THE IMPLEMENTATION OF OPTIMAL CONTROL

WITH

SENSITIVITY REDUCTION TO PLANT PARAMETER VARIATIONS

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SCOPE AND CONTENTS: The dual configuration is innovated as a new approach in sensitivity reduction. Three types of sensitivity due to variations in plant parameters are discussed. It has been shown that cost insensitive and terminal insensitive designs are indeed achievable by applying the dual configuration to implement the optimal control.

The theory has been developed for a general class of optimal systems and the linear systems with quadratic cost functionals have been analytically evaluated to illustrate the theory.

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NOMENCLATURE

$\forall$	for all
ε	Belongs to, an element of
f^T	Transpose of f
$\lVert y \rVert$	Norm of y ; if y is a vector, $\lVert y \rVert = \sqrt{\sum_i y_i^2}$ and if y is a matrix, $\lVert y \rVert = \sqrt{\sum_{i,j} y_{ij}^2}$
$\left.\frac{\partial J}{\partial w}\right\vert_{w_0}$	Differentiation of J with respect to w and all the arguments are then set to its nominal value
$\left.\frac{\partial H}{\partial x}\right\vert_{w_0}$	Differentiation of H with respect to x and all the arguments are then set to its optimal value
ΔJ	Incremental change of J
δJ	First variation of J
$\delta^2 J$	Second variation of J
Ω	Admissible control set
ϕ	State transition function
A,B	Linear system matrices
c	Constant scalar (vector)
c(t)	Output vector
D,R,Q	Symmetric matrices of the integrand of a cost functional
e(t)	Error vector

EM_w^x	Measure for error vector
f	Generalized system function
g(t)	Open loop portion of the dual implementation
h(t,y,M)	System function of the state sensitivity equation of the dual implementation
H	Hamiltonian
H(t)	Feedback function for closed loop implementation
I,J	Cost functionals
K	Terminal cost
i,j,k,l,m,r,r'	Indices, Subscripts
L, L_1, L_2	Integrands of the cost functional
M(t)	Feedback portion of the dual implementation
p	Costate vector
p*	Optimal costate vector (at nominal)
n	Dimension of the state vector
q	Dimension of the control vector
r	Dimension of the plant parameter
S_w^J	Cost sensitivity
$y(t), S_w^x$	State Sensitivity
$y(t_f), S_w^{t_f}$	Terminal (state) sensitivity
SM_w^J	Cost sensitivity measure
SM_w^x	State Sensitivity measure
$SM_w^{t_f}$	Terminal state sensitivity measure

S	k fold
t	time, $t_0 < t < t_f$
t_0, t_f	Initial and final times of the plant when operating
T_0, T_f	Initial and final times for the availability of the plant
T	time interval
u	control vector; control by dual implementation
u^*	The optimal control (at nominal)
u_0	control by open loop implementation
u_c	control by closed loop implementation
U	Collection of admissible control
v	Implementation vector
v_0	Implementation vector of open loop configuration
v_c	Implementation vector of closed loop configuration
w	Plant parameter or value of the plant parameter
x	state variable (trajectory)
x^*	Optimal state trajectory (at nominal)
x_0	State trajectory of open loop implementation
x_c	State trajectory of closed loop implementation
$\dot{x}$	Differentiation of x with respect to time

y_0	State sensitivity of open loop implementation
y_c	State sensitivity of closed loop implementation
y	State sensitivity of dual implementation
$\overline{\Omega}$	The closure of Ω

CHAPTER I

INTRODUCTION

Though considerable amount of research in Optimal Control Theory has been done since 1957, the first results concerning the sensitivity analysis of optimal systems were published only as recently as 1963 [11, 24]. Since then, a rapidly growing number of technical articles have appeared and as a result a very broad field of research was started.

Due to its short existence, this area of research is still in its infant stage. There are only a few general important results; many difficult problems have been uncovered but little has been done in the direction of obtaining the solutions.

Unlike automatic control systems, optimal control processes are associated with a given cost functional. When the Optimal Control Theory is considered as a set, the studies of the necessary and sufficient conditions for the optimal control and the sensitivity analysis can be regarded as the three major subsets. Sensitivity analysis in Optimal Control Theory consists of two major problems. If the implementation of the optimal control is not unique, comparison between different implementations in the light of sensitivity is one of the major topics in the field of sensitivity analysis. Recent research emphasises on the comparison between open- and closed-loop implementations. Another topic of sensitivity analysis is in the direction of reducing the sensitivity. Without innovating the configuration of two degrees of freedom, two approaches are possible. First, the given cost functional is modified and a new optimal control policy is consequently developed. The most popular approach is to include a scalar function of sensitivity in the given cost functional. Second, the given cost functional is not changed and the choice of implementation is based on the result of comparison as there are only two types of implementations available. In this thesis, a third approach is formulated. In contrast to the first approach, the given cost functional is unchanged so that the control thus implemented is optimal with respect to the given cost functional as desired. Differing from the second approach a prototype of implementation which is the configuration of two degrees of freedom is applied. The scope of the present work is limited in the direction of reducing various sensitivities without making any comparison between the suggested configuration with either the open- or the closed-loop implementations.

Pontryagin's Minimum Principle is regarded as the basic tool in determining the optimal control. Sufficient treatment of the control problem and the technique of obtaining the optimal control will be given in Chapter II. The physical and mathematical aspects of changes in the plant parameters are evaluated in Chapter III. In addition, the sensitivity is redefined since the term has often been misused and the need for clarification is obvious. Chapter IV introduces the configuration of two degrees of freedom in optimal control systems, and formulates the implementation problems where the technique developed in Chapter II is readily applicable. Chapter V exposes the field of sensitivity analysis using the linear system with a quadratic cost functional as an illustration and the idea developed in Chapter IV is applied to this special yet important system.

CHAPTER II

OPTIMAL CONTROL SYSTEMS

2.1 Introduction:

Knowledge of the physical world is based upon experiment and abstraction. The engineer examines specific physical systems with definite objectives in mind, while the theoretician attempts to discover the basic laws which govern and describe the behavior of physical systems in general.

In the role played by the engineer, a physical system is considered as a black box. Certain input "signals" to one black box are applied in order to observe and measure the resultant output "signals". The ultimate objective is the determination of an input which will produce an output with certain desired characteristics and which will minimize the "cost" of operation. A trial and error procedure may be applied to achieve the objective but except when one is very lucky, in general it would not work. The aim of this chapter is to supply a systematic technique for determining such an input and to discuss its characteristics.

2.2 Mathematical Description of Control Processes:

There are two different ways of describing control processes: (i) by means of state variables and

(ii) by transfer functions. Recent developments in optimal control theory are based on using vector differential equations as models for physical systems and rely heavily on the concept of state. In approximate terms, the state of a system may be defined as the minimum information about the system at some instant of time t_0 which, together with a specification of the input vector u(t) for all time subsequent to t_0, enables the computation of the output vector c(t) for all time subsequent to t_0. In other words, knowledge of the state at t_0 obviates the need for any information about the past behavior of the system for predicting its future. Knowledge of the output is generally not sufficient. The state may be regarded as separating the system's past from its future. This definition of state suffices for our later development; and for the precise definition one may refer to Zadeh and Desoer [52].

Let $u(t_0,t_f]$ denote all values of u(t) in the interval of $t_0<t\leq t_f$ and x(t) denote the state. The state concept can be expressed mathematically as

$$c(t) = c\{x(t_0),\ u(t_0,t]\} \qquad t>t_0 \qquad [2.2.1]$$

Equation (2.2.1) states that the future output behavior can be determined from a knowledge of (or equivalently, is a function of) the present state, and a specification of the future input signal. It is possible to show that, if certain care is taken in the mathematical definition of state, then

$$x(t) = x\{x(t_0), u(t_0, t]\} \quad \forall\, t>t_0 \qquad [2.2.2]$$

which means the future state behavior of the system also depends only on the present state and the future input. Equation (2.2.2) is better represented by

$$x(t)=\phi\{t;\ x(t_0), u(t_0, t]\} \quad \forall\, t>t_0 \qquad [2.2.3]$$

where ϕ is the transition function of time determined according to the knowledge of $x(t_0)$ and $u(t_0, t]$. The state $x(t)$ of a system in equation (2.2.3) contains sufficient information about the system; in usual practice, state $x(t)$ in the form of equation (2.2.3) is not available directly. A differential system is a dynamical system with the system state variable $x(t)$ described by a set of differential equations,

$$\dot{x}(t)=f(x(t), u(t), t) \qquad [2.2.4]$$

with $x(t_0)$ as initial point and ϕ in equation (2.2.3) as the solution to the vector differential equation (2.2.4).

By the existence theorem [40] for differential equation (2.2.4) the components of f and $\frac{\partial f}{\partial x}$ must be continuous on $R^n x R^q x T$ where n is the dimension of x(t) with $x(t_0) \varepsilon R^n$, q is the dimension of u(t) with a piecewise continuous function u(t) from T into R^q and T is the open time interval with $T=(T_0,T_f)$, $T_0<t_0<t<T_f$. Both n and q are assumed to be finite.

The control processes or physical plants, or simply systems, which are considered through all chapters are continuous-time dynamical differential systems described by the systems of differential equation (2.2.4) with $x(t_0)$ as initial point and equation (2.2.3) the solution. Sample-data system will be excluded.

All the developments that follow are based on the vector differential equations as models of the physical systems. In the older literature on control theory, however, the same systems are modeled by transfer functions. In the new approach, state variables, transition matrix, etc. are used and the mathematical tools are abstract linear algebra and differential equation theory. In the old approach, the key words are frequency response, pole zero pattern, etc. and the main mathematical tool is complex function theory. It is very unfortunate that the gap between the old and the new approaches become wider and wider, but no bridging of this increasing gap will be attempted in this thesis.

2.3 The Control Problem:

The basic ingredients of the (optimal) control problem are

1) a control process which is to be "controlled"
2) a cost functional or performance index which measures the effectiveness of a given "control action".
3) the objective of the control process
4) a set of constraints.

As discussed in the previous section, the control process under consideration is

$$\dot{x}(t)=f(x(t),u(t),t) \quad [2.3.1]$$

with the transition function ϕ given as

$$x(t)=\phi(t;x(t_0),u(t_0,t]) \quad [2.3.2]$$

The transition function ϕ depends on the control $u(t_0,t]$ when the initial point $x(t_0)$ is specified. If more than one control $u(t_0,t)$ in $R^q xT$ satisfies the objective of the control process, a choice must be made among the candidates. A cost functional is established for this purpose. In general, a cost function J is a scalar integral function and takes the form of

$$J(u,t_0,t_f)=K(x(t_f),t_f)+\int_{t_0}^{t_f} L(x(t),u(t),t)dt \quad [2.3.3]$$

where t_f is the final time. The integrand L is assumed to be a continuous real-valued function on $R^n x R^q x T$ and K is a continuous real valued function on $R^n x T$. The optimal control $u^*(t_0, t_f]$ of the control process associated with the cost functional J is the control which satisfies the constraint and the objective of the control process and which gives a minimum value for the cost functional J relatively with respect to the possible candidates. Since finding the maximum of a real-valued function is the same as finding the minimum of the negative of the function, it is clearly sufficient to consider only the minimization of the cost functional.

The objective of the control process specifies both the initial conditions and the final conditions for the state and time. The initial conditions supply the information about the initial time $t_0 > T_0$, and the initial state $x(t_0)$. Without loss of generality let the initial time be specified and the initial state be given as a point in R^n. The final conditions supply the information about the final time $t_f < T_f$, and the final state $x(t_f)$. The final time may or may not be specified while the final state can be a point, free, in a target set, in a moving target set or a moving point. This will be discussed in more detail in the next section.

Since the control process is described by equation (2.3.1), the constraint set, if there is any, must be described also in terms of either the control or the state. Hence there are two types of constraints, i.e. the control constraint and the state constraint. Let U_t be a closed, bounded and convex subset of R^q and denote the collection of the sets U_t by Ω; that is,

$$\Omega=\{U_t : t\varepsilon T=(T_0,T_f)\} \qquad [2.3.4]$$

Moreover, all the elements in U_t must be bounded and piecewise continuous. In general, the constraint is given as $u\varepsilon\Omega$ where the set Ω is described by equation (2.3.4). If $\Omega=R^q$, then the optimal control problem is unconstrained in control. Magnitude constraint can be expressed as

$$\Omega=\{U : t\varepsilon T\}$$

with $U = \{ \sum_{i=1}^{q} \mu_i e_i : |u_i| \leqslant M_i \}$

where M_is are given constants and $e_i \ldots e_q$ is the natural basis of R^q. Norm constraint is expressed as

$$U_t=\{u : ||u(t)|| \leqslant M\} \quad \forall t\varepsilon T=(T_0,T_f)$$

and $\Omega=\{U_t : t\varepsilon T\}$.

Similarly, the state constraint can be constructed.

With all the notions developed, the optimal control problem is formulated as

"To find a control u, $u\varepsilon\Omega$, which takes the initial set to the final set and which minimizes the cost functional (2.3.3)."

Note that the state constraint is excluded. If it is not the case suitable modifications are suggested by Berkovitz [5] and McGill [36] and the development is very similar to the unconstrainted one.

Major topics involving the optimal control problem are:

1) The existence and uniqueness of the optimal control
2) The techniques of obtaining the optimal control analytically or numerically if it exists.
3) The sensitivities of the optimal system, the comparison between different implementations in the light of sensitivities if the implementation of the optimal control is not unique and the reduction of sensitivities by implementation if it is possible.

2.4 Pontryagin's Minimum Principle:

In the early fifties, minimum time control laws were obtained for a variety of second and third order systems. In 1956, a principle, leading to the solution of the general problem of finding a control process, was hypothesized by Pontryagin on the basis of the results of work done by him, Boltyanskii and Gamkrelidze. This principle, which received the name, "Maximum Principle", was verified at first for individual types of systems and, in particular, in the case of linear systems. A year later the proof of optimality for minimum time control problem was completed by Boltyanskii and Gam-krelidze. In three years time, the maximum principle was extended to the general case of minimizing an arbitrary function of the integral function of variable systems and a detailed presentation of basic results was obtained by Pontryagin and his associates. [41] In this section, the celebrated Maximum principle[1] of Pontryagin will be given without proof. A rigorous derivation based on geometric arguments is presented by Pontryagin et.al. A less rigorous demonstration than Pontryagin's is given by Athans and Falb [2].

Consider a given n^{th} order control process with $(T_0, T_f)=T$ as interval of definition and with state equation

$$\dot{x}(t)=f(x(t),u(t),t) \qquad [2.4.1]$$

Ω is a given subset of R^q such that

$$u(t)\varepsilon\Omega \qquad \forall\ t\varepsilon T \qquad [2.4.2]$$

and $$u(t-)=u(t) \qquad \forall\ t\varepsilon T \qquad [2.4.3]$$

except at a finite number of points in time.

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1 Since the minimization not maximization of a cost functional is considered, it will be called the minimum Principle of Pontryagin.

The transition function for the given control process is

$$x(t)=\phi(t;x(t_0),u(t_0,t]) \qquad [2.4.4]$$

The cost functional is given as

$$J(u,t_0,t_f)=K(x(t_f),t_f)+\int_{t_0}^{t_f} L(x,u,t)dt \qquad [2.4.5]$$

where $L(x,u,t)$ is a real-valued function on $R^n x R^q x T$ and $K(x,t)$ is a real-valued function on $R^n x T$. It is assumed that the components of f, L, $\frac{\partial f}{\partial x}, \frac{\partial L}{\partial x}, \frac{\partial f}{\partial t}$, and $\frac{\partial L}{\partial t}$ are continous on $R_n x \bar{\Omega} x T$, where $\bar{\Omega} \varepsilon R^q$. Both the initial time t_0 and the initial state $\dot{x}(t_0)$ are given. By an admissible arc(x,u) it is meant that x(t) satisfies (2.4.4) and that the corresponding u satisfies (2.4.2) and (2.4.3). The Hamiltonian function H(x,p,u,t) is a real-valued function nxl vector x, the nxl vector p, the qxl vector u and t and is given by

$$H(x,p,u,t)=L(x,u,t)+f^T(x,u,t)p(t) \qquad [2.4.6]$$

The canonical system associated with the control process (2.4.1) is a $2n^{th}$ order system of differential equations:

$$\dot{x}(t) = \frac{\partial H(x,p,u,t)}{\partial p} \qquad [2.4.7]$$

$$\dot{p}(t) = \frac{-\partial H(x,p,u,t)}{\partial x} \qquad [2.4.8]$$

In view of the assumption made, the functions H and $\frac{\partial H}{\partial x}$ are continuous on $R^n x R^n x \bar{\Omega} x T$. Consequently by the existence theorem, there exist a state $\hat{x}$ and a costate $\hat{p}$ which are the solution of the canonical system corresponding to $\hat{u}$ in Ω. The arc $(x^*(t), u^*(t))$ is said to be optimal if $(x^*(t), u^*(t))$ is admissible and if the cost functional $J(u^*, t_0, t_f)$ is a minimum, or mathematically

$$J(u^*, t_0, t_f) = \min_{u \varepsilon \Omega} J(u, t_0, t_f) \qquad [2.4.9]$$

Pontryagin's Minimum Principle states that if u* is the optimal control, x* is the optimal state trajectory and p* is the optimal cost state trajectory corresponding to u*, then the Hamiltonian $H(x^*, p^*, u^*, t)$ is a minimum with respect to u; or mathematically

$$H(x^*, p^*, u^*, t) = \min_{u \varepsilon \Omega} H(x^*, p^*, u, t) \qquad [2.4.10]$$

or equivalently, $H(x^*, p^*, u^*, t) \leqslant H(x^*, p^*, u, t), \forall u \varepsilon \Omega,$ [2.4.11]

In the case where $\Omega = R^q$, that is no constraint, equation (2.4.10) can be replaced by

$$\frac{\partial H}{\partial u^*}(x^*, p^*, u^*, t) = 0 \qquad [2.4.12]$$

Note that the optimal trajectory x* and the optimal costate trajectory p* must satisfy the canonical system. With the given initial condition $x^*(t_0) = x(t_0)$ where both t_0 and $x(t_0)$ are given, n boundary conditions are required to determine $x^*(t)$ and $p^*(t)$ from the canonical system. These boundary conditions are supplied by the transversality condition which depends on the terminal set. Several cases which will be considered are:

1) t_f is fixed and $x(t_f)$ is free
2) t_f is fixed and $x(t_f)$ is a fixed point in R^n
3) t_f is fixed and $x(t_f)$ is in the target set S, a smooth k-fold in R^n given by $S=\{x: g_1(x)=0 \ldots g_{n-k}(x)=0\}$
4) t_f is free and $x(t_f)$ is free
5) t_f is free and $x(t_f)$ is a fixed point in R_n
6) t_f is free and $x(t_f)$ is in the target set S, a smooth k-fold in k^n given by $S=\{x: g_1(x)=0, \ldots g_{n-k}(x)=0\}$
7) t_f is free and $x(t_f)$ is in the moving target set S, a smooth k+1 fold in $R^n xT$ given by $S=\{x: g_1(x,t)=0 \ldots g_{n-k}(x,t)=0\}$
8) t_f is free and $x(t_f)$ is a moving point given by $S=\{(g(t),t)=t\varepsilon T\}$

In case 8, it is assumed that g(t) is a continuously differentiable function from T into R^n. In case 7, it is assumed that the functions $g_i(x,t)$, $\frac{\partial g_i}{\partial t}(x,t)$, and $\frac{\partial g_i}{\partial x}(x,t)$ are continuous on $R^n xT$ for all $i\varepsilon[1,n-k]$ and that the vectors $\frac{\partial g_i}{\partial x}(x,t)$ are linearly independent at each point of S. Corresponding to different terminal conditions, the real-valued function $K(x(t_f),t_f)$ is modified. As a function of the control u(t), the Hamiltonian, i.e. equation (2.4.10), is a minimum at u^*. The behavior of the Hamiltonian along the optimal trajectory depends on the terminal conditions.

Together with the transversality condition, Pontryagin's Minimum principle for case (1) to case (8) can be stated as

If $u^*(t)$ is the optimal control, then equation(2.4.10) must be satisfied where the canonical system is rewritten as

$$\dot{x}^*(t) = \left.\frac{\partial H}{\partial p}\right|_* \qquad [2.4.13]$$

$$\dot{p}^*(t) = -\left.\frac{\partial H}{\partial x}\right|_* \qquad [2.4.14]$$

and corresponding to the terminal condition (1) to (8), the following transversality condition must be met respectively:

1) $K=K(x)$ where K, $\frac{\partial K}{\partial x}$, and $\frac{\partial^2 K}{\partial x^2}$ are continuous, then

$$p^*(t_f) = \left.\frac{\partial K}{\partial x}\right|_{x^*(t_f)} \qquad [2.4.15]$$

$$H(t) = H^*(t_f) - \int_t^{t_f} \left.\frac{\partial H}{\partial t}\right|_* dt \qquad [2.4.16]$$

2) $K=0$ then there is no condition on $p^*(t_f)$ and equation (2.4.16) holds

3) $K=K(x)$ where K, $\frac{\partial K}{\partial x}$ and $\frac{\partial^2 K}{\partial x^2}$ are continuous, then

$$p^*(t_f) = \sum_{i=1}^{n-k} \alpha_i \left.\frac{\partial g_i}{\partial x}\right|_{x^*(t_f)} + \left.\frac{\partial K}{\partial x}\right|_{x^*(t_f)} \qquad [2.4.17]$$

where α_i are constants and equation (2.4.16) holds.

4) $K=K(x,t)$ where K, $\frac{\partial K}{\partial x}$, $\frac{\partial^2 K}{\partial x^2}$, $\frac{\partial K}{\partial t}$ and $\frac{\partial^2 K}{\partial t^2}$ are continuous, then

$$p^*(t_f) = \frac{\partial K}{\partial x}\bigg|_{*t_f} \qquad [2.4.18]$$

$$H^*(t_f) = -\frac{\partial K}{\partial t}\bigg|_{*t_f} \qquad [2.4.19]$$

$$H^*(t) = H^*(t_f) - \int_t^{t_f} \left[\frac{\partial H}{\partial t}\bigg|_* + \frac{\partial^2 K}{\partial t^2}\bigg|_* \right] dt \qquad [2.4.20]$$

5) K=0 then there is no condition on $p^*(t_f)$ and

$$H^*(t_f) = 0 \qquad [2.4.21]$$

and
$$H^*(t) = -\int_t^{t_f} \frac{\partial H}{\partial t}\bigg|_* dt \qquad [2.4.22]$$

6) K=K(x,t) where K(x,t) $\frac{\partial K}{\partial x}, \frac{\partial^2 K}{\partial x^2}, \frac{\partial K}{\partial t}$ and $\frac{\partial^2 K}{\partial t^2}$ are continuous

then

$$p^*(t_f) = \sum_{i=1}^{n-k} \alpha_i \frac{\partial g_i}{\partial x}\bigg|_* + \frac{\partial K}{\partial x}\bigg|_{*,t_f} \qquad [2.4.23]$$

and equations (2.4.19) and (2.4.20) hold.

7) K=K(x,t) where K(x,t) $\frac{\partial K}{\partial x}, \frac{\partial^2 K}{\partial x^2}, \frac{\partial K}{\partial t}$ and $\frac{\partial^2 K}{\partial t^2}$ are continuous then

$$p^*(t_f) = \sum_{i=1}^{n-k} \alpha_i \frac{\partial g_i}{\partial x}\bigg|_{*t_f} + \frac{\partial K}{\partial x}\bigg|_{*t_f} \qquad [2.4.24]$$

$$H^*(t_f) = \sum_{i=1}^{n-k} \alpha_i \frac{\partial g_i}{\partial x}\bigg|_{*,t_f} - \frac{\partial K}{\partial t}\bigg|_{*t_f} \qquad [2.4.25]$$

and equation (2.4.20) holds.

8) K=K(x,t) where K(x,t), $\frac{\partial K}{\partial x}(x,t)$, $\frac{\partial K}{\partial t}(x,t)$, $\frac{\partial^2 K}{\partial x^2}(x,t)$, $\frac{\partial^2 K}{\partial t^2}(x,t)$ and $\frac{\partial^2 K}{\partial x \partial t}(x,t)$ are continuous, then

there is no condition on $p^*(t_f)$ and

$$H^*(t_f) = p^*(t_f)^T \frac{dg}{dt}\bigg|_{*,t} \qquad [2.4.26]$$

and $$H^*(t) = p^*(t_f)^T \frac{dg}{dt}\bigg|_{*,t} - \int_t^{t_f} \frac{\partial H}{\partial t}\bigg|_* dt \qquad [2.4.27]$$

General optimal control problems will fall into one of the eight cases listed above. By no means, they will cover all the control problems, but similar conditions may be obtained. In the cases where the initial time and the initial state $x(t_0)$ are not specified as a point, initial transversality conditions can be similarily evaluated. When the control processes (2.4.1) and the cost function (2.4.5) do not explicitly depend on time, the Hamiltonian along the optimal trajectory must be zero when t_f is fixed and becomes a constant when t_f is free. From case (4) to case (8) where t_f is free, one additional condition is imposed on $H^*(t_f)$ and this locates the optimal final time.

In fact, Pontryagin's Minimum Principle represents a set of necessary conditions for optimality or more precisely, for local optimality. If there exists a control $\hat{u}(t)\varepsilon\Omega$ which satisfies all the necessary conditions as imposed, the control $\hat{u}(t)$ is said to be an extremal rather than an optimal. Note that the extremal is not necessarily the optimal. In the next section, a sufficient condition will be given, which represents a strengthening of the necessary conditions.

It has been proved [33] that for a linear system

$$\dot{x}(t)=A(t)x(t)+h(u,t) \qquad [2.4.28]$$

associated with a cost functional given by

$$J(u,t_0,t_f)=K(x(t_f),t_f) + \int_{t_0}^{t_f} \{L_1(x,t)+L_2(u,t)\} \qquad [2.4.29]$$

the extremal is also the optimal.

2.5 Sufficient Condition for Optimality:

In examing questions concerned with the theory of optimum systems, it is necessary to note the numerous works of R. Bellman, which are systematically presented in [4]. The method of "dynamic programming" developed by Bellman gives a new tool for the solution of the control problems which are closely associated with Pontryagin's minimum principle. In this section, the sufficient conditions for optimality will be stated in conjunction with Bellman's functional concept of dynamic programming.

Consider a control process in R^n

$$\dot{x}(t)=f(x(t),u(t),t) \qquad [2.5.1]$$

where the admissible controls, u(t), are all bounded and piecewise continuous function on a fixed finite time interval $T_0<t<T_f$ or $T=[T_0,T_f]$, with values in some restraint set $\Omega\varepsilon R^q$, and steering the initial state $x(t_0)$ to a target set $S\varepsilon R^n$. The cost is

$$J(u,t_0,t_f) = \int_{t_0}^{t_f} L(x,u,t)dt+K(x(t_f)) \qquad [2.5.2]$$

where f,K, and L are in class C' in all arguments.

Consider the Hamiltonian

$$H(x,p,u,t)=L(x,u,t)+f^T(x,u,t)p(t) \qquad [2.5.3]$$

Let $\hat{u}(t)$ be an extremal control with corresponding state $\hat{x}(t)$ and costate $\hat{p}(t)$ such that

$$H(\hat{x},\hat{p},\hat{u},\hat{t})= \min_{u\varepsilon\Omega} H(\hat{x},\hat{p},u,t) \qquad [2.5.4]$$

where $\hat{x}$ and $\hat{p}$ are determined from the canonical system, with the boundary conditions satisfying the transversality condition.

Assume that the control law $\hat{u}(t)$ is a feed-back one such that

$$\hat{u}(t)=\hat{u}(\hat{x},\hat{p},t) \qquad [2.5.5]$$

Substitute equation (2.5.5) into (2.5.2) and denote

$$\hat{J}(\hat{x},t_0,t_f)=J(\hat{u},t_0,t_f)=K(\hat{x}(t_f))+\int_{t_0}^{t_f} L(\hat{x},\hat{u}\ ,t)dt \qquad [2.5.6]$$

Consider a time-varying cost functional defined as

$$\hat{J}(\hat{x},t) = \int_{t}^{t_f} L(\hat{x},\hat{u}(\hat{x},\hat{p},t),t)dt \qquad [2.5.7]$$

Differentiating with respect to t, equation (2.5.7) yields

$$-L(\hat{x},\hat{u},t) = \frac{\partial \hat{J}}{\partial \hat{x}}(x,t)^{T} \dot{\hat{x}} + \frac{\partial \hat{J}}{\partial t}(x,t) \qquad [2.5.8]$$

Substituting equation (2.5.1) into (2.5.8) we have

$$\frac{\partial \hat{J}}{\partial t}(\hat{x},t)+L(\hat{x},\hat{u},t)+f^{T}(\hat{x},\hat{u},t)\frac{\partial J}{\partial \hat{x}}(\hat{x},t)=0 \qquad [2.5.9]$$

In view of the definition of Hamiltonian, equation (2.5.9) can be rewritten as

$$\frac{\partial \hat{J}}{\partial t}(\hat{x},t)+H(\hat{x},\frac{\partial \hat{J}}{\partial x}(\hat{x},t),\hat{u},t)=0 \qquad [2.5.10]$$

Equation (2.5.10) is known as the Hamiltonian-Jacobi-Bellman equation which is a partial differential equation for the function $\hat{J}(\hat{x},t)$. The boundary condition is

$$\hat{J}(\hat{x},t_f)=K(\hat{x}(t_f)) \quad \text{for } \hat{x}(t_f)\varepsilon S \qquad [2.5.11]$$

Assume that there exists a feed-back control law $\hat{u}$ and let $\hat{J}(\hat{x},t)$ be the solution of the Hamiltonian-Jacobi-Bellman equation with the boundary condition (2.5.10). Assume also that $(\hat{x},\hat{u})$ is an admissible arc with $\hat{x}(t_f)\varepsilon S$ and $u(x,t) = \hat{u}(\hat{x},\frac{\partial \hat{J}}{\partial \hat{x}}(\hat{x},t),t)$, then the control $\hat{u}$ is optimal with optimal trajectory $\hat{x}$ and with cost $\hat{J}^*(\hat{u},t_0,t_f)= \hat{J}(\hat{x}(t_0)t_0)$.

Combining equations (2.5.4) and (2.5.10), we have

$$\frac{\partial \hat{J}(\hat{x},t)}{\partial t} + \min_{u\varepsilon\Omega} \{L(\hat{x},\hat{u},t) + \frac{\partial \hat{J}}{\partial \hat{x}}(\hat{x},t)^{T} f(\hat{x},u,t)\}= 0$$

[2.4.12]

Upon discretizing equation (2.5.11), a recursive equation can be obtained. Hence the optimal cost trajectory can be approximated through the use of high speed computer.

The Hamilton -Jacobi-Bellman equations represent a requirement on the behavior of the cost. Analytically, the equation is often quite difficult to solve if not impossible. Hence the equation is most often used as a check on the optimality of a control derived from the necessary conditions as stated by Pontryagin's Minimum Principle.

CHAPTER III

SENSITIVITIES OF OPTIMAL CONTROL PROCESSES

3.1 Introduction:

When a given input is applied to a given plant, the output of the plant does not necessarily agree closely with the value that is expected. The expected output is obtained from the knowledge of the input-output relationship of the plant and is rather a theoretical value. Excluding the measurement errors that may be involved, the discrepancies of the experimental output from the theoretical can be accounted for by two categories of disturbances to the plant. The external disturbances to the plant are generally regarded as the noise and the internal perturbation is regarded as the plant parameter variation. The noise is considered as an additional input but discussion of its effects is beyond the scope of the present work.

The plant parameter is regarded as the independent variable upon which some plant arguments depend. The plant arguments are determined by the interest of the designer; they may be eigenvalues of the plant, for the cost or the terminal state. Sensitivity in the gross sense is defined as the change of the dependent argument due to the change of the plant parameter. Corresponding to various plant arguments, various sensitivities are defined. The purpose of this section is to introduce the definitions of various sensitivities and their measures.

3.2 The Physical and Mathematical Aspects of the Plant Parameters:

The physical plants in engineering differ widely in forms. In spite of the specific differences, a large class of engineering systems is described by a mathematical model. Certain differences are expected between the physical systems and its mathematical models. Generally, the correspondence between the mathematical model and its physical system is quite satisfactory; however, this is not always the case. To account for the discrepancies, a plant parameter w is included in the mathematical model as

$$\dot{x}(t,w) = f(x,u,t,w) \qquad [3.2.1]$$

and the physical plant is factitiously represented by

$$\dot{x}(t) = f(x,u,t) \qquad [3.2.2]$$

where x is the state trajectory and u is the control. The plant parameter is included due to the uncertainty involved during the process of obtaining the mathematical model through identification. If the plant is scrutinized in more detail, it is possible to determine the system function f in equation (3.2.1) as a function for the parameter w. Hence there exists a value w_0 such that the mathematical model

$$\dot{x}(t,w_0) = f(x,u,t,w_0) \qquad [3.2.3]$$

agrees closely enough with the physical model of equation (3.2.2). And the drifts of the plant parameter from w_0 to w will then take account of any uncertainties in the identification process. The mathematical model (3.2.3) is called the nominal plant and the physical plant (3.2.2) is represented by the mathematical model of equation (3.2.1) with the parameter w in the neighborhood of its nominal value w_0.

In some cases, the physical plant may have an exact mathematical model but the plant parameter is introduced when the elements of the physical plant are sensitive to environmental conditions. Some of the components of the physical system may be sensitive to temperature, humidity etc. Nominally, the physical system is assumed to be operated under certain temperature and humidity.

If the physical plant is constructed such that the character of each of its components can be evaluated, the corresponding mathematical model can be directly obtained. This procedure of analysis is always employed especially in passive network. It is well known that the labelled value of the components of the physical plant cannot be exact and in usual practice, the tolerances of the components are given by the manufacturer. In this case, the need of inserting the plant parameter is obvious and the way of obtaining the corresponding mathematical model

(3.2.2) is also easy.

From the physical point of view, the plant parameter arises from uncertainties in indentification, from environmental effect or from the inaccurate values of the components. Many other factors may be involved depending on the specific plant.

From the mathematical point of view, the plant parameters are divided into two categories according to the fashions that the plant parameter changes. Stochastic plant parameter [13] varies in an unpredictable fashion and and the magnitude of the plant parameter cannot be estimated at any time. On the contrary, a deterministic plant parameter is predictable at any time. In the present work, the discussion of the stochastic plant parameter will be ignored.

It is impractical to restrict the plant parameter to be a scalar. In many cases, there is more than one independent plant parameter in the system. And the plant parameter is considered as a vector quantity. The plant with a vector plant parameter will be called the multi-parameter system.

The deterministic plant parameters are further classified into two categories. Consider a plant parameter w in the time interval of $[T_0, T_f]$ and assume that the plant is operating in the time interval of $[t_0, t_f]$ where $[t_0, t_f]$

is a small line segment in $[T_0,T_f]$, i.e. $T_0<<t_0<t_f<<T_f$. If the plant parameter takes the value of w in the time interval of $[t_0,t_f]$ and has a negligible change around w in the whole interval of $[t_0,t_f]$, then the plant parameter is considered as constant in $[t_0,t_f]$. By a constant plant parameter, it is meant that the plant parameter takes a constant value in $[t_0,t_f]$ but may be time dependent on a large interval $[T_0,T_f]$. However, if the change of the plant parameter around a certain value is large enough not to be neglected in the interval of $[t_0,t_f]$, then the plant parameter is considered as time dependent. In investigating the physical plants due to the variation of plant parameters, incremental change of the time-varying plant parameter is considered while the change in constant plant parameter is assumed to be differential. In many aspects, the approach in either case is similar and discussion on the plants with time-varying plant parameter will not be exclusively made. Whenever required, important differences between time-varying and constant plant parameters will be emphasized.

Various problems involved in the physical plant yield various mathematical forms of the plant parameters. In general, the representation of the physical plant by equation (3.2.1) together with the nominal system of equation (3.2.3) describes more fully than equation (3.2.3).

Mathematical solution of equation (3.2.3) is possible for any given control and boundary conditions. In the analysis of practical systems, along with the obtaining of solutions, it is extremely important to have a knowledge of the variations of the solution with respect to plant parameters. Sensitivity analysis represents a further connection between the mathematical model and the physical system, and enables the engineer to apply the results from analyzing equations to physical systems with far greater dependability.

3.3 Sensitivities for Control Systems:

The idea of sensitivity was introduced by Bode in one of his fundemental works[6]. In a feedback circuit, the sensitivity S_{θ}^{T} for a system argument T is given by

$$S_{\theta}^{T} = \frac{\partial \ln \theta}{\partial \ln T} \qquad [3.3.1]$$

where θ is the gain through the complete system. The definition was modified and extended by Horowitz, [20] Truxal [48] and Mason [35] as

$$S_{\theta}^{T} = \frac{\partial \ln T}{\partial \ln \theta} \qquad [3.3.2]$$

where T is any system argument of interest and θ is any system variable. Equation (3.3.2) can be rewritten as

$$S_{\theta}^{T}=\frac{\partial T/T}{\partial \theta/\theta}$$ [3.3.3]

In words, the sensitivity of T with respect to θ is the percentage change in θ which causes the change in T. All changes are restricted to be differentially small.

Corresponding to equation (3.3.2), the sensitivity for argument T may be defined, alternatively, as

$$S_{\theta}^{T}=\frac{\partial T}{\partial \ln\theta}$$ [3.3.4]

It was used by Ur [49] in analyzing the locus of the closed loop root with respect to the variation in the open loop parameter.

Among several definitions of sensitivity, equation (3.3.3) is generally used; however its disadvantage is also well known. Whenever the system argument T or the parameter θ is a vector, the definition must be modified. Goldstein and Kuo [15] extended Mason's [35] definition of single parameter sensitivity significantly to the multiparameter case. Hakimi and Cruz [17] constructed some sensitivity measures with multiple parameter variations and Lee [34] introduced the concept of sensitivity group. All the works attempt to give a reasonable sensitivity measure for a scalar system argument T with multiparameter variations. Even so, the multiparameter sensitivity appears to be quite complicated and the extension to a vector system argument T is difficult.

Instead of considering sensitivity, Cruz and Perkins [7, 38] constructed the sensitivity matrix and its corresponding measure. It was modified by Kriendler and sensitivity was defined as

$$S_\theta^T = \frac{\partial T}{\partial \theta} \qquad [3.3.5]$$

Equations (3.3.3) and (3.3.5) are similar in form , but the concepts are different. Equation (3.3.3) expresses in terms of percent change. If the percent change in T is large while the percent change in θ is small, the sensitivity in equation (3.3.3) will be large. Therefore it can be concluded that the argument T is very sensitive to the parameter θ. Conclusion of this kind cannot be drawn by using definition (3.3.5). The change in T is small in percentage but may be large relative to the change in θ. Consequently, sensitivity from equation (3.3.5) is very large even if the argument T is not sensitive to the parameter θ. Sensitivity by equation (3.3.5) does not imply anything and the sensitivity measure must be developed. Basically, in Cruz and Perkins' approach, both the sensitivity and its measure are constructed for the purpose of comparison.

3.4 Sensitivities for Optimal Control Systems:

For the automatic systems, major analytical work was done in the frequency domain; therefore the sensitivities are functions of the complex frequency. For optimal control systems, all sensitivities are defined in time domain. The cost or the performance sensitivity was defined by Dorato [11] in 1963 and the terminal state sensitivity by Holtzman and Horing [19] in 1965. Kriendler [29] has elaborated a precise definition and the important implications for the term "state sensitivity". Various authors have been investigating sensitivity without carefully specifying the sensitivity they are referring to. Thus the term sensitivity has become more confusing than ever. This section is devoted to clarify the terms and to establish suitable measures for the comparison.

In one of the first works on the sensitivities of optimal control, Dorato suggested a definition of sensitivity for the cost functional (3.4.1).

$$J = J(w,t_0,t_f) = \int_{t_0}^{t_f} L(x(t,w),u(t,w),t)dt \qquad [3.4.1]$$

due to the change of the plant parameter w. The control process is given as

$$\dot{x}(t,w) = f(x(t,w),u(t,w),t,w) \qquad [3.4.2]$$

where x(t,w) is an nxl state vector

$u(t,w)$ is a qxl control vector

w is an rxl plant parameter vector.

At nominal, i.e. $w = w_0$, the cost functional has the value of $J(w_0,t_0,t_f)$ which is a minimum if the nominal control $u(t,w_0)$ is optimal. The variation of the cost functional due to a small change of the plant $\delta w, \delta w = w-w_0$ is

$$\Delta J(w) = J(w,t_0,t_f) - J(w_0,t_0,t_f) \qquad (3.4.3)$$

Expanding $J(w,t_0,t_f)$ around w_0 by Taylor series, it is seen that

$$J(w,t_0,t_f) = J(w_0,t_0,t_f) + \left.\frac{\partial J(w,t_0,t_f)}{\partial w}\right|_{w=w_0}^{T} \delta w + \frac{1}{2!}\,\delta w^T \left.\frac{\partial^2 J(w,t_0,t_f)}{\partial w^2}\right|_{w=w_0} \delta w + \dots \qquad (3.4.4)$$

Combining equations (3.4.3) and (3.4.4), the incremental change of the cost functional is expressed as

$$\Delta J(w) = \left.\frac{\partial J(w,t_0,t_f)}{\partial w}\right|_{w_0}^{T} \delta w + \frac{1}{2}\,\delta w^T \left[\frac{\partial^2 J(w,t_0,t_f)}{\partial w^2}\right]_{w_0} \delta w + \dots \qquad (3.4.5)$$

Denote

$$\delta J(w) = \frac{\partial J(w,t_0,t_f)}{\partial w}\Bigg|_{w_0}^{T} \delta w \qquad (3.4.6)$$

and $$\delta^2 J(w) = \frac{1}{2}\delta w^T \frac{\partial J(w,t_0,t_f)}{\partial w^2}\Bigg|_{w_0} \delta w \qquad (3.4.7)$$

Hence $\Delta J(w)$ becomes

$$\Delta J(w) = \delta J(w) + \delta^2 J(w) + \ldots\ldots \qquad (3.4.8)$$

For an infinitesimal change of w around w_0, or $\delta w \to 0$ the higher order variations are negligible. Or when the cost functional $J(w,t_0 t_f)$ has a Frechet derivative , the change of the cost functional is approximated by

$$\Delta J(w) \doteq \delta J(w) = \frac{\partial J(w\ ,t_0,t_f)}{\partial w}\Bigg|_{w_0}^{T} \delta w \qquad (3.4.9)$$

The cost sensitivity S_w^J is therefore defined as

$$S_w^J = \frac{\partial J(w,t_0,t_f)}{\partial w}\Bigg|_{w_0} \qquad (3.4.10)$$

In words, the cost sensitivity is the first partial derivative of the cost functional with respect to the plant parameter w at its nominal value w_0. It is important to note that the cost sensitivity indicates a meaningful relationship with the change of the cost functional only if w is close enough to w_0 or the cost functional has a

Frechet derivative. Without this hidden assumption, the cost sensitivity is not sufficient to give enough information about the change of the cost functional and confusion may arise.

By cost insensitivity, it is meant that

$$S_w^J = 0 \tag{3.4.11}$$

The term "insensitivity" must not be carried too far literally. By equation (3.4.9), cost insensitivity implies

$$\Delta J(w) \doteq 0 \tag{3.4.12}$$

Or in words, the change of cost functional is <u>approximately</u> equal to zero. As the higher order terms are neglected, $\Delta J(w)$ can not be identically zero even if cost insensitivity is achieved. But as far as first order approximation is concerned, the cost insensitivity is the most ideal case.

In the multiparameter cases, the cost sensitivity is a rx1 constant vector in R^r. If the implementation of the optimal control at nominal is not unique, several cost sensitivity vectors are obtained. For the purpose of comparing different implementations in the light of cost sensitivity, a scalar measure must be established. This measure will be called the cost sensitivity measure denoted by SM_w^J. Dorato [11] has proposed a cost sensitivity measure given by

$$SM_w^J = \| S_w^J \|^2 \qquad (3.4.13)$$

which is simply the squared norm of the cost sensitivity. When the plant parameter is a scalar, i.e. $r = 1$, the squared norm of the cost sensitivity becomes the square of the cost sensitivity.

The adopted definition for cost sensitivity suffers one major disadvantage, that is the change of the plant parameter w is small enough around w_0 or the cost functional has a Frechet derivative. Realizing this, Sinha and Atluri [3, 45, 46] abandoned the definition by Dorato and proposed another definition for cost sensitivity as

$$S_w^J = \frac{\Delta J(w)}{\Delta w} \qquad (3.4.14)$$

where Δw is the change of the plant parameter. Realistically, Sinha and Atluri's definition is more useful than Dorato's. However, two major difficulties may be introduced by definition (3.4.14). As $\Delta J(w)$ in equation (3.4.14) is given by equation (3.4.8), to develop some analytical resultsis very difficult if not impossible. Moreover, in the multiparameter case, definition (3.4.14) must be modified. Whatever modification will be made, the work to obtain the modified cost sensitivity will be very laborious.

The cost sensitivity in equation (3.4.10) is regarded as absolute because it is defined basically as the difference between the cost at w and that at w_0. Another definition of cost sensitivity that is of comparative nature was introduced by Rohrer and Sobral [42]. For a given plant parameter w, the relative cost sensitivity for the control u(t,w) is defined as the difference between the actual value of cost and that which would be obtained, if the control u(t,w) were optimal with respect to the plant parameter at w. The reason for this definition goes back to the philosophy of optimality with respect to a given cost functional. The control $u(t,w_0)$ is determined such that the cost functional (3.4.1) is minimized. If the implementation of $u(t,w_0)$ is not unique, there may exist an implementation such that the control u(t,w) is also optimal with respect to the cost functional (3.4.1) at w_0. According to the definition of comparative cost sensitivity, cost sensitivity reduction by implementation is to find an implementation u(t,w) such that $u(t,w_0)$ is optimal at w_0 and such that the value of the cost functional corresponding to u(t,w) is close to the minimum one at w. This is an interesting problem but it will not be considered here in the present work as the plant at w_0

is considered much more important than the plant at w. In general, the optimal (minimum) values of the cost function at both w_0 and w are very close to each other. Hence although the concepts of absolute and relative cost sensitivities are different, the results are about the same.

For clarification, the cost sensitivity is defined according to equation (3.4.10) and the measure is given by equation (3.4.13). Equation (3.4.11) is considered as the cost insensitive condition.

In many practical cases, the main interest of the control system engineer is centered upon the system's response or trajectory. Therefore, the deviation of the optimal state trajectory in the presence of the plant parameter variations is of great interest.

Let x(t,w) be the state trajectory at w and $x^*(t)$ be the nominal trajectory. Again by Taylor series expansion, x (t,w) can be written as

$$x(t,w) = x^*(t) + \left.\frac{\partial x(t,w)}{\partial w}\right|_{w_0}^{T} \delta w + \ldots \quad (3.4.15)$$

The error vector e(t) is therefore given by

$$e(t) = x(t,w) - x^*(t) = \left.\frac{\partial x(t,w)}{\partial w}\right|_{w_0}^{T} \delta w + \ldots.. \quad (3.4.16)$$

A measure for the error vector, EM_J^x , was first formulated by Cruz and Perkins [7] as

$$EM_w^x = \int_{t_0}^{t_f} ||e(t)||^2 \, dt \qquad (3.4.17)$$

The important assumption that the state trajectory has a Frechet derivative was made by Kriendler [28]. And the state sensitivity y(t) was defined as

$$y(t) = \left. \frac{\partial x(t,w)}{\partial w} \right|_{w_0} \qquad (3.4.18)$$

Note that $y(t)\delta w \doteq e(t)$ when the assumption is inserted; and the measure for error vector EM_w^x becomes as

$$EM_w^x = \int_{t_0}^{t_f} \delta w^T y^T(t) y(t) \delta w dt \qquad (3.4.19)$$

In comparing different implementations of the nominally optimal control, the change of the plant parameter δw is assumed to be identical for every possible implementation. Therefore the term δw appears in equation (3.4.19) is redundant and Kriendler [29] defined a measure for the state sensitivity y(t) as

$$SM_w^x = \int_{t_0}^{t_f} y^T(t) y(t) dt \qquad (3.4.20)$$

In the multi-parameter case, y(t) is an nxr matrix function of time. Hence the state sensitivity measure of equation (3.4.20) must be further modified. It is suggested here that the norm square of the matrix y(t) be the integrand replacing $y^T(t)y(t)$, or rewritten as

$$SM_{\bar{w}}^{x} = \int_{t_0}^{t_f} ||y(t)||^2 \, dt \qquad (3.4.21)$$

The sensitivity measure given by equation (3.4.21) is known as the integral state sensitivity.

It is important to note that the works by Cruz and Perkins and that by Kriendler are very closely related to each other. However some subtle differences between them must be heeded otherwise confusion may arise. [31, 47].

For clarification, the sensitivity is defined by equation (3.4.18) and its corresponding measure by (3.4.21). The problem of reducing state sensitivity by implementation is to find an implementation for a control such that it is optimal at nominal with respect to a given cost functional and such that the integral state sensitivity is relatively reduced.

In the area of sensitivity analysis in optimal control systems, the sensitivity of terminal condition is less emphasized. It has been shown in Chapter II that various terminal conditions could have been assumed by different control processes. In the case where $x(t_f)$ is free, the investigation of the terminal state sensitivity is rather redundant.

Denote $x^*(t)$ the optimal state trajectory at nominal and $x(t,w)$ the state trajectory in the presence of plant parameter variation. The terminal error vector is

$$e(t_f) = x(t_f,w) - x^*(t_f) = \left.\frac{\partial x(t,w)}{\partial w}\right|_{w_0,t_f}^{T} \delta w \quad + \ldots \tag{3.4.22}$$

Assume that the state trajectory has a Frechet derivative, then

$$e(t_f) \doteq \left.\frac{\partial x(t,w)}{\partial w}\right|_{w_0,t_f}^{T} \delta w \tag{3.4.23}$$

Again, in comparing different implementations, δw is identical for all possible implementations. Hence the terminal state sensitivity, or terminal sensitivity, is defined by Horing and Holtzman[19] as

$$S_w^{t_f} = \left.\frac{\partial x(t,w)}{\partial w}\right|_{w_0,t_f} = y(t_f) \tag{3.4.24}$$

In the multi-parameter case, the terminal sensitivity $S_w^{t_f}$ is an nxr constant matrix; the norm square of that matrix is then a suitable measure for terminal sensitivity or

$$SM_w^{t_f} = \| S_w^{t_f} \|^2 = \| y(t_f) \|^2 \qquad (3.4.25)$$

By terminal insensitivity, it is meant that

$$y(t_f) = 0 \qquad (3.4.26)$$

in which case, the terminal error vector $e(t_f)$ is approximately a null vector. As far as first order approximation is concerned, the terminal insensitivity is an ideal case.

Corresponding to the terminal sensitivity, the initial sensitivity is denoted by $y(t_0)$. Without loss of generality, the initial state for an optimal control process is a given point $x(t_0)$ with the initial time t_0 specified. In general, the initial point $x(t_0)$ can be set very accurately and is independent of the plant parameter. Hence, the initial insensitivity that

$$y(t_0) = 0 \qquad (3.4.27)$$

is always assumed.

By the sensitivity alone, it can be the cost sensitivity, the state sensitivity or the terminal sensitivity. To reduce the sensitivity by implementation is the problem of finding an implementation for the control such that the control is nominally optimal with respect to a given cost functional and such that the sensitivity measure is reduced relative to other possible implementations.

CHAPTER IV

REDUCTION OF SENSITIVITIES BY IMPLEMENTATION

4.1 Introduction:

As discussed in Chapter II, the optimal control input to a nominal plant can be obtained by current techniques available. But the way of achieving this determined optimal control, or, the implementation of the nominally optimal control is not restricted by any means. Generally three types of implementation schemes are possible. Open- and closed-loop implementations have been extensively studied while the implementation by two degrees of freedom does not appear to have been studied in the field of optimal control theory.

As will be shown later, the configuration of two degrees of freedom provides the designer with some flexibility, and because of this feature, it is possible to realize the reduction in the sensitivity of interest.

4.2 State Sensitivity Equation:

The term "state sensitivity equation", or "sensitivity equation" in a shorter form, was introduced by Kokotovic and Rutman [25, 43]. However, before putting forward the state sensitivity equation into the plant under consideration, certain modifications must be made and certain assumptions must be clarified.

Consider a plant (4.2.1)

$$\dot{x}(t,w) = f(x,u,t,w) \qquad (4.2.1)$$

where w is the plant parameter with nominal value at w_0. Denote $x^*(t)$ and $u^*(t)$ as the optimal trajectory and control for the nominal plant (4.2.2), respectively.

$$\dot{x}(t,w_0) = f(x,u,t,w_0) \qquad (4.2.2)$$

associated with a given cost $J(w,t_0,t_f)$. Define an implementation vector v(t) as the partial derivative of the control u(t,w) with respect to the plant parameter w as

$$v(t) = \left.\frac{\partial u(t,w)}{\partial w}\right|_{w = w_0} \qquad (4.2.3)$$

When the plant parameter drifts away from w_0 to w, the state trajectory will also deviate from the nominally optimal, i.e. $x^*(t)$. Partially differentiating equation (4.2.1) with respect to w, we get

$$\frac{\partial \dot{x}(t,w)}{\partial w} = \frac{\partial f}{\partial w} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial w} \qquad (4.2.4)$$

Setting the above equation at the nominal value, we have

$$\frac{\partial}{\partial w}\left(\frac{d\,x(t,w)}{dt}\right)\bigg|_{w=w_0} = \frac{\partial f}{\partial w}\bigg|_{w=w_0} + \frac{\partial f}{\partial x}\bigg|_{w=w_0}\frac{\partial x}{\partial w}\bigg|_{w=w_0}$$

$$+ \frac{\partial f}{\partial u}\bigg|_{w=w_0}\frac{\partial u}{\partial w}\bigg|_{w=w_0} \qquad (4.2.5)$$

Combining equations (4.2.3) and (3.4.18), equation (4.2.5) may be written as

$$\frac{\partial}{\partial w}\left(\frac{dx(t,w)}{dt}\right)\bigg|_{w_0} = \frac{\partial f}{\partial x}\bigg|_{w_0} y + \frac{\partial f}{\partial u}\bigg|_{w_0} v + \frac{\partial f}{\partial w}\bigg|_{w_0} \qquad (4.2.6)$$

where

$\frac{\partial f}{\partial w}\Big|_{w_0} = \frac{\partial f}{\partial w}\Big|_{w=w_0}$ and so forth. If the plant parameter w is time-invariant, the following equation holds.

$$\frac{\partial}{\partial w}\left(\frac{dx(t,w)}{dt}\right)\bigg|_{w_0} = \frac{d}{dt}\left(\frac{\partial x(t,w)}{\partial w}\bigg|_{w_0}\right) = \frac{dy(t)}{dt} = \dot{y}(t) \qquad (4.2.7)$$

Hence equation (4.2.6) can be rewritten as,

$$\dot{y}(t) = \frac{\partial f}{\partial x}\bigg|_{w_0} y + \frac{\partial f}{\partial u}\bigg|_{w_0} v + \frac{\partial f}{\partial w}\bigg|_{w_0} \qquad (4.2.8)$$

It is known that the state sensitivity depends on the implementation of the nominally optimal control and this is assured by equation (4.2.8). Since the relationship between y(t) and v(t) is linked by equation (4.2.8); it is therefore called the state sensitivity equation for the implementation v(t).

The functions $\left.\frac{\partial f}{\partial x}\right|_{w_0}$ and $\left.\frac{\partial f}{\partial u}\right|_{w_0}$ are known functions of $x^*(t)$ and $u^*(t)$. In order to evaluate the state sensitivity y(t) for any given v(t), the function $\left.\frac{\partial f}{\partial w}\right|_{w_0}$ must be given. In other words, it must be assumed that the plant (4.2.1) is known not only at the nominal value, which is equation (4.2.2) but also known as a function of the plant parameter, i.e. equation (4.2.1). This assumption is not restrictive at all, as the role of plant parameter can be approximated for most of the practical plants. In some cases where $\frac{\partial f}{\partial w}$ cannot be obtained, the approach will be different and will be discussed in a later section.

As a priori condition to the meaningfulness of the state sensitivity equation, the nominally optimal trajectory and control must be known. Hence the functions $\left.\frac{\partial f}{\partial x}\right|_{w_0}$ and $\left.\frac{\partial f}{\partial u}\right|_{w_0}$ can be regarded as known functions of time.

It must be emphasized that equation (4.2.7) holds only for time-invariant plant parameter. Consequently, the state sensitivity equation (4.2.8) is not valid for time-varying plant parameter. A new sensitivity function may be defined in terms of the first order change in the trajectory due to the variations in the time-varying parameters. This first order dispersion is given by the equation

$$\delta\dot{x} = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial w}\delta w \qquad (4.2.9)$$

where $\delta x \doteq x(t,w) - x(t,w_0)$, $\delta u = u(t,w) - u(t,w_0)$ and $\delta w = w-w_0$ for suitably small δw. In this case, one more assumption must be added, that is, the value of the plant parameter is given so that δw is known. With suitable modifications of the sensitivity measure, the development for time-varying and time-invariant plant parameter is very similar; and exclusive discussion on the time-varying plant parameter will be skipped.

4.3 Open- and Closed-Loop Implementations:

From current technique, the nominally optimal control $u^*(t)$ is obtained as a function of time. Corresponding to $u^*(t)$, there exists a physical system which realizes the signal $u^*(t)$. The direct feeding of the signal $u^*(t)$ into the plant is called the open-loop implementation as illustrated in Figure (1). When the plant parameter w drifts away from the nominal value, the state trajectory will change from $x^*(t)$ to $x(t,w)$

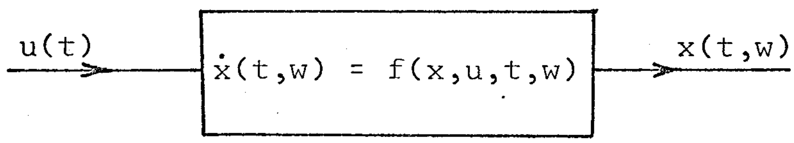


Figure 1.

Open-Loop Implementation

while the open-loop control $u^*(t)$ remains unchanged. Mathematically, it is expressed as

$$v_0(t) = 0 \qquad (4.3.1)$$

where $v_0(t)$ is the open-loop implementation vector as derived by equation (4.2.3). Substituting equation (4.3.1) into equation (4.2.8), the state sensitivity equation for open-loop implementation is

$$y_0(t) = \left.\frac{\partial f}{\partial x}\right|_{w_0} y_0 + \left.\frac{\partial f}{\partial w}\right|_{w_0} \qquad (4.3.2)$$

where $y_0(t)$ is the state sensitivity for open-loop implementation . If the initial condition that $y_0(t_0) = 0$ is assumed, then the state sensitivity $y_0(t)$ is determined from equation (4.3.2). If $u^*(t)$ is uniquely determined, then so is $y_0(t)$. Hence no flexibility can be obtained from the open-loop implementation.

Implementation by combining the state variables in a proper fashion and feeding the resultant into the plant is called the closed-loop implementation. Linear or non-linear feedbacks are possible. Generally, non-linear feedback control is written as

$$u(t,w) = h(t,x(t,w)) \qquad (4.3.3)$$

and linear feedback is

$$u(t,w) = H(t)x(t,w) \qquad (4.3.4)$$

Non-linear feedback is rather complicated, and little knowledge about it is available in the current field of optimal control theory. Hence it is excluded for the following discussion. Even for the linear feedback implementation, its existence is not guaranteed for most of the optimal systems. Figure (2) illustrates the closed-loop implementation with linear feedback.

When the plant parameter deviates from its nominal value, so does the state trajectory x(t,w). However the state variables are fed back as the input to the plant and may or may not regulate the plant in a desirable fashion. The implementation vector for the closed-loop implementation with linear feedback is

$$v_c(t) = H(t)y_c(t) \qquad (4.3.5)$$

where $y_c(t)$ is the state sensitivity for the closed-loop implementation with linear feedback. Combining equations (4.3.5) and (4.2.8), the state sensitivity equation for closed-loop implementation is

$$\dot{y}_c(t) = \left.\frac{\partial f}{\partial x}\right|_{w_0} y_c + \left.\frac{\partial f}{\partial u}\right|_{w_0} Hy_c + \left.\frac{\partial f}{\partial w}\right|_{w_0} \qquad (4.3.6)$$

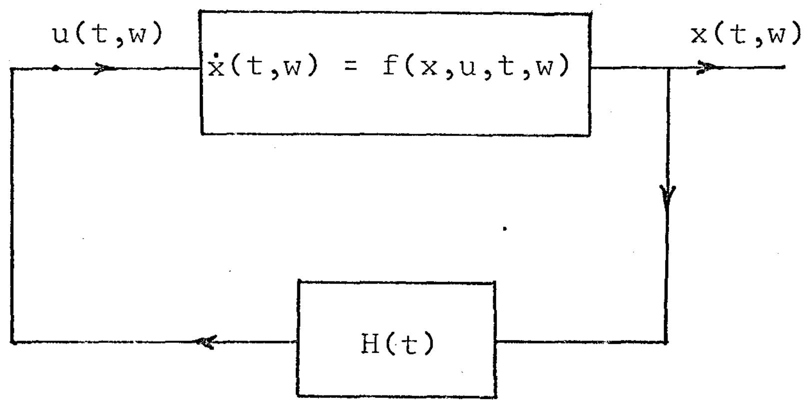


Figure 2.

Closed-Loop Implementation

In some cases, the feedback block H(t) is uniquely determined. With the initial condition that $y_c(t_0) = 0$, the closed-loop state sensitivity $y_c(t)$ is uniquely determined from equation (4.3.6). Hence both the existence of the closed-loop implementation and the flexibility from the configuration cannot be guaranteed, except for the case of linear systems with quadratic cost functionals.

4.4 The TDF (Two-degree of freedom) Implementation:

The application of the configuration of two degrees of freedom was demonstrated by Horowitz [20] in reducing the pole sensitivity of an automatic control system. In optimal control theory, current research has been involved with the comparison of open - and closed-loop implementations of a nominally optimal control. As shown in the previous section, the implementations by open- or closed-loop do not yield any flexibility for the designer to reduce the sensitivity of his interest. Here, the implementation by two degrees of freedom is introduced to accomplish this purpose.

The configuration of two degrees of freedom is a combination of the open - and closed-loop schemes; hence it is better called "the TDF configuration". As illustrated in Figure (3), the TDF implementation consists of two parts, the open loop portion g(t) and the feedback

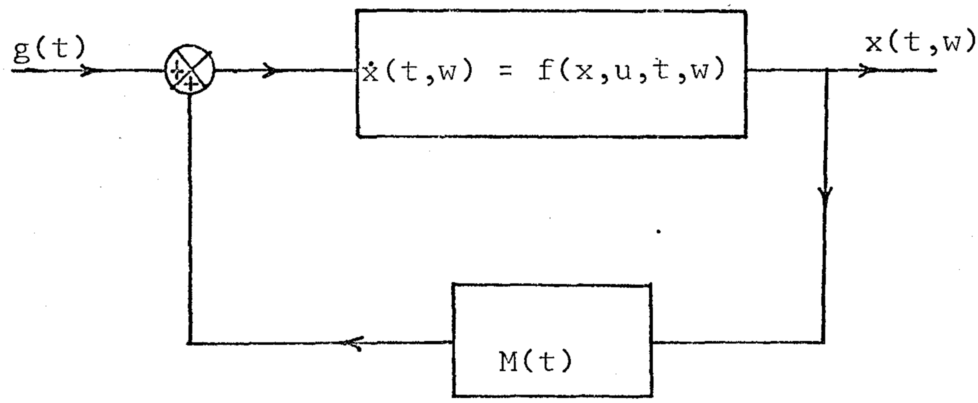


Figure 3.

TDF Implementation

portion M(t). At time t and value of w; the control u(t,w) is mathematically expressed as

$$u(t,w) = M(t)x(t,w) + g(t) \qquad (4.4.1)$$

where x(t,w) is the state trajectory corresponding to the control u(t,w). Here only linear feedback in x(t,w) is considered. Non-linear feedback is also possible; but it will not be considered here due to the complexity. It is understood that the state variables must be accessible for the designer, otherwise, the TDF implementation will have no practical meaning at all. Moreover, it is noted that both the open- and closed-loop portions are dependent only on time. Generally, three different types of the TDF implementation are possible,

$$u_2(t,w) = M(t,w)x(t,w) + g(t) \qquad (4.4.2)$$

$$u_3(t,w) = M(t)x(t,w) + g(t,w) \qquad (4.4.3)$$

$$u_4(t,w) = M(t,w)x(t,w) + g(t,w) \qquad (4.4.4)$$

with the corresponding implementation vectors as,

$$v_2(t) = M_W x^*(t) + M(t,w_0)y_2(t) \qquad (4.4.5)$$

$$v_3(t) = M(t,w_0)y_3(t) + g_W \qquad (4.4.6)$$

$$v_4(t) = M_W x^*(t) + M(t,w_0)y_4(t) + g_W \qquad (4.4.7)$$

where $M_w(t) = M_w = \left.\frac{\partial M}{\partial w}\right|_{w_0}$ and $g_w = \left.\frac{\partial g}{\partial w}\right|_{w_0} = g_w(t)$.

The plant parameter dependence of M and g may take two forms. If M and g are assumed to be implicit functions of the plant parameter, then the plant parameter must be physically available. This requirement appears to be impractical for most physical plants. Hence M and g can only be explicit functions of w; however, two problems may arise. The choice of the physical blocks of the plant on which M and g depend is rather arbitrary but important and there is no logical or systematic approach available. Hence the method of trial and error must be resumed. The physical block of the plant upon which M and g must depend may not be reconstructed. Foreseeing the difficulties involved, the investigation of the implementation schemes of equation (4.4.2) to (4.4.4) will be abandoned. Hence, by the TDF implementation, it is meant that the control is given by equation (4.4.1) of which the conditions that

$$\frac{\partial M(t)}{\partial w} = 0 \qquad (4.4.8)$$

$$\frac{\partial g(t)}{\partial w} = 0 \qquad (4.4.9)$$

must be satisfied by any pair of [M(t),g(t)].

Nominally, the control in equation (4.4.1) takes the value of

$$u(t,w_0) = M(t)x(t,w_0) + g(t) \qquad (4.4.10)$$

Now the control $u(t,w_0)$ must be optimal with respect to the nominal plant associated with a given cost functional and the trajectory $x(t,w_0)$ must be the nominally optimal state trajectory, or mathematically

$$u(t,w_0) = u^*(t) \qquad (4.4.11)$$

$$x(t,w_0) = x^*(t) \qquad (4.4.12)$$

Hence equation (4.4.10) yields

$$u^*(t) = M(t)x^*(t) + g(t) \qquad (4.4.13)$$

Equation (4.4.13) is the optimal condition at nominal and must be satisfied by any pair of [M(t),g(t)]. This condition relating M(t) and g(t) hence defines a set of [M(t),g(t)] in $R^{nxq} \times R^q \times [t_0,t_f]$ and this set will be called the admissible implementation set. Every element in the admissible implementation set must satisfy equation (4.4.13).

In the extreme cases, the element [0,g(t)] in the admissible implementation set denotes the open-loop implementation and the element [M(t),0] is the closed-loop implementation. Obviously, the admissible implementation set includes these two special elements; however, the elements in the set are far beyond exhausted. In other words, besides open- and closed-loop implementations, there are numerous TDF implementations to realize the nominally optimal control. For every given M(t), there corresponds a unique g(t) so that {M(t),g(t)} is in the set. However, the existence of an M(t) corresponding to a given g(t) is not guaranteed by equation (4.4.13). And even if the M(t) does exist, it is generally non-unique. Because of this non-bilateral relationship between M(t) and g(t), the application of the TDF implementation to an optimal control system, depends on the manipulation of the feedback portion M(t) so that certain specifications are met.

The implementation vector v(t) for the implementation equation (4.4.1) is given by

$$v(t) = M(t)y(t) \qquad (4.4.14)$$

where y(t) is the state sensitivity corresponding to u(t) in equation (4.4.1). Combining equation (4.4.14) and (4.2.8), the state sensitivity equation for the TDF implementation is

$$\dot{y}(t) = \left(\frac{\partial f}{\partial x}\bigg|_{w_0} + \frac{\partial f}{\partial u}\bigg|_{w_0} M\right)y(t) + \frac{\partial f}{\partial w}\bigg|_{w_0} \qquad (4.4.15)$$

With the assumption that $y(t_0) = 0$, the state sensitivity y(t) is determined for any given M(t). The time function $\left.\frac{\partial f}{\partial w}\right|_{w_0}$ is considered as the driving function in the state sensitivity equation (4.4.15). The feedback portion M(t) is not determined yet; it may be a function of y(t); hence equation (4.4.15) is not necessarily a linear differential equation. It will be shown later that M(t) is chosen in such a manner that various sensitivities will be reduced. Also, it is noted that the state sensitivity is independent of the open-loop portion g(t).

Replacing M(t) by H(t), equation (4.4.15) will be identical with equation (4.3.6). The difference between the two equations is that H(t) is used in the implementation of the nominally optimal control and M(t) is not; in fact, M(t) is not restricted by any means so far. In the dual implementation, g(t) is used to meet the optimal condition at nominal, while M(t) is applied to reduce various sensitivities. The following sections discuss how M(t) is utilized.

4.5 Cost Sensitivity:

It was proved by Pagurek [37] that the open- and closed-loop implementations yield identical cost sensitivity for linear systems under certain assumptions. The idea was accepted by Witsenhausen and the extension to larger classes of system with more generalized cost functional was successfully done [50]. Since then, modifications were made by Dunn [12], Youla and Dorato [51], Kokotovic and Heller [26], and Kokotovic, Heller & Sannuti [27]. All the work done compares the relative merits between open- and closed-loop implementations of the optimal control for various plant and different cost functionals. The cost sensitivity for the TDF implementation has not been evaluated, yet the development follows closely the work done by Kokotovic, Heller and Sannuti [27].

Consider the plant (4.5.1) associated with the cost functional (4.5.2) as,

$$\dot{x}(t,w) = f(x,u,t,w) \qquad (4.5.1)$$

$$J(w,t_0,t_f) = K(x(t_f)) + \int_{t_0}^{t_f} L(x,u,t,w)dt \qquad (4.5.2)$$

Note that the integrand of the cost functional can be a function of the plant parameter w. Assume that the initial time t_0 and initial state $x(t_0)$ are specified. The final time t_f and final state $x(t_f)$ will be discussed later. For the plant under consideration, there is no constraint on the control or the state trajectory. Let w_0 be the nominal value of the plant parameter w. It is assumed that the components of f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial w}$, L, $\frac{\partial L}{\partial x}$, $\frac{\partial L}{\partial t}$, and $\frac{\partial L}{\partial w}$ are continuous over the interval $[t_0, t_f]$.

For an unconstrained problem, the optimal control $u^*(t)$ at nominal must satisfy the following conditions,

$$\dot{p}^*(t) = -\frac{\partial H(u^*, w_0)}{\partial x^*} \qquad (4.5.3)$$

$$\frac{\partial H(u^*, w_0)}{\partial u^*} = 0 \qquad (4.5.4)$$

where $H(u^*, w_0) = L(x^*, u^*, t, w_0) + f^T(x^*, u^*, t, w_0)p^*(t)$ and $p^*(t)$ is the co-vector corresponding to the nominal system. In doing so, the plant (4.5.1) is set to nominal and the optimal control $u^*(t)$ is then obtained by Pontryagin's minimum principle. The procedure can be reversed by applying Pontryagin's minimum principle first then setting all the values at nominal. Define a Hamiltonian function as

$$H(u,w) = L(x,u,t,w) + f^T(x,u,t,w)p(t) \qquad (4.5.5)$$

where $p(t)$ is the co-vector at w. For every value of w, the corresponding co-vector is $p(t)$ and the optimal control $u(t,w)$ at w must satisfy,

$$\dot{p}(t) = -\frac{\partial H(u,w)}{\partial x} \qquad (4.5.6)$$

$$\frac{\partial H(u.w)}{\partial u} = 0 \qquad (4.5.7)$$

The control u(t,w) that satisfies equation (4.5.6) and (4.5.7) is the optimal control at w. However, this is more than necessary because it is required that the control is optimal at nominal only. Hence setting equations (4.5.6) and (4.5.7) at nominal, we have

$$\dot{p}(t) = - \left.\frac{\partial H(u,w)}{\partial x}\right|_{w_0} \tag{4.5.8}$$

$$0 = \frac{\partial H(u,w)}{\partial u} \quad w_0 \tag{4.5.9}$$

Now p(t) will be the co-vector at nominal. The partial derivative of the Hamiltonian in equation (4.5.5) with respect to w is given by

$$\frac{\partial H(u,w)}{\partial w} = \frac{\partial L(x,u,t,w)}{\partial w} + \frac{\partial f^T(w,u,t,w)}{\partial w}\, p(t) \tag{4.5.10}$$

The cost sensitivity is obtained by differentiating equation (4.5.2),

$$\frac{\partial J(w,t_0,t_f)}{\partial w} = \frac{\partial x(t_f)^T}{\partial w}\,\frac{\partial K(x(t_f))}{\partial x(t_f)} + \int_{t_0}^{t_f} \left\{\frac{\partial x}{\partial w}^T \frac{\partial L}{\partial x} + \frac{\partial u}{\partial w}^T \frac{\partial L}{\partial u} + \frac{\partial L}{\partial w}\right\} dt \tag{4.5.11}$$

Setting equation (4.5.11) to w_0, and substituting equations (3.4.18), (3.4.24) and (4.2.3) into the resultant equation, it yields

$$S_w^J = y(t_f)^T \frac{\partial K(x(t_f)}{\partial x(t_f)} + \int_{t_0}^{t_f} \{y^T \frac{\partial L}{\partial x}\bigg|_{w_0} + \frac{\partial L}{\partial w}\bigg|_{w_0}$$

$$+ v^T \frac{\partial L}{\partial u}\bigg|_{w_0} \}dt \tag{4.5.12}$$

Replacing L by $H(u,w) - f^T p^*$ where $p^*(t)$ is the co-vector at nominal, we have

$$S_w^J = y(t_f)^T \frac{\partial K(x(t_f))}{\partial x(t_f)} + \int_{t_0}^{t_f} \{y^T (\frac{\partial H}{\partial x}\bigg|_{w_0} - \frac{\partial f^T}{\partial x}\bigg|_{w_0} p^*)$$

$$+ v^T (\frac{\partial H}{\partial u}\bigg|_{w_0} - \frac{\partial f}{\partial u}\bigg|_{w_0}^T p^*) + \frac{\partial L}{\partial w}\bigg|_{w_0} \}dt \tag{4.5.13}$$

Combining equations (4.5.8) (4.5.9) and (4.5.13), we have

$$S_w^J = y(t_f)^T \frac{\partial K(x(t_f))}{\partial x(t_f)} - \int_{t_0}^{t_f} y^T \dot{p}^* dt$$

$$- \int_{t_0}^{t_f} \{y^T \frac{\partial f^T}{\partial x}\bigg|_{w_0} p^* + v^T \frac{\partial f^T}{\partial u}\bigg|_{w_0} p^* - \frac{\partial L}{\partial w}\bigg|_{w_0} \}dt$$

$$\tag{4.5.14}$$

Integrating the first integral of equation (4.5.14) by parts, we have

$$S_w^J = y(t_f)^T \frac{\partial K(x(t_f))}{\partial x(t_f)} - y^T p^*]_{t_0}^{t_f}$$

$$+ \int_{t_0}^{t_f} \{ (\dot{y} - \frac{\partial f}{\partial x}\Big|_{w_0} y - \frac{\partial f}{\partial u}\Big|_{w_0} v)^T p^* + \frac{\partial L}{\partial w}\Big|_{w_0} \} dt$$

(4.5.15)

Substituting equation (4.2.8), equation (4.5.15) is simplified to

$$S_w^J = y(t_f)^T \frac{\partial K(x(t_f))}{\partial x(t_f)} - y^T p^*]_{t_0}^{t_f} + \int_{t_0}^{t_f} \{ \frac{\partial f}{\partial w}^T\Big|_{w_0} p^* + \frac{\partial L}{\partial w}\Big|_{w_0} \} dt$$

(4.5.16)

Setting equation (4.5.10) at w_0 and combining with equation (4.5.16), we have

$$S_w^J = y(t_f)^T \frac{\partial K(x(t_f))}{\partial x(t_f)} - y^T p^*]_{t_0}^{t_f} + \int_{t_0}^{t_f} \frac{\partial H(u,w)}{\partial w}\Big|_{w_0} dt$$

(4.5.17)

Recalling that $y(t_0) = 0$ is assumed, equation (4.5.17) is rewritten as

$$S_w^J = - y(t_f)^T \{ p^*(t_f) - \frac{\partial K(x(t_f))}{\partial x(t_f)} \} + \int_{t_0}^{t_f} \frac{\partial H(u,w)}{\partial w}\Big|_{w_0} dt$$

(4.5.18)

With suitable boundary condition on $x(t_f)$, the optimal values of $x^*(t)$ and $u^*(t)$ at nominal are determined as known functions of time. As the functions $\left.\frac{\partial f}{\partial w}\right|_{w_0}$ and $\left.\frac{\partial L}{\partial w}\right|_{w_0}$ are functions of $x^*(t)$ and $u^*(t)$,

$\left.\frac{\partial H(u,w)}{\partial w}\right|_{w_0}$ is then a function of time and is independent of the implementation of the nominally optimal control. The values of $p^*(t_f)$ and $\frac{\partial K(x(t_f))}{\partial x(t_f)}$ can be calculated and are not accessible to be changed. In equation (4.5.18), the only quantity subjected to the designer's manipulation is the terminal sensitivity $y(t_f)$ which depends on the implementation of the nominally optimal control via the state sensitivity equation. In general, the integral term of equation (4.5.18) is non-zero, and $y(t_f)$ cannot be zero as far as cost insensitivity is concerned. Therefore it can be concluded that the cost insensitivity is achieved at the expense of terminal sensitivity. It is also obvious that the cost and terminal insensitivities cannot be realized at the same time in any implementation of optimal control at nominal.

It is known from equation (4.5.18) that the cost sensitivity depends also on $p^*(t_f)$, and $p^*(t_f)$ is related to $x^*(t_f)$ by the transversality condition. Therefore, the cost sensitivity depends also on how the boundary point $x^*(t_f)$

is specified. Four different categories of boundary conditions are analysed as follows.

1. <u>Case 1</u>: t_f is specified and $x(t_f)$ is a point given in R^n:

The term $K(x(t_f))$ in this case is zero. The optimal control problem at nominal becomes a two-point boundary-value problem, and $p^*(t_f)$ is fixed consequently. The cost sensitivity in this case is:

$$S_w^J = - y^T(t_f)p^*(t_f) + \int_{t_0}^{t_f} \left.\frac{\partial H(u,w)}{\partial w}\right|_{w_0} dt \qquad (4.5.19)$$

2. <u>Case 2</u>: t_f is specified and $x(t_f)$ is free:

Through the transversality condition, the value $p^*(t_f)$ is given by

$$p^*(t_f) = \frac{\partial K(x(t_f))}{\partial x(t_f)} \qquad (4.5.20)$$

The cost sensitivity in this case becomes

$$S_w^J = \int_{t_0}^{t_f} \left.\frac{\partial H(u,w)}{\partial w}\right|_{w_0} dt \qquad (4.5.21)$$

which is a predetermined constant (or vector) and does not depend on implementation.

3. <u>Case 3</u>: t_f is specified and $x(t_f)$ is an element of a k-fold in R^n:

If S is a smooth k-fold in R^n and the final condition is given as $x(t_f) \varepsilon S$, then the vector $p^*(t_f) - \frac{\partial K(x(t_f))}{\partial x(t_f)}$ must be transversal to the smooth k-fold. Let S be given as

$$z_i(x(t_f)) = 0 \qquad r = [1, \ldots n-k] \qquad (4.5.22)$$

then the boundary condition is given by

$$p^*(t_f) = \sum_{i=1}^{n-k} \alpha_i \frac{\partial z_i(x(t_f))}{\partial x(x(t_f))} + \frac{\partial K(x(t_f))}{\partial x(t_f)} \qquad (4.5.23)$$

where α's are constant. The cost sensitivity in this case is given by

$$S_w^J = -y^T(t_f) \sum_{i=1}^{n-k} \alpha_i \frac{\partial z_i(x(t_f))}{\partial x(t_f)} + \int_{t_0}^{t_f} \frac{\partial H(u,w)}{\partial w}\bigg|_{w_0} dt$$

4. <u>Case 4</u>: t_f is free

The final state can be fixed as in case 1, free as in case 2, or restricted to be in a given k-fold as in case 3. The corresponding cost sensitivities will take the same expressions respectively as if t_f is specified.

In the first three cases, the cost sensitivity does not depend on the implementation. In case 2 where $x(t_f)$ is free, this is true for $K(x(t_f)) = 0$ or $K(x(t_f)) \neq 0$. In other words, the cost sensitivity is the same for any implementation if $x(t_f)$ is free. This astonishing result was obtained by Pagurek [37], extended by Witsenhausen [50] and Dunn [12] and finally clarified by Dorato and Youla [51] and Kokotovic et al [26, 27].

It is very important to notice that equation (4.5.18) is a valid form for cost sensitivity only if equations (4.5.8) and (4.5.9) are true. In the cases where the control is under some constraints, equation (4.5.9) does not necessarily hold. Following closely the development outlined above, the cost sensitivity for the cases when there are control constraints is found to be

$$S_w^J = y^T(t_f)\left[\frac{\partial K(x(t_f))}{\partial x(t_f)} - p^*(t_f)\right] + \int_{t_0}^{t_f} \left\{ \frac{\partial H}{\partial w}\bigg|_{w_0} + v^T \frac{\partial H}{\partial u}\bigg|_{w_0} \right\} dt \tag{4.5.24}$$

To focus the problem is necessary as the cases in optimal control are so prodigious that it is simply impossible to evaluate case by case. Case 2 is beyond discussion in the light of cost sensitivity reduction by implementation as it was already shown that this is impossible. A typical problem of unconstrained control with fixed end point, i.e. case 1, will be considered in the following sections.

Cases 1 and 3 are very similar in many features and similar procedures will lead to similar results as expected.

4.6 Cost or Terminal Insensitivities with Integral State Sensitivity Reduction:

It was shown that additional freedom is available for the TDF implementation of equation (4.4.1). The state sensitivity is affected by the closed loop portion M(t) via equation (4.4.15). By suitable adjustment of M(t), a specific $y(t_f)$ may be obtained but this is not necessarily guaranteed.

To be more specific, the plant under consideration is

$$\dot{x}(t,w) = f(x,u,t,w) \tag{4.6.1}$$

and the cost functional is

$$J(w,t_0,t_f) = \int_{t_0}^{t_f} L(x,u,t,w)dt \tag{4.6.2}$$

with both initial t_0 and final t_f times specified, and both initial $x(t_0)$ and final $x(t_f)$ states given. The cost sensitivity has been proven to be

$$S_w^J = -y^T(t_f)p^*(t_f) + \int_{t_0}^{t_f} \left.\frac{\partial H(u,w)}{\partial w}\right|_{w_0} dt \tag{4.6.3}$$

$$\left.\frac{\partial H(u,w)}{\partial w}\right|_{w_0} = \left.\frac{\partial L(x,u,t,w)}{\partial w}\right|_{w_0} + \left.\frac{\partial f^T(x,u,t,w)}{\partial w}\right|_{w_0} p^*(t) \qquad (4.6.4)$$

Noting that both L and f are known functions of the plant parameter, and that $\left.\frac{\partial L}{\partial w}\right|_{w_0}$ and $\left.\frac{\partial f}{\partial w}\right|_{w_0}$ are known functions of time, the integral term in equation (4.6.3) is constant denoted by c as

$$c = \int_{t_0}^{t_f} \left.\frac{\partial H(u,w)}{\partial w}\right|_{w_0} dt \qquad (4.6.5)$$

The constant c will be a vector if w is a vector otherwise c is a scalar.

Recall that cost insensitivity implies $S_w^J = 0$, and it will be achieved if

$$y^T(t_f)p^*(t_f) = c \qquad (4.6.6)$$

Above equation is regarded as the mathematical specification for cost insensitivity in fixed end points optimal control problem. If the plant parameter w is a scalar, equation (4.6.6) defines a hyperplane in R^n spanned by the n-components of $y(t_f)$. If the plant parameter is a rxl vector, equation (4.6.6) can be rewritten in the component form as,

$$\sum_{j}^{n} y_{ji}(t_f)p_j{}^*(t_f) = c_i \qquad i\varepsilon[1,r] \qquad (4.6.7)$$

where $y_{ji}(t_f)$ are the elements of nxr matrix $y(t_f)$, $p_j{}^*(t_f)$ are the components of $p^*(t_f)$ and c_i are the components of the rx1 constant vector c. For every i, $i\varepsilon[1,r]$, equation (4.5.7) defines a hyperplane in R^{nxr} spanned by the elements of $y(t_f)$. The intersection of all these r hyper -planes defines a smooth rx(n-1) fold in R^{nxr}. This rx(n-1) fold will be called the cost insensitive fold in R^{nxr}. Note that the cost insensitive fold is a zero fold when n=1. This implies that $y(t_f)$ is a determined 1xr constant vector when n=1. In all other cases, there are always more unknowns in $y(t_f)$ than the number of equations derived from equation (4.6.6) as n>1,r>0 and nr>r where nxr is the number of unknown in $y(t_f)$ and r is the number of equations. In other words, the terminal sensitivity $y(t_f)$ is not uniquely determined by the cost insensitivity specification for systems with two or more state variables. For all cases, the cost-insensitive fold is non-empty.

Let h(t,y,M) be a nxr matrix defined by

$$h(t,y,M) = \{\frac{\partial f}{\partial x}\Big|_{w_0} + \frac{\partial f}{\partial u}\Big|_{w_0} M\}y + \frac{\partial f}{\partial w}\Big|_{w_0} \qquad (4.6.8)$$

and the state sensitivity equation (4.4.15) is rewritten as

$$\dot{y}(t) = h(t,y,M) \qquad (4.6.9)$$

with $y(t_0) = 0$. Recall that the integral state sensitivity is defined by

$$I(M) = \int_{t_0}^{t_f} ||y(t)||^2 \, dt \qquad (4.6.10)$$

The state sensitivity equation (4.6.9) is a function of an undetermined matrix M(t). In sensitivity reduction, the function M(t) is adjusted such that the sensitivity specification is met. Three types of sensitivity have been defined and the choice of the sensitivity to be reduced depends on the specification. Therefore, it is factitiously assumed that different sensitivity specifications are given.

When the interest is focused on the terminal sensitivity, the ideal case is to achieve terminal insensitivity, that is, $y(t_f) = 0$. It is, therefore, required to find a matrix function M(t) driving y(t) from $y(t_0)=0$ to $y(t_f) = 0$ via the state sensitivity equation (4.6.9). Note that the existence of such a matrix function M(t) must be assumed without further verification. It was proved by Holtzman and Horing [19] that for a class of linear feedback systems, there is a finite range of parameter variations which has no effect on the terminal condition. It was shown by Gadabassi et. al. [16] that for a class of linear time-invariant systems, it is impossible to achieve the insensitivity of a specified terminal condition by open loop implementation. However in applying the TDF implementation, it is distracting to prove that there exists at least one matrix function M(t) driving $y(t_0) = 0$ to a specified $y(t_f)$ via the differential system (4.6.9).

If the interest lies in the cost sensitivity, the best case that can be achieved is to have the cost insensitive design, that is, $S_w^J = 0$. It is therefore required to find a matrix function M(t) driving $y(t_0) = 0$ to a point $y(t_f)$ via the state sensitivity equation (4.6.9). The terminal point $y(t_f)$ is restricted to be in the cost insensitive hyperplane. Note that in both cost and terminal insensitive designs, the matrix function M(t) is adjusted such that a desirable $y(t_f)$ is obtained. Consequently, the cost and terminal insensitivities are mutually exclusive in general.

For the purpose of reducing the state sensitivity, it is required to find a matrix function M(t) which steers y(t)from $y(t_0) = 0$ to any point in R^{nxr} via the state sensitivity equation (4.6.9) and which minimizes the integral state sensitivity. This is a well defined control problem and the necessary conditions for the optimality of M(t) are given by Pontryagin's Minimum Principle.

In the cost and terminal insensitive design, there may exist numerous matrix functions M(t) which satisfy the insensitive specification. Among the possible candidates, it is desirable to have an optimal M(t), which yields a minimum value for the corresponding integral state sensitivity. Hence a mixed type of sensitivity specification can be formulated. The problem of insensitive design with integral state sensitivity reduction can be stated as to find a matrix function M(t) which steers y(t) from $y(t_0)=0$ to a specified terminal point $y(t_f)$ via the differential system (4.6.8) and (4.6.9) and which minimizes the integral state sensitivity (4.6.10). For cost insensitive implementation with integral state sensitivity reduction, the terminal point $y(t_f)$ is restricted to be in the cost insensitive hyperplane. In the case of terminal insensitive design with integral state sensitivity reduction, the terminal point $y(t_f)$ is specified as a null vector or matrix.

As a priori condition to all the problems which have been formulated, it must be assumed that the components of h, $\frac{\partial h}{\partial t}$, and $\frac{\partial h}{\partial y}$ are continuous in $R^{nxq} x R^{nxr} x [t_0,t_f]$. The Pontryagin's Minimum Principle will then supply the necessary conditions for the optimal implementation M(t). The basic schemes introduced in this section have been discussed in [53]. However, before expecting meaningful results or practical solution, the problems must be further refined.

4.7 Soft Constraint & Design Objectives:

In applying Pontryagin's minimum principle to the cost or terminal insensitive problems, the corresponding Hamiltonian functions are linear in M(t), which is the feedback portion of the TDF implementation. As the problem is actually a class of singular problems [2, 18, 21], the solution, if it exists, will have no practical value if there are no reasonable constraints imposed on M(t).

Realize that M(t) represents the signal in time domain as well as the physical system of which the impulse response is given by M(t) [8]. It is well known that the energy of any physical signals must be finite. Hence a reasonable constraint on M(t) is

$$\int_{t_0}^{t_f} ||M(t)||^2 \, dt \leq k \qquad (4.7.1)$$

where k is a finite constant. This basic constraint will be called the soft constraint. The value of k in the soft constraint is relatively arbitrary and carries no meaning in general and yet the behaviour of M(t) depends heavily on the constant. Unless it is required, it is advised to include the soft constraint in the performance index $I(M)$ in equation (4.6.10) as

$$I(M) = \int_{t_0}^{t_f} \{||y(t)||^2 + \beta||M(t)||^2\} \, dt \tag{4.7.2}$$

where β is a weighting constant which allows the designer to put a relative weight between $\int_{t_0}^{t_f} ||y(t)||^2 dt$ and $\int_{t_0}^{t_f} ||M(t)||^2 dt$. It is obvious that the soft constraint of equation (4.7.1) is, therefore, obviated.

In the cost insensitive implementation problem, the boundary condition on $y(t_f)$ is restricted to be in the smooth rx(n-1) fold in R^n as given by equation (4.6.6). Besides satisfying the cost insensitive specification, the terminal sensitivity can be relatively minimized. The first way of achieving terminal sensitivity reduction is to modify the performance index $I(M)$ (4.7.2) as

$$I(M) = \alpha||y(t_f)||^2 + \int_{t_0}^{t_f} \{||y(t)||^2 + \beta||M(t)||^2\} dt \tag{4.7.3}$$

where α is a positive weighting constant. Another possible way is to find an element $y^*(t_f)$ in the cost insensitive fold such that $||y^*(t_f)||^2$ is a minimum, then the boundary condition is equated to $y^*(t_f)$.

It has been clarified in section 4.6 that cost and terminal insensitive design can hardly be satisfied simultaneously in any implementation. The choice between the two insensitivities must be made by the designer. Besides satisfying either the cost or the terminal insensitivity, the feedback portion M(t) of the TDF implementation is chosen such that the integral state sensitivity is minimized. Moreover, for cost insensitive design, M(t) is chosen such that both the terminal and the integral state sensitivities are minimized.

To summarize, some flexibilities from the TDF implementation are accessible to the designer. A problem has been formulated and stated as, "to find a matrix function M(t) driving y(t) from $y(t_0)=0$ to a given $y(t_f)$ via system (4.6.8) and (4.6.9) and minimizing the performance index given by equation (4.7.3)". For terminal insensitivity, $y(t_f) = 0$ and $\alpha = 0$. For cost insensitivity $y(t_f) = y^*(t_f)$ and $\alpha = 0$ or $y(t_f)$ in the cost insensitive fold and $\alpha \neq 0$. The necessary conditions for M(t) to be the optimal implementation problem as formulated are obtained via Pontryagin's Minimum Principle. The open loop portion g(t) of equation (4.4.1) is obtained from equation (4.4.10) upon determining M(t).

The outlined scheme is systematic and generalized so that a minimum modification is required to meet various sensitivity specifications. The constants α and β in equation (4.7.3) are adjustable so that the designer is able to emphasize his interest. Also additional constraints on M(t) or y(t) are permissible.

However, as an a priori condition to the suggested scheme of design, the plant function f(x,u,t,w) must be a known function of the plant parameter w, so that the term $\left.\frac{\partial f}{\partial w}\right|_{w_0}$ in the state sensitivity equation can be evaluated. When the plant function f is known only at nominal i.e. only $f(x,u,t,w_0)$ is given, the whole suggested scheme will fail and another approach must be developed.

4.8 Preventive Design:

Basically, the plant function f(x,u,t,w) in equation (4.2.1) is a function of the plant parameter w. The role of w in f may be too expensive to identify or, in some cases impossible. Also, it may not be economical to obtain detailed knowledge about the plant parameter if the chance for the nominal system to change is fairly small and the effort to approximate the relation of w and f is tremendous. For all these situations, the only information supplied to the designer is the nominal plant and the function $\left.\frac{\partial f}{\partial w}\right|_{w_0}$ is not known. Hence for any given implementation scheme, i.e. given v(t) as a function of y(t), the state sensitivity y(t) cannot be predicted from the state sensitivity equation (4.2.8). Therefore, to achieve terminal insensitivity by designing a suitable implementation is hopeless.

As there is an unknown function in the state sensitivity equation, the state sensitivity y(t) cannot be controlled by the implementation of the nominally optimal control in a desirable manner. The cost sensitivity given by equation (4.5.19) depends on the terminal sensitivity $y(t_f)$. If $y(t_f)$ cannot be controlled by the implementation, nor can the cost sensitivity. It is true that the cost sensitivity cannot have any value as required. However the cost insensitivity can be achieved pointwise in time under fairly restrictive assumptions.

Consider a nominal plant (4.8.1)

$$\dot{x}(t,w_0) = f(x,u,t,w_0) \qquad (4.8.1)$$

which is also represented by

$$\dot{x}(t,w_0) = f(x,u,t,w)\Big|_{w_0} \qquad (4.8.2)$$

where the plant function f(x,u,t,w) in equation (4.8.2) is an unknown function of the plant parameter w. Let the cost functional associated with the nominal plant be

$$J(w,t_0,t_f) = \int_{t_0}^{t_f} L(x,u,t)dt \qquad (4.8.3)$$

As the role of the plant parameter is not given, the integrand $L(x,u,t)$ in equation (4.8.3) cannot be a function of w as in equation (4.5.2). Also the scalar functional $K(x(t_f))$ in equation (4.5.2) is zero. This is necessary for cost insensitivity by realizing the fact that the control of $y(t_f)$ is impossible. If $K(x(t_f))$ is non-zero, the cost sensitivity derived from equation (4.8.3) will depend on the terminal sensitivity and the insensitive specification is therefore impossible to achieve. It is assumed that the initial time t_0 and the initial state $x(t_0)$ are specified and the control is under no constraints. The components of f , $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t}$, $\frac{\partial L}{\partial x}$, $\frac{\partial L}{\partial t}$ and L are continuous in $R^n x R^q x[t_0,t_f]$ where n is the dimension of the state trajectory, $x(t,w_0)$ and q is the dimension of the control. Hence the necessary conditions for the optimal control are expressed as,

$$\left.\frac{\partial H(u,w)}{\partial u}\right|_{w_0} = 0 \qquad (4.8.4)$$

$$\dot{p}^*(t) = -\left.\frac{\partial H(u,w)}{\partial x}\right|_{w_0} \qquad (4.8.5)$$

where $H(u,w) = L(x,u,t) + f^T(x,u,t,w)p^*(t_0)$ and $p^*(t)$ is the costate vector at w_0. Note that the functions

$\left.\frac{\partial H(u,w)}{\partial u}\right|_{w_0}$ and $\left.\frac{\partial H(u,w)}{\partial x}\right|_{w_0}$ are defined even if the Hamiltonian function H(u,w) is not defined. The cost functional at time t is

$$J(w,t_0,t) = \int_{t_0}^{t} L(x,u,t)dt \qquad (4.8.6)$$

where $t_0 < t \leq t_f$. The pointwise cost sensitivity is defined as

$$S_w^J(t) = \left.\frac{\partial J(w,t_0,t)}{\partial w}\right|_{w_0} \qquad (4.8.7)$$

It is obvious that pointwise cost insensitivity implies cost insensitivity, or mathematically, $S_w^J(t) = 0$ implies $S_w^J(t_f) = S_w^J = 0$. The pointwise cost sensitivity is obtained by partially differentiating equation (4.8.6) and setting the resultant equation at w_0, and is expressed as

$$S_w^J(t) = \int_{t_0}^{t} \left\{ \left.\frac{\partial x}{\partial w}^T\right|_{w_0} \cdot \left.\frac{\partial L}{\partial x}\right|_{w_0} + \left.\frac{\partial u}{\partial w}^T\right|_{w_0} \left.\frac{\partial L}{\partial u}\right|_{w_0} \right\} dt \qquad (4.8.8)$$

Assume that the TDF implementation given in equation (4.4.1) is applied. Substituting equations (3.4.18), (4.4.14) into equation (4.8.8), we have

$$S_w^J(t) = \int_{t_0}^{t_f} y^T\{\frac{\partial L}{\partial x}\Big|_{w_0} + M^T\frac{\partial L}{\partial u}\Big|_{w_0}\}dt \qquad (4.8.9)$$

where y(t) is the state sensitivity corresponding to the TDF implementation. As indicated by equation (4.8.9), the pointwise cost sensitivity depends on the state sensitivity which cannot be determined. However, pointwise cost insensitivity is achieved without knowing y(t) if the condition,

$$\frac{\partial L}{\partial x}\Big|_{w_0} + M^T\frac{\partial L}{\partial u}\Big|_{w_0} = 0 \qquad (4.8.10)$$

is satisfied for all time in the interval of $(t_0, t_f]$. Equation (4.8.4) can be rewritten as

$$\frac{\partial L}{\partial u}\Big|_{w_0} + \frac{\partial f^T(x,u,t,w)}{\partial u}\Big|_{w_0} p^* = 0 \qquad (4.8.11)$$

Combining equations (4.8.11) and (4.8.10), we have

$$M^T\frac{\partial f\ (x,u,t,w)}{\partial u}\Big|_{w_0} p^* = \frac{\partial L}{\partial x}\Big|_{w_0} \qquad (4.8.12)$$

The functions $\frac{\partial L}{\partial x}\Big|_{w_0}$ and $\frac{\partial f}{\partial u}\Big|_{w_0}$ are functions of the optimal trajectory x*(t) and control u*(t) which can be regarded as known functions of time. Hence M(t) can be determined pointwisely in time from equation (4.8.12).

The qxl vector function of time $\frac{\partial f}{\partial u}\Big|_{w_0}^T p^*$, must not be a null vector for all time t, $t \epsilon (t_0, t_f]$ unless the time function $\left.\frac{\partial L}{\partial x}\right|_{w_0}$ is also a null vector at that specific point, in which case equation (4.8.12) is automatically satisfied independently of M(t). For all the cases discussed in section 4.5, it has been shown that the cost sensitivity is identical for all implementations in the admissible implementation set when $x(t_f)$ is free. As $K(x(t_f))$ is zero in the preventive design, $p^*(t_f)$ will be a null vector for nominal optimality. Hence it can be concluded that equation (4.8.12) does not hold in general when $x(t_f)$ is free.

Because of the condition that $\left.\frac{\partial f}{\partial u}\right|_{w_0} p^* \neq 0, \forall t \epsilon (t_0 t_f]$, the pointwise cost insensitive design is applicable only to a limited number of cases depending mainly on the nominally optimal characteristics. The shortage of the preventive design is expected since the knowledge about the plant is given at the very minimum. It is not required to know the plant function f as a function of the plant parameter w; the pointwise insensitive design can be applied practically to all optimal control problems provided that the restrictive condition is satisfied. As far as sensitivity reduction is concerned, the pointwise cost insensitive design is therefore the ultimate resort of the TDF configuration.

4.9 Conclusion:

The TDF implementation consists of two physical blocks represented mathematically as the open-loop portion g(t), and the closed-loop portion M(t). The feedback block M(t) is used to achieve insensitive design or to meet different sensitivity specifications and g(t) is used in the implementation of the nominally optimal control, upon knowing M(t). Depending on the knowledge about the plant, two approaches are derived. If the plant function is given as a function of plant parameter, cost or terminal insensitive designs are possible. Besides fulfilling the insensitive specification, the designer is able to reduce the integral state sensitivity relatively among the possible implementations. If only the nominal plant is known, the situation is not very optimistic. However, pointwise cost insensitive design may be achieved provided that a fairly restrictive condition is satisfied.

CHAPTER V

LINEAR SYSTEMS WITH QUADRATIC COST FUNCTIONALS

5.1 Introduction:

The techniques of determining the optimal control and the application of the dual configuration have been developed for a general system associated with a generalized cost functional. The systems considered in this chapter are linear and the cost functional quadratic. The sensitivity analysis in optimal control theory has been studied for a limited number of years yet some important results have been obtained, mostly concerning the linear systems with quadratic cost functionals. The purpose of this chapter is to explore the field of sensitivity analysis in optimal control theory by presenting the well-developed results and also to illustrate the new approach in sensitivity reduction which has been innovated in Chapter IV.

5.2 Open- and Closed-Loop Implementations:

One of the most powerful design techniques that has been fully developed to date deals with the design of the optimal control for a linear system, possibly time varying, with respect to a quadratic cost functional. The pioneering work in the area was done by Kalman [23]. The Hamilton-Jacobi-Bellmen equation is utilized as the method of attack.

Consider the linear system at nominal

$$\dot{x}(t,w_0) = A(t,w_0)x(t,w_0) + B(t,w_0)u(t,w_0) \qquad (5.2.1)$$

and the cost functional

$$J = \frac{1}{2}x^T(t_f,w_0)Fx(t_f,w_0) + \frac{1}{2}\int_{t_0}^{t_f}\{x^T(t,w_0)Q(t)x(t,w_0) + u^T(t,w_0)R(t)u(t,w_0)\}dt \qquad (5.2.2)$$

where $x(t,w_0)$ is the nxl state vector for the nominal system, $u(t,w_0)$ is the qxl control vector for the nominal system, and w_0 is the rxl nominal time invariant parameter vector.

It is assumed that F is a symmetric nxn positive semidefinite constant matrix and that Q(t) and R(t) are respectively nxn and rxr positive definite symmetric matrices. To simply the notations, let $x(t) = x(t,w_0)$, $u(t) = u(t,w_0)$, $A_0 = A(t,w_0)$ and $B_0 = B(t,w_0)$. Equation (5.2.1) and (5.2.2) are rewritten as

$$\dot{x}(t) = A_0 x(t) + B_0 u(t) \qquad (5.2.3)$$

$$J = \frac{1}{2} x^T(t_f) F x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \{x^T(t) Q(t) x(t) + u^T(t) R(t) n(t)\} dt \qquad (5.2.4)$$

Assume that the initial time t_0 and the initial state $x(t_0)$ are given. Define the Hamiltonian as

$$H = \frac{1}{2}[x^T(t) Q(t) x(t) + u^T(t) R(t) u(t)] + [A_0 x(t) + B_0 u(t)]^T p(t) \qquad (5.2.5)$$

where p(t) is the nx1 costate vector. The canonical system is

$$\dot{x}(t) = A_0 x(t) + B_0 u(t) \qquad (5.2.6)$$

$$\dot{p}(t) = -A_0^T p(t) - Q(t) x(t) \qquad (5.2.7)$$

There is no constraint on the control u(t), hence by Pontryagin's Minimum Principle, the optimal control is

$$u(t) = -R^{-1}(t) B_0{}^T(t) p(t) \qquad (5.2.8)$$

Denote the control of (5.2.8) as $u^*(t)$ and its corresponding canonical variables as $x^*(t)$ and $p^*(t)$ which are the solutions of equations (5.2.6) and (5.2.7).

For a feedback control law, let

$$p^*(t) = K(t) x^*(t) \qquad (5.2.9)$$

where K(t) is an nxn matrix to be determined. The optimal feedback control law is obtained by combining equations (5.2.9) and (5.2.8) as

$$u^*(t) = -R^{-1}(t)B_0^TK(t)x^*(t) \qquad (5.2.10)$$

Differentiating equation (5.2.9) and combining equations (5.2.6), (5.2.7) and (5.2.8), we have

$$[\dot{K}(t) + K(t)A_0 - K(t)B_0R^{-1}(t)B_0^TK(t) + A_0^TK(t) + Q(t)]x^*(t) = 0 \qquad (5.2.11)$$

Equation (5.2.11) must hold independently of the value of $x^*(t)$. This implies

$$\dot{K}(t) + K(t)A_0 + A_0^TK(t) + Q(t) - K(t)B_0R^{-1}(t)B_0^TK(t) = 0 \qquad (5.2.12)$$

which is the well-known Riccati equation. Note that K(t) is symmetric if the boundary condition on K(t) is also symmetric.

For sufficient condition, let $J^*(x,t)$ be

$$J^*(x^*,t) = \frac{1}{2}x^{*T}(t)K(t)x^*(t) \qquad (5.2.13)$$

The Hamilton-Jacobi-Bellman equation becomes

$$\frac{\partial}{\partial t}J^* + \underset{u(t)}{\mathrm{Min}}\{\frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t) + x^T(t)A_0^T\frac{\partial J^*}{\partial x^*} + u^T(t)B_0^T\frac{\partial J^*}{\partial x^*}\} = 0 \qquad (5.2.14)$$

Replacing u(t) in (5.2.4) by u* in (5.2.10), it can be easily shown that J*(x,t) in equation (5.2.13) is the solution for the Hamilton-Jacobi-Bellman equation provided that K(t) is symmetric.

Hence the optimal control is given by (5.2.15)

$$u^*(t) = -R^{-1}(t)B_0{}^T\frac{\partial J^*}{\partial x^*}$$

which is equation (5.2.10).

When $x(t_f)$ is not specified, the boundary condition for $p^*(t_f)$ is given by

$$p^*(t_f) = Fx(t_f) \qquad (5.2.16)$$

Comparing equations (5.2.16) and (5.2.9), it is obvious that

$$K(t_f) = F \qquad (5.2.17)$$

which is symmetric. Hence equation (5.2.13) is the solution for equation (5.2.14).

When $x(t_f)$ is given as a point and F = 0 there is no condition on $p^*(t_f)$. Hence both $p^*(t_0)$ and $p^*(t_f)$ are determined from the canonical system (5.2.6) and (5.2.7) corresponding to the optimal control (5.2.8). The boundary condition for the K(t) matrix will be

$$p^*(t_0) = K(t_0)x^*(t_0) \tag{5.2.18}$$

$$p^*(t_f) = K(t_f)x^*(t_f) \tag{5.2.19}$$

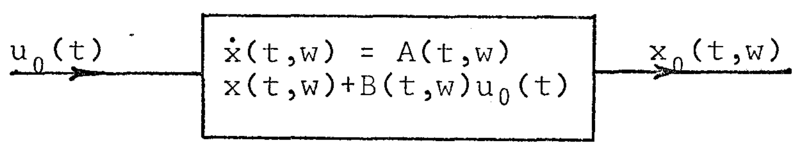
instead of the equation (5.2.17). Note that $K(t_f)$ will be no longer symmetric, and $J^*(x^*,t)$ in equation (5.2.13) will not be the solution for the Hamilton-Jacobi-Bellman equation. However, there still exists a matrix $K(t)$ which is the solution of the Riccati equation. In the case where $x^*(t_f) = 0$, it is well known that $K(t)\to\infty$ as $t\to t_f$. A practical disadvantage of this solution is the physical realizability and the extreme sensitivity of the feedback controller as t approaches to t_f.

Combining equations (5.2.10) and (5.2.6), the feedback system is represented by

$$\dot{x}^*(t) = G_0(t)x(t) \tag{5.2.20}$$

$$G_0(t) = A_0 - B_0R^{-1}(t)B_0^TK(t) \tag{5.2.21}$$

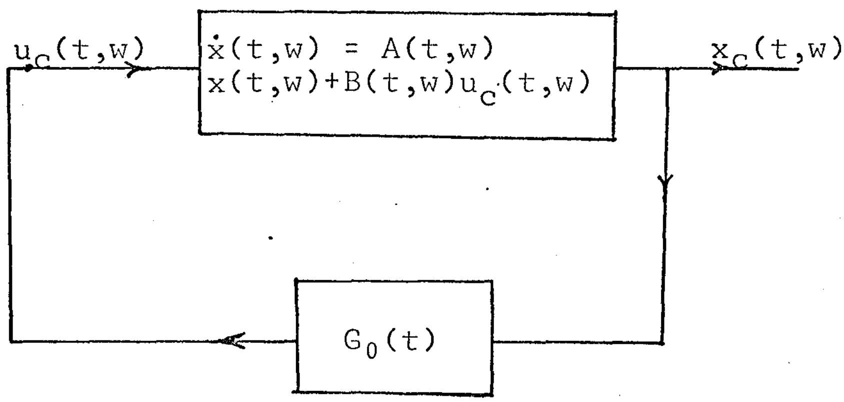
Hence, in general, for a linear system with quadratic cost functional, open- and closed-loop implementations are guaranteed. This is illustrated in figure (4) and (5). The two systems are said to be nominally equivalent.



$$u_0(t) = -R^{-1}(t)B(t,w_0)p^*(t)$$

Figure 4.

Open-Loop Implementation for Optimal Linear System with Quadratic Cost Functional.



$$G_0(t) = A(t,w_0) - B(t,w_0)R^{-1}(t)B^T(t,w_0)K(t)$$

$$u_c(t,w) = G_0(t)x_c(t,w)$$

Figure 5.

Closed-Loop Implementation

Optimal Linear System with

Quadratic Cost Functional.

5.3 Comparison between Open- and Closed-Loop Implementations:

Equation (5.2.1) represents the nominal system with the plant parameter at w_0. When the plant parameter changes from the nominal value w_0 to a value w, the linear system becomes

$$\dot{x}(t,w) = A(t,w)x(t,w) + B(t,w)u(t,w) \tag{5.3.1}$$

and the cost functional is

$$J(w,t_0,t_f) = \frac{1}{2}x^T(t_f,w)Fx^T(t_f,w) + \frac{1}{2}\int_{t_0}^{t_f}\{x^T(t,w)Q(t)x(t,w) + u^T(t,w)R(t)u(t,w)\}dt \tag{5.3.2}$$

As the open-loop control $u_0(t)$ is independent of the plant parameter, hence the corresponding implementation vector is zero, that is

$$u_0(t,w) = -R^{-1}(t)B_0{}^T p^*(t) \tag{5.3.3}$$

$$v_0(t) = \left.\frac{\partial u_0(t,w)}{\partial w}\right|_{w_0} = 0 \tag{5.3.4}$$

The closed-loop control $u_c(t,w)$ is rewritten as

$$u_c(t,w) = -R^{-1}(t)B_0{}^T K(t)x(t,w) \tag{5.3.5}$$

where $x(t,w_0) = x(t,w)\Big|_{w_0} = x^*(t)$. It is obvious that when w_0 changes to w, the closed-loop control will change too. The corresponding implementation vector $v_c(t)$ is

$$v_c(t) = \frac{\partial u_c(t,w)}{\partial w}\Big|_{w_0} = -R^{-1}(t)B_0^T K(t)y_c(t) \tag{5.3.6}$$

where $y_c(t)$ is the closed-loop state sensitivity.

The state sensitivity equation for the linear system is obtained by differentiating equation (5.3.1) with respect to w,

$$\dot{y}(t) = A_w x^*(t) + B_w u^*(t) + A_0 y(t) + B_0 v(t) \tag{5.3.7}$$

where $A_w = \frac{\partial A(t,w)}{\partial w}\Big|_{w_0}$ and $B_w = \frac{\partial B(t,w)}{\partial w}\Big|_{w_0}$

which are known functions of time. Putting equations (5.3.4) and (5.3.6) into (5.3.7), the state sensitivity equations for open- and closed-loop implementations are, respectively,

$$\dot{y}_0(t) = A_w x^*(t) + B_w u^*(t) + A_0 y_0(t) \tag{5.3.8}$$

$$\dot{y}_c(t) = A_w x^*(t) + B_w u^*(t) + A_0 y_c(t) - B_0 R^{-1} B_0^T K y_c(t) \tag{5.3.9}$$

where y_0 and y_c are respectively the state sensitivity for open- and closed-loop configurations. With the assumed initial condition that $y_0(t_0) = y_c(t_0) = 0$ it is obvious that $y_0(t) \neq y_c(t)$. In other words, $x_0(t,w) \neq x_c(t,w)$ where $x_0(t,w)$ and $x_c(t,w)$ are respectively the state trajectory corresponding to open- and closed-loop implementations.

With all the differences between the open- and closed-loop implementations, three types of sensitivity must be evaluated. It is further assumed that both the final time t_f and the final state $x(t_f)$ are given and that F in equation (5.2.2) is a null matrix.

Equation (4.5.19) is a general expression for the cost sensitivity. For linear systems (5.3.1) with the quadratic cost (5.3.2) with F = 0, the derivative of the Hamiltonian with respect to the plant parameter is given as

$$\left.\frac{\partial H}{\partial w}\right|_{w_0} = [A_w x^*(t) + B_w u^*(t)]^T p^*(t) \qquad (5.3.10)$$

Denote a vector c as

$$c = \int_{t_0}^{t_f} [A_w x^*(t) + B_w u^*(t)]^T p^*(t) dt \qquad (5.3.11)$$

which is a constant rx1 vector. By equation (4.5.19) the cost sensitivity for open-loop control $S_w^J(v_0)$ is directly obtained as

$$S_w^J(v_0) = -y_0^T(t_f)p^*(t_f) + c \qquad (5.3.12)$$

and the cost sensitivity for closed-loop control $S_w^J(v_c)$ is

$$S_w^J(v_c) = -y_c^T(t_f)p^*(t_f) + c \qquad (5.3.13)$$

Let $S(v_0,v_c)$ be a constant defined as

$$S(v_0,v_c) = \| S_w^J(v_0)\|^2 - \| S_w^J(v_c)\|^2 \qquad (5.3.14)$$

By the definition of the norm, $S(v_0,v_c)$ can be expanded to

$$S(v_0,v_c) = p^{*T}(t_f)\{y_0(t_f)y_0^T(t_f) - y_c(t_f)y_c^T(t_f)\}p^*(t_f)$$

$$- 2c^T\{y_0(t_f) - y_c(t_f)\}^T p^*(t_f) \qquad (5.3.15)$$

The values $p^*(t_f)$ and c are calculated and $y_0(t_f)$ and $y_c(t_f)$ are fixed and can be obtained from their corresponding state sensitivity equations. Hence the constant $S(v_0,v_c)$ is well defined. If $S(v_0,v_c) > 0$, then it can be concluded that closed-loop implementation is less sensitive to plant parameter variation with regard to the cost sensitivity than the open-loop implementation. However, the values c, $y_c(t_f)$, and $y_0(t_f)$ cannot be generalized and, in the light of cost sensitivity, the comparison of open- and closed-loop implementations depends on individual problems. Hence there is no general conclusion.

With respect to state sensitivity, it is well known that feedback configuration can provide a reduction of sensitivity to the variations of the plant parameters. Whether linear optimal systems provide the closed-loop sensitivity reduction has been answered affirmatively by Kalman [24] and Anderson [1] on the basis of the analogy of their results with the classical return difference. By applying a modification of the relationship between open- and closed-loop sensitivity, Cruz and Parkins [7] proved some results which are similar to Kalman's and Anderson's. Kriendler then applied all the results in the proof of closed loop sensitivity reduction [29].

For a linear time-invariant system with quadratic cost functional and a scalar plant parameter, the closed-loop implementation is less sensitive in view of the state sensitivity than open-loop implementation, or mathematically,

$$\int_0^{t'} y_c(t)Z(t)y_c(t)dt < \int_0^{t'} y_0(t)Z(t)y_0 dt \qquad (5.3.16)$$

where $Z = K^T B_0 R^{-1} B_0^T K$. The result can be extended to the case where Z = I and the time interval [0,t'] can be extended to $(-\infty, +\infty)$. Kriendler also succeeded in extending the result of sensitivity reduction by closed loop implementation to linear time-varying system with quadratic cost [30]. Attempt has been made to extend the result to non-linear systems [28].

With respect to terminal condition, little research has been done. It was proved by Holtzman and Horing [19] that for a class of feedback system with minimum control energy policy, that is, Q = 0, and F = 0, there is a finite range of parameter variations which have no effect on the terminal conditions. An interesting remark on this was made by Porter [39].

In general, current results indicate a more favorable side for closed-loop implementation in the aspect of state and terminal sensitivities. In cost sensitivity, general conclusion is impossible.

5.4 Current Approaches in Sensitivity Reduction:

In the previous section, current research asserts that optimally linear systems with closed-loop implementation are, in one sense or another, less sensitive than the equivalent open-loop systems. In spite of the fact that such a characteristic is a reassuring result, it actually does not solve a very practical question.

Assume the optimal control has been calculated and implemented with a closed-loop controller. The state sensitivity is calculated. Obviously it may or may not satisfy the desired specifications. If not, what can be done to improve the sensitivity? Gavrilovic and Petrovic [14] and Siljak and Dorf [44] suggested the introduction of sensitivity terms in the given cost functional J, for example,

$$I = \int_{t_0}^{t_f} \| y_c(t) \|^2 \, dt \qquad (5.4.1)$$

the design procedure then is to choose the control which minimizes the cost functional J_α

$$J_\alpha = J + \alpha I \qquad (5.4.2)$$

where α is a scalar. The design procedure is essentially based on computing the control and the resulting sensitivity for several values of α. By doing this, the designer may eventually satisfy the specification on the state sensitivity.

The basic scheme of introducing a cost function J_α has been modified by Kahne [22,32], D'Angelo, Moe and Hendricks [9] and Dompe and Dorf [10]. In Kahne's approach, it is assumed that the system matrix B in the plant equation (5.3.1) does not depend on the plant parameter and that the $\frac{\partial u(t,w)}{\partial w}$ in the state sensitivity equation is negligible. With these, the state sensitivity equation becomes

$$\dot{y}(t) = A_w(t)x(t) + A_0 y(t) \qquad (5.4.3)$$

together with the linear system

$$\dot{x}(t) = A(t,w)x(t) + B(t)u(t) \qquad (5.4.4)$$

where w is known value of the plant parameter. A cost functional including the sensitivity is suggested as

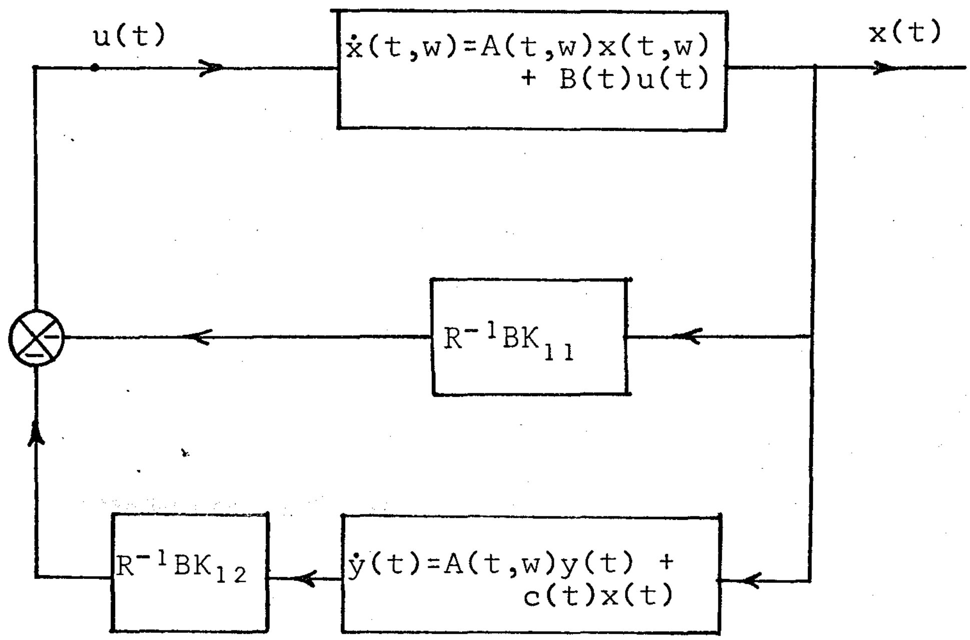
$$J_K = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) + y^T(t)D(t)y(t)\}dt \qquad (5.4.5)$$

where D(t) is a symmetric positive semi-definite nxn matrix and is at least twice differentiable in t. The optimal control is then determined by system (5.4.3) and (5.4.4) with respect to the cost functional J_K. Simulation of the optimal control is illustrated in Figure (6) and is given by

$$u(t) = -R^{-1}B^T(K_{11}x(t) + K_{12}y(t)] \qquad (5.4.6)$$

where K_{11} and K_{12} are the partitioned matrices of K and K is the solution of a modified Riccati equation.

D'Angelo, Moe and Hendricks have developed a sensitivity differential equation:



$$u(t) = -R^{-1}BK_{11}x(t) - R^{-1}BK_{12}y(t)$$

Figure 6.

Structure of Low State Sensitivity

Optimal Linear System by Kahne

$$\frac{\partial^i y(t,w)}{\partial w^i} = \sum_{r=0}^{i+1} \binom{i+1}{r} \frac{\partial^r A}{\partial w^r} \frac{\partial^{i+1-r} x(t,w)}{\partial w^{i+1-r}}$$

$$- \sum_{r'=0}^{i+1} \sum_{r=0}^{L} \sum_{q=0}^{i+1-r'} \sum_{m=0}^{q} \binom{i+1}{r'} \binom{i+1-r'}{q} \binom{q}{m}$$

$$\times \frac{\partial^{r'} B}{\partial w^{r'}} R^{-1} \frac{\partial^{q-m} B}{\partial w^{q-m}} \frac{\partial^m K}{\partial w^m} \frac{\partial^{r+i+1-r'-q} x(t,w)}{\partial w^{r+i+1-r'-q}}$$

(5.4.7)

where $\frac{\partial^i y(0,w)}{\partial w^i} = 0 \qquad i \varepsilon [0,L]$

and where $y(t,w) = \frac{\partial x(t,w)}{\partial w}$ and $K(t,w)$ is the feedback system to be determined. The optimal control is then assumed in the form of

$$u(t,w) = - \sum_{r=0}^{n} R^{-1}(t) B^T(t,w) K_{0v}(t) \frac{\partial^r s(t,w)}{\partial w^r}$$

where K_{0v} are symmetric partitioned matrices of K such that $\frac{\partial^m K_{0v}}{\partial w^m} = 0$ for $m \geq 1$. The plant equation (5.3.1) and the sensitivity differential equation (5.4.7) are used associated with a cost functional given by

$$J_0 = \frac{1}{2} \int_{t_0}^{t_f} [u^T R u + x^T Q x + S^T M S] dt$$

where M is a nLxnL symmetric positive semi-definite matrix.

The implementation of the optimal control is illustrated in Figure (7). More details about both approaches are given in references [9, 22].

In both approaches, the state sensitivity and its higher order derivatives are used to implement the optimal control. The realization of the functions $y, \frac{\partial y}{\partial w}, \ldots \frac{\partial^L y}{\partial w^L}$ merges as a new problem. Moreover, the assumptions in Kahne's approach are not realistic in many cases.

If the designer has the freedom to add to the given cost functional a term of state sensitivity as illustrated, he may as well have the freedom to use the following cost functional,

$$J = \frac{1}{2} \int_{t_0}^{t_f} (x^T Q x + \alpha u^T R u) dt \qquad (5.4.10)$$

where α is a constant. By trying several values for α, the designer may finally obtain the desired sensitivity.

In all the approaches discussed in this section, the original cost functional is modified to meet the sensitivity reduction requirement. Hence the control thus implemented will not be nominally optimal with respect to the given cost functions. In general, this is not desirable. Moreover, the reduction of cost

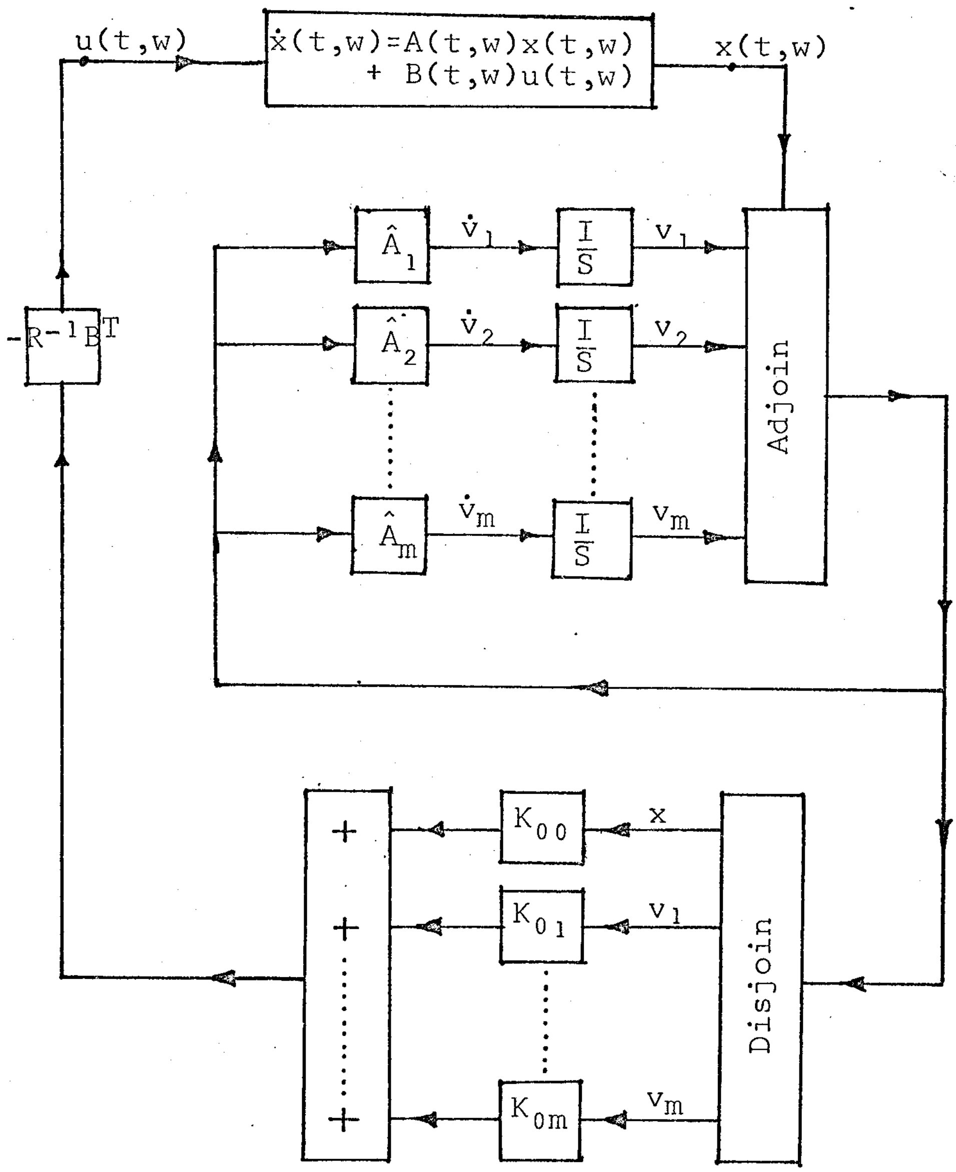


Figure 7.

Simulation of Optimal Control

by D'Angelo, Moe and Hendricks

sensitivity and terminal sensitivity have never been investigated. The next section illustrates the new approach in sensitivity reduction which has been innovated in Chapter IV.

5.5 Sensitivity Reduction by Implementation of Optimal Control:

It has been illustrated in Chapter IV that sensitivity reduction can be achieved by the application of the TDF implementation, i.e. the configuration with two degrees of freedom. Consider a linear system (5.2.1) with a nominal control $u^*(t)$ which is optimal with respect to a given quadratic cost functional (5.2.2). The implementation of the optimal control by TDF configuration is given by

$$u(t,w) = M(t)x(t,w) + g(t) \qquad (5.5.1)$$

where $x(t,w)$ is the state trajectory corresponding to the control $u(t,w)$ at any value of w. The time functions $M(t)$ and $g(t)$ are related by the condition that the control $u(t,w)$ must be optimal nominally, or mathematically $u(t,w_0) = u^*(t)$. Hence for any value of $M(t)$, the function $g(t)$ is determined from

$$g(t) = u^*(t) - M(t)x^*(t) \qquad (5.5.2)$$

where $x^*(t)$ is the known optimal state trajectory at nominal. The function M(t) is determined such that the insensitive condition is achieved whenever possible.

Let y(t) denote the state sensitivity corresponding to the TDF implementation (5.5.1) and the state sensitivity equation is

$$\dot{y}(t) = (A_0 + B_0 M)y(t) + f_w(t) \tag{5.5.3}$$

where $f_w(t) = A_w x^*(t) + B_w u^*(t)$ (5.5.4)

is a known function of time.

Assume that the control problem (5.2.1) and (5.2.2) is a fixed end-point problem. The cost sensitivity corresponding to the TDF implementation is

$$S_w^J(v) = -y^T(t_f)p^*(t_f) + c \tag{5.5.5}$$

where c is a rx1 vector given by equation (5.3.11). The cost insensitive condition is then given by

$$y^T(t_f)p^*(t_f) = c \tag{5.5.6}$$

Together with a cost functional

$$I(M) = \alpha\|y(t_f)\|^2 + \int_{t_0}^{t_f}\{\|y(t)\|^2 + \beta\|M(t)\|^2\}dt \tag{5.5.7}$$

an optimal implementation problem will be formulated as finding a matrix function M(t) driving $y(t_0) = 0$ to a given $y(t_f)$ via system (5.5.3) (5.5.4) and minimizing the cost function I(M). The boundary condition $y(t_f)$ is a nxr null matrix for the terminal insensitive design and $y(t_f)$ satisfies the condition (5.5.6) in the case of cost insensitive design.

The rest of the section is devoted in illustrating Pontryagin's minimum principle to solve the optimal implementation problem. For simplicity, assume that the plant is a scalar, i.e. r = 1. The feedback portion of the TDF implementation, M(t), is a qxn matrix function of time with the components $m_{kj}(t)$. Let y_i , a_{ij} , b_{ik} and f_i be the components of y(t), $A_0(t)$, $\bar{B}_0(t)$ and $f_w(t)$ respectively where i=1, ...n; j=1, ...n and k=1,...q . In component forms, equations (5.5.3) and (5.5.7) are rewritten as,

$$\dot{y}_i(t) = \sum_{j}^{n} a_{ij}y_j + \sum_{j,k}^{n,q} b_{ik}m_{kj}y_j + f_i \quad (5.5.8)$$

$$I(M) = \alpha\sum_{i}^{n} y_i^2(t_f) + \int_{t_0}^{t_f}\{\sum_{i}^{n} y_i^2(t) + \beta\sum_{k,j}^{q,n} m_{kj}^2(t)\}dt \quad (5.5.9)$$

Define a Hamiltonian as

$$H = \sum_{i}^{n} y_i^2(t) + \beta\sum_{k,j}^{q,n} m_{kj}^2(t) + \sum_{i,j}^{n,n} a_{ij}y_jz_i + \sum_{i}^{n} z_if_i + \sum_{i,j,k}^{n,n,q} z_ib_{ik}m_{kj}y_j \quad (5.5.10)$$

where $z_i(t)$ are the components of a nxl vector z(t). The cononical system is

$$\dot{z}_i(t) = -2y_i - \sum_{j}^{w} a_{ji}z_j - \sum_{j,k}^{n,q} z_j b_{jk} m_{ki} = -\frac{\partial H}{\partial y_i} \tag{5.5.11}$$

$$\dot{y}_i(t) = \sum_{i}^{n} a_{ij}y_j + \sum_{j,k}^{n,q} b_{ik} m_{kj} y_j + f_i = \frac{\partial H}{\partial z_i} \tag{5.5.12}$$

and the necessary condition for the optimality of m_{kj} is

$$0 = \frac{\partial H}{\partial m_{kj}} = 2\beta m_{kj} + \sum_{i}^{w} z_i b_{ik} y_j \tag{5.5.13}$$

Assume that $\beta > 0$, hence from equation (5.5.13), we have

$$m_{kj} = -\frac{1}{2\beta} \sum_{i}^{n} z_i b_{ik} y_j \tag{5.5.14}$$

Substituting equation (5.5.14) into (5.5.11) and (5.5.12), the canonical system becomes

$$\dot{y}_i(t) = \sum_{i}^{n} a_{ij}y_j - \frac{1}{2\beta} \sum_{jkl}^{nqn} b_{ik} z_l b_{lk} y_j + f_i \tag{5.5.15}$$

$$\dot{z}_i(t) = -2y_i - \sum_{j}^{n} a_{ji}z_i + \frac{y_i}{2\beta} \sum_{jkl}^{nqn} z_j b_{jk} z_l b_{lk} \tag{5.5.16}$$

Or in vector form, the optimal implementation M(t) is

$$M(t) = -\frac{1}{2\beta} B_0^T z y^T \tag{5.5.17}$$

associated with the canonical system,

$$\dot{y}(t) = (A_0 - \frac{1}{2\beta} B_0 B_0^T z y^T) y + f_w \tag{5.5.18}$$

$$\dot{z}(t) = -2y - A_0^T z + \frac{1}{2\beta} y z^T B_0 B_0^T z \tag{5.5.19}$$

Equations (5.5.17), (5.5.18) and (5.5.19) determines the feedback portion of the TDF implementation (5.5.1) and the open portion g(t) is given by equation (5.5.2). The n boundary conditions that $y(t_0) = 0$ are assumed. To solve the canonical system of differential equation (5.5.18) and (5.5.19), n more boundary condition are required. For terminal insensitivity, the boundary condition is $y(t_f) = 0$. For cost insensitivity, the transversality condition requires that

$$z(t_f) = \gamma p^*(t_f) + 2\alpha y(t_f) \tag{5.5.20}$$

where γ is a constant. The boundary condition for the canonical system is then given by equations (5.5.6) and (5.5.20). The cost insensitivity with terminal sensitivity reduction, the boundary condition on $y(t_f)$ is the point which is on the hyperplane (5.5.6) and which yields a minimum distance to the origin. In any case, it becomes a two-point boundary-value problem. Analytical solution is not likely and high-speed computer must be used.

The cost function I(M) in equation (5.5.9) contains two constants, i.e. α and β. These two constants must be fixed before the optimal implementation block M(t) is calculated. The constant α is not included in the system of differential equations (5.5.18) and (5.5.19) and therefore is not significant. The constant β is inserted in order to remove the problem involved in the singular extremal. From physical point of view the feedback gain $||M(t)||^2$ is limited when $\beta \neq 0$. By doing that, the soft constraint is then removed. In general the value for β is small enough so that the integral state sensitivity term becomes significant. However, as a result of primary investigation, too small a value for β will cause instability of the differential system (5.5.18) and (5.5.19).

In this section, the new approach has been applied to reduce the sensitivities for a linear systems with quadratic cost functional. Differing from Kahne's or D'Angelo's approaches, the state sensitivity is not used as the components of implementation. Also, the TDF configuration enables the designer to realize the possibility of both cost insensitive and terminal insensitive designs. However the development so far is beyond perfection. Many problems are still involved in the new approach, for example, the systematic way of obtaining a solution for the canonical system (5.5.18) and (5.5.19), which is actually a problem of two-point boundary-value searching.

CHAPTER VI

CONCLUSION

In this thesis, a new approach to sensitivity reduction in optimal control systems is introduced. Current researchers endeavour to modify the given cost functional in order to satisfy the sensitivity specification. The application of the TDF configuration removes the necessity of changing the original cost functional. Cost insensitive and terminal insensitive designs have, in the past, been ignored because of the apparent impossibility. However, it has been clearly shown here that it is indeed realizable. The application of the TDF implementation to the reduction of the integral state sensitivity is not so well developed as the other two. More information is required about the sensitivity specification and about the constraints concerning the physical feedback portion of the TDF configuration is required. Upon knowing this the TDF configuration is readily applicable with some modifications wherever required.

As noted in the introduction of the thesis, the field of sensitivity analysis in optimal control theory is relatively new. Moreover, the application of the TDF configuration in the implementation of a given optimal control has been demonstrated for the first time in this thesis. Because of these two facts, many results and concepts developed here are primitive and prospective in nature. Refinement and verification of the theory are obviously required.

As a result of introducing the TDF configuration, various new problems may arise. The one which is of immediate concern is the investigation of the so-called two-point boundary-value problem for non-linear systems with special emphasis on the numerical techniques. Here, the canonical system of equations (5.5.18) and (5.5.19) demands more attention. The comparison with the TDF configuration and the open- or closed-loop implementations is also a new topic in sensitivity analysis. Here it has already been shown that, in general, there exists an implementation of the optimal control by TDF configuration which is superior to the closed- or open-loop configurations in the aspects of cost or terminal sensitivities. However, in the light of state sensitivity, this is not necessarily true. Further investigation is required to supply the answer.

Besides applying the TDF configuration in reducing the sensitivity, the configuration can be used to solve the multi-optimality problem which can be stated as finding an implementation such that the control is optimal at more than one value of the plant parameter.

In conclusion, the TDF configuration is introduced as a new approach to sensitivity reduction. The potentiality of the configuration is far beyond exhausted and more effort is required to discover its full value.

REFERENCES

1. Anderson, B. D. O., "Sensitivity Improvement Using Optimal Control", Proc. IEE (London), Vol. 113, June, 1966, pp 1084-1086.
2. Athans, M. and Falb, P. L., "Optimal Control: An Introduction to the Theory and its Applications", McGraw-Hill Co., New York, 1966.
3. Atluri, S. R., "Sensitivity of Optimal Control Systems", M. Eng., (Thesis, 1966), McMaster University, Ontario, Canada.
4. Bellman, R., "Dynamic Programming", Princeton University Press, Princeton, New Jersey, 1957.
5. Berkovitz, L. D., "On Control Problems with Bounded State Variables", J. Math. Anal. Appl., Vol. 5, 1962, pp 488-498.
6. Bode, H. B., "Network Analysis and Feedback Amplifier Design", D. VanNostrand Co., Inc., 1945.
7. Cruz, J. B. and Perkins, W. R., "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design", IEEE Trans. on Automatic Control, Vol. AC-9, July, 1964, pp 216-222.
8. Dai, S. H., "Signal Representation in Vector Space", B. Sc. (Thesis, 1968), Ottawa University, Ottawa, Ontario, Canada.

9. D'Angelo, H., Moe, M. L., and Hendricks, T. C., "Trajectory Sensitivity of an Optimal Control Systems", Proc. of Fourth Annual Allerton Conf. on Circuit and System Theory, (The Uni. of Illinois, Urbana, Ill.) pp 489-498.

10. Dompe, R. and Dorf, R. C., "The design of Optimal Low Sensitivity Linear Control Systems", Proc. of Fifth Annual Allerton Conf. on Circuit and System Theory, (The Uni. of Illinois, Urbana, Ill.) pp 170-180.

11. Dorato, P., "On Sensitivity in Optimal Control Systems", IEEE Trans. on Automatic Control, Vol. AC-8, July, 1963, pp 256-257.

12. Dunn, J. C., "Further Results on the Sensitivity of Optimally Controlled Systems", IEEE Trans. on Automatic Control, Vol. AC-12, June, 1967, pp 324-326.

13. Gauthier, L. P. and Cadzow, J. A., "Performance Sensitivity of Optimal Control-Systems with Random Variable Parameters", IEEE Trans. on Automatic Control, Vol. AC-12, April 1967, pp 207-208.

14. Gavrilovic, M. and Petrovic, R., "On the Synthesis of the Least Sensitive Control", in <u>Sensitivity Methods in Control Theory</u>, Proc. International Symposium on Sensitivity, (Dubrovnik, Yugoslavia,

1964), L. Radanovic, Ed., New York, Pergamon, 1966, pp. 407-417.

15. Goldstein, A. J. and Kuo, F. F., "Multiparameter Sensitivity", IRE Trans. on Circuit Theory, Vol. CT-8, June, 1961, pp 177-178.

16. Guardabassi, G. Locatelli, A. and Rinaldi, S., "On the Insensitivity of the Terminal Conditions in Open-Loop Control Systems", IEEE Trans. on Automatic Control Vol. AC-13, Aug., 1968, pp 442-443.

17. Hakimi, S. L. and Cruz, J. B., "Measures of Sensitivity for Linear Systems with Large Multiple Parameter Variation", IRE Wescon Convention Record, Part 2, Aug., 1960, pp 109-115.

18. Kelly, H. J., Kopp, R. E., and Moyer, H. G., "Singular Extremals", in Topics in Optimization, G. Leitmann, Ed., Academic Press, 1967, pp 63-101.

19. Holtzman, J. M. and Horing, S., "The Sensitivity of Terminal Conditions of Optimal Control Systems to Parameter Variations", IEEE Trans. on Automatic Control, Vol. AC-10, Oct., 1965, pp 420-426.

20. Horowitz, I. M., "Synthesis of Feedback Systems", Academic Press, 1963.

21. Johnson, C. D., "Singular Solutions in Problems of Optimal Control", in Advances in Control Systems, C. T. Leondes, Ed., Vol. 2, Academic Press, 1965, pp 209-268.

22. Kahne, S. J., "Low Sensitivity Design of Optimal Linear Control Systems", Proc. of Fourth Annual Allerton Conf. on Circuit and System Theory, (The Uni. of Illinois, Urbana, Ill.) pp 499-507.

23. Kalman, R. E., "Contributions to the Theory of Optimal Control", Boletin de la Sociedad Mathematica Mexicana, Vol. 5, 1960, pp 102-119.

24. Kalman, R. E., "What is a Linear Control System Optimal?", Trans. ASME, J. Basic Engrg. Series D, Vol. 86, March, 1964, pp 51-60.

25. Kokotovic, P. V. and Rutman, R. S., "Sensitivity of Automatic Control Systems", Automation and Remote Control Vol. 26, 1965, pp 727-749.

26. Kokotovic, P. V. and Heller, J., "Direct and Adjoint Equations for Parameter Optimization", IEEE. Trans. on Automatic Control, Vol. AC-12, Oct., 1967, pp 609-610.

27. Kokotovic, P. V., Heller, J., and Sannuti, P., "Sensitivity Comparison of Optimal Controls", Int. J. of Control, Vol. 9, Jan., 1969, pp 111-115.

28. Kreindler, E., "On the closed-loop Sensitivity Reduction of Nonlinear Systems", Int. J. of Control, Vol. 6, Aug., 1967, pp 171-178.

29. Kreindler, E., "Closed Loop Sensitivity Reduction of Linear Optimal Control Systems", IEEE Trans. on Automatic Control, Vol. AC-13, June 1968, pp.254-262.

30. Kreindler, E., "Sensitivity of Time-Varying,Linear Optimal Control Systems", J. of Optimization Theory and Application, Vol. 3, No. 2, Feb., 1969, pp 98-106.

31. Kreindler, E., "On Trajectory Sensitivity in Optimal Control", Proc. IEEE, Vol. 57, No. 4, Apr., 1969, pp 695-696.

32. Lamont, G. B. and Kahne, S. J., "Time Domain Sensitivity Comparisons", Proc. of Fifth Annual Allerton Conference, (The Uni. of Illinois, Urbana, Ill.) pp 161-169.

33. Lee, E. B., "A sufficient Condition in the Theory of Optimal Control", J. SIAM on Control, Vol. 1, 1963, pp 241-245.

Lee, E. B. and Markus, L., "Foundations of Optimal Control Theory", John Wiley & Sons, Inc., New York, 1967.

34. Lee, S. C., "On Multiparameter Sensitivity", Proc. of the Third Annual Allerton Conf. on Circuit and

System Theory, (The Uni. of Illinois, Urbana, Ill.), pp 407-420.

35. Mason, S. J., "Feedback Theory - Some Properties of Signal Flow Graph", Proc. IRE, Vol. 41, Sept., 1953, pp 1144-1156.

36. McGill, R., "Optimal Control, Inequality State Constraints, and the Generalized Newton-Raphson Algorithm", J. SIAM on Control, Vol. 3, 1965, pp. 291-298.

37. Pagurek, B., "Sensitivity of the Performance of Optimal Linear Control Systems to Parameter Variations", Int. J. of Control, Vol. 1, Jan., 1965, pp 33-45

"Sensitivity of the Performance of Optimal Linear Control Systems to Parameter Variations", IEEE Trans. on Automatic Control, Vol. AC-10, Apr., 1965, pp 178-180.

"Sensitivity of the Performance of Optimal Control Systems to Plant Parameter Variations", in <u>Sensitivity Methods in Control Theory</u>, Proc. International Symposium on Sensitivity, (Dubrovnik , Yugoslavia, 1964), L. Radanovic, Ed., New York, Pergamon, 1966, pp 383-395.

38. Perkins, W. R., and Cruz, J. B., "The Parameter Variation Problem in State Feedback Control Systems", Trans. ASME, J. Basic Engrg., Series D, Vol. 87,

March, 1965, pp 120-124.

39. Porter, W. A., "On Singularities that Arise in a Class of Optimal Feedback Systems", IEEE Trans. on Automatic Control, Vol. AC-11, July, 1966, pp 617-619.

40. Pontryagin, L. S., "Ordinary Differential Equations", Addison-Welsey Pub. Co., Inc., Reading, Mass., 1962.

41. Pontryagin, L. S. Boltyanskii, V., Gamkreldize R., and Mishcheuko, E., "The Mathematical Theory of Optimal Processes", Interscience Pub., Inc., New York, 1962.

42. Rohrer, R. A. and Sobral, M. Jr., "Sensitivity Considerations in Optimal System Design", IEEE Trans. on Automatic Control, Vol. AC-10, Jan., 1965, pp 43-48.

43. Rutman, R. S. and Kokotovic, P. V., "On the Generalization of System Equation", Proc. of the Third Annual Allerton Conf. on Circuit and System Theory, (The Uni. of Illinois, Urbana, Ill.) pp 27-34.

44. Siljak, D. D. and Dorf, R. C., "On the Minimization of Sensitivity in Optimal Control Systems", Proc. of the Third Annual Allerton Conf. on Circuit and System Theory, (The Uni. of Illinois,

Urbana, Ill.) pp 225-229.

45. Sinha, N. K. and Atluri, S. R., "Sensitivity of Optimal Control Systems", Proceedings of the Fourth Allerton Conference on Circuit and System Theory, (The University of Illinois, Urbana, Illinois), pp. 508-516.

46. Sinha, N. K. and Atluri, S. R., "On the Sensitivity of Optimal Systems with Quadratic Performance Criteria ", IEEE Trans. on Automatic Control, Vol. AC-12, Apr., 1967, pp 208-209.

47. Sobral, M. Jr., "Sensitivity in Optimal Control Systems", Proc. of IEEE, Vol. 56, No. 10, Oct., 1968, pp. 1644-1652.

48. Truxal, J. C., "Control System Synthesis", McGraw-Hill Co., Inc., New York, 1955.

49. Ur, H., "Root Locus Properties and Sensitivity Relations in Control Systems", IRE Trans. on Automatic Control, Vol. AC-5, No. 1, Jan., 1960, pp 57-65.

50. Witsenhausen, H. S., "On the Sensitivity of Optimal Control Systems", IEEE Trans. on Automatic Control, Vol. AC-10, Oct., 1965, pp 495-496.

51. Youla, D. C. and Dorato, P., "On the Comparison of Sensitivities of Open-Loop and Closed-Loop Optimal Control Systems", IEEE Trans. on Automatic Control, Vol. AC-13, Apr., 1968, pp 186-188.

52. Zadeh, L. A. and Desoer, C. A., "Linear System Theory: The State Space Approach", McGraw-Hill Co., New York, 1963.

53. Sinha, N. K. and Dai, S. H., "Reduction of the Sensitivity of an Optimal Control System to Plant Parameter Variations", to be published in IEEE Trans. on Automatic Control, Vol. AC-15, No. 5, Oct., 1970.