_{ij}$ are quite complicated, we also present the approximate CIs for θ_{ij} based on the asymptotic distributions of the estimators for a large sample size. Finally, the parametric bootstrap method is used to construct CIs for θ_{ij}.

5.4.1 Exact confidence intervals

As in Section 4.4.1, in order to guarantee the invertibility of the pivotal quantities for the parameters θ_{ij}, we assume that the tail probability of $\hat{\theta}_{ij}$ presented in Corollary 5.3.3 is a monotonically increasing function of θ_{ij} for each $i, j = 1, 2$. Although this assumption can not be verified in an analytical way due to the complex structure of the exact distributions of the pivotal quantities, extensive numerical computations seem to support this monotonicity assumption (see Figure 5.1).

Let us now construct the exact $100(1-\alpha)\%$ CI for θ_{ij}, $i, j = 1, 2$. We first denote θ_{ij}^L and θ_{ij}^U for the lower and upper bounds of the two-sided CI for θ_{ij}, respectively. Then, by the monotonicity assumption, they are the unique solutions of the equations

$$Pr\Big[\hat{\theta}_{ij} > \hat{\theta}_{ij}^{obs}\Big] = \frac{\alpha}{2}$$

and

$$Pr\Big[\hat{\theta}_{ij} > \hat{\theta}_{ij}^{obs}\Big] = 1 - \frac{\alpha}{2},$$

respectively, where $\hat{\theta}_{ij}^{obs}$ is simply the observed value of the MLE of θ_{ij} in this case. Since θ_{ij}^L and θ_{ij}^U can not be expressed in an explicit form, they are numerically obtained by solving the following two non-linear equations using some iterative techniques such as the bisection method, Newton-Raphson method or Brent's method:

$$\begin{aligned} \frac{\alpha}{2} &= \sum_{\mathbf{n}\in\mathfrak{S}} \sum_{k=0}^{n_{i\cdot}} C_{\mathbf{n},jk}^{[i]}\, \Gamma\bigg(n_{ij}\Big(\frac{1}{\theta_{ij}^L} + \frac{1}{\theta_{ij'}}\Big)\big\langle \hat{\theta}_{ij}^{obs} - \tau_{ijk}\big\rangle\ ;\ n_{i\cdot}\bigg), \qquad (5.4.1) \\ 1 - \frac{\alpha}{2} &= \sum_{\mathbf{n}\in\mathfrak{S}} \sum_{k=0}^{n_{i\cdot}} C_{\mathbf{n},jk}^{[i]}\, \Gamma\bigg(n_{ij}\Big(\frac{1}{\theta_{ij}^U} + \frac{1}{\theta_{ij'}}\Big)\big\langle \hat{\theta}_{ij}^{obs} - \tau_{ijk}\big\rangle\ ;\ n_{i\cdot}\bigg) \qquad (5.4.2) \end{aligned}$$

for $i, j = 1, 2$ with $j' \neq j$, where τ_{ijk}, $C_{\mathbf{n},jk}^{[i]}$ and $\Gamma(\cdot\ ;\alpha)$ are as defined earlier. Note that the coefficients $C_{\mathbf{n},jk}^{[i]}$ in the above two equations are functions of $\boldsymbol{\theta}$. Hence, before solving for the confidence limits for fixed i and j, we replace θ_{ij} in $C_{\mathbf{n},jk}^{[i]}$ in an appropriate manner. That is, θ_{ij}^L is substituted for θ_{ij} in $C_{\mathbf{n},jk}^{[i]}$ of (5.4.1) and likewise θ_{ij}^U for θ_{ij} in $C_{\mathbf{n},jk}^{[i]}$ of (5.4.2). The observed values of the MLEs are also substituted for all the other unknown parameters in the expressions given above.

5.4.2 Approximate confidence intervals

In this subsection, we present an approximate method to construct CIs for θ_{ij} using the asymptotically optimal properties of the MLEs for large sample sizes. Although the exact method described in the preceding subsection is preferable, its computation again encounters some difficulties for large samples. On the other hand, the approximate method provides not only the computational ease but also a good probability coverage (close to the nominal level) when the sample size is large. This finding is further discussed in Section 5.5.

Let us first denote the (expected) Fisher information matrix of $\boldsymbol{\theta}$ by

$$\mathbf{I}_E(\boldsymbol{\theta}) = \left[I_{ij:i'j'}\right]_{i,j,i',j'=1,2},$$

where

$$I_{ij:i'j'} = -E\left[\frac{\partial l(\boldsymbol{\theta})}{\partial\theta_{ij}\partial\theta_{i'j'}}\right] = \begin{cases} E\left[-\dfrac{N_{ij}}{\theta_{ij}^2} + \dfrac{2U_i}{\theta_{ij}^3}\right] & , \ i = i' \text{ and } j = j' \\ 0 & , \ \textit{otherwise} \end{cases}$$

with U_1 and U_2 being as defined in (5.2.7) and (5.2.8), respectively. It is clear that $\mathbf{I}_E(\boldsymbol{\theta})$ is a diagonal matrix and by substituting $\hat{\theta}_{ij}$ for θ_{ij}, the observed Fisher information matrix of $\boldsymbol{\theta}$ is simply

$$\mathbf{I}_O(\boldsymbol{\theta}) = diag\left(\frac{n_{11}}{\hat{\theta}_{11}^2}, \frac{n_{12}}{\hat{\theta}_{12}^2}, \frac{n_{21}}{\hat{\theta}_{21}^2}, \frac{n_{22}}{\hat{\theta}_{22}^2}\right).$$

Upon inverting this matrix, we obtain the asymptotic variance of $\hat{\theta}_{ij}$ as

$$V_{ij} = \frac{\hat{\theta}_{ij}^2}{n_{ij}}, \qquad i,j = 1,2.$$

Since $\hat{\theta}_{ij}$ is asymptotically unbiased for θ_{ij}, we can then use

$$\frac{\hat{\theta}_{ij} - \theta_{ij}}{\sqrt{V_{ij}}}, \qquad i,j = 1,2$$

as a pivotal quantity for θ_{ij} to construct two-sided $100(1-\alpha)\%$ approximate CI for θ_{ij}, which is given by

$$\left(\max\left\{0, \hat{\theta}_{ij} - z_{\alpha/2}\sqrt{V_{ij}}\right\}, \ \hat{\theta}_{ij} + z_{\alpha/2}\sqrt{V_{ij}}\right), \qquad i,j = 1,2,$$

where $z_{\alpha/2}$ is the $(1-\alpha/2)$-th quantile of a standard normal distribution.

5.4.3 Bootstrap confidence intervals

We also construct the CIs for θ_{ij} using the bias-corrected and accelerated (BCa) percentile bootstrap method. Before we obtain the BCa percentile bootstrap CIs for θ_{ij}, the following algorithm is implemented to generate the bootstrap sample of size B based on the original Type-I censored sample of size $n_{\cdot\cdot}$:

Step 1 Given the stress change time point τ, the right censoring time point τ_c, the initial sample size n and the original Type-I censored sample of size $n_{\cdot\cdot}$, calculate $\hat{\theta}_{ij}$, the MLEs of θ_{ij} for $i, j = 1, 2$, from (5.2.9).

Step 2 Generate a random sample of (T_{11}, T_{12}) of size n, where T_{11} and T_{12} are independently from exponential distributions with mean parameters $\hat{\theta}_{11}$ and $\hat{\theta}_{12}$, respectively. For each pair of (T_{11}, T_{12}), take the minimum of the two values as well as the corresponding index of the minimum (*e.g.*, record 1 if T_{11} is smaller than T_{12}, else record 2). Let $\mathbf{T}_1$ be the vector of the minima and $\mathbf{C}_1$ be the vector of the indices.

Step 3 Sort the elements of $\mathbf{T}_1$ in an ascending order and permute the elements of $\mathbf{C}_1$ in a corresponding manner. Let $v_{1:n} < \cdots < v_{n:n}$ denote the ordered elements of $\mathbf{T}_1$.

Step 4 Find $n_{1\cdot}^*$ such that $v_{n_{1\cdot}^*:n} < \tau \leq v_{n_{1\cdot}^*+1:n}$. Then, for $1 \leq k \leq n_{1\cdot}^*$, set $t_{k:n}^*$ to be the value of $v_{k:n}$ and set n_{1j}^* to be the number of j's in the first $n_{1\cdot}^*$ elements of the permuted $\mathbf{C}_1$ for $j = 1, 2$ (*viz.*, $n_{11}^* + n_{12}^* = n_{1\cdot}^*$).

Step 5 Generate a random sample of (T_{21}, T_{22}) of size $\eta = n - n_{1\cdot}^*$, where T_{21} and T_{22} are independently from exponential distributions with mean parameters $\hat{\theta}_{21}$

and $\hat{\theta}_{22}$, respectively. Again, for each pair of (T_{21}, T_{22}), take the minimum of the two values as well as the corresponding index of the minimum. Let $\mathbf{T}_2$ be the vector of the minima and $\mathbf{C}_2$ be the vector of the indices.

Step 6 Sort the elements of $\mathbf{T}_2$ in an ascending order and permute the elements of $\mathbf{C}_2$ correspondingly. Let $w_{1:\eta} < \cdots < w_{\eta:\eta}$ denote the ordered elements of $\mathbf{T}_2$.

Step 7 Find $n_{2\cdot}^*$ such that $w_{n_{2\cdot}^*:\eta} < (\tau_c - \tau) \leq w_{n_{2\cdot}^*+1:\eta}$. Then, for $1 \leq k \leq n_{2\cdot}^*$, set $t_{n_{1\cdot}^*+k:n}^*$ to be the value of $(w_{k:\eta} + \tau)$. Also, set n_{2j}^* to be the number of j's in the first $n_{2\cdot}^*$ elements of the permuted $\mathbf{C}_2$ for $j = 1, 2$ (*viz.*, $n_{21}^* + n_{22}^* = n_{2\cdot}^*$).

Step 8 Based on τ, τ_c, n, n_{ij}^* and the ordered observations $\mathbf{t}^* = (t_{1:n}^*, \ldots, t_{n_{\cdot\cdot}^*:n}^*)$, calculate the new MLEs of θ_{ij}, denoted by $\hat{\theta}_{ij}^*$ for $i, j = 1, 2$, from (5.2.9).

Step 9 Repeat Steps 2-8 B times. Then, for fixed i and j, arrange all the values of $\hat{\theta}_{ij}^*$ in an ascending order to obtain the bootstrap sample

$$\left\{\hat{\theta}_{ij}^{*[1]} < \hat{\theta}_{ij}^{*[2]} < \cdots < \hat{\theta}_{ij}^{*[B]}\right\}, \qquad i, j = 1, 2.$$

With the bootstrap samples generated as above, we now obtain the two-sided $100(1-\alpha)\%$ BCa percentile bootstrap CI for θ_{ij} as

$$\left(\hat{\theta}_{ij}^{*[\alpha_{ij}B]},\ \hat{\theta}_{ij}^{*[\beta_{ij}B]}\right), \qquad i, j = 1, 2,$$

where

$$\alpha_{ij} = \Phi\left(\hat{z}_{0;ij} + \frac{\hat{z}_{0;ij} - z_{\alpha/2}}{1 - \hat{a}_{ij}(\hat{z}_{0;ij} - z_{\alpha/2})}\right)$$

and

$$\beta_{ij} = \Phi\left(\hat{z}_{0;ij} + \frac{\hat{z}_{0;ij} + z_{\alpha/2}}{1 - \hat{a}_{ij}(\hat{z}_{0;ij} + z_{\alpha/2})}\right);$$

see Efron and Tibshirani (1993). As usual, $\Phi(\cdot)$ denotes the CDF of the standard normal distribution and the value of the bias-correction $\hat{z}_{0;ij}$ is given by

$$\hat{z}_{0;ij} = \Phi^{-1}\left(\frac{\sum_{k=1}^{B} J\left(\hat{\theta}_{ij}^{*[k]} < \hat{\theta}_{ij}\right)}{B}\right), \qquad i,j = 1,2,$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal CDF and $J(\cdot)$ is an indicator function that takes on the value of 1 if the argument is true and 0 otherwise. A good estimate of the acceleration factor a_{ij} is suggested to be

$$\hat{a}_{ij} = \frac{\sum_{k=1}^{n_{ij}} \left(\hat{\theta}_{ij}^{(\cdot)} - \hat{\theta}_{ij}^{(k)}\right)^3}{6\left\{\sum_{k=1}^{n_{ij}} \left(\hat{\theta}_{ij}^{(\cdot)} - \hat{\theta}_{ij}^{(k)}\right)^2\right\}^{3/2}}, \qquad i,j = 1,2,$$

where $\hat{\theta}_{ij}^{(k)}$ is the MLE of θ_{ij} based on the original Type-I censored sample with the k-th observation deleted from the failures that occurred at the stress level s_i by the risk factor j (*i.e.*, the jackknife estimate) for $k = 1, 2, \ldots, n_{ij}$, and

$$\hat{\theta}_{ij}^{(\cdot)} = \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \hat{\theta}_{ij}^{(k)}, \qquad i,j = 1,2.$$

5.5 Numerical study

In order to evaluate the performance of all the different methods of constructing CIs discussed in Section 5.4, a Monte Carlo simulation study was conducted and the results are presented in this section. As in Section 4.5, the values of the parameters were chosen to be $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, and $\theta_{22} = 6.0$ to illustrate a particular scenario under which the increased stress level causes 50% loss of the mean time to failure by any single risk factor and the chance of a test unit to fail by the risk factor 1 is twice as high as the chance to fail by the risk factor 2 before or after the change in the stress levels. The initial sample size n was chosen to be 15 and 30, and several different choices were made for the stress change time point τ while the censoring time point τ_c was fixed at 6. Based on 1000 Monte Carlo simulations with $B = 1000$ bootstrap replications, the true coverage probabilities of the 90%, 95% and 99% CIs for θ_{ij} were determined. The results are presented in Tables 5.1-5.10 along with the estimated mean bounds and widths of the CIs from this simulation.

Again, it was observed that the lower bounds of the approximate CIs frequently hit below zero for small sample sizes or for high levels of confidence even though the parameters θ_{ij} can take only positive values in this setting. In order to make such intervals sensible ones, the negative lower bounds were all replaced by zero in Tables 5.3-5.5 and 5.8-5.10.

From Table 5.1, we clearly see that overall the exact method performs the best as its CIs possess the actual coverage probabilities to be much closer to the nominal levels than the other CIs based on different methods. On the other hand, the performance of the approximate CIs and the BCa bootstrap CIs is unsatisfactory for a small sample size as their actual coverage probabilities are quite below the specified nominal levels in most cases. A possible explanation for this may be due to the high degree of skewness in the exact distributions of $\hat{\theta}_{ij}$ and hence, a much larger sample size is required to justify the use of the asymptotic approach to construct CIs. Moreover, from Table 5.2, we observe that the widths of the CIs obtained from the approximate method and the parametric bootstrap method are unduly narrow compared to those of the exact CIs in general. Serious underestimation of the interval width again provides a reason for the poor probability coverages of the approximate

Table 5.1: Estimated Coverage Probabilities (in %) based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 15$, $\tau_c = 6$ and $B = 1000$

Nominal CL		90%			95%			99%		
Parameter	τ	Exact	Approx	Boot	Exact	Approx	Boot	Exact	Approx	Boot
θ_{11}	1	92.0	84.0	74.1	95.2	89.6	79.7	98.8	94.2	86.6
	2	91.5	88.7	85.6	95.3	91.6	91.0	98.5	95.9	94.6
	3	90.8	88.6	88.1	95.2	90.8	92.5	98.7	95.3	96.9
	4	91.6	88.7	88.3	95.4	92.1	92.7	99.1	95.4	97.7
	5	90.9	90.5	84.3	96.0	94.7	90.1	99.0	97.4	96.4
θ_{12}	1	88.2	85.3	80.4	90.3	85.4	85.4	97.9	85.7	92.7
	2	88.9	85.3	81.0	93.3	86.9	87.9	98.5	93.0	94.5
	3	89.3	87.0	80.6	94.1	90.3	86.3	98.0	93.9	92.8
	4	89.9	87.6	84.3	95.0	90.6	88.2	98.7	94.7	94.5
	5	90.5	90.8	84.0	95.4	92.9	88.3	99.3	96.5	92.9
θ_{21}	1	91.5	86.6	88.9	95.9	93.4	94.0	99.7	95.6	98.8
	2	94.1	87.8	90.9	98.3	91.9	94.7	99.9	94.9	98.8
	3	95.3	86.7	86.2	98.4	89.5	90.8	99.6	93.9	96.3
	4	95.8	87.5	77.2	97.5	90.2	84.3	99.8	94.0	94.2
	5	91.0	76.0	80.1	95.5	80.2	86.3	99.5	87.1	93.5
θ_{22}	1	93.1	86.5	85.1	95.6	89.6	89.5	99.4	93.6	93.0
	2	92.6	86.1	82.2	95.9	89.2	88.1	99.1	93.1	94.9
	3	91.8	83.4	78.3	95.1	85.9	86.7	99.2	90.0	96.2
	4	89.9	76.8	80.5	94.3	80.0	88.2	98.6	85.8	94.7
	5	89.4	53.8	54.3	94.0	58.9	60.8	98.6	69.6	67.4

Table 5.2: Average Widths of Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 15$, $\tau_c = 6$ and $B = 1000$

Nominal CL		90%			95%			99%		
Parameter	τ	Exact	Approx	Boot	Exact	Approx	Boot	Exact	Approx	Boot
θ_{11}	1	30.868	16.815	8.945	26.505	19.275	9.986	34.080	23.054	11.521
	2	17.356	15.729	15.198	19.795	17.838	18.194	31.710	21.264	23.288
	3	13.040	12.214	15.504	17.232	14.143	19.700	24.743	17.079	28.907
	4	11.954	10.359	13.233	15.046	12.170	17.622	21.694	15.020	28.635
	5	10.507	9.405	11.667	14.149	11.137	15.624	19.328	14.042	26.536
θ_{12}	1	42.403	24.880	8.360	41.416	27.551	9.313	40.139	33.122	10.843
	2	34.519	33.558	17.930	29.992	37.505	20.134	51.042	44.897	23.356
	3	28.931	34.661	24.861	32.019	38.845	28.748	44.985	46.352	34.330
	4	24.927	32.818	28.028	35.488	36.913	33.337	44.276	43.936	42.032
	5	23.048	32.222	29.640	34.237	36.427	36.230	45.010	43.335	47.546
θ_{21}	1	6.084	4.653	6.550	7.936	5.583	8.889	13.399	7.006	15.605
	2	9.240	5.651	7.709	11.006	6.570	10.274	18.334	7.959	16.401
	3	13.451	7.629	8.574	14.726	8.652	10.746	24.477	10.314	15.168
	4	17.170	8.669	7.468	18.735	9.700	8.885	28.966	11.593	11.332
	5	19.711	5.953	4.470	24.289	6.649	5.126	30.654	7.997	6.050
θ_{22}	1	20.265	15.135	16.793	23.941	17.377	20.594	32.537	20.691	27.735
	2	25.123	16.330	15.236	24.050	18.307	18.283	35.825	21.818	23.488
	3	28.526	15.136	12.829	24.525	16.917	14.965	39.752	20.245	18.351
	4	27.973	12.642	9.071	27.958	14.120	10.417	41.897	16.969	12.339
	5	23.034	7.040	4.700	29.391	7.869	5.312	35.902	9.488	6.120

Table 5.3: Average Bounds of 90% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 15$, $\tau_c = 6$ and $B = 1000$

Parameter	τ	Exact CI	Approximate CI	BCa Bootstrap CI
θ_{11}	1	(1.856, 32.725)	(0.142, 16.957)	(4.387, 13.332)
	2	(2.606, 19.962)	(0.718, 16.446)	(4.137, 19.335)
	3	(2.931, 15.971)	(1.368, 13.582)	(3.916, 19.420)
	4	(3.024, 14.978)	(1.854, 12.213)	(3.637, 16.870)
	5	(3.354, 13.861)	(2.224, 11.629)	(3.764, 15.432)
θ_{12}	1	(2.038, 44.440)	(0.036, 24.917)	(5.796, 14.156)
	2	(3.121, 37.641)	(0.215, 33.773)	(7.519, 25.449)
	3	(3.757, 32.687)	(0.520, 35.181)	(7.919, 32.780)
	4	(4.014, 28.941)	(0.842, 33.661)	(7.464, 35.491)
	5	(4.211, 27.259)	(1.184, 33.406)	(7.790, 37.429)
θ_{21}	1	(1.712, 7.796)	(1.018, 5.670)	(1.733, 8.283)
	2	(1.549, 10.789)	(0.695, 6.346)	(1.596, 9.304)
	3	(1.308, 14.759)	(0.322, 7.951)	(1.646, 10.220)
	4	(0.980, 18.150)	(0.083, 8.752)	(1.699, 9.167)
	5	(0.561, 20.272)	(0.008, 5.961)	(1.267, 5.737)
θ_{22}	1	(2.477, 22.742)	(0.540, 15.675)	(3.816, 20.610)
	2	(2.066, 27.189)	(0.257, 16.587)	(3.439, 18.675)
	3	(1.653, 30.180)	(0.100, 15.235)	(3.296, 16.125)
	4	(1.136, 29.109)	(0.023, 12.665)	(2.639, 11.711)
	5	(0.582, 23.616)	(0.001, 7.041)	(1.543, 6.242)

Table 5.4: Average Bounds of 95% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 15$, $\tau_c = 6$ and $B = 1000$

Parameter	τ	Exact CI	Approximate CI	BCa Bootstrap CI
	1	(1.458, 27.963)	(0.024, 19.300)	(3.890, 13.877)
	2	(2.221, 22.016)	(0.242, 18.079)	(3.613, 21.808)
θ_{11}	3	(2.591, 19.823)	(0.641, 14.784)	(3.506, 23.205)
	4	(2.771, 17.817)	(1.044, 13.214)	(3.284, 20.906)
	5	(2.849, 16.998)	(1.396, 12.533)	(3.411, 19.036)
	1	(1.585, 43.001)	(0.003, 27.554)	(5.063, 14.376)
	2	(2.485, 32.477)	(0.034, 37.538)	(6.463, 26.597)
θ_{12}	3	(3.156, 35.175)	(0.117, 38.962)	(6.840, 35.588)
	4	(3.649, 39.137)	(0.227, 37.140)	(6.450, 39.788)
	5	(3.758, 37.995)	(0.343, 36.769)	(6.815, 43.045)
	1	(1.518, 9.454)	(0.636, 6.219)	(1.523, 10.412)
	2	(1.336, 12.342)	(0.331, 6.901)	(1.366, 11.640)
θ_{21}	3	(1.085, 15.811)	(0.100, 8.752)	(1.358, 12.103)
	4	(0.789, 19.524)	(0.013, 9.713)	(1.376, 10.261)
	5	(0.457, 24.746)	(0.001, 6.650)	(1.029, 6.155)
	1	(2.108, 26.049)	(0.166, 17.543)	(3.227, 23.821)
	2	(1.699, 25.749)	(0.051, 18.358)	(2.862, 21.144)
θ_{22}	3	(1.274, 25.799)	(0.014, 16.931)	(2.706, 17.671)
	4	(0.883, 28.841)	(0.002, 14.122)	(2.144, 12.561)
	5	(0.477, 29.868)	(0.000, 7.869)	(1.256, 6.568)

Table 5.5: Average Bounds of 99% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 15$, $\tau_c = 6$ and $B = 1000$

Parameter	τ	Exact CI	Approximate CI	BCa Bootstrap CI
θ_{11}	1	(1.076, 35.156)	(0.000, 23.054)	(2.952, 14.473)
	2	(1.680, 33.391)	(0.007, 21.271)	(2.677, 25.965)
	3	(1.976, 26.719)	(0.053, 17.132)	(2.622, 31.529)
	4	(2.024, 23.718)	(0.152, 15.172)	(2.495, 31.130)
	5	(2.707, 22.035)	(0.257, 14.299)	(2.521, 29.057)
θ_{12}	1	(1.115, 41.253)	(0.000, 33.122)	(3.835, 14.678)
	2	(1.859, 52.900)	(0.000, 44.897)	(4.758, 28.114)
	3	(2.225, 47.210)	(0.001, 46.353)	(4.978, 39.309)
	4	(2.623, 46.899)	(0.003, 43.939)	(4.649, 46.682)
	5	(2.917, 47.927)	(0.007, 43.342)	(4.864, 52.411)
θ_{21}	1	(1.217, 14.616)	(0.103, 7.109)	(1.152, 16.757)
	2	(1.025, 19.359)	(0.027, 7.986)	(0.938, 17.339)
	3	(0.817, 25.294)	(0.002, 10.316)	(0.862, 16.030)
	4	(0.595, 29.562)	(0.000, 11.593)	(0.853, 12.185)
	5	(0.342, 30.996)	(0.000, 7.997)	(0.655, 6.705)
θ_{22}	1	(1.473, 34.010)	(0.003, 20.694)	(2.275, 30.010)
	2	(1.208, 37.032)	(0.001, 21.819)	(1.942, 25.430)
	3	(0.928, 40.680)	(0.000, 20.245)	(1.784, 20.135)
	4	(0.675, 42.572)	(0.000, 16.969)	(1.391, 13.729)
	5	(0.369, 36.271)	(0.000, 9.488)	(0.812, 6.933)

Table 5.6: Estimated Coverage Probabilities (in %) based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 30$, $\tau_c = 6$ and $B = 1000$

Nominal CL		90%		95%		99%	
Parameter	τ	Approx	Boot	Approx	Boot	Approx	Boot
θ_{11}	1	88.7	88.6	93.2	93.2	96.2	97.5
	2	89.7	91.6	93.5	94.3	95.6	98.4
	3	91.8	89.6	94.5	94.9	97.6	99.2
	4	90.4	89.1	93.5	94.2	96.6	99.1
	5	89.8	89.4	93.5	94.9	96.6	98.5
θ_{12}	1	87.2	89.1	91.8	92.7	95.3	95.2
	2	87.4	87.9	91.7	93.0	95.6	96.5
	3	89.3	87.6	93.0	93.7	96.3	97.7
	4	88.2	88.8	92.5	94.6	95.8	98.1
	5	89.6	89.6	93.1	94.7	96.2	97.9
θ_{21}	1	88.8	90.8	92.1	95.5	95.9	99.1
	2	89.9	90.9	93.6	95.2	95.8	98.7
	3	88.8	89.9	92.3	95.2	95.2	99.2
	4	89.7	88.5	92.9	92.7	95.7	97.2
	5	87.6	88.5	91.0	89.9	93.5	96.8
θ_{22}	1	89.9	89.9	92.0	94.7	95.6	98.7
	2	89.5	91.2	93.3	95.0	96.1	97.7
	3	88.3	86.1	91.8	93.1	95.6	96.9
	4	86.1	88.2	91.9	92.8	94.8	97.4
	5	88.5	85.8	92.0	93.8	93.7	98.6

Table 5.7: Average Widths of Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 30$, $\tau_c = 6$ and $B = 1000$

Nominal CL		90%		95%		99%	
Parameter	τ	Approx	Boot	Approx	Boot	Approx	Boot
θ_{11}	1	15.056	14.909	17.182	18.355	20.577	23.613
	2	8.982	12.057	10.629	16.269	13.482	27.324
	3	7.205	8.598	8.582	11.384	11.178	19.494
	4	6.325	7.381	7.536	9.522	9.885	15.838
	5	5.777	6.529	6.883	8.273	9.038	13.115
θ_{12}	1	34.066	18.522	38.123	20.906	45.590	23.947
	2	30.355	30.016	34.476	36.408	41.188	45.720
	3	24.681	30.481	28.491	39.138	34.475	56.013
	4	21.937	28.657	25.646	38.754	31.536	59.471
	5	19.326	27.838	22.790	37.332	28.497	60.826
θ_{21}	1	2.830	3.311	3.372	4.208	4.431	6.776
	2	3.627	4.267	4.317	5.647	5.624	10.062
	3	4.424	6.189	5.237	8.359	6.601	14.638
	4	6.887	7.948	7.891	10.245	9.476	15.304
	5	8.280	6.830	9.265	8.068	11.072	10.121
θ_{22}	1	9.613	13.785	11.333	18.765	14.170	32.297
	2	12.389	15.185	14.337	20.206	17.356	31.214
	3	16.001	16.493	18.106	20.360	21.600	27.598
	4	16.415	13.641	18.384	16.190	21.959	20.362
	5	12.224	7.868	13.651	9.042	16.407	10.715

Table 5.8: Average Bounds of 90% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 30$, $\tau_c = 6$ and $B = 1000$

Parameter	τ	Approximate CI	BCa Bootstrap CI
	1	(0.956, 16.012)	(4.111, 19.020)
	2	(2.361, 11.343)	(3.898, 15.955)
θ_{11}	3	(2.984, 10.189)	(3.950, 12.548)
	4	(3.284, 9.609)	(4.080, 11.461)
	5	(3.449, 9.226)	(4.156, 10.685)
	1	(0.333, 34.400)	(7.588, 26.110)
	2	(1.538, 31.893)	(7.765, 37.780)
θ_{12}	3	(2.720, 27.400)	(7.561, 38.041)
	4	(3.611, 25.548)	(7.625, 36.282)
	5	(4.239, 23.564)	(7.860, 35.698)
	1	(1.709, 4.539)	(2.029, 5.340)
	2	(1.493, 5.120)	(1.911, 6.178)
θ_{21}	3	(1.103, 5.527)	(1.788, 7.978)
	4	(0.538, 7.425)	(1.744, 9.692)
	5	(0.080, 8.360)	(1.851, 8.681)
	1	(2.113, 11.726)	(3.768, 17.553)
	2	(1.392, 13.781)	(3.529, 18.715)
θ_{22}	3	(0.643, 16.643)	(3.758, 20.251)
	4	(0.204, 16.618)	(3.599, 17.241)
	5	(0.021, 12.245)	(2.782, 10.650)

Table 5.9: Average Bounds of 95% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 30$, $\tau_c = 6$ and $B = 1000$

Parameter	τ	Approximate CI	BCa Bootstrap CI
θ_{11}	1	(0.383, 17.566)	(3.586, 21.941)
	2	(1.579, 12.208)	(3.558, 19.827)
	3	(2.297, 10.879)	(3.634, 15.019)
	4	(2.678, 10.215)	(3.774, 13.296)
	5	(2.896, 9.779)	(3.863, 12.135)
θ_{12}	1	(0.065, 38.188)	(6.509, 27.415)
	2	(0.576, 35.052)	(6.810, 43.218)
	3	(1.355, 29.846)	(6.741, 45.879)
	4	(2.052, 27.698)	(6.887, 45.641)
	5	(2.643, 25.433)	(7.134, 44.466)
θ_{21}	1	(1.438, 4.810)	(1.874, 6.083)
	2	(1.150, 5.467)	(1.741, 7.388)
	3	(0.715, 5.952)	(1.591, 9.950)
	4	(0.235, 8.127)	(1.500, 11.744)
	5	(0.013, 9.278)	(1.542, 9.609)
θ_{22}	1	(1.320, 12.654)	(3.394, 22.159)
	2	(0.678, 15.015)	(3.120, 23.326)
	3	(0.218, 18.324)	(3.224, 23.584)
	4	(0.042, 18.426)	(3.017, 19.207)
	5	(0.002, 13.653)	(2.305, 11.348)

Table 5.10: Average Bounds of 99% Confidence Intervals based on 1000 Simulations with $\theta_{11} = 6.0$, $\theta_{12} = 12.0$, $\theta_{21} = 3.0$, $\theta_{22} = 6.0$, $n = 30$, $\tau_c = 6$ and $B = 1000$

Parameter	τ	Approximate CI	BCa Bootstrap CI
θ_{11}	1	(0.025, 20.603)	(2.676, 26.289)
	2	(0.417, 13.899)	(2.819, 30.142)
	3	(1.050, 12.228)	(3.048, 22.543)
	4	(1.513, 11.399)	(3.217, 19.054)
	5	(1.822, 10.861)	(3.332, 16.447)
θ_{12}	1	(0.001, 45.591)	(4.870, 28.817)
	2	(0.037, 41.226)	(5.042, 50.762)
	3	(0.152, 34.627)	(5.203, 61.216)
	4	(0.363, 31.900)	(5.483, 64.954)
	5	(0.589, 29.086)	(5.684, 66.510)
θ_{21}	1	(0.909, 5.340)	(1.597, 8.373)
	2	(0.522, 6.146)	(1.431, 11.493)
	3	(0.182, 6.783)	(1.219, 15.858)
	4	(0.022, 9.498)	(1.036, 16.339)
	5	(0.000, 11.072)	(1.033, 11.154)
θ_{22}	1	(0.297, 14.467)	(2.730, 35.027)
	2	(0.072, 17.428)	(2.395, 33.609)
	3	(0.010, 21.610)	(2.322, 29.920)
	4	(0.000, 21.959)	(2.092, 22.453)
	5	(0.000, 16.407)	(1.576, 12.291)

CIs and the BCa bootstrap CIs.

Nevertheless, we realize from Table 5.6 that a larger sample size eventually improves the probability coverages for both the approximate CIs and the BCa bootstrap CIs. As the sample size grows, the larger computational time as well as the unstable precision (especially for θ_{1j}, $j = 1, 2$) becomes a problematic issue for constructing CIs by the exact method. Hence, based on the simulation study, we recommend the use of the bootstrap approach to construct CIs for θ_{ij} when the initial sample size is considerably large since it offers computational feasibility and also performs quite well in terms of probability coverages and widths for large sample sizes (*e.g.*, $n \geq 30$). But, for small sample sizes (say, n up to 30), the exact method developed here is the one that is recommended.

5.6 Illustrative example

We have simulated a Type-I censored sample from a simple step-stress test with two competing risks in order to illustrate the methods of inference described in the preceding sections. The dataset was generated with the same choices of the parameters as in Section 4.6 as follows:

$$\theta_{11} = 8.96,\ \theta_{12} = 12.18,\ \theta_{21} = 4.48,\ \theta_{22} = 4.06$$

along with the stress change time point $\tau = 3$ and the censoring time point $\tau_c = 6$ for an equal step duration. In this setup, when the stress level increases, there is a 50% decrease in the mean time to failure caused by the risk factor 1 and a 67% decrease in the mean time to failure caused by the risk factor 2. Also, at the initial stress level, there is a 58% chance for a test unit to fail by the risk factor 1 but it drops to 47% after the increment of the stress level at τ. From the initial sample size of $n = 25$, we observed a total of $n_{\cdot\cdot} = 23$ failure times (*i.e.*, 8% right censoring) and they are presented in Table 5.11.

From this dataset, we have $n_{11} = 7$, $n_{12} = 5$, $n_{21} = 5$, $n_{22} = 6$ and hence, the observed MLEs of θ_{ij} are found from (5.2.9) to be

$$\hat{\theta}_{11} = 8.299,\ \hat{\theta}_{12} = 11.620,\ \hat{\theta}_{21} = 3.855,\ \hat{\theta}_{22} = 3.213.$$

The CIs for θ_{ij} are also presented in Table 5.12 using all three methods described in Section 5.4. Since the exact CIs for θ_{ij} require the monotonicity of the tail probability functions of $\hat{\theta}_{ij}$, we provide the numerical justification of this assumption by plotting the tail probabilities with the observed values of the MLEs from the sample. From Figure 5.1, it is evident that all the plots show the monotonically increasing behavior of the tail probabilities with respect to the unknown parameter θ_{ij} for each $i, j = 1, 2$. In addition, the two horizontal lines corresponding to the values of $\alpha/2$ and $1 - \alpha/2$ for $\alpha = 0.10$ are overlaid in each plot of Figure 5.1 to illustrate how the exact CIs are constructed by inverting the tail probabilities of $\hat{\theta}_{ij}$. In the first plot of Figure 5.1, for example, the values of θ_{11} from the two intersecting points are the unique solutions of Eqs. (5.4.1) and (5.4.2), respectively, and together they provide the exact 90% CI for θ_{11}.

From Table 5.12, we observe that in comparison to the exact CIs, the approximate method always provides narrower CIs while the BCa bootstrap CIs are sometimes narrower and at other times wider. This is so since the coverage probabilities for the approximate CIs are significantly lower than the nominal levels while

Table 5.11: Type-I Censored Sample from $n = 25$ units on a Simple Step-stress Test with Two Competing Risks, $\tau = 3$ and $\tau_c = 6$

Stress Level 1 (before $\tau = 3$)		**Stress Level 2** (after $\tau = 3$)	
Failure Time	Failure Cause	Failure Time	Failure Cause
0.011	1	3.246	2
0.273	2	3.362	2
0.395	1	3.498	1
1.173	1	3.774	2
1.477	1	3.879	1
1.608	2	4.024	1
1.890	1	4.169	2
2.066	2	4.438	2
2.133	2	4.882	2
2.577	1	5.343	1
2.706	1	5.670	1
2.787	2		
$n_{1\cdot} = 12$		$n_{2\cdot} = 11$	

Table 5.12: Interval Estimation based on the Type-I Censored Step-stress Data in Table 5.11 with $B = 1000$

Parameter	CL	Exact CI	Approximate CI	BCa Bootstrap CI
	90%	(4.619, 16.642)	(3.140, 13.460)	(4.006, 15.052)
$\theta_{11} = 8.96$	95%	(4.186, 19.482)	(2.151, 14.448)	(3.539, 18.037)
	99%	(3.485, 27.218)	(0.219, 16.380)	(0.000, 21.909)
	90%	(5.870, 27.521)	(3.072, 20.167)	(6.142, 30.571)
$\theta_{12} = 12.18$	95%	(5.250, 33.818)	(1.435, 21.805)	(5.453, 33.018)
	99%	(4.269, 53.014)	(0.000, 25.005)	(4.530, 63.678)
	90%	(1.968, 9.215)	(1.019, 6.691)	(2.089, 11.984)
$\theta_{21} = 4.48$	95%	(1.749, 11.435)	(0.476, 7.235)	(1.889, 20.592)
	99%	(1.393, 18.968)	(0.000, 8.296)	(1.613, 35.663)
	90%	(1.735, 7.000)	(1.055, 5.370)	(1.663, 7.135)
$\theta_{22} = 4.06$	95%	(1.554, 8.424)	(0.642, 5.783)	(1.431, 9.482)
	99%	(1.257, 12.827)	(0.000, 6.591)	(1.165, 17.168)

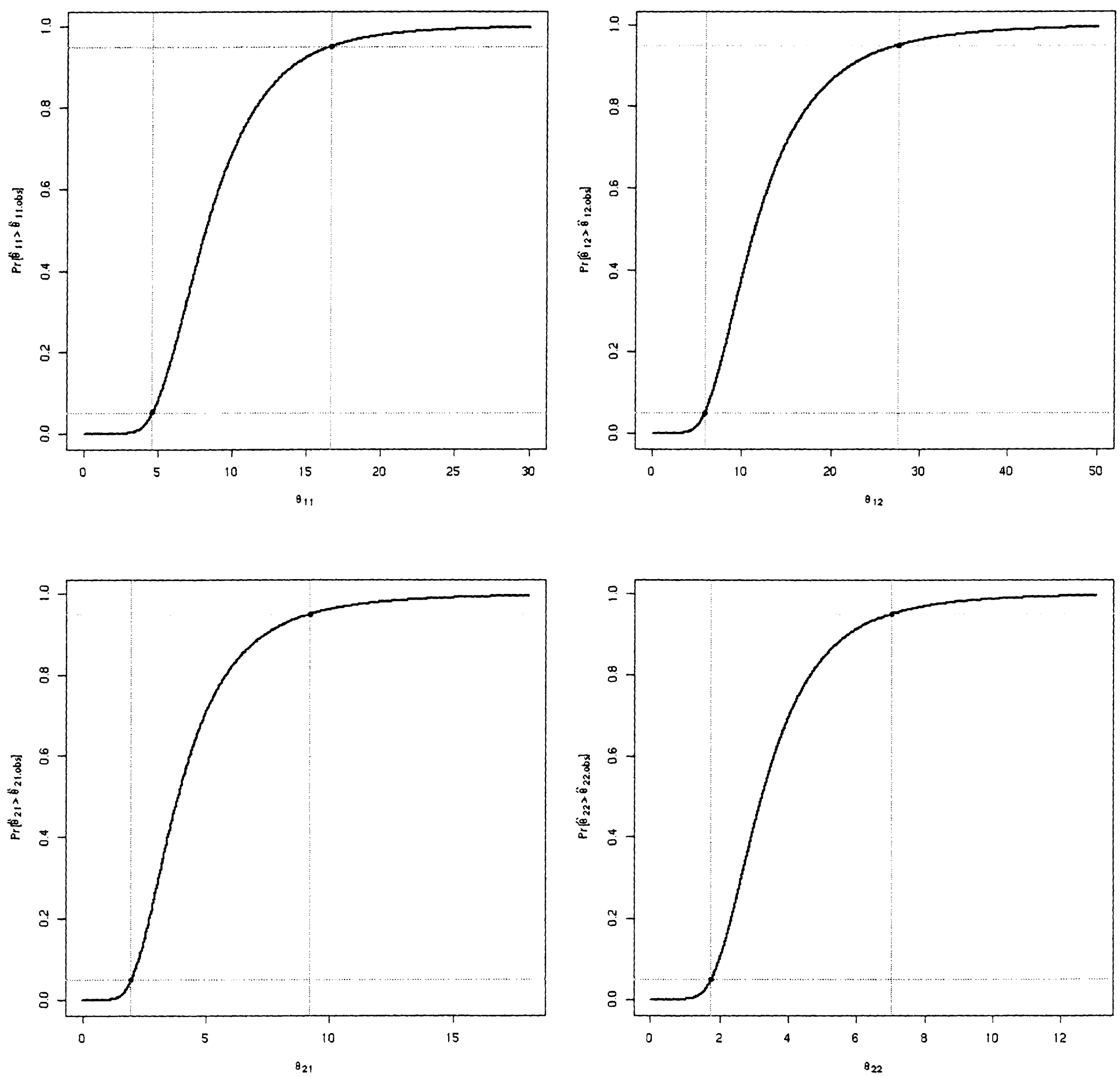

Figure 5.1: Plots of Tail Probabilities of $\hat{\theta}_{ij}$, $Pr\left[\hat{\theta}_{ij} > \hat{\theta}_{ij}^{obs}\right]$, with respect to θ_{ij} and Exact 90% Confidence Intervals for θ_{ij} from the Step-stress Data in Table 5.11

those for the bootstrap CIs seem to fluctuate around the nominal levels (see Tables 5.1 and 5.6). We also note from Table 5.12 that the CIs for θ_{11} are consistently narrower than those for θ_{12}. The primary reason for this is that when θ_{11} is smaller than θ_{12}, we expect a relatively large number of failures to occur before τ by the risk factor 1 than by the risk factor 2, resulting in lower variability in the estimation of θ_{11} than θ_{12}, and vice versa. The same intuition also applies to explain the wider CIs for θ_{21} compared to those for θ_{22} in Table 5.12 although the differences in this case are smaller.

5.7 Summary and conclusions

In this chapter, we have discussed the simple step-stress model under time constraint when the lifetimes corresponding to different risk factors have independent exponential distributions. We have derived the MLEs of the unknown mean parameters θ_{ij} under the assumption of a cumulative exposure model and their exact conditional distributions through the use of the CMGF. We have also proposed several different procedures for constructing CIs for θ_{ij}. We have then conducted a simulation study to assess the performance of all these procedures and a numerical example has been presented to illustrate all the methods of inference described in this chapter. Based on the results of the simulation study, our recommendation for constructing CIs for θ_{ij} is to use the exact method whenever possible, especially in the case of small sample sizes (*e.g.*, $n_{\cdot\cdot} \le n < 30$) since the other two methods are unsatisfactory in terms of probability coverages. For larger sample sizes, however, the BCa percentile bootstrap method is more appropriate because of its computational ease as well as for its improved probability coverages being close to the nominal levels.

Chapter 6

Future Research

In this chapter, we will outline the directions of some possible future research that are currently being considered following the research work presented in this thesis. We first describe some of the ways to generalize and extend the models considered here. Then, we will discuss further inferential issues which need to be investigated for the models considered here as well as for the models from other related topics in the context of ALT.

6.1 Generalizations and extensions

In Section 3.2, one of the crucial assumptions for constructing the k-level step-stress model with the two regression parameters α and β is that at each stress level, MTTF of a test unit is a log-linear function of stress. Although this log-linear link function is a simple and reasonable approach based on various empirical models studied in physical acceleration, it is of interest to consider other choices of link functions to explore different stress-dependent physical processes causing failure of a test unit. It is particularly desired to see how such a modification affects the results of the inference on the regression parameters and to search for some robust link functions for modeling the stress-response relationships.

In Chapters 4 and 5, we have developed the exact inference for the simple step-stress models under different censoring schemes, in particular, Type-I and Type-II censoring situations. As noted in Remarks 4.2.2 and 5.3.1, it is of natural interest to extend and generalize the proposed models to accommodate multiple stress levels (*i.e.*, $k > 2$) and multiple competing risks, and to develop the inferential methods in such situations. For more comprehensive inference under censoring, we are also interested in devising the inferential methods for these generalized models under more advanced forms of censoring schemes such as progressive Type-I / Type-II censoring and Type-I / Type-II hybrid censoring situations.

We also note that the proposed inference in Chapters 2, 4 and 5 is conditional based on the existence of the MLEs and this in turn introduces the complexity to the exact conditional distributions of the MLEs in all cases. One possible problem of interest is to see how this conditioning will change and improve the precision and accuracy of the estimation in the two-sample or multi-sample situations. That is, two or more samples are independently obtained from the identically designed simple step-stress test with the competing risk structure under some possible censoring, and we wish to develop efficient inference for the unknown parameters upon combining these samples together.

In this thesis, we have explored several inferential issues regarding the failure data from exponential distribution. Exponential distribution is a simple scale-parameter distribution which is characterized by its constant hazard rate and lack-of-memory (memoryless) property. It is thus used to model a long flat intrinsic failure portion of the bathtub curve where most test units spend most of their lifetimes, and it is also used to model the failure time of non-wearing down and non-repairable test units. Nevertheless, it lacks some flexibility in fitting a variety of failure time data mainly due to its simple structure. Therefore, it is desired to develop inferential methods for the step-stress models with the competing risk structure when the lifetime distributions of the different risk factors are identical or non-identical and when they are different from exponential. Some popular choices for the lifetime distributions will be Weibull, gamma, lognormal, etc. Due to the added complexity of the distribution models, however, it is not possible to devise methods of exact inference in these cases and consequently, the inference for the parameters will be done numerically based on the asymptotic normality of the MLEs; see, for example, Balakrishnan and Kateri (2008).

6.2 Related problems of interest

In order to construct the exact confidence intervals for the unknown mean parameters in Sections 2.5.1, 4.4.1 and 5.4.1, we have made a crucial assumption that the tail probability of the MLE is a monotonically increasing function of the parameter of interest. This is required to guarantee the invertibility of the pivotal quantity of the parameter and the same assumption was made by many authors for constructing the exact CIs in different contexts. However, the proof of this monotonicity could not be established due to the complicated structure of the pivotal quantities. Recently, Balakrishnan and Iliopoulos (2008) have derived a general method which can be used to show that the MLE of an exponential mean is stochastically increasing under different censoring schemes. Therefore, one of the main research problems which need to be solved is to establish the analytical proof of the required monotonicity for the simple step-stress models with the competing risk structure under different censoring schemes.

For the simple step-stress models discussed in this thesis, the main objective of the inference was the exact point and interval estimation under different censoring schemes. Another important topic in inference is the tests of hypotheses on the mean parameters. As mentioned in Remarks 4.2.1 and 5.2.1, no assumptions have been made about the relationships among the MTTFs of the risk factors at the different stress levels. Even for the simple step-stress model with no competing risk structure, the problem of testing a hypothesis in an exact way has not been solved yet in literature. Hence, we are currently developing the method of testing a hypothesis based on the exact conditional distribution of the likelihood ratio test statistic in order to test $H_0: \ \theta_1 = \theta_2$ against $H_a: \ \theta_1 > \theta_2$, where θ_i is the MTTF at the stress level s_i for $i = 1, 2$.

Since the exponential distribution is the underlying distribution of different risk factors in the models considered in Chapters 4 and 5, we are also interested in devising a method to test the goodness of fit (GOF) of the distribution from a real dataset collected. This is necessary for the model verification and to measure the validity of the assumed failure time distribution under the cumulative exposure model. In addition, it is also of interest to investigate the optimal censoring schemes and optimal stress change time points for these models using some suitable objective functions of choice.

Moreover, in Chapters 4 and 5, we have assumed the independence of the different risk factors in the simple step-stress models. In reality, however, it is likely that these risk factors interact with each other to cause the failure of a test unit and thus, to improve the previous models with a practical aspect, we should incorporate some sort of dependence or correlation structure among the risk factors into the model. One way to accomplish this is via the choice of some multivariate distributions, and we are currently in the process of constructing the simple step-stress model under the cumulative exposure assumption with the dependent competing risks based on the Marshall-Olkin bivariate exponential distribution. Among several multivariate exponential distributions, we have specifically selected the Marshall-Olkin multivariate distribution since this is the only distribution which reproduces the CDF for the simple exponential step-stress model (with no competing risks) as its marginal; see Balakrishnan and Basu (1995) for details.

For analyzing the failure time data from the step-stress tests, the implicit assumption is that the effect of the stress increase is instantaneous to the lifetime of a test unit. However, it may not be the case in practice and there may be certain amount of time taken until the stress change brings the full effect to the lifetime distribution of the test unit. Thus, it is of interest to work on how to model and analyze such lagging times of the stress change effect with the current framework of the step-stress models.

Another interesting problem for which the step-stress model gives insight into the modeling approach is as follows. We would like to test the reliability of a system which is composed of two components and the system fails only when both components fail. In a sense, this is a parallel system since the system failure time is the maximum of the component failure times. However, the characteristic feature of this system is that the failure of one component increases the stress load to the other component of the system. Thus, the surviving component inside the system will experience the step-stress condition at a random time point of the stress increase. It is of interest, therefore, to develop inferential methods to analyze the failure data in this situation.

Appendix A

Lemmas and Proof of Theorem in Chapter 2

Lemma A.1. *The joint probability mass function (JPMF) of* $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ *is given by*

$$p(\mathbf{n}) = \prod_{i=1}^{k} \binom{N_i}{n_i} \{F(\Delta_i)\}^{n_i} \{1 - F(\Delta_i)\}^{N_i - n_i}, \tag{A.1}$$

where $\Delta_i = \tau_i - \tau_{i-1}$ *and* $F(t)$ *is as defined in (2.2.2).*

Proof. (A.1) follows immediately by straightforward integration of the JPDF of $(\mathbf{n}, \mathbf{y})$ given in (2.2.3) with respect to $\mathbf{y}$. □

Corollary A.1. *Since* $0 \leq n_i \leq N_i$, *we easily see that*

$$\begin{aligned} n_1 &\sim Binomial\big(n,\ F(\Delta_1)\big), \\ \big(n_i \mid n_1, n_2, \ldots, n_{i-1}\big) &\sim Binomial\big(N_i,\ F(\Delta_i)\big) \end{aligned}$$

for $i = 2, 3, \ldots, k$.

Corollary A.2. *The probability of observing at least one failure is*

$$Pr\Big[D \geq 1\Big] = 1 - Pr\Big[D = 0\Big] = F\bigg(\sum_{i=1}^{k} \tau_i c_i^*\bigg)$$

with $c_k^* = n - \sum_{i=1}^{k-1} c_i^*$ *and* $F(t)$ *is as given in (2.2.2).*

Proof. Using the JPMF of $\mathbf{n}$ obtained in (A.1),

$$\begin{aligned} Pr\Big[D = 0\Big] &= Pr\Big[\mathbf{n} = \mathbf{0}_k\Big] = p(\mathbf{0}_k) \\ &= \prod_{i=1}^{k} \big\{1 - F(\Delta_i)\big\}^{n - \sum_{j=1}^{i-1} c_j^*} = \exp\bigg\{-\frac{1}{\theta}\sum_{i=1}^{k} \tau_i c_i^*\bigg\} \end{aligned} \tag{A.2}$$

which readily gives the result. □

Lemma A.2. *The CMGF of* $\hat{\theta}$*, conditioned on* $D \geq 1$*, is*

$$M_c(t) = \frac{\displaystyle\sum_{d=1}^{n}\Big(1 - \frac{t\theta}{d}\Big)^{-d} \sum_{\{\mathbf{n}:D=d\}} \prod_{i=1}^{k} \binom{N_i}{n_i} \bigg\{F\Big(\Delta_i\Big(1 - \frac{t\theta}{d}\Big)\Big)\bigg\}^{n_i} \bigg\{1 - F\Big(\Delta_i\Big(1 - \frac{t\theta}{d}\Big)\Big)\bigg\}^{N_i - n_i}}{1 - Pr\Big[D = 0\Big]} \tag{A.3}$$

for $t < 1/\theta$*, where* $\Delta_i = \tau_i - \tau_{i-1}$ *and* $F(t)$ *is as defined in (2.2.2).*

Proof. It can be shown that given $n_1, n_2, \ldots, n_i$, the random variables $(y_{i,l} - \tau_{i-1})$, $l = 1, 2, \ldots, n_i$, are distributed jointly as order statistics from a random sample of size n_i from a right-truncated exponential distribution at Δ_i. The PDF and CDF of this right truncated distribution are, respectively,

$$f_{RT;i}(t) = \frac{f(t)}{F(\Delta_i)} \qquad \text{and} \qquad F_{RT;i}(t) = \frac{F(t)}{F(\Delta_i)}$$

for $0 < t < \Delta_i$, where $f(t)$ and $F(t)$ are as given in (2.2.1) and (2.2.2). Then, via straightforward integration with respect to $\mathbf{y}$, we obtain

$$E\left[e^{t\hat{\theta}}\middle|D=d\right] = \frac{\sum_{\{\mathbf{n}:D=d\}}\prod_{i=1}^{k}\left(1-\frac{t\theta}{d}\right)^{-n_i}\binom{N_i}{n_i}\left\{F\left(\Delta_i\left(1-\frac{t\theta}{d}\right)\right)\right\}^{n_i}\left\{1-F\left(\Delta_i\left(1-\frac{t\theta}{d}\right)\right)\right\}^{N_i-n_i}}{\sum_{\{\mathbf{n}:D=d\}} p(\mathbf{n})} \tag{A.4}$$

for $t < 1/\theta$, and for $d = 1, 2, \ldots, n$,

$$Pr\left[D=d\middle|D\geq 1\right] = \left\{1 - Pr\left[D=0\right]\right\}^{-1}\sum_{\{\mathbf{n}:D=d\}} p(\mathbf{n}), \tag{A.5}$$

where $p(\mathbf{n})$ and $Pr\left[D=0\right]$ are as given in (A.1) and (A.2), respectively. Since we have from (2.4.1) that

$$M_c(t) \;=\; \sum_{d=1}^{n} E\left[e^{t\hat{\theta}}\middle|D=d\right]\times Pr\left[D=d\middle|D\geq 1\right], \tag{A.6}$$

(A.3) is readily obtained upon substituting the expressions of (A.4) and (A.5) in (A.6) along with Corollary A.2. □

Corollary A.3. *In the case of Type-I PC with equi-spaced time intervals (viz., $\Delta_i = \tau > 0$ for $i = 1, 2, \ldots, k$), the CMGF of $\hat{\theta}$, conditioned on $D \geq 1$, is*

$$M_c(t) = \sum_{d=1}^{n}\sum_{\{\mathbf{n}:D=d\}}\sum_{j=0}^{d} C_{\mathbf{n},j}^{[\theta]}\ \exp\left\{t\ \tau_{\mathbf{n},j}\right\}\left(1-\frac{t}{d/\theta}\right)^{-d}, \qquad t < \frac{1}{\theta}, \tag{A.7}$$

where $\tau_{\mathbf{n},j}$ and $C_{\mathbf{n},j}^{[\theta]}$ are as defined in (2.4.3) and (2.4.4), respectively.

Proof. After setting $\Delta_i = \tau > 0$ in (A.3), the above result is derived by binomial expansion of the term $\left(1-\exp\left\{-\frac{\tau}{\theta}\left(1-\frac{t\theta}{d}\right)\right\}\right)^{d}$. □

One remark is made for the equations given in (A.3) and (A.7) as well as in (2.4.2) since they all contain the notation of summation over the set $\{\mathbf{n} : D = d\}$. This set simply defines a collection of all the possible values $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ can take on satisfying the condition $D = d$ (*i.e.*, the total number of failure observations is d for $1 \leq d \leq n$). For numerical implementation of this summation, one suggested method is to expand it iteratively in the following way:

$$\sum_{\{\mathbf{n}:D=d\}} p(\mathbf{n}) = \sum_{n_1=\eta_1^L}^{\eta_1^U} \sum_{n_2=\eta_2^L}^{\eta_2^U} \cdots \sum_{n_{k-1}=\eta_{k-1}^L}^{\eta_{k-1}^U} p(\mathbf{n})\Bigg|_{n_k=\eta_k^U},$$

where

$$\begin{aligned} \eta_i^L &= \begin{cases} \eta_i^U & , N_i - c_i^* < \eta_i^U \\ 0 & , \textit{otherwise} \end{cases} \\ &= \eta_i^U \times J\left(n - d < \sum_{j=1}^{i} c_j^*\right), \\ \eta_i^U &= d - \sum_{j=1}^{i-1} n_j \end{aligned}$$

for $d = 1, 2, \ldots, n$. Here, $J(\cdot)$ is an indicator function as used in Section 2.5.3. Now, in order to obtain the exact conditional distribution of $\hat{\theta}$ in Theorem 2.4.1, we require the following lemma.

Lemma A.3. *Let Y be a gamma random variable with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$. That is,*

$$Y \sim Gamma(\alpha, \lambda)$$

with its PDF as given in (2.4.5). Then, for any arbitrary constant δ, the PDF of

$X = Y + \delta$ is given by

$$\gamma(x-\delta\ ;\ \alpha,\lambda) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)}(x-\delta)^{\alpha-1}e^{-\lambda(x-\delta)} & ,\ x > \delta \\ 0 & ,\ \textit{otherwise} \end{cases} \tag{A.8}$$

and its moment generating function is of the form

$$M_X(t) = e^{t\delta}\left(1 - \frac{t}{\lambda}\right)^{-\alpha}, \qquad t < \lambda. \tag{A.9}$$

Proof. (A.8) and (A.9) are derived readily from the properties of the gamma distribution and the definition of the moment generating function; see Johnson, Kotz and Balakrishnan (1994). □

Proof of Theorem 2.4.1: Applying the inversion theorem of a moment generating function to the result from Corollary A.3 in conjunction with Lemma A.3, we can obtain the exact conditional PDF of $\hat{\theta}$, given $D \geq 1$. Hence, the result. □

Appendix B

Proof of Lemmas and Theorems in Chapter 3

Proof of Lemma 3.3.1: The expression in (3.3.3) can be verified by induction. Following the usual convention, we have $E[N_1] = n$ for $i = 1$ in (3.3.3). Now, suppose that (3.3.3) holds for $i = i'$. Using (3.3.2) in conjunction with Property (1) and the fact that $N_{i+1} = N_i - n_i - c_i$, we have

$$\begin{aligned} E[N_{i'+1}] &= E[N_{i'} - n_{i'} - c_{i'}] \\ &= (E[N_{i'}] - E[n_{i'}])(1 - \pi^*_{i'}) \\ &= E[N_{i'}]S_{i'}(\tau)(1 - \pi^*_{i'}) \\ &= n \prod_{j=1}^{i'} S_j(\tau)(1 - \pi^*_j), \end{aligned}$$

which is precisely the expression in (3.3.3) for $i = i' + 1$. Hence, the result. □

Proof of Theorem 3.3.1: Using Properties (1) and (2), the expected value of U_i in (3.2.5) is

$$\begin{aligned} E[U_i] &= E\left[\sum_{j=1}^{n_i}(y_{i,j}-(i-1)\tau)\right] + (E[N_i]-E[n_i])\tau \\ &= E[n_i]\left(\theta_i - \tau\frac{S_i(\tau)}{F_i(\tau)}\right) + (E[N_i]-E[N_i]F_i(\tau))\tau \\ &= E[N_i]F_i(\tau)\theta_i = E[n_i]\exp(\alpha+\beta x_i), \end{aligned}$$

for $i = 1, 2, \ldots, k$. The matrix $\mathbf{I}_n(\alpha, \beta)$ can then be expressed as

$$\mathbf{I}_n(\alpha,\beta) = \begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} \\ I_{\alpha\beta} & I_{\beta\beta} \end{pmatrix},$$

where

$$\begin{aligned} I_{\alpha\alpha} &= E\left[-\frac{\partial^2}{\partial\alpha^2}l(\alpha,\beta)\right] &&= \sum_{i=1}^{k} E[U_i]\exp[-(\alpha+\beta x_i)] &&= \sum_{i=1}^{k} E[n_i] &&= n\sum_{i=1}^{k} A_i(\tau), \\ I_{\alpha\beta} &= E\left[-\frac{\partial^2}{\partial\alpha\partial\beta}l(\alpha,\beta)\right] &&= \sum_{i=1}^{k} E[U_i]\exp[-(\alpha+\beta x_i)]x_i &&= \sum_{i=1}^{k} E[n_i]x_i &&= n\sum_{i=1}^{k} A_i(\tau)x_i, \\ I_{\beta\beta} &= E\left[-\frac{\partial^2}{\partial\beta^2}l(\alpha,\beta)\right] &&= \sum_{i=1}^{k} E[U_i]\exp[-(\alpha+\beta x_i)]x_i^2 &&= \sum_{i=1}^{k} E[n_i]x_i^2 &&= n\sum_{i=1}^{k} A_i(\tau)x_i^2, \end{aligned}$$

with $A_i(\tau)$ redefined by using Lemma 3.3.1 as

$$A_i(\tau) = \frac{1}{n}E[n_i] = \frac{1}{n}E[N_i]F_i(\tau) = \left[\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*)\right]F_i(\tau).$$

□

Proof of Theorem 3.4.1: The first two derivatives of (3.4.2) with respect to τ are

$$\begin{aligned} \phi'(\tau) &= -(1+\xi)^2\frac{A_1'(\tau)}{[A_1(\tau)]^2} - \xi^2\frac{A_2'(\tau)}{[A_2(\tau)]^2}, \\ \phi''(\tau) &= -(1+\xi)^2\frac{A_1(\tau)A_1''(\tau)-2[A_1'(\tau)]^2}{[A_1(\tau)]^3} - \xi^2\frac{A_2(\tau)A_2''(\tau)-2[A_2'(\tau)]^2}{[A_2(\tau)]^3}, \end{aligned}$$

where

$$
\begin{aligned}
A_1(\tau) &= F_1(\tau) = 1 - S_1(\tau), \qquad A_1'(\tau) = \frac{1}{\theta_1} S_1(\tau), \qquad A_1''(\tau) = -\frac{1}{\theta_1^2} S_1(\tau), \\
A_2(\tau) &= F_2(\tau) S_1(\tau)(1-\pi_1^*) = (1 - S_2(\tau)) S_1(\tau)(1-\pi_1^*), \\
A_2'(\tau) &= \left[\frac{1}{\theta_2} S_2(\tau) - \frac{1}{\theta_1} F_2(\tau)\right] S_1(\tau)(1-\pi_1^*), \\
A_2''(\tau) &= -\left[\frac{1}{\theta_2^2} S_2(\tau) + \frac{2}{\theta_1\theta_2} S_2(\tau) - \frac{1}{\theta_1^2} F_2(\tau)\right] S_1(\tau)(1-\pi_1^*).
\end{aligned}
$$

Since $A_1(\tau) > 0$ and $A_1''(\tau) < 0$ for all $\tau > 0$, $A_1(\tau)A_1''(\tau) - 2[A_1'(\tau)]^2 < 0$, making the first term of $\phi''(\tau)$ positive for all $\tau > 0$. Furthermore, $A_2(\tau) > 0$ and

$$
\begin{aligned}
& A_2(\tau)A_2''(\tau) - 2[A_2'(\tau)]^2 \\
&= -F_2(\tau)\left[\frac{1}{\theta_2^2} S_2(\tau) + \frac{2}{\theta_1\theta_2} S_2(\tau) - \frac{1}{\theta_1^2} F_2(\tau)\right] [S_1(\tau)(1-\pi_1^*)]^2 \\
&\qquad -2\left[\frac{1}{\theta_2} S_2(\tau) - \frac{1}{\theta_1} F_2(\tau)\right]^2 [S_1(\tau)(1-\pi_1^*)]^2 \\
&= -\left\{\frac{1}{\theta_2^2} S_2(\tau)F_2(\tau) + \frac{2}{\theta_2^2}[S_2(\tau)]^2 - \frac{2}{\theta_1\theta_2} S_2(\tau)F_2(\tau) + \frac{1}{\theta_1^2}[F_2(\tau)]^2\right\} [S_1(\tau)(1-\pi_1^*)]^2 \\
&= -\left\{\frac{1}{\theta_2^2} S_2(\tau) + \left[\frac{1}{\theta_2} S_2(\tau) - \frac{1}{\theta_1} F_2(\tau)\right]^2\right\} [S_1(\tau)(1-\pi_1^*)]^2 < 0
\end{aligned}
$$

for all $\tau > 0$, making the second term of $\phi''(\tau)$ positive, too. Therefore, $\phi(\tau)$ is convex as $\phi''(\tau) > 0$ for all $\tau > 0$. Hence, τ_C^* satisfying $\phi'(\tau_C^*) = 0$ minimizes $\phi(\tau)$ for $k = 2$. □

Proof of Theorem 3.4.2: Differentiating (3.4.4) with respect to τ, we obtain

$$
\begin{aligned}
\delta'(\tau) &= \{A_1'(\tau)A_2(\tau) + A_1(\tau)A_2'(\tau)\}(x_2 - x_1)^2 \\
&= \frac{1}{\theta_1} F_1(\tau)F_2(\tau)\left\{Q_1^D + \frac{\theta_1}{\theta_2} Q_2^D - 1\right\} S_1(\tau)(1-\pi_1^*)(x_2 - x_1)^2,
\end{aligned}
$$

where $Q_1^D = \dfrac{S_1(\tau)}{F_1(\tau)}$ and $Q_2^D = \dfrac{S_2(\tau)}{F_2(\tau)}$. Now, we observe that $\delta(\tau)$ monotonically increases when $Q_1^D > 1$ or equivalently when $\tau < \theta_1 \log 2$ since $Q_1^D > 1$ implies $\delta'(\tau) > 0$. In other words, when the chance of a unit to survive the first stress level exceeds 50%, $\delta(\tau)$ increases. Besides, we see that $\theta_1 > \theta_2$ according to (3.2.1) from assumption (iii) because the MTTF decreases as the stress increases (*i.e.*, $x_1 < x_2$). Then, the following is true:

$$\theta_1 > \theta_2 \iff S_1(\tau) > S_2(\tau) \iff F_1(\tau) < F_2(\tau).$$

It is then obvious that $Q_1^D > Q_2^D$ and so, $Q_1^D + \dfrac{\theta_1}{\theta_2} Q_2^D < Q_1^D + \dfrac{\theta_1}{\theta_2} Q_1^D = Q_1^D \left(1 + \dfrac{\theta_1}{\theta_2}\right)$. If $Q_1^D \left(1 + \dfrac{\theta_1}{\theta_2}\right) < 1$ or equivalently if $\tau > \theta_1 \log\left(2 + \dfrac{\theta_1}{\theta_2}\right)$, $\delta'(\tau) < 0$ and $\delta(\tau)$ monotonically decreases. Since $\delta'(\tau)$ is absolutely continuous in τ, there is $\tau_D^* \in \left(\theta_1 \log 2, \theta_1 \log\left(2 + \dfrac{\theta_1}{\theta_2}\right)\right)$ such that $\delta'(\tau_D^*) = 0$. As $\delta'(\tau)$ changes the sign around τ_D^*, τ_D^* maximizes $\delta(\tau)$. □

Proof of Theorem 3.4.3: Differentiating (3.4.6) with respect to τ, we get

$$\begin{aligned} a'(\tau) &= A_1'(\tau)(1 + x_1^2) + A_2'(\tau)(1 + x_2^2) \\ &= \frac{1}{\theta_1} S_1(\tau)(1 + x_1^2) + \left[\frac{1}{\theta_2} S_2(\tau) - \frac{1}{\theta_1} F_2(\tau)\right] S_1(\tau)(1 - \pi_1^*)(1 + x_2^2) \\ &= \frac{1}{\theta_1} S_1(\tau) Q_2^A (1 - \pi_1^*)(1 + x_2^2), \end{aligned}$$

where $Q_1^A = \dfrac{1 + x_1^2}{(1 - \pi_1^*)(1 + x_2^2)}$ and $Q_2^A = Q_1^A - 1 + \left(1 + \dfrac{\theta_1}{\theta_2}\right) S_2(\tau)$. Now, observe that

$$a'(\tau) > 0 \iff Q_2^A > 0 \iff \theta_2 \log\left[\left(1 + \frac{\theta_1}{\theta_2}\right)(1 - Q_1^A)^{-1}\right] > \tau.$$

Therefore, $a(\tau)$ monotonically increases when $\tau < \theta_2 \log\left[\left(1+\frac{\theta_1}{\theta_2}\right)(1-Q_1^A)^{-1}\right]$ and decreases when $\tau > \theta_2 \log\left[\left(1+\frac{\theta_1}{\theta_2}\right)(1-Q_1^A)^{-1}\right]$. The turning point of $a(\tau)$ which maximizes $a(\tau)$ is then $\tau_A^* = \theta_2 \log\left[\left(1+\frac{\theta_1}{\theta_2}\right)(1-Q_1^A)^{-1}\right]$. From the derivation, however, τ_A^* is only defined when $0 < Q_1^A < 1$ or equivalently when $\pi_1^* < \frac{x_2^2 - x_1^2}{1+x_2^2}$. Otherwise, we have

$$Q_1^A \geq 1 \iff Q_2^A > 0 \iff a'(\tau) > 0,$$

and so $a(\tau)$ is ever increasing in τ. That is, τ_A^* is unbounded if $Q_1^A \geq 1$. □

Proof of Lemma 3.5.1: We shall establish this result by induction. Using (3.3.2) and Property (1), (3.5.1) with $i = k-1$ is

$$\begin{aligned}
Pr(N_k = 0|n_1, n_2, \ldots, n_{k-2}) &= Pr(N_{k-1} - n_{k-1} - c_{k-1} = 0|n_1, n_2, \ldots, n_{k-2}) \\
&= Pr((N_{k-1} - n_{k-1})(1-\pi_{k-1}^*) = 0|n_1, n_2, \ldots, n_{k-2}) \\
&= Pr(n_{k-1} = N_{k-1}|n_1, n_2, \ldots, n_{k-2}) \\
&= [F_{k-1}(\tau)]^{N_{k-1}} = [H_{k-1}(\tau)]^{N_{k-1}}.
\end{aligned}$$

Now, let us suppose that (3.5.1) holds for $i = i'+1$. Then, for $i = i'$, we get

$$\begin{aligned}
Pr(N_k = 0|n_1, n_2, \ldots, n_{i'-1}) &= \sum_{n_{i'}=0}^{N_{i'}} Pr(N_k = 0|n_1, n_2, \ldots, n_{i'-1}, n_{i'})p(n_{i'}|n_1, n_2, \ldots, n_{i'-1}) \\
&= \sum_{n_{i'}=0}^{N_{i'}} [H_{i'+1}(\tau)]^{N_{i'+1}} p(n_{i'}|n_1, n_2, \ldots, n_{i'-1}) \\
&= \sum_{n_{i'}=0}^{N_{i'}} [H_{i'+1}(\tau)]^{(N_{i'}-n_{i'})(1-\pi_{i'}^*)} \binom{N_{i'}}{n_{i'}} [F_{i'}(\tau)]^{n_{i'}} [S_{i'}(\tau)]^{N_{i'}-n_{i'}} \\
&= \left(F_{i'}(\tau) + S_{i'}(\tau)[H_{i'+1}(\tau)]^{1-\pi_{i'}^*}\right)^{N_{i'}} = [H_{i'}(\tau)]^{N_{i'}},
\end{aligned}$$

which completes the proof. □

Proof of Lemma 3.5.2: We shall establish this result by induction. When $i = 1$, (3.5.4) becomes

$$E\Big[N_1[H_1(\tau)]^{N_1}\Big] = E\Big[n[H_1(\tau)]^n\Big] = n[H_1(\tau)]^n,$$

which conforms to the lemma. Assuming now that the lemma holds for $i = i'$, the binomial theorem along with definition (3.5.2) yields the left-hand side of (3.5.4) at $i = i' + 1$ as

$$\begin{aligned}
&E\Big[N_{i'+1}[H_{i'+1}(\tau)]^{N_{i'+1}}\Big] \\
&\quad= E\Big[E\Big[(N_{i'} - n_{i'})(1-\pi^*_{i'})[H_{i'+1}(\tau)]^{(N_{i'}-n_{i'})(1-\pi^*_{i'})}\Big| n_1, n_2, \ldots, n_{i'-1}\Big]\Big] \\
&\quad= E\Big[N_{i'}\big(F_{i'}(\tau) + S_{i'}(\tau)[H_{i'+1}(\tau)]^{1-\pi^*_{i'}}\big)^{N_{i'}-1}\Big](1-\pi^*_{i'})[H_{i'+1}(\tau)]^{1-\pi^*_{i'}}S_{i'}(\tau) \\
&\quad= E\Big[N_{i'}[H_{i'}(\tau)]^{N_{i'}}\Big](1-\pi^*_{i'})\left(\frac{H_{i'}(\tau) - F_{i'}(\tau)}{H_{i'}(\tau)}\right) \\
&\quad= n[H_1(\tau)]^n\left[\prod_{j=1}^{i'}(1-\pi^*_j)\left(1 - \frac{F_j(\tau)}{H_j(\tau)}\right)\right],
\end{aligned}$$

which completes the proof. □

Proof of Theorem 3.5.1: By partitioning the support of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ into two mutually exclusive sets, $\{\mathbf{n} : N_k > 0\}$ and $\{\mathbf{n} : N_k = 0\}$, the conditional expectation of n_i, given $N_k > 0$, can be obtained as

$$\begin{aligned}
E_c[n_i] &= E[n_i | N_k > 0] \\
&= \frac{1}{Pr(N_k > 0)} \sum_{\{\mathbf{n}:N_k>0\}} n_i p_J(\mathbf{n}) \\
&= \frac{1}{1-[H_1(\tau)]^n} \left[E[n_i] - \sum_{\{\mathbf{n}:N_k=0\}} n_i p_J(\mathbf{n}) \right] \\
&= E[n_i] \frac{1-V_i(\tau)}{1-[H_1(\tau)]^n},
\end{aligned}$$

where $p_J(\mathbf{n})$ is the JPMF of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and

$$V_i(\tau) = \frac{1}{E[n_i]} \sum_{\{\mathbf{n}:N_k=0\}} n_i p_J(\mathbf{n}),$$

for $i = 1, 2, \ldots, k$. Note that $N_k = 0$ implies $n_k = 0$ since $0 \le n_k \le N_k$, and thus $V_k(\tau) = 0$. Now, from Property (1) and Lemma 3.3.1,

$$E[n_i] = E[N_i] F_i(\tau) = n \left[\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*) \right] F_i(\tau),$$

for $i = 1, 2, \ldots, k$. Using Lemma 3.5.1, (3.3.2), Property (1), (3.5.2), and Lemma 3.5.2 along with the property of iterated expectation and the binomial theorem, we obtain

$$\begin{aligned}
\sum_{\{\mathbf{n}:N_k=0\}} n_i p_J(\mathbf{n}) &= E\Big[n_i Pr(N_k = 0 | n_1, n_2, \ldots, n_i) \Big] \\
&= E\Big[E\Big[n_i [H_{i+1}(\tau)]^{N_{i+1}} \Big| n_1, n_2, \ldots, n_{i-1} \Big] \Big] \\
&= E\left[\sum_{n_i=1}^{N_i} n_i [H_{i+1}(\tau)]^{(N_i-n_i)(1-\pi_i^*)} \binom{N_i}{n_i} [F_i(\tau)]^{n_i} [S_i(\tau)]^{N_i-n_i} \right] \\
&= F_i(\tau) E\Big[N_i \big(F_i(\tau) + S_i(\tau)[H_{i+1}(\tau)]^{1-\pi_i^*} \big)^{N_i-1} \Big] \\
&= \frac{F_i(\tau)}{H_i(\tau)} E\Big[N_i [H_i(\tau)]^{N_i} \Big] \\
&= n[H_1(\tau)]^n \frac{F_i(\tau)}{H_i(\tau)} \prod_{j=1}^{i-1} (1-\pi_j^*) \left(1 - \frac{F_j(\tau)}{H_j(\tau)} \right),
\end{aligned}$$

for $i = 1, 2, \ldots, k-1$. Therefore, $V_i(\tau)$ is simplified, for $i = 1, 2, \ldots, k-1$, as

$$\begin{aligned}
V_i(\tau) &= \frac{1}{E[n_i]} \sum_{\{\mathbf{n}:N_k=0\}} n_i p_J(\mathbf{n}) \\
&= \frac{n[H_1(\tau)]^n \dfrac{F_i(\tau)}{H_i(\tau)} \displaystyle\prod_{j=1}^{i-1}(1-\pi_j^*)\left(1-\frac{F_j(\tau)}{H_j(\tau)}\right)}{n\left[\displaystyle\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*)\right] F_i(\tau)} \\
&= \frac{[H_1(\tau)]^n}{H_i(\tau)} \prod_{j=1}^{i-1} \frac{[H_{j+1}(\tau)]^{1-\pi_j^*}}{H_j(\tau)} \\
&= \frac{[H_1(\tau)]^{n-1}}{\prod_{j=1}^{i-1}[H_{j+1}(\tau)]^{\pi_j^*}},
\end{aligned}$$

which completes the proof of the theorem. □

Proof of Lemma 3.5.3: We shall establish this result by induction. First of all, it is valid when $i = 1$ since the left-hand side of (3.5.5) produces $E_c[N_1] = E_c[n] = n$ and its right-hand side yields $E[N_1]\dfrac{1 - H_1(\tau)V_1(\tau)}{1 - [H_1(\tau)]^n} = E[N_1] = E[n] = n$ as $V_1(\tau) = [H_1(\tau)]^{n-1}$. Let us then assume that (3.5.5) holds for $i = i'$. Using Theorem 3.5.1, (3.5.5) at $i = i' + 1$ leads to

$$\begin{aligned}
E_c[N_{i'+1}] &= E_c[N_{i'} - n_{i'} - c_{i'}] = (E_c[N_{i'}] - E_c[n_{i'}])(1 - \pi^*_{i'}) \\
&= \left(E[N_{i'}] \frac{1 - H_{i'}(\tau)V_{i'}(\tau)}{1 - [H_1(\tau)]^n} - E[n_{i'}] \frac{1 - V_{i'}(\tau)}{1 - [H_1(\tau)]^n} \right)(1 - \pi^*_{i'}) \\
&= \frac{E[N_{i'}]}{1 - [H_1(\tau)]^n}(S_{i'}(\tau) - (H_{i'}(\tau) - F_{i'}(\tau))V_{i'}(\tau))(1 - \pi^*_{i'}) \\
&= \frac{E[N_{i'}]}{1 - [H_1(\tau)]^n}(S_{i'}(\tau) - S_{i'}(\tau)[H_{i'+1}(\tau)]^{1-\pi^*_{i'}}V_{i'}(\tau))(1 - \pi^*_{i'}) \\
&= \frac{E[N_{i'}]S_{i'}(\tau)}{1 - [H_1(\tau)]^n}(1 - H_{i'+1}(\tau)V_{i'+1}(\tau))(1 - \pi^*_{i'}) \\
&\qquad\qquad \text{since } V_{i'}(\tau) = V_{i'+1}(\tau)[H_{i'+1}(\tau)]^{\pi^*_{i'}} \\
&= E[N_{i'+1}] \frac{1 - H_{i'+1}(\tau)V_{i'+1}(\tau)}{1 - [H_1(\tau)]^n} \\
&\qquad\qquad \text{since } E[N_{i'+1}] = E[N_{i'}]S_{i'}(\tau)(1 - \pi^*_{i'}).
\end{aligned}$$

Hence, the result. □

Proof of Theorem 3.5.2: Using Properties (1) and (2) and the results of Theorem 3.5.1 and Lemma 3.5.3, the expected value of (3.2.5), conditioned on $N_k > 0$, is

$$\begin{aligned}
E_c[U_i] &= E_c\left[\sum_{j=1}^{n_i} (y_{i,j} - (i-1)\tau) \right] + \tau E_c[N_i] - \tau E_c[n_i] \\
&= E_c[n_i]\left(\theta_i - \tau \frac{S_i(\tau)}{F_i(\tau)} \right) + \tau E_c[N_i] - \tau E_c[n_i] \\
&= E[N_i]F_i(\tau)\left(\frac{1 - V_i(\tau)}{1 - [H_1(\tau)]^n} \right)\left(\theta_i - \frac{\tau}{F_i(\tau)} \right) + \tau E[N_i]\left(\frac{1 - H_i(\tau)V_i(\tau)}{1 - [H_1(\tau)]^n} \right) \\
&= \frac{E[N_i]}{1 - [H_1(\tau)]^n}\Big[(1 - V_i(\tau))\theta_i F_i(\tau) + \tau(1 - H_i(\tau))V_i(\tau) \Big]
\end{aligned}$$

for $i = 1, 2, \ldots, k$. The Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$, given $N_k > 0$, is then expressed as

$$\mathbf{I}_n(\alpha,\beta) = \begin{pmatrix} I_{\alpha\alpha}^{cond} & I_{\alpha\beta}^{cond} \\ I_{\alpha\beta}^{cond} & I_{\beta\beta}^{cond} \end{pmatrix},$$

where

$$\begin{aligned}
I_{\alpha\alpha}^{cond} &= E_c\left[-\frac{\partial^2}{\partial\alpha^2} l(\alpha,\beta)\right] &&= \sum_{i=1}^{k} E_c[U_i]\exp[-(\alpha+\beta x_i)] &&= n\sum_{i=1}^{k} A_i(\tau),\\
I_{\alpha\beta}^{cond} &= E_c\left[-\frac{\partial^2}{\partial\alpha\partial\beta} l(\alpha,\beta)\right] &&= \sum_{i=1}^{k} E_c[U_i]\exp[-(\alpha+\beta x_i)]x_i &&= n\sum_{i=1}^{k} A_i(\tau)x_i,\\
I_{\beta\beta}^{cond} &= E_c\left[-\frac{\partial^2}{\partial\beta^2} l(\alpha,\beta)\right] &&= \sum_{i=1}^{k} E_c[U_i]\exp[-(\alpha+\beta x_i)]x_i^2 &&= n\sum_{i=1}^{k} A_i(\tau)x_i^2,
\end{aligned}$$

with $A_i(\tau)$ redefined by using the expression of $E_c[U_i]$ as

$$\begin{aligned}
A_i(\tau) &= \frac{1}{n}E_c[U_i]\exp[-(\alpha+\beta x_i)] = \frac{1}{n\theta_i}E_c[U_i]\\
&= \frac{E[N_i]}{n(1-[H_1(\tau)]^n)}\left[(1-V_i(\tau))F_i(\tau) + \frac{\tau}{\theta_i}(1-H_i(\tau))V_i(\tau)\right].
\end{aligned}$$

Substituting the result of Lemma 3.3.1 for $E[N_i]$ in $A_i(\tau)$ above, we obtain (3.5.7). □

Proof of Theorem 3.5.3: We shall establish this result by induction. Since $H_k(\tau) = 0$ and $H_{k-1}(\tau) = F_{k-1}(\tau) < 1$, the relation is clearly valid for $i = k$ and $i = k-1$. Assuming now that it holds for $i = i'+1$, we find

$$\begin{aligned}
0 \le H_{i'+1}(\tau) < 1 \quad &\Longleftrightarrow \quad 0 \le [H_{i'+1}(\tau)]^{1-\pi_{i'}^*} < 1 \qquad \text{since } 0 \le \pi_{i'}^* < 1\\
&\Longleftrightarrow \quad F_{i'}(\tau) \le F_{i'}(\tau) + S_{i'}(\tau)[H_{i'+1}(\tau)]^{1-\pi_{i'}^*} < F_{i'}(\tau) + S_{i'}(\tau)\\
&\Longleftrightarrow \quad F_{i'}(\tau) \le H_{i'}(\tau) < 1\\
&\Longrightarrow \quad 0 \le H_{i'}(\tau) < 1.
\end{aligned}$$

Hence, $H_i(\tau)$ is always bounded between 0 and 1 for $i = 1, 2, \ldots, k$. □

Appendix C

Lemmas and Proof of Theorems in Chapter 4

Lemma C.1. *The JPMF of* $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2)$, *where* $\mathbf{N}_i = (N_{i1}, N_{i2})$, $i = 1, 2$, *is given by*

$$Pr\big[\mathbf{N} = \mathbf{n}\big] = \binom{n}{n_{11}, n_{12}, n - n_{1\cdot}} \binom{n_{2\cdot}}{n_{21}} \left\{ \prod_{i,j=1}^{2} \pi_{ij}^{n_{ij}} \right\} \big\{F(\tau)\big\}^{n_{1\cdot}} \big\{1 - F(\tau)\big\}^{n - n_{1\cdot}} \tag{C.1}$$

for $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2)$, *where* $\mathbf{n}_i = (n_{i1}, n_{i2})$, $i = 1, 2$, *and* n_{ij} *are non-negative integers satisfying* $\sum_{i,j=1}^{2} n_{ij} = r \leq n$. *Here,* π_{ij} *are as defined in (4.2.4) and (4.2.5) and* $F(\tau)$ *is as given in (4.3.2).*

Proof. For fixed $\mathbf{n} = (n_{11}, n_{12}, n_{21}, n_{22})$, we let the realized values of $N_{i\cdot}$ be $n_{i\cdot} = n_{i1} + n_{i2}$ for $i = 1, 2$ still holding $n_{1\cdot} + n_{2\cdot} = r$. Then, by conditioning on $N_{1\cdot} = n_{1\cdot}$, we have

$$\begin{aligned} Pr\big[\mathbf{N} = \mathbf{n}\big] &= Pr\big[\mathbf{N} = \mathbf{n}\big|N_{1\cdot} = n_{1\cdot}\big]\ Pr\big[N_{1\cdot} = n_{1\cdot}\big] \\ &= Pr\big[\mathbf{N}_1 = \mathbf{n}_1\big|N_{1\cdot} = n_{1\cdot}\big]\ Pr\big[\mathbf{N}_2 = \mathbf{n}_2\big|N_{1\cdot} = n_{1\cdot}\big]\ Pr\big[N_{1\cdot} = n_{1\cdot}\big] \\ &= Pr\big[N_{11} = n_{11}\big|N_{1\cdot} = n_{1\cdot}\big]\ Pr\big[N_{21} = n_{21}\big|N_{2\cdot} = n_{2\cdot}\big]\ Pr\big[N_{1\cdot} = n_{1\cdot}\big], \end{aligned} \tag{C.2}$$

where the second equality results from the fact that given $N_{1\cdot} = n_{1\cdot}$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are independent. Now, it can be easily shown that

$$\begin{aligned} \big(N_{1j}\big|N_{1\cdot} = n_{1\cdot}\big) &\sim Binomial\big(n_{1\cdot},\ \pi_{1j}\big) \qquad \text{for } j = 1, 2, \\ \big(N_{2j}\big|N_{2\cdot} = n_{2\cdot}\big) &\sim Binomial\big(n_{2\cdot},\ \pi_{2j}\big) \qquad \text{for } j = 1, 2, \\ N_{1\cdot} &\sim Binomial\big(n,\ F(\tau)\big) \end{aligned}$$

for the model under consideration and thus, (C.1) follows by substituting the binomial probability mass functions in (C.2). $\square$

Using the result in Lemma C.1, we can further simplify the expression of the denominator in (4.3.3). Let us first denote $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2)$ where $\mathbf{m}_i = (m_{i1}, m_{i2})$ and $m_{i\cdot} = m_{i1} + m_{i2}$ for $i = 1, 2$. Analogous to n_{ij} and $n_{i\cdot}$, m_{ij} and $m_{i.}$ are non-negative integers which satisfy $\sum_{i,j=1}^{2} m_{ij} = m_{1\cdot} + m_{2\cdot} = r \leq n$. Then, using (C.1) along with the binomial theorem, we have

$$\sum_{\mathbf{m}\in\mathfrak{S}} Pr\big[\mathbf{N}=\mathbf{m}\big]$$

$$\begin{aligned}
&= \sum_{m_{1\cdot}=2}^{r-2}\sum_{m_{11}=1}^{m_{1\cdot}-1}\sum_{m_{21}=1}^{m_{2\cdot}-1}\binom{n}{m_{11}, m_{12}, n-m_{1\cdot}}\binom{m_{2\cdot}}{m_{21}}\left\{\prod_{i,j=1}^{2}\pi_{ij}^{m_{ij}}\right\}\{F(\tau)\}^{m_{1\cdot}}\{1-F(\tau)\}^{n-m_{1\cdot}} \\
&= \sum_{m_{1\cdot}=2}^{r-2}\binom{n}{m_{1\cdot}}\{F(\tau)\}^{m_{1\cdot}}\{1-F(\tau)\}^{n-m_{1\cdot}}\left\{\sum_{m_{11}=1}^{m_{1\cdot}-1}\binom{m_{1\cdot}}{m_{11}}\pi_{11}^{m_{11}}\pi_{12}^{m_{12}}\right\} \\
&\qquad\qquad\qquad\qquad\times\left\{\sum_{m_{21}=1}^{m_{2\cdot}-1}\binom{m_{2\cdot}}{m_{21}}\pi_{21}^{m_{21}}\pi_{22}^{m_{22}}\right\} \\
&= \sum_{m_{1\cdot}=2}^{r-2}\binom{n}{m_{1\cdot}}\{F(\tau)\}^{m_{1\cdot}}\{1-F(\tau)\}^{n-m_{1\cdot}}\big(1-\pi_{11}^{m_{1\cdot}}-\pi_{12}^{m_{1\cdot}}\big)\big(1-\pi_{21}^{m_{2\cdot}}-\pi_{22}^{m_{2\cdot}}\big). \qquad (C.3)
\end{aligned}$$

Lemma C.2. *The CMGF of $\hat{\theta}_{1j}$, conditioned on $\mathbf{N}\in\mathfrak{S}$, is*

$$M_{1j}(t) = \sum_{\mathbf{n}\in\mathfrak{S}}\sum_{k=0}^{n_{1\cdot}} C_{\mathbf{n},jk}^{[1]}\ \exp\Big\{t\ \tau_{1jk}\Big\}\left(1-\frac{t}{n_{1j}\Big(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\Big)}\right)^{-n_{1\cdot}}, \qquad t<\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right) \qquad (C.4)$$

for $j=1,2$, where τ_{1jk} and $C_{\mathbf{n},jk}^{[1]}$ are as defined in (4.3.5) and (4.3.6), respectively.

Proof. Under the life-testing scheme outlined in Section 4.2, let us denote

$$\Big\{0<T_{1:n}<T_{2:n}<\cdots<T_{N_{1\cdot}:n}<\tau\Big\}$$

for the ordered failure times before τ, where $N_{1\cdot}$ is the total number of failures before τ. Then, it can be shown that their joint distribution, conditioned on $\mathbf{N}_1=\mathbf{n}_1$, is identical to the joint distribution of all order statistics from a random sample of size $n_{1\cdot}=n_{11}+n_{12}$ from the right truncated distribution at τ; see Arnold *et al.* (1992) and David and Nagaraja (2003) for details. The PDF and CDF of this right truncated distribution are, respectively,

$$f_{RT}(t)=\frac{f(t)}{F(\tau)} \qquad \text{and} \qquad F_{RT}(t)=\frac{F(t)}{F(\tau)}$$

for $0 < t < \tau$, where $F(t)$ and $f(t)$ are as given in (4.2.1) and (4.2.2). Then, by using straightforward integration and binomial expansion, we derive

$$\begin{aligned}
E\left[e^{t\hat{\theta}_{1j}}\Big|\mathbf{N_1}=\mathbf{n_1}\right] &= \left\{F(\tau)\right\}^{-n_{1\cdot}}\left(1-\exp\left\{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}-\frac{t}{n_{1j}}\right)\tau\right\}\right)^{n_{1\cdot}} \\
&\quad \times \exp\left\{t\ (n-n_{1\cdot})\frac{\tau}{n_{1j}}\right\}\left(1-\frac{t}{n_{1j}\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)}\right)^{-n_{1\cdot}} \\
&= \left\{F(\tau)\right\}^{-n_{1\cdot}}\sum_{k=0}^{n_{1\cdot}}\binom{n_{1\cdot}}{k}(-1)^k\exp\left\{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)\tau k\right\} \\
&\quad \times \exp\left\{t\ \tau_{1jk}\right\}\left(1-\frac{t}{n_{1j}\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)}\right)^{-n_{1\cdot}} \qquad \text{(C.5)}
\end{aligned}$$

for $t < \left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)$, $j=1,2$. Since we have from (4.3.1) that

$$\begin{aligned}
M_{1j}(t) &= E\left[e^{t\hat{\theta}_{1j}}\Big|\mathbf{N}\in\mathfrak{S}\right] \\
&= \sum_{\mathbf{n}\in\mathfrak{S}} E\left[e^{t\hat{\theta}_{1j}}\Big|\mathbf{N_1}=\mathbf{n_1}\right]\times Pr\left[\mathbf{N}=\mathbf{n}\Big|\mathbf{N}\in\mathfrak{S}\right] \qquad \text{(C.6)}
\end{aligned}$$

for $j=1,2$, (C.4) is readily obtained upon using (C.5) and (4.3.3) in (C.6) along with Lemma C.1. □

Lemma C.3. *The CMGF of $\hat{\theta}_{2j}$, conditioned on $\mathbf{N}\in\mathfrak{S}$, is*

$$M_{2j}(t) = \sum_{\mathbf{n}\in\mathfrak{S}} C_{\mathbf{n}}^{[2]}\left(1-\frac{t}{n_{2j}\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)}\right)^{-n_{2\cdot}}, \qquad t<\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right) \qquad \text{(C.7)}$$

for $j=1,2$, where $C_{\mathbf{n}}^{[2]}$ is as defined in (4.3.9).

Proof. Again, under the life-testing scheme described in Section 4.2, let us denote

$$\left\{\tau \le T_{N_{1\cdot}+1:n} < T_{N_{1\cdot}+2:n} < \cdots < T_{r:n} < \infty\right\}$$

for the ordered failure times after τ. We also denote $N_{2\cdot} = r - N_{1\cdot}$ for the restricted number of failures after τ. Then, it can be shown that the conditional joint distribution of these failure times, given $\mathbf{N}_2 = \mathbf{n}_2$, is the same as the joint distribution of the order statistics from a Type-II right censored sample of size $n_{2\cdot} = n_{21} + n_{22}$ initially derived from a random sample of size $n - n_{1\cdot}$ (or $n - r + n_{2\cdot}$) with the left truncated distribution at τ; see Arnold *et al.* (1992) and David and Nagaraja (2003) for details. The PDF and CDF of this left truncated distribution are given, respectively, by

$$f_{LT}(t) = \frac{f(t)}{1 - F(\tau)} \qquad \text{and} \qquad F_{LT}(t) = \frac{F(t) - F(\tau)}{1 - F(\tau)}$$

for $\tau \leq t < \infty$, where $F(t)$ and $f(t)$ are as given in (4.2.1) and (4.2.2). Then, from the results on the distribution of spacings from an exponential distribution, it follows that

$$E\left[e^{t\hat{\theta}_{2j}}\Big|\mathbf{N}_2 = \mathbf{n}_2\right] = \left(1 - \frac{t}{n_{2j}\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)}\right)^{-n_{2\cdot}} \tag{C.8}$$

for $t < \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)$, $j = 1, 2$. From (4.3.1), we have

$$\begin{aligned} M_{2j}(t) &= E\left[e^{t\hat{\theta}_{2j}}\Big|\mathbf{N} \in \mathfrak{S}\right] \\ &= \sum_{\mathbf{n} \in \mathfrak{S}} E\left[e^{t\hat{\theta}_{2j}}\Big|\mathbf{N}_2 = \mathbf{n}_2\right] \times Pr\left[\mathbf{N} = \mathbf{n}\Big|\mathbf{N} \in \mathfrak{S}\right] \end{aligned}$$

for $j = 1, 2$ and thus, upon using (C.8) and (4.3.3) in conjunction with Lemma C.1, (C.7) immediately follows. $\square$

Proof of Theorems 4.3.1 and 4.3.2: Applying the inversion theorem of a moment generating function to the results from Lemmas C.2 and C.3 in conjunction with Lemma A.3 in Appendix A, we can obtain the exact conditional PDFs of $\hat{\theta}_{ij}$, given $\mathbf{N} \in \mathfrak{S}$ for $i, j = 1, 2$. Hence, the result. $\square$

Appendix D

Lemmas and Proof of Theorems in Chapter 5

Lemma D.1. *The JPMF of* $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2)$*, where* $\mathbf{N}_i = (N_{i1}, N_{i2})$*,* $i = 1, 2$*, is given by*

$$\begin{aligned} Pr\big[\mathbf{N} = \mathbf{n}\big] &= \binom{n}{n_{11}, n_{12}, n_{21}, n_{22}, n - n_{\cdot\cdot}} \left\{ \prod_{i,j=1}^{2} \pi_{ij}^{n_{ij}} \right\} \\ &\quad \times \big\{F(\tau)\big\}^{n_{1\cdot}} \big\{F(\tau_c) - F(\tau)\big\}^{n_{2\cdot}} \big\{1 - F(\tau_c)\big\}^{n - n_{\cdot\cdot}} \qquad \text{(D.1)} \end{aligned}$$

for $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2)$*, where* $\mathbf{n}_i = (n_{i1}, n_{i2})$*,* $i = 1, 2$*, and* n_{ij} *are non-negative integers such that* $\sum_{i,j=1}^{2} n_{ij} = n_{\cdot\cdot} \leq n$*. Here,* π_{ij} *are as defined in (5.2.4) and (5.2.5) and* $F(t)$ *is as given in (5.2.1).*

Proof. For fixed $\mathbf{n} = (n_{11}, n_{12}, n_{21}, n_{22})$, we let the realized values of $N_{i\cdot}$ be $n_{i\cdot} = n_{i1} + n_{i2}$ for $i = 1, 2$ and let the observed value of $N_{\cdot\cdot}$ be $n_{\cdot\cdot}$ still holding $n_{1\cdot} + n_{2\cdot} = n_{\cdot\cdot}$.

Then, by conditioning on $N_{1\cdot} = n_{1\cdot}$ and $N_{2\cdot} = n_{2\cdot}$, we have

$$\begin{aligned}
Pr\big[\mathbf{N} = \mathbf{n}\big] &= Pr\big[\mathbf{N} = \mathbf{n}\big|N_{1\cdot} = n_{1\cdot}, N_{2\cdot} = n_{2\cdot}\big]\ Pr\big[N_{1\cdot} = n_{1\cdot}, N_{2\cdot} = n_{2\cdot}\big] \\
&= Pr\big[\mathbf{N}_1 = \mathbf{n}_1\big|N_{1\cdot} = n_{1\cdot}, N_{2\cdot} = n_{2\cdot}\big]\ Pr\big[\mathbf{N}_2 = \mathbf{n}_2\big|N_{1\cdot} = n_{1\cdot}, N_{2\cdot} = n_{2\cdot}\big] \\
&\qquad \times\ Pr\big[N_{1\cdot} = n_{1\cdot}, N_{2\cdot} = n_{2\cdot}\big|N_{\cdot\cdot} = n_{\cdot\cdot}\big]\ Pr\big[N_{\cdot\cdot} = n_{\cdot\cdot}\big] \\
&= Pr\big[N_{11} = n_{11}\big|N_{1\cdot} = n_{1\cdot}\big]\ Pr\big[N_{21} = n_{21}\big|N_{2\cdot} = n_{2\cdot}\big] \\
&\qquad \times\ Pr\big[N_{1\cdot} = n_{1\cdot}\big|N_{\cdot\cdot} = n_{\cdot\cdot}\big]\ Pr\big[N_{\cdot\cdot} = n_{\cdot\cdot}\big], \qquad \text{(D.2)}
\end{aligned}$$

where the second equality results from the fact that given $N_{1\cdot} = n_{1\cdot}$ and $N_{2\cdot} = n_{2\cdot}$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are independent. Now, it can be easily shown that

$$\begin{aligned}
\big(N_{1j}\big|N_{1\cdot} = n_{1\cdot}\big) &\sim Binomial\big(n_{1\cdot},\ \pi_{1j}\big) \qquad \text{for } j = 1, 2, \\
\big(N_{2j}\big|N_{2\cdot} = n_{2\cdot}\big) &\sim Binomial\big(n_{2\cdot},\ \pi_{2j}\big) \qquad \text{for } j = 1, 2, \\
\big(N_{1\cdot}\big|N_{\cdot\cdot} = n_{\cdot\cdot}\big) &\sim Binomial\left(n_{\cdot\cdot},\ \frac{F(\tau)}{F(\tau_c)}\right), \\
N_{\cdot\cdot} &\sim Binomial\big(n,\ F(\tau_c)\big)
\end{aligned}$$

for the model under consideration and thus, (D.1) follows by substituting the binomial probability mass functions in (D.2). □

Using the result in Lemma D.1, we can further simplify the expression of the denominator in (5.3.3). Let us first denote $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2)$ where $\mathbf{m}_i = (m_{i1}, m_{i2})$ and $m_{i\cdot} = m_{i1} + m_{i2}$ for $i = 1, 2$. Analogous to n_{ij} and $n_{i.}$, m_{ij} and $m_{i.}$ are non-negative integers which satisfy $\sum_{i,j=1}^{2} m_{ij} = m_{1\cdot} + m_{2\cdot} = m_{\cdot\cdot} \leq n$. Then, using (D.1) along with the binomial theorem, we have

$$\sum_{\mathbf{m}\in\mathfrak{S}} Pr\big[\mathbf{N}=\mathbf{m}\big]$$

$$\begin{aligned}
&= \sum_{m_{..}=4}^{n}\sum_{m_{1.}=2}^{m_{..}-2}\sum_{m_{11}=1}^{m_{1.}-1}\sum_{m_{21}=1}^{m_{2.}-1}\binom{n}{m_{11},m_{12},m_{21},m_{22},n-m_{..}}\left\{\prod_{i,j=1}^{2}\pi_{ij}^{m_{ij}}\right\} \\
&\qquad\qquad \times\big\{F(\tau)\big\}^{m_{1.}}\big\{F(\tau_c)-F(\tau)\big\}^{m_{2.}}\big\{1-F(\tau_c)\big\}^{n-m_{..}} \\
&= \sum_{m_{..}=4}^{n}\binom{n}{m_{..}}\big\{1-F(\tau_c)\big\}^{n-m_{..}}\sum_{m_{1.}=2}^{m_{..}-2}\binom{m_{..}}{m_{1.}}\big\{F(\tau)\big\}^{m_{1.}}\big\{F(\tau_c)-F(\tau)\big\}^{m_{2.}} \\
&\qquad\qquad \times\left\{\sum_{m_{11}=1}^{m_{1.}-1}\binom{m_{1.}}{m_{11}}\pi_{11}^{m_{11}}\pi_{12}^{m_{12}}\right\}\left\{\sum_{m_{21}=1}^{m_{2.}-1}\binom{m_{2.}}{m_{21}}\pi_{21}^{m_{21}}\pi_{22}^{m_{22}}\right\} \\
&= \sum_{m_{..}=4}^{n}\binom{n}{m_{..}}\big\{1-F(\tau_c)\big\}^{n-m_{..}}\sum_{m_{1.}=2}^{m_{..}-2}\binom{m_{..}}{m_{1.}}\big\{F(\tau)\big\}^{m_{1.}}\big\{F(\tau_c)-F(\tau)\big\}^{m_{2.}} \\
&\qquad\qquad \times\big(1-\pi_{11}^{m_{1.}}-\pi_{12}^{m_{1.}}\big)\big(1-\pi_{21}^{m_{2.}}-\pi_{22}^{m_{2.}}\big). \qquad \text{(D.3)}
\end{aligned}$$

Lemma D.2. *The CMGF of $\hat{\theta}_{1j}$, conditioned on $\mathbf{N}\in\mathfrak{S}$, is*

$$M_{1j}(t)=\sum_{\mathbf{n}\in\mathfrak{S}}\sum_{k=0}^{n_{1.}}C_{\mathbf{n},jk}^{[1]}\ \exp\Big\{t\ \tau_{1jk}\Big\}\left(1-\frac{t}{n_{1j}\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)}\right)^{-n_{1.}},\qquad t<\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right) \tag{D.4}$$

for $j=1,2$, where τ_{1jk} and $C_{\mathbf{n},jk}^{[1]}$ are as defined in (5.3.5) and (5.3.6), respectively.

Proof. Under the life-testing scheme outlined in Section 5.2, let us denote

$$\Big\{0<T_{1:n}<T_{2:n}<\cdots<T_{N_{1.}:n}<\tau\Big\}$$

for the ordered failure times before τ, where $N_{1}.$ is the total number of failures before τ. Then, it can be shown that their joint distribution, conditioned on $\mathbf{N}_1=\mathbf{n}_1$, is identical to the joint distribution of all order statistics from a random sample of size $n_{1}. = n_{11}+n_{12}$ from the right truncated distribution at τ; see Arnold *et al.* (1992) and

David and Nagaraja (2003) for details. The PDF and CDF of this right truncated distribution are, respectively,

$$f_{RT}(t) = \frac{f(t)}{F(\tau)} \qquad \text{and} \qquad F_{RT}(t) = \frac{F(t)}{F(\tau)}$$

for $0 < t < \tau$, where $F(t)$ and $f(t)$ are as given in (5.2.1) and (5.2.2). Then, by using straightforward integration and binomial expansion, we derive

$$\begin{aligned} E\left[e^{t\hat{\theta}_{1j}}\middle|\mathbf{N}_1 = \mathbf{n}_1\right] &= \left\{F(\tau)\right\}^{-n_{1\cdot}}\left(1 - \exp\left\{-\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}} - \frac{t}{n_{1j}}\right)\tau\right\}\right)^{n_{1\cdot}} \\ &\quad \times \exp\left\{t\ (n - n_{1\cdot})\frac{\tau}{n_{1j}}\right\}\left(1 - \frac{t}{n_{1j}\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)}\right)^{-n_{1\cdot}} \\ &= \left\{F(\tau)\right\}^{-n_{1\cdot}}\sum_{k=0}^{n_{1\cdot}}\binom{n_{1\cdot}}{k}(-1)^k\exp\left\{-\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\tau k\right\} \\ &\quad \times \exp\left\{t\ \tau_{1jk}\right\}\left(1 - \frac{t}{n_{1j}\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)}\right)^{-n_{1\cdot}} \end{aligned} \tag{D.5}$$

for $t < \left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)$, $j = 1, 2$. Since we have from (5.3.1) that

$$\begin{aligned} M_{1j}(t) &= E\left[e^{t\hat{\theta}_{1j}}\middle|\mathbf{N} \in \mathfrak{S}\right] \\ &= \sum_{\mathbf{n}\in\mathfrak{S}} E\left[e^{t\hat{\theta}_{1j}}\middle|\mathbf{N}_1 = \mathbf{n}_1\right] \times Pr\left[\mathbf{N} = \mathbf{n}\middle|\mathbf{N} \in \mathfrak{S}\right] \end{aligned} \tag{D.6}$$

for $j = 1, 2$, (D.4) is readily obtained upon using (D.5) and (5.3.3) in (D.6) along with Lemma D.1. $\square$

Lemma D.3. *The CMGF of $\hat{\theta}_{2j}$, conditioned on $\mathbf{N} \in \mathfrak{S}$, is*

$$M_{2j}(t) = \sum_{\mathbf{n}\in\mathfrak{S}}\sum_{k=0}^{n_{2\cdot}} C_{\mathbf{n},jk}^{[2]}\ \exp\left\{t\ \tau_{2jk}\right\}\left(1 - \frac{t}{n_{2j}\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)}\right)^{-n_{2\cdot}}, \qquad t < \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right) \tag{D.7}$$

for $j = 1, 2$, where τ_{2jk} and $C_{\mathbf{n},jk}^{[2]}$ are as defined in (5.3.8) and (5.3.9), respectively.

Proof. Again, under the life-testing scheme described in Section 5.2, let us denote

$$\left\{\tau \leq T_{N_1.+1:n} < T_{N_1.+2:n} < \cdots < T_{N..:n} < \tau_c\right\}$$

for the ordered failure times between τ and τ_c. We also denote $N_2. = N.. - N_1.$ for the total number of failures between τ and τ_c. Then, it can be shown that the conditional joint distribution of these failure times, given $N_1. = n_1.$ and $\mathbf{N}_2 = \mathbf{n}_2$, is the same as the joint distribution of all order statistics obtained from a random sample of size $n_2. = n_{21} + n_{22}$ from the distribution left truncated at τ and right truncated at τ_c; see Arnold *et al.* (1992) and David and Nagaraja (2003) for details. The PDF and CDF of this doubly truncated distribution are given, respectively, by

$$f_{LRT}(t) = \frac{f(t)}{F(\tau_c) - F(\tau)} \qquad \text{and} \qquad F_{LRT}(t) = \frac{F(t) - F(\tau)}{F(\tau_c) - F(\tau)}$$

for $\tau \leq t < \tau_c$, where $F(t)$ and $f(t)$ are as given in (5.2.1) and (5.2.2). Then, by using straightforward integration and binomial expansion, it follows that

$$\begin{aligned}
&E\left[e^{t\hat{\theta}_{2j}}\middle| N_1. = n_1., \ \mathbf{N}_2 = \mathbf{n}_2\right] \\
&= \left\{F(\tau_c) - F(\tau)\right\}^{-n_2.}\left(1 - \exp\left\{-\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}} - \frac{t}{n_{2j}}\right)(\tau_c - \tau)\right\}\right)^{n_2.} \\
&\quad \times \exp\left\{-\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\tau\ n_2. + t\ (n - n..)\frac{(\tau_c - \tau)}{n_{2j}}\right\}\left(1 - \frac{t}{n_{2j}\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)}\right)^{-n_2.} \\
&= \left\{F(\tau_c) - F(\tau)\right\}^{-n_2.}\sum_{k=0}^{n_2.}\binom{n_2.}{k}(-1)^k \exp\left\{-\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)(\tau_c - \tau)k\right\} \\
&\quad \times \exp\left\{-\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\tau\ n_2. + t\ \tau_{2jk}\right\}\left(1 - \frac{t}{n_{2j}\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)}\right)^{-n_2.} \qquad \text{(D.8)}
\end{aligned}$$

for $t < \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)$, $j = 1, 2$. From (5.3.2), we have

$$\begin{aligned}
M_{2j}(t) &= E\left[e^{t\hat{\theta}_{2j}}\middle|\mathbf{N} \in \mathfrak{S}\right] \\
&= \sum_{\mathbf{n}\in\mathfrak{S}} E\left[e^{t\hat{\theta}_{2j}}\middle| N_1. = n_1., \ \mathbf{N}_2 = \mathbf{n}_2\right] \times Pr\left[\mathbf{N} = \mathbf{n}\middle|\mathbf{N} \in \mathfrak{S}\right]
\end{aligned}$$

for $j = 1, 2$ and thus, upon using (D.8) and (5.3.3) in conjunction with Lemma D.1, (D.7) immediately follows. $\square$

Proof of Theorems 5.3.1 and 5.3.2: Applying the inversion theorem of a moment generating function to the results from Lemmas D.2 and D.3 in conjunction with Lemma A.3 in Appendix A, we can obtain the exact conditional PDFs of $\hat{\theta}_{ij}$, given $\mathbf{N} \in \mathfrak{S}$ for $i, j = 1, 2$. Hence, the result. $\square$

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