INJECTIVITY IN CONGRUENCE DISTRIBUTIVE EQUATIONAL CLASSES

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By

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SCOPE AND CONTENTS: In this thesis, we study the concept of injectivity in equational classes of (universal) algebras and in particular we are concerned with congruence distributive equational classes that have enough injectives. We show that every reasonable equationally complete congruence distributive equational class has enough injectives and we describe them completely. We then examine what equational subclasses of Lattices, Heyting algebras, and pseudo-complemented lattices have enough injectives.

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INTRODUCTION

The concept of injectivity may be traced back to 1940 with Baer's initial results for Abelian groups and modules over a given ring, [1]. The first results in non-module types of algebras did not appear until Halmos, [20], described the injective Boolean algebras using Sikorski's lemma on the extension of homomorphisms. In recent years, there has been a plethora of results describing the injective algebras in particular equational classes (e.g. [2], [3], [4], [6], [11], [13]).

In [16], Eckmann and Schopf introduced the fundamental notion of essential extension and showed the basic relations that this concept had with injectivity in the equational class of all modules over a given ring. They developed the notion of injective hull (or envelope) which provided every module with a minimal injective extension or equivalently a maximal essential extension. It was not until 1967 that Banaschewski and Bruns, [9], showed that the same results held for Boolean Algebras and Berthiaume proved the analogous theorems for S-sets, [11].

With the advent of Category Theory, Baer's results were abstracted into what is called in Mitchell [26], a C_3 (Abelian) Category with a generator and the notions of injectivity and essential extension were put in their proper categorical setting. The questions of having enough injectives and of having injective hulls were formulated and some answers were provided in [7] and [15] when the first property implied the second.

It was remarked in [14] that an equational class, qua category, has in general the property that enough injectives is equivalent to having injective hulls. This allowed Banaschewski in [8] to transcribe his categorical conditions to equational classes and supply necessary and sufficient conditions for an equational class to have enough injectives (or equivalently injective hulls). It also showed that the categorical relationships between essential extensions and injectives, given in [16] for modules and in [9] for Boolean algebras, were true in any equational class with enough injectives.

In this dissertation we study the question of enough injectives for an equational class in more Universal Algebraic terms. Since an equational class is always determined uniquely by its subdirectly irreducible members, we try to determine conditions for the subdirectly irreducible algebras that will ensure enough injectives in the equational class. The powerful results of Jónsson in [22] allow us to develop some existence theorems for equational classes with distributive congruence relations. In particular, we can provide a completely Universal algebraic generalization of the results for distributive lattices [10] and Boolean Algebras [9]. We also provide some results that describe those equational subclasses of particular equational classes that have enough injectives.

In Chapter 1, we prove the results remarked in [14], analyze Banaschewski's conditions for their Universal Algebraic Content, and examine the relationships between subdirectly irreducible algebras and Banaschewski's conditions. This allows us to reduce his conditions considerably in the case of "nice" equationally complete equational classes.

In Chapter 2, we use Jónsson's results to obtain existence theorems for enough injectives in an algebraic setting that give us practical criteria for checking particular equational classes. We find that every congruence distributive, equationally complete equational class with non-trivial finite algebras has enough injectives (barring a certain pathological case). We then describe these injectives and to some degree the injective hulls.

In Chapter 3, we determine precisely what equational subclasses of lattices and Heyting algebras have enough injectives and supply partial results for the equational class of bounded pseudo-complemented lattices.

PRELIMINARIES

For the basic definitions and results in Universal Algebra, we refer the reader to [18] save for the following exception: an algebra may have an empty underlying set if (and only if) there are no nullary operations. A trivial algebra is one whose underlying set is either empty or a singleton.

For typographical convenience we will identify an algebra $\mathfrak{A} = (A, (f_i)_{i \in I})$ with its underlying set, A. This should cause no problems as we will always be working in an equational class of a given type. Classes of algebras (usually equational) will be written $\underset{\sim}{K}$ and the class operators of homomorphic images, subalgebras, isomorphic images, products, subdirect products, filtered or reduced products and ultraproducts will be respectively denoted by the following symbols: $\underline{H}$, $\underline{S}$, $\underline{I}$, $\underline{P}$, $\underline{P}_S$, $\underline{P}_F$ and $\underline{P}_U$.

We will indicate injective (= one to one) homomorphisms by a double-barred arrow $\Vdash\!\longrightarrow$ and surjective (= onto) ones by a double-headed arrow $\twoheadrightarrow$ à la Halmos [21].

The basic definitions and results of Category Theory are found in Mitchell [26] save for the usual exception that our reflections are his coreflections.

All of the notions from lattice theory may be found in Birkhoff [12], Rasiowa and Sikorski [29], and Szasz [30].

CHAPTER 1

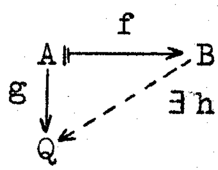
INJECTIVITY AND EQUATIONAL CLASSES

1. Injectives and Essential Extensions

Throughout this section, $\underset{\sim}{K}$ will be an arbitrary but fixed equational class of a given type. The algebras and diagrams in the definitions and diagrams of this section will be assumed to be in $\underset{\sim}{K}$.

1.1 DEFINITION: An algebra Q (in $\underset{\sim}{K}$) is called injective (in $\underset{\sim}{K}$) if for every extension f: $A \rightarrowtail B$ and every homomorphism g: $A \longrightarrow Q$, there exists a homomorphism h: $B \longrightarrow Q$ such that hf = g.

This is usually expressed diagramatically by:



1.2. DEFINITION: An algebra is called an absolute subretract if it is a retract of every extension.

Since the categorical monomorphisms of an equational class are exactly the injective homomorphisms or extensions, we may apply the basic categorical results directly.

1.3. THEOREM: The product of a family of injective algebras and a retract of an injective algebra is again injective. Also, every injective algebra is an absolute subretract.

Proof: ([26], p.70)

The first statement implies that the singleton algebra (= empty product) is always injective in any equational class.

1.4. DEFINITION: An extension $f: A \rightarrowtail B$ is called essential if for every homomorphism $g: B \longrightarrow C$, g is a monomorphism whenever gf is. f is a proper essential extension if it is not an isomorphism.

Since $\underset{\sim}{K}$ is an equational class (more precisely: $\underline{H}(\underset{\sim}{K}) \subseteq \underset{\sim}{K}$), we have an equivalent formulation in terms of $\Theta(B)$, the lattice of congruences on B, viz: the only congruence on B which separates (the points of) Imf (= f[A]) is Δ_B, the diagonal or identity congruence on B. If A is a subalgebra of B, and f is the natural embedding of A into B (written $f: A \leq B$, or just $A \leq B$), we write $A \leq_E B$, and say that A is large in B. Therefore $f: A \rightarrowtail B$ is essential iff $\mathrm{Imf} \leq_E B$. If we call a (homo)morphism essential, it will be implicitly assumed that it is also a monomorphism. The following remarks are clear.

1.5. Lemma: For monomorphisms $f: A \rightarrowtail B$ and $g: B \rightarrowtail C$, we have:

(1) If f and g are essential, so is gf

(2) If gf is essential, then so is g

1.6. Lemma: Let $f: A \rightarrowtail B$ be an arbitrary monomorphism and define $M(f) = \{\theta \in \Theta(B): \theta|\mathrm{Imf} = \Delta_B\}$ where $(\theta|\mathrm{Imf})$ is the congruence on Imf, $\theta \cap \mathrm{Imf}^2$. Then if $\varkappa_\theta: B \twoheadrightarrow B/\theta$

is the canonical homomorphism associated with Θ, $\kappa_\Theta \circ f$ is essential if and only if Θ is maximal in $M(f)$.

Proof: $\kappa_\Theta \circ f$ is essential iff it is a monomorphism and $\mathrm{Im}\ \kappa_\Theta \circ f \leq_E B/\Theta$. By the 2nd Isomorphism Theorem, this is equivalent to $\Theta \in M(f)$ and for all $\Psi \in \Theta(B)$, $\Theta \leq \Psi$ implies $\Psi \notin M(f)$.

1.7. THEOREM: In an equational class, every extension can be continued to an essential extension in the sense that given $f: A \rightarrowtail B$, there exists $g: B \twoheadrightarrow C$ such that gf is essential.

Proof: Using the notation of the previous lemma, $M(f)$ is inductive since for any universal algebra B, $\Theta(B)$ is an algebraic lattice.

1.8. THEOREM: An algebra is an absolute subretract (in $\underset{\sim}{K}$) if and only if it has no proper essential extensions (i.e. every essential extension is an isomorphism).

Proof: If A is an absolute subretract and $f: A \rightarrowtail B$ is essential, then there exists a $g: B \twoheadrightarrow A$ such that $gf = 1_A$. But then gf is a monomorphism hence so is g as f was assumed to be essential. Therefore f, being an inverse of an isomorphism, is also an isomorphism.

Conversely, assume that A has no proper essential extensions, and take $f: A \rightarrowtail B$. By 1.7, there exists a $g: B \twoheadrightarrow C$ such that gf is essential. Since A has no proper essential extensions, gf is an isomorphism and A is a retract of B.

Now suppose $f: A \rightarrowtail Q$ is an injective extension (i.e. Q is injective). By elementary diagram chasing, we see that for every essential extension $g: A \rightarrowtail E$ of A, $E \in \underline{IS}(Q)$. In fact $S(f) = \{ X \leq Q: \ Imf \leq_E X \}$ is a representative set of essential extensions of A.

1.9 Lemma: Let $f: A \rightarrowtail Q$ be an injective extension of A and define $S(f) = \{ X \leq Q: \ Imf \leq_E X \}$. Then for each $M \in S(f)$, M is injective if and only if M is maximal in S(f).

Proof: If M is injective, then M has no proper essential extensions by 1.8 and 1.3. Take $X \in S(f)$, $M \leq X$. Then the composition $A \leq M \leq X$ is essential and so by 1.5, $M \leq_E X$. Therefore the natural embedding $M \leq X$ is an isomorphism and $M = X$.

If M is maximal in S(f), we show that M has no proper essential extensions and hence is a retract of the injective algebra Q. For if $g: M \rightarrowtail B$ is essential, there exists a (mono)morphism $h: B \rightarrowtail Q$ such that $hg = j: M \leq Q$. But then $M \leq_E Imh \leq Q$ and since $A \leq_E M$ we have $A \leq_E Imh$ by 1.3 and $Imh \in S(f)$. By the maximality of M in S(f), M = Imh and therefore g is an isomorphism.

1.10. THEOREM: If an algebra has an injective extension, then it also has an essential injective extension.

Proof: If $f: A \rightarrowtail Q$ is an injective extension, then S(f) as defined above in 1.9 is easily seen to be inductive since the subalgebra lattice of any (universal) algebra is an algebraic closure system.

1.11. DEFINITION: An injective hull of an algebra A (in $\underset{\sim}{K}$) is an essential injective extension.

1.12. Corollary: An algebra has an injective hull iff it has an injective extension.

We close this section with the following categorical result.

1.13. THEOREM: Any two injective hulls of an algebra A are isomorphic over A.

Proof: ([26], p.88).

2. Equational Classes with Enough Injectives

By 1.3, every equational class has at least one (up to isomorphism) injective algebra, the singleton. Indeed for some equational classes, these are the only injectives (e.g. Groups and Lattices, [10]). Other equational classes, Abelian groups [1] and Boolean Algebras [20], have the following more interesting property.

1.14. DEFINITION: An equational class is said to have enough injectives if every algebra in the class has an injective extension (in the class).

This definition is purely categorical and by [14] and 1.12 we see that it is equivalent to the stronger categorical statement that every algebra has an injective hull. Let us note however that the two cases mentioned so far are not exhaustive in that there exists equational classes (e.g. Heyting algebras [6]) that have some (non-trivial) injectives but not enough. The following results are in Banaschewski [8].

1.15. <u>THEOREM</u>: If an equational class $\underset{\sim}{K}$ has enough injectives then for $A \in \underset{\sim}{K}$, T.F.A.E.:

(1) A is injective in $\underset{\sim}{K}$

(2) A is an absolute subretract in $\underset{\sim}{K}$

(3) A has no proper essential extensions in $\underset{\sim}{K}$

1.16. <u>THEOREM</u>: If an equational class $\underset{\sim}{K}$ has enough injectives, then for any extension $f: A \rightarrowtail B$ in $\underset{\sim}{K}$, T.F.A.E.:

(IH) B is injective and f is essential (i.e. injective hull)

(ME) f is essential and for any $g: B \rightarrowtail C$, if gf is essential, g is an isomorphism (i.e. B is a maximal essential extension)

(mQ) B is injective and for $g: A \rightarrowtail Q$, $h: Q \rightarrowtail B$, if $hg = f$ and Q is injective, h is an isomorphism (i.e. B is a minimal injective extension).

In [8], Banaschewski interpreted his categorical results from [7] to provide the following result.

1.17. <u>THEOREM</u>: An equational class $\underset{\sim}{K}$ has enough injectives if and only if every algebra in $\underset{\sim}{K}$ has a representative set of essential extensions and in $\underset{\sim}{K}$, qua category, pushouts preserve monomorphisms.

Because every morphism in an equational class can be factored through its image, the pushout criterium in this theorem is equivalent to the following two conditions:

1.18. <u>DEFINITION</u>: An equational class K satisfies the (weak) amalgamation property iff (AP): given monomorphisms in $\underset{\sim}{K}$ f_i: $A \rightarrowtail B_i$ $(i = 1,2)$, there exists an algebra C in $\underset{\sim}{K}$ and monomorphisms g_i: $B_i \rightarrowtail C$ $(i = 1,2)$ such that $g_1 f_1 = g_2 f_2$.

1.19. <u>DEFINITION</u>: An equational class $\underset{\sim}{K}$ satisfies the congruence restriction property if (CRP): for all $A \leq B \in \underset{\sim}{K}$, the restriction mapping $\theta \longmapsto \theta|A$ of $\Theta(B)$ into $\Theta(A)$ is surjective.

If $\underset{\sim}{K}$ satisfies (CRP) and $A \leq B \in K$, then every congruence $\theta \in \Theta(A)$ equals $\psi|A$ for some congruence $\psi \in \Theta(B)$. In particular then $\theta = \phi|A$ where ϕ is the smallest congruence on B containing θ. If $a,b \in A$, and $\theta_A(a,b)$ is the smallest congruence on A containing the pair (a,b), we must have that $\theta_A(a,b) = \theta_B(a,b)|A$. While we know of no counterexample at present it seems that for a given pair of algebras $A \leq B$, the fact that $\theta_A(a,b) = \theta_B(a,b)|A$ for all $a,b \in A$ should not necessarily imply that every congruence on A is the restriction of some congruence on B. Quantifying over an equational class however does give us the following:

1.20. <u>THEOREM</u>: If $\underset{\sim}{K}$ is an equational class, then T.F.A.E.:

(1) $\underset{\sim}{K}$ satisfies (CRP)

(2) $\underset{\sim}{K}$ satisfies the principal congruence restriction property (PCRP): For all $A \leqslant B \in \underset{\sim}{K}$ and all $a,b \in A$, $\theta_A(a,b) = \theta_B(a,b)|A$.

<u>Proof</u>: Assume $\underset{\sim}{K}$ satisfies (PCRP) and consider $A \leqslant B \in \underset{\sim}{K}$ and $f: A \twoheadrightarrow C$ with $\text{Ker } f = \{(x,y): f(x) = f(y)\} \in \Theta(A)$. Define:

$$E = \{(X,g): A \leqslant X \leqslant B \text{ and } g: X \twoheadrightarrow C \text{ such that } g|A = f\}$$

We give E the usual partial order by:

$$(X,g) \leqslant (Y,h) \text{ iff } X \leqslant Y \text{ and } h|Y = g$$

$(E,\leqslant)$ is clearly inductive so we may take (M,g) maximal in $(E,\leqslant)$. Note that $\text{Ker } g|A = \text{Ker } f$.

We define $P = \{\theta \in \Theta(B): \theta|M \leqslant \text{Ker } g\}$

We need the following properties of P and (M,g)

(a) $\theta \in P$ implies $M = [M]\theta$ where $[M]\theta = \{b \in B$: there exists $m \in M$ such that $(m,b) \in \theta\}$

Clearly $[M]\theta$ is a subalgebra of B and $M \leqslant [M]\theta$. Also by Gratzer [19], we may define $h: [M]\theta \twoheadrightarrow C$ extending g. Therefore $([M]\theta, h)$ E and by the maximality of (M,g), we must have $[M]\theta = M$ and $h = g$.

(b) P is directed:

Take $\theta, \psi \in P$ and $(a,b) \in (\theta \vee \psi) M$. Therefore there exists a sequence $a = x_0, x_1, \ldots, x_n = b$ in B such that:

$$x_i \ \theta \ x_{i+1} \qquad (i \text{ even})$$

$$x_i \ \psi \ x_{i+1} \qquad (i \text{ odd})$$

Since $a \in M$, $x_1 \in [a]\theta \subseteq M$ by (a) and it follows by easy finite induction that $x_i \in M$ for all $i = 0,1,2,..,n$. Therefore

$$(a,b) \in (\theta|M) \vee (\psi|M) \leqslant \text{Ker } g$$

and $\theta \vee \psi \in P$.

(c) $\bigvee P \in P$:

For $(\bigvee P)|M = \bigcup\{\theta|M\colon \theta \in P\} \leqslant \text{Ker } g$.

Therefore $\phi = \bigvee P$ is the largest congruence on B whose restriction to M is contained in Ker g.

But if $(a,b) \in \text{Ker } g$, then $\theta_M(a,b) = \theta_B(a,b)|M \leqslant \text{Ker } g$ by (PCRP). Therefore $\theta_B(a,b) \in P$ and hence $\theta_B(a,b) \leqslant \phi$. Therefore $\phi|M = \text{Ker } g$ and $\phi|A = \text{Ker } g|A = \text{Ker } f$. The converse is trivial.

1.21. Corollary: $\underset{\sim}{K}$ satisfies (CRP) iff for all $B \in \underset{\sim}{K}$ for all $a,b,c,d \in B, \theta_A(a,b) = \theta_B(a,b)|A$ where $A = \langle a,b,c,d \rangle$, the subalgebra generated by $\{a,b,c,d\}$.

Proof: The condition is clearly necessary by our theorem. Conversely take $A \leqslant B \in \underset{\sim}{K}$, $a,b \in A$ and $(c,d) \in \theta_B(a,b)|A$. Then for $C = \langle a,b,c,d \rangle \leqslant A \leqslant B$, we have $(c,d) \in \theta_B(a,b)|C \leqslant \theta_A(a,b)$.

While there exists a formulation of this last corollary in terms of the free algebras of an equational class, the formulation does not seem to lead to a "Mal'cev Type Condition", that is an equivalent condition concerning identities satisfied by the equational class.

We conclude this section with the following lemma.

1.22. Lemma: Let $\underset{\sim}{K}$ satisfy (CRP) and let $f: A \rightarrowtail B$ and $g: B \rightarrowtail C$ be monomorphisms in $\underset{\sim}{K}$ then gf is essential iff both f and g are essential.

Proof: By 1.5, we need only show that if gf is essential, then f is essential. Therefore let $h: B \longrightarrow D$ be such that hf is a monomorphism. Without loss of generality we may assume h is surjective. By (CRP), there exists a congruence Φ on C such that $\Phi|B = \operatorname{Ker} h$. If $\varkappa: C \twoheadrightarrow C/\Phi$ is the canonical homomorphism and $j: D \rightarrowtail C/\Phi$ is the natural map such that $jh = \varkappa g$ we have $jhf = \varkappa gf$ a monomorphism. By assumption, gf is essential and so $\varkappa$ is a monomorphism and hence also h.

3. Injectivity and Subdirectly Irreducible Algebras

Since every equational class $\underset{\sim}{K}$ is determined uniquely by $\underset{\sim}{K}_{SI}$, its class of subdirectly irreducible algebras, it would be of interest if one could relate injectivity to the subdirectly irreducibles. To obtain such a relation, we need the following lemma from [14].

1.23. <u>Lemma</u>: Every essential extension of a subdirectly irreducible algebra is again subdirectly irreducible.

<u>Proof</u>: Without loss of generality assume $S \leqslant_E T$ and that S is subdirectly irreducible. Let $(\Theta_i : i \in I)$ be a family of congruences on T whose meet (= intersection) is Δ_T. Then

$$\bigcap_{i \in I} (\Theta_i | S) = (\bigcap_{i \in I} \Theta_i)|S = \Delta_T|S = \Delta_S$$

Since S is subdirectly irreducible, $\Theta_i|S = \Delta_S$ for some $i \in I$ and since $S \leqslant_E T$, $\Theta_i = \Delta_T$ for this i.

Therefore if an equational class $\underset{\sim}{K}$ has enough injectives, the injective hull of every subdirectly irreducible is again subdirectly irreducible and in this sense there will be "enough injective subdirectly irreducibles". Conversely, since every algebra in $\underset{\sim}{K}$ is a subdirect product of algebras in $\underset{\sim}{K}_{SI}$, if every $S \in \underset{\sim}{K}_{SI}$ has an extension $T \in \underset{\sim}{K}_{SI}$ which is injective in $\underset{\sim}{K}$, $\underset{\sim}{K}$ will have enough injectives. This gives us the following and perhaps more applicable characterization theorem.

1.24. <u>THEOREM</u>: An equational class has enough injectives if and only if it has enough injective subdirectly irreducible algebras.

In particular cases when we know already the subdirectly irreducible algebras of an equational class, we have a viable procedure to demonstrate enough injectives. This method is essentially used in [4] and [10] and can even be applied to Abelian groups by using the characterization of the subdirectly irreducible Abelian groups found in Pierce [27].

Before proceeding with an immediate application of this approach, we must describe an interesting pathology that will occur if our equational class under consideration does not have any nullary operations defined in its type (i.e. if $\tau = (\lambda_i)_{i \in I}$, $\lambda_i > 0$ for all $i \in I$). If $\underset{\sim}{K}$ is such a class and has enough injectives, we note immediately that the empty map $f: \emptyset \rightarrowtail \{x\}$ is essential and that the singleton is the injective hull of $\emptyset$. If Q is a non-trivial injective in $\underset{\sim}{K}$ then there exists a morphism $g: \{x\} \longrightarrow Q$ extending the empty map $\emptyset \leq Q$. Therefore every injective algebra in K has a one element subalgebra.

This pathology becomes rather lucid when we consider $\underset{\sim}{K}$ to be the equational class of "Boolean algebras" considered as algebras of type (2,2,1) with operations join, meet and complementation (To the usual distributive lattice equations, add the De Morgan laws for complementation and the equation $x \vee x' = y \vee y'$). Now $\underset{\sim}{K}$ differs from $\underset{\sim}{B}$, the equational class of "real" Boolean algebras (where we consider 0,1 or both as defined nullary operations) only in that it contains $\emptyset$ as an algebra. However $\underset{\sim}{K}$ has only trivial injectives (!!!) since every injective must contain a one-element subalgebra.

To avoid this pathology we consider the following condition.

1.25. <u>DEFINITION</u>: An algebra A, will be called $\emptyset$-regular if $\emptyset \leq A$ implies A has a one element subalgebra.

1.26. <u>THEOREM</u>: Let $\underset{\sim}{K}$ be an (equationally complete) equational class which contains up to isomorphism only one subdirectly irreducible algebra S. Moreover assume that S is finite. Then $\underset{\sim}{K}$ has enough injectives if and only if $\underset{\sim}{K}$ satisfies (CRP) and S is $\emptyset$-regular.

<u>Proof</u>: Since S is the only subdirectly irreducible in $\underset{\sim}{K}$, S has no proper essential extensions in $\underset{\sim}{K}$ and must be at least an absolute subretract in $\underset{\sim}{K}$. Since $\underset{\sim}{K} = \underline{SP}(S)$, it will be enough to show S is injective. Since any non-trivial subalgebra of S must have a non-trivial subdirect representation by S and since S is finite, S can have at most only trivial proper subalgebras. Therefore to show S is injective, we need only consider the case where $\emptyset \neq A \leq B$ and $f: A \twoheadrightarrow S$ as S is $\emptyset$-regular. Since $\underset{\sim}{K}$ satisfies (CRP) there exists a congruence Θ on B with $\Theta|A = \text{Ker } f$ and a canonical $g: S \rightarrowtail B/\Theta$ such that $gf = \chi_\Theta|A$. But then there exists $h: B/\Theta \twoheadrightarrow S$ such that $hg = 1_S$ and hence $h \circ \chi_\Theta|A = hgf = f$ and S is injective.

Using this approach Mrs. E. Nelson has shown:

1.27. <u>Corollary</u>: Every equationally complete equational subclass of semi-groups has enough injectives.

Since every equational class with enough injectives must necessarily satisfy (CRP), we close this section with some necessary conditions for an equational class to satisfy (CRP). These will be used in Chapter 3.

1.28. THEOREM: Let $\underset{\sim}{K}$ be an equational class satisfying (CRP). Then

(1) Every large subalgebra of a subdirectly irreducible is again subdirectly irreducible.

(2) If $S \in \underset{\sim}{K}_{SI}$ and $a,b \in S$ such that $\theta_S(a,b) = \phi_S$, the least non-trivial congruence on S, then for all subalgebras $\langle a,b\rangle \leq X \leq S$, $X \leq_E S$ and hence by (1), X is subdirectly irreducible.

Proof: Take $S \leq_E T \in \underset{\sim}{K}_{SI}$ and let $(\theta_i : i \in I)$ be a family of congruences on S whose meet is Δ_S. Since K satisfies (CRP), there exists $\psi_i \in \Theta(T)$ such that $\psi_i | S = \theta_i$ for each $i \in I$ and we have:

$$\left(\bigcap_{i\in I}\psi_i\right)\Big|S = \bigcap_{i\in I}\theta_i = \Delta_S$$

Since $S \leq_E T$, $\bigcap_{i \in I} \psi_i = \Delta_T$ and therefore $\psi_i = \Delta_T$ for some $i \in I$. But then $\theta_i = \Delta_S$ for this i.

The second statement follows from 1.22 and the above since clearly $\langle a,b\rangle \leq_E S$.

CHAPTER 2

CONGRUENCE DISTRIBUTIVE EQUATIONAL CLASSES

1. Existence Theorems

While completely algebraic conditions may not be possible to ensure enough injectives in an arbitrary equational class, the fundamental results of Jónsson, [22], gives us some hope if we impose a further condition.

2.1. DEFINITION: An equational class is called congruence distributive if the congruence lattice of every algebra in the class is distributive.

2.2. DEFINITION: An algebra S is called self-injective if it is $\emptyset$ -regular and any homomorphism from a subalgebra of S into S extends to an endomorphism of S.

Clearly if S is injective in some equational class, S will be self-injective but in general, self-injectivity is independent of equational class considerations. Jónsson's Lemma ([22] or [18], p.244) allows us to formulate our main existence theorem.

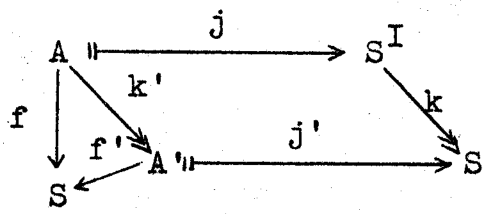
2.3. THEOREM: Let $\underset{\sim}{K}$ be a congruence distributive equational subclass of an equational class $\underset{\sim}{L}$. Assume further that $\underset{\sim}{K} = \underline{S}\,\underline{P}(S)$ where S is a finite subdirectly irreducible algebra whose non-empty subalgebras are either injective in $\underset{\sim}{L}$ or subdirectly irreducible. Then $\underset{\sim}{K}$ has enough injectives if and only if S is self-injective.

Proof: By the previous remarks, the condition is clearly necessary as S can have no essential extensions in $\underset{\sim}{K}$. Assume then that S is self-injective. Since $\underset{\sim}{K} = \underline{S}\,\underline{P}(S)$, it will be sufficient to show S is injective. Again since $\underset{\sim}{K} = \underline{S}\,\underline{P}(S)$ and also since S is $\emptyset$-regular, we may show this by finding extensions of homomorphisms in diagrams of the form

$$\begin{array}{l} j\colon A \leqslant S^I \\ \quad f\downarrow \\ \quad\ \ S \end{array}$$

where $A \neq \emptyset$.

Let $B = \operatorname{Im} f \leqslant S$. If B is injective in $\underset{\sim}{L}$, B is also injective in $\underset{\sim}{K}$ and we are done. If B is subdirectly irreducible, by Jónsson's Lemma, there exists an ultra filter U on I such that $\theta_U|A \leqslant \operatorname{Ker} f$ where $(x,y) \in \theta_U$ iff $E(x,y) = \{ i \in I\colon x_i = y_i \} \in U$. Since S is finite, S^I/θ_U is isomorphic to S and hence there exists a homomorphism $k\colon S^I \twoheadrightarrow S$ with $\operatorname{Ker} k = \theta_U$. By letting $A' = k[A] \leqslant S$ and $k'\colon A \twoheadrightarrow A'$ be $k|A$ we obtain the following communative diagram:



where f' is the canonical map determined by the fact that Ker k' $= \Theta_U | A \leqslant$ Ker f.

Since S is self injective, there exists an endomorphism g of S such that $g|A' = f'$ and therefore

$$(gk)|A = gkj = gj'k' = f'k' = f$$

and S is injective.

2.4. DEFINITION: Let $\underset{\sim}{R}(p^k)$ be the equational class of (communative) rings with unit generated by the Galois field of order p^k, $GF(p^k)$ (p prime and $k \geqslant 1$).

2.5. Corollary: For every prime p, and natural number $k \geq 1$, $\underset{\sim}{R}(p^k)$ has enough injectives.

Proof: By [25], $\underset{\sim}{R}(p^k)$ is congruence distributive and $\underset{\sim}{R}(p^k) = \underline{S}\,\underline{P}(GF(p^k))$. By ([31], p.117), the subrings of $GF(p^k)$ are exactly the Galois fields $GF(p^n)$ for $n|k$. Moreover for each $n|k$, there is a unique embedding $j: GF(p^n) \leqslant GF(p^k)$ which extends to the identity on $GF(p^k)$. Therefore $GF(p^k)$ is self-injective.

This corollary has also been obtained by Banaschewski (unpublished) by ring theoretical methods. Further applications will appear in Chapter 3. Our next result is really a corollary also but its importance warrants the more prestigous title of theorem.

2.6. THEOREM: Every equationally complete, congruence distributive equational class that contains a non-trivial Ø-regular finite algebra has enough injectives.

Proof: If $\underset{\sim}{K}$ is such an equational class, then K contains a finite Ø-regular subdirectly irreducible algebra, S, as a homomorphic image of the given finite algebra. Since $\underset{\sim}{K}$ is equationally complete, S is generic for $\underset{\sim}{K}$ and in fact by [22], S is up to isomorphism the only subdirectly irreducible algebra in $\underset{\sim}{K}$. It follows easily as in the proof of 1.26 that S has at most trivial subalgebras. S is clearly self-injective and since $\underset{\sim}{K} = \underline{S}\,\underline{P}(S)$, $\underset{\sim}{K}$ has enough injectives by 2.3.

2.7. Corollary: The following equational classes have enough injectives:

(1) Distributive lattices ([2] and [10])

(2) Boolean Algebras ([20])

(3) $\underset{\sim}{R}(p)$ (Banaschewski)

The connection between this theorem and 1.26 is given by the following lemma.

2.8. Lemma: Let $\underset{\sim}{K}$ be a congruence distributive equational class which contains (up to isomorphism) only finitely many subdirectly irreducible algebras $\{S_1, S_2, \ldots, S_n\}$ all of which are finite. Then $\mathbf{K}$ satisfies (CRP) iff for all $i = 1,\ldots,n$, the classes $\underline{H}\,\underline{S}(S_i)$ satisfy (CRP).

The proof is analogous to that of 2.3 in that we need only show that for $A \leq B \in \mathbf{K}$, every completely - meet - irreducible congruence on A is the restriction of a congruence on B. We obtain this by applying Jónsson's Lemma to the diagram

$$\begin{array}{ccc} A \leq B & \rightarrowtail & \prod_i^{1,n} S_i^{I_i} \\ \downarrow & & \\ S_j & & \end{array}$$

which reduces the problem to the stated condition as any ultra product of a finite family of finite algebras is isomorphic to one of the algebras.

If two equational classes of the same type have enough injectives, can we say the same for their join? In general the answer is no for in the equational class $\mathbf{R}(p^2) \vee \mathbf{R}(p^3)$, GF(p) has two distinct maximal essential extensions $GF(p^2)$ and $GF(p^3)$ hence there are not enough (in fact none) injective subdirectly irreducibles. We now derive a condition that assures us of a positive answer.

2.9. <u>DEFINITION</u>: Let $\mathbf{A}$ be an equational subclass of an equational class $\mathbf{B}$. The natural reflection of $\mathbf{B}$ into $\mathbf{A}$ is the functor $R: \mathbf{B} \longrightarrow \mathbf{A}$ defined by:

(i) $R(B) = B/\rho(B)$ where $\rho(B) = \bigwedge\{\theta \in \Theta(B) : B/\theta \in \underset{\sim}{A}\}$

(ii) For $u: B \longrightarrow C$, $R(u): R(B) \longrightarrow R(C)$ is the canonical morphism that makes the following diagram commute

$$\begin{array}{ccc} B & \xrightarrow{\;u\;} & C \\ {\scriptstyle \rho B}\big\downarrow & & \big\downarrow{\scriptstyle \rho C} \\ R(B) & \xrightarrow{\;R(u)\;} & R(C) \end{array}$$

where ρB is the canonical morphism $B \twoheadrightarrow B/\rho(B)$. R is actually the left adjoint to the natural inclusion functor $J: \underset{\sim}{A} \longrightarrow \underset{\sim}{B}$. The natural transformation $\rho = (\rho B)_{B \in \underset{\sim}{B}}: I_{\underset{\sim}{B}} \longrightarrow JR$ is the front adjunction and also is sometimes called the reflection.

2.10. Lemma: Let $\underset{\sim}{A} \subseteq \underset{\sim}{B}$ be equational classes. If the natural reflection $R: \underset{\sim}{B} \longrightarrow \underset{\sim}{A}$ preserves monomorphisms, then every essential extension of an algebra $A \in \underset{\sim}{A}$ in $\underset{\sim}{B}$ is also in $\underset{\sim}{A}$. Moreover every injective algebra in $\underset{\sim}{A}$ is injective in $\underset{\sim}{B}$.

Proof: Without loss of generality let $A \in \underset{\sim}{A}$ and $A \leq_E B \in \underset{\sim}{B}$. By applying R we see that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\;j\;} & B \\ {\scriptstyle \rho A = 1_A}\big\downarrow & & \big\downarrow{\scriptstyle \rho B} \\ R(A) & \xrightarrow{\;R(j)\;} & R(B) \end{array}$$

Since $R(j)$ is a monomorphism, $\rho B \circ j = R(j)$ is a monomorphism. But j is essential hence ρB is an isomorphism and $B \in \underset{\sim}{A}$.

The second claim is a consequence of ([26], p.136) (see also [8]) since R preserves monomorphisms.

2.11. <u>Lemma</u>: If $\mathbf{A} \subseteq \mathbf{B}$ are equational classes that satisfy (CRP), and if for all $A \in \mathbf{A}$, every essential extension of A in $\mathbf{B}$ is in $\mathbf{A}$, then the natural reflection $R: \mathbf{B} \longrightarrow \mathbf{A}$ preserves monomorphisms.

<u>Proof</u>: Consider the diagram

$$\begin{array}{ccc} A & \overset{j}{\rightarrowtail} & B \\ {\scriptstyle \rho A}\big\downarrow & & \\ R(A) & & \end{array}$$

with j the natural embedding. Since $\mathbf{B}$ satisfies (CRP), the set $M = \{ \theta \in \Theta(B) : \theta | A = \operatorname{Ker} \rho A \}$ is non-empty and is therefore inductive. If ψ is maximal in M, the canonical map $f: R(A) \rightarrowtail B/\psi$ is essential and then by assumption $B/\psi \in \mathbf{A}$. Therefore $\operatorname{Ker} \rho B = \rho(B) \leq \psi$ and if g is the canonical map $g: R(B) \twoheadrightarrow B/\psi$ we have $g \circ R(j) = f$ whence $R(j)$ is a monomorphism.

2.12. <u>THEOREM</u>: Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be equational classes of the same type, each containing only finitely many subdirectly irreducible algebras, all of which are finite. Assume further that $\mathbf{K} = \mathbf{K}_1 \vee \mathbf{K}_2$ is congruence distributive and that the natural reflections $R_i: \mathbf{K}_i \longrightarrow \mathbf{L} = \mathbf{K}_1 \cap \mathbf{K}_2$ $(i = 1,2)$ preserve monomorphisms. Then if $\mathbf{K}_1$ and $\mathbf{K}_2$ have enough injectives, so does $\mathbf{K}$.

Proof: Since $\underset{\sim}{K}$ is congruence distributive, the subdirectly irreducible algebras in $\underset{\sim}{K}$ are exactly those in $\underset{\sim}{K}_1$ or $\underset{\sim}{K}_2$. By 2.8, it follows that $\underset{\sim}{K}$ satisfies (CRP) since both $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ do. If the natural reflections $S_i: \underset{\sim}{K} \longrightarrow K_i$ preserve monomorphisms, then the subdirectly irreducibles that are injective in $\underset{\sim}{K}_i$ (i = 1,2) are also injective in $\underset{\sim}{K}$ by 2.10 and $\underset{\sim}{K}$ will have enough injectives. Therefore take $A \in \underset{\sim}{K}_1$ and $A \leqslant_E B \in K$. For every completely-meet-irreducible congruence θ on A, we can select a maximal member $\bar{\theta} \in M(\theta) = \{\psi \in \Theta(B): \psi | A = \theta\}$. Therefore the canonical map $f_\theta: A/\theta \rightarrowtail B/\bar{\theta}$ is essential. But by 1.23, $B/\bar{\theta}$ is then subdirectly irreducible hence $\bar{\theta}$ is completely-meet-irreducible. Moreover

$$\bigcap\{\bar{\theta}: \theta \text{ comp-meet-irreducible on } A\} \mid A = \Delta_A$$

and since $A \leqslant_E B$, this meet must be Δ_B. Therefore $B \in \underset{\sim}{K}_1$, iff $B/\bar{\theta} \in \underset{\sim}{K}_1$ for all of these $\bar{\theta}$'s. But if $S \in \underset{\sim}{K}_1$ is subdirectly irreducible and $S \leqslant_E T \in \underset{\sim}{K}_2$, we have $S \in \underset{\sim}{K}_1 \cap \underset{\sim}{K}_2 = \underset{\sim}{L}$ and since $R_2: \underset{\sim}{K}_2 \longrightarrow \underset{\sim}{L}$ preserves monomorphisms, $T \in \underset{\sim}{L} \subseteq \underset{\sim}{K}_1$ by 2.10. Therefore by 2.11, $S_1: \underset{\sim}{K} \longrightarrow \underset{\sim}{K}_1$ preserves monomorphisms. By symmetry, $S_2: \underset{\sim}{K} \longrightarrow \underset{\sim}{K}_2$ also preserves monomorphisms and therefore $\underset{\sim}{K}$ has enough injectives by 1.24.

2.13. Corollary: If $p_1, \ldots, p_n$ are distinct prime numbers, then

$$\bigvee_i^{1,n} \underset{\sim}{R}(p_i^{k_i})$$

has enough injectives for every choice of $k_1, \ldots, k_n \in N$.

Proof: If $p_1 \neq p_2$, $\underset{\sim}{R}(p_1^{k_1}) \cap \underset{\sim}{R}(p_2^{k_2}) = \underset{\sim}{T}$, the equational class of trivial rings. Therefore the natural reflections $R_i : \underset{\sim}{R}(p_i^{k_i}) \longrightarrow \underset{\sim}{T}$ trivially perserve monomorphisms. The result then follows easily by induction.

This corollary was proved by Banaschewski (unpublished) by ring-theoretical methods. Other applications appear in Chapter 3.

2. The Injectives in Equationally Complete Equational Classes.

Throughout this section, $\underset{\sim}{K}$ will be an equationally complete, congruence distributive equational class that contains a non-trivial $\emptyset$-regular finite algebra. By 2.6, we know that $\underset{\sim}{K} = \underline{S}\,\underline{P}(S)$ where S is the only (up to isomorphism) subdirectly irreducible algebra in $\underset{\sim}{K}$. Moreover S is finite and has at most trivial subalgebras. It is also clear that S is simple. We wish to describe the injectives in $\underset{\sim}{K}$ that exist by 2.6.

Let T be the subset of all $s \in S$ that are not images of nullary operations of τ, the type of $\underset{\sim}{K}$. If $2 \leq |S \setminus T|$, the cardinality of $S \setminus T$, then S has no subalgebras and in fact S will be the $\underset{\sim}{K}$-free algebra on $\emptyset$ generators. Moreover, the only endomorphism of S will be 1_S, the identity homomorphism. If $|S \setminus T| \leq 1$, we define an algebra S^{o} of type $\tau_o = \tau \,\dot{\cup}\, (\lambda_s)_{s \in T}$ (without loss of generality T and the domain of τ are disjoint) where $\lambda_s = 0$ for all $s \in T$ by adding to the original operations on S the nullary operations $f_s(\emptyset) = s$ for all $s \in T$. If $\underset{\sim}{K}^\circ$ is the equational class of type $\tau_\circ$ generated by S°, then it is clear that $\underset{\sim}{K}^\circ$ is equationally complete, congruence distributive, and that $\underset{\sim}{K}^\circ = \underline{S}\,\underline{P}(S^\circ)$. By 2.6, $\underset{\sim}{K}^\circ$ has enough injectives. In order to find the connection between the injectives in $\underset{\sim}{K}$ and $\underset{\sim}{K}^\circ$ we require the following unpublished result of S. Comer and B. Jónsson.

2.14. Lemma: Let S be a simple algebra and suppose that $\underset{\sim}{K} = \underline{H}\,\underline{S}\,\underline{P}(S)$ is a congruence distributive. Then for every set I, every congruence relation on S^I is induced by a filter on I. (i.e. every $\theta \in \Theta(S^I)$ is of the form $\theta = \theta_F$ for some filter F on I).

2.15. THEOREM: The injective algebras in $\underset{\sim}{K}^\circ$ are exactly the injectives in $\underset{\sim}{K}$ with the extra nullary operations suitably defined.

Proof: Every non-trivial injective in $\underset{\sim}{K}^\circ$ is a $\tau_\circ$-retract of some power of S° since $\underset{\sim}{K}^\circ = \underline{S}\,\underline{P}(S^\circ)$. By forgetting the added constants, this algebra will become a τ-retract of the same power of S and so will be injective in $\underset{\sim}{K}$.

Conversely if Q is a non-trivial injective in $\underset{\sim}{K}$, then Q is a retract of a power of S and so there exists a τ-homomorphism $f: S^I \twoheadrightarrow Q$. If $\Delta : S \rightarrowtail S^I$ is the embedding of S into the constant maps $I \longrightarrow S$, we have $f \cdot \Delta$ is a monomorphism since Ker f is induced by a filter

F on I and $E(\Delta(s), \Delta(t)) = \emptyset$ for $s \neq t$. Therefore f defines a τ_o-homomorphism $f_o: (S^\circ)^I \longrightarrow\!\!\!\!\rightarrow Q^\circ$ where Q° is the algebra Q with the added constants $(f \circ \Delta(s))_{s \in T}$. That Q° is injective in $\underset{\sim}{K}^\circ$ may be found in [27], p.109.

Without loss of generality we will assume for the remainder of this section that <u>S has no proper subalgebras</u>. As remarked before, this implies that S is the K-free algebra on $\emptyset$ generators and has only one endomorphism 1_S.

2.16. <u>DEFINITION</u>: $S: \underset{\sim}{B} \longrightarrow \underset{\sim}{K}$, where $\underset{\sim}{B}$ is the equational class of Boolean algebras (defined with nullary operations) is the functor given by:

(i) For $B \in \underset{\sim}{B}$, S(B) is the Boolean extension of S($\in$ K) by B (see [17] or [18]).

(ii) For $f: B \longrightarrow C$, $S(f): S(B) \longrightarrow S(C)$ is defined by the mapping $\alpha \longmapsto f \circ \alpha$. Since f is a Boolean homomorphism, $f \circ \alpha$ is a disjoint S-cover of C whenever α is a disjoint S-cover of B and S(f) is a homomorphism in $\underset{\sim}{K}$.

That S is a functor is implicit in [18] .

2.17. <u>DEFINITION</u>: $T: \underset{\sim}{K} \longrightarrow \underset{\sim}{B}$ is the functor defined by:

(i) For $A \in \underset{\sim}{K}$, T(A) is the field of subsets of $\underset{\sim}{K}(A,S)$, the set of all (surjective) homomorphisms of A into S, generated by the subsets

$X_A(a,F) = \{ f: A \longrightarrow S: f(a) \in F \}$ $(a \in A,\ F \subseteq S)$.

(ii) For $g: A \longrightarrow B$, $T(g): T(A) \longrightarrow T(B)$ is determined by the restriction to $T(A)$ of the map $g^*: P(\underset{\sim}{K}(A,S)) \longrightarrow P(\underset{\sim}{K}(B,S))$, the power sets of $\underset{\sim}{K}(A,S)$ (resp $\underset{\sim}{K}(B,S)$), that takes $X \longmapsto \{f: f \circ g \in X\}$. It follows easily that $T(g)(X_A(a,F)) = X_B(g(a),F)$.

Note that $\underset{\sim}{K}(A,S) \setminus X_A(a,F) = X_A(a,S \setminus F)$ and so the set of generators for TA is closed under complementation. We will write $X_A(a,s)$ instead of $X_A(a, \{s\})$. Also since S is injective in $\underset{\sim}{K}$, T preserves monomorphisms.

2.18. <u>DEFINITION</u>: For each $A \in \underset{\sim}{K}$, $\eta A: A \longrightarrow STA$ is the homomorphism defined by $\eta A(a): S \longrightarrow T(A)$, $\eta A(a)(s) = X_A(a,s)$.

Clearly $\eta A(a) \in S(TA)$ for each $a \in A$ and ηA is a homomorphism. Moreover, since $\underset{\sim}{K}(A,S)$ separates the points of A, ηA is a monomorphism for each $A \in \underset{\sim}{K}$.

2.19. <u>Lemma</u>: For each $B \in \underset{\sim}{B}$, the function $\epsilon B: TSB \longrightarrow B$ defined on the generators by $X_{SB}(\alpha, F) \longmapsto \bigvee_{s \in F} \alpha(s)$ ($\alpha \in SB$, $F \subseteq S$) is a surjective Boolean homomorphism of TSB onto B.

<u>Proof</u>: Take $\alpha_i \in SB$ and $F_i \subseteq S$, $i = 1,\ldots,n$ such that $c = \bigwedge_{i}^{1,n} \bigvee_{s \in F} \alpha_i(s) > 0$ and take $f: B \longrightarrow\!\!\!\!\rightarrow 2$ such that $f(c) = 1$. Then for $Sf: SB \longrightarrow\!\!\!\!\rightarrow S(2)$ we have

$$S(f)(\alpha_i) = f \circ \alpha_i : S \longrightarrow B \longrightarrow\!\!\!\!\rightarrow 2 \qquad (i = 1,2,\ldots,n)$$

Now for each $i = 1,2,\ldots,n$, $c \leq \bigvee_{s \in F_i} \alpha_i(s)$ and therefore there exists an $s_i \in F_i$ such that $f(\alpha_i(s_i)) = 1$. Therefore $S(f)(\alpha_i)$ is the element of $S(2)$ associated under the natural isomorphism, σ, of $S(2)$ and S with $s_i \in F_i$. But then

$$\sigma \circ S(f) \in \bigcap_{i}^{1,n} X_{SB}(\alpha_i, F_i)$$

Therefore $\bigcap_{i}^{1,n} X_{SB}(\alpha_i, F_i) = \emptyset$ implies $\bigwedge_{i}^{1,n} \varepsilon B\,(X_{SB}(\alpha_i, F_i)) = 0$ and εB is a well defined Boolean homomorphism.

Since S contains at least two distinct elements $s \neq t$, we may define for each $b \in B$; $\alpha_b \in S(B)$ by:

$$\alpha_b(x) = \begin{cases} b & x = s \\ b' & x = t \\ 0 & \text{otherwise} \end{cases}$$

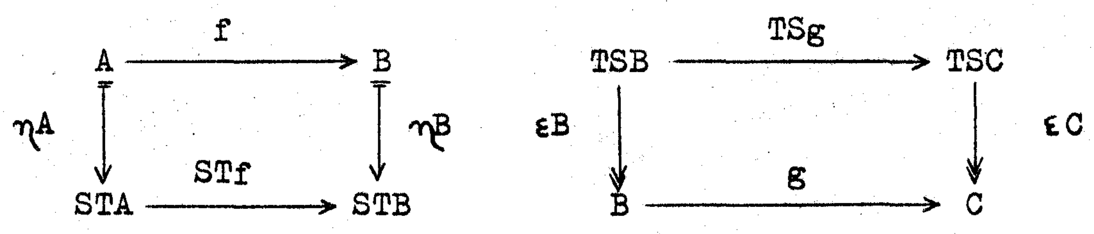
Then $\varepsilon B(X_{SB}(\alpha_b, s)) = \alpha_b(s) = b$ and εB is surjective.

2.20. <u>THEOREM</u>: The functors $\underset{\sim}{K} \underset{S}{\overset{T}{\rightleftarrows}} \underset{\sim}{B}$ are adjoint covariant functors with front adjunction

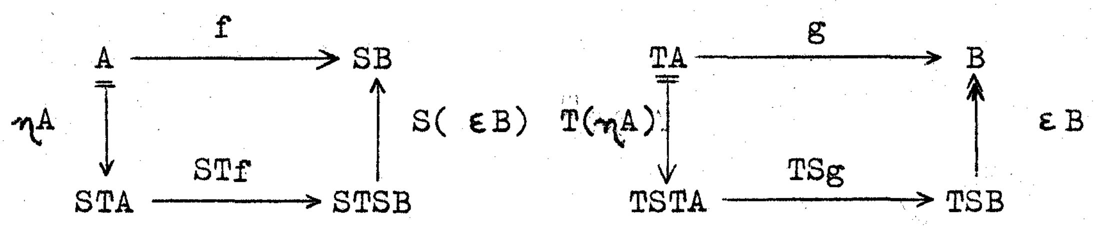
$\eta = (\eta A)_{A \in \underset{\sim}{K}} : I_{\underset{\sim}{K}} \longrightarrow ST$ and back adjunction

$\varepsilon = (\varepsilon B)_{B \in \underset{\sim}{B}} : TS \longrightarrow I_{\underset{\sim}{B}}$.

<u>Proof</u>: We need only show that the following four diagrams are always communative in their respective categories.



(a) (b)



(c) (d)

Re (a): $(STf \circ \eta A)(a) = STf(\eta A(a)) = Tf \circ \eta A(a) \quad (a \in A)$.

Therefore for all $s \in S$

$$\begin{aligned}(STf \circ \eta A)(a)(s) &= (Tf \circ \eta A(a))(s) = Tf(X_A(a,s)) \\ &= X_B(f(a),s) = (\eta B \circ f)(a)(s)\end{aligned}$$

Re (c): For all $a \in A$ and all $s \in S$:

$$\begin{aligned}(S\,\varepsilon B \circ STf \circ \eta A)(a)(s) &= (S(\varepsilon B \circ Tf)(\eta A(a)))(s) \\ &= (\varepsilon B \circ Tf \circ \eta A(a))(s) \\ &= \varepsilon B(X_B(f(a),s)) \\ &= f(a)(s)\end{aligned}$$

The remaining two are done similarly.

2.21. <u>Lemma</u>: $\varepsilon : TS \longrightarrow I_{\underset{\sim}{B}}$ is a natural equivalence.

Proof: Since every Boolean algebra is a subalgebra of a power set algebra P(I) for some set I, it is sufficient to show $\varepsilon P(I)$ is an isomorphism for each set I and then apply diagram (b). Now by [18], p.147, there is a canonical isomorphism $f: S^I \rightarrowtail\!\!\!\twoheadrightarrow S(P(I))$ given by

$f(\alpha): s \longmapsto E(\alpha, \Delta(s)) = \{ i \in I: \alpha(i) = s \}$. Therefore we need only show that $T(S^I) \xrightarrow{Tf} TS(P(I)) \xrightarrow{\varepsilon P(I)} P(I)$ is a monomorphism.

Since S has only one endomorphism, 1_S, every homomorphism $u: S^I \twoheadrightarrow S$ is determined uniquely by an ultrafilter U_u on I by 2.14. This is given by:

$$u(\alpha) = s \quad \text{iff } E(\alpha, \Delta(s)) \in U_u$$

Therefore

$$(\varepsilon P(I) \circ Tf)(X_{S^I}, (\alpha, F)) = \bigcup_{s \in F} f(\alpha)(s)$$

$$= \bigcup_{s \in F} E(\alpha, \Delta(s))$$

and

$$\bigcap_{j}^{1,n} X_{S^I}(\alpha_j, F_j) \neq \emptyset$$

iff there exists an ultrafilter U on I such that

$$\bigcap_{j}^{1,n} \bigcup_{s \in F_j} E(\alpha_j, \Delta(s)) \in U$$

iff

$$\bigcap_{j}^{1,n} \bigcup_{s \in F_j} E(\alpha_j, \Delta(s)) \neq \emptyset$$

Therefore by [29], $\varepsilon P(I) \circ Tf$ is an isomorphism.

2.22. Corollary: S is a full and faithful functor (i.e. the correspondence $\underset{\sim}{B}(B,C) \longrightarrow \underset{\sim}{K}(SB,SC)$ is a bijection).

2.23. Corollary: ηS^I is an isomorphism for each set I.

Consider $f\colon S^I \rightarrowtail\!\!\!\twoheadrightarrow S(P(I))$ in diagram (c).

2.24. Corollary: For every $A \in \underline{H}\,\underline{P}(S)$, ηA is an isomorphism.

Proof: For $g\colon S^I \twoheadrightarrow A$, by diagram (a), $\eta A \circ g = STg \circ \eta S^I$ which is surjective as STg and ηS^I are. Therefore ηA is also surjective.

2.25. THEOREM: The injective algebras in $\underset{\sim}{K} = \underline{S}\,\underline{P}(S)$ are exactly (up to isomorphism) the Boolean extensions of S by complete Boolean algebras.

Proof: Since T preserves monomorphisms, S preserves injectives (see [8]). Conversely if Q is a non-trivial injective in $\underset{\sim}{K}$, Q is a retract of some power of S say $Q \overset{f}{\rightarrowtail} S^I \overset{g}{\twoheadrightarrow} Q$ with $gf = 1_Q$. By applying T, we have $TQ \overset{Tf}{\rightarrowtail} T(S^I) \overset{Tg}{\twoheadrightarrow} TQ$ as a retract. But $T(S^I) \cong TS(P(I)) \cong P(I)$ is a complete Boolean algebra. Therefore TQ is a complete Boolean algebra and since $Q \in \underline{H}\,\underline{P}(S)$, we have by 2.24, $Q \overset{\eta Q}{\rightarrowtail\!\!\!\twoheadrightarrow} S(TQ)$.

2.26. Corollary: The finite injective algebras in $\underset{\sim}{K}$ are exactly the finite powers of S.

For S(B) is finite iff B is finite.

2.27. Corollary: ηA is essential for each $A \in \underset{\sim}{K}$.

Proof: If $f: A \rightarrowtail SB$ is a injective hull of A, we have by diagram (c), $S(\varepsilon B) \circ STf \circ \eta A = f$ is essential. Therefore ηA is essential by 1.22.

2.28. Lemma: For each $A \in \underset{\sim}{K}$ and each $B \in \underset{\sim}{B}$, $A \overset{f}{\rightarrowtail} SB$ is an injective hull of A in $\underset{\sim}{K}$ iff $TA \overset{Tf}{\rightarrowtail} TSB \overset{\varepsilon B}{\rightarrowtail\!\!\!\twoheadrightarrow} B$ is an injective hull of TA in $\underset{\sim}{B}$.

Proof: If B is injective in $\underset{\sim}{B}$, then SB is injective in K by 2.25. Conversely, if SB is injective in $\underset{\sim}{K}$ then again by 2.25, $SB \cong SC$ for some injective Boolean algebra C. But by 2.21, we have $B \cong TSB \cong TSC \cong C$ and so B is also injective. Therefore since εB is an isomorphism we need only show that f is essential iff Tf is essential.

Take $u: B \twoheadrightarrow C$ such that $u \circ \varepsilon B \circ Tf$ is a monomorphism. By applying S, we have that $S(u \circ \varepsilon B \circ Tf) \circ \eta A = Su \circ f$ is a monomorphism. If f is essential, then Su is a monomorphism hence so is u as a full and faithful functor reflects monomorphisms.

Take $g: SB \twoheadrightarrow C$ such that $g \circ f$ is a monomorphism. Since T preserves monomorphisms, $Tg \circ Tf = T(g \circ f)$ is a monomorphism. If Tf is essential, Tg is a monomorphism and therefore $g = S(\varepsilon B \circ Tg) \circ \eta SB$ is also a monomorphism.

2.29. <u>Corollary</u>: The injective hull of each $A \in \underset{\sim}{K}$ is given by $A \overset{\eta A}{\rightarrowtail} STA \overset{Sf}{\rightarrowtail} SB$ where $f\colon TA \rightarrowtail B$ is the MacNeille completion of TA.

For by [9], the injective hull of a Boolean algebra is its MacNeille completion.

2.30. <u>Corollary</u>: The injective hulls of an algebra $A \in \underset{\sim}{K}$ are unique up to unique isomorphism over A.

For S is full and faithful and therefore any isomorphism of injective hulls SB and SC of A over A are images under S of an isomorphism of the injective hulls B and C of TA and there is only one such here by [9].

CHAPTER 3

EQUATIONAL SUBCLASSES WITH ENOUGH INJECTIVES

1. Lattices

As has been noted already, the equational classes of groups, lattices, and Heyting algebras do not have enough injectives. However each does have an equational subclass that does, viz: Abelian groups, distributive lattices and Boolean Algebras respectively. The natural question then arises whether one can describe all equational subclasses of a given equational class that have enough injectives. In this section we answer this question for lattices, using Banaschewski's theorem 1.17, to simplify the original answer in [14] .

3.1. Lemma: A lattice is subdirectly irreducible iff it is an essential extension of a two element chain.

Proof: By 1.23, the condition is sufficient as any two element chain is a subdirectly irreducible lattice. Conversely, let S be a subdirectly irreducible lattice with least non-trivial congruence ϕ_S. Then there exists a $u < v$ in S such that $\theta_S(u,v) = \phi_S$ and clearly $\langle u,v \rangle \leq_E S$.

3.2. THEOREM: Let $\underset{\sim}{K}$ be an equational class of lattices. Then T.F.A.E.:

(1) $\underset{\sim}{K}$ has a non-trivial injective

(2) $\underset{\sim}{K}$ has enough injectives

(3) $\underset{\sim}{K} = \underset{\sim}{D}$, the equational class of distributive lattices.

<u>Proof</u>: (1) $\Longrightarrow$ (2): If Q is a non-trivial injective in $\underset{\sim}{K}$, then Q contains a two element chain and therefore the two element chain $2 (= 0 < 1)$ has an injective extension. But then the injective hull of 2 exists and must contain up to isomorphism all subdirectly irreducibles in $\underset{\sim}{K}$ by 3.1. Therefore $\underset{\sim}{K}$ has enough (namely one) injective subdirectly irreducibles and (2) follows from 1.24.

(2) $\Longrightarrow$ (3): If $\underset{\sim}{K} \neq \underset{\sim}{D}$, then $\underset{\sim}{K}$ contains either M_5, the modular five element lattice, or N_5, the non-modular five-element lattice. But both these subdirectly irreducibles contain non subdirectly irreducible large sublattices and therefore $\underset{\sim}{K}$ does not satisfy (CRP) by 1.28. Therefore $\underset{\sim}{K}$ does not have enough injectives.

(3) $\Longrightarrow$ (1): 2 is a non-trivial injective in $\underset{\sim}{D}$.

2. <u>Heyting Algebras</u>

A Heyting algebra is a bounded relatively pseudo-complemented lattice considered as a universal algebra of type (2,2,2,0,0) with operations $(\vee, \wedge, \rightarrow, 0, 1)$ where

$a \longrightarrow b$ is the relative pseudo-complement of a in b ($x \leq a \rightarrow b$ iff $a \wedge x \leq b$). $\underset{\sim}{H}$, the class of all Heyting algebras, is equational and the basic results may be found in [28] under the alias of pseudo-Boolean algebras. Of prime importance in our considerations is the following result. ([28], p.63-65).

3.3. THEOREM: For every $A \in \underset{\sim}{H}$, the correspondence $\theta \longmapsto [1]\theta$ is a lattice isomorphism between the congruence lattice $\Theta(A)$ and $\underline{F}(A)$, the lattice of all filters on A.

3.4. Corollary 1: $A \in \underset{\sim}{H}$ is subdirectly irreducible iff 1 is completely-join-irreducible.

For $A \in \underset{\sim}{H}_{SI}$ iff $F_o = \bigcap(\underline{F}(A) \setminus \{\{1\}\})$ is a proper filter. But then $F_o = \{s_A, 1\}$ where $s_A = \bigvee(A \setminus \{1\})$.

3.5. Corollary 2: $\underset{\sim}{H}$ satisfies (CRP).

For if $A \leq B$ and $F \in \underline{F}(A)$, then $G = \{b \in B\colon \exists a \in F \ni a \leq b\} \in \underline{F}(B)$ and $A \cap G = F$.

3.6. Corollary 3: For $A \leq B$ in $\underset{\sim}{H}$, T.F.A.E.:

(1) $A \leq_E B$

(2) For all $F \in \underline{F}(B)$ ($F \cap A = \{1\}$ implies $F = \{1\}$)

(3) For all $b \in B$, $b < 1$ implies there exists an $a \in A$ with $b \leq a < 1$.

For (2) is just the restatement of (1) in terms of filters and (3) is just the contrapositive of (2).

3.7. <u>Lemma</u>: $A \in \underset{\sim}{H}_{SI}$ iff $A = 2$, the two element Boolean algebra, or A is an essential extension of 3, the three element chain $0 < \lambda < 1$.

<u>Proof</u>: If A is a non-Boolean subdirectly irreducible, then $\lambda_A = \bigvee(A \setminus \{1\}) > 0$ and the monomorphism $f: 3 \rightarrowtail A$ by $0 \longmapsto 0$, $\lambda \longmapsto \lambda_A$ and $1 \longmapsto 1$ is an essential Heyting homomorphism by 3.6. The converse follows from 1.23, as 3 is clearly subdirectly irreducible.

This lemma is analogous to 3.1 and so we obtain the following theorem for Heyting algebras.

3.8. THEOREM: Let $\underset{\sim}{K}$ be an equational subclass of $\underset{\sim}{H}$ distinct from $\underset{\sim}{B}$, the Boolean algebras ($a \longrightarrow b = a' \vee b$). Then T.F.A.E.:

(1) $\underset{\sim}{K}$ contains a non-Boolean injective algebra

(2) $\underset{\sim}{K}$ has enough injectives

(3) $\underset{\sim}{K} = \underline{S}\,\underline{P}(H)$ for some non-Boolean $H \in \underset{\sim}{H}_{SI}$ and H is injective in $\underset{\sim}{K}$.

The proof is entirely analogous to that of 3.2 where H will be the injective hull of 3.

We now wish to determine which $H \in \underset{\sim}{H}_{SI}$ satisfy condition (3) of this theorem. Therefore throughout the remainder of this section we let $\underset{\sim}{K} = \underline{S}\,\underline{P}(H)$ be an equational class of Heyting algebras distinct from $\underset{\sim}{B}$ that has enough injectives.

3.9. Lemma: 4, the four element chain $0 < f < e < 1$ is not in $\underset{\sim}{K}$.

Proof: Assume $4 \in K$ and let $j: 3 \leq 4$ be the essential embedding (i.e. $j(e) = e$). Let $\mu_0: 3 \longrightarrow H$ be the essential embedding of 3 into H (i.e. $\mu_0(e) = e_H = \bigvee(H \setminus \{1\}) \prec 1$). Therefore there exists a monomorphism (j is essential!) $\sigma_0: 4 \rightarrowtail H$ with $\sigma_0 j = \mu_0$. We define inductively for $n \geq o$:

$$\mu_{n+1}: 3 \rightarrowtail H \qquad \mu_{n+1}(e) = \sigma_n(f)$$

$$\sigma_{n+1}: 4 \rightarrowtail H \qquad \text{with } \sigma_{n+1} \circ j = \mu_{n+1}$$

Let $e_n = \mu_n(e)$, $n \geq o$. Then since all σ_n are monomorphisms

$$e_{n+1} = \mu_{n+1}(e) = \sigma_n(f) < \sigma_n(e) = \mu_n j(e)$$
$$= \mu_n(e) = e_n$$

Moreover

$$e_n \longrightarrow e_{n+1} = \sigma_n(e \longrightarrow f) = \sigma_n(f) = e_{n+1}.$$

Therefore H contains all finite chains as subalgebras and since $\underset{\sim}{K}$ is equational, $\underset{\sim}{K}$ contains all chains. In particular $\underset{\sim}{K}$ contains all successor ordinals considered as Heyting algebras in their natural partial order and therefore $\underset{\sim}{K}$ has a proper class of non-isomorphic essential extensions of 3, a contradiction on 1.17.

For every Boolean algebra $B(= (B,(\vee,\wedge,',o,e)))$, we let B^1 be the set $B \cup \{1\}$ $(1 \notin B)$ with the partial order $x \leq y$ iff $y = 1$ or $x \leq y$ in B. It is clear that B^1 is a lattice and in fact a Heyting algebra where:

$$x \longrightarrow y = \begin{cases} 1 & , \quad x \leq y \\ y & , \quad x = 1 \\ x' \vee y, & \text{otherwise} \end{cases}$$

In [24], Lee noted that this lattice is a distributive pseudo-complement lattice but both these observations are just special cases of the notion of star sums defined in [6].

3.10. <u>DEFINITION</u>: For each $n \geq o$, $A_n = P(n)^1$, where $P(n)$ is a power set Boolean algebra on an n-element set (Note: $A_o \cong 2$ and $A_1 \cong 3$).

3.11. <u>Lemma</u>: $H = A_n$ for some natural number $n \geq o$.

<u>Proof</u>: By 3.9, $4 \notin \underset{\sim}{K}$ and since $\underset{\sim}{K} = \underline{S}\,\underline{P}(H)$, this is equivalent to $4 \notin H$. Therefore H has only two dense elements $\{e_H, 1\}$. Since $x \leq e_H$ for all $x \in H \setminus \{1\}$, we have for all $x \neq o$, e_H, 1:

$$x \vee x^* = e_H = x^{**} \vee x^*$$

where $x^* = x \longrightarrow o$. Since every Heyting algebra is a distributive lattice, we have $x = x^{**}$ for all $x \neq e$ and therefore $[o,e_H]$ is a Boolean algebra under the restricted partial order and $H = [o,e_H]^1$.

Now if $[o,e_H]$ is infinite, $\underset{\sim}{K}$ contains all A_n, $n \geq o$ and therefore all B^1, for $B \in \underset{\sim}{B}$. But this would give a proper <u>class</u> of non-isomorphic essential extensions of 3, contradicting 1.17.

If $\underset{\sim}{A}_n$ is the equational subclass of H generated by A_n, we see that $\underset{\sim}{A}_n = \underline{S}\,\underline{P}(A_n)$ as every proper homomorphic image of an A_k is a Boolean algebra. Moreover all subalgebras of A_n are of the form A_k, $k \leq n$ and so $\underset{\sim}{A}_n$ satisfies the conditions of 2.3.

3.12. <u>Lemma</u>: A_n is self-injective iff $n \leq 2$.

<u>Proof</u>: It is clear that A_1 and A_2 are self injective. If $n \geq 3$, let $a_1, a_2, a_3, \ldots, a_n$ be the atoms of A_n. Consider A_2 as the subalgebra $\{o, a_1, a_1^*, e, 1\}$ of A_n ($e = \bigvee_i^{1,n} a_i$ and $a_1^{**} = a_1$) and define $f\colon A_2 \rightarrowtail A_n$ by:

$$o \longmapsto o,\; a_1 \longmapsto a_1^*,\; a_1^* \longmapsto a_1,\; e \longmapsto e \text{ and } 1 \longmapsto 1$$

If A_n were self injective, there exists an endomorphism $g\colon A_n \longrightarrow A_n$ extending f.

Since $g(e) = f(e) \neq f(1) = g(1)$, g is a monomorphism Since $o < a_2 < a_1^*$ $(a_2 \vee a_3 \leq a_1^*)$, we must have

$$o = g(o) < g(a_2) < g(a_1{}^*) = f(a_1{}^*) = a_1$$

which is a contraction on the fact that a_1 is an atom.

Combining these lemmas, we obtain a complete characterization of the equational classes of Heyting algebras with enough injectives.

3.13. <u>THEOREM</u>: The only equational classes of Heyting algebras with enough injectives are $\underset{\sim}{B}$, $\underset{\sim}{A}_1$, and $\underset{\sim}{A}_2$. Moreover every other equational subclass of $\underset{\sim}{H}$ has exactly the injective Boolean algebras as its injective algebras.

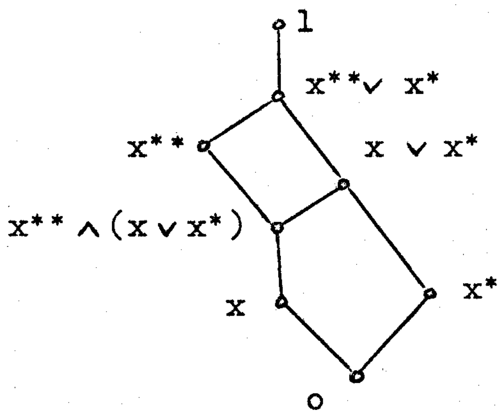
3. <u>Pseudo-Complemented Lattices</u>

We call a pseudo-complemented lattice a *-algebra when it is considered as a universal algebra of type (2,2,1,0,0) with operations ($\vee$, $\wedge$, *, 0, 1) ($x \leq a^*$ iff $a \wedge x = o$). From [5], we can infer that $\underset{\sim}{P}$, the class of all *-algebras is indeed an equational class with the Boolean algebras, $\underset{\sim}{B}$ as its only equationally complete subclass, Lee, [24], has characterized the lattice of equational classes of distributive *-algebras where he considers the Heyting algebras (A_n: $n \geq o$) defined in 3.10 only as *-algebras. When so considered, we will write B_n; $n \geq o$ to avoid ambiguity with section 2. Since lemma 3.12 and the remarks before make no explicit use of the "arrow" operation, we may conclude immediately:

3.15. <u>THEOREM</u>: The only equational classes of distributive *-algebras with enough injectives are $\underset{\sim}{B}$, $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$, where $\underset{\sim}{B}_i = \underline{S}\,\underline{P}(B_i)$, the equational subclass of $\underset{\sim}{P}$ generated by B_i, (i = 1,2).

In order to extend these results, we require some more information about the lattice of equational subclasses of P.

3.16. <u>Lemma</u>: FP(1), the free *-algebra on one generator is given by the following diagram:



<u>Proof</u>: The diagram determines a *-algebra with one generator x and in any *-algebra we have:

$$x \vee x^* \leq (x^{**} \wedge (x \vee x^*)) \vee x^* \leq x \vee x^*$$

$$(x^{**} \wedge (x \vee x^*))^* = x^* \qquad \text{since}$$

$$x \leq x^{**} \wedge (x \vee x^*) \leq x^{**}$$

Therefore the only identities satisfied in the above *-algebra are consequences of the defining equations of $\underset{\sim}{P}$.

By the congruences $\Theta(o,x^*)$, $\Theta(x,x^{**})$ on FP(1), we see that B_1 and B_2 are retracts of FP(1) and hence are projective in $\underset{\sim}{P}$. The other non-Boolean subdirectly irreducible image of FP(1) is N_5 considered as a *-algebra by the congruence $\Theta(x \vee x^*, 1)$. It is clear that there exists largest equational subclasses of $\underset{\sim}{P}$ that do not contain B_1, B_2 and N_5 respectively. These are given by the equations $x \vee x^* = 1$, $x^{**} \vee x^* = 1$ and $x = x^{**} \wedge (x \vee x^*)$ respectively and are denoted by $\underset{\sim}{P}|B_1$, $\underset{\sim}{P}|B_2$ respectively $\underset{\sim}{P}|N_5$.

3.17. <u>Lemma</u>: A *-algebra A is Boolean iff neither $B_1 \leq A$ nor $N_5 \leq A$.

<u>Proof</u>: If $B_1 \nleq A$, then A satisfies the equation $x \vee x^* = 1$. If A is non-Boolean, there exists an $a \in A$ with $a < a^{**}$. Therefore $a^{**} \neq 1$ and $N_5 \cong \{o, a, a^*, a^{**}, 1\} \leq A$.

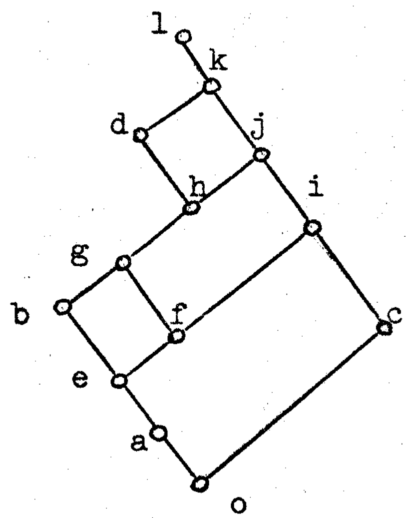
It is clear that $\underset{\sim}{N_5}$ and $\underset{\sim}{B_1}$ are covers of $\underset{\sim}{B}$ and by this last lemma, every non-Boolean equational subclass of $\underset{\sim}{P}$ must contain either $\underset{\sim}{N_5}$ or $\underset{\sim}{B_1}$.

3.18. <u>THEOREM</u>: The injectives in $\underset{\sim}{P}$ are exactly the injective Boolean algebras.

<u>Proof</u>: The Glivenko-Stone theorem implies that the natural reflection $R: \underset{\sim}{P} \longrightarrow \underset{\sim}{B}$ preserves monomorphisms. By 2.10 then, the injective Boolean algebras are injective in $\underset{\sim}{P}$. By 3.17, we may prove the theorem by showing that B_1 and N_5 have arbitrarily large essential extensions. Since $B_1 \leq_E B^1$ for every Boolean algebra B, B_1 has arbitrarily large essential extensions. For each cardinal α, let $N_5(\alpha) = P(\alpha) \dot{\cup} \{a\}$ where $\emptyset < a < \{\beta\}$ for some $\beta \in \alpha$. Then $N_5 \cong \{\emptyset, a, \{\beta\}, \alpha \setminus \{\beta\}, \alpha\}$ which is a large subalgebra of $N_5(\alpha)$.

If $S \in \underset{\sim}{P}_{SI}$, then the least congruence on S must be smaller than the Glivenko-Stone congruence $\bigvee\{\theta(x,x^{**}): x \in S\}$ and so must be generated by a pair (a,b) with $a < b$ and $a^* = b^*$. If $S \in \underset{\sim}{K}$ and $\underset{\sim}{K}$ has enough injectives, then $\underset{\sim}{K}$ satisfies (CRP) and $\langle a,b\rangle \leq_E S$ and is subdirectly irreducible. Therefore the following *-algebra is of some interest.

3.19. <u>THEOREM</u>: $FP(2: x \wedge y = x, x^* = y^*)$, the free *-algebra on two generators $\{x,y\}$ satisfying the relations $x \leq y$ and $x^* = y^*$ is given by the following diagram:



<u>Proof</u>: Since $FP(2: x \wedge y = x, x^* = y^*) \cong FP(2)/\theta$ where $\theta = \theta(x \wedge y, x) \vee \theta(x^*,y^*)$ we define $a = [x]\theta$, $b = [y]\theta$, $c = [x^*]\theta = [y^*]\theta$, etc. Then if $M = \{p \in FP(2): m \in u$ for some $u \in \{o,a,b,c,d,e,f,g,h,i,j,k,l\}\}$ we have by easy algebraic induction that $M = FP(2)$ and the diagram defines the partial order of $FP(2: x \wedge y = x, x^* = y^*)$.

3.20. <u>Corollary</u>: If $\underset{\sim}{K}$ is an equational class of *-algebras that satisfies (CRP), then for every $S \in \underset{\sim}{K}_{SI}$ and every pair $s,t \in S$, $s < t$ and $\Phi_S = \Theta(s,t)$, $\langle s,t \rangle$ is a subdirectly irreducible homomorphic image of $FP(2\colon x \wedge y = x, x^* = y^*)$ by the map $a \longmapsto s$ and $b \longmapsto t$. Moreover the least non-trivial congruence on $\langle s,t \rangle$ is generated by (s,t).

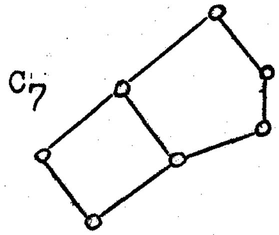
Now the completely-meet-irreducible congruences on $P = FP(2\colon x \wedge y = x,\ x^* = y^*)$ are given by the following partitions:

$$\Theta_1\colon \{\{o,c\},\ \{a,e,f,i\},\ \{b,d,g,h,j,k,l\}\}$$
$$\Theta_2\colon \{\{b,d,e,f,g,h\},\ \{i,j,k,l\},\ \{o\},\ \{a\},\ \{c\}\}$$
$$\Theta_3\colon \{\{o\},\ \{a,e,f\},\ \{b,g\},\ \{c\},\ \{d,h\},\ \{i\},\ \{j,k,l\}\}$$
$$\Theta_4\colon \{\{o,c\},\ \{a,b,e,f,g,h,i,j\},\ \{d,k,l\}\}$$
$$\Theta_5\colon \{\{o\},\ \{a,b,d,e,f,g,h\},\ \{c\},\ \{i,j,k\},\ \{l\}\}$$
$$\Theta_6\colon \{\{o\},\ \{a,b,e\},\ \{c\},\ \{d,f,g,h\},\ \{i,j,k,l\}\}$$
$$\Theta_7\colon \{\{o,c\},\ \{a,b,d,e,f,g,h,i,j,k,l\}\}$$
$$\Theta_8\colon \{\{o,a,b,d,e,f,g,h\},\ \{c,i,j,k,l\}\}$$

and if we let $P_i = P/\Theta_i$, $i = 1,2,\ldots,8$ we have

$$P_7 \cong P_8 \cong 2, \quad P_1 \cong P_4 \cong B_1, \quad P_5 \cong B_2, \quad P_6 \cong P_2 \cong N_5$$

and P_3 isomorphic to the following *-algebra



We are interested in those i = 1,...,8 for which the following two properties hold:

(1) θ_i separates (a,b)

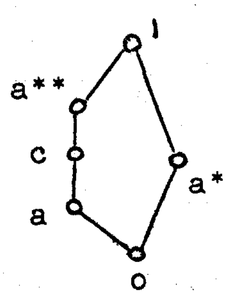
(2) The least non-trivial congruence on P_i is generated by ($[a]\theta_i$, $[b]\theta_i$).

By direct inspection we see that i = 1 and i = 2 are the only possibilities whence:

3.21. THEOREM: If $\underset{\sim}{K}$ is an equational class of *-algebras that satisfies (CRP), then every non-Boolean $S \in \underset{\sim}{K}_{SI}$ is an essential extension of B_1 or N_5. Moreover if $\Phi_S = \theta(a,b)$ with $a < b$ we have $a^{**} = b$ and $a \prec a^{**}$ (a is covered by a**).

Proof: We need only show the second part.

If $a < b$ and $\theta(a,b) = \Phi_S$, then $\langle a,b\rangle \cong P_1$ or P_2 and in both cases $a^{**} = b$. If $a < c < a^{**}$ then S contains as a large subalgebra either the four element chain $o < a < c < 1$ or the *-algebra



neither of which is subdirectly irreducible, contradicting 1.28.

3.22. Corollary: If $\underset{\sim}{K}$ is an equational class of *-algebras with enough injectives, then $\underset{\sim}{K}$ has at most three injective subdirectly irreducibles namely 2, and the injective hulls of B_1 and N_5.

If we restrict our attention now to equational subclasses of $\underset{\sim}{P}|N_5$, we obtain the following theorem in the same way as 3.2 and 3.8.

3.23. THEOREM: For equational class $\underset{\sim}{K} \subseteq \underset{\sim}{P}|N_5$ that satisfies (CRP), T.F.A.E.:

(1) $\underset{\sim}{K}$ has a non-Boolean injective algebra

(2) $\underset{\sim}{K}$ has enough injectives

(3) $\underset{\sim}{K} = \underline{S}\,\underline{P}(H)$ where H is a non-Boolean injective subdirectly irreducible in $\underset{\sim}{K}$.

Our characterization theorem now follows.

3.24. THEOREM: The only equational subclasses of $\underset{\sim}{P}|N_5$ that have enough injectives are $\underset{\sim}{B}$, $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$.

Proof: $\underset{\sim}{K} = \underline{S}\,\underline{P}(H)$ for some $H \in (\underset{\sim}{P}|N_5)_{SI}$. If $\underset{\sim}{K} \neq \underset{\sim}{B}$, then H is non-Boolean and $B_1 \leq_E H$ by some dense element $e \prec 1$. If $u \in H$, $u^* = o$ and $e \nleq u$, then $\{o, e \wedge u, e, 1\}$ forms a non-subdirectly irreducible large subalgebra of H. By 1.28 then, H has only two dense elements $\{e, 1\}$.

Suppose there exists an $a \in H \setminus \{o,e,1\}$ with $a \vee a^* = 1$. Then without loss of generality we may assume that $a = a^{**}$ and $a^* \nleq e$. Now:

$$(a^* \wedge e)^* = (a^{***} \wedge e^{**})^* = a^{**} = a$$

so $a \vee (a^* \wedge e)$ is dense.

If $a \vee (a^* \wedge e) = 1$, then the subalgebra $\{o,a,a^*,a^* \wedge e,1\}$ is isomorphic to N_5, a contradiction on the fact that $\underset{\sim}{K} \subseteq \underset{\sim}{P}|N_5$.

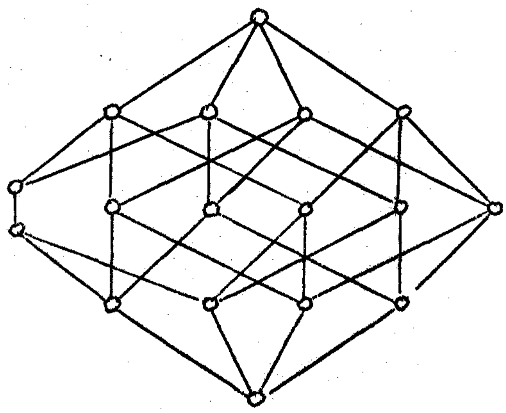
If $a \vee (a^* \wedge e) = e$, then the subalgebra $\{o,a,a^*,a^* \wedge e,e,1\}$ is isomorphic to $2xB_1$, and is a large subalgebra of H that is not subdirectly irreducible, a contradiction on 1.28.

Therefore for all $a \in H \setminus \{o,e,1\}$, $a \vee a^* = e$ and $a = a^{**} \wedge (a \vee a^*) = a^{**} \wedge e = a^{**}$.

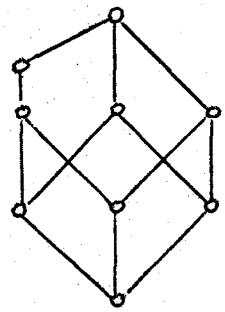
But then H is of the form B^1 and hence K is an equational class of distributive *-algebras. The result then follows from 3.15.

If we restrict ourselves to $\underset{\sim}{P}|B_1$, we have only the partial result:

3.25. <u>THEOREM</u>: The equational subclasses of $\underset{\sim}{P}|B_1$ generated by N_5, N_9, N^9, N_{17} have enough injectives where N^9 is the dual of N_9 and the others are given by the diagrams:



N_{17}



N_9

<u>Proof</u>: Since every subalgebra of N_5 and N_{17} is either subdirectly irreducible or Boolean (hence injective in P !) we may apply 2.3 as both N_5 and N_{17} are self-injective.

The conjecture is that these are the only equational subclasses of $\underset{\sim}{P}|B_1$ with enough injectives but so far we have no proof.

3.26. <u>THEOREM</u>: The join of $\underset{\sim}{N}_5$, $\underset{\sim}{N}_9$, $\underset{\sim}{N}^9$, or $\underset{\sim}{N}_{17}$ with $\underset{\sim}{B}_1$ or $\underset{\sim}{B}_2$ has enough injectives.

<u>Proof</u>: Since the respective intersection is always $\underset{\sim}{B}$, the natural reflections into $\underset{\sim}{B}$ preserve monomorphisms and the result follows from 2.12.

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