ON MODULAR EQUATIONAL CLASSES

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SCOPE AND CONTENTS: This thesis determines necessary and sufficient conditions for an equational class to be modular and finds a modular property that is equivalent to permutability.

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TABLE OF CONTENTS

ON MODULAR EQUATIONAL CLASSES

1. Introduction:

Let us call an equational class K of algebras permutable if and only if every two congruence relations on any K-algebra $\mathfrak{A}$ are permutable. We will call K modular (distributive) if the congruence lattice of every K-algebra is modular (distributive). Mal'cev [1] has given a set of equations which characterize permutable equational classes and Jónsson [2] has done the same for the case of distributivity. In this paper, we give a set of equations which characterize a modular equational class. We give a definition of n-modularity suggested by our equations and show that 2-modularity is equivalent to permutability.

2. The Characterization of Modularity:

For any algebra $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ..., we will use the respective upper case Latin letters A, B, C, ... to indicate the algebra's underlying set. For an algebra $\mathfrak{A}$ and $x,y \in A$, we let $\Theta(x,y)$ be the smallest congruence relation on $\mathfrak{A}$ that contains (x,y).

<u>THEOREM 1</u>: For an equational class K of algebras, the following are equivalent:

(a) K is modular

(b) There is a natural number n and a sequence of terms m_i, i = 0, 1, ..., n in four variables such that for every K-algebra $\mathfrak{A}$ and all $a,b,c,d \in A$

(M1) $m_0(a,b,c,d) = a$ and $m_n(a,b,c,d) = d$

(M2) $m_i(a,b,b,a) = a \quad i = 0, 1, \ldots, n$

(M3) $m_i(a,b,b,d) = m_{i+1}(a,b,b,d)$ (i even)

(M4) $m_i(a,a,d,d) = m_{i+1}(a,a,d,d)$ (i odd)

Without loss of generality, we assume K to be non-trivial.

(a)$\Longrightarrow$(b). Let $\mathfrak{F}$ be an algebra which is K-freely generated by the four element set $\{a,b,c,d\}$. We define congruence relations on $\mathfrak{F}$ by:

$$\Theta = \Theta(b,c) \qquad \psi = \Theta(a,b) \vee \Theta(c,d) \qquad \phi = \Theta(a,d) \vee \Theta(b,c)$$

By (a) we have $(a,d) \in \phi \wedge (\psi \vee (\phi \wedge \Theta)) = (\phi \wedge \psi) \vee (\phi \wedge \Theta)$.

It follows that for some natural number n there is a sequence $u_0, u_1, \ldots, u_n$ in $\mathfrak{F}$ satisfying

(1) $u_0 = a$, $u_n = d$

(2) $u_i (\phi \wedge \Theta) u_{i+1}$ (i even)

(3) $u_i (\phi \wedge \psi) u_{i+1}$ (i odd).

Since $\mathfrak{F}$ is generated by $\{a,b,c,d\}$, there exists a sequence of terms $m_0, m_1, \ldots, m_n$ in four variables such that

$$u_i = m_i(a,b,c,d) \quad (i = 0, 1, 2, \ldots, n)$$

Since every homomorphism of the term algebra in four variables into a K-algebra factors through $\mathfrak{F}$ in such a way that the variables are mapped into a,b,c,d respectively, it is enough to show that the above identities hold in $\mathfrak{F}$ for the free generators a,b,c,d.

(M1) follows easily from (1).

(M2): From (1), (2) and (3) above, it follows that $m_i(a,b,c,d)\ \phi\ a$ holds for all i = 0, 1, ..., n. This together with $a\ \phi\ d$ and $b\ \phi\ c$ gives us $m_i(a,b,b,a)\ \phi\ a$. But the natural homomorphism of $\mathfrak{F}$ onto the factor algebra $\mathfrak{F}/\phi$ maps the subalgebra of $\mathfrak{F}$ generated by $\{a,b\}$ isomorphically and identifies $m_i(a,b,b,a)$ and a. Therefore,

$$m_i(a,b,b,a) = a \quad (i = 0, 1, \ldots, n)$$

(M3): For i even, we get from (2) that $m_i(a,b,c,d)\ \Theta\ m_{i+1}(a,b,c,d)$. Since $b\ \Theta\ c$, this gives $m_i(a,b,b,d)\ \Theta\ m_{i+1}(a,b,b,d)$. Again the natural homomorphism of $\mathfrak{F}$ onto $\mathfrak{F}/\Theta$ maps the subalgebra of $\mathfrak{F}$ generated by $\{a,b,d\}$ isomorphically and identifies $m_i(a,b,b,d)$ and $m_{i+1}(a,b,b,d)$. Therefore,

$$m_i(a,b,b,d) = m_{i+1}(a,b,b,d) \quad (i \text{ even})$$

The proof of (M4) is similar.

(b) $\Longrightarrow$ (a): Let Θ, Ψ, ϕ be congruence relations on a K-algebra $\mathfrak{A}$ satisfying $\Theta \subseteq \phi$. We have to show $(\Theta \vee \Psi) \wedge \phi \leq \Theta \vee (\Psi \wedge \phi)$. For each k ε N, let $\Lambda_k = \Psi \circ \Theta \circ \cdots \circ \Theta \circ \Psi$ (2k+1 factors). Then $(\Theta \vee \Psi) \wedge \phi = \bigcup_{k \in N} (\phi \cap \Lambda_k)$. Hence it suffices to show that $\phi \cap \Lambda_k \subseteq \Theta \vee (\Psi \wedge \phi)$ for every natural number k. We show this by induction over k.

For $k = 0$, this is obvious. For every k, the relation Λ_k is reflexive, symmetric and compatable with all operations. It follows easily that it is also compatable with all polynomials on $\mathfrak{A}$.

For $k > 1$, then $(a,d) \varepsilon \phi \cap \Lambda_{k+1} = \phi \cap (\psi \circ \Theta \circ \Lambda_k)$ implies that there exists elements $b, c \varepsilon A$ such that

$$a \ \phi \ d, \ a \ \Lambda_k b, \ b \ \Theta \ c, \ c \ \psi \ d.$$

Since $\Theta \leq \phi$ and $\psi \leq \Lambda_k$, we also have

$$b \ \phi \ c \qquad c \ \Lambda_k \ d.$$

Define $u_i = m_i(a,b,c,d)$ $(i = 0, 1, \ldots, n)$. By (M1) $a = u_0$ and $u_n = d$.

For i even we have:

$$u_i = m_i(a,b,c,d) \ \Theta \ m_i(a,b,b,d) = m_{i+1}(a,b,b,d) \ \Theta \ u_{i+1}$$

and hence

(4) $u_i \ \Theta \ u_{i+1}$ (i even)

For each i, we have $u_i \ \phi \ m_i(a,b,b,a) = a$ and $a = m_i(a,a,a,a) \ \phi$ $\phi \ m_i(a,a,d,d)$. Therefore,

(5) $u_i \ \phi \ m_i(a,a,d,d)$ $(i = 0, 1, \ldots, n)$

For i odd $u_i \ \Lambda_k \ m_i(a,a,d,d) = m_{i+1}(a,a,d,d) \ \Lambda_k \ u_{i+1}$.

By combining with (5) we have

$$u_i \ \phi \cap \Lambda_k \ m_i(a,a,d,d) = m_{i+1}(a,a,d,d) \ \phi \cap \Lambda_k \ u_{i+1} \quad \text{(i odd)}$$

By induction hypothesis, $\phi \cap \Lambda_k \subseteq \Theta \vee (\psi \wedge \phi)$ and this gives:

(6) $u_i \; \Theta \vee (\psi \wedge \phi) \; u_{i+1}$ (i odd)

This, together with (4) yields

$$(a,d) \in \Theta \vee (\Theta \vee (\psi \wedge \phi)) = \Theta \vee (\psi \wedge \phi)$$

which was to be proved.

3. A Relation between Permutability and Modularity.

As mentioned in the introduction, we define an equational class to be n-modular for some $n \in N$ if there exists a sequence of $n + 1$ terms in four variables satisfying statement (b) in Theorem 1. Clearly if K is modular, K is n-modular for some $n \in N$. Conversely if K is n-modular for any $n \in N$, K is modular.

<u>THEOREM 2</u>: An equational class is permutable if and only if it is 2-modular.

If K is permutable, then by [1] there exists a term p in three variables satisfying $p(a,a,b) = b$ and $p(a,b,b) = a$ in every K-algebra. We define

$m_0(a,b,c,d) = a,$

$m_1(a,b,c,d) = p(a,p(b,c,d),d)$

$m_2(a,b,c,d) = d.$

(M1) is satisfied by definition, and:

$m_1(a,b,b,a) = p(a,p(b,b,a),a) = p(a,a,a) = a$

$m_1(a,b,b,d) = p(a,p(b,b,d),d) = p(a,d,d) = a = m_0(a,b,b,d)$

$m_1(a,a,b,b) = p(a,p(a,b,b),b) = p(a,a,b) = b = m_2(a,a,b,b)$

Therefore, $\{m_0, m_1, m_2\}$ satisfy (M1) to (M4) and K is 2-modular.

If K is 2-modular, then by Theorem 1, there exists m_0, m_1, and m_2 satisfying the properties (M1) to (M4). Define

$$p(a,b,c) = m_1(a,b,c,c)$$

$p(a,a,b) = m_1(a,a,b,b) = m_2(a,a,b,b) = b$ by (M4) and (M1)

$p(a,b,b) = m_1(a,b,b,b) = m_0(a,b,b,b) = a$ by (M3) and (M1).

Therefore, K is permutable.

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