DEPENDENCE OF NUCLEAR ENERGY ON AXIAL ASYMMETRY

DEPENDENCE OF NUCLEAR ENERGY ON AXIAL ASYMMETRY AND ITS CONSEQUENCES

By

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SCOPE AND CONTENTS:

An investigation is carried out to determine whether or not on the basis of theory one expects axial asymmetry in heavy nuclei. The average one-body potential well is taken as an anisotropic harmonic oscillator, modified by the usual $\underset{\sim}{\ell} \cdot \underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ forces. The method of matrix diagonalisation is used to find the eigenvalues. Residual interaction between nucleons is also treated in the pairing approximation. The calculations show that prolate axial symmetry is favoured in each of the nuclei considered. It is possible to indicate the underlying physical reasons for this result.

Assuming equilibrium prolate axial symmetry an attempt is made to understand the collective excited levels in the transition region, i.e. the region between spherical nuclei and permanently deformed nuclei. A model calculation is done to show how the levels can be obtained.

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INTRODUCTION

The shell model introduced by Mayer and Jensen in 1949 proved to be a major breakthrough in the theory of nuclear structure. In this model each nucleon is supposed to move independently in some sort of spherical potential generated by all other nucleons. A single-particle spin-orbit force is postulated to give the sequence of magic numbers. If the strength of the spin-orbit force is suitably chosen then this model adequately explains many phenomena; for example, the ground state, the excited states, etc.

It soon became evident, however, that this description was incomplete. One piece of evidence came from the quadrupole moments of some nuclei. If one assumes that nucleons always fill up orbits in a spherical potential then the theoretical prediction of quadrupole moments of nuclei fall short of experimental values in very many cases. In such cases, it must be assumed that nucleons fill up orbits in some aspherical potential. Since nuclear forces are short-range, this aspherical potential is presumably due to some aspherical shape of the surface. This shape can be described by some parameters which we call the collective coordinates.

Various phenomena point out the fact that one should incorporate in the theory the dynamics of these coordinates. Many features of fission phenomena can be understood in terms of the motion of these collective coordinates. Also nuclei seem to have energy levels which are collective in nature, i.e. which cannot be explained simply in terms of nucleon

excitation. One might say then that these are the energies associated with the collective coordinates. One is thus led to a description of the nucleus as a shell structure which is capable of performing oscillations.

This naturally leads to a generalisation of the shell model; not only do we allow for energy levels which are associated with collective coordinates but also the equilibrium shape of the nucleus may be non-spherical. Extended shell model levels for such deformed nuclei have been calculated by several authors (Ni 55, Go 56, Ne 60). One still retained one basic feature of the shell model; that nucleons fill up orbits independently in some sort of potential.

This last assumption is somewhat in error. The nucleus is a many-body system interacting via a two nucleon potential (not much work has been done to examine effects of a possible three or more body force). There may be an attractive short range part of the basic two nucleon force which cannot be incorporated in an overall single-particle potential. This residual interaction may have strong influences on some nuclear properties. The need for examining the effects of such possible and plausible forces came from various sources (Bo, Mo, Pi 58). On the basis of the single-particle model one would expect to see low-lying intrinsic excitations in nuclei, odd or even-even. These low-lying single-particle excitations are absent in the case of even-even nuclei. This suggests that there are strong forces which prevent the breaking of pairs of nucleons. This force cannot be entirely diagonal since then it cannot prevent the occurrence of low-lying two-particle excitations at about twice the single particle energies corresponding to exciting the pair as a whole. The necessity of treating this residual interaction was seen also from considerations of moments of

inertia of nuclei. Theoretical calculations on the basis of the hydrodynamic model consistently gave too low values for moments of inertia. Calculations based on the simple single-particle model gave too high values for the moments of inertia, namely that of a rigid body. It was shown that two-particle correlations will reduce this value considerably.

The need to treat such a short-range two-body force was known in solid state physics and Bardeen, Cooper and Schrieffer's theory of superconductivity treated this type of force in an elegant way (Ba, Co, Sc 57). In this theory, a short-range residual interaction strongly binds any two electrons moving with opposite momenta and in singlet states (Cooper pairs). By virtue of this force electrons are continually scattered from one state of zero total momentum to another.

Following the methods of superconductivity, the formal apparatus of tackling the problem in the nuclear case was set up by Belyaev (Be 59). Since then it has been applied by very many workers to predict moments of inertia (Ni, Pr 61), energy levels of nuclei (Ki, So 60), equilibrium deformations of nuclei (Bè, Sz 61), etc.

Most of the work in determining equilibrium deformation has assumed axial symmetry as a premise to start from. Our work has been to examine if nuclei in the deformed region would prefer axial asymmetry or not according to the present set up of the theory. Residual interaction has been treated by the method of the pairing model. Some attempt is also made to obtain the collective spectra in the transition region, i.e. the region between deformed nuclei and spherical nuclei.

CHAPTER 1: NUCLEAR COLLECTIVE MOTION

This chapter provides a basis for the calculations done in chapter 5. Here we are concerned with the dynamics of the collective coordinates. Numerical estimates of the various parameters entering our equations can be made on the basis of the hydrodynamic model but one should recognise that the estimates can at best be very crude.

Let us write with Bohr (Bo 52) the equation of the surface of the nucleus as

$$R = R_o \left[1 + \sum_{\lambda,\mu} \alpha_{\lambda\mu} Y_{\lambda\mu} \right] \qquad 1.1$$

Here the $\alpha_{\lambda\mu}$'s are the collective coordinates. They are dynamical variables, i.e. they change with time. $Y_{\lambda\mu}$'s are normalised spherical harmonics with Condon-Shortley phase. Since both R and R_o are real and $Y_{\lambda\mu}$'s in general are complex, $\alpha_{\lambda\mu}$'s in general are complex.

The problem is to study the dynamics of the nucleus as $\alpha_{\lambda\mu}$'s change with time. It can be shown that a motion of α_0 leads to a change of the nuclear volume. This is the "breathing mode" of the nucleus (Fe 57) and the energies associated with such modes are expected to be high; motion of α_1 merely leads to a centre of mass motion (Pr 62). Thus the lowest order to consider is α_2 ; octupole vibrations associated with α_3 have also been found but we will not be concerned with them; these occur at higher energies.

Accordingly we write

$$R = R_o \left[1 + \sum_{\mu} \alpha_{2\mu} Y_{2\mu} \right] \qquad 1.2$$

Let us initially assume that the deviation from sphericity is small. We can then write the potential energy associated with such coordinates as

$$V = \frac{1}{2} C \sum_{\mu} |\alpha_{2\mu}|^2 \qquad 1.3$$

The kinetic energy will be

$$T = \frac{1}{2} B \sum |\dot{\alpha}_{2\mu}|^2 \qquad 1.4$$

The energy values of a Hamiltonian which is a sum of eqns. 1.3 and 1.4 can be written as

$$E = \left(\sum_{\mu} n_{\mu} + \frac{5}{2} \right) \hbar \omega \qquad 1.5$$

with

$$\omega = \sqrt{\frac{C}{B}}$$

We could refer to the integral number n_μ as the number of phonons of character 2μ. A phonon of 2μ character has total spin 2, Z-axis spin projection μ and parity even. Elementary considerations show that a state with no phonon is 0+, with 1 phonon is 2+, with 2 phonons is 0+, 2+, 4+, with 3 phonons is 0+, 2+, 3+, 4+, 6+, etc. The spectrum of such an idealised Hamiltonian is equispaced (Fig. 1.1).

Let us now transform the Hamiltonian to a coordinate system fixed in the nucleus. Such a procedure is clearly convenient to treat nuclei which have non-zero equilibrium deformation. For the body-fixed system we choose a coordinate system whose axes coincide with the principal axes of

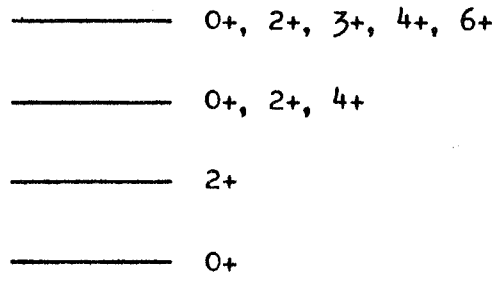


Fig. 1.1. Level Spectrum corresponding to eqn. 1.5.

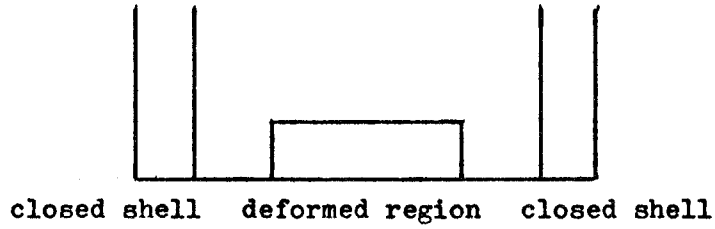


Fig. 1.2. Near closed shells we expect spectrum corresponding to Fig. 1.1.

In deformed regions we expect spectrum corresponding to Fig. 1.3.

the nucleus. The nuclear surface in this coordinate system can be written as

$$R = R_o\left[1 + \sum_\nu a_{2\nu} Y_{2\nu}(\theta', \phi')\right] \qquad 1.6$$

The $a_{2\nu}$'s can be expressed in terms of the $\alpha_{2\nu}$'s by means of the relation

$$a_{2\nu} = \sum_\mu \alpha_{2\mu} D^2_{\mu\nu} \quad (\theta_i) \qquad 1.7$$

Here the θ_i's are the three Eulerian angles specifying the orientation of the body-fixed system with respect to the space-fixed system. The matrix elements $D^2_{\mu\nu}$ are related to those used by Rose (Ro 57) by the following equation $D^2_{\mu\nu} = i^{\mu-\nu} D^{2*}_{\mu\nu}$ (Rose).

The inverse relation is

$$\alpha_{2\mu} = \sum_\nu a_{2\nu} D^{2*}_{\mu\nu} \quad (\theta_i) \qquad 1.8$$

With our choice of coordinate systems $a_{2,1} = a_{2,-1} = 0$; $a_{2,2} = a_{2,-2}$.

Thus a_o, a_2 (or a_{-2}) and the three Eulerian angles take over the role of the five α_μ's.

It is convenient to make a further substitution

$$\begin{aligned} a_o &= \beta \cos\gamma \\ a_2 &= a_{-2} = \frac{1}{\sqrt{2}} \beta \sin\gamma \end{aligned} \qquad 1.9$$

The equation of the surface (1.6) then becomes

$$R = R_o\left[1 + \beta\cos\gamma\, Y_{20} + \frac{1}{\sqrt{2}}\beta\sin\gamma\,(Y_{2,2} + Y_{2,-2})\right] \qquad 1.10$$

We can easily see the physical significance of β and γ.

The total deformation is given by

$$\sum_{\mu} |\alpha_{2\mu}|^2 = \sum_{\nu} a_{2\nu}^2 = \beta^2 \quad ;$$

thus β is a measure of total deformation.

The signigicance of γ can be seen by writing the three body-fixed axes explicitly.

$$R_Z = R(0,\phi') = R_o\left[1 + \sqrt{\frac{5}{4\pi}}\,\beta\cos\gamma\right]$$

$$R_X = R(\tfrac{\pi}{2}, 0) = R_o\left[1 + \sqrt{\frac{5}{4\pi}}\,\beta\cos(\gamma - \tfrac{2\pi}{3})\right] = R_o\left[1 - \sqrt{\frac{5}{4\pi}}\,\beta\cos(\gamma + \tfrac{\pi}{3})\right] \qquad 1.11$$

$$R_Y = R(\tfrac{\pi}{2}, \tfrac{\pi}{2}) = R_o\left[1 + \sqrt{\frac{5}{4\pi}}\,\beta\cos(\gamma - \tfrac{4\pi}{3})\right] = R_o\left[1 - \sqrt{\frac{5}{4\pi}}\,\beta\cos(\gamma - \tfrac{\pi}{3})\right]$$

The above equations show that for $\gamma = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ the nucleus is a prolate ellipsoid of revolution about Z, X and Y axes respectively; for $\gamma = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ it is an oblate ellipsoid of revolution about Y, Z and X axes respectively. At any other value of γ it is an ellipsoid with three unequal axes.

It is also worthwhile to note the following symmetry properties.

γ, $\gamma + \frac{2\pi}{3}$ and $\gamma + \frac{4\pi}{3}$ describe the same nuclear surface and can be obtained from one another by an interchange of the axes.

γ and $-\gamma$ define the same nuclear shape. Therefore the γ-dependence of any quantity which is a function of the nuclear shape only is given by $f(\cos 3\gamma)$.

We now return to the problem of transforming the Hamiltonian defined by 1.3 and 1.4 to the body-fixed system.

To transform 1.4 we first use the relation 1.8 and then to obtain the quantum mechanical formulation of the problem, replace time derivatives by differential operators by standard techniques (Pa 33).

The kinetic energy term is then given by

$$T = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\,\beta^4\,\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\right] + \sum_{K=1}^{3}\frac{\hbar^2 Q_K^2}{8B\beta^2\sin^2(\gamma - \frac{2\pi}{3}K)} \qquad 1.12$$

The Q_K's are the usual angular momentum operators.

The potential energy is transformed to

$$V = \tfrac{1}{2}C\beta^2 \qquad 1.13$$

Solution of the Hamiltonian defined by 1.12 and 1.13 was given for the spin I = 0 case by Bohr (Bo 52); it was extended to all I-states by Bès (Bè 59). One, of course, gets the spectrum shown in Fig. 1.1. The solutions can be written as

$$f_{n_\beta}(\beta)\sum_{K=0}^{I} g_K^I(\gamma)\,|IMK\rangle$$

where K is restricted to even values and $|IMK\rangle$ is a normalised wave function given by

$$\left[\frac{2I+1}{16\pi^2(1+\delta_{Ko})}\right]^{\frac{1}{2}}\left(D^I_{MK} + (-)^I D^I_{M,-K}\right) \qquad 1.14$$

A potential energy function $\frac{1}{2}C\beta^2$ minimizes at $\beta = 0$ (spherical shape). The above Hamiltonian is thus valid only for nuclei with spherical

equilibrium shape. Thus one would expect the above Hamiltonian to be applicable to nuclei near the closed shells. Even near the closed shells anharmonic terms in the potential (1.3) may spoil the simple equispaced spectrum (Ke, Sh 62).

Between two closed shells there are regions in the nuclear periodic table which show distinct rotational spectra. The existence of rotational spectra is associated with a deformation of shape (Ke 59). In such a case we must write

$$V = \frac{1}{2} \sum_{\mu} C_{\mu} \left| (\alpha_{\mu} - \alpha_{\mu}(0)) \right|^2$$

and a priori we have no information about C. In a general case we then write

$$V = V(\beta, \gamma) \qquad 1.14$$

and try to solve the Hamiltonian given by the sum of 1.12 and 1.14. It is to be noted that we are limiting ourselves to even-even nuclei. Our treatment so far has completely ignored a particular degree of freedom, namely, intrinsic excitation. This appears to be valid for low energy excitation of an even-even system, but for an odd-even system one must also include the motion of the last odd nucleon. This will introduce some complications since there will be coupling between the odd-nucleon and the even-even core. Since our work is concerned with even-even systems we do not pursue this point further.

The solution of the Hamiltonian given by 1.12 and 1.14 is obtained by making an essentially Born-Oppenheimer approximation. One assumes that the nucleus is strongly stable about some equilibrium deformation (β_o, γ_o). Most authors consider $\gamma_o = 0$. The Hamiltonian becomes

$$H = -\frac{\hbar^2}{2B}\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{2}D(\beta-\beta_o)^2$$

$$-\frac{\hbar^2}{2B}\frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{2}A(\gamma-\gamma_o)^2$$

$$-\frac{\hbar^2}{8B\beta^2}\left[\frac{Q_1^2}{\sin^2(\gamma-\frac{2\pi}{3})} + \frac{Q_2^2}{\sin^2(\gamma-\frac{4\pi}{3})} + \frac{Q_3^2}{\sin^2\gamma}\right] + V(\beta_o,\gamma_o) \qquad 1.15$$

The eigenfunctions of 1.15 are normalized in a space with volume element $\beta^4\, d\beta\, |\sin 3\gamma|\, d\gamma\, d\Omega$ where $d\Omega = \sin\theta\, d\theta\, d\phi\, d\Psi$; $V(\beta_o, \gamma_o)$ is a constant and can be left out.

Let $\Phi = \beta^2\psi$ where ψ is the wavefunction of 1.15. This leads to the eigenvalue equation

$$H'\Phi = E\Phi$$

where

$$H' = -\frac{\hbar^2}{2B}\left(\frac{\partial^2}{\partial\beta^2} - \frac{2}{\beta^2}\right) + \frac{1}{2}D(\beta-\beta_o)^2$$

$$-\frac{\hbar^2}{2B\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{2}A(\gamma-\gamma_o)^2$$

$$+\frac{\hbar^2}{8B\beta^2}\left[\frac{Q_1^2}{\sin^2(\gamma-\frac{2\pi}{3})} + \frac{Q_2^2}{\sin^2(\gamma-\frac{4\pi}{3})} + \frac{Q_3^2}{\sin^2\gamma}\right] \qquad 1.16$$

Assume now that the amplitudes are small and $\gamma_o = 0$. Hence in 1.16 we replace $\sin 3\gamma$ by 3γ, β by β_o, $\cos\gamma$ by 1 in the appropriate places. This leads to

$$H' = H_{rot.} + H_\beta + H_\gamma + U_1 + U_2 \qquad 1.17$$

with

$$H_{rot.} = \frac{\hbar^2}{2\mathcal{I}} (Q^2 - Q_3^2) \; ; \quad \mathcal{I} = 3B\beta_o^2$$

$$H_\beta = -\frac{\hbar^2}{2B} \frac{\partial^2}{\partial\beta^2} + \frac{1}{2} D(\beta - \beta_o)^2 + \frac{\hbar^2}{B\beta_o^2}$$

$$H_\gamma = -\frac{\hbar^2}{2B} \left(\frac{1}{\gamma} \frac{\partial}{\partial\gamma} \gamma \frac{\partial}{\partial\gamma} - \frac{Q_3^2}{4\gamma^2} \right) + \frac{1}{2} A\gamma^2 \qquad 1.18$$

$$U_1 = -\frac{\gamma}{\sqrt{3}} \frac{\hbar^2}{\mathcal{I}} (Q_1^2 - Q_2^2)$$

$$U_2 = \frac{\partial}{\partial\beta} \left(\frac{\hbar^2}{2\mathcal{I}} \right)_{\beta=\beta_o} (Q^2 - Q_3^2)(\beta - \beta_o)$$

The terms U_1 and U_2 represent rotation vibration interaction and are small. The effects of these can be taken into account by perturbation methods. Neglecting these the wavefunction is given by $\phi = f(\beta)\, g(\gamma)\, D(\theta)$ normalised in a space with volume element $d\beta\, \gamma\, d\gamma\, d\Omega$.

The eigenvalues of $H_{rot.}$ are

$$E_{rot.} = \frac{\hbar^2}{2} \left[I(I+1) - K^2 \right] \qquad 1.19$$

The wavefunctions are the D-matrix elements referred to earlier.

Neglecting zero-point energy the eigenvalues of H_β are

$$E_\beta = n_\beta \hbar \omega_\beta \qquad \text{where} \qquad \omega_\beta = \sqrt{\frac{D}{B}} \qquad 1.20$$

The eigenfunctions of H_β are

$$f_{n_\beta}(\beta) = \left[\frac{\left(\frac{B\omega_\beta}{\hbar\pi} \right)^{1/2}}{2^{n_\beta} n_\beta !} \right]^{1/2} e^{-\frac{B\omega_\beta}{2\hbar}\beta^2} H_{n_\beta}\left(\sqrt{\frac{B\omega_\beta}{\hbar}}\,\beta \right) \qquad 1.21$$

Neglecting zero point energy the eigenvalues of H_γ are

$$E_\gamma = n_\gamma \hbar\omega_\gamma \; ; \qquad \omega_\gamma = \sqrt{\frac{A}{B}} \qquad 1.22$$

For a given n_γ, K takes on value

$$\frac{K}{2} = n_\gamma, \; n_\gamma - 2, \; \ldots \geq 0$$

The wavefunctions are

$$g_{K,n_\gamma} = \frac{2B\omega_\gamma}{\hbar} \frac{\left(\frac{n_\gamma - K/2}{2} \right)!}{\left[\left(\frac{n_\gamma + K/2}{2} \right)! \right]^3} e^{-\frac{B\omega_\gamma}{2\hbar}\gamma^2}$$

$$\times \left(\frac{B\omega_\gamma}{\hbar}\gamma^2 \right)^{K/4} L^{K/2}_{\frac{n_\gamma + K/2}{2}} \left(\frac{B\omega_\gamma}{\hbar}\gamma^2 \right) \qquad 1.23$$

The last factor is a Laguerre polynomial.

The level spectrum to be expected is shown in Fig. 1.3. The pattern is indeed widely seen in the deformed region of the nuclear periodic table, i.e. $152 \leq A \leq 188$ and $A \geq 228$. This provides one of the major triumphs of the Bohr-Mottelson theory. It is to be understood, however, that the theory in its simple form will not provide all the information that we might

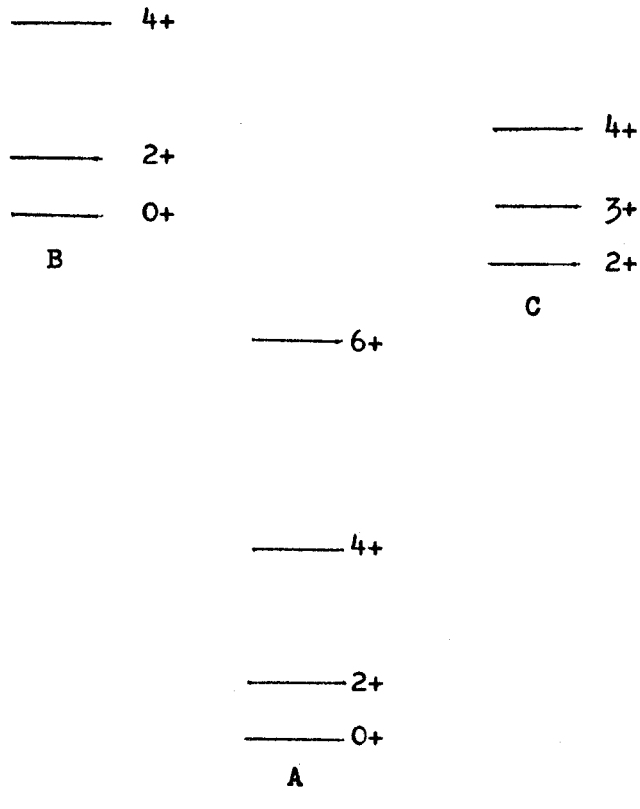


Fig. 1.3. Level Spectrum to be expected in deformed regions. The levels have been shifted sideways for convenience.

A corresponds to ground state rotational levels.

B corresponds to rotational levels built on β-vibration.

C corresponds to rotational levels built on γ-vibration.

want to have. For example we have used the same B for all the three parts of the Hamiltonian namely $H_{rot.}$, H_{β} and H_{γ}. One can estimate B from the hydrodynamic model, but the nucleus is not a classical fluid and the value of B must be very closely linked with the microscopic features that a given nucleus has. In particular, we expect the three B's for the three degrees of freedom to be different. Similarly, one may argue that the restoring forces C's must be obtained likewise from a microscopic theory. Recently such attempts have been made with remarkable success. These attempts include prediction of moments of inertia (Ni, Pr 61), evaluation of β and γ-vibrational levels, of the B's and C's as well as electromagnetic transition probabilities between various levels (Bè 61, Ma 62, Ur, Za 62).

An alternative approach to explain nuclear spectra

Bohr-Mottelson theory presupposes the equilibrium value of γ to be 0 and explains the 2nd 2+ level as being associated with γ-vibration (see Fig. 1.3; Al, Bo, Hu, Mo, Wi 56). An alternative theory to explain the levels associated with γ-vibrational levels (on the Bohr-Mottelson theory) was proposed by Davydov (Da, Fi 58). In his model γ is not a dynamical variable, but a fixed parameter. If γ is not 0 or $\pi/3$, the nucleus is an asymmetric top. The problem thus reduces to finding the eigenvalues and eigenfunctions of an asymmetric top. Each value of J will have $2J+1$ different energy levels but not all of these are allowed.

The wave-function is single-valued in the quantities α_μ but there are various choices of the body-fixed axes all of which give the same α_μ's but different θ, Ψ, ϕ and γ (β is invariant). The wavefunction

must be invariant under any change of the body-fixed system which does not change the α_μ's. In terms of group theory only those wavefunctions which transform according to the symmetric representation of group D_2 are allowed.

This gives, for the first few spins, the following states

J = 1	no level
J = 2	2 levels
J = 3	1 level
J = 4	3 levels
J = 5	2 levels
J = 6	4 levels.

The spacings of the rotational levels depend on the three moments of inertia which can be expressed in terms of γ by the following relation

$$\mathcal{J}_K = \frac{1}{4B\beta^2 \sin^2(\gamma - \frac{2\pi}{3}K)}$$

Thus it is possible to plot the energy levels as a function of γ. It should be noted that for $\gamma = 0$, Davydov theory allows only one set of even spin-states. The ratio of the energies of the 2nd 2+ state and the 1st 2+ state should provide an information of the value of γ. One then hopes to obtain other information by using this value of γ. For example, one could calculate theoretically the rates of electric quadrupole transition.

The Davydov theory has been extensively applied (Da, Ro 59, Ma, Ke 60) and at the present time enjoys as much success as the Bohr-Mottelson theory. One would like to understand, however, why such a model should work. Whether nuclei are axially symmetric or not depends on the form of the $V(\beta, \gamma)$ function. It is possible to compute the potential energy function

from a more microscopic theory and determine if the function minimizes at some non-zero value of γ. Further the potential energy function must be fairly stable around the non-zero equilibrium value so that it makes sense to talk of a fixed γ-value when discussing levels which have energies of the order of 1 MeV. It is hoped that the situation will become clearer when more experimental evidence is obtained. At the present time, however, both the theories exist side by side, often supplementing each other.

CHAPTER 2: INDIVIDUAL STATES OF NUCLEONS IN NUCLEI

Basically, the nucleus is a many-body system and one would expect to obtain nucleonic states in a nucleus via the two-body force. Such an attempt is quite ambitious and early in the development of nuclear physics the question was by-passed by postulating a single one-particle field which a nucleon feels. It is, however, understood now why such a theory works. Many body calculations with nuclear matter show that the "healing distance" (Go, Wa, We 58) within the nucleus is short so that it makes sense to talk of approximately non-interacting particles. Of course, the particles are not entirely free, rather they are immersed in a one-body potential which binds them to the nuclear volume.

Two simple potentials which were studied are the harmonic oscillator

$$V = -V_o + \frac{1}{2} m \omega^2 r^2$$

and the square well

$$V = -V_o , \quad r < R$$

$$= 0 , \quad r > R$$

In the square well, the high degeneracy of the harmonic oscillator states is removed and states of high angular momentum move down in energy. The actual nuclear case is probably somewhere in between these two extremes. It was clear that neither the harmonic oscillator nor the square well nor the intermediate case provide an adequate potential because they produce magic numbers in the wrong places. A spin orbit

force of the form $C\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ (with $C < 0$) is required to produce the correct sequence of magic numbers.

Thus a reasonable potential to assume is the following

$$- V_o + \frac{1}{2} m\omega^2 r^2 + C\underset{\sim}{\ell}\cdot\underset{\sim}{s} + D\,\underset{\sim}{\ell}^2 \qquad 2.1$$

Where the last term ($D < 0$) has been added as a correction to the harmonic oscillator approximation.

A potential energy of the form $\frac{1}{2} m\omega^2 r^2$ is attributable to a spherical nucleus. Nuclear forces are short range and hence we expect the equipotentials to follow the surface. It was mentioned earlier that there are regions in the nuclear periodic table where nuclei have deformation. This is exemplified by the occurrence of rotational spectra, which in turn imply the existence of axes. Thus in a general case our harmonic oscillator approximation must take the form

$$\frac{1}{2} m\,\omega_x^2\, x'^2 + \frac{1}{2} m\,\omega_y^2\, y'^2 + \frac{1}{2} m\,\omega_z^2\, z'^2 .$$

The case which has been extensively studied is that of axial symmetry for which $\omega_x = \omega_y \neq \omega_z$. In the following only Nilsson's calculations will be reviewed, since this is the most dependable result we have. Nilsson introduces a parameter δ defined in the following way

$$\omega_x^2 = \omega_o^2 \left(1 + \frac{2}{3}\delta\right) = \omega_y^2$$

$$\omega_z^2 = \omega_o^2 \left(1 - \frac{4}{3}\delta\right).$$

The assumption that the nuclear volume be independent of deformation leads to $\omega_x\,\omega_y\,\omega_z = \mathring{\omega}_o^3$ where $\mathring{\omega}_o$ is the value of

$\omega_o(\delta)$ for $\delta = 0$; δ is related to the quantity β defined in chapter 1 in the following way

$$\delta \approx .95\beta$$

By introducing a coordinate transformation

$$X \equiv \sqrt{\frac{m\omega_o}{\hbar}}\, X' \quad \text{etc.}$$

the Hamiltonian

$$H = -\frac{h^2}{2m}\nabla'^2 + \frac{1}{2}m(\omega_x^2 X'^2 + \omega_y^2 Y'^2 + \omega_z^2 Z'^2) + C\underset{\sim}{\ell}\cdot\underset{\sim}{s} + D\underset{\sim}{\ell}^2 \qquad 2.2$$

transforms to

$$H_o + H_\delta + C\underset{\sim}{\ell}\cdot\underset{\sim}{s} + D\underset{\sim}{\ell}^2$$

where

$$H_o = \hbar\omega_o \frac{1}{2}\left[-\nabla^2 + r^2\right]$$

and

$$H_\delta = -\delta\hbar\omega_o \frac{4}{3}\sqrt{\frac{\pi}{5}}\, r^2 Y_{20}$$

Nilsson then chooses a representation with basis vectors $|N\ell\Lambda\rangle|s\Sigma\rangle$ where $|s\Sigma\rangle$ is the spin part, with Σ the s_z projection. N is the usual 3-dimensional harmonic oscillator principal quantum number; Λ is the 3-axis projection of $\underset{\sim}{\ell}$. In this representation, H_o and $\underset{\sim}{\ell}^2$ are diagonal and $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ has off diagonal elements but connects only basis vectors with the same N. The operator $r^2 Y_{20}$ has off-diagonal elements and moreover connects states with different N values (the difference in N being an even number). In the major part of his work, Nilsson neglects this coupling between different N values.

The H_o part of the Hamiltonian in 2.2 contributes merely equal diagonal elements for a given N, and hence it is necessary to consider only

$$H - H_o = -\delta \hbar \omega_o \frac{4}{3}\sqrt{\frac{\pi}{5}}\, r^2 Y_{20} + C\underset{\sim}{\ell}\cdot\underset{\sim}{s} + D\underset{\sim}{\ell}^2$$

Defining now

$$\chi \equiv -\frac{1}{2}\frac{C}{\hbar \mathring{\omega}_o}$$

$$\mu \equiv \frac{2D}{C} \qquad 2.3$$

and

$$\eta \equiv \frac{\delta \omega_o(\delta)}{\chi \mathring{\omega}_o}$$

one has

$$H - H_o = \chi \hbar \mathring{\omega}_o \left[-\eta \frac{4}{3}\sqrt{\frac{\pi}{5}}\, r^2 Y_{20} - 2\underset{\sim}{\ell}\cdot\underset{\sim}{s} - \mu\underset{\sim}{\ell}^2 \right]$$

The final calculations now consist in diagonalizing the Hamiltonian $H - H_o$ for each N. This gives the energy values and the wavefunctions in an axially symmetric potential well.

The values of χ and μ are chosen so as to reproduce the shell model states at $\delta = 0$. The value of $\mathring{\omega}_o$ is obtained by taking the mean value of r'^2 for all nucleons to be equal $\frac{3}{5}(1.2 \times 10^{-13} A^{1/3})^2$ cm^2 which gives $\hbar \mathring{\omega}_o = 41\, A^{-1/3}$ MeV. The value of η is fixed at 0.05. In this first calculations Nilsson considers the μ values given in table 2.1. In his appendix A Nilsson suggests a method of reducing the errors in calculation which arise from the coupling of different N values. This is the method followed in our calculations. Consider the following transformation

$$\xi = X' \sqrt{\frac{m\omega_x}{\hbar}} \; ; \quad \eta = Y' \sqrt{\frac{m\omega_y}{\hbar}} \; ; \quad \zeta = Z' \sqrt{\frac{m\omega_z}{\hbar}}$$

applied to the Hamiltonian in 2.2. This gives

$$H = H_\xi + H_\eta + H_\zeta + C\underset{\sim}{\ell}\cdot\underset{\sim}{s} + D\underset{\sim}{\ell}^2 \qquad 2.4$$

where

$$H_\xi = \tfrac{1}{2}\hbar\omega_x \left(-\frac{\partial^2}{\partial\xi^2} + \xi^2 \right) \quad \text{etc.}$$

We note that in the absence of $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ or $\underset{\sim}{\ell}^2$ force the solutions are $|n_\xi\rangle |n_\eta\rangle |n_\zeta\rangle$ with $n_\xi + n_\eta + n_\zeta = N$ a good quantum number.

The idea now is to treat only that part of the $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ forces which conserves the principal quantum number. Accordingly an operator $\underset{\sim}{\ell}_t$ is defined with components

$$(\underset{\sim}{\ell}_t)_\xi = -i\left(\eta\frac{\partial}{\partial\zeta} - \zeta\frac{\partial}{\partial\eta}\right) \text{etc.}$$

The perturbation term to consider then is

$$H_{\text{pert.}} = C(\underset{\sim}{\ell} - \underset{\sim}{\ell}_t)\cdot\underset{\sim}{s} + D(\underset{\sim}{\ell}^2 - \underset{\sim}{\ell}_t^2).$$

Nilsson shows explicitly that the effect of $H_{\text{pert.}}$ is small. Apart from the smallness of $H_{\text{pert.}}$, it is even questionable which Hamiltonian H or H_t is a better description for a deformed nucleus. $\underset{\sim}{\ell}^2$ is the proper correction for a spherical nucleus but there is no reason why a spherically symmetric correction is the right one for a non-spherical potential well.

The Nilsson model has been extensively applied with tremendous success. The original calculations were modified for some shells, namely different μ values were employed. Some level shifts were also introduced (Mo, Ni 59). These were done chiefly by comparing experimental levels in odd-A nuclei. This is in the correct spirit of the calculations which aim at reproducing as many experimental evidence as one can. The model has

withstood a host of experimental facts; prediction of spin and parity of levels, prediction of single-particle magnetic moment, electromagnetic transition rate between different intrinsic states, etc. It has also been used to determine equilibrium deformations of nuclei. For this, one fills up the lowest levels at each deformation with the given number of nucleons. Assuming no residual interaction, the energy of the nucleus is the sum of the energies of these levels. By doing the calculation at all deformations, one can then locate the minimum in energy and hence determine the equilibrium deformation. Of course, Nilsson's calculations are done only for two extreme γ values, 0 and $\frac{\pi}{3}$, and hence cannot answer the question whether some nuclei are non-axial or not. However, it can tell whether or not the prolate shape is preferred to the oblate shape. Earlier calculations by Mottelson and Nilsson had indicated that the prolate is favoured to the oblate in most cases. Some attempts will be made later to show why this was so.

The question of axial asymmetry in nuclei is an interesting one. Davydov's theory completely hinges on it. Extensive calculations by Newton (Ne 60) for low mass nuclei had indicated that in some cases axial asymmetry might be favoured. Newton however did not take any residual interaction into account. It is, therefore, important to see what the theory predicts for heavy nuclei, specifically, in the region $152 \leq A \leq 188$. Also of interest is to know what modifications will be introduced by the pairing forces.

Table 2.1

μ-values used by Nilsson in his 1955 paper; χ = .05

Shells	μ value
N = 0,1,2	μ = 0
N = 3	μ = .35
N = 4	μ = .45, .55 (for protons)
N = 5,6	μ = .45
N = 7	μ = .40

CHAPTER 3: A SPECIAL KIND OF RESIDUAL INTERACTION: THE PAIRING MODEL

The simplest version of the shell model is that of particles moving independently of each other in an over-all single particle potential. More sophisticated treatments take two body correlations into account to various degrees; in this way one has the extreme single particle model, the single particle model and the individual particle model. The usual approach is to treat the closed shell as inert and introduce a two-body force between particles outside the closed shell. One can then reduce the problem to that of matrix diagonalization (Ku 56).

For deformed nuclei the level density becomes more uniform and closed shells are on the same footing as the outside levels. One, therefore, looks for some method of treating a large part of the residual interaction without having to take recourse to matrix diagonalization. As mentioned earlier the residual interaction that we wish to take into account must reproduce the energy gap that is observed in even-even nuclei. This of course at once reminds one of the electron-electron correlation via phonon interaction in superconductivity.

Further, the two-body force in nuclei appears to have the following feature. Consider a nucleus which is a combination of two closed shells plus or minus two particles (Mo 59). The observed levels of the nucleus are then obtained by combining configurations $|jm_1\rangle$ and $|jm_2\rangle$ where the $|jm\rangle$'s are the shell model states. There are many states obtainable from the combination of $|jm_1\rangle$ and $|jm_2\rangle$ and in the absence of any residual

interaction they will all be degenerate. Experiments indicate that of all the various spin states permissible, the 0+ state is by far the lowest. One would then attempt, as a first approximation, to treat a force which has non-zero matrix elements between states like $|jm, j-m, J=0\rangle$ only, where the symbol $|jm, j-m, J=0\rangle$ means that $|jm\rangle$ and $|j-m\rangle$ combine to give total spin 0.

The problem is best treated by methods of second quantization. The following notation will be employed. We label the states by ν ; $-\nu$ will then indicate the time-reversed state. We at once consider deformed nuclei; for spherical nuclei, one has a few particles in one j-shell which is degenerate. This degenerate case can be exactly solved (Ke 61); or one can also use the approximation method of deformed nuclei.

For convenience, ν will stand for the j_3 component in Nilsson states; the $-\nu$ state then has $-j_3$ as the 3 component.

According to the approximation we have wished to make, we consider the Hamiltonian

$$H = \sum_{\nu} \mathcal{E}_{\nu}(a^{+}_{\nu}a_{\nu} + a^{+}_{-\nu}a_{-\nu}) + \sum_{\nu,\nu'} \langle \nu, -\nu, J=0|v|\nu', -\nu', J=0\rangle\, a^{+}_{\nu}a^{+}_{-\nu}a_{\nu'}a_{-\nu'} \qquad 3.1$$

where a^{+}_{ν}, a_{ν} are the creation and annihilation operators respectively of the state ν ; the symbol $|\nu, -\nu, J=0\rangle$ means that the states $|\nu\rangle$ and $|-\nu\rangle$ have combined to total spin $J = 0$.

The next simplifying approximation is obtained by setting

$$\langle \nu, -\nu, J=0|v|\nu', -\nu', J=0\rangle = -G$$

where G gives the strength of the pairing interaction.

We now try to define quasi-particles which already embody in themselves a large part of the pairing interaction. For this a Bogoliubov-Valatin canonical transformation is applied. We define

$$\alpha_\nu \equiv U_\nu a_\nu - V_\nu a^+_{-\nu}$$
$$\beta_\nu \equiv U_\nu a_{-\nu} + V_\nu a^+_\nu \qquad 3.2$$

where U_ν , V_ν are numbers and satisfy the condition

$$U^2_\nu + V^2_\nu = 1 \qquad 3.3$$

In order to obtain results in the system of quasi-particles, it is convenient to remove the restriction that the number of nucleons be a constant of motion. We thus consider an auxiliary Hamiltonian

$$H' = H - \lambda N \quad \text{where } N = \sum_\nu (a^+_\nu a_\nu + a^+_{-\nu} a_{-\nu})$$

and fix λ by requiring that $\langle \Phi_o | N | \Phi_o \rangle = \eta$, the given number of nucleons. $|\Phi_o\rangle$ is, by definition, the ground state.

The inverse relations for eqns. 3.2 are

$$a_\nu = U_\nu \alpha_\nu + V_\nu \beta^+_\nu$$
$$a_{-\nu} = U_\nu \beta_\nu - V_\nu \alpha^+_\nu \qquad 3.4$$

With these relations the Hamiltonian H' becomes

$$H' = U + H_{20} + H_{11} + H_{int.} \qquad 3.5$$

where

$$U = \sum (\epsilon_\nu - \lambda)\, 2V^2_\nu - \Delta^2/G - G \sum V^4_\nu \; ; \; \Delta = G \sum U_\nu V_\nu \qquad 3.6$$

$$H_{20} = \sum \left[(\epsilon_\nu - \lambda)\, 2U_\nu V_\nu - \Delta(U^2_\nu - V^2_\nu) - 2G\, U_\nu V^3_\nu \right] (\alpha^+_\nu \beta^+_\nu + \beta_\nu \alpha_\nu) \qquad 3.7$$

$$H_{11} = \sum \left[(\varepsilon_\nu - \lambda)(U_\nu^2 - V_\nu^2) + 2U_\nu V_\nu - G V_\nu^3 (U_\nu^2 - V_\nu^2) \right] (\alpha_\nu^+ \alpha_\nu + \beta_\nu^+ \beta_\nu) \quad 3.8$$

$H'_{int.}$ contains products of four quasiparticle operators. Since we are interested in going to a system of independent (or at the worst, weakly interacting) particles, we will neglect this term. The effect of this term has been estimated to be small (Be 59, Ni, Pr 60).

To obtain a system of free quasiparticles we next set $H_{20} = 0$ identically. This gives

$$(\varepsilon_\nu - \lambda)\, 2\, U_\nu V_\nu - \Delta(U_\nu^2 - V_\nu^2) - 2G U_\nu V_\nu^3 = 0 \quad 3.9$$

Defining $\tilde{\varepsilon}_\nu \equiv \varepsilon_\nu - G V_\nu^2$ one obtains from 3.9

$$V_\nu^2 = \frac{1}{2}\left(1 - \frac{\tilde{\varepsilon}_\nu - \lambda}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta^2}}\right) \quad 3.10$$

$$U_\nu^2 = \frac{1}{2}\left(1 + \frac{\tilde{\varepsilon}_\nu - \lambda}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta^2}}\right) \quad 3.11$$

Defining $E_\nu \equiv \sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta^2}$, we obtain

$$H' = U + \sum E_\nu (\alpha_\nu^+ \alpha_\nu + \beta_\nu^+ \beta_\nu) \quad 3.12$$

The Lagrange multiplier λ is to be eliminated now. For this, express the number operator $N = (a_\nu^+ a_\nu + a_{-\nu}^+ a_{-\nu})$ in terms of quasiparticle operators. By relations 3.4.

$$N = \sum 2V_\nu^2 + \sum (U_\nu^2 - V_\nu^2)(\alpha_\nu^+ \alpha_\nu + \beta_\nu^+ \beta_\nu) + 2 \sum U_\nu V_\nu (\alpha_\nu^+ \beta_\nu^+ + \beta_\nu \alpha_\nu)$$

The ground-state is the zero quasi-particle state so that λ is eliminated by the following equation

$$\langle \Phi_o | N | \Phi_o \rangle = \sum 2V_\nu^2 = \sum \left(1 - \frac{\tilde{\varepsilon}_\nu - \lambda}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta^2}} \right) = n$$

(n = the given number of particles).

The energy of the ground state is simply

$$\langle \Phi_o | H | \Phi_o \rangle = \sum 2V_\nu^2 - \frac{\Delta^2}{G} \qquad 3.13$$

The higher states for an even-even system can be constructed by creating two, four, etc. quasi-particles. A two quasi-particle state would be $\alpha_\nu^+ \alpha_{\nu'}^+ |\Phi_o\rangle$ and its energy with respect to the ground state is given by $E_\nu + E_{\nu'}$; this difference is at least 2Δ; hence Δ can be defined as the energy gap.

For an odd nucleus, the ground state could be denoted by $\alpha_\nu^+ |\Phi_o\rangle$ where $|\Phi_o\rangle$ is the even-even ground state. An excited state is then given by $\alpha_{\nu'}^+ |\Phi_o\rangle$ and the difference in energy is simply $E_{\nu'} - E_\nu$; no gap will be exhibited.

We note in passing that for G = 0 (and consequently $\Delta = 0$), equations 3.10 and 3.11 give

$$V_\nu^2 = 1, \quad U_\nu^2 = 0 \quad \text{for} \quad \varepsilon_\nu < \lambda$$

$$V_\nu^2 = 0, \quad U_\nu^2 = 1 \quad \text{for} \quad \varepsilon_\nu > \lambda$$

Thus λ is the chemical potential and one gets back the usual zero temperature Fermi distribution. For non-zero G, however, the distribution is smeared out (Fig. 3.1).

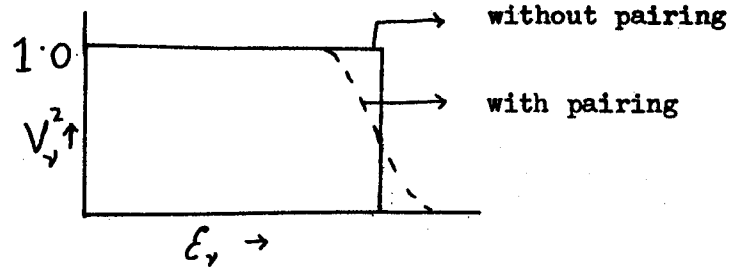


Fig. 3.1. Effect of the pairing correlation on the distribution function.

Finally, we wish to write the zero quasi-particle state in terms of our older particles. It is easy to show that the state which obeys the relation

$$\alpha_\nu |\Phi_o\rangle = (U_\nu a_\nu - V_\nu a^+_{-\nu}) |\Phi_o\rangle = 0$$

$$\beta_\nu |\Phi_o\rangle = (U_\nu a_{-\nu} + V_\nu a^+_\nu) |\Phi_o\rangle = 0$$

is given by

$$|\Phi_o\rangle = \prod_\nu (U_\nu + V_\nu a^+_\nu a^+_{-\nu}) |0\rangle$$

where $|0\rangle$ is the true vacuum.

Mottelson (Mo 59) arrives at the above formulae from a slightly different approach. We wish to solve the Hamiltonian

$$H = \sum_\nu \mathcal{E}_\nu (a^+_\nu a_\nu + a^+_{-\nu} a_{-\nu}) - G \sum_{\nu,\nu'} a^+_\nu a^+_{-\nu} a_{\nu'} a_{-\nu'} \qquad 3.14$$

The most direct thing to do is to construct an operator $A = \sum A_\nu a^+_\nu a^+_{-\nu}$ where the A_ν's will depend upon the energies of the levels as well as the strength of the pairing interaction. If we are trying to create an N (N = even) particle system, we should operate $A^{N/2}$ on true

vacuum. However, construction of the operator A is difficult. Hence we try the next best thing. We begin with a trial wave function

$$|\Phi_o\rangle = \prod_\nu (U_\nu + V_\nu a^+_\nu a^+_{-\nu})|0\rangle \qquad 3.15$$

which is easy to work with but does not conserve the number of particles. This is the price we must pay for starting with an easy state vector. It is then necessary to introduce an auxiliary Hamiltonian $H' = H - \lambda N$ and fix λ, as mentioned earlier. Mottelson shows that the condition $\delta \langle \Phi_o | H' | \Phi_o \rangle = 0$ leads to all the equations deduced earlier.

The pairing force has only a small effect on the binding energy; however, it is important to include the pairing force in calculations of equilibrium deformation. We can elucidate this point further. Without the pairing interaction, the states are filled up to the Fermi level; the levels above have no influence at all. With pairing the distribution is smeared out, and the behaviours of the levels just above the Fermi sea also contribute to the total behaviour. In considering equilibrium deformation one is faced with small changes of energy and contributions of the levels above the Fermi sea may play an important part.

We notice from 3.9 that the pairing force contributes a term to the self-consistent field, viz. $G U_\nu V_\nu^3$ and does in a sense cause a renormalization of the single particle energies ε_ν of the deformed field. It originates from the diagonal part of the pairing Hamiltonian and therefore has the character of a self-energy term. From its nature, we see that it affects mostly those energy levels close to the Fermi level, but is otherwise small. One may take the experimental single-particle energies as the renormalized energies; in a theoretical calculation one can also define the

energies calculated to be the renormalized energies or just ignore the term for the reason explained above.

Finally, the strength of the pairing interaction G should be chosen so as to reproduce the increase in the ground-state binding energy of even-even system. One can also estimate the value of Δ from the position of the first excited two quasi-particle state . The value of G also depends upon the number of states in which pairwise scattering is allowed to take place. Belyaev has shown that states far removed from the Fermi surface contribute to the wavefunction only through a renormalization of G. Bès and Szymanski (Bè, Sz 61) in their investigation of axially symmetric deformation considered scattering between 24 levels near the Fermi level. We have used the same values as they have: namely $AG_n = 26.5$ MeV. and $AG_p = 32.2$ MeV. The neutrons and protons occupy different levels in all the nuclei that we consider; hence we do not consider any pairing interaction between neutrons and protons.

CHAPTER 4: CALCULATIONS ON THE QUESTION OF AXIAL ASYMMETRY IN NUCLEI

It is well known that the collective model is remarkably successful in describing various nuclear properties in the region $152 \leq A \leq 188$. Assuming axial symmetry, the equilibrium deformation in this region has been calculated previously without any residual interaction (Mo, Ni 59) and also with residual interaction (Bè, Sz 61). For a group of nuclei in this region $\beta \approx 0.3$ from these calculations (β, γ were defined in chapter 1). The results are in good agreement with experiments.

We wish to see if $\beta = 0.3$, $\gamma = 0$ is a consistent assumption for these nuclei; that is, at this β value, whether a non-zero γ-value is energetically more favourable or not.

Specifically the nuclei chosen are Sm^{154}, Gd^{156}, Gd^{158}, Gd^{160}, Dy^{162}, Dy^{164}, Er^{166}, Er^{168} and Yb^{172}. The calculations are envisaged in two stages. First, single particle levels will be calculated for $\beta = 0.3$, $\gamma = 0 \left(\frac{\pi}{12}\right) \frac{\pi}{3}$; $\gamma = 0$ is the prolate shape and $\gamma = \frac{\pi}{3}$ is the corresponding oblate shape. One can then calculate equilibrium deformation by doing Mottelson-Nilsson (Mo, Ni 59) type calculation. Second, residual interaction will be included using the method of the pairing interaction.

Calculation of single-particle levels

The equation to solve is

$$\left[-\frac{\hbar^2}{2m} \nabla'^2 + \frac{1}{2} m \left(\omega_x^2 X'^2 + \omega_y^2 Y'^2 + \omega_z^2 Z'^2\right) + C\underset{\sim}{\ell}\cdot\underset{\sim}{s} + D\underset{\sim}{\ell}^2 \right] \Psi = E\Psi \qquad 4.1$$

The following transformation is made

$$\xi = X' \sqrt{\frac{m\omega_x}{\hbar}} \; ; \qquad \eta = Y' \sqrt{\frac{m\omega_y}{\hbar}} \; ; \qquad \zeta = Z' \sqrt{\frac{m\omega_z}{\hbar}}$$

Equation 4.1 transforms to

$$(H_\xi + H_\eta + H_\zeta + C\underset{\sim}{\ell}\cdot\underset{\sim}{s} + D\underset{\sim}{\ell}^2)\,\Psi = E\Psi \qquad 4.2$$

where

$$H_\xi = \tfrac{1}{2}\hbar\omega_x \left(-\frac{\partial^2}{\partial\xi^2} + \xi^2 \right) \quad \text{etc.}$$

The idea is to replace $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ in eqn. 4.2 by $\underset{\sim}{\ell}_t\cdot\underset{\sim}{s}$ and $\underset{\sim}{\ell}_t^2$ where $\underset{\sim}{\ell}_t$ is the angular momentum in the pseudospherical coordinates. This is the method of appendix A in Nilsson's work (Ni 55);(see also chapter 2).

Nuclear forces are short range, so we postulate that equipotentials follow the shape of the surface. This gives

$$\omega_x = \frac{\omega_o}{1 - \sqrt{\frac{5}{4\pi}}\,\beta\cos(\gamma + \frac{\pi}{3})}$$

$$\omega_y = \frac{\omega_o}{1 - \sqrt{\frac{5}{4\pi}}\,\beta\cos(\gamma - \frac{\pi}{3})} \qquad 4.3$$

$$\omega_z = \frac{\omega_o}{1 + \sqrt{\frac{5}{4\pi}}\,\beta\cos\gamma}$$

ω_o is fixed by requiring the condition of volume conservation (chapter 2); $\omega_x\omega_y\omega_z = \mathring{\omega}_o^3$; $\hbar\mathring{\omega}_o = 41A^{-1/3}$ MeV. This completely defines the frequencies.

It is convenient to define two other deformation parameters B and Γ and use the following parametrization (Ne 60) instead.

$$\omega_x = \omega(B,\Gamma)\left[1 + B\cos\left(\Gamma + \tfrac{\pi}{3}\right)\right]$$
$$\omega_y = \omega(B,\Gamma)\left[1 + B\cos\left(\Gamma - \tfrac{\pi}{3}\right)\right] \qquad 4.4$$
$$\omega_z = \omega(B,\Gamma)\left[1 - B\cos\Gamma\right]$$

The values of B, Γ and $\omega(B,\Gamma)$ will be taken from Newton's work. They are listed in table 4.1.

The Hamiltonian we consider can now be broken up into two parts H_o and H' with

$$H_o = \tfrac{1}{2}\hbar\omega(B,\Gamma)\left(-\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\zeta^2} + \xi^2 + \eta^2 + \zeta^2\right) \qquad 4.5$$

and

$$H' = \tfrac{1}{4}\hbar\omega(B,\Gamma)\,B\cos\Gamma\left(-\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2} + 2\frac{\partial^2}{\partial\zeta^2} + \xi^2 + \eta^2 - 2\zeta^2\right)$$
$$-\tfrac{\sqrt{3}}{4}\hbar\omega(B,\Gamma)\,B\sin\Gamma\left(-\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \xi^2 - \eta^2\right) + C\underset{\sim}{\ell_t}\cdot\underset{\sim}{S} + D\underset{\sim}{\ell_t^2} \qquad 4.6$$

<u>Choice of representation</u>:

The basis vectors chosen are

$$\phi_{N\ell j\Omega} = R_{n\ell}(\rho)\,\chi_{\ell j\Omega}(\phi,\theta,s)$$

$$\rho = (\xi^2 + \eta^2 + \zeta^2)^{1/2}$$

$$R_{n\ell}(\rho) = \frac{\sqrt{2}\{\Gamma(n+1)\}^{1/2}}{[\Gamma(n+\ell+3/2)]^{3/2}}\; e^{-\frac{\rho^2}{2}}\; \rho^{\ell}\; L_n^{\ell+1/2}\;;\; n = \frac{N-\ell}{2}$$

The factor $L_n^{\ell+1/2}$ is a Laguerre polynomial with the Morse and Feshbach definition (Mo, Fe 53).

$$\chi_{\ell j \Omega} = (\ell\ 1/2\ \ \Omega-1/2\ \ 1/2 \mid \ell\ 1/2\ j\ \Omega)\ Y_{\ell\ \Omega-1/2}\ S_{1/2\,1/2}$$

$$+ (\ell\ 1/2\ \ \Omega+1/2\ -\ 1/2 \mid \ell\ 1/2\ j\ \Omega)\ Y_{\ell\ \Omega+1/2}\ S_{1/2\,-1/2}$$

The Clebsch-Gordan coefficients and the spherical harmonics have Condon and Shortley phases.

For a given N, the only non-zero matrix elements of H_o in this representation are all diagonal and equal. Hence, we need consider H' only.

Of H', $\underset{\sim}{\ell}_t \cdot \underset{\sim}{s}$ and $\underset{\sim}{\ell}_t^2$ have diagonal elements only with values

$$\frac{j(j+1) - \ell(\ell+1) - s(s+1)}{2} \quad \text{and} \quad \ell(\ell+1)$$

(there is no need to keep the subscript on ℓ) respectively.

Further, Nilsson shows (Ni 55, appendix A)

$$\left(\phi_{N\ell\Lambda\Sigma} \left| -\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2} + 2\frac{\partial^2}{\partial\zeta^2} + \xi^2 + \eta^2 - 2\zeta^2 \right| \phi_{N'\ell'j'\Sigma'}\right)$$

$$= \delta_{NN'}\left(\phi_{N\ell\Lambda\Sigma} \left| 2(\xi^2 + \eta^2 - 2\zeta^2) \right| \phi_{N'\ell'\Lambda'\Sigma'}\right)$$

Since $\phi_{N\ell j\Omega}$ is a linear combination of $\phi_{N\ell\Lambda\Sigma}$, it follows that the relation holds in this representation also. By a similar token

$$\left(\phi_{N\ell j\Omega} \left| -\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \xi^2 - \eta^2 \right| \phi_{N\ell j\Omega}\right)$$

$$= \delta_{NN'}\left(\phi_{N\ell j\Omega} \left| 2(\xi^2 - \eta^2) \right| \phi_{N\ell j\Omega}\right).$$

Remembering that matrix elements between states with the same N only are to be calculated, we can then write H' as

$$H' = -2\sqrt{\frac{\pi}{5}}\hbar\omega\, B \cos\Gamma\ \rho^2 Y_{20} - 2\sqrt{\frac{\pi}{5}}\hbar\omega\, B \sin\Gamma\ \rho^2\ \frac{Y_{22} + Y_{2-2}}{2}$$

$$+ C\underset{\sim}{\ell}_t \cdot \underset{\sim}{s} + D\underset{\sim}{\ell}_t^2.$$

Now

$$\langle N \ell j \Omega | \rho^2 Y_{2\mu} | N \ell' j' \Omega' \rangle = \langle N \ell | \rho^2 | N \ell' \rangle \langle \ell j \Omega | Y_{2\mu} | \ell' j' \Omega' \rangle$$

$$\langle N \ell | \rho^2 | N \ell \rangle = (N + 3/2)$$

$$\langle N, \ell - 2 | \rho^2 | N \ell \rangle = \sqrt{(N - \ell + 2)(N + \ell + 1)}$$

$$\langle \ell j \Omega | Y_{2\mu} | \ell' j' \Omega' \rangle = \sqrt{\frac{5}{4\pi}} (j' 2 \Omega' \mu | j' 2 j \Omega)(j 2 - 1/2 \; 0 | j 2 j' - 1/2)$$

Since μ is 0 or ± 2, we see that a state Ω will be connected to itself, and $\Omega \pm 2$ only .

For the nuclei we have chosen neutron levels have to be calculated up to N = 6 and proton levels up to N = 5. For N = 6, the matrix is of order 28 x 28, the basis vectors being $|6\ 13/2\ 6\ 13/2\rangle$, $|6\ 13/2\ 6\ 9/2\rangle$ ..... $|6\ 13/2\ ,\ 6\ -11/2\rangle$; $|6\ 11/2\ 6\ 9/2\rangle$, $|6\ 11/2\ 6\ 5/2\rangle$ ..... $|6\ 11/2\ 6\ -11/2\rangle$; $|6\ 9/2\ 4\ 9/2\rangle$; $|6\ 9/2\ 4\ 5/2\rangle$ ..... $|6\ 9/2\ 4\ -7/2\rangle$; $|6\ 7/2\ 4\ 5/2\rangle$ ..... $|6\ 7/2\ 4\ -7/2\rangle$; $|6\ 5/2\ 2\ 5/2\rangle$ ..... $|6\ 5/2\ 2\ -3/2\rangle$; $|6\ 3/2\ 2\ 1/2\rangle$, $|6\ 3/2\ 2\ -3/2\rangle$ and $|6\ 1/2\ 0\ 1/2\rangle$. For N = 5 the matrix is of order 21 x 21, for N = 4 it is 15 x 15 and for N = 3 it is 10 x 10.

For the coefficients C and D we take the values recommended by Nilsson and Prior (Ni, Pr 60). It is well known that different values of D are needed to calculate neutron and proton levels. Further, the same level shifts which they suggest have also been made. These are listed in table 4.2. With these assignments of parameters, we can use Newton's result for N = 0, 1, 2, 3, 4 (neutrons) and for N = 0, 1, 2 (protons). N = 5, 6 levels for neutrons and N = 3, 4, 5 levels for protons were calculated in the

IBM 7090 computer in the University of Toronto. The well-known method of Jacobi was used for matrix diagonalisation.

Since five γ-values were chosen, this then corresponds to diagonalising twenty-five matrices. Actually thirty matrices were diagonalised, $\gamma = \frac{\pi}{6}$ being always calculated twice to guarantee accuracy of calculations as will be explained now. Eqn. 4.3 shows that $(\beta, \frac{\pi}{6})$ and $(-\beta, \frac{\pi}{6})$ correspond to the same physical situation. It is then advantageous to calculate $\omega(B, \Gamma)$, B and Γ for both $(\beta, \frac{\pi}{6})$ and $(-\beta, \frac{\pi}{6})$ and obtain energy values using both the set of parameters. In this way it was checked that the energy values calculated for the large matrices are, at the worst, off by 1 in the fifth decimal place. A check with Newton's results was also made and the agreement was exact.

The energy values are tabulated in appendix 1. To them is to be added $(N + \frac{3}{2})\hbar\omega$ where N is the relevant principal quantum number; it is to be emphasized that ω is deformation dependent and has to be taken from table 4.1. (Note the difference from Newton's tables.) The levels give the unshifted energy values.

To save space, the wavefunctions have not been given, although they were calculated. They can, however, be obtained on writing to the author.

Equilibrium deformation without pairing

The top curves in figs. 4.1 to 4.9 correspond to the plot $\sum \varepsilon_\nu (\beta_0, \gamma)$ as a function of γ for the nine nuclei mentioned. Each energy level contains two nucleons, one nucleon corresponding to the time reversed state. The different curves within each diagram correspond to different intrinsic configurations. It is seen that in each of the cases

$\gamma = 0$ is favoured to any other γ-value. This is not in entire agreement with a previous work reported by Dutt and Mukherjee (Du, Mu 62) and we proceed to show why it is not so.

1. Dutt and Mukherjee consider the same parametrization as is made in eqn. 4.4 but consider B to be independent of γ. Actually B is a function of both β and γ, hence the range of energy values they obtain really belong to some complex β-γ contour. While this is the major difference, there are minor differences also.
2. They use different values for D, the coefficient of the $\underset{\sim}{\ell}^2$ force.
3. They completely neglect the N = 6 shell for neutrons.
4. No level shift is made in their work.

Equilibrium deformation with pairing

To include the pairing correlation we now solve

$$H = \sum_{\nu} \varepsilon_{\nu} (a^{+}_{\nu} a_{\nu} + a^{+}_{-\nu} a_{-\nu}) - G \sum_{\nu,\nu'} a^{+}_{\nu} a^{+}_{-\nu} a_{\nu'} a_{-\nu'} \qquad 4.6$$

where the ε_{ν}'s are the energy values we have already calculated; $-\nu$ corresponds to the time reversed state ν, a^{+}_{ν} and a_{ν} are the creation and annihilation operators respectively of the state ν and G is the strength of the pairing interaction. The ground state energy of eqn. 4.6, as is well known from the methods of the pairing model (see chapter 3), is given by

$$E = \sum \varepsilon_{\nu} 2V^2_{\nu} - \frac{\Delta^2}{G} \qquad 4.7$$

where

$$V^2_{\nu} = \frac{1}{2}\left(1 - \frac{\varepsilon_{\nu} - \lambda}{\sqrt{(\varepsilon_{\nu} - \lambda)^2 + \Delta^2}}\right) \qquad 4.8$$

and

$$\Delta^2 = G \sum U_\nu V_\nu \; ; \qquad U_\nu = \sqrt{1 - V_\nu^2} \tag{4.9}$$

λ is to be eliminated from the condition

$$\sum \left(1 - \frac{\varepsilon_\nu - \lambda}{\sqrt{(\varepsilon_\nu - \lambda)^2 + \Delta^2}} \right) = n \tag{4.10}$$

where n is the given number of particles.

For the nuclei we consider, neutrons and protons occupy different states and are not correlated by the pairing interaction. The problem thus separates out into two systems, the neutron system (the N-system) and the proton system (the P-system). Different pairing strengths are to be attributed to the different systems. We use the values given by Bès and Szymanski; they allow scattering between 24 levels near the Fermi sea and fix G_n at $\frac{26.5}{A}$ MeV. and G_p at $\frac{32.2}{A}$ MeV.

In our case, it was seen earlier that even without the residual interaction all the nuclei considered favoured prolate symmetry. Since the residual interaction basically leads to configuration mixing, it is obvious that the conclusions will not be changed except possibly at the end of the mass region considered.

As examples, pairing model calculations were carried out on three nuclei: Sm^{154}, Dy^{162} and Yb^{172}. To calculate E of eqn. 4.7, we need to know λ and Δ^2. Eqn. 4.9 can be rewritten as

$$\frac{2}{G} = \sum \frac{1}{\sqrt{(\varepsilon_\nu - \lambda)^2 + \Delta^2}} \tag{4.11}$$

Eqns. 4.10 and 4.11 were solved in the Bendix G-15D computer at McMaster University to find λ and Δ^2. The trick lies in noting that n is determined chiefly by λ and G is determined chiefly by Δ^2. One can, therefore, use an iterative procedure. λ and Δ^2 having been determined, it is trivial to calculate E as given by eqn. 4.7.

The curves obtained after doing the pairing model calculations are given in the figures at the end of this chapter. It will be seen that different intrinsic configurations lose their meaning and the curves become considerably flattened.

Some comments are to be made on the use of eqn. 4.7. It might be argued that the nuclear potential is really generated by two body forces and hence we should have a factor $\frac{3}{4}$ in front of the summation in eqn. 4.7. Here, however, we adopt the point of view of Bès and Szymanski (Bè, Sz 61; see also Mo 57) which is as follows.

Superimposed upon the harmonic oscillator there is of course a central binding field. If we include the factor $\frac{3}{4}$ this means that we are putting a constraint on the binding field, namely that we do not allow it to vary.

On the other hand, it is known that the factor $\frac{3}{4}$ must not be present when calculating nuclear excitation spectra. In this case, the change in the potential energy ($\frac{1}{2}$ the sum of individual changes) is increased by an exactly equal amount which comes from the change in the field.

The least ambiguous way, therefore, seems to be to treat the change in deformation as a many-particle excitation. It should be added that calculations were also done with a factor $\frac{3}{4}$ and all our conclusions about equilibrium deformation remained unchanged.

It is well known that under the condition of volume conservation coulomb energy favours prolate to oblate (Mo, To 54). For the nuclei considered coulomb energy difference between prolate and oblate shapes is $\approx 0.2\ \hbar\omega_0$; from quite general considerations this energy will increase like $-\cos 3\gamma$ between $\gamma = 0$ and $\gamma = \frac{\pi}{3}$.

Discussion of the results obtained

Having thus convinced ourselves that with the usual description of nuclear potential and residual interaction the above nuclei will favour prolate axial symmetry, we now ask: what are the features of the theory that lead consistently to prolate equilibrium deformation?

A simple model to consider is a three-dimensional harmonic oscillator without any $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ or $\underset{\sim}{\ell}^2$ force. Zaikin (Za 59) has shown that in this model some nuclei will have axial asymmetry. Unpublished calculations of Gursky (Gu 55) had also shown this. If one restricts oneself to only prolate and oblate shapes, then in the first half of a shell a prolate shape will be favoured and in the last half, an oblate shape.

Lemmer and Weisskopf have given semiquantitative arguments as to how this behaviour can change because of an $\underset{\sim}{\ell}^2$ force (Le, We 61). For convenience of calculations, they use a force ηr^4 where η is positive. Nilsson's calculations with a $D\underset{\sim}{\ell}^2$ force ($D < 0$) pulls higher angular momentum states down; an ηr^4 term pushes the lower angular momentum states up and thus the level ordering is affected in a similar way. An explicit calculation is now done for $N = 4$ shell using perturbation theory. Fixing the value of η by comparing with D, the authors show that the symmetry between prolate and oblate shapes is destroyed and that prolate continues to be favoured throughout most of the shell.

Let us first do a model calculation for our nuclei assuming no $\underset{\sim}{\ell} \cdot \underset{\sim}{s}$ or $\underset{\sim}{\ell}^2$ force and no level shift (the level shifts made in the exact calculations are unimportant in deciding the question of equilibrium deformation). With the model Hamiltonian, the proton systems in our nuclei are filling up the last half of the N = 4 shell and the neutrons are at about the middle of the N = 5 shell. One would thus expect the oblate shape to be energetically more favourable than the prolate shape. As examples of magnitudes of preference, in Sm^{154}, Dy^{162} and Er^{168} the oblate is the more stable state by an amount $\approx 1.2\, \hbar \overset{\circ}{\omega}_0$.

Thus the $\underset{\sim}{\ell}^2$ force is an important factor in determining axial symmetry in so far as it affects the energy levels in an unfilled shell. There is, however, another factor which is crucial for all our calculations.

For the nuclei considered N = 0, 1, 2 and 3 shells are always completely filled up by both protons and neutrons. We now show that the contribution to energy from these completely filled shells is such as to favour the prolate shape.

Using the eqn. 4.3 and the condition for volume conservation

$$\omega_0 = \overset{\circ}{\omega}_0 \left[1 - \frac{5}{16\pi}\, \beta^2 + \frac{1}{12} \left(\frac{5}{4\pi}\right)^{3/2} \beta^3 \cos 3\gamma \right]$$

ω_x, ω_y and ω_z can now be completely determined in terms of $\overset{\circ}{\omega}_0$, β and γ (from eqn. 4.3).

First assume that there are no $\underset{\sim}{\ell} \cdot \underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ forces present. The states are then labelled by $|n_x, n_y, n_z\rangle$. The contribution from a completely filled shell N to the energy is

$$E^T(N) = \sum_{\text{all states in N}} \left[(n_x + \tfrac{1}{2})\,\hbar\omega_x + (n_y + \tfrac{1}{2})\,\hbar\omega_y + (n_z + \tfrac{1}{2})\,\hbar\omega_z \right]$$

$$= 2\,\frac{(N+1)(N+2)(2N+3)}{12}\,\hbar\omega_0^{\circ}\left[3 + \frac{15}{16\pi}\beta^2 - \frac{1}{2}\left(\frac{5}{4\pi}\right)^{3/2}\beta^3 \cos 3\gamma\right] \quad 4.12$$

The factor of 2 in eqn. 4.12 comes from spin degeneracy. The expression shows that $\gamma = 0$ will be favoured.

We now prove that these considerations apply even when $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ forces are present. For convenience imagine the matrix of the Hamiltonian formed in the $|jm\rangle$ representation, so that the $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ forces are entirely diagonal. The total contribution to the energy is just the sum of the diagonal elements of the matrix. The sum of the diagonal elements from the oscillator part is still given by eqn. 4.12; the diagonal elements from $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ do not depend on the deformation. Thus the conclusions made earlier remain unchanged.

The difference in energy from closed shells N = 0, 1, 2, 3 (both neutrons and protons) add up to $\approx .67\,\hbar\omega_0^{\circ}$. A look at the curves in figs. 4.1 to 4.9 will show that this is crucial for obtaining prolate equilibrium deformation. However, in the absence of $\underset{\sim}{\ell}$-dependent forces, the unfilled shell may favour the oblate shape by an amount which is greater than $0.67\,\hbar\omega_0^{\circ}$ and hence the influence of the filled shells may not be sufficient to reverse the result. But $\underset{\sim}{\ell}^2$ force does substantially reduce the amount of which a partially filled shell may favour the prolate shape; it does this by its influence on the order of the level whereby higher $\underset{\sim}{\ell}$-states are filled sooner.

The following example is instructive. Consider nucleons filling up N = 4 shell. We will do a Lemmer-Weisskäpf type calculation but instead

of an r^4 force we will use the usual $\underset{\sim}{\ell}^2$ force. To start with, let us forget about contributions from N = 0, 1, 2 and 3 shells.

Without the $\underset{\sim}{\ell}^2$ and $\underset{\sim}{\ell} \cdot \underset{\sim}{s}$ forces, we now calculate the energy at $\beta = 0.3$ and $\beta = -0.3$ for each case as N = 4 fills up. We denote by ΔE the energy difference between the oblate and the prolate for a given number of nucleons. Straight forward calculation now gives $\Delta E_{min.} = -1.18\ \hbar\omega_o^\circ$ which occurs when there are 18 nucleons.

To include now the effects of the $\underset{\sim}{\ell} \cdot \underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ forces we use Nilsson's asymptotic solutions for single-particle energies and wavefunctions. For the large value of β we are considering the approximation is not bad (See Mo, Ni 59). The wavefunctions are denoted by $| n_z, n_\perp, \Lambda, \Sigma \rangle$ where

$$n_\perp = n_x + n_y$$

$$\Lambda = \text{3-axis projection of } \underset{\sim}{\ell}$$

and $$\Sigma = \text{3-axis spin projection.}$$

The energy values are

$$(n_z + \tfrac{1}{2})\ \hbar\omega_z + (n_\perp + 1)\ \hbar\omega_\perp + C\,\Lambda\Sigma + D\,(\Lambda^2 + 2n_\perp n_z + 2n_z + n_\perp)$$

where C and D are the coefficients of the $\underset{\sim}{\ell} \cdot \underset{\sim}{s}$ and the $\underset{\sim}{\ell}^2$ force respectively. We now again calculate ΔE for $C = -0.1\ \hbar\omega_o^\circ$ and $D = -0.0275\ \hbar\omega_o^\circ$ as the shell fills up. Now, however, $\Delta E_{min.}$ is only $-.52\ \hbar\omega_o^\circ$ which occurs for 24 nucleons. In fact, all marked preferences for the oblate are smoothed out and the background of N = 0, 1, 2, 3 shells becomes decisive.

The kind of behaviour from closed shells is not peculiar to potentials of the form 4.3 only. Rather it is a property of volume conservation. For example, Davydov and Zaikin (Da, Za 59) consider the following potential.

$$\omega_k = \mathring{\omega}_o \, e^{-\xi_k} \; ; \qquad k = 1 \text{ to } 3$$

$$\xi_k = \sqrt{\frac{5}{4\pi}} \, \beta \cos\left(\gamma - \frac{2\pi}{3} k\right)$$

A similar calculation as above will again show that the prolate will be favoured to the oblate to third order in β.

An alternative approach: The self-consistent method

Since this behaviour of closed shells is pertinent to volume conservation methods, it is worthwhile to find out how the closed shells will behave under a different method of calculation of equilibrium. The method we have used so far will be referred to as the volume conservation method. The alternative method we will discuss will be referred to as the self-consistent method. It was first suggested by Belyaev (Be 59) and has been discussed by Kerman (Ke 60) and Bès (Be 61) and Bès and Szymanski (Bè, Sz 61).

The self-consistent method starts by postulating that there are quadrupole-quadrupole forces between nucleons; if it were not so, the nucleus would always be spherical, its energy and wavefunctions being determined by the modified isotropic harmonic oscillator field and the pairing force. A two-body quadrupole force will be difficult to handle, hence we do the next best thing; we assume that a quadrupole field exists and each nucleon interacts with this field. Thus we consider the following Hamiltonian

$$H_i = H_{oi} - kQq_i \qquad 4.13$$

where

$$H_{oi} = \frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 r_i^2$$

and

$$q_i = 2Z_i^2 - X_i^2 - Y_i^2$$

The value of k will depend upon the strength of the quadrupole interaction. This Hamiltonian describes an axially symmetric system but for an orientation it is sufficient to compare the oblate and prolate cases.

The self-consistency condition on the quadrupole moment is given by

$$\langle \Phi_o | \sum_i q_i | \Phi_o \rangle = Q \qquad 4.14$$

where

$$\sum_i H_i \Phi_o = E \Phi_o$$

To satisfy eqn. 4.14 we introduce a Lagrange multiplier and find the eigenstate Φ of an auxiliary Hamiltonian

$$\sum H_i' \Phi = E' \Phi \qquad 4.15$$

where

$$H_i' = H_{oi} - kQq_i - \hat{\mu} q_i \qquad 4.16$$

The value of $\hat{\mu}$ at each Q is determined by the analogue to eqn. 4.14, viz.

$$\langle \Phi | \sum q_i | \Phi \rangle = Q$$

When the real system is in state Φ which is not one of its energy eigenstates, the expectation value of its quadrupole moment is Q and the energy expectation value of the system is

$$E = \langle \Phi | \sum H_i' + \tfrac{1}{2} kQ^2 + \hat{\mu} Q | \Phi \rangle = \langle \phi | \sum H_i' | \Phi \rangle + \tfrac{1}{2} kQ^2 + \hat{\mu} Q \qquad 4.17$$

where $\frac{1}{2} kQ^2$ has been added to prevent counting of pairs twice.

It can be shown (Bè 61) that at equilibrium deformation, i.e. $\partial E/\partial Q = 0$, the Lagrange multiplier vanishes and the Hamiltonian used to generate the wavefunctions is the actual Hamiltonian of eqn. 4.13. Hence if we take the value of Q given by $\partial E/\partial Q = 0$, the function Φ of eqn. 4.15 is actually Φ_o and we have found a stationary state with a self-consistent value of Q.

Let us apply these results to a nucleus which has a few shells filled up and some nucleons in an unfilled shell. Suppose the total system has equilibrium deformation Q_T.

The energy of the nucleus at Q_T is then

$$E(Q_T) = \langle \Phi_o | \sum (H_{oi} - kQ_T q_i) | \Phi_o \rangle + \tfrac{1}{2} kQ^2 \qquad 4.18$$

The auxiliary Hamiltonian H' for the corresponding deformation of opposite sign, i.e. $-Q_T$ is

$$H_i' = H_{oi} + (kQ_T - \hat{\mu})\, q_i$$

where $\hat{\mu}$ is chosen to give

$$\langle \Phi | \sum q_i | \Phi \rangle = -Q_T \qquad 4.19$$

The energy expectation value at this deformation is

$$E(-Q_T) = \langle \Phi | \sum H_{oi} + (kQ_T - \hat{\mu}) \sum q_i | \Phi \rangle + \tfrac{1}{2} kQ_T^2 + \hat{\mu} \langle \Phi | \sum q_i | \Phi \rangle \qquad 4.20$$

Of course this state Φ is not a stationary state of the nucleus and rapidly changes in time; if the nucleus is prepared in this state with moment $-Q_T$, it will eventually go to the state Q_T, since this is, by definition, the equilibrium state. The expectation value $E(-Q_T)$ of eqn. 4.20 is of course larger than the energy $E(Q_T)$ of the stationary state. We ask, however, the question: what is the contribution of the closed shells to the energy difference between the two configurations?

The difference in energy values is

$$E(-Q_T) - E(Q_T) = E'(-Q_T) - E(Q_T) + \hat{\mu} \langle \Phi \mid \sum q_i \mid \Phi \rangle$$

$$= E'_{C\ell}(-Q_T) - E_{C\ell}(Q_T) + \hat{\mu} \langle \Phi \mid \sum_{C\ell} q_i \mid \Phi \rangle$$

$$+ E'_{un}(-Q_T) - E_{un}(Q_T) + \hat{\mu} \langle \Phi \mid \sum_{un} q_i \mid \Phi \rangle$$

where the subscripts $C\ell$ and un refer to closed and unfilled shells respectively, and

$$E'_{C\ell}(-Q_T) = \langle \Phi \mid \sum_{C\ell} H'_i \mid \Phi \rangle \quad \text{etc.}$$

We may consider the contributions from closed shells to be

$$\delta E_{C\ell} = E'_{C\ell}(-Q_T) - E_{C\ell}(Q_T) + \hat{\mu} \langle \Phi \mid \sum_{C\ell} q_i \mid \Phi \rangle \qquad 4.21$$

Now first assume that there are no $\underset{\sim}{\ell} \cdot \underset{\sim}{s}$ or $\underset{\sim}{\ell}^2$ forces. Then

$$E'_{C\ell}(-Q_T) = 2 \frac{(N+1)(N+2)(2N+3)}{12} \hbar\omega \left[3 - \frac{3(kQ_T - \hat{\mu})^2}{m^2\omega^4} \right] \qquad 4.22$$

$$E_{C\ell}(Q_T) = 2 \frac{(N+1)(N+2)(2N+3)}{12} \hbar\omega \left[3 - \frac{3k^2Q_T^2}{m^2\omega^4} \right] \qquad 4.23$$

$$\langle \Phi | \sum q_i | \Phi \rangle = -2 \frac{(N+1)(N+2)(2N+3)}{6} \frac{\hbar}{m\omega} \left[3 \frac{kQ_T - \hat{\mu}}{m\omega^2} \right] \qquad 4.24$$

From eqns. 4.21 to 4.24 we obtain

$$\delta E_{C\ell} = \frac{(N+1)(N+2)(2N+3)}{2} \frac{\hat{\mu}^2}{m^2\omega^4} \qquad 4.25$$

If we now wish to include the effects of the $\underset{\sim}{\ell} \cdot \underset{\sim}{s}$ and $\underset{\sim}{\ell}^2$ forces for the above calculations we can do so by exploiting again Nilsson's asymptotic solutions (Ni 56). It is then easy to verify that all the above calculations remain unchanged. Thus the closed shells follow the preference of the total system and the simple conclusions of the volume-conservation method no longer hold good.

Thus it is not at all abvious that the self-consistent method of calculation will always predict prolate equilibrium. A self-consistent calculation will proceed in the following stages.

1. Now we must also include $(Y_{2,2} + Y_{2,-2})$ type of deformation and the Hamiltonian H_i is generalised to the form

$$H_i = H_{oi} - kQ \cos\gamma (2Z_i^2 - X_i^2 - Y_i^2) - kQ \sin\gamma \sqrt{3} (X_i^2 - Y_i^2)$$

The auxiliary Hamiltonian is

$$H_i' = H_{oi} - \mu(2Z_i^2 - X_i^2 - Y_i^2) - \sigma\sqrt{3} (X_i^2 - Y_i^2) \qquad 4.26$$

The first objective is to change the numerical values of μ and σ in steps and calculate the energy values in terms of ω. Wavefunctions are also needed. A consistent description will be obtained by restricting μ to positive values only; $\sigma\sqrt{3}$ need not exceed the value 3μ.

2. Pairing model calculations are done using the above single-particle energies and wavefunctions. In particular, the ground state energy and wavefunction, E' and $|\Phi\rangle$ are determined.

3. The energy is given by

$$E = E' + \hat{\mu} Q \cos\gamma + \hat{\sigma} Q \sin\gamma + \frac{1}{2} kQ^2 \qquad 4.27$$

where

$$\hat{\mu} = \mu - kQ\cos\gamma \; ; \quad \hat{\sigma} = \sigma - kQ\sin\gamma \; .$$

We, however, need to know k in terms of ω which is done following the method of Mottelson (Mo 59).

4. We require that at equilibrium (for which $\mu = kQ\cos\gamma_{eq.}$ and $\sigma = kQ\sin\gamma_{eq.}$) the equipotential has the same shape as the density distribution. This gives

$$\frac{1 - \frac{4\mu}{m\omega^2}}{1 + \frac{2\mu}{m\omega^2} - \frac{2\sigma\sqrt{3}}{m\omega^2}} = \frac{\langle\Phi_o|\sum x_i^2|\Phi_o\rangle}{\langle\Phi_o|\sum z_i^2|\Phi_o\rangle}$$

$$\frac{1 - \frac{4\mu}{m\omega^2}}{1 + \frac{2\mu}{m\omega^2} + \frac{2\sigma\sqrt{3}}{m\omega^2}} = \frac{\langle\Phi_o|\sum y_i^2|\Phi_o\rangle}{\langle\Phi_o|\sum z_i^2|\Phi_o\rangle} \qquad 4.28$$

Of the various $|\Phi\rangle$'s that are supposed to have been calculated, we pick up the one that satisfies eqn. 4.28. We have then determined $kQ\cos\gamma_{eq.}$ and $kQ\sin\gamma_{eq.}$ in terms of ω (alternatively, equilibrium values of μ and σ in terms of ω).

5. The condition

$$\langle \Phi | \sum 2z_i^2 - x_i^2 - y_i^2 | \Phi \rangle = Q \cos \gamma$$

$$\langle \Phi | \sum \sqrt{3} (x_i^2 - y_i^2) | \Phi \rangle = Q \sin \gamma$$

now allows us to determine $Q \cos \gamma_{eq.}$ and $Q \sin \gamma_{eq.}$ in units of $\frac{\hbar}{m\omega}$. From step 4 now we will have the value of k in terms of ω.

Thus there is only one parameter to determine, namely ω. Alternatively, if we know k we can determine the value of ω. Bès and Szymanski suggest a value $k = 102\, A^{-5/3} \left(\frac{m\dot{\omega}_o}{\hbar}\right)^2$ MeV. No actual calculation with this value has been performed yet.

Note that for axial asymmetry, eqn. 4.28 must give $\sigma \neq 0$.

The self-consistent method is much harder than the volume conservation method because the energy is calculated somewhat indirectly. Specifically, eqn. 4.28 will be difficult to solve because of the very complicated nature of $|\Phi_o\rangle$, the ground state wavefunction. We expect, however, the volume conservation method to be a good description of the nucleus since it is found that the "breathing modes" of the nucleus occur at high energy. A self-consistent method which also conserves the volume has not been formulated; this will, to say the least, make calculations very complicated.

Table 4.1

Parameters B, Γ and ω (eqn. 4.4), taken from Newton's work (Ne 60); (β, γ) and $(-\beta, \frac{1}{3}\pi - \gamma)$ correspond to the same physical situation.

β		$\dot{\gamma} = 0$	$\gamma = \frac{1}{12}\pi$	$\gamma = \frac{1}{6}\pi$
0.3	B	0.1728776	0.1786267	0.1917971
	sin Γ	0.0	0.327238	0.579354
	cos Γ	1.0	0.944942	0.815076
	ω	1.008029	1.008372	1.009199
-0.3	B	-0.2090109	-0.2041195	-0.1917971
	sin Γ	0.0	0.197858	0.416200
	cos Γ	1.0	0.980231	0.909273
	ω	1.010372	1.010028	1.009199

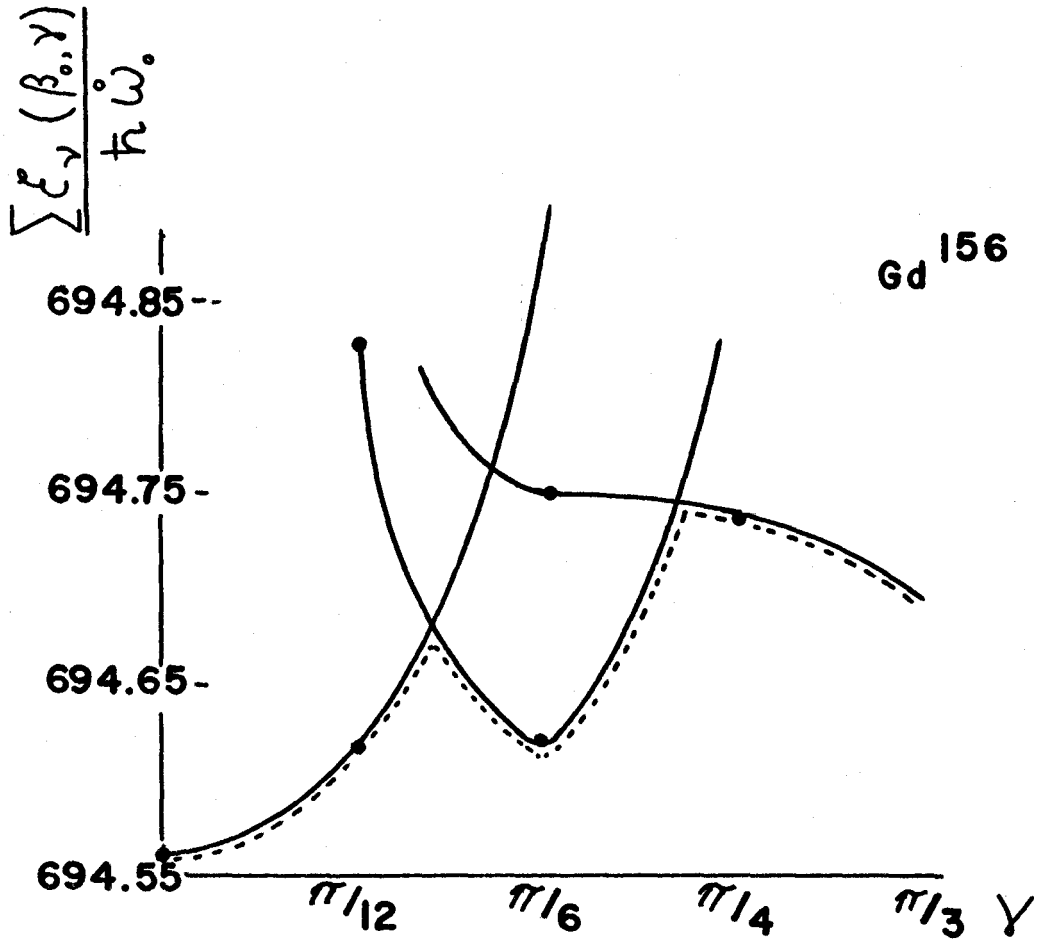
Table 4.2

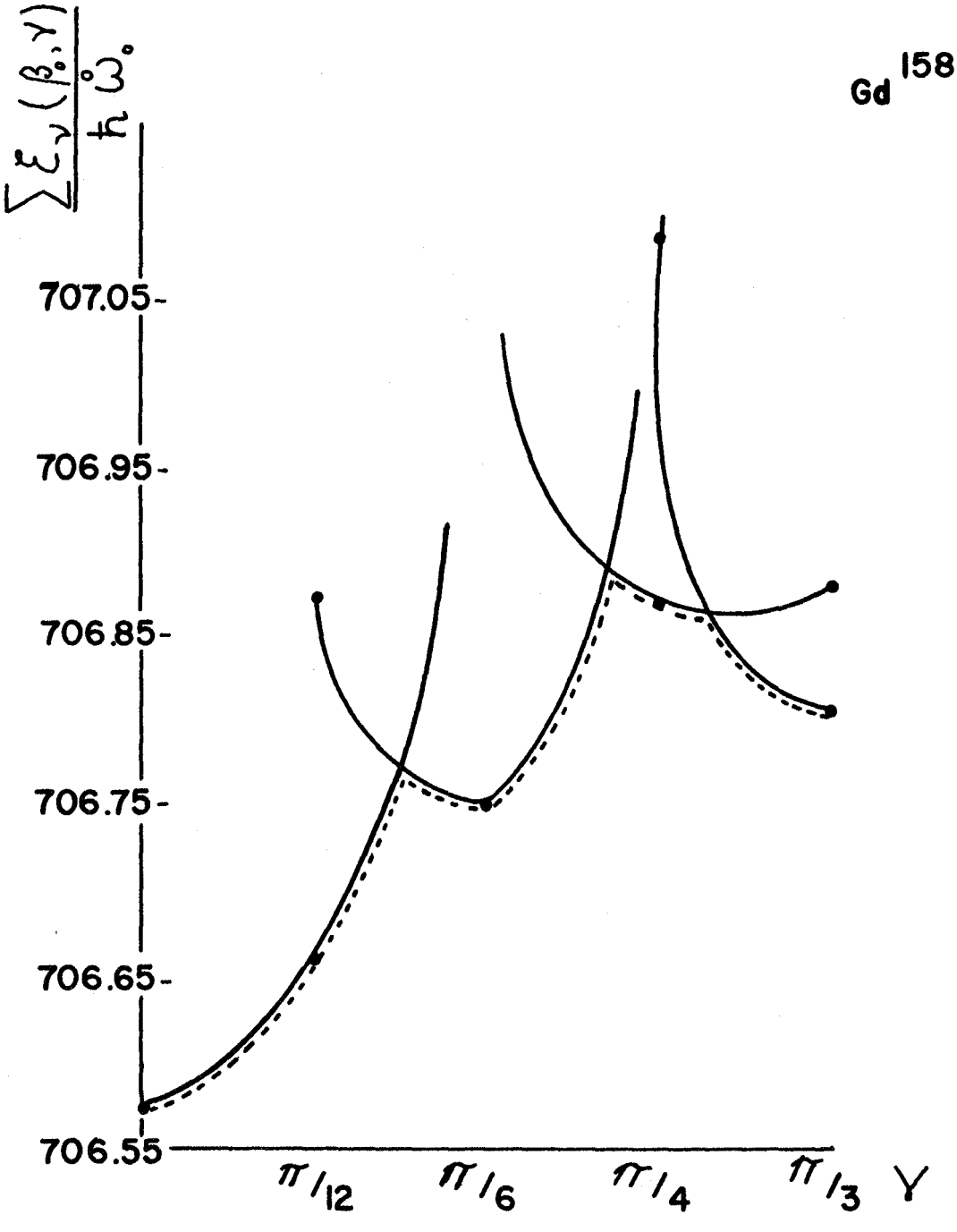
Parameters defining the single particle level spectrum employed in the calculations. μ was defined in chapter 2. The corresponding values of D are also given. The parameter C is fixed at $-0.1\ \hbar\omega_0$ which corresponds to $\chi = 0.05$.

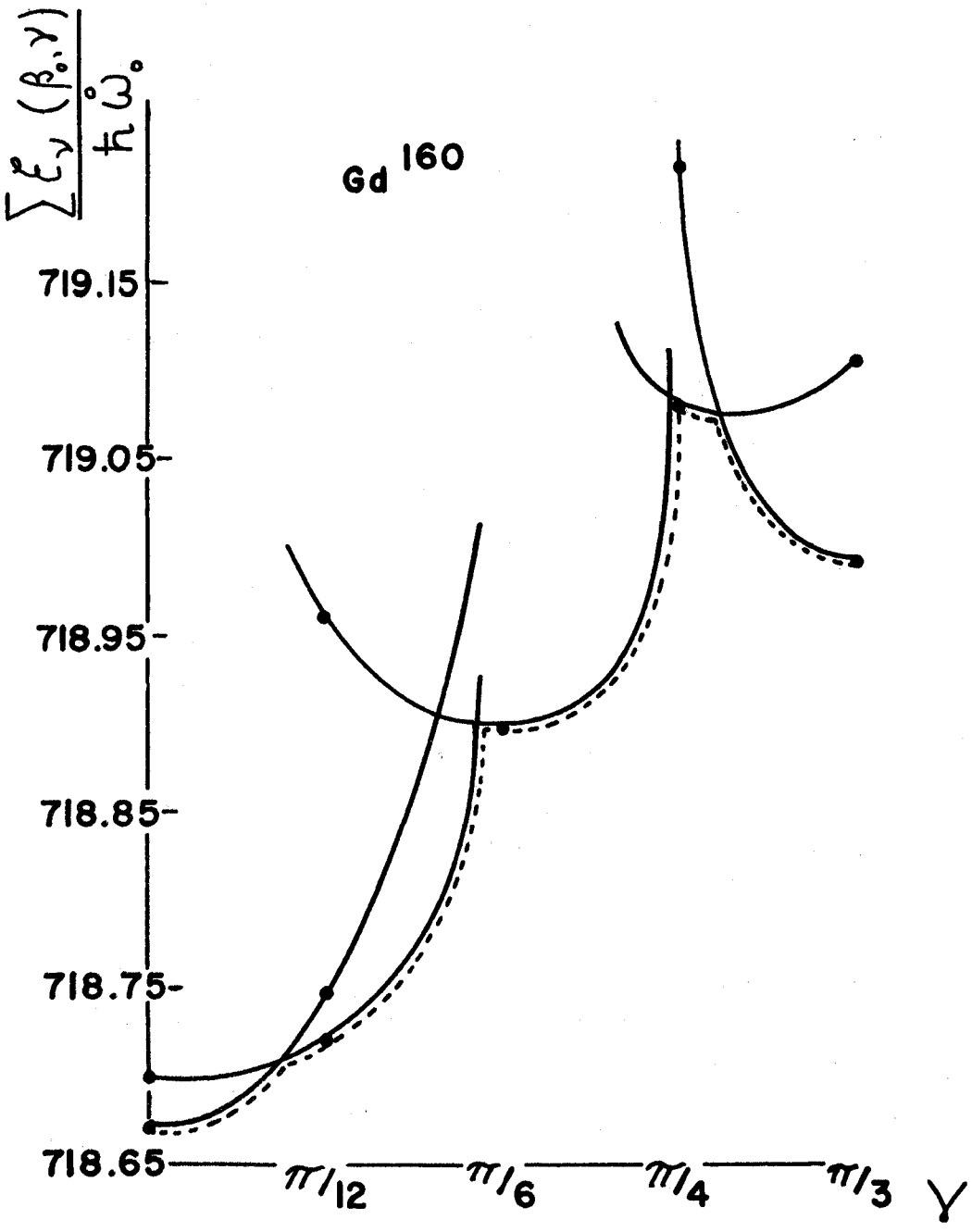
	Shells	μ	D	Additional shifts in units of h_0
Protons	N = 0,1,2	0	0	- -
	N = 3	.45	-.0225	- -
	N = 4	.55	-.0275	- -
	N = 5	.55	-.0275	$h_{11/2}$: -.075
				Others: +.1
Neutrons	N = 0,1,2	0	0	- -
	N = 3	.35	-.0175	- -
	N = 4	.45	-.0225	- -
	N = 5	.45	-.0225	- -
	N = 6	.45	-.0225	$i_{13/2}$ unchanged
				Others: +.15

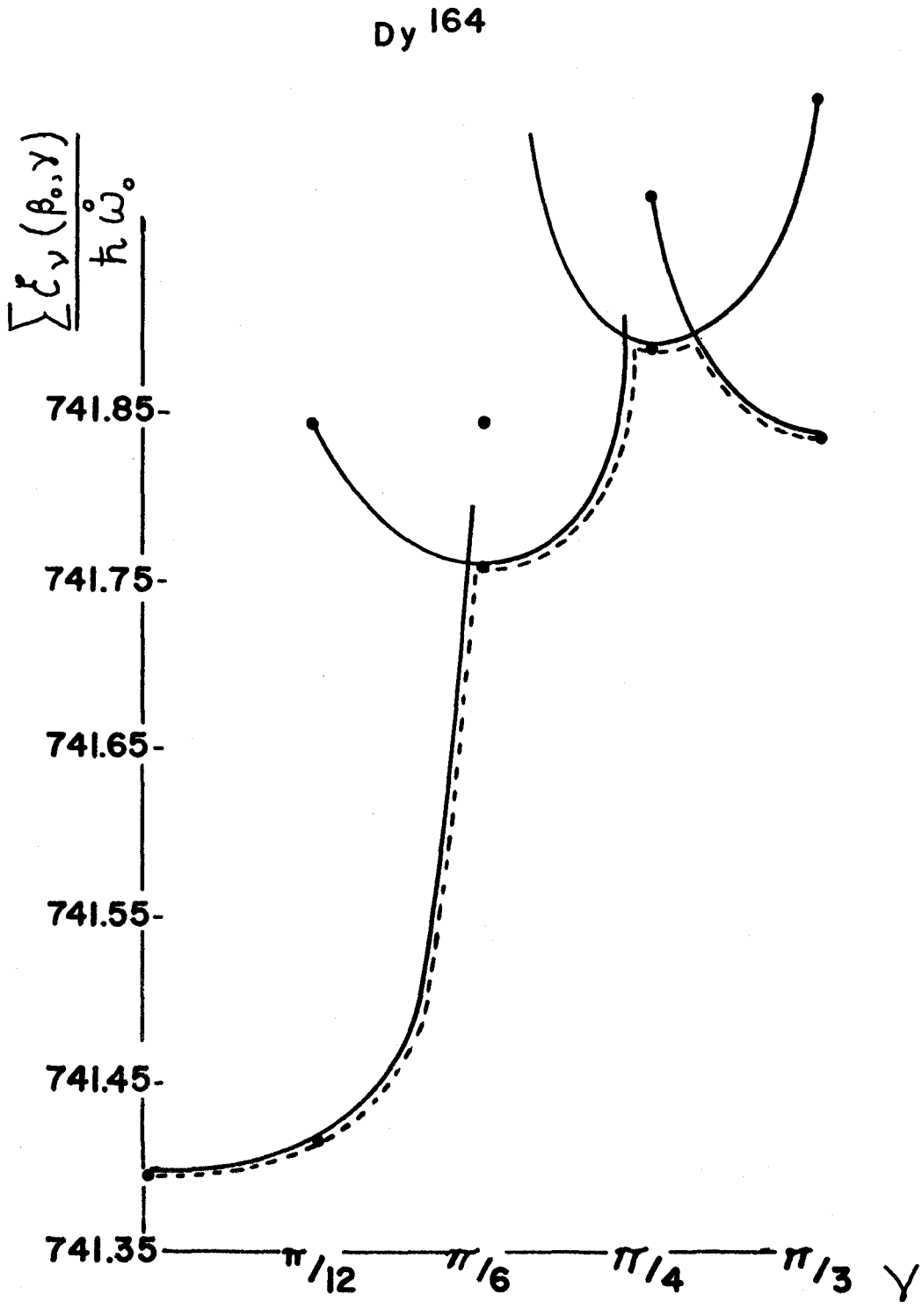
Captions for figs. 4.1 to 4.9 (p. 55 to 63)

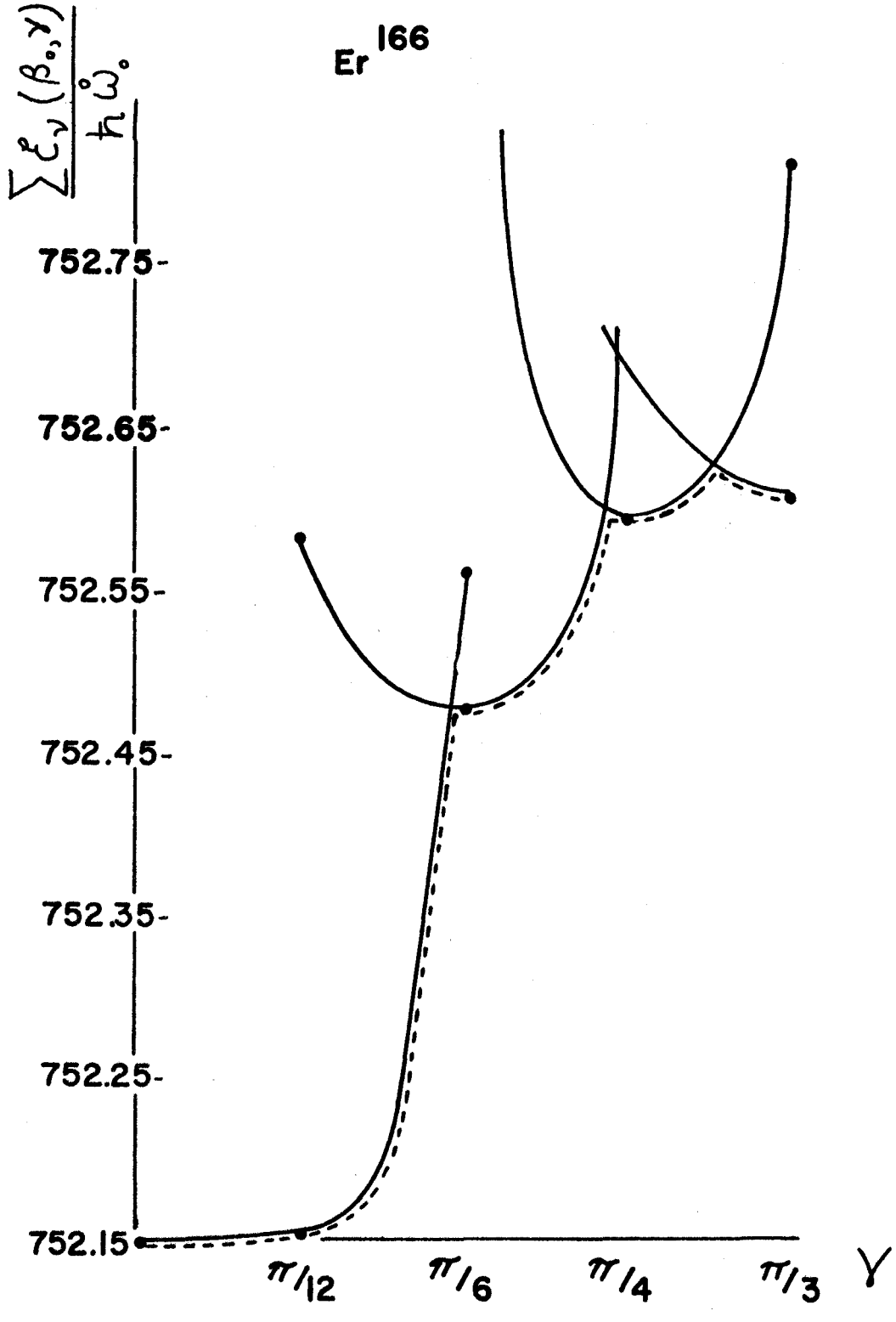
The first six figures show the plot of $\sum \mathcal{E}_\nu(\beta_0=.3,\gamma)$ against γ for various nuclei. The different curves within each diagram correspond to different intrinsic configurations . The lower diagrams in the last three figures correspond to the curves obtained after including the pairing correlation.

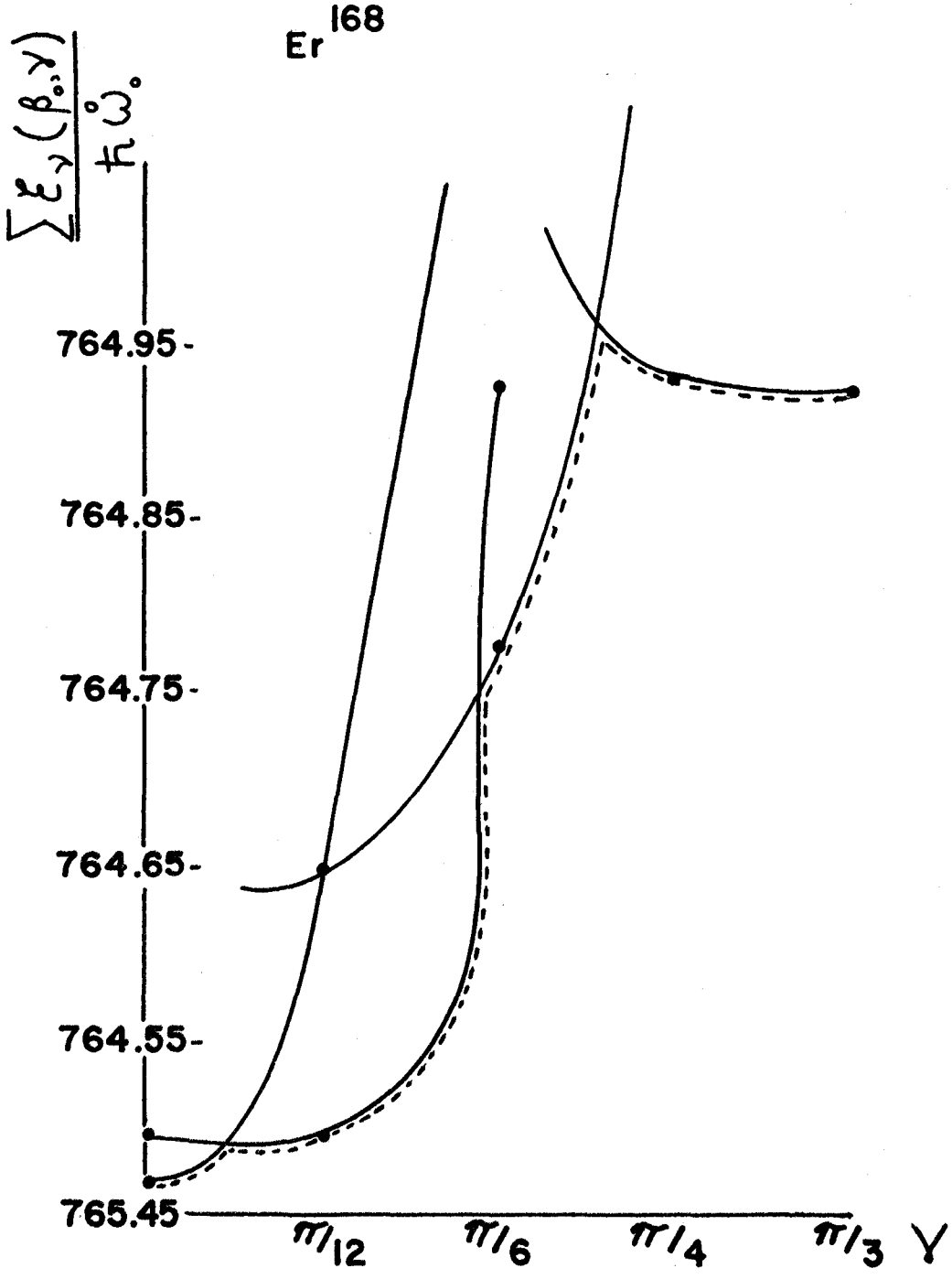

$\frac{\sum \varepsilon_\nu(\beta_0,\gamma)}{\hbar\omega_0}$
Gd 156
694.85
694.75
694.65
694.55
π/12
π/6
π/4
π/3
γ

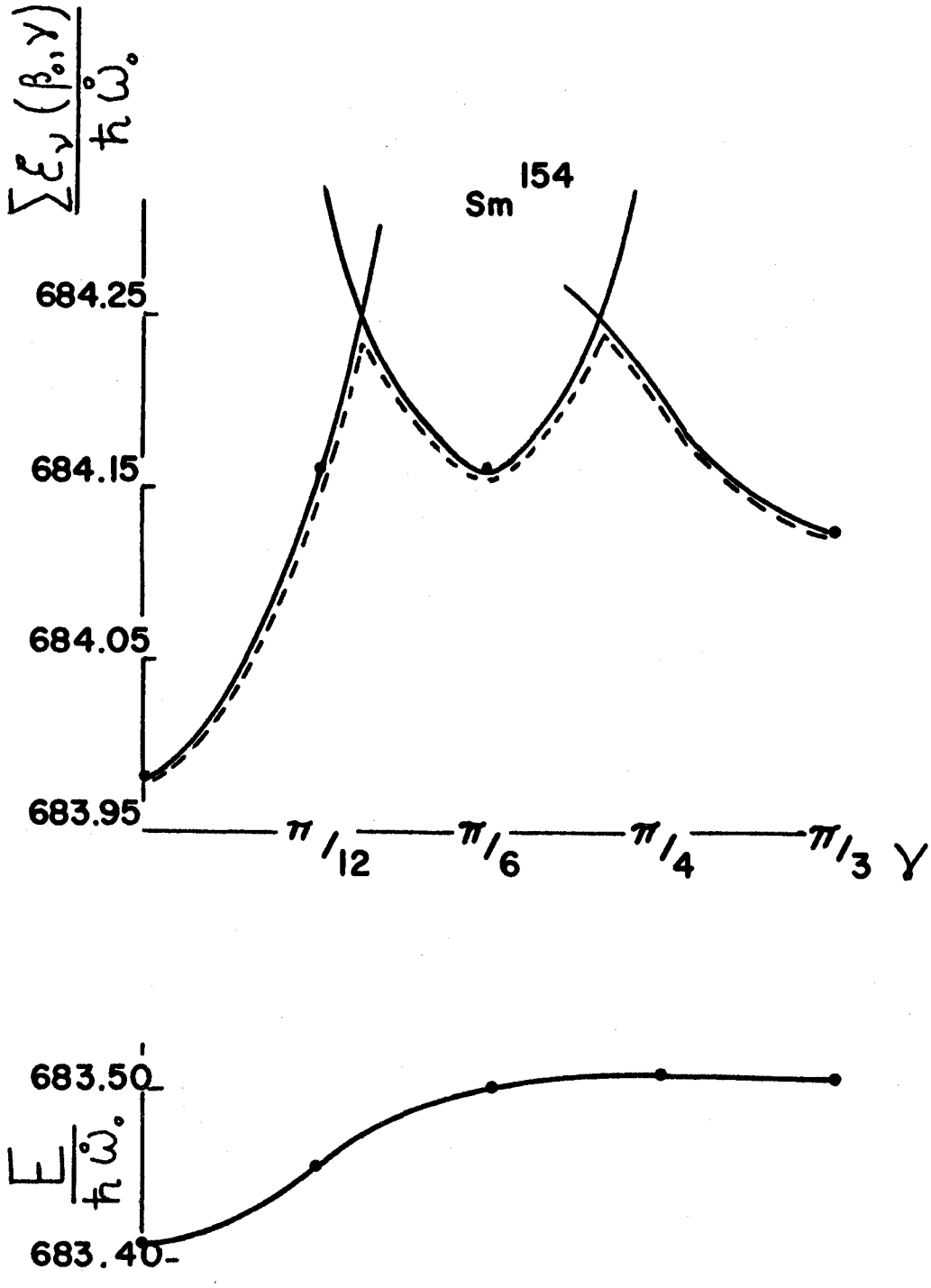

$\frac{\sum \mathcal{E}_\nu(\beta_0,\gamma)}{\hbar\omega_0}$
Gd 158
707.05
706.95
706.85
706.75
706.65
706.55
π/12
π/6
π/4
π/3
γ

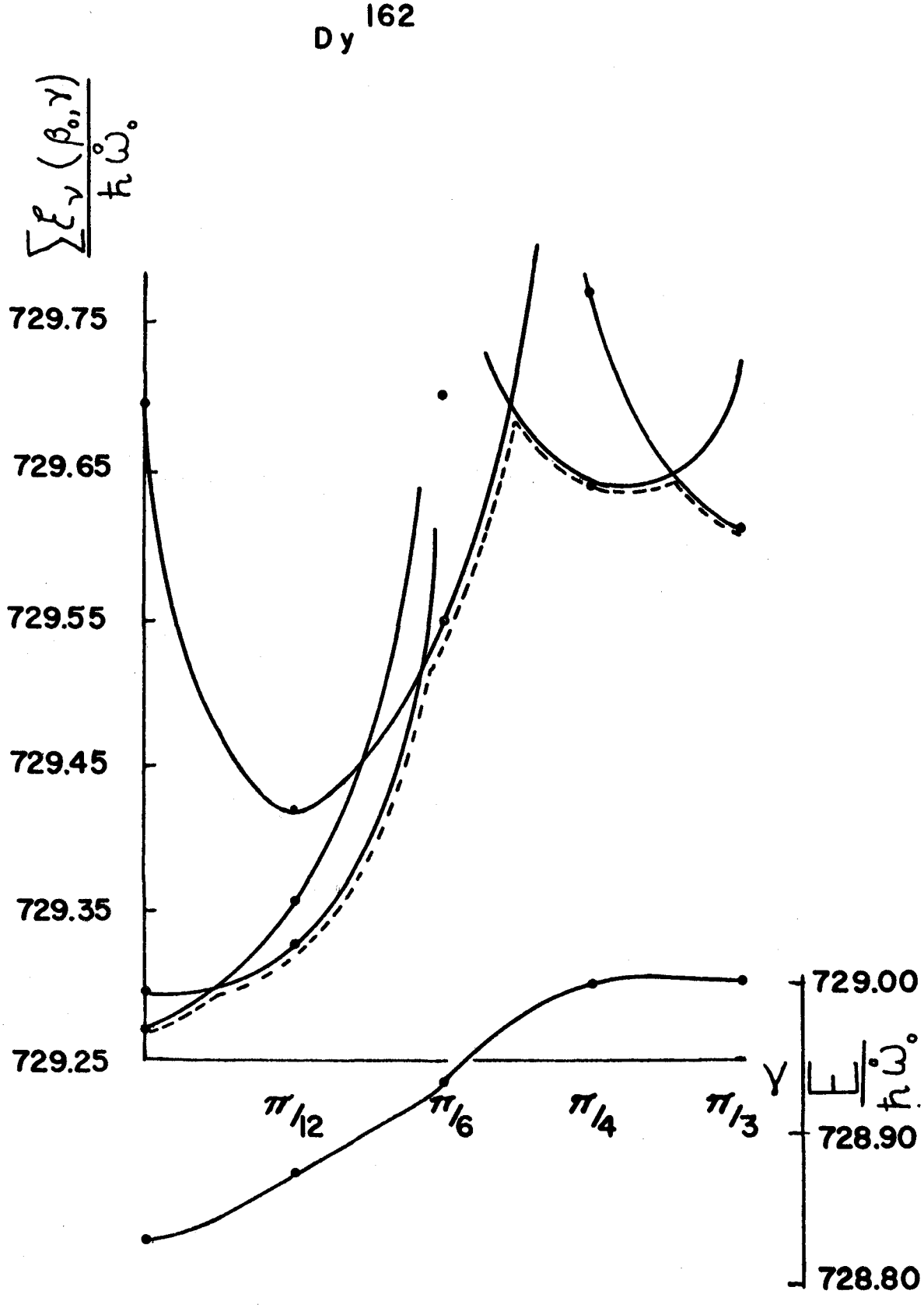

$\sum \mathcal{E}_\gamma (\beta_0, \gamma) \Big|_{\hbar\omega_0}$
Gd 160
719.15
719.05
718.95
718.85
718.75
718.65
π/12
π/6
π/4
π/3
γ

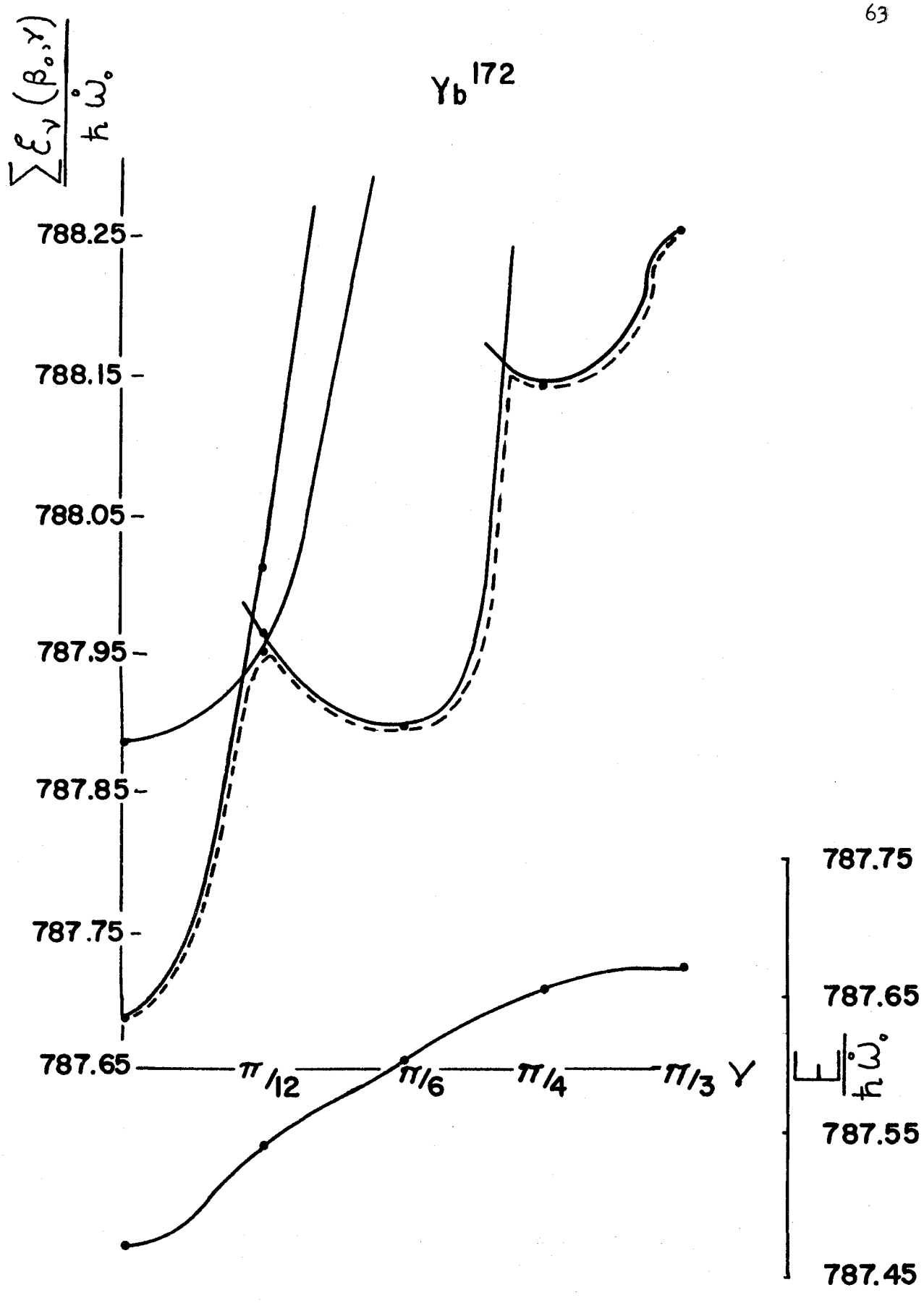

Dy 164
$\frac{\sum \varepsilon_\nu(\beta_0,\gamma)}{\hbar\omega_0}$
741.85
741.75
741.65
741.55
741.45
741.35
π/12
π/6
π/4
π/3
γ


$\frac{\sum \varepsilon_\nu(\beta_0,\gamma)}{\hbar\omega_0}$
Er^{166}
752.75
752.65
752.55
752.45
752.35
752.25
752.15
$\pi/12$
$\pi/6$
$\pi/4$
$\pi/3$
γ


Er 168
$\sum \varepsilon_\gamma (\beta_0, \gamma)$
$\hbar\omega^3$
764.95
764.85
764.75
764.65
764.55
765.45
π/12
π/6
π/4
π/3
γ


$\frac{\sum \varepsilon_\nu(\beta_0,\gamma)}{\hbar\omega_0}$
Sm 154
684.25
684.15
684.05
683.95
$\pi/12$
$\pi/6$
$\pi/4$
$\pi/3$
γ
$\frac{E}{\hbar\omega_0}$
683.50
683.40


Dy 162
$\frac{\sum \varepsilon_\nu(\beta_0,\gamma)}{\hbar\omega_0}$
729.75
729.65
729.55
729.45
729.35
729.25
$\pi/12$
$\pi/6$
$\pi/4$
$\pi/3$
γ
729.00
728.90
728.80


$\sum \varepsilon_\nu(\beta_0,\gamma)$ / $\hbar\omega_0$
Yb^{172}
788.25
788.15
788.05
787.95
787.85
787.75
787.65
$\pi/12$
$\pi/6$
$\pi/4$
$\pi/3$
γ
$I/\hbar\omega_0$
787.75
787.65
787.55
787.45

CHAPTER 5: A MODEL CALCULATION FOR γ-SOFT NUCLEI

The earlier calculations (in chapter 4) have indicated that from the theory we generally expect heavy nuclei to be axially symmetric. It is therefore of interest to look at a particular region of the nuclear periodic table where the asymmetric rotator theory of the nucleus has been extensively applied (Ke, Ma 59). The region of interest is $A \approx 188$. An attempt will be made to obtain the collective levels from the Bohr-Mottelson theory.

The collective Hamiltonian of an even-even system is (see chapter 1)

$$\left[-\frac{\hbar^2}{2B}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\,\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\,\frac{\partial}{\partial\gamma}\right) + \sum_{k=1}^{3}\frac{\hbar^2 Q_k^2}{8B\beta^2\sin^2(\gamma-\frac{2\pi}{3}k)} + V(\beta,\gamma)\right]\Psi = E\Psi \qquad 5.1$$

$V(\beta,\gamma)$ is the potential energy and is to be obtained from a microscopic theory. For example, we obtained $V(\beta_o,\gamma)$ for a few nuclei in the previous chapter. By repeating the calculation for all β values we could obtain $V(\beta,\gamma)$ numerically.

It would be nice to obtain an analytic expression for $V(\beta,\gamma)$; then we should have some means of solving equation 5.1. For a realistic one body nuclear potential this is impossible. The usual trick then is to expand $V(\beta,\gamma)$ around β_o, γ_o.

$$V(\beta,\gamma) = \frac{1}{2} D(\beta-\beta_o)^2 + \frac{1}{2} A\gamma^2 \qquad 5.2$$

Note that we are already assuming $\gamma_o = 0$.

Of course, potential energy curves are not simple parabolas; however, so long as $V(\beta,\gamma)$ is a steeply rising function around $(\beta_o,0)$ the approximation is good. Taking a time-dependent approach, the system is spending most of its time around $(\beta_o, 0)$ and third and higher order terms in the Taylor series expansion are relatively unimportant. A criterion of how steeply rising is the function $V(\beta, \gamma)$ is how high (in energy) the β-and γ-vibrational levels are. The γ-vibrational levels are, however, quite low for nuclei around A $\approx$ 188. It is thus obvious that higher order terms in γ have to be considered.

We now consider the following expansion. From quite general considerations, $V(\beta,\gamma)$ is a function of cos 3γ (see chapter 1). We, therefore, can write the following

$$V(\beta,\gamma) = \frac{1}{2} D(\beta-\beta_o)^2 + A\cos 3\gamma + C\cos^2 3\gamma + \ldots. \qquad 5.3$$

A, C, etc. will in general, be functions of β. Such an expansion may be helpful if it is found that odd powers of cos 3γ have similar effects. Similarly, for even powers of cos 3γ, one may gain an insight by merely considering $\cos^2 3\gamma$.

Fig. 5.1 shows the potential energy contours using eqn. 5.3. We have $\beta_o = .3$ $A = -\beta\hbar\omega$; $C = \ldots = 0$; $\omega^2 \equiv D/B$ where D is defined in eqn. 5.3 and B is the mass parameter in eqn. 5.1. With representative values of D and B we set $\hbar\omega = .76$ MeV.

It will be seen that a term proportional to cos 3γ reproduces the gross features of the potential energy contours as given in the literature

(see for example, Wi, Je 56). "A" is chosen to be negative so that the minimum occurs at $\gamma = 0$.

Method of calculation

From eqns. 5.1 and 5.3 we write now

$$H = T_{kin.} + \frac{1}{2} D(\beta - \beta_o)^2 + A \cos 3\gamma + C \cos^2 3\gamma + \ldots \qquad 5.4$$

We now diagonalise the above Hamiltonian in the representation whose basis functions are eigenfunctions of the following equation

$$\left[T_{kin.} + \frac{1}{2} D(\beta - \beta_o)^2 \right] \Psi = E\Psi \qquad 5.5$$

Eqn. 5.5 is the γ-unstable spectrum considered by Wilets and Jean (Wi, Je 56).

Setting $\Psi(\beta, \gamma, \theta_i) \equiv f(\beta)\, \phi(\gamma, \theta_i)$ we get

$$E\left[\beta^2 f(\beta)\right] = \left[\frac{\hbar^2}{2B} \left(-\frac{\partial^2}{\partial \beta^2} + \frac{(\lambda+1)(\lambda+2)}{2} \right) + \frac{1}{2} D(\beta - \beta_o)^2 \right] \left\{ \beta^2 f(\beta) \right\} \qquad 5.6$$

and

$$\left[-\frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} + \frac{1}{4} \sum_{k=1}^{3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)} \right] \phi(\gamma, \theta_i) = \lambda(\lambda+3)\, \phi(\gamma, \theta_i) \qquad 5.7$$

Here λ is an integer; it is degenerate with respect to the angular momentum for the first few states as follows:

$\lambda = 0$	$I = 0$
$\lambda = 1$	$I = 2$
$\lambda = 2$	$I = 2, 4$
$\lambda = 3$	$I = 0, 3, 4, 6$
$\lambda = 4$	$I = 2, 4, 5, 6 \ldots$
$\lambda = 5$	$I = 2, 4, 5, 6 \ldots$

The construction of the γ-dependent part of the wavefunction has been done by D. R. Bès (Bè 59). The wavefunctions can be written as $\phi_{\lambda IM}$ where M is the 3-axis projection of angular momentum I in the space-fixed system.

The β-part is constructed as follows.

Putting $\omega^2 = D/B$; $x = (B\omega/\hbar)^{1/2}$; $\epsilon = E/\hbar\omega$, we get

$$\epsilon F(x) = \frac{1}{2}\left[-\frac{\partial^2}{\partial x^2} + \frac{(\lambda+1)(\lambda+2)}{x^2} + (x - x_0)^2 \right] F(x) \qquad 5.8$$

An approximate solution of eqn. 5.8 is readily obtained by expanding the effective potential v(x) about the minimum x'

$$v(x) = \frac{1}{2}\left[\frac{(\lambda+1)(\lambda+2)}{x^2} + (x - x_0)^2 \right] \approx v(x') + \frac{\omega'^2}{2}(x - x')^2$$

$$\epsilon = (n_\beta + \tfrac{1}{2})\,\omega' + v(x') \qquad 5.9$$

The wavefunctions are

$$F(x) = H_{n_\beta}\left[(x - x')\,\omega'\right] \exp\left\{ -\tfrac{1}{2}(x - x')^2\,\omega' \right\}$$

$$f(\beta) = N \frac{F(x)}{x^2} = \frac{N\hbar^2}{B\omega\beta^2} H_{n_\beta}\left\{\omega' \left(\frac{B\omega}{\hbar}\right)^{1/2} (\beta-\beta')\right\} \exp\left\{-\frac{B\omega\omega'}{2\hbar}(\beta-\beta')^2\right\}$$

where N is a normalisation constant.

The volume element of β is $\beta^4 d\beta$; that of γ is $|\sin 3\gamma| d\gamma$.

Effects of a cos 3γ term

As a model calculation we set $B = \hbar^2$ (.00756 MeV.$^{-1}$), D = 76 MeV.; $\beta_0 = .3$ and $A = -2\hbar\omega\beta$. Cutting off all basis functions beyond $\lambda = 5$, $n_\beta = 0$ and $\lambda = 3$, $n_\beta = 1$ we get the spectrum shown in fig. 5.2. It was checked that for the small γ-dependent term we consider, the neglect of higher basis functions is justified. Also the spectrum is not sensitive to the power of β chosen in "A". It will be noticed that the small γ-dependent term already brings out the features of rotation-vibration spectra. By means of a small γ-dependent term we are able to increase the ratio of the energy of the first 4+ state to that of the first 2+ state from 2.35 to 2.7. In the limit of complete β and γ stability this ratio is 3.33 (usual rotational spectrum). Wilets and Jean have shown that for a completely β-stable but completely γ-unstable nucleus this ratio is 2.5. However, if all the γ-stability comes from a cos 3γ term, the second 2+ level must occur much higher than the first 4+ level. This is not always the case for nuclei in the transition region (La, Bo 61).

Effects of a cos^2 3γ term

The introduction of a cos^2 3γ term can help us out of this difficulty. The trends due to this term can be observed by using second order perturbation theory and neglecting the coupling between $n_\beta = 0$ and $n_\beta = 1$ bands (this

coupling is small in our case). The ratio of the energy of the first 4+ state to the first 2+ state still increases but the first 4+ state is raised above the second 2+ state. By assuming a form $A \cos 3\gamma + B \cos^2 3\gamma$ with $A < 0$ and $B > 0$, one can obtain equilibrium prolate symmetry as well as low γ-vibration. Further the $I = 3$ level which is too high under the influence of a $\cos 3\gamma$ term alone will move down also.

The influence of higher orders powers of $\cos 3\gamma$ is not expected to alter these qualitative features. This is because all odd powers of $\cos 3\gamma$ connect the same states as $\cos 3\gamma$ does ($\Delta\lambda$ = odd); all even powers of $\cos 3\gamma$ connect the same states as $\cos^2 3\gamma$ does ($\Delta\lambda$ = even). Thus the effects are expected to be similar.

The outcome of the above analysis seems to be the following; In order to get a high ratio (high compared to the value 2.5 which is obtained for a completely β-stable but completely γ-unstable spectrum) of the first 4+ to the first 2+ level as well as a low-lying second 2+ state we must invoke a $\cos^2 3\gamma$ term as well as a $\cos 3\gamma$ term. A highly idealised case would be to consider the well shown in fig. 5.4 which roughly corresponds to a superposition of $A \cos 3\gamma + C \cos^2 3\gamma$.

The ratio of the first 4+ to the first 2+ depends mainly upon the depth V_o whereas the occurrence of the second 2+ level will depend mainly on the slope of $V(\gamma)$ and "a". In the case shown in the figure the energy of the second 2+ state varies roughly as $1/a^2$ where "a" is the spread of the potential (Sc 49). For γ-stable nuclei like Dy the quantity $V_o \equiv V(\gamma = \frac{\pi}{3}) - V(\gamma = 0) \approx 2.5$ to 4 MeV.; the $-A \cos 3\gamma$ term that we considered gave $V_o = .46$ MeV. The spectra in the region A = 190 indicate that for these nuclei V_o is somewhere in between the value considered in the calculations and the typical value of γ-stable nuclei.

Discussion

It thus appears from the above semi-quantitative arguments that it will be possible to obtain the level spectrum of γ-soft nuclei (for example Os^{190} or Pt^{192}) from the Bohr-Mottelson theory. Closer scrutiny, however, reveals that a difficulty lies in explaining the rate of electromagnetic quadrupole transition from the second 2+ to the ground state. This transition is completely forbidden in the γ-unstable spectrum (Wi, Je 56). A $\cos 3\gamma$ term destroys this forbiddenness but a $\cos^2 3\gamma$ term does not violate this forbiddenness. The spectra of γ-stable nuclei (for example, Er^{166}) suggests that the $\cos 3\gamma$ term is the dominant term in the potential energy expansion for these nuclei. For Os^{190} or Pt^{192} a $\cos^2 3\gamma$ term must also be important; hence the above mentioned transition rate should be weak in Os^{190}. It is known, however, that this transition rate in Os^{190} is not any weaker than the rates in other γ-stable nuclei (Bè 61).

It has come to our notice that Tamura and Komai were able to fit the spectrum of Cd^{114} as well as transition rate by means of a $\cos 3\gamma$ term only (Ta, Ko 59).

The first step in understanding the spectra in the transition region must proceed by calculating the shape of $V(\beta, \gamma)$. This can be done by the method given in chapter 4. Having obtained $V(\beta, \gamma)$ one can then proceed to break it up into the form $A \cos 3\gamma + C \cos^2 3\gamma$ by fitting the coefficients A, C etc. An alternative method would be to exploit single particle levels at $\gamma = 0$ and use a perturbation theory. This method is given in appendix 2.

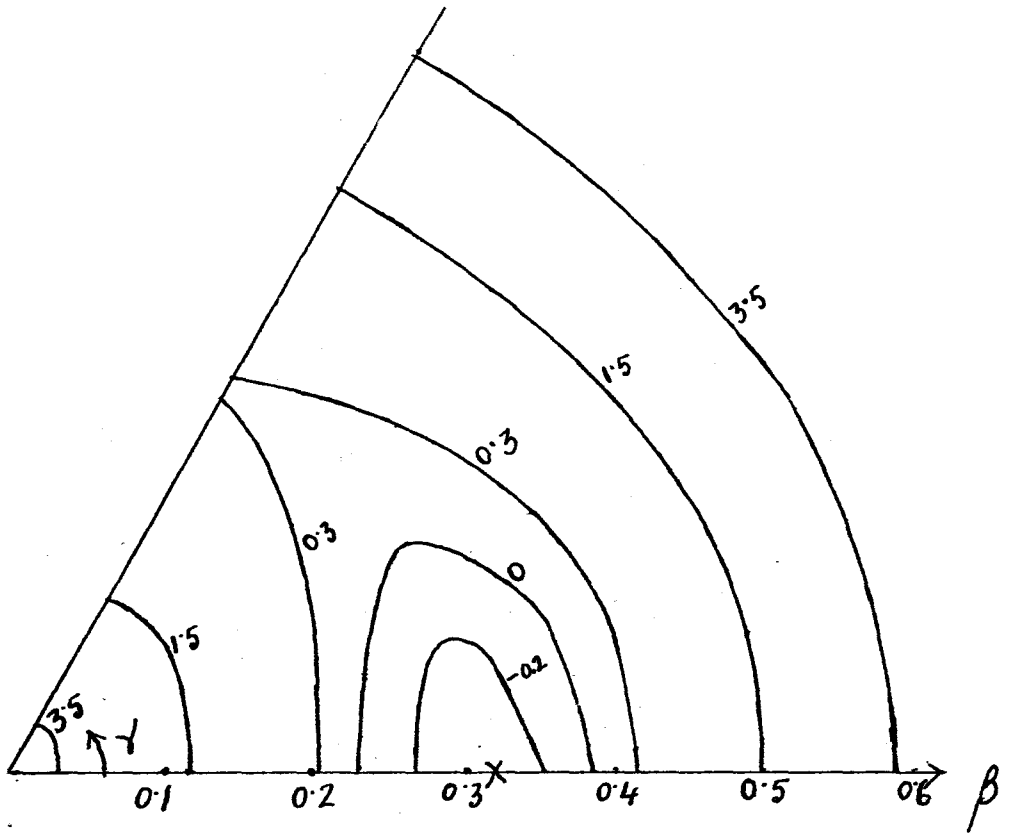


Fig. 5.1. Potential energy contours (eqn. 5.3) in units of $\hbar\omega$ (=.76 MeV.). We have $\beta_0 = .3$; $A = -\beta\hbar\omega$; $C = \ldots = 0$. The minimum has value $-.31\ \hbar\omega$ and occurs at the point marked by the cross.

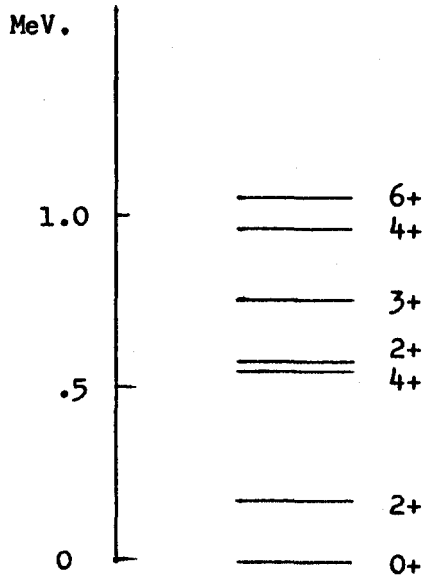


Fig. 5.2. Spectrum of Osmium 190.

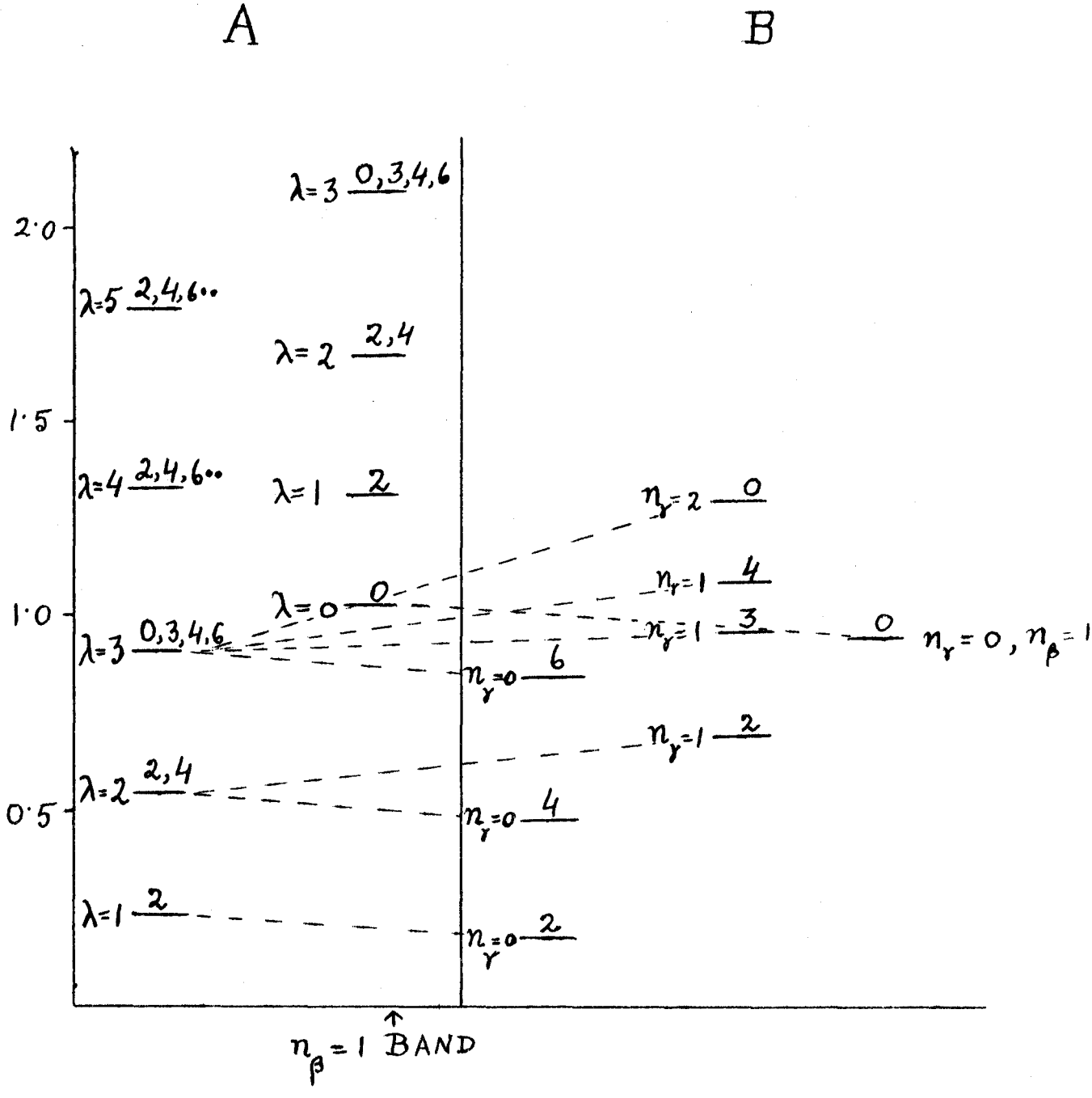


Fig. 5.3. Part A is the γ-unstable spectrum.

Part B is obtained by including $-2\hbar\omega \cos 3\gamma$ in the potential energy.

Energies plotted in units of $\hbar\omega$ = .76 MeV.

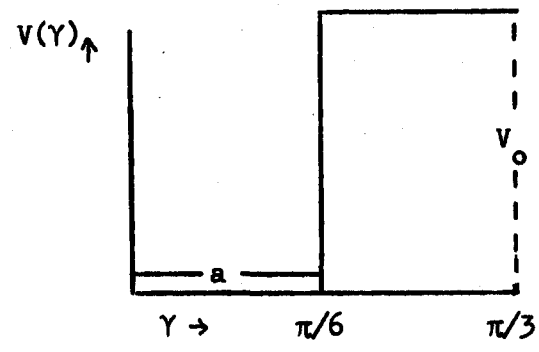


Fig. 5.4. A highly idealised example to consider is the potential well shown above. This, of course, only roughly corresponds to a form A cos 3 γ + C $\cos^2$ 3 γ. However, this examplifies why a harmonic approximation will be bad for obtaining the second 2+ state in γ-soft nuclei. See also the text, chapter 5, page 69.

CONCLUSIONS

Using the method of volume conservation, the only method that has been used in any calculation of nuclear single-particle levels so far, we have tried to examine theoretically if one expects axial asymmetry in nuclei or not. The conclusion is that, at least for heavy nuclei, prolate axial symmetry is likely to be favoured. The result is traced to three causes: a) presence of an $\underset{\sim}{\ell}^2$ force; b) the cumulative behaviour of the shells which are filled completely; c) the coulomb interaction energy which favours the prolate shape. The last effect is not so important if the pairing correlation is not included. After the effects of the pairing correlation are taken into account, the energy curves flatten out considerably and the coulomb energy becomes important. In the opinion of the author, there is no guarantee that the same conclusions would be reached if one tried to calculate the equilibrium deformation by the self-consistent method. The results that we obtain are, however, good to the extent that volume conservation holds good.

Assuming prolate axial symmetry, a model calculation has then been performed to understand the nuclear spectra in the transition region from the Bohr-Mottelson theory. No data fitting has been attempted; rather, the aim has been to understand the trends. The conclusion is that it will be possible to obtain the positions of energy levels although a simultaneous agreement with electromagnetic transition rates may pose a problem.

Appendix 1

By moving across a row we can follow a single-partidle energy level as a function of γ . The columns refer, from left to right, to

$$\gamma = 0,\ \frac{1}{12}\pi,\ \frac{1}{6}\pi,\ \frac{1}{4}\pi \text{ and } \frac{1}{3}\pi.$$

N = 5; μ = .45; neutron levels				
-0.48934	-0.52530	-0.62269	-0.75265	-0.85641
-0.95272	-0.96643	-0.99358	-0.89010	-0.95310
-0.24069	-0.28093	-0.32115	-0.34006	-0.31665
-0.00379	-0.05628	-0.21628	-0.08530	-0.04036
-1.28629	-1.26323	-1.42338	-1.44499	-1.45295
-0.73797	-0.67064	-0.87606	-0.85155	-0.84799
-0.54977	-0.59211	-0.57698	-0.50057	-0.47643
-0.04023	0.10139	0.25310	0.36922	0.42459
0.29360	0.34580	0.74876	0.87069	0.91360
-1.37809	-1.39506	-1.22227	-1.19766	-1.19114
-0.93999	-0.92058	-0.65055	-0.55031	-0.57650
-0.77092	-0.81002	-0.82605	-0.80241	-0.78811
-0.39184	-0.30963	-0.18944	-0.64779	-0.59917
-0.05724	-0.09942	-0.06196	-0.02288	0.00987
0.40109	0.56839	0.12750	0.08857	0.05103
-1.13945	-1.14038	-1.13057	-1.10054	-1.08900
-0.50267	-0.51122	-0.48793	-0.44267	-0.40855
-0.29128	-0.33969	-0.45392	-0.39458	-0.51058
0.31766	0.18584	0.41146	0.45084	0.45117
-0.73429	-0.75918	-0.82194	-0.98986	-0.89835
0.04422	-0.01129	-0.11605	-0.21538	-0.29497

N = 6; μ = .45; neutron levels The first 7 levels from the top are the $i_{13/2}$ levels				
-0.72220	-0.76021	-0.86391	-1.25821	-1.22514
-1.19218	-1.20768	-1.24273	-1.34047	-1.31186
-1.55176	-1.54043	-1.50140	-1.47042	-1.46376
-1.74764	-1.78291	-1.83140	-1.86605	-1.87854
-1.67568	-1.64394	-1.61499	-1.60579	-1.60396
-1.38809	-1.39188	-1.38783	-1.00534	-1.12704
-0.96903	-0.99580	-1.06525	-1.14450	-1.15602
-0.08678	-0.14005	-0.27648	-0.41982	-0.70499
-0.62843	-0.65991	-0.68494	-0.66100	-0.57674
-0.43074	-0.45863	-0.58516	-0.70708	-0.70835
0.28006	0.23604	0.90085	0.20403	0.17201
-1.06689	-1.14226	-1.18402	-1.18324	-1.18635
-0.91657	-1.21231	-1.16604	-1.15714	-1.15141
-0.39690	-0.25957	0.53845	0.51650	0.58256
-0.09649	-0.04161	0.37536	0.59913	0.60519
0.45834	0.68331	0.22284	-0.26059	-0.21249
-1.22671	-0.99684	-0.96067	-0.41783	-0.38249
-1.10029	-0.94030	-0.90236	-0.84076	-0.82514
-0.72096	-0.64703	-0.52388	-0.91021	-0.86676
-0.41774	-0.47461	-0.42598	-0.36798	-0.32625
0.02693	0.44349	-0.24886	-0.13095	-0.23040
0.44663	0.12904	-0.10108	0.02976	0.08699
-0.86307	-0.87023	-0.84244	-0.78841	-0.76243
-0.68996	-0.72119	-0.80130	-0.85662	-0.85443
-0.06523	-0.18992	0.03079	1.04753	1.09911
0.24503	0.18755	-0.02697	0.09081	0.12926
-0.36855	-0.41201	-0.49904	-0.56359	-0.60617
-0.14508	-0.20006	-0.34154	-0.54175	-0.52439

N = 3; μ = .45; proton levels				
-0.75187	-0.74858	-0.74202	-0.73785	-0.73677
-0.60085	-0.60138	-0.59198	-0.55852	-0.53033
-0.40462	-0.38034	-0.34033	-0.29512	-0.27644
-0.39782	-0.42262	-0.46232	-0.49863	-0.50172
-0.15860	-0.18284	-0.27510	-0.37156	-0.43522
-0.14896	-0.20212	-0.21702	-0.22137	-0.20022
-0.12700	-0.06003	0.01937	0.08398	0.11694
0.14285	0.34240	0.44835	0.52070	0.54623
0.16922	0.06844	0.16886	0.16626	0.15597
0.25266	0.16207	-0.03281	-0.11288	-0.16344

N = 4; μ = .55; proton levels				
-0.64399	-0.66283	-0.70855	-0.74762	-0.73221
0.02966	-0.02733	-0.17028	-0.30733	-0.39525
-1.02022	-1.00614	-1.15297	-1.16701	-1.17236
-0.51289	-0.56690	-0.58534	-0.56318	-0.54322
-0.24509	-0.24304	-0.40270	-0.44580	-0.42683
0.27393	0.42710	0.56849	0.25484	0.25072
-1.12820	-1.13639	-0.97185	-0.94050	-0.93137
-0.73356	-0.71721	-0.68019	-0.65347	-0.64658
-0.51252	-0.43515	-0.34845	-0.27780	-0.25036
-0.10952	-0.03346	0.07894	0.16796	0.21004
0.28246	0.21915	0.23989	-0.08746	-0.12857
-0.85318	-0.86016	-0.86933	-0.85532	-0.83820
-0.25636	-0.31816	-0.22605	-0.19985	-0.16981
0.05594	0.01681	-0.03493	0.66475	0.69870
-0.40147	-0.43127	-0.51168	-0.61722	-0.69971

N = 5; μ = 0.55; proton levels The first 6 levels from the top are the $h_{11/2}$ levels				
-0.63934	-0.67092	-0.75698	-0.87355	-0.96795
-1.10036	-1.10931	-1.12479	-1.01035	-1.00852
-1.42128	-1.39535	-1.56547	-1.59296	-1.60295
-1.50376	-1.52815	-1.36059	-1.34487	-1.34114
-1.28213	-1.27774	-1.25460	-1.21898	-1.20768
-0.88429	-0.90465	-0.95616	-1.10984	-1.07641
-0.38747	-0.42925	-0.53713	-0.72988	-0.64286
-0.06937	-0.11486	-0.21364	-0.44092	-0.38503
-0.85980	-0.78222	-1.00031	-0.99712	-0.99799
-0.63305	-0.68556	-0.74153	-0.54322	-0.51045
-0.10820	0.04002	0.18832	0.30233	0.35478
0.27167	0.53759	0.71500	0.83563	0.87814
-1.03051	-1.01863	-0.92152	-0.90050	-0.89153
-0.86002	-0.91211	-0.69316	-0.63589	-0.61432
-0.46092	-0.38584	-0.27835	-0.62678	-0.69164
-0.10197	-0.16934	-0.14501	-0.10700	-0.07105
0.38018	0.29597	0.35405	0.38991	0.39148
-0.64096	-0.62657	-0.59774	-0.18130	-0.14269
-0.36651	-0.40165	-0.42826	-0.31088	-0.41941
0.25387	0.13971	0.05895	0.01103	-0.02780
-0.10578	-0.15112	-0.29106	-0.46482	-0.57499

Appendix 2.

The coefficients of the $\cos 3\gamma$ and $\cos^2 3\gamma$ terms in the expansion of equation 5.3 can be obtained by exploiting single particle solutions at $\gamma=0$ in the following way.

The single particle Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\nabla'^2 + \frac{1}{2}m\left(\omega_x^2 x'^2 + \omega_y^2 y'^2 + \omega_z^2 z'^2\right) + C\,\underset{\sim}{l}\cdot\underset{\sim}{s} + D\,\underset{\sim}{l}^2 \qquad \text{A.2.1}$$

Since the objective is merely to demonstrate how the expansion terms come about, the following calculations will be done to first order in β only. Using the relation

$$\omega_x = \omega_0\left[1-\sqrt{\frac{5}{4\pi}}\,\beta\cos\left(\gamma+\pi/3\right)\right]^{-1} \qquad \text{etc. and introducing}$$

a coordinate transformation $X \equiv \sqrt{\frac{m\omega_0}{\hbar}}\,x'$

$$H = -\frac{1}{2}\hbar\omega_0\left(\nabla^2 - r^2\right) + C\,\underset{\sim}{l}\cdot\underset{\sim}{s} + D\,\underset{\sim}{l}^2$$
$$+ \frac{1}{2}\hbar\omega_0\left[-\sqrt{\frac{5}{4\pi}}\,\beta\cos\gamma\left(2z^2-x^2-y^2\right) - \sqrt{\frac{5}{4\pi}}\,\beta\sin\gamma\sqrt{3}\left(x^2-y^2\right)\right]$$

Suppose now that we know the solution at β_0, $\gamma = 0$ and want to generate solutions at β_0, $\gamma \neq 0$. Then

$$\Delta E = \frac{1}{2}\hbar\omega_0\left[-\sqrt{\frac{5}{4\pi}}\,\beta_0\left(\cos\gamma-1\right)\langle 2z^2-x^2-y^2\rangle - \sqrt{\frac{5}{4\pi}}\,\beta_0\sin\gamma\,\langle\sqrt{3}\left(x^2-y^2\right)\rangle\right] \qquad \text{A.2.2}$$

We use eqn. A.2.2. for the range $\gamma = 0$ to $\pi/3$ and for the range $\gamma = \pi/3$ to $2\pi/3$, the perturbative increment can be regarded as

$$\Delta E = \frac{1}{2}\hbar\omega_0\left[-\sqrt{\frac{5}{4\pi}}\,\beta_0\left\{\cos\left(\frac{2\pi}{3}-\gamma\right)-1\right\}\langle 2z^2-x^2-y^2\rangle\right.$$
$$\left. - \sqrt{\frac{5}{4\pi}}\,\beta_0\left\{\sin\left(\frac{2\pi}{3}-\gamma\right)\right\}\langle\sqrt{3}\left(x^2-y^2\right)\rangle\right] \qquad \text{A.2.3}$$

This means that in the range $\gamma = \pi/3$ to $2\pi/3$ We look at the level that is obtained by an interchange of the X and Z axes.

The function which is given by eqn. A.2.2 in the region $\gamma = 0$ to $\pi/3$ and by eqn. A.2.3 in the region $\gamma = \pi/3$ to $2\pi/3$ can now be Fourier expanded in the form $f(\beta_0) + \sum k_n(\beta_0) \cos n3\gamma$ where the boundary condition implies that $f(\beta_0) + \sum k_n(\beta_0) = 0$

The physics of the problem is such that We prefer to have the expansion in powers of cos 3γ. This means we write the expansion in the form

$$f(\beta_0) + \sum_{n=0}^{\infty} K_{2n+1} \cos^{2n+1} 3\gamma + \sum_{n=1}^{\infty} K_{2n} \cos^{2n} 3\gamma$$

which for our problem can be approximated as

$$f(\beta_0) + u(\beta_0) \cos 3\gamma + W(\beta_0) \cos^2 3\gamma$$

with

$$u(\beta_0) = \sum K_{2n+1}$$

and

$$W(\beta_0) = \sum K_{2n}$$

Thus we write the energy as

$$\mathcal{E}(\beta_0,\gamma) = \mathcal{E}_0(\beta_0, \gamma=0) + f(\beta_0) + u(\beta_0) \cos 3\gamma + W(\beta_0) \cos^2 3\gamma$$

where

$$f(\beta_0) + u(\beta_0) + W(\beta_0) = 0$$

Generalising now

$$\mathcal{E}(\beta,\gamma) = \mathcal{E}(\beta_0, \gamma=0) + f(\beta_0)$$
$$+ \frac{\partial}{\partial\beta}(\mathcal{E}+f)\Big|_{\beta_0,\gamma=0} (\beta-\beta_0) + \frac{1}{2}\frac{\partial^2}{\partial\beta^2}(\mathcal{E}+f)\Big|_{\beta_0,\gamma=0} (\beta-\beta_0)^2$$
$$+ \left[u(\beta_0) + \frac{\partial u}{\partial\beta}\Big|_{\beta_0,\gamma=0} (\beta-\beta_0) + \ldots\right] \cos 3\gamma$$
$$+ \left[W(\beta_0) + \frac{\partial W}{\partial\beta}\Big|_{\beta_0,\gamma=0} (\beta-\beta_0) + \ldots\right] \cos^2 3\gamma$$

By suitable choice of β_0 it will be possible to have $\sum \frac{\partial}{\partial \beta}(\varepsilon+f)\Big|_{\beta_0,\gamma=0} = 0$

To include now the pairing force we have to consider

$$E = \sum 2\varepsilon_\nu V_\nu^2 - \Delta^2/G$$

where

$$\sum 2V_\nu^2 = \sum \left(1 - \frac{\varepsilon_\nu - \lambda}{\sqrt{(\varepsilon_\nu-\lambda)^2+\Delta^2}}\right) = N \qquad \text{A.2.4}$$

and

$$\frac{2}{G} = \sum \frac{1}{\sqrt{(\varepsilon_\nu-\lambda)^2+\Delta^2}} \qquad \text{A.2.5}$$

Now

$$E(\beta_0,\gamma) = \sum 2\varepsilon_\nu(\beta_0,\gamma) V_\nu^2(\beta_0,\gamma) - \Delta'^2/G$$

where $\varepsilon_\nu(\beta_0,\gamma)$ is known from the previous expansion.

By a similar token we write

$$\lambda'(\beta_0,\gamma) = \lambda(\beta_0,\gamma=0) + g(\beta_0) + a(\beta_0)\cos 3\gamma + b(\beta_0)\cos^2 3\gamma$$

$$\Delta'^2(\beta_0,\gamma) = \Delta^2(\beta_0,\gamma=0) + h(\beta_0) + c(\beta_0)\cos 3\gamma + d(\beta_0)\cos^2 3\gamma$$

where g and h are not additional unknowns but satisfy the condition

$$g(\beta_0) + a(\beta_0) + b(\beta_0) = 0$$

$$h(\beta_0) + c(\beta_0) + d(\beta_0) = 0$$

Since N and G are constants independent of deformation , eqns. A.2.4 and A.2.5 now determine a, b, c and d. In the eqns.A.2.4 and A.2.5 we put equal to zero the coefficients of cos 3γ and $\cos^2 3\gamma$. One then has to solve two simultaneous equations.

Extension to the case where both β and γ vary is straightforward.

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