EXTENSIONS OF A PARTIALLY ORDERED SET

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SCOPE AND CONTENTS: In this thesis we introduce the concept of a dense extension of a partially ordered set and study some of the properties of the resulting class of extensions. In particular we study the dense distributive extensions, dense Boolean extensions and dense meet continuous extensions of distributive, Boolean and meet continuous lattices respectively.

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INTRODUCTION

The theory of extensions of partially ordered sets started with MacNeille's celebrated paper [13], in which certain special extensions of partially ordered sets were introduced and which contains in particular the well known MacNeille completion, generalizing Dedekind's famous construction of the reals from the rationals. While the MacNeille completion preserves desirable properties of lattices in certain cases (for example the property of being Boolean) it fails to do so in other cases (for example in the case of distributive lattices). This led to the attempt to extend and systematize MacNeille's ideas [1, 3, 15]. In order to get adequate results these authors restricted themselves to the case of join (meet) dense extensions. The theory thus obtained lacks (contrary to the theory of partially ordered sets as a whole) the property of being self dual. One starting point of this thesis was the attempt to develop a self dual extension theory including the join and meet dense extensions as special cases. The class of "dense" extensions introduced here has these properties.

Another starting point of this thesis was the observation that equations in a partially ordered set (also including those involving infinitely many variables) are in general not preserved in arbitrary extensions of it. Thus Funayama [8] gave an example of a distributive lattice whose MacNeille completion is not modular answering a question posed by MacNeille. Also Dilworth and McLaughlin [7] gave an example of an infinitely meet distributive lattice whose MacNeille completion is not modular. In addition P. Crawley [5] gave an example of a distributive lattice which has no complete completely faithful extensions (E, w) where E is a modular lattice. We have tried here to find necessary and sufficient conditions for the existence of extensions which preserve equations holding in the underlying set. The classes of partially ordered sets for which we have obtained results in this direction include distributive, Boolean and meet continuous lattices. A brief synopsis of the material in this thesis follows.

In Chapter 0 we collect together the basic definitions and results which we utilize in the ensuing chapters.

In Chapter I we introduce the concept of a dense extension of a partially ordered set and observe that this class of extensions include the classes of join dense and meet dense extensions. We obtain a complete survey of dense extensions by proving that each dense extension is equivalent to an unique canonical dense extension. Further we prove the equivalence theorem for dense extensions, namely, any two dense extensions which are injectively smaller than each other are equivalent. In addition we give conditions under which a given canonical dense extension is complete, respectively

$\mathfrak{m}$-$\mathfrak{n}$ faithful. Finally we introduce the concepts of dense (join dense, meet dense) kernels of an arbitrary extension and study the properties these inherit from the given extension.

In Chapter II we study dense extensions of certain special lattices. We establish that for each infinite cardinal m the join dense extension provided by the m complete lower ends of a meet continuous lattice is meet continuous. In particular we deduce Crawley's result [5] that the injectively largest completely faithful join dense extension of a meet continuous lattice is a meet continuous extension and in addition we establish that up to equivalence this is the only such join dense complete completely faithful meet continuous extension. Further we obtain necessary and sufficient conditions for a complete completely faithful canonical join dense extension of a distributive lattice to be a distributive extension. We also show in this chapter that a Dedekind-MacNeille extension of a Boolean lattice is up to equivalence the only finitely faithful Boolean extension which is a join and meet completion. We further give an example of an infinitely meet distributive lattice L whose Dedekind-MacNeille extension is not meet continuous and observe that L has no complete finitely meet faithful meet dense meet continuous extensions.

The various dense kernels of an extension introduced in Chapter I cannot be suitably described in the category whose objects are all extensions of a partially ordered set and whose maps are order preserving homomorphisms. In order to overcome this difficulty we consider in Chapter III several more restricted categories whose objects are extensions of a given partially ordered set. We obtain for example that the injective join dense kernel of an extension is up to equivalence the injectively largest join dense extension which is injectively smaller than the given extension in the category whose objects are extensions and whose maps are join preserving order homomorphisms or join dense. We also study the relation between the injective and projective orderings in suitable categories of complete dense extensions. Finally in this chapter we obtain a categorical characterization of the injectively largest completely faithful join dense extension of a meet continuous lattice.

CHAPTER 0

PRELIMINARIES.

This chapter is a collection of all the basic definitions and results which will be needed in the ensuing chapters.

1. Lattices and Homomorphisms.

A general reference for the definitions and the results in this section is Birkhoff [2].

A partially ordered set is a pair $(P, \leq)$ where P is a set and $\leq$ is a binary relation in P which satisfies

(P1) For all x, $x \leq x$ (Reflexive)

(P2) If $x \leq y$ and $y \leq x$ then $x = y$ (Antisymmetric)

(P3) If $x \leq y$ and $y \leq z$ then $x \leq z$ (Transitive)

If $(P, \leq)$ and $(Q, \leq)$ are any two partially ordered sets a mapping f from P into Q is called an order homomorphism if $x \leq y$ implies $f(x) \leq f(y)$. If further for an order homomorphism f we have $f(x) \leq f(y)$ implies $x \leq y$ then f is called an order isomorphism.

Given a relation $\leq$ on X one obtains a relation $\overset{*}{\leq}$ on X called its converse by requiring $x \overset{*}{\leq} y$ if and only if $y \leq x$. It follows by inspection of (P1) - (P3) that the following Duality Principle holds, namely, the converse of any partial ordering is a partial ordering.

One can manufacture from given partially ordered sets new ones. For example given $(P, \leq)$ we obtain its dual $(P, \overset{*}{\leq})$ where $\overset{*}{\leq}$ is the converse of the given relation on P. Further if $(P, \leq)$ and $(Q, \leq)$ are given we obtain two new partially ordered sets called respectively their ordinal sum and ordinal product as follows:

(1) Put $P \oplus Q = P \cup Q$ where we assume without loss of generality that P, Q are disjoint. We partially order $P \oplus Q$ by requiring that $a \leq b$ retains its original meaning if a, b are both in P or in Q and that $a \leq b$ holds for all a in P, b in Q.

(2) $P \circ Q = \{(p, q) / p \in P, q \in Q\}$. We partially order $P \circ Q$ by placing $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 < p_2$ or $p_1 = p_2$ and $q_1 \leq q_2$.

A partially ordered set in which for every pair of elements x, y we have either $x \leq y$ or $y \leq x$ is called totally ordered or a chain. It is clear that the ordinal sum and ordinal product of chains are again chains. A chain C is called dense-in-itself if given $a < b$ in C there exists a c in C satisfying $a < c < b$.

Let R be the chain of all rational numbers.

Theorem 1: Any countable chain is isomorphic with a subchain of R. Any countable chain which is dense-in-itself is isomorphic with either R, $R \oplus 1$, $1 \oplus R$, or $1 \oplus R \oplus 1$. In particular the chains $R \oplus R$, $R \circ R$ are both isomorphic to R.

A lattice is a partially ordered set $(P, \leq)$ any two of whose elements have a greatest lower bound or "meet" $x \wedge y$, and least upper bound or "join" $x \vee y$. A sublattice of a lattice L is a subset which contains with any two elements their join and their meet. We remark that a subset of a lattice may be a lattice with respect to the ordering on L without being a sublattice of L. An element o of a lattice L is called a <u>zero element</u> if $o \leq x$ for each x in L. Similarly an element e of L is called an <u>unit element</u> if $x \leq e$ for each x in L.

Let L be a lattice with zero element 0 and unit element e. An element y in L is called the <u>complement</u> of an element x in L if and only if $x \wedge y = o$ and $x \vee y = e$. L is called <u>complemented</u> if all of its elements have complements.

A lattice L is called <u>distributive</u> if and only if it satisfies one of the following <u>equivalent</u> conditions

(L1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(L2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(L3) $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$

Criteria for establishing a lattice to be distributive is given in the following theorem a proof of which may be found in Curry [6].

<u>Theorem 2</u>: Let L be a lattice.

The following conditions are equivalent:

(1) L is distributive

(2) $a \wedge (b \vee c) \leq (a \wedge b) \vee c$

(3) $a \wedge b \leq c$, $a \leq b \vee c$ implies $a \leq c$.

(4) $a \wedge b \leq b \wedge c$, $a \vee b \leq b \vee c$ implies $a \leq c$.

A <u>complemented distributive</u> lattice is called a <u>Boolean lattice</u>. In general, if an element of a lattice has a complement this need not be unique. However, in a <u>distributive lattice an element has at most one complement</u>. In particular every element of a Boolean lattice has an unique complement.

<u>Theorem 3:</u> Every Boolean lattice L is infinitely distributive, that is, for each a in L and each subset X of L if $\bigvee^{L} X$ exists then $\bigvee a \wedge X$ exists and $a \wedge \bigvee^{L} X = \bigvee^{L} a \wedge X$ and if $\bigwedge_{L} X$ exists then $\bigwedge_{L} a \vee X$ exists and $a \vee \bigwedge_{L} X = \bigwedge_{L} a \vee X$.

A partially ordered set in which every subset has a meet and a join is called a <u>complete lattice.</u>

<u>Theorem 4:</u> Let P be a partially ordered set. The following conditions are equivalent:

(1) Every subset of P has a join.

(2) Every subset of P has a meet.

Hence a partially ordered set is a complete lattice if and only if one of (1), (2) hold.

Let L be a complete lattice and R a sublattice of L. R will be called a $\left\{\begin{matrix}\text{meet -}\\ \text{join -}\end{matrix}\right\}$ <u>complete sublattice of L</u> if and only if $\left\{\begin{matrix}\bigwedge_{R} H\\ \bigvee^{R} H\end{matrix}\right\}$ exists for every subset H of R and coincides with $\left\{\begin{matrix}\bigwedge_{L} H\\ \bigvee^{L} H\end{matrix}\right\}$.

R is called a <u>complete sublattice</u> of L if and only if it is both a <u>meet and join complete sublattice of L.</u> We remark that a subset of a complete lattice may be a complete lattice with respect to the given ordering without being a complete sublattice. Examples of complete lattices are furnished by <u>closure systems</u>. A <u>closure system</u> is a pair $(E, \mathfrak{A})$ where E is a set and $\mathfrak{A}$ is a collection of subsets of E such that $\mathcal{L} \subseteq \mathfrak{A}$ implies $\bigcap \mathcal{L} \in \mathfrak{A}$. We remark that $E \in \mathfrak{A}$ since E is the intersection of the empty collection.

Theorem 5: If $(E, \mathfrak{A})$ is a closure system then $\mathfrak{A}$ partially ordered by inclusion is a complete lattice in which an arbitrary meet is set intersection but in general arbitrary joins is not set union.

A mapping f from a lattice R into a lattice S is called a meet homomorphism if for any x, y in R we have $f(x \wedge y) = f(x) \wedge f(y)$; it is called a join homomorphism if $f(x \vee y) = f(x) \vee f(y)$ for every x, y in R. A meet and a join homomorphism is called a lattice homomorphism. Further a mapping f from R into S is called a join complete lattice homomorphism if and only if (1) f is a meet homomorphism (2) for every subset H of R such that $\bigvee^{R} H$ exists we have $\bigvee^{S} f(H)$ exists and $f(\bigvee^{R} H) = \bigvee^{S} f(H)$ and it is called a meet complete lattice homomorphism if (3) it is a join homomorphism and (4) for every subset H of R such that $\bigwedge_{R} H$ exists $\bigwedge_{S} f(H)$ exists and $f(\bigwedge_{R} H) = \bigwedge_{S} f(H)$.

2. Categories and Functors.

In this section we give the definition of a category and some related concepts. A general reference for the definitions given here is MacLane [12]

A category $\mathcal{C}$ consists of a class of objects and with each pair X, Y of objects a set H(X, Y) called the set of maps $f: X \to Y$ such that for any three objects X, Y, Z in $\mathcal{C}$ there is given a mapping $H(X, Y) \times H(Y, Z) \to H(X, Z)$ denoted by $(f.g) \rightsquigarrow g.f$ which satisfies (1) $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ then $h.(g.f) = (h.g).f$. (2) For each object X in $\mathcal{C}$ there exists a map e_X in H(X, X) such that $e_X.f = f$ for all $f \in H(Y,X)$ and $f.e_X = f$ for all f in H(X, Y).

An element in $H(A, A)$ for any A in $\mathcal{C}$ is called an <u>identity map</u>. The objects of $\mathcal{C}$ are in one to one correspondence $A \to H(A, A)$ with the set of <u>identities</u>.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A function T which maps the objects of $\mathcal{C}$ into the objects of $\mathcal{D}$ and, in addition, assigns to each map f in $\mathcal{C}$ a map $T(f)$ in $\mathcal{D}$ is called a <u>covariant functor</u> from $\mathcal{C}$ into $\mathcal{D}$ if the following conditions are satisfied:

(1) If f is in $H(A, B)$ then $T(f)$ is in $H(T(A), T(B))$ for any A, B in $\mathcal{C}$.

(2) If e_A is in $H(A, A)$ then $T(e_A) = e_{T(A)}$ for any A in $\mathcal{C}$

(3) If f is in $H(A, B)$ g in $H(B, C)$ for any A, B, C in $\mathcal{C}$ then $T(g \cdot f) = T(g)\, T(f)$.

Further T is called a <u>contravariant functor</u> from $\mathcal{C}$ to $\mathcal{D}$ if the above conditions are replaced by

(1^1) If f is in $H(A, B)$ then $T(f)$ is in $H(T(B), T(A))$ for any A, B in $\mathcal{C}$.

(2^1) If e_A is in $H(A, A)$ then $T(e_A) = e_{T(A)}$ for any A in $\mathcal{C}$.

(3^1) If f is in $H(A, B)$, g in $H(B, C)$ for any A, B, C in $\mathcal{C}$ then $T(g \cdot f) = T(f)\, T(g)$.

If T is a functor from $\mathcal{C}$ to $\mathcal{D}$ and S is a functor from $\mathcal{D}$ into $\mathcal{E}$ then they may be composed in the obvious manner to form a functor ST from $\mathcal{C}$ into $\mathcal{E}$. If T, S have same (opposite) variance then ST is covariant (contravariant). In view of property (2) above we see that a functor T is completely determined by the function T defined for maps only. Thus a covariant functor is essentially a homomorphism of the maps of $\mathcal{C}$ to the maps of $\mathcal{D}$ subject to the condition that the identities be mapped to the identities. One functor that always exists is the identity functor $I_{\mathcal{C}}$ defined from $\mathcal{C}$ into $\mathcal{C}$ which keeps each object and map of $\mathcal{C}$ fixed.

A category $\mathcal{C}$ will be called equivalent to a category $\mathcal{D}$ if and only if there exists a covariant (contravariant) functor S from $\mathcal{C}$ into $\mathcal{D}$ and a covariant (contravariant) functor T from $\mathcal{D}$ into $\mathcal{C}$ such that $TS = I_{\mathcal{C}}$ and $ST = I_{\mathcal{D}}$. A map f in $\mathcal{C}$ from P to Q is called a monomorphism if and only if u, v are any two maps from R into P with $f \circ u = f \circ v$ then $u = v$; it is called an epimorphism if and only if for any maps u, v from Q into R with $u \circ f = v \circ f$ then $u = v$.

CHAPTER I

DENSE EXTENSIONS OF A PARTIALLY ORDERED SET.

In this chapter we introduce the concept of a dense extension of a partially ordered set. The class of these extensions will be found to include the classes of join dense and meet dense extensions studied in [1, 3, 15]. Whereas the notions of meet dense respectively join dense extensions are not self-dual the notion of dense extension will be seen to be self-dual in the sense that if (R, w) is a dense extension of P then (R*, w) will be a dense extension of P* where P*, R* are the duals of P, R respectively. We will introduce on the class of all extensions an injective and a projective ordering and study the basic relations between them. We will establish as well in this chapter a canonical form for dense extensions in terms of a system of lower and upper end pairs. In addition we will study complete dense extensions and various classes of faithful dense extensions and state pertinent results about their canonical forms.

1. Some Preliminary Definitions.

In this section we introduce the definitions of the various dense extensions, injective and projective orderings and state the Duality Principle for extensions and some other basic results.

Let P be a partially ordered set.

Definition 1: An extension of P is a pair (E, w) consisting of a partially ordered set E and an order isomorphism w from P into E.

Let (R, w) and (S, π) be any two extensions of P.

Definition 2: (R, w) is injectively smaller than (S, π) (or (S, π) is injectively larger than (R, w)), written (R, w) $\leq_i$ (S, π) if and only if there exists an order isomorphism f from R into S such that $f \cdot w = \pi$. Further (R, w) is said to be equivalent to (S, π), written (R, w) $\cong$ (S, π), if and only if there exists an order isomorphism f from R onto S such that $f \cdot w = \pi$.

Dually we give

Definition 3: (R, w) will be called projectively larger than (S, π) (or (S, π) is projectively smaller than (R,w)), in symbols, (R, w) $\geq_p$ (S, π) if and only if there exists an order epimorphism f from R onto S with $f \cdot w = \pi$.

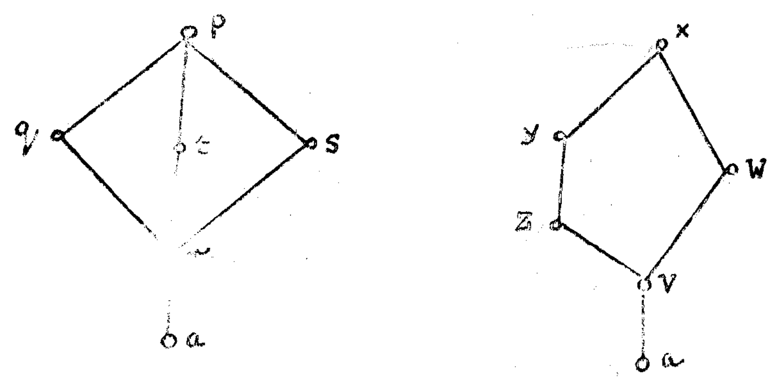
The relationship between the above two orderings will be studied in suitable categories later. We now, however, state some basic properties of these orderings which are immediate consequences of the definitions.

Proposition 1: The relations of injective, respectively, projective orderings are quasi-orderings whereas the relation $\cong$ is an equivalence relation. Further both orderings are compatible with the relation $\cong$, that is, if (R, w) $\leq_i$ (S, π) ((R, w) $\geq_p$ (S, π)) and if (R, w) $\cong$ (R^1, w^1), (S, π) $\cong$ (S^1, π^1) then (R^1, w^1) $\leq_i$ (S^1, π^1) ((R^1, w^1) $\geq_p$ (S^1; w^1)). In addition if (R, w) $\cong$ (S, π) then (R, w) $\leq_i$ (S, π) and (S, π) $\leq_i$ (R, w) ((R, w) $\geq_p$ (S, π) and (S, π) $\geq_p$ (R, w)).

In general if (R, w) $\leq_i$ (S, π) ((R, w) $\geq_p$ (S, π)) then the order isomorphism f (order epimorphism f) from R into S (R onto S) with $f \cdot w = \pi$ need not be unique. Further (R, w) need not be projectively smaller (injectively larger) than (S, π). In addition any two extensions which are injectively or projectively smaller than each other need not be equivalent. We give examples of these facts.

Example 1. Let $P = [0, 1]$, $R = [0, 3)$, $S = [0, 3]$ be intervals of the real numbers and let i be the identity mapping from P into R, S respectively. Then clearly $(R, i) \leq_i (S, i)$ by the identity mapping from R but (R, i) is not projectively smaller than (S, i) since S has an unit element and R does not.

Example 2. Let $P = \{a\}$ be a one element set and let $R = \{a, p, q, r, s, t\}$, $S = \{a, v, w, x, y, z\}$ be sets containing P whose order relations are given by the following diagrams.



Let i be the identity mapping from P. Then $(R, i) \geq_p (S, i)$ for clearly the mapping f defined by $f(a) = a$, $f(p) = x$, $f(q) = y$, $f(r) = v$, $f(s) = w$, $f(t) = z$ is an order epimorphism with $f \cdot i = i$. Notice that the mapping g given by $g(a) = a$, $g(p) = x$, $g(q) = z$, $g(r) = v$, $g(s) = w$, $g(t) = y$ also makes $(R, i) \geqslant_\rho (S, i)$ but $f \neq g$. Further (S, i) is not injectively smaller than (R, i). Otherwise there would exist an order isomorphism h from S into R with $h(a) = a$. But then since R and S have the same number of elements h would be also onto. But then the lattices R - P, S - P would be isomorphic. However, R - P is modular and S - P is not and this contradiction establishes our claim.

Example 3. [Bruns, 3]
Let P, (R, i), (S, i) be as in example 1. Then $(R, i) \leq_\iota (S, i)$. Further $(S, i) \leq_\iota (R, i)$ since one has that $([0, 2], i) \cong (S, i)$ (by defining $f(r) = r$ if r in P and $f(r) = 2r-1$ if r in $[1, 2]$) and $([0, 2], i) \leq_\iota (R, i)$. However, (R, i) is not equivalent to (S, i) since S has an unit element and R does not.

Example 4. Let R be the chain of rational numbers, S any partially ordered set and N the chain of integers. Consider the ordinal products R.R and R.N. Form the ordinal sums $P = R.R \oplus S$ and $Q = R.N \oplus S$. Then the identity mapping i from S makes $(R.R \oplus S, i)$ and $(R.N \oplus S, i)$ into extensions of S. The mapping f from P into Q given by $f = i$ on S and $f(r, s) = (r, [s])$ where $[s]$ is the greatest integer less equal s is clearly an order epimorphism from P onto Q with $f.i = i$, that is, $(P, i) \geqslant_\rho (Q, i)$. By Theorem 1 of Chapter 0 there exists an order isomorphism g from R onto R.R. Define h from Q to P by $h = i$ on S and for any (r, n) in R.N, put $h(r, n) = g(r)$. Then since g is an order isomorphism onto R.R we have immediately that h is an order epimorphism from Q onto P with h.i = i, that is, (Q, i) $\geqslant_\rho$(P, i). However, (P, i) is not equivalent with(Q, i). Otherwise we would have R.R order isomorphic with R.N. This, however, is a contradiction since R.R is dense-in-itself but clearly R.N is not. Hence (P, i), (Q, i) are projectively larger than each other but are not equivalent.

Definition 4: A subset S of a partially ordered set P will be called dense in P if and only if for any x, y in P with $x \nleq y$ there exists a s in S with $s \leq x$ and $s \nleq y$ or $s \geqslant y$ and $s \ngeqslant x$. S is said to be join dense in P if and only if every x in P is a join in P of elements of S. S is said to be meet dense in P if and only if every x in P is a meet in P of elements of S.

Remark 1: It follows immediately from the above definition that if S is join dense (meet dense) in P then S is dense in P. Further if S is join dense (meet dense) in P then for any $x \nleq y$ in P there exists a s in S with $s \leq x$, $s \nleq y$ ($s \geq y$, $s \ngeqslant x$) but the converse is not true. For example take P = $\{a, b, c\}$ a three element chain with largest element a and smallest element c and take S = $\{b\}$. Then clearly S is neither join dense nor meet dense in P but S is dense in P.

Definition 5: An extension (E, w) of a partially ordered set P will be called a dense extension of P if and only if w(P) is dense in E. It is called a join dense (meet dense) extension of P if and only if w(P) is join dense (meet dense) in P.

In view of the above remark it follows immediately that every join dense (meet dense) extension is also a dense extension. The relation between extensions of P and the extensions of its dual P* is given by the following

<u>Proposition 2</u>: (<u>Duality Principle for Extensions</u>). The mapping f which attaches to each extension (R, w) of a partially ordered set P the extension (R*, w) of the dual P*, where R* is the dual of R, is a one to one mapping of the class of all extensions of P onto the class of all extensions of P*. Moreover we have the following properties:

(1) $(R, w) \cong (S, \pi)$ if and only if $(R^*, w) \cong (S^*, \pi)$.

(2) $(R, w) \leq_i (S, \pi)$ $((R, w) \geq_\rho (S, \pi))$ if and only if $(R^*, w) \leq_i (S^*, \pi)$ $((R^*, w) \geq_\rho (S^*, \pi))$.

(3) (R, w) is a join dense (meet dense) extension of P if and only if (R*, w) is a meet dense (join dense) extension of P*.

(4) (R, w) is a dense extension of P if and only if (R*, w) is a dense extension of P*.

<u>Proposition 3</u>: Let (R, w) be any extension injectively smaller than a dense extension (S, π) of P. Then (R, w) is a dense extension. In particular all extensions of P equivalent to a dense extension are dense extensions.

<u>Proof</u>: (R, w) injectively smaller than (S, π) implies the existence of an order isomorphism f from R into S with $f.w = \pi$. Take any a, b

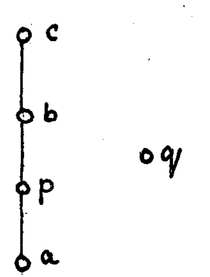
in R with $a \nleq b$. But then $f(a) \nleq f(b)$ and since $\pi(P)$ is dense in S there exists a x in P with $\pi(x) \leq f(a)$ and $\pi(x) \nleq f(b)$ or $\pi(x) \geq f(b)$ and $\pi(x) \ngeq f(a)$. Finally since $f.w = \pi$ and f is an order isomorphism we have from the previous line that $w(x) \leq a$ and $w(x) \nleq b$ or $w(x) \geq b$ but $w(x) \ngeq a$, that is, w(P) is dense in R. Thus (R, w) is a dense extension and this completes the proof.

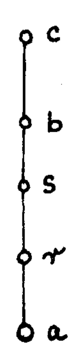
Remark 2: [Bruns, 3].

In a similar manner one can establish that any extension injectively smaller than a join dense (meet dense) extension is join dense (meet dense).

The dual situation to that mentioned in Proposition 3 is not true in general; namely if (R, w) is projectively smaller than a dense extension then (R, w) need not be dense. The following examples illustrate this fact.

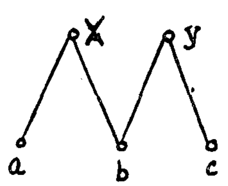
Example 5: Let $P = \{a, b, c\}$ be a three element chain with smallest element a and greatest element c. Let $R = \{a, b, c, p, q\}$ and $S = \{a, b, c, r, s\}$ be two sets containing P whose order relations are described by the following diagrams:

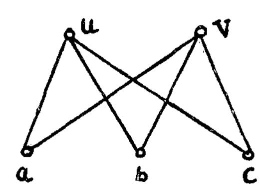




Then (R, i), (S, i), i being the identity map from P are extensions of P. Also (R, i) is clearly a dense extension of P, but (S, i) is not a dense extension since $s \not\leq r$ but each element of P comparable with s is comparable with r. Consider the mapping f from (R, i) onto (S, i) which maps P identically and f(p) = r, f(q) = s. Then of course f is an order epimorphism with $f \cdot i = i$, but (S, i) is not dense.

<u>Example 6</u>: Let P = {a, b, c} be a totally unordered set and let R = {a, b, c, x, y} , S = {a, b, c, u, v} be sets containing P whose orderings are given by the diagrams below:





Then (R, i) is a join dense extension of P, but (S, i) is not a join dense extension, i being the identity map. But clearly $(R, i) \geq_p (S, i)$.

In at least one case the situation mentioned in the above example does not occur as the following proposition shows.

<u>Proposition 4</u>: Let (R, w) be a dense extension projectively smaller than a join dense extension (S, π). Then (R, w) is a join dense extension.

<u>Proof</u>: (R, w) projectively smaller than (S, π) implies there exists an order epimorphism f from S onto R with $f.\pi = w$. Take any r in R

and suppose $f(s) = r$. Now we claim $r = \bigvee^{R}\{w(x) / w(x) \leq r\}$. Take any c in R, $c \geqslant w(x)$ for each $w(x) \leq r$. Suppose $c \ngeq r$. Then since (R, w) is a dense extension we have a y with $w(y) \geq c$ but $w(y) \ngeq r$. But then $\pi(y) \ngeq s$ and since (S, π) is join dense there exists a z with $\pi(z) \leq s$ but $z \nleq y$, that is, $y \ngeq z$ with $w(z) \leq r$. But from the previous lines $y \geq x$ for each x with $w(x) \leq r$. This contradiction establishes our claim and also the proposition.

2. A Canonical Form for Dense Extensions.

In this section we obtain a canonical form for dense extensions and establish the basic equivalence theorem for dense extensions which states that any two dense extensions which are injectively smaller than each other are equivalent.

Definition 6: A subset A of a partially ordered set P will be called a lower end if and only if x in A and $y \leq x$ imply y in A. Dually a subset E of P will be called an upper end if and only if x in E and $y \geqslant x$ imply y in E.

Examples of lower and upper ends of particular importance are the principal lower ends, respectively the principal upper ends generated by a single element of P. Thus for any x in P the principal lower end generated by x is $(\leftarrow x] = \{y / y \in P,\ y \leq x\}$ and the principal upper end generated by x is $[x \rightarrow) = \{y / y \in P,\ y \geq x\}$.

Let $\mathcal{O}_0$, $\mathcal{E}_0$ respectively be the collection of all lower and upper ends, $\mathcal{O}(P)$, $\mathcal{E}(P)$ respectively the collection of all principal lower and upper ends of a partially ordered set P. Let Ω be the Cartesian product of $\mathcal{O}_o$, $\mathcal{E}_o$, that is,

(i) $\Omega = \{(A, E) / A \in \mathcal{O}_o, E \in \mathcal{E}_o\}$

(ii) $\Delta = \{((\leftarrow x], [x\rightarrow)) / x \in P\}$

For any subset S of P write

(iii) $MaS = \{y / y \in P, \ y \geq s$ for each s in $S\}$

(iv) $MiS = \{y / y \in P, \ y \leq s$ for each s in $S\}$

For our purposes we are mainly interested in the sets contained in the following subset of Ω.

(v) $\mathcal{X}_o = \{(A, E) / (A, E) \in \Omega$ with $E \subseteq MaA\}$

We remark that $\mathcal{X}_o$ contains Δ and also that

(vi) $\mathcal{X}_o = \{(A, E) / (A, E) \in \Omega$ with $A \subseteq MiE\}$; for A is a subset of MiE if and only if E is a subset of MaA.

Further for any subset $\mathcal{X}$ of Ω define

(vii) $pr_1(\mathcal{X}) = \{A / A \in \mathcal{O}_o$, there exists $E \in \mathcal{E}_o$ with $(A, E) \in \mathcal{X}\}$

(viii) $pr_2(\mathcal{X}) = \{E / E \in \mathcal{E}_o$, there exists $A \in \mathcal{O}_o$ with $(A, E) \in \mathcal{X}\}$

(ix) For any pair of elements (A_1, E_1), (A_2, E_2) in $\mathcal{X}$ put $(A_1, E_1) \leq (A_2, E_2)$ if and only if $A_1 \subseteq A_2$ and $E_1 \supseteq E_2$.

Also for any subsets $\mathcal{X}$ of Ω containing Δ, $\mathcal{O}$ of $\mathcal{O}_o$ containing $\mathcal{O}(P)$, $\mathcal{E}$ of $\mathcal{E}_o$ containing $\mathcal{E}(P)$ define the following mappings from P into these subsets:

(x) α from P into $\mathcal{X}$ given by $\alpha(x) = ((\leftarrow x], [x\rightarrow))$.

(xi) α_1 from P into $\mathcal{O}$ given by $\alpha_1(x) = (\leftarrow x]$.

(xii) α_2 from P into $\mathcal{E}$ given by $\alpha_2(x) = [x\rightarrow)$.

We now have clearly

<u>Proposition 5</u>: The relation $\leqq$ is a partial ordering on Ω and the pair $(\mathcal{X}, \alpha)$ where $\mathcal{X}$ is any subset containing Δ is an extension of P.

For any $\mathcal{O}\!\mathcal{l}$, $\mathcal{E}$ satisfying, $\mathcal{O}\!\mathcal{l}(P) \subseteq \mathcal{O}\!\mathcal{l} \subseteq \mathcal{O}\!\mathcal{l}_o$, $\mathcal{E}(P) \subseteq \mathcal{E} \subseteq \mathcal{E}_o$ define

(xiii) $\sigma(\mathcal{O}\!\mathcal{l}) = \{(A, MaA) / A \in \mathcal{O}\!\mathcal{l}\}$

(xiv) $\delta(\mathcal{E}) = \{(MiE, E) / E \in \mathcal{E}\}$

The following definition will be useful in formulating results.

<u>Definition 7</u>: Any subset $\mathcal{X}$ of $\mathcal{X}_o$ containing Δ will be called an <u>admissible subset</u> of $\mathcal{X}_o$. The subsets $\sigma(\mathcal{O}\!\mathcal{l})$, $\delta(\mathcal{E})$ will be called respectively the <u>join admissible</u>, <u>meet admissible</u> subsets induced respectively by $\mathcal{O}\!\mathcal{l}$, $\mathcal{E}$.

The importance of admissible subsets is realized in the following proposition.

<u>Proposition 6</u>: Any extension $(\mathcal{X}, \alpha)$ where $\mathcal{X}$ is a admissible subset is a dense extension. In particular $(\mathcal{X}_o, \alpha)$ is a dense extension.

<u>Proof</u>: Take (A, E), (B, F) in $\mathcal{X}$ and suppose $(A, E) \nleq (B, F)$. This gives either $A \nsubseteq B$ or $E \nsupseteq F$. Now if $A \nsubseteq B$ then there exists x in A but x not in B. Then since $\mathcal{X}$ is admissible we have $E \subseteq MaA \subseteq Ma\,(\leftarrow x] = \alpha_2(x)$. Hence $\alpha(x) \leq (A, E)$ but since x is not in B, $\alpha(x) \nleq (B, F)$. Dually if $E \nsupseteq F$ there exists x with $\alpha(x) \geq (B, F)$ but $\alpha(x) \ngeq (A, E)$.

Thus $\alpha(P)$ is dense in $\mathcal{X}$ and $(\mathcal{X}, \alpha)$ is a dense extension. This completes the proof.

Proposition 7: The extensions $(\sigma(\mathfrak{A}), \alpha)$, $(\delta(\mathcal{E})), \alpha)$ are respectively join dense, meet dense. In particular the extension $(\sigma(\mathfrak{A}_0), \alpha)$ is join dense and $(\delta(\mathcal{E}_0), \alpha)$ is meet dense.

Proof: Take any element (A, MaA) in $\sigma(\mathfrak{A})$. Then of course (A, MaA) $\geq$ $\alpha(x)$ for each x in A. Further for any (B, MaB) in $\sigma(\mathfrak{A})$ which is greater than or equal to $\alpha(x)$ for each x in A we have $B \supseteq A$ and thus MaB $\subseteq$ MaA, that is (B, MaB) $\geq$ (A, MaA). Thus (A, MaA) $= \bigvee \{\alpha(x) / x \in A\}$ By the duality principle we have the rest of the proposition and this completes the proof.

The above propositions allow us to make the following definitions.

Definition 8: We shall call any extension of the form $(\mathcal{X}, \alpha)$ where $\mathcal{X}$ is an admissible subset a canonical dense extension. The extensions of the form $(\sigma(\mathfrak{A}), \alpha)$ will be called canonical join dense extensions. Dually extensions of the form $(\delta(\mathcal{E}), \alpha)$ will be called canonical meet dense extensions.

Let P be any partially ordered set and (E, w) any extension of P. For any s in E define

(1) $L(s, w) = \{x / x \in P, w(x) \leq s\}$.

(2) $U(s, w) = \{x / x \in P, w(x) \geq s\}$.

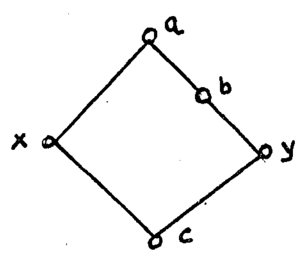
The following observation will be useful.

Lemma 1: (E, w) is a dense extension of P if and only if for all s, t in E : $L(s, w) \subseteq L(t, w)$ and $U(s, w) \supseteq U(t, w)$ imply $s \leq t$.

Proof: Suppose (E, w) is dense. Then if $s \nleq t$ for some s, t in E since w(P) is dense in E we have immediately $L(s, w) \nsubseteq L(t, w)$ or $U(s, w) \nsupseteq U(t, w)$. Conversely if the condition holds and $s \nleq t$ then we have by assumption $L(s, w) \nsubseteq L(t, w)$ or $U(s, w) \nsupseteq U(t, w)$ which gives that (E, w) is dense completing the proof.

Lemma 1 implies that if (E, w) is a dense extension of P then for all s, t in E if $L(s, w) = L(t, w)$, $U(s, w) = U(t, w)$ then $s = t$. The converse, however, is not true.

Example 7: Let $P = \{a, b, c\}$ be a three element set contained in $E = \{a, b, c, x, y\}$. Let the ordering in E be given by the following diagram:



Then certainly (E, i) is an extension of P, i being the identity map. Further we have in E the condition if $L(s, i) = L(t, i)$ and $U(s, i) = U(t, i)$ then $s = t$ for any s, t in E. However, (E, i) is not dense for $y \nleq x$ but $L(y, i) \subseteq L(x, i)$ and $U(x, i) \subseteq U(y, i)$.

Lemma 2: If (R, w) is any extension injectively smaller than a dense extension (S, π) of P then there exists exactly one order isomorphism f from R into S with $f.w = \pi$.

Proof: Since (R, w) is injectively smaller than (S, π) there exists an order isomorphism f from R into S with $f.w = \pi$. Let g be another order isomorphism from R into S with $g.w = \pi$. Then for any r in R we claim $L(f(r), \pi) = L(g(r), \pi)$ and $U(f(r), \pi) = U(g(r), \pi)$. To see this take any x in P with $\pi(x) \leq f(r)$, then $w(x) \leq r$ since $f.w = \pi$, and f is an order isomorphism. But then $g(w(x)) = \pi(x) \leq g(r)$ and thus $L(f(r), \pi) \subseteq L(g(r), \pi)$. The reverse inclusion follows similarly and by duality we have $U(f(r), \pi) = U(g(r), \pi)$. Then by Lemma 1 since (S, π) is dense we have $f(r) = g(r)$. Hence $f = g$ and this completes the proof.

Corollary 1: If $(\mathcal{X}, \alpha)$ is any canonical dense extension of a partially ordered set then there exists precisely one order isomorphism from $(\mathcal{X}, \alpha)$ to $(\mathcal{X}_0, \alpha)$ which keeps Δ fixed, namely the identity mapping.

Remark 3: [Bruns, 3] If (R, w) is any extension injectively smaller than a join dense extension (S, π) then the unique order isomorphism of Lemma 2 from R into S is given by $f(r) = \bigvee^S \{\pi(x) / x \in P, w(x) \leq r\}$ and in case (S, π) is meet dense it is given by $f(r) = \bigwedge_S \{\pi(x) / x \in P, w(x) \geq r\}$.

The following theorem translates the injective ordering on canonical dense extensions to that of set inclusion.

Theorem 1: Let $(\mathcal{X}, \alpha)$, $(\mathcal{Y}, \alpha)$ be canonical dense extensions. Then $(\mathcal{X}, \alpha)$ is injectively smaller than $(\mathcal{Y}, \alpha)$ if and only if $\mathcal{X} \subseteq \mathcal{Y}$.

Proof: If $(\mathcal{X}, \alpha)$ is injectively smaller than $(\mathcal{Y}, \alpha)$ then there exists an order isomorphism f from $\mathcal{X}$ into $\mathcal{Y} \subseteq \mathcal{X}_0$ with $f.\alpha = \alpha$. Then from Corollary 1 we infer that f must be the identity mapping from $\mathcal{X}$ into $\mathcal{Y}$, that is, $\mathcal{X} = f(\mathcal{X}) \subseteq \mathcal{Y}$. Conversely if $\mathcal{X} \subseteq \mathcal{Y}$ then the identity mapping f from $\mathcal{X}$ into $\mathcal{Y}$ is an order isomorphism with $f.\alpha = \alpha$. Thus $(\mathcal{X}, \alpha)$ is injectively smaller than $(\mathcal{Y}, \alpha)$. This completes the proof of the theorem.

Corollary 2: Any two canonical dense extensions are equivalent if and only if their canonical subsets are equal.

The next theorem obtains a complete survey of dense extensions in terms of canonical dense extensions.

Theorem 2: Let (R, w) be a dense extension of a partially ordered set P. Then (R, w) is equivalent to exactly one canonical dense extension $(\mathcal{X}, \alpha)$. The admissible subset of this extension is given by $\mathcal{X} = \{(L(r, w), U(r, w)) \,/\, r \in R\}$.

Proof: Note first that $\mathcal{X}$ is an admissible subset for $L(w(x), w) = (\leftarrow x]$, $U(w(x), w) = [x \rightarrow)$ and hence Δ is a subset of $\mathcal{X}$. Also since w is an order isomorphism L(r, w) is a lower end and U(r, w) is an upper end. Further $U(r, w) \subseteq MaL(r, w)$ since if $w(y) \geq r$ then $w(y) \geq w(x)$ for each x in L(r, w), that is, $y \geq x$ for each x in L(r, w). Now consider the mapping f from R into $\mathcal{X}$ given by f(r) = (L(r, w), U(r, w)). Then clearly f is onto $\mathcal{X}$. Also since (R, w) is a dense extension Lemma 1 immediately gives that f is an order isomorphism and from the first line we get $f \cdot \alpha = \alpha$. Hence (R, w) is equivalent to $(\mathcal{X}, \alpha)$. If $(\mathcal{Y}, \alpha)$ is another canonical dense extension equivalent to (R, w) then it follows that $(\mathcal{X}, \alpha)$, $(\mathcal{Y}, \alpha)$ are equivalent. This means that $\mathcal{X} = \mathcal{Y}$ by Corollary 2. This completes the proof.

As a consequence of Theorem 3 we have the following theorem which does not hold for extensions in general.

Theorem 3: (Equivalence Theorem for Dense Extensions). Let (R, w), (S, π) be any two dense extensions of a partially ordered set P. If $(R, w) \leq_i (S, \pi)$ and $(S, \pi) \leq_i (R, w)$ then (R, w) is equivalent to (S, π).

Proof: By Theorem 2 there exist unique canonical dense extensions $(\mathcal{X}, \alpha)$, $(\mathcal{Y}, \alpha)$ which are respectively equivalent to (R, w), (S, π). Our hypotheses then give $(\mathcal{X}, \alpha) \leq_i (\mathcal{Y}, \alpha)$ and $(\mathcal{Y}, \alpha) \leq_i (\mathcal{X}, \alpha)$. By Theorem 1 we then have $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$, that is, $\mathcal{X} = \mathcal{Y}$. Thus $(\mathcal{X}, \alpha)$, $(\mathcal{Y}, \alpha)$ are equivalent and hence so are (R, w) and (S, π). This completes the proof.

Remark 4: An alternative proof of Theorem 3 without reference to canonical dense extensions may be given as follows. Since $(R, w) \leq_i (S, \pi)$ and $(S, \pi) \leq_i (R, w)$ there exist by Lemma 2 unique order isomorphisms f from R into S and g from S into R with $f.w = \pi$ and $g.\pi = w$. Then f.g is an order isomorphism from S into S and $(f.g).\pi = f.w = \pi$. The identity mapping i on S is also an order isomorphism with $i.\pi = \pi$. But by Lemma 2 since (S, π) is a dense extension there is only one order isomorphism from S to S which keeps $\pi(P)$ fixed. Thus $f.g = i$ and this means that f is onto. Hence (R, w) and (S, π) are equivalent.

<u>Corollary 3</u>: Every join dense extension (R, w) is equivalent to exactly one canonical join dense extension $(\sigma(\mathfrak{A}), \alpha)$. Dually every meet dense extension is equivalent to exactly one canonical meet dense extension $(\delta(\mathcal{E}), \alpha)$.

<u>Proof</u>: If (R, w) is join dense then in particular (R, w) is dense. Hence by Theorem 2 there exists an unique canonical dense extension $(\mathfrak{X}, \alpha)$ equivalent to (R, w). Then by Remark 2 $(\mathfrak{X}, \alpha)$ is also join dense. Hence for each (A, E) in $\mathfrak{X}$ we have $(A, E) = \bigvee\{\alpha(x) / \alpha(x) \leq (A, E)\}$. Take y in MaA, then $\alpha(y) \geq \alpha(x)$ for each x in A, that is, $\alpha(y) \geq \alpha(x)$ for each x with $\alpha(x) \leq (A, E)$. Hence $\alpha(y) \geq (A, E)$ and thus y is in E. But since $\mathfrak{X}$ is admissible $E \subseteq$ MaA, in all, E = MaA. Thus (A, E) = (A, MaA), that is, $\mathfrak{X}$ is the unique join admissible subset induced by $pr_1(\mathfrak{X})$. The second half of the corollary follows by duality. This completes the proof.

<u>Remark 5</u>: The canonical representations given in Corollary 5 may be effectively reduced to representations just in terms of systems of lower ends, respectively upper ends as derived in Bruns [3], Banaschewski [1]. Partially order all subsystems $\mathfrak{A}$ of $\mathfrak{A}_0$ containing $\mathfrak{A}(P)$ by inclusion. Then a pair $(\mathfrak{A}, \alpha_1)$ is a join dense extension

clearly equivalent with $(\sigma(\mathcal{O}\!\mathcal{l}), \alpha)$ under the obvious map. Dually if we partially order all subsystems $\mathcal{E}$ of $\mathcal{E}_o$ containing $\mathcal{E}(P)$ by exclusion then all pairs $(\mathcal{E}, \alpha_2)$ are meet dense extensions. Any such pair is then clearly equivalent to $(\delta(\mathcal{E}), \alpha)$ under the obvious map. For this reason from now on when we refer to canonical join dense (canonical meet dense) extensions we will mean either the extensions $(\mathcal{O}\!\mathcal{l}, \alpha_1)$ $((\mathcal{E}, \alpha_2))$ or the extensions $(\sigma(\mathcal{O}\!\mathcal{l}), \alpha)$, $(\delta(\mathcal{E}), \alpha)$ induced by these and which are equivalent to them.

The equivalence Theorem 3 tells us that the collection γ of all canonical dense extensions forms a partially ordered set under the injective ordering. The main features of this partially ordered set is listed in the next theorem.

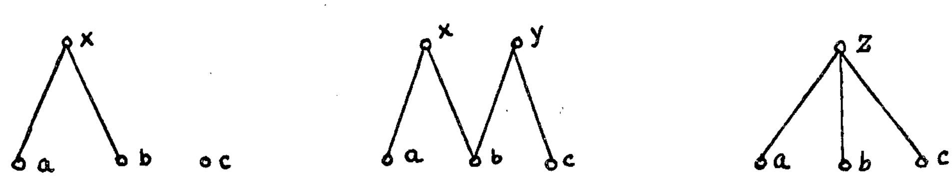
Theorem 4: The collection γ of all canonical dense extensions of a partially ordered set is partially ordered by the injective ordering. In this ordering, it is a complete, atomic, Boolean lattice. The operations of meet, join, complement in this lattice are respectively, set intersection, set union and $\complement \mathcal{X} = (\mathcal{X}_o - \mathcal{X}) \cup \Delta$. The unit element of this lattice is $(\mathcal{X}_o, \alpha)$ and the zero element is the trivial extension (Δ, α). Further $(\mathcal{X}_o, \alpha)$ is the largest canonical dense extension of a partially ordered set.

Proof: Let $\mathcal{G}$ denote the collection of admissible subsets of $\mathcal{X}_o$. The mapping h from γ into $\mathcal{G}$ given by $h((\mathcal{X}, \alpha)) = \mathcal{X}$ is clearly an order isomorphism from γ onto $\mathcal{G}$, $\mathcal{G}$ being ordered by inclusion.

Further the mapping g from $\mathcal{C}$ into the power set of $\mathcal{X}_0 - \Delta$ given by $g(\mathcal{X}) = \mathcal{X} - \Delta$ is an order isomorphism of $\mathcal{C}$ onto the power set $\mathcal{P}(\mathcal{X}_0 - \Delta)$, the power set being ordered by inclusion. Hence the composite map g.h from γ onto $\mathcal{P}(\mathcal{X}_0 - \Delta)$ is an order isomorphism. Since $\mathcal{P}(\mathcal{X}_0 - \Delta)$ is a complete, atomic, Boolean lattice so is γ. This completes the proof.

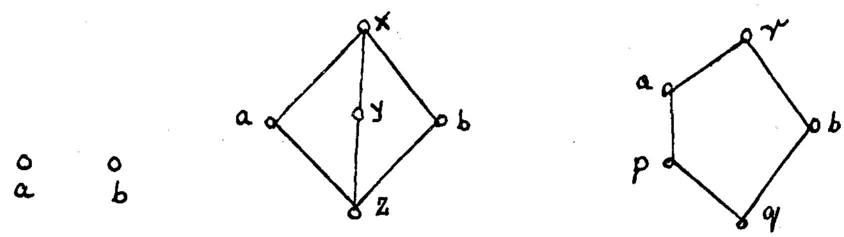
We bring this section to a close by giving the following examples which show that in general no pleasant relationships exist between the injective and projective orderings even for dense or join dense extensions.

Example 8: Let $P = \{a, b, c\}$ be a totally unordered set and let $Q = \{a, b, c, x\}$, $R = \{a, b, c, x, y\}$ and $S = \{a, b, c, z\}$ be three sets containing P whose order relations are given by the following diagrams:



Then (Q, i), (R, i), (S, i), i the identity map from P are clearly join dense extensions of P. Further, clearly $(Q, i) \leq_i (R, i)$ but (Q, i) is not projectively smaller than (R, i) since y is comparable to b, c in R but no element of Q is comparable to both b, c in Q. Again $(R, i) \geqslant_p (S, i)$ by the mapping which keeps P fixed and maps x, y to z. However, (S, i) is not injectively smaller than (R, i) since z $\geq$ a, b, c in S but no element of R is greater than, or equal to a, b, c.

<u>Example 9</u>: Let $P = \{a, b\}$, $R = \{a, b, x, y, z\}$, $S = \{a, b, p, q, r\}$ be sets whose ordering are given as follows:



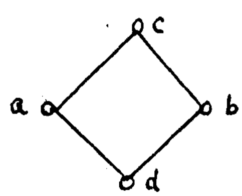
Then (R, i), (S, i) are clearly dense extensions of P. Also the mapping f which keeps P fixed and maps x to r, z to q, y to p is an order epimorphism onto S with f.i = i. However, (R, i) is not injectively smaller than (S, i) for if so then R, S being finite sets the lattices R, S would be isomorphic. However, R is modular and S is not. Note that in this case R and S are even complete lattices.

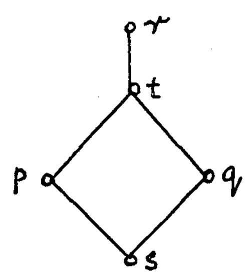
3. <u>$\mathcal{M}$-$\mathcal{N}$ Faithful Dense Extensions.</u>

In this section we study $\mathcal{M}$ join faithful, $\mathcal{N}$ meet faithful extensions of a partially ordered set. We give necessary and sufficient conditions for a given canonical dense extension to be $\mathcal{M}$ join faithful, respectively $\mathcal{N}$ meet faithful.

In general if (R, w) is an extension of a partially ordered set P and M is a subset of P which has a join in P then the set $w(M)$ need not have a join in R and even it did it could be that $w(\bigvee^P M) \neq \bigvee^R w(M)$. The following example illustrates this fact.

Example 10: Let $P = \{a, b, c, d\}$ and $E = \{p, q, r, s, t\}$ be two sets whose order relations are given by the following diagrams:





Define w from P into E by $w(a) = p$, $w(b) = q$, $w(c) = r$, $w(d) = s$. Then w is clearly an order isomorphism of P into E and thus (E, w) is an extension of P. Now $M = \{a, b\}$ has join c in P whereas $\bigvee^E w(M) = t \neq w(c) = r$.

The situation mentioned in the above example motivates the following definition. Let $\mathfrak{M}$ be any system of subsets of a partially ordered set P.

Definition 9: An extension (R, w) of P is called $\mathfrak{M}$ join faithful if and only if for each M in $\mathfrak{M}$ which has a join in P the set $w(M)$ has a join in R and $w(\bigvee^P M) = \bigvee^R w(M)$. Correspondingly (R, w) is called $\mathfrak{M}$ meet faithful if and only if for each M in $\mathfrak{M}$ which has a meet in P the set $w(M)$ has a meet in R and $w(\bigwedge_P M) = \bigwedge_R w(M)$. In case $\mathfrak{M}$ is the collection of all finite subsets of P we will say (R, w) is finitely join faithful, respectively finitely meet faithful; if $\mathcal{M}$ is the collection of all subsets of P we will say (R, w) is completely join faithful, respectively completely meet faithful. Further we will say (R, w) is finitely faithful (completely faithful) if and only if it is finitely meet and join faithful (completely meet and join faithful).

In order to obtain descriptions of the canonical dense extensions which are $\mathcal{M}$ join faithful, $\mathcal{M}$ meet faithful we make the following definition.

Definition 10: A lower end A of P will be called a $\mathcal{M}$ lower end if and only if for each M in $\mathcal{M}$ such that $M \subseteq A$ and M has a join in P then $\bigvee^{P} M \in A$. Similarly an upper end E of P will be called a $\mathcal{M}$ upper end if and only if for each M in $\mathcal{M}$ if $M \subseteq E$ and the meet in P of M exists then $\bigwedge_{P} M \in E$. In particular if $\mathcal{M}$ is the collection of all finite subsets of P then a $\mathcal{M}$ lower end is called an ideal and a $\mathcal{M}$ upper end is called a filter. Similarly if $\mathcal{M}$ is the collection of all subsets of P then a $\mathcal{M}$ lower end will be called a complete lower end and a $\mathcal{M}$ upper end will be called a complete upper end.

Let $\mathcal{M}$ and $\mathcal{N}$ be arbitrary systems of subsets of P. Let $\mathcal{O}_{\mathcal{M}}(P)$, $\mathcal{E}_{\mathcal{N}}(P)$ be the collection of all $\mathcal{M}$ lower ends, $\mathcal{N}$ upper ends respectively of P. We define

(1) $\mathcal{X}(\mathcal{M}, \mathcal{N}) = \{(A, E)/(A, E) \in \mathcal{X}_o , A \in \mathcal{O}_{\mathcal{M}}(P), E \in \mathcal{E}_{\mathcal{N}}(P)\}$

Then in particular when $\mathcal{N}$ is the empty collection we have

(2) $\mathcal{X}(\mathcal{M},\phi) = \{(A, E)/(A, E) \in \mathcal{X}_o, A \in \mathcal{O}_{\mathcal{M}}(P), E \in \mathcal{E}_o\}$ and when $\mathcal{M}$ is the empty collection we have

(3) $\mathcal{X}(\phi,\mathcal{N}) = \{(A, E)/(A, E) \in \mathcal{X}_o, A \in \mathcal{O}_o, E \in \mathcal{E}_{\mathcal{N}}(P)\}$

Theorem 5: A canonical dense extension $(\mathcal{X},\alpha)$ of P is $\mathcal{M}$ join and $\mathcal{N}$ meet faithful if and only if $\mathcal{X}$ is a subset of $\mathcal{X}(\mathcal{M},\mathcal{N})$. In particular $(\mathcal{X}(\mathcal{M},\mathcal{N}),\alpha)$ is the largest canonical dense $\mathcal{M}$ join and $\mathcal{N}$ meet faithful extension of P.

Proof: Suppose $(\mathcal{X},\alpha)$ is $\mathcal{M}$ join and $\mathcal{N}$ meet faithful and take (A, E) in $\mathcal{X}$. Take M in $\mathcal{M}$, $M \subseteq A$ such that $\vee M$ exists. Then $\alpha(x) \leq (A, E)$ for each x in M. This implies that $(A, E) \geq \bigvee_{x \in M} \alpha(x) = \alpha(\vee M)$ since $(\mathcal{X},\alpha)$ is $\mathcal{M}$ join faithful. This means that $\vee M \in A$. Hence A is a $\mathcal{M}$ lower end and by duality we get E is a $\mathcal{N}$ upper end. Thus $\mathcal{X} \subseteq \mathcal{X}(\mathcal{M},\mathcal{N})$. Next suppose $\mathcal{X} \subseteq \mathcal{X}(\mathcal{M},\mathcal{N})$. Take any M in $\mathcal{M}$ such that the join in P of M exists. Certainly $\alpha(\vee M) \geq \alpha(M)$. Now suppose $(A, E) \in \mathcal{X}$ such that $(A, E) \geq \alpha(M)$. This implies $x \in A$ for each x in M, that is, $M \subseteq A$. Then since A is a $\mathcal{M}$ lower end we have $\vee M \in A$. This means $(A, E) \geq \alpha(\vee M)$. Hence $\alpha(\vee M) = \vee \alpha(M)$, that is, $(\mathcal{X},\alpha)$ is $\mathcal{M}$ join faithful and by duality we get $(\mathcal{X},\alpha)$ is $\mathcal{N}$ meet faithful. This completes the proof.

Corollary 4: A canonical dense extension $(\mathcal{X},\alpha)$ of P is $\mathcal{M}$ join faithful if and only if $\mathcal{X} \subseteq \mathcal{X}(\mathcal{M},\phi)$ and is $\mathcal{N}$ meet faithful if and only if $\mathcal{X} \subseteq \mathcal{X}(\phi,\mathcal{N})$. In particular $(\mathcal{X}(\mathcal{M},\phi), \alpha)$ is the largest canonical dense extension which is $\mathcal{M}$ join faithful and $(\mathcal{X}(\emptyset,\mathcal{M}),\alpha)$ is the largest canonical dense extension which is $\mathcal{M}$ meet faithful.

4. Complete Dense Extensions.

In this section we study complete canonical dense extensions of a partially ordered set. We will give necessary and sufficient conditions on the admissible subset $\mathcal{X}$ of a canonical dense extension $(\mathcal{X},\alpha)$ in order that it be complete. Further we give here necessary and sufficient conditions for an arbitrary family in $\mathcal{X}$ to have a join or a meet in $\mathcal{X}$.

Definition 11: An extension (E, w) of a partially ordered set P is called complete if and only if E is a complete partially ordered set.

An important complete extension is the following Dedekind-Macneille extension of a partially ordered set.

Definition 12: An extension (E, w) of P is called a Dedekind-Macneille extension if and only if (E, w) is a complete meet and join dense extension of P.

Definition 13: A lower end A of P is called normal if and only if A = MiMa A. Dually an upper end E of P is called normal if and only if E = Ma Mi E.

Let $\mathcal{O}_N(P)$, $\mathcal{E}_N(P)$ respectively be the collection of all normal lower ends and normal upper ends of P. Of course every principal lower end, respectively principal upper end is normal. Among the properties of a Dedekind-Macneille extension [see Bruns, 3] we note that it is upto equivalence the injectively smallest complete extension, it has for its canonical forms the equivalent extensions $(\mathfrak{O}_N, \alpha_1)$, (ξ_N, α_2) and further that it is completely faithful.

The largest canonical dense extension $(\mathcal{X}_0, \alpha)$ is seen to be complete for if $(A_i, E_i)_{i \in I}$ is an arbitrary family in $\mathcal{X}_0$ then $(\bigcup_{i \in I} A_i, \bigcap_{i \in I} E_i)$ is an element of $\mathcal{X}_0$ and thus also the least upper bound of this family. Thus in view of the remarks just made we have that if $(\mathcal{X}, \alpha)$ is a complete canonical dense extension then $\sigma(\mathfrak{O}_N) \subseteq \mathcal{X} \subseteq \mathcal{X}_0$.

Proposition 8: Let $(\mathcal{X}, \alpha)$ be any canonical dense extension and (A_i, E_i) an arbitrary family in $\mathcal{X}$. The following statements are equivalent:

(1) $\bigvee (A_i, E_i)$ exists in $\mathcal{X}$

(2) $(\bigcap_i pr_1 Ma(\{(A_i, E_i) / i \in I\}), \bigcap_i E_i)$ belongs to $\mathcal{X}$

In this case $\bigvee_{i \in I} (A_i, E_i) = (\bigcap_i pr_1 Ma(\{(A_i, E_i) / i \in I\}), \bigcap_i E_i)$

Proof: (1) implies (2): Suppose $\bigvee_i (A_i, E_i) = (A, E)$. Put $B = \bigcap_i pr_1 Ma(\{(A_i, E_i) / i \in I\})$. Then $A \in pr_1 Ma(\{(A_i, E_i) / i \in I\})$ and thus $A \supseteq B$. Further if we take $C \in pr_1 Ma(\{(A_i, E_i) / i \in I\})$ then there exists U in $pr_2(\mathcal{X})$ with (C, U) in $\mathcal{X}$ and $(C, U) \geq (A_i, E_i)_{i \in I}$, and thus $(C, U) \geq (A, E)$. Thus $C \supseteq A$ and this gives $A = B$. Also $E \subseteq E_i$ for each $i \in I$ and thus $E \subseteq \bigcap E_i$. Next if we take x in $\bigcap_{i \in I} E_i$ then $\alpha(x) \geq (A_i, E_i)$ for each $i \in I$ and then $\alpha(x) \geq (A, E)$. Hence

$x \in E$ and we have $E = \bigcap E_i$. Thus $(B, \bigcap E_i) = (A, E)$ belongs to $\mathcal{X}$.

(2) implies (1): Put $(A, E) = (\bigcap_i \mathrm{pr}_1 \mathrm{Ma}(\{(A_i, E_i) / i \in I\}, \bigcap E_i)$.

Then (A, E) belongs to $\mathcal{X}$ by assumption and clearly is an upper bound of the given family. Take (B, U) in $\mathcal{X}$, $(B, U) \geqslant (A_i, E_i)$ for each i in I. Then $U \subseteq \bigcap E_i$, that is, $U \subseteq E$ and $B \in \mathrm{pr}_1 \mathrm{Ma}(\{(A_i, E_i) / i \in I\})$.

This means that B contains A. Hence $(B, U) \geqslant (A, E)$. Thus $\bigvee^{\mathcal{X}}_{i \in I}(A_i, E_i)$ exists. This completes the proof.

Dually we obtain

<u>Proposition 9</u>: Let $(\mathcal{X}, \alpha)$ be any canonical dense extension of P and let $(A_i, E_i)_{i \in I}$ be an arbitrary family in $\mathcal{X}$. The following statements are equivalent:

(1) $\bigwedge_{i \in I} (A_i, E_i)$ exists in $\mathcal{X}$.

(2) $(\bigcap_{i \in I} A_i, \bigcap_i \mathrm{pr}_2 \mathrm{Mi}(\{(A_i, E_i) / i \in I\})$ belongs to $\mathcal{X}$. In this case $\bigwedge_{i \in I} (A_i, E_i) = (\bigcap_{i \in I} A_i, \bigcap \mathrm{pr}_2 \mathrm{Mi}(\{(A_i, E_i) / i \in I\}))$.

<u>Corollary 5</u>: Let $(\mathcal{X}, \alpha)$ be any canonical dense extension of P and let S be any subset of P. The following statements are equivalent:

(1) $\bigvee^{\mathcal{X}} \alpha(S)$ exists

(2) $(A, \mathrm{Ma}A)$ belongs to $\mathcal{X}$ where A is the smallest lower end in $\mathrm{pr}_1(\mathcal{X})$ containing S. In this case $(A, \mathrm{Ma}A) = \bigvee^{\mathcal{X}} \alpha(S)$.

Corollary 6: Let $(\mathcal{X}, \alpha)$ be any canonical dense extension and S any subset of P.

The following statements are equivalent:

(1) $\bigwedge^{\mathcal{X}} \alpha(S)$ exists

(2) $(MiE, E) \in \mathcal{X}$ where E is the smallest upper end in $pr_2(\mathcal{X})$ containing S.

Remark 6: It follows immediately from Propositions 8 and 9 that if $(\mathcal{X}, \alpha)$ is a complete canonical dense extension then $pr_1(\mathcal{X})$, $pr_2(\mathcal{X})$ are closure systems.

We now obtain the following characterization for a canonical dense extension to be complete.

Theorem 6: A canonical dense extension $(\mathcal{X}, \alpha)$ is complete if and only if (1) $pr_2(\mathcal{X})$ is a closure system and (2) for each U in $pr_2(\mathcal{X})$, the subset $\mathcal{X}(U) = \{(A, E) \in \mathcal{X} / E \supseteq U\}$ is a complete lattice, the ordering in $\mathcal{X}(U)$ being the restriction of the ordering of $\mathcal{X}$.

Proof: Suppose $(\mathcal{X}, \alpha)$ is a complete extension. Then by Remark 6 $pr_2(\mathcal{X})$ is a closure system. Next take any family $(A_i, E_i)_{i \in I}$ in $\mathcal{X}(U)$. Let $(A, E) = \bigvee^{\mathcal{X}} (A_i, E_i)$.

Then by Proposition 8, $E = \bigcap E_i$ and since each E_i contains U so does E. But then $(A, E) \in \mathcal{X}(U)$ and $\mathcal{X}(U) \subseteq \mathcal{X}$. Hence (A, E) is the join in $\mathcal{X}(U)$ as well. Hence $\mathcal{X}(U)$ is a complete lattice. Conversely suppose the given conditions hold. Let (A_i, E_i) be an arbitrary family in $\mathcal{X}$.

Let $U = \bigcap E_i$; Then since $pr_2(\mathcal{X})$ is a closure system $U \in pr_2(\mathcal{X})$. Consider the complete lattice $\mathcal{X}(U)$. Then (A_i, E_i) belongs to $\mathcal{X}(U)$ for each i. Let $(A, E) = \bigvee_{i \in I}^{\mathcal{X}(U)} (A_i, E_i)$. Then $E \supseteq U$ and if x in E then we have $\alpha(x) \geq (A_i, E_i)$ for each i and thus $x \in \bigcap E_i = U$. Hence $E = U$. Let (B, V) in $\mathcal{X}$ be an upper bound of the family. Then $V \subseteq U$ and this implies $\mathcal{X}(V) \supseteq \mathcal{X}(U)$. Let (C, W) be the join of (A_i, E_i) in $\mathcal{X}(V)$. Then we have that $(A, E) = (A, U) = \bigvee^{\mathcal{X}(U)}(A_i, E_i)$ $\geq \bigvee^{\mathcal{X}(V)}(A_i, E_i) = (C, W)$ which implies $U \subseteq W$. But if $x \in W$ then $\alpha(x) \geq (A_i, E_i)$ for each i, and thus $x \in U$. Hence $U = W$ and this gives $(C, W) = (C, U) \in \mathcal{X}(U)$ and thus $(C, W) \geq (A, E)$ for (A, E) is the least upper bound in $\mathcal{X}(U)$ of the given family. Hence (C, W) = (A, E). Finally $(B, V) \geq \bigvee^{\mathcal{X}(V)}(A_i, E_i) = \bigvee^{\mathcal{X}(U)}(A_i, E_i) = (A, E)$. Hence (A, E) is the join in $\mathcal{X}$ as well and hence $\mathcal{X}$ is a complete lattice. This completes the proof.

Dually we obtain

<u>Theorem 7</u>: A canonical dense extension $(\mathcal{X}, \alpha)$ is complete if and only if (1) $pr_1(\mathcal{X})$ is a closure system and (2) for each $B \in pr_1(\mathcal{X})$, the subset $\mathcal{X}(B) = \{(A, E) / A \supseteq B, (A, E) \in \mathcal{X}\}$ is a complete lattice under the restriction of the ordering of $\mathcal{X}$.

5. Dense Kernels

In this section we associate with each extension (E, w) certain dense, respectively join dense, meet dense extensions, study the relationships among them and note various properties of (E, w) which are inherited by these extensions.

Let P be a partially ordered set and (E, w) an extension of P.

Definition 14: An element a in E will be called join dense (meet dense) if and only if it is the join (meet) in E of images of elements of P less than or equal to (greater than or equal to) a. a is called dense in E if and only if $a \nleq b$ for any b in E implies $L(a, w) \nsubseteq L(b, w)$ or $U(a, w) \nsupseteq (b, w)$ and $b \nleq a$ for any b in E implies $L(b, w) \nsubseteq L(a, w)$ or $U(b, w) \nsupseteq U(a, w)$.

We introduce the following systems of lower and upper ends of P:

(1) $K_p (E, w) = \{(L(a, w), U(a, w)) \ / \ a \in E\}$.

(2) $J_p (E, w) = \{L(a, w) \ / \ a \in E\}$.

(3) $M_p (E, w) = \{U(a, w) \ / \ a \in E\}$.

We further introduce the following subsets of E:

(4) $K_i (E, w) = \{a \ / \ a \in E, \text{ a dense}\}$.

(5) $J_i (E, w) = \{a \ / \ a \in E, \text{ a join dense}\}$.

(6) $M_i (E, w) = \{a \ / \ a \in E, \text{ a meet dense}\}$.

When no ambiguity can occur as to which extension is being referred to we will use the symbols K_p, J_p, M_p, J_i, K_i and M_i without mentioning the extension.

Definition 15: The extensions (K_p, α), (J_p, α_1), (M_p, α_2) will be called respectively the projective dense, the projective join dense, the projective meet dense kernels of the extension (E, w).

Definition 16: The extensions (K_i, w) (J_i, w), (M_i, w) will be called respectively the injective dense, injective join dense, injective meet dense kernels of (E, w).

The following properties are easily verified:

(i) $K_p^2 \cong K_p$, $J_p^2 \cong J_p$, $M_p^2 \cong M_p$

$K_i^2 \cong K_i$, $J_i^2 \cong J_i$, $M_i^2 \cong M_i$

(ii) K_p (E, w) $\cong$ (E, w) if and only if (E, w) is dense.

J_p (E, w) $\cong$ (E, w) if and only if (E, w) is join dense

M_p (E, w) $\cong$ (E, w) if and only if (E, w) is meet dense.

Similarly the injective dense, join dense, meet dense kernels of an extension (E, w) are equivalent to (E, w) if and only if (E, w) is respectively dense, join dense, meet dense.

Proposition 10: The projective dense, join dense, meet dense kernels of (E, w) are projectively smaller than (E, w). Further if $(B, \varphi) \leq_i (C, \psi)$ then

(1) $K_p(B, \varphi) \leq_i K_p(C, \psi)$, (2) $J_p(B, \varphi) \leq_i J_p(C, \psi)$ and (3) $M_p(B, \varphi) \leq_i M_p(C, \psi)$. In particular equivalent extensions have equivalent projective dense, projective join dense, projective meet dense kernels.

Proof: Clearly the mapping f from (E, w) into K_p(E, w) given by f(a) = (L(a,w), U(a,w)) is an order epimorphism with $fw = \alpha$. Thus (E, w)

$\geqslant_p K_p$ (E, w) and the rest of the first part of the proposition follows similarly. If $(B, \varphi) \leq_i (C, \psi)$ then there exists an order isomorphism g from B into C with $g.\varphi = \psi$. Define h from $K_p(B, \varphi)$ into $K_p(C, \psi)$ by

$h((L(b, \varphi), U(b, \varphi)) = (L(g(b), \psi), U(g(b), \psi))$. Since g is an order isomorphism with $g.\varphi = \psi$ so is h with $h.\alpha = \alpha$. Hence $K_p(B, \varphi) \leq_i K_p(C, \psi)$. The appropriate modification of the mapping h establishes (2) and by duality we have (3). This completes the proof.

It is clear from the definitions of the injective kernels that the injective (dense, join dense, meet dense) kernels of a given extension are injectively smaller than the extension. This fact combined with property (ii) at once gives

Corollary 7: For any extension (E, w) of P, (1) $J_i(E, w) \leq_i J_p(E, w)$ and (2) $M_i(E, w) \leq_i M_p(E, w)$, (3) $K_i(E, w) \leq_i K_p(E, w)$.

In general the injective (meet dense, join dense) kernels are not equivalent with the projective (meet dense, join dense) kernels. Further an analagous statement to the second part of Proposition 10 does not hold for injective kernels in general as the following example shows.

Example 11: Let $P = \{a, b\}$ be a totally unordered set and $Q = \{a, b, x\}$, $R = \{a, b, y, z\}$ be two sets whose order relations are given by the following diagrams:

Clearly (Q, w), (R, w) are extensions of P, w being the identity map from P. Further of course $(Q, w) \leq_i (R, w)$. But $J_i(Q, w)$ is not injectively smaller than $J_i(R, w)$ since $J_i(R, w) = P$, a two element set and $J_i(Q, w)$ is equal to (Q, w) and Q is a three element set. Further $J_i(R, w)$ is not equivalent with $J_p(R, w)$ since clearly $J_p(R, w)$ is equivalent with (Q, w).

In at least one important special case injective (join, meet) kernels are equivalent respectively with projective (join, meet) kernels.

Proposition 11: Let (E, w) be a complete extension of P. Then (1) J_i (E, w) is equivalent with J_p (E, w) and (2) M_i (E, w) is equivalent with M_p (E, w).

Proof: It is enough to prove (1), the rest follows by duality. Consider the mapping h from J_p into J_i given h (L(a, w)) = $\bigvee^E \{w(x) \,/\, w(x) \leq a\}$. Since (E, w) is complete, h is well defined and further the image under h of elements of J_p clearly lie in J_i. Also if c is a join dense element then h(L(c, w)) = c. But, as is easily seen if h(L(a,w)) = c then L(a,w) = L(c,w).

Thus h is an order isomorphism with $h \circ \alpha = w$. Hence $J_p \leq_i J_i$. Since the reverse holds by Corollary 7 we have the proposition.

In view of Proposition 11 for a complete extension (E, w) we will speak of the join (meet) kernel of (E, w) and refer to the injective (join, meet) kernel of(E, w) which we now denote by J(E, w), M(E, w).

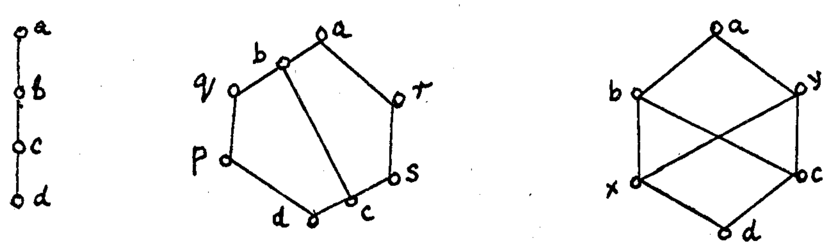
We now obtain the following characterization of the join (meet) kernel of a complete extension.

Theorem 8: The join dense (meet dense) kernel of any complete extension is a complete extension and is the injectively largest join dense (meet dense) extension which is injectively smaller than (E, w).

Proof: Since an arbitrary join of join dense elements of E is again a join dense element it follows that J(E, w) is a complete extension. Further if (S, π) is join dense and $(S, \pi) \leq_i (E, w)$ then by Propositions 10 and 11 we have $J_i\ (S, \pi) \leq_i J_p\ (S, \pi) \leq_i J\ (E, w)$. By the duality principle we have the theorem for the meet dense kernel of (E, w) as well. This completes the proof.

Example 11 shows that no corresponding characterization exists for join (meet) dense kernels of arbitrary extensions. We will study the case of join (meet) dense kernels of arbitrary extensions in a suitable category in Chapter 3. Further projective (injective) dense kernels of a complete extension need not be complete and the projective dense kernel need not be injectively smaller than the complete extension.

<u>Example 12</u>: Let $P = \{a, b, c, d\}$, $Q = \{a, b, c, d, p, q, r, s\}$ $R = \{a, b, c, d, x, y\}$ be sets whose orderings are as follows:



Then (Q, w), w the identity map from P is a <u>complete extension of P</u> and (R, w) is the projective dense kernel of (Q, w). We observe that <u>(R, w) is not complete</u> for the two element set $\{x, c\}$ has no join in R. Further (R, w) is not injectively smaller than (Q, w). Otherwise, if they existed an order isomorphism f from R into Q which kept P fixed then f must necessarily map x to one of p, q and y to one of r, s. However, x, y are related in R but p, q are related to neither of r, s. Hence no such f exists.

<u>Proposition 12:</u> If (E, w) is a $\mathcal{M}$ join, $\mathcal{N}$ meet faithful extension of P then the injective, respectively, projective (dense, join dense, meet dense) kernels of (E, w) are $\mathcal{M}$ join, $\mathcal{N}$ meet faithful.

Proof: We prove the proposition for the projective join dense kernel. The rest follows in a similar fashion. By Corollary 4 to Theorem 5 we must show L(a, w) is a $\mathcal{M}$ complete lower end. Take any M in $\mathcal{M}$ such that $\bigvee M$ exists and $M \subseteq L(a, w)$. Since (E, w) is $\mathcal{M}$ join faithful we have $w(\bigvee M) = \overset{E}{\bigvee}\{w(x) \,/\, x \in M\}$. Now clearly $a \geq w(M)$ and thus $a \geq \overset{E}{\bigvee} w(M) = w(\bigvee M)$. Hence $\bigvee M \in L(a, w)$. By duality one obtains that U(a, w) is a $\mathcal{N}$ complete upper end. Hence $J_p(E, w)$ is $\mathcal{M}$ join, $\mathcal{N}$ meet faithful. This completes the proof.

CHAPTER II

DENSE EXTENSIONS OF SPECIAL LATTICES.

This chapter is devoted to the study of dense extensions of meet continuous, Boolean and distributive lattices respectively. We establish that for each infinite cardinal m the collection of all m complete lower ends of a meet continuous lattice provides a join dense meet continuous extension. In particular we prove that there is up to equivalence precisely one complete completely faithful join dense meet continuous extension of a meet continuous lattice. We establish further that a Dedekind-Macneille extension of a Boolean lattice is up to equivalence the only finitely faithful Boolean extension which is a meet and join completion and obtain criteria for certain completely faithful dense extensions of a Boolean lattice to be infinitely meet distributive extensions. In addition we obtain a criterion other than that obtained by Funayama [8] for the Dedekind-Macneille extension of a distributive lattice to be a distributive extension. We use this criterion to obtain a proof of the well known result which states that the Dedekind-Macneille extension of a Boolean lattice is a Boolean extension.

1. Completions and Full Extensions.

In this section we introduce full extensions and join (meet) completions of a partially ordered set. We note the existence of an injectively smallest up to equivalence, full extension and give different characterizations of it.

Let P be a partially ordered set.

Definitional: An extension (E, w) of P will be called a join (meet) completion if and only if the image of every subset of P has a join (meet) in E.

Proposition 1: Let $(\mathcal{X}, \alpha)$ be a canonical dense extension of P. Then the following statements are equivalent:

(1) $(\mathcal{X}, \alpha)$ is a join completion.

(2) $\mathcal{X}$ contains the join admissible subset induced by $pr_1(\mathcal{X})$ and $pr_1(\mathcal{X})$ is a closure system.

In particular if $(\mathcal{X}, \alpha)$ is join dense then $(\mathcal{X}, \alpha)$ is a complete extension if and only if $(\mathcal{X}, \alpha)$ is a join completion.

Proof: (1) implies (2): If $(\mathcal{X}, \alpha)$ is a join completion then by Corollary 5 of the previous chapter it follows that $\mathcal{X}$ contains the join admissible subset induced by $pr_1(\mathcal{X})$. Next let $(A_i)_{i \in I}$ be an arbitrary family in $pr_1(\mathcal{X})$ and put $A = \bigcap_{i \in I} A_i$; Take $E_i \in pr_2(\mathcal{X})$ such that $(A_i, E_i) \in \mathcal{X}$ for each $i \in I$. Then $(A_i, E_i) \geqslant \bigvee \alpha(A) = (B, MaB)$, where by Corollary 5, B is the smallest lower end in $pr_1(\mathcal{X})$ containing A. But on the other hand from the previous line $B \subseteq A_i$ for each $i \in I$. Hence $A = B$ and $A \in pr_1(\mathcal{X})$ as required.

(2) implies (1): Take any subset S of P. Since $pr_1(\mathcal{X})$ is a closure system we know that the join of $\alpha_1(S)$ exists in $pr_1(\mathcal{X})$. Let this join be A. Then by our assumption $(A, MaA) \in \mathcal{X}$ and then by Corollary 5, $(A, MaA) = \bigvee^{\mathcal{X}} \alpha(S)$. Hence $(\mathcal{X}, \alpha)$ is a join completion and this

completes the proof.

Dually we have the following

<u>Proposition 2</u>: Let $(\mathcal{X}, \alpha)$ be a canonical dense extension of P. Then the following statements are equivalent:

(1) $(\mathcal{X}, \alpha)$ is a meet completion.

(2) $\mathcal{X}$ contains the meet admissible subset induced by $pr_2(\mathcal{X})$ and $pr_2(\mathcal{X})$ is a closure system. In particular if $(\mathcal{X}, \alpha)$ is meet dense then $(\mathcal{X}, \alpha)$ is a complete extension if and only if $(\mathcal{X}, \alpha)$ is a meet completion.

<u>Remark 1</u>: If (E, w) is a join (meet) completion then the injective and projective join (meet) dense kernels of (E, w) are equivalent. Observe that the mapping h from J_p into E given by $h(L(a, w)) = \bigvee^{E} \{w(x) \ / \ w(x) \leq a\}$ is actually onto J_i and is an order isomorphism with $h \circ \alpha_1 = \alpha_1$. Thus $J_p \leq_i J_i$ and since $J_i \leq_i J_p$ by Corollary 7 we have $J_i \cong J_p$. Hence <u>we will speak of the join (meet) dense kernel</u> of a join (meet) completion (E, w) and denote it by (J(E, w), w) ((M(E, w)), w) and if no confusion prevails by (J, w) ((M, w)) where J (respectively M) are the collection of join dense (meet dense) elements of E.

For any partially ordered set P and any sets $\mathcal{O}$, $\mathcal{E}$ of lower, upper ends satisfying $\mathcal{O}(P) \subseteq \mathcal{O} \subseteq \mathcal{O}_0$ and $\mathcal{E}(P) \subseteq \mathcal{E} \subseteq \mathcal{E}_0$ let us define:

(1) $P \otimes P = \{(x, y) \ / \ x, y \in P, x \leq y\}$. We partially order this set by the componentwise ordering.

(2) $\mathcal{X}_{(\mathcal{O}, \mathcal{E})}(P) = \{(A, E) \ / \ A \in \mathcal{O}, E \in \mathcal{E}, \ A \subseteq MiE\}$.

(3) $\mathcal{X}_N(P) = \{(A, E) / A \in \mathcal{O}\!\!\iota_N(P),\ E \in \mathcal{E}_N(P),\ A \subseteq MiE\}$.

(4) A mapping j from P into $P \otimes P$ given by $j(x) = (x, x)$.

It is clear that $(P \otimes P, j)$ is a <u>dense extension</u> of P. We now introduce the following definitions.

<u>Definition 2</u>: The admissible subset $\mathcal{X}_{(\mathcal{O}\!\iota,\mathcal{E})}(P)$ will be called the <u>full admissible subset</u> induced by $\mathcal{O}\!\iota$ and $\mathcal{E}$.

<u>Definition 3</u>: An extension (E, w) of P will be called <u>full</u> if and only if (1) (E, w) is dense, (2) $(P \otimes P, j) \leq_i (E, w)$ (3) if (S, π) is any extension with $J_p(S, \pi) \leq_i J_p(E, w)$, $M_p(S, \pi) \leq_i M_p(E, w)$ satisfying (1) and (2) then $(S, \pi) \leq_i (E, w)$.

A description for a canonical dense extension to be full is given by

<u>Proposition 3</u>: A canonical dense extension $(\mathcal{X}, \alpha)$ is full if and only if $\mathcal{X}$ is the full admissible subset induced by $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$.

<u>Proof</u>: Suppose $(\mathcal{X}, \alpha)$ is full and let $\mathcal{Y}$ be the full admissible subset induced by $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$. Then $\mathcal{X} \subseteq \mathcal{Y}$ and since $(\mathcal{Y}, \alpha)$ is dense, clearly injectively larger than $(P \otimes P, j)$ and satisfies $J_p(\mathcal{Y}) = J_p(\mathcal{X})$, $M_p(\mathcal{Y}) = M_p(\mathcal{X})$ we must have, $(\mathcal{X}, \alpha)$ being full, that $(\mathcal{Y}, \alpha) \leq_i (\mathcal{X}, \alpha)$. This means by Theorem 1 of the first chapter that $\mathcal{Y} \subseteq \mathcal{X}$. Hence $\mathcal{X} = \mathcal{Y}$. Conversely, let $\mathcal{X}$ be the full admissible set induced by $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$. Then certainly $(P \otimes P, j) \leq_i (\mathcal{X}, \alpha)$ and if $(\mathcal{Y}, \alpha)$ is any canonical dense extension with $pr_1(\mathcal{Y}) \subseteq pr_1(\mathcal{X})$, $pr_2(\mathcal{Y}) \subseteq pr_2(\mathcal{X})$, then immediately $\mathcal{Y}$ is contained in $\mathcal{X}$. This is seen to be true since $\mathcal{Y}$ is contained in the full admissible subset

induced by $pr_1(\mathcal{Y})$ and $pr_2(\mathcal{Y})$ which because of our assumption about $\mathcal{Y}$ is contained in $\mathcal{X}$. Hence $(\mathcal{X}, \alpha)$ is full. This completes the proof.

Definition 4: A canonical dense extension $(\mathcal{X}, \alpha)$ of P will be called a full canonical extension if and only if $\mathcal{X}$ is the full admissible subset induced by $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$.

Proposition 4: A full canonical extension $(\mathcal{X}, \alpha)$ is complete if and only if $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$ are closure systems.

Proof: If $(\mathcal{X}, \alpha)$ is complete then by Propositions 1 and 2 it follows that $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$ are closure systems. Conversely, assume $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$ are closure systems and let $(A_i, E_i)_{i \in I}$ be an arbitrary family in $\mathcal{X}$. Let A be the join of $(A_i)_{i \in I}$ in $pr_1(\mathcal{X})$ and E be the join of $(E_i)_{i \in I}$ in $pr_2(\mathcal{X})$. Then since $pr_2(\mathcal{X})$ is a closure system we must have $E = \bigcap E_i \subseteq \bigcap MaA_i$. Hence if x belongs to E, we have $\alpha_1(x) \geq A_i$ for each i, that is, $\alpha(x) \geq A$. Hence x belongs to MaA. Thus $E \subseteq MaA$ and $(\mathcal{X}, \alpha)$ being full we have by Proposition 3 that (A, E) belongs to $\mathcal{X}$ and is clearly the least upper bound of the family. Hence $(\mathcal{X}, \alpha)$ is a complete extension and this completes the proof.

Let (E, w) be a meet and join completion of P.

Definition 5: By the join component of an element a in E we mean the largest join dense element in E less than or equal to a. By the meet component of an element a in E we mean the smallest meet dense element in E greater than or equal to a.

<u>Definition 6</u>: An extension (E, w) of P will be called a <u>generalized Dedekind-Macneille</u> extension if and only if (1) it is full (2) it is complete (3) the join component of each a in E is meet dense and dually the meet component of each a in E is join dense.

A description of a generalized Dedekind-Macneille extension in terms of a full canonical extension is obtained in the following

<u>Proposition 5</u>: An extension (E, w) of P is a generalized Dedekind-Macneille extension if and only if (E, w) is equivalent with ($\mathcal{X}_N(P), \alpha$).

<u>Proof</u>: Suppose (E, w) is a generalized Dedekind-Macneille extension of P. Then by Proposition 3 (E, w) is equivalent to a full canonical extension ($\mathcal{X}, \alpha$). Since (E, w) is complete by Proposition 4, $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$ are closure systems and thus respectively contain $\mathcal{O}_N(P)$, $\mathcal{E}_N(P)$. Since the join component of any (A, E) in $\mathcal{X}$, which by Corollary 5 of the previous chapter is (A, MaA), is meet dense, we have by Corollary 6 of the previous chapter that A = MiE for a suitable E in $pr_2(\mathcal{X})$. But then $A \in \mathcal{O}_N(P)$. Since (A, E) was arbitrary in $\mathcal{X}$ this means that $pr_1(\mathcal{X}) = \mathcal{O}_N(P)$. By duality we have $pr_2(\mathcal{X}) = \mathcal{E}_N(P)$, in all, $\mathcal{X} = \mathcal{X}_N(P)$. Conversely since $\mathcal{X}_N(P)$ is by Propositions 3 and 4 full and complete and since for any $(A, E) \in \mathcal{X}_N(P)$, the join component (A, MaA), the meet component (MiE, E) are respectively meet dense, join dense we have that $(\mathcal{X}_N(P), \alpha)$ is a generalized Dedekind-Macneille extension. Finally since (E, w) is equivalent with ($\mathcal{X}_N(P), \alpha$) we have that (E, w) is a generalized Dedekind-Macneille extension. This completes the proof.

Proposition 6: A generalized Dedekind-Macneille extension is up to equivalence the injectively smallest full complete extension of P.

Proof: It is enough to work with canonical dense extensions. Suppose $(\mathcal{X}, \alpha)$ is any complete full canonical extension. Then by Proposition 4, $pr_1(\mathcal{X})$, $pr_2(\mathcal{X})$ are closure systems containing $\mathcal{O}(P)$, $\mathcal{E}(P)$ respectively. Hence $pr_1(\mathcal{X}) \supseteq \mathcal{O}_N(P)$, $pr_2(\mathcal{X}) \supseteq \mathcal{E}_N(P)$. Thus $(\mathcal{X}, \alpha)$ being full by Proposition 3 we get $\mathcal{X} \supseteq \mathcal{X}_N(P)$. Hence by Proposition 5 $(\mathcal{X}, \alpha)$ is injectively larger than a generalized Dedekind-Macneille extension. This completes the proof.

A further characterization is given in the following

Proposition 7: A generalized Dedekind-Macneille extension is up to equivalence the injectively largest dense extension in the class of all dense extensions whose projective join and meet dense kernels are equivalent.

Proof: Suppose $(\mathcal{X}, \alpha)$ is a canonical dense extension. Then its projective join dense kernel is given by $(pr_1(\mathcal{X}), \alpha_1)$ and its projective meet dense kernel by $(pr_2(\mathcal{X}), \alpha_2)$. If these are equivalent it follows that each is a meet and a join dense extension. Hence $pr_1(\mathcal{X}) \subseteq \mathcal{O}_N(P)$, $pr_2(\mathcal{X}) \subseteq \mathcal{E}_N(P)$. Hence $\mathcal{X}$ being contained in the full admissible subset induced by $pr_1(\mathcal{X})$ and $pr_2(\mathcal{X})$ is contained in $\mathcal{X}_N(P)$. Hence $(\mathcal{X}, \alpha) \leq_i (\mathcal{X}_N(P), \alpha)$ and this completes the proof.

An immediate corollary of the last two Propositions is the following.

Corollary 1: A generalized Dedekind-Macneille extension is up to equivalence the only full complete extension whose projective join and meet dense kernels are equivalent.

2. Meet Continuous Dense Extensions.

In this section we will prove that for every non-empty cardinal number m the join dense extension provided by the system of all m complete lower ends of a meet continuous lattice is meet continuous. In particular we shall deduce P. Crawley's result [5] that the largest complete completely faithful join dense extension of a meet continuous lattice is meet continuous and further we shall show that up to equivalence this is the only one. In addition we will obtain criteria for a Dedekind-Macneille extension of a meet continuous lattice to be a meet continuous lattice.

We begin with the following proposition which will be useful later.

Proposition 8: Let (E, w) be any join completion of a partially ordered set P and (R, π) any join dense extension injectively larger than the join kernel of (E, w). Then there exists exactly one join complete order homomorphism f from R into E with $f.\pi = w$. This f is given by $f(r) = \bigvee^{E} \{ w(x) \,/\, \pi(x) \leq r \}$.

Proof: Suppose f and g are any two join complete order homomorphisms from R into E with $f.\pi = g.\pi = w$. Then for any r in R we have $f(r) = f(\bigvee^{R} \{ \pi(x) \,/\, \pi(x) \leq r \}) = \bigvee^{E} \{ w(x) \,/\, \pi(x) \leq r \} =$ $g(\bigvee^{R} \{ \pi(x) \,/\, \pi(x) \leq r \}) = g(r)$. Hence f is unique if it exists.

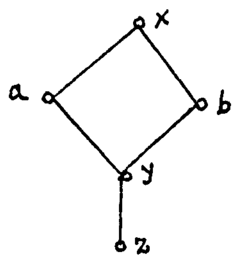
Now define a mapping f from R into E by $f(r) = \bigvee^{E}\{w(x) \,/\, \pi(x) \leq r\}$. Since the join kernel (J, w) of (E, w) is injectively smaller than (R, π) there exists an order isomorphism g from J into R with $g.w = \pi$. Now for any join dense element s in J we have by the definition of f, $f(g(s)) = \bigvee^{E}\{w(x) \,/\, \pi(x) \leq g(s)\} = s$, since g is an order isomorphism with $g.w = \pi$. If T is any subset of P such that $\bigvee^{R} \pi(T)$ exists then clearly $f(\bigvee^{R} \pi(T)) \geq \bigvee^{E} w(T)$. If $c \geq w(T)$ for any c in E, then the join component b of c is also an upper bound of w(T). Then $g(b) \geq \pi(T)$ and thus $g(b) \geq \bigvee^{R} \pi(T)$. Thus $f(g(b)) = b \geq f(\bigvee^{R} \pi(T))$. In all $f(\bigvee^{R} \pi(T)) = \bigvee^{E} w(T)$. Finally if S is any subset of R such that $\bigvee^{R} S$ exists then we have $f(\bigvee^{R} S) = f(\bigvee^{R}\{\pi(x) \,/\, \pi(x) \leq s,\ s \in S\}) = \bigvee^{E}\{w(x) \,/\, \pi(x) \leq s,\ s \in S\} = \bigvee^{E} f(S)$, using the fact that (R, π) is a join dense extension. This completes the proof.

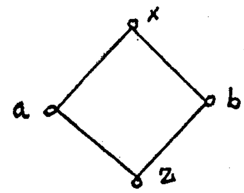
Dually one gets

Proposition 9: Let (E, w) be any meet completion of a partially ordered set P. If (R, π) is any meet dense extension of P which is injectively larger than the meet kernel of (E, w) then there exists precisely one meet complete homomorphism f from R into E with $f.\pi = w$.

In general the mapping f of Proposition 8 need not carry meets into meets. The following example illustrates this fact.

Example 1: Let $P = \{a, b\}$ be a totally unordered set, and $Q = \{a, b, x, y, z\}$, $R = \{a, b, x, z\}$ be sets whose order relations are given by the following diagrams:





Then (Q, i), (R, i) are extensions of P, i being the identity map. Further (Q, i) is a join completion of P and (R, i) is the join kernel of (Q, i). Then the unique map f by Proposition 8 maps $a \wedge_R b = z$ into z and $f(a) \wedge_E f(b) = a \wedge_E b = y$ and $y \neq z$.

Definition 7: A lattice L is called meet continuous if and only if for every up-directed family $(x_i)_{i \in I}$ such that $\bigvee^L_{i \in I} x_i$ exists we have $y \wedge \bigvee^L_{i \in I} x_i = \bigvee^L_{i \in I} y \wedge x_i$ for each y in L. Dually L is called join continuous if and only if for every down directed family $(x_i)_{i \in I}$, $y \vee \bigwedge_{L,\, i \in I} x_i = \bigwedge_{L,\, i \in I} y \vee x_i$ for every y in L.

Definition 8: A lattice L is called infinitely meet distributive if and only if for every y in L and every subset X of L such that $\bigvee^L X$ exists we have $y \wedge \bigvee^L X = \bigvee^L y \wedge X$. Dually L is called infinitely join distributive if and only if for every y in L and every subset X of L such that $\bigwedge_L X$ exists we have $y \vee \bigwedge_L X = \bigwedge_L y \vee X$.

Definition 9: An extension (E, w) of L will be called a meet continuous extension if and only if E and L are meet continuous lattices. (E, w) will be called a join continuous extension of L if and only if E and L are join continuous lattices. Similarly (E, w) will be called an infinitely meet (join) distributive extension if E and L are infinitely meet (join) distributive.

Proposition 10: Let (E, w) be any finitely faithful meet continuous join completion of a meet continuous lattice. Then the join dense kernel of (E, w) is a complete finitely faithful meet continuous extension.

Proof: Let (J, w) be the join dense kernel of (E, w). Since the join in E of arbitrary sets of join dense elements is again join dense it follows from Proposition 1 that (J, w) is complete. Further by Proposition 12 of the previous chapter we know that (J, w) is finitely faithful. Hence for any a, b in J the sets L(a, w), L(b, w) are up-directed and we have since E is meet continuous

$a \wedge_E b = \bigvee^E \{w(x) \,/\, w(x) \leq a\} \wedge_E \bigvee^E \{w(y) \,/\, w(y) \leq b\} = \bigvee^E \{w(x \wedge y) \,/\, w(x) \leq a,\ w(x) \leq b\}$.

Hence $a \wedge_E b$ belongs to J since it is join dense. But then $a \wedge_E b = a \wedge_J b$, that is, J is a join complete sublattice of E and hence J is meet continuous.

Dually we have

Proposition 11: Let (E, w) be any finitely faithful join continuous meet completion of a join continuous lattice. Then the meet dense kernel of (E, w) is a complete finitely faithful join continuous extension.

Let P be any partially ordered set and m any infinite cardinal.

Definition 10: We call a lower end A of P m-complete if and only if for every subset S of A such that $|S| \leq m$ and $\bigvee S$ exists the join $\bigvee S$ belongs to A. Dually we call an upper end E of P m complete if and only if for every subset S of E such that $|S| \leq m$ and $\bigwedge S$ exists the meet $\bigwedge S$ belongs to E.

We note that a lower (upper) end is complete as defined in the previous chapter if and only if it is m-complete for every non-empty cardinal m.

(1) Let $\mathcal{O}_m(P)$ be the set of all m complete lower ends of P.

(2) Let $\mathcal{E}_m(P)$ be the set of all m complete upper ends of P.

(3) Let $\mathcal{O}_K(P)$ be the set of all complete lower ends of P.

(4) Let $\mathcal{E}_K(P)$ be the set of all complete upper ends of P.

In view of Corollary 4 of the previous chapter we know that the extension $(\mathcal{O}_m(P), \alpha_1)$ is the injectively largest $\mathcal{M}$ join faithful canonical join dense extension of P and further that $(\mathcal{E}_m(P), \alpha_2)$ is the injectively largest $\mathcal{M}$ meet faithful canonical meet dense extension of P where $\mathcal{M} = \{S/\ S \subseteq P,\ |S| \leq m\}$. Further we know that $(\mathcal{O}_K(P), \alpha_1)$ is the injectively largest canonical join dense completely faithful extension of P and dually $(\mathcal{E}_K(P), \alpha_2)$ is the injectively largest canonical meet dense completely faithful extension of P. Each of these extensions is of course complete.

The following lemma gives a useful description of the join and meet operations in the complete lattice of m-complete lower ends.

Lemma 1: Let $(A_i)_{i \in J}$ be an arbitrary family of m-complete lower ends of a meet continuous lattice L. Then

(i) $\bigvee_{i \in J} A_i = \{x \,/\, x = \overset{L}{\bigvee} S,\ |S| \leq m,\ S \text{ up-directed and } S \subseteq \overset{I}{\bigvee}_{i \in J} A_i\}$

where $\overset{I}{\bigvee}$ refers to the join in the lattice of ideals of L. Further for any two m complete lower ends of L,

(ii) $A \wedge B = \{a \wedge b \,/\, a \in A,\ b \in B\}$.

Proof: The left side of (i) certainly contains the right side. We thus need only show that the right side of (i) is a m-complete lower end. Take any set T which is contained in the right side of (i) with $|T| \leq m$ and such that $\overset{L}{\bigvee} T$ exists. Then for each t in T there exists an up-directed set $S_t \subseteq \overset{I}{\bigvee}_{i \in J} A_i$ with $|S_t| \leq m$ and

$t = \overset{L}{\bigvee} S_t$.

Let C be the set of all finite joins from $S = \bigcup_{t \in T} S_t$. Then clearly

$\overset{L}{\bigvee} T = \overset{L}{\bigvee} C$, $C \subseteq \overset{I}{\bigvee}_{i \in J} A_i$, C is up-directed and $|C| \leq \sum_{t \in T} |S_t| \leq m$.

Hence $\overset{L}{\bigvee} T$ belongs to the right side of (i). Further take any x in L and suppose $x \leq \overset{L}{\bigvee} S$ where $\overset{L}{\bigvee} S$ belongs to the right side of (i). Then since L is meet continuous we have $x = x \wedge \overset{L}{\bigvee} S = \overset{L}{\bigvee} x \wedge S$. Since S is up-directed so is the set $x \wedge S$ and $|x \wedge S| \leq |S| \leq m$. Thus x belongs to the right side of (i) and therefore it is a m-complete lower end. (ii) follows immediately since $A \wedge B = A \cap B$. This completes the proof.

<u>Proposition 12</u>: Let L be a meet continuous lattice. Then the extension $(\mathfrak{A}_m(L), \alpha_1)$ is a meet continuous extension of L. Further if $(\mathfrak{A}, \alpha_1)$ is a complete canonical join dense extension injectively smaller than $(\mathfrak{A}_m(L), \alpha_1)$ then $(\mathfrak{A}, \alpha_1)$ is a meet continuous extension if and only if there exists a join complete lattice homomorphism f from $\mathfrak{A}_m(L)$ onto $\mathfrak{A}$ which maps $\mathfrak{A}$ identically.

<u>Proof</u>: Let $(A_i)_{i \in J}$ be any arbitrary up-directed family and let A be any element of $\mathfrak{A}_m(L)$. Take any a in A and any x in $\bigvee_{i \in J} A_i$. Then by the Lemma there exists an up-directed set S, such that $|S| \leq m$, $S \subseteq \bigvee^{I}_{i \in J} A_i$ with $x = \bigvee^{L} S$. Since L is meet continuous we then have $a \wedge x = \bigvee^{L} a \wedge S$. Now since each s in S belongs to $\bigvee^{I}_{i \in J} A_i$ there exists a finite subset $F_s \subseteq \bigcup_{i \in J} A_i$ with $s \leq \bigvee F_s$. But then since the family is up-directed we must have that F_s is contained in some suitable member A_s of the family. Hence $a \wedge S \subseteq \bigvee^{I}_{i \in J} A \wedge A_i$ and since $|a \wedge S| \leq m$ we have, by the Lemma, that $a \wedge x$ belongs to $\bigvee_{i \in J} A \wedge A_i$. Thus again by the Lemma it follows that $A \wedge \bigvee_{i \in J} A_i \subseteq \bigvee_{i \in J} A \wedge A_i$, that is, $(\mathfrak{A}_m(L), \alpha_1)$ is meet continuous. Further if $(\mathfrak{A}, \alpha_1)$ is a

join complete lattice homomorphic image of ($\mathcal{O}\!\mathcal{l}_m(L), \alpha_1$) then since the latter is meet continuous so is the former. Conversely consider the mapping f from $\mathcal{O}\!\mathcal{l}_m(L)$ onto $\mathcal{O}\!\mathcal{l}$ given by $f(A) = \bigvee^{\mathcal{O}\!\mathcal{l}} \{\alpha_1(x) \;/\; x \in A\}$. Then by Proposition 8, f is a join complete order homomorphism and since $\mathcal{O}\!\mathcal{l} \subseteq \mathcal{O}\!\mathcal{l}_m(L)$ we have that f restricted to $\mathcal{O}\!\mathcal{l}$ is the identity. Also since ($\mathcal{O}\!\mathcal{l}$, α_1) is meet continuous we have for any A, B $\in \mathcal{O}\!\mathcal{l}_m$

$f(A) \wedge f(B) = \bigvee^{\mathcal{O}\!\mathcal{l}} \{\alpha_1(x) \;/\; x \in A\} \wedge \bigvee^{\mathcal{O}\!\mathcal{l}} \{\alpha_1(y) \;/\; y \in B\} =$

$\bigvee^{\mathcal{O}\!\mathcal{l}} \{\alpha_1(x \wedge y) \;/\; x \in A, \;\; y \in B\} = f(A \wedge B)$. Thus f has the desired properties and this completes the proof.

<u>Corollary 2</u>: Crawley [5] . The extension ($\mathcal{O}\!\mathcal{l}_K(L), \alpha_1$) of a meet continuous lattice L is meet continuous.

<u>Proof</u>: Take a cardinal m with $m > |L|$. Then clearly $\mathcal{O}\!\mathcal{l}_K(L) = \mathcal{O}\!\mathcal{l}_m(L)$ and the corollary follows from Proposition 12.

Dually one obtains

<u>Proposition 13</u>: Let L be any join continuous lattice. Then the extension ($\mathcal{E}_m(L), \alpha_2$) is join continuous. If ($\mathcal{E}, \alpha_2$) is a complete canonical meet dense extension injectively smaller than ($\mathcal{E}_m(L), \alpha_2$) then ($\mathcal{E}, \alpha_2$) is join continuous if and only if there exists a meet complete lattice homomorphism f from $\mathcal{E}_m(L)$ onto $\mathcal{E}$ which maps $\mathcal{E}$ identically.

<u>Corollary 3</u>: If L is infinitely meet distributive then the extensions ($\mathcal{O}\!\mathcal{l}_m(L), \alpha_1$), ($\mathcal{O}\!\mathcal{l}_K(L), \alpha_1$) are infinitely meet distributive. Dually if L is infinitely join distributive then the extensions ($\mathcal{E}_m(L), \alpha_2$), ($\mathcal{E}_K(L), \alpha_2$) are infinitely join distributive.

<u>Proof</u>: Take any A, B, C in $\mathcal{O}\!\mathcal{l}_m(L)$. By Proposition 12 $\mathcal{O}\!\mathcal{l}_m(L)$ is meet

continuous. Hence we must show that it is distributive. Take any a in A, x in $B \vee C$. Then by the Lemma $x = \bigvee S$ where $S \subseteq B \overset{I}{\vee} C$ and $|S| \leq m$. Since L is distributive we have that for each s in S there exists b_s, c_s in B, C with $s = b_s \vee c_s$. Thus $a \wedge x = \bigvee a \wedge S = \bigvee (a \wedge b_s) \vee (a \wedge c_s) \in (A \wedge B) \vee (A \wedge C)$. This shows that $(\mathfrak{A}_m(L), \alpha_1)$ is distributive. We get the rest by duality. This completes the proof.

<u>Proposition 14</u>: Let L be any meet continuous lattice and (E, w) any complete completely join faithful, finitely meet faithful meet continuous extension of L. Then the join kernel of (E, w) is equivalent to $(\mathfrak{A}_K(L), \alpha_1)$.

<u>Proof</u>: Let J be the collection of all join dense elements of E. Consider the mapping f from $\mathfrak{A}_K(L)$ into J given by $f(A) = \overset{E}{\bigvee} \{w(x) \,/\, x \in A\}$. f is clearly an order homomorphism with $f \cdot \alpha_1 = w$. Suppose $f(A) \leq f(B)$. Then for any x in A we have $w(x) \leq \overset{E}{\bigvee} \{w(y) \,/\, y \in B\}$ and this implies, since E is finitely meet faithful and meet continuous, that $w(x) = \overset{E}{\bigvee} \{w(x \wedge y) \,/\, y \in B.\}$ Thus $x = \overset{L}{\bigvee} \{x \wedge y \,/\, y \in B\}$ and since B is a complete lower end we have x belongs to B. Thus f is an order isomorphism and hence $(\mathfrak{A}_K(L), \alpha_1) \leq_i (J, w)$. However, by Proposition 12 of the previous chapter we have that (J, w) is completely faithful. Thus (J, w) $\leq_i (\mathfrak{A}_K(L), \alpha_1)$ for $(\mathfrak{A}_K(L), \alpha_1)$ is the injectively largest completely faithful extension. Hence by the equivalence theorem for dense extensions we have the proposition. This completes the proof.

Combining Proposition 14 and Corollary 2 we get

Theorem 1: The extension ($\mathfrak{A}_K(L)$, α_1) is the only canonical join dense complete completely faithful meet continuous extension of a meet continuous lattice L.

Proof: By Corollary 2 ($\mathfrak{A}_K(L)$, α_1) is a meet continuous join dense extension and it is certainly complete completely faithful. Now if ($\mathfrak{A}$, α) is any complete completely faithful canonical join dense extension which is meet continuous then by Proposition 14 we have at once that $\mathfrak{A} = \mathfrak{A}_K(L)$. This completes the proof.

Theorem 2: The extension($\mathcal{E}_K(L)$, α_2) is the only canonical meet dense complete completely faithful join continuous extension of a join continuous lattice L.

We have immediately

Corollary 4: The extension ($\mathfrak{A}_K(L)$, α_1) is the only canonical join dense complete completely faithful infinitely meet distributive extension of an infinitely meet distributive lattice L and dually the extension ($\mathcal{E}_K(L)$, α_2) is the only canonical meet dense complete completely faithful infinitely join distributive extension of an infinitely join distributive lattice L.

Theorems 1 and 2 imply respectively that in general a Dedekind-Macneille extension of a meet (join) continuous lattice will fail to be meet (join) continuous since in general not every complete lower (upper) end is normal. Dilworth and McLaughlin [7] have given an example of an infinitely meet distributive lattice whose Dedekind-Macneille completion is not even modular. The following is an example of an infinitely meet distributive lattice whose Dedekind-Macneille completion is not meet continuous.

Example 2: Let $P = [0, 1)$ be the half open unit interval of the reals in the usual ordering and let $Q = P \times P$, the Cartesian product of P with itself, partially ordered under the component wise ordering. Then Q is clearly a lattice in which finite meets and arbitrary existing joins are computed component wise. Hence Q is infinitely meet distributive. Consider the following subset of Q:

$$A = \left\{(x, y) \,/\, x, y \in P,\ 0 \leq x \leq \tfrac{1}{2},\ 0 \leq y < 1\right\}.$$

Clearly A is a complete lower end. However, A is not normal since $MaA = \emptyset$ and thus $Mi\,MaA = Q \neq A$. Hence by Theorem 1 it follows that the Dedekind-Macneille completion of Q is not meet continuous.

Remark 2: For any subset S of a partially ordered set L we have

(1) $MiMaS = \bigcap\left\{C / S \subseteq C,\ C \in \mathcal{O}_N(L)\right\}$ and

(2) $MaMiS = \bigcap\left\{E / S \subseteq E,\ E \in \mathcal{E}_N(L)\right\}$.

Call the right side of (1) Q. Then certainly Q is a subset of Mi MaS. On the other hand if $x \in$ Mi MaS then take any normal lower end C with $C \supseteq S$. Then $MiMaC = C \supseteq MiMaS$ and thus Q contains MiMaS. Thus (1) holds and we get (2) dually.

We now establish the following criterion using this observvation.

Theorem 3: Let L be a meet continuous lattice. Then the following statements are equivalent:

(1) A Dedekind-Macneille extension of L is meet continuous.

(2) For every directed subset S of L if $x \in$ Mi MaS then $x = \bigvee x \wedge S$.

(3) $\mathfrak{A}_N(L) = \mathfrak{A}_K(L)$, that is every complete lower end is normal.

Proof: (1) implies (2): Take any directed subset S of L and suppose $x \in$ Mi MaS. Suppose $y \geq S$. Then $\alpha_1(y) \geq \overset{\mathfrak{A}_N}{\bigvee} \{\alpha_1(s) \ / \ s \in S\} = C$, say. Then $\alpha_1(y) \geq C \geq \alpha_1(S)$ and thus by Remark 2 we get $\alpha_1(x) \leq C$. Then using (1) we have $\alpha_1(x) = \overset{\mathfrak{A}_N}{\bigvee} \{\alpha_1(x \wedge s) \ / \ s \in S\}$ and this implies $x = \bigvee x \wedge S$.

(2) implies (3): Let A be any complete lower end and take x in MiMaA. Then by (2) $x = \bigvee x \wedge A$ and since $x \wedge A \subseteq A$ and A is complete we get that x belongs to A. Thus A = MiMaA, that is, A is normal.

(3) implies (1): This is clear by Theorem 1. This completes the proof.

Dually we obtain

Theorem 4: Let L be any join continuous lattice. Then the following statements are equivalent:

(1) A Dedekind-Macneille completion of L is join continuous.

(2) For every directed subset S of L if $x \in$ MaMiS then $x = \bigwedge x \vee S$.

(3) $\mathcal{E}_N(L) = \mathcal{E}_K(L)$, that is, every complete upper end is normal.

Corollary 5: Let L be an infinitely meet distributive lattice. Then the following statements are equivalent:

(1) A Dedekind-Macneille completion of L is infinitely meet distributive.

(2) For every subset S of L if $x \in \text{MiMaS}$ then $x = \bigvee x \wedge S$.

(3) $\mathfrak{A}_N(L) = \mathfrak{A}_K(L)$, that is, every complete lower end is normal.

Corollary 6: A necessary and sufficient condition that there exist a finitely meet faithful complete meet dense meet continuous extension of a meet continuous lattice L is that for every directed subset S of L if $x \in \text{MiMaS}$ then $x = \bigvee x \wedge S$.

Proof: If the condition holds then by Theorem 3 the Dedekind-Macneille completion of L is meet continuous. On the other hand, if (E, w) is a finitely meet faithful meet continuous extension of L, then by Proposition 10 the join dense kernel of (E, w) is meet continuous. However, since (E, w) is meet dense its join dense kernel is equivalent to a Dedekind-Macneille completion. Then using Theorem 3 we have the corollary.

Similarly one has

Corollary 7: A necessary and sufficient condition that there exist a finitely meet faithful complete meet dense infinitely meet distributive extension of an infinitely meet distributive lattice L is that for every subset S of L if $x \in \text{MiMaS}$ then $x = \bigvee x \wedge S$.

Remark 3: It follows from Corollary 6 that the lattice Q of Example 2 is an example of a lattice which is infinitely meet distributive but has no complete finitely meet faithful meet dense meet continuous extensions.

3. Boolean Lattices and Dense Extensions.

In this section we prove that every complete lower end of a Boolean lattice L is normal and use this fact and its dual to obtain a characterization of dense complete completely faithful infinitely meet (join) distributive extensions of L. Further we establish that up to equivalence the Dedekind-Macneille completion of L is the only finitely faithful meet and join complete Boolean extension. In addition we obtain a proof of the Stone-Glivenko theorem which states that the Dedekind-Macneille completion of a Boolean lattice is a Boolean extension.

Let B be any Boolean lattice with zero element 0 and unit element e. For any x in B let x' be the complement of x. Further for any subset X of B we put

(1) $X' = \{x' \,/\, x \in X\}$

(2) $\mathrm{Mi}\, X' = \{y \,/\, y \in B,\ y \leq X'\}$

(3) $\mathrm{Ma}\, X' = \{y \,/\, y \in B,\ y \geq X'\}$

Remark 4: For any lower end A of B, the normal end MiMaA has a complement in $\mathcal{O}\!\iota_N$ which is given by $\mathrm{Mi}A'$. Dually for every upper end E of B the normal upper end MaMiE has a complement in $\mathcal{E}_N$ and this is given by $\mathrm{Ma}E'$.

Let L be any lattice with zero element 0 and unit element e. Consider the following conditions on L:

(CI) For every x, z and every subset S of L such that $z \geq x \wedge S$ there exists a t in L with $z \vee t \geq S$ and $t \wedge x = 0$.

(CII) For every x, z and every subset T of L such that $z \leq x \vee T$ there exists a s in L with $z \wedge s \leq T$ and $s \vee x = e$.

<u>Proposition 15</u>: Let L be an infinitely meet distributive lattice with zero element 0 satisfying (CI). Then the Dedekind-Macneille completion of L is infinitely meet distributive.

<u>Proof</u>: By Corollary 5 of the previous section it is enough to show that every complete lower end is normal. Let A be a complete lower end and take any x in MiMaA. Put $B = \{a \,/\, a \in A,\ a \leq x\}$. Let y in L be any upper bound of B. Then for each b in A we have $b \wedge x$ belongs to B and thus $b \wedge x \leq y$. But then by (CI) there exists a t in L with $t \vee y \geq b$ for each b in A and such that $t \wedge x = 0$. This gives that $x \leq t \vee y$, that is, $x = (x \wedge t) \vee (x \wedge y) = x \wedge y \leq y$. Thus x is the least upper bound of B. Since A is a complete lower end we then have that x belongs to A. This completes the proof.

Dually we obtain,

<u>Proposition 16</u>: Let L be an infinitely join distributive lattice with unit element e satisfying (CII). Then the Dedekind-Macneille extension of L is infinitely join distributive.

<u>Corollary 8</u>: Let L be any Boolean lattice. Then $\mathfrak{A}_N(L) = \mathfrak{A}_K(L)$, $\mathcal{E}_N(L) = \mathcal{E}_K(L)$, that is, every complete lower (upper) end of L is a normal lower (upper) end.

<u>Proof</u>: Take any x, z in L and let S be any subset of L with $z \geq x \wedge S$.

Then $z \vee x' \geq S$ since L is distributive and $x \wedge x' = 0$. Thus L satisfies (CI) and by duality (CII) as well. Then applying first Propositions 15, 16 and then Corollary 5 we have our result. This completes the proof.

<u>Corollary 9</u>: (Stone-Glivenko) [2]. The Dedekind-Macneille extension ($\mathfrak{A}_N(L), \alpha_1$) of a Boolean lattice L is a Boolean extension, that is, $\mathfrak{A}_N(L)$ is a Boolean lattice.

<u>Proof</u>: Since L satisfies (CI) and (CII) we have by Propositions 15 and 16 that $\mathfrak{A}_N(L)$ is infinitely distributive. We observed in Remark 4 that $\mathfrak{A}_N(L)$ is complemented. Hence, in all, it is a Boolean lattice. This completes the proof.

<u>Corollary 10</u>: The canonical full extension ($\mathcal{X}_N(L), \alpha$) of a Boolean lattice is the only canonical full complete completely faithful extension of L. It is further an infinitely distributive extension.

<u>Proof</u>: First of all ($\mathcal{X}_N(L), \alpha$) by Proposition 6 and Theorem 5 of the previous chapter has the properties mentioned. Further if $(\mathcal{X}, \alpha)$ is any full complete completely faithful canonical extension then by Proposition 6 we get that $\mathcal{X}_N \subseteq \mathcal{X}$ and by Theorem 5 of the previous chapter $\mathcal{X} \subseteq \mathcal{X}_K(L)$, the full admissible subset induced by $\mathfrak{A}_K(L)$, $\mathcal{E}_K(L)$. But then by Corollary 8 we have that $\mathcal{X}_K(L) = \mathcal{X}_N(L)$. Hence $\mathcal{X} = \mathcal{X}_N(L)$. The rest follows from Corollary 9. This completes the proof.

Let P be a partially ordered set with zero element 0 and unit element e. Let (E, w) be an extension of P.

Definition 11: An element a of E will be called join null if and only if L(a, w) $= \{o\}$ and it will be called meet null if and only if U(a, w) $= \{e\}$

The following lemma will be useful.

Lemma 2: Let L be a Boolean lattice and $(\mathcal{X}, \alpha)$ a complete canonical dense extension such that $\mathcal{X}$ is a distributive lattice. Then each element of $\mathcal{X}$ is a join of its join component and a join null element. Dually each element of $\mathcal{X}$ is a meet of its meet component with a meet null element.

Proof: Take any (A, E) in $\mathcal{X}$. Then its join component by Corollary 5 of the previous chapter is (A, MaA). Also since $\mathcal{X}$ is complete, $\mathcal{X}$ contains the join admissible subset induced by $\mathcal{O}_N(L)$. Thus $(MiA', Ma\,MiA') \in \mathcal{X}$. Now clearly $MaA \cap Ma\,MiA' = \{e\}$, e the unit of L. Thus $(A, E) \leq (A, MaA) \vee (MiA', Ma\,MiA')$. Then since $\mathcal{X}$ is distributive we have $(A,E) = (A, MaA) \vee ((A,E) \wedge (MiA', MaMiA'))$. But since $A \cap MiA' = \{0\}$, we have that $((A, E) \wedge (MiA', Ma\,MiA'))$ is a join null element. The remainder of the lemma follows by duality. This completes the proof.

We now obtain the following description for certain complete completely faithful canonical dense extensions of a Boolean lattice to be infinitely meet distributive extensions.

Theorem 5: Let $(\mathcal{X}, \alpha)$ be a complete completely faithful canonical dense extension of a Boolean lattice L. Suppose that the join null elements of $\mathcal{X}$ form a complete lattice and the meet null elements of $\mathcal{X}$ form a lattice. Then the following conditions are equivalent:

(1) $(\mathcal{X}, \alpha)$ is an infinitely meet distributive extension.

(2) $\mathcal{X}$ is a distributive lattice.

(3) $\mathcal{X}$ is a join complete sublattice of $\mathcal{X}_N(L)$.

Proof: (1) implies (2) clearly, moreover by Corollary 10 we have that (3) implies (1). Thus we must show that (2) implies (3). Take an arbitrary family $(A_i, E_i)_{i \in I}$ in $\mathcal{X}$ and let $(A, E) = \bigvee^{\mathcal{X}}_{i \in I}(A_i, E_i)$.

By Lemma 2 there exist join null elements $((0), U_i)$, $((0), U)$ in $\mathcal{X}$ such that for each $i \in I$, $(A_i, E_i) = (A_i\ MaA_i) \vee ((0), U_i)$, where $(o) = \alpha_1(o)$ and $(A, E) = (A, MaA) \vee ((o), U)$. Let $\bigvee^{\mathcal{X}}(A_i, MaA_i) = (B, MaB)$, where $B = \bigvee^{\mathcal{O}_N}_{i \in I} A_i$ and since by assumption the join of join null elements is join null let $\bigvee^{\mathcal{X}}((o), U_i) = ((o), V)$. It is clear that A contains B. Suppose (C, MaC) is any join dense element of $\mathcal{X}$ less than or equal to (A, E) with $(C, MaC) \wedge (B, MaB) = \alpha(o)$, the zero element of $\mathcal{X}$. Then since $\mathcal{X}$ is distributive we get $(C, MaC) = (C, MaC) \wedge ((B, MaB) \vee ((o), V)) \leq ((o), V)$ and this gives that $C = \alpha_1(o)$. Thus $\alpha(o)$ is the only join dense element of $\mathcal{X}$ with $\alpha(o) \leq (A, E)$ and such that $\alpha(o) \wedge (B, MaB) = \alpha(o)$. Since $(\mathcal{X}, \alpha)$ is completely faithful and since $\mathcal{O}_K(L) = \mathcal{O}_N(L)$, $\mathcal{E}_K(L) = \mathcal{E}_N(L)$ we indeed have that $\mathcal{X} \subseteq \mathcal{X}_N(L)$. Then A, B both belong to $\mathcal{O}_N(L)$. Let D in $\mathcal{O}_N(L)$ be the relative complement of B in A. Then $A = B \overset{\mathcal{O}_N}{\vee} D$ and $B \wedge D = \alpha_1(o)$. But then since

(D, MaD) $\leq$ (A, E) we have from the above that D $=\alpha_1(o)$ and thus A = B. Thus the joins in $\mathcal{X}$ and $\mathcal{X}_N(L)$ coincide. By duality we establish that finite meets in $\mathcal{X}$, $\mathcal{X}_N(L)$ coincide. This completes the proof.

Dually we obtain

Theorem 6: Let $(\mathcal{X}, \alpha)$ be a complete completely faithful canonical dense extension of a Boolean lattice L. Suppose that the meet null elements of $\mathcal{X}$ form a complete lattice and the join null elements form a lattice. Then the following conditions are equivalent:

(1) $(\mathcal{X}, \alpha)$ is an infinitely join distributive extension.

(2) $\mathcal{X}$ is a distributive lattice.

(3) $\mathcal{X}$ is a meet complete sublattice of $\mathcal{X}_N(L)$.

Definition 12: An extension (E, w) of a Boolean lattice L will be called a Boolean extension if and only if E is a Boolean lattice.

The following lemma is useful in determining the dense Boolean extensions of a Boolean lattice.

Lemma 3: Let $(\mathcal{X}, \alpha)$ be any finitely faithful canonical dense Boolean extension of L which is a join and meet completion. Let (A, E) be any element of $\mathcal{X}$ and let (B, F) be its complement in $\mathcal{X}$. Then (1) Mi MaB = Mi MaE$'$ and (2) Ma MiF = Ma MiA$'$.

Proof: Since $(\mathcal{X}, \alpha)$ is finitely faithful we have, for any x in L, that $\alpha(x) \vee \alpha(x') = \alpha(e)$, $\alpha(x) \wedge \alpha(x') = \alpha(o)$ and thus $\alpha(x)' = \alpha(x')$. Now the join component (A, MaA) of (A, E) being a join dense element has for its complement the meet dense element $\bigwedge\{\alpha(x') \,/\, x \in A\} =$ (Mi A', U) for some suitable U in $pr_2(\mathcal{X})$ with Mi U = Mi A'. Since (A, MaA) is the join component of (A, E) we have that its complement (MiA', U) is the meet component of (B, F). But the meet component of (B, F) is (MiF, F). Thus MiF = MiA', that is, MaMiF = MaMiA'. The rest of the lemma follows by duality. This completes the proof.

We conclude this section by establishing

<u>Theorem 7</u>: A Dedekind-Macneille extension of a Boolean lattice L is up to equivalence the only finitely faithful dense Boolean extension which is a meet and join completion.

<u>Proof</u>: Firstly a Dedekind-Macneille extension of L has the mentioned properties. Next suppose $(\mathcal{X}, \alpha)$ is any finitely faithful dense Boolean meet and join completion of L. Take any (A, E) $\in \mathcal{X}$. Then since $\mathcal{X}$ is distributive we have by Lemma 2 a join null element ((o), U) in $\mathcal{X}$ such that (A, E) = (A, MaA) $\vee$ ((o), U) where o is the zero element of L and (o) $= \alpha_1$(o), the lower end consisting of the zero element only. Let (B, F) be the complement of ((o), U). Then by Lemma 3 we have that $F \subseteq$ MaMiF $\subseteq$ MaMi $\{o\}'$ $= \{e\}$, e the unit of L. Hence $F = \{e\}$. But then clearly ((o), U) $\leq$ (B, F) and thus ((o), U) $= \alpha$(o), the zero element of $\mathcal{X}$. Hence (A, E) = (A, MaA), that is, each element of $\mathcal{X}$ is join dense. By duality we then have that each element is meet dense as well. Finally since $(\mathcal{X}, \alpha)$ is a join (meet) completion we get that $(\mathcal{X}, \alpha)$ is equivalent with a Dedekind-Macneille completion. This completes the proof.

4. Distributive Lattices and Dense Extensions.

In this section we determine the complete completely faithful join (meet) dense distributive extensions of an infinitely meet (join) distributive lattice. We obtain also a criterion, different from Funayama's [8], for the Dedekind-Macneille extension of a distributive lattice to be a distributive extension. We further give necessary and sufficient conditions for a given join dense completely faithful extension of a distributive lattice to be a distributive extension.

Definition 13: An extension (E, w) of a distributive lattice L will be called a distributive extension if and only if E is a distributive lattice.

Proposition 17: Let $(\mathfrak{A}, \alpha_1)$ be any completely faithful canonical join dense distributive extension of a distributive lattice L. Then $\mathfrak{A}$ is a sublattice of $\mathfrak{A}_K(L)$, the lattice of complete lower ends of L.

Proof: Since $(\mathfrak{A}, \alpha_1)$ is a join dense completely faithful extension we have that $\mathfrak{A} \subseteq \mathfrak{A}_K(L)$. Hence for any A, B in $\mathfrak{A}$ we have $A \overset{\mathfrak{A}}{\vee} B \supseteq A \overset{\mathfrak{A}_K}{\vee} B$. Take any y in $A \overset{\mathfrak{A}}{\vee} B$. Then $\alpha_1(y) \leq A \overset{\mathfrak{A}}{\vee} B$ and since $\mathfrak{A}$ is a distributive lattice we have $\alpha_1(y) = (\alpha_1(y) \wedge_{\mathfrak{A}} A) \overset{\mathfrak{A}}{\vee} (\alpha_1(y) \wedge_{\mathfrak{A}} B)$. Hence $\alpha_1(y) = \overset{\mathfrak{A}}{\bigvee}(\alpha_1(y \wedge a) \vee \alpha_1(y \wedge b))$, the join being taken over all a in A, $b \in B$. Thus $\alpha_1(y) = \overset{\mathfrak{A}}{\bigvee}\{\alpha_1((y \wedge a) \vee (y \wedge b)) / a \in A, b \in B\}$. This immediately gives $y = \overset{L}{\bigvee} \{(y \wedge a) \vee (y \wedge b) / a \in A, b \in B\}$.

Hence y belongs to $A \overset{\mathfrak{O}_K}{\vee} B$. Of course the meet in both lattices is set intersection. This completes the proof.

Dually we get

Proposition 18: Let $(\mathcal{E}, \alpha_2)$ be any completely faithful canonical meet dense distributive extension of a distributive lattice L. Then $\mathcal{E}$ is a sublattice of $\mathcal{E}_K(L)$, the lattice of complete upper ends of L.

Corollary 11: Let $(\mathfrak{O}, \alpha_1)$ be any canonical join dense completely faithful ~~distributive~~ extension of an infinitely meet distributive lattice L. Then $(\mathfrak{O}, \alpha_1)$ is a distributive extension if and only if $\mathfrak{O}$ is a sublattice of $\mathfrak{O}_K(L)$. Dually let $(\mathcal{E}, \alpha_2)$ be any canonical meet dense completely faithful ~~distributive~~ extension of an infinitely join distributive lattice L. Then $(\mathcal{E}, \alpha_2)$ is a distributive extension if and only if $\mathcal{E}$ is a sublattice of $\mathcal{E}_K(L)$.

Theorem 8: Let L be a distributive lattice. A Dedekind-Macneille extension (E, w) of L is a distributive extension if and only if for all x, y in L, a in E we have $w(x) \wedge (w(y) \vee a) \leq w(x \wedge y) \vee (w(x) \wedge a)$.

Proof: If (E, w) is a distributive extension then the condition clearly holds. Conversely suppose that the condition holds. Take any a, b, c in E with $a \wedge b \leq a \wedge c$, $a \vee b \leq a \vee c$. To show that E is a distributive lattice we must show that $b \leq c$. Take any x, y in L with $w(x) \leq b$ and $w(y) \geq c$. Then $w(x) \wedge a \leq w(y) \wedge a$, $w(x) \vee a \leq w(y) \vee a$. Then $w(x) \leq w(x) \wedge (w(y) \vee a) = w(x \wedge y) \vee (w(x) \wedge a)$ using our assumption.

Then $w(x) \leq w(x \wedge y) \vee (w(y) \wedge a) \leq w(y)$. Thus $w(x) \leq \bigwedge \{w(y) / w(y) \geq c\} = c$ since c is a meet dense element. Hence $c \geq \bigvee \{w(x) / w(x) \leq b\} = b$ since b is a join dense element. Thus $b \leq c$ and (E, w) is a distributive extension. This completes the proof.

<u>Corollary 12</u>: Let L be a distributive lattice. A necessary and sufficient condition for there to exist a complete completely faithful dense distributive extension whose join and meet dense kernels are equivalent is that for any x, y in L and any normal lower end A we have $\alpha_1(x) \wedge (\alpha_1(y) \overset{\mathfrak{A}_N}{\vee} A) \leq \alpha_1(x \wedge y) \overset{\mathfrak{A}_N}{\vee} (\alpha_1(x) \wedge A)$.

<u>Proof</u>: If the stated condition holds then it follows by Theorem 8 that the Dedekind-Macneille extension is distributive and this is an extension of the desired type. Conversely let $(\mathcal{X}, \alpha)$ be a canonical dense extension of the desired type. Then by Proposition 7, $\mathcal{X} \subseteq \mathcal{X}_N(L)$. Since $\sigma(\mathfrak{A}_N)$ is a complete sublattice of $\mathcal{X}_N(L)$ we get that the set J of join dense elements of $\mathcal{X}$ equals the set M of meet dense elements of $\mathcal{X}$. But then the equal subsets J, M form a sublattice of $\mathcal{X}$. Since $\mathcal{X}$ is distributive then so is J and the extension (J, α) is a distributive Dedekind-Macneille extension of L. Hence by Theorem 8 the condition holds and this completes the proof.

<u>Corollary 13</u>: (Stone-Glivenko) [2]. A Dedekind-Macneille extension of a Boolean lattice L is a Boolean extension.

Proof: (1) For any y in L and any normal lower end A we have: $\alpha_1(y') \wedge A = \alpha_1(y') \wedge (\alpha_1(y) \vee A)$ where the unadorned join, meet symbols refer to the lattice $\mathfrak{A}_N(L)$, and y' is the complement of y. To see this take any z in L with $z \geq y' \wedge A$. Then $z \vee y \geq y \vee (y' \wedge a) \geq a$ for each a in A. Thus $\alpha_1(z \vee y) \geq \alpha_1(y) \vee A$. But then $\alpha_1(z) \geq \alpha_1(y') \wedge \alpha_1(y \vee z) \geq \alpha_1(y') \wedge (\alpha_1(y) \vee A)$. Hence $\alpha_1(y') \wedge A \supseteq \alpha_1(y') \wedge (\alpha_1(y) \vee A)$ and since the reverse inclusion is clear we have (1).

(2) For any x in L we have: $\alpha_1(x) \vee A = \alpha_1(x) \underset{\mathcal{J}(L)}{\vee} A$, where $\mathcal{J}(L)$ is the lattice of all ideals of L. Take any z with $\alpha_1(z) \leq \alpha_1(x) \vee A$. Then using (1) we have $\alpha_1(z \wedge x') \leq \alpha(x') \wedge A \leq A$. Thus $\alpha_1(z) \leq \alpha_1(x) \vee \alpha_1(z \wedge x') \leq \alpha_1(x) \underset{\mathcal{J}(L)}{\vee} A$. Hence (2) is established.

(3) For any x, y in L, using (2) and the fact that the lattice of ideals of L is distributive we have that $\alpha_1(x) \wedge (\alpha_1(y) \vee A) = \alpha_1(x) \wedge (\alpha_1(y) \underset{\mathcal{J}(L)}{\vee} A) = \alpha_1(x \wedge y) \underset{\mathcal{J}(L)}{\vee} (\alpha_1(x) \wedge A) = \alpha_1(x \wedge y) \vee (\alpha_1(x) \wedge A)$. Hence the lattice $\mathfrak{A}_N(L)$ is distributive. We saw in the last section (Remark 4) that it was complemented. Hence $\mathfrak{A}_N(L)$ is a Boolean lattice. This completes the proof.

Let L be a lattice, $\mathfrak{A}$ any system of ideals of L containing $\mathfrak{A}(P)$ and $\mathcal{E}$ any system of filters of L containing $\mathcal{E}(P)$. In what follows $\overset{I}{\vee}$ will refer to the join in the lattice of ideals of L, $\overset{K}{\vee}$ will refer to the join in the lattice of complete lower ends of L and unadorned join, meet symbols will refer to the join, meet respectively in $\mathfrak{A}$, $\mathcal{E}$. Further $\underset{F}{\wedge}$ will refer to the meet in the lattice of filters of L and $\bigwedge_K$ will refer to the meet in the lattice of complete upper ends of L.

<u>Definition 14</u>: We call L <u>$\mathcal{O}\!\mathcal{l}$-continuous</u> if and only if for every x in L, A, B in $\mathcal{O}\!\mathcal{l}$ if $S \subseteq A \bigvee^{I} B$ such that $\bigvee^{L} S$ exists then there exists a set T such that $\bigvee^{L} T$ exists, T is contained both in $A \bigvee^{I} B$ and $\alpha_1(x)$ and $x \wedge \bigvee^{L} S = \bigvee^{L} T$. Dually we call L <u>$\mathcal{E}$-continuous</u> if and only if for every x in L, E, G in $\mathcal{E}$ if $S \subseteq E \bigwedge_F G$ such that $\bigwedge_L S$ exists then there exists a set T such that $\bigwedge_L T$ exists, T is contained both in $E \bigwedge_F G$ and $\alpha_2(x)$ and $x \vee \bigwedge_L S = \bigwedge_L T$.

The following lemma will be useful.

<u>Lemma 4</u>: Let L be a distributive lattice and $(\mathcal{O}\!\mathcal{l}, \alpha_1)$ any completely faithful canonical join dense distributive extension of L. Then for any A, B in $\mathcal{O}\!\mathcal{l}$ we have $A \vee B = \{x \,/\, x \in L,\ x = \bigvee S,\ S \subseteq A \bigvee^{I} B\}$.

<u>Proof</u>: Let us call the set on the right side of the above equality Q. Since $(\mathcal{O}\!\mathcal{l}, \alpha_1)$ is completely faithful we have that $\mathcal{O}\!\mathcal{l}$ consists of complete lower ends and hence $A \vee B$ contains Q. Now take y in L with $\alpha_1(y) \leq A \vee B$. Then since $\mathcal{O}\!\mathcal{l}$ is a distributive lattice we get $\alpha_1(y) = (\alpha_1(y) \wedge A) \vee (\alpha_1(y) \wedge B)$. But then $\alpha_1(y) = \bigvee\{\alpha_1(y \wedge a) \vee \alpha_1(y \wedge b) \,/\, a \in A,\ b \in B\}$.

Thus $\alpha_1(y) = \bigvee\{\alpha_1((y \wedge a) \vee (y \wedge b)) \,/\, a \in A,\ b \in B\}$ and this gives $y = \bigvee^{L}\{(y \wedge a) \vee (y \wedge b) \,/\, a \in A,\ b \in B\}$. The set $S = \{(y \wedge a) \vee (y \wedge b) \,/\, a \in A,\ b \in B\}$ clearly is contained in $A \bigvee^{I} B$. Hence y belongs to Q and this completes the proof.

Theorem 9: Let $(\mathfrak{A}, \alpha_1)$ be any completely faithful canonical join dense extension of a distributive lattice L. Then $(\mathfrak{A}, \alpha_1)$ is a distributive extension if and only if (1) $\mathfrak{A}$ is a sublattice of $\mathfrak{A}_K(L)$ and (2) L is $\mathfrak{A}$-continuous.

Proof: Suppose $(\mathfrak{A}, \alpha_1)$ is a distributive extension. By Proposition 17 we then have that $\mathfrak{A}$ is a sublattice of $\mathfrak{A}_K(L)$. Further take any x in L, A, B in $\mathfrak{A}$ and let S be a subset of $A \overset{I}{\vee} B$ such that $\overset{L}{\vee} S$ exists. Then clearly $x \wedge \overset{L}{\vee} S$ belongs to $\alpha_1(x) \wedge (A \vee B) = (\alpha_1(x) \wedge A) \vee (\alpha_1(x) \wedge B)$. Then by Lemma 4 there exists a set T contained in $(\alpha_1(x) \wedge A) \overset{I}{\vee} (\alpha_1(x) \wedge B) = \alpha_1(x) \wedge (A \overset{I}{\vee} B)$ such that $\overset{L}{\vee} T$ exists and $x \wedge \overset{L}{\vee} S = \overset{L}{\vee} T$. But this means that L is $\mathfrak{A}$-continuous. Conversely suppose that (1) and (2) both hold. Take any A, B in $\mathfrak{A}$ and put $Q = \{x \,/\, x \in L,\ x = \vee S,\ S \subseteq A \overset{I}{\vee} B\}$. Then Q is certainly contained in $A \overset{K}{\vee} B$, but since L is $\mathfrak{A}$-continuous we have immediately that Q is a complete lower end. Hence $A \overset{K}{\vee} B = Q$ and by (1) $A \vee B = Q$. Now take any x, y in L such that y belongs to $\alpha_1(x) \wedge (A \vee B)$. Then by the above $y \leq x \wedge \overset{L}{\vee} S$ for some suitable $S \subseteq A \overset{I}{\vee} B$. But since L is $\mathfrak{A}$-continuous there exists a $T \subseteq A \overset{I}{\vee} B$, and $\alpha_1(x)$ such that $\overset{L}{\vee} T$ exists and equals y. Now since $T \subseteq \alpha_1(x) \wedge (A \overset{I}{\vee} B)$ we have that it is contained in $(\alpha_1(x) \wedge A) \overset{I}{\vee} (\alpha_1(x) \wedge B)$. This means that y belongs to $(\alpha_1(x) \wedge A) \overset{K}{\vee} (\alpha_1(x) \wedge B)$. Then using (1) we have that $\mathfrak{A}$ is distributive. This completes the proof.

Dually we get

<u>Theorem 10:</u> Let $(\mathcal{E}, \alpha_2)$ be any completely faithful canonical meet dense extension of a distributive lattice L. Then $(\mathcal{E}, \alpha_2)$ is a distributive extension if and only if (1) $\mathcal{E}$ is a sublattice of $\mathcal{E}_K(L)$ and (2) L is $\mathcal{E}$-continuous.

CHAPTER III

EXTENSIONS IN CATEGORIES.

In this chapter we study various categories whose objects are extensions of a given partially ordered set. We will obtain here a categorical characterization for the injective and projective join (meet) dense kernels of an extension in a suitable category. Further we study the relations between the injective and the projective orderings and obtain also a categorical characterization of the injectively largest completely faithful join dense extension of a meet continuous lattice.

1. Injective and Projective Kernels.

Let P be a partially ordered set and let (R, w), (S, π) be extensions of P.

Definition 1: By a join preserving map from (R, w) into (S, π) over P we mean a mapping f from R into S such that for every subset M of P if the join $\bigvee^{R} w(M)$ exists then $f(\bigvee^{R} w(M)) = \bigvee^{S}\pi(M)$. Dually by a meet preserving map from (R, w) into (S, π) over P we mean a mapping f from R into S such that for every subset M of R such that $\bigwedge_{R} w(M)$ exists we have that $f(\bigwedge_{R} w(M)) = \bigwedge_{S}\pi(M)$.

Definition 2: By a left isotone map over P from (R, w) into (S, π) we mean a mapping f from R into S such that (1) $f.w = \pi$, and (2) f is an order homomorphism such that $f(w(x)) \leq f(a)$ implies $w(x) \leq a$ for any x in P, a in R. Similarly by a right isotone map over P from (R, w) into (S, π) we mean a mapping f from R into S such that (3) $f.w = \pi$ and (4) f is an order homomorphism with $f(w(x)) \geq f(a)$ implies $w(x) \geq a$ for any x in P, a in R.

Definition 3: A mapping f from (R, w) into (S, π) will be called join dense if and only if (i) f is an order homomorphism from R into S with $f.w = \pi$ and (ii) for every join dense element s in S and every subset M of P if $s \geq \pi(M)$ then there exists a in R such that $s \geq f(a) \geq \pi(M)$. Similarly a mapping f from (R, w) into (S, π) will be called meet dense if and only if (iii) f is an order homomorphism from R into S with $f.w = \pi$ and (iv) for every meet dense element s in S and every subset M of L such that $s \leq \pi(M)$ there exists an element a in R with $s \leq f(a) \leq \pi(M)$.

Remark 1: If f is a join dense mapping from (R, w) into (S, π) then f maps R onto the join dense elements of S. To see this take any join dense element c in S. Then clearly $c \geq \pi(x)$ for each x in $L(c, \pi)$ and since f is join dense there exists an element a in R such that $c \geq f(a) \geq \pi(x)$ for each x in $L(c, \pi)$. But then $c \geq f(a) = \bigvee^{S} \{\pi(x) / \pi(x) \leq c\} = c$ since c is join dense. Hence $f(a) = c$ and f maps onto the join dense elements of S.

Similarly if f is a meet dense mapping then f maps onto the meet dense elements of f.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be categories whose objects are extensions of P. The maps of $\mathcal{A}$ are join preserving over P or join dense. The maps of $\mathcal{B}$ are meet preserving over P or meet dense. The maps of $\mathcal{C}$ are left-isotone over P and those of $\mathcal{D}$ are right-isotone over P.

Let $\mathcal{M}$ be any category whose objects are extensions of P.

Definition 4: We say (R, w) is injectively smaller than (S, π) in $\mathcal{M}$, written $(R, w) \leq_i (S, \pi)$ in $\mathcal{M}$, if and only if there exists an order isomorphism f in $\mathcal{M}$ from R into S with $f.w = \pi$. Similarly we say (R, w) is projectively smaller than (S, π) in $\mathcal{M}$, written $(R, w) \leq_p (S, \pi)$ in $\mathcal{M}$, if and only if there exists an order epimorphism f in $\mathcal{M}$ from S onto R with $f.\pi = w$. Finally we say (R, w) is equivalent to (S, π) in $\mathcal{M}$, written $(R, w) \cong (S, \pi)$ in $\mathcal{M}$ if and only if there exists an order isomorphism f in $\mathcal{M}$ from R onto S with $f.w = \pi$.

Theorem 1: The injective join dense kernel of an extension (E, w) is up to equivalence in $\mathcal{A}$ the injectively largest join dense extension in $\mathcal{A}$ injectively smaller in $\mathcal{A}$ than the extension (E, w).

Proof: Let $J_i(E, w)$ be the injective join dense kernel of (E, w). Then the set J_i consists of join dense elements of E. Thus the identity mapping from J_i into E is clearly a join dense mapping which makes $J_i(E, w)$ injectively smaller than (E, w) in $\mathcal{A}$. Now suppose that (R, π) is a join dense extension which is injectively smaller in $\mathcal{A}$ than (E, w). Then there exists an order isomorphism f in $\mathcal{A}$ from R into E with $f\circ\pi = w$. If f is join preserving over P then for any r in R we have that

$f(r) = f(\bigvee^{R}\{\pi(x) / \pi(x) \leq r\}) = \bigvee^{E}\{w(x) / \pi(x) \leq r\}$

which means that f(r) belongs to J_i. Then $(R, w) \leq_i J_i(E, w)$ in $\mathcal{A}$. If f is a join dense map then by Remark 1 f maps onto J_i and again certainly $(R, w) \leq_i J_i(E, w)$ in $\mathcal{A}$. This completes the proof.

<u>Theorem 2</u>: The injective meet dense kernel of an extension (E, w) is up to equivalence in $\mathcal{B}$ the injectively largest meet dense extension in $\mathcal{B}$ injectively smaller in $\mathcal{B}$ than the extension (E, w).

Further we have

<u>Theorem 3:</u> The projective join dense kernel of an extension (E, w) is up to equivalence in $\mathcal{C}$ the projectively largest join dense extension in $\mathcal{C}$ projectively smaller in $\mathcal{C}$ than (E, w).

<u>Proof</u>: Recall that $J_p(E) = \{L(a, w) / a \in E\}$. The mapping f from E to $J_p(E)$ given by $f(a) = L(a, w)$ is certainly an order epimorphism. Further $f(w(x)) = \alpha_1(x)$ and $\alpha_1(x) \leq f(a)$ certainly implies that $w(x) \leq a$. Hence f is left isotone and $(J_p(E), \alpha_1) \leq_p (E, w)$ in $\mathcal{C}$. Next suppose that $(R, \pi) \leq_p (E, w)$ in $\mathcal{C}$ where (R, π) is any join dense extension of P. Then there exists a left isotone map g from E onto R. Define h from $J_p(E)$ into R by $h(f(a)) = g(a)$. Since g is left isotone and (R, π) is a join dense extension we get that $f(a) = f(b)$ implies $g(a) = g(b)$. Thus h is well defined and since g, f are onto maps so is h.

Next suppose that $f(a) \leq f(b)$. Take any x in P with $\pi(x) \leq g(a)$. Then since g is left isotone we get that $w(x) \leq a$. But then $w(x) \leq b$ and thus $\pi(x) \leq g(b)$. Since (R, π) is a join dense extension this means that $g(a) \leq g(b)$. Further it is clear that $h(f(w(x))) = \pi(x)$ and h is left isotone since g is left isotone. In all $(J_p(E, w), \alpha_1) \geq_p (R, \pi)$ in $\mathcal{C}$. This completes the proof.

Dually we obtain

Theorem 4: The projective meet dense kernel of an extension (E, w) is up to equivalence in $\mathcal{D}$ the projectively largest meet dense extension in $\mathcal{D}$ which is projectively smaller in $\mathcal{D}$ than (E, w).

2. Injective and Projective Orderings.

In this section we study the relations between the injective and projective orderings in suitable categories of complete dense extensions.

Let $\mathcal{C}$ be the category whose objects are full complete extensions of a partially ordered set P and whose maps are join and meet preserving over P, let $\mathcal{C}_1$, be the category whose objects are complete join dense extensions of P and whose maps are join preserving over P, let $\mathcal{D}_1$, be the category whose objects are complete join dense extensions of P and whose maps are order isomorphisms and finally let $\mathcal{D}$ be the category whose objects are full complete extensions of P and whose maps are order isomorphisms.

<u>Proposition 1</u>: Let (R, w), (S, π) be any objects in $\mathcal{C}_1$. Then $(R, w) \leq_p (S, \pi)$ in $\mathcal{C}_1$ if and only if $(R, w) \leq_i (S, \pi)$ in $\mathcal{D}_1$.

<u>Proof</u>: Suppose (R, w) is projectively smaller than (S, π) in $\mathcal{C}_1$. Then there exists a join preserving order epimorphism f over P from S on to R. Consider the mapping h from R in to S defined by $h(r) = \bigvee^S \{\pi(y) \mid y \in P,\ w(y) \leq r\}$. h is clearly an order homomorphism from R into S with $h.w = \pi$. Take any a, b in R and suppose $a \nleq b$. Then since (R, w) is a join dense extension there exists a x in P with $w(x) \leq a$ but $w(x) \nleq b$. Since f is join preserving over P we indeed have that $f(\bigvee^S \{\pi(y) \mid w(y) \leq b\}) = b$ using also the fact that b is a join dense element. Hence $w(x) \nleq f(\bigvee^S \{\pi(y) \mid w(y) \leq b\})$. But this implies that $\pi(x) \nleq \bigvee^S \{\pi(y) \mid w(y) \leq b\} = h(b)$. Thus we have that $h(a) \nleq h(b)$. Hence h is an order isomorphism from R into S with $h.w = \pi$. This means that (R, w) is injectively smaller than (S, π) in $\mathcal{D}_1$.

Conversely, suppose that (R, w) is injectively smaller than (S, π) in $\mathcal{D}_1$. Then there exists an order isomorphism g from R into S with $g.w = \pi$. Consider the mapping k from S into R defined by $k(s) = \bigvee^R \{r \mid r \in R,\ g(r) \leq s\}$. Then for any r in R we have $k(g(r)) = \bigvee^R \{t \mid t \in R,\ g(t) \leq g(r)\} = \bigvee^R \{t \mid t \in R,\ t \leq r\} = r$ using the fact that g is an order isomorphism. Now take any subset M of P. Then by the definition of k we have that $k(\bigvee^S \pi(M)) = \bigvee^R \{w(x) \mid \pi(x) \leq \bigvee^S \pi(M)\}$ which is certainly an upper bound of $w(M)$. Suppose that $r \geq w(M)$. Then $g(r) \geq \pi(M)$ and thus $g(r) \geq \bigvee^S \pi(M)$. In all, $(k.g)(r) = r \geq k(\bigvee^S \pi(M))$.

Hence k is join preserving over P and since $k \circ g$ is the identity on R we have that k is onto R. Thus (R, w) is projectively smaller than (S, π) in $\mathcal{C}_1$. This completes the proof.

__Corollary 1__: Let (R, w), (S, π) be any objects in $\mathcal{C}_1$. Then (R, w) is equivalent to (S, π) in $\mathcal{C}_1$ if and only if $(R, w) \leq_p (S, \pi)$ in $\mathcal{C}_1$ and $(S, \pi) \leq_p (R, w)$ in $\mathcal{C}_1$.

__Remark 2__: Let $\mathcal{C}_2$ be the category whose objects are __complete meet dense extensions__ of a partially ordered set P and whose maps are meet preserving over P. Let $\mathcal{D}_2$ be the category whose objects are __complete meet dense extensions__ of P and whose maps are __order isomorphisms__. Then by duality we obtain that (R, w) $\leq_p (S, \pi)$ in $\mathcal{C}_2$ if and only if $(R, w) \leq_i (S, \pi)$ in $\mathcal{D}_2$. In particular (R, w) is equivalent to (S, π) in $\mathcal{C}_2$ if and only if $(R, w) \leq_p (S, \pi)$ in $\mathcal{C}_2$ and $(S, \pi) \leq_p (R, w)$ in $\mathcal{C}_2$.

<u>Proposition 2</u>: Let (R, w), (S, π) be any two full complete extensions. Then $(R, w) \leq_p (S, \pi)$ in $\mathcal{C}$ if and only if $(R, w) \leq_i (S, \pi)$ in $\mathcal{D}$.

<u>Proof</u>: Suppose $(R, w) \leq_i (S, \pi)$ in $\mathcal{D}$. Then there exists an order isomorphism f from R into S with $f.w = \pi$. Define g from S into R by $g(s) = \bigvee^R \{ r \,/\, r \in R, f(r) \leq s \}$. Then for any r in R, $g(f(r)) = \bigvee^R \{ s \,/\, s \in R, f(s) \leq f(r) \} = \bigvee^R \{ s \,/\, s \in R, s \leq r \} = r$. Hence $g.f$ is the identity on R and thus g is an order epimorphism onto R with $g.\pi = w$. Take any subset T of P and put $s = \bigwedge_S \{ \pi(x) \,/\, x \in T \}$. Then $g(s) \leq w(T)$. Further if $r \leq w(T)$ then $f(r) \leq \pi(T)$ and thus $f(r) \leq s$. Then $(gf)(r) = r \leq g(s)$. Hence g is meet preserving over P and by duality one has that g is join preserving over P. Hence g is a map of $\mathcal{C}$ and $(R, w) \leq_p (S, \pi)$ in $\mathcal{C}$. Conversely, suppose $(R, w) \leq_p (S, \pi)$ in $\mathcal{C}$.

Let $J(R)$, $J(S)$ be respectively the join dense elements of R and S and $M(R)$, $M(S)$ be respectively the meet dense elements of R and S. Let w_1, w_2 be the mappings from P into $J(R) \otimes M(R)$, $J(S) \otimes M(S)$ respectively given by $w_1(x) = (w(x), w(x))$, $w_2(x) = (\pi(x), \pi(x))$. Then since the extensions (R, w), (S, π) are full and since the extensions $(J(R) \otimes M(R), w_1)$, $(J(S) \otimes M(S), w_2)$ are dense we have at once by the definition of a full extension that (R, w) is equivalent to $(J(R) \otimes M(R), w_1)$ and (S, π) is equivalent to $(J(S) \otimes M(S), w_2)$ in $\mathcal{C}$. Since $(R, w) \leq_p (S, \pi)$ in $\mathcal{C}$ there exists a join and meet preserving map f from S onto R. But then clearly f carries J(S) onto J(R) and M(S) onto M(R). Thus by Proposition 1 we have that the join kernels and meet kernels satisfy, $(J(R), w) \leq_i (J(S), w)$ in $\mathcal{C}_1$ and $(M(R), w) \leq_i (M(S), w)$ in $\mathcal{C}_2$. This implies in the obvious manner that

$(R, w) \cong (J(R) \otimes M(R), w_1) \leq_i (J(S) \otimes M(S), w_2) \cong (S, \pi)$ in $\mathcal{D}$. In all, $(R, w) \leq_i (S, \pi)$ in $\mathcal{D}$ and this completes the proof.

Corollary 2: Let (R, w) and (S, π) be any two full complete extensions. Then (R, w) is equivalent to (S, π) in $\mathcal{C}$ if and only if $(R, w) \leq_p (S, \pi)$ in $\mathcal{C}$ and $(S, \pi) \leq_p (R, w)$ in $\mathcal{C}$.

3. Categorical Characterization of ($\mathcal{O}_K(L)$, α_1).

In this section we obtain a categorical description of the extension ($\mathcal{O}_K(L), \alpha_1$) where L is a meet continuous lattice analagous to the description of a Dedekind-Macneille extension of a partially ordered set obtained in [4].

Let L be a meet continuous lattice and let (R, w), (S, π) be extensions of L.

Definition 5: By a join continuous order homomorphism f from (R, w) into (S, π) over L we mean a mapping f from R into S such that for every up-directed set M of L if $\bigvee^R w(M)$ exists then $f(\bigvee^R w(M)) = \bigvee^S \pi(M)$.

Let $\mathcal{M}$ be the category whose objects are completely join faithful, finitely meet faithful meet continuous extensions of L and whose maps are left isotone, join continuous order homomorphisms over L.

Let (R, w) and (S, π) be arbitrary objects in $\mathcal{M}$.

Definition 6: (R, w) is called an essential extension of L if and only if for any map f in $\mathcal{M}$ from (R, w) into (S, π) we have that $f(a) \leq f(b)$ implies $a \leq b$ for any a, b in R.

Remark 3: Let $\mathcal{C}$ be a category whose objects are extensions of L. Let (E, w), (R, π) be arbitrary objects in $\mathcal{C}$. Then (E, w) is called an essential extension [4] if and only if for any map f in $\mathcal{C}$ from E into R such that if f.w is an order isomorphism from L into R then f is an order isomorphism. In the category $\mathcal{M}$ any map f in $\mathcal{M}$ from (E, w) into (R, π) satisfies $f.w = \pi$. Hence the requirement that f.w be an order isomorphism from L is automatically satisfied and is thus omitted from the above definition.

Note that the extension ($\mathcal{O}_K(L)$, α_1) determined by the complete lower ends of L is by Theorem 1 of the previous chapter a meet continuous extension. It is also completely faithful and thus belongs to $\mathcal{M}$. In addition note that for each object (E, w) of $\mathcal{M}$ the mapping w is in $\mathcal{M}$.

A description of essential extensions in $\mathcal{M}$ is obtained in the following proposition.

Proposition 3: An object (E, w) in $\mathcal{M}$ is an essential extension of L if and only if (E, w) is injectively smaller in $\mathcal{M}$ than the extension $(\mathfrak{A}_K(L), \alpha_1)$

Proof: Suppose (E, w) is injectively smaller than $(\mathfrak{A}_K(L), \alpha_1)$ in $\mathcal{M}$. Then (E, w) is a join dense extension. Take any map f from (E, w) into (R, π) in $\mathcal{M}$. Suppose $f(a) \leq f(b)$. Take x in L such that $w(x) \leq a$. Then $f(w(x)) = \pi(x) \leq f(b)$. Since f is left-isotone we get that $w(x) \leq b$. Then since (E, w) is a join dense extension we have that $a \leq b$. Hence (E, w) is an essential extension of L. Conversely, suppose that (E, w) is an essential extension of L. Consider the mapping f from E into $\mathfrak{A}_K(L)$ given by

$f(a) = \bigvee^{\mathfrak{A}_K} \{ \alpha_1(x) \mid w(x) \leq a \}$. f is clearly an order homomorphism with $f.w = \alpha_1$. Take any up-directed set M contained in L such that $\bigvee^{E} w(M)$ exists. Take any y in L such that $w(y) \leq \bigvee^{E} w(M)$. Then since E is meet continuous and finitely meet faithful we have that $w(y) = \bigvee^{E} \{ w(x \wedge y) \mid y \in M \}$. Thus $y = \bigvee^{L} \{ x \wedge y \mid y \in M \}$.
Then since $(\mathfrak{A}_K(L), \alpha_1)$ is completely faithful we have that
$\alpha_1(y) = \bigvee^{\mathfrak{A}_K} \{ \alpha_1(x \wedge y) \mid y \in M \} \leq \bigvee^{\mathfrak{A}_K} f(w(M))$. Hence f is a join continuous order homorphism over L. Further suppose that $\alpha_1(x) \leq f(a)$. Then $\alpha_1(x) \leq \bigvee^{\mathfrak{A}_K} \{ \alpha_1(y) \mid w(y) \leq a \}$. Then since $(\mathfrak{A}_K(L), \alpha_1)$ is a meet continuous, completely faithful extension we get that $\alpha_1(x) = \bigvee^{\mathfrak{A}_K} \{ \alpha_1(x \wedge y) \mid w(y) \leq a \}$.
But then $x = \bigvee^{L} \{ x \wedge y \mid w(y) \leq a \}$. Since (E, w) is a completely join faithful extension we then have that

$w(x) = \bigvee^{E} \{ w(x \wedge y) \mid w(y) \leq a \} \leq a$. Hence f is left-isotone, that is, f is a map in $\mathcal{M}$. Since (E, w) is an essential extension this means that f is an order isomorphism. Hence $(E, w) \leq_i (\mathcal{O}_K(L), \alpha_1)$ in $\mathcal{M}$ and this completes the proof.

Definition 7: An extension (E, w) of L in $\mathcal{M}$ will be called an injective extension of L if and only if for every pair of objects (A, φ), (B, ψ) in $\mathcal{M}$ such that $(A, \varphi) \leq_i (B, \psi)$ in $\mathcal{M}$ and any map h in $\mathcal{M}$ from (A, φ) to (E, w) extends to a map f in $\mathcal{M}$ from (B, ψ) into (E, w), that is, $f \cdot g = h$ where g is an order isomorphism in $\mathcal{M}$ from A into B.

Theorem 5: The following are equivalent for an extension (E, w) of L in $\mathcal{M}$.

(1) (E, w) is the injectively largest join dense completely faithful extension of L.

(2) (E, w) is an essential, injective extension of L in $\mathcal{M}$.

(3) (E, w) is a minimal injective extension of L in $\mathcal{M}$.

(4) (E, w) is a maximal essential extension of L in $\mathcal{M}$.

Proof: (1) implies (2): (E, w) is an essential extension by Proposition 3. Take (A, φ), (B, ψ) in $\mathcal{M}$ and let g be an order isomorphism from A into B in $\mathcal{M}$. Let h be any map in $\mathcal{M}$ from (A, φ) into (E, w). Define a mapping f from (B, ψ) into (E, w) by $f(b) = \bigvee^E \{ w(x) / \psi(x) \leq b \}$. Since (E, w) is the injectively largest join dense completely faithful extension we have that $(E, w) \cong (\mathcal{O}_K(L), \alpha_1)$ in $\mathcal{M}$ and hence (E, w) is a complete extension. Hence f is well-defined. Further if $w(y) \leq f(b)$ then since (E, w) is meet continuous we get that $y = \bigvee^{L} \{x \wedge y \,/\, \psi(x) \leq b\}$. Then $\psi(y) = \bigvee^{B} \{\psi(x \wedge y) \,/\psi(x) \leq b\}$ since (B, ψ) is completely join faithful. Thus $\psi(y) \leq b$ and f is left-isotone. Take any up-directed subset M of L and suppose $\bigvee^{B} \psi(M)$ exists. Then since f is left-isotone we get immediately that f is a join continuous order homomorphism over L for $w(y) \leq f(\bigvee^{B} \psi(M))$ gives as before that $w(y) \leq \bigvee^{E} w(M)$. Further for any a in A, clearly $f(g(a)) \leq h(a)$. Next take any y with $w(y) \leq h(a)$. Then since h is left-isotone we get that $\varphi(y) \leq a$. Then $g(\varphi(y)) = \psi(y) \leq g(a)$. But then $w(y) \leq f(g(a))$. Since $h(a)$ is a join dense element we have that $h(a) \leq f(g(a))$, that is, $f.g = h$ and (E, w) is an injective extension of L in $\mathcal{M}$.

(2) implies (3): Let (R, π) be another injective extension of L with $(R, \pi) \leq_{i} (E, w)$ in $\mathcal{M}$. Let i be the identity mapping on R. Then since (E, w) is essential, there exists a map f in $\mathcal{M}$ from (E, w) into (R, π) with $f.g = i$ where g is an order isomorphism in $\mathcal{M}$ from (R, π) into (E, w). But then f is onto R, and since (E, w) is essential we have further that f is an order isomorphism. Thus $(R, \pi) \cong (E, w)$ in $\mathcal{M}$.

(3) implies (4): Let (R, π) in $\mathcal{M}$ be an essential extension of L and suppose $(E, w) \leq_{i} (R, \pi)$ in $\mathcal{M}$. Then there exists an order isomorphism g in $\mathcal{M}$ from (E, w) into (R, π) with $g.w = \pi$. Let i be the identity mapping on E. Then in view of (3) there exists a map f in $\mathcal{M}$ from R into E with $f.g = i$. But then f is onto E and

since (R, π) is essential we must have that f is an order isomorphism. Thus (E, w) is equivalent to (R, π) in $\mathcal{M}$. To see that (E, w) is an essential extension consider the extension $(\mathfrak{A}_K(L), \alpha_1)$ of L. We know from the implication (1) implies (2) that it is injective. Now consider the mapping w from L into E which is a map of $\mathcal{M}$. Then this mapping in view of (3) extends to a mapping f from $\mathfrak{A}_K(L)$ into E with $f.\alpha_1 = w$. Since by Proposition 3 $(\mathfrak{A}_K(L), \alpha_1)$ is an essential extension we get that f is an order isomorphism. Then by (3) since $(\mathfrak{A}_K(L), \alpha_1)$ is injective we have that f maps onto E. Thus (E, w) is an essential extension.

(4) implies (1): This follows immediately from Proposition 3 and the fact that $(\mathfrak{A}_K(L), \alpha_1)$ is the injectively largest join dense completely faithful extension of L.

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