DIAGONAL REPRESENTATION OF THE DOUBLY STOCHASTIC LIMIT

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By

BISWANATH DATTA , M.Sc.(Cal.)

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AUTHOR: Biswanath Datta

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SCOPE AND CONTENTS:

The main result of this thesis is the following theorem. If A is a non-negative symmetric matrix, then there exists a diagonal matrix D such that D A D is doubly stochastic, if and only if A has total support. The relevant theory is discussed and some other results of similar nature are also obtained, including a sufficient and necessary condition for the uniqueness of D above.

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CHAPTER I

INTRODUCTION

After estimating the entries in a strictly positive doubly stochastic matrix, Sinkhorn observed that the row and column sums of the matrix A of the observed values are not equal to 1 due to the inherent errors of the experiment.

To remedy this situation he scaled the rows of A to obtain a row stochastic matrix A_1 and then scaled the columns of A_1 to obtain a column stochastic matrix A_2. Continuing this procedure, he generated a sequence A_1, A_2, A_3... by alternately scaling the rows and colums of matrices. He observed, and eventually proved [5] that $S = \lim_{n \to \infty} A_n$ exists and is doubly stochastic. The doubly stochastic matrix S can be directly obtained from A by a single scaling of each row and column of A. This is equivalent to pre and post multiplying A with properly chosen diagonal matrices D_1 and D_2.

Later, Sinkhorn and Knopp [7] proved that if A is a non-negative square matrix and the sequence A_1, A_2, A_3... is generated as above, then $S = \lim_{n \to \infty} A_n$ exists if and only if A has at least one positive diagonal. Furthermore, S can be represented as $S = D_1 A D_2$, where D_1 and D_2 are diagonal matrices with positive main diagonals, if and only if A has total support. They also proved that D_1 and D_2 are unique up to a scalar multiple if and only if A is fully indecomposable.

Some of the results of Sinkhorn and Knopp quoted above were also obtained by Brualdi, Parter and Schneider [1] who used different techniques in their proof.

A similar problem was considered earlier by Marcus and Newman [3]. Given a non-negative symmetric matrix A, under what condition does there exist a diagonal matrix D such that D A D is doubly stochastic?

Marcus and Newman [3] gave the following sufficient conditions: A is strictly positive or A is positive semidefinite without a zero row.

Brualdi, Parter and Schneider [1] gave the weaker sufficient condition: A has a strictly positive main diagonal.

None of the above conditions is nesessary, as illustrated by $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In this thesis we prove that the condition: "A has total support" is sufficient and necessary.[1)]

In Chapter II we review the relevant theory of non-negative matrices and present a theorem which serves as a tool in the proof of our main result in Chapter IV. Chapter III deals with stochastic matrices and also contains some other results which may be of

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1) After this thesis was written and the main result published in [9], it was brought to our attention that Marshall and Olkin [10] obtained yet another sufficient condition, namely that the matrix is copositive.

independent interest. Here we also explain how to construct all pairs (D_1, D_2) of diagonal matrices such that D_1 A D_2 is doubly stochastic, from a single given pair with this property, and establish the relationship between D_1 and D_2 when A is symmetric.

In Chapter IV, we prove our main result, the DAD theorem for symmetric matrices and give a necessary and sufficient condition for the uniqueness of D.

CHAPTER II

NON-NEGATIVE MATRICES

2.1 Decomposable and Indecomposable matrices

Definitions. A real square matrix is positive (non-negative) if its entries are all positive (non-negative). We write $A > 0$ ($A \geqslant 0$) to indicate that the matrix A is positive (non-negative). An n-square matrix $P = (p_{ij})$ is a permutation matrix if there exists a permutation σ of the first n natural numbers such that P is defined by

$$p_{ij} = \begin{cases} 1 \text{ if } j = \sigma(i) \\ 0 \text{ if } j \neq \sigma(i) \end{cases}$$

A permutation matrix $P = (p_{ij})$ with all $p_{ii} = 1$ is called an identity matrix and is denoted by I. It follows from the definition of a permutation matrix that (i) The transpose of a permutation matrix is a permutation matrix. (ii) The product of two permutation matrices of the same order is again a permutation matrix. (iii) The inverse of a permutation matrix is equal to its transpose.

A square matrix A is said to be decomposable if there exists a permutation matrix P such that $P A P^T = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix}$ where A_1 and A_2 are square matrices and 0 is a matrix of zeros.

If such a permutation matrix P does not exist, then A is indecomposable.

Decomposable and indecomposable matrices are also called reducible and irreducible respectively in the literature.

2.1.1 If $A \geqslant 0$ is an n-square indecomposable matrix, then

$$(I + A)^{n-1} > 0.$$

Proof. It suffices to show that $(I + A)^{n-1}\, y > 0$ for any non-null n-component vector $y \geqslant 0$.

$$\text{Let } z = (I + A)\, y.$$

We will show that z has more positive components than y does.

Assuming the contrary, from $z \geqslant y$ (component wise) follows that the i^{th} component of z is positive if and only if the i^{th} component of y is positive. Choosing a suitable permutation matrix P, we can write $P\, y = \begin{pmatrix} 0 \\ y_1 \end{pmatrix}$ and $P\, z = \begin{pmatrix} 0 \\ y_2 \end{pmatrix}$, where y_1 and y_2 are strictly positive vectors of the same dimension, say k. Let us write the matrix $P\, A\, P^T$ in the form $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are square matrices and A_{22} is of order k.

Then we have $\begin{pmatrix} 0 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 \\ y_1 \end{pmatrix}$.

This gives $A_{12}\, y_1 = 0$. Since $y_1 > 0$, we must have $A_{12} = 0$, violating the indecomposiblity of A. Since y has at least one positive component, $(I + A)\, y$ has at least two, and by induction $(I + A)^i\, y$ has at least i+1 for $i = 1, 2, \ldots, n-1$. In particular we have $(I + A)^{n-1}\, y > 0$ for arbitrary non-null y.

2.1.2 Let $A \geqslant 0$ be indecomposable. Then for any given i and j, there exists an integer $m > 0$ such that $a_{ij}^{(m)} > 0$ where $a_{ij}^{(m)}$ denotes the (i,j) entry of A^m.

Proof. First consider the case when $i \neq j$. Let

$$B = (b_{ij}) = (I + A)^{n-1} = A^{n-1} + \binom{n-1}{1} A^{n-2} + \ldots + I \qquad \ldots(1)$$

Since by 2.1.1 the (i,j) entry of B is positive, at least one term in the above expansion has a positive (i,j) entry. Hence $a_{ij}^{(m)} > 0$ for some $1 \leqslant m \leqslant n-1$.

It remains to show that for arbitrary $1 \leqslant i \leqslant n$, $a_{ii}^{(m)} > 0$ for some m. The case when $a_{ii} > 0$ is trivial. Suppose that $a_{ij} > 0$ (A being indecomposable can not have a zero row). Let m' be such that $a_{ij}^{(m')} > 0$. Then choosing $m = m' + 1$ we have $a_{ii}^{(m)} > 0$.

2.1.3 Let $A \geqslant 0$ be an n-square Indecomposable matrix with strictly positive main diagonal. Then $A^{n-1} > 0$.

Proof. The nonzero places of a product of non-negative matrices are determined by the nonzero places of the factors. Hence the nonzero places of A^{n-1} are those of $(I+A)^{n-1}$ and hence $A^{n-1} > 0$.

2.2 Perron-Frobenius theorem

We now state (without proof) some fundamental results on non-negative matrices. Perron proved these results for positive matrices in 1907 and Frobenius extended them to non-negative matrices in 1912.

Theorem 2.1 (Perron and Frobenius)

Let $A \geqslant 0$ be an n-square indecomposable matrix, then

(i) A has a positive eigenvalue ρ of multiplicity one.

(ii) The moduli of all other eigenvalues of A are less than or equal to ρ.

(iii) There exists a positive eigenvector X corresponding to ρ.

(iv) If the number of eigenvalues of A of modulus ρ is h and $h > 1$, then there exists a Permutation matrix P such that

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & A_{h-1,h} \\ A_{h,1} & 0 & 0 & \dots & 0 \end{pmatrix} \quad \dots(1)$$

where the diagonal zero submatrices are square.

2.3 Primitive and Imprimitive matrices

Definitions: The quantity ρ defined in theorem 2.1 is called the maximal root of A. Let h be the number of eigenvalues of modulus ρ .

Then A is called primitive, if $h = 1$ and imprimitive otherwise.

In the latter case h is called the index of imprimitivity. By definition, primitive and imprimitive matrices are necessarily indecomposable.

An imprimitive matrix with index of imprimitivity $h > 1$ is sometimes called 'cyclic matrix of index h'.

2.3.1 Every positive integer power of a primitive matrix is primitive.

Proof Let A be a primitive matrix with maximal root ρ and let m be any positive integer. Then, since ρ is a simple eigenvalue of A and is the only eigenvalue of A of its own modulus, ρ^m is a simple eigenvalue of A^m and is the only eigenvalue of A^m of modulus ρ^m. Thus we need to prove only that A^m is indecomposable.

Assuming the contrary, let us suppose that for some ν, A^ν is decomposable. Then we can assume that

$$A^\nu = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \qquad \ldots(1)$$

where A_1 and A_3 are square.

Since A is indecomposable, by theorem 2.1, there exists a positive eigenvector y corresponding to the maximal root ρ such that

$$A\,y = \rho\, y.$$

Then $$A^\nu y = \rho^\nu y \qquad \ldots(2)$$

Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, where $y_1 > 0$ and $y_2 > 0$.

Then from (1) and (2) we have

$$A_3\, y_2 = \rho^\nu\, y_2.$$

This shows that ρ^ν is an eigen-value of A_3.

Since A^T is also indecomposable and has ρ as its maximal root, again by theorem 2.1 we have $A^T z = \rho z$, for some $z > 0$.

Repeating the entire process on A^T, we conclude as before that ρ^ν is also an eigenvalue of A_1^T and hence of A_1.

Being an eigenvalue of both A_1 and A_3, ρ^ν must be a multiple eigenvalue of A^ν, which is a contradiction.

2.3.2 If A is primitive, then there is a power of A which is positive:

$$A^\nu > 0 \quad \text{for some} \quad \nu \geq 1$$

Proof Since A is primitive, by definition it is indecomposable. Hence by 2.1.2, there exists a positive integer m_1 such that

$$A^{m_1} = (a_{ij}^{(m_1)}) \quad \text{has} \quad a_{11}^{(m_1)} > 0.$$

Again, since A^{m_1} is primitive by 2.3.1 and hence indecomposable, for some m_2

$$(A^{m_1})^{m_2} = (b_{ij}^{(m_1 m_2)}) \quad \text{has} \quad b_{11}^{(m_1 m_2)} > 0$$

as well as $b_{22}^{(m_1 m_2)} > 0$.

Continuing in this way, we see that there exists a positive integer m such that $A^m = (a_{ij}^{(m)})$ has all $a_{ii}^{(m)} > 0$.

Therefore by 2.1.3, $(A^m)^{n-1} > 0$.

Taking $\nu = m(n-1)$, we have the desired result.

2.3.3 If $A \geq 0$ is an imprimitive matrix with index of imprimitivity $h > 1$, then $h = 2$ if A is symmetric.

Proof By the Perron-Frobenius theorem, there exists a permutation matrix P such that PAP^T has the form 2.2(1).

If A is symmetric, then so is PAP^T and the form matrix in 2.2 (1) is symmetric only if $h = 2$.

2.3.4 Let $A \geqslant 0$ be an imprimitive matrix with index of imprimitivity $h > 1$. Then there exists a permutation matrix P such that $PA^hP^T = \dot{\sum}_{i=1}^{h} A_i$, where each A_i is Primitive. (Here $\dot{\sum}$ designates direct sum as in [3]).

Proof The direct computation of the powers of PAP^T of 2.2(1) gives us $PA^hP^T = \dot{\sum}_{i=1}^{h} A_i$. It can be shown that each A_i has the same set of eigenvalues. We need to prove that each A_i is primitive. Assuming the contrary, let some A_i be imprimitive. Then A^h has at least (h+1) eigenvalues of modulus ρ^h, where ρ is the maximal root of A. This implies that A has at least (h+1) eigenvalues of modulus ρ which contradicts the hypothesis that A is an imprimitive matrix with index of imprimitivity h.

2.4 Support and total support

Definition Let $A = (a_{ij})$ be a non-negative n-square matrix. We say that A has support if there exists a permutation σ of the first n natural numbers such that $\prod_{i=1}^{n} a_{i\,\sigma(i)} > 0$. A has total support if it has at least one nonzero entry and for each $a_{ij} > 0$ there exists a permutation σ such that $j = \sigma(i)$ and $\prod_{r=1}^{n} a_{r\,\sigma(r)} > 0$.

2.5 Fully indecomposable matrices

A square matrix $A \geqslant 0$ is said to be Partly decomposable if there exist permutation matrices P and Q such that PAQ can be written in the form:

$$P\,A\,Q = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}$$

where A_1 and A_3 are square matrices and 0 is a matrix of zeros.

If no such P and Q exist, then A is said to be <u>fully indecomposable</u>.

It follows from the definitions that every decomposable matrix is partly decomposable and every fully indecomposable matrix is indecomposable.

Obviously, if a n-square matrix $A \geqslant 0$ is partly decomposable, then it contains a p by $(n-p)$ zero sub-matrix, for some positive integer p.

<u>2.5.1</u> Let A be a n-square partly decomposable matrix with total support, then there exist permutation matrices P and Q such that

$$P\,A\,Q = \sum_{i=1}^{k} A_i$$

where each A_i is fully indecomposable.

<u>Proof</u> By definition there exist permutation matrices P_1 and Q_1 such that $P_1\,A\,Q_1 = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix}$ where A_1 and A_2 are square. Since A has total support and A_1 and A_2 are square, $A_3 = 0$ and both A_1 and A_2 have total support.

Thus $P_1\,A\,Q_1 = A_1 \dot{+} A_2$ where A_1 and A_2 are square matrices with total support ($A_1 \dot{+} A_2$ is the direct sum of A_1 and A_2). If either A_1 or A_2 is partly decomposable, it can be decomposed similarly.

Thus the repeated application of the above argument shows that there exist permutation matrices P and Q such that $P A Q$ has the desired form.

<u>2.5.2</u> A symmetric primitive matrix A with total support is fully indecomposable.

<u>Proof</u>. Suppose that A is not fully indecomposable. Then there exist permutation matrices P and Q such that $P A Q = A_1 \dot{+} A_2$, where both A_1 and A_2 are square.

Let $S = P A Q$. Then $S S^T = P A^2 P^T = A_1 A_1^T \dot{+} A_2 A_2^T$. This shows that A^2 is decomposable, which is a contradiction, because from the primitivity of A follows the primitivity and indecomposability of A^2.

<u>Corollary</u>. If A is a symmetric partly decomposable matrix with total support, then A is either decomposable or an imprimitive matrix with index of imprimitivity $h = 2$.

<u>Proof</u>. By 2.5.2 A cannot be primitive. Hence A is either decomposable or an indecomposable matrix with index of imprimitivity $h > 1$ and by 2.3.3, $h = 2$.

<u>2.5.3</u> Let $A \geq 0$ be a symmetric imprimitive matrix with total support. Then there exists a permutation matrix P such that

$$P A P^T = \begin{pmatrix} O & A_1 \\ A_1^T & O \end{pmatrix}$$

where the diagonal zero submatrices are square and A_1 is fully indecomposable.

Proof Since A is symmetric, by 2.3.3 its index of imprimitivity is 2. So, by the Perron-Frobenius theorem there exists a Permutation matrix P such that

$$P\,A\,P^T = \begin{pmatrix} 0 & A_1 \\ A_1^T & 0 \end{pmatrix}$$

where A_1 is a square submatrix and 0 is a square matrix of zeros. We now show that A_1 is fully indecomposable.

Assuming the contrary there exist permutation matrices P_1 and Q_1 such that $P_1\,A_1\,Q_1 = B_1 \dotplus B_2$, where B_1 and B_2 are square matrices with total support.

Then $A_1 = P_1^T\,(B_1 \dotplus B_2)\,Q_1^T$ and

$$A_1\,A_1^T = P_1^T\,(B_1 \dotplus B_2)\,Q_1^T\,Q_1\,(B_1^T \dotplus B_2^T)\,P_1$$

$$= P_1^T\,(B_1B_1^T \dotplus B_2B_2^T)P_1.$$

This shows that $A_1\,A_1^T$ is decomposable.

Since $P\,A^2\,P^T = (\,P\,A\,P^T)^2 = A_1\,A_1^T \dotplus A_1^T\,A_1$ there exists a permutation matrix S such that $S\,A^2\,S^T$ is the direct sum of more than two indecomposable matrices. This contradicts the fact that the index of imprimitivity of A is 2.

2.5.4 Combining 2.5.2 with 2.5.3, we can now state the following.

Theorem 2.2. Let $A \geqslant 0$ be a symmetric indecomposable matrix with total support. Then either A is fully indecomposable or else there exists a permutation matrix P such that $P A P^T = \begin{pmatrix} O & A_1 \\ A_1^T & O \end{pmatrix}$ where A_1 is fully indecomposable.

As a corollary of the above theorem we obtain the following.

Theorem 2.3. Let A be a symmetric non-negative matrix with total support. Then there exists a permutation matrix P such that $P A P^T$ is the direct sum of matrices $A_1, A_2, \ldots A_k$, where each A_i is either fully indecomposable or else is of the form $A_i = \begin{pmatrix} O_i & B_i \\ B_i^T & O_i \end{pmatrix}$ where B_i is fully indecomposable and O_i is a matrix of zeros.

CHAPTER III

STOCHASTIC MATRICES

Definitions: A non-negative square matrix A is called row-stochastic (column stochastic) if all its row (column) sums are equal to one.

A is doubly stochastic if it is both row stochastic and column stochastic.

3.1. Some fundamental results on doubly stochastic matrices

The following theorem, due to König [3] forms the basis of many results known about doubly stochastic matrices.

Theorem 3.1 (König). Every doubly stochastic matrix has a positive diagonal.

Corollary 1 (Birkhoff). Every doubly stochastic matrix is the convex combination of permutation matrices.

Proof. Let S be a doubly stochastic matrix. We then have to show that there exist permutation matrices $P_1, P_2, \ldots P_k$ and positive numbers $\lambda_1, \lambda_2, \ldots \lambda_k$ such that $S = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k$ where $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$.

Since S is a doubly stochastic matrix, it has a positive diagonal. Let $s_{1\,\sigma(1)}, s_{2\,\sigma(2)} \ldots s_{n\,\sigma(n)}$ be one such. If S is not a permutation matrix, then taking $\lambda_1 = \min\limits_{1 \leq i \leq n} s_{i\,\sigma(i)}$ we can write $S = \lambda_1 P_1 + (1 - \lambda_1) R_1$, where P_1 is a permutation matrix having 1's in the positions $(i, \sigma_{(i)})$, $i = 1, 2, \ldots n$ and R_1 is a doubly stochastic matrix.

We observe that the doubly stochastic matrix R_1 contains more zeros than S.

We can then decompose R_1 in the same way and get $S = \lambda_1 P_1 + \lambda_2 P_2 + (1 - \lambda_1 - \lambda_2) R_2$, where P_1 and P_2 are permutation matrices and R_2 is doubly stochastic.

Continuing this way, we get finally

$$S = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_K P_K + R_K$$

where R_k is a matrix of zeros and $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$.

<u>Corollary 2</u>. Every doubly stochastic matrix has total support.

<u>Proof</u>. The proof follows immediately from the Corollary 1, because every doubly stochastic matrix S is the convex combination of Permutation matrices $P_1, \ldots P_k$ and each P_i has total support.

<u>Corollary 3</u>. (Perfect and Mirsky). Let A be an arbitrary matrix of order n. Then there exists a doubly stochastic matrix S of order n such that the non-zero places of S are precisely the non-zero places of A, if and only if A has total support.

<u>Proof</u>. If A has total support, then for such non-zero place (i, j) of A there exists a permutation matrix which is non-zero at (i, j). Taking such permutation matrices (one for each non-zero entry of A) and taking their arithmetic mean, we obtain the desired doubly stochastic matrix.

On the other hand, we know from Corollary 2 that every doubly stochastic matrix A has total support.

3.2 Doubly stochastic limit and its diagonal represenation

Definition. Let $A \geqslant 0$ be a square matrix with a positive diagonal. Then the sequence of non-negative matrices $\{A_k\}$ obtained from A by alternately normalizing the rows and columns of A converge to a limit S which is doubly stochastic. This limit is called the doubly stochastic limit of A.

We recall Sinkhorn and Knopp's result that the diagonal representation of the doubly stochastic limit S in the form $D_1 A D_2$, where D_1 and D_2 are diagonal matrices with positive main diagonals is possible if and only if A has total support and D_1 and D_2 are unique up to a scalar factor if and only if A is fully indecomposable [7].

The above result is used to prove the following.

Lemma 3.1. Let a non-negative n-square matrix M be the direct sum of k fully indecomposable matrices M_i, $i = 1,\ldots k$ and let $D_1 = \sum_{i=1}^{k} S_i$ and $D_2 = \sum_{i=1}^{k} T_i$ be diagonal matrices such that $D_1 M D_2$ is doubly stochastic. Then if D_1^* and D_2^* are diagonal matrices with the property that $D_1^* M D_2^*$ is also doubly stochastic, there exist positive numbers $\alpha_1, \alpha_2 \ldots \alpha_k$ such that

$$D_1^* = \sum_{i=1}^{k} \alpha_i S_i \quad \text{and} \quad D_2^* = \sum_{i=1}^{k} \frac{1}{\alpha_i} T_i.$$

Proof. Since $D_1 M D_2 = \sum_{i=1}^{k} S_i M_i T_i$ is doubly stochastic, so is each $S_i M_i T_i$.

Let $D_1^* = \sum_{i=1}^{k} X_i$ and $D_2^* = \sum_{i=1}^{k} Y_i$

Then, since $D_1^* M D_2^*$ is also doubly stochastic, we similarly conclude that so is each $X_i M_i Y_i$.

It follows that $S_i M_i T_i = X_i M_i Y_i$, $i = 1, 2 \ldots \kappa$. Moreover, since each M_i is fully indecomposable, by the uniqueness part of the theorem of Sinkhorn and Knopp [7], there exist positive numbers $\alpha_1, \alpha_2 \ldots \alpha_k$ such that $X_i = \alpha_i S_i$

$$\text{and, } Y_i = \frac{1}{\alpha_i} T_i \qquad i = 1, 2 \ldots \kappa.$$

3.2.1. Assuming that A is a partly decomposable matrix with total support, we will show how to generate all pairs (G_1, G_2) of diagonal matrices such that $G_1 A G_2$ is doubly stochastic, given any pair (D_1, D_2) with the same property.

There exist permutation matrices P and Q such that $A^* = P A Q$ is the direct sum of κ fully indecomposable matrices $A_1, A_2 \ldots A_\kappa$.

Let $S = D_1 A D_2$, then $D_1' A^* D_2'$ is also doubly stochastic, where $D_1' = P D_1 P^T$ and $D_2' = Q^T D_2 Q$. So, we may write $D_1' = \sum_{i=1}^{\kappa} S_i$ and $D_2' = \sum_{i=1}^{k} T_i$ where S_i and T_i are diagonal matrices of the same order as the order of A_i.

Now, if (D_1'', D_2'') is any other pair of diagonal matrices such that $D_1'' A^* D_2''$ is also doubly stochastic, then by Lemma 3.1, there exists positive numbers $\alpha_1, \alpha_2 \ldots \alpha_K$ such that

$$D_1'' = \sum_{i=1}^{\kappa} \alpha_i S_i \text{ and } D_2'' = \sum_{i=1}^{k} \frac{1}{\alpha_i} T_i.$$

Now let $G_1 = P^T \left(\sum_{i=1}^{k} \alpha_i S_i \right) P$ and let

$$G_2 = Q \left(\sum_{i=1}^{k} \frac{1}{\alpha_i} T_i \right) Q^T,$$

Then $G_1 A G_2$ is doubly stochastic.

Theorem 3.2. Let M be an indecomposable symmetric non-negative matrix with total support and let D_1 and D_2 be diagonal matrices with positive main diagonals such that $D_1 M D_2$ is doubly stochastic. Then $D_2 = p D_1$ for some $p > 0$.

Proof. By theorem 2.2, either M is fully indecomposable or else there exists a permutation matrix P such that $P M P^T = M^* = \begin{pmatrix} 0 & M_1 \\ M_1^T & 0 \end{pmatrix}$, where M_1 is fully indecomposable.

We consider the two cases separately.

Case 1. M is fully indecomposable.

Let $D_1 M D_2 = S$.

Then since $S^T = D_2 M D_1$ is also doubly stochastic and D_1 and D_2 are unique up to a scalar multiple, it follows that $D_2 = p D_1$ for some $p > 0$.

Case 2. M is not fully indecomposable.

Let D_1' and D_2' are diagonal matrices with positive main diagonals such that $D_1' M^* D_2'$ is doubly stochastic, where D_1' and D_2' are defined as follows:

$$D_1' = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad D_2' = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

Then $S = D_1' M^* D_2' = \begin{pmatrix} 0 & S_1 M_1 T_2 \\ S_2 M_1^T T_1 & 0 \end{pmatrix}$

Here $S_1 M_1 T_2$ and $S_2 M_1^T T_1$ are doubly stochastic and so is $(S_2 M_1^T T_1)^T = T_1 M_1 S_2$.

Since M_1 is fully indecomposable, it follows that

$$T_1 = p\, S_1 \quad \text{and} \quad T_2 = p\, S_2 \quad \text{for some } p > 0.$$

Hence $\quad D'_2 = p\, D'_1 \quad$ for some $\quad p > 0.$

Now, taking $D_1 = P^T D'_1 P$ and $D_2 = P^T D'_2 P$, we have the required result.

Remark. When A is decomposable and $D_1 A D_2$ is doubly stochastic, the relation between D_1 and D_2 is exhibited as follows:

There exists a permutation matrix P such that $P A P^T$ is the direct sum of indecomposable matrices $A_1, A_2, \ldots A_k$.

Let $D_1^{(1)}, \ldots D_1^{(k)}$ and $D_2^{(1)} \ldots D_2^{(k)}$ be the diagonal matrices such that $D_1^{(i)} A_i D_2^{(i)}$ is doubly stochastic. Then taking $D_1 = P^T (\sum_{i=1}^{k} D_1^{(i)})P$ and $D_2 = P^T (\sum_{i=1}^{k} D_2^{(i)})P$, we see that $D_1 A D_2$ is doubly stochastic and in view of the above theorem, each $D_2^{(i)}$ is a scalar multiple of $D_1^{(i)}$.

3.3 The diagonal equivalence of non-negative matrices to non-negative matrices with prescribed rows.

In this section, we extend a result of Sinkhorn about strictly positive matrices to the case of certain types of non-negative matrices.

As a tool we use a sufficient condition of Brualdi, Parter and Schneider [1] for the existence of a diagonal matrix D such that $D A D$ is row stochastic, quoted in Chapter I.

3.3.1. In [6] Sinkhorn proved that corresponding to each strictly positive matrix A there exist a unique row stochastic matrix of the form $D A D$, where D is a diagonal matrix with positive entries.

As a corollary he proved the following: Corresponding to each positive n-square matrix A and each set of positive real numbers $p_1, p_2, \dots p_n$ there is a unique matrix of the form $D A D$ with row sums $p_1, p_2, \dots p_n$, where D is a positive diagonal matrix.

We generalize this for non-negative matrices.

Theorem 3.3. Let A be a non-negative n-square matrix with a positive main diagonal and let $p_1, \dots p_n$ be given positive numbers. Then there exists a unique matrix of the form $D A D$ with row sums $p_1, p_2, \dots p_n$, where D is a diagonal matrix with positive main diagonal.

Proof. Let $D_1 = dg\,(p_1, p_2, \dots p_n)$ and let $B = D_1^{-1} A$. Then B has a positive main diagonal. By [1], there exists a positive diagonal matrix D such that $D B D$ is row stochastic. Let $S = D B D$. Then $D_1 S = D_1 D B D = D_1 D D_1^{-1} A D = D_1 D_1^{-1} D A D = D A D$.

CHAPTER IV

THE DIAGONAL EQUIVALENCE OF A NON-NEGATIVE SYMMETRIC MATRIX TO A DOUBLY STOCHASTIC MATRIX

4.1 The D A D Theorem

In this section, we establish a necessary and sufficient condition on a non-negative symmetric matrix A such that there exists a diagonal matrix D with positive main diagonal with the property that $D A D$ is doubly stochastic.

Definition. A non-negative matrix A has Property D if there exists a diagonal matrix D with positive main diagonal such that $D A D$ is doubly stochastic.

Theorem 4.1. Let $A \geq 0$ be a symmetric matrix. Then A has Property D if and only if A has total support.

Proof. Since every doubly stochastic matrix has total support, it follows that if A has property D, then A has total support.

Now, let us assume that A is a non-negative symmetric matrix with total support. Then we distinguish between the two cases.

Case 1. A is indecomposable.

By theorem 3.2 there are diagonal matrices D_1 and D_2 with positive main diagonals such that $D_1 A D_2$ is doubly stochastic and $D_2 = \rho D_1$ for some $\rho > 0$.

If we now choose $D = \sqrt{\rho}\, D_1$, then $D A D$ is doubly stochastic.

Case 2. A is decomposable.

Then there exists a permutation matrix P such that $P A P^T = \sum_{i=1}^{k} {}^{\oplus} A_i$, where each A_i is a symmetric indecomposable matrix with total support.

By Case 1, there exist diagonal matrices $D_1, D_2 \dots D_k$ such that $D_i A_i D_i$ is doubly stochastic, for $i = 1, 2, \dots k$.

Now let $D = P^T (\sum_{i=1}^{k} {}^{\oplus} D_i) P$. Then $D A D = P^T \left(\sum_{i=1}^{k} {}^{\oplus} D_i \sum_{i=1}^{k} {}^{\oplus} A_i \sum_{i=1}^{k} {}^{\oplus} D_i \right) P$ is doubly stochastic and the theorem is proved.

Remark. It is easy to see that the result of Brualdi, Parter and Schneider quoted in Chapter I about the existence of a non-negative diagonal matrix D such that D A D is doubly stochastic, follows as a special case of theorem 4.1.

For, if a symmetric matrix $A = (a_{ij}) \geq 0$ has a positive main diagonal, then every non-zero element a_{ij} $(i \neq j)$ is associated with a positive diagonal consisting of a_{ij}, a_{ji} and all other main diagonal entries of A excepting a_{ii} and a_{jj}.

4.2. Uniqueness in the D A D theorem

Let $A \geq 0$ be a symmetric matrix with total support and let $D > 0$ and $G \geq 0$ be diagonal matrices such that D A D and G A G are doubly stochastic. Then, since the doubly stochastic limit of A is unique, it follows that $D A D = G A G$.

It is therefore natural to ask when D is unique. We deal with this problem in the present section.

<u>Lemma 4.1</u>. Let $A \geqslant 0$ be a symmetric fully indecomposable matrix. Then there exists a unique diagonal matrix $D \geqslant 0$ such that $D A D$ is doubly stochastic.

<u>Proof</u>. The existence of D is a part of theorem 4.1. We prove only uniqueness here.

Let $D \geqslant 0$ and $G \geqslant 0$ be two diagonal matrices such that $D A D$ and $G A G$ are both doubly stochastic.

Let $B = D A D$. then $G A G = G D^{-1} B D^{-1} G$. Since A is fully indecomposable, so is B and by the theorem of Sinkhorn and Knopp, there exists a positive number α such that $G D^{-1} = \alpha I$ and $D^{-1} G = \frac{1}{\alpha} I$, where I is an identity matrix of the same order as the order of B.

The above equalities are satisfied only if $\alpha = 1$ and therefore $D = G$.

<u>Theorem 4.2</u>. Let $A \geqslant 0$ be a symmetric matrix with total support and let $D \geqslant 0$ be a diagonal matrix such that $D A D$ is doubly stochastic. Then D is unique if and only if there exists a permutation matrix P such that $P A P^T$ is the direct sum of fully indecomposable matrices.

<u>Proof</u>. Let $P A P^T = A_1 \dotplus A_2 \dotplus \dots \dotplus A_K$ and let $D = P^T (D_1 \dotplus D_2 \dotplus \dots \dotplus D_k) P$, where the order of D_i is the same as the order of A_i.

Now, $D A D$ is doubly stochastic if and only if $P D A D P^T = (P D P^T)(P A P^T)(P D P^T)$ is doubly stochastic.

But this is doubly stochastic if and only if each of the matrices $D_i A_i D_i$ is doubly stochastic, for $i = 1, 2, \dots k$.

Clearly, D is unique if and only if each D_i is unique. If each A_i is fully indecomposable, then the uniqueness of each D_i follows from Lemma 4.1.

If some A_i is not fully indecomposable, then by theorem 2.3. $A_i = \begin{pmatrix} 0 & B_i \\ B_i^T & 0 \end{pmatrix}$, where B_i is fully indecomposable and '0' is a matrix of zeros. We may then write $D_i = D_{i1} \dotplus D_{i2}$, where $D_{i1} B_i D_{i2}$ is doubly stochastic.

In this case for an arbitrary $\alpha > 0$ we can define

$$G_i = \alpha_i D_{i1} \dotplus \frac{1}{\alpha_i} D_{i2} \text{ and}$$

$$G = P^T (D_1 \dotplus D_2 \dotplus \dots \dotplus D_{i-1} \dotplus G_i \dotplus D_{i+1} \dotplus \dots \dotplus D_k) P$$

Then $G A G$ is doubly stochastic.

<u>Remark</u>. From the proof of the theorem 4.2 we can see that if D is not unique, then there exists a simple relationship between D and all other diagonal matrices $G \geqslant 0$ such that $G A G$ is doubly stochastic.

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