THE SQUARE WAVE TRANSFORMATION OF POINT PROCESSES

A STUDY OF THE SQUARE WAVE TRANSFORMATION
OF
POINT PROCESSES

BY

PAUL DOUGLAS DOWLING, B.Sc.

A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

(October) 1967

ACKNOWLEDGMENTS

The research reported in this thesis was done while the author was a graduate student in mathematics at McMaster University. It was directed by Professor I. Z. Chorneyko of the Mathematics Department.

The author acknowledges his indebtedness to Professor Chorneyko for suggesting this problem and for his invaluable advice, criticism and guidance throughout the entire period in which this research was carried out.

Finally a vote of thanks to my mother who patiently typed pages of equations and symbols Greek to her, and to Mr. E. Funke of the National Research Council for helpful conversations regarding instrumentation.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

A process consisting of a series of events occurring in continuous time when interest is concentrated on the individual occurrences of the events themselves constitutes a point process. The emissions from a radioactive source or accidents occurring in time are examples of a series of point events, with the events being distinguished only by their positions in time.

In this thesis stationary processes, in which the time origin is an arbitrary point, will be dealt with, and for simplicity it is assumed that there is zero probability that two or more events occur simultaneously. There are basically two ways of looking at point processes; in terms of the number of events occurring in fixed time intervals or in terms of intervals between events.

The purpose of this thesis is to study various methods of analysing point processes by means of the square wave transformation. Figure 1-1 and Figure 1-2 are examples of a point process and the square wave transformation of it.

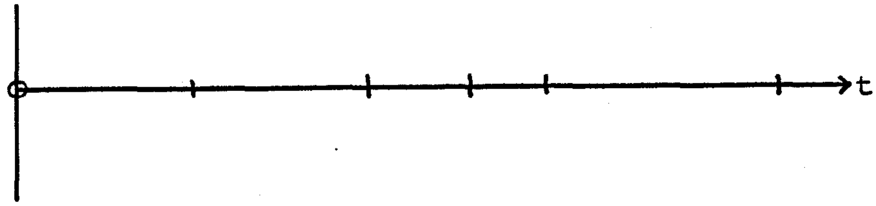

Fig. 1-1 Point Process X(t)

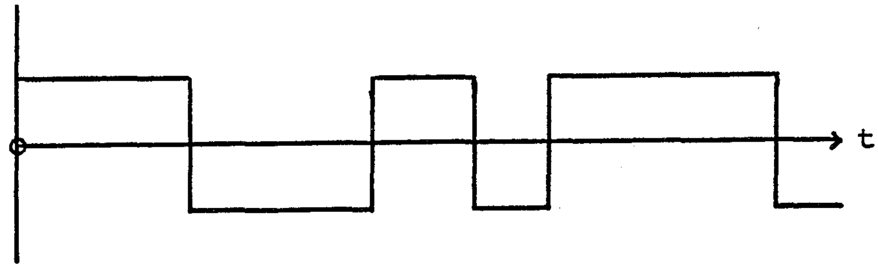

Fig. 1-2 Square Wave of the Point Process X(t)

A point process can easily be transformed into a square wave by means of a flip-flop device at each pulse. The resulting square wave can be fed into an analyser to calculate the auto-covariances and spectral density function. The ease of instrumentation for calculation of information from the square wave makes it very practical to work with.

In chapter (2) of this thesis the basic definitions and notations are described. Probability analysis of point processes is dealt with in chapter (3) and spectral analysis in chapter (4). The square wave transformation is introduced in chapter (4) and is analysed using spectral analysis in order to extract the statistical properties of the original point process. In chapter (5) the computational results and conclusions from utilizing the methods discussed in chapters

(3) and (4) are summarized. Appendix A gives a detailed account of the experimental time series employed and the computational results from using the methods in chapter (3). Appendix B consists of the corresponding spectral analysis of the different point processes. All computations were done on the I.B.M. 7040 computer and the FORTRAN lV programs used in the time series analysis are exhibited in Appendix C.

CHAPTER 2

DEFINITIONS AND NOTATIONS

A stochastic process is defined as a collection $\{X(t), t \varepsilon T\}$ of random variables. The set T is called the index set of the process. No restriction is placed on the nature of T. The two important cases are when $T=\{0, \pm 1, \pm 2, \ldots.\}$ or $T=\{0,1,2,\ldots.\}$ in which case the stochastic process is said to be a discrete parameter process or when $T=\{t: -\infty < t < \infty\}$ or $T=\{t: t \geq 0\}$, the continuous parameter process.

A special case of a stochastic process is the point process which can be defined in the following manner. Let a sequence of events occur at the instants t_i where $i \geq 1$, $t_{i+1} > t_i$ and $t_1 > 0$. Then, this time sequence will be denoted by $X(t)$, $t \varepsilon \{t_1, t_2, \ldots.\}$, a discrete parameter process, so that if $X(t)$ is a stochastic process which describes the time of occurrence of events which are considered to occur for an infinitesimal duration, then $X(t)$ is a point process. For example, consider events occurring periodically in time with period B. Then $X(t)$, $t \varepsilon \{t_1, t_2, \ldots. | t_i = i \cdot B\}$ is a point process.

In the analysis of any process only a finite record of observations is available, so that the index set of the process can be considered as the time interval $(0,T]$. The truncated sample function is defined by $X_T(t)$ as $X_T(t) = X(t) \quad 0<t\leq T$

$= 0$ elsewhere

Let n be the number of events in the observation period T, and let τ be any fixed time interval length. Then n(t) is the number of events occurring in the time interval (0,t] and $n_i(\tau)$ is the number of events in the interval $((i-1)\tau, i\tau]$.

Unless otherwise stated, the time series is assumed to be stationary. This means that all statistical properties depend upon differences $X_T(t_i)-X_T(t_j)$ rather than on the time points $X_T(t_i)$ and $X_T(t_j)$ themselves. Let A_1, A_2, ...be arbitrary sets on the real axis, and T_hA_1, T_hA_2, ...be the sets obtained by translating through h. Let N(A) be the number of events in A. The point process is stationary if the two sets of random variables

$$N(A_1),\ N(A_2),\ldots\ldots\ldots\ldots,N(A_k);$$

$$N(T_hA_1),\ N(T_hA_2),\ldots\ldots\ldots,N(T_hA_k)$$

have the same joint distribution for all initial sets A_1, A_2,... A_k and all real h and all k=1, 2,.........

The time series $X_T(t)$ has the sample mean value function

$$\mu=\frac{1}{T}\int_0^T X_T(t)\,dt$$

$$=\frac{1}{T}\sum_{t=1}^{T} X_T(t)$$

for the continuous and discrete value case respectively.

The sample autocovariance function $R_T(k)$ is defined by

$$R_T(k) = E\{[X_T(t)-\mu]\cdot[X_T(t+k)-\mu]\}$$

$$= \frac{1}{T}\int_0^{T-|k|} [X_T(t)-\mu]\cdot[X_T(t+k)-\mu]\,dt \qquad |k|<T$$

$$= 0 \qquad |k|\geq T$$

and

$$R_T(k) = \frac{1}{T}\sum_{t=1}^{T-|k|} [X_T(t)-\mu]\cdot[X_T(t+|k|)-\mu] \qquad k=0,\ \pm 1,\ldots,\pm(T-1)$$

$$= 0 \qquad k=\pm T,\ \pm(T+1),\ldots.$$

The <u>sample spectral density function</u> is defined by

$$f_T(\omega) = \frac{1}{2\pi}\int_{-T}^{T} e^{-ik\omega}\, R_T(k)\,dk = \frac{1}{\pi}\int_0^{T} \cos k\omega\, R_T(k)\,dk$$

or

$$f_T(\omega) = \frac{1}{2\pi}\sum_{k=-T}^{T} e^{-ik\omega}\, R_T(k) = \frac{1}{2\pi} R_T(0) + \frac{1}{\pi}\sum_{k=1}^{T} \cos k\omega\, R_T(k)$$

for the continuous and discrete parameter cases respectively.

Let $\lambda(t)$, the number of events per unit time, be the (probability) rate of occurrence of an event. The Poisson process is an important stochastic process which serves as a mathematical model for empirical phenomena like the arrival of calls at a telephone exchange, the emission of particles from a radioactive source, and the occurrence of serious coal-mining accidents. Consider point events occurring singly in time with the rate of occurrence $\lambda(t) = \lambda$, a constant. If $N(t,t+\Delta t)$ is the number of events in the interval $(t,t+\Delta t]$ then assume that, as $\Delta t \to 0^+$

$$\text{prob}\ \{N(t,t+\Delta t) = 0\} = 1-\lambda\Delta t+0(\Delta t),$$

$$\text{prob}\ \{N(t,t+\Delta t) = 1\} = \lambda\Delta t+0(\Delta t),$$

so that prob $\{N(t,t+\Delta t)>1\} = 0(\Delta t)$,

where $0(\Delta t)$ denotes a function tending to zero more rapidly than Δt. Also assume that $N(t,t+\Delta t)$ is independent of occurrences in $(0,T]$. A stochastic process of point events satisfying these conditions is called a Poisson process of rate λ.

CHAPTER 3

PROBABILITY ANALYSIS

In this chapter various methods of analyzing the intervals between events and the number of events in fixed time intervals are considered in order to determine if the point process is random, or generated by some probability mechanism or follows some pattern which can be determined from a sample of the process.

GRAPHICAL METHODS

(1) The intervals between successive events x_i are plotted as a function of i or against the time at the midpoint of the interval. Trends in the interval length will be indicated. This is equivalent to plotting the time interval between pulses of the square wave. Fig. 3-1

(2) If the intervals between events are independent of time then x_i versus i will form a scatter diagram. Then a frequency polygon or histogram can be formed to determine the probability distribution (if it exists) of the interval lengths. Fix a time interval length x such that $mx = \max_{i=1,\dots,n} x_i$, where it is suggested that $12 \leq m \leq 25$, depending upon the sample size n.

Then count the number $n_j(x)$ of interval lengths x_i that belong in the interval $((j-1)x, jx]$, $j=1,\ldots\ldots,m$. Plot $n_j(x)$ versus j for a frequency polygon or form a bar graph of $n_j(x)$ as base for a histogram. The histogram approximates the density function for large n. Fig. 3-2

RANDOM SERIES

If the series of events is random, that is the times of occurrences are independently and identically distributed from the uniform distribution, then it is called a Poisson process since the number of events in a fixed time length has the Poisson distribution. The Poisson process when graphed reveals the following,

(1) x_i versus i will be a scatter diagram showing that x_i is independent of i,

(2) the histogram formed for the interval lengths between events will approximate an exponential density function. This is proved as follows. Assume that X(t) is a Poisson process with rate of occurrence λ. Take a new time origin at t_o. If t_o+Z is the time of the first event after t_o, the random variable Z is independent of whether an event occurs at t_o and of occurrences before t_o.

Let $P(x) = \text{prob}\{Z>x\}$. To determine the distribution of Z,

$$\begin{aligned} \text{let } P(x+\Delta x) &= \text{prob}\{Z>x+\Delta x\} \qquad\qquad \Delta x>0 \\ &= \text{prob}\{Z>x \quad \text{and no event occurs in } (t_o+x,\ t_o+x+\Delta x]\} \\ &= \text{prob}\{Z>x\}\text{prob}\{\text{no event occurs in } (t_o+x, t_o+x+\Delta x] \mid Z>x\} \end{aligned}$$

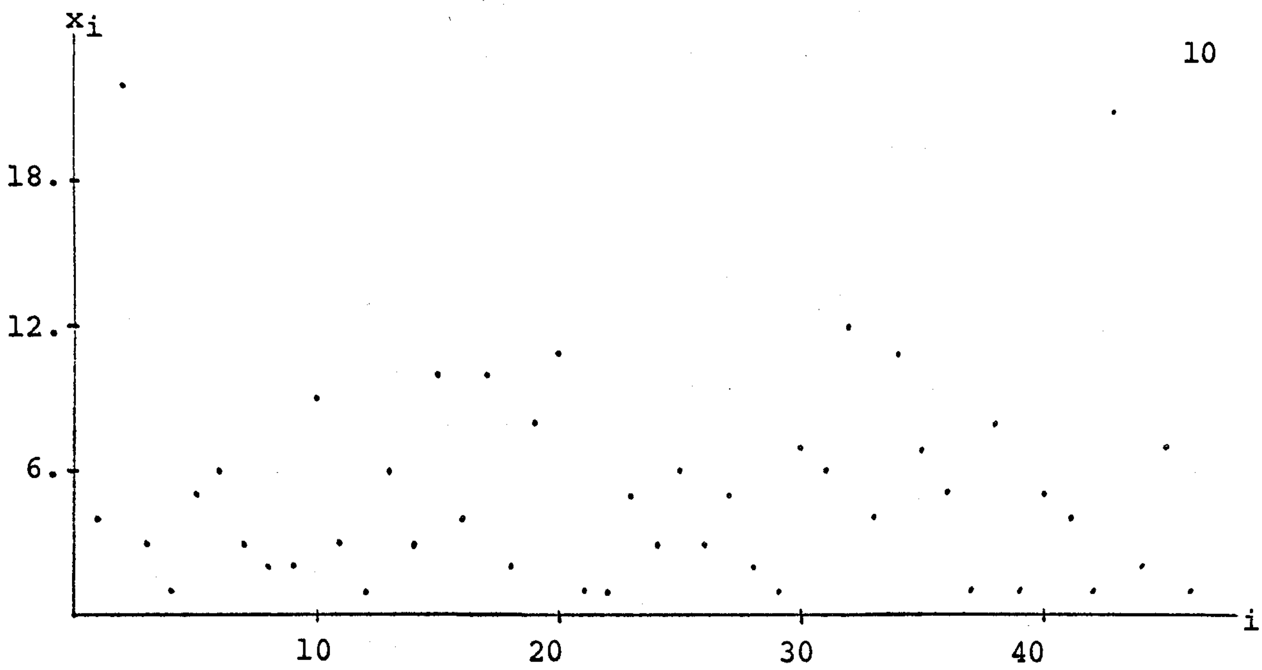

Fig. 3-1 Interval lengths x_i versus i for a point process X(t)

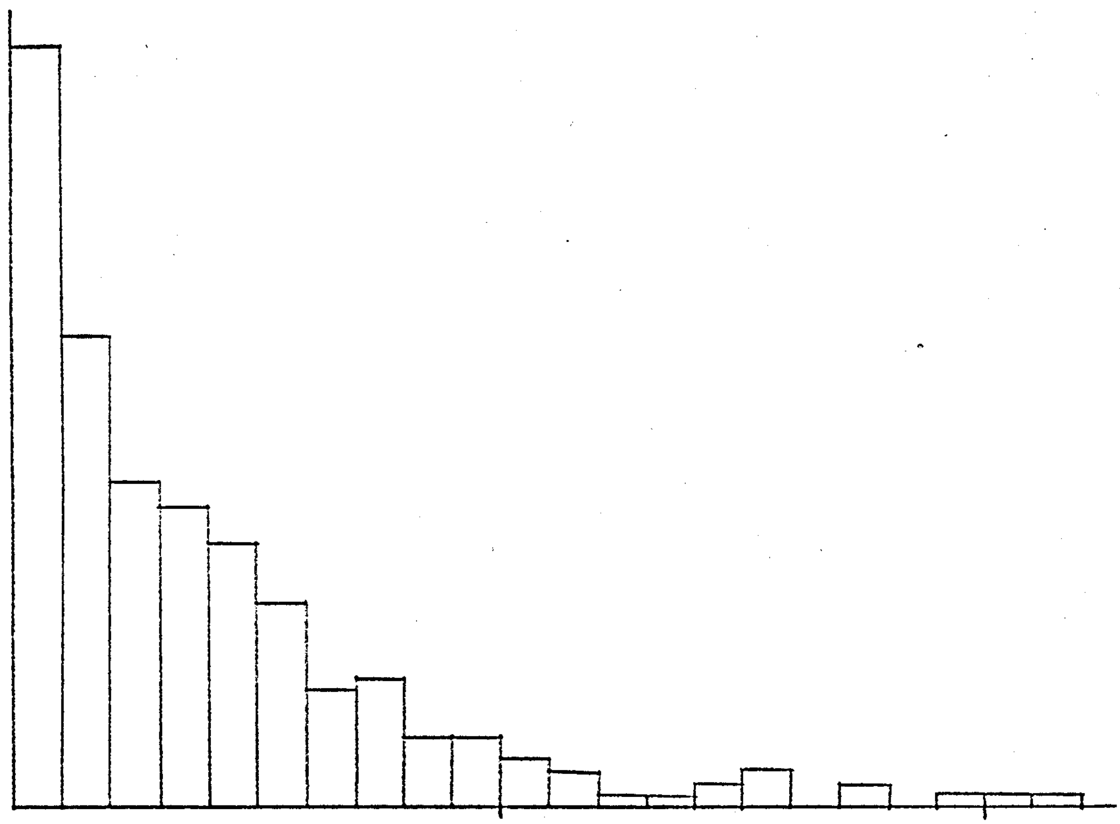

Fig. 3-2 Histogram of interval lengths of a point process X(t)

$$P(x+\Delta x) = P(x)\{1-\lambda\Delta x+0(\Delta x)\}$$
$$= P(x) - \lambda P(x)\Delta x + P(x)0(\Delta x)$$

Then

$$\lim_{\Delta x \to 0} \frac{P(x+\Delta x) - P(x)}{\Delta x} = P'(x) = -\lambda P(x)$$

$$\frac{dP(x)}{P(x)} = -\lambda dx$$

$$\log_e P(x) = -\lambda x+k$$

$$P(x) = Ke^{-\lambda x}$$

Since $P(0) = \text{prob}\{Z>0\} = 1$, $P(x) = e^{-\lambda x}$

The distribution function of Z is $1 - e^{-\lambda x}$, the probability density function is $\frac{d}{dx}(1-e^{-\lambda x}) = \lambda e^{-\lambda x}$ $(x>0)$

PROPERTIES OF RANDOM PROCESSES

If the time series is random then inferences can be made about λ. The intervals between successive events have the exponential distribution with probability density function $f(x) = \lambda e^{-\lambda x}$, and rate of occurrence λ.

Let $x_1, x_2, \ldots\ldots, x_m$ be m intervals from this distribution. Let $s = \sum_{i=1}^{m} x_i$ and $\bar{x} = s/m$. Then s has the probability density function

$$f(s) = \frac{\lambda(\lambda s)^{m-1}e^{-\lambda s}}{(m-1)!} \qquad s>0$$

a gamma distribution, and $\bar{x}$ is distributed as

$$g(\bar{x}) = \frac{m\lambda(m\lambda\bar{x})^{m-1}e^{-m\lambda\bar{x}}}{(m-1)!} \qquad \bar{x}>0$$

These results are verified by finding the moment generating function of the random variables s and x_i.

$M_s(t) = E\{e^{ts}\}$ is the moment generating function of the random variable s.

$M_s(t) = M_{\sum_i x_i}(t) = \prod_i M_{x_i}(t)$ since s is the sum of m independent and identically distributed random variables x_i.

Now

$$M_x(t) = E\{e^{xt}\} = \int_0^\infty e^{xt}\lambda e^{-\lambda x}dx$$

$$= \frac{\lambda}{\lambda-t}\int_0^\infty (\lambda-t)e^{-(\lambda-t)x}dx$$

$$= \frac{\lambda}{\lambda-t}$$

Then

$$M_s(t) = \prod_i M_{x_i}(t) = \left(\frac{\lambda}{\lambda-t}\right)^m$$

Now, if two random variables have the same moment generating function then these random variables have the same probability distribution (by the uniqueness theorem for moment generating functions).

The random variable y with the distribution

$$h(y) = \frac{1}{\Gamma(\alpha)B^{\alpha}} y^{\alpha-1} e^{-y/B} \quad 0<y<\infty$$

$$= 0 \text{ elsewhere}$$

where $\alpha>0$, $B>0$, $\Gamma(\alpha) = (\alpha-1)!$ has the gamma distribution, $M_y(t) = (1-Bt)^{-\alpha}$, $t<1/B$.

Now $M_S(t) = \left(\frac{1}{1-t/\lambda}\right)^m = (1-t/\lambda)^{-m}$

and is the moment generating function of a random variable with the gamma distribution where $\alpha=m$, $B=1/\lambda$

Then

$$f(s) = \frac{1}{(m-1)!\ (1/\lambda)^m} s^{m-1} e^{-\lambda s} \qquad 0<s<\infty$$

$$= \frac{\lambda(\lambda s)^{m-1} e^{-\lambda s}}{(m-1)!} \qquad 0<s<\infty$$

$M_{\bar{X}}(t) = M_{\frac{s}{m}}(t) = M_S(t/m)$ is the moment generating function for the random variable $\bar{X}$.

$$M_S(t/m) = (1-t/m\lambda)^{-m} \text{ , then } \alpha = m,\ B = 1/m\lambda$$

and

$$g(\bar{X}) = \frac{m\lambda(m\lambda\bar{X})^{m-1} e^{-m\lambda\bar{X}}}{(m-1)!} \qquad \bar{X}>0$$

The following properties can be established:

(1) $g(\bar{X})$ has mean $1/\lambda$ and variance $1/(m\lambda^2)$,

(2) as m increases $g(\bar{X})$ becomes normally distributed,

(3) $\bar{X}$ is a sufficient estimator for $1/\lambda$,

(4) $2\lambda\bar{x}$ is distributed as χ^2 with 2 degrees of freedom,

(5) $2m\lambda\bar{x}$ is distributed as χ^2 with 2m degrees of freedom.

Using this information a confidence interval for λ with probability $1-2\alpha$ is

$$\frac{\chi^2_{1-\alpha,2m}}{2m\bar{x}} < \lambda < \frac{\chi^2_{\alpha,2m}}{2m\bar{x}}$$

If there are two random series of events with m_1 and m_2 intervals and means $\bar{x}_1$ and $\bar{x}_2$ respectively, then the hypothesis that $\lambda_1 = \lambda_2$ can be tested by the F distribution with $2m_1$ and $2m_2$ degrees of freedom. Then $F = \lambda_1\bar{x}_1/(\lambda_2\bar{x}_2)$ with $2m_1$ and $2m_2$ degrees of freedom. If there are more than two rates of occurrence then the F test can not be used to test the homogeneity of the λ_i. A special application of the χ^2 test, known as Bartlett's test, may be applied.

Considerable computation can be saved by applying an F test to the largest and smallest variance before Bartlett's test. If the F test indicates that the largest variance is not significantly different from the smallest one, then it is reasonable to assume that the variances lying in between do not differ significantly from the smallest one. The application of the χ^2 and F tests to the analysis of the time intervals depends on the assumption of homogeneity (λ is not a function of time). There are several tests of homogeneity, one of which is the g test, another is the previously mentioned Bartlett's test that all λ_i are equal. These tests, which are based on interval lengths, can be used as tests of randomness.

TESTS OF RANDOMNESS

(1) g test

Let x_m be the largest among m independent intervals and let $\bar{x}$ be the mean interval length. The statistic $g = \frac{x_m}{m\bar{x}}$ has a probability relation which has been compiled by Fisher[1]. To determine whether the length of an interval is significant under the hypothesis of randomness, a significance level α is selected and g_α is computed from tables[2]. g is calculated and if $g > g_\alpha$ then the longest interval between events is significant and the series is not random.

Since this test is based on the largest among the m independent intervals, then it is possible that measurement errors or some other conditions have created an interval length which is an outlier. If the graph of x_i versus i indicates an interval length much larger than any other, it is possible that the g test will reject the hypothesis of randomness if this value is used for x_m even though the sample is homogeneous. If the sample containing such a large deviation is not representative, or if the occurrence of such a

1 Fisher, R.A. Proc. Roy. Soc. A,125,54, (1929)

2 Fisher, R.A. Contributions to Mathematical Statistics, Chapman and Hall, London, (1950)

large deviation is unlikely in a sample from the population in question, then it is necessary to reject this observation. However, if large deviations occur in a number of samples the presence of additional factors of intermittent character may be responsible.

(2) Bartlett's Test

If the sequence of intervals is divided up into K sets of m_i successive intervals where v_i^2 is the estimate of variance from sample i, and $\bar{v}^2$ is the estimate of the pooled variance, $\bar{v}^2 = \frac{\sum_{i=1}^{K} m_i v_i^2}{\sum_{i=1}^{K} m_i}$

then Bartlett's test requires the calculation of

$$\bar{\chi}^2 = 2.3026 \{\log_{10}\bar{v}^2 \sum_{i=1}^{K} (m_i-1) - \sum_{i=1}^{K} (m_i-1)\log_{10} v_i^2\}/C$$

where $C = 1+\left(\sum_{i=1}^{K}\frac{1}{m_i-1} - \frac{1}{m-K}\right)/3(K-1)$, $\quad m = \sum_{i=1}^{K} m_i$

The distribution of $\bar{\chi}^2$ is approximately χ^2 with K-1 degrees of freedom and the approximation is reasonably accurate if the $m_i-1\geqq 5$, i=1, 2..., K.

When all samples are of the same size m, then

$$\bar{\chi}^2 = 2.3026[(m-1)K\log_{10}\bar{v}^2 - (m-1)\sum_{i=1}^{K} \log_{10} v_i^2]/C$$

where $C = 1+(K+1)/[3K(m-1)]$

If $\bar{\chi}^2<\chi^2$ with K-1 degrees of freedom, then the hypothesis of homogeneity is accepted.

If a set of interval lengths are independent and exponentially distributed with parameter λ, then the mean and variance of the interval lengths are $1/\lambda$ and $1/\lambda^2$, respectively. Divide a sequence of interval lengths up into K sets of m successive intervals. Let $\bar{x}_i$ be the mean interval length of the ith set, and let the intervals be independent and exponentially distributed with parameter λ_i. The variance, v_i^2, of this set can be estimated by $\bar{x}_i^2$, since $\bar{x}_i$ is a sufficient estimator for $1/\lambda_i$. Now, under the hypothesis that all the λ_i are equal, the K sets will constitute a set of Km interval lengths which are exponentially distributed with parameter λ. The variance of this pooled series, $\bar{v}^2$, can be estimated by $\left(\frac{1}{K}\sum_{i=1}^{K}\bar{x}_i\right)^2$ since the parameter λ will have $\frac{1}{K}\sum_{i=1}^{K}\bar{x}_i$ as an estimator of $1/\lambda$.

The hypothesis that all the λ_i are equal can be tested by computing $\bar{\chi}^2$ and

$$\bar{\chi}^2 = 2.3026[2(m-1)\,K\log\left(\frac{1}{K}\sum_{i=1}^{K}\bar{x}_i\right) - 2(m-1)\sum_{i=1}^{K}\log\bar{x}_i]/C$$

$$= 2.3026(2m-2)\,K\,[\log\left(\frac{1}{K}\sum_{i=1}^{K}\bar{x}_i\right) - \frac{1}{K}\sum_{i=1}^{K}\log\bar{x}_i]/C$$

where $C = 1 + (K+1)/[3K(m-1]$

<u>DEPARTURE FROM RANDOMNESS</u>

<u>TRENDS AND CYCLES</u>

There are two types of sequences in time. One is a slowly moving function of time which is often called a trend, and is exemplified by a polynomial of fairly low degree,

$$s(t) = a_0 + a_1 t + \ldots + a_q t^q \quad , \; t=t_1, t_2, \ldots, t_n.$$

Another type of sequence is cyclical, such as a finite Fourier series,

$$s(t) = b_0 + \sum_{i=1}^{q} (b_i \cos\lambda_i t + c_i \sin\lambda_i t), \; t=t_1, t_2, \ldots, t_n.$$

Trends or cycles in the rate of occurrence or number of occurrences can occur as functions of time and the interval lengths between successive events as a function of the interval index.

Consider a point process, where the time interval between the events occurring at time t_i and t_{i+1} is $x_{i+1} = 2x_i$ and $x_1 = B$, a constant. Then the interval lengths of the process are $x_1 = B$, $x_2 = 2B, \ldots, x_j = 2^{j-1}B, \ldots$.
For events occurring at the times $t_1, t_2, \ldots, t_j, \ldots$ then the time of the jth event at t_j is

$$t_j = \sum_{i=1}^{j} 2^{i-1}B = B(2^j - 1)$$

The jth event occurs at time t_j so that $n(t_j) = j$.
Then

$$t_j = B(2^{n(t_j)} - 1) \qquad \text{or}$$

$$n(t) = [\log_2\{(B+t)/B\}]$$

where $[\log_2\{(B+t)/B\}]$ denotes the largest integer less than or equal to $\log_2\{(B+t)/B\}$.

Then, $n(t) = g(t) = [\log_2\{(B+t)/B\}]$ and

$$x_i = h(i) = 2^{i-1}B.$$

If a function $s(t)$, which is a polynomial in t of specified degree q, is assumed, then the problem is to estimate the coefficients a_0, $a_1, \ldots, a_q$ on the basis of observations $X(t_1)$, $X(t_2), \ldots, X(t_n)$ of the sample series. In the case of the Fourier series the problem is to estimate the coefficients b_0, b_i and $c_i, i = 1, 2, \ldots, q$.

In either case the estimation can be done by the method of least squares, the estimates of the parameters being the values of the constants which minimize $\sum_{i=1}^{n} \{X(t_i) - s(t_i)\}^2$

INTERVALS WITH PROBABILITY DISTRIBUTIONS

If the sample from a point process is independent of time then it can be specified by the intervals between events, and these intervals $\{x_i\}_{i=1}^{n}$ can be used to construct a histogram in order to infer information about the probability distribution of the x_i. A χ^2 test can then be used to determine the goodness of fit of the theoretical distribution to the sampled data.

SUPERPOSITION OF PERIODIC SERIES OF EVENTS

If the series of events is periodic with period B between successive events, then $n_i(\tau)$ versus i will approximate a straight line with constant ordinate value, and $n(t)$ versus t will be $n(t) = [t/B]$, where the square brackets denote the largest integer less than or equal to the argument. The graph of x_i versus i will be a series of points with constant ordinate value B.

Now if the series is generated by the superposition of several periodic sources, then only the pooled output can be used to determine the periods B_i. If the series is long and the number of sources is small, it is possible to determine the B_i exactly and to assign each event to its appropriate source.

This can be done by forming the histogram for the interval lengths between successive events. This will be bounded by a point concentration about B_1, the smallest of the B_i. The graph of x_i versus i will give the exact value of the upper bound B_1. Next, find an interval of length B_1 and from it build up the output of the first source by repeated additions and subtractions of B_1. Delete this set of events from the pooled series of events and analyse the remaining events to find the next smallest period.

As soon as the frequencies become very small or if two or more of the smallest frequencies are very close together, then this method is not practical. The frequency distribution of intervals is insensitive since the frequency curve is very nearly exponential except when the number of frequencies is small, or the frequencies are far apart. In order to detect whether the interval distribution is exponential (Poisson process) or merely a pooled output from periodic sources, then the variance time curve analysis is necessary.

VARIANCE TIME CURVE

Let V(t) be the variance of the number of events occurring in time (0, T]. If the series is random with mean rate of occurrence λ then $V(t) = \lambda t$. To find V(t) for the pooled output of periodic sources first consider a single source with period B_i. Let $\gamma_i = 1/B_i$. Then $\gamma_i t = n_i + \alpha_i$ where n_i is an integer, $0 \leq \alpha_i < 1$. Taking observations at equidistant intervals t_1, t_2,.... then an interval of length t_j contains either n_i or n_i+1 events from this source and the limiting frequency of intervals containing n_i+1 events is α_i since $\gamma_i t_j = \gamma_i t_{j-1} + \alpha_i$. Let Y = 1 be the occurrence of n_i+1 events and Y = 0 the occurrence of n_i events. Prob $\{Y=1\} = \alpha_i$, Prob $\{Y = 0\} = 1-\alpha_i$ and Y has the binomial distribution with mean α_i and variance $\alpha_i(1-\alpha_i)$.

Since the different sources are independent, then for N sources $V(t) = \sum_{i=1}^{N} \alpha_i(1-\alpha_i)$. If t is very large compared with B_i then $\alpha_i(1-\alpha_i) << t\gamma_i$ so that

$V(t) << \sum_{i=1}^{N} t\gamma_i = t\lambda$ where $1/\lambda$ is the mean interval between successive events.

As t increases, α_i takes each value between 0 and 1 equally often giving $\alpha_i(1-\alpha_i)$ an average value of 1/6. For large t, V(t) oscillates about an average of N/6. Therefore a graph of V(t) versus t will differentiate between a random process and a pooled series from N periodic sources.

In order to calculate V(t) the series is divided into m intervals of length τ ($\tau \approx 2T/n$ provided good results in estimating the number of sources for the examples in Appendix A) such that $n_i(\tau)$ is small.

Let $\quad y_i = n_i(\tau)$

Let $\quad U_1^{\,r} = y_1 + y_2 + \ldots\ldots\ldots\ldots + y_r$

$$U_2^{\,r} = y_1 + y_2 + \ldots\ldots\ldots\ldots + y_{r+1}$$

$$U^r_{m-r+1} = y_{m-r+1} + \ldots\ldots\ldots\ldots + y_m$$

Calculate $\quad \sum_{i=1}^{M} (U_i^{\,r})^2, \quad \sum_{i=1}^{M} U_i^{\,r}, \quad \left(\sum_{i=1} \frac{U_i^{\,r}}{M} \right)^2$

and $\quad s = \sum_{i=1}^{M} (U_i^{\,r})^2 - \left(\sum_{i=1}^{M} U_i^{\,r}/M \right)^2 \quad$ where $M = m-r+1$

Now calculate $\quad \hat{V}(r\tau) = \left(\frac{3M}{3M^2 - 3Mr + r^2 - 1} \right) s$

$\hat{V}(r\tau)$ is the estimate of the variance of the number of events in an interval $r\tau$. Now plot $\hat{V}(r\tau)$ versus $r\tau$ or r for a number of r to obtain an estimate of the variance time curve.

CHAPTER 4

SPECTRAL ANALYSIS

Since the point process $X(t)$ is defined only at the time points t_1, t_2,....., it is necessary to transform the one-dimensional sequence into a discrete or continuous two-dimensional stochastic process so that the autocovariance function and corresponding spectral density function can be calculated.

The sequence of intervals $\{X_1, X_2, ...\}$ between successive events can be used to describe the point process, the time parameter being the serial number of the event. By dividing the time axis into a large number of narrow intervals of width dt, and counting the number of events in each interval, a new process, $dN(t)$, is obtained. Another way of studying the process is to convert the sequence of pulses or events into a square wave by means of a flip-flop device at each pulse.

INTERVAL LENGTHS

The sequence of interval lengths $\{X_1, X_2, ...\}$ has the mean value function $u(t) = E\{X_i\} = u$ and variance $V(X_i) = \sigma^2$. This sequence is considered as a stationary real-valued process in discrete time. With a finite sample $\{X_1, X_2, \ldots, X_n\}$ the sample autocovariance function $R_T(K)$ is defined as

$$R_T(K) = \frac{1}{n-|K|}\sum_{i=1}^{n-|K|}(X_i-u)\cdot(X_{i+|K|}-u) \qquad K=0, \pm 1, \ldots, \pm(n-1)$$

and the corresponding spectral density function $f_T(\omega)$ is defined as

$$f_T(\omega) = \frac{1}{2\pi}\sum_{K=-(n-1)}^{n-1} e^{-iK\omega} \quad R_T(K)$$

$$= \frac{1}{2\pi} R_T(0) + \frac{1}{\pi}\sum_{K=1}^{n-1}\cos K\omega\, R_T(K)$$

where $R_T(0) = \sigma^2$

OCCURRENCE RATE

Consider the process $\{dN(t)\}$ where $dN(t)$ is the number of events in the interval $(t,t+dt]$.

Let $\quad E\left\{\frac{dN(t)}{dt}\right\} = u(t) = u.$

The covariance function for this stationary stochastic process is defined by $R(K) = E\left(\frac{dN(t)\cdot dN(t+K)}{(dt)^2}\right) - u^2, \quad K>0.$

For $K<0 \quad R(K) = R(-K)$ and for $K=0$, $R(0) = \sigma^2$ where

$\sigma^2 = E\left(\left[\frac{dN(t)}{dt}\right]^2\right) - u^2.$ For all K, the complete covariance function is $\bar{R}(K) = \sigma^2\delta(K) + R(K)$ where $\delta(K)$ is the Dirac delta function, the probability density function of a probability distribution located entirely at the point

K=0. The spectral density function for dN(t) is defined by

$$\mathcal{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iK\omega}\,\bar{R}(K)\,dK.$$

However for any time interval (0,T], only a finite number of intervals of length dt are used, transforming {dN(t)} into a discrete valued process. The autocovariance function and spectral density function would be calculated as described for the sequence of interval lengths.

SQUARE WAVE

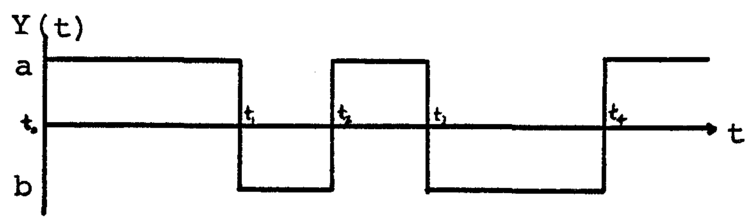

Fig. 4-1

The point process X(t) defined at $t_1, t_2, \ldots, t_n$ in the interval (0,T] can be transformed into a continuous process Y(t) by a square wave. Then Y(t) takes the values a and b alternately at each occurrence for the length of time until the next occurrence (Figure 4-1). Now the problem is to derive information about the original process X(t) by correlation and spectral analysis of the square wave Y(t). In order to determine if spectral analysis can yield as much information as probability analysis, a probability distribution for the occurrence of interval lengths or sign changes can not be assumed. Then, it is necessary to calculate the sample mean value function and autocovariance function and then the spectral density function from the observed time series.

Without loss of generality, it can be assumed that the transformation of the time series to a square wave begins with Y(t) assigned the value a. For the process Y(t), the sample mean value function is easily calculated,

$$u = \frac{1}{T}\int_0^T Y(t)\,dt = \frac{1}{T}\left(\sum_{i=1}^{n-[n/2]} aX_{2i-1} + \sum_{i=1}^{[n/2]} bX_{2i}\right)$$

where [n/2] denotes the largest integer less than or equal to n/2. The sample autocovariance function $R_T(K)$ is

$$R_T(K) = \frac{1}{T}\int_0^{T-|K|} [Y(t)-u]\cdot[Y(t+|K|)-u]\,dt \qquad |K|<T$$

$$= \frac{1}{T}\int_0^{T-|K|} Y(t)\cdot Y(t+|K|)\,dt - u^2 \qquad |K|<T$$

Since Y(t) and Y(t+|K|) only assume the values a and b

then $\quad Y(t)\cdot Y(t+|K|) = \begin{cases} a^2 \\ b^2 \\ a\cdot b \end{cases}$

and $\displaystyle \frac{1}{T}\int_0^{T-|K|} Y(t)\cdot Y(t+|K|)\,dt = \frac{1}{T}\{r_1(K)\cdot a^2+r_2(K)\cdot b^2 + [T-|K|-r_1(K)-r_2(K)]\cdot ab\}$

where $r_1(K)$ and $r_2(K)$ are to be determined.

The only problem in determining $R_T(K)$ is that of calculating $r_1(K)$ and $r_2(K)$, which is computationally difficult. $r_1(K)$ and $r_2(K)$ can be calculated for any particular value of K or at equidistant intervals ΔK, where ΔK can be taken as small as required. Then $R_T(K)$ is not restricted as in the discrete valued case of interval lengths, to a maximum number of estimated values equal to the total number of point events.

$$R_T(0) = \frac{1}{T}\int_0^T Y(t)\cdot Y(t)\,dt - u^2$$

$$= \frac{1}{T}\left(\sum_{i=1}^{n-[n/2]} a^2 X_{2i-1} + \sum_{i=1}^{[n/2]} b^2 X_{2i}\right) - u^2$$

where [n/2] denotes the largest integer less than or equal to the argument $n/2$.

It is necessary to calculate the spectral density function from the sample autocovariances without any assumptions about the distribution of interval lengths. Once a method for estimating the spectral density function is determined, its effectiveness can be determined by comparing these results to the spectrum obtained by taking the Fourier transform of the theoretical autocovariance function.

In order to estimate the autocovariances of the square wave the following method is suggested. For $0<K<T$ fix a value of K and determine the minimum value of the integers m_1

and m_2 such that $K < \sum_{i=1}^{m_1} X_i$ and $K \le \sum_{i=1}^{m_2} X_{n-i+1}$.

Let $\sum_{i=1}^{m_1} X_i - K = W_{m_1}$ and $\sum_{i=1}^{m_2} X_{n-i+1} - K = V_{n-m_2+1}$ so that

$$V = (X_1, X_2, \ldots, X_{n-m_2}, V_{n-m_2+1}) = (V_1, V_2, \ldots, V_{n-m_2+1})$$

$$W = (W_{m_1}, X_{m_1+1}, \ldots X_n) = (W_{m_1}, W_{m_1+1}, \ldots, W_n)$$

where $V_i = X_i \quad i = 1, \ldots, n-m$

$W_j = X_j \quad j = m+1, \ldots, n$

Now arrange the $n-m_2+1$ and $n-m_1+1$ elements in columns

V_1	W_{m_1}
V_2	W_{m_1+1}
.	.
.	.
.	.
V_{n-m_2+1}	W_n

The graphic representation of V and W is given by Figure 4-2.

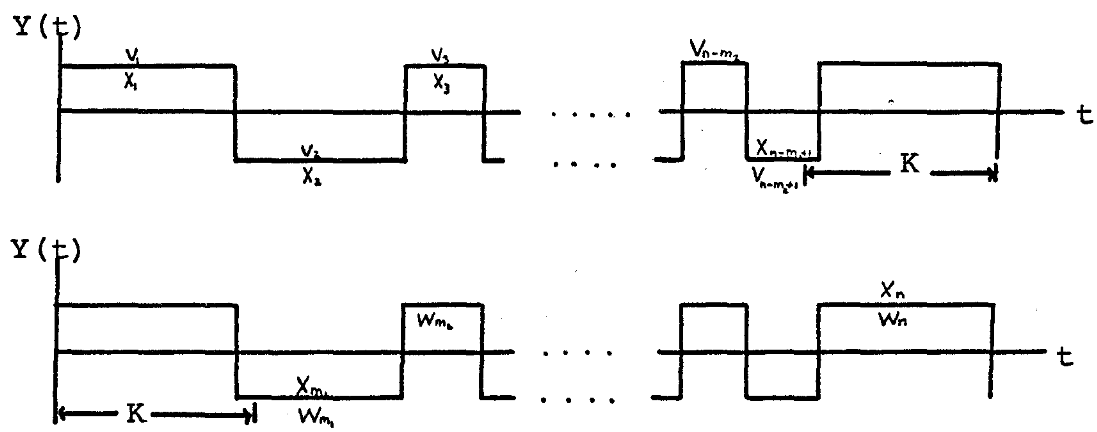

Fig. 4-2 The square wave as denoted by V and W respectively.

Take the two elements of the first row, V_1 and W_{m_1}, pick the smallest element and record it and the indices 1 and m_1 of these two interval lengths. Subtract the smaller element i.e. $\min\{V_1, W_{m_1}\}$ from the two interval lengths in the first row, V_1 and W_{m_1}. Either one element is now zero, or both are if $V_1 = W_{m_1}$. Any interval length in the first row reduced to zero by subtracting the smaller element from it is replaced by the next interval length in that column. If $V_1-\min\{V_1, W_{m_1}\}$ is zero it is replaced by V_2, and if $W_{m_1}-\min\{V_1, W_{m_1}\}$ is zero it is replaced by W_{m_1+1} and the new interval index or indices are recorded. If $V_1-\min\{V_1, W_{m_1}\}$ or $W_{m_1}-\min\{V_1, W_{m_1}\}$ is not zero then the index of this reduced interval length remains the same. For example

$$V = (2.0_1, 2.0_2, 1.0_3, 2.0_4, 2.0_5, 1.0_6, 0.5_7)$$

$$W = (0.5_1, 2.0_2, 1.0_3, 2.0_4, 2.0_5, 1.0_6, 1.0_7, 1.0_8)$$

are obtained from a point process with eight events where $X_1 = 2.0$, $X_2 = 2.0$, $X_3 = 1.0$, $X_4 = 2.0$, $X_5 = 2.0$, $X_6 = 1.0$, $X_7 = 1.0$, $X_8 = 1.0$ and $K = 1.5$.

Then

V	W
2.0_1	0.5_1
2.0_2	2.0_2
1.0_3	1.0_3
2.0_4	2.0_4
2.0_5	2.0_5
1.0_6	1.0_6
0.5_7	1.0_7
	1.0_8

$\min\{V_1, W_{m_1}\} = \min\{2.0_1, 0.5_1\} = 0.5$ with indices (1,1)

$V_1-\min\{V_1, W_{m_1}\} = 2.0_1 - 0.5 = 1.5_1$

$W_{m_1} - \min\{V_1, W_{m_1}\} = 0.5_1 - 0.5 = 0$, so that $W_{m_1+1} = 2.0_2$ replaces $W_{m_1} = 0.5_1$.

The new array is

V	W
1.5_1	2.0_2
2.0_2	1.0_3
1.0_3	2.0_4
2.0_4	2.0_5
2.0_5	1.0_6
1.0_6	1.0_7
0.5_7	1.0_8

Now the procedure is repeated taking 1.5_1 and 2.0_2 as the new entries in the first row.

$\min\{1.5_1, 2.0_2\} = 1.5$ with indices (1,2),

$1.5_1 - \min\{1.5_1, 2.0_2\} = 0$, and element $V_2 = 2.0$ is moved into the first row.

$2.0_2 - \min\{1.5_1, 2.0_2\} = 0.5_2$, and this is the reduced interval length for the second column.

The new array is

V	W
2.0_2	0.5_2
1.0_3	1.0_3
2.0_4	2.0_4
2.0_5	2.0_5
1.0_6	1.0_6
0.5_7	1.0_7
	1.0_8

and the same procedure is repeated on the elements 2.0_2 and 0.5_2. At each step the values of the smallest element and the interval indices are recorded and are used to calculate $r_1(K)$ and $r_2(K)$. If both indices are odd, then both intervals were from $Y(t) = a$, and if both are even they are from $Y(t) = b$. The sum of these minimum lengths with both indices odd is the value of $r_1(K)$ and the sum with both indices even is the value of $r_2(K)$. For a series with a large number of occurrences this method involves long tedious computations. With the aid of a computer $R_T(K)$ can be easily calculated for various values of K. A computer program for this is given in Appendix C.

From the preceding example the following was obtained,

minimum value	indices	
0.5	1 1	odd
1.5	1 2	
0.5	2 2	even
1.0	2 3	
0.5	2 4	even
1.0	3 4	
0.5	4 4	even
1.5	4 5	
0.5	5 5	odd
1.0	5 6	
0.5	5 7	odd
0.5	6 7	
0.5	6 8	even
0.5	7 8	

and $\quad r_1(K) = 1.5, \qquad r_2(K) = 2.0$.

Then $\quad R_T(K) = \frac{1}{T}\{r_1(K)a^2+r_2(K)b^2+[T-|K|-r_1(K)-r_2(K)]\ ab\}$

$$R_T(1.5) = \frac{1}{12}\,[1.5 + 2 + (12-3.5)(-1)]$$

$$= -5/12$$

where $\quad a = -b = 1.$

CHAPTER 5

COMPUTATIONAL RESULTS AND CONCLUSIONS

Probability analysis, using the methods in Chapter 2, reveals that no single test for randomness is effective, but that a combination such as Bartlett's test and the variance time curve analysis is necessary. If the intervals between events have a probability distribution, this can be effectively determined by a histogram and a χ^2 test of fit. The pooled output from several periodic sources can be distinguished from a Poisson process by means of the variance time curve, and the number of periodic sources can be determined. Unless the number of periods is small and the periods far apart, the histogram analysis of the pooled output will not reveal any information about the constituent periods. The computational results for several examples appear in Appendix A.

Spectral analysis of the interval lengths and occurrence rate yielded very little information. However, a spectral analysis of the square wave produced some very interesting results. While spectral analysis of the square wave did not determine if a process was random or not, it did distinguish between the pooled output of several periodic sources and a random or "near random" point process. The "near random" process was the result of splicing two random processes with rates of occurrence λ_1 over (0,t) and λ_2 over (t, 2t) to form a single point process over (0, 2t). While spectral analysis of the square wave could not distinguish between the random and "near random" series, no study was made to find out if they could be distinguished, and if so, under what conditions.

For the examples with the intervals having a probability distribution, spectral analysis of the square wave revealed a function which closely approximated the density function of the interval lengths. The autocovariance function can be calculated in terms of the inverse Laplace transform of a function involving the Laplace transform of the interval lengths [3]. However, some examples dealt with in Appendix B have truncated distributions, since the interval lengths have to be positive, and the Laplace transforms of these functions, as well as most density functions, have not been tabulated. The spectral density functions for the square wave of time series with the different density functions can be calculated and graphed, and the resulting graphs used to identify the type of point process.

The only available information from probability analysis of a pooled series from several periodic sources is the number of constituent periodic series. The method of spectral analysis to determine constituent frequencies was then applied to the pooled series in order to see if this would locate the periods. Spectral analysis of the interval lengths and occurrence rate of the pooled series revealed little information, but analysis of the square wave revealed peak frequencies ω_j, located in the range $(0, \pi)$. Furthermore, it was observed that the peak frequencies ω_j/π corresponded to a linear combination of the occurrence rates of the periodic sources and that the number of peaks was related to the number of periods. After examining the spectrum for many pooled series (almost 20, with some examples given in Appendix B) and encountering the same results in all of them, the following conjecture was made. It appears that for N periodic sources with periods B_i and rates of occurrence λ_i ($i=1, 2,\ldots N$; N small), spectral analysis of the square wave of the pooled output results in the occurrence of 2^{N-1} peaks of $f_T(\omega)$ at frequencies ω_j, $j=1, 2\ldots, 2^{N-1}$. If $\hat{\lambda}_j = \omega_j/\pi$, then each $\hat{\lambda}_j$ is a function of $\lambda_1, \lambda_2, \ldots, \lambda_N$ and is equal to the absolute value of a linear combination of the λ_i,

$$\hat{\lambda}_j = |\alpha_1\lambda_1+\alpha_2\lambda_2+\ldots+\alpha_N\lambda_N| \qquad j=1, 2, \ldots, 2^{N-1}$$

$$\text{where} \quad \alpha_1 = 1 \qquad \text{and} \quad \alpha_K = \begin{cases} +1 \\ -1 \end{cases} \quad K=2,\ldots., N.$$

The 2^{N-1} possible sign changes for the α_K, $K=2,\ldots.., N$ account for the 2^{N-1} peak frequencies.

The proof of this conjecture is hindered by the fact that no one has produced a model to describe the superposition series of several periodic sources. A model to describe the pooled output from two periodic sources with periods B_1 and B_2, $B_1<B_2$, is $X(t) = Y(t)+R(t)$ where $Y(t)$ is a periodic point process with period $(B_1+B_2)/(B_2-B_1)$ and $R(t)$ is a point process with the intervals between events having a uniform distribution over $(0, B_1)$. This was obtained by analysis of the frequency of the interval length B_1 from the pooled output. Further analysis in this direction may prove rewarding, but it is possible that no closed expression exists to describe the pooled series since the interval lengths tend to an exponential distribution as the number of sources increases [4]. The effect of increasing the number of sources has not been investigated yet, examples with N=2, 3, and 4 periodic sources are given in Appendix B, and all computational results pertaining to Chapter 4.

APPENDIX A

In order to evaluate the information obtainable from probability and spectral analysis, several types of point processes were generated and analysed.

TABLE A-1

EXAMPLE	TYPE OF PROCESS	NUMBER OF EVENTS
P1	Poisson process with mean rate of occurrence $\lambda = 0.167$	250
P2	Point process consisting of a Poisson process with $\lambda_1 = 0.203$ over the time period (0,600] and a Poisson process with $\lambda_2 = 0.297$ over (600,1200]	300
P3	Poisson process with mean rate of occurrence $\lambda = 0.208$	250
S1	Superposition of 4 periodic point processes with periods 2.00, 2.27, 5.15, 8.23	250
S2	Superposition of 4 periodic point processes with periods 1.93, 2.89, 3.83, 3.96	250

EXAMPLE	TYPE OF PROCESS	NUMBER OF EVENTS
G1	Point process with the intervals between events Gaussian distributed with density function $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \text{EXP}\{-1/2(x-u)^2/\sigma^2\}$ where $u = 1.50$ and $\sigma = 0.50$	250
G2	Point process with the intervals between events Gaussian distributed with $u = 6.0$ and $\sigma = 2.0$	250
R1	Point process with the intervals between events having a Rayleigh distribution with density function $f(x) = \frac{x}{r^2} \text{EXP}\{-1/2(x/r)^2\}$ where $r^2 = 2.0$	250
R2	Point process with the intervals between events having a Rayleigh distribution with $r^2 = 5.0$	250

Examples P1, P2, P3, were generated using a table of random numbers. In order to obtain a point process with Gaussian distributed interval lengths, a computer subroutine was used to obtain a series of random Gaussian distributed numbers with mean zero and any standard deviation, σ. The subroutine generates two uniformly distributed numbers on (0,1), x and y. These are used to give two elements of the list as $(-2 \log x)^{1/2} \cos 2\pi y$ and $(-2 \log x)^{1/2} \sin 2\pi y$. By a translation of the origin to -3σ, negative numbers were eliminated and a mean $u = 3\sigma$ was obtained. The interval lengths with a Rayleigh distribution were generated by using the probability integral transform $y = \int_0^x f(t)\,dt$ where $f(t) = \frac{t}{r^2} e^{-\frac{1}{2}(t/r)^2}$. This transformation maps the positive real line into the interval from zero to one, and y has the uniform distribution. Then $y = F(x) = \int_0^x f(t)\,dt$ and $y = 1-e^{-\frac{1}{2}(x/r)^2}$ yields $x = 2r^2 \cdot \log y$. Random numbers on the interval (0, 1) were generated by a subroutine which computed a string of pseudorandom numbers z_i by the relation $z_i = az_{i-1}$ (modulo m). Values of x were calculated by solving the equation $x = \left|\sqrt{2r^2 |\log y|}\right|$ (y takes on the values z_i).

GRAPHICAL METHODS

For each example the interval lengths x_i were plotted versus i, but only for the first 40 or 50 interval lengths since these were considered as a representative sample from each point process. If the graph of x_i versus i indicated that the interval lengths were independent of time, then the histogram of interval lengths was formed, (Figure A-1 to Figure A-9).

Examination of the graphs for each example reveals several examples which appear to be Poisson processes, P1, P2, P3, S1 and S2. For each one of these examples x_i versus i is

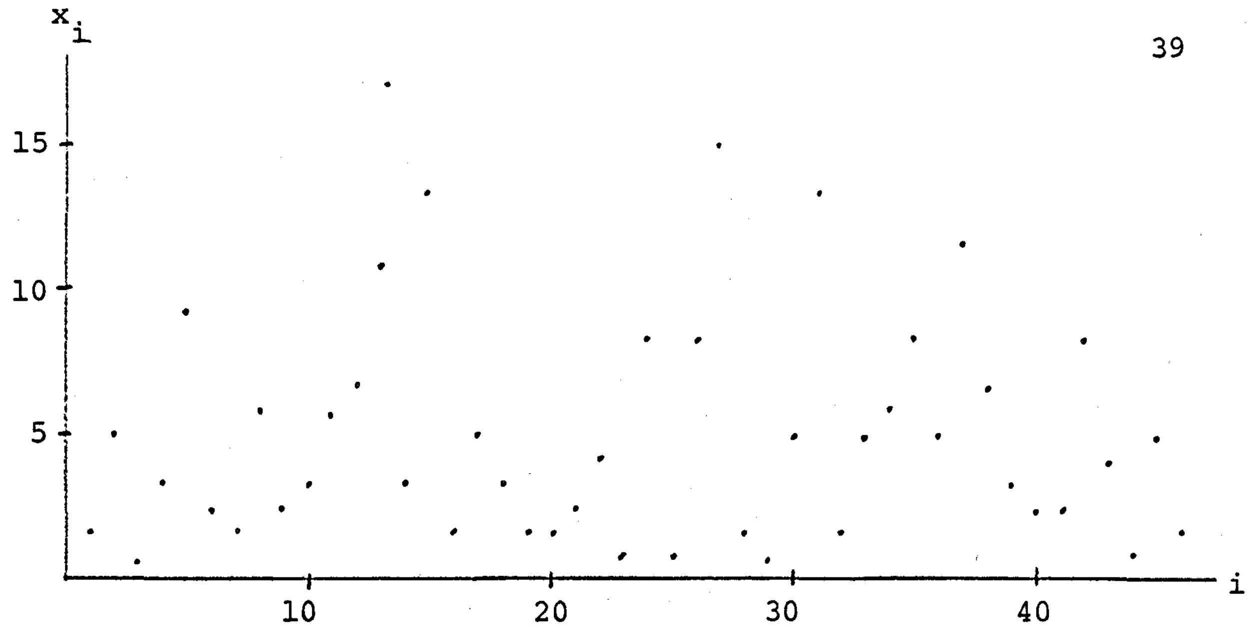

Fig. A-1(a) Interval lengths versus index for example P1.

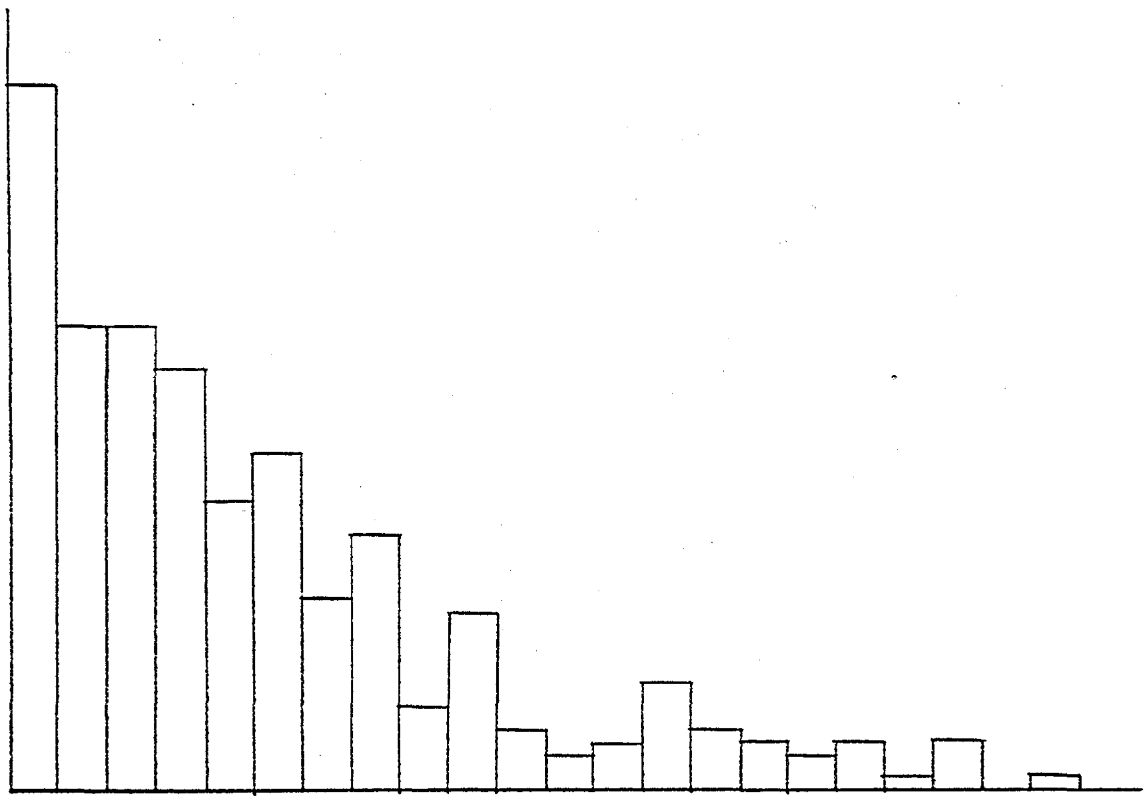

Fig. A-1(b) Histogram of interval lengths for example P1.

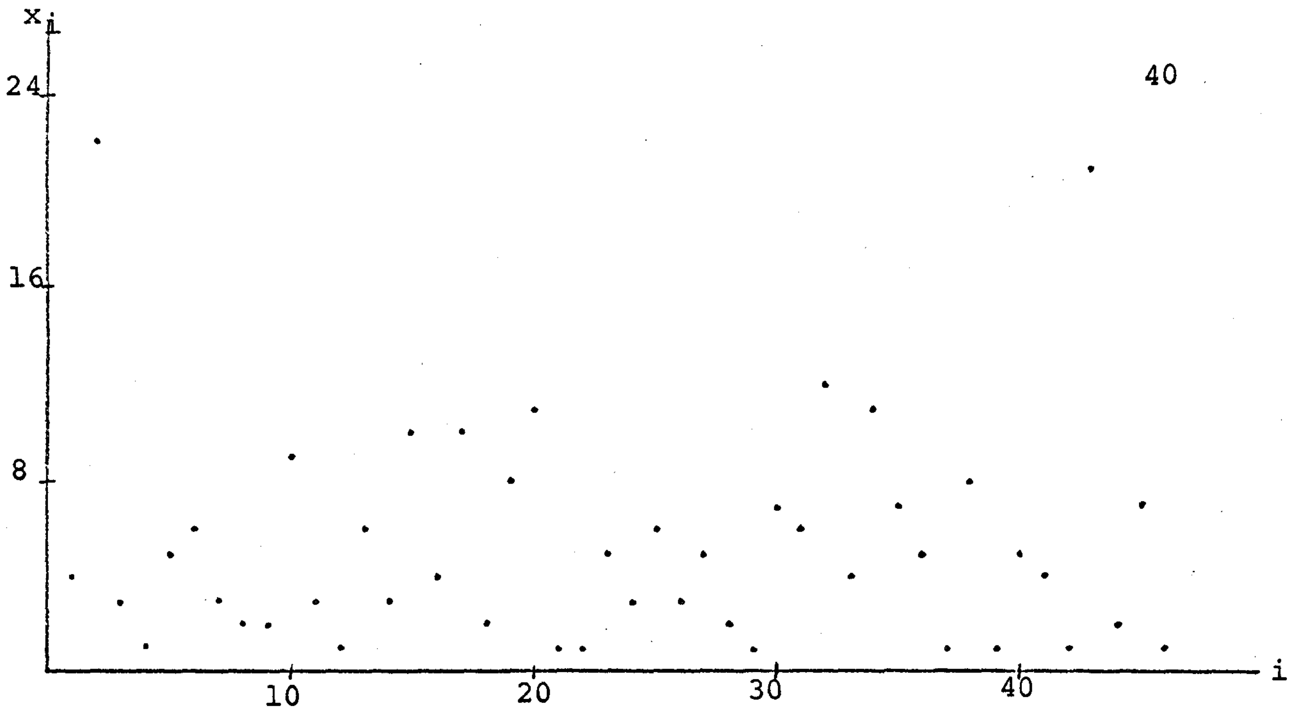

Fig. A-2(a) Interval lengths versus index for example P2.

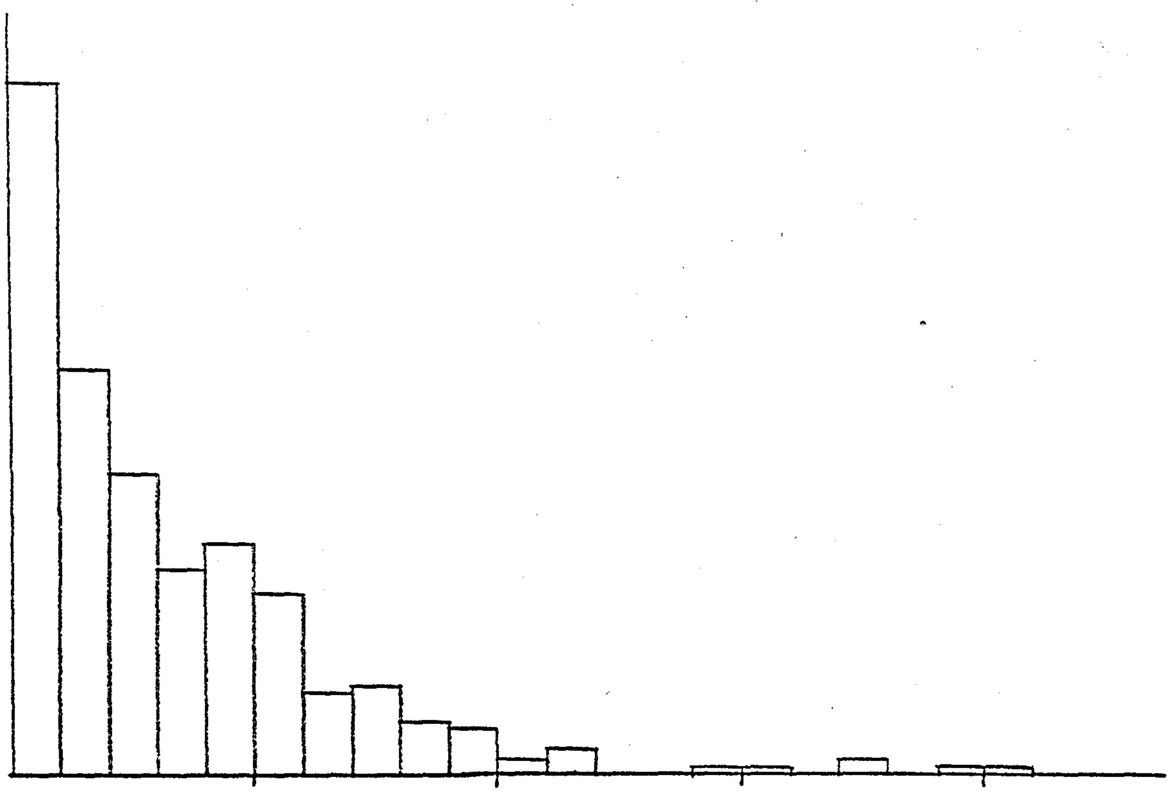

Fig. A-2(b) Histogram of interval lengths for example P2.

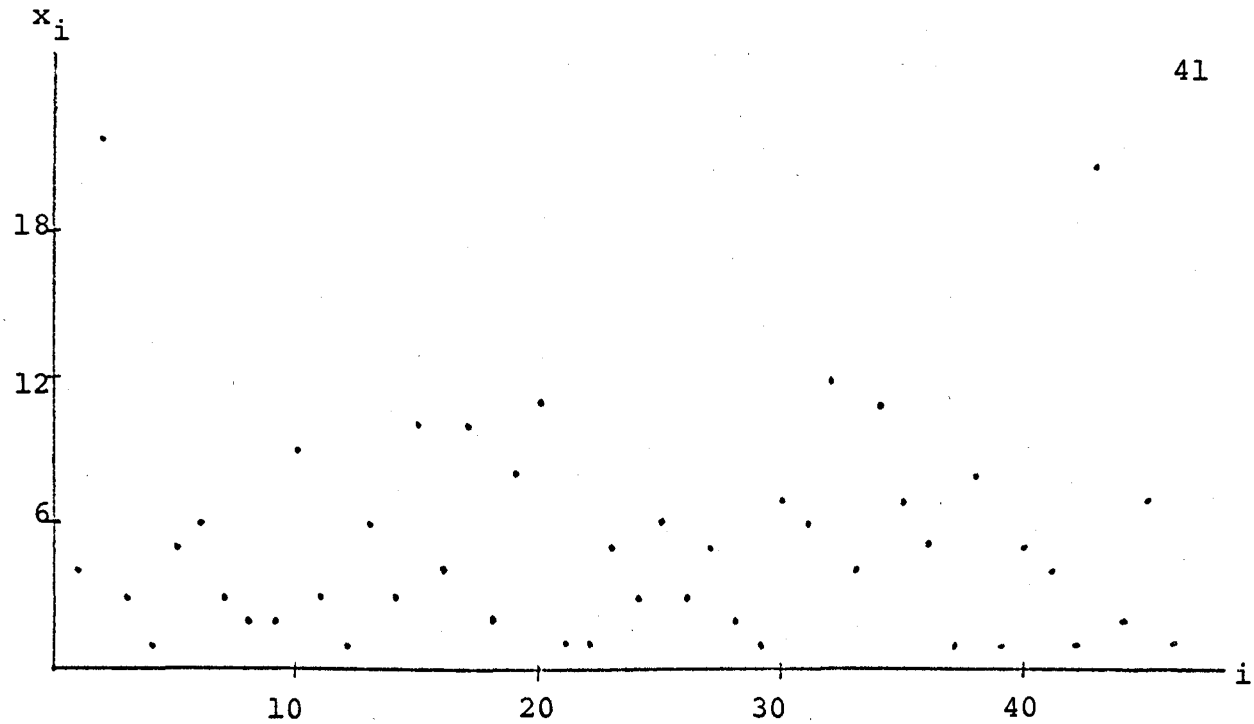

Fig. A-3(a) Interval lengths versus index for example P3.

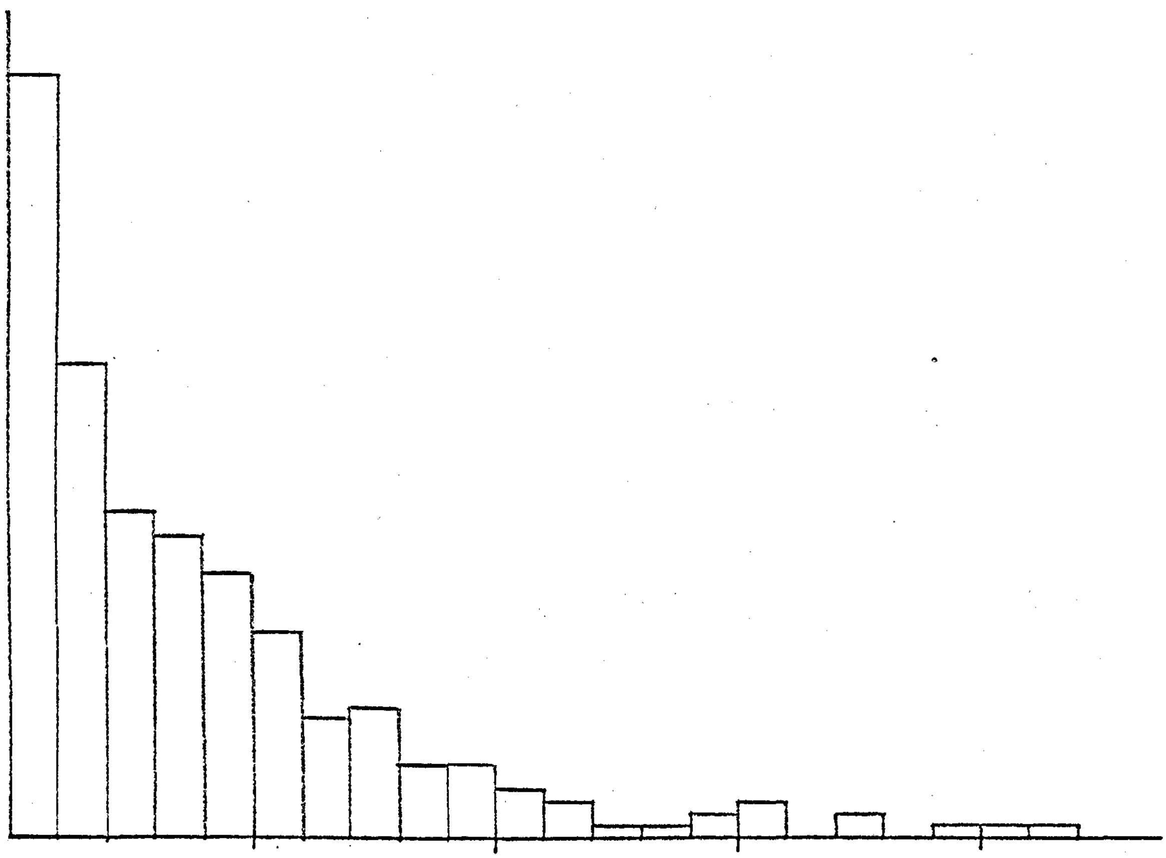

Fig. A-3(b) Histogram of interval lengths for example P3.

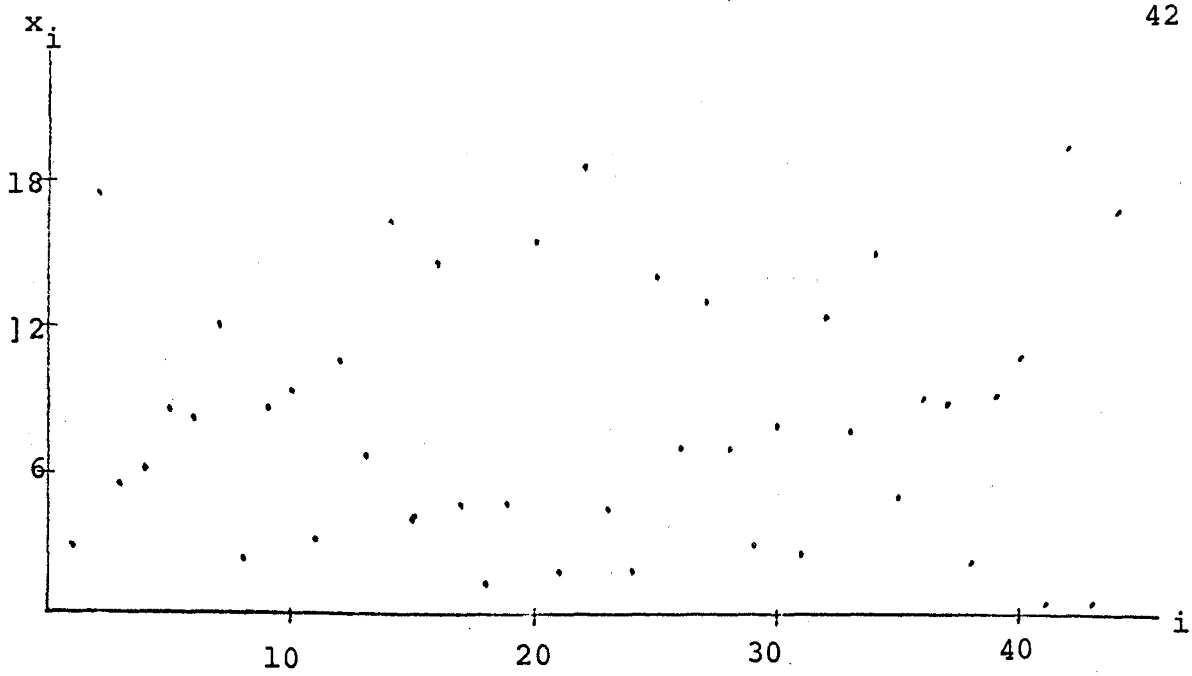

Fig. A-4(a) Interval lengths versus index for example S1.

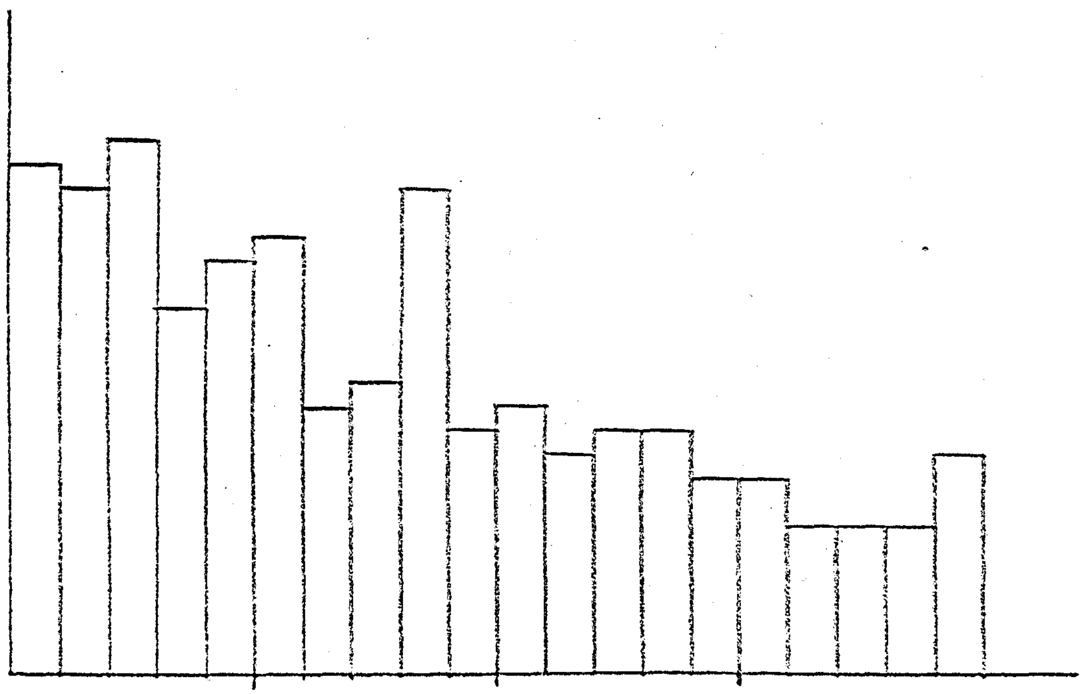

Fig. A-4(b) Histogram of interval lengths for example S1.

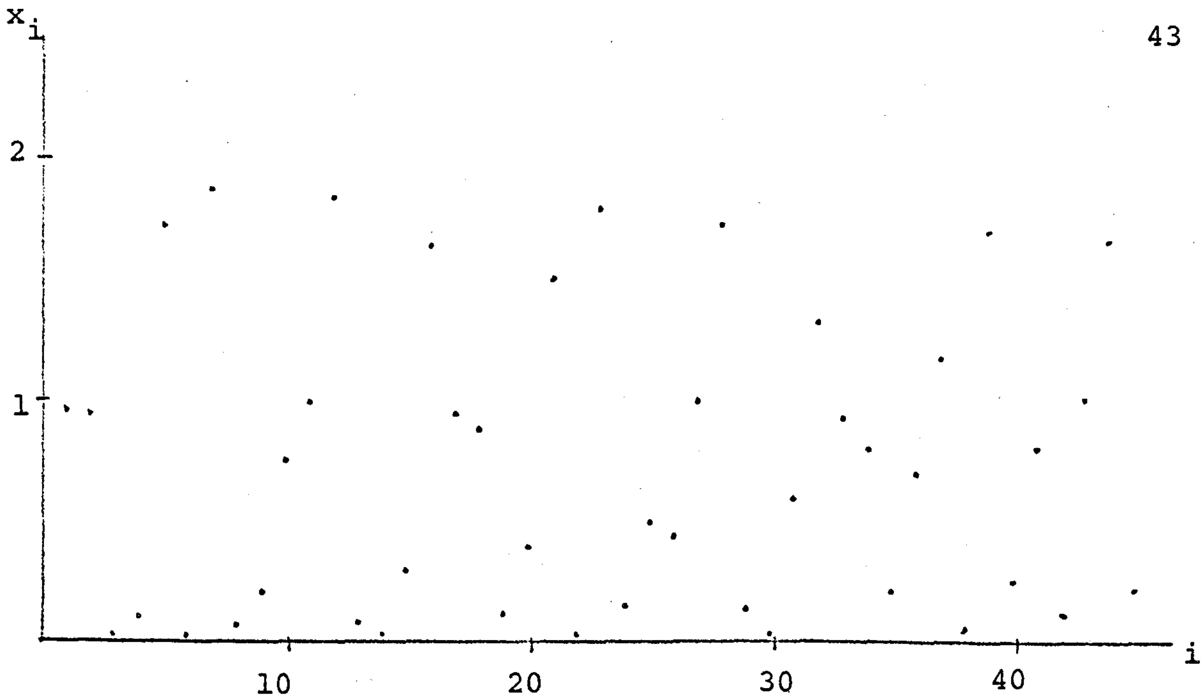

Fig. A-5(a) Interval lengths versus index for example S2.

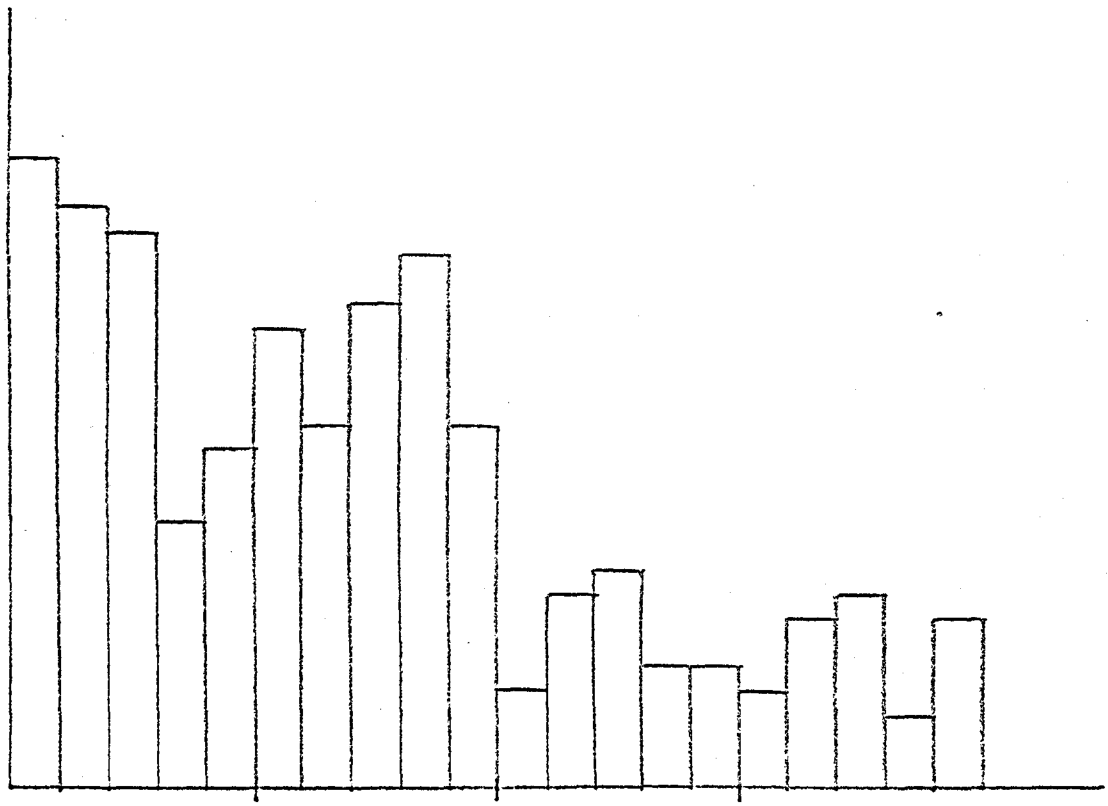

Fig. A-5(b) Histogram of interval lengths for example S2.

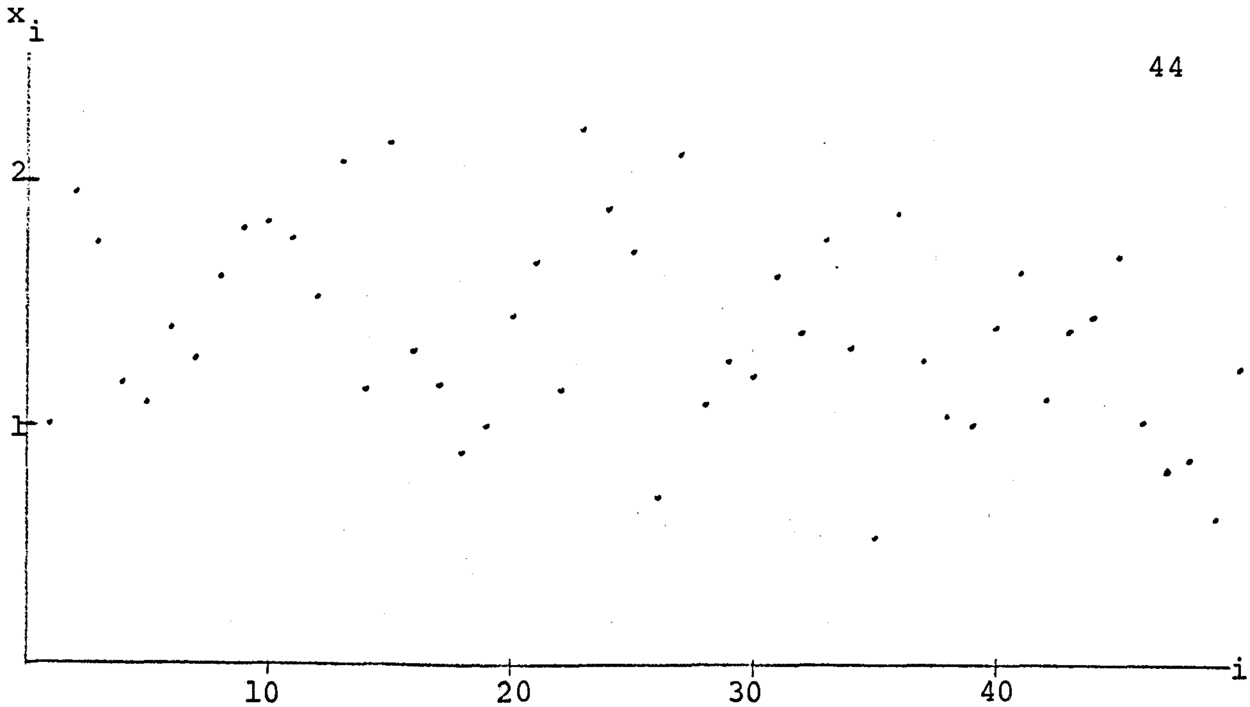

Fig. A-6(a) Interval lengths versus index for example G1.

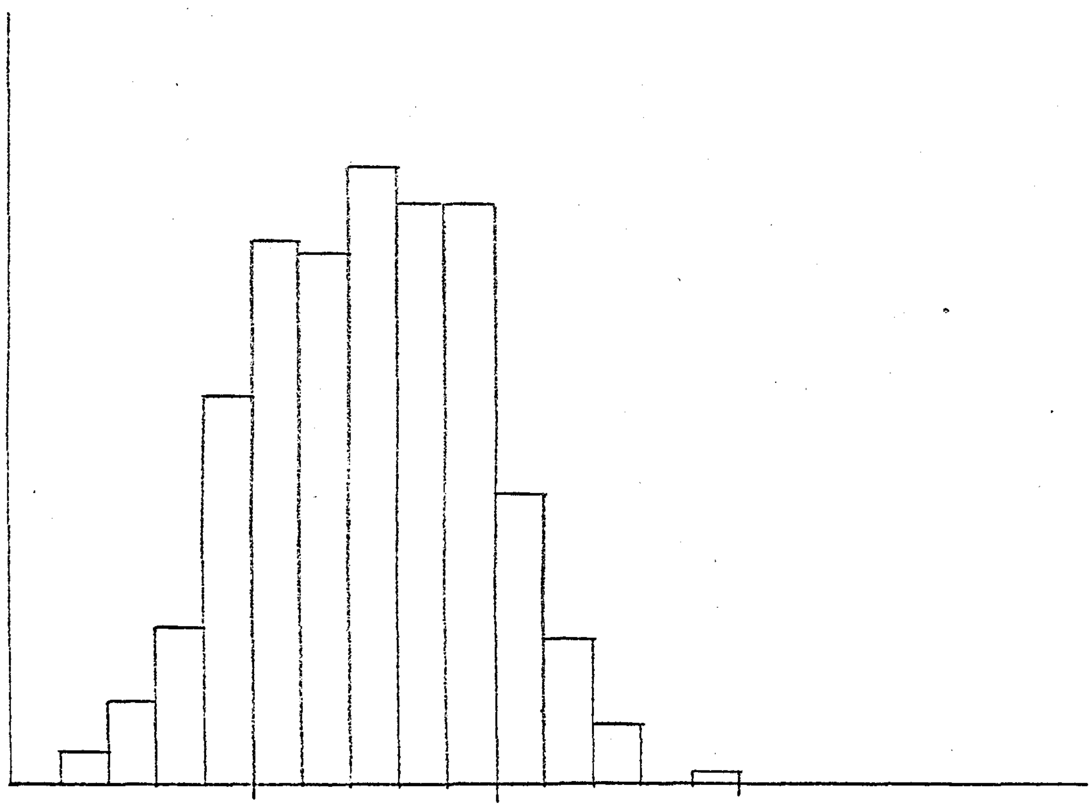

Fig. A-6(b) Histogram of interval lengths for example G1.

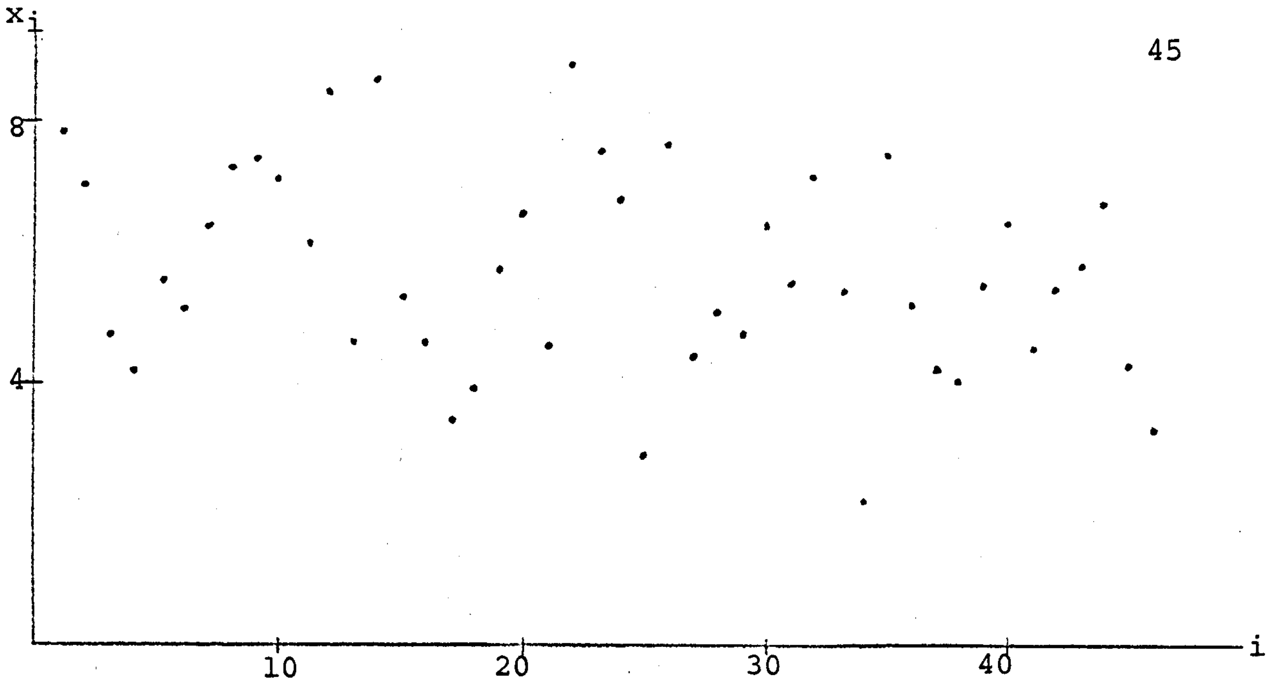

Fig. A-7(a) Interval lengths versus index for example G2.

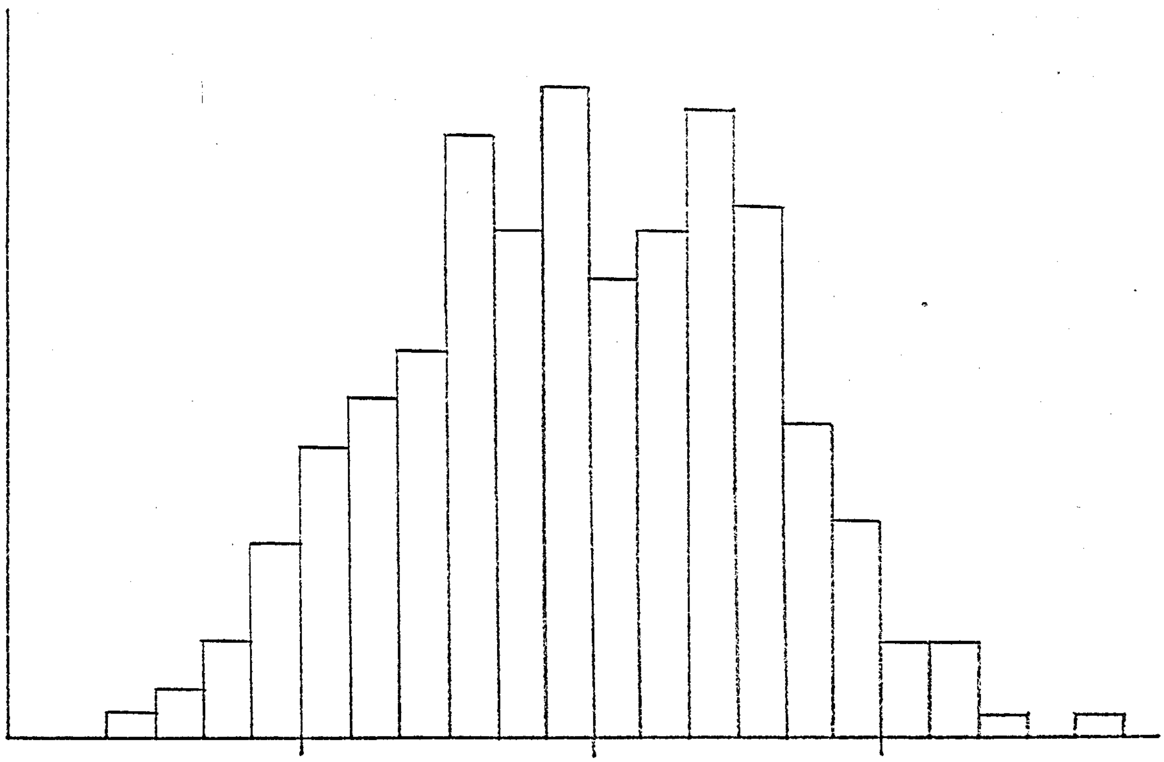

Fig. A-7(b) Histogram of interval lengths for example G2.

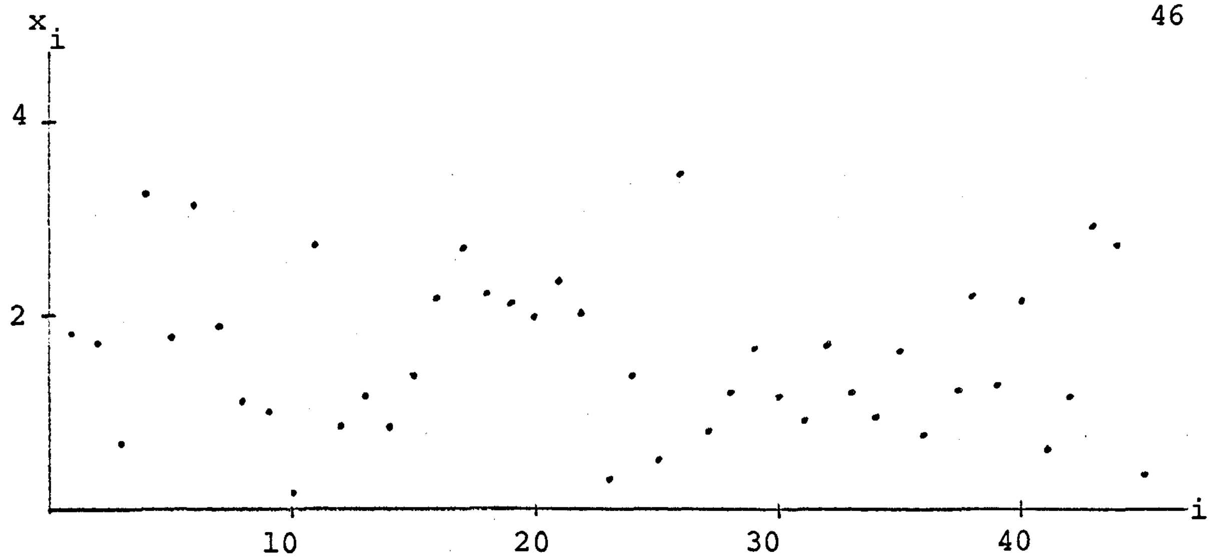

Fig. A-8(a) Interval lengths versus index for example R1.

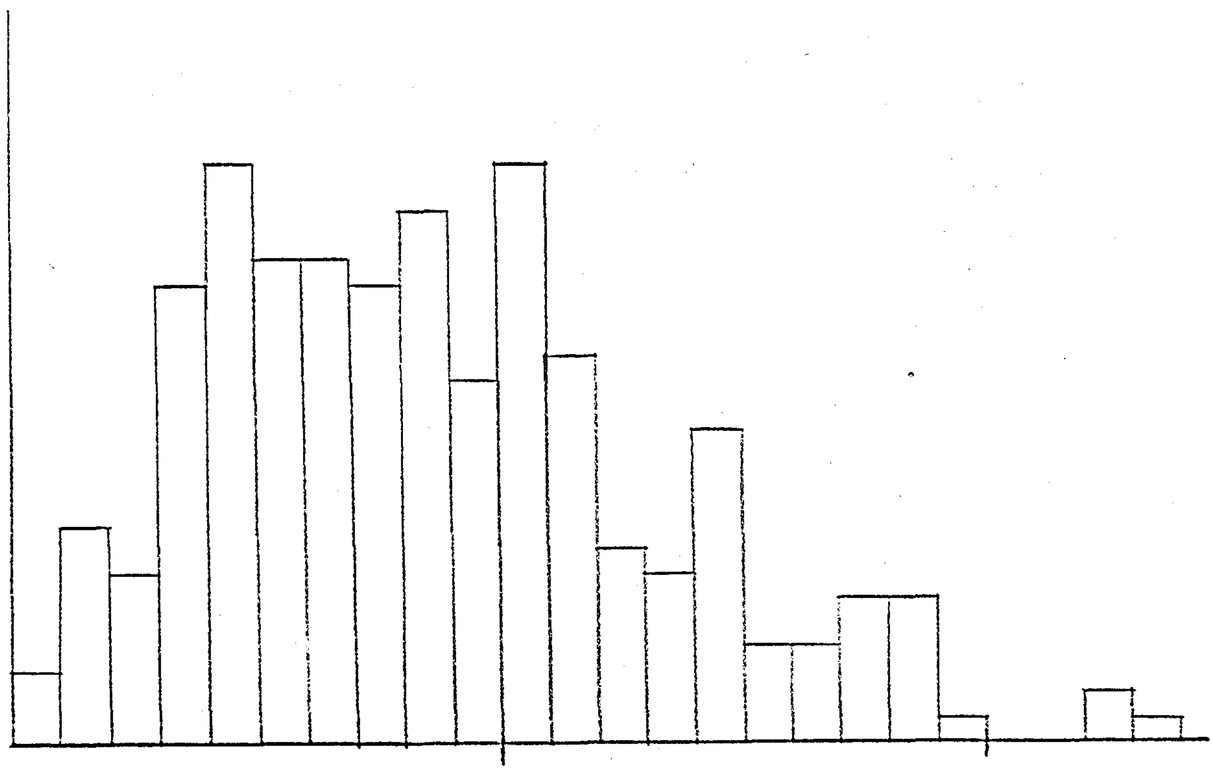

Fig. A-8(b) Histogram of interval lengths for example R1.

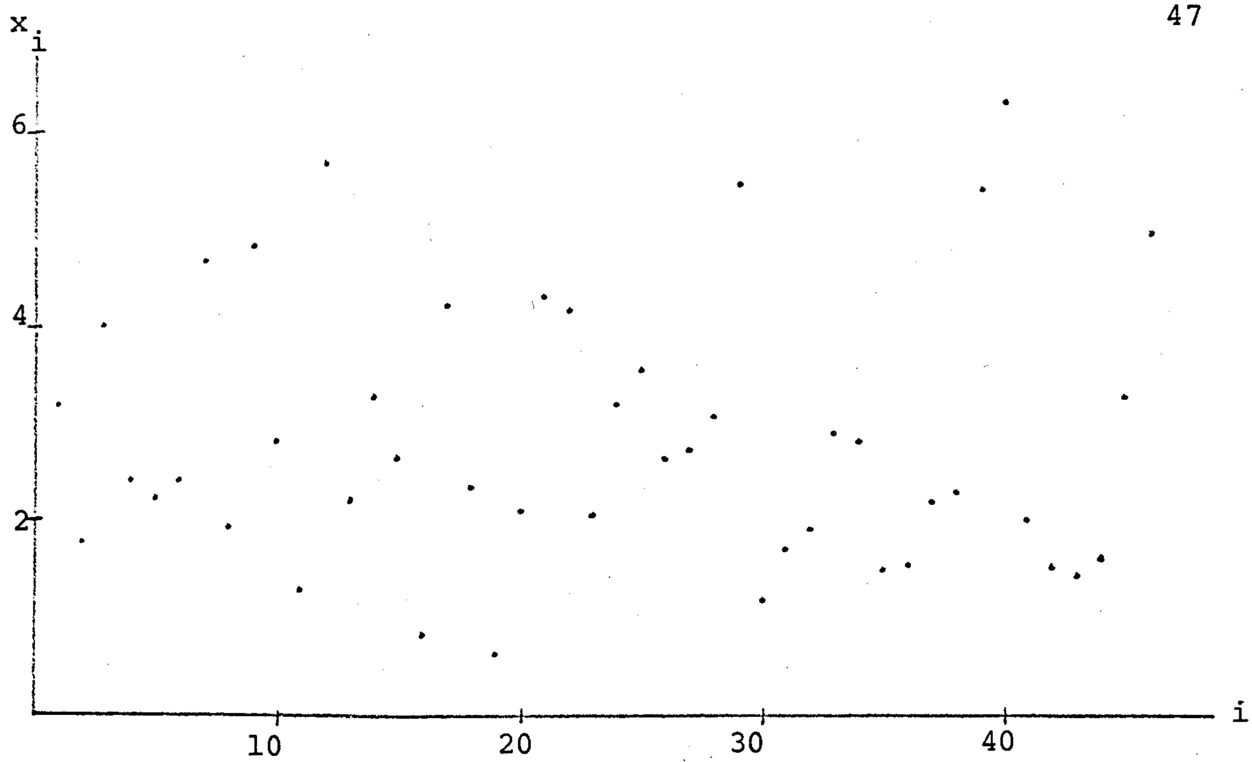

Fig. A-9(a) Interval lengths versus index for example R2.

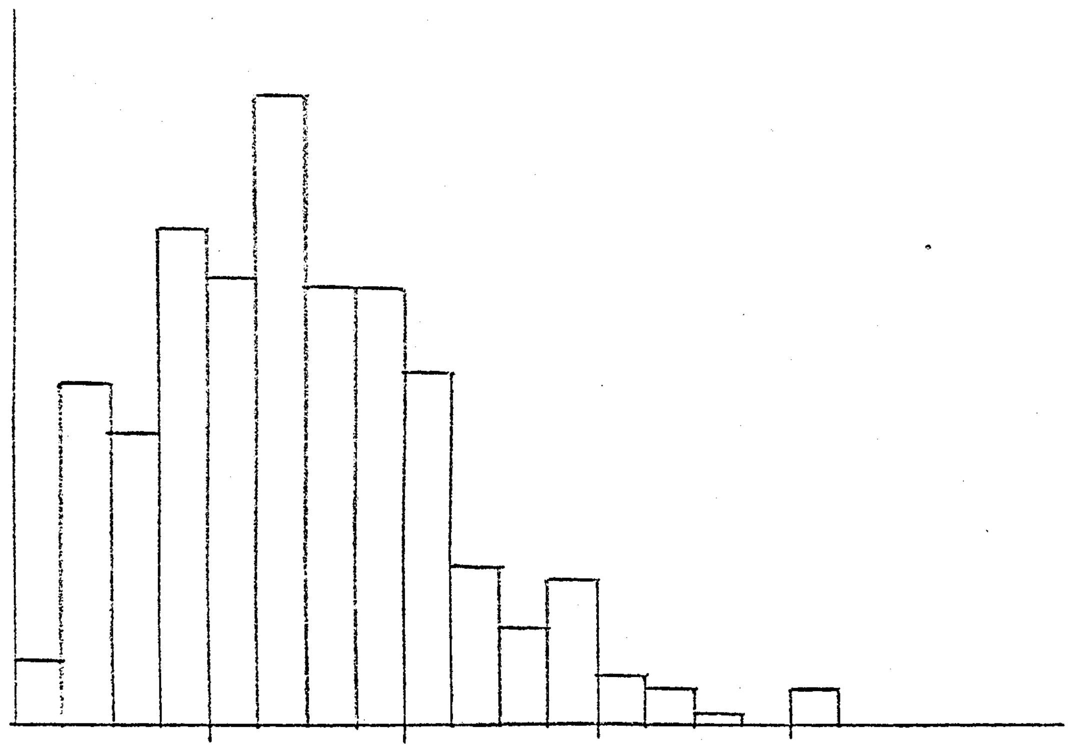

Fig. A-9(b) Histogram of interval lengths for example R2.

a scatter diagram and the histogram of interval lengths approximates the probability density function of the exponential distribution. Tests for randomness were then tried in an attempt to identify the type of process as Poisson or not. The test based on interval lengths, Bartlett's test, was tried first and then compared with other tests based on the number of occurrences in an interval length.

TESTS FOR RANDOMNESS

BARTLETT'S TEST

Each example was divided up into K = 10 equal series of m = n/K interval lengths, and

$\bar{x}_i = \sum_{j=1}^{m} \frac{x_j}{m}$ $\quad i = 1\ldots\ldots, K$ was calculated for the K sets of interval lengths. Then χ^2 was calculated and the hypothesis that all the λ_i are equal was tested at the 5% level by calculating $\chi^2_{.95,K-1}$ with K-1 degrees of freedom.

$$\chi^2 = 2.3026[2(m-1)\log(\frac{1}{K}\sum_{i=1}^{K}\bar{x}_i) - 2(m-1)\sum_{i=1}^{K}\log\bar{x}_i]/C$$

$$C = 1 + (K+1)/[3K(m-1)]$$

The results are tabulated along with the results from other tests of randomness (TABLE A-3).

KOLMOGOROV-SMIRNOV TEST

This test for randomness is a measure of the maximum deviation of n(t) from the straight line joining the points (0, 0) and (T, n). For a sample of size n in a continuum (0, T]

$$n \cdot D_n = \max_{0 \le t \le T} | n(t) - n \cdot t/T| \quad \text{is calculated.}$$

To test the hypothesis of randomness a significance level α is selected, and a critical value of $E_{n\alpha}$, which is tabulated for particular n and α, is selected. If $n \cdot D_n > E_{n\alpha}$, the hypothesis that the series is random is rejected. Instead of finding $E_{n\alpha}$ for a particular n and α from tables the approximations

$$E_{n.95} \simeq 1.358\sqrt{n}$$

$$E_{n.99} \simeq 1.628\sqrt{n}$$

were used.

RECTANGULAR DISTRIBUTION TEST

The time interval (0,T] was divided up into m = 25 equal time intervals of length τ for each example. O_i, the number of events observed in the interval $\big((i-1)\tau,\ i\tau\big]$ $i=1,\ldots, m$ and $E = n \cdot \tau/T$ were calculated. Then

$$\chi^2 = \sum_{i=1}^{m} (O_i - E)^2 / E \quad \text{was evaluated.}$$

The upper and lower limits of χ^2 for a 95% confidence interval and m-1 = 24 degrees of freedom are $\chi^2_{.975,24} = 39.4$ and $\chi^2_{.025,24} = 12.4$ respectively.

VARIANCE TIME CURVE

In order to distinguish between a Poisson process and the pooled output of several periodic sources the variance time curve was calculated for the examples whose histogram of interval lengths appeared exponentially distributed. The series was divided up into small intervals of length τ, and $y_i = n_i(\tau)$ was calculated.

$U_i^r = y_i + y_{i+1} \dots + y_{i+r-1}$ was calculated for $i=1, 2, \dots, M$ and $r = 1, 2, \dots, 15$ where $M = 100$.

Then

$$\hat{V}(r\tau) = \left[\sum_{i=1}^{M} (U_i^r)^2 - \left(\sum_{i=1}^{M} U_i^r/M\right)^2\right] \cdot [3M/(3M^2-3Mr+r^2-1)]$$

was tabulated and graphed (TABLE A-2, Figure A-10(a) to A-10(d))

TABLE A-2

r	$\hat{V}(r\tau)$ EX P1	$\hat{V}(r\tau)$ EX P2	$\hat{V}(r\tau)$ EX P3	$\hat{V}(r\tau)$ EX S1	$\hat{V}(r\tau)$ EX S2
1	2.11	2.61	2.43	0.56	0.89
2	3.68	6.47	5.46	0.68	0.59
3	5.43	10.3	8.17	0.63	0.67
4	6.61	14.9	10.9	0.61	0.61
5	7.88	20.9	13.8	0.54	0.54
6	9.19	27.9	17.1	0.78	0.69
7	11.3	35.3	20.0	0.67	0.90
8	13.4	43.9	23.1	0.24	0.37
9	15.9	53.0	25.4	0.58	0.80
10	18.2	61.4	26.8	0.67	0.58
11	20.0	70.2	28.1	0.81	0.77
12	22.2	80.4	29.4	0.75	0.88
13	23.2	91.5	30.3	0.52	0.46
14	25.0	102.	30.4	0.88	0.83
15	26.8	113.	31.5	0.76	0.84

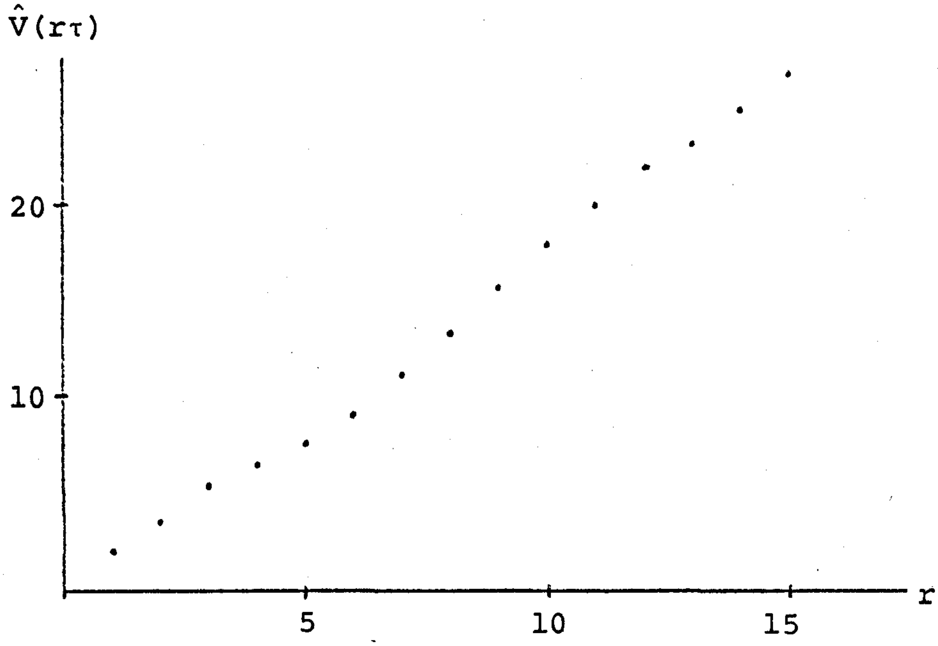

Fig. A-10(a) Variance time curve for example P1.

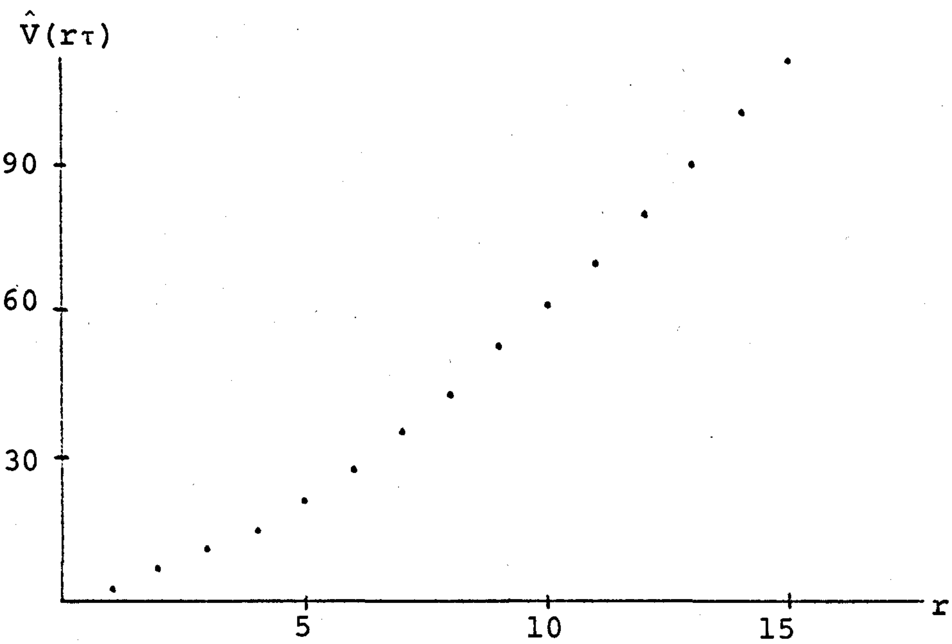

Fig. A-10(b) Variance time curve for example P2.

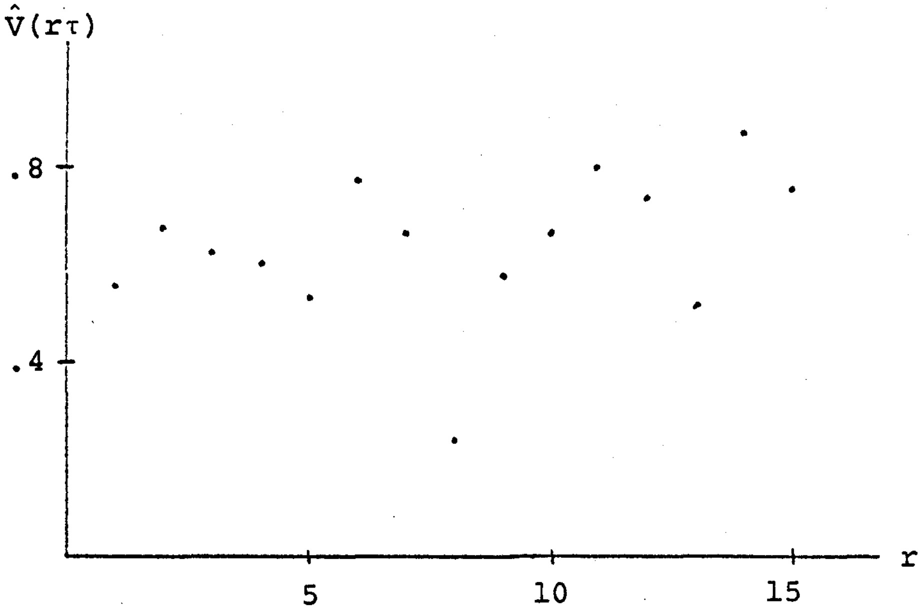

Fig. A-10(c) Variance time curve for example S1

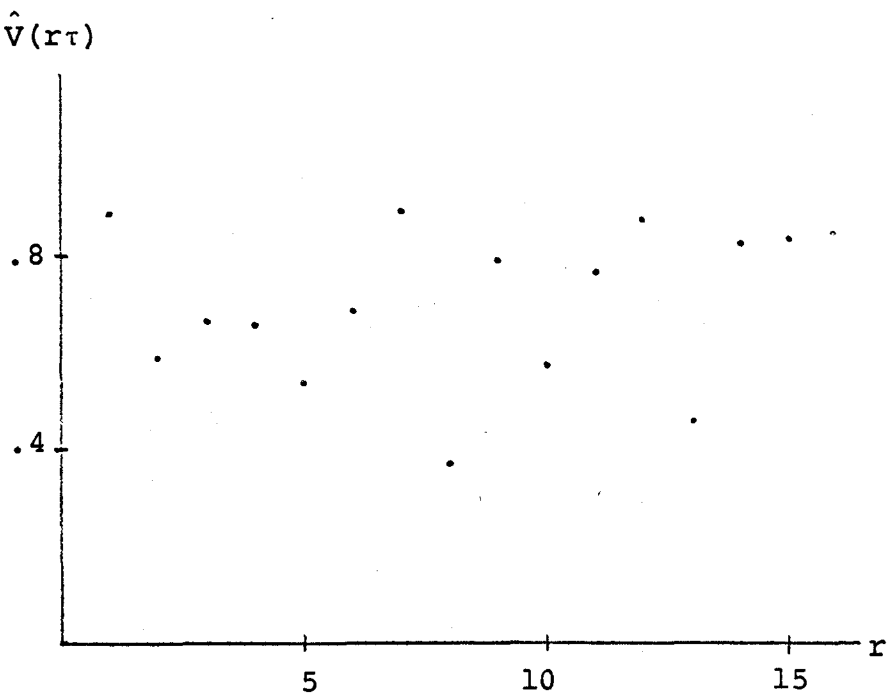

Fig. A-10(d) Variance time curve for example S2.

The results for the various tests for randomness are tabulated in order to compare their accuracy. (TABLE A-3)

TABLE A-3

Example	Bartlett's Test χ^2	Kolmogorov-Smirnov Test $n \cdot D_n$	.95 lim.	.99 lim.	Rect. Dist. Test χ^2	Variance Time Curve
P1	6.80	13.2	21.5	25.7	14.8	Linear
P2	19.58	35.0	23.5	28.2	29.2	Linear
P3	5.96	15.7	21.5	25.7	20.8	Linear
S1	0.26	2.31	21.5	25.7	1.40	Oscil-lates
S2	0.22	2.70	21.5	25.7	2.40	Oscil-lates

The .95 upper bound for Bartlett's Test is 16.92 and the .95 bounds for the rectangular distribution test are [12.4, 39.4].

It is quite apparent that there is no single test of randomness that is effective for all point processes, and that a combination of tests such as the variance time curve and Bartlett's test is necessary.

INTERVALS WITH PROBABILITY DISTRIBUTION

The histogram of the following examples indicate that the intervals may have a probability distribution.

TABLE A-4

Example	Probability Distribution	Density Function	Mean	Variance
P1	EXPONENTIAL	$f(x) = \lambda e^{-\lambda x}$ $x>0$ $= 0$ otherwise	$1/\lambda$	$1/\lambda^2$
P2	EXPONENTIAL on (0,600]	$f(x)$ where $\lambda=\lambda_1$	$1/\lambda_1$	$1/\lambda_1^2$
	EXPONENTIAL on (600,1200]	$f(x)$ where $\lambda=\lambda_2$	$1/\lambda_2$	$1/\lambda_2^2$
P3	EXPONENTIAL	-	-	-
G1	GAUSSIAN	$g(x) = \frac{e^{-\frac{1}{2}\{(\frac{x-\mu}{\sigma})^2\}}}{\sqrt{2\pi}\sigma}$ $x>0$ $=0 \quad x\leq 0$	μ	σ^2
G2	GAUSSIAN	-	-	-
R1	RAYLEIGH	$h(x) = \frac{x}{r^2} e^{-(x/r)^2}$ $x>0$ $= 0 \quad x\leq 0$	$r\sqrt{\pi/2}$	$r^2(2-\frac{\pi}{2})$
R2	RAYLEIGH	-	-	-

The sample mean $\overline{x}$ and sample variance $S_x{}^2$ were calculated for the sequence of interval lengths $\{x_i\}_{i=1}^{n}$ for each example where

$$\overline{x} = \sum_{i=1}^{n} \frac{x_i}{n},$$

$$S_x{}^2 = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n-1},$$

The statistics $\overline{x}$ and $S_x{}^2$ were used as estimates of the mean and variance respectively of the theoretical distribution. A χ^2 test of fit was then used to determine the goodness of fit of the histogram to the theoretical distribution. Example P1 is worked out in detail.

Ex. P1 $\overline{x} = 5.996$, $\lambda = 1/\overline{x} = 0.167$, sample size $n = 250$

The exponential curve to fit the histogram would be $f(x) = 0.167e^{-0.167x}$. For the interval length $L = [a,b)$ let $P(L)$ equal the area under $f(x)$ for $a \leq x < b$.

Then

$$P(L) = \int_a^b f(x)\,dx = e^{-\lambda a} - e^{-\lambda b}$$

$$= e^{-0.167a} - e^{-0.167b}$$

The expected number of interval lengths L is calculated as $nP(L)$, and the observed number of interval lengths of length L are recorded. (TABLE A-5)

TABLE A-5

Interval Length L = [a b)	Probability of L P(L)	Expected Number E	Observed Number O
[0,1.0)	0.154	38.5	44
[1.0,2.0)	0.130	32.5	29
[2.0,3.0)	0.110	27.5	29
[3.0,4.0)	0.093	23.3	26
[4.0,5.0)	0.079	19.3	18
[5.0,6.0)	0.067	16.8	21
[6.0,7.0)	0.056	14.0	12
[7.0,8.0)	0.048	12.0	16
[8.0,9.0)	0.041	10.3	5
[9.0,10.0)	0.034	8.5	11
[10.0,11.0)	0.029	7.3	4
[11.0,12.0)	0.024	6.0	2
[12.0,13.0)	0.021	5.3	3
[13.0,14.0)	0.017	4.3	7
[14.0,15.0)	0.015	3.8	4
[15.0,16.0)	0.013	3.3	3
[16.0,17.0)	0.011	2.8	2
[17.0,18.0)	0.009	2.3	3
[18.0,19.0)	0.007	1.8	1
[19.0,20.0	0.007	1.8	3
[20.0,21.0)	0.005	1.3	0

O, E and $|O-E|$ are tabulated in K cells. When the class frequency E is less than five the adjoining cells are added to ensure a class frequency of five or more. (TABLE A-6)

TABLE A-6

Observed O	44	29	29	26	18	21	12	16	5	11	4	2	3	14	5	7
Expected E	39	33	28	23	19	17	14	12	10	9	7	6	5	8	6	7
$\lvert O-E \rvert$	5	4	1	3	1	4	2	4	5	2	3	4	2	6	1	0

K = 16 cells Calculate $\chi^2 = \sum \frac{(O-E)^2}{E} \approx 16.5$

If $\chi^2 > \chi^2_{K-p-1,.95}$ the hypothesis that the interval lengths are exponentially distributed is rejected at the 5% level. The number of degrees of freedom, K-p-1, consists of the number of cells K, minus the number p of parameters estimated, minus one.

$\chi^2_{14,.95} = 23.7$

The results for all examples were tabulated. (TABLE A-7).

Under the hypothesis that the point processes have a probability distribution, the χ^2 test of fit accepts the hypothesis that the histogram of the interval lengths is a good fit to the theoretical distribution.

TABLE A-7

Example	Mean	Variance	Sample Mean	Sample Variance	Test of fit χ^2	.95 lim.
P1	6.00	36.0	6.00	29.35	16.5	23.7
P2	4.93					
	over (0.600]	24.3	4.00	13.86	18.7	21.0
	3.37					
	over (600,1200]	11.4				
P3	4.81	23.1	4.79	22.69	13.6	22.4
G1	1.50	0.25	1.48	0.22	8.8	15.5
G2	6.00	4.0	5.93	3.46	9.3	22.4
R1	1.80	0.86	1.74	0.84	16.2	26.3
R2	2.80	2.15	2.93	2.34	12.6	19.7

SUPERPOSITION OF EVENTS

Consider the point process constructed by pooling three periodic point processes with periods 1.40, 3.63 and 7.81. The graph of x_i versus i is bounded at $x_i = 1.40$ and the histogram of interval lengths reveals a bound at one end, maximum of the x_i (Fig. A-11). The series of events with period B = 1.40 can be deleted from the pooled series. Then the graph of x_i versus i is bounded above by $x_i = 3.63$ and this series of events with period B = 3.63 can be deleted from the pooled series. The constituent periods can be determined and each point can be assigned to its proper source,

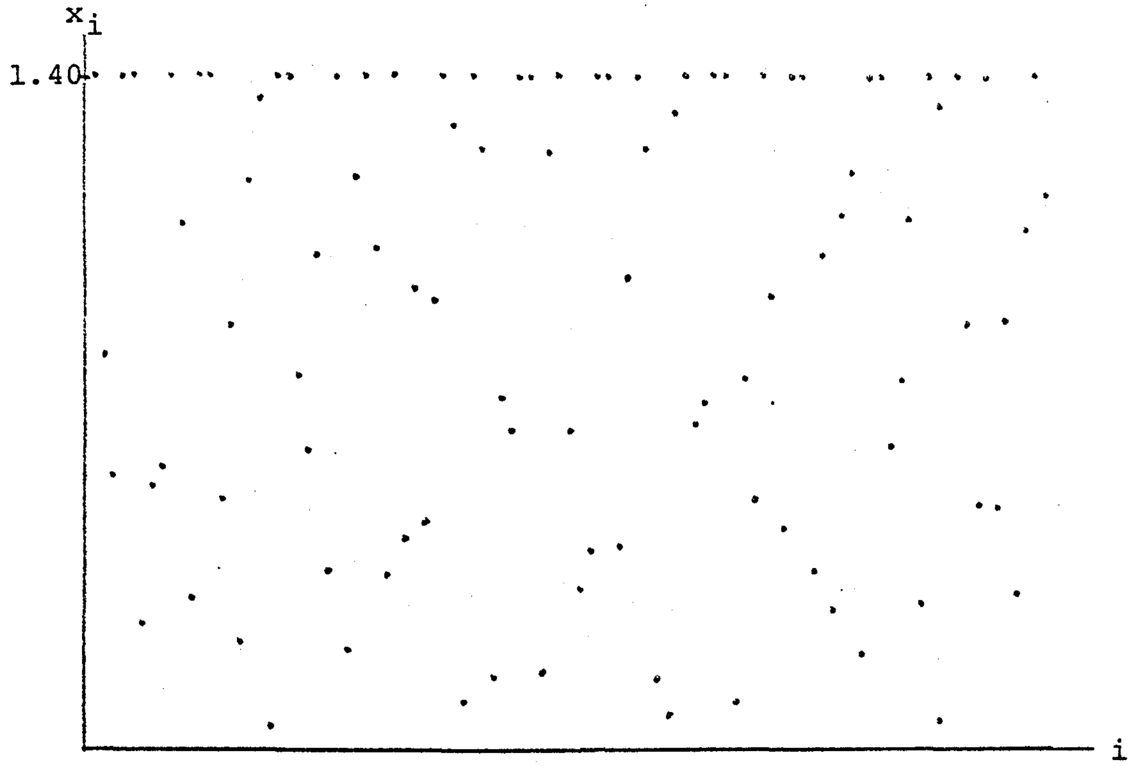

Fig. A-11(a) Interval lengths versus index for a pooled series.

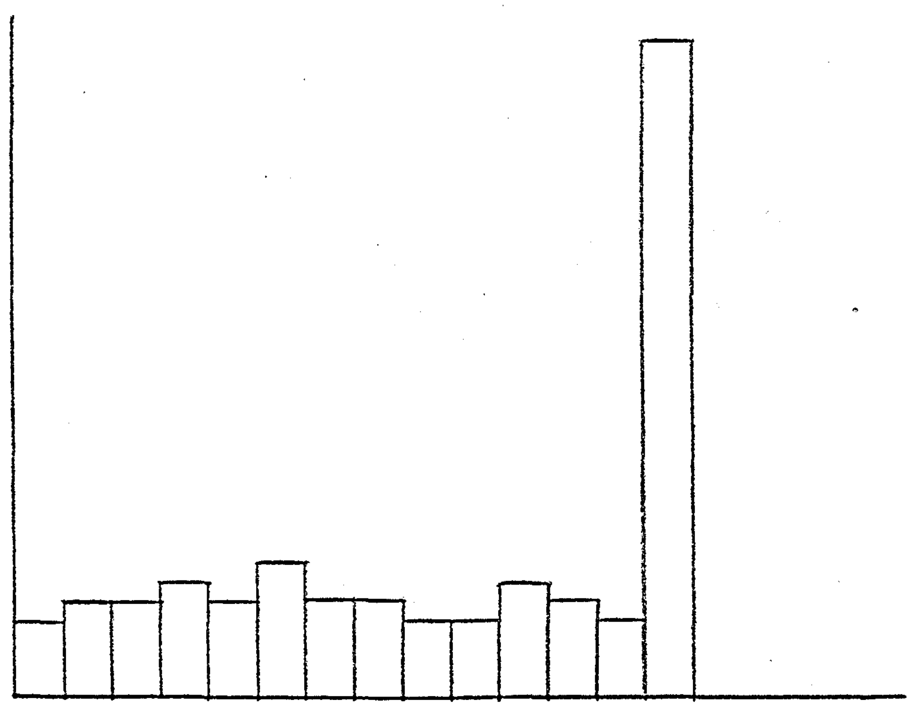

Fig. A-11(b) Histogram of interval lengths for the pooled series.

but only if the periods are not close together.

Series S1 and S2 are not bounded by a particular value of x_i, so that the periods can not be found in the above manner. However, the number of periodic sources can be found by employing the variance time curve. S1 oscillates about 0.65 and S2 about 0.68. Since $N/6 = 0.65$, S1 is the pooled output of $N = 4$ periodic sources and S2 is also the pooled output of 4 periodic sources. This is the only information that can be derived from these processes by this method of analysis.

APPENDIX B

SPECTRAL ANALYSIS

INTERVAL LENGTHS

The sequence of interval lengths $\{X_i\}_{i=1}^{n}$ with mean μ and variance $\sigma^2 = R_T(0)$ has the sample autocovariance function

$$R_T(K) = \frac{1}{n-|K|} \sum_{i=1}^{n-|K|} (X_i-\mu)(X_{i+|K|} - \mu) \qquad |K|=0,\ 1\ \ldots,n-1.$$

$R_T(K)$ was calculated using the statistics $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$ as estimates of the mean μ and variance σ^2 respectively. Then the spectral density function

$$f_T(\omega) = \frac{1}{2\pi} R_T(0) + \frac{1}{\pi}\sum_{K=1}^{n-1} \cos K\omega R_T(K)$$

or

$$\frac{f_T(\omega)}{\sigma^2} = \frac{1}{2\pi}\left\{1 + 2\sum_{K=1}^{n-1} \cos K\omega \frac{R_T(K)}{R_T(0)}\right\}$$

was estimated at equidistant intervals of ω for $0 \leq \omega \leq \pi$, so that values of $f_T(\omega_j)$ were recorded for $\omega_j = 2\pi j/n$, $j=0,1,..,n/2$. $R_T(K)$ was estimated for $K=1, 2,..,n/5$ and $K=1, 2,...,n-1$. Using $n-1$ autocovariances resulted in negative values of $f_T(\omega_j)$ for values of ω_j near π, while using only twenty per cent of the autocovariances reduced this variability of $f_T(\omega_j)$. The estimate of $f_T(\omega)/\sigma^2$ versus ω was graphed for several of the examples. (Fig. B-1 to B-6)

Spectral analysis of the interval lengths of the examples revealed very little information about the frequency distribution of the interval lengths. For the superposition of several periodic sources one or more peak frequencies was located. For example Fig. B-5, the superposition of two periodic sources with periods $B_1 = 2.0$, $B_2 = 2.27$, and $\lambda_1 = 0.500$, $\lambda_2 = 0.441$ where $\lambda_i = 1/B_i$ $i=1,2$ had one peak frequency at ω_j where $j=117$, $n=250$. Then $\omega_j/\pi = 0.936$ and $\lambda_1+\lambda_2 = 0.941$. However, no relationship between the peak frequencies and rates of occurrence of the constituent periodic sources could be determined for the pooled output of three and four periodic sources. For example, Fig. B-6 records two peak frequencies and $\omega_j/\pi = \begin{cases} 0.704 & j=88 \quad (n=250). \\ 0.952 & j=119 \end{cases}$

This example is the superposition of three periodic sources with $B_1=1.93$, $B_2=2.89$, $B_3=8.27$ and $\lambda_1+\lambda_2+\lambda_3=0.985$, $\lambda_1+\lambda_2-\lambda_3=0.743$. Example S1 (Fig. B-2) with four periodic sources has only two peak frequencies revealed and

$$\omega_j/\pi = \begin{cases} 0.704 & j=88 \quad (n=250). \\ 0.800 & j=100 \end{cases}$$

However, no linear combination $\alpha_1\lambda_1+\alpha_2\lambda_2+\dots+\alpha_N\lambda_N$, where $\alpha_i = \begin{cases} +1 \\ -1 \end{cases}$ and N is the number of periodic sources, was close to the values of ω_j/π.

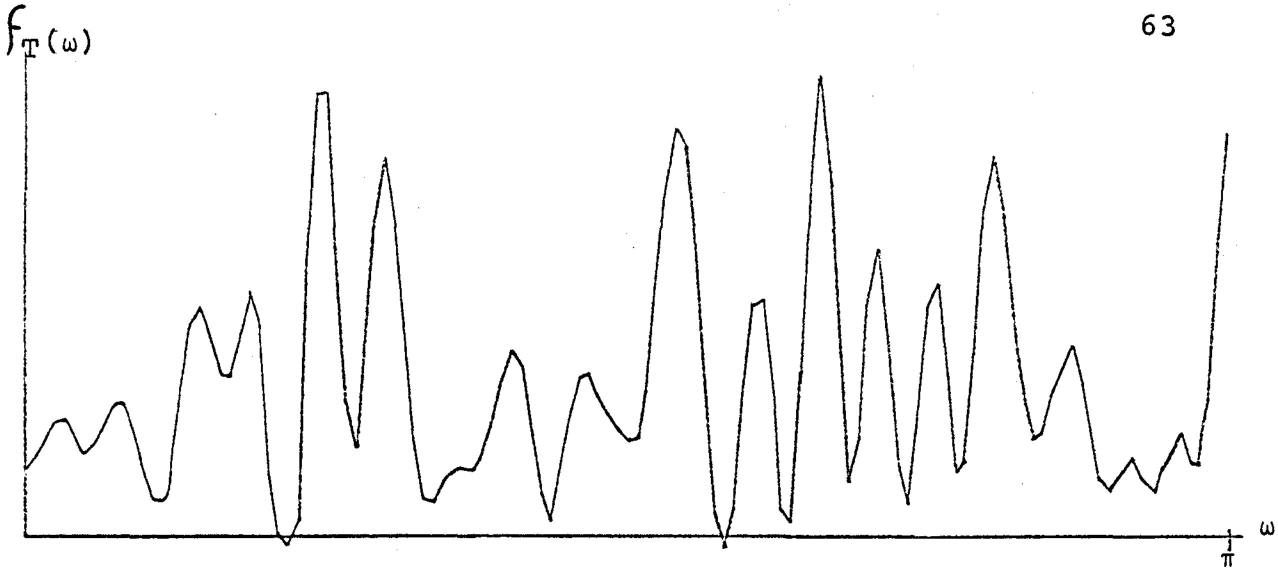

Fig. B-1 Spectral density function of the interval lengths for example P1.

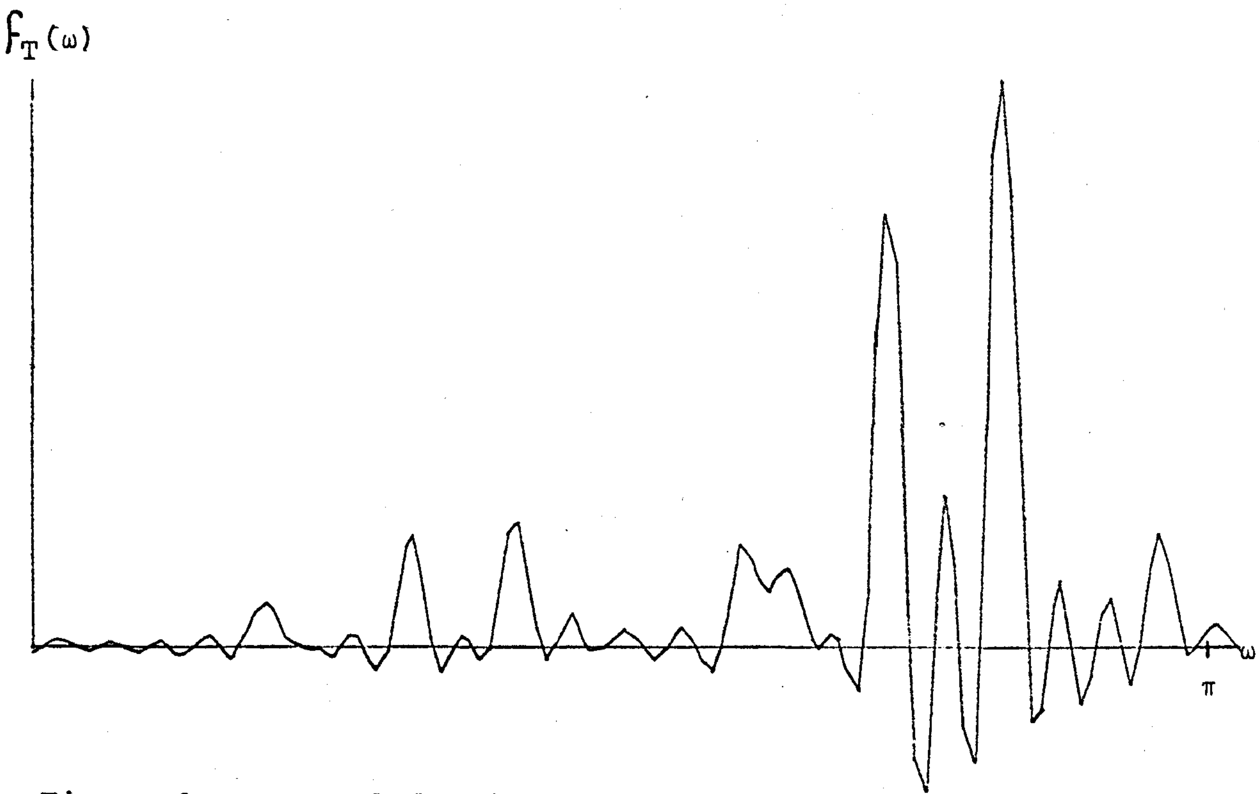

Fig. B-2 Spectral density function of the interval lengths for example S1.

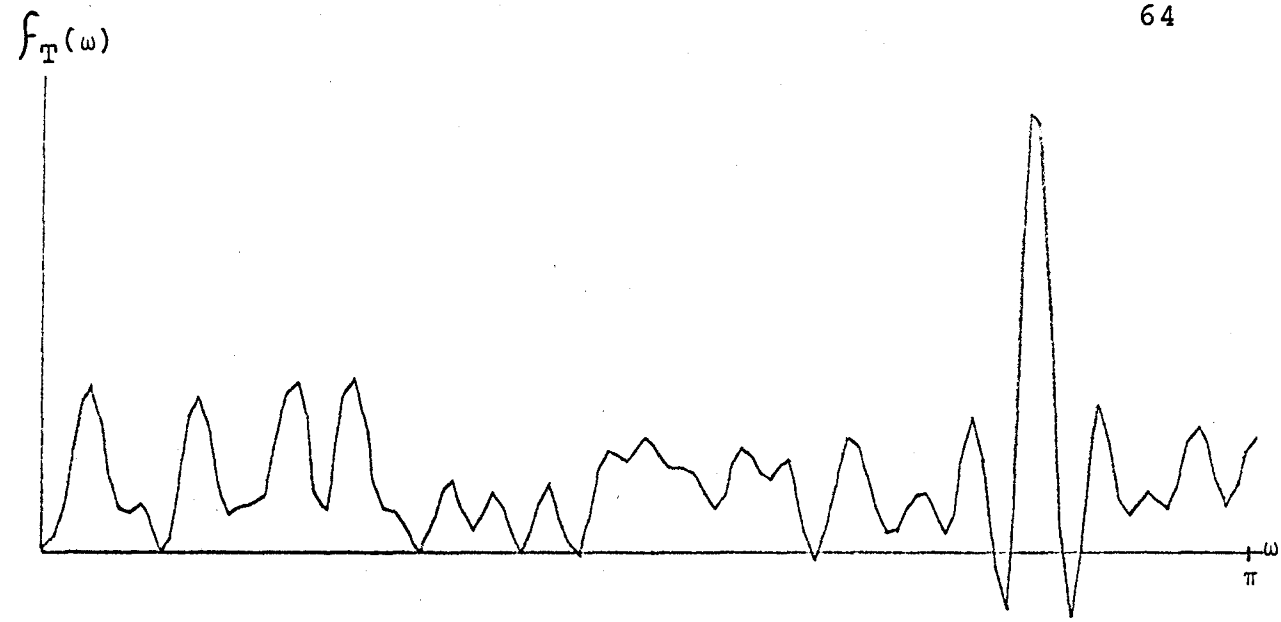

Fig. B-3 Spectral density function of the interval lengths for example G1.

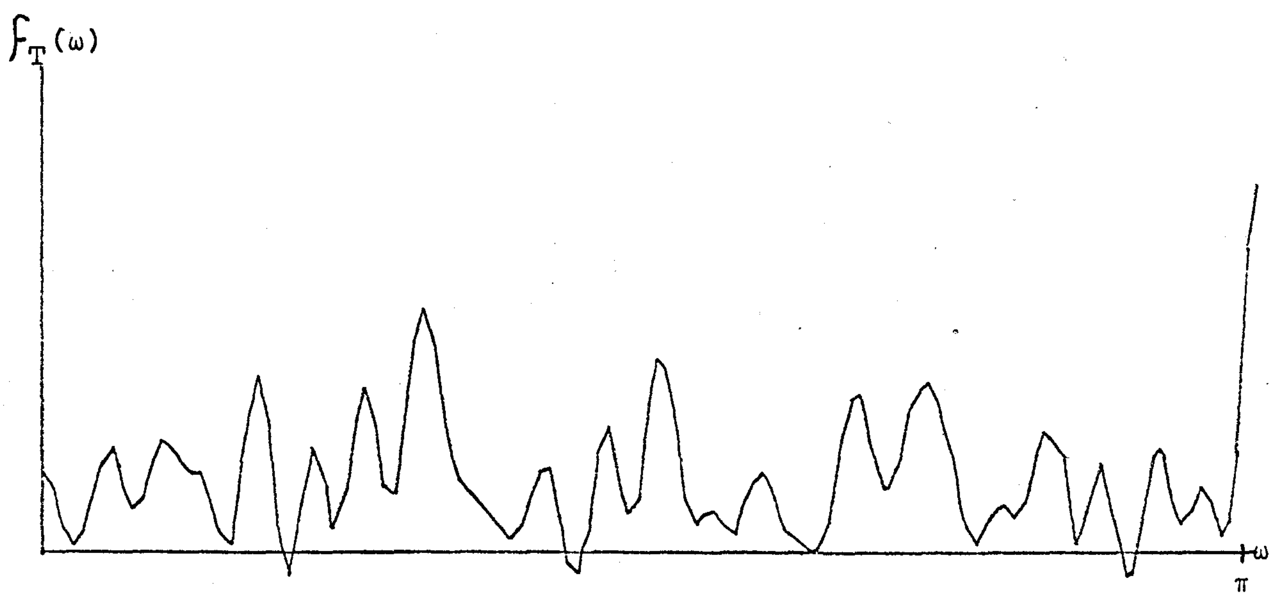

Fig. B-4 Spectral density function of the interval lengths for example R2.

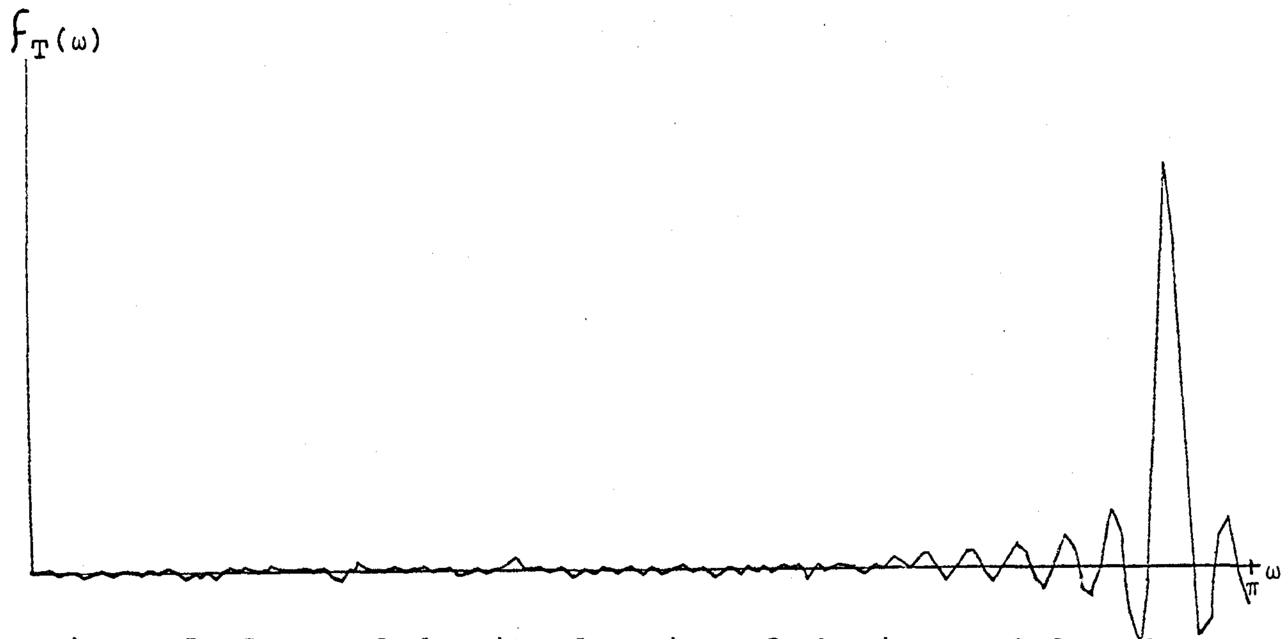

Fig. B-5 Spectral density function of the interval lengths for a pooled series from 2 periodic sources.

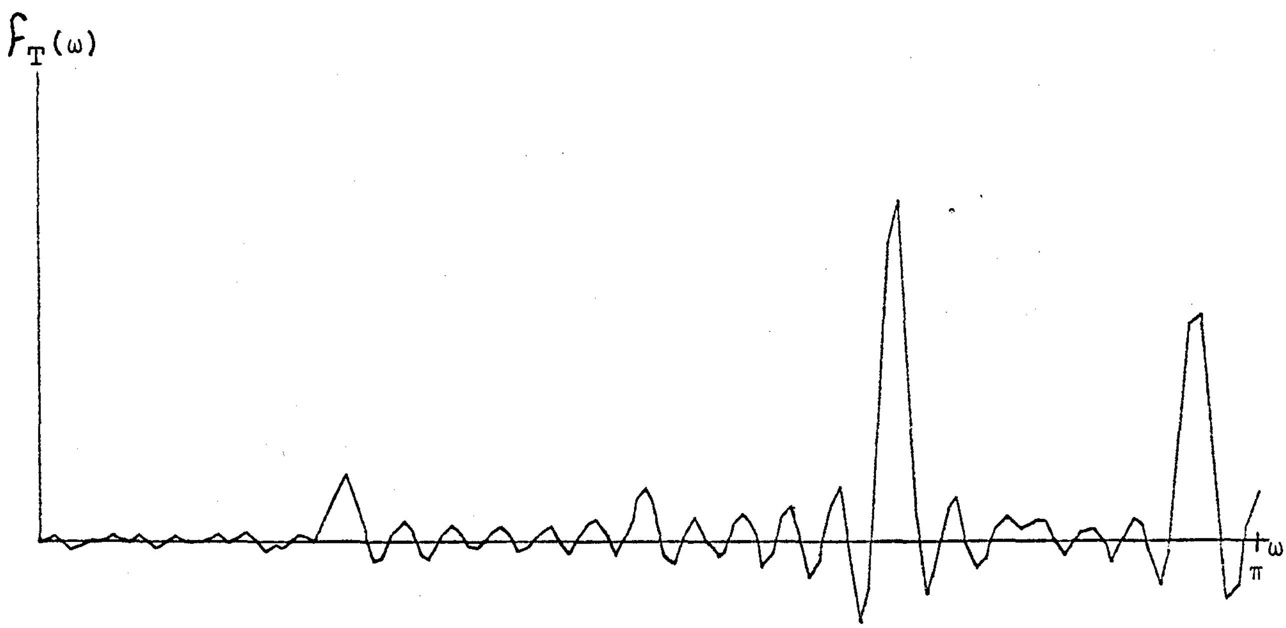

Fig. B-6 Spectral density function of the interval lengths for a pooled series from 3 periodic sources.

OCCURRENCE RATE

The process $\{dN(t)\}$ was obtained by dividing up the point process $X(t)$ into n intervals of equal length τ and counting the number of occurrences $n_i(\tau)$ in the ith interval $i=1, 2, \ldots, n$. Then

$$\left\{\frac{dN(t)}{dt}\right\} = \left\{\frac{n_i(\tau)}{\tau}\right\}_{i=1}^{n}$$

$$\mu = \frac{\sum_{i=1}^{n} n_i(\tau)}{n\tau}$$

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{n_i(\tau)}{\tau} - \mu\right)^2 = R_T(0)$$

$$R_T(K) = \frac{1}{n-|K|}\sum_{i=1}^{n-|K|}\left(\frac{n_i(\tau)}{\tau} - \mu\right)\left(\frac{n_{i+|K|}(\tau)}{\tau} - \mu\right)$$

and $$\frac{f_T(\omega_j)}{\sigma^2} = \frac{1}{2\pi}\left\{1+2\sum_{K=1}^{n-1}\cos\omega_j K\frac{R_T(K)}{\sigma^2}\right\} \qquad \omega_j = 2\pi j/n \quad j = 0, 1, \ldots, n/2$$

were calculated. The estimate of $f_T(\omega)/\sigma^2$ versus ω was graphed for several of the examples (Fig. B-7 to B-9). Very little information was obtained by this method of analysis other than the observation that the spectrum of the occurrence rate resembled the spectrum of the interval lengths for the superposition of several periodic sources.

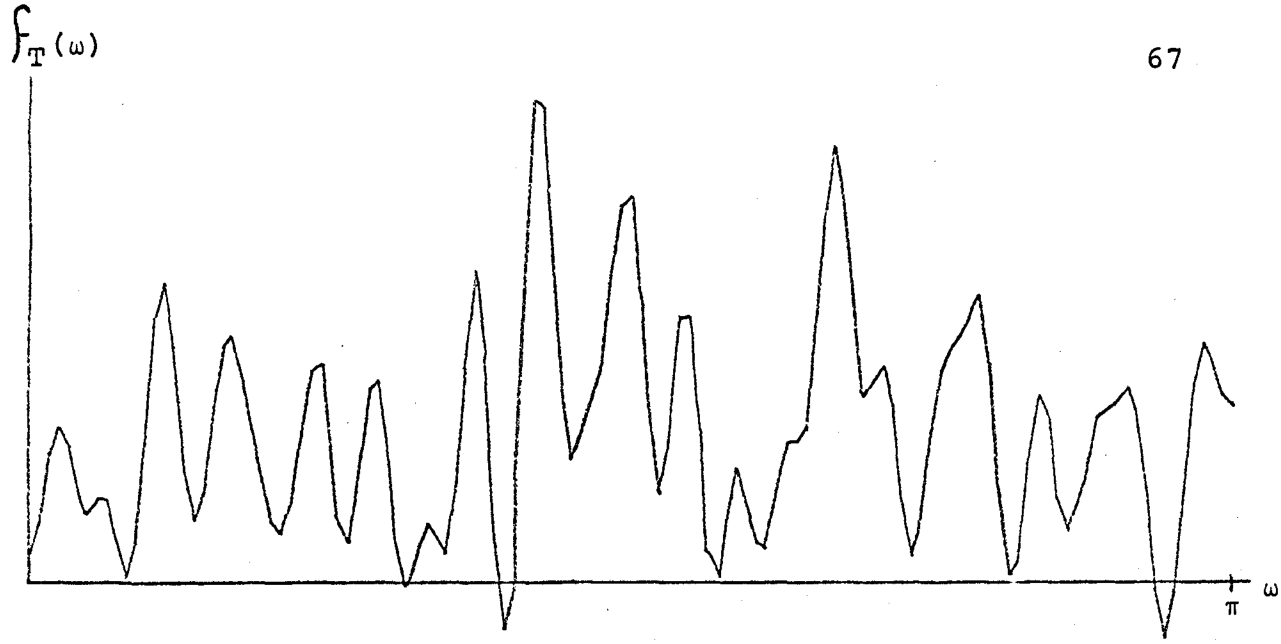

Fig. B-7 Spectral density function of the occurrence rate for example P1.

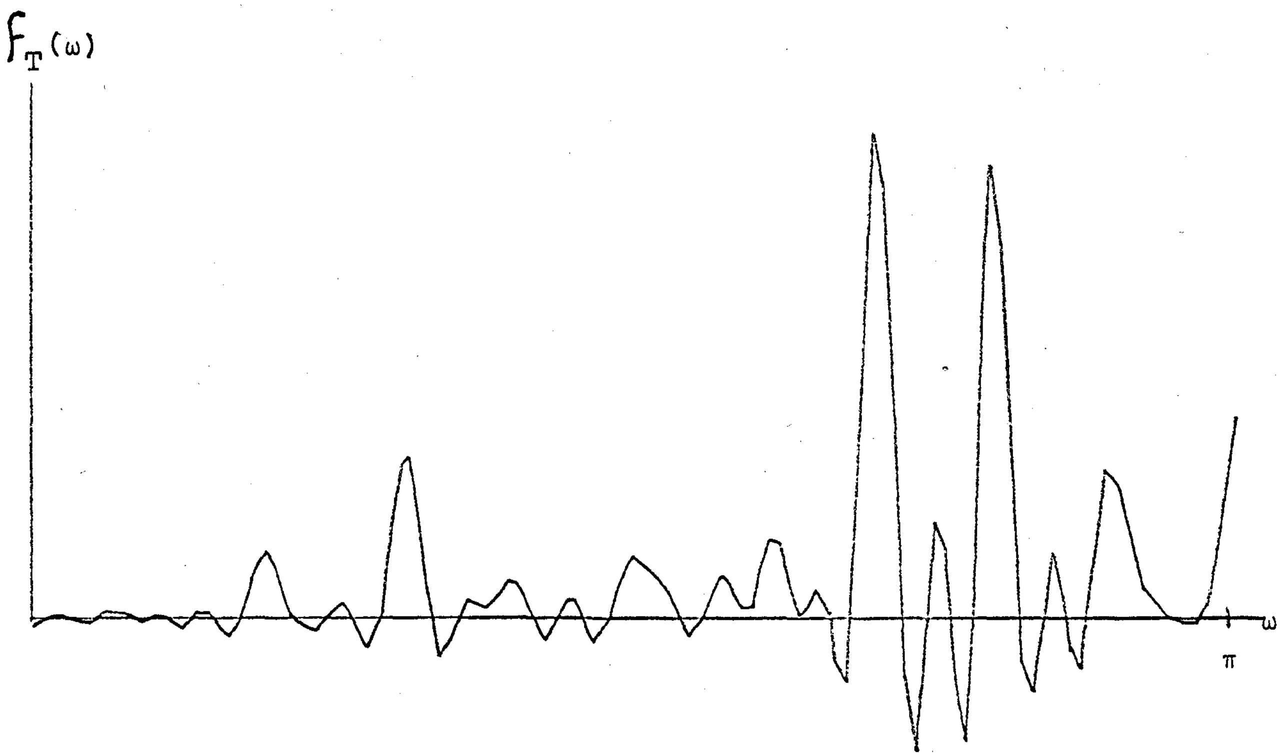

Fig. B-8 Spectral density function of the occurrence rate for example S1.

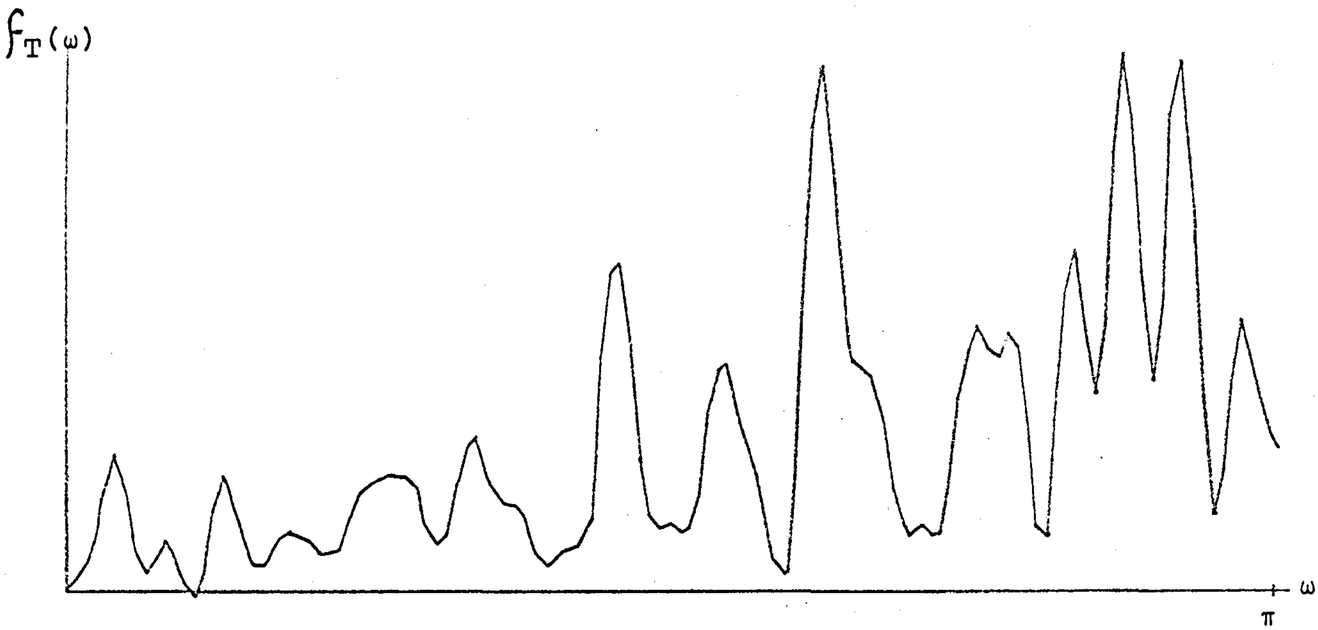

Fig. B-9 Spectral density function of the occurrence rate for example G1.

SQUARE WAVE

The spectral density function

$$\frac{f_T(\omega_j)}{\sigma^2} = \frac{1}{2\pi}\left\{1+2\sum_K \frac{R_T(K)\cos K\omega}{\sigma^2}\right\} \qquad \omega_j = 2\pi j/n, \quad j=0,1,2,\ldots n/2$$

was calculated using n=250 values of $R_T(K)$ at equidistant intervals of length $\Delta K \simeq T/5n$. This was done for square wave values a=1, b=-1 and a=1, b=0 for all examples. There was no significant difference in the spectral density estimates obtained except about very low frequencies. This was attributed to the mean μ which dominates the contribution to $f(\omega)$ at zero frequency as well as frequencies close to zero. The effect is reduced by subtracting the sample mean from the autocovariances but some effect still persists. When a=1, b=-1 the sample mean tends to be closer to zero than when a=1 and b=0. For this reason only, the results of spectral analysis are given for a=1, b=-1.

$\frac{f_T(\omega_j)}{\sigma^2}$ was calculated using the first 16 autocovariances only, (Fig. B-10 to B-16)

Spectral analysis of the square wave transformation appears to yield more information about the point process and the frequency distribution of its interval lengths than the other two methods of spectral analysis. Some of the experimental results can be compared with the theoretical results. The autocovariance function R(K) can be written as

$$R(K) = \lim_{M\to\infty} \frac{1}{2\pi} \int_{-M}^{M} T(x)T(x+K)\,dx$$

where T(x) is the square wave. It has been shown by F. Brooks and N. Diamantides that

$$R(K) = \frac{1}{2} - \frac{1}{2u} L^{-1}\left\{\frac{1-f^*(s)}{s^2(1+f^*(s))}\right\}$$

where $f^*(s)$ denotes the Laplace transform of the density function $f(x)$ of the interval lengths, u denotes the mean of $f(x)$, and L^{-1} denotes the inverse Laplace transform.

SUPERPOSITION OF PERIODIC SERIES OF EVENTS

For the examples of the superposition of periodic sources, peaks were observed in the square wave spectral

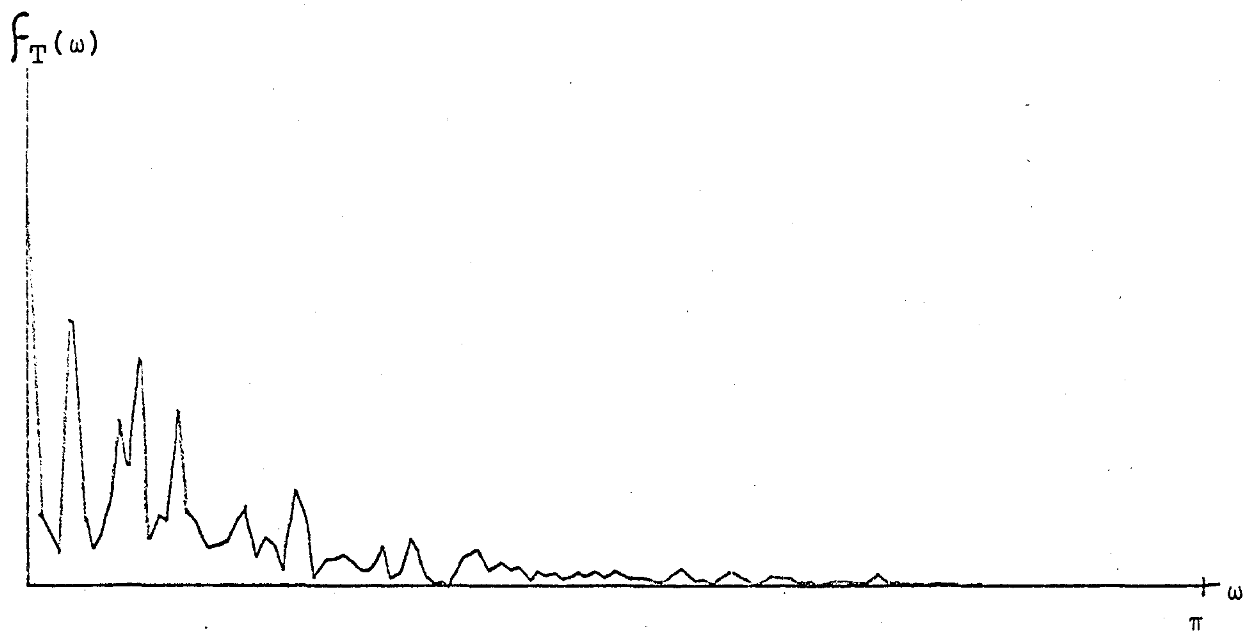

Fig. B-10(a) Spectral density function of the square wave for example P1.

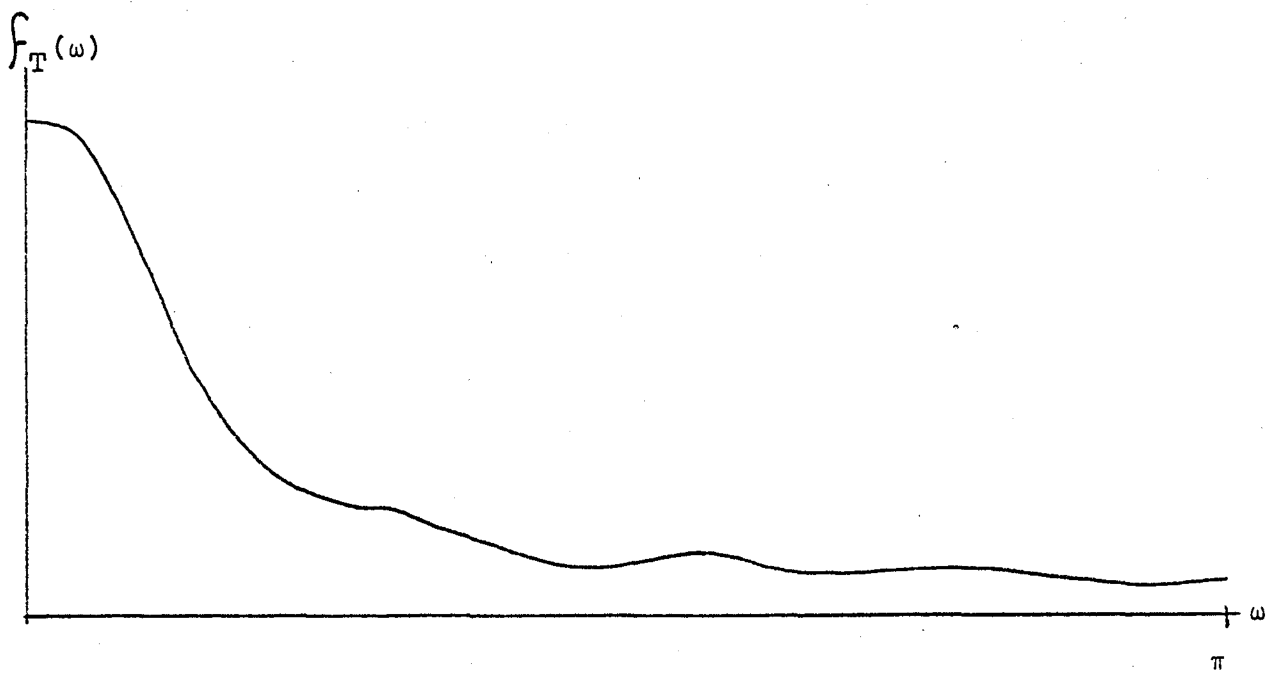

Fig. B-10(b) Spectral density function of the square wave for example P1 (16 autocovariances).

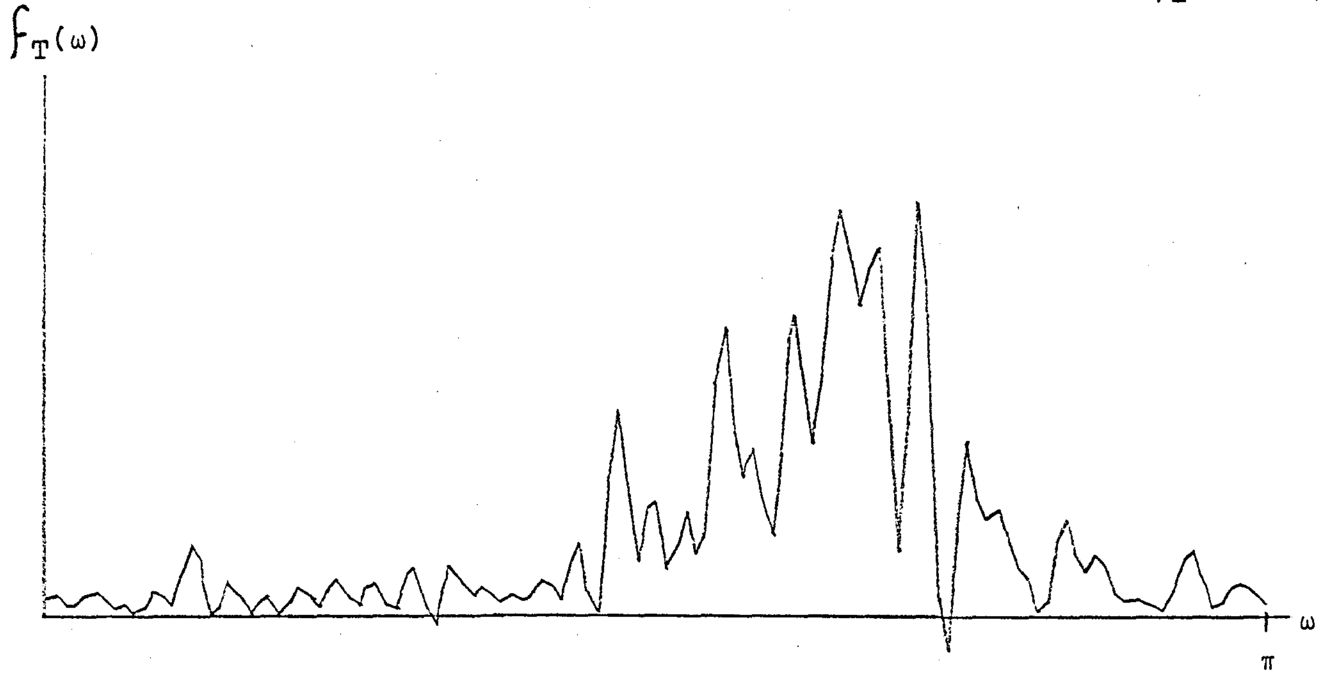

Fig. B-11(a) Spectral density function of the square wave for example G1.

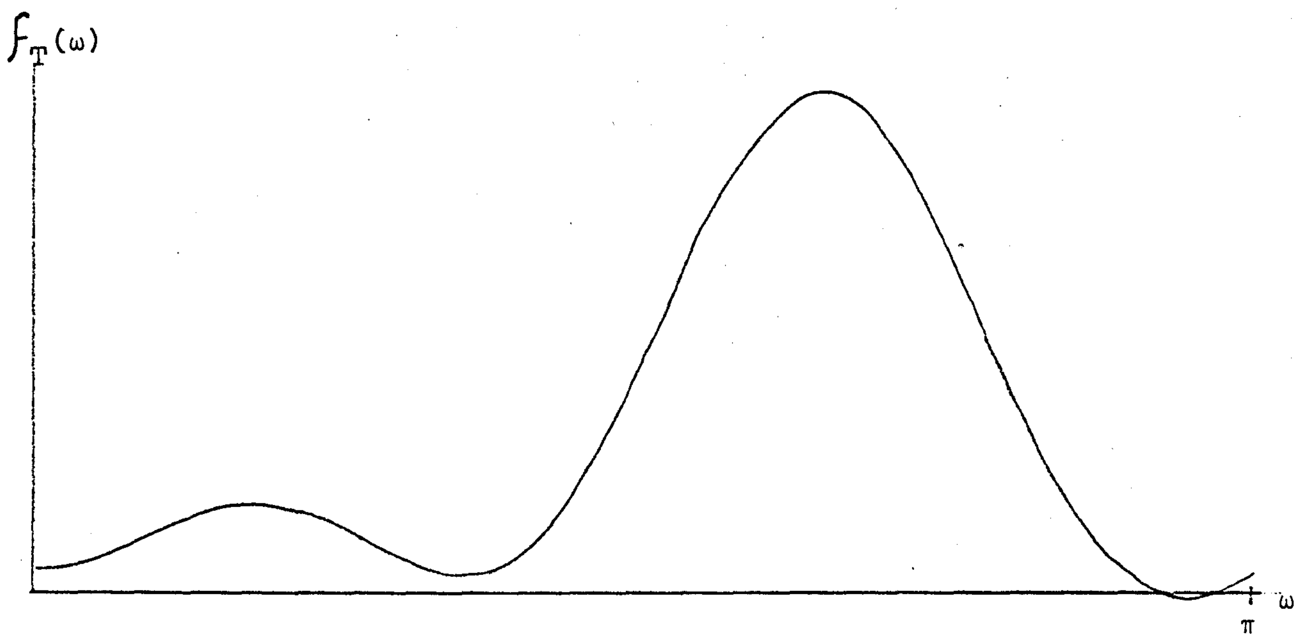

Fig. B-11(b) Spectral density function of the square wave for example G1 (16 autocovariances).

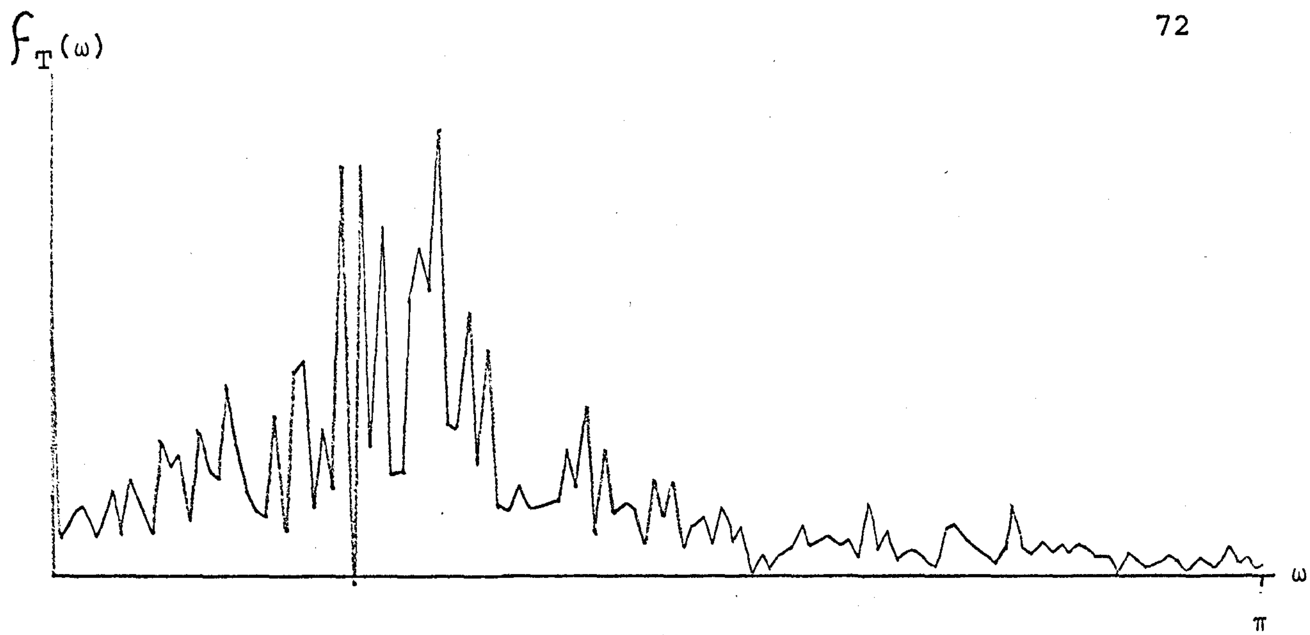

Fig. B-12(a) Spectral density function of the square wave for example R2.

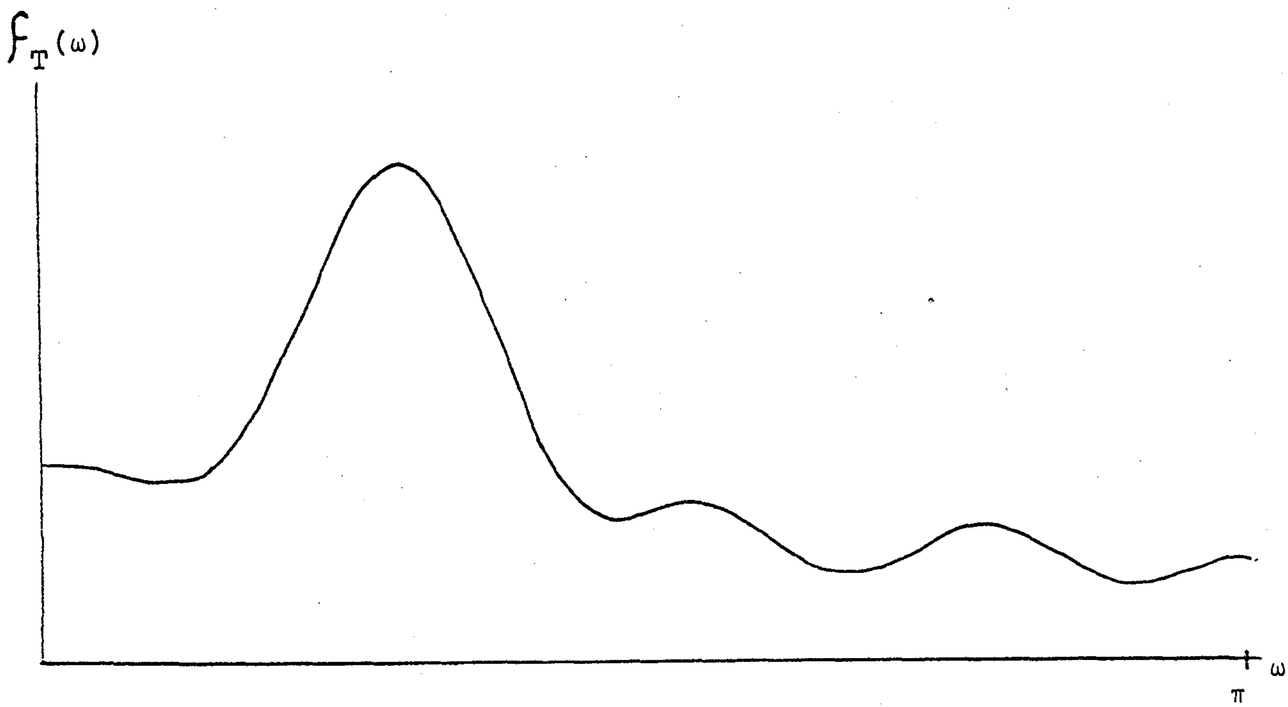

Fig. B-12(b) Spectral density function of the square wave for example R2 (16 autocovariances).

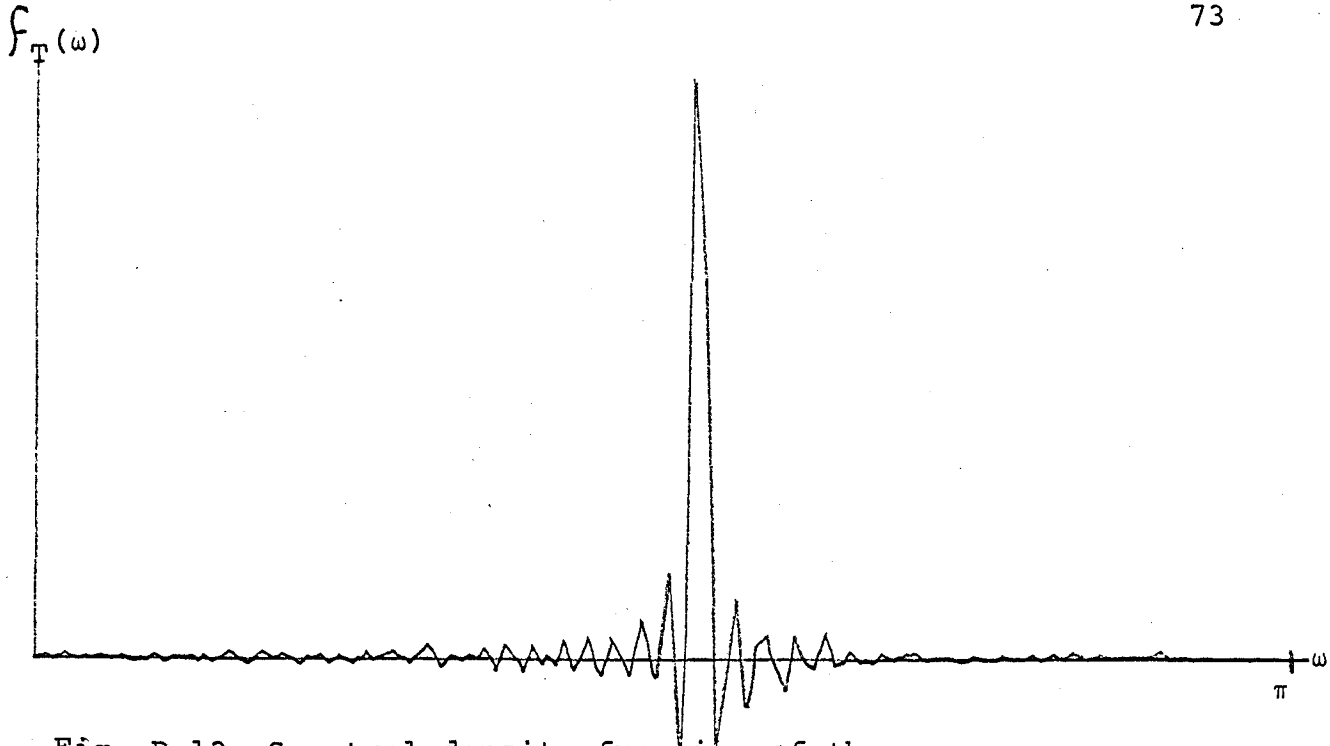

Fig. B-13 Spectral density function of the square wave for a periodic process.

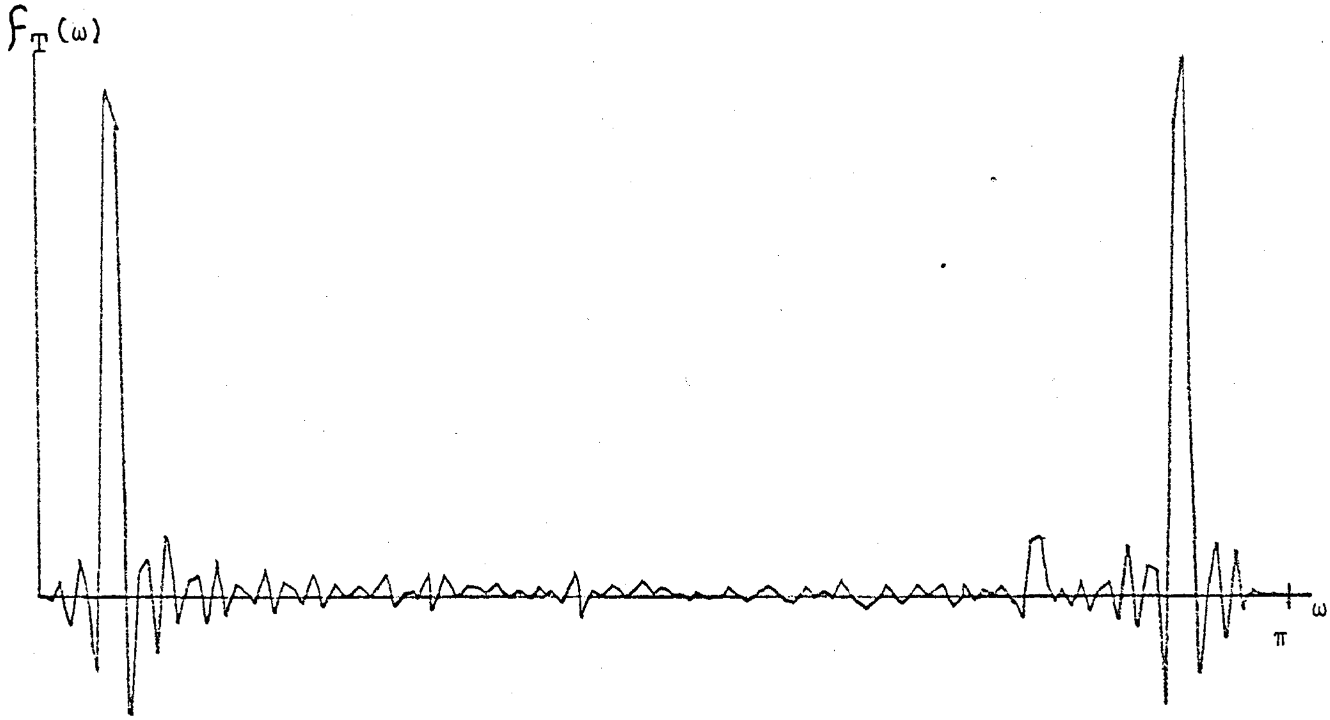

Fig. B-14 Spectral density function of the square wave for the pooled series from 2 periodic sources.

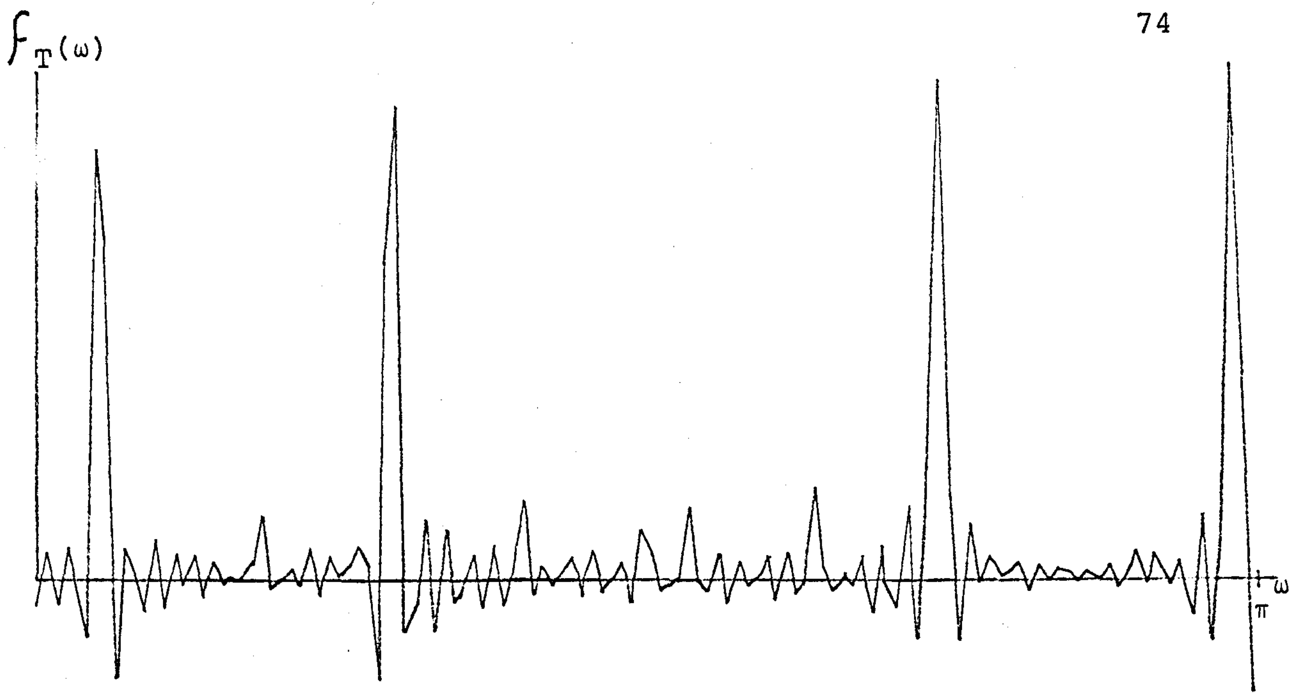

Fig. B-15 Spectral density function of the square wave for a pooled series from 3 periodic sources.

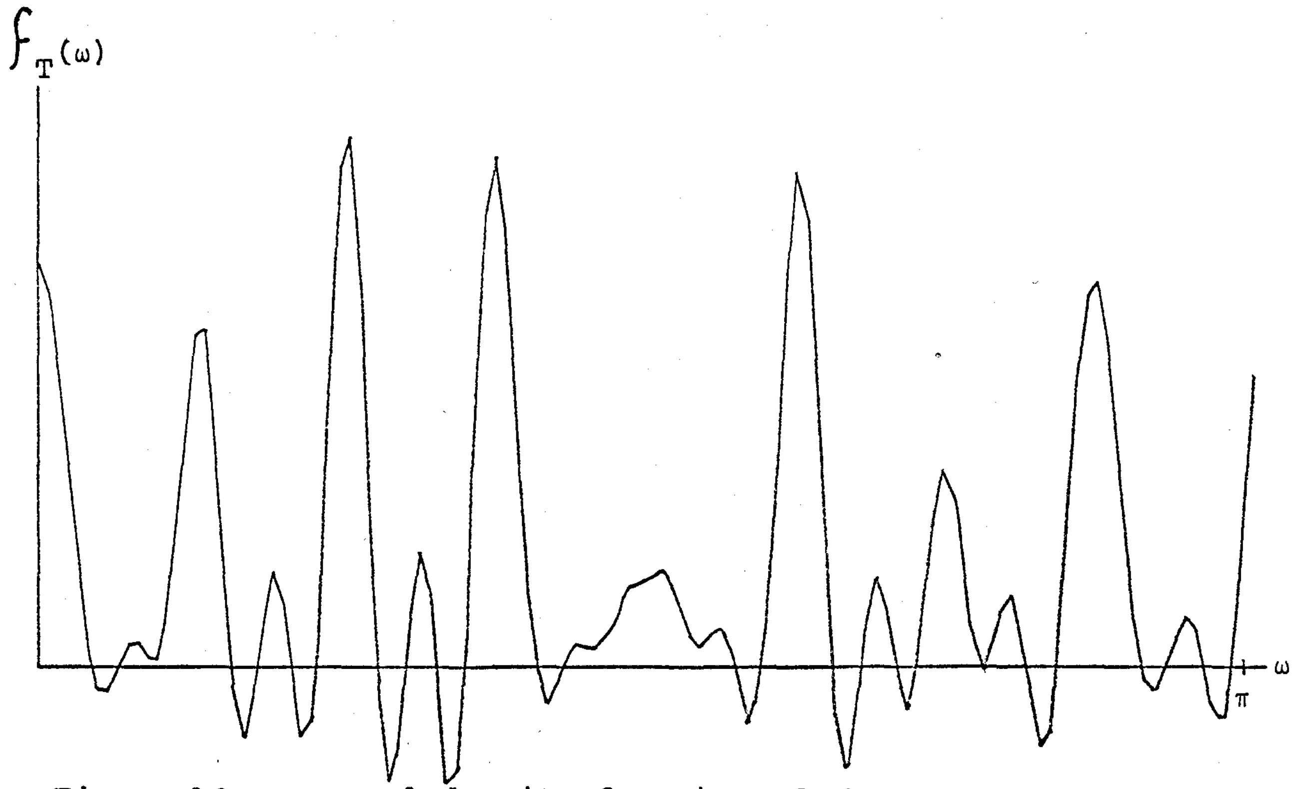

Fig. B-16 Spectral density function of the square wave for example S1.

estimates as was the case with interval and occurrence rate analysis, but more peaks were obtained. In order to discover the relationship (if any) of these peaks to the constituent periods, the superposition process was studied for examples with one, two, three and then four periodic sources.

For a single periodic source with period B and rate of occurrence $\lambda = 1/B$, the square wave can be considered as an approximation of the sine curve $Z(t) = \sin\omega t$ where $\omega = \pi/\Delta t$ radians per unit time. Then the occurrences at the time instants B, 2B,....are the zeroes of $Z(t) = \sin\omega t, t>0$ where $\omega = \pi/\Delta t = \pi/B = \pi\lambda$. (Fig. B-17)

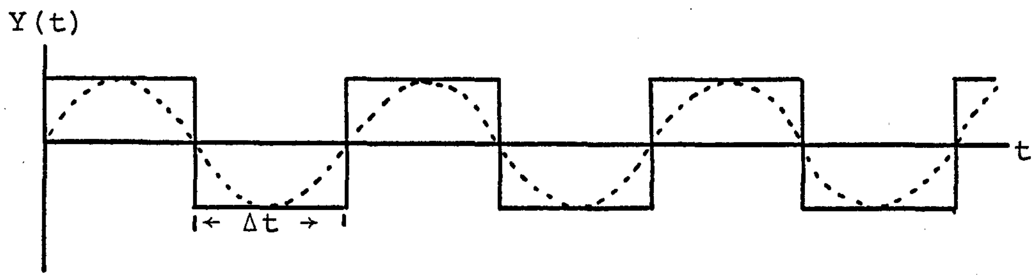

Fig. B-17 Square wave approximation of a sine wave.

For a single periodic source of period B the spectrum $f(\omega)$ consists of a single peak at $\omega = \pi/B$, and information about the rate of occurrence and period is directly available. Fig. B-13 is the spectrum of the periodic point process with B=1.83, calculated for values of ω where $\omega = \frac{2\pi j}{n}$, $j=0,1,\ldots,n/2$ and $n=250$. The peak occurred at $j=68$ and since $\omega = \pi\lambda$ $\lambda \approx \frac{2.68}{250} = 0.544$. Then $B \simeq 1/0.544 = 1.84$

and the period can accurately be estimated.

For two periodic sources with periods $B_1 = 2.00$ and $B_2 = 2.27$ and mean rates of occurrence $\lambda_1 = 0.500$ and $\lambda_2 = 0.441$, spectral analysis of the pooled output revealed two peaks at ωj where j=7 and j=118. (Fig. B-14)
Let $\hat{\lambda}_1 = \omega_j/\pi$, j=7, and $\hat{\lambda}_2 = \omega_j/\pi$, j=118. Then $\hat{\lambda}_1 = 0.056$
and $\hat{\lambda}_2 = 0.944$
and $\hat{\lambda}_1 \simeq \lambda_1 - \lambda_2 = 0.059$

$\hat{\lambda}_2 \simeq \lambda_1 + \lambda_2 = 0.941$

since $\omega_j = \dfrac{2\pi j}{n}$, j=0, 1,...., n/2, n=250.

Fig. B-15 is an example of three periodic sources pooled together: $B_1 = 1.93$ $\lambda_1 = 0.518$; $B_2 = 2.89$, $\lambda_2 = 0.346$; $B_3 = 8.27$, $\lambda_3 = 0.121$. Spectral analysis of the square wave of the pooled output revealed peaks at frequencies ω_j where j=6, 37, 93 and 123 (n=250). Then $\hat{\lambda}_1 = 0.048$, $\hat{\lambda}_2 = 0.296$, $\hat{\lambda}_3 = 0.744$ and $\hat{\lambda}_4 = 0.984$.

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.985 \simeq \hat{\lambda}_4$$
$$\lambda_1 + \lambda_2 - \lambda_3 = 0.743 \simeq \hat{\lambda}_3$$
$$\lambda_1 - \lambda_2 + \lambda_3 = 0.293 \simeq \hat{\lambda}_2$$
$$\lambda_1 - \lambda_2 - \lambda_3 = 0.051 \simeq \hat{\lambda}_1$$

Since $\omega = \pi\lambda$ and ω can be measured by $\mathcal{F}(\omega)$ only in the range $[0,\pi]$ then λ can be measured for $0 \le \lambda \le 1$. Since values of $\lambda > 1$ are possible, then the original time series X(t) of length T can be converted to time cT, where c is a positive constant. Then each periodic series of period B_i is transformed into a periodic series of period cB_i with rate of occurrence $\lambda_i = 1/cB_i$. For $c>1$, there exists a number c_0 such that $\lambda_i<1$ for all $c>c_0$, so that any rate of occurrence can be decreased in magnitude by lengthening the period of the original time series. Similarly, any rate of occurrence can be increased for $c<1$. Since $\pi\sum_i \lambda_i$ is the largest frequency obtained in spectral analysis of the square wave of the pooled series, then if $\sum_i \lambda_i>1$ this frequency is outside the measurable range $[0,\pi]$. Then, it is necessary to convert the pooled output in time T to a pooled output in time cT. Then each constituent periodic series is converted to time cT, so that c can be taken sufficiently large to ensure that $c^{-1}\pi\sum_i \lambda_i<\pi$. For example, consider the two periodic series with periods $B_1=1.21$, $B_2=1.38$ and mean rates of occurrence $\lambda_1=0.826$ and $\lambda_2=0.725$ respectively. Then $\lambda_1+\lambda_2 = 1.551$ and $\lambda_1-\lambda_2 = 0.101$. Since $\sum_i \lambda_i>1$, then only one peak frequency should be discovered initially. Spectral analysis of the square wave revealed a single peak at ω_j, j=13, so that $\hat{\lambda}_1= 2\pi\left(\frac{13}{250}\right)$ $= 0.104$. However, the variance time curve indicated two sources so that the other peak must be at a near zero frequency or greater than $\omega=\pi$. Converting the original series to time cT where c=2.0 revealed two peaks at ω_j, j=6 and 97. Then $c^{-1}\hat{\lambda}_1 = 0.048$ and $c^{-1}\hat{\lambda}_2 = 0.776$ so that $\hat{\lambda}_1 = 0.096$ and $\hat{\lambda}_2 = 1.552$. Once again $\lambda_1+\lambda_2\simeq\hat{\lambda}_2$

$$\lambda_1-\lambda_2\simeq\hat{\lambda}_1 .$$

Since the frequency ω_j; j=6 is very close to zero, then $c^{-1}\hat{\lambda}_1$ will not give a value of $\hat{\lambda}_1$ that is as accurate as that obtained in time T. To determine λ_1 and λ_2 the equations

$$\lambda_1-\lambda_2 = 0.104$$

$$\lambda_1+\lambda_2 = 1.552$$

should be used. The constituent periods B_1 and B_2 can then be determined.

Spectral analysis of the square wave of example S1 in time T revealed peaks at ω_j where j=17, 32, 47, 78, 109, (Fig. B-16). Peaks were found at j=8, 16, 24, 39, 54, 63, 79 for the series in time cT with c=2.0 (Fig. B-18(a)) and at j=7, 66 for time cT with c=0.25 (Fig. B-18(b)). The corresponding values of $\hat{\lambda} = \omega_j/\pi$ (n=250) were

j	$c^{-1}\hat{\lambda}$	$\hat{\lambda}$
7	0.056	0.014
66	0.528	0.132
32	-	0.256
47	-	0.376
78	-	0.624
109	-	0.872
63	0.504	1.008
79	0.632	1.264

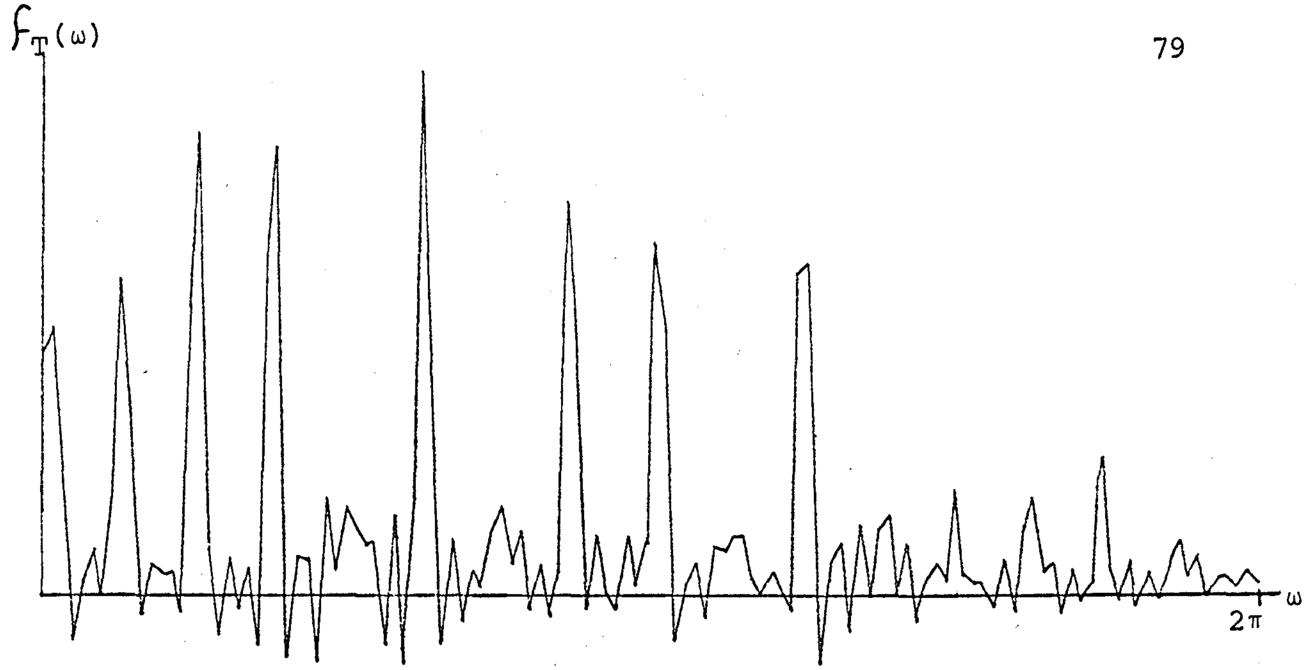

Fig. B-18(a) Spectral density function of the square wave for example S1 in time 2T.

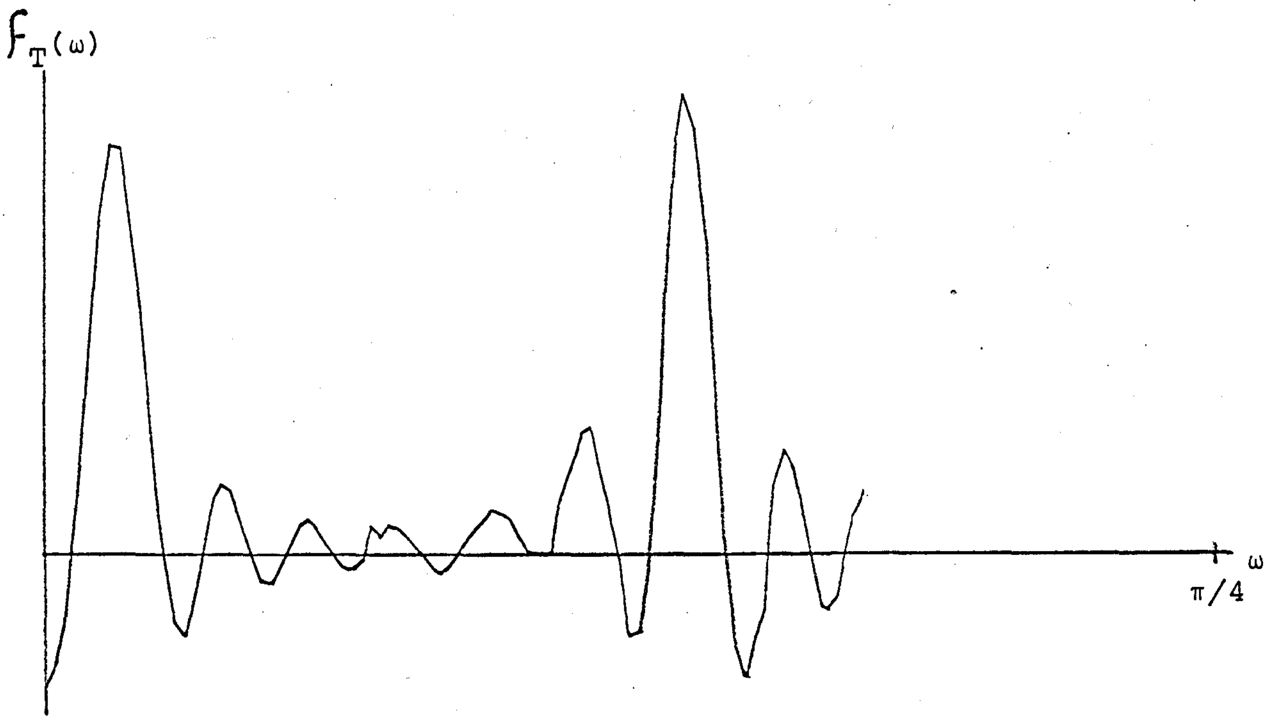

Fig. B-18(b) Spectral density function of the square wave for example S1 in time T/4.

Then

$$\hat{\lambda}_1 = 0.014 \simeq |\lambda_1-\lambda_2-\lambda_3+\lambda_4| = 0.015$$
$$\hat{\lambda}_2 = 0.132 \simeq \lambda_1-\lambda_2+\lambda_3-\lambda_4 = 0.131$$
$$\hat{\lambda}_3 = 0.256 \simeq |\lambda_1-\lambda_2-\lambda_3-\lambda_4| = 0.255$$
$$\hat{\lambda}_4 = 0.376 \simeq \lambda_1-\lambda_2+\lambda_3+\lambda_4 = 0.375$$
$$\hat{\lambda}_5 = 0.624 \simeq \lambda_1+\lambda_2-\lambda_3-\lambda_4 = 0.625$$
$$\hat{\lambda}_6 = 0.872 \simeq \lambda_1+\lambda_2-\lambda_3+\lambda_4 = 0.869$$
$$\hat{\lambda}_7 = 1.008 \simeq \lambda_1+\lambda_2+\lambda_3-\lambda_4 = 1.013$$
$$\hat{\lambda}_8 = 1.264 \simeq \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 1.257$$

It appears that for N periodic sources, N small, with periods $B_1<B_2<\ldots.<B_N$ and rates of occurrence $\lambda_1, \lambda_2, \ldots \lambda_N$ respectively, then spectral analysis of the square wave of the pooled output results in the occurrence of peaks of $f(\omega)$ at frequencies $\omega_1, \omega_2, \ldots.$ If $\hat{\lambda}_i = \omega_i/\pi$ then each $\hat{\lambda}_i$ is a function of $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $\hat{\lambda}_i$ is equal to the absolute value of a linear combination of the λ_j, $j=1, \ldots, N$;

$$\hat{\lambda}_i = |\alpha_1\lambda_1+\alpha_2\lambda_2+\ldots.+\alpha_N\lambda_N|$$

where $\alpha_1 = 1$ and $\alpha_K = \begin{cases} +1 \\ -1 \end{cases}$ $K=2,\ldots,N.$

There are 2^{N-1} possible combinations of $\alpha_1, \ldots, \alpha_N$ which accounts for the occurrence of 2^{N-1} peaks of $f(\omega)$ at $\omega_1, \omega_2, \omega_3, \ldots, \omega_{2^{N-1}}$; each frequency corresponding to a $\hat{\lambda}_i$, $i=1,\ldots,2^{N-1}$.

Using the variance time curve, the number N of periodic sources can be determined and spectral analysis of the square wave could be used to locate the 2^{N-1} peak frequencies. Then a series of equations would have to be solved for by ordering the $\hat{\lambda}_i$, $i=1,\ldots, 2^{N-1}$. With N=4 and 8 peak frequencies, then by ordering and relabelling $\hat{\lambda}_1 \le \hat{\lambda}_2 \le \ldots < \hat{\lambda}_8$ where

$$\hat{\lambda}_8 = \lambda_1+\lambda_2+\lambda_3+\lambda_4$$
$$\hat{\lambda}_7 = \lambda_1+\lambda_2+\lambda_3-\lambda_4$$
$$\hat{\lambda}_6 = \lambda_1+\lambda_2-\lambda_3+\lambda_4$$
$$\hat{\lambda}_5 = \begin{cases} \lambda_1-\lambda_2+\lambda_3+\lambda_4 \\ \text{or } \lambda_1+\lambda_2-\lambda_3-\lambda_4 \quad \text{and} \quad \hat{\lambda}_4 = \lambda_1-\lambda_2+\lambda_3+\lambda_4 \end{cases}$$

This yields two sets of possible equations, one of which is linearly dependent so that another equation has to be substituted for $\hat{\lambda}_5 = \lambda_1+\lambda_2-\lambda_3-\lambda_4$. As N increases the number of possible sets of equations also increases, and the autocovariance function tends to an exponential function.

If the $\hat{\lambda}_i$, $i=1,\ldots 2^{N-1}$ are ordered and relabelled (if necessary) so that $\hat{\lambda}_1 < \hat{\lambda}_2 <, \ldots, < \hat{\lambda}_{2^{N-1}-1} < \hat{\lambda}_{2^{N-1}}$, then

$$\lambda_N = \frac{\hat{\lambda}_{2^{N-1}} - \hat{\lambda}_{2^{N-1}-1}}{2}$$

and $$\lambda_{N-1} = \frac{\hat{\lambda}_{2^{N-1}} - \hat{\lambda}_{2^{N-1}-2}}{2}.$$

All that is necessary is to determine the three largest frequencies and to use these to calculate estimates of λ_N and λ_{N-1}, and then B_N and B_{N-1}. Then by calculating the jth order differences in interval lengths X_{ij} where

$$X_{ij} = X(t_{i+j}) - X(t_i) \qquad i=1,\ 2,\ \ldots,\ n-j \qquad j<n$$
$$j=1,\ 2,\ \ldots..$$

and comparing these with the estimates of B_N and B_{N-1} the exact periods and occurrences corresponding to the time series with these periods can be determined. Considering example S1

$$\lambda_N \simeq \frac{1.264-1.008}{2} = 0.128$$

$$\lambda_{N-1} \simeq \frac{1.264-0.872}{2} = 0.196$$

$$B_N \simeq 1/\lambda_N = 7.81$$

$$B_{N-1} \simeq 1/\lambda_{N-1} = 5.10$$

and the differences X_{ij} were calculated to determine the periods exactly. The 6th order differences $X(t_{i+6})-X(t_i)$ located a period of length 5.15, part of the difference table is given in TABLE B-1.

TABLE B-1

X_{ij} $j=6$	OCCURRENCES
4.85	
5.15	$X(t_{57}) - X(t_{51})$
5.67	
.	
.	
4.00	
5.15	$X(t_{63}) - X(t_{57})$
4.33	
.	
.	
6.00	
5.15	$X(t_{69}) - X(t_{63})$
4.75	

The periodic series with period 5.15 can then be eliminated from the pooled process. Although the estimate of B_4=8.23 is not very good this could be improved. Since $\hat{\lambda}_8$=1.264 the original series could be converted to time cT where c=1.5, yielding a more accurate estimate of $\hat{\lambda}_8$. Even this is not necessary since one periodic source has been eliminated from the pooled series. Spectral analysis of the deleted pooled series will only locate 2^{N-2} peak frequencies now, and the remaining periods can be found by working with successive deleted series.

It appears that a model to describe the pooled output of several periodic processes would be helpful, but as yet no such model has been found in the available journals. A model to describe the special case of two pooled periodic processes is now given. Future analysis in this direction might prove rewarding.

Let $X_1(t)$ be a periodic point process with period B_1 and $X_2(t)$ a periodic point process with period B_2, where $B_1<B_2$ and B_1, B_2 are mutually irrational. Let X(t) be the point process obtained by pooling $X_1(t)$ and $X_2(t)$. In the case of a pooled output of two periodic point processes, it has been observed that there is a pattern to the occurrence of the interval length equal to the smallest period B_1.

For example consider $X_1(t)$, $X_2(t)$ defined at t_1, t_2,.. such that

$$X_1(t_j) = j\cdot B_1$$

$$X_2(t_j) = j\cdot B_2 \qquad \text{for } j=1,\ 2,.....$$

When $B_1=2.00$, $B_2=2.27$ and $X_i=X(t_{i+1})-X(t_i)$ $i=1, 2, \ldots.$

$X_{15}= 2.00$

$X_{30}= 2.00$

$X_{47}= 2.00$

$X_{62}= 2.00$

$X_{79}= 2.00$

$X_{94}= 2.00$

$X_{109}= 2.00$

The first occurrence of $X_i=B_1$ is X_{15}, the next X_{30} etc. so that if n_1 and n_2 are the number of occurrences of $X_1(t)$ and $X_2(t)$ between occurrences of $X_i=B_1$ then the following is observed

OCCURRENCE OF B_1	INTERVAL	n_1	n_2
1	X_{15}	8	7
2	X_{30}	8	7
3	X_{47}	9	8
4	X_{62}	8	7
5	X_{79}	9	8
6	X_{94}	8	7
7	X_{109}	8	7
8	X_{126}	9	8
9	X_{141}	8	7
10	X_{158}	9	8

$$X_{15} = X_{n_1+n_2}$$

$$X_{30} = X_{2(n_1+n_2)}$$

$$X_{47} = X_{3(n_1+n_2)+2}$$

$$X_{62} = X_{4(n_1+n_2)+2}$$

$$X_{79} = X_{5(n_1+n_2)+4}$$

The Kth occurrence of B_1 is the interval length $X_{K(n_1+n_2)+m}$ where m is a non-negative integer. It appears that some form of periodicity exists which determines the index $i=K(n_1+n_2)+m$ of the Kth interval length with $X_i=B_1$

$X_1(t_j) = j \cdot B_1$ is the time of the jth occurrence of B_1 and

$X_2(t_j) = j \cdot B_2$ is the time of the jth occurrence of B_2.

Consider $\bar{X}_i = X_2(t_i) - X_1(t_i)$

Then $\bar{X}_i = i \cdot B_2 - i \cdot B_1 = i \cdot (B_2 - B_1)$ so that $\{\bar{X}_i\}$ is an increasing sequence of interval lengths, which increases by a factor of B_2-B_1.

When $\bar{X}_i > B_1$ then $\quad i(B_2-B_1) > B_1$

$$iB_2 > (i+1)B_1$$

$$X_2(t_i) > X_1(t_{i+1})$$

Since $B_1 < B_2$, then $\quad X_1(t_i) < X_2(t_i)$

For $\bar{X}_i > B_1$, $\quad X_1(t_i) < X_1(t_{i+1}) < X_2(t_i)$

and for $\bar{X}_i \not> B_1$, $\quad X_2(t_{i-1}) < X_1(t_i)$.

Then the first occurrence of an interval length equal to B_1 occurs when $\bar{X}_i > B_1$,

$$i(B_2-B_1) > B_1$$

$$i > B_1/(B_2-B_1).$$

Let $\alpha_2 = B_1/(B_2-B_1)$. Then $i_1^{(2)} = [\alpha_2]$ is the smallest integer which denotes the number of interval lengths from $X_2(t)$ before the first interval of length B_1. The number of interval lengths from $X_1(t)$ for an interval of length B_1 will be the largest integer j satisfying $j \cdot B_1 < i \cdot B_2$.

Then

$$j \cdot B_1 < \left(\frac{B_1}{B_2-B_1}\right) \cdot B_1$$

$$j < \left(\frac{B_1}{B_2-B_1}\right) \cdot \frac{B_2}{B_1}$$

$$j < B_2/(B_2-B_1)$$

Let $\alpha_1 = B_2/(B_2-B_1)$. Then $i_1^{(1)} = [\alpha_1]$ is the integer which denotes the number of interval lengths from $X_1(t)$ to get the first interval of length B_1.

Then $i_1^{(1)} + i_1^{(2)} = [\alpha_1] + [\alpha_2]$ is the number of interval lengths necessary to obtain the first interval length $X_i = B_1$, $i = i_1^{(1)} + i_1^{(2)}$.

Since $\alpha_1 = B_2/(B_2-B_1) = 1 + B_1/B_2 + B_1^2/B_2(B_2-B_1)$ and

$\alpha_2 = B_1/B_2 + B_1^2/B_2(B_2-B_1)$ then

$\alpha_1 = \alpha_2 + 1$ and $[\alpha_1] = [\alpha_2+1] = [\alpha_2] + 1$.

The first occurrence of $X_i = B_1$ occurs when $\bar{X}_i > B_1$ and $\bar{X}_i < 2B_1$. If $\bar{X}_i > 2B_1$ then $X_2(t_i) > X_1(t_{i+2})$ and the interval of length B_1 has occurred at least twice.

$$i(B_2-B_1) > 2B_1$$

$$i > \frac{2B_1}{B_2-B_1} = 2\alpha_1$$

so that $i_2^{(2)} = [2\alpha_2]$ is the number of interval lengths from the process $X_2(t)$ before the second occurrence of $X_i=B_1$. In general then the Kth occurrence of an interval of length B_1 will occur only when $\bar{X}_i > KB_1$ and $i > \frac{KB_1}{B_2-B_1} = K\alpha_2$.

Then $i_K^{(2)} = [K\alpha_2]$ denotes the number of occurrences of interval lengths from the process $X_2(t)$ before the Kth occurrence of $X_i = B_1$ and $i_K^{(1)} = [K\alpha_1]$ the number of interval lengths for $X_i = B_1$. The occurrence of the Kth interval length B_1 is $X_{i_K^{(1)}+i_K^{(2)}} = X_{[K\alpha_1]+[K\alpha_2]}$.

Since $[K\alpha_1] + [K\alpha_2] \neq [K(\alpha_1+\alpha_2)]$ in general then the occurrence of the interval of length B_1 is not periodic with period $\alpha_1+\alpha_2 = (B_1+B_2)/(B_2-B_1)$. However α_1 and α_2 can be written as $\alpha_1 = n_1+s$ and $\alpha_2 = n_2+s$ where n_1, n_2 are non-negative integers and $0 \le s < 1$. Then

$[K\alpha_1] + [K\alpha_2] = [K(n_1+s)] + [K(n_2+s)] = K(n_1+n_2) + 2[Ks]$

$[K(\alpha_1+\alpha_2)] = [K(n_1+n_2+2s)] = K(n_1+n_2) + [2Ks]$.

Writing Ks as $Ks = n+r$ where n is a non-negative integer and $0 \le r < 1$ then $2[Ks] = 2n$ and $[2Ks] = [2n+2r] = 2n+[2r]$.

$$0<2r<2 \quad \text{so that} \quad [2r] = \begin{cases} 0 & 0<r<\tfrac{1}{2} \\ 1 & \tfrac{1}{2}\le r<1 \end{cases}$$

Thus $[2Ks] = 2[Ks]$ or $2[Ks]+1$

For all intents and purposes then, we can consider $[K\alpha_1] + [K\alpha_2] \simeq [K(\alpha_1+\alpha_2)]$ so that the occurrence of B_1 is periodic with period $(B_1+B_2)/(B_2-B_1)$.

Assume that the periods B_i, $(i=1,\ldots,N)$ from periodic sources are positive numbers and are mutually irrational in the sense that there exists no set of positive or negative integers n_i, not all zero, such that $\sum_{i=1}^{N} n_i B_i = 0$. The generalized form of Weyl's theorem[3] states that if $\{\alpha\}$ denotes the fractional part of α and if $\alpha_1, \alpha_2, \ldots, \alpha_K$ are irrational numbers themselves mutually irrational, then the sequences $[\{n\alpha_1\}], \ldots,$ $[\{n\alpha_K\}]$ are independently uniformly distributed over (0, 1) for $n=1, 2, \ldots$

For any pooled output of periodic sources the largest interval length possible is equal to the smallest period B_1. If this upper bound B_1 is removed then the sequence of interval lengths is uniformly distributed over the interval $(0, B_1)$.

For the case of the superposition of two periodic sources then the pooled series $X(t)$ can be expressed as

$$X(t) = Y(t)+Z(t)$$

where $Y(t)$ is a periodic point process with period $(B_1+B_2)/(B_2-B_1)$ and $Z(t)$ is a point process with the intervals between events having the uniform distribution over the interval $(0, B_1)$.

[3] Weyl, H. (1916) Math. Ann. 77,313.

APPENDIX C

```
C      TO CALCULATE THE AUTOCOVARIANCES AND SPECTRAL DENSITY FUNCTION
C           OF THE SQUARE WAVE WITH VALUES W1 AND W2
C        TIME IS THE LENGTH OF THE TIME SERIES X(T), AND AUTOCOVARIANCES
C         WILL BE CALCULATED FROM 0 TO HMAX AT INTERVALS OF HK
      TIME=370.43
      HMAX=74.0
      HK=0.30
      W1=1.0
      W2=-1.0
      DIMENSION X(500),A(1000), B(1000), C(1000), D(1000), SDF(500,15),
     1 V(1500),T(600)
C         READ IN THE N INTERVAL LENGTHS X(I).
      N=250
      READ(5,2) (X(I),I=1,N)
    2 FORMAT(F8.3)
      II=0
      DUM=0.
      DO 10 I=1,N,2
   10 DUM=X(I)+DUM
```

```
      SMEAN=(W1*DUM+W2*(TIME-DUM))/TIME
      VAR=(W1*W1*DUM+W2*W2*(TIME-DUM))/TIME-SMEAN*SMEAN
      WRITE(6,15) SMEAN,VAR
   15 FORMAT(1H-,20X,7H MEAN =,F10.5,10X,11H VARIANCE =,F10.5)
      WRITE(6,18)
   18 FORMAT(1H-,32X,3H K=,10X,7H COV(K))
C       CALCULATE M1=KP1 AND M2=KP2
      HH=HK
  320 L2=0
      RK=0.
      DO 20 I=1,N
      RK=X(I)+RK
      IF(HH.GT.HMAX) GO TO 400
      IF(HH.LT.RK) GO TO 30
      IF(L2.EQ.0) GO TO 25
      L2=0
      GO TO 20
   25 L2=1
   20 CONTINUE
   30 KP1=I
      KMB=N-KP1+1
      B(1)=RK-HH
      JJ=KP1+1
      DO 40 J=JJ,N
```

```
      K=J-KP1+1
   40 B(K)=X(J)
      SK=0.
      DO 80 I=1,N
      M=N-I+1
      SK=X(M)+SK
      IF(HH.LE.SK) GO TO 90
   80 CONTINUE
   90 KP2=M
      A(KP2)=SK-HH
      JJ=KP2-1
      DO 100 J=1,JJ
  100 A(J)=X(J)
C         CALCULATE THE AUTOCOVARIANCES AND STORE IN D(II)
      L1=0
      J=1
      K=1
      Y1=0.
      Y2=0.
      Y1Y2=0.
      R=A(J)
      S=B(K)
  200 Z=AMIN1(R,S)
      IF(L1.EQ.L2) GO TO 110
      Y1Y2=Y1Y2+Z
      GO TO 130
```

```
110 IF(L1.EQ.0) GO TO 120
    Y2=Y2+Z
    GO TO 130
120 Y1=Y1+Z
130 R=R-Z
    S=S-Z
    IF(R.EQ.0.) GO TO 140
    K=K+1
    IF(K.GT.KMB) GO TO 300
    S=B(K)
    IF(L2.EQ.0) GO TO 150
    L2=0
    GO TO 200
150 L2=1
    GO TO 200
140 J=J+1
    IF(J.GT.KP2) GO TO 300
    R=A(J)
    IF(L1.EQ.0) GO TO 160
    L1=0
    GO TO 200
160 L1=1
    GO TO 200
300 COV=Y1*W1*W1+Y2*W2*W2+Y1Y2*W1*W2
    COV=COV/(TIME-HH)-SMEAN*SMEAN*(1.0-HH/TIME)
```

```
 310 FORMAT(1H ,20X,2F20.8)
     II=II+1
     C(II)=HH
     D(II)=COV
     HH=HH+HK
     IF(HH.GT.HMAX) GO TO 400
     GO TO 320
C     TO COMPUTE ESTIMATES OF THE SPECTRAL DENSITY FUNCTION
C    IF WEIGHTING IS TO BE USED, TO COMPARE WITH THE FIRST ESTIMATE
C       USING THE WEIGHTS 1 - K/M  WHERE M IS THE NUMBER OF LAGS, K=1,M
C    THEN PUT  MAX=1  .  LAGS FROM 2 TO LMAX WILL BE DONE IN  STEPS
C          OF 2
 400 MAX=1
     LMAX=64
     L=1
     PIE=3.14159265
     G=HMAX/HK
     M=INT(G)
     NN=126
 430 DO 410 J=1,NN
     SUM=0.
     DO 420 I=1,M
     R=(FLOAT(J-1)*C(I)*2.0*PIE)/250.0
     S=COS(R)
     SUM=(D(I)*S)/VAR + SUM
```

```
420 CONTINUE
    SDF(J,L) = (1.0+2.0*SUM)/(2.0*PIE)
410 CONTINUE
    IF(MAX.LT.1) GO TO 440
    IF(L.EQ.1) GO TO 490
    IF(M.GE.LMAX) GO TO 440
490 M=8*L
    L=L+1
    GO TO 430
440 WRITE(6,450)
450 FORMAT(1H-,15X,4H J= ,20X,27H SPECTRAL DENSITY ESTIMATES)
    WRITE(6,460)
460 FORMAT(1H-,30X,13H NO WEIGHTING,5X,15H WITH WEIGHTING)
    DO 470 J=1,NN
    JJ=J-1
470 WRITE(6,480)JJ, (SDF(J,I), I=1,L)
480 FORMAT(1H-,15X,I4,12X,9F10.5)
    STOP
    END
```

```
C        TO COMPUTE THE MEAN  VARIANCE  AND AUTO-COVARIANCES OF A PROCESS
C                                X(T)
C        N IS THE TOTAL NUMBER OF OBSERVATIONS OF X(T)
C       M IS THE NUMBER OF AUTO-COVARIANCES TO BE CALCULATED, M = N-1  IS
C            THE MAXIMUM NUMBER.   COV(N) = VARIANCE IN THIS PROGRAM
      DIMENSION X(1000), COV(1000), SDF(400,20), V(2000), T(500)
      N=250
      M=N/5
C        READ IN THE N INTERVAL LENGTHS OR OCCURRENCE RATES X(I).
      READ(5,2) (X(I),I=1,N)
    2 FORMAT(F20.8)
      SMEAN=0.
      DO 10 J=1,N
   10 SMEAN=SMEAN+X(J)
      SMEAN=SMEAN/FLOAT(N)
      DO 20 K=1,M
      NP=N-K
      DUM=0.
      DO 30 J=1,NP
      I=J+K
   30 DUM = DUM + (X(J)-SMEAN)*(X(I)-SMEAN)
   20 COV(K)=DUM/FLOAT(NP)
      VAR=0.
```

```
      DO 40 J=1,N
      DUM=X(J)-SMEAN
      DUM=DUM*DUM
   40 VAR=DUM+VAR
      COV(N)=VAR/FLOAT(N)
      WRITE(6,50)
   50 FORMAT(1H-,20X,12H THE MEAN IS,20X,16H THE VARIANCE IS,20X,20H THE
     1 COVARIANCES ARE)
      WRITE(6,60) SMEAN,COV(N)
   60 FORMAT(1H-,10X,F20.8,10X,F20.8,30X,3H K=)
      DO 65 K=1,M
   65 WRITE(6,70) K,COV(K)
   70 FORMAT(1H-,90X,I5,3X,F20.8)
C       TO COMPUTE ESTIMATES OF THE SPECTRAL DENSITY FUNCTION
C     IF WEIGHTING IS TO BE USED, TO COMPARE WITH THE FIRST ESTIMATE
C        USING THE WEIGHTS 1 - K/M  WHERE M IS THE NUMBER OF LAGS, K=1,M
C     THEN PUT  MAX=1  .  LAGS FROM 2 TO LMAX WILL BE DONE IN  STEPS
C            OF 2
      MAX=1
      LMAX=16
      L=1
      PIE=3.14159265
      NN=N/2+1
  130 DO 110 J=1,NN
```

```
      SUM=0.
      DO 120 K=1,M
      R=(FLOAT(J-1)*FLOAT(K)*2.0*PIE)/FLOAT(N)
      S=COS(R)
      SUM=(COV(K)*S)/COV(N) +SUM
      IF(L.LT.2) GO TO 120
      SUM=(1.0-FLOAT(K)/FLOAT(M))*SUM
  120 CONTINUE
      SDF(J,L) = (1.0+2.0*SUM)/(2.0*PIE)
  110 CONTINUE
      IF(MAX.LT.1) GO TO 140
      IF(L.EQ.1) GO TO 190
      IF(M.GE.LMAX) GO TO 140
  190 M=2*L
      L=L+1
      GO TO 130
  140 WRITE(6,150)
  150 FORMAT(1H-,15X,4H J= ,20X,27H SPECTRAL DENSITY ESTIMATES)
      WRITE(6,160)
  160 FORMAT(1H-,30X,13H NO WEIGHTING,5X,15H WITH WEIGHTING)
      DO 170 J=1,NN
      JJ=J-1
  170 WRITE(6,180)JJ, (SDF(J,I), I=1,L)
  180 FORMAT(1H-,15X,I4,12X,9F10.5)
```

BIBLIOGRAPHY

[1] Bartlett, M. S. (1963) The Spectral Analysis of Point Processes, J.R.S.S. B 25, 264-295.

[2] Blackman, R. B. and Tukey, J. W. (1958) The Measurement of Power Spectra, Dover Publications Inc., New York.

[3] Brooks, F. and Diamantides, N. (1963) A Probability Theorem For Random Two-Valued Functions with Application to Autocorrelations, Siam Review Vol. 5, 1, 33-40.

[4] Cox, D. R. (1954) On the Superposition of Several Strictly Periodic Sequences of Events, Biometrika 40, 354-360.

[5] Cox, D. R. (1955) Some Statistical Methods Connected with Series of Events, J.R.S.S. B 27, 332-337.

[6] Cox, D. R. and Miller, H. D. (1965) The Theory of Stochastic Processes, J. Wiley and Sons Inc., New York.

[7] Jenkins, G. M. (1961) General Considerations in the Analysis of Spectra, Technometrics 3, 133-166.

[8] Maguire, B. A. (1952) The Time Intervals Between Industrial Accidents, Biometrika 39, 168-180.

[9] McFadden, J. A. (1962) On the Lengths of Intervals in Stationary Point Processes, J.R.S.S. B 25, 413-431.

[10] Parzen, E. (1961) Mathematical Considerations in the Estimation of Spectra, Technometrics 3, 167-190.

[11] Shapiro, H. S. and Silverman, R. A. (1960) Alias Free Sampling of Random Noise, J. Soc. Indust. Appl. Math. 8, 225-248.

[12] Solodovnikov, V. V. (1965) Statistical Dynamics of Linear Automatic Control Systems, D. Van Nostrand Co. Ltd., Toronto.