FIBRED CATEGORIES I

FIBRED CATEGORIES AND THE THEORY OF STRUCTURES

(PART I)

By

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A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

May, 1966

DOCTOR OF PHILOSOPHY (1966) (Mathematics)

McMASTER UNIVERSITY Hamilton, Ontario.

TITLE: Fibred Categories and the Theory of Structures - (Part I)

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NUMBER OF PAGES: V, 164

SCOPE AND CONTENTS: This THESIS comprises the core of Chapter I and a self-contained excerpt from Chapter II of the author's work "Fibred Categories and the Theory of Structures". As such, it contains a recasting of "categorical algebra" on the (BOURBAKI) set-theoretic frame of GROTHENDIECK-SONNERuniverses, making use of the GROTHENDIECK structural definition of category from the beginning. The principle novelties of the presentation result from the exploitation of an intrinsic construction of the arrow category $\underset{\sim}{C}^2$ of a $\underset{\sim}{\mathfrak{U}}$ -category $\underset{\sim}{C}$. This construction gives rise to the adjunction of a (canonical) ($\underset{\sim}{\mathfrak{U}}$ -$\underset{\sim}{CAT}$)-category structure to the couple $(\underset{\sim}{C}^2,\underset{\sim}{C})$, for which the consequent category structure supplied the couple $(\underset{\sim}{CAT}(\underset{\sim}{T},\underset{\sim}{C}^2),\underset{\sim}{CAT}(\underset{\sim}{T},\underset{\sim}{C}))$ for each category $\underset{\sim}{T}$, is simply that of natural transformations of functors (which as such are nothing more than functors into the arrow category).

ACKNOWLEDGEMENTS

The author wishes to express grateful thanks to his advisor, Professor G. Bruns, without whose continued support he would never have got to the stage of writing this thesis, but whose high standards of scholarship he fears he has not always reached.

To Professor G. Sabidussi and McMaster University for personal and financial support, respectively.

To Professor B. Banaschewski, who introduced him to categories and functors (and to Professor F. D. Quigley who introduced him to Bourbaki and Professor Banaschewski).

To Professor S. Takahashi who introduced him to the work of Grothendieck and the notion of representability.

In addition, he cannot fail to acknowledge his profound debt to Professor A. Grothendieck himself whose papers have inspired almost everything the author of this thesis has written (but whose responsibility for any lack of comprehension on the part of the author is, of course, non-existent).

Last, but far from least, he realises at this late hour that it is almost impossible to adequately thank both Mrs. Maureen von Lieres, and my wife Geraldine, for their infinite patience and consummate skill in translating incomprehensible heiroglyphics into a form at least more open to evaluation.

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TO E. P. MOSELEY

- who introduced me to mathematics

& TO MY FATHER

- who made it possible for me to make it possible

INTRODUCTION

Fibred Categories and the Theory of Structures, of which this THESIS forms most of Chapter I and a short excerpt of Chapter II, has as its purpose the fulfilment of the preferatory promise made in DUSKIN (1963) to "rewrite BOURBAKI'S ⟨⟨structures⟩⟩ (1957) either in terms of ⟨⟨ categories and functors ⟩⟩ or vice-versa" by rewriting them both in terms of each other - at the same time. This curious feature is made possible by the "technical device" of (GROTHENDIECK-SONNER-TARSKI) (c.f. GABRIEL (1962), SONNER(1962), TARSKI(1960), ETC.) ⟨⟨ universes ⟩⟩ axiomatically adjoined in a compatible fashion to, in this case, BOURBAKI'S THÉORIE DES ENSEMBLES. The remainder of the mathematical theory has then a "model" which is "closed" under all "set-theoretic operations" with, as the only restriction, that these operations be indexed by a member of the universe $\mathfrak{U}$.

For a given universe $\mathfrak{U}$, the meta-mathematical theory of structures can be given a formal functorial mathematical treatment at least within the totality of sets of $\mathfrak{U}$. To carry this process out is the purpose of CHAPTER III.

In that context the BOURBAKI notion of ⟨⟨ initial structure ⟩⟩, for instance, becomes a quite special case of the elegant and useful generalization ⟨⟨ inverse image of a f[illegible] morphisms by a functor defined by GROTHENDIECK (1962) and whose terminology I have largely adopted, but which I can happily assert I discovered for myself independently in 1963 (abstracting BOURBAKI (1957)). The attendant notions of ⟨⟨ fibred-category ⟩⟩ and ⟨⟨ *co-fibred* category ⟩⟩ of GROTHENDIECK (1962)there are, naturally occurring notions in a remarkably wide variety of situations (none of which happen to be given as examples by GROTHENDIECK, curiously enough).

Implicit in the functorial version of this "theory of structures" is the *lack* of any necessity to *postulate* that the objects of the base category be only "unstructured" members of $\mathfrak{U}$; the *definition* being possible quite often without any such supposition (the representability of the structure is quite another thing!). The, by now, well known notion of group in the category of topological spaces, for example, forms an excellent example of the this notion, for which a combination of the methods of LINTON (1965) and LAWVERE (1965) with those of GROTHENDIECK lead to quite satisfactory results at least in the case of "algebraic structures".

In reverse order, Chapter II, of which only a very small excerpt has been included here, studies the "formal calculus of binary relations" by itself. It is interesting to note that a considerable portion of this calculus can be carried out in categories which do not even have finite products with *pre-correspondences* which are not even (the representations of) graphs. The section included here is only the ⟨⟨ *naive* ⟩⟩ and ⟨⟨ unprojected ⟩⟩ theory, which, in spite of its simplicity, is remarkably useful.

With the aid of "projection" defined by means of additional assumptions one can obtain as much as one desires of the RIGUET (1948) – BOURBAKI "calculus". It thus offers a non-abelian generalization of the useful (partly abelian) theory of PUPPE (1962)+ McLANE (1964).

REMARK While one is giving these credits, it should be mentioned that RIGUET (1948) observed that the "multiplication in a BRANT-Groupoid was a "difunctional relation". Now a difunctional relation (RIGUET'S term) is nothing other than a fibre-product (with slightly less stringent requirements(i.e. of the form $\langle\langle \hat{g}^{1}\cdot h \rangle\rangle$, with h, g, quasi-functions, rather than functions). Consequently he may be said to be a god father at least of the structural definition of a category used here, and invented (in a different context) by GROTHENDIECK).

Chapter I, which follows this introduction, starts from the "structural definition" of category and develops most of the "general theory" of categories and functors through this definition.

It seems to be implicit with this sort of definition that a ⟨⟨ natural transformation of functors ⟩⟩ should have something to do with the arrows of the arrow category which are traditionally represented as commutative squares, and this is indeed the case. The construction of the arrow category given here does not require the notion of natural transformation of functor in its definition, and indeed seems to be the very defining object for this notion. The resulting functorization of this *sine qua non* of category theory seems to have useful consequences and seems to lead to other considerations, which will be explored in Chapter III. In any case, we have included in Chapter I, and thus in this thesis, all of the relevant (as well as some of the irrelevant, we fear) functorial notions and have tied them together, hopefully, once-and-for-all.

Chapter O, which has been omitted here, simply reviews the set-theoretic formalism used in the work which has been in nearly all cases that of BOURBAKI or GROTHENDIECK. The set theoretic propositions which are proved in (O) are easily found in BOURBAKI or established by the reader himself without difficulty. The next paragraph of comments in this brief introduction may be of help to the reader unfamiliar with the notion of ⟨⟨ fibre product ⟩⟩ - which appears to be the one genuinely unifying notion of Chapter I. For this concept we use the term cartesian square (of GROTHENDIECK) rather than "pull-back-diagram" often preferred by American authors. This last usage has been dictated purely by considerations of ease of translation, at least from French to English and back, where "cartesian" is obviously more satisfactory (even if possibly less evocative in some contexts).

NOTE ON FIBRE-PRODUCTS OF SETS : Let A and B be sets and $f : A \longrightarrow X$, $g : B \longrightarrow X$ be a pair of applications. The graph $g^{-1} \circ f = \{ (a,b) / f(a) = g(b) \} \subseteq A \times B$ is called the fibre product (or fibred-product, if one prefers) of A with B over X and is usually denoted by ⟨⟨ $A \times_X B$ ⟩⟩ or ⟨⟨ $A \times_{f,g} B$ ⟩⟩ or simply ⟨⟨ $g^{-1} \cdot f$ ⟩⟩. It inherits from the product A x B nearly all of its properties such as associativity, commutativity, etc. For the purposes of category theory its usefulness seems to be unlimited as will be apparent in the course of Chapter I. It is usually convenient to call a square (D) of sets and applications cartesian provided the (so called) lifted application, $a \boxtimes b : R \longrightarrow A \times B$ defined by $\langle\langle\, r \rightsquigarrow (a(r), b(r)) \,\rangle\rangle$ defines a bijection of R onto $A \times_X B$.

$$\begin{array}{ccc} R & \xrightarrow{b} & B \\ {\scriptstyle a}\downarrow & (D) & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

Keep in mind that if g admits a section, then so does a; we will often denote such forced applications by the use of $\langle\langle\ *\ \rangle\rangle$. Thus in DEFINITION (1.0.1) $I^*(\underset{\sim}{C})$ is such a defined application induced by the section $I(\underset{\sim}{C})$.

CHAPTER I

§1 CATEGORICAL ALGEBRA

1.0 $\underset{\sim}{\mathfrak{U}}$- CATEGORIES

DEFINITION (1.0.1) A category $\underset{\sim}{C}$ is a couple consisting of a set $\mathcal{O}b(\underset{\sim}{C})$, called the set of objects of $\underset{\sim}{C}$, and a set $\mathcal{F}l(\underset{\sim}{C})$, called the set of arrows (or morphisms) of $\underset{\sim}{C}$ supplied with the following structure:

$(SC)_I$ a couple $(\sigma_0(\underset{\sim}{C}), \sigma_1(\underset{\sim}{C})) : \mathcal{F}l(\underset{\sim}{C}) \rightrightarrows \mathcal{O}b(\underset{\sim}{C})$ of applications, called, respectively the source and target applications of $\underset{\sim}{C}$,

$(SC)_{II}$ an application $\underset{\sim}{I}(\underset{\sim}{C}) : \mathcal{O}b(\underset{\sim}{C}) \longrightarrow \mathcal{F}l(\underset{\sim}{C})$ called the identity assignment of $\underset{\sim}{C}$, and

$(SC)_{III}$ an application $\mu(\underset{\sim}{C}) : \mathcal{F}l(\underset{\sim}{C}) \underset{\sigma_1,\sigma_0}{\times} \mathcal{F}l(\underset{\sim}{C}) \longrightarrow \mathcal{F}l(\underset{\sim}{C})$

called composition (or multiplication) of arrows in $\underset{\sim}{C}$, which is required to satisfy the following axioms:

$(AC)_I$ $\sigma_0(\underset{\sim}{C}) \circ \mu(\underset{\sim}{C}) = \sigma_0(\underset{\sim}{C}) \cdot pr_1$ and $\sigma_1(\underset{\sim}{C}) \cdot \mu(\underset{\sim}{C}) = \sigma_1(\underset{\sim}{C}) \cdot pr_2$

$(AC)_{II}$ $\mu(\underset{\sim}{C}) \circ (\underline{id}_{\mathcal{F}l(\underset{\sim}{C})} \times \mu(\underset{\sim}{C})) = \mu(\underset{\sim}{C}) \cdot (\mu(\underset{\sim}{C}) \times \underline{id}_{\mathcal{F}l(\underset{\sim}{C})})$;

$(AC)_{III}$ $\sigma_0(\underset{\sim}{C}) \circ \underset{\sim}{I}(\underset{\sim}{C}) = \underline{id}_{\mathcal{O}b(\underset{\sim}{C})} = \sigma_1(\underset{\sim}{C}) \cdot \underset{\sim}{I}(\underset{\sim}{C})$; and

$(AC)_{IV}$ $\mu(\underset{\sim}{C}) \circ \underset{\sim}{I}(\underset{\sim}{C})^{*0} = \underline{id}_{\mathcal{F}l(\underset{\sim}{C})} = \mu(\underset{\sim}{C}) \circ \underset{\sim}{I}(\underset{\sim}{C})^{*1}$.

In order to facilitate the interpretation of these axioms we shall investigate them in detail.

$(SC)_I$ says that any arrow f in $\mathbf{C}$ has a (uniquely determined) source and target defined by $\sigma_o(\mathbf{C})$ (f) and $\sigma_1(\mathbf{C})$ (f) respectively. We will use ⟨⟨f: $A \to B$⟩⟩ as an abbreviation for ⟨⟨$f \in \mathfrak{Fl}(\mathbf{C})$ and $\sigma_o(f) = A$ and $\sigma_1(f) = B$⟩⟩ and will denote by $\mathbf{C}(A,B)$ (or $\mathrm{Hom}_{\mathbf{C}}(A,B)$ or $B(A)$) the set of all arrows of $\mathbf{C}$ with source A and target B.

$(SC)_{II}$ says that each object A in $\mathbf{C}$ has associated with it an arrow $\mathbf{I}(\mathbf{C})(A)$ called the <u>identity arrow</u> of the object A (which will denote by I_A), whose source and target is simply the object A (by $(AC)_{III}$).

Since every arrow has a source and target we may consider the set $\mathfrak{Fl}(\mathbf{C}) \underset{\sigma_1,\ \sigma_o}{\times} \mathfrak{Fl}(\mathbf{C})$ of those couples (f, g) of arrows such that the target of f coincides with the source of g. $(SC)_{III}$ says that for such couples one has a law of composition defined which assigns to each such couple (f, g) an arrow $\mu(\mathbf{C})$ (f, g) called the <u>composition of the arrow</u> f <u>with the arrow</u> g and ordinarily denoted by gf. $(AC)_I$ then says that the following diagrams of sets and applications commute:

(1.0.1.1)
$$\begin{tikzcd} \mathfrak{Fl}(\mathbf{C}) \times_{\sigma_1,\sigma_o} \mathfrak{Fl}(\mathbf{C}) \arrow[r, "pr_2"] \arrow[d, "\mu"'] & \mathfrak{Fl}(\mathbf{C}) \arrow[d, "\sigma_1"] \\ \mathfrak{Fl}(\mathbf{C}) \arrow[r, "\sigma_1"'] & \mathfrak{Ob}(\mathbf{C}) \end{tikzcd}$$

(1.0.1.2)
$$\begin{tikzcd} \mathfrak{Fl}(\mathbf{C}) \times_{\sigma_1,\sigma_o} \mathfrak{Fl}(\mathbf{C}) \arrow[r, "\mu"] \arrow[d, "pr_1"'] & \mathfrak{Fl}(\mathbf{C}) \arrow[d, "\sigma_o"] \\ \mathfrak{Fl}(\mathbf{C}) \arrow[r, "\sigma_o"'] & \mathfrak{Ob}(\mathbf{C}) \end{tikzcd}$$

i.e. that the target of the composed arrow gf is the same as that g and the source of the composed arrow is the same as that of f. In short that if $f : A \to B$ and $g : B \to C$ be given then gf is defined and $gf : A \to C$.

$(AC)_{II}$ says that the following diagram commutes:

$$(1.0.1.2)\qquad \begin{tikzcd} \mathcal{F}\ell(\underset{\sim}{C}) \underset{\sigma_1,\sigma_0}{\times} \mathcal{F}\ell(\underset{\sim}{C}) \underset{\sigma_1,\sigma_0}{\times} \mathcal{F}\ell(\underset{\sim}{C}) \arrow[r, "\mu \times id_{\mathcal{F}\ell(C)}"] \arrow[d, "id_{\mathcal{F}\ell(C)} \times \mu"'] & \mathcal{F}\ell(\underset{\sim}{C}) \arrow[d, "\mu"] \\ \mathcal{F}\ell(\underset{\sim}{C}) \underset{\sigma_1,\sigma_0}{\times} \mathcal{F}\ell(\underset{\sim}{C}) \arrow[r, "\mu"] & \mathcal{F}\ell(\underset{\sim}{C}) \end{tikzcd}$$

i.e. that given a triple (f,g,h) such that gf and hg be defined then $h(gh) = (hg)f$. In more familiar terms: <u>composition of arrows is associative</u> (whenever defined).

$(AC)_{III}$ allows us to define the applications $(\underset{\sim}{I}(\underset{\sim}{C})^{*0}, \underset{\sim}{I}(\underset{\sim}{C})^{*1}) : \mathcal{F}\ell(\underset{\sim}{C}) \rightrightarrows \mathcal{F}\ell(\underset{\sim}{C}) \underset{\sigma_1,\sigma_0}{\times} \mathcal{F}\ell(\underset{\sim}{C})$ by $f \rightsquigarrow (I_{\sigma_0(f)}, f)$ and $f \rightsquigarrow (f, I_{\sigma_1(f)})$ respectively. $(AC)_{IV}$ says that $fI_{\sigma_0(f)} = f$ and $I_{\sigma_1(f)} f = f$ whatever be $f \in \mathcal{F}\ell(\underset{\sim}{C})$, i.e. that the identity arrows behave as identity elements under the composition whenever that composition be defined.

(1.0.2) The observations of the preceding paragraph show that we could have defined a category $\underset{\sim}{C}$ as a non void set $\mathcal{O}b(\underset{\sim}{C})$ of <u>objects</u> such that for each couple (A,B) of objects one is given a set $\underset{\sim}{C}(A,B)$ called the set of <u>arrows of</u> A <u>into</u> B, provided that these sets of arrows are <u>composable</u> through the donation for each triple (A,B,C) of objects of $\underset{\sim}{C}$ of a <u>law of composition</u> $\mu : \underset{\sim}{C}(A,B) \times \underset{\sim}{C}(B,C) \rightarrow \underset{\sim}{C}(A,C)$ which is required to be <u>associative</u>, i.e. $f : A \rightarrow B$, $g : B \rightarrow C$, $h : C \rightarrow D$ implies that $h(gh) = (hg)f$, where $\langle\langle f : A \rightarrow B \rangle\rangle$ is here defined by $\langle\langle f \in \underset{\sim}{C}(A,B) \rangle\rangle$, and have for each $A \in \mathcal{O}b(\underset{\sim}{C})$ an arrow $I_A \in \underset{\sim}{C}(A,A)$ such that $I_A f = f$ and $gI_A = g$ in each case that these compositions be defined.

One may then define the set of arrows of $\underset{\sim}{C}$ by $\mathcal{A}\ell(\underset{\sim}{C}) = \coprod_{(A,B)\in \mathcal{O}b(C)\times\mathcal{O}b(C)} \underset{\sim}{C}(A,B)$ or under the additional assumption that $(A,B) \neq (A',B')$ implies $\underset{\sim}{C}(A,B) \cap \underset{\sim}{C}(A',B') = \emptyset$, as $\bigcup_{(A,B)\in\mathcal{O}b(C)\times\mathcal{O}b(C)} \underset{\sim}{C}(A,B)$. In either case, the effect is simply to fulfil the "condition essentially of prudence" that each arrow have a uniquely determined source and target, i.e. that the source and target functions be definable. (In this definition for $f \in \underset{\sim}{C}(A,B)$, A is defined to be source of f and B the target of f).

To pass from one definition to the other, one simply notes that, in the first case the functions σ_0 and σ_1 define an application $\sigma_0 \blacksquare \sigma_1 : \mathcal{F}\ell(\underset{\sim}{C}) \to \mathcal{O}b(\underset{\sim}{C}) \times \mathcal{O}b(\underset{\sim}{C})$ by $f \rightsquigarrow (\sigma_0(f), \sigma_1(f))$ such that $\underset{\sim}{C}(A,B) = \overline{\sigma_0 \blacksquare \sigma_1}^{-1}\left\langle \{(A,B)\} \right\rangle$, while in the second case $f \in \mathcal{F}\ell(\underset{\sim}{C})$ implies that $f \in \underset{\sim}{C}(A,B)$ for some unique couple $(A,B) \in \mathcal{O}b(\underset{\sim}{C}) \times \mathcal{O}b(\underset{\sim}{C})$ so that $f \rightsquigarrow (A,B)$ is functional with $\sigma_0(f) = A$, $\sigma_1(f) = B$ then defining the source and target functions.

(1.0.3) In either of the preceding definitions, the set of objects of $\underset{\sim}{C}$ and the set of identity arrows are in a one-to-one correspondence with each other. If one is really algebraically inclined one may identify the objects with the identity arrows and rephrase the definition of category in the following fashion: A category is a set (whose elements are called arrows) in which a partial composition is defined which satisfies the following axioms:

(i) h(gh) is defined iff (hg)f is defined and then h(gh) =(hg)f = hgf;

(ii) if hg and gf are defined then hgf is defined;

(iii) if f is an arrow then there exist arrows u and u' such that u'f and fu are defined and u'f = fu = f.

We leave it for the interested reader to recover this definition of an abstract or "non-objective" category from either of the preceding definitions and to convince himself of their equivalence.

REMARK (1.0.4) The three definitions of category given in the preceding paragraph have been (implicitly) formalized within the ⟨⟨theory of sets⟩⟩ where they are entirely equivalent. If one looks at them more closely, however, one can find essential differences which are perhaps worth pointing out.

The third definition is formalizable completely outside of the theory of sets in a satisfactory manner by taking an equality theory (first order functional calculus with equality) with couples, adjoining two substantive signs ⟨⟨σ and τ⟩⟩ each of weight one (called source and target), one substantive sign ⟨⟨ $\cdot$ ⟩⟩ of weight two (called multiplication) and adding as axioms the formal counter-parts of (i), (ii), and (iii) of (1.0.3). (σ and τ allow one to say when one desires the multiplication to be defined). The resulting "first order" theory may properly be called the theory of the multiplication of a category. It is quite tedious and when properly done can expand a three line set theoretic proof into a three page ⟨⟨algebraic proof⟩⟩ with no increase in content. Its proofs comprise the familiar ⟨⟨finite arrow-diagram chases⟩⟩ which occur in many expositions of elementary category theory.

The second definition occupies an intermediate formal position and is (in intended content) the original definition used by Eilenberg and McLane in their paper (Eilenberg-McLane, 1942) which marks the official nascence of the subject. It is appropriate when one has the desire to formalize the notion of ⟨⟨category⟩⟩ within a "Gödel-Bernays type" theory which makes a "set-class" distinction. In such a theory the predicate ⟨⟨is a group⟩⟩, for instance, is class-collective and one can speak of the ⟨⟨class of all groups⟩⟩. The predicate ⟨⟨is a homomorphism of groups⟩⟩ is set-collective for couples of groups so that one can speak of the set of all group homomorphisms between two groups. The operation of composition of group homomorphisms then defines a category "structure" on the class of all groups.

The same observations are of course valid for any species of structure with morphisms and also within this system one can even speak of the class of all sets and indeed of the category of all sets with set applications as morphisms (i.e. arrows). One can even speak of the graph of a composition and identity preserving function between such classes as a perfectly legitimate notion, but thereafter one encounters difficulties; for example a naive generalization would be to consider the category of all such categories with such class functions as its arrows, but such an obvious general situation has been specifically prohibited by the class-set distinction of the theory ⟨⟨sets are members of classes, proper classes are not members of anything⟩⟩, so that such an obvious ⟨⟨next step⟩⟩ is formally prohibited for precisely those cases in which it would promise to be interesting. (The situation is actually even worse than it appears here. In Gödel-Bernays the usual method of distinguishing functions from different sets which have the same graph by considering the couple consisting of graph together with a "set of arrival" is prohibited in these cases, for couples in G-B cannot have proper classes as their projections).

The first definition (which is implicit in Grothendieck (1961) and Gabriel (1963)) marks a determination to consider the notion of ⟨⟨category⟩⟩ as a "full-fledged" species of structure in its own right within the general cadre of "partial algebraic structures", <u>and</u> at the same time allow it to function in the context for which it was originally designed. This is accomplished by means of ⟨⟨universes⟩⟩ and ⟨⟨ $\mathfrak{U}$-<u>categories</u>⟩⟩.

<u>DEFINITION</u> (1.0.5) Let $\mathfrak{U}$ be a universe.
A category $\mathbf{C}$ is called a $\mathfrak{U}$-<u>category</u> provided $\mathfrak{Ob}(\mathbf{C}) \subseteq \mathfrak{U}$ and for each $(X,Y) \in \mathfrak{Ob}(\mathbf{C}) \times \mathfrak{Ob}(\mathbf{C})$ the set $\mathbf{C}(X,Y)$ of arrows of $\mathbf{C}$ with source X and target Y is a <u>member</u> of $\mathfrak{U}$ (i.e. for all $(X,Y) \in \mathfrak{Ob}(\mathbf{C}) \times \mathfrak{Ob}(\mathbf{C})$, $\mathbf{C}(X,Y) \in \mathfrak{U}$).

If $\mathfrak{U}$ is a universe, then it is "closed" under all of the usual set-theoretic operations applied to families of its members provided they are <u>indexed</u> by some member of $\mathfrak{U}$. Consequently $\mathfrak{U}$ behaves <u>with respect to its members</u> as if it were the "set of all sets". The definition of a $\mathfrak{U}$-category with its sets of arrows as <u>members</u> of $\mathfrak{U}$ is entirely analogous to the Eilenberg-McLane requirement that $\mathbf{C}(A,B)$ be a ⟨⟨set⟩⟩ for each couple of objects in $\mathbf{C}$. It will soon become clear that one may reason with $\mathfrak{U}$-categories using very little reference to $\mathfrak{U}$.

<u>EXAMPLES</u> (1.0.6) -1° A category with exactly one object <u>is</u> a <u>semigroup with unit</u> (i.e. a monoid); a category for which the squares (1.0.1.1) and (1.0.1.2) are cartesian (i.e. every arrow is an *isomorphism*) is a (Brandt) <u>groupoid</u>; a groupoid with exactly one object is a <u>group</u>; an *additive category* is a <u>ringoid</u>; a ringoid with one object is a <u>ring</u>.

2° Let R be the graph of a pre-order (i.e. reflexive and transitive) relation on a set E. Define as source and target the canonical projections of R onto E. Let I be the diagonal application $I : E \to R$ and $\mu : R \times R \to R$ the application defined by the assignment $\langle\langle((a,b),\ (b,c)) \rightsquigarrow (a,c)\rangle\rangle$, then E supplied with this structure is a category with E as its set of objects and R as its set of arrows. In particular an equivalence relation on a set defines a category structure on E; any (partially) ordered set and any lattice is a category. Any set E is supplied with a category structure by the diagonal Δ_E and its projections; such a category is called a discrete category.

3° Let $\mathfrak{U}$ be a universe; the set $\mathfrak{U}$ is the set of objects of a category ENS-$\mathfrak{U}$ whose arrows are simply applications of sets in $\mathfrak{U}$ and whose composition is composition of functions. The resulting $\mathfrak{U}$-category is called the category of $\mathfrak{U}$-sets. If no specific reference is made to $\mathfrak{U}$, this category will be called the category of sets (and set-applications) and will be denoted by (ENS).

4° A species of structure Σ with morphisms defines for any universe $\mathfrak{U}$ the category of ($\mathfrak{U}$ -) sets supplied with a structure of species Σ. The resulting category is called the ($\mathfrak{U}$-) category of Σ-structured sets. In particular one has the categories of ($\mathfrak{U}$-) groups and homomorphisms (Gr); abelian ($\mathfrak{U}$-) groups and homomorphisms (Ab); topological ($\mathfrak{U}$-) spaces and continuous maps (TopSp); of pointed sets and pointed applications (ENS_0). A pointed set is a couple consisting of a set E and an element ξ_0 such that $\xi_0 \in E$ (ξ_0 is called the base point of the pointed set E); a pointed application of two pointed sets in simply an application which preserves the base points.

5° If C is a $\mathfrak{U}$ -category of topological spaces and continuous maps, we may define a new category whose objects are again topological spaces, but whose arrows are homotopy classes of such continuous maps and whose composition is defined by means of the homotopy class of the composite of representatives from a given couple of homotopy classes. (This, incidentally, gives an example of a "large category" whose arrows need not be applications).

6^o Let $\underset{\sim}{C}$ be a category; $X \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C})$. We form the category $\underset{\sim}{C}/X$ of <u>objects</u> (of $\underset{\sim}{C}$) <u>above</u> X as follows: the objects of $\underset{\sim}{C}/X$ are the arrows of $\underset{\sim}{C}$ whose target is X; if $\xi_1 : T_1 \to X$ and $\xi_2 : T_2 \to X$ are two objects of $\underset{\sim}{C}/X$ we define $\underset{\sim}{C}/X(\xi_1, \xi_2)$ to be the set of all arrows f of $\underset{\sim}{C}(T_1, T_2)$ such that $\xi_2 f = \xi_1$. Composition of such X-morphisms is that induced by $\underset{\sim}{C}$. By abuse of notation one often refers to objects of $\underset{\sim}{C}/X$ by their source alone and writes $\underset{\sim}{C}/X(T_1, T_2)$ for $\underset{\sim}{C}/X(\xi_1, \xi_2)$. The arrow ξ_1 is then referred to as the <u>structural map</u> of the object T_1 in $\underset{\sim}{C}/X$. Note that if $\underset{\sim}{C}$ is a $\underset{\sim}{\mathcal{U}}$-category, then so is $\underset{\sim}{C}/X$ whatever be $X \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C})$.

7^o Let $\underset{\sim}{C}$ be a category. 1st (DEF): defines the <u>arrow category</u> of $\underset{\sim}{C}$ (denoted by abuse of language by $\underset{\sim}{\mathfrak{Fl}}(\underset{\sim}{C})$) as that category whose objects are the <u>fibre systems</u> of $\underset{\sim}{C}$, i.e. triples (p, X, Y) consisting of a couple (X, Y) of objects of $\underset{\sim}{C}$ together with an arrow $p : X \to Y$. An arrow f in $\underset{\sim}{\mathfrak{Fl}}(\underset{\sim}{C})$ with source $p_1 = (p_1, X_1, Y_1)$ target $p_2 = (p_2, X_2, Y_2)$ is a couple $(f_1\ f_2)$ of arrows of $\underset{\sim}{C}$ such that $p_2 f_1 = f_2 p_1$. In other words such that the diagram

(1.0.6.1)
$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "p_1"'] & X_2 \arrow[d, "p_2"] \\ Y_1 \arrow[r, "f_2"'] & Y_2 \end{tikzcd}$$

commutes. Composition in $\underset{\sim}{\mathfrak{Fl}}(\underset{\sim}{C})$ is the obvious one. Note here that we could just as well have defined $\underset{\sim}{C}/X$ as having as its objects the <u>fibre</u>

<u>systems of</u> $\underset{\sim}{C}$ with <u>base</u> X and having as arrows those couples of morphisms of $\mathcal{F}\ell(\underset{\sim}{C})$ whose second projection was I_X.

<u>REMARK</u> (1.0.6.2) It is amusing to observe the three definitions of <u>⟨⟨category⟩⟩</u> parallel the axiomatization of most algebraic structures, for example, that of groups: As the notion of group gradually became known as a distinct entity one first axiomatized the notion of the <u>multiplication of a group</u>. When this was understood it sufficed as long as one was only interested in groups, "one at a time". Gradually one became more interested in the interrelation of groups with one another and the concept of "<u>group homomorphism</u>"took shape. As long as one was only interested in how "groups behaved among themselves" the Bernays-Gödel definable "class of all groups" was sufficient for all purposes. Once this "theory of groups" became familiar, however, it was natural to inquire about the interaction of groups with other types of structures and the notion of category and functor then assumed a natural role in the study of such interrelations. So long as one was only interested in "one such interrelation at a time", the Eilenberg-McLane formulation within "Bernays-Gödel" was more or less adequate. It is only when one begins to study whole "classes" of such interactions in their own right that these formulations become inadequate. The notion of <u>universe</u> then offers the least modification of any existing system to allow such a study.

<u>DEFINITION</u> (1.0.7) If $\underset{\sim}{C}$ is a category, the <u>dual</u> (or <u>opposite category</u>) $\underset{\sim}{C}^{(op)}$ of $\underset{\sim}{C}$ is that category whose objects are those of $\underset{\sim}{C}$ and whose arrows are also those of $\underset{\sim}{C}$, but whose target and source applications have been interchanged relative to $\underset{\sim}{C}$. Composition in $\underset{\sim}{C}^{(op)}$ is of course that of $\underset{\sim}{C}$ after this interchange.

In more precise terms: given a category $\underset{\sim}{C}$ with source and target applications $\sigma_0(\underset{\sim}{C})$ and $\sigma_1(\underset{\sim}{C})$, respectively, and composition $\mu(\underset{\sim}{C}) : \mathcal{F}\ell(\underset{\sim}{C})_{\sigma_1}\times_{\sigma_0}\mathcal{F}\ell(\underset{\sim}{C}) \longrightarrow \mathcal{F}\ell(\underset{\sim}{C})$, define $\underset{\sim}{C}^{(op)}$ by $\mathcal{O}b(\underset{\sim}{C}^{(op)}) = \mathcal{O}b(\underset{\sim}{C})$,

$\mathcal{F}\ell(\underset{\sim}{C}^{(op)}) = \mathcal{F}\ell(\underset{\sim}{C})$; $\sigma_0(\underset{\sim}{C}^{(op)}) = \sigma_1(\underset{\sim}{C})$ and $\sigma_1(\underset{\sim}{C}^{(op)}) = \sigma_0(\underset{\sim}{C})$,

$\mu(\underset{\sim}{C}^{(op)}) = \mu(\underset{\sim}{C})\circ\varkappa : \mathcal{F}\ell(\underset{\sim}{C}^{(op)}) \times_{\sigma_1(C^{(op)}),\sigma_0(C^{(op)})} \mathcal{F}\ell(\underset{\sim}{C}^{(op)})) \xrightarrow{\varkappa} \mathcal{F}\ell(\underset{\sim}{C}) \times_{\sigma_1(C),\sigma_0(C)} \mathcal{F}\ell(\underset{\sim}{C}) \xrightarrow{\mu} \mathcal{F}\ell(\underset{\sim}{C}) = \mathcal{F}\ell(\underset{\sim}{C}^{(op)})$,

where $\varkappa$ is the canonical bijection defined by the assignment ⟨⟨$(f,g) \rightsquigarrow (g,f)$⟩⟩.

As this is nothing more than a precise description of the construction of an opposite algebraic structure from a given one, we can simply say that $\underset{\sim}{C}^{(op)}$ is $\underset{\sim}{C}$ supplied with its opposite structure. The fact that the axioms of a category are self-dual for interchange of σ_o and σ_1 shows that $\underset{\sim}{C}^{(op)}$ is a category (anti-isomorphic to $\underset{\sim}{C}$) and the relation ⟨⟨f : A⟶B in C⟩⟩ is equivalent to ⟨⟨f : B⟶A in $\underset{\sim}{C}^{(op)}$⟩⟩. This latter relation is referred to as ⟨⟨reversing the arrows⟩⟩, and one can speak of $\underset{\sim}{C}^{(op)}$ as having been obtained from $\underset{\sim}{C}$ by reversing the arrows. The importance of $\underset{\sim}{C}^{(op)}$ rests on the following meta-theorem called the

PRINCIPAL OF DUALITY (1.0.8). Any term or relation of the theory of categories is a term or relation of the theory of categories after interchange of the terms σ_o and σ_1. Any theorem of the theory of categories is a theorem of the theory of categories after the interchange of σ_o with σ_1.

This fact allows us to state any notion or assertion for an arbitrary category $\underset{\sim}{C}$ and know that duality gives a corresponding dual notion or assertion in $\underset{\sim}{C}^{(op)}$. One always has $(\underset{\sim}{C}^{(op)})^{(op)} = \underset{\sim}{C}$ and $\underset{\sim}{C}$ is a $\underset{\sim}{\mathcal{U}}$-category if and only if $\underset{\sim}{C}^{(op)}$ is a $\underset{\sim}{\mathcal{U}}$-category.

In general we will follow the convention of referring to the dual of a term which has been defined by means of $\underset{\sim}{C}^{(op)}$ by prefixing the term by ⟨⟨CO- ⟩⟩, e.g. *product and coproduct , kernel and cokernel*, etc.

(1.1) SPECIAL MORPHISMS

DEFINITION (1.1.1) For any couple (A,T) of objects in $\underset{\sim}{C}$, recall that A(T) designates the set of arrows in $\underset{\sim}{C}$ with source T and target A.

Let $f : A \to B$ be an arrow in $\mathcal{C}$. For each $T \in \mathrm{Ob}(\mathcal{C})$ define $f(T) : A(T) \longrightarrow B(T)$ by $f(T)\ (\xi) = f\xi$ and $T(f) : T(B) \longrightarrow T(A)$ by $T(f)(\mu) = \mu f$. $f : A \to B$ is called a

<u>monomorphism</u> if for all $T \in \mathrm{Ob}(\mathcal{C})$, $f(T) : A(T) \to B(T)$ is an injection; an
<u>epimorphism</u> if for all $T \in \mathrm{Ob}(\mathcal{C})$, $T(f) : T(B) \longrightarrow T(A)$ is an injection; a
<u>bi-morphism</u> if f is both a monomorphism and an epimorphism; a
<u>retraction</u> if for all $T \in \mathrm{Ob}(\mathcal{C})$, $f(T) : A(T) \longrightarrow B(T)$ is a surjection; a
<u>section</u> if for all $T \in \mathrm{Ob}(\mathcal{C})$, $T(f) : T(B) \longrightarrow T(A)$ is a surjection; an
<u>isomorphism</u> if f is both a retraction and a section.

<u>PROPOSITION</u> (1.1.2) Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be arrows in $\mathcal{C}$, so that $gf : A \longrightarrow C$ and $I_A : A \longrightarrow A$ be defined. Let $f(T) : A(T) \longrightarrow B(T)$ and $g(T) : B(T) \longrightarrow C(T)$ be defined as in (1.1.1) for any $T \in \mathrm{Ob}(\mathcal{C})$. Similarly for any $T \in \mathrm{Ob}(\mathcal{C})$, let $T(f) : T(B) \longrightarrow T(A)$ and $T(g) : T(C) \longrightarrow T(B)$ be defined as in (1.1.1). Then for any $T \in \mathrm{Ob}(\mathcal{C})$, one has that $g(T) \circ f(T) = gf(T) : A(T) \longrightarrow B(T)$, $I_A(T) = I_{A(T)} : A(T) \longrightarrow A(T)$, and that $T(f) \circ T(g) = T(gf) : T(C) \longrightarrow T(A)$, $T(I_A) = I_{T(A)} : T(A) \longrightarrow T(A)$.

Let $x \in A(T)$, then by definition $x : T \longrightarrow A$ is an arrow in $\mathcal{C}$. $f(T)(x) = fx : T \longrightarrow B$ and $g(T)\ (fx) = g(fx) : T \longrightarrow C$. But $g(fx) = (gf)x = gf(T)(x)$ by the associativity of composition. The proofs of the remaining assertions are equally trivial.

<u>COROLLARY</u> (1.1.3) (a) $f : A \longrightarrow B$ is a monomorphism iff given any T and any couple $(x,y) : T \rightrightarrows A$ such that $fx = fy$, one has $x = y$. (b) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are monomorphisms then $gf : A \longrightarrow C$ is monomorphism. (c) If $gf : A \longrightarrow C$ is a monomorphism then

f is a monomorphism. (d) If $f : A \to B$ is a section, then f is a monomorphism; in particular every isomorphism is a monomorphism. (e) If $f : A \to B$ is a section and also an epimorphism, then f is an isomorphism. (f) f is a monomorphism iff f is an epimorphism in $\underset{\sim}{C}^{(op)}$.

Condition (a) is nothing more than the statement that for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $f(T)$ is an injection, i.e. $f(T)(x) = f(T)(y) \Longrightarrow x = y$, $x, y \in A(T)$. Similarly condition (f) is immediate from the definition of $\underset{\sim}{C}^{(op)}$. Since the composition of two injections is an injection. PROPOSITION (1.1.2) says that for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $gf(T) : A(T) \to C(T)$ is an injection, i.e. that $gf : A \to C$ is a monomorphism, which thus gives (b). Since the composition of two functions is injective only if the first one is an injection, we have (c). If f is a section then by definition, $T(f) : T(B) \to T(A)$ is surjective for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, in particular $A(f) : A(B) \to A(A)$ is surjective. Consequently there exists an arrow $r : B \to A$ such that $rf = A(f)(r) = I_A$. (1.1.2) thus implies that for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $r(T) \circ f(T) : A(T) \to A(T)$ is equal to $I_{A(T)}$ which is an injection for all T. (c) then requires that f be a monomorphism. To obtain (e) note that if $f : A \to B$ is a section and also an epimorphism, then for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $T(f) : T(B) \to T(A)$ is a bijection and in particular the arrow $r : B \to A$ such that $rf = I_A$ is unique. Now $(fr)f = I_B f$ and f is an epimorphism, hence $fr = I_B$ and for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$; $f(T) : A(T) \to B(T)$ is a surjection (in fact, a bijection).

<u>COROLLARY</u> (1.1.3 dual) (a) $f : A \to B$ is an epimorphism iff given any couple $(x,y) : B \rightrightarrows T$ such that $xf = yf$, one has $x = y$; (b) if $f : A \to B$ and $g : B \to C$ are epimorphisms, then gf is an epimorphism; (c) If $gf : A \to C$ is an epimorphism, then so is $g : B \to C$

an epimorphism; (d) If f is a retraction then f is an epimorphism; every isomorphism is an epimorphism; (e) $f : A \to B$ is a retraction and also a monomorphism, then f is an isomorphism; (f) f is an epimorphism iff f is a monomorphism in $\underset{\sim}{C}^{(op)}$.

COROLLARY (1.1.4) $f : A \to B$ is a retraction iff there exists an arrow $g : B \to A$ such that $gf = I_B$. If such a g exists, then it is necessarily a section. (i.e. $f : A \to B$ is a retraction iff f admits a section g). Dually, $f : A \to B$ is a section iff f admits a retraction (i.e. there exists a $g : B \to A$ such that $gf = I_A$). f is an isomorphism iff there exists $g : A \to B$ such that $gf = I_A$ and $gf = I_B$. If f is an isomorphism then for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $f(T) : A(T) \to B(T)$ and $T(f) : T(B) \to T(A)$ are bijections. If f(T) or T(f) is a bijection for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, then f is an isomorphism. If f and g are sections (resp. retractions), then gf is a section (resp. retraction). If gf is a section then f is a section; if gf is a retraction, then g is a retraction.

This COROLLARY is nothing more than a summary of a portion of the proof of (1.1.3) and its dual together with a similar observation on surjective applications.

EXAMPLES (1.1.5) - 1° In the category ($\underset{\sim}{ENS}$) monomorphisms coincide with injective applications and epimorphisms with surjective applications. In fact, in ($\underset{\sim}{ENS}$) an application is injective iff it is a section, and (on the axiom of choice) a surjection iff it is a retraction (i.e. admits a section). Here every bimorphism is an isomorphism.

-2° In the category ($\underset{\sim}{Gr}$) monomorphisms are group homomorphisms whose underlying set application is injective. Similarly epimorphisms are homomorphisms whose underlying set application is surjective. Every bimorphism here is an isomorphism. In $(\underset{\sim}{Ab})$ a morphism is a section iff it is a direct summand.

-3° In the category $(\underset{\sim}{TopSp}(\text{-}T_2))$ of separated topological spaces, monomorphisms are those continuous maps whose underlying set application is injective. The epimorphisms are those continuous maps whose images are a dense subspace of the target. The canonical injection of a dense subspace into its target gives an example of a bimorphism which is not an isomorphism.

-4° In the category $(\underset{\sim}{TopSp})$ let $f : Co\text{-}Im(f) \longrightarrow Im(f)$ be the continuous map deduced from some map $f : A \longrightarrow B$. Then f is a bimorphism which has a bijection as its underlying set application. f is not, in general, an isomorphism.

(1.2) SPECIAL OBJECTS

DEFINITION (1.2.1) Let $\underset{\sim}{C}$ be a $\underset{\sim}{U}$-category; $I \in \underset{\sim}{U}$, and $(P \xrightarrow{p_\iota} A)_{\iota\in I}$ a family of arrows in $\underset{\sim}{C}$ all with source P. The object P supplied with the family $(p_\iota)_{\iota\in I}$ is said to define a representation of the product of the family of objects $(A_\iota)_{\iota\in I}$ with the family $(p_\iota)_{\iota\in I}$ as its canonical projections provided that for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, the set application $\underset{\iota\in I}{\boxtimes} p_\iota(T) : P(T) \longrightarrow \prod_{\iota\in I} A_\iota(T)$ defined by the assignment

$$\langle\langle \quad x \rightsquigarrow (p_\iota x)_{\iota\in I} \rangle\rangle \text{ is a bijection.}$$

PROPOSITION (1.2.2) If $(P, (p_\iota)_{\iota\in I})$ defines a representation of the product of the family $(A_\iota)_{\iota\in I}$ then the object P supplied with its projections is unique (up to a unique isomorphism).

Let $(P', (p'_\iota)_{\iota\in I})$ also define a representation of the product of the family $(A_\iota)_{\iota\in I}$. Then by definition the applications

$$\overset{-1}{\underset{\iota\in I}{\boxtimes} p_\iota}(P')\circ \underset{\iota\in I}{\boxtimes} p'_\iota(P') : P'(P') \longrightarrow \prod_{\iota\in I} A_\iota(P') \to P(P') \text{ and}$$

$$\overset{-1}{\underset{\iota\in I}{\boxtimes} p'_\iota}(P)\circ \underset{\iota\in I}{\boxtimes} p_\iota(P) : P(P) \to \prod_{\iota\in I} A_\iota(P) \longrightarrow P'(P)$$

are bijective, in particular $r = \overset{-1}{\underset{\iota\in I}{\boxtimes} p'_\iota}(P)\circ \underset{\iota\in I}{\boxtimes} p_\iota(P)(I_P) : P \longrightarrow P'$ and $r' = \overset{-1}{\underset{\iota\in I}{\boxtimes} p_\iota}(P')\circ \underset{\iota\in I}{\boxtimes} p'_\iota(P')(I_{P'}) : P' \longrightarrow P$ exist and $r\circ r' = I_{P'}$, $r'\circ r = I_P$. The last two equalities result from the injectivity of the lifted applications.

(1.2.3) Since a representation of the product of the family $(A_\iota)_{\iota\in I}$ is unique up to a unique isomorphism, among the isomorphism classes of "objects above $(A_\iota)_{\iota\in I}$" there is exactly one to which such a representation belongs, if it exists. If it exists, then a canonical representative of this isomorphism class is defined to be "the" <u>product of the family</u> $(A_\iota)_{\iota\in I}$ and is denoted by $(\prod_{\iota\in I} A_\iota, (pr_\iota)_{\iota\in I})$. The morphisms pr_ι are called the <u>canonical projections</u> of the (by abuse of language) <u>product</u> $\prod_{\iota\in I} A_\iota$. Note that the family $(pr_\iota)_{\iota\in I}$ need have none of its members as an actual epimorphism.

<u>DEFINITION</u>(1.2.4) A $\underset{\sim}{\mathfrak{U}}$-category $\underset{\sim}{C}$ is said to <u>admit</u> (arbitrary) <u>products of families of objects</u> (indexed by members of $\underset{\sim}{\mathfrak{U}}$) if given any family $(A_\iota)_{\iota\in I}$ of objects of $\underset{\sim}{C}$, $I\in\underset{\sim}{\mathfrak{U}}$ there exists an object $\prod_{\iota\in I} A_\iota$ supplied with a family $(pr_\iota : \prod_{\iota\in I} A_\iota \to A_\iota)_{\iota\in I}$ of arrows in $\underset{\sim}{C}$ such that for $T\in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C})$, the application

(1.2.4.1) $\underset{\iota\in I}{\boxtimes} pr_\iota(T) : (\prod_{\iota\in I} A_\iota)(T) \to \prod_{\iota\in I} A_\iota(T)$,
defined by $\langle\langle x \rightsquigarrow (pr_\iota x)_{\iota\in I} \rangle\rangle$ is a bijection.

(1.2.5) Let $(f_\iota)_{\iota\in I} \in \prod_{\iota\in I} A_\iota(T)$ be given, and suppose that the product of the family $(A_\iota)_{\iota\in I}$ exists, then the unique morphism $\overset{-1}{\underset{\iota\in I}{\boxtimes} pr_\iota}(T)((f_\iota)_{\iota\in I})$ with source T and target $\prod_{\iota\in I} A_\iota$ will be denoted by $\underset{\iota\in I}{\boxtimes} f_\iota$.

For each $\iota \in I$, $pr_\iota \blacksquare_{\iota\in I} f_\iota = f_\iota$.

(1.2.6) Let $(A_\iota)_{\iota\in I}$ and $(B_\iota)_{\iota\in I}$ be two families of objects of $\underset{\sim}{C}$ indexed by $I \in \underset{\sim}{\mathfrak{U}}$ for which the product exists. Furthermore, let $(f_\iota : A_\iota \to B_\iota)_{\iota\in I}$ be a family of morphisms of $\underset{\sim}{C}$ so that for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $(f_\iota(T) : A_\iota(T) \to B_\iota(T))_{\iota\in I}$ is the associated family of set applications, and $\prod_{\iota\in I} f_\iota(T) : \prod_{\iota\in I} A_\iota(T) \to \prod_{\iota\in I} B_\iota(T)$ the application which they define through the assignment $\langle\langle (x_\iota)_{\iota\in I} \rightsquigarrow (f_\iota x_\iota)_{\iota\in I}\rangle\rangle$. We have assumed that the products $\prod_{\iota\in I} A_\iota$ and $\prod_{\iota\in I} B_\iota$ exist (together with their projections) so that the application $p : \prod_{\iota\in I} A_\iota(\prod_{\iota\in I} A_\iota) \xrightarrow{\sim} \prod_{\iota\in I} A_\iota(\prod_{\iota\in I} A_\iota) \to \prod_{\iota\in I} B_\iota(\prod_{\iota\in I} A_\iota) \xrightarrow{\sim} \prod_{\iota\in I} B(\prod_{\iota\in I} A_\iota)$ is defined by composition with the representation bijections.

The element $p\,(I_{\prod_{\iota\in I} A_\iota}) : \prod_{\iota\in I} A_\iota \to \prod_{\iota\in I} B_\iota$ will be designated by $\prod_{\iota\in I} f_\iota$ and will be called the <u>product of the family of morphisms</u> $(f_\iota : A_\iota \to B_\iota)_{\iota\in I}$. For each $\iota \in I$, $pr_\iota \prod_{\iota\in I} f_\iota = f_\iota pr_\iota$ so that for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, the following diagram of set applications is commutative:

$$(1.2.6.1) \qquad \begin{tikzcd} \prod_{\iota\in I} f_\iota(T) : (\prod_{\iota\in I} A_\iota)(T) \arrow[r] \arrow[d, "\blacksquare_{\iota\in I} pr_\iota(T)"', "\wr"] & (\prod_{\iota\in I} B_\iota)(T) \arrow[d, "\wr", "\blacksquare_{\iota\in I} pr_\iota(T)"'] \\ (\prod_{\iota\in I} f_\iota)(T) : \prod_{\iota\in I} A_\iota(T) \arrow[r] & \prod_{\iota\in I} B_\iota(T) \end{tikzcd}$$

<u>PROPOSITION</u> (1.2.7) Under the conditions of (7.2.6), if the family of arrows $(f_\iota : A_\iota \to B_\iota)_{\iota\in I}$ is such that each f_ι is a monomorphism, then $\prod_{\iota\in I} f_\iota : \prod_{\iota\in I} A_\iota \to \prod_{\iota\in I} B_\iota$ is a monomorphism. If the family $(f_\iota)_{\iota\in I}$ is such that each f_ι is a retraction, then $\prod_{\iota\in I} f_\iota$ is a retraction.

This is a result of the definition (1.1.1) and the commutativity of (1.2.6.1).

<u>PROPOSITION</u> (1.2.8) Let $(A_\iota)_{\iota\in I}$ be a family for which the product exists and $J \subseteq I$ be a subset of I for which $(A_\iota)_{\iota\in J}$ also

admits a product. For each $T \in \mathfrak{U}(\underset{\sim}{C})$ such that $\prod_{\iota\in I} A_\iota(T) \neq \emptyset$ the application $pr_J(T): \prod_{\iota\in I} A_\iota(T) \to \prod_{\iota\in J} A_\iota(T)$ defined by $\langle\langle (x_\iota)_{\iota\in I} \rightsquigarrow (x_\iota)_{\iota\in J} \rangle\rangle$ is surjective and by composition with the representation bijections, gives rise to an arrow, $pr_J : \prod_{\iota\in I} A_\iota \to \prod_{\iota\in J} A_\iota$, called <u>the projection of index</u> J on the <u>partial product</u> $\prod_{\iota\in J} A_\iota$. pr_J is a retraction provided $\prod_{\iota\in I} A_\iota(\prod_{\iota\in J} A_\iota) \neq \emptyset$.

For each $T \in \mathfrak{U}(\underset{\sim}{C})$, the following diagram is commutative:

(1.2.8.1)
$$\begin{array}{ccc}
pr_J(T) : (\prod_{\iota\in I} A_\iota)(T) & \longrightarrow & (\prod_{\iota\in J} A_\iota)(T) \\
\downarrow \wr & & \downarrow \wr \\
pr_J(T) : \prod_{\iota\in I} A_\iota(T) & \longrightarrow & \prod_{\iota\in J} A_\iota(T)
\end{array}$$

<u>COROLLARY</u> (1.2.9) In order that any canonical projection $pr_\lambda : \prod_{\iota\in I} A_\iota \to A_\lambda$ of a product $\prod_{\iota\in I} A_\iota$ be a retraction it is necessary and sufficient that for each $\iota\in I$, $A_\iota(A_\lambda) \neq \emptyset$.

This is obtained from (1.2.8) applied to the case that $J=\{\lambda\}$, since if for all $\iota\in I$, $A_\iota(A_\lambda) \neq \emptyset$, then $\prod_{\iota\in I} A_\iota(A_\lambda) \neq \emptyset$.

(1.2.9.1)
$$\begin{array}{ccc}
pr_\lambda(T) : (\prod_{\iota\in I} A_\iota)(T) & \longrightarrow & A_\lambda(T) \\
\downarrow \wr & & \downarrow \wr \\
pr_\lambda(T) : \prod_{\iota\in I} A_\iota(T) & \longrightarrow & A_\lambda(T)
\end{array}$$

<u>PROPOSITION</u> (1.2.10) [<u>ASSOCIATIVITY OF PRODUCTS</u>] Let $I \neq \emptyset$ be a member of $\underset{\sim}{\mathfrak{U}}$ and $(J_\lambda)_{\lambda\in L}$ be a partition of I. Further, let $(A_\iota)_{\iota\in I}$ be a family of objects of $\underset{\sim}{C}$ such that for each $\lambda\in L$, the product $\prod_{\iota\in J_\lambda} A_\iota$ exists. Under these conditions the product $\prod_{\iota\in I} A_\iota$ exists if and only if the product $\prod_{\lambda\in L}(\prod_{\iota\in J_\lambda} A_\iota)$ exists and then $\prod_{\iota\in I} A_\iota$ is isomorphic to $\prod_{\lambda\in L}(\prod_{\iota\in J_\lambda} A_\iota)$.

For each $T \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})$, the application of $\prod_{\iota\in I} A_\iota(T)$ onto $\prod_{\lambda\in L}(\prod_{\iota\in J_\lambda} A_\iota(T))$ defined by composition of the canonical bijection $(\langle\langle\ x \rightsquigarrow (pr_{J_\lambda} \cdot x)_{\lambda\in L}\ \rangle\rangle)$ with the representation bijections is bijective. Consequently the product of the family $(A_\iota)_{\iota\in I}$ is representable if and only if the product of the family $(\prod_{\iota\in J_\lambda} A_\iota)_{\lambda\in L}$ is representable, in which case they are canonically isomorphic.

(1.2.11) If a category admits products in each case that the index set is <u>finite</u>, will say that $\underset{\sim}{C}$ admits <u>finite products</u>. In this case if $I \neq \emptyset$, it suffices to postulate the existence of the product of a couple of objects in $\underset{\sim}{C}$, since induction will then give the existence for any $n \geq 2$. In this case <u>associativity</u> may be established directly via the canonical associativity bijection arising from $\langle\langle\ (a,(b,c)) \rightsquigarrow ((a,b),c)\ \rangle\rangle$ and <u>commutativity</u> via the canonical bijection defined by $\langle\langle\ (x,y) \rightsquigarrow (y,x)\ \rangle\rangle$.

(1.2.12) If the index set I is finite and in fact <u>empty</u>, then the set $\prod_{\iota\in\phi} A_\iota(T)$ is one element set $\{\phi\}$ whatever be T. The product of such an empty family, if it exist, is called a ("the", up to isomorphism) <u>final</u> (or <u>terminal</u>) <u>object</u> in the category $\underset{\sim}{C}$ and will be denoted by $\mathbb{1}$. Thus if $\underset{\sim}{C}$ admits a final object, then for each $T \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})$ the trivial constant application $\varkappa : \mathbb{1}(T) \xrightarrow{\sim} \{\phi\}$ is a bijection. (i.e. for each object T in $\underset{\sim}{C}$ there exists one and only one arrow with source T and target $\mathbb{1}$. This arrow will be denoted by $\mathbb{1}_T : T \rightarrow \mathbb{1}$).

<u>PROPOSITION</u> (1.2.13) Let $\underset{\sim}{C}$ admit a final object $\mathbb{1}$. For any object Y in C let $\mathbb{1}_Y : Y \rightarrow \mathbb{1}$ be the canonical arrow. The following are then equivalent:

(a) $\mathbf{1}_Y : Y \longrightarrow \mathbf{1}$ is a retraction;

(b) for all $T \in \mathfrak{Ob}(\underset{\sim}{C})$, $Y(T) \neq \emptyset$;

(c) $Y(\mathbf{1}) \neq \emptyset$.

If $\mathbf{1}_Y$ is a retraction, then for all $T \in \mathfrak{Ob}(\underset{\sim}{C})$, $\mathbf{1}_Y(T) : Y(T) \longrightarrow \mathbf{1}(T)$ is surjective. $\mathbf{1}(T) \neq \emptyset$, hence $Y(T) \neq \emptyset$ for all $T \in \mathfrak{Ob}(\underset{\sim}{C})$. $Y(T) \neq \emptyset$ for all T implies that $Y(\mathbf{1}) \neq \emptyset$. If $Y(\mathbf{1}) \neq \emptyset$, then $\mathbf{1}_Y s = I_{\mathbf{1}}$ for some $s \in Y(\mathbf{1})$. (Any arrow $s : \mathbf{1} \longrightarrow Y$ is necessarily a section associated with $\mathbf{1}_Y$).

<u>DEFINITION</u> (1.2.14) Let $(a : A \longrightarrow Q,\ b : B \longrightarrow Q)$ be a couple of arrows in $\underset{\sim}{C}$ with the same target Q and $(d_o : R \longrightarrow A,\ d_1 : R \rightarrow B)$ a couple of arrows in $\underset{\sim}{C}$ with source R such that $ad_o = bd_1$. i.e. such that the following diagram is commutative:

(1.2.14.1)
$$\begin{tikzcd} R \arrow[r, "d_1"] \arrow[d, "d_o"'] & B \arrow[d, "b"] \\ A \arrow[r, "a"'] & Q \end{tikzcd}$$

Under these conditions the object R supplied with the couple (d_o, d_1) is said to <u>define a representation</u> of the <u>fibre product of</u> A <u>with</u> B <u>over</u> Q provided that for each $T \in \mathfrak{Ob}(\underset{\sim}{C})$, the application $d_o(T) \boxtimes d_1(T)$ of $R(T)$ into $A(T) \times B(T)$ defined by the assignment

$$\langle\langle \quad x \rightsquigarrow (d_o x, d_1 x) \quad \rangle\rangle$$

defines a bijection of R(T) onto the fibre product

$$A(T) \underset{a(T),\, b(T)}{\times} B(T) = \{(u,v) \mid (u,v) \in A(T) \times B(T) \text{ and } au = bv\} .$$

Such a representation, if it exist, is unique (up to a unique isomorphism) and a canonically selected representative will be $(A \times_Q B, (p_o,p_1))$ or $(A \,{}_a\times_b B, (p_o,p_1))$. With the usual abuse of language the object $A \times_Q B$ is often called the <u>fibre product</u> and the couple (p_o,p_1) referred to as the <u>first</u> and <u>second projections</u> or <u>structural arrows</u> of the fibre product.

Recall that in Chapter 0, a commutative square such as (1.2.14.7) of sets and applications was called <u>cartesian</u> provided that $d_o \times d_1 : R \to A \times B$ defined a bijection of R onto $A \,{}_a\times_b B$. Consequently (1.2.14) may be reformulated as

<u>DEFINITION</u> (1.2.15) Let $\underset{\sim}{C}$ be a category and 'D' (1.2.15.1)

(1.2.15.1)
$$\begin{tikzcd} R \arrow[r, "d_1"] \arrow[d, "d_o"'] \arrow[dr, phantom, "(D)"] & B \arrow[d, "b"] \\ A \arrow[r, "a"'] & Q \end{tikzcd}$$

(1.2.15.2)
$$\begin{tikzcd} R(T) \arrow[r, "d_1(T)"] \arrow[d, "d_o(T)"'] \arrow[dr, phantom, "(D(T))"] & B(T) \arrow[d, "b(T)"] \\ A(T) \arrow[r, "a(T)"'] & Q(T) \end{tikzcd}$$

a <u>square</u> 'D' in $\underset{\sim}{C}$ is then called <u>cartesian</u> (in $\underset{\sim}{C}$) (or a <u>pull-back diagram</u>) provided that for each $T \in \mathfrak{Ob}(\underset{\sim}{C})$, the square of sets and applications 'D(T)' (1.2.15.2) is <u>cartesian</u>.

(1.2.16) If $\underset{\sim}{C}$ is a $\underset{\sim}{\mathfrak{U}}$-category, then for any object Q in $\underset{\sim}{C}$, the category $\underset{\sim}{C}/Q$ of objects above Q (or <u>fibre systems</u> with <u>base</u> Q) is also a $\underset{\sim}{\mathfrak{U}}$-category, in which all theorems concerning products are applicable. In $\underset{\sim}{C}/Q$, however, the product $((A,a)\times(B,b),p)$ of a couple of objects (A,a) and (B,b) exists if and only if in $\underset{\sim}{C}$, the <u>fibre product</u> $(A \,{}_a\times_b B, (p_o,p_1))$ exists. (This is, of course, the origin of the term "fibre product"). In other words for all objects (T,u) in $\underset{\sim}{C}/Q$, $T \quad Q$,

(1.2.16.1)

$$\underset{\sim}{C}/Q((T,u),\ (A\ x_Q B, ap_o)) \xrightarrow{\sim} \underset{\sim}{C}/Q\ ((T,u),\ (A,a))\ x\ \underset{\sim}{C}/Q\ ((T,u),(B,b)).$$

Consequently, associativity and (for couples) commutativity of fibre products holds (or may be just as easily established directly from the definition (1.2.14)).

(1.2.17) In the square (1.2.15.1), the couple (d,a) may be regarded as a morphism of the arrow d_o into the arrow b in the category $\underset{\sim}{\mathcal{F}\ell}(\underset{\sim}{C})$. If the square (1.2.14.1) is cartesian in $\underset{\sim}{C}$ we will say that the couple (d,a) is a cartesian morphism of d_o into d_1 (in $\underset{\sim}{\mathcal{F}\ell}(\underset{\sim}{C})$). The fact that a fibre product in $\underset{\sim}{C}$ is just a product in $\underset{\sim}{C}/Q$ leads to the terminology of calling an arrow $f : A \to Q$ squarable if given any arrow $h : T \to Q$, the fibre product $T\ x_Q A$ exists.

NOTE: In the immediately succeeding propositions, the squares (C_1) (C_2), $(\bar{C}_1)$ $(C_1 \cdot C_2)$ all refer to those of (1.2.17.1) in some category $\underset{\sim}{C}$.

(1.2.17.1)
$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(C_1)"] & B \arrow[d, "\beta"] \\ A' \arrow[r, "a'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "a"'] \arrow[dr, phantom, "(\bar{C}_1)"] & A' \arrow[d, "a'"] \\ B \arrow[r, "\beta"'] & B' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "b"] \arrow[d, "\beta"'] \arrow[dr, phantom, "(C_2)"] & C \arrow[d, "\gamma"] \\ B' \arrow[r, "b'"'] & C' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "ba"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(C_1 \cdot C_2)"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "b'a'"'] & C' \end{tikzcd}$$

PROPOSITION (1.2.18) The following statements are theorems:

(a) (C_1) is cartesian if and only if $(\bar{C}_1)$ is cartesian;

(b) if (C_1) is cartesian and (C_2) is cartesian, then $(C_2 \cdot C_1)$ is cartesian;

(c) if $(C_2 \cdot C_1)$ is cartesian and (C_2) is cartesian, then (C_1) is cartesian.

These propositions are immediate consequences of the definition of cartesian square and their counterparts in the theory of sets (O, PROPOSITION

<u>PROPOSITION</u> (1.2.19) If (C_1) is a cartesian square in $\underset{\sim}{C}$, one has the following implications:

(a) if a' is a monomorphism, then a is a monomorphism;

(b) if a' is a retraction, then a is a retraction;

(c) if a' is an isomorphism, then a is an isomorphism.

These implications are immediate consequences of the definitions and their counterparts in the theory of sets (O, PROPOSITION

<u>COROLLARY</u> (1.2.20) If (C_1) is a cartesian square in $\underset{\sim}{C}$, one has the following implications:

(a) if β is a monomorphism, then α is a monomorphism;

(b) if β is a retraction, then α is a retraction;

(c) if β is an isomorphism, then α is an isomorphism.

If (C_1) is cartesian then $(\bar{C}_1)$ is cartesian (PROP. 1.2.18(a)), and (1.2.19) is applicable.

<u>DEFINITION</u> (1.2.21) Let $(a_1, a_2) : A \rightrightarrows B$ be a couple of arrows in $\underset{\sim}{C}$ with the same source A and same target B. Let $\iota : K \to A$ be an arrow with target A such that $a_1 \iota = a_2 \iota$. Under these conditions, the object K supplied with the morphism ι is said to <u>define a representation of the kernel</u> (or <u>equaliser</u>) <u>of the couple of morphisms</u> (a_1, a_2) provided that for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$ the application $\iota(T) : K(T) \to A(T)$ (defined by

$$« \quad \iota(T)(x) = \iota x \quad »)$$

defines a bijection of $K(T)$ onto the subset $\mathrm{Ker}(a_1(T), a_2(T))$ of $A(T)$ consisting of those $f : T \to A$ such that $a_1 f = a_2 f$.

Such a representation, if it exist, is unique (up to a unique isomorphism) and a canonically selected representative will be called "the" kernel of the couple (a_1, a_2) and will be denoted by $(\mathrm{Ker}(a_1,a_2), \iota)$ (or $(\mathrm{Eq}(a_1,a_2), \iota)$). The arrow $\iota : \mathrm{Ker}(a_1,a_2) \longrightarrow A$ is then necessarily a monomorphism (by definition of monomorphism) so that the usual abuse of language leads to calling the object $\mathrm{Ker}(a_1,a_2)$ the kernel of (a_1,a_2) and ι the canonical injection of the kernel into the object A.

Recall that in Chapter 0 (   ) a diagram such as

$$(1.2.21.1) \qquad K \xrightarrow{\iota} A \underset{a_1}{\overset{a_2}{\rightrightarrows}} B$$

of sets and set-applications was called exact provided $a_2\iota = a_1\iota$ and ι defined (by means of $x \rightsquigarrow \iota(x)$) a bijection of K onto the subset $\mathrm{Ker}(a_2,a_1)$ consisting of all $a \in A$ such that $a_2(a) = a_1(a)$. Consequently DEFINITION (1.2.21) may be reformulated as

DEFINITION (1.2.22). In a category $\underset{\sim}{C}$ a diagram of the form

$$K \xrightarrow{\iota} A \underset{a_1}{\overset{a_2}{\rightrightarrows}} B$$

is called exact (in $\underset{\sim}{C}$), provided that for all $T \in \mathcal{O}\ell(\underset{\sim}{C})$, the diagram (1.2.22.1) of sets and applications

$$(1.2.22.1) \qquad K(T) \xrightarrow{\iota(T)} A(T) \underset{a_1(T)}{\overset{a_2(T)}{\rightrightarrows}} B(T)$$

is exact.

PROPOSITION (1.2.23) Consider the following diagram (1.2.23.1) of objects and arrows in a category $\underset{\sim}{C}$.

(1.2.23.1)
$$\begin{tikzcd} (\mathcal{E}): & A \arrow[r, "a"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(D)"] & A' \arrow[r, shift left, "a_2"] \arrow[r, shift right, "a_1"'] \arrow[d, "\beta"] & A'' \arrow[d, "\gamma"] \\ (\mathcal{F}): & B \arrow[r, "b"] & B' \arrow[r, shift left, "b_2"] \arrow[r, shift right, "b_1"'] & B'' \end{tikzcd}$$

We suppose that this diagram is sequentially commutative, i.e. (1) $b\alpha = \beta a$; (2) $\gamma a_2 = b_2\beta$; (3) $\gamma a_1 = b_1\beta$; (4) $a_2 a = a_1 a$; (5) $b_2 b = b_1 b$. Under these conditions, the following implications are true:

(a) If the square (D) is cartesian, ($\mathcal{F}$) exact implies ($\mathcal{E}$) exact;

(b) ($\mathcal{E}$) exact, and ($\mathcal{F}$) exact, and γ a monomorphism implies that (D) is cartesian.

The proposition is an immediate consequence of the definitions and it's set-theoretic counterpart (0, PROPOSITION

REMARK (1.2.24) The concepts of ⟨⟨kernel of a couple of arrows⟩⟩, ⟨⟨product of a couple of objects⟩⟩ and ⟨⟨fibre product of a couple of arrows⟩⟩ and their ⟨⟨infinite⟩⟩ counterparts are not unrelated. Indeed, the canonical bijections of Chapter 0, as yet unused, suggest precisely this interdependence and could be used in much the same way as in the proof of the associativity of products to produce a precise formulation. We prefer, however, to delay this study until we have a more general notion of ⟨⟨representability⟩⟩ at our disposal and will content ourselves here with a statement of some of the dual definitions and propositions of (1.2).

DEFINITION (1.2.25) (1.2.1 dual). Let $\underset{\sim}{C}$ be a $\underset{\sim}{\mathfrak{U}}$-category; $I \in \underset{\sim}{\mathfrak{U}}$, and $(A_\iota \xrightarrow{i_\iota} S)_{\iota \in I}$ a family of arrows in $\underset{\sim}{C}$ all with target S. The object S supplied with the family $(i_\iota)_{\iota\in I}$ is said to define a co-representation of the product of the family $(A_\iota)_{\iota\in I}$ with the family $(i_\iota)_{\iota\in I}$ as its canonical co-projections (or canonical injections) provided that for each $T \in \mathcal{O}b(\underset{\sim}{C})$, the application $\underset{\iota\in I}{\boxtimes} T(i_\iota): T(S) \longrightarrow \prod_{\iota\in I} T(A_\iota)$ defined by the assignment ⟨⟨ $x \rightsquigarrow (xi_\iota)_{\iota\in I}$ ⟩⟩ is a bijection.

(1.2.26) Such a co-representation of the product is unique up to a unique isomorphism and a canonically selected representative, if it exist, is called "the" co-product (or sum) of the family $(A_\iota)_{\iota\in I}$ and is denoted by $(\coprod_{\iota\in I} A_\iota, (in_\iota)_{\iota\in I})$ (or $(\sum_{\iota\in I} A_\iota, (in_\iota)_{\iota\in I})$). A category is said to admit (arbitrary) co-products of families of objects provided such a co-representation always exists. The defining "commutation formula" for coproducts is of course

$$(1.2.26.1) \qquad \underset{\iota\in I}{\boxtimes} T(in_\iota) \ : T(\coprod_{\iota\in I} A_\iota) \xrightarrow{\sim} \prod_{\iota\in I} T(A_\iota), \ T \in \mathcal{O}b(\underset{\sim}{C}).$$

For any family $(f_\iota)_{\iota\in I}$ of arrows in $\prod_{\iota\in I} T(A_\iota)$, the unique arrow $\overleftarrow{\underset{\iota\in I}{\boxtimes} T}(in_\iota)((f_\iota)_{\iota\in I})$ will be denoted by $\underset{\iota\in I}{\boxplus} f_\iota : \coprod_{\iota\in I} A_\iota \longrightarrow T$; the coproduct of a family $(f_\iota : A_\iota \longrightarrow B_\iota)_{\iota\in I}$ by $\coprod_{\iota\in I} f_\iota : \coprod_{\iota\in I} A_\iota \longrightarrow \coprod_{\iota\in I} B_\iota$. In this case if each of the f_ι is an epimorphism then so is $\coprod_{\iota\in I} f_\iota$ and if each of the f_ι is a section then so is $\coprod_{\iota\in I} f_\iota$.

(1.2.27) In cases when the index set I is finite the product is sometimes denoted by ⟪$A_1 \times .. \times A_n$⟫; the dual notation for coproduct then is usually ⟪$A_1 + \ldots + A_n$⟫. If one uses the perhaps preferable notation of ⟨⟨ $A_1 \pi \cdots \pi A_n$ ⟩⟩ then the dual is ⟨⟨ $A_1 \amalg \cdots \amalg A_n$ ⟩⟩.

(1.2.28) [1.2.12 dual). In the case of a void index set, the coproduct is called "the" initial (or co-terminal) object in $\underset{\sim}{C}$ and a canonical representative is denoted by ⟨⟨ $\emptyset$ ⟩⟩. The unique canonical arrow with source $\emptyset$ and target T will be then denoted by $\emptyset_T : \emptyset \longrightarrow T$. We will defer until later the discussion of the properties of $\emptyset$ in relation to those of $\mathbb{1}$.

DEFINITION (1.2.29) (1.2.14 and 1.2.15 dual) The couple $(Q, (a,b))$ of (1.2.14.1) is said to define a co-representation of the fibre product of (d_o, d_1) provided that for each $T \in \mathfrak{O}r(\underset{\sim}{C})$, the diagram

(1.2.29.1)
$$\begin{tikzcd} T(Q) \arrow[r, "T(b)"] \arrow[d, "T(a)"'] & T(B) \arrow[d, "T(d_1)"] \\ T(A) \arrow[r, "T(d_o)"] & T(R) \end{tikzcd}$$

of sets and applicationsis cartesian.

Any two such co-representations are, as usual, canonically isomorphic (in $\underset{\sim}{C}^{(op)}/(A,B)$). If such a representation exist in $\underset{\sim}{C}$, a canonically selected representative will be denoted by $(A+_R B, (\iota_o, \iota_1)$ or $(A \amalg_{d_o, d_1} B, (\iota_o, \iota_1))$ or $(A +_R B, (\iota_o, \iota_1))$ etc. and will be called "the" fibre coproduct (or fibre sum) of A and B (under B). The square (1.2.14.1) is said to be co-cartesian (or a push-out diagram) under these circumstances, and the duals of the propositions (1.2.18) - (1.2.20) are all applicable. For example one has

PROPOSITION (1.2.30) (1.2.19 dual) . If the square (C_1) (1.2.17.1) is co-cartesian in $\underset{\sim}{C}$, the following implications hold:

(a) if α is an epimorphism, then ρ is an epimorphism;

(b) if α is a section, then ρ is a section;

(c) if α is an isomorphism, then ρ is an ismmorphism.

DEFINITION (1.2.31) (1.2.21 and 1.2.22 dual) . The couple (Q, ν) in the diagram

(1.2.31.1)
$$A \overset{a_2}{\underset{a_1}{\rightrightarrows}} B \xrightarrow{\nu} Q$$

is said to define a co-representation of the kernel of (a_1, a_2) in $\underset{\sim}{C}$ provided that for each $T \in \underset{\sim}{\mathcal{Q}}(\underset{\sim}{C})$, the diagram

$$(1.2.31.2) \qquad T(Q) \xrightarrow{T(\nu)} T(B) \underset{T(a_1)}{\overset{T(a_2)}{\rightrightarrows}} T(A)$$

of sets an application is exact.

Any two such representations are canonically isomorphic (in $\underset{\sim}{C}^{(\varphi)}/B$) and if such a representation exists in $\underset{\sim}{C}$, a canonically selected representative will be denoted by $(\mathrm{CoKer}(a_1, a_2), \nu)$ or $(\mathrm{Co\text{-}Eq}(a_1, a_2), \nu)$ and will be called "the" cokernel (or co-equalizer) of the couple (a_1, a_2). The diagram (1.2.31.1) is then referred to as co-exact (or simply exact) under these circumstances.

EXAMPLES (1.2.32) - 1° In the category ($\underset{\sim}{ENS}$), the product always exists and is simply the cartesian product supplied with its canonical projections; the coproduct is the "set-sum" or disjoint union supplied with its canonical injections; the initial object is simply the empty set $\emptyset$ and the final object the one element set $\{\emptyset\}$; the fibre product is again the fibre product; the fibre sum the disjoint union modulo the equivalence relation generated by the lifted application (into the product); the kernel of a couple simply the kernel; the co-kernel, the target of the couple modulo the equivalence relation generated by the lifted map (into the product).

- 2° In the category ($\underset{\sim}{Gr}$), the product is the cartesian product with its usual group structure; the co-product is the (Kurôs) free-join; the kernel is the set-kernel with the unique subgroup structure which it inherits from the source; the cokernel is the target modulo the congruence generated by the lifted map; the fibre product, the kernel of the projections of the product sequentially composed with the given homomorphisms; the fibre sum, the free join with the images amalgamated (i.e. amalgamated sum). Initial and final objects are isomorphic (as the one element group).

- 3° In the category (Ab), the product is the cartesian product; the sum the subgroup of the product usually called the (external direct sum, $\bigoplus_{\iota\in I} A_\iota$); finite products and coproducts coincide (as direct sum $A \oplus B$); kernels are difference kernels (Ker (a - b) (= Ker (a - b, 0))); cokernels are difference cokernels (Coker (a - b) (=Coker (a - b, 0))). Fibre sum, and product are ker and coker of homs into or out of direct sums. The zero group is both initial and final.

- 4° In the category of commutative (formerly "anti-commutative") R-algebras, finite sums correspond to tensor products ($A \otimes B$). In graded algebras, the product is a graded subring of the product of the underlying rings. In the category of rings with unit ($0 \neq 1$) and unitary ring homs, the initial and final object is the two element ring $\{0,1\}$.

- 5° In the category (TopSp), the "special and cospecial objects" (i.e. limits*) are the same as those for (ENS) supplied with the appropriate initial or final topology. If one restricts the maps to closed mappings only, one loses the general existence of products (the projections are not, in general closed). For normal spaces general existence of products is lost altogether. For connected spaces, one loses sums, but regains them if one only considers pointed spaces (the sum is then the disjoint union with base points identified). For T_2-spaces, the simple construction of cokernels and fibre sums is lost.

using *proper maps* only, one loses products but retains *fibre products*. For *compact* T_2-*spaces*, the *sum* is the *Stone-Čech compactification* of the *disjoint union*.

- 6° In any category $\underset{\sim}{C}$, if $X \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})$, then the category $\underset{\sim}{C}/X$ has products iff $\underset{\sim}{C}$ has fibre products, sums iff $\underset{\sim}{C}$ has sums, and, in any case *has* a final object, i.e. (X, I_X).

- 7° In any category $\underset{\sim}{C}$, the squares (D) and ($\bar{D}$) are *both* cartesian and cocartesian:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "I_A"'] \arrow[dr, phantom, "(D)"] & B \arrow[d, "I_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "I_A"] \arrow[d, "f"'] \arrow[dr, phantom, "(D)"] & A \arrow[d, "f"] \\ B \arrow[r, "I_B"'] & B \end{tikzcd}$$

For each $T \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})$, the application $I_A(T) \boxtimes f(T) : A(T) \longrightarrow A(T) \times B(T)$ defines a bijection of $A(T)$ onto the *graph* of the application $f(T) : A(T) \longrightarrow B(T)$. The application $f(T) \boxtimes I_A(T) : A(T) \longrightarrow B(T) \times A(T)$ defines a bijection onto the graph $\overset{-1}{f}(T) \subseteq B(T) \times A(T)$.

- 8° In $(\underset{\sim}{ENS})$ the square (I) is *both* cartesian and *cocartesian*.

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "(I)"] & B \arrow[d] \\ A \arrow[r] & A \cup B \end{tikzcd}$$

The arrows are, of course, the canonical injections.

- 9° In any$_*$abelian* category (for example, the category $(\underset{\sim}{Ab})$) a sequence

$$(\mathfrak{J}) : \quad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is *exact* if and only if the square $(\mathfrak{J})$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "(3)"] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

is both cartesian <u>and</u> co-cartesian.

<u>REMARKS</u> (1.2.32.1) The examples 1°-5° should convince the reader familiar with any one of the categories cited that the "special objects" of (1.2) are the fundamental objects used in the classical theory of these structures to produce "new objects" from "old ones", and as such are fundamental to the study of the corresponding theory. That such familiar objects should have the same "categorical"description is one of the justifications of the study of category theory.

The examples should also convince the reader that objects which play the "same categorical role" differ from category to category and may, in fact, have this as their <u>only</u> similarity. By the same token, it should also be clear that the "same" object may play a quite <u>disimilar</u> role with only "slight" change of category, and finally, be aware that although a familiar construction of one of these special objects may fail to give the desired special object, this alone is <u>no</u> indication that the category in question does not posses such an object (e.g. the categorical product in "most" elementary cases is constructed from the cartesian product. The cartesian product of graded algebras is not a graded algebra, the product still exists, however (Ex.4°). Dually, the disjoint union of compact T_2 spaces is not compact, the sum still exists, however (Ex.5°)).

For a mathematically sophisticated reader, these "cautionary" remarks are unnecessary. Such a reader is aware of the "algebraic flavor" of the definitions of the special objects and would no more expect an object which plays the rôle, say, of a product in some category to continue to do so in some "extension category", than he would that a unit in some submonoid of semigroups would necessarily also be a unit for the whole semigroup.

(1.3) $\mathfrak{U}$ - Functors and the Category CAT - $\mathfrak{U}$

DEFINITION (1.3.1) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A couple $F = (\mathfrak{ob}(F), \mathfrak{fl}(F))$ consisting of an application $\mathfrak{ob}(F) : \mathfrak{Ob}(\mathcal{C}) \longrightarrow \mathfrak{Ob}(\mathcal{D})$, called the object function, and an application $\mathfrak{fl}(F) : \mathfrak{Fl}(\mathcal{C}) \longrightarrow \mathfrak{Fl}(\mathcal{D})$ called the arrow function, is called a functor (or morphism of categories) with source $\mathcal{C}$ and target $\mathcal{D}$ provided the following conditions are satisfied:

$$(\mathrm{AMC})_{\mathrm{I}} \quad \mathfrak{ob}(F)\cdot\sigma_0(\mathcal{C}) = \sigma_0(\mathcal{D})\cdot\mathfrak{fl}(F) \quad \text{and} \quad \mathfrak{ob}(F)\cdot\sigma_1(\mathcal{C}) = \sigma_1(\mathcal{D})\cdot\mathfrak{fl}(F);$$

$$(\mathrm{AMC})_{\mathrm{II}} \quad \mu(\mathcal{D})\circ(\mathfrak{fl}(F) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{fl}(F)) = \mathfrak{fl}(F)\cdot\mu(\mathcal{C});$$

$$(\mathrm{AMC})_{\mathrm{III}} \quad \mathfrak{fl}(F)\cdot \mathbb{I}(\mathcal{C}) = \mathbb{I}(\mathcal{D})\cdot\mathfrak{ob}(F).$$

$(\mathrm{AMC})_{\mathrm{I}}$ states that the application

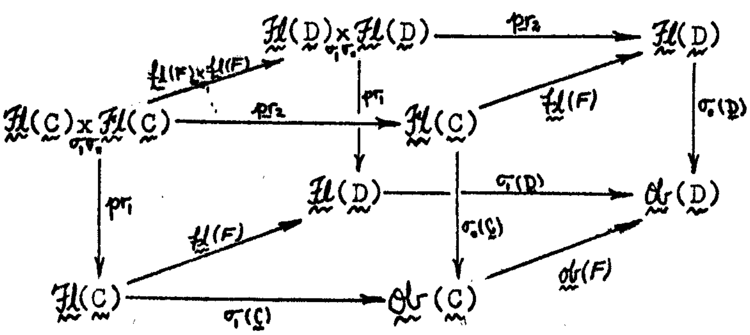
$$\mathfrak{fl}(F) \times \mathfrak{fl}(F) : \mathfrak{Fl}(\mathcal{C}) \times \mathfrak{Fl}(\mathcal{C}) \longrightarrow \mathfrak{Fl}(\mathcal{D}) \times \mathfrak{Fl}(\mathcal{D})$$

(defined by the assignment $\langle\langle\ (f,g) \rightsquigarrow (\mathfrak{fl}(F)(f), \mathfrak{fl}(F)(g))\ \rangle\rangle$ defines by restriction an application

$$\mathfrak{fl}(F)_{\sigma_1}\!\times_{\sigma_0}\mathfrak{fl}(F) : \mathfrak{Fl}(\mathcal{C})_{\sigma_1}\!\times_{\sigma_0}\mathfrak{Fl}(\mathcal{C}) \longrightarrow \mathfrak{Fl}(\mathcal{D})_{\sigma_1}\!\times_{\sigma_0}\mathfrak{Fl}(\mathcal{D})$$

so that the cubic diagram (1.3.1.1) of sets and applications is commutative. $(\mathrm{AMC})_{\mathrm{II}}$ says that this application is compatible with the compositions

(1.3.1.1)



of $\mathbf{C}$ and $\mathbf{D}$, i.e. that the diagram (1.3.1.2) is commutative. $(\mathrm{AMC})_{\mathrm{III}}$ says that the arrow and

(1.3.1.2)
$$
\begin{tikzcd}
\mathcal{A}r(\mathbf{C}) \times_{\sigma,\tau} \mathcal{A}r(\mathbf{C}) \arrow[r, "\mathcal{A}r(F) \times \mathcal{A}r(F)"] \arrow[d, "\mu(\mathbf{C})"'] & \mathcal{A}r(\mathbf{D}) \times_{\sigma,\tau} \mathcal{A}r(\mathbf{D}) \arrow[d, "\mu(\mathbf{D})"] \\
\mathcal{A}r(\mathbf{C}) \arrow[r, "\mathcal{A}r(F)"] & \mathcal{A}r(\mathbf{D})
\end{tikzcd}
$$

(1.3.1.3)
$$
\begin{tikzcd}
\mathcal{A}r(\mathbf{C}) \arrow[r, "\mathcal{A}r(F)"] \arrow[d, "I(\mathbf{C})"'] & \mathcal{A}r(\mathbf{D}) \arrow[d, "I(\mathbf{D})"] \\
\mathcal{O}b(\mathbf{C}) \arrow[r, "\mathcal{O}b(F)"] & \mathcal{O}b(\mathbf{D})
\end{tikzcd}
$$

object functions are also compatible with the identity assignments, i.e. that the diagram (1.3.1.3) is commutative.

With the customary abuse of notation of denoting the object function and the arrow function with the same letter F, our usual notation compresses these conditions into the following convenient form:

given $f : A \to B$ an arrow in $\mathbf{C}$, then $F(f) : F(A) \to F(B)$ is an arrow in $\mathbf{D}$ by $(\mathrm{AMC})_{\mathrm{I}}$; $F(gh) = F(g)F(f)$ for any composable couple (f,g) in $\mathbf{C}$ by $(\mathrm{AMC})_{\mathrm{II}}$; and $F(I_A) = I_{F(A)}$ for any object A in $\mathbf{C}$ by $(\mathrm{AMC})_{\mathrm{III}}$. In short, a <u>functor is a source and target, composition and identity preserving mapping of categories</u> and will be denoted by ⟨⟨$F : C \longrightarrow D$⟩⟩.

(1.3.2) Using the Eilenberg-McLane definition of category (1.0.2), the definition of functor would lead to ⟨⟨ a <u>functor</u> $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$ consists of a function $F : \mathfrak{Ob}(\underset{\sim}{C}) \longrightarrow \mathfrak{Ob}(\underset{\sim}{D})$, called the object function, and a family $(F_{(A,B)})_{(A,B) \in \mathfrak{Ob}(\underset{\sim}{C}) \times \mathfrak{Ob}(\underset{\sim}{C})}$ of functions $F_{(A,B)} : \underset{\sim}{C}(A,B) \longrightarrow \underset{\sim}{D}(F(A),F(B))$, $(A,B) \in \mathfrak{Ob}(\underset{\sim}{C}) \times \mathfrak{Ob}(\underset{\sim}{C})$ which preserve composition and identities whenever defined ⟩⟩.

(1.3.3) The definition of functor is conformal with the general definition of <u>morphism</u> of a <u>species of structure</u>. In particular, a functor $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$ defines an <u>isomorphism</u> of $\underset{\sim}{C}$ <u>with</u> $\underset{\sim}{D}$ provided the object and arrow functions are <u>bijections</u>.

<u>NOTE</u>: The notion of ⟨⟨ <u>isomorphism of categories</u> ⟩⟩ should <u>not be confused</u> with the more important notion of ⟨⟨ <u>equivalence of categories</u> ⟩⟩ to be defined later (1.3.11).

<u>PROPOSITION</u> (1.3.4) If $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$ and $G : \underset{\sim}{D} \longrightarrow \underset{\sim}{E}$ are functors, then $G \cdot F : \underset{\sim}{C} \longrightarrow \underset{\sim}{E}$, defined by $\mathfrak{Ob}(G \circ F) = \mathfrak{Ob}(G) \circ \mathfrak{Ob}(F)$ and $\mathfrak{Fl}(G \cdot F) = \mathfrak{Fl}(G) \cdot \mathfrak{Fl}(F)$, is a functor with source $\underset{\sim}{C}$ and target $\underset{\sim}{E}$, said to be obtained by <u>composition</u> of F and G. For any category $\underset{\sim}{C}$ one has the <u>identity functor</u> $I_{\underset{\sim}{C}} : \underset{\sim}{C} \longrightarrow \underset{\sim}{C}$ defined by $I_{\underset{\sim}{C}}(X) = X$ for all $X \in \mathfrak{Ob}(\underset{\sim}{C})$ and $I_{\underset{\sim}{C}}(f) = f$ for all $f \in \mathfrak{Fl}(\underset{\sim}{C})$.

(1.3.5) Let $\underset{\sim}{\mathfrak{U}}$ be a universe, then in some universe $\mathfrak{U}^*$ such that $\underset{\sim}{\mathfrak{U}} \in \underset{\sim}{\mathfrak{U}}^*$ (whose existence is guaranteed by the ⟨⟨ axiom of universes ⟩⟩) one has the set $\underset{\sim}{\text{CAT}}$-$\underset{\sim}{\mathfrak{U}}$ of all $\underset{\sim}{\mathfrak{U}}$-categories. Proposition (1.3.3) then states that $\underset{\sim}{\text{CAT}}$-$\underset{\sim}{\mathfrak{U}}$ has a category structure provided we take as <u>morphisms</u> functors between $\underset{\sim}{\mathfrak{U}}$-categories. We will designate by $\underset{\sim}{\text{Hom}}\ (\underset{\sim}{C},\underset{\sim}{D})$ or $(\underset{\sim}{\mathfrak{U}}\text{-})\underset{\sim}{\text{CAT}}(\underset{\sim}{C},\underset{\sim}{D})$ the set (member of $\underset{\sim}{\mathfrak{U}}^*$) of all such $\underset{\sim}{\mathfrak{U}}$-<u>functors</u>

with source the $\mathfrak{U}$-category $\underset{\sim}{C}$ and target the $\mathfrak{U}$-category $\underset{\sim}{D}$. If no explicit reference need by made to $\mathfrak{U}$ we will designate this category simply by $(\underset{\sim}{CAT})$.

DEFINITION (1.3.6) A functor $F : \underset{\sim}{C}^{(op)} \longrightarrow \underset{\sim}{D}$ is called a contra-variant functor from $\underset{\sim}{C}$ to $\underset{\sim}{D}$. Its defining characteristic with respect to $\underset{\sim}{C}$ is simply ⟨⟨ $F(gf) = F(f)\ F(g)$ for any composable couple (g,f) in $\underset{\sim}{C}$ ⟩⟩. A functor $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$ sometimes is called a co-variant functor. Recall that $\underset{\sim}{C}$ is a $\mathfrak{U}$-category iff $\underset{\sim}{C}^{(op)}$ is a $\mathfrak{U}$-category (cf. 1.0.7).

DEFINITION (1.3.7) Let $(\underset{\sim}{C}_\iota)_{\iota\in I}$ be a family of $\mathfrak{U}$-categories with $I \in \mathfrak{U}$. We define the product of family $(\underset{\sim}{C}_\iota)_{\iota\in I}$ of $\mathfrak{U}$-categories to be that $\mathfrak{U}$-category $\prod_{\iota\in I}\underset{\sim}{C}_\iota$ whose objects are the members of the set $\prod_{\iota\in I}\mathfrak{Ob}(\underset{\sim}{C}_\iota)$ and whose arrows are the members of set $\prod_{\iota\in I}\mathfrak{Fl}(\underset{\sim}{C}_\iota)$ (with composition defined by $\mu(f,g) = (\mu(C_\iota)\ (f_\iota, g_\iota))_{\iota\in I}$) supplied with the family $(\underset{\sim}{pr}_\iota : \prod_{\iota\in I}\underset{\sim}{C}_\iota \longrightarrow \underset{\sim}{C}_\iota)_{\iota\in I}$ of canonical projection functors (defined in the obvious fashion).

It should be observed that $\prod_{\iota\in I}\underset{\sim}{C}_\iota$ supplied with the family of functors $(pr_\iota)_{\iota\in I}$ is the product of the family $(\underset{\sim}{C}_\iota)_{\iota\in I}$ in the category $\underset{\sim}{CAT}$-$\mathfrak{U}$.

DEFINITION (1.3.8) A multifunctor is a functor F whose source is the product $\prod_{\iota\in I}\underset{\sim}{C}_\iota$ of some family of categories. If the ι^{th} factor is of the form $\underset{\sim}{C}_\iota^{(op)}$ for some category $\underset{\sim}{C}_\iota$, then F may be said to be contravariant on $\underset{\sim}{C}_\iota$, and by a pleonasm, covariant on C_ι^{op}. The most important case is where I is finite and all of the members of the family $(\underset{\sim}{C}_\iota)_{\iota\in I}$ are of the form $\underset{\sim}{C}$ or $\underset{\sim}{C}^{(op)}$ for some fixed category $\underset{\sim}{C}$, a functor $F : \prod_{\iota\in I}\underset{\sim}{C}_\iota \longrightarrow \underset{\sim}{D}$ is then called a multifunctor on $\mathbf{C}$ and the variance becomes of interest. If $I = \{1,2\}$ then F is called a bifunctor on $\mathbf{C}$.

DEFINITION (1.3.9) If $\mathbf{C}$ and $\mathbf{D}$ are categories each supplied with a functor to a category $\mathbf{E}$, we define the fibre product of $\mathbf{C}$ with $\mathbf{D}$ over $\mathbf{E}$ to be that category $\mathbf{C} \times_{\mathbf{E}} \mathbf{D}$ whose objects are $\mathfrak{Ob}(\mathbf{C}) \times_{\mathfrak{Ob}(\mathbf{E})} \mathfrak{Ob}(\mathbf{D})$, whose arrows are $\mathfrak{A}(\mathbf{C}) \times_{\mathfrak{A}(\mathbf{E})} \mathfrak{A}(\mathbf{D})$, and whose composition is that inherited from $\mathbf{C} \times \mathbf{D}$, supplied with the two projection functors $pr_1 : \mathbf{C} \times_{\mathbf{E}} \mathbf{D} \to \mathbf{C}$ and $pr_2 : \mathbf{C} \times_{\mathbf{E}} \mathbf{D} \to \mathbf{D}$.

If $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ are $\mathfrak{U}$-categories, then $\mathbf{C} \times_{\mathbf{E}} \mathbf{D}$ is a $\mathfrak{U}$-category and the definition of fibre product is conformal with that of fibre product in $\mathbf{CAT}$-$\mathfrak{U}$.

(1.3.10) The one point category $\mathbf{1}$, $\mathfrak{Ob}(\mathbf{1}) = \{\emptyset\}$, $\mathfrak{A}(\mathbf{1}) = \{(\emptyset,\emptyset)\}$ (or its isomorphic equivalent) is the final object (or final category) in $\mathbf{CAT}$-$\mathfrak{U}$, in all cases.

DEFINITION (1.3.11) Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and for each couple $(A,B) \in \mathfrak{Ob}(\mathbf{C}) \times \mathfrak{Ob}(\mathbf{C})$, let

$$F_{(A,B)} \;:\; \mathbf{C}(A,B) \longrightarrow \mathbf{D}(F(A),\, F(B))$$

be the restriction of $\mathfrak{A}(F)$ to $\mathbf{C}(A,B)$. $F : \mathbf{C} \longrightarrow \mathbf{D}$ is said to be

faithful provided for each $(A,B) \in \mathfrak{Ob}(\mathbf{C}) \times \mathfrak{Ob}(\mathbf{C})$, $F_{(A,B)}$ is injective;

full provided for each $(A,B) \in \mathfrak{Ob}(\mathbf{C}) \times \mathfrak{Ob}(\mathbf{C})$, $F_{(A,B)}$ is surjective;

fully faithful provided for each $(A,B) \in \mathfrak{Ob}(\mathbf{C}) \times \mathfrak{Ob}(\mathbf{C})$, $F_{(A,B)}$ is bijective;

<u>separating</u> (or <u>definitive</u>) provided F is faithful and <u>weakly quasi-injective</u> (i.e. $f : A \xrightarrow{\sim} B$ and $F(f) : F(A) = F(B)$ implies $f : A = B$ or equivalently, $F(f) = I_{F(\sigma_0(f))}$ and f an isomorphism implies $f = I_{\sigma_0(f)}$);

<u>embedding</u> provided $\mathfrak{H}(F)$ (and hence also $\mathfrak{Ob}(F)$) is injective; and an

<u>equivalence</u> provided F is fully faithful and <u>quasi-surjective</u> (i.e. for all $X \in \mathfrak{Ob}(\underset{\sim}{D})$, there exists an $A \in \mathfrak{Ob}(\underset{\sim}{C})$ such that $F(A) \xrightarrow{\sim} X$); and an

<u>isomorphism</u> provided $\mathfrak{Ob}(F)$ and $\mathfrak{H}(F)$ are bijective.

If F is faithful, then it does <u>not</u> necessarily follow that $\mathfrak{H}(F)$ is injective. If $\mathfrak{H}(F)$ is injective, then F is of course faithful and, moreover, $\mathfrak{Ob}(F)$ is then also injective, so that F is even an embedding under this hypothesis. Actual isomorphism of "large" categories are rare; equivalences are much more common and are an entirely satisfactory substitute for isomorphism in nearly all interesting cases.

<u>DEFINITION</u> (1.3.12) A category $\underset{\sim}{C}$ is said to be a <u>subcategory</u> of a category $\underset{\sim}{D}$ provided that $\mathfrak{Ob}(\underset{\sim}{C}) \subseteq \mathfrak{Ob}(D)$, $\mathfrak{H}(\underset{\sim}{C}) \subseteq \mathfrak{H}(\underset{\sim}{D})$, and the couple $i_{\underset{\sim}{C}} = (\mathfrak{Ob}(i), \mathfrak{H}(i))$ consisting of the canonical inclusion applications is a functor. $i_{\underset{\sim}{C}} : \underset{\sim}{C} \hookrightarrow \underset{\sim}{D}$ is then called the <u>canonical inclusion functor</u>.

<u>DEFINITION</u> (1.3.13) Let $\underset{\sim}{C}$ be a subcategory of $\underset{\sim}{D}$ with $i_{\underset{\sim}{C}} : \underset{\sim}{C} \hookrightarrow \underset{\sim}{D}$ the inclusion functor. The subcategory $\underset{\sim}{C}$ is said to be

<u>full</u> provided $\iota_{\underset{\sim}{C}}$ is full (and hence fully faithful);

<u>dense</u> (or <u>representative</u>) provided $\iota_{\underset{\sim}{C}}$ an equivalence; a

<u>seive</u> provided $X \in \underset{\sim}{Ob}(\underset{\sim}{C})$ and $f : T \longrightarrow X$, implies

$f \in \underset{\sim}{Fl}(\underset{\sim}{C})$; and a

<u>seive with respect to</u> $\mathcal{C}(\subseteq \underset{\sim}{Fl}(D))$ (or $\mathcal{C}$-<u>seive</u>) provided

$X \in \underset{\sim}{Ob}(\underset{\sim}{C})$ and $(f : T \rightarrow X) \in \mathcal{C}$ implies $f \in \underset{\sim}{Fl}(\underset{\sim}{C})$.

(1.3.14) Some authors require that a ⟨⟨ full subcategory ⟩⟩ be a seive with respect to <u>isomorphisms</u> in $\underset{\sim}{D}$. A seive is always a full subcategory and, as all full subcategories, is <u>completely determined by its set of objects</u>. It should be kept in mind that the <u>image</u> ($\underset{\sim}{Ob}(F)$ $\langle \underset{\sim}{Ob}(\underset{\sim}{C}) \rangle$, $\underset{\sim}{Fl}(F)$ $\langle \underset{\sim}{Fl}(\underset{\sim}{C}) \rangle$) of some functor $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$ is <u>not necessarily</u> a <u>subcategory</u> of $\underset{\sim}{D}$. (It is if $\underset{\sim}{Ob}(F)$ is injective, for instance), or more generally provided the relation

⟨⟨ $(f,g) \in \underset{\sim}{Fl}(\underset{\sim}{C})_{\sigma_1}\times_{\sigma_0}\underset{\sim}{Fl}(\underset{\sim}{C})$ ⟩⟩ is <u>equivalent</u> to the relation

⟨⟨ $(F(f), F(g)) \in \underset{\sim}{Fl}(\underset{\sim}{D})_{\sigma_1} \times_{\sigma_0} \underset{\sim}{Fl}(\underset{\sim}{D})$ ⟩⟩ .

<u>EXAMPLES</u> (1.3.15) - 1° Let $\underset{\sim}{C}$ be a $(\underset{\sim}{U})$-category, PROPOSITION (1.1.2) shows that for each $X \in \underset{\sim}{Ob}(C)$, the assignment

⟨⟨ $T \rightsquigarrow X(T)$, $f \rightsquigarrow X(f)$, $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $f \in \underset{\sim}{Fl}(\underset{\sim}{C})$ ⟩⟩

defines a functor $h_X : \underset{\sim}{C} \longrightarrow (\underset{\sim}{ENS})$, called the canonical (<u>contra-variant</u>) hom-functor defined by $X \in \underset{\sim}{Ob}(\underset{\sim}{C})$), and the assignment

⟨⟨ $T \rightsquigarrow T(X)$, $f \rightsquigarrow f(X)$, $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$ $f \in \underset{\sim}{Fl}(\underset{\sim}{C})$ ⟩⟩

a functor $h'_X : \underset{\sim}{C} \longrightarrow (\underset{\sim}{ENS})$, called the canonical (<u>co-variant</u>) <u>hom-functor</u> defined by $X \in \underset{\sim}{Ob}(\underset{\sim}{C})$).

The assignment

$$\langle\langle \quad (X,Y) \rightsquigarrow \underset{\sim}{C}\,(X,Y) \quad \rangle\rangle$$

then defines (in obvious fashion) a functor hom : $\underset{\sim}{C}^{(op)} \times \underset{\sim}{C} \longrightarrow (\underset{\sim}{ENS})$ which is then called the <u>canonical (hom) bi-functor</u> from $\underset{\sim}{C}$ into $(\underset{\sim}{ENS})$, "contra-variant in the first variable, co-variant in the second.

- 2° Let X and Y be objects in $\underset{\sim}{C}$ and $f : X \longrightarrow Y$ an arrow in $\underset{\sim}{C}$. f then defines a <u>functor</u> $f_* : \underset{\sim}{C}/X \longrightarrow \underset{\sim}{C}/Y$ from the category of objects above X into those above Y. (1.0.6 Ex. 6° and 7°), by $f(U,\xi) = (U, f\xi)$ and $f_*(\mathfrak{V}) = \mathfrak{U}$, called the <u>direct image by</u> f. If f is squareable (1.2.17) then f defines a functor $f^* : \underset{\sim}{C}/Y \longrightarrow \underset{\sim}{C}/X$ by $\langle\langle \; (V,\mu) \rightsquigarrow (X \underset{f,\mu}{\times} Y,\ pr_1) \; \rangle\rangle$ called the <u>inverse image</u> (or <u>change of base</u>), by f.

The category $\underset{\sim}{C}/X$ is always supplied with its "<u>inclusion</u>" functor $i_{\mathcal{K}} : \underset{\sim}{C}/X \quad \underset{\sim}{C}$ by $\langle\langle \; (T,\mu) \rightsquigarrow T,\ x \rightsquigarrow x \; \rangle\rangle$.

- 3° If $\underset{\sim}{C}$ admits products and/or sums, then

$\langle\langle (X_\iota)_{\iota\in I} \rightsquigarrow \prod_{\iota\in I} X_\iota,\ (f_\iota)_{\iota\in I} \rightsquigarrow \prod_{\iota\in I} f_\iota \; \rangle\rangle$ and $\langle\langle \; (X_\iota)_{\iota\in I} \rightsquigarrow \coprod_{\iota\in I} X_\iota,\ (f_\iota)_{\iota\in I} \rightsquigarrow \amalg f_\iota \; \rangle\rangle$ define functors $\prod_{\iota\in I} : \underset{\sim}{C}^I \longrightarrow \underset{\sim}{C}$ and $\coprod_{\iota\in I} : \underset{\sim}{C}^I \longrightarrow \underset{\sim}{C}$.

- 4° A <u>monoid</u> or <u>unitary ring homomorphism</u> defines a functor for the corresponding categories (1.0.6 Ex.1°). Any <u>monotone</u> (<u>increasing</u>) function defines a functor for any pre-ordered set (<u>qua</u> category (1.0.6. Ex.2°); a monotone (decreasing) function, a contra-variant functor. In particular, the functions $\hat{f} : \underset{\sim}{\mathfrak{P}}(E) \longrightarrow \underset{\sim}{\mathfrak{P}}(F)$ and $\overset{-1}{f} : \mathfrak{P}(F) \longrightarrow \underset{\sim}{\mathfrak{P}}(E)$ define covariant functors of the <u>categories</u>

$\mathfrak{P}(E)$. Any open or cont. map is a functor from the categories OUV (S) of open sets of some topological spaces. A functor $F : \mathrm{OUV}^{(op)}(S) \longrightarrow C$ is called a pre-sheaf (with values in C).

- 5° The assignments $\langle\langle\, E \rightsquigarrow \mathfrak{P}(E),\ f \rightsquigarrow \hat{f}\ \rangle\rangle$ and $\langle\langle\, E \rightsquigarrow \mathfrak{P}(E),\ f \rightsquigarrow \hat{f}^{-1}\ \rangle\rangle$ define functors $(\mathrm{ENS}) \xrightarrow{\mathfrak{P}} (\mathrm{ENS})$ and $(\mathrm{ENS})^{(op)} \longrightarrow (\mathrm{ENS})$.

- 6° Any category C defines by a species of structure with morphisms has a functor $X \rightsquigarrow \underline{X}$ which deletes part or all of the structure and is called the canonical projection (or "forgetful") functor defined through Σ. « $X \rightsquigarrow \underline{X}$ » is in general faithful (and in fact separating). For example the category (Gr) has its underlying base set functor $\text{▬} : (\mathrm{Gr}) \longrightarrow (\mathrm{ENS})$.

- 7° The assignment $X \rightsquigarrow F(X)$ of any set to the free-group on X defines a functor $F : (\mathrm{ENS}) \longrightarrow (\mathrm{Gr})$. Similarly for polynomial-algebras, free monoid algebras, ringsof quotients, symmetrizations, etc. all define functors. Stone-Cech compactification, completion of a uniform space, etc. in topology give examples of numerous examples functor defining object mappings.

- 8° The classical homology or co-homology theories are defined of certain functors from some category of topological spaces into some appropriate "algebraic" category; the relativised homology or co-homology theories as functors on some appropriate subcategory of the arrow category of (TopSp); similarly for the classical homotopy theories.

- 9° As examples of subcategories one has the full subcategories of (Ab) in (Gr) and $(\mathrm{TopSp}\text{-}T_2)$ in (TopSp); restriction to isomorphisms or other "composition closed" type of function give numerous examples of non full subcategories. As an example of a seive, take $\underset{\sim}{C}/X$ and some (Y, ξ) in $\underset{\sim}{C}/X$. The set of all (T, μ) such that there exists an X-morphism $f : T \longrightarrow Y$ forms the set of objects of a seive in $\underset{\sim}{C}/X$ (said to be associated with (Y, ξ)).

(1.4) TRANSFORMATIONS OF ($\mathfrak{U}$-) FUNCTORS - THE CATEGORY $\mathrm{Hom}_{\mathfrak{U}}(\mathbf{C},\mathbf{D})$ (= $\mathbf{CAT}(\mathbf{C},\mathbf{D})$).

Z (1.4.1) If $\mathbf{C}$ and $\mathbf{D}$ are ($\mathfrak{U}$-) categories (objects of $\mathbf{CAT}$ (-$\mathfrak{U}$)), set $\mathbf{CAT}$ ($\mathbf{C},\mathbf{D}$) of all functors $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a member of some universe $\mathfrak{U}^*$ which has $\mathfrak{U}$ as an element and <u>not</u> in general itself a subset or even of the same cardinality as some subset of $\mathfrak{U}$. Consequently for any $\mathfrak{U}$-category $\mathbf{C}$, the functor defined by the assignment

$$\langle\langle \quad \mathbf{T} \rightsquigarrow \mathbf{CAT}\ (\mathbf{T},\mathbf{C}) \quad \rangle\rangle$$

is a functor from $\mathbf{CAT}$-$\mathfrak{U}$ into $\mathbf{ENS}$-$\mathfrak{U}^*$.

We now proceed to supply $\mathbf{CAT}$ ($\mathbf{T},\mathbf{C}$) with an important (canonical) $\mathfrak{U}^*$-category structure.

(1.4.2) Recall that the <u>arrow category</u> $\mathbf{C}^2$ of a ($\mathfrak{U}$-) category is defined (1.0.6. Ex 7°) as having as its objects $\mathcal{O}b(\mathbf{C}^2)$ the set $\mathcal{F}l(\mathbf{C})$ and having as its own arrows $\mathcal{F}l(\mathbf{C}^2)$ the set $\mathcal{F}l^2(\mathbf{C}) = (\mathcal{F}l(\mathbf{C}) {}_{\sigma_1,\tau_0}\!\times \mathcal{F}l(\mathbf{C})) \underset{\mu(C),\mu(C)}{\times} (\mathcal{F}l(\mathbf{C}) {}_{\tau_1,\sigma_0}\!\times \mathcal{F}l(\mathbf{C}))$ consisting of the "<u>commutative squares</u>" of $\mathbf{C}$. The source and target applications are simply the iterations of the first and second projections corresponding to the assignments

$$\langle\langle \quad ((p_0,q_1),\ (p_1,q_0)) \rightsquigarrow p_0 \ \rangle\rangle \text{ and } \langle\langle \quad ((p_0,q_1),(p_1,q_0)) \rightsquigarrow q_0 \ \rangle\rangle$$

for a square $((p_0,q_1),\ (p_1,q_0)) \in \mathcal{F}l^2(\mathbf{C})$. An arrow $\varphi : p_0 \Rightarrow q_0$ in $\mathbf{C}^2$ then may be considered as the couple of "top and bottom arrows" in the commutative square diagram

(1.4.2.1)
$$\begin{tikzcd} T \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & B \arrow[d, "q_0"] \\ A \arrow[r, "q_1"'] & V \end{tikzcd}$$
.

The source of φ is the arrow p_o and the target q_o. Multiplication ($\mu(\underset{\sim}{C}^{2})$) is defined for "properly coincident squares" through the multiplication in $\underset{\sim}{C}$ by the assignment

$$\langle\langle\ ((p_o,q_1),\ (p_1,q_o)),\ ((q_o,r_1),\ (s_1,r_o)) \rightsquigarrow ((p_o,r_1q_1),\ (s_1p_1,r_o))\ \rangle\rangle,$$

which is just the description of "lateral adjunction" of the diagrams φ and ψ , i.e.

(1.4.2.1)
$$\langle\langle\ \begin{tikzcd} T \arrow[r, "p_1"] \arrow[d, "p_o"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & B \arrow[r, "s_1"] \arrow[d, "q_o"'] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & V \arrow[d, "r_o"] \\ A \arrow[r, "q_1"'] & U \arrow[r, "r_1"'] & W \end{tikzcd} \rightsquigarrow \begin{tikzcd} T \arrow[r, "s_1\, p_1"] \arrow[d, "p_o"'] \arrow[dr, phantom, "\overset{\psi\cdot\varphi}{\Longrightarrow}"] & Y \arrow[d, "r_o"] \\ A \arrow[r, "r_1q_1"'] & W \end{tikzcd}\ \rangle\rangle.$$

Taking as the identity assignment the obvious one,

$$\langle\langle\quad (p_o\ :\ A \longrightarrow B) \rightsquigarrow ((p_o,I_B),\ (I_A,p_o))\quad \rangle\rangle,$$

it is a matter of trivial verification that under this multiplication, $\mathfrak{M}(\underset{\sim}{C}^{2})$ supplies the set $\mathfrak{M}(\underset{\sim}{C})$ with a "natural" category structure.

(1.4.3) The <u>category</u> $\underset{\sim}{C}^2$ is a $\underset{\sim}{\mathfrak{U}}$-category if and only if $\underset{\sim}{C}$ is. To see this (and for other purposes as well) it is helpful to note that for a fixed couple (p_o, q_o) of objects in $\underset{\sim}{C}^2$, the set of <u>arrows in</u> $\underset{\sim}{C}^2$ which have source $p_o\ :\ T \rightarrow A$ and target $q_o\ :\ B \rightarrow U$ may be identified with the <u>fibre product</u>

(1.4.3.1)
$$\begin{tikzcd} U(A) \times B(T) \arrow[r] \arrow[d] \arrow[dr, phantom, "{U(p_o),\ q_o(T)}"] & B(T) \arrow[d, "q_o(T)"] \\ U(A) \arrow[r, "U(p_o)"] & U(T) \end{tikzcd}\quad .$$

which is certainly a member of $\underset{\sim}{\mathfrak{U}}$ if $U(A)$ and $B(T)$ are

<u>DEFINITION</u> (1.4.3) [GROTHENDIECK (1961 - TDTE III)]

Let $\underset{\sim}{C}$ be a category. A system $C = (\mu\,;\; F \overset{\sigma_0}{\underset{\sigma_1}{\rightrightarrows}} O, I)$ consisting of objects F and O in $\underset{\sim}{C}$ and arrows $(\sigma_0, \sigma_1) : F \rightrightarrows O$, $O \xrightarrow{I} F$, $\mu : F {}_{\sigma_1}\!\times_{\sigma_0} F \longrightarrow F$ is said to be a <u>category</u> <u>in</u> $\underset{\sim}{C}$ (or a $\underset{\sim}{C}$-<u>category</u>) (with F as its $\underset{\sim}{C}$-<u>arrows</u>, O as its $\underset{\sim}{C}$-objects, <u>etc</u>.) provided the corresponding arrows and objects which occur in the equations of the <u>DEFINITION OF CATEGORY</u> (1.0.1) all exist in $\underset{\sim}{C}$ and give rise to valid equations in $\underset{\sim}{C}$ (with composition in $\underset{\sim}{C}$ replacing composition of applications), e.g. providing the following diagrams exist and are commutative in $\underset{\sim}{C}$:

(1.4.3.1)
$$
\begin{tikzcd}
F {}_{\sigma_1}\!\times_{\sigma_0} F \arrow[r, shift left, "\pi_2"] \arrow[d, "\mu"'] & F \arrow[l, shift left, "I^*"] \arrow[d, "\sigma_1"] \\
F \arrow[r, shift left, "\sigma_1"] & O \arrow[l, shift left, "I"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F {}_{\sigma_1}\!\times_{\sigma_0} F \arrow[r, "\mu"] \arrow[d, shift left, "\pi_1"] & F \arrow[d, shift left, "\sigma_0"] \\
F \arrow[u, shift left, "I^*"] \arrow[r, "\sigma_0"] & O \arrow[u, shift left, "I"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F {}_{\sigma_1}\!\times_{\sigma_0} F {}_{\sigma_1}\!\times_{\sigma_0} F \arrow[r, "\mu \times 1_F"] \arrow[d, "1_F \times \mu"'] & F {}_{\sigma_1}\!\times_{\sigma_0} F \arrow[d, "\mu"] \\
F {}_{\sigma_1}\!\times_{\sigma_0} F \arrow[r, "\mu"] & F
\end{tikzcd}
.
$$

In analogous fashion one may define a $\underset{\sim}{C}$-<u>functor of</u> $\underset{\sim}{C}$-<u>categories</u> as a couple (f, f') of morphisms of $\underset{\sim}{C}$, $f : O_1 \longrightarrow O_2$, $f' : F_1 \longrightarrow F_2$ which satisfy the $\underset{\sim}{C}$-analogues of $(AMC)_I$ – $(AMC)_{III}$ (1.3.1) i.e. so that the diagrams

(1.4.3.2)
$$
\begin{tikzcd}
F_1 \arrow[r, "f'"] \arrow[d, shift right, "\sigma_1^1"'] \arrow[d, shift left, "\sigma_0^1"] & F_2 \arrow[d, shift right, "\sigma_1^2"'] \arrow[d, shift left, "\sigma_0^2"] \\
O_1 \arrow[r, "f"'] & O_2
\end{tikzcd}
\qquad
\begin{tikzcd}
F_1 \arrow[r, "f'"] & F_2 \\
O_1 \arrow[u, "I_1"] \arrow[r, "f"'] & O_2 \arrow[u, "I_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F_1 {}_{\sigma_1}\!\times_{\sigma_0} F_1 \arrow[r, "f' \times f'"] \arrow[d, "\mu_1"] & F_2 {}_{\sigma_1}\!\times_{\sigma_0} F_2 \arrow[d, "\mu_2"] \\
F_1 \arrow[r, "f'"'] & F_2
\end{tikzcd}
$$

are all sequentially commutative. The composition of $\underset{\sim}{C}$-functors (being again a $\underset{\sim}{C}$-functor) gives rise to the <u>Category</u> <u>of</u> $\underset{\sim}{C}$-<u>categories</u> <u>with</u> $\underset{\sim}{C}$-<u>functors as morphisms</u>.

<u>LEMMA</u> (1.4.4) If $C = (\mu \,;\, F \overset{\sigma_0}{\underset{\sigma_1}{\rightrightarrows}} O, I)$ is a $\underset{\sim}{C}$-category, then for each $T \in \mathfrak{Ob}(\underset{\sim}{C})$, $\tilde{C}(T) = (\tilde{\mu}(T);\ F(T) \overset{\sigma_0(T)}{\underset{\sigma_1(T)}{\rightrightarrows}} O(T),\ I(T))$ is a category (with $O(T)$ as its set of objects, $F(T)$ as its set of arrows, <u>etc</u>.). Moreover, for each $f \in T(U)$, $C(f) = (O(f) : O(T) \longrightarrow O(U),\ F(f) : F(T) \longrightarrow F(U))$ is a functor $C(f) : C(T) \longrightarrow C(U)$.

By the definition of $\underset{\sim}{C}$-category, the definition of the applications $\sigma_1(T)$, $\sigma_0(T)$ etc., and of <u>fibre-product</u> in $\underset{\sim}{C}$, one has that for each $T \in \mathfrak{Ob}(\underset{\sim}{C})$ all of the diagrams of sets and applications corresponding to (1.4.3.1), e.g.

(1.4.4.1)
$$\begin{tikzcd}
\hat{\mu}(T) : F(T) \times_{\sigma_0(T),\sigma_1(T)} F(T) \arrow[r, "\sim"] \arrow[d, "pr_1"'] & F \times_{\sigma_1,\sigma_0} F(T) \arrow[r, "\mu(T)"] \arrow[d, "pr_1(T)"] & F(T) \arrow[d, "\sigma_0(T)"] \\
\sigma_0(T) : F(T) = & F(T) \arrow[r, "\sigma_0(T)"] & O(T)
\end{tikzcd}$$
,

are commutative. (Here $\hat{\mu}(T)$ is $\widehat{pr_1(T) \boxtimes pr_2}^{-1}(T) \cdot \mu(T)$). Hence, with the multiplication $\tilde{\mu}(T)$, $(O(T), F(T))$ is supplied with a category structure in the sense of <u>DEFINITION</u> (1.0.1), so that $C(T)$ is a category for each $T \in \mathfrak{Ob}(\underset{\sim}{C})$.

To verify that $C(f)$ is a functor for $f \in T(U)$, observe that the diagrams such as

(1.4.4.2)
$$\begin{tikzcd}
F(T) \arrow[r, "F(f)"] \arrow[d, "\sigma_0(T)"'] & F(U) \arrow[d, "\sigma_0(U)"] \\
O(T) \arrow[r, "O(f)"] & O(U)
\end{tikzcd}$$

are commutative merely because of the <u>associativity of composition in $\underset{\sim}{C}$</u> : for $x \in F(T)$, $O(f) \cdot \sigma_0(T) = (\sigma_0 x)f = \sigma_0(xf) = \sigma_0(U) \cdot F(f)(x)$. The verification of the other axioms $(AMC)_I$ — $(AMC)_{IV}$ is equally trivial.

LEMMA (1.4.4) can be paraphrased with the substitution

⟨⟨ . . . for each $T \in \mathfrak{Ob}(\underset{\sim}{C})$, $C(T) = (\mu(T);\ F(T) \rightrightarrows O(T), I(T))$ "is" a category (up to a unique isomorphism) . . . ⟩⟩ and thus replace $\tilde{\mu}(T)$ with the given $\mu(T)$.

NOTE : The converse of (1.4.4) is also true and will be proved later in a more general context.

COROLLARY (1.4.5) If $F = (f, f')$ is a $\underset{\sim}{C}$-functor of $\underset{\sim}{C}$-categories with source $C_1 = (\mu_1;\ F_1 \rightrightarrows O_1, I_1)$ and target $C_2 = (\mu_2;\ F_2 \rightrightarrows O_2, I_2)$, then for each $T \in \mathfrak{Ob}(\underset{\sim}{C})$, the couple $F(T) = (f(T), f'(T))$ is a functor from the category $C_1(T)$ into the category $C_2(T)$.

The associativity of composition again gives commutativity of the evaluated diagrams corresponding to (1.4.32) in $(\underset{\sim}{ENS})$, so that, by definition of functor (1.3.1), the corollary holds. Coherence is again a consequence of (1.2.6).

(1.4.5.1)
$$\begin{tikzcd}
F_1 \overset{\mu_1(T)}{\underset{\sigma_1,\sigma_0}{\ddot{\times}}} F_1(T) \arrow[r, "f' \times f'(T)"] \arrow[d, "\wr"] & F_2 \overset{\mu_2(T)}{\underset{\sigma_1,\sigma_0}{\ddot{\times}}} F_2(T) \arrow[d, "\wr"] \\
F_1(T) \underset{\sigma_1(T),\sigma_0(T)}{\times} F_1(T) \arrow[r, "f'(T) \times f'(T)"] \arrow[d, "\tilde{\mu}_1(T)"] & F_2(T) \underset{\sigma_1(T),\sigma_0(T)}{\times} F_2(T) \arrow[d, "\tilde{\mu}_2(T)"] \\
F_1(T) \arrow[r, "f'(T)"] & F_2(T)
\end{tikzcd}$$

DEFINITION (1.4.6) A $\underset{\sim}{\mathfrak{U}}$-category $\underset{\sim}{C}$ is called $\underset{\sim}{\mathfrak{U}}$-small if its set of objects (and hence also its set of arrows, since it is a $\underset{\sim}{\mathfrak{U}}$-category) is a member of $\underset{\sim}{\mathfrak{U}}$, i.e., $\mathfrak{Ob}(\underset{\sim}{C}) \in \underset{\sim}{\mathfrak{U}}$.

If $\mathbf{C}$ is a $\mathbf{U}$-category and C is a $\mathbf{C}$-category, then the category $C(T)$ of (1.4.4) is $\mathbf{U}$-small.

(1.4.7) For any category $\mathbf{C}$, let $\mathbf{C}^2$ be its arrow category. $\mathbf{C}^2$ is supplied with the following <u>canonical functors</u>:

(1.4.7.1)
$$\begin{tikzcd}
\mathbf{C}^2 \times_{\mathbf{C}} \mathbf{C}^2 \arrow[r, shift left, "\mu(\mathbf{C})"] \arrow[r, shift right, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & \mathbf{C}^2 \arrow[d, shift left, "\sigma_0(\mathbf{C})"] \\
\mathbf{C}^2 \arrow[r, "\sigma_1(\mathbf{C})"] & \mathbf{C} \arrow[u, shift left, "\mathbf{I}(\mathbf{C})"]
\end{tikzcd}$$

1° a couple of $(\sigma_0(\mathbf{C}), \sigma_1(\mathbf{C})) : \mathbf{C}^2 \rightrightarrows \mathbf{C}$ called the <u>source</u> and <u>target</u> functors defined by

$$\langle\langle \quad p_0 \rightsquigarrow \sigma_0(\mathbf{C})(p_0),\ ((p_0, q_1), (p_1, q_0)) \rightsquigarrow p_1 \quad \rangle\rangle \text{ and}$$

$$\langle\langle \quad p_0 \rightsquigarrow \sigma_1(\mathbf{C})(p_0),\ ((p_0, q_1), (p_1, q_0)) \rightsquigarrow q_1 \quad \rangle\rangle;$$

2° $\mathbf{I}(\mathbf{C}) : \mathbf{C} \longrightarrow \mathbf{C}^2$ called the <u>identity assignment</u>, defined by $\langle\langle \ X \rightsquigarrow I_X,\ f \rightsquigarrow ((I_X, f), (f, I_X)) \ \rangle\rangle$, and

3° $\mu(\mathbf{C}) : \mathbf{C}^2 \times_{\sigma_1(\mathbf{C}), \sigma_0(\mathbf{C})} \mathbf{C}^2 \longrightarrow \mathbf{C}^2$ called the <u>multiplication</u>, defined by

$$\langle\langle \ (p_0, s_0) \rightsquigarrow s_0 p_0,\ ((p_0, q_1), (p_1, q_0)), ((s_0, r_1), (q_1, r_0)) \rightsquigarrow ((s_0 p_0, r_1), (p_1, r_0 q)) \rangle\rangle.$$

It is easily verified that these are indeed functors for they correspond in the square (1.4.2.1) to the assignments of top and bottom arrows for source and target (which are functors because of the "lateral multiplication" which is defined in $\mathbf{C}^2$) and in 3°, to the "vertical composition" of squares:

(1.4.7.2) $$\left\langle\!\left\langle\;\begin{tikzcd} T \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "q_0"] \\ A \arrow[r, "q_1"] \arrow[d, "s_0"'] & U \arrow[d, "r_0"] \\ C \arrow[r, "r_1"'] & V \end{tikzcd} \rightsquigarrow \begin{tikzcd} T \arrow[r, "p_1"] \arrow[d, "s_0 p_0"'] & B \arrow[d, "r_0 q_0"] \\ C \arrow[r, "r_1"'] & V \end{tikzcd}\;\right\rangle\!\right\rangle$$

which is the <u>same</u> as that of C^2 using the <u>conversion bijection</u>,

$$(1.4.7.3) \quad \langle\langle\ ((p_0,q_1),\ (p_1 q_0)) \rightsquigarrow ((p_1,q_0),\ (p_0,q_1))\ \rangle\rangle.$$

<u>LEMMA</u> (1.4.8) Let $\mathbf{C}$ be a $\mathfrak{U}$-category and $\mathbf{C}^2$ its arrow category, so that both $\mathbf{C}$ and $\mathbf{C}^2$ are objects of $\mathfrak{U}$-$\mathbf{CAT}$, then

$$\mathbf{\underline{C}} = (\ \ \boldsymbol{\mu}(\mathbf{C});\ (\sigma_0(\mathbf{C}),\ \sigma_1(\mathbf{C})) : \mathbf{C}^2 \rightrightarrows \mathbf{C},\ \mathbf{I}(\mathbf{C}))$$

is a ($\mathfrak{U}$ -$\mathbf{CAT}$)- category.

The functoriality of the structure functions having been verified together with the knowledge of unrestricted existence of fibre products in $\mathfrak{U}^*$CAT in (1.4.6) and (1.3.9), the verification of $(AC)_I$ - $(AC)_{IV}$ is straightforward and left to the reader.

<u>THEOREM</u> (1.4.9) For each $\mathfrak{U}$-category $\mathbf{T}$, the set $\mathbf{CAT}\ (\mathbf{T},\mathbf{C})$ is the set of objects of a category structure (in some $\mathfrak{U}^* \ni \mathfrak{U}$) whose arrows are the members of the set $\mathbf{CAT}\ (\mathbf{T},\mathbf{C}^2)$ with the applications $(\mathbf{CAT}(\mathbf{T},\ \sigma_0(\mathbf{C})),\ \mathbf{CAT}\ (\mathbf{T},\ \sigma_1(\mathbf{C}))\ :\ \mathbf{CAT}\ (\mathbf{T},\mathbf{C}^2) \rightrightarrows \mathbf{CAT}(\mathbf{T},\mathbf{C})$ as its source and target functions, and $\mathbf{CAT}\ (\mathbf{T},\ \boldsymbol{\mu}(\mathbf{C}))\ :\ \mathbf{CAT}\ (\mathbf{T},\mathbf{C}^2) \times_{\mathrm{CAT}(T,\sigma_1),\,\mathrm{CAT}(T,\sigma_0)} \mathbf{CAT}\ (\mathbf{T},\mathbf{C}^2) \longrightarrow \mathbf{CAT}(\mathbf{T},\mathbf{C}^2)$ as its multiplication. Supplied with this structure $\mathbf{CAT}(\mathbf{T},\mathbf{C})$ is not in general a $\mathfrak{U}$-category, but is always a small $\mathfrak{U}^*$-category. For any functor $F : \mathbf{U} \longrightarrow \mathbf{T}$, $\mathbf{CAT}\ (F,\mathbf{C})\ :\ \mathbf{CAT}\ (\mathbf{T},\mathbf{C}) \rightarrow \mathbf{CAT}\ (\mathbf{U},\mathbf{C})$ is the object mapping of a functor; whose arrow mapping is $\mathbf{CAT}(F,\mathbf{C}^2)\ :\ \mathbf{CAT}(\mathbf{T},\mathbf{C}^2) \longrightarrow \mathbf{CAT}(\mathbf{U},\mathbf{C}^2)$.

The theorem is a corollary of LEMMA (1.4.4) applied to the category $\mathfrak{U}$-CAT using the ($\mathfrak{U}$-CAT) - category $\underset{\sim}{C} = (\mu(\underset{\sim}{C});$ $(\sigma_0(\underset{\sim}{C}), \sigma_1(\underset{\sim}{C})); \underset{\sim}{C}^2 \rightrightarrows \underset{\sim}{C}, \; I(\underset{\sim}{C}))$ of LEMMA (1.4.8).

$$\begin{tikzcd}[column sep=huge]
\underset{\sim}{C}^2(\underset{\sim}{T}) \underset{\sigma_1,\sigma_0}{\times} \underset{\sim}{C}^2(\underset{\sim}{T}) \arrow[r, "\underset{\sim}{C}^2(F)_{\sigma_1}\times_{\sigma_1}\underset{\sim}{C}^2(F)"] \arrow[d, shift right=4, "pr_1"'] \arrow[d, "\mu"] \arrow[d, shift left=4, "pr_2"] & \underset{\sim}{C}^2(\underset{\sim}{U}) \underset{\sigma_1,\sigma_0}{\times} \underset{\sim}{C}^2(\underset{\sim}{U}) \arrow[d, shift right=4, "pr_1"'] \arrow[d, "\mu"] \arrow[d, shift left=4, "pr_2"] \\
\underset{\sim}{C}^2(\underset{\sim}{T}) \arrow[r, "\underset{\sim}{C}^2(F)"] \arrow[d, shift right=2, "\sigma_1"'] \arrow[d, shift left=2, "\sigma_0"] & \underset{\sim}{C}^2(\underset{\sim}{U}) \arrow[d, shift right=2, "\sigma_1"'] \arrow[d, shift left=2, "\sigma_0"] \\
\underset{\sim}{C}(\underset{\sim}{T}) \arrow[r, "\underset{\sim}{C}(F)"] & \underset{\sim}{C}(\underset{\sim}{U})
\end{tikzcd} \tag{1.4.9.1}$$

<u>DEFINITION</u> (1.4.10) The category produced in THEOREM (1.4.9) with objects $\underset{\sim}{CAT}$ $(\underset{\sim}{C},\underset{\sim}{D})$ and arrows $\underset{\sim}{CAT}$ $(\underset{\sim}{C},\underset{\sim}{D}^2)$ will be denoted by «$\underline{\underline{CAT}}$ $(\underset{\sim}{C},\underset{\sim}{D})$ (or $\underline{\underline{Hom}}$ $(\underset{\sim}{C},\underset{\sim}{D})$)» and will be called the <u>category of functors</u> with source $\underset{\sim}{C}$ and target $\underset{\sim}{D}$ (as opposed to the set of functors $\underset{\sim}{CAT}$ $(\underset{\sim}{C},\underset{\sim}{D})$). The arrows in this category will be called <u>morphisms</u> (or <u>natural transformations</u>) <u>of functors</u>. Unless otherwise noted, it will always be this category structure to which we refer in reference to $\underset{\sim}{CAT}$ $(\underset{\sim}{C},\underset{\sim}{D})$.

If we denote composition of natural transformation as ⟨⟨ $\theta\cdot\psi$ ⟩⟩, ⟨⟨ $\underline{\underline{CAT}}$ $(F,\underset{\sim}{C})$ is a functor ⟩⟩ is simply

$$\boxed{\langle\langle \quad (\theta\cdot\psi)\,F = \theta F\cdot \psi F \quad \rangle\rangle.}$$

<u>PROPOSITION</u> (1.4.11) The following statements are equivalent:

1° $\varphi : \underset{\sim}{T} \to \underset{\sim}{C}^2$ and $\sigma_0(\underset{\sim}{C})\varphi = F : \underset{\sim}{T} \to \underset{\sim}{C}$ and $\sigma_1(\underset{\sim}{C})\varphi = G : \underset{\sim}{T} \to \underset{\sim}{C}$.

(⟨⟨ φ is an <u>arrow of the category</u> $\underline{\underline{CAT}}$ $(\underset{\sim}{T},\underset{\sim}{C})$ with source **F and target G** ⟩⟩).

2° $F : \underset{\sim}{T} \to \underset{\sim}{C}$ and $G : \underset{\sim}{T} \to \underset{\sim}{C}$ are functors and,

$(\varphi(T) : F(T) \longrightarrow G(T))$ $T \in \mathcal{O}b(\underset{\sim}{C})$ is a family of morphisms in

$\underset{\sim}{C}$ such that for $f : T \to U$ in $\underset{\sim}{T}$, the diagram

(1.4.11.1)
$$\begin{tikzcd}
F(T) \arrow[r, "F(f)"] \arrow[d, "\varphi(T)"] & F(U) \arrow[d, "\varphi(U)"] \\
G(T) \arrow[r, "G(f)"] & G(U)
\end{tikzcd}$$

of objects and arrows in $\underset{\sim}{C}$ is commutative.

(⟨⟨ φ is a <u>natural transformation of</u> F <u>into</u> G ⟩⟩).

For the proof of this equivalence, observe that the definition of ⟨⟨ functor ⟩⟩ of ⟨⟨ arrow category of $\underset{\sim}{C}$ ⟩⟩ (1.4.2) determines the form of any functor $\varphi : \underset{\sim}{T} \to \underset{\sim}{C}^2$. If $f : T \to U$ is an arrow in $\underset{\sim}{T}$, then $\varphi(f) : \varphi(T) \Longrightarrow \varphi(U)$ must be an arrow in $\underset{\sim}{C}^2$ and hence we may write

$$\varphi(f) : \begin{tikzcd}
\sigma_0(\varphi(T)) \arrow[r, "u"] \arrow[d, "\varphi(T)"'] \arrow[dr, phantom, "\varphi(T) \Longrightarrow \varphi(U)"] & \sigma_0(\varphi(U)) \arrow[d, "\varphi(U)"] \\
\sigma_1(\varphi(T)) \arrow[r, "t"] & \sigma_1(\varphi(U))
\end{tikzcd}$$

as the commutative square (($\varphi(T)$, t), (u, $\varphi(U)$)) which the image of f under the arrow function of φ . The source and target functors then yield the assignments

⟨⟨ $(f : T \to U) \rightsquigarrow \sigma_0(\varphi(T) \to \sigma_0(\varphi(U))$ and $(f : T \to U) \rightsquigarrow \sigma_1(\varphi(T)) \to \sigma_1(\varphi(T))$ ⟩⟩,

which we may rewrite as

$$\langle\langle\ f : T \to U \rightsquigarrow F(T) \xrightarrow{F(f)} F(U) \text{ and } f : T \to U \rightsquigarrow G(T) \xrightarrow{G(f)} G(U)\ \rangle\rangle$$

if we let $F(T) = \sigma_o(\varphi(T))$ and $F(f) = u$ and $G(T) = \sigma_1(\varphi(T))$, $G(f) = t$. F and G are functors as the composition of the functors $\underset{\sim}{\sigma}_o(\underset{\sim}{C})$ and $\underset{\sim}{\sigma}_1(\underset{\sim}{C})$ with φ.

Conversely given functors F and G which satisfy 2° for some family $(\varphi(T))_{T \in \mathcal{O}b(\underset{\sim}{T})}$ of arrows in $\underset{\sim}{C}$, the assignment

$$\langle\langle\quad T \rightsquigarrow \varphi(T),\ f \rightsquigarrow ((\varphi(T), G(f)), F(f), \varphi(U))\quad \rangle\rangle$$

defines a functor $\varphi : \underset{\sim}{T} \to \underset{\sim}{C}^2$ which clearly satisfies 1°.

(1.4.12) It is convenient to note the form of the composition of natural transformations in its evaluated description. Let F, G and H be functors from $\underset{\sim}{C}_1 \to \underset{\sim}{C}_2$ and $\varphi : F \longrightarrow G$, $\psi : G \longrightarrow H$ natural transformations, then $\psi \cdot \varphi : F \longrightarrow H$ is the natural transformation obtained by the "vertical composition" of squares

$$(1.4.12.1)\ \langle\langle \begin{array}{ccc} F(T) & \xrightarrow{F(f)} & F(U) \\ {\scriptstyle \varphi(T)}\downarrow & & \downarrow{\scriptstyle \varphi(U)} \\ G(T) & \xrightarrow{G(f)} & G(U) \\ {\scriptstyle \psi(T)}\downarrow & & \downarrow{\scriptstyle \psi(U)} \\ H(T) & \xrightarrow{H(f)} & H(U) \end{array} \rightsquigarrow \begin{array}{ccc} F(T) & \xrightarrow{F(f)} & F(U) \\ {\scriptstyle \psi(T)\varphi(T)}\downarrow & & \downarrow{\scriptstyle \psi(U)\varphi(U)} \\ H(T) & \xrightarrow{H(f)} & H(U) \end{array} \rangle\rangle$$

for any objects T, U in $\underset{\sim}{C}_1$ and any $f : T \to U$ (which is, of course, just the definition of the "functorial composition" $\underset{\sim}{\mu}(\underset{\sim}{C}) : \underset{\sim}{C}^2 {}_{\underset{\sim}{\sigma}_1}\!\times_{\underset{\sim}{\sigma}_o} \underset{\sim}{C}^2 \longrightarrow \underset{\sim}{C}^2$ (1.4.6.3°).

<u>DEFINITION</u> (1.4.13) A natural transformation $\varphi : F \longrightarrow G$ of functors is called an <u>isomorphism of</u> F <u>with</u> G provided it has an inverse

in $\underline{\underline{\mathrm{CAT}}}\,(\mathbf{C},\mathbf{D})$. (i.e. there exists $\psi : G \longrightarrow F$ such that $\varphi\cdot\psi = I_F$ and $\psi\cdot\varphi = I_G$).

In order that φ be an isomorphism it is necessary and sufficient that for all $T \in \mathfrak{Ob}(\mathbf{C})$, $\varphi(T) : F(T) \longrightarrow G(T)$ be an isomorphism in $\mathbf{D}$.

<u>EXAMPLES</u> (1.4.13) - 1° Let $\mathbf{C}$ be a category, X, Y objects in $\mathbf{C}$ and $f : X \longrightarrow Y$ an arrow in $\mathbf{C}$. For each $T \in \mathfrak{Ob}(\mathbf{C})$, let $f(T) : X(T) \longrightarrow Y(T)$ be the application defined by $\langle\langle\, f(T)(x) = fx,\ x \in X(T) \,\rangle\rangle$. The assignment $\langle\langle\, T \rightsquigarrow f(T),\ T \in \mathfrak{Ob}(\mathbf{C}) \,\rangle\rangle$ then defines a natural transformation h_f of the functor h_X into the functor h_Y. The functoral morphism $h_f : h_X \longrightarrow h_Y$ is said to be <u>associated with</u> (or <u>induced by</u>) $f \in Y(X)$. Similarly for each $T \in \mathfrak{Ob}(\mathbf{C})$, let $T(f) : T(Y) \longrightarrow T(X)$ be the application defined by $\langle\langle\, x \rightsquigarrow xf \,\rangle\rangle$. The assignment $\langle\langle\, T \rightsquigarrow T(f) \,\rangle\rangle$ then defines a natural transformation h'_f of the functor h'_Y into the functor h'_X said to be <u>associated with</u> (or <u>induced by</u>) $f \in Y(X)$. In other words for each $(T,U) \in \mathfrak{Ob}(\mathbf{C}) \times \mathfrak{Ob}(\mathbf{C})$ and each $\xi \in U(T)$, the diagrams

$$
(1.4.13.1)\quad
\begin{array}{ccc}
X(U) & \xrightarrow{X(\xi)} & X(T) \\
{\scriptstyle f(T)}\downarrow & & \downarrow{\scriptstyle f(U)} \\
Y(U) & \xrightarrow[\;]{Y(\xi)} & Y(T)
\end{array}
\qquad\qquad
(1.4.12.2)\quad
\begin{array}{ccc}
T(X) & \xrightarrow{\xi(Y)} & U(Y) \\
{\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle U(f)} \\
T(X) & \xrightarrow{\xi(X)} & U(X)
\end{array}
$$

of sets and applications are commutative.

2° Let $\mathbf{C}$ be a category and $(X \times Y,\ p_o, p_1)$ a representation of the product of X and Y (1.2.1). For each $T \in \mathfrak{Ob}(\mathbf{C})$, let $p_o(T) \boxtimes p_1(T) : X \times Y\ (T) \longrightarrow X(T) \times Y(T)$ be the application defined by ⟨⟨ $x \rightsquigarrow (p_0 x, p_1 x)$ ⟩⟩. The assignment ⟨⟨ $T \rightsquigarrow p_o(T) \boxtimes p_1(T)$ ⟩⟩ then defines a functorial isomorphism of the functor $h_{X \times Y}$ with the product functor $h_X \times h_Y$ (defined by $h_X \times h_Y(T) = h_X(T) \times h_Y(T) (= X(T) \times Y(T))$ by definition.

3^o Two classical examples of natural versus "un-natural " isomorphism are those of the isomorphism of a finite abelian group G with its double character group D(D(G)) $\iota : I_{\underset{\sim}{Ab}} \longrightarrow D \cdot D : \underset{\sim}{Ab}^{fin} \longrightarrow \underset{\sim}{Ab}^{fin}$ which is natural in contrast to the always present isomorphism of $G \xrightarrow{\sim} D(G)$ which depends on choice of generators and is not functoral. Parallel is the natural isomorphism of a finite dimensional vector space with its double dual $V \xrightarrow{\sim} \mathrm{Hom}\,(\mathrm{Hom}(V,R),R)$ versus the un-natural but always present isomorphism of $V \xrightarrow{\sim} V^*$ which exists from the equality of their dimension.

4^o All of the "canonical maps" of Chapter 0 are natural when the functors are restricted to ($\underset{\sim}{ENS}$-$\underset{\sim}{\mathcal{U}}$).

REMARK (1.4.13) It may be argued that the sequence of LEMMATA leading to THEOREM (1.4.8) is a ridiculously cumbersome method of showing that the set of functors between two categories has a category structure with natural transformations as morphisms. (The conventional procedure is to define ⟨⟨ natural transformation of F into G ⟩⟩ via ($1.4.9.2^o$) and take (1.4.10 as the definition of ⟨⟨ composition of natural transformation ⟩⟩). The justification of this technique is in its "internalization" of ⟨⟨ functorial morphisms ⟩⟩ within ($\underset{\sim}{CAT}$) and the fact that the procedure is applicable elsewhere. For certain other purposes it seems also to "naturalise" ⟨⟨ operations on natural transformations ⟩⟩ and, as it is basically a ⟨⟨ simplicial technique ⟩⟩, suggests reasons for their apparent importance.

The notion of ⟨⟨ natural transformation ⟩⟩, however arrived at, is the fundamental idea of the theory, and, in point of fact, it was to explain (or rather describe) their occurence in mathematics that led EILENBERG and MacLANE in their initial paper to, as FREYD put it, "define ⟨⟨ category ⟩⟩ so that one could define ⟨⟨ functor ⟩⟩ and define functor ⟩⟩ so that one could define ⟨⟨ natural transformation⟩⟩".

Whether or not ⟨⟨ categories and functors ⟩⟩ actually suceeds in explaining "natural transformations in their natural habitat" is an entirely different matter. It can be argued quite strongly that even with "all of this machinery", E. WITT'S quip, "Oh, "natural map", everybody knows what that is - but nobody can define it!" is still substantially correct.

(1.5) (*) - COMPOSITION OF TRANSFORMATIONS OF FUNCTORS

(1.5.1) Let $\mathbf{C}$ be a $\mathfrak{U}$-category and $\mathbf{C}^2$ its arrow category with $\mathbf{C} = ($ $\mu(C)$; $C^2 \underset{\sigma_1(C)}{\overset{\sigma_0(C)}{\rightrightarrows}} C$, $\mathbb{I}(C)$ the system of categories and functors which define its canonical ($\mathfrak{U}$-CAT) category structure (1.4.6) (which we will abbreviate as the diagram

(1.5.1.1)
$$\begin{tikzcd} \mathbf{C}^2 \times_{\sigma_1,\sigma_0} \mathbf{C}^2 \arrow[r, shift left=2, "pr_0"] \arrow[r, "\mu" description] \arrow[r, shift right=2, "pr_1"'] & \mathbf{C}^2 \arrow[r, shift left, "\sigma_0"] \arrow[r, shift right, "\sigma_1"'] & \mathbf{C}). \end{tikzcd}$$

If $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a functor, we define the <u>canonical extension of</u> F <u>to the arrow category</u> to be that functor $F^2 : \mathbf{C}^2 \longrightarrow \mathbf{D}^2$ defined by the assignment

(1.5.1.2)
$$\left\langle\!\left\langle \begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \rightsquigarrow \begin{tikzcd} F(A) \arrow[d, "F(f)"] \\ F(B) \end{tikzcd}, \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & A' \arrow[d, "g"] \\ B \arrow[r, "b"] & B' \end{tikzcd} \rightsquigarrow \begin{tikzcd} F(A) \arrow[r, "F(a)"] \arrow[d, "F(f)"'] & F(A') \arrow[d, "F(g)"] \\ F(B) \arrow[r, "F(b)"] & F(B') \end{tikzcd} \right\rangle\!\right\rangle$$

F is trivially a functor because of the "lateral composition" of squares in $\mathbf{C}^2$ and, moreover, $I_{\mathbf{C}}{}^2 = I_{\mathbf{C}^2}$, $(GF)^2 = G^2F^2$.

More interestingly the functor F^2 equally well preserves "vertical composition of squares"; <u>the couple</u> (F, F^2) <u>defines a</u> ($\mathfrak{U}$-CAT)-<u>functor</u> F <u>of the</u> ($\mathfrak{U}$-CAT) <u>category</u> $\mathbf{C}$ <u>into the</u> ($\mathfrak{U}$-CAT) <u>category</u> $\mathbf{D}$ (1.3.1) i.e., the following diagram is sequentially commutative:

(1.5.1.3)
$$\begin{tikzcd} \mathbf{C}^2 \times_{\sigma_1,\sigma_0} \mathbf{C}^2 \arrow[rr, "F^2 \times F^2"] \arrow[d, shift right=2, "pr_1"'] \arrow[d, "\mu" description] \arrow[d, shift left=2, "pr_0"] & & \mathbf{D}^2 \times_{\sigma_1,\sigma_0} \mathbf{D}^2 \arrow[d, shift right=2, "pr_1"'] \arrow[d, "\mu" description] \arrow[d, shift left=2, "pr_0"] \\ \mathbf{C}^2 \arrow[rr, "F^2"] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] & & \mathbf{D}^2 \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] \\ \mathbf{C} \arrow[rr, "F"] & & \mathbf{D} \end{tikzcd}$$

Consequently, COROLLARY (1.4.5) gives rise to a corresponding corollary of THEOREM (1.4.9) which we state as

<u>COROLLARY</u> (1.5.2). For any functor $F : \mathbf{C} \longrightarrow \mathbf{D}$, and any category $\mathbf{T}$ in $(\mathbf{CAT})$, $\mathbf{CAT}(\mathbf{T},F) : \mathbf{CAT}(\mathbf{T},\mathbf{C}) \longrightarrow \mathbf{CAT}(\mathbf{T},\mathbf{D})$ defines (with $\mathbf{CAT}(\mathbf{T},F^2) : \mathbf{CAT}(\mathbf{T},\mathbf{C}^2) \longrightarrow \mathbf{CAT}(\mathbf{T},\mathbf{D}^2)$ as its arrow function) a functor $\underline{\underline{CAT}}(\mathbf{T},F) : \underline{\underline{CAT}}(\mathbf{T},\mathbf{C}) \longrightarrow \underline{\underline{CAT}}(\mathbf{T},\mathbf{D})$ of the corresponding categories of functors with natural transformations as morphisms.

i.e. $\boxed{\langle\langle \;\; F^2(\theta \cdot \psi) = F^2\theta \circ F^2\psi \;\; \rangle\rangle.}$

(1.5.3) The existence of the canonical extension $F^2 : \mathbf{C}^2 \longrightarrow \mathbf{D}^2$ for any functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ allows a convenient "equational restatement" of the definition of natural transformation.

Let F and G be functors, $F, G : \mathbf{C} \longrightarrow \mathbf{D}$, $\psi : F \longrightarrow G$ a functorial morphism, i.e.

$$\langle\langle \quad (F,G) : C \xrightarrow{\psi} D^2 \underset{\sigma_1^D}{\overset{\sigma_0^D}{\rightrightarrows}} D \quad \rangle\rangle$$

Since $\sigma_1^D F^2 = F \;\; \sigma_1^C = (\sigma_0^D \psi)\, \sigma_1^C = \sigma_0^D(\psi\, \sigma_1^C)$, $\psi\sigma_1^C : \mathbf{C}^2 \longrightarrow \mathbf{D}^2$ and $F^2 : \mathbf{C}^2 \longrightarrow \mathbf{D}^2$, are the projections of a composable couple $(\psi\sigma_1^C, F^2)$ of arrows from the category $\underline{\underline{CAT}}(\mathbf{C}^2,\mathbf{D})$. Similarly, $G^2 : \mathbf{C}^2 \longrightarrow \mathbf{D}^2$ and $\psi\sigma_0^C : \mathbf{C}^2 \longrightarrow \mathbf{D}^2$ are those of the couple $(G^2, \psi\sigma_0^C)$. Moreover, the coincidence of the functorial multiplication $\mu(\mathbf{D})$ with that of $\mathbf{D}$ after conversion and the fact that the arrows of $\mathbf{C}$ are the objects of $\mathbf{C}^2$, gives that

(1.5.3.1) $\boxed{\psi\, \sigma_1^C \circ F^2 = G^2 \circ \sigma_0^C.}$

(1.5.3.2)
$$\begin{tikzcd}
\sigma_0^D F^2 = F\,\sigma_0^C \arrow[r, "\psi\sigma_0^C"] \arrow[d, "F^2"] & G\,\sigma_0^C = \sigma_0^D G^2 \arrow[d, "G^2"'] \\
\sigma_1^D F^2 = F\,\sigma_1^C \arrow[r, "\psi\sigma_1^C"] & G\,\sigma_1^C = \sigma_1^D G^2
\end{tikzcd}$$

On evaluation, the equation (1.5.3) is nothing more than the assertion that, for all $\mathfrak{O}\!\!\!\sim(\underset{\sim}{C}^2)$, $f : A \longrightarrow B$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\psi(A)"] & F(B) \arrow[d, "\psi(B)"] \\
G(A) \arrow[r, "G(f)"] & G(B)
\end{tikzcd}$$

is commutative.

(1.5.4) Implicit in (1.5.1) is the observation that for any $\underset{\sim}{\mathfrak{U}}$-category $\underset{\sim}{T}$, the construction of the arrow category $\underset{\sim}{T}^2$ (1.4.2) is functorial, i.e. the assignments

$$\langle\langle \quad \underset{\sim}{T} \rightsquigarrow \underset{\sim}{T}^2,\ F \rightsquigarrow F^2 \quad \rangle\rangle \text{ define a functor}$$

$$\underset{\sim}{2} : (\underset{\sim}{\mathfrak{U}}\text{-})\ \underset{\sim}{CAT} \longrightarrow (\underset{\sim}{\mathfrak{U}}\text{-})\ \underset{\sim}{CAT}$$ (which is in fact an embedding, since the functor $\underset{\sim}{I}(\underset{\sim}{C})$ is a section).

If THEOREM (1.4.9) is applied to the canonical objects and arrows of a diagram in ($\underset{\sim}{CAT}$) such as

(1.5.4.1)
$$\begin{tikzcd}
\underset{\sim}{C}^2 \arrow[r, dashed, shift left, "F^2"] \arrow[r, dashed, shift right, "G^2"'] \arrow[d, shift right, "\underset{\sim}{\sigma}_1^C"'] \arrow[d, shift left, "\underset{\sim}{\sigma}_0^C"] & \underset{\sim}{D}^2 \arrow[d, shift right, "\underset{\sim}{\sigma}_1^D"'] \arrow[d, shift left, "\underset{\sim}{\sigma}_0^D"] \\
\underset{\sim}{C} \arrow[r, dashed, shift left, "F"] \arrow[r, dashed, shift right, "G"'] \arrow[ur, dashed, "\psi"] & \underset{\sim}{D}
\end{tikzcd}$$

one obtains the structure applications of the small ($\mathfrak{U}^{\pm}$)functors $\underset{\sim}{D}(\underset{\sim}{\sigma}_1^C)$, $\underset{\sim}{D}(\underset{\sim}{\sigma}_0^C)$: $\underset{\sim}{D}(\underset{\sim}{C}) \rightrightarrows \underset{\sim}{D}(\underset{\sim}{C}^2)$, viz:

(1.5.4.2)
$$\begin{tikzcd}[column sep=huge]
\underset{\sim}{D}^2(\underset{\sim}{C}) \times_{\sigma_1\sigma_0} \underset{\sim}{D}^2(\underset{\sim}{C}) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[d, shift right=2] \arrow[d, "\mu"] \arrow[d, shift left=2] & & \underset{\sim}{D}^2(\underset{\sim}{C}^2) \times_{\sigma_1\sigma_0} \underset{\sim}{D}^2(\underset{\sim}{C}^2) \arrow[d, shift right=2] \arrow[d, "\mu"'] \arrow[d, shift left=2] \\
\underset{\sim}{D}^2(\underset{\sim}{C}) \arrow[rr, shift left, "\underset{\sim}{D}^2(\sigma_0^{\underset{\sim}{C}})"] \arrow[rr, shift right, "\underset{\sim}{D}^2(\sigma_1^{\underset{\sim}{C}})"'] \arrow[d, shift right, "\underset{\sim}{\sigma_1}^{\underset{\sim}{D}}(\underset{\sim}{C})"'] \arrow[d, shift left, "\sigma_2^{\underset{\sim}{D}}(\underset{\sim}{C})"] & & \underset{\sim}{D}^2(\underset{\sim}{C}^2) \arrow[d, shift right, "\sigma_1^{D}(C)"'] \arrow[d, shift left, "\sigma_0^{\underset{\sim}{D}}(\underset{\sim}{C}^2)"] \\
\underset{\sim}{D}(\underset{\sim}{C}) \arrow[rr, shift left, "\underset{\sim}{D}(\sigma_1^{\underset{\sim}{C}})"] \arrow[rr, shift right, "\underset{\sim}{D}(\sigma_1^{C})"'] \arrow[urr, "\underset{=}{2}_{DC}"] & & \underset{\sim}{D}(\underset{\sim}{C}^2)
\end{tikzcd}$$

where the application $\underset{=}{2}_{DC} : \underset{\sim}{D}(\underset{\sim}{C}) \hookrightarrow \underset{\sim}{D}^2(\underset{\sim}{C}^2)$ is the restriction of the arrow application of $\underset{=}{2} : (\underset{\sim}{CAT}) \longrightarrow (\underset{\sim}{CAT})$.

<u>PROPOSITION</u> (1.5.5) For any couple $(\underset{\sim}{D}, \underset{\sim}{C})$ of $\underset{\sim}{\mathcal{U}}$-categories, the applications defined by the assignments

$$\langle\langle \quad F \rightsquigarrow F^2, \quad \psi \rightsquigarrow (F^2, \psi\underset{\sim}{\sigma_1}^{\underset{\sim}{C}}, \psi\underset{\sim}{\sigma_0}^{\underset{\sim}{C}}, G^2), \; F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}, \; \psi : F \longrightarrow G \quad \rangle\rangle$$

define a natural transformation $\underset{=}{2}_{\underset{\sim}{CD}}$ of the (canonical) functor $\underset{=}{CAT}(\sigma_0^{\underset{\sim}{C}}, \underset{\sim}{D}) : \underset{=}{CAT}(\underset{\sim}{C}, \underset{\sim}{D}) \longrightarrow \underset{=}{CAT}(\underset{\sim}{C}^2, \underset{\sim}{D})$ into the (canonical) functor $\underset{=}{CAT}(\underset{\sim}{\sigma_1}^{C}, \underset{\sim}{D}) : \underset{=}{CAT}(\underset{\sim}{C}, \underset{\sim}{D}) \longrightarrow \underset{=}{CAT}(\underset{\sim}{C}^2, \underset{\sim}{D})$.

$$\underset{=}{2}_{\underset{\sim}{CD}} : \underset{=}{CAT}(\sigma_0^{\underset{\sim}{C}}, \underset{\sim}{D}) \longrightarrow \underset{=}{CAT}(\sigma_1^{\underset{\sim}{C}}, \underset{\sim}{D}).$$

The equation (1.5.3.1) gives the verification that $(F^2, \psi\underset{\sim}{\sigma_1}^{\underset{\sim}{C}}, \psi\underset{\sim}{\sigma_0}^{\underset{\sim}{C}}, G^2)$ is indeed a commutative square, i.e. a morphism with source F^2 and target G^2 in the arrow category of $\underset{=}{CAT}(\underset{\sim}{C}^2, \underset{\sim}{D})$, for any natural transformation

$$(\psi : F \longrightarrow G) \in \underset{\sim}{\mathcal{M}}(\underset{=}{CAT}(\underset{\sim}{C}, \underset{\sim}{D})) = \underset{\sim}{CAT}(\underset{\sim}{C}, \underset{\sim}{D}^2).$$

On evaluation, one obtains

(1.5.5.1)
$$\begin{tikzcd}
\underline{\underline{CAT}}(\sigma_0^{C}, D)(F) \arrow[r, "\underline{\underline{CAT}}(\sigma_0^{C}, D)(\psi)"] \arrow[d, "2_{CD}(F)"'] & \underline{\underline{CAT}}(\sigma_0^{C}, D)(G) \arrow[d, "2_{CD}(G)"] \\
\underline{\underline{CAT}}(\sigma_1^{C}, D)(F) \arrow[r, "\underline{\underline{CAT}}(\sigma_1^{C}, D)(\psi)"'] & \underline{\underline{CAT}}(\sigma_1^{C}, D)(G)
\end{tikzcd} .$$

(1.5.6) In the appendix to GODEMENT (1958) there are listed "cinq règles de calcul fonctoriel", which have proved quite useful in the manipulation of functors and natural transformations. We will interpret them here in the sense of DEFINITION (1.4.9) so that given functors $(F,G) : C \rightrightarrows D$, the relation $\langle\langle \theta : F \to G \rangle \in \mathfrak{Fl}(\underline{\underline{CAT}}(C,D))$ is equivalent to $\langle\langle\ \theta : C \to D^2$ and $\sigma_0(D)\theta = F$, $\sigma_1(D)\theta = G\ \rangle\rangle$. Composition of functors will be denoted by simple juxtaposition $\langle\langle\ UV\ (V : C \to D,\ U : D \to E\ )\ \rangle\rangle$ and composition of the natural transformations by $\langle\langle \psi \cdot \varphi\ (\ \varphi : F \to G,\ \ \psi : G \to H\ )\ \rangle\rangle$. The GODEMENT operations $\langle\langle\ *\ \rangle\rangle$ are then defined by
$\langle\langle\ V * \theta = V^2\, \theta$ ("fore-substitution") and
$\theta * V = \theta\, U$ ("aft-substitution") $\rangle\rangle$ where sources and targets of both functors and transformations are those specified in the diagram

(1.5.6.1)
$$\begin{tikzcd}
 & & D^2 \arrow[r, "V^2"] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] & E^2 \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] \\
B \arrow[r, "U"] & C \arrow[ur, "\theta"] \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & D \arrow[r, "V"] & E
\end{tikzcd} .$$

We now state and prove the "five rules" noting that if the original proofs were trivial, the "functoriality of the definition of natural transformation" renders their correspondents somewhat "more than trivial".

We leave it to the reader to specify the appropriate sources and targets (easily done with aid of diagrams such as (1.5.3.2) and (1.5.3.3).

(I) $(UV) * \theta = (UV)^2\, \theta = (U^2V^2)\, \theta = U^2\, (V^2\theta) = U^2(V^2 * \theta) = U*(V*\theta)$

(II) $\theta*(UV) = \theta(UV) = (\theta U)V = (\theta * U)*V$, and

(III) $(U*\theta)*V = (U^2\theta)*V = (U^2\theta)V = U^2\, (\theta V) = U*(\theta * V) + U*\theta*V$,

since θ, U, V, V^2, U^2 are functors and $(U\,V)^2 = U^2V^2$.

COROLLARY (1.5.2) and THEOREM (1.4.9) gives

(IV) $U*(\theta'\cdot\theta'')*V = U^2(\theta\cdot\theta'')V = (U^2\theta'\cdot U^2\theta'')V = (U^2\theta' V)\circ(U^2\theta'' V) = (U*\theta'*V)\circ(U*\theta*V)$.

(1.5.3.1) and (IV) give

(V) $(\psi*G)\circ(U*\varphi) = (\psi G)\cdot(U^2\varphi) = (\psi\sigma_1\varphi)\cdot(U^2\varphi) = (\psi\sigma_1\cdot U^2)\varphi = (V^2\cdot\psi\sigma_0)\varphi = (V^2\varphi)\cdot(\psi F) = (V*\varphi)\circ(\psi*F)$.

(1.5.6.2)
$$\begin{tikzcd}
C^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] & D^2 \arrow[r, shift left, "U^2"] \arrow[r, shift right, "V^2"'] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] & E^2 \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] \\
C \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] \arrow[ur, "\varphi"] & D \arrow[r, shift left, "U"] \arrow[r, shift right, "V"'] \arrow[ur, "\psi"'] & E
\end{tikzcd}$$

(1.6) REPRESENTATION OF SET-VALUED FUNCTORS

(1.6.1) Let $\underset{\sim}{C}$ be a $(\underset{\sim}{\mathfrak{U}})$-category and $F : \underset{\sim}{C}^{(op)} \longrightarrow (\underset{\sim}{ENS})$ be a functor. We will denote by $\langle\langle\, \mathcal{H}om(h_X, F) \,\rangle\rangle$ the set (member of $\underset{\sim}{\mathfrak{U}}^*$) of all functorial morphisms $\varphi : h_X \longrightarrow F$ with source the ("contravariant hom") functor h_X associated with the object X in $\underset{\sim}{C}$ (1.3.15 Ex 1°) and target the functor F. $\mathcal{H}om(h_X, F) = \underline{\underline{CAT}}\,(\underset{\sim}{C}^{op}, (\underset{\sim}{ENS}))(h_X, F)$.

If $\varphi : h_X \longrightarrow F$ is such a natural transformation, then by definition, for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $\varphi(T) : X(T) \longrightarrow F(T)$ is an application of $h_X(T)$ $(=X(T))$ into $F(T)$. In particular, the set $X(X)$ is not empty, and the application $\varphi(X) : X(X) \longrightarrow F(X)$ defines a unique element $\varphi(X)(I_X)$ of $F(X)$. The same is true for any $\varphi \in \underset{\sim}{\mathcal{H}om}(h_X, F)$ and hence the assignment $\langle\langle\ \varphi \rightsquigarrow \varphi(X)\ (I_X)\ \rangle\rangle$ defines an application

$$\Psi : \quad \underset{\sim}{\mathcal{H}om}(h_X, F) \longrightarrow F(X)$$

If $\xi \in F(X)$ and $f \in X(T)$ in $\underset{\sim}{C}$, then since $F(f) : F(X) \longrightarrow F(T)$ is an application of sets, $F(f)\ (\xi)$ is a well defined element of $F(T)$, and the consequent assignment $\langle\langle\ f \rightsquigarrow F(f)(\xi)\ \rangle\rangle$ will define a function $\xi(T) : X(T) \longrightarrow F(T)$. Moreover, the subsequent assignment $\langle\langle\ T \rightsquigarrow \xi(T)\ \rangle\rangle$ defines a natural transformation $\underline{\xi} : h_X \longrightarrow F$.

To see this last assertion, it is sufficient to note that given any $g \in T(U)$,

$$F(g)\cdot\xi(T)(f) = F(g)(F(f)(\xi)) = F(g)\circ F(f)(\xi) = F(fg)(\xi) = F(X(f))(\xi) = \underline{\xi}(U)\cdot X(g)(f)$$

for any $f \in X(T)$, i.e. the square

(1.6.1.1)
$$\begin{tikzcd} X(T) \arrow[r, "X(g)"] \arrow[d, "\underline{\xi}(T)"'] & X(U) \arrow[d, "\underline{\xi}(U)"] \\ F(T) \arrow[r, "F(g)"] & F(U) \end{tikzcd}$$

commutes for any choice of $g \in T(U)$.

Since $\underline{\xi}$ is a natural transformation for any choice of $\xi \in F(X)$, the assignment $\langle\langle\ \xi \rightsquigarrow \underline{\xi}\ \rangle\rangle$ defines an application

$$\Phi : \ F(X) \longrightarrow \underset{\sim}{\mathcal{H}om}(h_X, F).$$

THEOREM (1.6.2) [YONEDA-GROTHENDIECK EVALUATION LEMMA ("Yoneda")]. For any category $\underset{\sim}{C}$, any object X in $\underset{\sim}{C}$, and any functor $F : \underset{\sim}{C}^{(\circ)} \longrightarrow (\underset{\sim}{ENS})$, the application Ψ of (1.6.1), with Φ as its reciprocal, is a bijection of the set $\underset{\sim}{\mathcal{H}om}(h_X, F)$ of all natural transformations of the contravariant hom-functor h_X into the functor F, onto the set F(X) defined by evaluation of the functor F at X. $\Psi : \underset{\sim}{\mathcal{H}om}(h_X, F) \xrightarrow{\sim} F(X)$.

All that remains is to verify that $\Psi \cdot \Phi$ and $\Phi \cdot \Psi$ are the identities on their respective sources. Calculation for $\varphi \in \underset{\sim}{\mathcal{H}om}(h_X, F)$; $\Phi \cdot \Psi\ (\varphi)\ (T) = \Phi(\varphi(X)(I_X))(T) : X(T) \longrightarrow F(T)$.
But $\Phi(\varphi(X)(I_X))(T)(x) = F(x)(\varphi(X)(I_X)) = \varphi(T)(X(x)(I_X)) = \varphi(T)(x)$, for any $x \in X(T)$, since φ is natural, hence $\Phi \cdot \Psi(\varphi)(T) = \varphi(T)$ and $\Phi \cdot \Psi\ (\varphi) = \varphi$.

Calculation for $\xi \in F(X)$: $\Psi \cdot \Phi\ (\xi) = \underline{\xi}(X)(I_X) = F(I_X)(\xi) = I_{F(X)}(\xi) = \xi$.

COROLLARY (1.6.3) The application of the set Y(X) into the set Y(X) into the set $\underset{\sim}{\mathcal{H}om}(h_X, h_Y)$ defined by the assignment ⟨⟨ $f \rightsquigarrow h_f$ ⟩⟩ (1.4.12 Ex 1°) is bijective.

On replacing the functor F by h_Y in (1.6.2), one obtains $\underset{\sim}{\mathcal{H}om}(h_X, h_Y) \xrightarrow{\sim} h_Y(X) = Y(X)$.

COROLLARY (1.6.4) Every functorial isomorphism $\underline{f} : h_X \xrightarrow{\sim} h_Y$ is induced by a unique isomorphism $f : X \xrightarrow{\sim} Y$.

By definition of isomorphism (1.1.1), ⟨⟨ $f : X \xrightarrow{\sim} Y$ is an isomorphism ⟩⟩ is equivalent to ⟨⟨ $h_f : h_X \xrightarrow{\sim} h_Y$ is an isomorphism ⟩⟩ and (1.6.3) asserts that $\underline{f}$ is of the form h_f for some $f : X \longrightarrow Y$.

DEFINITION (1.6.5) [GROTHENDIECK (1959) TDTE II]

Let $F : C^{(op)} \longrightarrow (ENS)$ be a functor. A couple (X, ξ) consisting of an object X in C and an element $\xi \in F(X)$ is said to define a representation of the functor F in C provided that for each $T \in Ob(C)$, the application of sets $\xi(T) : X(T) \longrightarrow F(T)$ defined by the assignment ⟨⟨ $x \rightsquigarrow F(X)(\xi)$ ⟩⟩ is a bijection. The thus defined natural isomorphism $\xi : h_X \longrightarrow F$ is then called the representation of F defined by (X, ξ) and a functor which admits such a representation is said to be representable with (X, ξ) (or, by abuse of language, X) as its representative.

A representable functor, then, is nothing but a functor which is functorially isomorphic to a "contravariant hom" functor h_X, for some object X in C. In the light of (1.6.4), the objects which occur in any couple of representations of a given functor are necessarily isomorphic, and a canonically selected representative (by means of the τ-operator, for instance, if such exist) may then be called "the" representation of F. One speaks of the representation as being "unique, up to a unique isomorphism".

Because of this "quasi-unicity" it is useful to consider the full subcategory of $CAT(C^{(op)}, (ENS))$ consisting of the representable functors (and all natural transformations between them). Every functor of the form h_X is a number of this subcategory (by means of the identity isomorphism) and moreover the assignment ⟨⟨ $X \rightsquigarrow h_X$ and $f \rightsquigarrow h_f$ ⟩⟩ defines a functor h (covariant!) with the category C as its source and $CAT(C^{(op)}, (ENS))$ as its target. (1.6.3) now says that the functor h is fully faithful and the definition of representable functor allows us to conclude

<u>THEOREM</u> (1.6.6) [GROTHENDIECK]. The canonical functor

$$h : \underset{\sim}{C} \longrightarrow \underline{\underline{CAT}}\,(\underset{\sim}{C}^{(op)}, (\underset{\sim}{ENS}))$$

is fully faithful and defines an <u>equivalence</u> of the category $\underset{\sim}{C}$ with the full subcategory of its target consisting of the (contravariant) <u>representable functors</u>.

(1.6.7) As this now stands, this is a simple consequence of the definition of equivalence given in (1.3.11). The full import of ⟨⟨ equivalence ⟩⟩ will become apparent later. It will suffice here to remark that it is because of this equivalence, that we are justified in using the "transfers" ⟨⟨ $f : X \longrightarrow Y$ " $\leftrightsquigarrow$ " for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $f(T) : X(T) \longrightarrow Y(T)$ ⟩⟩ and reasoning <u>almost exclusively set-theoretically</u>. However, we can say now that

If $E : \underset{\sim}{C}_1 \longrightarrow \underset{\sim}{C}_2$ defines an equivalence of $\underset{\sim}{C}_1$ with $\underset{\sim}{C}_2$ (1.3.11), then there exists a functor $F : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$ such that F is a quasi-inverse of E, i.e. there exist functorial isomorphisms $\xi : EF \xrightarrow{\sim} I_{\underset{\sim}{C}_2}$ and $\varphi : FE \xrightarrow{\sim} I_{\underset{\sim}{C}_1}$.

If E is an equivalence, then, by definition, the application of $\underset{\sim}{C}_1(A,B) \longrightarrow \underset{\sim}{C}_2(E(A),E(B))$ defined by ⟨⟨ $f \rightsquigarrow E(f)$ ⟩⟩ is a bijection and, moreover, every object X in $\underset{\sim}{C}_2$ is isomorphic to an object of the form E(C) for some object C in $\underset{\sim}{C}_1$. If one such object F(X) is selected for each X in C_2, along with one isomorphism $\xi_X : E\,(F(X)) \xrightarrow{\sim} X$, then the assignment ⟨⟨ $X \rightsquigarrow F(X)$ ⟩⟩ defines a function

$\underset{\sim}{Ob}(F) : \underset{\sim}{Ob}(\underset{\sim}{C}_2) \longrightarrow \underset{\sim}{Ob}(\underset{\sim}{C}_1)$, and then $\underset{\sim}{C}_1(F(X), F(X')) \xrightarrow{\sim} \underset{\sim}{C}_2(E(F(X)),E(F(X')))$ and in turn $\underset{\sim}{C}_2\,(E(F(X)), E(F(X'))) \xrightarrow{\sim} \underset{\sim}{C}_2(X,X')$ by means of the selected isomorphisms $\xi_X : E(F(X)) \xrightarrow{\sim} X$ and $\xi_{X'} : E(F(X')) \xrightarrow{\sim} X'$.

We now can define an application $\underset{\sim}{\mathfrak{L}}(F) : \underset{\sim}{\mathfrak{M}}(\underset{\sim}{C}_2) \longrightarrow \underset{\sim}{\mathfrak{M}}(\underset{\sim}{C}_1)$ by means of

$$\langle\langle\ f : X \longrightarrow X' \rightsquigarrow \xi_{X'}^{-1} f \xi_X : E(F(X)) \longrightarrow E(F(X')) \rightsquigarrow \hat{E}^{-1}(\xi_{X'}^{-1} f \xi_X) : F(X) \longrightarrow F(X')\ \rangle\rangle.$$

The resulting assignments $\langle\langle\ X \rightsquigarrow F(X)\ ,\ f \rightsquigarrow F(f)\ \rangle\rangle$ then define a unique functor $F : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$ such that

$$\xi : EF \longrightarrow I_{\underset{\sim}{C}_2} \quad \text{and} \quad \varphi : FE \longrightarrow I_{\underset{\sim}{C}_1}$$

If $\underset{\sim}{C}_2$ is the image of some fully faithful functor as it is here in (1.6.6), then φ may be taken as simply the identity isomorphism.

(1.6.8) One also has, by duality, the notion of a <u>co-representation of a</u> (covariant) <u>functor</u> $F : \underset{\sim}{C} \longrightarrow (\underset{\sim}{ENS})$ and thus also of a <u>co-representable functor</u> (i.e. a (covariant) functor which is isomorphic to a (covariant) "hom-functor" h'_X for some $X \in \underset{\sim}{Ob}(\underset{\sim}{C})$. We shall leave the explicit formalization of this notion to the interested reader.

<u>NOTE</u> We defer all examples of these notions until the end of (1.8.26).

Z The "co-"terminology of (1.6.8) is in conflict with that of MacLANE (1965).

(1.6.9) For $F : \underset{\sim}{C}^{(op)} \longrightarrow (\underset{\sim}{ENS})$, define the <u>category of</u> "objects of $\underset{\sim}{C}$ above F" to be that full subcategory $\underset{\sim}{C}/F$ of $\underline{\underline{CAT}}\ (\underset{\sim}{C}^{op}\ ,\ (\underset{\sim}{ENS}))/F$ (c.f. (1.0.6 Ex 6°) whose objects are the functors of the form h_X above F for some $X \in \underset{\sim}{Ob}(\underset{\sim}{C})$, i.e. an object of $\underset{\sim}{C}/F$ is a couple $(h_X, \underline{\theta})$ where $\underline{\theta} : h_X \longrightarrow F$ is a functorial morphism, and an arrow of $\underset{\sim}{C}/F$ is a functorial morphism $h_f : h_X \longrightarrow h_Y$ such that $\underline{\xi} \cdot h_f = \underline{\theta}$, where $\underline{\theta}$ and $\underline{\xi}$ are the structural maps in

(1.6.9.1)
$$\begin{tikzcd} & h_Y \arrow[d, "\xi"] \\ h_X \arrow[ur, "h_f"] \arrow[r, "\underline{\theta}"'] & F \end{tikzcd}$$

Define the <u>category of</u> F-<u>pointed objects of</u> $\underset{\sim}{C}$ [EHRESMANN (1957), KAN (1958)] as that category $\underset{\sim}{C}^*/F$ whose objects are couples (X,θ) where $X \in \mathfrak{Ob}(\underset{\sim}{C})$ and $\theta \in F(X)$ and whose arrows are morphisms of $\underset{\sim}{C}$ such that $F(f)(\xi) = \theta$, i.e. ⟨⟨ $f' : (X,\xi) \longrightarrow (Y,\theta)$ ⟩⟩ is equivalent to ⟨⟨ $f : X \longrightarrow Y$ and $F(f)(\xi) = \theta$ ⟩⟩. Composition in $\underset{\sim}{C}^*/F$ is just that of $\underset{\sim}{C}$ used in the obvious fashion. A final object in $\underset{\sim}{C}^*/F$ will be called a <u>universal point</u> [MacLANE (1965)].

<u>PROPOSITION</u> (1.6.10) On the basis of the definitions of (1.6.9), the following statements are equivalent for a functor $F : \underset{\sim}{C}^{(op)} \longrightarrow (\underset{\sim}{ENS})$:

1° F is representable;

2° The category $\underset{\sim}{C}/F$ has a final object;

3° The category $\underset{\sim}{C}^*/F$ has a universal point.

If F be representable, let $\underline{\theta} : h_X \xrightarrow{\sim} F$ be the representation isomorphism. Consequently for any functor $T : \underset{\sim}{C}^{(op)} \longrightarrow (\underset{\sim}{ENS})$, $\underset{\sim}{\mathcal{H}om}(T,h_X) \xrightarrow{\sim} \underset{\sim}{\mathcal{H}om}(T,F)$, in particular for any h_T, $\underset{\sim}{\mathcal{H}om}(h_T,h_X) \xrightarrow{\sim} \underset{\sim}{\mathcal{H}om}(h_T,F)$ so that h_X is final. (In fact F is final in $\underline{\underline{CAT}}(\underset{\sim}{C}^{op}, (\underset{\sim}{ENS}))/F$ so that $h_X \xrightarrow{\sim}$ final object $\Rightarrow$ h_X final in $\underline{\underline{CAT}}(\underset{\sim}{C}^{op}, (\underset{\sim}{ENS})/F \Rightarrow$ h_X final in $\underset{\sim}{C}/F$). If h_X is final in $\underset{\sim}{C}/F$ then (X,θ) is final in $\underset{\sim}{C}^*/F \xrightarrow{\sim} \underset{\sim}{C}/F$ and conversely. If $h_X \xrightarrow{\theta} F$ is final in $\underset{\sim}{C}/X$ then $\underset{\sim}{\mathcal{H}om}(h_T,h_X) \xrightarrow{\sim} \underset{\sim}{\mathcal{H}om}(h_T,F)$ for any h_T, which gives by the "Yoneda" that $X(T) \xrightarrow{\sim} F(T)$ for all $T \in \mathfrak{Ob}(\underset{\sim}{C})$.

(1.7) <u>UNIVERSAL MAPPING PROBLEMS</u>

(1.7.0) The notion of ⟨⟨ representability ⟩⟩ lies at the base of the notion of ⟨⟨ solution of a universal mapping problem ⟩⟩ and is indeed (but for trivial modifications) equivalent to it. Partly for historical reasons, and partly because its terminology is evocative when used with the usual abuses of language, we shall redevelop these notions formalized first by SAMUEL (1948) and BOURBAKI (1957) conformally with the definition (1.0.1) of category.

<u>DEFINITION</u> (1.7.1) Let $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ be ($\underset{\sim}{\mathfrak{U}}$-) categories. An <u>association</u> $\underset{\sim}{A}$ <u>of</u> $\underset{\sim}{C}_1$ <u>with</u> $\underset{\sim}{C}_2$ is a set $\underset{\sim}{\mathcal{Fl}}(\underset{\sim}{A})$ called the <u>set of arrows of the association</u> (or $\underset{\sim}{C}_1$-$\underset{\sim}{C}_2$ <u>arrows</u>), supplied with the following structure:

$(SA)_I$ applications $\sigma(\underset{\sim}{A}) : \underset{\sim}{\mathcal{Fl}}(\underset{\sim}{A}) \longrightarrow \underset{\sim}{\mathcal{Ob}}(\underset{\sim}{C}_1)$ and $\tau(\underset{\sim}{A}) : \underset{\sim}{\mathcal{Fl}}(\underset{\sim}{A}) \longrightarrow \underset{\sim}{\mathcal{Ob}}(\underset{\sim}{C}_2)$ called the <u>source</u> and <u>target applications</u> of $\underset{\sim}{A}$;

$(SA)_{II}$ applications $\eta_1(\underset{\sim}{A}) : \underset{\sim}{\mathcal{Fl}}(C_1) \underset{\sigma_1(C_1), \underset{\sim}{\sigma}(A)}{\times} \underset{\sim}{\mathcal{Fl}}(\underset{\sim}{A}) \longrightarrow \underset{\sim}{\mathcal{Fl}}(\underset{\sim}{A})$ and $\eta_2(\underset{\sim}{A}) : \underset{\sim}{\mathcal{Fl}}(\underset{\sim}{A}) \times \underset{\sim}{\mathcal{Fl}}(\underset{\sim}{C}_2) \longrightarrow \underset{\sim}{\mathcal{Fl}}(\underset{\sim}{A})$, called C_1-$\underset{\sim}{A}$ and $\underset{\sim}{A}$-C_2 <u>multiplication</u> (or <u>composition</u>); which is required to satisfy the following axioms:

$(AA)_I$ $\tau(\underset{\sim}{A})\,\eta_1(\underset{\sim}{A}) = \tau(\underset{\sim}{A}) \circ pr_2$ and
$\sigma_o(C_1) \cdot pr_1 = \sigma(\underset{\sim}{A}) \circ \eta_1(\underset{\sim}{A})$;

$(AA)_{II}$ $\eta_1(\underset{\sim}{A}) \cdot I^*(\underset{\sim}{C}_1) = \underline{id}_{\underset{\sim}{\mathcal{Fl}}(\underset{\sim}{A})}$;

$(AA)_{III}$ $\eta_1(\underset{\sim}{A}) \cdot (\mu(\underset{\sim}{C}_1) \hat{\times} \underline{id}) = \eta_1(\underset{\sim}{A}) \cdot (\underline{id} \hat{\times} \eta_1(\underset{\sim}{A}))$

$(AA)_{IV}$ $\sigma(\underset{\sim}{A}) \cdot \eta_2(\underset{\sim}{A}) = \sigma(\underset{\sim}{A}) \cdot pr_1$ and
$\sigma_1(C_2) \cdot pr_2 = \tau(\underset{\sim}{A}) \cdot \eta_2(\underset{\sim}{A})$;

$(AA)_{V} \quad \eta_2(\underset{\sim}{A}) \circ I^*(\underset{\sim}{C}_2) = \underline{id}(\mathcal{H}(\underset{\sim}{A}));$

$(AA)_{VI} \quad \eta_2(\underset{\sim}{A}) \cdot (\underline{id} \mathbin{\hat{\times}} \mu_2(\underset{\sim}{C}_2)) = \eta_2(\underset{\sim}{A}) \circ (\eta_2(\underset{\sim}{A}) \mathbin{\hat{\times}} \underline{id});$

$$(AA)_{VII} \quad \eta_1(\underset{\sim}{A}) \cdot (\underline{id} \mathbin{\hat{\times}} \eta_2(\underset{\sim}{A})) = \eta_2(\underset{\sim}{A}) \cdot (\eta_1(\underset{\sim}{A}) \mathbin{\hat{\times}} \underline{id}).$$

Axioms $(AA)_I$-$(AA)_{III}$ assert the commutativity of the squares in the diagram

$$\begin{tikzcd}
\mathcal{H}(\underset{\sim}{C}_1) \times_{\sigma_1,\sigma_0} \mathcal{H}(\underset{\sim}{C}_1) \times_{\sigma_1,\sigma} \mathcal{H}(\underset{\sim}{A}) \arrow[r, "\mu(\underset{\sim}{C}_1) \hat{\times} \underline{id}"] \arrow[d, "\underline{id} \times \eta_1(\underset{\sim}{A})"'] \arrow[dr, phantom, "\boxed{\mathrm{III}}"] & \mathcal{H}(\underset{\sim}{C}_1) \times_{\sigma_1,\sigma} \mathcal{H}(\underset{\sim}{A}) \arrow[r, "pr_2"] \arrow[d, "\eta_1(\underset{\sim}{A})"] \arrow[dr, phantom, "\boxed{\mathrm{I}}"] & \mathcal{H}(\underset{\sim}{A}) \arrow[d, "\tau(\underset{\sim}{A})"] \\
\mathcal{H}(\underset{\sim}{C}_1) \times_{\sigma_1,\sigma} \mathcal{H}(\underset{\sim}{A}) \arrow[r, "\eta_1(\underset{\sim}{A})"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\boxed{\mathrm{I'}} \quad \boxed{\mathrm{II}}"] & \mathcal{H}(\underset{\sim}{A}) \arrow[r, "\tau(\underset{\sim}{A})"] \arrow[d, "\sigma(\underset{\sim}{A})"] & Ob(\underset{\sim}{C}_2) \\
\mathcal{H}(\underset{\sim}{C}_1) \arrow[r, shift left, "\sigma_1(\underset{\sim}{C}_1)"] & Ob(\underset{\sim}{C}_1) \arrow[l, shift left, "I(\underset{\sim}{C}_1)"]
\end{tikzcd} \tag{1.7.1.1}$$

and that $\eta_1(\underset{\sim}{A})$ is a retraction with $\langle\langle \alpha \rightsquigarrow I^*(\underset{\sim}{C}_1)(\alpha) = (I_{\sigma(\alpha)}, \alpha) \rangle\rangle$ as a section. In other words, that an external composition

$$\langle\langle \ (f, \alpha) \rightsquigarrow \eta_1(A)(f, \alpha) = \alpha \cdot f \ \rangle\rangle$$

is defined for couples (f, α) consisting of an arrow from $\underset{\sim}{C}_1$ and an arrow from $\underset{\sim}{A}$ such that the target of f coincides with source of α and that the source and target of the composite is that of f and α respectively.
For this composition, the identities of $\underset{\sim}{C}_1$ act as identities with the $\underset{\sim}{C}_1$ - $\underset{\sim}{C}_2$ arrows and; further, this composition is associative in the sense

that one has ⟨⟨ $\alpha\cdot(fg) = (\alpha\cdot f)\cdot g$ ⟩⟩ whenever defined.

The axioms $(AA)_{IV}$ - $(AA)_{VI}$ assert that the squares in diagram

(1.7.1.2)
$$\begin{tikzcd}[column sep=large]
\mathfrak{Fl}(\mathbf{A}) \underset{\tau,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[r, "\mathrm{id}\,\hat{\times}\,\mu(\mathbf{C}_2)"] \arrow[d, "\eta_2(\mathbf{A})\hat{\times}\mathrm{id}"'] \arrow[dr, phantom, "\boxed{\mathrm{VI}}"] & \mathfrak{Fl}(\mathbf{A}) \underset{\tau,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[r, "pr_2"] \arrow[d, shift left, "\eta_2(\mathbf{A})"] \arrow[dr, phantom, "\boxed{\mathrm{IV,\ V}}"] & \mathfrak{Fl}(\mathbf{C}_2) \arrow[d, shift left, "\sigma_1(\mathbf{C}_2)"'] \\
\mathfrak{Fl}(\mathbf{A}) \underset{\tau\,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[r, "\eta_2(\mathbf{A})"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\boxed{\mathrm{IV}}"] & \mathfrak{Fl}(\mathbf{A}) \arrow[r, "\tau(\mathbf{A})"] \arrow[u, shift left, "\mathrm{I}^*(\mathbf{C}_2)"] \arrow[d, "\sigma(\mathbf{A})"] & \mathfrak{Ob}(\mathbf{C}_2) \arrow[u, shift left=2, "\mathrm{I}(\mathbf{C}_2)"'] \\
\mathfrak{Fl}(\mathbf{A}) \arrow[r, "\sigma(\mathbf{A})"] & \mathfrak{Ob}(\mathbf{C}_1) &
\end{tikzcd}$$

all commute with $\eta_2(\mathbf{A})$ a retraction.

The statements are readily translated into ⟨⟨ another external composition ⟨⟨ $(\alpha, x) \rightsquigarrow \eta_2(\alpha, x) = x*\alpha$ ⟩⟩ is defined for couples (α, x) consisting of a $\mathbf{C}_1$-$\mathbf{C}_2$ arrow and an arrow in $\mathbf{C}_2$ such that the target of a coincides with the source of x; that the target of the composite arrow is that of x while the source, that of a; that the identities of $\mathbf{C}_2$ act as identities in this composition; and that this composition is associative in the sense that ⟨⟨ $(xy)*\alpha = x*(y*\alpha)$ ⟩⟩ when defined ⟩⟩.

The last axiom $(AA)_{VII}$ that the diagram

(1.7.1.3)
$$\begin{tikzcd}[column sep=huge]
\mathfrak{Fl}(\mathbf{C}_1) \underset{\sigma_1,\sigma}{\times} \mathfrak{Fl}(\mathbf{A}) \underset{\tau,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[r, "\eta_1(\mathbf{A})\,\hat{\times}\,\mathrm{id}"] \arrow[d, "\mathrm{id}\,\hat{\times}\,\eta_2(\mathbf{A})"'] \arrow[dr, phantom, "\boxed{\mathrm{VII}}"] & \mathfrak{Fl}(\mathbf{A}) \underset{\tau,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[d, "\eta_2(\mathbf{A})"] \\
\mathfrak{Fl}(\mathbf{C}_1) \underset{\sigma_1,\sigma}{\times} \mathfrak{Fl}(\mathbf{A}) \arrow[r, "\eta_1(\mathbf{A})"] & \mathfrak{Fl}(\mathbf{A})
\end{tikzcd}$$

is commutative, i.e. that the two external composition compatibly associate with each other (⟨⟨ $x * (\alpha \cdot f)) = (x * \alpha) \cdot f$ ⟩⟩ whenever defined).

(1.7.2) The diagrams occuring in the definition of an association will all be folded into the following single sequentially commutative diagram (in which the structural maps for $\mathbf{C}_1$ and $\mathbf{C}_2$ also occur):

(1.7.2.1)

$$
\begin{tikzcd}[column sep=large]
 & & \mathfrak{Fl}(\mathbf{A}) \underset{\tau,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[r, "pr_2"] \arrow[d, shift right, "id \times \mu(C_2)"'] \arrow[d, shift left, "\eta_2(A) \times id"] & \mathfrak{Fl}(\mathbf{C}_2) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[d, shift right=2, "pr_1"'] \arrow[d, "\mu(C_2)"] \arrow[d, shift left=2, "pr_2"] \\
 & \mathfrak{Fl}(\mathbf{C}_1) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{A}) \underset{\tau,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[r, "1^{(A)}", "\eta_1 \times id"'] \arrow[d, shift right, "id \times \eta_2(A)"'] \arrow[d, shift left, "pr_1"] \arrow[dr, shift left] \arrow[dr, shift right] & \mathfrak{Fl}(\mathbf{A}) \underset{\tau,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_2) \arrow[r, "pr_2"] \arrow[d, shift right, "\eta_2(A)"'] \arrow[d, shift left, "pr_1"] & \mathfrak{Fl}(\mathbf{C}_2) \arrow[d, shift right, "\sigma_1(C_2)"'] \arrow[d, shift left, "\sigma_0(C_2)"] \\
\mathfrak{Fl}(\mathbf{C}_1) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_1) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{A}) \arrow[r, shift left, "\mu(C_1) \times id"] \arrow[r, shift right, "id \times \eta_1(A)"'] \arrow[d, "pr_2"] & \mathfrak{Fl}(\mathbf{C}_1) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{A}) \arrow[r, shift left, "pr_2"] \arrow[r, shift right, "\eta_1(A)"'] \arrow[d, "pr_1"] & \mathfrak{Fl}(\mathbf{A}) \arrow[r, "\tau(\mathbf{A})"] \arrow[d, "\sigma(\mathbf{A})"] & \mathfrak{Ob}(\mathbf{C}_2) \\
\mathfrak{Fl}(\mathbf{C}_1) \underset{\sigma_1,\sigma_0}{\times} \mathfrak{Fl}(\mathbf{C}_1) \arrow[r, shift left=2] \arrow[r, "\mu(C_1)"] \arrow[r, shift right=2] & \mathfrak{Fl}(\mathbf{C}_1) \arrow[r, shift left, "\sigma_0(\mathbf{C}_1)"] \arrow[r, shift right, "\sigma_1(\mathbf{C}_1)"'] & \mathfrak{Ob}(\mathbf{C}_1) &
\end{tikzcd}
$$

The horizontally (resp. vertically) enclosed portion of (1.7.2.1) expresses the notion of a <u>category of operators</u> (<u>operating</u> on the left (resp.right) in the sense of EHRESMANN (1957) and that of <u>linked categories</u> of SONNER (1963) .

(1.7.3) If $\mathbf{A}$ is an association of $\mathbf{C}_1$ with $\mathbf{C}_2$, we shall let $\mathbf{A}(A,X)$ or $X[A]$ be the fibre above the couple $(A,X) \in \mathfrak{Ob}(\mathbf{C}_1) \times \mathfrak{Ob}(\mathbf{C}_2)$.

$\underset{\sim}{A}(A,X) = \{\alpha: A \longrightarrow X \mid \alpha \in \underset{\sim}{\mathfrak{Fl}}(\underset{\sim}{A}),\ \sigma(\alpha) = A,\ \tau(\alpha) = X\}$ and say that the association $\underset{\sim}{A}$ is a $\underset{\sim}{\mathfrak{U}}$-<u>association</u> provided for all $(A,X) \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_1) \times \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2)$, $\underset{\sim}{A}(A,X) \in \underset{\sim}{\mathfrak{U}}$ ("Property $\underset{\sim}{\mathfrak{U}}$ ").

Let $\underset{\sim}{A}$ be a $\underset{\sim}{\mathfrak{U}}$-association of $\underset{\sim}{C}_1$ with $\underset{\sim}{C}_2$ and let $\alpha: A' \longrightarrow X$ be a member of $\underset{\sim}{\mathfrak{Fl}}(\underset{\sim}{A})$, $f: A' \longrightarrow A$ an arrow of $\underset{\sim}{C}_1$, and $x: X \longrightarrow X'$ an arrow in $\underset{\sim}{C}_2$. We define the following applications by the indicated assignments:

1° $\alpha^{\cdot}(A') : \underset{\sim}{C}_1(A',A) \longrightarrow \underset{\sim}{A}(A',X)$ by $\langle\langle \alpha \rightsquigarrow \alpha\cdot \quad \rangle\rangle$,
2° $\alpha^{*}(X') : \underset{\sim}{C}_2(X,X') \longrightarrow \underset{\sim}{A}(A,X')$ by $\langle\langle x \rightsquigarrow x * \alpha \quad \rangle\rangle$,
3° $\underset{\sim}{A}(f,X) : \underset{\sim}{A}(A,X) \longrightarrow \underset{\sim}{A}(A',X)$ by $\langle\langle \alpha \rightsquigarrow \alpha\cdot f \quad \rangle\rangle$, and
4° $\underset{\sim}{A}(A,X) : \underset{\sim}{A}(A,X) \longrightarrow \underset{\sim}{A}(A,X')$ by $\langle\langle \alpha \rightsquigarrow x * \alpha \rangle\rangle$.

(1.7.4) In passing we note that the entire system for a $\underset{\sim}{\mathfrak{U}}$-association may be given an immediate "EILENBERG-MacLANE translation" as follows:

"For each couple $(A,X) \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_1) \times \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2)$, one is given a set $\underset{\sim}{A}(A,X)$ such that for each triple $(A,X,Y) \in \underset{\sim}{\mathfrak{Ob}}(C_1) \times \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2) \times \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2)$ there is defined a composition

$$\langle\langle \quad ((\alpha, x) \rightsquigarrow x * \alpha) \ : \underset{\sim}{A}(A,X) \times \underset{\sim}{C}_2(X,Y) \longrightarrow \underset{\sim}{A}(A,Y) \quad \rangle\rangle$$

which satisfies the following axioms,

(I) - Being given $x_1 \in \underset{\sim}{C}_2(X,Y)$, $x_2 \in \underset{\sim}{C}_2(Y,Y'')$, and $\alpha \in \underset{\sim}{A}(A,X)$, one has $(x_2\, x_1) * \alpha = x_2(x_1 * \alpha)$;

(II) - Being given $I_X \in \underset{\sim}{C}_2(X,X)$ and $\alpha \in \underset{\sim}{A}(A,X)$, one has $I_X * \alpha = \alpha$; further for each triple $(A, B, X) \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_1)^2 \times \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2)$ there is defined a composition ......

there followeth the sequence of the analogues of $(AA)_I$ – $(AA)_{III}$ and $(AA)_{VII}$. "

Note also that SWAN (1964) has given an "effectively" first order formulation of the axioms expressing this notion. They read

"1° if $f : A' \longrightarrow A$ is in $\underset{\sim}{C}_1$, $x : X \longrightarrow X'$ in $\underset{\sim}{C}_2$, and $\alpha : A \longrightarrow X$ is a $\underset{\sim}{C}_1$-$\underset{\sim}{C}_2$ map, then $h \cdot f : A' \longrightarrow X$ and $x * h : A \longrightarrow X'$ are defined and are $\underset{\sim}{C}_1$-$\underset{\sim}{C}_2$ maps;

2° $f_1 : A'' \longrightarrow A'$, $f_2 : A' \longrightarrow A$ in $\underset{\sim}{C}_1$, $x_1 : X \longrightarrow X'$, $x_2 : X' \longrightarrow X''$ in $\underset{\sim}{C}_2$ and $\alpha : A \longrightarrow X$ a $\underset{\sim}{C}_1$-$\underset{\sim}{C}_2$ map, then $\alpha(f_2 f_1) = (\alpha \cdot f_2) \cdot f_1$, $(x_2 x_1) * \alpha = x_2 * (x_1 * \alpha)$, and $x_1 * (\alpha \cdot f_2) = (x_1 * \alpha) f_2$; and

3° if f and g are identities, then $\alpha \cdot f = \alpha$, $g * \alpha = \alpha$ ".

It will soon become clear that the existence of these translations is almost the only remaining significant aspect of the entire hotion (Its use in practice being supplanted by the more elegant - if less intuitive - notions of representability and adjunction).

DEFINITION (1.7.5) Let $\underset{\sim}{C}$ be a category and

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\mathrm{I}"] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"] \arrow[dr, phantom, "\mathrm{II}"] & A_3 \arrow[d, "f_2"] \\ B_1 \arrow[r, "b_1"'] & B_2 \arrow[r, "b_2"'] & B_3 \end{tikzcd} \qquad (1.7.5.1)$$

a diagram in $\underset{\sim}{C}$ composed of two commutative squares I and II and their composite II $\cdot$ I = III.

If both I and the composite III are cartesian, I will be called a cartesian complement of II, and a cartesian factor or component of (the cartesian square) III.

If II is given, then the cartesian square I is determined up to a unique isomorphism by the specification of b_1 alone. Consequently, we may refer to the arrow b_1 as defining a cartesian complement of II.

DEFINITION (1.7.6) Let $\underset{\sim}{A}$ be an association of $\underset{\sim}{C}_1$ with $\underset{\sim}{C}_2$. $\alpha \in \mathfrak{Fl}(\underset{\sim}{A})$ will be called σ-universal (resp. σ- contra-universal) provided the inclusion $\{\alpha\} \subseteq \mathfrak{Fl}(\underset{\sim}{A})$, defines a cartesian complement of the square IV (resp. the converse square $\overset{\frown}{\mathrm{IV}}$) of (1.7.1.2); and τ-universal (resp. τ-contra-universal) provided it defines a cartesian complement of the square $\overset{\frown}{\mathrm{I}}^{-1}$ (resp. $\overset{\frown}{\mathrm{I}}$) of (1.7.1.1).

PROPOSITION (1.7.7) For $\alpha \in \mathfrak{Fl}(\underset{\sim}{A})$, one has the following immediate equivalences:

1° $\alpha : A \longrightarrow X$ is τ-universal (resp. τ-contra-universal) iff given any $A' \in \mathfrak{Ob}(\underset{\sim}{C})$, and any $\theta : A' \longrightarrow X$, there exists a unique $f : A' \longrightarrow A$ (resp. $f : A \longrightarrow A'$) such that $\alpha \cdot f = \theta$ (resp. $\theta \cdot f = \alpha$).

2° $\alpha : A \longrightarrow X$ is σ-universal (resp. σ-contra-universal) iff given any $X' \in \mathfrak{Ob}(\underset{\sim}{C}_2)$ any $\theta : A \longrightarrow X'$ there exists a unique $x : X \longrightarrow X'$ (resp. $x : X' \longrightarrow X$) such that $x * \alpha = \theta$ (resp. $x * \theta = \alpha$).

NOTE: The justification for the assertions of (1.7.0) lie in the following sequence of observations.

In these propositions, $(SA)_I$ and $(SA)_{II}$ refer to the structure applications and $(AA)_I$ - $(AA)_{VII}$ to the axioms occuring in the definition (1.7.1) of an association of categories $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$. It will be assumed the association in reference is always a $\underset{\sim}{\mathfrak{U}}$-association. Other applications needed will be those of (1.7.3) -1° -4°. In most cases the proofs are trivial and are omitted.

PROPOSITION (1.7.8) If $\underset{\sim}{\mathfrak{A}} = (\underset{\sim}{\mathfrak{Hl}}(\underset{\sim}{\mathfrak{A}}),\ \sigma(\underset{\sim}{\mathfrak{A}}),\ \tau(\underset{\sim}{\mathfrak{A}}))$ and $\eta_1(\underset{\sim}{\mathfrak{A}})$ be given as in (1.7.1) and satisfy $(AA)_I$ $-(AA)_{III}$ together with property "$\underset{\sim}{\mathfrak{U}}$" of (1.7.2), (⟨⟨ $\underset{\sim}{\mathfrak{A}}$ is a τ -association of $\underset{\sim}{C}_1$ with $\underset{\sim}{C}_2$ ⟩⟩), then the applications defined by the assignments ⟨⟨ $A \rightsquigarrow \underset{\sim}{\mathfrak{A}}(A,X)$, $f \rightsquigarrow \underset{\sim}{\mathfrak{A}}(f,X)$ ⟩⟩, define, for any $X \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2)$, a functor $\underset{\sim}{\mathfrak{A}}(\ \bullet\ ,X) : \underset{\sim}{C}_1^{(op)} \longrightarrow (\underset{\sim}{ENS})$. For any $\alpha : A \longrightarrow X$, the application defined by ⟨⟨ $A' \rightsquigarrow \alpha^{\bullet}(A')$ ⟩⟩ is a natural transformation $\alpha^{\bullet} : h_A \longrightarrow \underset{\sim}{\mathfrak{A}}(\ \bullet\ ,X)$.

In order that the functor $\underset{\sim}{\mathfrak{A}}(\ \bullet\ ,X) : \underset{\sim}{C}_1^{(op)} \longrightarrow (ENS)$ be representable, it is necessary and sufficient that there exist an $\alpha : A \longrightarrow X$ in $\underset{\sim}{\mathfrak{Hl}}(\underset{\sim}{\mathfrak{A}})$ which is τ -universal.

The first assertion is nothing more than an immediate application of the associativity and identity behaviour, guaranteed the multiplication $\eta_1(\underset{\sim}{\mathfrak{A}})$ by $(AA)_{II}$ and $(AA)_{III}$, to the applications $A \rightsquigarrow \underset{\sim}{\mathfrak{A}}(A,X)$, $f \rightsquigarrow \underset{\sim}{\mathfrak{A}}(f,X)$, and $A' \rightsquigarrow \alpha^{\bullet}(A')$, whose definitions themselves are allowed by $(AA)_I$ and $\eta_1(\underset{\sim}{\mathfrak{A}})$.

For the second part note that in (1.7.7), ⟨⟨ $\alpha : A \longrightarrow X$ is just the translation of ⟨⟨ for all $A' \in \mathfrak{Ob}(\mathcal{C}_1)$, $\alpha^*(A') : \mathcal{C}_1(A',X) \longrightarrow A(A',X)$ is a bijection ⟩⟩. The converse is a trivial application of the "Yoneda" Lemma (1.6.2).

<u>COROLLARY</u> (1.7.9) If A is a $\mathcal{U}$-association, then the <u>source</u> of any τ-universal arrow in $\mathcal{Fl}(\mathcal{A})$ is unique up to a unique isomorphism in $\mathcal{C}_1$.

<u>PROPOSITION</u> (1.7.10) If $\mathcal{A} = (\mathcal{Fl}(\mathcal{A}), \sigma(\mathcal{A}), \tau(\mathcal{A}))$ and $\eta_2(\mathcal{A})$ be given as in (1.7.1) and satisfy $(AA)_{IV} - (AA)_{VI}$ together with property "$\mathcal{U}$" of (1.7.2) (⟨⟨ $\mathcal{A}$ is a σ-<u>association</u> of $\mathcal{C}_1$ with $\mathcal{C}_2$ ⟩⟩), then the applications defined by the assignments ⟨⟨ $X \rightsquigarrow \mathcal{A}(A,X)$, $x \rightsquigarrow \mathcal{A}(A,x)$ ⟩⟩ define, for <u>any</u> $A \in \mathfrak{Ob}(\mathcal{C}_1)$, a functor $\mathcal{A}(A,\,.\,) : \mathcal{C}_2 \longrightarrow \mathcal{ENS}\text{-}\mathcal{U}$. For any $\alpha : A \longrightarrow X$, the application defined by ⟨⟨ $X' \rightsquigarrow \alpha^*(X')$ ⟩⟩ defines a natural transformation $\alpha^* : h'_X \longrightarrow \mathcal{A}(A,\,.\,)$.

In order that $\mathcal{A}(A,.) : \mathcal{C}_2 \longrightarrow (\mathcal{ENS})$ be co-representable, for some $A \in \mathfrak{Ob}(\mathcal{C}_1)$, it is necessary and sufficient that there exist an $\alpha : A \longrightarrow X$ which is σ-universal.

<u>COROLLARY</u> (1.7.11) If A is a σ-association, then the target of any σ-universal arrow in $\mathcal{Fl}(\mathcal{A})$ is unique up to a unique isomorphism in $\mathcal{C}_2$.

<u>PROPOSITION</u> (1.7.12) If $\mathcal{A}$ is both a σ and a τ-association, then $(AA)_{VII}$ gives that for each $x : X \longrightarrow X'$ in $\mathcal{C}_2$, the application defined by ⟨⟨ $A \rightsquigarrow \mathcal{A}(A,x)$ ⟩⟩ defines a natural transformation $\mathcal{A}(\,.\,,x) : \mathcal{A}(\,.\,,X) \longrightarrow \mathcal{A}(\,.\,,X')$; and that for each $f : A' \longrightarrow A$,

the application defined by $\langle\langle\ X \rightsquigarrow \mathcal{A}(f,X)\ \rangle\rangle$ defines a natural transformation $\mathcal{A}(f,\,.\,) : \mathcal{A}(A,\,.\,) \longrightarrow \mathcal{A}(A',\,.\,)$.

<u>COROLLARY</u> (1.7.13) If $\mathcal{A}$ is an association of $\mathbf{C}_1$ with $\mathbf{C}_2$, then the applications defined by $\langle\langle\ X \rightsquigarrow \mathcal{A}(\,.\,,X),\ x \rightsquigarrow \mathcal{A}(\,.\,,x)\ \rangle\rangle$ define a <u>functor</u> $\mathcal{A} : \mathbf{C}_2 \longrightarrow \underline{\mathbf{CAT}}\ (\mathbf{C}_1^{(op)},\ (\mathbf{ENS})$, and the applications defined by $\langle\langle\ A \rightsquigarrow \mathcal{A}(A,\,.\,),\ f \rightsquigarrow \mathcal{A}(f,\,.\,)\ \rangle\rangle$ define a functor $\mathcal{A}' : \mathbf{C}_1^{(op)} \longrightarrow \underline{\underline{\mathrm{CAT}}}\ (\mathbf{C}_2,\ (\mathbf{ENS}))$.

<u>LEMMA</u> (1.7.14) (FUNDAMENTAL ADJUNCTION OF $\mathbf{CAT}$)

Let $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ be $\mathcal{U}$-categories. There exists a canonical bijection

$$\mathcal{E}(\mathbf{C},\mathbf{D},\mathbf{E}) \ :\ \mathbf{CAT}\ (\mathbf{C}\times\mathbf{D},\mathbf{E}) \longrightarrow \mathbf{CAT}\ (\mathbf{C},\ \underline{\underline{\mathrm{CAT}}}\ (\mathbf{D},\ \mathbf{E})).$$

Let $B : \mathbf{C} \times \mathbf{D} \longrightarrow \mathbf{E}$ be a bifunctor. For each $C \in \mathcal{O}b(\mathbf{C})$, $B(C,D) \in \mathcal{O}b(E)$ whatever be $D \in \mathcal{O}b(D)$ and similarly $B(C,d) : B(C,D) \longrightarrow B(C,D')$ in $\mathcal{F}l(\mathbf{E})$ whatever be $d : D \longrightarrow D'$ in $\mathbf{D}$; moreover if $c : C \longrightarrow C'$, then $B(c,D) : B(C,D) \longrightarrow B(C',D)$ is an arrow in $\mathbf{E}$, whatever be $D \in \mathcal{O}b(\mathbf{D})$. Consequently, the "bifunctoriality" of B gives that the assignments $\langle\langle\ D \rightsquigarrow B\ (C,D),\ d \rightsquigarrow B(C,d)\ \rangle\rangle$ define a functor $B(C,\,.\,) : \mathbf{D} \longrightarrow \mathbf{E}$ for each choice of $C \in \mathcal{O}b(\mathbf{C})$ and that $\langle\langle\ D \rightsquigarrow B\ (c,D)\ \rangle\rangle$ defines a natural transformation $B(c,\,.\,) :\ B(C,\,.\,) \longrightarrow B(C',\,.\,)$. This, in turn, leads to the observation that the assignment $\langle\langle\ C \rightsquigarrow B\ (C\ .\ ),\ \ c \rightsquigarrow B\ (c,\,.\,)\ \rangle\rangle$ defines a functor $\mathcal{E}(\mathbf{C},\mathbf{D}\ ,\ \mathbf{E})\ (B) : \mathbf{C} \longrightarrow \underline{\underline{\mathrm{CAT}}}\ (\mathbf{D},\mathbf{E})$ for each choice of $B \in \mathbf{CAT}\ (\mathbf{C}\times\mathbf{D},\mathbf{E})$.

Reciprocally let $A : \mathbf{C} \longrightarrow \underline{\underline{\mathrm{CAT}}}\ (\mathbf{D},\mathbf{E})$ be a functor; then for any $C \in \mathcal{O}b(\mathbf{C})$, $A(C) : \mathbf{D} \longrightarrow \mathbf{E}$ is a functor and for any $c : C \longrightarrow C'$ in $\mathcal{F}l(\mathbf{C})$, $A(c) : A(c) \longrightarrow A(c')$ is a natural transformation.

Consequently for any $d : D \longrightarrow D'$ in $\mathcal{Fl}(\mathbf{D})$, the square

(1.7.14.1)
$$\begin{tikzcd} A(C)(D) \arrow[r, "A(C)(d)"] \arrow[d, "A(c)(D)"] & A(C)(D') \arrow[d, "A(c)(D')"] \\ A(C')(D) \arrow[r, "A(C')(d)"] & A(C')(D') \end{tikzcd}$$

is commutative. We now define a bifunctor $\overset{-1}{\mathcal{E}}(C,D,E)\ (A) : \mathbf{C} \times \mathbf{D} \longrightarrow \mathbf{E}$ by the assignment $\langle\langle\ (C,D) \rightsquigarrow A(C)(D),\ (c,d) \rightsquigarrow A(C')(d)\cdot A(C)(D)\ \rangle\rangle$.

We leave it for the reader to check that $\overset{-1}{\mathcal{E}}(C,D,E)(A)$ is a bifunctor and is indeed the reciprocal of $\mathcal{E}(E,D,E)$ (and conversely).

COROLLARY (1.7.15) There exists a canonical bijection

$$\mathcal{E}'(E,D,E) : \mathbf{CAT}\ (\mathbf{D} \times \mathbf{C},\ \mathbf{E}) \xrightarrow{\sim} \mathbf{CAT}\ (\mathbf{C},\ \mathbf{CAT}\ (\mathbf{D},\mathbf{E})).$$

$\mathcal{E}'(C,D,E)$ is obtained by composing $\mathcal{E}(C,D,E)$ with the canonical bijection of $\mathbf{CAT}\ (\mathbf{D} \times \mathbf{C},\mathbf{E})$ onto $\mathbf{CAT}\ (\mathbf{C} \times \mathbf{D},\mathbf{E})$ deduced from the isomorphism of $\mathbf{C} \times \mathbf{D}$ with $\mathbf{D} \times \mathbf{C}$.

COROLLARY (1.7.16) If $\mathbf{A}$ is an association of $\mathbf{C}_1$ with $\mathbf{C}_2$, then the assignments $\langle\langle\ (A,X) \rightsquigarrow \mathbf{A}(A,X),\ (f,x) \rightsquigarrow \mathbf{A}(f,x)\ \rangle\rangle$ define a bifunctor $\mathbf{A} : \mathbf{C}_1^{(op)} \times \mathbf{C}_2 \longrightarrow (\mathbf{ENS})$.

(1.7.16) is a direct application of (1.7.15) to the functor $\mathbf{A} : \mathbf{C}_2 \longrightarrow \mathbf{CAT}\ (\mathbf{C}_1^{(op)}\ ,\ (\mathbf{ENS}))$ of (1.7.13).

COROLLARY (1.7.17) Let $\mathbf{ASS}\ (\mathbf{C}_1,\mathbf{C}_2)$ be the (not necessarily $\mathfrak{U}$-) set of all $\mathfrak{U}$-associations of $\mathbf{C}_1$ with $\mathbf{C}_2$. There exist canonical bijections

$$\Xi : \mathbf{ASS}\ (\mathbf{C}_1,\mathbf{C}_2) \xrightarrow{\sim} \overset{\times\!\times}{\mathbf{CAT}}\ (\mathbf{C}_1 \times \mathbf{C}_2,\ (\mathbf{ENS})) \xrightarrow{\sim} \mathbf{CAT}(\mathbf{C}_2,\ \mathbf{CAT}(\mathbf{C}_1^{(op)},\ (\mathbf{ENS})),$$

where the $\langle\langle \times\!\times \rangle\rangle$ indicates consideration of "pairwise disjoint functors" only.

COROLLARY (1.7.16) establishes the first application which satisfies $\langle\langle \ast\ast \rangle\rangle$ since the source and target relations are functional (c.f.(1.0.2) for an entirely analogous restriction).
For the reciprocal: given a bifunctor $\mathcal{A}^* : \mathbf{C}_1^{(op)} \times \mathbf{C}_2 \longrightarrow (\mathbf{ENS})$, such that $(C,X) \neq (C',X')$ implies $\mathcal{A}^*(C,X) \cap \mathcal{A}^*(C',X') = \emptyset$ define $\mathcal{A}$ by $\mathcal{H}(\mathcal{A}) = \bigcup_{(C,X)\in \mathcal{O}b(C_1)\times \mathcal{O}b(C_2)} \mathcal{A}^*(C,X)$; then $\alpha \in \mathcal{H}(\mathcal{A})$ implies the existence of a unique couple (C,X) such that $\alpha \in \mathcal{A}^*(C,X)$; define $\sigma(\mathcal{A})(\alpha) = C$, $\tau(\mathcal{A})(\alpha) = X$.
Finally define $\eta_1(\mathcal{A})(f,\alpha) = \mathcal{A}^*(f,x)(\alpha)$, $f : C' \longrightarrow C$, $\alpha \in \mathcal{A}^*(C,X)$, and $\eta_2(\mathcal{A})(\alpha,x) = \mathcal{A}^*(A,x)(\alpha)$, $\alpha \in \mathcal{A}^*(A,X)$, $x : X \longrightarrow X'$,
The axioms are immediately satisfied.

COROLLARY (1.7.18) If $\mathcal{A}$ is an association of $\mathbf{C}_1$ with $\mathbf{C}_2$ such that for each $X \in \mathcal{O}b(\mathbf{C}_2)$, the functor $\mathcal{A}(\,.\,,X) \in \underline{\mathrm{CAT}}(\mathbf{C}_1^{(op)}, (\mathbf{ENS}))$ be representable (or what amounts to the same, that there exists a τ-universal arrow $\alpha \in \mathcal{H}(\mathcal{A})$ with target X for each $X \in \mathcal{O}b(\mathbf{C}_2)$), then there exists a functor $G : \mathbf{C}_2 \longrightarrow \mathbf{C}_1$ (and up to a unique functorial isomorphism, only one such functor), such that for each $A \in \mathcal{O}b(\mathbf{C}_1)$, $\mathbf{C}_1(A,G(X)) \xrightarrow{\sim} \mathcal{A}(A,X)$ (naturally).

Dually, if for each $A \in \mathcal{O}b(\mathbf{C}_1)$, the functor $\mathcal{A}(A,\ ) : \mathbf{C}_2 \longrightarrow (\mathbf{ENS})$ is (co-) representable (or what amounts to the same, if there exists a σ-universal arrow in $\mathcal{H}(\mathcal{A})$ with source A, for each $A \in \mathcal{O}b(C_1)$), then there exists one (and up to a functorial isomorphism, only one), functor $F : \mathbf{C}_1 \longrightarrow \mathbf{C}_2$ such that for each $X \in \mathcal{O}b(\mathbf{C}_2)$, $\mathcal{A}(A,X) \xrightarrow{\sim} \mathbf{C}_2(F(A),X)$ (naturally).

If for each $X \in \mathcal{O}b(\mathbf{C}_2)$, $\mathcal{A}(\,.\,,X)$ is representable, then the image of the functor $\mathcal{A} : \mathbf{C}_2 \longrightarrow \underline{\underline{\mathrm{CAT}}}(\mathbf{C}_1^{(op)}, (\mathbf{ENS}))$ is contained within the full subcategory $\underset{\sim}{\mathrm{Rep}}\,(\underset{\sim}{C}_1)$ of the (contra-variant) representable functors of $\underset{\sim}{C}_1$. By THEOREM (1.6.6), this category is equivalent to $\underset{\sim}{C}_1$ and by (1.6.7) has a functor $\mathcal{J} : \underset{\sim}{\mathrm{Rep}}\,(\underset{\sim}{C}_1) \longrightarrow \underset{\sim}{C}_1$ defined by any selected system of representatives. Consequently, $G = \mathcal{J}\underset{\sim}{\mathcal{A}} : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$ and has the desired properties.

An entirely analogous argument suffices for the other case.

<u>COROLLARY</u> (1.7.19) Let $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ be ($\underset{\sim}{\mathcal{U}}$-) categories and $\underset{\sim}{\mathcal{A}} : \underset{\sim}{C}_1^{(op)} \times \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{(\mathrm{ENS})}$ a bifunctor such that for each $X \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2)$, the functor $\underset{\sim}{\mathcal{A}}\,(\ .\ ,X) : \underset{\sim}{C}_1 \longrightarrow \underset{\sim}{(\mathrm{ENS})}$ is representable with $(G(X), \xi_X)$ as a selected representative. Then there exists a unique <u>functor</u> $G : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$ which has $(G(X))_{X \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2)}$ as its object function for which the assignment $\langle\langle\ (A,X) \rightsquigarrow \xi_X(A) : \underset{\sim}{C}_1(A,G(X)) \xrightarrow{\sim} \underset{\sim}{\mathcal{A}}(A,X)\ \rangle\rangle$ defines a functorial isomorphism $\xi : H_{\underset{\sim}{C}_1}\,(I_{\underset{\sim}{C}_1^{\circ}} \times G) \xrightarrow{\sim} \underset{\sim}{\mathcal{A}}$.

Dually, if the bifunctor $\underset{\sim}{\mathcal{A}}$ be such that for each $A \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_1)$, the functor $\underset{\sim}{\mathcal{A}}(A,\ .\ ) : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{(\mathrm{ENS})}$ be a co-representable, with $(F(A),\ \varphi_A)$ as a selected representative, then there exists a unique functor $F : \underset{\sim}{C}_1 \longrightarrow \underset{\sim}{C}_2$ with $(F(A))_{A \in \underset{\sim}{\mathfrak{Ob}}(C_1)}$ as its object function for which the assignment $\langle\langle\ (A,X) \rightsquigarrow \varphi_A(X) : \underset{\sim}{C}_2(F(A),X) \xrightarrow{\sim} \underset{\sim}{\mathcal{A}}(A,X)\ \rangle\rangle$ defines a functorial isomorphism $\varphi : H_{\underset{\sim}{C}_2}\,(F^{(op)} \times I_{\underset{\sim}{C}_2}) \xrightarrow{\sim} \underset{\sim}{\mathcal{A}}$.

If $\underset{\sim}{\mathcal{A}}$ is the given bifunctor, then it defines (on separation) an association of $\underset{\sim}{C}_1$ with $\underset{\sim}{C}_2$ by (1.7.16), and (1.7.18) is applicable. The functor $\mathcal{J}$ is uniquely determined by a specified system of representatives so that one can assert uniqueness on this basis.

(1.7.20) This completes our formal discussion of universal mapping problems (the search for τ-universal or σ-universal arrows for some given association of categories) by showing that the entire notion itself can be given a functorial description (1.7.8), (1.7.10), (1.7.12) and then reduced to a representability problem (the search for a representation of some set-valued functor).

COROLLARY (1.7.17) shows, on the other hand, that any such representability problem can be used to define a couple of universal mapping problems which will be soluble if and only if the given representation problem admits a solution, so that what one really has is a "dictionary" for translation of one type of problem into another. We have used our representability theorems to specifically make observations about the form of solutions of "universal mapping problems" but in all cases these could have been made directly, reasoning only on the basis of the axioms for an association of categories. The reader is invited to do this, should he be so inclined.

COROLLARY (1.7.18) suggests consideration of the following notion (due to KAN [1958]), which, it will turn out, is also "translationally equivalent" to the other two:

(1.8) ADJUNCTION OF CATEGORIES - ADJOINT FUNCTORS

(1.8.1) If $\underset{\sim}{A}$ is an association of $\underset{\sim}{C}_2$ with $\underset{\sim}{C}_1$ (1.7.1) in which both conditions of (1.7.18) are verified, then there exist functors $F : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$ and $G : \underset{\sim}{C}_1 \longrightarrow \underset{\sim}{C}_2$ such that the bijections

$\varphi(A,X) : \mathbf{C}_2(A,G(X)) \xrightarrow{\sim} \mathfrak{A}(A,X)$ and $\psi(A,X) : \mathfrak{A}(A,X) \xrightarrow{\sim} \mathbf{C}_1(F(A),X)$ define, by composition, a bijection $\Sigma(A,X) : \mathbf{C}_1(F(A),X) \xrightarrow{\sim} \mathbf{C}_2(A,G(X))$, which is functorial and, in fact, an isomorphism of the functor $H_{\mathbf{C}_1}(F^{(op)} \times I_{\mathbf{C}_1})$ with the functor $H_{\mathbf{C}_2}(I_{\mathbf{C}_2^{(op)}} \times G)$, where $H_{\mathbf{C}_i}$ is the canonical "hom"- bifunctor, i.e., <u>up to an isomorphism</u>, the square

(1.8.1.1)
$$\begin{tikzcd}
\mathbf{C}_2^{(op)} \times \mathbf{C}_1 \arrow[r, "I_{\mathbf{C}_2^{(op)}} \times G"] \arrow[d, "F^{(op)} \times I_{\mathbf{C}_1}"'] \arrow[dr] & \mathbf{C}_2^{(op)} \times \mathbf{C}_2 \arrow[d, "H_{\mathbf{C}_2}"] \\
\mathbf{C}_1^{(op)} \times \mathbf{C}_1 \arrow[r, "H_{\mathbf{C}_1}"] & (\mathbf{ENS})
\end{tikzcd}$$

of categories and functors is comutative.

This is a frequently occuring situation and leads us to

<u>DEFINITION</u> (1.8.2). [KAN (1958)] Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be ($\mathfrak{U}$-) categories. A triple $\aleph = (F,G,\varphi)$ consisting of functors $F : \mathbf{C}_2 \longrightarrow \mathbf{C}_1$ and $G : \mathbf{C}_1 \longrightarrow \mathbf{C}_2$ supplied with a functorial isomorphism $\varphi : H_{\mathbf{C}_1}(F^{(op)} \times I_{\mathbf{C}_1}) \xrightarrow{\sim} H_{\mathbf{C}_2}(I_{\mathbf{C}_2^{(op)}} \times G)$ is called an <u>adjunction of</u> $\mathbf{C}_2$ <u>with</u> $\mathbf{C}_1$. If $\aleph = (F,G,\varphi)$ is an adjunction, then F is said to be a <u>co-adjoint</u> of G, and G an <u>adjoint</u> of F.

This amounts to asserting the existence of a family

$$\left(\varphi(A,X) : \mathbf{C}_1(F(A),X) \xrightarrow{\sim} \mathbf{C}_2(A,G(A)) \right)_{(A,X) \in \mathfrak{Ob}(\mathbf{C}_1) \times \mathfrak{Ob}(\mathbf{C}_2)}$$

of bijections such that for couples $(f,g) \in \mathfrak{Fl}(\mathbf{C}_2) \times \mathfrak{Fl}(\mathbf{C}_1)$, the square

$$(1.8.2.1)\qquad \begin{tikzcd} \underset{\sim}{C}_1(F(A),X) \arrow[r, "\underset{\sim}{C}_1(F(f)\text{,}x)"] \arrow[d, "\wr", "\varphi(A,X)"'] & \underset{\sim}{C}_1(F(A'),X') \arrow[d, "\wr"', "\varphi(A',X')"] \\ \underset{\sim}{C}_2(A,G(X)) \arrow[r, "\underset{\sim}{C}_2(f\text{,}G(x))"] & \underset{\sim}{C}_2(A',G(X')) \end{tikzcd}$$

is commutative.

We will use the notation

$$\langle\langle\ \varphi : F \dashv G : (\underset{\sim}{C}_1, \underset{\sim}{C}_2)\ \rangle\rangle$$

to indicate that the triple (F, G, φ) defines an <u>adjunction of $\underset{\sim}{C}_2$ with $\underset{\sim}{C}_1$</u>.

<u>REMARK ON NOTATION AND TERMINOLOGY</u> (1.8.3) As is quite often the case, what is natural in one context is unnatural in another. The terminology used here for adjunction is consistent with that used in EILENBERG-MOORE (1965) and, we feel, with that used in this thesis (in spite of any apparent conflict). It is inconsistent with that of KAN (1958) where F was called the (left-) adjoint of G and G the right-adjoint of F. As "left-right" terminology is of little mnemonic value, the "left-right" distinction was gradually abandoned and F was called <u>the</u> adjoint of G and, sadly, G the co-adjoint of F in, for example, MacLANE (1965). This latter use <u>was consistent</u> with MacLANE'S previous definition of a <u>contra-variant</u> functor as being "co-representable", but unfortunately, this usage was already in conflict with the now standard term ⟨⟨ "co"-product ⟩⟩, defined by means of a "representation" of a <u>co-variant product</u> functor. With this as a norm, objects which are defined by means of "representations" of covariant functors "get "co-"applied to them - ergo "co-representation". In an adjunction, G, is usually given and then F is defined by means of a representation of the <u>co-variant</u> functor ⟨⟨ $X \rightsquigarrow \underset{\sim}{C}_2(A,G(X))$ ⟩⟩, hence ⟨⟨ F is co-adjoint to G ⟩⟩.

Historically, "universal mapping problems" arose before the notions of "category and functor" were well known. There the common abuses of language lead to the functorially properly defined elements of $\underset{\sim}{C}_2(A,G(X))$ being considered as <u>quasi-morphisms</u> of A into X. This puts a "natural orientation" into the situation and A becomes, quite naturally, a member of the "first" category" and X a member of the "second" category - thus an "association of C_1 with C_2". At the same time one begins to realize that quite often G is actually an inclusion functor $\underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$ which makes the source of G the "natural" choice for the "first" category and the target of G, the natural choice for the "second" category. This leads to simpler notation in adjunctions and has been "adopted" here.

In summary, our "apologia for our use of ⟨⟨ co- ⟩⟩" runs as follows:

It is "natural" to speak of an object P which satisfies the relation

$$\langle\langle \text{ for all } T \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}),\ \underset{\sim}{C}(T,P) \xrightarrow{\sim} \prod_{\iota \in I} \underset{\sim}{C}(T,P_\iota) \ \rangle\rangle$$

as defining a representation of a product since it will then have all of the properties attributed to what are commonly called "products" in the majority of well known categories, as well as having the set theoretic product " $\prod$ " appear in its defining relation. (Unfortunately the functor $\langle\langle\ T \rightsquigarrow \prod_{\iota\in I} \underset{\sim}{C}(T,P_\iota)\ \rangle\rangle$ is contravariant). It is then "natural" to call an object in $\underset{\sim}{C}^{op}$ which defines a representation of a product in $\underset{\sim}{C}^{op}$, to be, qua object in $\underset{\sim}{C}$, as defining a co-representation of a product (in $\underset{\sim}{C}$) and hence be a "co-product" in $\underset{\sim}{C}$. It will then have the defining property

$$\langle\langle \text{ for all } T \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}),\ \underset{\sim}{C}(P,T) \xrightarrow{\sim} \prod_{\iota \in I} \underset{\sim}{C}(P_\iota,\ T) \ \rangle\rangle$$

and can then be considered as a "representation of a co-variant functor" (or a co-representation of a contra-variant functor). Whatever the terminology $\langle\langle\ T \rightsquigarrow \prod_{\iota\in I} \underset{\sim}{C}(P_\iota,\ T)\ \rangle\rangle$ is co-variant. The extension of this reasoning leads to ⟨⟨ co-representation ⟩⟩ as we have defined it, and co-adjoint etc. and is consistent with "co-kernel and co-image" in the theory of modules.

(1.8.4) Let $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ be categories $F : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$, $G : \underset{\sim}{C}_1 \longrightarrow \underset{\sim}{C}_2$ functors, and $\varphi : H_{\underset{\sim}{C}_1}(F^{op} \times I_{\underset{\sim}{C}_1}) \longrightarrow H_{\underset{\sim}{C}_2}(I_{\underset{\sim}{C}_2^{op}} \times G)$ a natural transformation, so that

$$\varphi(A,F(A)) : \underset{\sim}{C}_1(F(A),\ F(A)) \longrightarrow \underset{\sim}{C}_2(A,GF(A))$$

be defined with $\Phi(A) \underset{DEF}{=} \varphi(A,F(A))(I_{F(A)}) : A \longrightarrow G(F(A))$ an arrow in $\underset{\sim}{C}_2$ for each $A \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_2)$. Similarly let

$$\psi :\ H_{\underset{\sim}{C}_2}(I_{\underset{\sim}{C}_2^{op}} \times G) \longrightarrow H_{\underset{\sim}{C}_1}(F^{op} \times I_{\underset{\sim}{C}_1})$$

also be a natural transformation so that

$$\Psi(G(X),X) : \underset{\sim}{C}_2(G(X),G(X)) \longrightarrow \underset{\sim}{C}_1(F(G(X)),X)$$

is defined and $\Psi(X) = \Psi(G(X),X)\,(I_{G(X)}) : F(G(X)) \longrightarrow X$ is an arrow in $\underset{\sim}{C}_1$ for each $X \in \mathcal{O}b(\underset{\sim}{C}_1)$.

PROPOSITION (1.8.5) [SHIH (1958)] With the notation and applications defined in (1.8.4), one has the assertions:

1° The application defined by the assignment $\langle\langle\ A \rightsquigarrow \Phi(A) : A \longrightarrow G(F(A))\ \rangle\rangle$ defines a natural transformation $\Phi : I_{\underset{\sim}{C}_2} \quad GF$. The application defined by $\langle\langle\ \varphi \rightsquigarrow \Phi\ \rangle\rangle$ is a bijection of $\mathcal{H}om(H_{\underset{\sim}{C}_1}(F^{(op)} \times I_{\underset{\sim}{C}_1}), H_{\underset{\sim}{C}_2}(I_{\underset{\sim}{C}_2^{op}} \times G))$ onto $\mathcal{H}om(I_{\underset{\sim}{C}_2}, GF)$.

2° The application defined by the assignment $\langle\langle\ X \rightsquigarrow \Psi(X) : F(G(X)) \longrightarrow X\ \rangle\rangle$ defines a natural transformation $\Psi : FG \longrightarrow I_{\underset{\sim}{C}_1}$. The application defined by $\langle\langle\ \psi \rightsquigarrow \Psi\ \rangle\rangle$ is a bijection of $\mathcal{H}om(H_{\underset{\sim}{C}_2}(I_{\underset{\sim}{C}_2^{op}} \times G), H_{\underset{\sim}{C}_1}(F^{op} \times I_{\underset{\sim}{C}_1}))$ onto $\mathcal{H}om(FG, I_{\underset{\sim}{C}_1})$.

3° $\psi \cdot \varphi = \underline{id}(H_{\underset{\sim}{C}_1}(F^{op} \times I_{\underset{\sim}{C}_1}) \Longleftrightarrow (\Psi F) \circ (F^2 \Phi) = \underline{id}(F) : F \xrightarrow{F\Phi} FGF \xrightarrow{\Psi F} F.$

4° $\varphi \cdot \psi = \underline{id}(H_{\underset{\sim}{C}_2}(I_{\underset{\sim}{C}_2^{op}} \times G) \Longleftrightarrow (G^2\Psi) \circ (\Phi G) = \underline{id}(G) : G \xrightarrow{\Phi G} GFG \xrightarrow{G\Psi} G.$

5° $F \dashv G$ (F is co-adjoint to G') $\Longleftrightarrow$ there exist functorial morphisms $\Phi : I_{\underset{\sim}{C}_2} \longrightarrow GR$ and $\Psi : FG \longrightarrow I_{\underset{\sim}{C}_1}$ such that

$$(\Psi F) \circ (F^2 \Phi) = \underline{id}(F) \text{ and } (G^2 \Psi) \circ (\Phi G) = \underline{id}(G).$$

The first part of 1° and 2° is trivial; for the second part, let $G(F(A),X) : \underset{\sim}{C}_1(F(A),X) \longrightarrow \underset{\sim}{C}_2(GF(A),G(X))$ be the restriction of the arrow function of G and let

$$\underset{\sim}{C}_2(\Phi(A), G(X)) : \underset{\sim}{C}_2(GF(A), G(X)) \longrightarrow \underset{\sim}{C}_2(A,G(X)$$

be the function deduced from $\Phi(A)$. Then

$$\varphi(A,X) : \underset{\sim}{C}_1(F(A),X) \xrightarrow{\mathcal{A}\ell(F)} \underset{\sim}{C}_2(G(F(A),G(X)) \xrightarrow{\underset{\sim}{C}_2(\Phi(A),G(X))} \underset{\sim}{C}_2(A,G(X)),$$

defined by composition, gives rise to the inverse for Ψ. The construction is similar for 2°. The proof of 3° and 4° is straightforward, given the information in 1° and 2°, and 5° is just the definition (1.8.2) applied with 3° and 4°.

<u>DEFINITION</u> (1.8.5.1) The transformations $\Psi : FG \longrightarrow I_{\underset{\sim}{C}_1}$ and $\Phi : I_{\underset{\sim}{C}_2} \longrightarrow GF$ of (1.8.5) are referred to as <u>universal</u> GF <u>and</u> FG <u>junctions</u> (MacLANE 1965) or, respectively, the <u>back</u> and <u>front adjunctions of</u> F <u>to</u> G <u>associated with the adjunction isomorphism</u> φ. One sometimes writes $\varphi \sim (\Phi, \Psi)$ in this context to indicate this situation and the essential equivalence (EILENBERG-MOORE 1965).

<u>PROPOSITION</u> (1.8.6) In order that a functor $F : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$ admit an adjoint it is necessary and sufficient that the functor defined by $\langle\langle\ A \rightsquigarrow \underset{\sim}{C}_1(F(A),X)\ \rangle\rangle$ be representable for each $X \in \mathcal{O}b(\underset{\sim}{C}_1)$.

If F admits an adjoint, then there exists a functorial isomorphism φ such that $\varphi(A,X) : \underset{\sim}{C}_1(F(A),X) \xrightarrow{\sim} \underset{\sim}{C}_2(A,G(X))$, which is simply the assertion that

$$(G(X), \quad \overbrace{\varphi\,(G(X),X)}^{-1}\ (I_{G(X)}))$$

defines a representation of the functor defined by

$$\langle\langle \quad A \rightsquigarrow \underset{\sim}{C}_1(F(A)),X) \quad \rangle\rangle,$$

<u>for each</u> $X \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_1)$.

Conversely, given a functor $F : \underset{\sim}{C}_2 \longrightarrow \underset{\sim}{C}_1$, one has a bifunctor $A = H_{\underset{\sim}{C}_1}(F^{(o)} x\ I_{\underset{\sim}{C}_1}) : \underset{\sim}{C}_2^{(op)} x\ \underset{\sim}{C}_1 \longrightarrow (\underset{\sim}{ENS})$ which is representable (on restriction) for each $X \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}_1)$. COROLLARY (1.7.19) is then applicable and assures the existence of a functor $G : \underset{\sim}{C}_1 \longrightarrow \underset{\sim}{C}_2$ and an <u>isomorphism</u> $\varphi : H_{\underset{\sim}{C}_2}(I_{\underset{\sim}{C}_2^*} x\ G) \xrightarrow{\sim} H_{\underset{\sim}{C}_1}(F^{(o)} x\ I_{C_2})$, i.e. the existence of an adjunction with G adjoint to F.

Dually one has the

<u>PROPOSITION</u> (1.8.7). In order that a functor $G : \underset{\sim}{C}_1 \longrightarrow \underset{\sim}{C}_2$ admit a co-adjoint it is necessary and sufficient that the (covariant) functor defined by $\langle\langle \quad X \rightsquigarrow \underset{\sim}{C}_2(A,G(X)) \quad \rangle\rangle$ be "representable" (i.e. co-representable) for each $A \in \underset{\sim}{\mathfrak{Ob}}\ (\underset{\sim}{C}_2)$.

<u>COROLLARY</u> (1.8.8) For any functor F, if F admit an adjoint (resp. co-adjoint) G, then G is unique up to functorial isomorphism and may be spoken of as "the" adjoint (resp. co-adjoint) of F.

(1.8.9) The previous two propositions show how any <u>adjoint problem</u> (the search for an adjoint or co-adjoint for some functor F) can be referred to a representability or universal mapping problem. In a sense, the translation can be made to go in the other direction also. Specifically, we make

<u>DEFINITION</u> (1.8.10). Let $G : \underset{\sim}{C}_1 \longrightarrow \underset{\sim}{C}_2$ be a functor. We shall say that "the" <u>co-adjoint of</u> G <u>is defined at</u> $A \in \underset{\sim}{Ob}(\underset{\sim}{C}_2)$ provided that there exists an object $F(A) \in \underset{\sim}{Ob}(\underset{\sim}{C}_1)$ and a family $(\varphi_A(X))_{X \in \underset{\sim}{Ob}(\underset{\sim}{C}_1)}$ of bijections

$$\varphi_A(X) : \underset{\sim}{C}_1\,(F(A),X) \xrightarrow{\sim} \underset{\sim}{C}_2(A,G(X))$$

which "are natural in X".

Now let $G : \underset{\sim}{C}^{(op)} \longrightarrow (\underset{\sim}{ENS})$ be a functor; in order that G be representable, it is necessary and sufficient that the adjoint of G be defined at $\{\phi\} \in \underset{\sim}{Ob}(\underset{\sim}{ENS})$. Then one will have a natural bijection

$$\hat{\varphi}(X) : \underset{\sim}{C}(X, F(\{\phi\})) = \underset{\sim}{C}^{(op)}\,(F(\{\phi\}),X) \xrightarrow{\sim} (\underset{\sim}{ENS})(\{\phi\}, G(X)) \xrightarrow{\sim} G(X), (X \in \underset{\sim}{Ob}(\underset{\sim}{C}))$$

which defines a representation of G. (Definition (1.1.10) simply asserts the representability of $\langle\langle\, X \rightsquigarrow C_2(A,G(X)) \,\rangle\rangle$).

(1.8.11) Systematic use of the arrow category (1.4.2) and its canonical functors (1.4.7) can be used to give an "element free" description of adjunction which may prove useful in some contexts. The following proposition represents one such formulation. In its proof we use the SHIH characterization (1.8.5) to establish its equivalence, although this could be done directly. Its principal difference from (1.8.5) is the elimination of the necessity of postulating ab initio the existence any natural transformations.

PROPOSITION (1.8.12) Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be $\mathfrak{U}$-categories supplied with functors $S : \underset{\sim}{B} \longrightarrow \underset{\sim}{A}$ and $T : \underset{\sim}{A} \longrightarrow \underset{\sim}{B}$. In order that S be a co-adjoint of T, it is necessary and sufficient that the square I of

(1.8.12.1)
$$\begin{tikzcd}
\underset{\sim}{B}^2 \times \underset{\sim}{A}^2 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\boxed{\mathrm{I}}"] & \underset{\sim}{A}^2 \arrow[r, "\sigma_1^{\underset{\sim}{A}}"] \arrow[d, "\sigma_o^{\underset{\sim}{A}} \boxtimes T\sigma_1^{\underset{\sim}{A}}"] \arrow[dr, phantom, "\boxed{\mathrm{III}}"] & \underset{\sim}{A} \arrow[d, "T"] \\
\sigma_1^{\underset{\sim}{B}} : \underset{\sim}{B}^2 \arrow[r, "S\sigma_o^{\underset{\sim}{B}} \boxtimes \sigma_1^{\underset{\sim}{B}}"] \arrow[d, "\sigma_o^{\underset{\sim}{B}}"'] \arrow[dr, phantom, "\boxed{\mathrm{III}}"] & \underset{\sim}{A} \times \underset{\sim}{B} \arrow[r, "pr_2"'] \arrow[d, "pr_1"] & \underset{\sim}{B} \\
\underset{\sim}{B} \arrow[r, "S"'] & \underset{\sim}{A} &
\end{tikzcd}$$

be a cartesian complement (1.7.5) each of the squares II and III.

For the sufficiency note that, by definition, I is a cartesian complement of II and III if and only if for any category $\underset{\sim}{C} \in \mathfrak{Ob}(\mathrm{CAT})$, the diagram

(1.8.12.2)
$$\begin{tikzcd}
\underset{\sim}{B}^2(\underset{\sim}{C}) \times \underset{\sim}{A}^2(\underset{\sim}{C}) \arrow[r] \arrow[d, "\underset{\sim}{A}(\underset{\sim}{C}) \times \underset{\sim}{B}(\underset{\sim}{C})"'] \arrow[dr, phantom, "\mathrm{I}(\underset{\sim}{C})"] & \underset{\sim}{A}^2(\underset{\sim}{C}) \arrow[r, "\sigma_1^{\underset{\sim}{A}}(\underset{\sim}{C})"] \arrow[d, "\sigma_o^{\underset{\sim}{A}}(\underset{\sim}{C}) \boxtimes T\sigma_1^{\underset{\sim}{A}}(\underset{\sim}{C})"] \arrow[dr, phantom, "\mathrm{III}(\underset{\sim}{C})"] & \underset{\sim}{A}(\underset{\sim}{C}) \arrow[d, "T(\underset{\sim}{C})"] \\
\sigma_1^{\underset{\sim}{B}}(\underset{\sim}{C}) : \underset{\sim}{B}^2(\underset{\sim}{C}) \arrow[r, "S\sigma_o^{\underset{\sim}{B}}(\underset{\sim}{C}) \boxtimes \sigma_1^{\underset{\sim}{B}}(\underset{\sim}{C})"'] \arrow[d] \arrow[dr, phantom, "\mathrm{II}(\underset{\sim}{C})"] & \underset{\sim}{A}(\underset{\sim}{C}) \times \underset{\sim}{B}(\underset{\sim}{C}) \arrow[r] \arrow[d] & \underset{\sim}{B}(\underset{\sim}{C}) \\
\underset{\sim}{B}(\underset{\sim}{C}) \arrow[r, "S(\underset{\sim}{C})"'] & \underset{\sim}{A}(\underset{\sim}{C}) &
\end{tikzcd}$$

of $\mathfrak{U}^*$- sets and applications has $\mathrm{II}(\underset{\sim}{C})\cdot\mathrm{I}(\underset{\sim}{C})$ and $\mathrm{III}(\underset{\sim}{C})\cdot\mathrm{I}(\underset{\sim}{C})$ as cartesian squares. In particular, evaluation at $\underset{\sim}{A}$ gives

(1.8.12.3)
$$\begin{tikzcd}
\underset{\sim}{B}^2(\underset{\sim}{A}) \underset{\underset{\sim}{A}(\underset{\sim}{A}) \times \underset{\sim}{B}(\underset{\sim}{A})}{\times} \underset{\sim}{A}^2(\underset{\sim}{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\mathrm{I}(\underset{\sim}{A})}"] & \underset{\sim}{A}^2(\underset{\sim}{A}) \arrow[r, "1^{\underset{\sim}{A}}(\underset{\sim}{A})"] \arrow[d, "\sigma_0^{\underset{\sim}{A}}(\underset{\sim}{A}) \boxtimes T\sigma_1^{\underset{\sim}{A}}(\underset{\sim}{A})"'] \arrow[dr, phantom, "\boxed{\mathrm{III}(A)}"] & \underset{\sim}{A}(\underset{\sim}{A}) \arrow[d, "T(\underset{\sim}{A})"] \\
\underset{\sim}{B}^2(\underset{\sim}{A}) \arrow[r, "S\sigma_0^{\underset{\sim}{B}}(A) \boxtimes \sigma_1^{\underset{\sim}{B}}(A)"'] \arrow[d, "{}_{o}\underset{\sim}{B}(A)"'] \arrow[dr, phantom, "\boxed{\mathrm{II}(\underset{\sim}{A})}"] & \underset{\sim}{A}(A)\times\underset{\sim}{B}(A) \arrow[r] \arrow[d] & \underset{\sim}{B}(\underset{\sim}{A}) \\
\underset{\sim}{B}(\underset{\sim}{A}) \arrow[r, "S(\underset{\sim}{A})"'] & \underset{\sim}{A}(\underset{\sim}{A}) &
\end{tikzcd}$$

with $\mathrm{II}(\underset{\sim}{A})\cdot\mathrm{I}(\underset{\sim}{A})$ and $\mathrm{III}(\underset{\sim}{A})\cdot\mathrm{I}(\underset{\sim}{A})$ both cartesian. Now the identity transformation $1_{\underset{\sim}{B}}T$ of T is an element of $\underset{\sim}{B}^2(\underset{\sim}{A})$ and the identity functor $I_{\underset{\sim}{A}}$ of the category $\underset{\sim}{A}$ is a member of $\underset{\sim}{A}(\underset{\sim}{A})$ with the property that $\sigma_1^{\underset{\sim}{B}} 1_{\underset{\sim}{B}}T = TI_A$ so that the couple $(1_{\underset{\sim}{B}}T, I_{\underset{\sim}{A}})$ is an element of $\underset{\sim}{B}^2(\underset{\sim}{A}) \underset{\underset{\sim}{B}(\underset{\sim}{A})}{\times} \underset{\sim}{A}(\underset{\sim}{A})$. Since $\mathrm{II}(\underset{\sim}{A})\cdot\mathrm{I}(\underset{\sim}{A})$ is cartesian, there exists a unique couple $(1_{\underset{\sim}{B}}T, \Psi)$ in $\underset{\sim}{B}^2(\underset{\sim}{A}) \underset{\underset{\sim}{A}\times\underset{\sim}{B}(\underset{\sim}{A})}{\times} \underset{\sim}{A}^2(\underset{\sim}{A})$ such that $\sigma_1^A\Psi = I_{\underset{\sim}{A}}$. The source of Ψ is equal to $S\,\sigma_o^{\underset{\sim}{B}} 1_{\underset{\sim}{B}}T$, which is simply ST; we have thus produced the existence of a natural transformation $\Psi : ST \longrightarrow I_{\underset{\sim}{A}}$.

Evaluation of (1.8.12.2) for the category $\underset{\sim}{B}$ gives, <u>mutatis mutandis</u> the existence of a unique transformation $\Phi : I_{\underset{\sim}{B}} \longrightarrow TS$ using the couple $(I_{\underset{\sim}{B}}, 1_{\underset{\sim}{A}}S)$ and the fact that $\mathrm{II}\cdot\mathrm{I}$ is cartesian.

So far we have only used the surjective property of cartesian squares without using the essential unicity of the produced couples (i.e. only the III∘I and II∘I were $_*$<u>pre-cartesian</u>*). We now bring this additional property into play with the observation that the couples $(\Phi T, T^2\Psi)$ and $(S^2\Phi, \Psi S)$ are composable in ($\underline{\underline{\mathrm{CAT}}}$) and moreover, since one has the chain of equalities

$$\langle\langle\langle\; \sigma_0^{A}\Psi, T\sigma_1^{A}\Psi) = (ST, T) = (ST, T\sigma_1^{A}) = (S\sigma_0^{B}\Phi T, \sigma_1^{B}T^2\Psi) = (S\sigma_0^{B}(T^2\Psi\circ\Phi T), \sigma_1^{B}(T^2\Psi\circ\Phi T))\rangle\rangle\rangle$$

both the couple $(T^2\Psi\circ\Phi T, \Psi)$ and the couple $(1_{\mathbf{B}}T, \Psi)$ are members of $\mathbf{B}^2(\mathbf{A}) \underset{\mathbf{A}(\mathbf{A})\times\mathbf{B}(\mathbf{A})}{\times} \mathbf{A}^2(\mathbf{A})$. Now the application defined by

$$\langle\langle\; (Z \rightsquigarrow (\sigma_0^{B}\,\mathrm{pr}_1 Z,\ \mathrm{pr}_2 Z)\;\rangle\rangle$$

is an injection of $\mathbf{B}^2(\mathbf{A}) \underset{\mathbf{A}(\mathbf{A})\times\mathbf{B}(\mathbf{A})}{\times} \mathbf{A}^2(\mathbf{A})$ into $\mathbf{B}(\mathbf{A}) \times \mathbf{A}^2(\mathbf{A})$ since II $\circ$ I is cartesian, and since

$$(\sigma_0^{B}(T^2\Psi\cdot\Phi T), \Psi) = (\sigma_0^{B}\Phi T, \Psi) = (T, \Psi) = (\sigma_0^{B}1_{\mathbf{B}}T, \Psi),$$

one has that $T^2\Psi\circ\Phi T = 1_{\mathbf{B}}T$. A corresponding argument for (1.8.12.3) evaluated at B, along with the injectivity the defining bijection for the cartesian square III $\cdot$ I gives that $\Psi S\cdot S^2\Phi = 1_{\mathbf{A}}S$.

We have thus produced the existence of natural transformations $\Phi : I_{\mathbf{B}} \longrightarrow TS$ and $\Psi : ST \longrightarrow I_{\mathbf{A}}$ such that $T^2\Psi\circ\Phi T = 1_{\mathbf{B}}T$ and $\Psi S\cdot S^2\Phi = 1_{\mathbf{A}}S$; consequently, (1.8.5.5°) allows us to conclude that S is a co-adjoint of T.

For necessity, let us suppose that S is a co-adjoint of T and thus that the functors Φ and Ψ exist and satisfy the requirements of (1.8.5.5°). The transform of the functorial diagram (1.8.12.7) evaluated at some category $\mathbf{C}$ then has the additional structure of functors $\Phi(\mathbf{C}) : \mathbf{B}(\mathbf{C}) \longrightarrow \mathbf{B}^2(\mathbf{C})$ and $\Psi(\mathbf{C}) : \mathbf{A}(\mathbf{C}) \longrightarrow \mathbf{A}^2(\mathbf{C})$ as well as the always present functors $S^2(C) : B^2(C) \longrightarrow A^2(C)$, $T^2(C) : A^2(C) \longrightarrow B^2(C)$, $1_A(C)$ and $1_B(C)$ which we show in the diagram

(1.8.12.4)
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
B^2(C) \underset{A(C) \times B(C)}{\times} A^2(C) \arrow[r] \arrow[d] & A^2(C) \arrow[r, shift left] \arrow[d] \arrow[dl, shift left, "T^2(C)"] \arrow[dl, phantom, shift right=4, "\mathrm{I}(C)"'] \arrow[dr, phantom, "\mathrm{III}(C)"] & A(C) \arrow[l, shift left, "1_A(C)"'] \arrow[l, shift left=3, "\Psi(C)"] \arrow[d] \\
B^2(C) \arrow[r] \arrow[ur, shift left, "S^2(C)"] \arrow[d] \arrow[dr, phantom, "\mathrm{II}(C)"] & A(C) \times B(C) \arrow[r] \arrow[d] & B(C) \\
B(C) \arrow[u, shift left=3, "1_B(C)"] \arrow[u, shift right=3, "\Phi(C)"'] \arrow[r, "S(C)"'] & A(C) &
\end{tikzcd}
$$

where the unlabelled arrows are those of (1.8.12.2). We must show that $I(C)$ is a cartesian complement of $II(C)$ and $III(C)$.

To this end, let $F : C \longrightarrow B$ be a functor and $\xi : C \longrightarrow A^2$ a natural transformation such that $\sigma_0^A \xi = SF$ (i.e. $\xi : SF \longrightarrow \sigma_1^A \xi$ in $\underline{\underline{CAT}}\,(C, A)$) so that the couple $(F, \xi) \in B(C) \times A^2(C)$. It follows immediately that $T^2(C)\,(\xi)\ (= \overset{A(C)}{T^2 \xi} : TSF \longrightarrow T\,\sigma_1^A\ )$ and $\Phi(C)\,(F)\ (= \Phi F : F \longrightarrow TSF)$ are defined, so that $T^2\xi \circ \Phi F : F \longrightarrow TSF \longrightarrow T\,\sigma_1^A$ is an element of $B^2(C)$ with the property that $\sigma_0^B(T^2 \xi \circ \Phi F) = \sigma_0^B \Phi F = F$. Since

$S\ \sigma_0^B(T^2\xi \circ \Phi F) = S\ \sigma_0^B \Phi F = SF = \sigma_0^A \xi$ and

$\sigma_1^B\ (T^2\xi \circ \Phi F) = \sigma_1^B\ T^2\xi = T\ \sigma_1^A \xi$, the couple $(T^2\xi \cdot \Phi\ F, \xi)$ is an element of $B^2(C) \underset{A(C)\times B(C)}{\times} A^2(C)$ with the sought after property.

In other words, we have thus proved that

$$\sigma_0^B\ pr_1 \boxtimes pr_2 : B^2(C) \underset{A(C) \times B(C)}{\times} A^2(C) \longrightarrow B(C) \times A^2(C)$$

defines a surjection of $B^2(C) \underset{A(C)\times B(C)}{\times} A^2(C)$ onto $B(C) \underset{A(C)}{\times} A^2(C)$. The injectivity follows using the relation ⟨⟨ $T^2\Psi \cdot \Phi\ T = 1_B T$ ⟩⟩.

An entirely similar argument using the application defined by ⟨⟨ $(X, G) \rightsquigarrow (X, \Psi\ G \cdot S^2X)$ ⟩⟩ gives the remainder of the proof.

COROLLARY (1.8.13) In order that S be a co-adjoint of T it is necessary and sufficient that for any $\mathfrak{U}$-category C, the functor $S(C) : \underline{CAT}\ (C,B) \longrightarrow \underline{CAT}\ (C,A)$ be a co-adjoint of the functor $T(C) : \underline{CAT}\ (C,A) \longrightarrow \underline{CAT}\ (C,B)$ in $\underline{CAT} - \mathfrak{U}$, i.e. that one have a family of natural $\mathfrak{U}^*$- bijections such that the diagram

(1.8.13.1)

$$\begin{array}{ccc}
(\underline{CAT}\ (C,B)[\underline{CAT}\ (C,S)\ (F),G] =) \mathcal{H}om(S(C)(F),G) & \xrightarrow{\mathcal{H}om(S(C)(\varphi),\theta)} & \mathcal{H}om(S(C)(F'),G') \\
\downarrow 2 & & \downarrow 2 \\
(\underline{CAT}\ (C,A)[F, \underline{CAT}(C,T)(G)] =) \mathcal{H}om(F,T(C)(G)) & \xrightarrow{\mathcal{H}om(\varphi,T(C)(\theta))} & \mathcal{H}om(F',T(C)(G'))
\end{array}$$

always commutes for any F, G, φ, θ (having the appropriate source and target).

We give the proof in one direction and leave the converse to the reader.

Suppose that S is a co-adjoint of T, then there exist natural transformations $\Phi : I_{\underset{\sim}{B}} \longrightarrow TS$ and $\Psi : ST \longrightarrow I_{\underset{\sim}{A}}$ such that for any category $\underset{\sim}{C}$, the application defined by the assignment

$$\langle\langle\ \xi \rightsquigarrow T^2\xi \circ \Phi F \ \rangle\rangle$$

is a bijection of $\mathcal{H}om(S(\underset{\sim}{C})(F),G)$ onto $\mathcal{H}om(F,T(\underset{\sim}{C})(G))$. We claim that this application is natural. In other words that given $\theta : G' \longrightarrow G$ and $\varphi : F \longrightarrow F'$, one has the equality

$$\langle\langle\ T^2\,\theta \circ (T^2\xi \cdot \Phi F') \cdot \varphi \ = T^2\,(\theta \cdot \xi \cdot S^2\varphi\ ) \circ \Phi F\ \rangle\rangle$$

obtained by the chase around the obverse square of (1.8.13.1) for an arbitrary $\xi : SF' \longrightarrow G'$. But this equality is immediate for it follows by composition from the equality

$$\langle\langle\ T^2S^2\varphi \cdot \Phi = \Phi F' \circ I^2_{\underset{\sim}{B}}\varphi\ \rangle\rangle$$

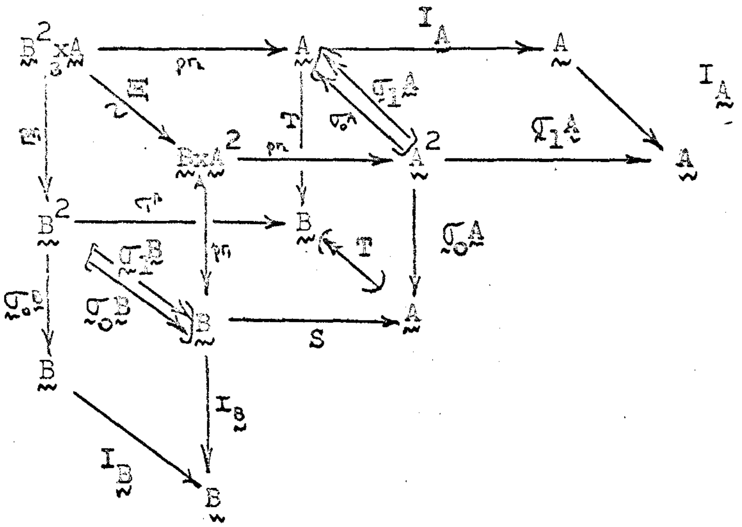
which is a simple application of (1.5.6.V) to the transformations $\varphi : F \longrightarrow F'$, $\Phi : I_B \longrightarrow TS$, and $T^2S^2 : I_{\underset{\sim}{B}} \longrightarrow I_{\underset{\sim}{B}}$.

(1.8.13.2)
$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "\Phi F"'] & F' \arrow[d, "\Phi F'"] \\
TSF \arrow[r, "T^2S^2"'] & TSF'
\end{tikzcd}$$

<u>COROLLARY</u> (1.8.14) In order that S be a co-adjoint of T it is necessary and sufficient that there exist a **category** isomorphism $\Xi : \underset{\sim}{B}^2 \underset{\sigma_1^{B}, T}{\times} \underset{\sim}{A} \xrightarrow{\sim} \underset{\sim}{B} \underset{S, \sigma_0^{A}}{\times} \underset{\sim}{A}^2$ such that

$$pr_1 \Xi = \sigma_0^{B}\, pr_1 \quad \text{and} \quad \sigma_1^{A}\, pr_2 \Xi = pr_2 .$$

(1.8.14.1)



$$\boxed{\underset{\sim}{A} \,\square\, S \xrightarrow{\sim} \overset{\frown 1}{T} \,\square\, \underset{\sim}{B}}$$

The category $\underset{\sim}{B}^2 \times_{A\times B} \underset{\sim}{A}^2$ of (1.8.12) is nothing more than a representation in ($\underset{\sim}{CAT}$) of the graph of precisely such a functorial isomorphism. N.B. This general notion will be investigated in detail in Chapter II and we defer until then more discussion of (1.8.14). The diagram (1.8.14.7) asserts that the $_*$(cartesian) composite* of the $_*$pre-correspondence* B with the$_*$ converse of T^* is isomorphic to S_*(cartesian) composed* with A, which will be abbreviated as in the "boxed" formula of (1.8.14.7)

COROLLARY (1.8.15) If $S : \underset{\sim}{B} \longrightarrow \underset{\sim}{A}$ is a co-adjoint of $T : \underset{\sim}{A} \longrightarrow \underset{\sim}{B}$ and $R : \underset{\sim}{C} \longrightarrow \underset{\sim}{B}$ is a co-adjoint of $Q : \underset{\sim}{B} \longrightarrow \underset{\sim}{C}$, then $SR : \underset{\sim}{C} \longrightarrow \underset{\sim}{A}$ is a co-adjoint of $QT : \underset{\sim}{A} \longrightarrow \underset{\sim}{C}$.

By (1.8.12) we have that the diagrams

(1.8.15.1)
$$\begin{tikzcd}
\underset{\sim}{B}^2 \times_{A\times B} \underset{\sim}{A}^2 \arrow[r, "\sigma_0^B"] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & \underset{\sim}{A}^2 \arrow[r, "\sigma_1^A"] \arrow[d] \arrow[dr, phantom, "\mathrm{III}"] & \underset{\sim}{A} \arrow[d, "T"] \\
\sigma_1^B : \underset{\sim}{B}^2 \arrow[r] \arrow[d, "\sigma_0^B"'] \arrow[dr, phantom, "\mathrm{II}"] & \underset{\sim}{A}\times\underset{\sim}{B} \arrow[r] \arrow[d] & \underset{\sim}{B} \\
\underset{\sim}{B} \arrow[r, "S"'] & \underset{\sim}{A} &
\end{tikzcd}$$

and

(1.8.15.2)
$$\begin{tikzcd}
\underset{\sim}{C}^2 \times_{B\times C} \underset{\sim}{B}^2 \arrow[r, "\sigma_0^B"] \arrow[d] \arrow[dr, phantom, "\mathrm{I}^*"] & \underset{\sim}{B}^2 \arrow[r, "\sigma_1^B"] \arrow[d] \arrow[dr, phantom, "\mathrm{III}^*"] & \underset{\sim}{B} \arrow[d, "Q"] \\
\underset{\sim}{C}^2 \arrow[r] \arrow[d, "\sigma_0^C"'] \arrow[dr, phantom, "\mathrm{II}^*"] & \underset{\sim}{B}\times\underset{\sim}{C} \arrow[r] \arrow[d] & \underset{\sim}{A} \\
\underset{\sim}{C} \arrow[r, "R"'] & \underset{\sim}{B} &
\end{tikzcd}$$

of categories and functors with structural morphisms as in (1.8.12.1) have I and I* as <u>cartesian complements</u> of their respective "complementary squares". From these two diagrams we obtain the diagram

(1.8.15.3)

$$\begin{tikzcd}
\mathbf{C}^2 \times \mathbf{B}^2 \times_{\mathbf{B}^2} \mathbf{A}^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{VI}"] & \mathbf{B}^2 \times_{\mathbf{A}\times\mathbf{B}} \mathbf{A}^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & \mathbf{A}^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{III}"] & \mathbf{A} \arrow[d, "T"] \\
\mathbf{C} \times_{\mathbf{B}\times\mathbf{C}} \mathbf{B}^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}^*"] & \mathbf{B}^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{V}"] & \mathbf{A}\times\mathbf{B} \arrow[r] \arrow[d, "I_A \times Q"] \arrow[dr, phantom, "\mathrm{IV}"] & \mathbf{B} \arrow[d, "Q"] \\
\mathbf{C}^2 \arrow[r] \arrow[d, "q^{\mathbf{C}}_{o}"'] \arrow[dr, phantom, "\mathrm{II}^*"] & \mathbf{B} \times \mathbf{C} \arrow[r, "S \times I_{\mathbf{C}}"'] \arrow[d, "pr_1"] \arrow[dr, phantom, "\mathrm{IV}^*"] & \mathbf{A}\times\mathbf{C} \arrow[r] \arrow[d, "pr_1"] & \mathbf{C} \\
\mathbf{C} \arrow[r, "R"'] & \mathbf{B} \arrow[r, "S"'] & \mathbf{A} &
\end{tikzcd}$$

of categories and functors where VI is cartesian and the remaining squares are as labelled. Now IV*∘ V = II and IV·V = III*, hence by hypothesis IV·V·I* is cartesian, and since III∘I is cartesian and VI is cartesian, III∘I·VI is cartesian so that the upper block of (1.8.15.3) is cartesian and the corner block $((V\cdot I^*)^{-1}\cdot(I\cdot VI)^{-1})^{-1}$ is cartesian. N.B. the notation is that of morphismsin the arrow category of ($\mathbf{CAT}$)), so that this corner block is a cartesian complement of its complementary square. In exactly the same fashion it is also a cartesian complement of IV*∘II* and the theorem is proved.

<u>REMARK</u> (1.8.16) We have presented the proof of the trivial proposition (1.8.15) in this curious fashion merely as an amusing exercise in the "calculus of cartesian squares". Its usual "element-wise" proof is scarcely over a few lines long.

Using the techniques of Chapter II, where it will be proved that cartesian composition ⟨⟨ ∘ ⟩⟩ of pre-correspondences* behaves much like composition of correspondences of sets, the proof of (1.8.15) on the basis of (1.8.14) is then,

$$\langle\langle\ \mathbf{A}\circ S \rightleftharpoons T^{-1}\circ\mathbf{B} \text{ and } \mathbf{B}\circ R \rightleftharpoons Q^{-1}\circ\mathbf{C} \Rightarrow T^{-1}\circ\mathbf{B}\circ R \rightleftharpoons T^{-1}\circ Q^{-1}\circ\mathbf{C} \Rightarrow \mathbf{A}\circ S\circ R \rightleftharpoons \widehat{QT}^{-1}\circ\mathbf{C} \Leftrightarrow \mathbf{A}\circ SR \rightleftharpoons \widehat{QT}^{-1}\circ\mathbf{C}\ \rangle\rangle$$

(1.8.17) For any $X \in \mathrm{Ob}(\mathbf{C})$, one has the trivial <u>one-element</u> subcategory $\mathbf{X}$ of $\mathbf{C}$ defined by $\mathrm{Ob}(\mathbf{X}) = \{X\}$ and $\mathrm{Fl}(\mathbf{X}) = \{I_X\}$. The category $\mathbf{C}/_X$ of objects of $\mathbf{C}$ above X is then isomorphic, the graph of the fibre product $\mathbf{X} \times_{\mathbf{C}} \mathbf{C}^2$ so that the square (1.8.17.1)

$$\text{(1.8.17.1)}\quad \begin{tikzcd} \mathbf{C}/_X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathbf{C}^2 \arrow[d, "Q_1^{\mathbf{C}}"] \\ \mathbf{X} \arrow[r] & \mathbf{C} \end{tikzcd} \qquad \text{(1.8.17.2)}\quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "q"] \\ X \arrow[r, "I_X"] & X \end{tikzcd}$$

of categories and functors is cartesian. $\mathbf{C}/_X$ is thus isomorphic to a subcategory of $\mathbf{C}^2$ in which any arrow has the form of (1.8.17.2). We shall habitually identify these two categories. For these categories it is customary to abuse notation and write $\mathbf{C}/_X(A,B)$ for a typical "hom" set in $\mathbf{C}/_X$ and speak of the arrows $f : A \longrightarrow B$ in $\mathbf{C}$ such that ⟨⟨ $gf = h$ ⟩⟩ as X-morphisms (or arrows) from A into B.

With the same abuse of language we speak of a commutative square such as

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "h"'] & A' \arrow[d, "h'"] \\ B \arrow[r, "b"'] & B' \end{tikzcd}$$

(i.e. arrow in $\mathbf{C}^2$ with source h and target h') as consisting of the <u>morphism</u> a <u>above the morphism</u> b and speak of such arrows a as being b-<u>morphisms</u> in $\underset{\sim}{C}$.

Consider now the commutative diagram of categories and functors

$$
(1.8.17.3) \qquad
\begin{tikzcd}
\underset{\sim}{C}/_A \arrow[r, "\alpha"] \arrow[d, "pr_2 = i^C_A"] & \underset{\sim}{D}/_X \arrow[d, "pr_2 = i^D_X"] \\
\underset{\sim}{C}^2 \arrow[d, "\sigma^C_0"] & \underset{\sim}{D}^2 \arrow[d, "\sigma^D_0"] \\
\underset{\sim}{C} \arrow[r, "F"'] & \underset{\sim}{D}
\end{tikzcd}
$$

defined for some functors F and α and objects A and X as specified. Then we have

<u>PROPOSITION</u> (1.8.18). If we designate by $\langle\langle\ \underset{\sim\sim\sim}{CAT}_F(\underset{\sim}{C}/_A,\ \underset{\sim}{D}/_X)\ \rangle\rangle$ the set of all functors $\alpha : \underset{\sim}{C}/_A \longrightarrow \underset{\sim}{D}/_X$ such that ($F/_X =_{\mathrm{DEF}}$) $F\sigma^C_0 i_A = \sigma^D_0 i_X \alpha$, then there exists a one-one correspondence of $\underset{\sim\sim\sim}{CAT}_F(\underset{\sim}{C}/_A,\ \underset{\sim}{D}/_X)$ with the set $\underset{\sim}{D}(F(A), X)$ of all arrows in $\underset{\sim}{D}$ with source F(A) and target X.

Let $\alpha : \underset{\sim}{C}/_A \longrightarrow \underset{\sim}{D}/_X$ be such an F-functor. Then the object I_A in $\underset{\sim}{C}/_A$ exists and $\alpha(I_A) : F(A) \longrightarrow X$ is an arrow in $\underset{\sim}{D}$ (object of $\underset{\sim}{D}/_X$) whose source is necessarily F(A) and whose target necessarily X. The assignment $\langle\langle\ \alpha \rightsquigarrow \alpha(I_A)\ \rangle\rangle$ then defines a function $\varepsilon : \underset{\sim\sim\sim}{CAT}_F(\underset{\sim}{C}/_A,\ \underset{\sim}{D}/_X) \longrightarrow \underset{\sim}{D}(F(A), X)$.

Now let $\xi \in \underset{\sim}{D}(F(A), X)$ and hence be an object in $\underset{\sim}{D}/_X$ whose source in $\underset{\sim}{D}$ is the image of the source of the final object I_A of $\underset{\sim}{C}/_A$.

Define a functor $\alpha_\xi : \underset{\sim}{C}/_A \longrightarrow \underset{\sim}{D}/_X$ by means of the object assignment ⟨⟨ $u \rightsquigarrow \xi F(u)$ ⟩⟩ which carries an object $u : T \longrightarrow A$ in $\underset{\sim}{C}/_A$ into the object $\xi F(u) : F(T) \longrightarrow X$ in $\underset{\sim}{D}/_X$ and, implicitly, an arrow (u, I_A, f, u') in $\underset{\sim}{C}/_A$ into $(\alpha(u), I_X, F(f), \alpha(u'))$. Since one always has that the relation

$$\langle\langle\ u'f = u \implies F(u')F(f) = F(u)\ \rangle\rangle$$

is valid for any functor $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$, the square $(\alpha(u), I_X, F(f), \alpha(u'))$ is indeed an arrow in $\underset{\sim}{D}/_X$ and we have by means of the assignment ⟨⟨ $\xi \rightsquigarrow \alpha_\xi$ ⟩⟩ defined a function $\rho : \underset{\sim}{D}(F(A), X) \longrightarrow \underset{\sim}{CAT}_F(\underset{\sim}{C}/_A, \underset{\sim}{D}/_X)$.

We claim that $\varepsilon\rho = \mathcal{U}$ and $\rho\varepsilon = \mathcal{U}$. Since one always has the relations

$$\langle\langle\ \varepsilon\rho(\xi) = \alpha_\xi(I_A) = \xi F(I_A) = \xi I_{F(A)} = \xi\ \rangle\rangle \text{ and}$$

$$\langle\langle\ \rho\varepsilon(\alpha) = \alpha_{\alpha(I_A)} \text{ and } \alpha_{\alpha(I_A)}(u) = \alpha(I_A)F(u) = \alpha(u) \text{ for all } u \in \underset{\sim}{Ob}(\underset{\sim}{C}/_A)\ \rangle\rangle$$

for arbitrary ξ and α the claim is established and the proposition is valid.

<u>COROLLARY</u> (1.8.19) There exists a canonical bijection of the $\underset{\sim}{Hom}(h_A, H_{\underset{\sim}{D}}(F^o \times X)$ of natural transformations of the functor ⟨⟨ $T \rightsquigarrow h_A(T) = \underset{\sim}{C}(T, A)$ ⟩⟩ into the functor

⟨⟨ $T \rightsquigarrow \underset{\sim}{D}(F(T), X)$ ⟩⟩, onto the set $\underset{\sim}{CAT}_F(\underset{\sim}{C}/_A, \underset{\sim}{D}/_X)$ of (1.8.18).

By the "Yoneda" $\mathcal{H}om(h_A, H_{\underset{\sim}{D}}(F' \times \underset{\sim}{X})) \longrightarrow H_{\underset{\sim}{D}}(F \times \underset{\sim}{X})(A) = \underset{\sim}{D}(F(A), X)$ and composition with ρ gives the desired bijection.

In "dual" fashion, for the category $A/_{\underset{\sim}{C}}$ of objects of $\underset{\sim}{C}$ below A ($\in \underset{\sim}{Ob}(\underset{\sim}{C})$), defined by means of the cartesian square

(1.8.19.1),
$$\begin{tikzcd} A/_{\underset{\sim}{C}} \arrow[r, "i'_A"] \arrow[d] & \underset{\sim}{C}^2 \arrow[d, "\underset{\sim}{\sigma}^{\underset{\sim}{C}}_o"] \\ (\{\underset{\sim}{A}\} =)\ \underset{\sim}{A} \arrow[r, "c_A"'] & \underset{\sim}{C} \end{tikzcd}$$

we obtain

PROPOSITION (1.8.20). If we designate by $\langle\langle\ \underset{\sim}{CAT}_F(A/_{\underset{\sim}{C}}, X/_{\underset{\sim}{D}})\ \rangle\rangle$ the set of all functors $\beta : A/_{\underset{\sim}{C}} \longrightarrow X/_{\underset{\sim}{D}}$ such that $F\,\underset{\sim}{\sigma}^{C}_1\, i'_A = \sigma^{D}_1\, i'_X\,\beta$, then one has a (canonical) bijection

(1.8.20.1) $$\varphi(A,X) : \underset{\sim}{CAT}_F(A/_{\underset{\sim}{C}}, X/_{\underset{\sim}{D}}) \longrightarrow \underset{\sim}{D}(X, F(A))$$

of the set of such F-functors onto the set of arrows of $\underset{\sim}{D}$ with source X and target F(A).

(1.8.20.2)
$$\begin{tikzcd} A/_{\underset{\sim}{C}} \arrow[r, "\beta"] \arrow[d, "i'_A"'] & X/_{\underset{\sim}{D}} \arrow[d, "i'_X"] \\ \underset{\sim}{C}^2 \arrow[d, "\sigma^{\underset{\sim}{C}}_1"'] & \underset{\sim}{D}^2 \arrow[d, "\sigma^{\underset{\sim}{D}}_1"] \\ \underset{\sim}{C} \arrow[r, "F"'] & \underset{\sim}{D} \end{tikzcd}$$

The proof is entirely similar to that of (1.8.18) and is left to the reader.

COROLLARY (1.8.21) There exists a canonical bijection of $\underset{\sim}{CAT}_F(A/_{\underset{\sim}{C}}, X/_{\underset{\sim}{D}})$ onto the set $\underset{\sim}{\mathcal{H}om}(h'_A, H_{\underset{\sim}{D}}(I_{\underset{\sim}{C}} \times F))$ of all natural transformations of the functor $\langle\langle\, T \rightsquigarrow h'_A(T) = \underset{\sim}{C}(A,T) \,\rangle\rangle$ into the functor $\langle\langle\, T \rightsquigarrow \underset{\sim}{D}(X, F(T)) \,\rangle\rangle$.

REMARK (1.8.22) The corollaries (1.8.19) and (1.8.21) can be obtained directly as well as the Yoneda Lemma in the following fashion:

LEMMA (1.8.22.1) Let $\underset{\sim}{C}$ be a category, $X \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})$ and $F : \underset{\sim}{C} \longrightarrow (\underset{\sim}{ENS})$ be a functor. There exists a (canonical) bijection of $\underset{\sim}{CAT}_F\,(X/\underset{\sim}{C},\ \{\emptyset\}/(\underset{\sim}{ENS}))$ onto the set $\underset{\sim}{\mathcal{H}om}(h'_X, F)$ of natural transformations of the (co-variant) hom-functor defined by $X \in \underset{\sim}{\mathrm{Ob}}\,(\underset{\sim}{C})$ into the functor F.

Let $\varphi : \underset{\sim}{C} \longrightarrow (\underset{\sim}{ENS})^2$ be a transformation such that $\sigma_0 \varphi = h'_X$ and $\sigma_1 \varphi = F$. There exists a unique functor $\iota_X : X/\underset{\sim}{C} \xrightarrow{X} \{\emptyset\}/(\underset{\sim}{ENS})$ such that $\sigma_1 i'_{\{\emptyset\}} \iota_X = h'_X \sigma_0^C i'_{X_0}$, defined by the assignment $\langle\langle\ f \rightsquigarrow \{\emptyset, I_X\} \ :\ \{\emptyset\} \longrightarrow X(X)\ \rangle\rangle$. Define $\alpha : X/\underset{\sim}{C} \longrightarrow \{\emptyset\}/(\underset{\sim}{ENS})$ as the lifted arrow defined by the composition in $\underline{CAT}$ which yields $F^2\, i'_X \circ \varphi \sigma_0^C\, i'_X \circ i'_{\{\emptyset\}} \iota_X$.
Then $i'_{\{\emptyset\}}\alpha = F^2 i'_X \circ \varphi \sigma_0^C i'_X \circ i'_{\{\emptyset\}} \iota_X$ AND $\sigma_1 i'_{\{\emptyset\}}\alpha = \sigma_1 F^2 i'_X = F \sigma_1^C i'_X$
as desired.

For the reciprocal, let $\theta : X/\underset{\sim}{C} \longrightarrow \{\emptyset\}/(\underset{\sim}{ENS})$ be such that $\sigma_1 i_{\{\emptyset\}}\, \theta = F \sigma_0^C i_X$. Then since the square $(\sigma_1^C i'_X,\ h'_X,\ h'^2_X \cdot \iota_X,\ \sigma_1 i'_{\{\emptyset\}})$ is cartesian, the standard argument of the Yoneda Lemma defines the functor $\varphi : \underset{\sim}{C} \longrightarrow \underset{\sim}{ENS})^2$ as desired. (Simply use the functors $F^{2 \cdot C}\theta(I_X)$ and $h'^{2}_X \cdot \iota_X$). As the lemma is no more than a "twisting" of the Yoneda-Lemma, the remainder is left to the reader.

COROLLARY (1.8.22.2) Then exists a canonical bijection of $\underset{\sim}{\mathcal{H}om}(h'_X, F)$ onto $F(X)$. Simply use the above lemma with PROPOSITION (1.8.20). One then has the system of bijections

$$\langle\langle\ \underset{\sim}{\mathcal{H}om}(h'_X, F) \xrightarrow{\sim} \underset{\sim}{CAT}_F(X/_{\underset{\sim}{C}},\ \{\emptyset\}/(\underset{\sim}{ENS}) \xrightarrow{\sim} \mathcal{F}(\{\emptyset\},\ F(X)) \xrightarrow{\sim} F(X).$$

PROPOSITION (1.8.23) Let $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$ be a functor of $\underset{\sim}{\mathfrak{U}}$-categories and $X \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{D})$. In order that adjoint of F be defined at $X \in \mathrm{Ob}(\underset{\sim}{D})$, it is necessary and sufficient that there exist an $A \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})$ and a functor $\alpha : \underset{\sim}{C}/_A \longrightarrow \underset{\sim}{D}/_X$ such that the square

(1.8.17.3)
$$\begin{tikzcd}
\underset{\sim}{C}/_A \arrow[r, "\alpha"] \arrow[d, "i_A"] & \underset{\sim}{D}/_X \arrow[d, "i_X"] \\
\underset{\sim}{C}^2 \arrow[d, "\sigma_0^C"] & \underset{\sim}{D}^2 \arrow[d, "\sigma_0^D"] \\
\underset{\sim}{C} \arrow[r, "F"] & \underset{\sim}{D}
\end{tikzcd}$$

of categories and functors be cartesian.

Suppose that (1.8.17.3) is cartesian and for any $T \in \mathfrak{Ob}(\mathbf{C})$, define a function

$$\alpha(T) : \mathbf{C}(T,A) \longrightarrow \mathbf{D}(F(T),A) \text{ by } \langle\langle \quad \alpha(T)(u) = \alpha(u) \quad \rangle\rangle$$

Although that application defined by $\langle\langle \ T \rightsquigarrow \alpha(T) \ \rangle\rangle$ defines a natural-transformation is a consequence of (1.8.19), we shall reprove this fact directly in the interest of clarity.

We must show that the diagram

(1.8.23.1.)
$$\begin{tikzcd}
\mathbf{C}(T,A) \arrow[r, "\mathbf{C}(f,A)"] \arrow[d, "\alpha(T)"'] & \mathbf{C}(U,A) \arrow[d, "\alpha(U)"] \\
\mathbf{D}(F(T),X) \arrow[r] & \mathbf{D}(F(U),X)
\end{tikzcd}$$

is commutative for any choice of $f : U \longrightarrow T$ in $\mathbf{C}$. To this end, let $u : T \longrightarrow A$ be an element of $\mathbf{C}(T,A)$ and, a forteori, an object in $\mathbf{C}/_A$. For any $f : U \longrightarrow T$ in $\mathbf{C}$, $uf \in \mathbf{C}(U,A)$ and thus $(uf,\ I_A,\ f,\ u)$

(1.8.23.1) $\langle\langle$
$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "uf"'] & T \arrow[d, "u"] \\
A \arrow[r, "I_A"'] & A
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
F(U) \arrow[r, "F(f)"] \arrow[d, "\alpha(uf)"] & F(T) \arrow[d, "\alpha(u)"] \\
X \arrow[r, "I_X"'] & X
\end{tikzcd}
\quad \rangle\rangle$$

is an arrow in $\mathbf{C}/_A$ with source uf and target u. Now (1.8.17.3) is commutative and α is a functor, so that $(\ \alpha(uf),\ I_X, F(f),\ \alpha(u))$ is a commutative square in $\mathbf{D}/_X$ and the arrow assignment is as in

(1.8.20.2). But this asserts that both $\alpha(u)$ and $\alpha(uf)$ are as desired in $\underset{\sim}{D}(F(T),X)$ and $\underset{\sim}{D}(F(U),X)$, respectively and, moreover, $\underset{\sim}{D}(F(f),X)(\alpha(u)) = F(f)\alpha(u) = \alpha(uf)$. Thus $\langle\langle\ T \rightsquigarrow \alpha(T)\ \rangle\rangle$ is natural.

But (1.8.17.3) is cartesian and given any $x \in \underset{\sim}{D}(F(T),X)$, one has $x \in \mathfrak{Ob}(\underset{\sim}{D}/_X)$ with $(T, x) \in \mathfrak{Ob}(\underset{\sim}{C} \underset{\underset{\sim}{D}}{\times} \underset{\sim}{D}/_X)$. Consequently, there exists a unique $z \in \mathfrak{Ob}(\underset{\sim}{C}/_A)$ such that $\sigma_{\underset{\sim}{C}/A}(z) = T$ and $\alpha(z) = x$, i.e. there exists a unique $z \in \underset{\sim}{C}(T,A)$ such that $\alpha(T)(z) = x$. Thus if the square (1.8.17.3) is cartesian, $\alpha(T)$ is <u>bijective</u> for any T and the necessity is established.

Reciprocally (and we have again made a direct proof), given the existence of a family $(\alpha(T))_{T\in\mathfrak{Ob}(\underset{\sim}{C})}$ of bijections $\alpha(T) : \underset{\sim}{C}(T,A) \longrightarrow \underset{\sim}{D}(F(T),X)$ which is "natural in T" define an application $\mathfrak{Ob}(\alpha) : \mathfrak{Ob}(\underset{\sim}{C}/_A) \longrightarrow \mathfrak{Ob}(\underset{\sim}{D}/_X)$ by means of the assignment

$$\langle\langle (u{:}T \longrightarrow A) \rightsquigarrow \alpha(T)(u) : F(T) \longrightarrow X\ \rangle\rangle.$$

Since α is a natural transformation, the application of $\mathfrak{Fl}(\underset{\sim}{C}/_A)$ into $\mathfrak{Fl}(\underset{\sim}{D}/_X)$ defined by the assignment

$$\langle\langle\ (ux, I_A, x, u) \rightsquigarrow (\alpha(U)(ux), I_X, F(x), \alpha(T)(u)) = (\alpha(T)(u)F(x), I_X, \alpha(T)(u))$$

does carry commutative squares into commutative squares and hence we have defined a functor $\underset{\sim}{\alpha} : \underset{\sim}{C}/_A \longrightarrow \underset{\sim}{D}/_X$ which makes (1.8.17.3) commutative. If the natural transformation α is bijective then the functor $\underset{\sim}{\alpha}$ clearly is cartesian above F.

PROPOSITION (1.8.24) In order that the co-adjoint of F be defined at $X \in Ob(\underset{\sim}{D})$ it is necessary and sufficient that there exist an object A and a functor ρ such that the square (1.8.20.2) be cartesian.

COROLLARY (1.8.25) Let $F : \underset{\sim}{C} \longrightarrow (\underset{\sim}{ENS})$ (resp. $F : \underset{\sim}{C}^{op} \longrightarrow (\underset{\sim}{ENS})$) be functors. In order that F be co-representable (resp. representable) it is necessary and sufficient that the square (1.8.25.1) (resp. (1.8.25.2)) be cartesian for some $X \in \underset{\sim}{Ob}(\underset{\sim}{C})$ and some functor α.

(1.8.25.1)
$$
\begin{tikzcd}
X/\underset{\sim}{C} \arrow[r, "\alpha"] \arrow[d] & \{\emptyset\}/(\underset{\sim}{ENS}) \arrow[d] \\
\underset{\sim}{C}^{2} \arrow[d, "\sigma_1^{\underset{\sim}{C}}"'] & (\underset{\sim}{ENS})^{2} \arrow[d, "\sigma_1"] \\
\underset{\sim}{C} \arrow[r, "F"] & (\underset{\sim}{ENS})
\end{tikzcd}
$$

(1.8.25.2)
$$
\begin{tikzcd}
X/\underset{\sim}{C}^{op} \arrow[r, "\alpha"] \arrow[d] & \{\emptyset\}/(\underset{\sim}{ENS}) \arrow[d] \\
\underset{\sim}{C}^{op} \arrow[d, "\sigma^{C}"'] & (\underset{\sim}{ENS})^{2} \arrow[d, "\sigma_1"] \\
\underset{\sim}{C}^{op} \arrow[r, "F"] & (\underset{\sim}{ENS})
\end{tikzcd}
$$

DEFINITION (1.8.26) Let $F : C \longrightarrow D$ be a functor and X an object of D. The (canonical) σ_1^{D}- (resp σ_0^{D}-) representation category for F relative to X is the fibre product category $C \times_{F,\sigma_1^{D},\iota_X} X/D$ (resp. $C \times_{F,\sigma_0^{D},\iota_X} D/X$) (Any category equivalent to such a fibre product category will be called a representation category).

EXAMPLE (1.8.26.1) The category C/F of (1.6.9) is simply the image of the σ_1-representation category of the canonical hom-functor h relative the given functor F. i.e. the square

$$
\begin{tikzcd}
C/F \arrow[r, hook] \arrow[dd] & \underline{\underline{CAT}}(C^{\circ}, (ENS))/F \arrow[d] \\
 & \underline{\underline{CAT}}(C^{\circ}, (ENS))^{2} \arrow[d, "\sigma_0"] \\
C \arrow[r, hook, "h"] & \underline{\underline{CAT}}(C^{\circ}, (ENS))
\end{tikzcd}
\tag{1.8.26.2}
$$

is cartesian. The reader is invited to reprove (1.6.10) on this new basis.

EXAMPLES (1.8.27) - 1° Let C be the category (ENS) and B be the category (Gr) of groups. Every group G has associated with it its underlying base set, $\underline{G}$. If $\mathcal{F}(S,\underline{G})$ is the set of all functions from some set S into $\underline{G}$, then the assignment ⟨⟨ $G \rightsquigarrow \mathcal{F}(S,\underline{G})$ ⟩⟩ defines a functor from (Gr) into (ENS) which is (co-) representable with the free group F(S) on S as the representing object and the application $b : S \longrightarrow \underline{F(S)}$ which assigns to each generator $s \in S$ its image in the free group as the defining transform, so that one has a natural bijection of the set $\mathrm{Gr}\,(F(S), G)$ onto $\mathcal{F}(S, \underline{G})$. But this is just the assertion that given any function α from S into the underlying base of some group G there exists a unique group homomorphism $f : F(S) \longrightarrow G$ such that $fb = \alpha$.

i.e. that $b : S \longrightarrow F(S)$ is a σ-<u>universal quasi-morphism</u> for the association $\mathcal{A}$ of (ENS) with (Gr) defined by $\mathcal{H}(\mathcal{A}) = \bigcup_{S \in \mathrm{ENS},\, G \in \mathrm{Gr}} \mathcal{F}(S, \underline{G})$ in the obvious fashion. Whence commeth the familiar diagram

$$
\text{(D)} \qquad
\begin{tikzcd}
F(S) \arrow[dr, dashed, "f"] & \\
S \arrow[u, "b"] \arrow[r, "\alpha"'] & G
\end{tikzcd}
$$

which asserts, with the usual abuse of language, that every function from the set of generators into some group G admits a <u>unique extension</u> to a <u>homomorphism</u> of the free group $F(S)$ on S into the group G, and thus preserves the set of generators.

This is clearly possible for any set S so what we really have is a natural isomorphism of bifunctors.

$$
\alpha(S,G) : \mathrm{Gr}\,(F(S),\, G) \longrightarrow \mathcal{F}(S, \underline{G}.)\ (= \mathrm{ENS}(S, \underline{G}))
$$

which asserts that the <u>functor</u> $F : (\mathrm{ENS}) \longrightarrow (\mathrm{Gr})$ is <u>co-adjoint</u> to the functor ("underlying base set" or "forgetful" functor) «–» $- : (\mathrm{Gr}) \longrightarrow (\mathrm{ENS})$.

2^o The description of the free group is typical of all such free-objects. For example, the functors which assign to each set "the" free polynomial algebra in the given set of indeterminates is co-adjoint to the underlying base set functor. Similarly for free-semigroups free monoids, etc. they are all characterised, upto a unique isomorphism, as co-adjoints of the respective underlying base set functors.

3^o In most of the examples in 2^o, it is clear that one need not "forget all of the structure available", but rather have projection (or, if one prefers, "inclusion") functors of, say, monoids within groups and algebras. These projections are also functorial and one has that, for example, the functor <u>tensor algebra</u> of a given R-module is co-adjoint to the projection functor of <u>R-algebras</u> into R-algebras. Similarly the functor <u>semigroup algebra of a semigroup</u> is co-adjoint to the projection functor of R algebras into semigroups. Such examples as these are legion. LAWVERE (1963) has demonstrated the existence of this type of co-adjoint in the wide class of structures including the above cases which he characterises by means of "<u>algebraic theories</u>". <u>Fields of fractions</u> and their generalizations provide many examples of representations.

4^o Let $\underset{\sim}{\mathfrak{P}}^{-1} : (\underset{\sim}{\mathrm{ENS}})^{(op)} \longrightarrow (\underset{\sim}{\mathrm{ENS}})$ be the (contra-variant) power set functor defined by $\langle\langle\ E \rightsquigarrow \underset{\sim}{\mathfrak{P}}(E),\ f \rightsquigarrow \overset{-1}{f}\ \rangle\rangle$. $\underset{\sim}{\mathfrak{P}}^{-1}$ is representable and one has for each set E that $\underset{\sim}{\mathfrak{F}}(E,2) \xrightarrow{\sim} \mathfrak{P}(E)$ by means of the <u>characteristic functions of subsets</u> of E.

5^o Let $\underset{\sim}{\mathrm{Corr}}(\cdot\ ,X) : (\underset{\sim}{\mathrm{ENS}})^* \longrightarrow (\underset{\sim}{\mathrm{ENS}})^{*}$ be the functor which assigns to each set T the set $\underset{\sim}{\mathrm{Corr}}(T,X)$ of all graphs of correspondences of T with X (i.e. subsets of T x X). $\underset{\sim}{\mathrm{Corr}}(\ \ ,X)$ is representable and one has $\underset{\sim}{\mathfrak{F}}(T, \underset{\sim}{\mathfrak{P}}(X)) \xrightarrow{\sim} \underset{\sim}{\mathrm{Corr}}(T,X)$ for each T, and, in fact, for each X, which is characteristic of the covariant power set functor.

6^o The examples from general topology are just as numerous. Nearly all of those derived from the projection functors derived from the various separation axioms admit co-adjoints and their constructions are readily accessible in the exercises of N. BOURBAKI (1965). One should mention also that the Stone-Čech Compactification of a completely regular space is co-adjoint to the projection functor which carries completely regular spaces into compact spaces. Completions of uniform spaces give another such example. The reader will find many other examples in any text on general topology. For a general discussion one should also see KENNISON (1964).

7^o In category theory itself we mention but two of many. If $\underset{\sim}{C}$ is some $\underset{\sim}{\mathcal{U}}$-category and $f : X \longrightarrow Y$ in $\underset{\sim}{\mathcal{Fl}}(\underset{\sim}{C})$, then one always has the direct image functor (1.3.15. Ex 2^o) $f_* : \underset{\sim}{C}/_X \longrightarrow \underset{\sim}{C}/_Y$. In order that f_* admit an adjoint it is necessary and sufficient that f be squareable (1.2.17). The functor $f^* : \underset{\sim}{C}/_Y \longrightarrow \underset{\sim}{C}/_X$ defined by taking the fibre product with f and supplying it with its projection then defines the adjoint. Clearly the adjoint is defined at each $g \in \underset{\sim}{\mathcal{Ob}}(\underset{\sim}{C}/_Y)$ for which the fibre product with f exists in $\underset{\sim}{C}$.

Another example is provided in $(\underset{\sim}{\mathrm{CAT}})$ by the assignment of each category to its <u>opposite</u>. The definition of <u>contravariant</u> functor on $\underset{\sim}{C}$ as a functor on $\underset{\sim}{C}^{(op)}$ is already an indentification (by definition) arising from the (trivial) solution to a co-representation problem.

(1.9) LIMITS OF DIAGRAMS

(1.9.1) If $\underset{\sim}{C}$ is a non-void category, then one always has the <u>constant functors</u> from the "one point" category $\underset{\sim}{1}$ into $\underset{\sim}{C}$. Such a constant functor c_X is completely determined by the value X of its object function and defines an isomorphism of $\underset{\sim}{1}$, the final object in $(\underset{\sim}{CAT})$, onto the "one point" <u>subcategory</u> $\underset{\sim}{X}$ of $\underset{\sim}{C}$, defined by $\underset{\sim}{\mathfrak{Ob}}(X) = \{X\}$ and $\underset{\sim}{\mathfrak{H}}(X) = \{I_X\}$.

Any category $\underset{\sim}{A}$ in $(\underset{\sim}{CAT})$ has a unique functor $\varphi_{\underset{\sim}{A}} : \underset{\sim}{A} \longrightarrow \underset{\sim}{1}$. Let $F : \underset{\sim}{A} \longrightarrow \underset{\sim}{C}$ be a functor; we shall say that F is a <u>constant functor with value</u> X $(\in \mathfrak{Ob}(\underset{\sim}{C}))$ provided F has the form $\langle\langle\, c_X : \underset{\sim}{A} \xrightarrow{\varphi_A} \underset{\sim}{1} \xrightarrow{c_X^1} \underset{\sim}{C} \,\rangle\rangle$ for $X \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C})$. The constant functors with source $\underset{\sim}{1}$ and target $\underset{\sim}{C}$ are simply the functorial <u>sections</u> associated with $\varphi_{\underset{\sim}{C}} : \underset{\sim}{C} \longrightarrow \underset{\sim}{1}$.

(1.9.2) If $\underset{\sim}{C}$ is a $\underset{\sim}{\mathfrak{U}}$-category and $\underset{\sim}{\Sigma}$ is a <u>small</u> $\underset{\sim}{\mathfrak{U}}$-category (1.4.6), then the category $\underline{\underline{CAT}}(\underset{\sim}{\Sigma}, \underset{\sim}{C})$ is itself also a $\underset{\sim}{\mathfrak{U}}$-category. In particular the category $\underset{\sim}{1}$ is a small $\underset{\sim}{\mathfrak{U}}$-category, whatever be $\underset{\sim}{\mathfrak{U}}$ $(\neq \emptyset)$, and one has an <u>isomorphism of categories</u>

(1.9.2.1) $$c : \underset{\sim}{C} \xrightarrow{\sim} \underline{\underline{CAT}}\,(\underset{\sim}{1}, \underset{\sim}{C})$$

defined by ⟨⟨ $X \rightsquigarrow c_X$, $f \rightsquigarrow c_f$ ⟩⟩. Hence

⟨⟨ any category can be represented as a category of functors with natural transformation as morphisms . ⟩⟩

Moreover, if $\mathbf{D}$ is any non-void $\mathcal{U}$-category, then $\mathbf{CAT}(\mathbf{1},\mathbf{D}) \neq \emptyset$ and $\varphi_{\mathbf{D}} : \mathbf{D} \longrightarrow \mathbf{1}$ is a retraction, with the functor $\mathbf{CAT}(\varphi_{\mathbf{D}},\mathbf{C}) : \mathbf{CAT}(\mathbf{1},\mathbf{C}) \longrightarrow \mathbf{CAT}(\mathbf{D},\mathbf{C})$ consequently defining, by composition with $c : \mathbf{C} \longrightarrow \mathbf{CAT}(\mathbf{1},\mathbf{C})$, an embedding (1.3.11) of $\mathbf{C}$ into the category $\mathbf{CAT}(\mathbf{D},\mathbf{C})$, which we will denote by

(1.9.2.2) $$\langle\langle \quad c : \mathbf{C} \hookrightarrow \mathbf{CAT}(\mathbf{D},\mathbf{C}) \quad \rangle\rangle$$

and call the (canonical) <u>constant functor embedding</u>. (If $\mathbf{D}$ is $\mathcal{U}$-small, then c is an arrow in $\mathbf{CAT}\text{-}\mathcal{U}$). If $F : \mathbf{C} \longrightarrow \mathbf{E}$ is a functor, then $\mathbf{CAT}(\mathbf{D},F) : \mathbf{CAT}(\mathbf{D},\mathbf{C}) \longrightarrow \mathbf{CAT}(\mathbf{D},\mathbf{E})$ carries the constant functor c_X into the constant functor $Fc_X = c_{F(X)}$.

(1.9.3) Operating for the moment in some universe $\mathcal{U}^*$ to which $\mathcal{U}$ belongs, we can form the arrow category $\mathbf{CAT}(\mathbf{\Sigma},\mathbf{C})^2$ of the small $\mathcal{U}^*$-category $\mathbf{CAT}(\mathbf{\Sigma},\mathbf{C})$ and obtain the construction

(1.9.3.1)
$$\begin{tikzcd}
c^2 : \mathbf{C}^2 \arrow[r, hook] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] & \mathbf{CAT}(\mathbf{\Sigma},\mathbf{C})^2 \; (= \mathbf{C}(\mathbf{\Sigma})^2) \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_0"] \\
c : \mathbf{C} \arrow[r, hook] & \mathbf{CAT}(\mathbf{\Sigma},\mathbf{C}) \; (= \mathbf{C}(\mathbf{\Sigma}))
\end{tikzcd}$$

in the usual fashion.

<u>DEFINITION</u> (1.9.4) The category $\mathbf{C} \underset{c,\,\sigma_0}{\times} \underline{\underline{\mathrm{CAT}}}(\Sigma, \mathbf{C})^2$ supplied with its projection functors will be called the category $\mathbf{Proj}(\Sigma, \mathbf{C})$ of <u>projective systems</u> of $\mathbf{C}$ <u>with scheme</u> Σ. For a given functor $\vartheta : \Sigma \to \mathbf{C}$, the representation category for c relative to F (1.8.26), $\mathbf{C}_{c,\sigma_0}\times \underline{\underline{\mathbf{C}(\Sigma)}}^2/\vartheta$, will be called the category $\mathbf{Proj}(\Sigma, \mathbf{C})_{/\vartheta}$ of <u>projective systems</u> of $\mathbf{C}$ <u>with target</u> ϑ. By abuse of language, a functorial morphism $\xi : c_X \longrightarrow \vartheta$ ($pr_2(X, \xi)$ from $\mathbf{Proj}(\Sigma, \mathbf{C})_{/\vartheta}$) will be called a <u>projective system</u> of <u>source</u> X and <u>target</u> ϑ .

If Σ is a $\mathfrak{U}$-small category, so that $\underline{\underline{\mathrm{CAT}}}(\Sigma, \mathbf{C})$ is a $\mathfrak{U}$-category, then Σ is called a <u>diagram scheme</u> with <u>vertices</u> $\mathfrak{Ob}(\Sigma)$ and <u>arrows</u> $\mathfrak{Fl}(\Sigma)$, and a functor $\vartheta : \Sigma \longrightarrow \mathbf{C}$ is called a <u>diagram in</u> $\mathbf{C}$ of <u>scheme</u> Σ . The constant functors $c_X : \Sigma \longrightarrow \mathbf{C}$, for some $X \in \mathfrak{Ob}(\mathbf{C})$, are then called the <u>constant diagram functors</u> (on Σ).

The squares

(1.9.4.1)
$$\begin{tikzcd}
\mathbf{Proj}(\Sigma, \mathbf{C}) \arrow[r, hook] \arrow[d] & \underline{\underline{\mathrm{CAT}}}(\Sigma, \mathbf{C})^2 \arrow[d, "\sigma_0"] \\
\mathbf{C} \arrow[r, hook, "c"] & \underline{\underline{\mathrm{CAT}}}(\Sigma, \mathbf{C})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{Proj}(\Sigma, \mathbf{C})_{/\vartheta} \arrow[r, hook] \arrow[dd] & \underline{\underline{\mathrm{CAT}}}(\Sigma, \mathbf{C})_{/\vartheta} \arrow[d, "i_\vartheta"] \\
 & \underline{\underline{\mathrm{CAT}}}(\Sigma, \mathbf{C})^2 \arrow[d] \\
\mathbf{C} \arrow[r, hook, "c"] & \underline{\underline{\mathrm{CAT}}}(\Sigma, \mathbf{C})
\end{tikzcd}$$

are cartesian, by definition; moreover, since c is embedding, pr_2 is also an embedding so that $\underset{\sim}{\mathrm{Proj}}\,(\underset{\sim}{\Sigma},\underset{\sim}{C})$, for example, <u>may be considered as the</u> subcategory <u>inverse image</u> of $c\langle \underset{\sim}{C}\rangle$ by $\underset{\sim}{\sigma}_0$. We will make this identification whenever convenient.

<u>DEFINITION</u> (1.9.5) Let $\vartheta : \underset{\sim}{\Sigma} \longrightarrow \underset{\sim}{C}$ be a functor. An object L in $\underset{\sim}{C}$ supplied with a projective system ξ of source L and target ϑ will be called <u>a projective limit of</u> ϑ provided the functor $\underset{\sim}{C}/_L \xrightarrow{\xi'} \underline{\underline{CAT}}(\underset{\sim}{\Sigma},\underset{\sim}{C})/_\vartheta$ canonically defined by ξ (1.8.18) is cartesian above c.

Any couple of projective limits of ϑ are uniquely isomorphic; a canonically selected representative, if such exist, will be called the projective limit and be denoted by ⟨⟨ $\underleftarrow{\ell im}\,\vartheta$ $(= (\underset{\sim}{ob}\,(\underleftarrow{\ell im}\,\vartheta),\ \underset{\sim}{rep}\,(\underleftarrow{\ell im}\,\vartheta)))$ ⟩⟩ By abuse of language, the object pr_1 ($\underleftarrow{\ell im}\,\vartheta$) of $\underset{\sim}{C}$ will also be called the projective limit of ϑ and be denoted by ⟨⟨ $\underleftarrow{\ell im}\,\vartheta$ ⟩⟩,

If the projective limit of ϑ exist in $\underset{\sim}{C}$, one has by definition that the square I of

$$(1.9.5.1)\qquad \begin{tikzcd}
\underset{\sim}{C}/_L \arrow[r, "\xi'"] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & \underline{\underline{CAT}}(\Sigma,\underset{\sim}{C})/_\vartheta \arrow[r] \arrow[d] & \underline{\underline{CAT}}(\Sigma,\ \underline{\underline{CAT}}(\underset{\sim}{C}^o,(\underset{\sim}{ENS})))/_h \arrow[d] \\
\underset{\sim}{C}^2 - \arrow[d, "\underset{\sim}{\sigma}_0"] & \underline{\underline{CAT}}(\underset{\sim}{\Sigma},\underset{\sim}{C})^2 \arrow[r, "\underline{\underline{h}}(\Sigma)^2"] \arrow[d, "\underset{\sim}{\sigma}_0"] & \underline{\underline{CAT}}(\Sigma,\underline{\underline{CAT}}\,(\underset{\sim}{C}^o,(\underset{\sim}{ENS})))^2 \arrow[d] \\
\underset{\sim}{C} \arrow[r] & \underline{\underline{CAT}}(\underset{\sim}{\Sigma},\underset{\sim}{C}) \arrow[r, "\underline{\underline{h}}(\Sigma)"] & \underline{\underline{CAT}}(\Sigma,\underline{\underline{CAT}}(\underset{\sim}{C}^{op},(\underset{\sim}{ENS})))
\end{tikzcd}$$

of categories and functors be cartesian (i.e. ξ' is <u>cartesian above</u> c).

In view of (1.8.25) this simply signifies (KAN 1958) that the adjoint of c be defined at $\vartheta : \underset{\sim}{\Sigma} \longrightarrow \underset{\sim}{C}$, or, in other words, that one has a functorial isomorphism $(\xi(T))_{T \in Ob(C)}$ such that

$$\xi(T) \quad : \underset{\sim}{C}\,(T, \sqcup) \xrightarrow{\sim} \underset{\sim\sim}{\mathcal{H}om}(c_T, \vartheta\,) = \underset{\sim\sim}{Proj}\,(T, \vartheta\,)$$

whatever be $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$.

(1.9.6) For a fixed T, it is possible to explicitly determine the form of the set $\underset{\sim\sim}{\mathcal{H}om}(c_T, \vartheta\,)$ in a very useful fashion.

Any $\varphi \in \underset{\sim\sim}{\mathcal{H}om}(c_T, \vartheta\,)$ is by definition a functor $\varphi : \underset{\sim}{\Sigma} \longrightarrow \underset{\sim}{C}^2$ such that $\underset{\sim}{\sigma}_0{}^{C}\varphi = c_T$ and $\underset{\sim}{\sigma}_1{}^{C}\varphi = \vartheta$. It is sufficient to consider the object function of φ whose graph is then $(\varphi_u : T \to \vartheta_u)_{u \in \underset{\sim}{Ob}(\underset{\sim}{\Sigma})}$, i.e. $\varphi \in \prod_{u \in Ob(\Sigma)} \underset{\sim}{C}(T, \vartheta_u)$, (where the set-theoretic product is known to exist at least in $\underset{\sim}{\mathcal{U}}^*$, and indeed in $\underset{\sim}{\mathcal{U}}$, provided $\underset{\sim}{\Sigma}$ is $\underset{\sim}{\mathcal{U}}$-small).

Moreover, for any couple $(u,v) \in \underset{\sim}{Ob}(\underset{\sim}{\Sigma}) \times Ob\,(\underset{\sim}{\Sigma})$ and any $\alpha \in \underset{\sim}{\Sigma}(u,v)$, we must have that the square $(\varphi(u)\,,\ \vartheta_\alpha\,,\ I_T, \varphi(v))$ is commutative. $\langle\langle\ \vartheta_\alpha\varphi(u) = \varphi(v)\, I_T = \varphi(v)\ \rangle\rangle$.

(1.9.6.1.)
$$\begin{tikzcd} T \arrow[r, "I_T"] \arrow[d, "\varphi(u)"'] & T \arrow[d, "\varphi(v)"] \\ \vartheta_u \arrow[r, "\vartheta_\alpha"] & \vartheta_v \end{tikzcd}$$

This reduces to nothing more than the assertion that the couple ($\varphi(u)$, $\varphi(v)$ be an element of the graph of the function $\vartheta_\alpha(T) : \vartheta_u(T) \longrightarrow \vartheta_v(T)$ for any choice of $\alpha \in \underset{\sim}{\Sigma}(u,v)$. The set of all such couples is then the set $\bigcap_{\alpha \in \Sigma(u,v)} \underset{\sim}{Gr}(\vartheta_\alpha(T)) \subseteq \vartheta_u(T) \times \vartheta_v(T)$ which itself is easily seen to be the fibre product (in ($\underset{\sim}{ENS}$))

(1.9.6.2)

$$\begin{tikzcd}
A_u(T) \times_{\vartheta_\alpha(T), \Delta_{u,v}(T)} A_v(T) \arrow[rr] \arrow[d] & & A_v(T) \arrow[d, "\Delta_{u,v}(T) = \boxtimes_{\alpha \in \Sigma(u,v)} \underline{id}_\alpha(A_v(T))"] \\
A_u(T) \arrow[rr, "\boxtimes_{\alpha \in \Sigma(u,v)} \vartheta_\alpha(T)"] & & (A_v(T))^{\underset{\sim}{\Sigma}(u,v)}
\end{tikzcd}$$

where $(A_v(T))^{\underset{\sim}{\Sigma}(u,v)}$ is the product of $\underset{\sim}{\Sigma}(u,v)$ copies of $A_v(T)$ (the "cartesian power"); Δ is the application "diagonal" defined by $\langle\langle\ f \rightsquigarrow (f_\alpha)_{\alpha \in \underset{\sim}{\Sigma}(u,v)},\ f_\alpha = f$ for each $\alpha\ \rangle\rangle$, and $\boxtimes_{\alpha \in \underset{\sim}{\Sigma}(u,v)} \vartheta_\alpha(T)$ is that defined by $\langle\langle\ g \rightsquigarrow (\vartheta_\alpha g)_{\alpha \in \underset{\sim}{\Sigma}(u,v)}\ \rangle\rangle$.

The "phenomenon" of (1.9.6.1) must occur for any couple $(u,v) \in \underset{\sim}{Ob}(\underset{\sim}{\Sigma}) \times \underset{\sim}{Ob}(\underset{\sim}{\Sigma})$. This can be assured by observing that for each such couple (u,v) we have the applications (at least in $\underset{\sim}{\mathcal{U}}^*$)

$$\langle\langle\quad \prod_{u \in Ob(\Sigma)} A_u(T) \xrightarrow{pr_u} A_u(T) \xrightarrow{\boxtimes \vartheta_\alpha(T)} (A_v(T))^{\underset{\sim}{\Sigma}(u,v)} \quad\rangle\rangle$$

and

$$\langle\langle\ \prod_{u \in Ob(\Sigma)} A_u(T) \xrightarrow{pr_v} A_v(T) \xrightarrow{\Delta_{u,v}^{(T)}} (A_v(T))^{\underset{\sim}{\Sigma}(u,v)}\ \rangle\rangle$$

defined by composition and, consequently, the applications

$$\langle\langle\ \epsilon_1(T) = \underset{(u,v)\in \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})\times \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})}{\boxtimes} \left(\underset{\alpha\in\Sigma(u,v)}{\boxtimes} \vartheta_\alpha(T)\cdot \underset{\sim}{pr}_u \right) : \prod_{u\in \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})} A_u(T) \longrightarrow \prod_{(u,v)\in \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})\times \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})} (A_v(T))^{\underset{\sim}{\Sigma}(u,v)}\ \rangle\rangle$$

and

$$\langle\langle\ \epsilon_2(T) = \underset{(u,v)\in \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})\times \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})}{\boxtimes} \left(\Delta^{(T)}_{(u,v)}\cdot pr_v\right) : \prod_{u\in \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})} A_u(T) \longrightarrow \prod_{(u,v)\in \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})\times \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})} (A_v(T))^{\underset{\sim}{\Sigma}(u,v)}\ \rangle\rangle$$

have the property that any $\varphi\in\prod_{u\in\mathcal{O}b(\Sigma)} A_u(T)$ such that $\epsilon_1(T)(\varphi) = \epsilon_2(T)(\varphi)$ satisfies the condition expressed by (1.9.6.1) for any $(u,v)\in \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma}) \times \underset{\sim}{\mathcal{O}b}(\underset{\sim}{\Sigma})$ and any $\alpha\in \underset{\sim}{\Sigma}(u,v)$.

The subset Ker $(\epsilon_1(T), \epsilon_2(T))$ of $\prod_{u\in\mathcal{O}b(\Sigma)} A_u(T)$ thus generalises the well known notion of ⟨⟨ the projective limit of a family of sets and applications ⟩⟩ (take for $\underset{\sim}{\Sigma}$ the opposite category associated with a pre-ordered set (1.0.6 Ex.2°)), and will be called the (<u>set-theoretic</u>) <u>projective limit of the</u> ($\underset{\sim}{\mathcal{U}}^{(*)}$) <u>diagram</u> $h\vartheta_T : \underset{\sim}{\Sigma} \longrightarrow (\underset{\sim}{ENS})$ and be denoted by ⟨⟨ $\varprojlim h\vartheta\ (T)$ ⟩⟩ or ⟨⟨ $\varprojlim\ ((A_u(T))_{u\in\mathcal{O}b(\Sigma)},\ (\vartheta_\alpha(T))_{\alpha\in\mathcal{F}l(\Sigma)})$ ⟩⟩ or some variant of same. In any cases we will have that

$$(1.9.6.3)\qquad \varprojlim h\vartheta\ (T) \subseteq \prod_{u\in\mathcal{O}b(\Sigma)} A_u(T) \underset{\epsilon_2(T)}{\overset{\epsilon_1(T)}{\rightrightarrows}} \prod_{(u,v)\in\mathcal{O}b(\Sigma)\times\mathcal{O}b(\Sigma)} (A_v(T))^{\underset{\sim}{\Sigma}(u,v)}$$

is exact in $(\underset{\sim}{ENS})$, and that "by definition"

$$(1.9.6.4)\qquad \langle\langle\ \underset{\sim}{Proj}(c_T,\vartheta) = \varprojlim h\vartheta\ (T)\ \rangle\rangle$$

is an explicit description of the set of all natural transformations of a constant diagram functor into a diagram ϑ as originally desired.

(1.9.7) Returning to our original problem we see that we have shewn that a necessary and sufficient condition for the existence of the projective limit of a functor $\vartheta : \underset{\sim}{\Sigma} \quad \underset{\sim}{C}$ is that the $\underset{\sim}{\mathfrak{U}}^*$-functor defined by $\langle\langle\ T \rightsquigarrow \varprojlim h\vartheta\ (T)\ \rangle\rangle$ be representable, in which case we will have a functorial isomorphism which has the property that for each $T \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})$,

$$\rho(T) : \underset{\sim}{C}(T, \varprojlim \vartheta\ ) \xrightarrow{\sim} \mathrm{Proj}\,(c_T, \vartheta\ ) \xrightarrow{\sim} \varprojlim h\vartheta\ (T) = \varprojlim\ (\underset{\sim}{C}(T,\vartheta_\lambda), \underset{\sim}{C}(T,\vartheta_\mu\ ))$$

is a bijection.

Conditions for this to hold are easily found provided we <u>restrict overselves to $\underset{\sim}{\mathfrak{U}}$ -diagrams</u>, i.e. to the requirement that $\underset{\sim}{\Sigma}$ be a <u>small</u> $\underset{\sim}{\mathfrak{U}}$-category. In fact (1.9.6.3) gives just what we want, namely,

<u>PROPOSITION</u>(1.9.8). [MARANDA (1963) FREYD (1960), FOLKLORE (??)] In order that the projective limit of any <u>diagram</u> $\vartheta : \underset{\sim}{\Sigma} \longrightarrow \underset{\sim}{C}$ on any <u>diagram scheme</u> $\underset{\sim}{\Sigma}$ (i.e. small $\underset{\sim}{\mathfrak{U}}$ -category) exist in a given $\underset{\sim}{\mathfrak{U}}$-category $\underset{\sim}{C}$, it is necessary and sufficient that the category $\underset{\sim}{C}$

(a) admit the product of any family $(A_\iota)_{\iota \in I}$ of its objects whose index set I is a member of $\underset{\sim}{\mathfrak{U}}$ (1. 2.4) and

(b) admit kernels of couples of arrows (1.2.21).

If $\underset{\sim}{\Sigma}$ is a small $\underset{\sim}{\mathfrak{U}}$ -category the two set-theoretic products which occur in (1.9.6.3) are both indexed by a member of $\underset{\sim}{\mathfrak{U}}$ and hence $\langle\langle\ T \rightsquigarrow \prod_{u \in \mathrm{Ob}(\Sigma)} A_u(T)\ \rangle\rangle$ is a functor into $\underset{\sim}{\mathrm{ENS}}\ -\underset{\sim}{\mathfrak{U}}$. Consequently if $\underset{\sim}{C}$ admit such ($\underset{\sim}{\mathfrak{U}}$-) products then the two product functors are representable and the arrows $\epsilon_1(T)$ and $\epsilon_2(T)$ then define via

$$\langle\langle\ T \rightsquigarrow \epsilon_1(T),\ T \rightsquigarrow \epsilon_2(T)\ \rangle\rangle\ ,$$

natural transformations of representable functors which are then represented by a couple of arrows with source and target the representations of the given products. If $\underset{\sim}{C}$ then admits kernels of couples, we have that the functor $\langle\langle\ T \rightsquigarrow \underleftarrow{\ell im}\, \lambda\vartheta\ (T)\ \rangle\rangle$ is representable and that the projective limit of ϑ exists, as desired.

Conversly, let $\underset{\sim}{\Sigma}$ be a small discrete (no arrows other than the identity arrows) $\underset{\sim}{\mathfrak{U}}$-category. A diagram on $\underset{\sim}{\Sigma}$ is completely determined by, and may be identified with, nothing more than a family of objects of $\underset{\sim}{C}$ indexed by $\underset{\sim}{Ob}(\underset{\sim}{\Sigma})$. A projective limit of such a "discrete diagram" is just the product of the family of objects defined by the diagram. Now any family of objects of $\underset{\sim}{C}$ whose index set is an element of $\underset{\sim}{\mathfrak{U}}$ defines a small discrete category together with a diagram which may be identified with the given family. By hypothesis the projective limit of such a diagram always exists, hence the category admits ($\underset{\sim}{\mathfrak{U}}$ -indexed) products. The reasoning for kernels of couple proceeds in the same fashion using the finite category whose objects are simply the members of the set $\{0,1\}$ and whose arrows are the members of the set $\{\ (0,0),\ (1,1)\ (0,1,1),\ (0,1,0)\ \}$ with no non-trivial multiplications, (or any category isomorphic to same).

$$\langle\langle\quad 0 \xrightarrow{(0,0)} 0 \underset{(0,1,1)}{\overset{(0,1,0)}{\rightrightarrows}} 1 \xrightarrow{(1,1)} 1 \quad\rangle\rangle$$

A diagram on this scheme has as its graph in $\underset{\sim}{C}$ a "diagram" of the form

$$\langle\langle\quad A_0 \underset{f_{(0,1,1)}}{\overset{f_{(0,1,0)}}{\rightrightarrows}} A_1 \quad\rangle\rangle$$

(where the identity arrows have been omitted) and a natural transformation of a constant diagram functor on this same scheme has the consequent form

$$\left\langle\!\left\langle \quad \begin{array}{ccc} T & \xrightarrow{\quad I_T \quad} & T \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ A & \xrightarrow[\quad f_i \quad]{} & A_1 \end{array} \qquad i = (0,1,0),\ (0,1,1) \quad \right\rangle\!\right\rangle,$$

which requires that $f_{(0,1,0)}\, f = g = f_{(0,1,1)} f$, or, in other words, that for each $T \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C})$, $f \in \operatorname{Ker}(f_o(T),\ f_1(T))$. A projective limit of this "kernel diagram" is then simply a representation of the functor defined by $\langle\langle\ T \rightsquigarrow \operatorname{Ker}(f_o(T),\ f_1(T))\ \rangle\rangle$. If such limits always exist, then the category $\underset{\sim}{C}$ must admit kernels of couples of arrows.

<u>PROPOSITION</u> (1.9.9) The following statements are equivalent for any $\underset{\sim}{\mathfrak{U}}$-category $\underset{\sim}{C}$:

1^o the category $\underset{\sim}{C}$ admits finite projective limits (i.e. the projective limit exists in $\underset{\sim}{C}$ for any diagram on any diagram scheme whose objects and arrows constitute <u>finite</u> sets);

2^o $\underset{\sim}{C}$ admits fibre products and possesses a final object (i.e. in the last case, the <u>void projective limit</u> is representable);

3^o $\underset{\sim}{C}$ admits finite products and kernels of couples of arrows;

4^o $\underset{\sim}{C}$ admits finite products and fibre products.

The proof is trivial in the light of (1.9.8). Suffice it to observe that given the existence of fibre products and a final object, there exists a canonical isomorphism of the fibre product of any couple of objects over the (unique) arrows into the final object and the categorical product of the couple of objects (since for each T, $A(T) \times B(T) \xrightarrow{\sim} A(T) \underset{\varphi_A(T),\varphi_B(T)}{\times} B(T)$ ("the fibre product over a one element set is simply the product").

Given finite products and kernels then one has fibre products, since one always has that

$$\langle\langle\ A(T) \underset{f(T),g(T)}{\times} B(T) = \mathrm{Ker}\,(f(T)pr_1(T),\ g(T)pr_2(T)) \subseteq \begin{tikzcd}[row sep=small] & B(T) \arrow[dr] & \\ A(T)\times B(T) \arrow[ur] \arrow[dr] & & C(T) \\ & A(T) \arrow[ur] & \end{tikzcd}\ \rangle\rangle;$$

And, finally, that since for all T, the square

$$(1.9.9.1) \qquad \begin{tikzcd} \mathrm{Ker}(f(T),g(T)) \arrow[r] \arrow[d, hook] & B(T) \arrow[d, hook, "\Delta_{B(T)}"] \\ A(T) \arrow[r, "f(T)\boxtimes g(T)"] & B(T)\times B(T) \end{tikzcd}$$

is cartesian, the existence of finite products and fibre products gives that of kernels of couples.

(1.9.10) The last theorem (1.9.9) makes it immediately possible to substitute "fibre-products" for "kernels" in (1.9.8) and obtain the same result. (This, of course, could have been proved directly, if we had desired). The dual assertion, leading to the existence of <u>inductive</u>, (<u>direct</u>, or co-projective (?)) <u>limits</u> is clear.

(1.9.11) A category which satisfies any of the equivalent conditions of (1.9.8) will be called ($\varprojlim$)-<u>complete</u>. The importance of the notion of limit will become readily apparent as well as the prime importance of the study of the preservation of limits under various functors. We remark at present on only two important aspects of the problem.

PROPOSITION (1.9.12) If $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$ is a functor, then

1° if F is the adjoint of some functor $G : \underset{\sim}{D} \longrightarrow \underset{\sim}{C}$,

F preserves (projective) limits (whenever they exist); and

2° if F is the co-adjoint of some functor $G : \underset{\sim}{D} \longrightarrow \underset{\sim}{C}$,

then F preserves co-limits (whenever they exist).

We prove 1°. Consider $F(\varprojlim \vartheta) \in \underset{\sim}{Ob}(\underset{\sim}{D})$; for any $T \in \underset{\sim}{Ob}(\underset{\sim}{D})$, $\underset{\sim}{D}(T, F(\varprojlim \vartheta)) \xrightarrow{\sim} \underset{\sim}{C}(G(T), \varprojlim \vartheta) \xrightarrow{\sim} \varprojlim (\underset{\sim}{C}(G(T), \vartheta_\alpha)) \xrightarrow{\sim} \varprojlim(D(T, F(\vartheta_\alpha)))$, which is the assertion that $F(\varprojlim \vartheta)$ is the limit of the diagram $F\vartheta : \underset{\sim}{\Sigma} \longrightarrow \underset{\sim}{D}$. This is abbreviated as ⟨⟨ $F(\varprojlim \vartheta) = \varprojlim F\vartheta$ ⟩⟩ and we sometimes say that F commutes with projective limits.

(1.9.13) We leave it to the reader to assure himself of the proper behaviour of the notion of ⟨⟨ equivalence of categories ⟩⟩ and preservation of the limits and co-limits by means of the following assertion and the SHIH characterization of adjoints (1.8.5)

⟨⟨ $\underset{\sim}{C}$ is equivalent to $\underset{\sim}{D}$ iff there exist functors $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$, $G : \underset{\sim}{D} \longrightarrow \underset{\sim}{C}$ and isomorphisms φ, ψ such that $\psi : GF \xrightarrow{\sim} Id, \varphi : FG \xrightarrow{\sim} Id$. ⟩⟩

LEMMA (1.9.14) Let $F : \underset{\sim}{C} \longrightarrow \underset{\sim}{D}$ be a functor and $T \in \underset{\sim}{Ob}(\underset{\sim}{D})$. There exists a (canonical) bijection of the set $\underset{\sim}{\mathcal{H}om}(c_T, F)$ of arrows in $\underset{\sim}{CAT}(\underset{\sim}{C}, \underset{\sim}{D})$ with source the constant functor c_T and target F, onto the set $\underset{\sim}{CAT}_{\underset{\sim}{D}}(\underset{\sim}{C}, T/_{\underset{\sim}{D}}) = \underset{\sim}{CAT}_{I_{\underset{\sim}{D}}}(\underset{\sim}{C}, T/_{\underset{\sim}{D}})$.

By definition, the set $\underset{\sim}{\mathcal{H}om}(c_T, F)$ is the set of functors $\theta : C \longrightarrow D^2$ such that $\underset{\sim}{\sigma}^{\underset{\sim}{D}}_o \theta = c_T$ and $\underset{\sim}{\sigma}^{\underset{\sim}{D}}_1 \theta = F$. By definition of the functor c_T (1.9.1), $c_T = c^1_T \varphi_C$, so that given any $\theta \in \underset{\sim}{\mathcal{H}om}(c_T, F)$

the couple $(\theta, \varphi_c) \in \underset{\sim}{D}^2(C) \underset{\sigma_1(c), c_T}{\times} \{\varphi_c\} \xrightarrow{\sim} T/_{D}(\underset{\sim}{C})$; hence there exists a unique $\theta' : \underset{\sim}{C} \longrightarrow T/_{\underset{\sim}{D}}$ such that $\tilde{i}'_T\theta' = \theta$. The remainder is left to the reader.

N.B. It is customary to restrict the terminology of "diagram" and" limit of a diagram" to $\underset{\sim}{\mathcal{U}}$-diagrams as we have, although in this type of treatment there is no theoretical distinction between a "functor" and a "diagram". It does have practical merit, though, if our aim is to describe actually occuring situations in the "large categories" which one encounters in practice. We have left the notion of ⟨⟨ projective limit ⟩⟩ outside of this restriction in order to draw attention to the following connection between ⟨⟨ projective limits ⟩⟩ and ⟨⟨ (co-) representable functors ⟩⟩ :

PROPOSITION (1.9.15) [BENABOU (1965)] Let $F : \underset{\sim}{C} \longrightarrow (\underset{\sim}{ENS})$ be a functor, $\underset{\sim}{R}(F) = \underset{\sim}{C} \underset{\underset{\sim}{ENS}}{\times} \{\emptyset\}/_{(\underset{\sim}{ENS})}$ the σ_1-representation category (1.8.26) of F relative to $\{\emptyset\}$, and $K : \underset{\sim}{R}(F) \longrightarrow \underset{\sim}{C}$ the "first-projection" functor. In order that F be (co-)representable, it is necessary and sufficient that the functor K posses a projective limit and that F be compatable with this limit $(\mu F(\varprojlim K) \xrightarrow{\sim} \varprojlim FK)$.

We establish this result by our methods: If F is (co-) representable, then (1.8.5.1) is cartesian for some $X \in \underset{\sim}{Ob}(\underset{\sim}{C})$ and functor α above F. But this gives the existence of a unique isomorphism $\tilde{\mu}^{-1}$ of $X/_{\underset{\sim}{C}}$ with $\underset{\sim}{R}(F)$ such that $\mu K = \underset{\sim}{\sigma}_1(\underset{\sim}{C}) i^1_X = X/_{\sigma_1^c}$.

Since (1.9.14) is applicable, for any $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, (1.8.20) gives the system

$$\langle\langle \; \underset{\sim}{\mathcal{H}om}(c_T, K) \xrightarrow{\sim} \underset{\sim}{CAT}_{I_{\underset{\sim}{C}}}(\underset{\sim}{R}(F), T/\underset{\sim}{C}) \xrightarrow{\sim} \underset{\sim}{CAT}_{I_C}(X/\underset{\sim}{C}, T/\underset{\sim}{C}) \xrightarrow{\sim} \underset{\sim}{C}(T, X) \; \rangle\rangle$$

which asserts that X is a projective limit of K. In addition, since one has for any set E the chain of isomorphisms

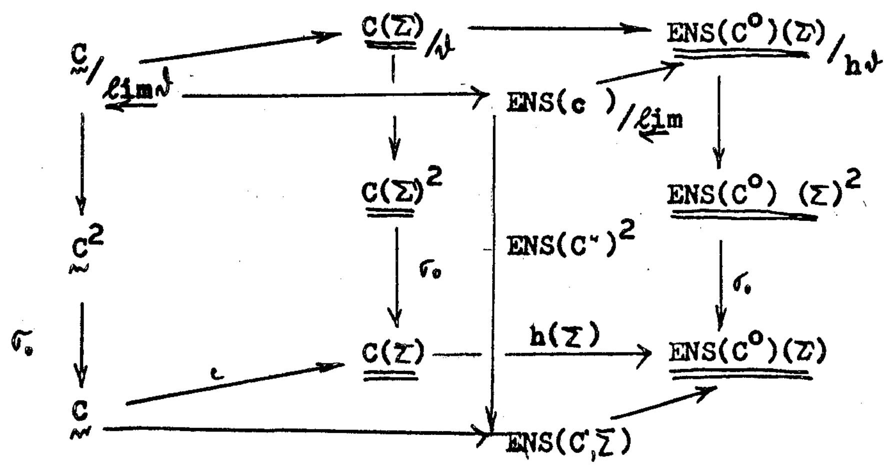
$$\langle\langle \mathcal{H}om(c_E, FK) \xrightarrow{\sim} \underset{\sim}{CAT}_{I_\alpha}(\underset{\sim}{R}(F),\ E/_{(\underset{\sim}{ENS})}) \xrightarrow{\sim} \underset{\sim}{CAT}_F\ (A/_{\underset{\sim}{C}},\ E/_{(\underset{\sim}{ENS})}) \xrightarrow{\sim} \mathcal{F}(E, F(A)) \ \rangle\rangle,$$

by application of (1.8.20) again, F is compatable with this limit.

Conversely, if X is an "F-compatible" projective limit for K, then(1.9.14)and (1.8.20) give that the identity of X must define an F-functor and also a $\underset{\sim}{C}$-functor of the cartesian product category $\underset{\sim}{R}(F)$ with the category $X/_{\underset{\sim}{C}}$. The compatibility of F with this limit then forces the so-defined functor to be an isomorphism, which establishes that (1.8.25.1) is cartesian and completes the proof.

It should be clear that if a category admits limits on some diagram scheme then such limits may be transferred to any functor category which has the given category as target, in a "point-wise" fashion by means of $(\varprojlim_{\alpha\in\Sigma} (F_\alpha))(T) = \varprojlim_{\alpha\in\Sigma} (F_\alpha(T))$, for each T in the source category. It is a consequence of (1.9.6) that $\underset{\sim}{ENS}$-$\underset{\sim}{\mathfrak{U}}$ admits $\underset{\sim}{\mathfrak{U}}$-limits for arbitrary diagrams and thus that any functor category over $(\underset{\sim}{ENS})$ will have limits as well. We shall leave it for the reader to re-establish the claims of (1.9.7), for example, by two simple observations using the cartesian squares of the diagram

(1.9.16.1)



REMARK (1.9.16) PROPOSITION (1.9.16) and its dual completes a sequence of "translational equivalences" of the various notions of ⟨⟨representability⟩⟩ which have been established in § 1. For example, we might tie them together with the statement that in the course of § 1, we have proved that the following statements are equivalent:

1° $F : \underset{\sim}{C}^{op} \longrightarrow (\underset{\sim}{ENS})$ is representable (1.6.8)

(GROTHENDIECK(1959));

2° $\underset{\sim}{C}/_F$ ($\rightsquigarrow$ $\underset{\sim}{REP}$(h) $\underline{rel}$ F) has a final object (1.6.9)

3° $\underset{\sim}{C}^*/_F$ ($\rightsquigarrow$ $\underset{\sim}{REP}$ (F) rel $\{\emptyset\}$) has a universal point (1.6.9)

(EHRESMANN(1957) - KAN(1958) - FLEISHER(1962);

4° the τ-association of $\underset{\sim}{C}^o$ with $(\underset{\sim}{ENS})$ defined by F

has a τ-universal arrow (1.7.8)

(BOURKAKI 1957, SAMUEL 1948); SONNER (1963), SWANN (1958))

5° the cO-adjoint of F is defined at $\{\emptyset\}$

(KAN 1958);

6° the square (1.8.25.2) is cartesian;

7° the square (1.8.26.2) is cartesian (with "F" replaced by "X");

8° the projection functor from the representation category of F has a projective limit in $\underset{\sim}{C}^{op}$

(= inductive limit in $\underset{\sim}{C}$) and F is compatable with this limit (1.9.15), (BENABOU(1965)

As one can readily see, most of the above are trivial variations on a theme which seems to have sprung, in part, from the notion of ⟨⟨ universal mapping problem ⟩⟩. The latter seems to have first been expressed as such by SAMUEL (1948) and, with a more abstract language at his disposal, by BOURBAKI (1957). This work was "ready-made" for the language of categories and functors and so was used by EHRESMANN (1957) and SWANN (1958), for instance. GROTHENDIECK (1957) seems the first to have given the notion of "universal mapping problem" a definitively elegant formulation as well as noted the remarkable attendant simplicity of the formal treatment of the notion in his terms.

KAN (1958), with the language of functors at his disposal, first seems to have abstracted the natural occurence of most universal problems into his beautiful theory of adjoint functors. KAN seems to have made most of his observations from actual practice rather than as any actual categorical abstraction of "universal mapping problems".

The essential equivalence of all of these notions seems to have been in mind almost from the beginning and notes to this effect for pairs of the notions which occur on the above list have been embedded in papers (e.g. SWANN (1958), FREYD (1961), or published separately (e.g. FLEISHER (1962), SONNER (1964)); others will probably continue to come to light for some time yet.

The first person, to the authors personal knowledge, to call attention to the equivalence of all of the major notions which occur here was TAKAHASHI (1962), who pointed out to the author the early paper of SWANN (1958) as well as introduced him to the concept of "representability". Such is the superficial dissimilarity of the variations on this theme, that the author (1963) was unaware of the notion of representation even after having excitedly discovered that "Bourbaki's universal mapping problem, could be formulated in terms of categories and functors and dualized ; and then every adjoint-situation just expressed the successful solution of a couple of dual universal mapping problems"; the result was nothing more nor less than that arrived at in (1.8.1) through the representation bijections themselves It is most probable that the author is not alone on this well-travelled road.

The references and citations of this section are not meant to be in any way definitive; they simply acknowledge the author's personal awareness of priorities. To all who have been inadvertently slighted as well as those who have been cited, but can easily protest, "but that's not at all what I did - and, anyway, you have the date wrong", I send a "mathematician's-all-saints'-day prayer" to you and ask pardon.

(1.10) DECOMPOSITION OF ARROWS AND FUNCTORS

(1.10.1) Let $f : A \longrightarrow B$ be an arrow in $\underset{\sim}{C}$. For each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, the fibre product in $(\underset{\sim}{ENS})$

$$\text{(1.10.1)} \qquad \begin{array}{ccc} A(T) \underset{f(T),f(T)}{\times} A(T) & \xrightarrow{\quad pr_2 \quad} & A(T) \\ \Big\downarrow pr_1 & & \Big\downarrow f(T) \\ A(T) & \xrightarrow[\quad f(T) \quad]{} & B(T) \end{array}$$

is simply the set of all couples $(x,y) \in A(T) \times A(T)$ such that $f(T)(x) = f(T)(g)$, or in other words, just the <u>graph of the equivalence relation associated with</u> $f(T)$.

If the functor defined by $\langle\langle\ T \rightsquigarrow A(T) \underset{f(T)f(T)}{\times} A(T)\ \rangle\rangle$ is representable, then the equivalence relation associated with $f(T)$ has a representation $(A \underset{f,f}{\times} A, pr_1, pr_2)$ in $\underset{\sim}{C}$ which will have the universal mapping property $\langle\langle$ given any couple of arrows $(x,y) : T \rightrightarrows A$ such that $fx = fy$, there exists a unique arrow $\theta : T \longrightarrow A \underset{f,f}{\times} A$ for which $pr_1\theta = x$ and $pr_2\theta = y$ $\rangle\rangle$.

If f is a monomorphism, for example, then for all $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $f(T) : A(T) \longrightarrow B(T)$ is an injection and the equivalence relation associated with $f(T)$ has as its graph simply the diagonal $\Delta_{A(T)}$ of $A(T) \times A(T)$. Now the functor defined by the diagonal is always representable, a representation being defined by the assignment $\langle\langle\ x \rightsquigarrow (x,x)\ \rangle\rangle$. Consequently, since the condition

《 the equivalence relation associated with $f(T)$ has as its graph the diagonal 》 is, in fact, equivalent to 《 $f(T)$ is an injection 》, we have as an immediate result

PROPOSITION (1.10.2). A necessary and sufficient condition that a morphism $f : A \to B$ in $\underset{\sim}{C}$ be a monomorphism is that the square

$$(1.10.2.1)\qquad \begin{array}{ccc} A & \xrightarrow{I_A} & A \\ {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

be cartesian.

One has an analogous result for epimorphisms and co-cartesian squares.

(1.10.3) If we carry an arrow $f : A \to B$ into $(\underset{\sim}{ENS})$ and decompose it in the usual fashion we obtain for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, the quotient set $A(T)/_{R(f(T)}$ supplied with its canonical surjection $A(T) \xrightarrow{\nu(T)} A(T)/_{Req(f(T)}$. The application $\nu(T)$ defines the set-theoretic cokernel of the couple of projections from the fibre product $A(T) \underset{f(T),f(t)}{\times} A(T)$, so that the diagram

$$A(T) \underset{f(T),f(T)}{\times} A(T) \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} A(T) \xrightarrow{\nu(T)} A(T)/_{A(T)\underset{f(T),f(T)}{\times}A(T)}$$

is (co-) exact in $(\underset{\sim}{ENS})$. Moreover, the equivalence relation associated with $\nu(T)$ has the same graph as that associated with $f(T)$, namely $A(T) \underset{f(T),f(T)}{\times} A(T)$.

Unfortunately, in order that the functor defined by $\langle\langle\ T \rightsquigarrow A(T)/A(T) \times_{f(T),f(T)} A(T)\ \rangle\rangle$ be representable, it is necessary and sufficient that one have a <u>retraction</u> $\mu : A \longrightarrow Q$ whose associated functorial equivalence relation has the same graph as that of $f(T)$ (and hence also of $\nu(T)$).

This is simply a consequence of the fact that for all T, $\nu(T)$ is surjective, hence, if representable, only with the aid of an arrow $\mu : A \longrightarrow Q$ for which there exists an arrow $s : Q \longrightarrow A$ such that $\mu s = I_Q$, i.e. a retraction.

This is clearly much too strong for most practical situations, in which one quite often always has some ability to "pass to the quotient", but seldom has that the resulting quotient map <u>admits a section</u>.

<u>DEFINITION</u> (1.10.4) [GROTHENDIECK TDTE II (1959)]
An arrow $f : A \longrightarrow B$ is called an <u>effective epimorphism</u> provided that the fibre product $(A_{f}\times_{f} A,\ 1_f,\ 2_f)$ exists in $\underset{\sim}{C}$ and f is a co-kernel of the couple $(1_f, 2_f)$.

(1.10.5) In the next chapter the notion of a <u>correspondence</u> <u>in $\underset{\sim}{C}$</u> will be investigated in detail. For the moment let us agree to call a <u>correspondence</u> (in $\underset{\sim}{C}$) <u>of an object</u> A <u>with itself</u> a representation of a functorial correspondence in <u>CAT</u> $(\underset{\sim}{C}^{o}, \underset{\sim}{(ENS)})$. In other words an object R supplied with a couple of arrows $(a,b) : R \rightrightarrows A$ such that for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, the application of sets $a(T) \times b(T) : R(T) \longrightarrow A(T) \times A(T)$ defined, as usual, by $\langle\langle\ x \rightsquigarrow (ax, bx)\ \rangle\rangle$ is injective. (Without this condition $(R,(a,b))$ will be called a pre-correspondence). The set $R(T)$ then has a bijection <u>onto the</u> graph of a correspondence of $A(T)$ with itself in $\underset{\sim}{(ENS)}$ for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$.

In order that the graph $a(T) \boxtimes b(T) \langle R(T) \rangle$ be the graph of a reflexive relation for each T, it is necessary and sufficient that the diagonal $\Delta_{A(T)}$ be contained in $a(T) \boxtimes b(T)\langle R(T)\rangle$, which, in the light of comments made in (1.10.1), simply amounts to the existence in $\underset{\sim}{C}$ of an arrow $s : A \longrightarrow R$ such that $a s = I_A$ and $bs = I_A$.

Under these conditions, the correspondence $R \rightrightarrows A$ will be said to be (the graph of) a reflexive relation in $\underset{\sim}{C}$.

PROPOSITION (1.10.5) If $R \rightrightarrows A$ is a reflexive relation in $\underset{\sim}{C}$ then the cokernel Cok (a,b) of the couple (a,b) exists iff the fibre-co-product $A \underset{a,b}{+} A$ exists, in which case they are isomorphic.

For any $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, consider the set $\mathrm{Ker}\,(T(a),T(b)) \subseteq T(A) \rightrightarrows T(R)$ consisting of those arrows $z : A \longrightarrow T$ such that $za = zb$, as well as the set $T(A) \underset{T(a),T(b)}{\times} T(A) \subseteq T(A) \times T(A)$ consisting of those couples (x,y) of arrows for which ⟨⟨xa = yb⟩⟩. By hypothesis (R,a,b) is reflexive hence there exists an arrow $s : A \longrightarrow R$ such that $as = I_A$ and $bs = I_A$. The relation ⟨⟨ $xa = yb$ ⟩⟩ always implies the relation ⟨⟨ $xas = ybs$ ⟩⟩, which is then in this case equivalent to ⟨⟨ $x = y$ ⟩⟩. The fibre product $T(A) \underset{T(a),T(b)}{\times} T(A)$ then must be contained in the diagonal and the application defined by ⟨⟨ $z \rightsquigarrow (z,z)$ ⟩⟩ will consequently define a bijection of Ker (T(a), T(b)) onto the set $T(A) \underset{T(a),T(b)}{\times} T(A)$ and thus force the mutual representability of the functors under consideration.

(1.10.6) The graph of an equivalence relation is certainly reflexive and since we have no reason to prefer one representation over another we are led to corresponding the effective epimorphisms of $\underset{\sim}{C}$ with those squares in $\underset{\sim}{C}$ of the form

(1.10.6.1)
$$\left\langle\!\left\langle \begin{tikzcd} A'' \arrow[r, "{}^2f"] \arrow[d, "{}^1f"'] & A' \arrow[d, "f"] \\ A' \arrow[r, "f"'] & A \end{tikzcd} \right\rangle\!\right\rangle$$

which are both <u>cartesian</u> and <u>co-cartesian</u> (i.e. which are <u>bi-cartesian</u>). We will usually "fold" such squares into the form

(1.10.6.2)
$$\langle\!\langle \; A'' \overset{{}^1f}{\underset{{}^2f}{\rightrightarrows}} A' \xrightarrow{f} A \; \rangle\!\rangle$$

and occasionally refer to the resulting diagram as a ⟨⟨ <u>short exact sequence terminating on</u> A ⟩⟩.

(1.10.7) The squares of $\underset{\sim}{C}$, as arrows in the arrow category $\underset{\sim}{C}^2$, themselves become the objects of the arrow category of $\underset{\sim}{C}^2$ which will denote by $\langle\langle\ \underset{\sim}{C}^3\ \rangle\rangle$.

The objects of $\underset{\sim}{C}^3$ are then the arrows of $\underset{\sim}{C}^2$, i.e. the commutative squares of $\underset{\sim}{C}$ such as

(1.10.7.1)
$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\Rightarrow"] & A' \arrow[d, "g"] \\ B \arrow[r, "b"'] & B' \end{tikzcd}$$
or (1.10.7.3)
$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "h"'] \arrow[dr, phantom, "\Rightarrow"] & C' \arrow[d, "i"] \\ D \arrow[r, "d"'] & D' \end{tikzcd}$$

while the arrows of $\underset{\sim}{C}^3$ are those "cubes" of $\underset{\sim}{C}$ which give rise to the same factorization of a morphism in $\underset{\sim}{C}^2$, i.e. members of

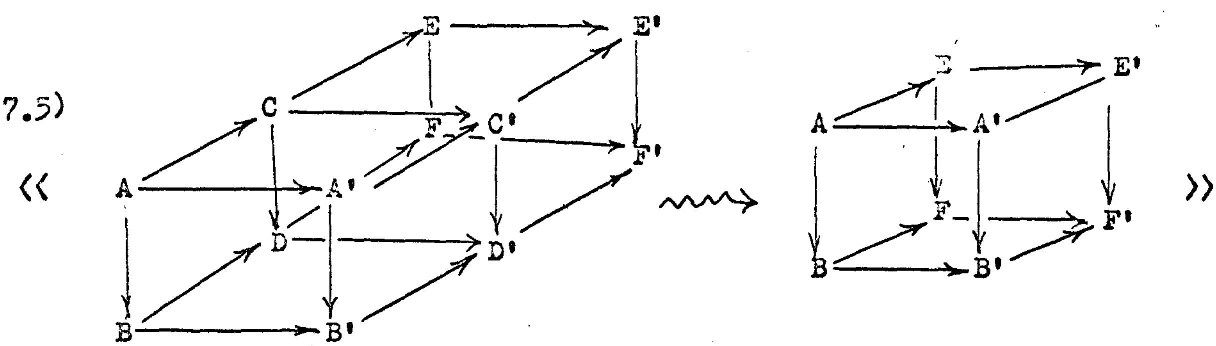
$$(\mathcal{Fl}(\underset{\sim}{C}^2) \underset{\sigma_1(C^2),\,\sigma_0(C^2)}{\times} \mathcal{Fl}(\underset{\sim}{C}^2)) \underset{\mu(C^2),\,\mu(C^2)}{\times} (\mathcal{Fl}(\underset{\sim}{C}^2) \underset{\sigma_1(C^2),\,\sigma_0(C^2)}{\times} \mathcal{Fl}(\underset{\sim}{C}^2)),$$ and any of

which would have as typical form a commutative cubic diagram such as

(1.10.7.4)
$$\begin{tikzcd} & C \arrow[rr, "c"] \arrow[dd, "h"] & & C' \arrow[dd, "i"] \\ A \arrow[ur, "\alpha"] \arrow[rr, "a"] \arrow[dd, "f"'] & & A' \arrow[ur, "\alpha'"] \arrow[dd, "g"] & \\ & D \arrow[rr, "d"] & & D' \\ B \arrow[ur, "\beta"] \arrow[rr, "b"'] & & B' \arrow[ur, "\beta'"'] & \end{tikzcd}$$

and be an arrow in $\mathbf{C}^3$ with source the square (1.10.7.1) and target the square (1.10.7.3). The multiplication in $\mathbf{C}^3$ is defined, as for the arrow category of any category, through the composition in $\mathbf{C}^{3-1}$ which amounts here again to "lateral adjunction" of cubes as in

(1.10.7.5)



REMARK (1.10.7.6) It is of course obvious that the actual depiction of such cubes need never be formally made. The propositions which occur in (1.4.2) et. seq. are entirely abstract in character and owe their validity only to the set-theoretic properties of fibre products and applications. The arrows $\mathfrak{Fl}(\mathbf{C}^2)$ of the arrow category $\mathbf{C}^2$ of any category $\mathbf{C}$ have been here defined as the members of the set

$$(\mathfrak{Fl}(\mathbf{C}) \underset{\sigma_1(c),\sigma_0(c)}{\times} \mathfrak{Fl}(\mathbf{C})) \underset{\mu(c),\mu(c)}{\times} (\mathfrak{Fl}(\mathbf{C}) \underset{\sigma_1(c)\rho_1(c)}{\times} \mathfrak{Fl}(\mathbf{C}))$$

and, as such, are simply "couples of couples". That one can, by definition, call the members of this set "commutative squares" and draw a little square picture which depicts a member of this set is certainly convenient, but is, of course, entirely irrelevant to the formal proof of the propositions.

Equality of arrows in $\mathbf{C}^2$ is then simple equality of couples. Thus it is well to remember in $\mathbf{C}^3$ that the arrows are quadruples of quadruples, relative to $\mathbf{C}$, and thus that in the cubic representation of such a gadjet, say (1.10.7.4), all of the side faces are commutative as arrows of $\mathbf{C}^2$ and, furthermore, must have the property that they define the same arrow under the composition in $\mathbf{C}^2$. Thus, in (1.10.7.4) the "four sides" define the same square

$$S = (f,\ \beta' b,\ \alpha' a,\ i) = (f,\ d\beta\ ,\ c\alpha\ ,\ i\ );$$

it follows that

$$\langle\langle \quad \beta' b = d\beta \quad \text{and} \quad \alpha' a = c\alpha \quad \rangle\rangle$$

or, in other words, that the <u>top and bottom faces are commutative as well</u>.

(1.10.8) $\mathbf{C}^3$, as usual, has two <u>functors</u> (source, and target) into $\mathbf{C}^2$ which correspond respectively in the diagram (1.10.7.5), to the direct projection of the "left hand face" (ABC D) and "right hand face" (A' B' D' C') into $\mathbf{C}^2$, $\mathbf{C}^3$ thus, by composition with the source and target functors of $\mathbf{C}^2$ into $\mathbf{C}$, has <u>four functors</u> into $\mathbf{C}$, as well its <u>functorial multiplication</u>, which corresponds in (1.10.7.5) to the "horizontal adjunction of cubes" to the primed (') face. This makes $\mathbf{C}^3$ into a ($\mathfrak{U}$ -CAT) - category and gives rise to the formal notion of a "natural transformation of natural transformations" defined simply as a <u>functor into the category</u> $\mathbf{C}^3$. By composition with source and target functors in $\mathbf{C}^2$ and the bi-furcation which occurs in $\mathbf{C}^2$, a typical such transformation of transformations would have as form, functors E, F, G, H, in $\mathbf{CAT}(\mathbf{T},\mathbf{C})$ and transformations

$\varphi : E \longrightarrow F$, $\quad \psi : G \longrightarrow H$, such that for any $f : A \longrightarrow B$ in $\mathbf{T}$, the cube with the "arrow functions $\xi(A) = (\xi_1(A), \xi_2(A))$"

(1.10.8.1)

$$\begin{tikzcd}
& E(B) \arrow[rr, "\xi(B)"] \arrow[dd] & & G(B) \arrow[dd, "\psi(B)"] \\
E(A) \arrow[ur, "E(f)"] \arrow[rr, "\xi_1(A)"] \arrow[dd, "\varphi(A)"] & & G(A) \arrow[ur, "G(f)"] \arrow[dd, "\psi(A)"] & \\
& F(B) \arrow[rr] & & H(B) \\
F(A) \arrow[ur] \arrow[rr, "\xi_2(A)"] & & H(A) \arrow[ur, "H(f)"] &
\end{tikzcd}$$

is commutative, or in other words, a morphism of the arrow category of $\mathbf{CAT}(\mathbf{T},\mathbf{C})$.

(1.10.9) We now wish to consider the subcategory $\mathbf{DEX}(\mathbf{C}^2)$ of $\mathbf{C}^3$ whose objects are the bi-cartesian squares associated with the effective epimorphisms of $\mathbf{C}$ and which arise as fibre products in $\mathbf{C}^2$. A typical arrow in this subcategory thus has the form of

(1.10.9.1)
$$
\begin{tikzcd}
 & B'' \arrow[rr, "2_g"] \arrow[dd] & & B' \arrow[dd, "g"] \\
A'' \arrow[ur, "0"] \arrow[rr, "2_f"] \arrow[dd, "1_f"'] & & A' \arrow[ur, "n"] \arrow[dd] & \\
 & B' \arrow[rr, "g"] & & B \\
A' \arrow[ur, "n"] \arrow[rr, "f"'] & & A \arrow[ur, "m"'] &
\end{tikzcd}
$$

which we will fold into the sequentially commutative diagram

(1.10.10.1)
$$
\begin{tikzcd}
S_A : \arrow[d] & A'' \arrow[r, shift left, "1_f"] \arrow[r, shift right, "2_f"'] \arrow[d, "0"] & A' \arrow[r, "f"] \arrow[d, "n"] & A \arrow[d, "m"] \\
S_B : & B'' \arrow[r, shift left, "1_g"] \arrow[r, shift right, "2_g"'] & B' \arrow[r, "g"'] & B
\end{tikzcd}
$$

or

(1.10.10.2)
$$
\begin{tikzcd}
(m,n,0): S_A \arrow[r] \arrow[d, phantom, ".."] & S_B \arrow[d, phantom, ".."] \\
A'' \arrow[r, "0"] \arrow[d, shift right, "2_f"'] \arrow[d, shift left, "1_f"] & B'' \arrow[d, shift right, "2_g"'] \arrow[d, shift left, "1_g"] \\
A' \arrow[r, "n"] \arrow[d, "f"'] & B' \arrow[d, "g"] \\
A \arrow[r, "m"'] & B
\end{tikzcd}
$$

and refer to as a <u>morphism</u> of the <u>exact sequence</u> (i.e. square) S_A into the exact sequence S_B. The composition of such "morphisms of sequences" is simply that of $\underset{\sim}{C}^3$ in the usual fashion.

(1.10.10) For the present now consider the subcategory $\underset{\sim}{DEX}(\underset{\sim}{C}^2)/_X$ of $\underset{\sim}{DEX}$ $(\underset{\sim}{C}^2)$ (consisting) of those arrows for which the target functor $\underset{\sim}{\sigma}^{t^\vee}_1 : \underset{\sim}{C}^3 \longrightarrow \underset{\sim}{C}^2$ is contained in some category $\underset{\sim}{C}/_X$ of arrows above an object X in $\underset{\sim}{C}$. The sequence representation is then as in

(1.10.11).

$$
\begin{tikzcd}
S' \arrow[d, "X"] & : A'' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A' \arrow[r] \arrow[d] & X \arrow[d, "I_X"] \\
S_X & : B'' \arrow[r, shift left] \arrow[r, shift right] & B' \arrow[r] & X
\end{tikzcd}
$$

<u>DEFINITION</u> (1.10.12) [GROTHENDIECK TDTE II (1959)]

An <u>effective epimorphism</u> $A'' \rightrightarrows A' \longrightarrow A$ is said to be <u>universal</u> provided that given any object $g : T \longrightarrow A$ in $\underset{\sim}{C}/_A$, the fibre product $(T' = T\,{}_g\!\times_f A', \; pr_1, \; pr_2)$ exists and pr_1 is an effective epimorphism. The morphism pr_1 will be said to have been obtained by <u>change of base</u> by g.

(1.10.12.1)

$$
\begin{tikzcd}
T'' \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "I''"] & A'' \arrow[d, shift left] \arrow[d, shift right] \\
T' \arrow[r] \arrow[d, "pr_1"'] \arrow[dr, phantom, "I"] & A' \arrow[d, "f"] \\
T \arrow[r, "g"'] & A
\end{tikzcd}
$$

<u>LEMMA</u> (1.10.13) If f is a universal effective epimorphism and f^* is a morphism obtained by change of base by g, then f^* is a universal effective epimorphism.

Let $h : U \longrightarrow T$ be an object in $\underset{\sim}{C}/_T$, then gh is an object in $\underset{\sim}{C}/_A$ for which the fibre product (U', pr_1^U, pr_2^U) of U with A' must exist. But $fpr_2 = gh\, pr_1$

(1.10.13.1)
$$
\begin{tikzcd}
U' \arrow[rr, bend left, "pr_2^U"] \arrow[r, dashed, "r"] \arrow[d, "pr_1^U"'] \arrow[dr, phantom, "\boxed{\mathrm{II}}"] & T' \arrow[r] \arrow[d, "f^*"] \arrow[dr, phantom, "\boxed{\mathrm{I}}"] & A' \arrow[d, "f"] \\
U \arrow[r, "h"] & T \arrow[r, "g"] & A
\end{tikzcd}
$$

and the square I of (1.10.13.1) is cartesian, hence there exists a unique arrow $r : U' \longrightarrow T'$ which makes the resulting square II commutative. As the composition of the two squares is cartesian II must be cartesian, which completes the lemma when one realizes that if a square such as I is cartesian then each of the separate squares which occur in II" as in (1.10.12.1) must also be cartesian as obtained from the fibre product in $\underset{\sim}{C}^2$.

<u>DEFINITION</u> (1.10.14) Let $f : B \longrightarrow A$ be an arrow in $\underset{\sim}{C}$ and $S \overset{d_0}{\underset{d_1}{\rightrightarrows}} A$ a couple of arrows in $\underset{\sim}{C}$ with target A and source S (i.e. a <u>pre-correspondence</u>* of A with itself). We will call the <u>inverse image</u> $(f^*S, \bar{d}_0, \bar{d}_1)$ <u>of</u> $(S, (d_0, d_1)$ <u>by</u> f, a representation in $\underset{\sim}{C}$,

if it exist, of the functor defined by the assignment

$$\langle\langle\ T \rightsquigarrow S(T) \underset{d_0(T)=d_1(T)\ ,\ f(T)\times f(T)}{\times} (B(T) \times B(T))\ \rangle\rangle.$$

$$(1.10.14.1) \qquad \begin{tikzcd}
S(T) \times B(T)\ (T) \arrow[r] \arrow[d] & B(T) \times B(T) \arrow[d, "f(T) \times f(T)"] \\
S(T) \arrow[r, "d_0(T)\quad d_1(T)"'] & A(T) \times A(T)
\end{tikzcd}$$

N.B. The obvious generalization of this concept will be discussed in Chapter II.

<u>PROPOSITION</u> (1.10.15) In order that the inverse image of a couple $S \underset{d_0}{\overset{d_1}{\rightrightarrows}} A$ exist, it is sufficient that any combination of the fibre products which occur in the diagram

$$(1.10.15.1) \qquad \begin{tikzcd}
f^*S = B \times S \underset{S}{\times} B \arrow[r, "f_0^{**}"] \arrow[d, "f_1^{**}"', "f"] & S \underset{d_1,f}{\times} B \arrow[r, "d_1^*"] \arrow[d, "f_1"] & B \arrow[d, "f"] \\
B \underset{f,d_0}{\times} S \arrow[r, "f_0^*"] \arrow[d, "d_0^*"] & S \arrow[r, "d_1"] \arrow[d, "d_0"'] & A \\
B \arrow[r, "f"] & A &
\end{tikzcd}$$

and lead to its corner, exist in $\underset{\sim}{C}$.

The lifted edge maps define the representation in an obvious fashion (together with the "middle projection" into S). (In the language of Chapter II the inverse image here is simply the precorrespondence $\hat{f}^{-1} \circ S \circ f$ (   ).

<u>COROLLARY</u> (1.10.16) If (S, d_0, d_1) is an equivalence relation in $\underset{\sim}{C}$, (i.e. define a representation of the graph of an equivalence relation in $(\underset{\sim}{ENS})$) for which the inverse image by f exist, then $(f^*S, \bar{d}_0, \bar{d}_1)$ is an equivalence relation in $\underset{\sim}{C}$.

For each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$ the square

$$\begin{tikzcd} S(T) \arrow[r, "d_1(T)"] \arrow[d, "d_0(T)"'] & A(T) \arrow[d, "\nu(T)"] \\ A(T) \arrow[r, "\nu(T)"] & A(T)/S(T) \end{tikzcd}$$

of sets and applications is cartesian (with $\nu(T)$ the quotient map). The evaluation at T of (1.10.15.1) is thus also cartesian and thus f^*S defines a representation in $\underset{\sim}{C}$ of the equivalence relation whose graph is $((\nu(T) \cdot f(T))^{-1} \circ (\nu(T) \cdot f(T)) \subseteq B(T) \times B(T)$.

<u>PROPOSITION</u> (1.10.17) (after GABRIEL (1964))

Let $\mathcal{S} = (S \overset{d_0}{\underset{d_1}{\rightrightarrows}} A)$ be a reflexive pre-correspondence in $\underset{\sim}{C}$, (i.e. there exists an $s : A \longrightarrow S$ such that $d_0 s = I_A = d_1 s$) and $f : B \longrightarrow A$ a universal effective epimorphism. Then: 1° the inverse image $f^*\mathcal{S} = (f^*S, d_0, d_1)$ of $\mathcal{S} = (B, (d_0, d_1)$ by f exists; and 2° the co-kernel $(Cok(d_0, d_1), \nu)$ of the couple (d_0, d_1) exists if and only if the co-kernel of the couple (d_0, d_1) exists, in which case they are isomorphic with $\nu f : B \xrightarrow{f} A \xrightarrow{\nu} Cok(d_0, d_1)$ defining the isomorphism.

The existence of $f^*\mathcal{S}$ follows immediately from (1.10.15) where we see from the universality of f that $\tilde{f}(= f_0^{*} f_1^{**} = f_1^{*} f_0^{**})$ obtained by as composition of epimorphisms is again an epimorphism.

We thus have the sequentially commutative diagram

(1.10.17.1)

$$\begin{tikzcd} & B\times B \arrow[dl, "\Delta"'] \arrow[d, shift right, "1_f"'] \arrow[d, shift left, "2_f"] \\ f^*S \arrow[r, shift left, "\bar{f}_0"] \arrow[r, shift right, "\bar{f}_1"'] \arrow[d, "\tilde{f}"'] & B \arrow[d, "f"] \\ S \arrow[r, shift left, "d_o"] \arrow[r, shift right, "d_1"'] & A \end{tikzcd}$$

where Δ is the unique arrow arising from the couple ($sf1_f, (1_f, 2_f)$) which is such that

$$\langle\langle\ (d_o\ sf\ 1_f,\ d_1\ sf\ 1_f) = (f1_f\ \ f2_f)\ \rangle\rangle$$

since $d_o sf1_f = f1_f$ and $d_1 sf1_f = d_1 sf2_f = f2_f$.

Consequently, for each $T \in \mathfrak{A}(\underset{\sim}{C})$, we have the sequentially commutative diagram

(1.10.17.2)

$$\begin{tikzcd} \mathrm{Ker}(T(d_o)T(d_1)) \arrow[r] \arrow[d, "\tilde{T}(f)"] & T(A) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "T(f)"] & T(S) \arrow[d, "T(\tilde{f})"] \\ \mathrm{ker}(T(f),T(f_1)) \arrow[r] & T(B) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & T(f^*S) \arrow[dl, "T(\Delta)"] \\ & T(B\times B) & \end{tikzcd}$$

in $(\underset{\sim}{ENS})$ with both $T(f)$ and $T(\check{f})$ injective, and $\widetilde{\widetilde{T(f)}}$ as the injection defined by $T(f)$ by restriction of the graph of $T(f)$.

Now let $x : B \longrightarrow T$ be such that $xf_o = xf_1$ and hence be such that $\langle\langle x\bar{f}_o\Delta = x\bar{f}_1\Delta \rangle\rangle$ and consequently $\langle\langle x1_f = x2_f \rangle\rangle$. The effectivity of f then ensures the existence of an arrow $\theta : A \longrightarrow T$ such that $\theta f = x$. The injectivity of $T(f)$ then ensures that the square (D) is cartesian by (1.2.23(b)) which then completes the proof by demonstrating that $T(f)$ is a bijection for each $T \in \underset{\sim}{\mathcal{U}}(\underset{\sim}{C})$.

<u>COROLLARY</u> (1.10.18) The composition of a couple of composable universal effective epimorphisms is a universal effective epimorphism.

Consider the diagram

(1.10.18.)

$$
\begin{tikzcd}
 & & B \times B \arrow[dll, "\Delta"'] \arrow[d, shift right, "1_f"'] \arrow[d, shift left, "2_f"] & \\
f^*(A \times_Q A) \arrow[rr, shift left, "d_o^*"] \arrow[rr, shift right, "d_1^*"'] \arrow[d, "\bar{f}"'] & & B \arrow[d, "f"] \arrow[dr, "gf"] & \\
A \times_Q A \arrow[rr, shift left, "1_g"] \arrow[rr, shift right, "2_g"'] & & A \arrow[r, "g"'] & Q
\end{tikzcd}
$$

with arrows constructed as in (1.10.17), which exists in $\underset{\sim}{C}$ since f is a universal effective epimorphism. (1.10.17) then asserts that (B,gf) defines a representation of the cokernel of (d_o^*, d_1^*). We claim that $f^*(A \times_Q A) \overset{d_o^*}{\underset{d_1^*}{\rightrightarrows}} B$ is the fibre product of $gf : B \longrightarrow Q$ with itself and immediately establish the claim by referring to the cartesian diagram used in the construction (1.10.15) of $f^*(R_{\mathcal{E}q}(g))$

(1.10.18.2)

$$\begin{tikzcd}
f^*(A \times_Q A) \arrow[r] \arrow[d] & A \times_Q A \times_A B \arrow[r] \arrow[d, dash] & B \arrow[d, dash, "f"] \\
B \times_B A \times_Q A \arrow[r] \arrow[d] & A \times_Q A \arrow[r] \arrow[d] & A \arrow[d, "g"] \\
B \arrow[r, "f"'] & A \arrow[r, "g"'] & Q
\end{tikzcd}$$

with the square in the bottom corner also cartesian by definition of g.

REMARK (1.10.13.3) This last shows that an effective epimorphism preceded by a universal effective epimorphism is at least an effective epimorphism.

If, in addition g is also universal, then let $h : T \longrightarrow Q$ be an arbitrary arrow in $\mathcal{C}/_Q$. We then have the diagram

(1.10.18.4)

$$\begin{tikzcd}
T^{**} \arrow[r, "f^*"] \arrow[d] & T^* \arrow[r, "g^*"] \arrow[d] & T \arrow[d, "h"] \\
B \arrow[r, "f"] & A \arrow[r, "g"] & Q
\end{tikzcd}$$

consisting of the two cartesian squares whose existence is insured by the universality existence requirements improved on g and f, but g^* is then effective and f^* is universal effective by (1.10.13). The preceding part of the proof shows that g^*f^* is effective which was to have been shown.

II THEORY OF CORRESPONDENCES

2.0 PRE-CORRESPONDENCES IN A $\underset{\sim}{\mathcal{U}}$-CATEGORY

DEFINITION 2.0.1) Let $\underset{\sim}{C}$ be a $\underset{\sim}{\mathcal{U}}$-category. A quadruplet

$$\mathcal{R} = (\underset{\sim}{Gr}(\mathcal{R}), (\underset{\sim}{str}_1(\mathcal{R}), \underset{\sim}{str}_2(\mathcal{R})), \underset{\sim}{\sigma}(\mathcal{R}), \underset{\sim}{\tau}(\mathcal{R}))$$

consisting of objects $\underset{\sim}{Gr}(\mathcal{R})$, $\underset{\sim}{\sigma}(\mathcal{R})$ and $\underset{\sim}{\tau}(\mathcal{R})$ of $\underset{\sim}{C}$ and arrows $\underset{\sim}{str}_1(\mathcal{R}) : \underset{\sim}{Gr}(\mathcal{R}) \longrightarrow \underset{\sim}{\sigma}(\mathcal{R})$ and $\underset{\sim}{str}_2(\mathcal{R}) : \underset{\sim}{Gr}(\mathcal{R}) \longrightarrow \underset{\sim}{\tau}(\mathcal{R})$ of $\underset{\sim}{C}$ is called a <u>pre-correspondence of</u> $A = \underset{\sim}{\sigma}(\mathcal{R})$ with $B = \underset{\sim}{\tau}(\mathcal{R})$ <u>with</u> (<u>pre</u>-) <u>graph</u> $G = \underset{\sim}{Gr}(\mathcal{R})$ and <u>first</u> and <u>second structural</u> arrows (or <u>projections</u>) $d_o = \underset{\sim}{str}_1(\mathcal{R})$ and $d_1 = \underset{\sim}{str}_2(\mathcal{R})$. $\underset{\sim}{\sigma}(\mathcal{R})$ is then called the <u>object of departure</u> (or <u>source</u>) of $\mathcal{R}$ and $\underset{\sim}{\tau}(\mathcal{R})$ the <u>object</u> of <u>arrival</u> (or <u>target</u>) of $\mathcal{R}$.

We will write ⟨⟨ $\mathcal{R} = (G^{d_o}_{d_1} : A \rightharpoonup B)$ ⟩⟩ or, if no confusion is possible ⟨⟨ $\mathcal{R} : A \rightharpoonup B$ ⟩⟩, as an abbreviation for ⟨⟨ $\mathcal{R}$ is a pre-correspondence with source A, target B, and graph G ⟩⟩ and will use as standard, a diagram of the form

$$\langle\langle \begin{tikzcd} G \arrow[r, "d_1"] \arrow[d, "d_o"'] & B \\ A \arrow[ur, "\mathcal{R}"'] & \end{tikzcd} \rangle\rangle \quad \text{or} \quad \langle\langle \begin{tikzcd} & G \arrow[dl, "d_o"'] \arrow[dr, "d_1"] & \\ \mathcal{R} : A \arrow[rr, harpoon] & & B \end{tikzcd} \rangle\rangle$$

(usually with ⟨⟨ $\xrightharpoonup{\mathcal{R}}$ ⟩⟩ ommitted) to indicate a pre-correspondence of A with B with graph G and structural arrows (d_o, d_1).

Any couple $(d_o, d_1) \in \mathcal{H}\ell(\underset{\sim}{C})_{\sigma,\iota(t)} \times \mathcal{H}\ell(\underset{\sim}{C})_{\tilde{\sigma},\iota(t)}$ then determines a unique pre-correspondence (in an evident fashion). We will shortly identify the precorrespondences of $\underset{\sim}{C}$ with the members of this set in a satisfactory fashion.

(2.0.2) For each $T \in \mathrm{Ob}(\underset{\sim}{C})$, a precorrespondence $\mathcal{R} = (R, (d_o, d_1), A, B)$ defines an application

$$\mathcal{R}(T) = d_o(T) \boxtimes d_1(T) : R(T) \longrightarrow A(T) \times B(T)$$

with image $\mathcal{R}_*(T) \subseteq A(T) \times B(T)$, in $(\underset{\sim}{ENS})$ by the standard assignment $\langle\langle\, f \rightsquigarrow (d_o f, d_1 f) \,\rangle\rangle$. If, for all $T \in \mathrm{Ob}(\underset{\sim}{C})$, the application $\mathcal{R}(T)$ is injective, we will call $\mathcal{R}$ a correspondence (of A with B) in $\underset{\sim}{C}$. Thus if A x B exists in $\underset{\sim}{C}$ the precorrespondences of A with B become identifiable with objects above A x B, and the correspondences of A with B with those monomorphisms in $\underset{\sim}{C}$ whose target is A x B. In any case, for a correspondence $\mathcal{R}$, $\mathcal{R}(T)$ defines a bijection of R(T) onto the graph of a correspondence of A(T) with B(T) for each $T \in \mathrm{Ob}(\underset{\sim}{C})$.

<u>DEFINITION</u> (2.0.3) If $\mathcal{R} = (R, (d_o, d_1), A, B)$ is a precorrespondence in $\underset{\sim}{C}$, we define the <u>converse</u> (or <u>inverse</u> or <u>reciprocal</u>) of $\mathcal{R}$ to be that precorrespondence $\overset{-1}{\mathcal{R}} = (R, (d_1, d_o), B, A)$.

For any precorrespondence $\mathcal{R}$, one has that $(\overset{-1}{\mathcal{R}})^{-1} = \mathcal{R}$ and that for each $T \in \mathrm{Ob}(C)$, $\overset{-1}{\mathcal{R}}(T) : R(T) \longrightarrow B(T) \times A(T)$, is equal to $\underset{\sim}{\kappa}(T) \circ R(T)$, where $\underset{\sim}{\kappa}(T) : A(T) \times B(T) \xrightarrow{\sim} B(T) \times A(T)$ is the "commutativity bijection" defined by $\langle\langle\, (a,b) \rightsquigarrow (b,a) \,\rangle\rangle$.

Thus, if $\mathcal{R}$ is a correspondence, then $\overset{-1}{\mathcal{R}}(T)$ defines a bijection of

R(T) onto the graph

$$\widehat{\mathcal{R}_*(T)}^{-1} = \{ (a,b) \mid (b,a) \in \mathcal{R}(T) \quad \langle R(T)\rangle \subseteq B(T) \times A(T)\},$$

converse of $\mathcal{R}_*(T) \subseteq A(T) \times B(T)$.

<u>DEFINTION</u> (2.0.4) If $\mathcal{R} = (R, (d_o, d_1), A, B)$ and $\mathcal{S} = (S, (s_o, s_1), A, B)$ are pre-correspondences in $\underset{\sim}{C}$, each with the same source and same target, we shall say that $\mathcal{R}$ is <u>equivalent</u> to $\mathcal{S}$, and write ⟨⟨ $\mathcal{R} \cong \mathcal{S}$ ⟩⟩, provided there exists an isomorphism ξ of the graph of $\mathcal{R}$ with the graph of $\mathcal{S}$ such that $s_o\xi = d_o$ and $s_1\xi = d_1$.

(2.0.4.1)
$$\begin{tikzcd}
R \arrow[rr, "s_1"] \arrow[dr, "\xi"] \arrow[dd, "s_o"'] & & B \arrow[dr, "I_B"] & \\
& S \arrow[rr, "d_1"] \arrow[dd, "d_o"] & & B \\
A \arrow[dr, "I_A"'] & & & \\
& A & &
\end{tikzcd}$$

This implies, in particular, the relation (2.0.4.2)
⟨⟨ for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, $d_o(T) \boxtimes d_1(T)\ \langle R(T)\rangle = s_o(T) \boxtimes s_1(T) \langle S(T)\rangle (\subseteq A(T) \times B(T))$ ⟩⟩.
By itself, this last condition is only equivalent to

(2.0.4.3) ⟨⟨there exists arrows $\xi : R \longrightarrow S$ and $\mu : S \longrightarrow R$ such that
$d_o\xi = s_o$, $d_1\xi = s_1$, $s_o\mu = d_o$, and $s_1\mu = d_1$ ⟩⟩

However, if $\mathcal{R}$ and $\mathcal{S}$ are correspondences, there can exist at most one such arrow, ξ, which then must also be a monomorphism; thus for correspondences, (2.0.4.2) or its equivalent (2.0.4.3), is a necessary and sufficient condition for equivalence of correspondences.

<u>DEFINITION</u> (2.0.5) If $\mathcal{R} = (R, (d_0, d_1), A, B)$ and $\mathcal{S} = (S, (s_0, s_1), A, B)$ are pre-correspondences of A with B, We will say that $\mathcal{R}$ is <u>contained in</u> $\mathcal{S}$, and write $\langle\langle\ \mathcal{R} \subsetneq \mathcal{S}\ \rangle\rangle$ provided there exists a monomorphism $\mu : \mathcal{R} \hookrightarrow \mathcal{S}$ in $\underset{\sim}{C}$ which commutes with the structural arrows of $\mathcal{R}$.

If $\mathcal{R} \subsetneq \mathcal{S}$ (or in fact with the mere existence of a commuting morphism μ), one has the containment

$$(2.0.5.1)\ \langle\langle \text{ for each } T \in \underset{\sim}{Ob}(\underset{\sim}{C}),\ d_0(T) \pi d_1(T) \langle R(T) \rangle \subseteq s_0(T) \pi s_1(T) \langle S(T) \rangle (\subseteq A(T) x B(T))\ \rangle\rangle.$$

If $\mathcal{R} : A \longrightarrow B$ is a correspondence, the condition (2.0.5.1.) is equivalent to $\langle\langle\ \mathcal{R} \subsetneq \mathcal{S}\ \rangle\rangle$.

The relation $\langle\mathcal{R} \subsetneq \mathcal{S}\rangle$ clearly induces a pre-order relation on the set of isomorphism classes of pre-correspondences, and an order relation on the set of isomorphism classes of correspondences of A with B. We will refer to the relation $\langle\langle\ \mathcal{R} \subsetneq \mathcal{S}\ \rangle\rangle$ as the <u>natural</u> (pre-) <u>ordering of the</u> (pre-) <u>correspondences of</u> A <u>with</u> B.

<u>DEFINITION</u> (2.0.6) If $\mathcal{R} = (R, (d_0, d_1), A, B)$ and $\mathcal{S} = (S, (s_0, s_1), B, C)$ are pre-correspondences in $\underset{\sim}{C}$ for which the fibre product $(R \underset{d_1, s_0}{x} S, pr_1, pr_2)$ of $\mathcal{R}$ with $\mathcal{S}$ (over B) exists in $\underset{\sim}{C}$, we will say that $\mathcal{R}$ is (<u>fibre</u>-or <u>cartesian</u>-) <u>composable with</u> $\mathcal{S}$, with the (unprojected) <u>fibre-composition</u> of $\mathcal{R}$ <u>with</u> $\mathcal{S}$ defined by

$$\langle\langle\quad \mathcal{S} \circ \mathcal{R} = (R \underset{d_1, s_0}{x} S,\ \ (d_0 pr_1, s_1 pr_2),\ A,\ C)\ \ \rangle\rangle$$

$$(2.0.6.1)\qquad \begin{tikzcd} R \underset{B}{\times} S \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & S \arrow[r, "s_1"] \arrow[d, "s_o"] & C \\ R \arrow[r] \arrow[d, "d_o"'] & B & \\ A & & \end{tikzcd}$$

The resulting "composition" depends on the choice of representation of the fibre product and here is only presumed defined up to isomorphism (i.e. equivalence) of pre-correspondences.

<u>PROPOSITION</u> (2.0.7) (ASSOCIATIVITY OF FIBRE-COMPOSITION) Let $\mathcal{R} : A \longrightarrow B$, $\mathcal{S} : B \longrightarrow C$, and $\mathcal{T} : C \longrightarrow D$ be pre-correspondences in $\underset{\sim}{C}$ such that $\mathcal{S} \hat{\circ} \mathcal{R} : A \longrightarrow C$ and $\mathcal{T} \hat{\circ} \mathcal{S} : B \longrightarrow D$ be defined in $\underset{\sim}{C}$. Then $\mathcal{T} \hat{\circ} (\mathcal{S} \hat{\circ} \mathcal{R}) : A \longrightarrow D$ is defined if and only if $(\mathcal{T} \hat{\circ} \mathcal{S}) \hat{\circ} \mathcal{R}$ is defined, and in which case $\mathcal{T} \hat{\circ} (\mathcal{S} \hat{\circ} \mathcal{R}) \cong (\mathcal{T} \hat{\circ} \mathcal{S}) \hat{\circ} \mathcal{R}$.

Canonical associativity of fibre products gives the desired result, since for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, the applications

$$\langle\langle (R \underset{d_1,s_o}{\times} S(T)) \times T(T) \xrightarrow{\sim} (R(T) \underset{d_1(T),s_o(T)}{\times} S(T)) \underset{s_1(T)pr_2,t_o(T)}{\times} T(T) \overset{\alpha(T)}{\xrightarrow{\sim}} R(T) \underset{d_1(T),s_o(T)pr_1}{\times} (S(T) \underset{s_1(T),t_o(T)}{\times} T(T) \xrightarrow{\sim} R(T) \underset{d_1(T),s_o(T)s_1,t_o}{\times} (S \times T(T)) \rangle\rangle$$

are bijective, with $\alpha(T)$ simply the bijection defined by $\langle\langle ((x,y),z) \rightsquigarrow (x,(y,z)) \rangle\rangle$. Then the functor defined by

$$\langle\langle T \rightsquigarrow (R \underset{d_1,s_o}{\times} S(T)) \underset{s_1pr_1,t_o(T)}{\times} T(T) \rangle\rangle$$

is representable iff that defined by

$$\langle\langle T \rightsquigarrow R(T) \underset{d_1(T),s_o(T)}{\times} (S \underset{s_1,t_o}{\times} T(T) \rangle\rangle$$

is representable, in which case the representatives are isomorphic, in fact, isomorphic to $(R \times_{d_1,s_0} S) \times_{pr_2,pr_1} (S \times_{s_1,t_0} T) \xrightarrow{\sim} R \times_{d_1,s_0,s_1,t_0} S \times T)$.

$$\begin{tikzcd}
R \times_B S \times_C T \arrow[r] \arrow[d] & S \times_C T \arrow[r] \arrow[d] & T \arrow[r, "t_1"] \arrow[d, "t_0"] & D \\
R \times_B S \arrow[r, "pr_2"] \arrow[d] & S \arrow[r, "s_1"] \arrow[d, "s_0"] & C & (\mathcal{T}) \\
R \arrow[r, "d_1"] \arrow[d, "d_0"'] & B & (\mathcal{S}) & (\mathcal{T}\cdot\mathcal{S}) \\
A & (\mathcal{R}) & (\mathcal{S}\cdot\mathcal{R}) & (\mathcal{T}\cdot\mathcal{S}\cdot\mathcal{R})
\end{tikzcd} \tag{2.0.7.1}$$

DEFINITION (2.0.8) For any object A in $\underset{\sim}{C}$ one has the identity correspondence $\underline{\mathcal{J}d}(A) = (A, (I_A, I_A), A, A) : A \longrightarrow A$.

$\underline{\mathcal{J}d}(A)$ has the property that given any pre-correspondences $\mathcal{R} : A \longrightarrow B$ and $\mathcal{S} : B \longrightarrow A$, $\mathcal{R} \circ \underline{\mathcal{J}d}(A) : A \longrightarrow B$ and $\underline{\mathcal{J}d}(A) \cdot \mathcal{S} : B \longrightarrow A$ are both defined and $\underline{\mathcal{J}d}(A) \cdot \mathcal{S} \cong \mathcal{S}$; $\mathcal{R} \cdot \underline{\mathcal{J}d}(A) = \mathcal{R}$.

(2.0.9) The preceding observations show that the pre-correspondences of C form a "partial category" or define a bifunctor in the naive sense toward which the preceding sequence of lemmas head in a natural fashion. We will make this precise shortly and investigate the actual structure involved here in detail; but first let us continue in a naive fashion and see where it leads.

(2.1) NATURAL CORRESPONDENCES OF SET VALUED FUNCTORS (I)

DEFINITION (2.1.1) Let $\underset{\sim}{C}$ be a $\mathfrak{U}$-category and $\mathrm{CAT}(\underset{\sim}{C}^{op}, (\mathrm{ENS}))$ the set of all contravariant functors from $\underset{\sim}{C}$ into the category ENS-$\mathfrak{U}$. For $F, G \in \mathrm{CAT}(\underset{\sim}{C}^{op}, (\mathrm{ENS}))$, a natural correspondence of F with G is a triple $\Gamma = (\chi, F, G)$ where $\chi = (\chi(T))_{T \in \mathrm{Ob}(\underset{\sim}{C})}$ is a family of sets satisfying the following conditions:

1° for each $T \in \mathrm{Ob}(\underset{\sim}{C})$, $\chi(T) \subseteq F(T) \times G(T)$;

2° for each $f \in \mathrm{Fl}(\underset{\sim}{C})$, $f \in T(U)$, $F(f) \times G(f) \langle \chi(T) \rangle \subseteq \chi(U)$.

If $\Gamma = (\chi, F, G)$ is such a natural correspondence, we will call χ, the graph, F the source, and G the target of Γ, and abbreviate the relation ⟨⟨ Γ is a natural correspondence of F with G, with graph χ ⟩⟩ by ⟨⟨ $\Gamma = (\chi : F \longrightarrow G)$ ⟩⟩. Moreover, for a fixed couple (F,G) of functors, we will identify the correspondence with its graph.

REMARK (2.1.2) The effect of conditions 1° and 2° is to make the assignment

$$\langle\langle\ T \rightsquigarrow \chi(T),\ f \rightsquigarrow F(f) \times G(f) \mid \chi(T)\ \rangle\rangle$$

define a functor from $\underset{\sim}{C}^{op}$ into (ENS); so that we could just as well define a natural correspondence of F with G as the product functor supplied with a distinguished subfunctor.

EXAMPLE (2.1.3) A natural transformation of F into G is simply a natural correspondence of F with G whose graph is that of an application of F(T) into G(T), for each $T \in \mathrm{Ob}(\underset{\sim}{C})$.

(2.1.4) The set $\mathbf{CAT}\,(\mathbf{C}^{(op)}, (\mathbf{ENS}))$ is supplied with a natural category structure if we take as its set of arrows natural <u>correspondences</u> of functors and define composition of natural correspondences by

$$\chi_2 \circ \chi_1 : F \longrightarrow H,\ \chi_2 \circ \chi_1(T) = \chi_2(T) \circ \chi_1(T),\ T \in \mathrm{Ob}(\mathbf{C}),\ \text{for}$$

$$\chi_1 : F \longrightarrow G,\ \chi_2 : G \longrightarrow H.$$

This category will be denoted by $\mathbf{CAT}^{\wedge}(\mathbf{C}^{(op)}, (\mathbf{ENS}))$.

We will denote by « $\mathcal{H}om\langle F,G\rangle$ » the set of all natural correspondences of F with G, and by « $\mathcal{H}om(F,G)$ » the subset of $\mathcal{H}om\langle F,G\rangle$ consisting of the natural transformations of F into G. Thus the category of functors and natural transformations becomes a subcategory of the category of functors and natural correspondences of functors.

We order the set $\mathcal{H}om\langle F,G\rangle$ by means of the relation

$$\langle\langle \text{ for each } T \in \mathrm{Ob}(\mathbf{C}),\ \chi_1(T) \subseteq \chi_2(T) \ \rangle\rangle,$$

which we will denote by « $\chi_1 \leq \chi_2$ ». The properties of functions then give most of the Boolean operations on the set $\mathcal{H}om\langle F,G\rangle$, with the notable exception of relative complementation, whose stability is not ensured by 1° and 2°.

(2.1.5) Before this line is further developed, we extend the Yoneda-Grothendieck evaluation in this context. To this end let X be an object in $\underset{\sim}{C}$, h_X the contravariant "hom"-functor defined by

$$\langle\langle\ h_X(T) = X(T),\ h_X(f) = X(f),\ T \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}),\ f \in T(U)\ \rangle\rangle\ ;$$

and $\chi : h_X \longrightarrow F$ a natural correspondence of h_X with some functor $F \in \underset{\sim}{CAT}\,(\underset{\sim}{C}^{op}\ ,\ (\underset{\sim}{ENS}))$.

By definition, for each $T \in \underset{\sim}{\mathfrak{Ob}}(C)$, one has $\chi(T) \subseteq X(T) x F(T)$, and in particular $\chi(X) \subseteq X(X)\ x\ F(X)$. The cut of the graph $\chi(X)$ at $\{I_X\} \subseteq X(X)$ is the subset (possibly empty) $\chi(T)\left\langle\ \{\ I_X\ \}\ \right\rangle \subseteq F(X)$, consisting of those $\xi \in F(X)$ such that $(I_X, \xi) \in \chi(X)$, and hence the assignment $\langle\langle\ X \rightsquigarrow \chi(X)\left\langle\{I_X\}\right\rangle\ \ \rangle\rangle$ defines a canonical application

(2.1.5.1) $$\boxed{\mathcal{H}\ :\ \underset{\sim}{\mathcal{H}om}\langle h_X, F\rangle \longrightarrow \mathfrak{P}(F(X)),}$$

such that if $\chi_1 \leqslant \chi_2$, then $\mathcal{H}(\chi_1) \subseteq \mathcal{H}(\chi_2)$.

Inversely, let $S \in \mathfrak{P}(F(X))$, i.e. $S \subseteq F(X)$. For each $T \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C})$ and each $f \in X(T)$, $F(f) : F(X) — F(T)$. Consequently, one may define a subset $\chi_S(T)$ of $X(T)\ x\ F(T)$ by

$$\chi_S(T) = \left\{\ (f,y)\ \middle|\ \ (\exists\ s \in S)(y = F(f)(s))\right\}$$

PROPOSITION (2.1.6) $\chi_S = (\chi_S(T))_{T \in Ob(C)}$ is the graph of a natural correspondence of h_X with F.

If $S = \emptyset$, the proposition is trivial, otherwise if $g \in T(U)$, then

$$X(g) \times F(g)\,((f,y)) = (X(g)(f), F(g)(y)) = (fg, F(g)(F(f)(s)))$$

$$= (fg, F(fg)(s))$$

which is an element of $\chi_S(U)$.

The assignment « $S \rightsquigarrow \chi_S$ » thus defines a canonical application,

(2.1.6.1) $$\boxed{\varphi : \mathcal{P}(F(X)) \longrightarrow \mathcal{H}om\langle h_X, F\rangle,}$$

such that if $S_1 \subseteq S_2$, then $\varphi(S_1) \leqslant \varphi(S_2)$.

PROPOSITION (2.1.7) The application $\mathcal{H}$ is a retraction with φ as a distinguished section, so that $\mathcal{P}(F(X))$ is identifiable with the set of equivalence classes of $\mathcal{H}om\langle h_X, F\rangle$ under the relation « χ_1 and χ_2 have the same cut at X ».

The calculation of $\mathcal{H}\circ\varphi\,(S)$ for $S \subseteq F(X)$ gives

$$\mathcal{H}\varphi\,(S) = \chi_S(X)\left\langle \{I_X\} \right\rangle = \left\{ s \in F(X) \,\middle|\, (I_X, F(I_X)(s)) \in \chi_S(X) \right\} = S.$$

i.e. $\mathcal{H}\circ\varphi = I_{\mathcal{P}(F(X))}$.

<u>THEOREM</u> (2.1.8) The couple of applications

$$(\mathcal{H}, \varphi) \ : \ \underset{\sim\sim}{\mathcal{H}om}\langle h_X, F\rangle \rightleftarrows \mathfrak{P}(F(X))$$

defines an (<u>interior</u>) Galois correspondence, i.e.

1° $\chi_1 \leqslant \chi_2$ implies $\mathcal{H}(\chi_1) \subseteq \mathcal{H}(\chi_2)$ and $S_1 \subseteq S_2$ implies $\varphi(S_1) \leqslant \varphi(S_2)$;

2° $\mathcal{H} \circ \varphi(S) = S$ and $\varphi \circ \mathcal{H}(\chi) \subseteq \chi$.

All that remains is the calculation of $\varphi \circ \mathcal{H}(\chi)$ for $\chi : h_X \longrightarrow F$: $\varphi \circ \mathcal{H}(\chi) = \varphi(\chi(X)\langle\{I_X\}\rangle)$. For $T \in \underset{\sim}{\mathcal{O}b}(\underset{\sim}{C})$, let

$$\psi(T) = \varphi \circ \mathcal{H}(\chi)(T) = \{(f,y) \mid (\exists s)(s \in \chi(X)\langle\{I_X\}\rangle$$

$$\text{and } y = F(f)(s)\} \ .$$

The relation ⟨⟨ $s \in \chi(X)\langle\{I_X\}\rangle$ ⟩⟩ is equivalent to ⟨⟨ $(I_X, s) \in \chi(X)$ ⟩⟩ and χ is natural, so that for each $f \in T(X)$, $X(f) \times F(f)\langle\chi(X)\rangle \subseteq \chi(T)$. Thus if $(f,y) \in \psi(T)$, then $(f,y) = (X(f)(I_X), F(f)(s)) = X(f) \times F(f)(I_X, s) \in \chi(T)$. In other words,

$$\psi(T) = \varphi \circ \mathcal{H}(\chi)(T) \subseteq \chi(T) \text{ for each } T \in \underset{\sim\sim}{\mathcal{O}b}(\underset{\sim}{C});$$

which is, by definition, the relation ⟨⟨ $\varphi \circ \mathcal{H}(\chi) \leqslant \chi$ ⟩⟩.

<u>COROLLARY</u> (2.1.9) [Yoneda Grothendieck Evaluation]

The restriction of the application $\mathcal{H} : \underset{\sim\sim}{\mathcal{H}om}\langle h_X, F\rangle \longrightarrow \mathfrak{P}(F(X))$ to the set $\underset{\sim}{\mathcal{H}om}(h_X,F)$ defines a bijection of $\underset{\sim}{\mathcal{H}om}(h_X,F)$ onto the set of one-element subsets of $\mathfrak{P}(F(X))$ and hence onto the set $F(X)$. The reciprocal of this bijection is the application φ composed with the canonical injection $\langle\langle\ \xi \rightsquigarrow \{\xi\}\ \rangle\rangle$ with its target restricted to $\underset{\sim}{\mathcal{H}om}(h_X,F)$.

(2.1.9.1)
$$\begin{array}{ccccc}
\underset{\sim}{\mathcal{H}om}(h_X,F) & \xrightarrow{\mathcal{H}} & \mathfrak{P}(F(X)) & \xrightarrow{\varphi} & \underset{\sim}{\mathcal{H}om}(h_X,F) \\
\uparrow & & \uparrow & & \uparrow \\
\underset{\sim}{\mathcal{H}om}(h_X F) & \xrightarrow{\sim} & F(X) & \xrightarrow{\sim} & \underset{\sim}{\mathcal{H}om}(h_X,F)
\end{array}$$

It suffices to remark that if χ is a functional correspondence then it is non-empty and its cut at X is a one element set. Conversely, if we start with a one element set, then the correspondence defined by φ is single valued and everywhere defined.

<u>COROLLARY</u> (2.1.10) The assignment

$$\langle\langle\quad X \rightsquigarrow h_X,\ f \rightsquigarrow (f(T))_{T\in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})}\quad\rangle\rangle$$

defines an embedding of the category $\underset{\sim}{C}$ into the category $\underset{\sim}{\mathrm{CAT}}^{\wedge}(\underset{\sim}{C}^{op}, (\underset{\sim}{\mathrm{ENS}}))$ and an equivalence of $\underset{\sim}{C}$ with a full subcategory of $\underset{\sim}{\mathrm{CAT}}(\underset{\sim}{C}^{op}, (\underset{\sim}{\mathrm{ENS}}))(\subseteq \underset{\sim}{\mathrm{CAT}}(\underset{\sim}{C}^{\circ}(\underset{\sim}{\mathrm{ENS}})))$.

If $\chi : F \longrightarrow G$ is a natural correspondence, then χ^{-1} defined by

$$\chi^{-1}(T) = \{\ (x,y)\ \mid\ (y,x) \in \chi(T)\} \subseteq\ G(T) \times F(T),\ T \in \underset{\sim}{\mathrm{Ob}}(\underset{\sim}{C})$$

is a natural correspondence of G with F and the application defined by $\langle\langle\ \chi \rightsquigarrow \chi^{-1}\quad\rangle\rangle$

is a bijection of $\mathcal{H}om\langle F,G\rangle$ onto $\mathcal{H}om\langle G,F\rangle$. Moreover, if $\chi : G'$ $\quad$ G, then χ defines, by composition, an <u>application</u> $\mathcal{H}om\langle \chi, F\rangle$ of $\mathcal{H}om\langle F,G\rangle$ into $\mathcal{H}om\langle G',F\rangle$ by $\langle\langle\ \theta \rightsquigarrow \theta\circ\chi\ \rangle\rangle$, and $\chi^{-1} : G \longrightarrow G'$ defines an application, $\mathcal{H}om\langle F, \chi^{-1}\rangle$, of $\mathcal{H}om\langle F,G\rangle$ into $\mathcal{H}om\langle F,G'\rangle$, by $\langle\langle\ \psi \rightsquigarrow \chi^{-1}\cdot\psi\ \rangle\rangle$. Since $(\chi^{-1}\circ\psi)^{-1} = \psi^{-1}\circ(\chi^{-1})^{-1} = \psi^{-1}\circ\chi$ and $(\chi^{-1}\circ\theta^{-1})^{-1} = (\theta^{-1})^{-1}\circ(\chi^{-1})^{-1} = \theta\circ\chi$, we have that the following diagram commutes in both directions:

$$\begin{tikzcd}
\mathcal{H}om\langle F, \chi^{-1}\rangle : \mathcal{H}om\langle F,G\rangle \arrow[r] \arrow[d, leftrightarrow, "\wr" swap, "-1"] & \mathcal{H}om\langle F,G'\rangle \arrow[d, leftrightarrow, "\wr" swap, "-1"] \\
\mathcal{H}om\langle \chi, F\rangle : \mathcal{H}om\langle G,F\rangle \arrow[r] & \mathcal{H}om\langle G',F\rangle
\end{tikzcd} \tag{2.1.10.1}$$

<u>COROLLARY</u> (2.1.11) The assignment

$$\langle\langle\ X \rightsquigarrow h_X,\ f \rightsquigarrow (f^{-1}(T))_{T\in \mathcal{O}b(\underset{\sim}{C})}\ \rangle\rangle$$

defines an embedding of the category $\underset{\sim}{C}^{(op)}$ into <u>CAT</u> $\check{}(\underset{\sim}{C}^{(op)}, (\underset{\sim}{ENS}))$ and an equivalence of $\underset{\sim}{C}^{(op)}$ with a subcategory whose arrows are those natural correspondences χ such that χ^{-1} is a function.

This corollary, elementary though it may be, does show that concepts defined in $\underset{\sim}{C}^{(op)}$ may be given an interpretation in $\underset{\sim}{CAT}(\underset{\sim}{C}^{(op)}, (\underset{\sim}{ENS}))$. For example, the requirement that an arrow $\alpha : X$ $\quad$ Y be an <u>epimorphism</u> in $\underset{\sim}{C}$ is trivially translated into $\langle\langle$ for any couple (χ_1, χ_2) of natural correspondences from a functor h_Z into h_Y such that $\alpha^{-1}\cdot\chi_1 = \alpha^{-1}\cdot\chi_2$, if χ_1^{-1} and

χ_2^{-1} are both functional, then $\chi_1 = \chi_2$. This follows since ⟨⟨ $\alpha^{-1} \cdot \chi_1 = \alpha^{-1} \circ \chi_2$ ⟩⟩ is equivalent to ⟨⟨ $\chi_1^{-1} \circ \alpha = \chi_2^{-1} \circ \alpha$ ⟩⟩ and the requirement that χ_1^{-1} and χ_2^{-1} be functional makes them both arise from the unique arrows $g_1 = \chi_1^{-1} \langle \{I_Y\} \rangle$, $g_2 = \chi_2^{-1} \langle \{I_Y\} \rangle$, so that all that we have really said is that $g_1\alpha = g_2\alpha$ implies that $g_1 = g_2$. The interest of this would be considerably diminished were it not for the fact that the various varieties of ⟨⟨ universal ⟩⟩ epimorphisms defined by Grothendieck apparently have similar interpretations in terms of ⟨⟨ cancellation requirements ⟩⟩ less restrictive than those of the above example.

(2.2) REPRESENTABILITY OF CORRESPONDENCES - RELATIVE REPRESENTABILITY

LEMMA (2.2.1) $\mathcal{R} = (R,(d_0,d_1),A,B)$ is a correspondence iff for each

$$T \in \underset{\sim}{\mathfrak{Ob}}(\underset{\sim}{C}),\quad d_0(T) \boxtimes d_1(T) : R(T) \longrightarrow A(T) \times B(T)$$

defines a bijection of $R(T)$ onto the natural correspondence

$$d_1(T) \circ d_0^{-1}(T) \quad : \quad A(T) \longrightarrow B(T).$$

The lemma is immediate since, in ($\underset{\sim}{ENS}$), one has the chain of equivalences

$$\langle\langle\ (x,y) \in d_1 \cdot d_0^{-1} \Leftrightarrow (\exists k \in R)((x,k) \in d_0^{-1} \text{ and}$$

$$(k,y) \in d_1) \Leftrightarrow (\exists\, k \in R)\ (d_0 \boxtimes d_1(k) = (x,y))\ \rangle\rangle.$$

Hence, for each $T \in \underset{\sim}{\mathcal{U}}(\underset{\sim}{C})$, $d_o(T) \boxtimes d_1(T) \langle R(T) \rangle = d_1 d_o^{-1}(T)$;
and if $\mathcal{R}$ is a correspondence, then $R(T) \xrightarrow{\sim} d_1\ d_o^{-1}(T)$.

(2.2.1.2)

$$\begin{tikzcd}
R \arrow[r, "d_1"] \arrow[d, "d_o"'] \arrow[dr, phantom, "(D)"] & B \arrow[d, "u_o"] \\
A \arrow[r, "u_1"'] & Q
\end{tikzcd}$$

LEMMA (2.2.2) In order that the square (D) of (2.2.1.2) be cartesian in $\underset{\sim}{C}$ it is necessary and sufficient that the couple (d_o, d_1) define a representation of the natural correspondence $h_{u_o}^{-1} \cdot h_{u_1} : h_A \longrightarrow h_B$.

This again is immediate, for in $(\underset{\sim}{ENS})$, we have that for each $T \in \underset{\sim}{\mathcal{U}}(\underset{\sim}{C})$ $A(T) \underset{u_1(T), u_o(T)}{\times} B(T) = u_o^{-1}(T) \cdot u_1(T)$.

$(x,y) \in A(T) \underset{u_1(T), u_o(T)}{\times} B(T) \Leftrightarrow \langle\langle\ u_1 x = u_o y = c\ \rangle\rangle \Leftrightarrow$ (there exists c) such that

$(x,c) \in u_1(T)$ and $(c,y) \in u_o^{-1}(T) \Leftrightarrow (x,y) \in u_o^{-1} \cdot u_1(T)$, so that the functor $\langle\langle\ T \rightsquigarrow A(T) \underset{u_1(T), u_o(T)}{\times} B(T)\ \rangle\rangle$ is representable if and only if the functor $\langle\langle\ T \rightsquigarrow u_o^{-1}(T) \cdot u_1(T)\ \rangle\rangle$ is representable.

Conjoining these results, we have the

PROPOSITION (2.2.3) (D) is cartesian in $\underset{\sim}{C}$ if and only if $(R, (d_o, d_1))$ is a correspondence and $d_1 \circ d_o^{-1} = u_o^{-1} \circ u_1 (\xrightarrow{\sim} (R, (d_o, d_1)))$.

DEFINITION (2.2.3) [GROTHENDIECK] Let $u : F \longrightarrow G$ be a natural transformation in $\underline{\underline{CAT}}$ $(\underset{\sim}{C}^{opp}$, $(\underset{\sim}{ENS}))$. Call u

<u>relatively representable</u> (<u>or F representable above G</u>) provided that given any $X \in \underset{\sim}{Ob}(C)$ and any natural transformation $\eta_X : h_X \longrightarrow G$, the correspondence $u^{-1} \circ \eta_X : h_X \longrightarrow F$ has a representable graph (in $\underset{\sim}{C}$).

<u>REMARK</u> (2.2.3.1) The above definition is not identical to that given in GROTHENDIECK (1960). The two are essentially the same, however, and rather than proving their equivalence, we prefer to simply develop one of the Grothendieck representability criteria in keeping with the local terminology; if anything, the proofs are simpler. Suffice it for us to remark that given any couple of objects (T,u), (U,v) in $\underset{\sim}{C}/_S$,

$$\mathrm{Hom}_S\ ((T,u),\ (U,v)) = u^{-1} \cdot v(T) \left\langle \{ I_T \} \right\rangle$$

and
$$\underset{\sim}{Ob}(\underset{\sim}{C}/_S) = \bigcup_{T \in \underset{\sim}{Ob}(\underset{\sim}{C})} S(T) .$$

In view of the remarks concerning cartesian squares, $u : F \longrightarrow G$ is relatively representable provided that the functor

$$\langle\langle T \rightsquigarrow h_X(T) \underset{G(T)}{\times} F(T) \rangle\rangle$$

is representable for any choice of $X \in \underset{\sim}{Ob}(\underset{\sim}{C})$ and transformation

$$\eta_X : h_X \longrightarrow G.$$

<u>DEFINITION</u> (2.2.4) Let $z_o : A \longrightarrow B$, $z_1 : A' \longrightarrow B'$ be a couple of arrows in $\underset{\sim}{C}$.

Define <u>the inverse image of the precorrespondence</u> $\mathfrak{X} = (X, (d_o, d_1), B, B')$ <u>by the couple</u> (z_o, z_1) as a representation, if it exist, of the functor $\langle\langle T \rightsquigarrow X(T) \underset{d_o \boxtimes d_1}{\times} A(T) \underset{z_o \times z_1(T)}{\times} A'(T) \rangle\rangle$ in $\underset{\sim}{C}$, i.e. so that for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, the square

(2.2.4.1)

$$\begin{tikzcd}
(z_o, z_1)^*(\ast)(T) \arrow[r] \arrow[d] & A(T) \times A'(T) \arrow[d, "z_o(T) \times z_1'(T)"] \\
X(T) \arrow[r, "d_o(T) \quad d_1(T)"] & B(T) \times B(T)
\end{tikzcd}$$

is cartesian.

(2.2.5) It is not difficult to see that in order that the inverse image by (z_o, z_1) exist for <u>any</u> pre-correspondence, and hence equivalently that $h_{z_o} \times h_{z_1}$ be relatively representable, it sufficies that $\underset{\sim}{C}$ admit fibre products. The representation then being given by $(A \times X_X \times A', z_o z_1^{**}, (d_o^* z_1^{**}, d_1^* z_o^{**}))$.

(2.2.5.1)

$$\begin{tikzcd}
A \times A' \arrow[r] \arrow[d, "z_1^{\vee}"'] & X \times_B A \arrow[r] \arrow[d, "z_1^*"'] & A' \arrow[d, "z_1"] \\
A \times X \arrow[r] \arrow[d, "d_o^*"'] & X \arrow[r, "d_1"] \arrow[d, "d_o"] & B' \\
A \arrow[r, "z_o"] & B &
\end{tikzcd}$$

N.B. This in no way requires that $h_B \times h_{B'}$, or $h_A \times h_{A'}$ be representable.

<u>LEMMA</u> (2.2.6) Suppose that F and G be representable by $(Y', \xi_{Y'})$ and (Y, ξ_Y), respectively, and let $\eta_X : h_X \longrightarrow G$ be a natural transformation. If $f : Y' \longrightarrow Y$ is the arrow $\xi^{-1}_{Y'}(Y') \cdot u(Y')\langle \xi_{Y'} \rangle$ $Y(X')$ and $g : X \longrightarrow Y$ is the arrow $\xi_Y(X) \cdot \eta_X(X)\ \langle I_X \rangle \in Y(X)$, then $u^{-1} \cdot \eta_X$ is representable iff $f^{-1} \cdot g$ is representable, i.e. iff the fibre product $X \underset{g,f}{\times} Y'$ exist in $\underset{\sim}{C}$.

For each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$, the square

$$(2.2.6.1)\qquad \begin{array}{ccc} \xi_{Y'}(T) : F(T) & \longrightarrow & Y'(T) \\ \Big\downarrow u(T) & & \Big\downarrow f(T) \\ y(T) : G(T) & \longrightarrow & Y(T) \end{array}$$

is cartesian, for $\xi_Y \cdot u = f \cdot \xi_{Y'}$ implies that

$$\xi_Y^{-1} \circ \xi_Y \cdot u \cdot \xi_{Y'}^{-1} = \xi_Y^{-1} \cdot f \circ \xi_{Y'} \cdot \xi_{Y'}^{-1}$$

and hence that $u \cdot \xi_Y^{-1} = \xi_Y^{-1} \cdot f$, or equivalently, $\xi_{Y'} \cdot u^{-1} = f^{-1} \cdot \xi_Y$. Moreover, $\xi_Y \cdot \eta_X = g$, and thus if u^{-1}, η_Y is representable, then $\xi_{Y'}\, u^{-1} \cdot \eta_Y$ ($= f^{-1} \cdot \xi_Y \cdot \eta_Y = f^{-1} \circ g$) is representable.

<u>COROLLARY</u> (2.2.7) In order that a natural transformation of representable functors be relatively representable it is necessary that for any X in $\underset{\sim}{C}$ and any $g : X \longrightarrow Y$ the fibre product $f^{-1} \circ g : X \longrightarrow Y'$ exist in $\underset{\sim}{C}$ (i.e. $f : Y' \longrightarrow Y$ be squarable in $\underset{\sim}{C}/_Y$).

<u>LEMMA</u> (2.2.8) If G is representable and u relatively representable, then F is representable and the arrow defined by u is squareable in ($\underset{\sim}{C}/_Y$).

Let (Y, ξ_Y), define the representation of G, so that $\xi_Y(T) : Y(T) \longrightarrow G(T)$ is a bijection for each $T \in \underset{\sim}{Ob}(\underset{\sim}{C})$. Now u relatively representable implies that $u^{-1}(T) \cdot \xi_Y(T) : Y(T) \longrightarrow F(T)$ is representable and one has the square D(T)

$$(2.2.8.1) \qquad \begin{tikzcd} & Y'(T) \arrow[r, "p_1(T)"] \arrow[d, "p_o(T)"'] \arrow[dr, phantom, "D(T)"] & F(T) \arrow[d, "u(T)"] \\ \xi_Y(T) : & Y(T) \arrow[r] & G(T) \end{tikzcd}$$

cartesian with its left hand corner a representable functor and ξ_Y (T) a bijection for each $T \in \underset{\sim}{Ob}(C)$. But D(T) cartesian, and ξ_Y an isomorphism, implies that p_1 be an isomorphism, i.e. G is representable through the bijection $Y'(T) \xrightarrow{p_1(T)} F(T)$.

We have at this point reproved the first elementary Grothendieck representability criterion, which can be stated here as

THEOREM (2.2.9). Let $u : F \longrightarrow G$ be a functorial morphism in $\underline{\underline{CAT}}$ (C^{op} , (ENS)), and suppose that G be representable. Then in order that u be relatively representable, it is necessary and sufficient that F be representable and that the arrow in $\underset{\sim}{C}$ defined by u be squareable.

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