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# SOME GENERAL METHODS OF SOLUTION APPLIED TO DIFFERENTIAL EQUATIONS ARISING FROM PROBLEMS IN MATHEMATICAL PHYSICS.

by

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# INTRODUCTION

In what follows I have selected a few problems arising in Mathematical Physics and solved them by methods which are general in their application. Since all but the simplest problems give rise to partial differential equations I have included one chapter only, the first, in which the equations arising are ordinary differential equations. This first chapter is composed of a group of problems on the motion of helical springs, and is included because it illustrates the application of operational methods to ordinary differential equations with constant coefficients.

In treating the solution of partial differential equations no general or complete solution has been obtained except in Chapter 2 , and in problems of the type treated the particular solution which satisfies the initial and boundary conditions of the problem can usually be obtained without first deriving a general or a complete solution.

I have chosen problems to illustrate various methods of solution . Thus Chapter 2 illustrates the application of Fourier Series and also the method of solution of partial differential equations by operators. This chapter serves to compare operational methods of solution of partial differential equations with other and older methods. Chapters 3 and 4 illustrate the application of Bessel Functions and Legendre Polynomials.

In each of chapters 3, 4, and 5 I have assumed that a particular solution exists in the form of a product of two functions, each of which is a function of a different independant variable. This assumption is often successful in problems of the type treated and is justified by its success.

I have derived the differential equations studied and included those assumptions made from the theory of Physics as well as statements of any approximations to actuality desirable for the solution of the equation. Some of the initial and boundary conditions have been chosen so as to simplify solution but I have attempted to chose conditions compatible with actuality and to simplify the problems by neglecting factors which have a negligible bearing on the result. Solutions obtained should therefore bear to actuality a relation consistent with the reasonableness of assumptions made and approximations included in the working.

I have included the pure mathematical treatment of functions applied to the problems by footnote references to the source of the result used or by a statement of the results in appendices to the chapters. Wherever possible I have treated each solution fully, applying well known results only where desirable for brevity and otherwise obtaining the solution from a full study of the particular functions arising. By so doing I have attempted to make each chapter, though dealing only with a particular problem, illustrative of a large body of similar problems.

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## CONTENTS

( Page numbers are given in brackets after sub-headings. )

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## CHAPTER I.

### Application of operational methods to ordinary differential equations with constant coefficients.

The solution of ordinary differential equations with constant coefficients can readily be effected by operational methods which are fully explained in " Operational Methods in Applied Mathematics ", by H. S. Carslaw and J. C. Jaeger ( Oxford University Press ).[I] Reference should be made to this book for a comprehensive treatment of the operational methods used in this chapter. Some important results and a brief explanation of the methods used are contained in appendices to the chapter.

The following problems are concerned with the motion of light helical springs. They are representative of a large number of similar problems arising in the various branches of Mathematical Physics.

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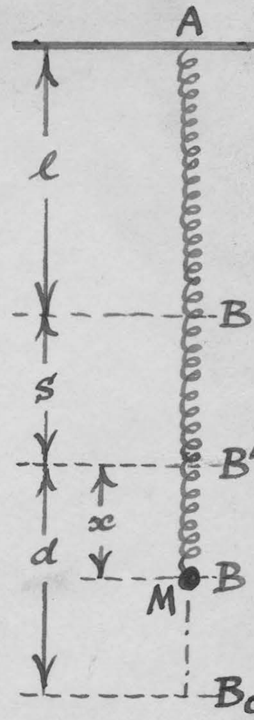
I. Hereafter this book will be refered to as O.M.inA.M.

Motion of a helical spring.




Suppose that one end of a light helical spring ( or light elastic chord ) is attached to a fixed point of support at $A$ and that the spring hangs vertically in a state of static equilibrium with a mass $M$ attached to the other end of the spring, $B'$, the elongation of the spring being $s$. If an additional force is applied to $M$ to produce a further extension $d$, the mass being displaced to $B_o$ under the action of this force and held in static equilibrium at $B_o$, and at time $t=0$ this force is removed, then the mass $M$ will start to move.

In order to simplify discussion of the subsequent motion of $M$, whilst at the same time retaining conditions which will approximate to those of actuality, I shall make the following assumptions :-

(i) That the mass $M$ attached to the spring is large compared with the mass of the spring, and hence that the mass of the spring may be neglected in comparison with $M$.

(ii) That the spring is perfectly elastic, so that material stresses and frictions may be neglected, and Hooke's Law holds for the spring. Hence the extension of the spring is proportional to the applied force.

(iii) That the medium in which the spring moves is such that resistance to the motion of the mass $M$ is proportional to the velocity, and that the medium offers negligible resistance to the flexing of the spring. i.e. Viscous damping occurs, the damping force being proportional to the velocity of $M$ and opposite in direction to the direction of displacement of $M$.

In the original state of static equilibrium we have from (ii) above

$$Mg = ks \qquad \text{I.I}$$

where $k$ is the modulus of elasticity of the spring.

At time $t>0$ let the displacement of $M$ below $B'$ be $x$, reckoned positive in a downwards direction. At time $t=0$ we have

$$x = d, \qquad \frac{dx}{dt} = 0 \qquad \text{I.2}$$

At time $t>0$ the forces acting on $M$ are

(a) $Mg$ acting vertically downwards.

(b) Tension $T$ in the spring acting vertically upwards.

(c) Viscous damping proportional to the velocity, say $h\frac{dx}{dt}$, acting in the direction opposite to the displacement of $M$.

Hence applying Newton's second Law of Motion we get

$$\begin{aligned} M\frac{d^2x}{dt^2} &= Mg - T - h\frac{dx}{dt} \\ &= Mg - k(x+s) - h\frac{dx}{dt} \\ &= -kx - h\frac{dx}{dt} \end{aligned}$$

since $Mg = ks$ from I.I.

i.e. $$\left(D^2 + \frac{h}{M}D + \frac{k}{M}\right)x = 0$$

or $$\left(D^2 + 2bD + a^2\right)x = 0, \quad t>0. \qquad \text{I.3}$$

where $b = \frac{h}{2M} > 0$ and $a^2 = \frac{k}{M} > 0$.

Motion in a vacuum.

If the motion takes place in a vacuum, or if the damping force is negligible, equation I.3 becomes

$$\left(D^2 + a^2\right)x = 0, \quad t>0. \qquad \text{I.4}$$

with $x = d$, $Dx = 0$, when $t = 0$.

Multiplying I.4 by $e^{-pt}$ $(p>0)$ and intergrating with respect to $t$ from $0$ to $\infty$, we get the Subsidary Equation[2]

2 See Appendices A and B.

$$(p^2 + a^2)\bar{x} = pd$$

Thus $$\bar{x} = d\left\{\frac{p}{p^2 + a^2}\right\}$$

Therefore $$x = d \cos at \qquad \text{I.5}$$

The motion is therefore simple harmonic motion with amplitude $d$ and period given by

$$T = \frac{2\pi}{a} = 2\pi\sqrt{\frac{M}{k}} \qquad \text{I.6}$$

Thus the period depends on the modulus of elasticity of the spring, $k$, the larger $k$ is the smaller being the period and the greater the frequency.

Viscous damping.

If viscous damping is not negligible equation I.3 is

$$(D^2 + 2bD + a^2)x = 0, \quad t > 0$$

with $x = d$, $Dx = 0$, when $t = 0$.

The Subsidiary Equation is

$$(p^2 + 2bp + a^2)\bar{x} = (p + 2b)d$$

Thus $$\bar{x} = d\left\{\frac{p + 2b}{p^2 + 2bp + a^2}\right\}$$

Three possible cases arise for consideration determined by the nature of the roots of $p^2 + 2bp + a^2 = 0$

(i) $b^2 - a^2 > 0$, roots being real and distinct.

(ii) $b^2 - a^2 = 0$, roots being real and equal.

(iii) $b^2 - a^2 < 0$, roots being complex and distinct.

Case (i)----- $b^2 - a^2 > 0$

Let the roots be $m_1 = -b + \sqrt{b^2 - a^2}$ and $m_2 = -b - \sqrt{b^2 - a^2}$, so that $m_2 < m_1 < 0$

The Subsidiary Equation is

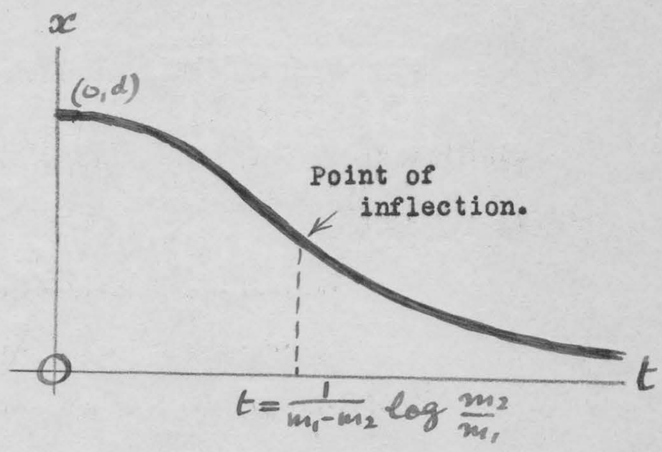
$$(p - m_1)(p - m_2)\bar{x} = (p + 2b)d$$

Hence
$$\bar{x} = d\left\{\frac{p+2b}{(p-m_1)(p-m_2)}\right\} = d\left\{\frac{p-2(m_1+m_2)}{(p-m_1)(p-m_2)}\right\}$$
$$= \frac{d}{m_1-m_2}\left\{\frac{-m_2}{p-m_1} + \frac{m_1}{p-m_2}\right\}$$

Therefore
$$x = \frac{d}{m_1-m_2}\left\{-m_2 e^{m_1 t} + m_1 e^{m_2 t}\right\}$$

$$Dx = -\frac{m_1 m_2 d}{m_1-m_2}\left\{e^{m_1 t} - e^{m_2 t}\right\}$$
$< 0$ for $t > 0$, since $m_2 < m_1 < 0$.
$= 0$ for $t = 0$.
$$D^2 x = -\frac{m_1 m_2 d}{m_1 - m_2}\left\{m_1 e^{m_1 t} - m_2 e^{m_2 t}\right\}$$
$= 0$ for $m_1 e^{m_1 t} = m_2 e^{m_2 t}$
$$t = \frac{1}{m_1 - m_2}\log\frac{m_2}{m_1}$$

The graph of the displacement of $M$ against $t$ is of the form shown in the following diagram.




The displacement $x$ approaches zero as $t$ increases. i.e. $M$ approaches $B'$ at a steadilly decreasing speed and its distance from $B'$ becomes infinitesimally small as $t \to \infty$.

<u>Case (ii) ----- $b^2 = a^2$. $m_1 = m_2 = -b < 0$.</u>

The Subsidiary Equation is
$$(p+b)^2\,\bar{x} = (p+2b)\,d$$

Hence
$$\bar{x} = d\left\{\frac{1}{p+b} + \frac{b}{(p+b)^2}\right\}$$

Therefore $$x = d\left\{e^{-bt} + bt\,e^{-bt}\right\}$$
$$= d(1+bt)\,e^{-bt}$$

Hence for $t>0$, $x>0$ and $\lim\limits_{t\to\infty} \dfrac{d(1+bt)}{e^{bt}} = 0.$

Also $$Dx = d\left\{\frac{b\,e^{bt} - b\,e^{bt}(1+bt)}{e^{2bt}}\right\} = -\frac{db^2 t}{e^{bt}}.$$
$< 0$ for $t>0$.
$= 0$ for $t=0$.
$$D^2x = -db^2\left\{\frac{e^{bt} - bt\,e^{bt}}{e^{2bt}}\right\} = -\frac{db^2(1-bt)}{e^{bt}}.$$
$= 0$ for $t = 1/b$
$< 0$ for $0 \leq t < 1/b$
$> 0$ for $t > 1/b$

The graph of the displacement of $M$ against $t$ is of a form similar to that for Case (i), the point of inflection being at $t = \frac{1}{b}$.

$x$ remains positive and approaches zero as $t$ increases.

$M$ approaches $B'$ at an increasingly slow speed.

This type of motion is called " dead beat ". If the retarding force is decreased by an arbritrarily small amount, so that $b^2 - a^2 < 0$, we have the third case for which the motion is oscillatory

Case (iii) ----- $b^2 - a^2 < 0 \;(a > b > 0)$.

The roots are complex and different. The Subsidiary Equation is
$$(p^2 + 2bp + a^2)\,\bar{x} = (p + 2b)\,d$$
$$\therefore \quad \bar{x} = d\left\{\frac{p+2b}{(p+b)^2 + a^2 - b^2}\right\}$$
$$= d\left\{\frac{p+b}{(p+b)^2 + (\sqrt{a^2-b^2})^2} + \frac{b}{(p+b)^2 + (\sqrt{a^2-b^2})^2}\right\}$$

Therefore
$$x = d\left\{\cos\sqrt{a^2-b^2}\,.t + \frac{b}{\sqrt{a^2-b^2}}\sin\sqrt{a^2-b^2}\,t\right\}e^{-bt} = K e^{-bt}\sin(ct+\alpha)$$

where $c = \sqrt{a^2-b^2}$, $K = \dfrac{da}{c}$ and $\alpha = \tan^{-1}\dfrac{c}{b}$.

The solution represents an oscillatory motion with an amplitude $Ke^{-bt}$ which decreases exponentially. The period is

$$T = \frac{2\pi}{c} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

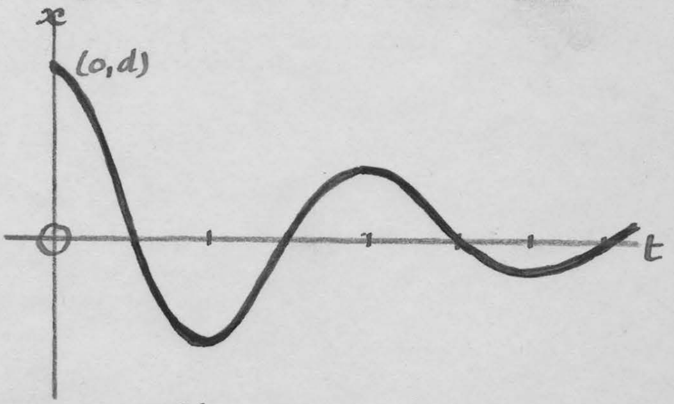
$$Dx = Ke^{-bt}\left\{c\cos(ct+\alpha) - b\sin(ct+\alpha)\right\}$$

$$= 0 \text{ for } t = 0$$

$$= 0 \text{ for } t > 0 \text{ when } \tan(ct+\alpha) = \frac{c}{b} = \tan\alpha$$

i.e. for $t = \frac{n\pi}{c}$, where $n$ is an integer.

The graph of the displacement of $M$ against $t$ is of the form shown in the following diagram.




The amplitude steadily decreases as $t$ increases and approaches zero as $t$ tends to infinity. The period of oscillation, $\frac{2\pi}{\sqrt{a^2-b^2}}$, remains constant.

In the undamped case the period was $\frac{2\pi}{a}$ and so the period is seen to be increased by damping.

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Forced vibration of a helical spring.

In the preceeding discussion it was supposed that the point of support of the spring, $A$, was fixed, but $A$ may move in accordance with some law which gives the displacement of $A$ as a function of the time $t$. Suppose that the displacement of $A$ is given by

$$y = f(t)$$

where $y$ is measured positive in a downward direction. The resulting motion of the spring is said to be a forced vibration.

If at time $t > 0$ the displacement of $M$ from its position of static equilibrium is $x$ then the actual extension of the spring is

$$s + x - y$$

The equation of motion of $M$ is

$$M\frac{d^2x}{dt^2} = Mg - k(s+x-y) - h\frac{dx}{dt}$$

$$= -k(x-y) - h\frac{dx}{dt}, \text{ since } Mg = ks \text{ from I.1}$$

$$= -k(x - f(t)) - h\frac{dx}{dt}$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{h}{M}\frac{dx}{dt} + \frac{k}{M}x = \frac{k}{M}f(t)$$

i.e. $$(D^2 + 2bD + a^2)x = a^2 f(t), \quad t>0 \qquad \text{I.7}$$

where $b = \frac{h}{2M} > 0$ and $a^2 = \frac{k}{M} > 0$.

An interesting case arises when $a^2 - b^2 > 0$ ( i.e. $a > b > 0$ ) and $f(t)$ is a simple harmonic motion of period $\frac{2\pi}{\omega}$.[3] Then

$$f(t) = A \sin \omega t$$ ( Where the amplitude $A$ is constant )

and I.7 becomes

$$(D^2 + 2bD + a^2)x = a^2 A \sin \omega t, \quad t>0 \qquad \text{I.8}$$

$$x = d, \quad Dx = 0 \quad \text{for} \quad t = 0$$

The Subsidiary Equation is

$$(p^2 + 2bp + a^2)\bar{x} = a^2 A \int_0^\infty e^{-pt} \sin \omega t \, dt + (p+2b)d$$

$$= \frac{a^2 A \omega}{p^2 + \omega^2} + (p+2b)d$$

$$\therefore \quad \bar{x} = \frac{a^2 A \omega}{(p^2+\omega^2)(p^2+2bp+a^2)} + \frac{(p+2b)d}{p^2+2bp+a^2}$$

The second term on the right hand side contributes to the solution that result obtained for the preceeding problem, Case (iii), and this contribution is due to free vibration. As $t$ becomes great the contribution of this term becomes and thereafter remains small and a steady state of motion will be reached to which its contribution is negligible. The final steady state of motion can therefore be considered as determined by

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3 Two other cases arise as in the preceeding problem and may be treated similarly.

$$\bar{x} = \frac{a^2 A\omega}{(p^2+\omega^2)(p^2+2bp+a^2)}$$

Expressing the right-hand side in partial fraction form we get[4]

$$\bar{x} = \frac{a^2 A\omega}{(a^2-\omega^2)^2+4b^2\omega^2}\left\{\frac{(a^2-\omega^2)-2bp}{p^2+\omega^2} + \frac{2bp+4b^2+(\omega^2-a^2)}{p^2+2bp+a^2}\right\}$$

$$= \frac{a^2 A\omega}{(a^2-\omega^2)+4b^2\omega^2}\left\{\frac{(a^2-\omega^2)-2bp}{p^2+\omega^2}\right\} + \frac{2ba^2 A\omega}{(a^2-\omega^2)^2+4b^2\omega^2}\left\{\frac{p+\frac{4b^2+(\omega^2-a^2)}{2b}}{p^2+2bp+a^2}\right\}$$

The second term on the right-hand side is similar to the term already neglected. From analogy to Case (iii) of the preceeding problem its solution is of the form

$$K' e^{-bt} \sin(ct+\alpha')$$

where $K'$ and $\alpha'$ are constants and $c=\sqrt{a^2-b^2}$.

Hence as $t$ becomes great the contribution of this term to the final steady state may be neglected. The final steady state is thus determined by

$$\bar{x} = \frac{a^2 A\omega}{(a^2-\omega^2)^2+4b^2\omega^2}\left\{\frac{(a^2-\omega^2)-2bp}{p^2+\omega^2}\right\}$$

$$= \frac{a^2 A}{(a^2-\omega^2)^2+4b^2\omega^2}\left\{(a^2-\omega^2)\frac{\omega}{p^2+\omega^2} - 2b\omega\frac{p}{p^2+\omega^2}\right\}$$

Therefore

$$x = \frac{a^2 A}{(a^2-\omega^2)^2+4b^2\omega^2}\left\{(a^2-\omega^2)\sin\omega t - 2b\omega\cos\omega t\right\}$$

$$= P\sin(\omega t-\beta) \qquad 1.9$$

where $P = \dfrac{a^2 A}{\sqrt{(a^2-\omega^2)^2+4b^2\omega^2}}$ and $\tan\beta = \dfrac{2b\omega}{a^2-\omega^2}$

Thus equation 1.9 represents the steady state of motion reached when contributions due to free vibration have become negligible. It is a sinusoidal motion of frequency $\frac{\omega}{2\pi}$ and of amplitude $P$. If $\omega$ is large the amplitude is small and the effect of the impressed force is small.

If $\omega=a$ the amplitude is $\dfrac{a^2 A}{2ab} = A\sqrt{\dfrac{kM}{h}}$

4 See Appendix C.

Thus the amplitude may become large if the damping resistance coefficient $h$ is small when compared with $kM$.

For any given medium the maximum amplitude occurs when

$$(a^2-\omega^2)^2 + 4b^2\omega^2 \text{ is a minimum.}$$

Since $a$ and $b$ are positive constants for any given medium we get a minimum for

$$\frac{d}{d\omega}\left\{(a^2-\omega^2)^2 + 4b^2\omega^2\right\} = 0$$

$$-4\omega(a^2-\omega^2) + 8b^2\omega = 0$$

$$\omega(a^2-\omega^2-2b^2) = 0$$

$\omega = 0$ gives a maximum. Hence for real $\omega$ $(a^2 > 2b^2)$ the minimum is given by

$$\omega = \sqrt{a^2-2b^2} = \sqrt{\frac{2Mk-h^2}{2M^2}}$$

Thus if the impressed frequency equals the natural frequency of vibration of the spring the amplitude may become dangerously large and the displacement of $M$ become so great as to break the spring. e.g. The natural frequency of the spring considered is $\frac{1}{T} = \frac{\sqrt{a^2-b^2}}{2\pi}$, and if $\omega^2 = a^2-b^2$ ( Which is greater than the minimum given by $\omega^2 = a^2-2b^2$ ) then the impressed frequency $\frac{\omega}{2\pi}$ can equal $\frac{\sqrt{a^2-b^2}}{2\pi}$. When the impressed frequency is equal to the natural frequency of vibration we get the phenomenon of resonance. It is to obviate the possibility that the frequency of the impulses generated by marching men may strike that of a part of the structure of a bridge over which they are marching that a column of soldiers is ordered to break step. The Biblical account of the fall of the walls of Jericho may also be explained if the seven priests blowing on seven trumpets struck a sonic vibration identical with that of a natural frequency of vibration of the structure of a part of the walls.

If the viscous damping is neglected ( i.e. $h=0$ and therefore $b=0$ )

and if $\omega = a$ equation I.8 becomes

$$(D^2+a^2)x = a^2 A \sin at, \quad t>0 \tag{I.10}$$

$$x=d, \; Dx=0 \quad \text{for} \quad t=0$$

The Subsidiary Equation is

$$(p^2+a^2)\bar{x} = a^2 A \int_0^\infty e^{-pt} \sin at \, dt + pd$$

$$= \frac{a^2 A a}{p^2+a^2} + pd$$

$$\bar{x}' = \frac{a^3 A}{(p^2+a^2)^2} + \frac{pd}{p^2+a^2}$$

Therefore

$$x = \frac{a^3 A}{2a^3}(\sin at - at\cos at) + d\cos at$$

$$= \frac{A}{2}\sin at + \left(d - \frac{Aa}{2}t\right)\cos at \tag{I.II}$$

Again if $x=0$, $Dx=0$ when $t=0$

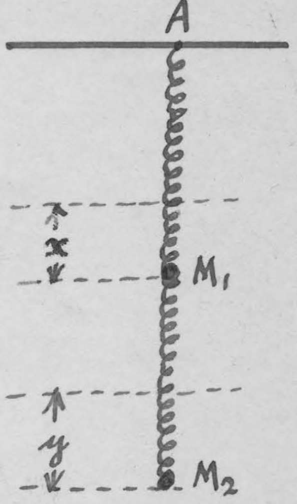
$$x = \frac{A}{2}(\sin at - at\cos at) \tag{I.12}$$

Both I.II and I.12 represent vibrations compounded from a first term of constant amplitude $\frac{A}{2}$ and a second term the amplitude of which increases with $t$ and can exceed any preassigned value if $t$ is taken sufficiently large. It would therefore appear that it is possible to produce infinite amplification and hence an infinite force. In practice the resistance is never zero and although forced vibrations may become large when the resistance is small the amplitude can never become infinitely large.

**Simultaneous Ordinary Differential Equations with constant coefficients.**

Consider masses $M_1$ and $M_2$ suspended from two light perfectly elastic springs as shown in the diagram. The point $A$ is fixed. Suppose that the moduli of elasticity of the springs are $k_1$ and $k_2$ respectively, and let $s_1$ and $s_2$ be the extensions of the springs in the position of static equilibrium. At time $t>0$ let the displacements of the masses from this equilibrium position be $x$ and $y$ respectively. At time $t=0$ let

$$x = x_0\,, \quad y = y_0\,, \quad Dx = Dy = 0.$$




If the motion takes place in a medium which offers negligible resistance to the motion we have for the lower spring

$$M_2 D^2 y = M_2 g - k_2(s_2 + y - x)$$

$$= -k_2(y-x) \quad \text{since} \quad M_2 g = k_2 s_2$$

and for the upper spring

$$M_1 D^2 x = M_1 g + k_2(y-x) - k_1(s_1 + x)$$

$$= k_2(y-x) - k_1 x \quad \text{since} \quad M_1 g = k_1 s_1$$

These equations can be written

$$b^2 x - (D^2 + b^2) y = 0 \qquad \text{I.13}$$

$$(D^2 + a^2 + b^2 m) x - m b^2 y = 0 \qquad \text{I.14}$$

where $a^2 = k_1/M_1$ , $b^2 = k_2/M_2$ and $M_2/M_1 = m$

The Subsidiary Equations are

$$b^2 \bar{x} - (p^2 + b^2)\bar{y} = -p y_0 \qquad \text{I.15}$$

$$(p^2 + a^2 + b^2 m)\bar{x} - m b^2 \bar{y} = p x_0 \qquad \text{I.16}$$

Multiply I.15 by $mb^2$, I.16 by $(p^2+b^2)$ and subtract

$$(p^2+b^2)(p^2+a^2+b^2 m)\bar{x} - m b^4 \bar{x} = p(p^2+b^2)x_0 + m b^2 p y_0$$

$$\left[p^4 + \{a^2 + b^2(1+m)\}p^2 + a^2 b^2\right]\bar{x} = p^3 x_0 + p b^2 (x_0 + m y_0)$$

Hence $$\bar{x} = \frac{p^3 x_0 + pb^2(x_0 + my_0)}{p^4 + \{a^2 + b^2(1+m)\}p^2 + a^2b^2}$$

Consider the roots of the denominator of the right-hand side above

$$p^4 + \{a^2 + b^2(1+m)\}p^2 + a^2b^2 = 0$$

gives $p^2 = -\frac{1}{2}\left[a^2 + b^2(1+m) \mp \sqrt{\{a^2+b^2(1+m)\}^2 - 4a^2b^2}\right.$.

Hence there are 4 roots which are equal in magnitude but opposite in sign in pairs. Let them be $\pm i\omega_1$ and $\pm i\omega_2$, where $i = \sqrt{-1}$ and

$$\omega_1 = \sqrt{\frac{1}{2}\left[a^2 + b^2(1+m) - \sqrt{\{a^2+b^2(1+m)\}^2 - 4a^2b^2}\right]}$$

$$\omega_2 = \sqrt{\frac{1}{2}\left[a^2 + b^2(1+m) + \sqrt{\{a^2+b^2(1+m)\}^2 - 4a^2b^2}\right]}$$

Hence we have

$$\bar{x} = \frac{p^3 x_0 + pb^2(x_0 + my_0)}{(p^2+\omega_1^2)(p^2+\omega_2^2)}$$

$$= \frac{1}{\omega_1^2 - \omega_2^2}\left[\left\{\frac{\omega_1^2 p}{p^2+\omega_1^2} - \frac{\omega_2^2 p}{p^2+\omega_2^2}\right\}x_0 + b^2(x_0+my_0)\left\{\frac{p}{p^2+\omega_2^2} - \frac{p}{p^2+\omega_1^2}\right\}\right]$$

$$= \frac{1}{\omega_1^2 - \omega_2^2}\left[\frac{\{(\omega_1^2 - b^2)x_0 - mb^2y_0\}}{p^2+\omega_1^2} - \frac{\{(\omega_2^2 - b^2)x_0 - mb^2y_0\}}{p^2+\omega_2^2}\right]p.$$

$$= \frac{1}{\omega_1^2 - \omega_2^2}\left[A\frac{p}{p^2+\omega_1^2} - B\frac{p}{p^2+\omega_2^2}\right]$$

where $A = (\omega_1^2 - b^2)x_0 - mb^2y_0$

$B = (\omega_2^2 - b^2)x_0 - mb^2y_0$

Therefore $$x = \frac{1}{\omega_1^2 - \omega_2^2}\left[A\cos\omega_1 t - B\cos\omega_2 t\right] \qquad \text{I.17}$$

and from I.14 $$y = \frac{1}{mb^2}(D^2 + a^2 + b^2m)x$$

$$= \frac{1}{mb^2(\omega_1^2-\omega_2^2)}\left[A(a^2+b^2m-\omega_1^2)\cos\omega_1 t - B(a^2+b^2m-\omega_2^2)\cos\omega_2 t\right]$$

$$= \frac{1}{mb^2(\omega_1^2-\omega_2^2)}\left[A(\omega_2^2 - b^2)\cos\omega_1 t - B(\omega_1^2 - b^2)\cos\omega_2 t\right] \qquad \text{I.18}$$

The motions of $M_1$ and $M_2$ are each composed of two simple harmonic motions. The frequency of those for $M_1$ and $M_2$ are the same but their amplitudes are different.

Verification that I.17 and I.18 satisfy I.13

$$(D^2+b^2)y = \frac{1}{mb^2(\omega_1^2-\omega_2^2)}\left[\begin{array}{l} A(\omega_2^2-b^2)(b^2-\omega_1^2)\cos\omega_1 t \\ -B(\omega_1^2-b^2)(b^2-\omega_2^2)\cos\omega_2 t \end{array}\right]$$

$$= \frac{-(\omega_1^2-b^2)(\omega_2^2-b^2)}{mb^2(\omega_1^2-\omega_2^2)}\left[A\cos\omega_1 t - B\cos\omega_2 t\right]$$

$$= -\frac{\omega_1^2\omega_2^2 - b^2(\omega_1^2+\omega_2^2) + b^4}{mb^2}\,x$$

$$= -\frac{a^2b^2 - b^2\{a^2+b^2(1+m)\}+b^4}{mb^2}\,x = b^2x$$

Also when $t=0$

$$x = \frac{A-B}{\omega_1^2-\omega_2^2} = \frac{(\omega_1^2-\omega_2^2)\,x_0}{\omega_1^2-\omega_2^2} = x_0$$

$$y = \frac{A(\omega_2^2-b^2)-B(\omega_1^2-b^2)}{mb^2(\omega_1^2-\omega_2^2)} = \frac{mb^2y_0(\omega_1^2-\omega_2^2)}{mb^2(\omega_1^2-\omega_2^2)} = y_0$$

and $Dx = Dy = 0$

I.17 and I.18 are therefore solutions of the problem.

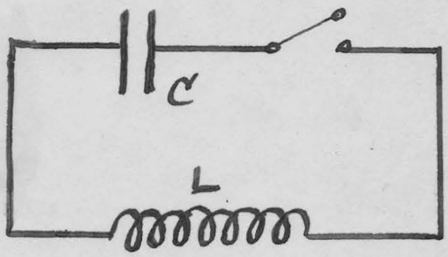
---

This method of solution may be extended to problems where there are more than two springs connected in the manner of the above problem. There will be as many equations as springs. Viscous damping may also be considered. These extensions lead to equations which may be solved by the operational methods employed above, but the algebra involved will be more arduous.

---

Electrical problems similar to the preceeding mechanical problems.

A problem analogous to that of the motion of a helical spring in a medium which offers negligible resistance to the motion is that of the discharge of the condenser in the simple electrical circuit shown, the resistance of the circuit being negligible.




A condenser of capacity $C$ is discharged through an induction coil of inductance $L$ when the circuit is closed. The circuit is assumed to offer negligible resistance to the passage of current.

The charge $Q$ on the plates of the condenser and the potential difference $V$ of the plates are connected by the equation

$$Q = CV$$

The current $I$ flowing through the coil is given by $I = -\frac{dQ}{dt}$

Since the resistance $R$ has been assumed to be negligible and the E.M.F. is $L\frac{dI}{dt}$ we have $V - L\frac{dI}{dt} = 0$

i.e. $\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$

$$(D^2 + a^2)Q = 0, \quad t > 0 \qquad \text{I.19}$$

where $a^2 = 1/CL$

When $t = 0$, $Q = Q_0$, $DQ = 0$.

Equation I.19 is similar to I.4 above and it has the solution

$$Q = Q_0 \cos\frac{t}{\sqrt{CL}}$$

with period of oscillation given by $T = 2\pi\sqrt{CL}$

A problem analogous to that of viscous damping of a helical spring arises if the resistance $R$ is not negligible. The voltage equation is

$$V - L\frac{dI}{dt} - IR = 0$$

i.e. $$\frac{Q}{C} + L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R = 0$$

$$(D^2 + 2bD + a^2)Q = 0, \quad t > 0 \qquad 1.20$$

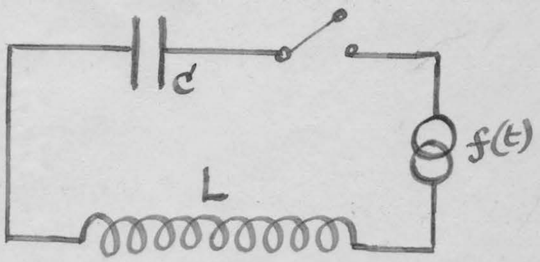
where $b = \frac{R}{2L}$, $a^2 = \frac{1}{CL}$

When $t = 0$, $Q = Q_0$, $DQ = 0$,

Equation 1.20 is similar to equation 1.3 above and its solution will be similar to that of 1.3

A problem analogous to that of forced vibration of a helical spring is that of a condenser placed in series with a source of E.M.F. and which discharges through an induction coil. The voltage equation is

$$V - L\frac{dI}{dt} - IR = f(t)$$




which gives

$$(D^2 + 2bD + a^2)Q = a^2 f(t)$$

where $f(t)$ is the impressed E.M.F. given as a function of $t$.

If the impressed E.M.F. is alternating and of the form

$$f(t) = E_0 \sin \omega t$$

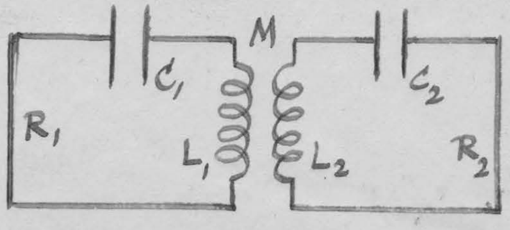
we have

$$(D^2 + 2bD + a^2)Q = a^2 E_0 \sin \omega t, \quad t > 0 \qquad 1.21$$

When $t = 0$, $Q = Q_0$, $DQ = 0$

Equation 1.21 is similar to 1.8 above and leads to a solution similar to 1.9.

An example in electrical theory of a problem giving rise to ordinary simultaneous differential equations with constant coefficients is provided by currents $I_1$ and $I_2$ flowing in coupled circuits as shown in the diagram. The currents $I_1$ and $I_2$ satisfy the following differential equations



$$M D^2 I_1 + \left(L_2 D^2 + R_2 D + \frac{1}{C_2}\right) I_2 = 0$$
$$M D^2 I_2 + \left(L_1 D^2 + R_1 D + \frac{1}{C_1}\right) I_1 = 0$$

If it is assumed that $R_1$ and $R_2$ are negligible the solution of these equations is similar to the solution of equations I.13 and I.14 above.

It will therefore be seen that electical problems and mechanical problems often lead to similar differential equations.

---

## APPENDIX A

### Laplace Transformations.

If $x$ is a function of $t$ and $p$[5] is a real positive number large enough to make the integral converge

$$\bar{x}(p) = \int_0^\infty e^{-pt} x(t)\,dt$$

defines $\bar{x}(p)$, the Laplace Transform of $x(t)$.

The following table lists some forms of $\bar{x}(p)$ and the corresponding $x(t)$.

| | $\bar{x}(p) = \int_0^\infty e^{-pt} x(t)\,dt$ | $x(t)$ |
|---|---|---|
| I | $\frac{1}{p}$ | $1$ |
| 2 | $1/p^n$ | $\frac{t^{n-1}}{\lfloor n-1}$ |
| 3 | $\frac{1}{p-a}$ | $e^{at}$ $(p > R(a))$ |
| 4 | $\frac{a}{p^2+a^2}$ | $\sin at$ |
| 5 | $\frac{p}{p^2+a^2}$ | $\cos at$ |
| 6 | $\frac{a}{p^2-a^2}$ | $\sinh at$, $(p > \lvert a \rvert)$ |
| 7 | $\frac{p}{p^2-a^2}$ | $\cosh at$ $(p > \lvert a \rvert)$ |
| 8 | $\frac{p}{(p^2+a^2)^2}$ | $\frac{t}{2a} \sin at$ |
| 9 | $\frac{1}{(p^2+a^2)^2}$ | $\frac{1}{2a^3}(\sin at - at\cos at)$ |

In the above table the paramrter $a$ is real except in 3 where it may be complex. Results I to 7 may be obtained directly by integration. 8 is obtained by differentiating both sides of 5 with respect to $a$, amd 9 is similarly obtained from 4.

---

5 Thus if $x(t) = e^{at}$ then for the Laplace Transform to exist we must have $p > a$, but if $x(t) = e^{t^2}$ the Laplace Transform does not exist.

It can be shown that if $p+a>0$, then $\bar{x}(p+a)$ exists[6] and

$$\bar{x}(p+a) = \int_0^\infty e^{-pt} e^{-at} x(t)\, dt$$

Hence a table similar to the above table can be made connecting $\bar{x}(p+a)$ and $e^{-at} x(t)$. In fact the above table will serve if $p$ is replaced by $p+a$ in the first column and each result in the second column is multiplied by $e^{-at}$.

e.g. From 2, $\dfrac{1}{(p+a)^n}$ is the Laplace Transform of $e^{-at} \dfrac{t^{n-1}}{\lfloor n-1}$.

The method of application of the above transformations is briefly explained in Appendix B.

---

## APPENDIX B.

**Brief explanation of operational methods of solution applicable to Chapter I.**

A comprehensive treatment of the methods of solution together with proofs of the validity of assumptions made in the following condensed treatment is given in O.M. in A.M., Chapter I.

Let $x$ be a function of $t$ and $p$ a real positive number. Assuming that $\lim_{t\to\infty}(e^{-pt}x) = 0$ and that $\int_0^\infty e^{-pt} x\, dt$ exists when $p$ is greater than some fixed positive number, then

$$\int_0^\infty e^{-pt} Dx\, dt = \left[e^{-pt}x\right]_0^\infty + p\int_0^\infty e^{-pt} x\, dt = -x_0 + p\bar{x} \qquad \text{(i)}$$

where $x_0$ is the value of $x$ when $t=0$.

Again assuming that $\lim_{t\to\infty}(e^{-pt}Dx) = 0$, we have

$$\int_0^\infty e^{-pt} D^2x\, dt = \left[e^{-pt}Dx\right]_0^\infty + p\int_0^\infty e^{-pt} Dx\, dt$$

$$= -x_1 + p(-x_0 + p\bar{x})$$

$$= -(px_0 + x_1) + p^2\bar{x} \qquad \text{(ii)}$$

where $x_1$ is the value of $Dx$ when $t=0$.

---

6 See O.M. in A.M., Chapter I, para.3, Theorem IV.

Hence by induction, making similar assumptions as to $\lim_{t\to\infty}\left(e^{-pt}D^2x\right)$ etc., we get

$$\int_0^\infty e^{-pt}D^r x\,dt = -\left(p^{r-1}x_0 + p^{r-2}x_1 + \cdots + p\,x_{r-2} + x_{r-1}\right) + p^r\bar{x} \qquad \text{(iii)}$$

Now consider the differential equation

$$\left(D^n + a_1 D^{n-1} + \cdots\cdots + a_{n-1}D + a_n\right)x = F(t) \qquad \text{(iv)}$$

where $a_1\, a_2 \cdots\cdots a_n$ are constants and $x_0\, x_1 \cdots\cdots x_{n-1}$ are the values of $x, Dx, \cdots\cdots D^{n-1}x$ when $t=0$ .

Multiply (iv) by $e^{-pt}$, integrate with respect to $t$ from $0$ to $\infty$ and apply results (i), (ii) and (iii). We get

$$\begin{aligned}
\left(p^n + a_1 p^{n-1} + \cdots + a_{n-1}p + a_n\right)\bar{x}\\
&= \left(p^{n-1}x_0 + p^{n-2}x_1 + \cdots\cdots + p\,x_{n-2} + x_{n-1}\right)\\
&\quad + a_1\left(p^{n-2}x_0 + \cdots\cdots + p\,x_{n-3} + x_{n-2}\right)\\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad + a_{n-2}\left(p x_0 + x_1\right)\\
&\quad + a_{n-1}x_0\\
&\quad + \int_0^\infty e^{-pt}F(t)\,dt
\end{aligned}$$

This is called the Subsidiary Equation. The integral on the right-hand side is evaluated from a table of Laplace Transformations.[7] Say it is $\varphi(p)$ . Then we have

$$\bar{x} = \frac{\left(p^{n-1}x_0 + \cdots + x_{n-1}\right) + a_1\left(p^{n-2}x_0 + \cdots + x_{n-2}\right) + \cdots\cdots + a_{n-2}\left(p x_0 + x_1\right) + a_{n-1}x_0 + \varphi(p).}{p^n + a_1 p^{n-1} + \cdots\cdots + a_{n-1}p + a_n}$$

The right-hand side is now split into partial fractions each of which is of the form of a known Laplace Transformation, and hence $x$ is evaluated.

e.g. For the type of equation 1.7 dealt with in this chapter

$$\left(D^2 + 2bD + a^2\right)x = a^2 f(t)$$

we get the Subsidiary Equation

---

7 Such a table is given in O.M. in A.M., Appendix V.

$$(p^2 + 2bp + a^2)\,\bar{x} = px_0 + x_1 + 2bx_0 + a^2\int_0^\infty e^{-pt} f(t)\,dt$$

Hence

$$\bar{x} = \frac{px_0 + x_1 + 2bx_0 + \varphi(p)}{p^2 + 2bp + a^2}$$

The right-hand side may be split into partial fractions each of the form of one of the Laplace Transformations listed in Appendix A, or of the form of these transformations with $p$ replaced by $p+a$ $(p+a>0)$. $x(t)$ can then be evaluated as the sum of these seperated transforms from the table given in Appendix A.[8]

---

8 For proof see O.M.in A.M., Chapter I, para.3, Theorems I to IV.

## APPENDIX C

( See page 9 )

Write

$$\frac{1}{(p^2+\omega^2)(p^2+2bp+a^2)} = \frac{Bp+C}{p^2+\omega^2} + \frac{Dp+E}{p^2+2bp+a^2}$$

Hence

$$\begin{aligned} B + D &= 0 \\ 2bB + C + E &= 0 \\ a^2B + 2bC + \omega^2 D &= 0 \\ a^2C + \omega^2 E &= 1 \end{aligned}$$

The determinant arising from the matrix of the left-hand side of this system of equations is

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 2b & 1 & 0 & 1 \\ a^2 & 2b & \omega^2 & 0 \\ 0 & a^2 & 0 & \omega^2 \end{vmatrix} = \begin{vmatrix} 1 & -2b & 1 \\ 2b & \omega^2-a^2 & 0 \\ a^2 & 0 & \omega^2 \end{vmatrix} = \omega^2(\omega^2-a^2) + 4b^2\omega^2 - a^2(\omega^2-a^2)$$

$$= (\omega^2-a^2)^2 + 4b^2\omega^2 > 0.$$

$$\therefore \; B\left\{(\omega^2-a^2)^2+4b^2\omega^2\right\} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2b & \omega^2 & 0 \\ 1 & a^2 & 0 & \omega^2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \omega^2 & 2b & 0 & 0 \\ 0 & a^2 & 1 & \omega^2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 2b & 0 & 0 \\ a^2 & 1 & \omega^2 \end{vmatrix}$$

$$= -2b$$

Hence

$$B = \frac{-2b}{(\omega^2-a^2)^2+4b^2\omega^2} = -D$$

and

$$C\left\{(\omega^2-a^2)^2+4b^2\omega^2\right\} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 2b & 0 & 0 & 1 \\ a^2 & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & \omega^2 \end{vmatrix} = \begin{vmatrix} 0 & -2b & 1 \\ 0 & \omega^2-a^2 & 0 \\ 1 & 0 & \omega^2 \end{vmatrix} = a^2-\omega^2$$

$$\therefore \; C = \frac{a^2-\omega^2}{(\omega^2-a^2)^2+4b^2\omega^2}$$

Again

$$E = \frac{1-a^2C}{\omega^2} = \frac{1}{\omega^2}\left\{\frac{(\omega^2-a^2)^2+4b^2\omega^2-a^2(a^2-\omega^2)}{(\omega^2-a^2)^2+4b^2\omega^2}\right\} = \frac{4b^2+(\omega^2-a^2)}{(\omega^2-a^2)^2+4b^2\omega^2}.$$

## CHAPTER 2

# Application of Fourier Series and of operational methods of solution to partial differential equations.

### Small transverse vibrations of a light elastic string.

Consider a perfectly elastic string of length $l$ feet and weight $W$ lbs. per unit of length which is stretched taut between two fixed points, the origin and the point $L\ (l, o)$ on the $x$-axis.

Suppose that the string offers no resistance to bending and that it is initially distorted into a curve having the equation

$$y = f(x)$$

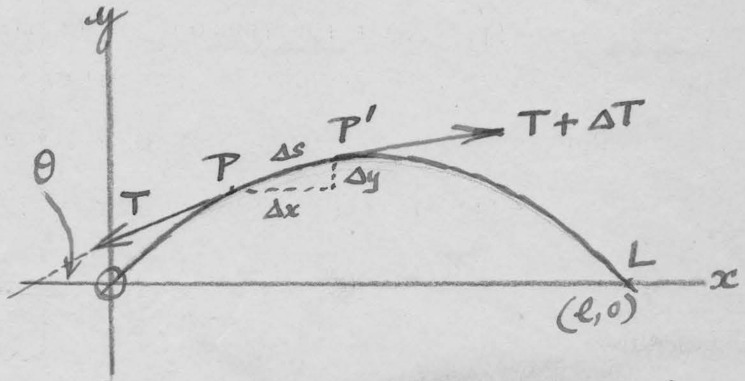
where $f(x)$ is a single valued finite and continuous function of $x$ for $o < x < l$.

I shall assume :-

(i) that the displacement of the string is so small compared with the length of the string $l$ that the length of the string may be taken as $l$ for any position of the string.

(ii) that the tension $T$ of the string is so large compared with the weight of the string $Wl$ that gravitational forces acting on the string can be neglected.

The component of $T$ in the direction of the $x$-axis may be considered to be constant, and the displacement of a point $P(x,y)$ of the string in the direction of the $x$-axis is negligible compared with its displacement in the direction of the $y$-axis. Thus the vibration can be considered as completely determined by the component vibration parallel to the $y$-axis.




At time $t$ consider a segment of string of length $\Delta s$ between the points $P(x,y)$ and $P'(x+\Delta x, y+\Delta y)$.

Let the tensions at $P$ and $P'$ be $T$ and $T+\Delta T$ respectively and let the line of action of $T$ be inclined at an angle $\theta$ to the positive direction of the $x$-axis.

Since the horizontal components of the tensions at $P$ and $P'$ are sensibly equal the difference in tensions at $P$ and $P'$, $\Delta T$, equals the difference in the vertical components of the tensions at $P$ and $P'$. Also the $y$-coordinate of $P$ is a continuous function of the distance $x$ and time $t$.

The vertical component of the tension at $P$ is

$$(T\sin\theta)_P = \left(T\lim_{\Delta s\to 0}\frac{\Delta y}{\Delta s}\right) = T\frac{\partial y(x,t)}{\partial s}$$

and the vertical component of the tension at $P'$ is

$$(T\sin\theta)_{P'} = T\frac{\partial y(x+\Delta x,t)}{\partial s}$$

Since the displacement of the string from the equilibrium position along the $x$-axis is small the square of the slope $\left(\frac{\partial y}{\partial x}\right)^2$ at any point of the string can be neglected in comparison with unity. Hence

$$\frac{\partial y(x,t)}{\partial s} = \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\frac{\partial y(x,t)}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}} \doteq \frac{\partial y(x,t)}{\partial x}$$

Similarly $$\frac{\partial y(x+\Delta x,t)}{\partial x} \doteq \frac{\partial y(x+\Delta x,t)}{\partial x}$$

The resultant force on the element $PP'$ of the string is therefore

$$T\frac{\partial y(x+\Delta x,t)}{\partial x} - T\frac{\partial y(x,t)}{\partial x}$$

By Newton's second law of motion this resultant force equals the product of the mass of the element $PP'$ (which was initially of length $\Delta x$ and therefore of mass $\frac{W}{g}\Delta x$ and has its centroid at a point $x_1$ midway between $x$ and $x+\Delta x$) and the acceleration in the direction of the $y$-axis. Hence

$$\frac{W}{g}\Delta x\left(\frac{\partial^2 y(x,t)}{\partial t^2}\right)_{x_1} = T\left\{\frac{\partial y(x+\Delta x,t)}{\partial x} - \frac{\partial y(x,t)}{\partial x}\right\}$$

$$\therefore \quad \left(\frac{\partial^2 y(x,t)}{\partial t^2}\right)_{x_1} = \frac{Tg}{W}\left\{\frac{\frac{\partial y(x+\Delta x,t)}{\partial x} - \frac{\partial y(x,t)}{\partial x}}{\Delta x}\right\}$$

and as $\Delta x \to 0$, $x_1 \to x$ and $\left\{\frac{\partial y(x+\Delta x,t)}{\partial x} - \frac{\partial y(x,t)}{\partial x}\right\} \to \frac{\partial^2 y}{\partial x^2}$

Passing to the limit we get

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}, \quad t > 0 \qquad 2.I$$

where $y$ is a continuous function of $x$ and $t$ and $a^2 = \frac{Tg}{W}$.

This differential equation is therefore that of the vibrating string.

Substitute $u = x+at$, $v = x-at$ in 2.I. We have

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

$$\frac{\partial y}{\partial t} = a\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$\frac{\partial^2 y}{\partial t^2} = a^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

$$\therefore \quad \frac{\partial^2 y}{\partial t^2} - a^2\frac{\partial^2 y}{\partial x^2} = -4\frac{\partial^2 y}{\partial u \partial v}$$

Therefore 2.I becomes

$$\frac{\partial^2 y}{\partial u \partial v} = 0, \quad t > 0 \qquad 2.2$$

Integrating 2.2 with respect to $v$ gives

$$\frac{\partial y}{\partial u} = F(u)$$

and integrating with respect to $u$ gives

$$y = \int F(u)\,du + \varphi(v) = \psi(u) + \varphi(v)$$

i.e. $$y = \psi(x+at) + \varphi(x-at) \qquad 2.3$$

where $\psi$ and $\varphi$ are arbitrary functions.

2.3 is the general solution of 2.1 . For our particular problem we need to find functions $\psi$ and $\varphi$ which satisfy the following initial and boundary conditions

(a) $y = f(x)$ , $t=0$ .

(b) $\frac{dy}{dt} = 0$ , $t=0$ . 2.4

(c) $y=0$ when $x=0$ , $t \geqslant 0$ .

(d) $y=0$ when $x=l$ , $t \geqslant 0$ .

A particular solution of 2.1 is given by writing

$$\psi(x+at) = A \sin k(x+at)$$
$$\varphi(x-at) = A \sin k(x-at)$$

The equation $y = A \sin kx$ represents a sinusoidal wave of amplitude $A$ and wave length $\lambda = \frac{2\pi}{k}$ .

Replacing $x$ by $x-at$ moves the curve $at$ units in the positive direction of the $x$-axis, and replacing $x$ by $x+at$ moves the curve $at$ units in the negative direction of the $x$-axis.

Thus $y = A \sin k(x-at)$ and $y = A \sin k(x+at)$ represent sinusoidal waves moving towards the positive and the negative directions of the $x$-axis respectively each with speed $a$ .

The equation

$$y = A \sin k(x+at) + A \sin k(x-at)$$
$$= 2A \sin kx \cos kat \qquad 2.5$$

results from the superimposition of the above two sinusoidal waves and represents a standing wave of amplitude $2A\cos kat$ varying with the time $t$ in a simple harmonic manner, and for which stationary points are given by

$$x = \frac{n\pi}{k}, \quad (n = 0, 1, 2, \cdots\cdots)$$

The solution 2.5 can be made to satisfy the boundary conditions 2.4(c) and 2.4(d) by writing

$$k = \frac{n\pi}{l} \quad \text{( where } n \text{ is an integer )}$$

Then 2.5 becomes

$$y = 2A\cos\frac{n\pi a}{l}t\,\sin\frac{n\pi}{l}x \qquad 2.6$$

and a solution in the form of an infinite series can be obtained by summing solutions of the type 2.6 for values of $n$ from $1$ to $\infty$. [I]

i.e. $$y = \sum_{n=1}^{\infty} A_n\cos\frac{n\pi a}{l}t\,\sin\frac{n\pi}{l}x \qquad 2.7$$

is a solution satisfying 2.I and the boundary conditions of our problem.

When $t=0$ the initial conditions 2.4(a) and 2.4(b) have also to be satisfied. From 2.7

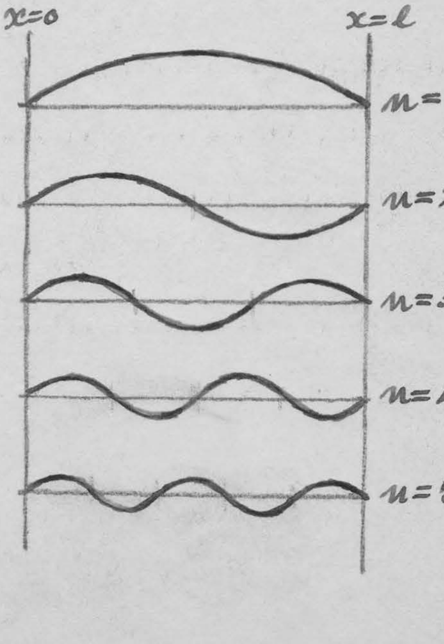
$$\frac{\partial y}{\partial t} = -\sum_{n=1}^{\infty} A_n\frac{n\pi a}{l}\sin\frac{n\pi a}{l}t\,\sin\frac{n\pi}{l}x$$

$$= 0 \quad \text{when} \quad t = 0$$

---

I From 2.7 it is seen that the string can vibrate in an infinite number of sinusoidal shapes, each shape corresponding to a certain frequency.
The lowest frequency, given by $n = 1$ is called the fundamental tone of the string and the higher frequencies are known as overtones or harmonics.

Rough sketches to illustrate the fundamental and first four overtones are facing. The rapid decrease in wave length is very noticable.



Thus 2.7 satisfies initial condition 2.4(b). It remains to ensure that $y = f(x)$ when $t = 0$. When $t = 0$ 2.7 becomes

$$y = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \qquad 2.8$$

Now $f(x)$ was defined to be a single valued finite and continuous function of $x$ for $0 < x < l$. If further $f(x)$ has only a finite number of maxima and minima in this interval then $f(x)$ can be expanded in a convergent half range Fourier Series of the form of 2.8.[2] Thus if

$$f(x) = \sum_{r=1}^{\infty} A_r \sin \frac{r\pi x}{l}$$

and each term of this series is multiplied by $\sin \frac{n\pi x}{l}$ we get on integrating between the limits $0$ to $l$

$$\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx = \sum_{r=1}^{\infty} \int_0^l A_r \sin \frac{r\pi x}{l} \sin \frac{n\pi x}{l}\, dx$$

The term on the right-hand side containing $A_n$ is

$$A_n \int_0^l \left(\sin \frac{n\pi x}{l}\right)^2 dx = \frac{A_n}{2} \int_0^l \left(1 - \cos \frac{2n\pi x}{l}\right) dx$$

$$= \frac{A_n}{2} \left[ x - \frac{l}{2n\pi} \sin \frac{2n\pi x}{l} \right]_0^l = \frac{l}{2} A_n$$

whilst the term on the right-hand side containing any other coefficient $A_s$ is

$$A_s \int_0^l \sin \frac{s\pi x}{l} \sin \frac{n\pi x}{l}\, dx = \frac{A_s}{2} \int_0^l \left\{ \cos \frac{(n-s)\pi x}{l} - \cos \frac{(n+s)\pi x}{l} \right\} dx$$

$$= \frac{A_s}{2} \left[ \frac{l}{(n-s)\pi} \sin \frac{(n-s)\pi x}{l} - \frac{l}{(n+s)\pi} \sin \frac{(n+s)\pi x}{l} \right]_0^l$$

$= 0$, since $n$ and $s$ are integers.

$$\therefore \frac{l}{2} A_n = \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

i.e. $$A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx \qquad 2.9$$

---

2 These conditions imposed on $f(x)$ are sufficient but not necessary for the expansion in a half range Fourier Series. They are satisfied in practice for all physical problems of this nature.

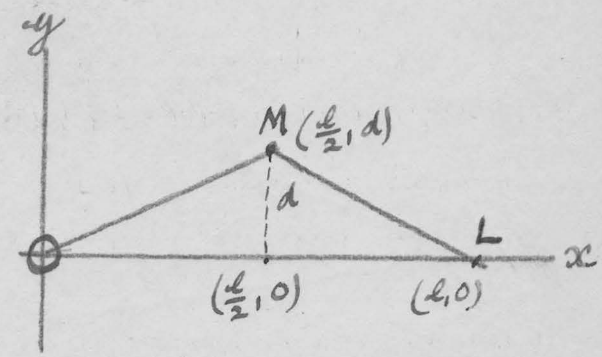
Thus we have a solution of our problem which satisfies 2.I and initial and boundary conditions 2.4 in the form of the infinite series

$$y = \frac{2}{l}\sum_{n=1}^{\infty}\left\{\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\right\}\cos\frac{n\pi a t}{l}\sin\frac{n\pi x}{l} \qquad 2.10$$

---

The function $f(x)$ which determines the initial distortion of the string may assume numerous forms consistent with the conditions placed upon it. An interesting case arises when the string is plucked or bowed initially. This case is of practical importance also.

Suppose that the stretched string is plucked at its middle point. ( Plucking the string at any other point leads to a similar solution with slightly more involved working ) The initial distortion will be considered as determined by the equations




$$y = \frac{2d}{l}x \quad \text{for } 0 < x < \frac{l}{2}$$
$$y = \frac{2d}{l}(l-x) \quad \text{for } \frac{l}{2} < x < l$$

Then from equation 2.10 we have

$$A_n = \frac{2}{l}\left\{\int_0^{l/2}\frac{2d}{l}x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2d}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{4d}{l^2}\left\{\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_l^{l/2} x\sin\frac{n\pi x}{l}\,dx + l\int_{l/2}^{l}\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{4d}{l^2}\left\{\left[-\frac{l}{n\pi}x\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{l/2} + \Big[\text{SAME}\Big]_l^{\frac{l}{2}} - l\left[\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_{l/2}^{l}\right\}$$

$$\therefore A_n = \frac{4d}{l^2}\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{n\pi}\cos n\pi - \frac{l^2}{n\pi}\cos n\pi + \frac{l^2}{n\pi}\cos\frac{n\pi}{2}\right\}$$

$$= \frac{4d}{l^2}\cdot\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{8d}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore A_1 = \frac{8d}{\pi^2},\quad A_2 = 0,\quad A_3 = -\frac{8d}{3^2\pi^2},\quad A_4 = 0,\ \ldots \text{ etc.}$$

The required solution is therefore

$$y = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi a}{l}t\ \sin\frac{n\pi}{l}x$$

$$= \frac{8d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi a t}{l}\sin\frac{n\pi}{l}x$$

$$= \frac{8d}{\pi^2}\left\{\cos\frac{\pi a t}{l}\sin\frac{\pi x}{l} - \frac{1}{3^2}\cos\frac{3\pi a t}{l}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi a t}{l}\sin\frac{5\pi x}{l} + \cdots\cdots + \frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi a t}{l}\sin\frac{n\pi x}{l} + \cdots\right\}$$

$$= \frac{8d}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\cos\frac{(2n-1)\pi a t}{l}\sin\frac{(2n-1)\pi}{l}x \qquad 2.11$$

By comparison with $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$, which is a convergent series, it can be shown that 2.11 is convergent for $0 < x < l$ and $t > 0$.

The first term of 2.11 is $\frac{8d}{\pi^2}\cos\frac{\pi a t}{l}\sin\frac{\pi x}{l}$, which represents a vibration of wave length $2l$ and amplitude $8d/\pi^2$. This is the fundamental vibration of the string and gives the fundamental tone.

The second term represents a vibration of wave length $\frac{2}{3}l$ and amplitude $\frac{8d}{9\pi^2}$, and is therefore a third harmonic. The third term is a fifth harmonic of amplitude $\frac{8d}{25\pi^2}$. Succeeding terms represent seventh, ninth, eleventh, etc. harmonics with rapidly decreasing amplitudes $\frac{1}{49}$ th, $\frac{1}{81}$ st, $\frac{1}{121}$ st, etc. of that of the fundamental vibration. The rapid decrease in the amplitudes of succeeding overtones leads to a predominance of the fundamental tone. The displacement of the string at any time as determined by 2.II will therefore approximate to that given by the fundamental vibration as a very rough first approximation. The exact shape assumed by the vibrating string at any time $t$ is discussed below on page 36.

Solutions can be found to comply with initial distortions into many other curves, or when the initial conditions are different to those of the above problem. Thus if initially every point of the stretched string is given an initial velocity normal to the string and of magnitude $F(x)$, $F(x)$ being a finite continuous function of $x$, then the initial conditions

$$y = 0, \quad \frac{\partial y}{\partial t} = F(x), \quad \text{when} \quad t = 0$$

will lead to the expansion of $F(x)$ in a convergent half range Fourier Series.

The particular case where the stretched string is initially plucked at its middle point may also be solved by operational methods. Operational methods applied to partial differential equations are very extensively treated in O.M.in A.M. to which book I am indebted for the following treatment of this particular problem. The methods used illustrate the power of this method of treatment to solve particular problems without resource to any generalised result such as 2.IO.

**Operational method of solution of the problem of a stretched light elastic string of length $l$, initially plucked at its middle point.**

Equation 2.1 and the initial and boundary conditions 2.4 as applied to our particular problem may be restated as follows

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \qquad 2.12$$

$$\left.\begin{aligned} y &= \frac{2d}{l}x, \quad 0 < x < \tfrac{l}{2}, \quad t = 0 \\ y &= \frac{2d}{l}(l-x), \quad \tfrac{l}{2} < x < l, \quad t = 0 \end{aligned}\right\} \qquad 2.13$$

$$\frac{\partial y}{\partial t} = 0, \quad 0 < x < l, \quad t = 0 \qquad 2.14$$

$$y = 0 \quad \text{for} \quad x = 0 \ \text{and} \ x = l, \quad t > 0 \qquad 2.15$$

Multiply 2.12 by $e^{-pt}$ ( $p > 0$ ) and integrate with respect to $t$ from $0$ to $\infty$. Using 2.13 and 2.14 we get the Subsidiary Equation[3]

$$a^2 \frac{d^2 \bar{y}}{dx^2} - p^2 \bar{y} = -p f(x), \quad 0 < x < l \qquad 2.16$$

where $f(x) = \frac{2d}{l}x$ for $0 \leq x \leq l/2$

$f(x) = \frac{2d}{l}(l-x)$ for $\frac{l}{2} \leq x \leq l$

Let

$$\bar{y} = A \cosh qx + B \sinh qx \qquad 2.17$$

where $q = p/a$, and $A$ and $B$ are functions of $x$.

Differentiate 2.17 with respect to $x$. We get

$$\frac{d\bar{y}}{dx} = q(A \sinh qx + B \cosh qx)$$

if

$$\frac{dA}{dx}\cosh qx + \frac{dB}{dx}\sinh qx = 0$$

Differentiating again with respect to $x$ gives

$$\frac{d^2\bar{y}}{dx^2} = q^2(A \cosh qx + B \sinh qx) + q\left(\frac{dA}{dx}\sinh qx + \frac{dB}{dx}\cosh qx\right)$$

---

3 See Appendices A and B to Chapter I.

Hence

$$\frac{d^2\bar{y}}{dx^2} - q^2\bar{y} = q\left(\frac{dA}{dx}\sinh qx + \frac{dB}{dx}\cosh qx\right)$$

and so 2.16 is satisfied by 2.17 if

$$\frac{dA}{dx}\sinh qx + \frac{dB}{dx}\cosh qx = -\frac{f(x)}{a}$$

$$\frac{dA}{dx}\cosh qx + \frac{dB}{dx}\sinh qx = 0$$

From these last two equations we get

$$\frac{dA}{dx} = \frac{f(x)}{a}\sinh qx \qquad 2.18$$

$$\frac{dB}{dx} = -\frac{f(x)}{a}\cosh qx$$

But from 2.15 when $x=0$, $\bar{y}=0$. Hence from 2.17 when $x=0$, $A=0$;

and from 2.18

$$A(x) = \frac{1}{a}\int_0^x f(z)\sinh qz\,dz \qquad 2.19$$

Again from 2.15 when $x=l$, $\bar{y}=0$ ; and hence from 2.17

$$A(l)\cosh ql + B(l)\sinh ql = 0$$

$$\therefore\ B(l) = -\frac{1}{a}\coth ql\int_0^l f(z)\sinh qz\,dz$$

and from 2.18

$$B(x) = \frac{1}{a}\left\{\int_x^l f(z)\cosh qz\,dz - \coth ql\int_0^l f(z)\sinh qz\,dz\right\} \qquad 2.20$$

Then from 2.17, 2.19 and 2.20 we get

$$a\bar{y} = \cosh qx\int_0^x f(z)\sinh qz\,dz$$

$$+\sinh qx\left\{\int_x^l f(z)\cosh qz\,dz - \frac{\cosh ql}{\sinh ql}\int_0^l f(z)\sinh qz\,dz\right\}$$

$$\therefore\ a\sinh ql\,\bar{y} = (\cosh qx\sinh ql - \sinh qx\cosh ql)\int_0^x f(z)\sinh qz\,dz$$

$$+\sinh qx\int_x^l f(z)(\cosh qz\sinh ql - \sinh qz\cosh ql)\,dz$$

$$= \sinh q(l-x)\int_0^x f(z)\sinh qz\,dz + \sinh qx\int_x^l f(z)\sinh q(l-z)\,dz$$

Substituting for $f(x)$ from 2.13 we get for $0 \le x \le \frac{l}{2}$

$$\frac{la}{2d}\sinh ql\,\bar{y} = \sinh q(l-x)\int_0^x z\sinh qz\,dz + \sinh qx\int_x^{\frac{l}{2}} z\sinh q(l-z)\,dz + \sinh qx\int_{\frac{l}{2}}^{l}(l-z)\sinh q(l-z)\,dz$$

$$= \sinh q(l-x)\left[\frac{z\cosh qz}{q} - \frac{\sinh qz}{q^2}\right]_0^x + \sinh qx\left[-\frac{z\cosh q(l-z)}{q} + \frac{\sinh q(l-z)}{q^2}\right]_x^{\frac{l}{2}} + \sinh qx\left[-\frac{(l-z)\cosh q(l-z)}{q} + \frac{\sinh q(l-z)}{q^2}\right]_{\frac{l}{2}}^{l}$$

$$= \sinh q(l-x)\left\{\frac{x}{q}\cosh qx - \frac{1}{q^2}\sinh qx\right\} + \sinh qx\left\{-\frac{l}{2q}\cosh\frac{ql}{2} - \frac{1}{q^2}\sinh\frac{ql}{2} + \frac{x}{q}\cosh q(l-x) + \frac{1}{q^2}\sinh q(l-x) + \frac{l}{2q}\cosh\frac{ql}{2} - \frac{1}{q^2}\sinh\frac{ql}{2}\right\}$$

$$= \frac{x}{q}\left\{\sinh q(l-x)\cosh qx + \cosh q(l-x)\sinh qx\right\} - \frac{2}{q^2}\sinh qx\sinh\frac{ql}{2}$$

$$= \frac{x}{q}\sinh ql - \frac{2}{q^2}\sinh qx\sinh\frac{ql}{2}$$

$$\therefore \frac{l}{2d}\bar{y} = \frac{x}{p} - \frac{a}{p^2}\frac{\sinh\frac{px}{a}}{\cosh\frac{pl}{2a}} \qquad 2.21$$

since $p = aq$

Similarly for $\frac{l}{2} \le x \le l$ we get

$$\frac{l}{2d}\bar{y} = \frac{l-x}{p} - \frac{a}{p^2}\frac{\sinh\frac{p}{a}(l-x)}{\cosh\frac{pl}{2a}} \qquad 2.22$$

Equation 2.22 may be obtained from 2.21 by replacing $x$ by $l-x$ , and so in order to find $y$ we need consider 2.21 only. From 2.21 we have

$$\frac{l}{2d}\bar{y} = x - \frac{a}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}\sinh\frac{\lambda x}{a}}{\lambda^2\cosh\frac{\lambda l}{2a}}\,d\lambda \qquad 2.33$$

where the first term on the right-hand side is obtained by the methods of Chapter I and the second term by using an Inversion Theorem, viz. that if

$$\bar{x}(p) = \int_0^\infty e^{-pt}x(t)\,dt, \qquad R(p)>0$$

then

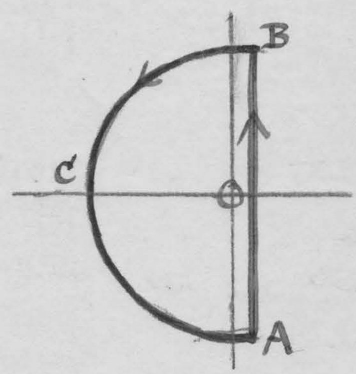
$$x(t) = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\,\bar{x}(\lambda)\,d\lambda$$

where $\gamma$ is a constant greater than the real part of all the singularities of $\bar{x}(\lambda)$.[4]

Now consider the integral

$$\frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}\sinh\frac{\lambda x}{a}}{\lambda^2\cosh\frac{\lambda l}{2a}}\,d\lambda$$

taken over the closed circuit shown in the diagram, the circle being of radius $\frac{2n\pi a}{l}$ .




The poles of the integrand are at $\lambda=0$ and $\lambda = \frac{\pm i(2n-1)\pi a}{l}$ $(n=1,2,3,\cdots)$.

Hence the circle does not pass through any pole of the integrand.

As $n\to\infty$ the integral over the arc $BCA$ tends to zero.[5] Hence the integral can be replaced by the limit when $n\to\infty$ of this integral over the closed circuit $ABCA$ .

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4 ~~For~~ the conditions to be satisfied by the functions and formal proofs of this theorem see O.M.in A.M.,Chapter IV, paras.28-30.

5 For proof see O.M.in A.M.,Chapter V, para.43.

We now obtain the value of the integral as an infinite series by applying the Theory of Residues.

The pole at $\lambda=0$ gives

The pole at $\lambda=\frac{i(2n-1)\pi a}{l}$ gives

$$\frac{e^{\frac{i(2n-1)\pi at}{l}}\sinh\frac{i(2n-1)\pi x}{l}}{\frac{d}{d\lambda}\left[\lambda^2\cosh\frac{\lambda l}{2a}\right]_{\lambda=\frac{i(2n-1)\pi a}{l}}}$$

$$=\frac{e^{\frac{i(2n-1)\pi at}{l}}\sin\frac{(2n-1)\pi x}{l}}{\left[2\lambda\cosh\frac{\lambda l}{2a}+\frac{l}{2a}\lambda^2\sinh\frac{\lambda l}{2a}\right]_{\lambda=\frac{i(2n-1)\pi a}{l}}}$$

$$=\frac{e^{\frac{i(2n-1)\pi at}{l}}\sin\frac{(2n-1)\pi x}{l}}{\frac{2i(2n-1)\pi a}{l}\cos\frac{(2n-1)\pi}{2}-\frac{(2n-1)^2\pi^2 a}{2l}\sin\frac{(2n-1)\pi}{2}}$$

$$=(-1)^n\frac{2l}{\pi^2 a}\frac{\sin\frac{(2n-1)\pi x}{l}}{(2n-1)^2}e^{\frac{i(2n-1)\pi at}{l}}$$

Hence the pair of poles at $\pm\frac{i(2n-1)\pi a}{l}$ give

$$\frac{(-1)^n}{(2n-1)^2}\frac{4l}{\pi^2 a}\sin\frac{(2n-1)\pi x}{l}\cdot\frac{1}{2}\left\{e^{\frac{i(2n-1)\pi at}{l}}+e^{-\frac{i(2n-1)\pi at}{l}}\right\}$$

$$=\frac{(-1)^n}{(2n-1)^2}\frac{4l}{\pi^2 a}\sin\frac{(2n-1)\pi}{l}x\cos\frac{(2n-1)\pi a}{l}t$$

Therefore from 2.2I we have

$$\frac{ly}{2d}=x-\left\{x+\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi}{l}x\cos\frac{(2n-1)\pi at}{l}\right.$$

$$\therefore y=\frac{8d}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\frac{(2n-1)\pi}{l}x\cos\frac{(2n-1)\pi a}{l}t \qquad 2.24$$

Which is the same result as 2.II .

For $\frac{l}{2}\leq x\leq l$ replace $x$ in 2.24 by $l-x$ .

<u>The shape assumed by the string as $t$ changes.</u>

From 2.24 it is seen that $y$ is periodic in $t$ of period $2l/a$ . Hence the form of the motion for the interval $0 < at \leq 2l$ will be repeated periodically and it will be sufficient to discuss the changing shape of the string during one interval only. Equation 2.23 gives

$$\frac{l}{2d}\, y = x - \frac{a}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda^2}\,\frac{\sinh\frac{\lambda x}{a}}{\cosh\frac{\lambda l}{2a}}\, d\lambda$$

Now

$$\frac{\sinh\frac{\lambda x}{a}}{\cosh\frac{\lambda l}{2a}} = \frac{e^{\frac{\lambda x}{a}} - e^{-\frac{\lambda x}{a}}}{e^{\frac{\lambda l}{2a}} + e^{-\frac{\lambda l}{2a}}} = e^{-\frac{\lambda}{a}\left(\frac{l}{2}-x\right)}\left\{\frac{1-e^{-\frac{2\lambda x}{a}}}{1+e^{-\frac{\lambda l}{a}}}\right\}$$

$$= \left\{e^{-\frac{\lambda}{a}\left(\frac{l}{2}-x\right)} - e^{-\frac{\lambda}{a}\left(\frac{l}{2}+x\right)}\right\}\left(1+e^{-\frac{\lambda l}{a}}\right)^{-1}$$

$$= \left\{e^{-\frac{\lambda}{a}\left(\frac{l}{2}-x\right)} - e^{-\frac{\lambda}{a}\left(\frac{l}{2}+x\right)}\right\}\sum_{m=0}^{\infty}(-1)^m e^{-\frac{m\lambda l}{a}}$$

$$\therefore \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda^2}\,\frac{\sinh\frac{\lambda x}{a}}{\cosh\frac{\lambda l}{2a}}\, d\lambda$$

$$= \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{e^{\frac{\lambda}{a}\left(at-\frac{l}{2}+x\right)} - e^{\frac{\lambda}{a}\left(at-\frac{l}{2}-x\right)}\right\}\sum_{m=0}^{\infty}(-1)^m e^{-\frac{m\lambda l}{a}}\,\frac{d\lambda}{\lambda^2} \qquad 2.25$$

Since the series in the integrand of the right-hand side of 2.25 is uniformly convergent it can be integrated term by term and the right-hand side of 2.25 becomes

$$\frac{1}{2i\pi}\sum_{m=0}^{\infty}(-1)^m\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{e^{\frac{\lambda}{a}\left[at-(m+\frac{1}{2})l+x\right]} - e^{\frac{\lambda}{a}\left[at-(m+\frac{1}{2})l-x\right]}\right\}\frac{d\lambda}{\lambda^2} \qquad 2.26$$

It can be proved that[6]

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{a\lambda}\frac{d\lambda}{\lambda^2} = 2i\pi a \quad \text{when } a>0$$
$$= 0 \quad \text{when } a \leq 0 \qquad 2.27$$

The result 2.27 is now applied to successive terms of the series 2.26 .

Consider $t$ (i) in the interval $0 < at \leq \frac{l}{2}$

and (ii) in the interval $\frac{l}{2} < at \leq l$

For the interval (i) applying 2.27 the term $e^{\frac{\lambda}{a}[at-\frac{l}{2}+x]}$ in 2.26 gives zero for $x \leq \frac{l}{2}-at$, and $\frac{1}{a}(at-\frac{l}{2}+x)$ when $x > \frac{l}{2}-at$.

The term $e^{\frac{\lambda}{a}[at-\frac{l}{2}-x]}$ gives zero, and all other terms give zero.

Hence from 2.23

$$\frac{l}{2d}y = x \quad \text{when } x \leq \frac{l}{2}-at$$
$$= x-(at-\frac{l}{2}+x) \quad \text{when } x > \frac{l}{2}-at$$

i.e.
$$= \frac{l}{2}-at \quad \text{when } x > \frac{l}{2}-at$$

Therefore

$$y = \frac{2d}{l}x \quad \text{when } x \leq \frac{l}{2}-at$$
$$= \frac{2d}{l}\left(\frac{l}{2}-at\right) \quad \text{when } x > \frac{l}{2}-at \qquad 2.28$$
$$= 0 \quad \text{when } at = \frac{l}{2}$$

For the interval (ii) the term $e^{\frac{\lambda}{a}(at-\frac{l}{2}+x)}$ gives $\frac{1}{a}(at-\frac{l}{2}+x)$.

The term $e^{\frac{\lambda}{a}(at-\frac{l}{2}-x)}$ gives $\frac{1}{a}(at-\frac{l}{2}-x)$ when $x < at-\frac{l}{2}$ ant it gives zero when $x \geq at-\frac{l}{2}$. All other terms give zero.

Hence from 2.23

$$\frac{l}{2d}y = x-\left\{(at-\frac{l}{2}+x)-(at-\frac{l}{2}-x)\right\} \quad \text{when } x < at-\frac{l}{2}$$
$$= x-(at-\frac{l}{2}+x) \quad \text{when } x \geq at-\frac{l}{2}$$

---

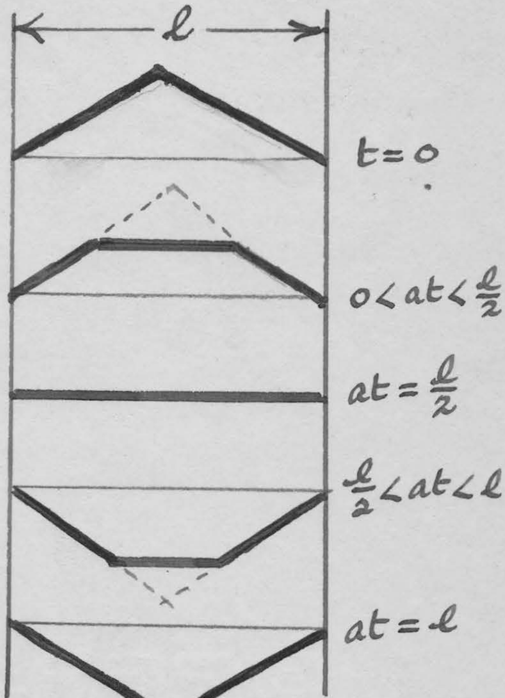
6 See O.M. in A.P., Chapter IV , Para.44.

Therefore

$$
\begin{aligned}
y &= -\frac{2d}{l}x && \text{when} \quad x < at - \frac{l}{2} \\
&= \frac{2d}{l}\left(\frac{l}{2} - at\right) && \text{when} \quad x \geq at - \frac{l}{2} \\
&= -\frac{2d}{l}x && \text{when} \quad at = l
\end{aligned}
\qquad 2.29
$$

For $l < at \leq 2l$ the motion takes place in the reverse order.

From 2.28 and 2.29 the form the string assumes as $t$ changes is seen to be as sketched in the diagram below. Except for $t = \frac{l}{2a}$ or a multiple of $\frac{l}{2a}$ the form of the string is made up of three straight parts. The outer parts have the same gradients as the two parts of the string at time $t = 0$, and the middle part is parallel to the $x$-axis. This middle part moves perpendicular to $x$-axis with velocity $\frac{2da}{l}$, whilst its ends move with velocity $a$ parallel to the $x$-axis.




$\frac{\partial^2 y}{\partial x^2} = 0$ except at the angles formed by the parts of the string as it changes shape, and $\frac{\partial^2 y}{\partial t^2} = 0$ everywhere. Hence the equation 2.12, or 2.1, is satisfied except at these points.

If the string is plucked at a point $x = c$ ( $c < l$ ) the same method as above may be used with intervals

$$0 < at \leq c \quad , \quad c < at \leq l - c \quad , \quad l - c < at < l \quad .$$

In the general case the form assumed by the string consists of two or more straight parts at any time $t$, except when it is passing through the position of static equilibrium along the $x$-axis.

## CHAPTER 3

### Application of Bessel Functions to the solution of partial differential equations of the second order .

#### Small vibrations of a circular membrane .

Consider a thin perfectly elastic circular membrane of radius $a$ and density $\rho$ units per unit of area . Suppose that it is fixed at its edges, is under a tension $T$ and is stretched taut and flat in the $xy$-plane with its centre at the origin . The tension $T$ will be uniform if the force exerted across a straight line of unit length in the plane of the membrane is independant of the orientation of that line.

Suppose that initially each point of the membrane is slightly displaced through a small distance $z$ normal to the $xy$-plane . Air resistance to subsequent motion will be neglected .

By reasoning similar to that used to derive the differential equation satisfied by a light elastic string for small vibrations it can be shown that if the membrane is distorted from the position of static equilibrium and released at time $t = 0$ then the subsequent motion of any point of the membrane may be considered as consisting entirely of vibration normal to the $xy$-plane . A consideration of the forces acting on an element of the membrane leads to the differential equation :-

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right), \quad t > 0 \qquad 3.I$$

where $c^2 = \frac{T}{\rho}$

We need to find a function $z(x,y)$ which satisfies 3.I and a set of boundary and initial conditions. In order to simplify the problem, in addition to neglecting air resistance, I shall assume that

(i) the initial distortion is such that the membrane has the form of a surface of revolution with the $Z$-axis as its geometrical axis.

(ii) for all values of $t > 0$ the surface of the membrane is a surface of revolution symmetrical about the $Z$-axis.

These assumptions suggest the use of cylindrical co-ordinates, and writing

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

equation 3.I becomes [I]

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right), \quad t > 0 \qquad 3.2$$

Since the membrane is of radius $a$ and is fixed at its edges, $z = 0$ for $r = a$, $t \geq 0$.

Since the form of the membrane has been assumed to be a surface of revolution symmetrical about the $Z$-axis at any time $t$ its equation must be a function of $r$ alone and independant of $\theta$. Let it be $z = F(r,t)$ and suppose that when $t = 0$ its equation becomes $z = f(r)$. We will also assume that at time $t = 0$, $\frac{\partial z}{\partial t} = 0$.

Equation 3.2, reduced since $z$ is independant of $\theta$, and the initial and boundary conditions of our problem may be restated as follows

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r}\right), \quad t > 0. \qquad 3.3$$

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I See appendix A to this chapter.

(a) $z = f(r)$ when $t = 0$ .

(b) $\frac{\partial z}{\partial t} = 0$ when $t = 0$ . 3.4

(c) $z = 0$ for $r = a$ and $t \geqslant 0$ .

Assume that a solution of 3.3 exists which is the product of two functions, the first of which is a function of $r$ alone and the second a function of $t$ alone.

i.e. $$z = R(r)\,T(t)$$ 3.5

Substituting from 3.5 in 3.3 we get

$$R\frac{d^2T}{dt^2} = c^2 T\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right)$$

$$\frac{1}{T}\frac{d^2T}{dt^2} = c^2\left(\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr}\right)$$

The left-hand member of this last equation is a function of $t$ alone and the right-hand member is a function of $r$ alone . Therefore the equation can hold in general only if the members on either side each are equal to the same constant, say $-\alpha^2$ , and we have

$$\frac{d^2T}{dt^2} + \alpha^2 T = 0$$ 3.6

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \frac{\alpha^2}{c^2}R = 0$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \beta^2 R = 0$$ 3.7

where $\beta = \alpha/c$ . Equation 3.6 has the complete solution

$$T = A\cos\alpha t + B\sin\alpha t$$ 3.8

Writing $x = \beta r$ in 3.7 we have since

$$\frac{dR}{dx} = \frac{1}{\beta}\frac{dR}{dr}, \qquad \frac{d^2R}{dx^2} = \frac{1}{\beta^2}\frac{d^2R}{dr^2}$$

that 3.7 becomes $$\frac{1}{\beta^2}\frac{d^2R}{dr^2} + \frac{1}{r\beta^2}\frac{dR}{dr} + R = 0$$

i.e. $$x\frac{d^2R}{dx^2}+\frac{dR}{dx}+xR=0 \qquad 3.9$$

which is a Bessel equation of order zero, and the form of solution is well known. In this problem however I shall work out the solution without reference to known results and from first principles.

Assume that a solution of 3.9 exists in the form of an infinite power series

$$R = x^{\alpha}(a_0+a_1x+a_2x^2+\ldots\ldots) \text{ , where } a_0 \neq 0 \text{ .}$$

$$= \sum_{n=0}^{\infty} a_n x^{\alpha+n}$$

$$\frac{dR}{dx} = \sum_{n=0}^{\infty}(\alpha+n)a_n x^{\alpha+n-1}$$

$$\frac{d^2R}{dx^2} = \sum_{n=0}^{\infty}(\alpha+n)(\alpha+n-1)a_n x^{\alpha+n-2}$$

Substituting in 3.9 we get

$$\sum_{n=0}^{\infty}(\alpha+n)(\alpha+n-1)a_n x^{\alpha+n-2}+\sum_{n=0}^{\infty}(\alpha+n)a_n x^{\alpha+n-1}+\sum_{n=0}^{\infty}a_n x^{\alpha+n+1}=0$$

Equating the coefficient of the lowest power of $x$ to zero ( i.e. of $x^{\alpha-1}$ ) we get the indicial equation

$$a_0\alpha(\alpha-1)+a_0\alpha=0$$

$$a_0\alpha^2=0$$

and if $a_0 \neq 0$ , then $\alpha=0$ .

Equating the coefficient of the next power of $x$ to zero, we get

$$(\alpha+1)\alpha\, a_1+(\alpha+1)a_1=0$$

$$a_1(\alpha+1)^2=0$$

But $\alpha=0$ . $\therefore\ a_1=0$ .

The general equation obtained by equating the coefficient of $x^{n+1}$ to zero is

$$(n+2)(n+1)\,a_{n+2} + (n+2)a_{n+2} + a_n = 0$$

$$\therefore\ a_{n+2} = -\frac{a_n}{(n+2)^2}$$

which holds for $n = 0, 1, 2, \cdots\cdots$ . Hence, since $a_1 = 0$ ,

$$a_3 = a_5 = a_7 = \cdots\cdots = a_{2n-1} = \cdots\cdots = 0$$

$$a_2 = -\frac{a_0}{2^2},\quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2.4^2},\ \cdots\ a_{2n} = (-1)^n \frac{a_0}{2^{2n}(\underline{|n})^2}$$

$$\therefore R = a_0\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots\right\} = a_0 J_0(x) = a_0 J_0(\beta r) \qquad 3.10$$

$$\therefore\ z = J_0(\beta r)\left\{C_1 \cos\alpha t + C_2 \sin\alpha t\right\} \qquad 3.11$$

where the $a_0$ has been absorbed in the new constants $C_1$ and $C_2$ .

Applying initial condition 3.4(b) to 3.11 we get

$$J_0(\beta r)\frac{C_2}{\alpha} = 0 \qquad \therefore\quad C_2 = 0$$

and 3.11 reduces to

$$z = C_1 J_0(\beta r) \cos\alpha t \qquad 3.12$$

The boundary condition 3.4(c) gives for all values of

$$C_1 J_0(\beta a) \cos\alpha t = 0$$

Therefore $\beta$ must be chosen so that $J_0(\beta a) = 0$ . i.e. $\beta a$ must be one of the infinite number of positive roots of $J_0(x) = 0$ .[2]

Suppose that $\beta a$ is the $n^{th.}$ root of $J_0(x) = 0$ , and denote it by $\beta_n a$ , where $\beta_n a = x_n$ ( $n^{th.}$ root of $J_0(x)$ )

$$\therefore\ \beta_n = \frac{x_n}{a} \qquad 3.13$$

Now $\beta = \frac{\alpha}{c}$ . $\therefore \alpha = c\beta_n$ and 3.12 becomes

$$z = C_n J_0(\beta_n r) \cos c\beta_n t \qquad 3.14$$

where $C_n$ is now used to denote the arbitrary constant .

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2 See Appendix B to this chapter .

3.14 satisfies 3.3 and the initial and boundary conditions 3.4(b) and 3.4(c). The sum of any number of such solutions for different integral values of $n$ will also satisfy these conditions , and the following is therefore a solution which satisfies them

$$z = \sum_{n=1}^{\infty} C_n J_0(\beta_n r) \cos c\beta_n t \qquad 3.15$$

The initial condition 3.4(a) remains to be satisfied. When $t=0$ 3.15 becomes

$$z = \sum_{n=1}^{\infty} C_n J_0(\beta_n r)$$

which must be identical with $f(r)$ . i.e. the coefficients $C_n$ must be so chosen that if $\beta_i a$ is any one of the infinite number of positive roots of $J_0(\beta r) = 0$ then

$$f(r) = \sum_{i=1}^{\infty} C_n J_0(\beta_i r)$$

Multiply both sides of this last equation by $J_0(\beta_n r)\, r\, dr$ , where $\beta_n a$ is the $n^{th.}$ root of $J_0(x)=0$ [3]. Then integrate between the limits zero to $r$ . We have

$$\int_0^r f(r) J_0(\beta_n r)\, r\, dr = \int_0^r \sum_{i=1}^{\infty} C_n J_0(\beta_i r) J_0(\beta_n r)\, r\, dr$$

$$= \int_0^r \sum_{i=1}^{\infty} C_n \sqrt{r}\, J_0(\beta_i r) . \sqrt{r}\, J_0(\beta_n r)\, dr$$

$$= \sum_{i=1}^{\infty} \frac{C_n r}{\beta_i^2 - \beta_r^2} \left\{ \beta_r J_0(\beta_i r) J_0'(\beta_n r) - \beta_i J_0(\beta_n r) J_0'(\beta_i r) \right\} \qquad 3.16$$

Now put $r = a$ , and since $J_0(\beta_n a) = 0 = J_0(\beta_i a)$ we have

$$\int_0^a f(r) J_0(\beta_m r)\, r\, dr = 0 \quad \text{for } n \neq i \qquad 3.17$$

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3 See Appendix C to this chapter .

Equation 3.16 gives

$$(\beta_i^2-\beta_n^2)\int_0^{r}\sum_{i=1}^{\infty} C_n\sqrt{r}\,J_0(\beta_i r).\sqrt{r}\,J_0(\beta_n r)\,dr$$

$$= r\sum_{i=1}^{\infty} C_n\left\{\beta_r J_0(\beta_i r)\,J_0'(\beta_n r)-\beta_i J_0(\beta_n r)\,J_0'(\beta_i r)\right\}$$

Differentiating both sides of this last equation with respect to we get

$$2\beta_i\int_0^{r}\sum_{i=1}^{\infty} C_n\sqrt{r}\,J_0(\beta_i r).\sqrt{r}\,J_0(\beta_n r)\,dr+(\beta_i^2-\beta_n^2)\int_0^{r}\sum_{n=1}^{\infty} C_n r^2 J_0'(\beta_i r)\,J_0(\beta_n r)\,dr$$

$$= r\sum_{i=1}^{\infty}\left\{+\beta_r J_0'(\beta_i r)\,J_0'(\beta_n r) - J_0(\beta_n r)\,J_0'(\beta_i r) - r\beta_i J_0(\beta_n r)\,J_0''(\beta_i r)\right\}$$

Now put $\beta_i=\beta_n$ and $r = a$ . Since $J_0(\beta_n r)=0$ we get

$$2\beta_r\int_0^{a}\sum_{n=1}^{\infty} C_n\sqrt{r}\,J_0(\beta_n r).\sqrt{r}\,J_0(\beta_n r)\,dr = a^2\beta_r C_n\left[J_0'(\beta_n a)\right]^2$$

i.e. $$\int_0^{a}\sum_{n=1}^{\infty} C_n r\left[J_0(\beta_n r)\right]^2 dr = \frac{a^2}{2}C_n\left[J_0'(\beta_n a)\right]^2 \qquad 3.18$$

for $i=n$ .

Thus from 3.16, 3.17 and 3.18 we have

$$\int_0^{a} f(r)\,J_0(\beta_n r)\,r\,dr = \frac{a^2}{2}\left[J_0'(\beta_n a)\right]^2 C_n$$

$$\therefore\ C_n = \frac{2}{a^2\left[J_0'(\beta_n a)\right]^2}\int_0^{a} f(r)\,J_0(\beta_n r)\,r\,dr$$

and the following is the required solution of the problem

$$Z = \sum_{n=1}^{\infty}\left\{\frac{2\int_0^{a} f(r)\,J_0(\beta_n r)\,r\,dr}{a^2\left[J_0'(\beta_n a)\right]^2}\right\}J_0(\beta_n r)\,\mathrm{Cos}\,c\beta_n t \qquad 3.19$$

If the initial distortion is into a cone symmetrical about the $Z$-axis, then

$$f(r) = d\left(1-\frac{r}{a}\right)$$

where $d$ is the displacement of the centre of the membrane in the direction of the $Z$-axis . A light membrane may be so distorted by pressure applied normally at its centre. Again uniform pressure acting in the direction of the $Z$-axis and exerted equally all over the surface of the membrane would appear to give an initial distortion of the form of a catenary of revolution.

# APPENDIX A .

Consider the equation $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

Write $$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{aligned} r &= (x^2+y^2)^{\frac{1}{2}} \\ \theta &= \tan^{-1}\frac{y}{x} \end{aligned}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial z}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 z}{\partial \theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{\partial z}{\partial\theta}\frac{\partial^2\theta}{\partial x^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \frac{\partial\theta}{\partial x} = \frac{-y/x^2}{1+(y/x)^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{-r\cos\theta\frac{\partial\theta}{\partial x} + \sin\theta\frac{\partial r}{\partial x}}{r^2} = \frac{2}{r^2}\sin\theta\cos\theta$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial r^2}\cos^2\theta - \frac{2}{r}\frac{\partial^2 z}{\partial r\partial\theta}\cos\theta\sin\theta + \frac{\partial z}{\partial r}\frac{\sin^2\theta}{r}$$
$$+ \frac{\partial^2 z}{\partial\theta^2}\frac{\sin^2\theta}{r^2} + \frac{2}{r^2}\frac{\partial z}{\partial\theta}\cos\theta\sin\theta.$$

Similarly $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2}\sin^2\theta + \frac{2}{r}\frac{\partial^2 z}{\partial r\partial\theta}\cos\theta\sin\theta + \frac{\partial z}{\partial r}\frac{\cos^2\theta}{r}$$
$$+ \frac{\partial^2 z}{\partial\theta^2}\frac{\cos^2\theta}{r^2} - \frac{2}{r^2}\frac{\partial z}{\partial\theta}\cos\theta\sin\theta.$$

and the above equation becomes

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}\right)$$

## APPENDIX B.

The Bessel Function of order $n$ is defined by

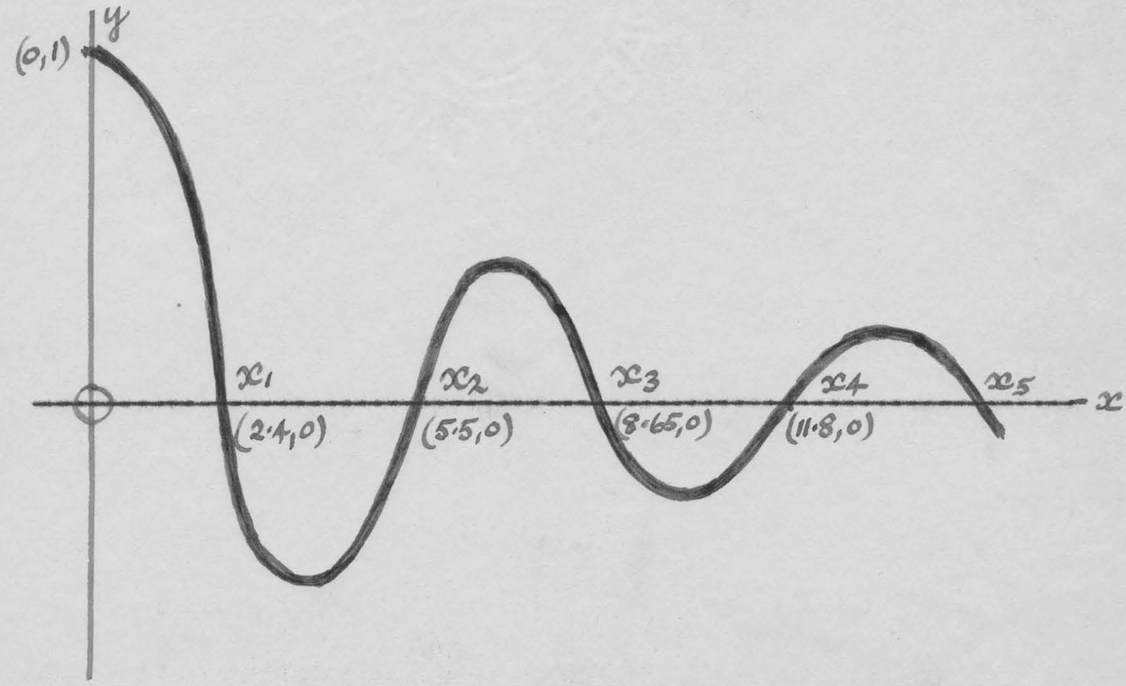
$$J_n(x) = \frac{x^n}{2^n \lfloor n} - \frac{x^{n+2}}{2^{n+2}\lfloor n+1} + \frac{x^{n+4}}{2^{n+4}\lfloor 2 \lfloor n+2} + \cdots + \frac{(-1)^k x^{n+2k}}{2^{n+2k}\lfloor k \lfloor n+k} + \cdots\cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+n}}{2^{2k+n}\lfloor k \lfloor n+k}$$

Hence

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(\lfloor k)^2}$$

A rough graph of $y = J_0(x)$ follows. The equation $J_0(x) = 0$ has an infinite number of positive roots and the approximate values of the first four of these roots are shown on the graph. It is worth noting that the roots are spaced at approximately $\pi$ units apart.



## APPENDIX C

The following theory is used on pages 44 and 45.

(i) $J_n(x) = 0$ has infinitely many positive roots, say $\lambda_1, \lambda_2, \cdots \lambda_n \cdots$.

(ii) $y = \sqrt{x}\, J_n(\lambda x)$ is a solution of the equation

$$4x^2 y'' + (4\lambda^2 x^2 - 4n^2 + 1)\, y = 0$$

Putting $u = \sqrt{x}\, J_n(\lambda x)$ and $v = \sqrt{x}\, J_n(\mu x)$ as two such solutions

$$4x^2 u'' + (4\lambda^2 x^2 - 4n^2 + 1)\, u = 0$$

and

$$4x^2 v'' + (4\mu^2 x^2 - 4n^2 + 1)\, v = 0$$

Multiply the first of these equations by $v$ and the second by $u$ and sub--tract. We get

$$-(\lambda^2 - \mu^2)\, uv = u''v - v''u$$

Integrate this equation between the limits $x = 0$ to $x = x$ giving

$$-(\lambda^2 - \mu^2) \int_0^x uv\, dx = \Big[u'v\Big]_0^x - \int_0^x u'v'\, dx - \Big[v'u\Big]_0^x + \int_0^x u'v'\, dx$$

$$= \Big[u'v - v'u\Big]_0^x$$

i.e.

$$(\lambda^2 - \mu^2) \int_0^x \sqrt{x}\, J_n(\lambda x) . \sqrt{x}\, J_n(\mu x)\, dx$$

$$= x \Big[\mu J_n(\lambda x)\, J_n'(\mu x) - \lambda J_n(\mu x)\, J_n'(\lambda x)\Big] \qquad \text{(A)}$$

Put $\lambda = \lambda_i$, $\mu = \lambda_j$, where $\lambda_i$ and $\lambda_j$ are roots of $J_n(x) = 0$ and $\lambda_i \neq \lambda_j$.

Then

$$\int_0^x \sqrt{x}\, J_n(\lambda_i x) . \sqrt{x}\, J_n(\lambda_j x)\, dx$$

$$= \frac{x}{\lambda_i^2 - \lambda_j^2} \Big[\lambda_j J_n(\lambda_i x)\, J_n'(\lambda_j x) - \lambda_i J_n(\lambda_j x)\, J_n'(\lambda_i x)\Big]$$

Now put $x=1$ . Since $J_n(\lambda_i) = J_n(\lambda_j) = 0$ we have

$$\int_0^1 \sqrt{x}\, J_n(\lambda_i x)\sqrt{x}\, J_n(\lambda_j x)\, dx = 0 \quad \text{if } i \neq j \qquad \text{(B)}$$

Differentiate equation (A) partially with respect to $\lambda$ . We get

$$2\lambda \int_0^x x\, J_n(\lambda x)\, J_n(\mu x)\, dx + (\lambda^2-\mu^2)\int_0^x x^2 J_n(\mu x)\, J_n'(\lambda x)\, dx$$
$$= x\left[\mu x\, J_n'(\lambda x)\, J_n'(\mu x) - J_n(\mu x)\, J_n'(\lambda x) - \lambda x\, J_n(\mu x)\, J_n''(\lambda x)\right]$$

Now put $\lambda = \mu = \lambda_i$ and $x=1$ . Since $J_n(\lambda_i)=0$ we have

$$\int_0^1 \sqrt{x}\, J_n(\lambda_i x) . \sqrt{x}\, J_n(\lambda_i x)\, dx = \frac{1}{2}\left[J_n'(\lambda_i)\right]^2 \text{ if } i = j \qquad \text{(C)}$$

Results (A), (B) and (C) above are of importance in application to pages 44 and 45.

## CHAPTER 4

# Application of Legendre Polynomials to the solution of partial differential equations of the second order.

### Conduction of heat in a solid body.

Consider a solid body of uniform density $\rho$ composed of a substance of uniform specific heat $c$. Let the temperature at a point $P(x,y,z)$ in the body be $u$ degrees at time $t>0$. It will be assumed that $u$ is a continuous function of position and time, and that no source or sink of heat exists in the body.

Experiments show that heat flows from points at higher temperature to points at lower temperature.

Consider an arbitrarilly chosen volume $V$ of the body bounded by a closed surface $S$. The amount of heat $\Delta H$ which crosses an element of surface $\Delta S$ in time $\Delta t$ will be assumed to be proportional to the greatest rate of decrease of the temperature $u$. i.e.

$$\Delta H = k \, \Delta S \, \Delta t \left| \frac{\partial u}{\partial n} \right| \qquad 4.1$$

where $k$ is the coefficient of thermal conductivity of the substance of the body (Calories/cm.sec.$C^{o}$ in C.G.S. units) and $\frac{\partial u}{\partial n}$ is the rate of change of $u$ in the direction of the normal to that level surface, $u = Const.$, which passes through $\Delta S$.

Let $\bar{q}$ be the vector representing the maximum rate of flow of heat at any point of the body. Then

$$\bar{q} = -k \operatorname{grad} u$$

where grad $u$ ( or $\nabla u$ ) is directed normally to the level surface $u = Const.$ in the direction of increasing $u$, and the negative sign is taken to direct $\bar{q}$ in the direction of decreasing $u$.

The total amount of heat $H$ flowing outwards across the closed surface $S$ from the volume $V$ in time $\Delta t$ is given from 4.1 by

$$H = -\Delta t \int_S k \frac{\partial u}{\partial n} dS$$

$$= \Delta t \int_S q_n dS \qquad 4.2$$

where $q_n$ is the component of $\bar{q}$ normal to $S$ at any point on it.

Now, to increase the temperature of an element of volume $\Delta V$ by $\Delta u$ an amount of heat equal to the product of the mass of the element. its specific heat and the increase in temperature must be supplied. Hence

$$\Delta H = \rho \Delta V c \Delta u = \rho c \Delta V \frac{\partial u}{\partial t} \Delta t$$

and the total loss of heat from the volume $V$ in time $\Delta t$ is therefore

$$H = -\Delta t \int_V \frac{\partial u}{\partial t} c\rho \, dV \qquad 4.3$$

Equating the right-hand sides of 4.2 and 4.3 we get

$$\int_S q_n dS = -\int_V \frac{\partial u}{\partial t} c\rho \, dV$$

and applying the Divergence Theorem to the left-hand side of this equation we get [I]

$$\int_V \operatorname{div} \bar{q} \, dV = -\int_V \frac{\partial u}{\partial t} c\rho \, dV$$

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I For statement of the Divergence Theorem used here see Appendix A.

But $\bar{q} = -k \operatorname{grad} u$ and so

$$\int_V \left\{ \operatorname{div}(-k \operatorname{grad} u) + c\rho \frac{\partial u}{\partial t} \right\} dV = 0$$

i.e.

$$\int_V \left\{ -k \nabla^2 u + c\rho \frac{\partial u}{\partial t} \right\} dV = 0$$

Now $V$ was chosen arbitrarilly, and since the integrand above is a contin--uous function the integral can only vanish if the integrand equals zero. If this were not so $V$ could be so chosen as to enclose a region throughout which the integrand has a constant sign. Hence

$$-k \nabla^2 u + c\rho \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho} \nabla^2 u$$

and if $c$ and $\rho$ are both constant[2]

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u, \quad t > 0 \qquad 4.4$$

where $a^2 = \frac{k}{c\rho} = const.$

For a steady distribution of temperature 4.4 reduces to the Laplace Equation

$$\nabla^2 u = 0$$

i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad t > 0 \qquad 4.5$$

---

2 The assumption that $c$ and $\rho$ are constant, and therefore independant of the temperature $u$, cannot be accepted in general but is reasonable in certain cases. In fact $c$ and $\rho$ vary slowly with the temperature and if the temperature range is not large this variation will be neglig--able. In the problem studied later in this chapter a small temperature range is considered and the assumption is therefore reasonable.

Again equation 4.4 will not hold if sources or sinks of heat are contained in the body. Thus if sources of heat are continuously dist--ributed throughout the body a term must be added to the right-hand side giving

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u + \frac{f(x,y,z,t)}{c\rho}$$

where $f$ is a function representing the strength of the sources.

Equations 4.4 and 4.5 must be solved subject to the initial and boundary conditions of a particular problem . For example if the surface of the body is so insulated that no heat leaves the body then at the surface $\frac{\partial u}{\partial n} = 0$ . Again the body may radiate heat from its surface and be enclosed in an evacuated container, the inner surface of which is kept at a constant temper--ature. Then applying Stephan's Law we have at the surface

$$k\frac{\partial u}{\partial n} = \sigma\left(u^4 - u_1^4\right)$$

where $\sigma$ is Stephan's surface constant for the body and $u_1$ the constant temperature of the inner surface of the container.

If the initial and surface conditions are known it can be proved that the problem of finding the temperature at any point of the body at time $t > 0$ has an unique solution . In practice the application of initial and boundary conditions present considerable difficulty in any but the simpler problems. To illustrate the application of Legendre Polynomials I have chosen the following problem :-

Flow of heat in a solid sphere.

Consider a solid sphere of unit radius and uniform density , specific heat and thermal conductivity. Suppose that there exists a steady state of temperature distribution, the sphere being immersed in media which maintain one hemispherical half of its surface at a constant temperature of $0^\circ$ C. and the other half of the surface at a constant temperature of $1^\circ$ C.

Since the flow of heat is steady the temperature $u$ is independant of $t$ and therefore $\frac{\partial u}{\partial t} = 0$ , and $u$ is a solution of 4.5 .

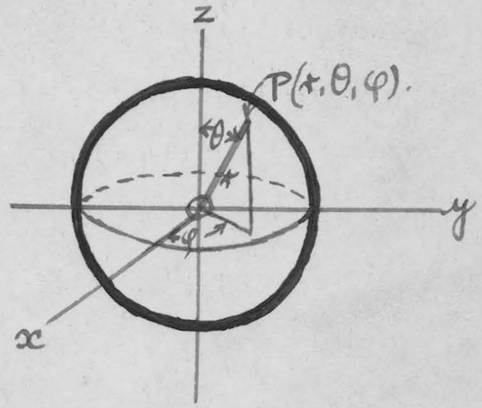
We are dealing with a sphere and this suggests the use of spherical co-ordinates . Writing

$$x = +\sin\theta\cos\varphi \;,\quad y = +\sin\theta\sin\varphi \;,\quad z = +\cos\theta$$

equation 4.5 becomes

$$\frac{r\,\partial^2(ur)}{\partial r^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=0,\quad t>0 \qquad 4.6$$

Chose the plane which seperates the unequally heated hemispheres as the $xy$-plane. The temperature at any point $P(r,\theta,\varphi)$ will be independ--ant of $\varphi$, and the temperature distrib--ution to the right and left of the $xz$-plane will be symmetrical.




It will therefore be sufficient to consider only that part of the sphere which lies to the right of the $xz$-plane. Since $u$ is independant of $\varphi$ equation 4.6 becomes

$$\frac{r\,\partial^2(ur)}{\partial r^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0,\quad t>0 \qquad 4.7$$

and a solution of 4.7 must be found to satisfy the boundary conditions

(a) When $r=1$, $u=1$ for $0<\theta<\frac{\pi}{2}$, $t\geq 0$

(b) When $r=1$, $u=0$ for $\frac{\pi}{2}<\theta<\pi$, $t\geq 0$ 4.8

Assume that a solution of 4.7 can be found which is the product of two functions, one of which is a function of $r$ alone and the other a function of $\theta$ alone.

i.e. $$u = R(r)\,\Theta(\theta).$$

Substituting this solution in 4.7 we get

$$r\,\Theta\,\frac{d^2(Rr)}{dr^2}+\frac{1}{\sin\theta}R\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

$$\frac{r}{R}\frac{d^2(Rr)}{dr^2}=-\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\frac{1}{\sin\theta},\quad t>0 \qquad 4.9$$

The left-hand side of this equation is a function of $r$ alone and the right-hand side is a function of $\theta$ alone. Thus the equality can only hold if the members on either side are each equal to the same constant.

Suppose that this constant is $a^2$ . Then we have

$$r\frac{d^2(Rr)}{dr^2} - a^2R = 0$$

$$r\frac{d}{dr}\left(R + r\frac{dR}{dr}\right) - a^2R = 0$$

$$r\left(2\frac{dR}{dr} + r\frac{d^2R}{dr^2}\right) - a^2R = 0$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - a^2R = 0, \quad t>0 \qquad 4.10$$

which is an equation of the Euler-Cauchy type.

Write $r = e^z$, $\frac{dr}{dz} = e^z = r$ , $\frac{dz}{dr} = e^{-z} = \frac{1}{r}$ .

$$\frac{dR}{dr} = \frac{dR}{dz}\frac{dz}{dr} = e^{-z}\frac{dR}{dz} = \frac{1}{r}\frac{dR}{dz}$$

$$\frac{d^2R}{dr^2} = \frac{d}{dz}\left(\frac{dR}{dz}\right)\frac{dz}{dr} = \frac{d}{dz}\left(e^{-z}\frac{dR}{dz}\right)\frac{1}{r} = \frac{1}{r^2}\left(\frac{d^2R}{dz^2} - \frac{dR}{dz}\right)$$

Substituting in equation 4.10 we have

$$\frac{d^2R}{dz^2} - \frac{dR}{dz} + 2\frac{dR}{dz} - a^2R = 0$$

$$\frac{d^2R}{dz^2} + \frac{dR}{dz} - a^2R = 0$$

The general solution is

$$R = A\,e^{m_1z} + B\,e^{m_2z} \quad (A \text{ and } B \text{ are constants})$$

$$= A\,r^{m_1} + B\,r^{m_2} \quad (\text{since } e^z = r\,.)$$

where $m_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + a^2}$ and $m_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} + a^2}$

Therefore $m_1 + m_2 = -1$ and $m_2 = -(m_1 + 1)$

$$m_1m_2 = \frac{1}{4} - \left(\frac{1}{4} + a^2\right) = -a^2$$

$$\therefore \quad a^2 = -m_1m_2 = m_1(m_1 + 1)$$

Hence the general solution can be written

$$R = Ar^{m} + Br^{-(m+1)} \tag{4.11}$$

where $m(m+1) = a^2$.

The right-hand side of 4.9 gives

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + a^2\Theta = 0$$

Writing $a^2 = m(m+1)$ this becomes

$$\frac{1}{\sin\theta}\left\{\sin\theta\frac{d^2\Theta}{d\theta^2} + \cos\theta\frac{d\Theta}{d\theta}\right\} + m(m+1)\Theta = 0$$

Write $\cos\theta = x$. We have

$$\frac{d\Theta}{d\theta} = -\sin\theta\frac{d\Theta}{dx}$$

$$\frac{d^2\Theta}{d\theta^2} = -\frac{d}{dx}\left(\frac{d\Theta}{dx}\sin\theta\right) = \sin\theta\left(\frac{d^2\Theta}{dx^2}\sin\theta + \frac{d\Theta}{dx}\cos\theta\right)$$

Hence the equation becomes

$$\sin^2\theta\frac{d^2\Theta}{dx^2} - 2\cos\theta\frac{d\Theta}{dx} + m(m+1)\Theta = 0$$

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + m(m+1)\Theta = 0 \tag{4.12}$$

which is a Legendre Equation with a singularity at $x = 1$ and at $x = -1$

Assume a solution of 4.12 in the form of a power series

$$\Theta = x^{\alpha}(a_0 + a_1x + \cdots\cdots + a_ix^i + \cdots\cdots) \tag{4.13}$$

$$= \sum_{i=0}^{\infty} a_i x^{\alpha+i}$$

$$\frac{d\Theta}{dx} = \sum_{i=0}^{\infty} a_i(\alpha+i)x^{\alpha+i-1}$$

$$\frac{d^2\Theta}{dx^2} = \sum_{i=0}^{\infty} a_i(\alpha+i)(\alpha+i-1)x^{\alpha+i-2}$$

Substituting in 4.12 we get

$$(1-x^2)\sum_{i=0}^{\infty} a_i(\alpha+i)(\alpha+i-1)\,x^{\alpha+i-2} - 2x\sum_{i=0}^{\infty} a_i(\alpha+i)\,x^{\alpha+i-1}$$

$$+\,m(m+1)\sum_{i=0}^{\infty} a_i\,x^{\alpha+i} = 0$$

$$\sum_{i=0}^{\infty} a_i(\alpha+i)(\alpha+i-1)\,x^{\alpha+i-2} - \sum_{i=0}^{\infty} a_i(\alpha+i)(\alpha+i-1)\,x^{\alpha+i}$$

$$-\,2\sum_{i=0}^{\infty} a_i(\alpha+i)x^{\alpha+i} + m(m+1)\sum_{i=0}^{\infty} a_i x^{\alpha+i} = 0 \qquad 4.14$$

Equating the coefficient of the lowest power of $x$ to zero ( i.e. the coefficient of $x^{\alpha-2}$ ) we get the indicial equation

$$a_0\alpha(\alpha-1) = 0 \qquad 4.15$$

Hence if $a_0 \neq 0$ then either $\alpha = 0$ or $1$ .

Taking $\alpha = 0$ and equating the coefficient of $x^{\alpha-1}$ in 4.14 to zero gives

$a_0(\alpha+1)\alpha = 0$ since $\alpha = 0$ .

$a_1$ is arbitrary. Take $a_1 \neq 0$ .

Equating the coefficient of $x^n$ in 4.14 to zero gives

$$a_{n+2}(n+2)(n+1) - a_n\{2n + n(n-1) - m(m+1)\} = 0$$

$$\therefore\ a_{n+2}(n+2)(n+1) = -a_n(m^2+m-n^2-n)$$

$$= -a_n(m-n)(m+n+1)$$

$$\therefore\ a_{n+2} = -\frac{(m-n)(m+n+1)\,a_n}{(n+2)(n+1)}$$

Hence

$$a_2 = -\frac{m(m+1)}{2}\,a_0$$

$$a_4 = -\frac{(m-2)(m+3)}{4.3}\,a_2 = \frac{m(m-2)(m+1)(m+3)}{\lfloor 4}\,a_0$$

$$a_6 = -\frac{m(m-2)(m-4)(m+1)(m+3)(m+5)}{\lfloor 6}\,a_0$$

- - - - - - - - - - - etc.

and $$a_3 = -\frac{(m-1)(m+2)}{\underline{|3}} a_1$$

$$a_5 = -\frac{(m-3)(m+4)}{5.4} a_3 = \frac{(m-1)(m-3)(m+2)(m+4)}{\underline{|5}} a_1$$

$$a_7 = -\frac{(m-1)(m-3)(m-5)(m+2)(m+4)(m+6)}{\underline{|7}} a_1$$

- - - - - - - - - - - - - - - - - - - - - - - - - - - - etc.

Substituting in 4.I3 for the coefficients in terms of $a_0$ and $a_1$ we get

$$\Theta = a_0\left\{1 - \frac{m(m+1)}{\underline{|2}}x^2 + \frac{m(m-2)(m+1)(m+3)}{\underline{|4}}x^4 - \cdots\right\}$$
$$+ a_1\left\{x - \frac{(m-1)(m+2)}{\underline{|3}}x^3 + \frac{(m-1)(m-3)(m+2)(m+4)}{\underline{|5}}x^5 - \cdots\right\}$$

4.I6

The first series on the right-hand side of 4.I6 is an even function of $x$ and the second is an odd function of $x$. These two series are therefore linearly independant and $a_0$ and $a_1$ are arbitrary constants. The series converge for $-1 < x < 1$ and 4.I6 is thus a general solution of 4.I2 for $|x| < 1$.

For $\alpha = 1$ we get the same solution but with $a_1 = 0$. Thus the solution 4.I6 includes the solution given by taking $\alpha = 1$.

If $m$ is an even integer the first series in 4.I6 terminates and reduces to a polynomial, whilst if $m$ is an odd integer the second series similarly reduces to a polynomial. If $a_0$ and $a_1$ are so adjusted as to make these polynomials unity when $x = 1$ we get a set of Legendre polynomials, $P_m(x)$ or $P_m(\cos\theta)$ since $x = \cos\theta$.[3]

---

3 The subscripts of the $P$'s indicate the values of $m$, and each is a particular solution of 4.I2 for $m$ the value of the subscript. Thus from 4.I6

$$P_0(x) \equiv 1, \quad P_1(x) \equiv x, \quad P_3(x) \equiv \tfrac{3}{2}x^2 - \tfrac{1}{2},$$
$$P_3(x) \equiv \tfrac{5}{2}x^3 - \tfrac{3}{2}x, \quad P_4(x) \equiv \tfrac{35}{8}x^4 - \tfrac{15}{4}x^2 + \tfrac{3}{8},$$
$$P_5(x) \equiv \tfrac{63}{8}x^5 - \tfrac{35}{4}x^3 + \tfrac{15}{8}x, \ldots\ldots \text{etc.}$$

The values of Legendre Polynomils are tabulated for varius values of $x$. They are also called Surface Zonal Harmonics.

Hence with 4.II we get a solution of 4.7 given by

$$u = \left\{ A r^{m} + B r^{-(m+1)} \right\} P_m(\cos\theta)$$

or rather two particular solutions

$$u = A r^{m} P_m(\cos\theta)$$

and $$u = B r^{-(m+1)} P_m(\cos\theta)$$

The second of these particular solutions is inapplicable since it becomes infinite as $r$ approaches zero. The particular solution applicable is therefore

$$u = A r^{m} P_m(\cos\theta) \qquad 4.I7$$

and the expression for the temperature at any point inside the sphere must be built up from terms of the type of 4.I7 with a positive integer. Each term of the following series satisfies 4.7

$$u = \sum_{m=0}^{\infty} A_m r^{m} P_m(\cos\theta) \qquad 4.I8$$

When $r = 1$ 4.I8 becomes

$$u = \sum_{m=0}^{\infty} A_m P_m(\cos\theta)$$

and if we determine the constants $A_m$ so as to satisfy the boundary conditions 4.8 we shall obtain a solution of our problem.

If we write $u(\theta) = F(x)$ the problem becomes that of expanding $F(x)$ in the interval $(-1, 1)$ in a series of Legendre polynomials.

i.e. $$F(x) = \sum_{m=0}^{\infty} A_m P_m(x) \qquad 4.I9$$

where $F(x) = 0$ for $-1 < x < 0$ and $F(x) = 1$ for $0 < x < 1$.

Multiply both sides of 4.I9 by $P_n(x)\,dx$ and integrate between the limits $-1$ to $1$ for $x$. We get for $n$ an integer

$$\int_{-1}^{1} F(x)\, P_n(x)\, dx = \int_{-1}^{1} \sum_{m=0}^{\infty} A_m P_m(x) P_n(x)\, dx$$

$$= \frac{2A_n}{2n+1}$$

Since $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$ if $m \neq n$ and $\int_{-1}^{1} A_n [P_n(x)]^2 dx = \frac{2A_n}{2n+1}$

Therefore

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} F(x)\, P_n(x)\, dx$$

$$= \frac{2n+1}{2} \int_{0}^{1} P_n(x)\, dx$$

Hence using the values of $P_0(x)$, $P_1(x)$, etc. given in footnote 3, page 58 we have

$$A_0 = \frac{1}{2}\int_0^1 P_0(x)\, dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

$$A_1 = \frac{3}{2}\int_0^1 P_1(x)\, dx = \frac{3}{2}\int_0^1 x\, dx = \frac{3}{4}.$$

$$A_2 = \frac{5}{2}\int_0^1 P_2(x)\, dx = \frac{5}{2}\int_0^1 \left(\frac{3}{2}x^2 - \frac{1}{2}\right) dx = 0.$$

$$A_3 = \frac{7}{2}\int_0^1 P_3(x)\, dx = \frac{7}{2}\int_0^1 \left(\frac{5}{2}x^3 - \frac{3}{2}x\right) dx = -\frac{1.7}{2.8}.$$

$$A_4 = \frac{9}{2}\int_0^1 P_4(x)\, dx = \frac{9}{2}\int_0^1 \left(\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}\right) dx = 0.$$

$$A_5 = \frac{11}{2}\int_0^1 P_5(x)\, dx = \frac{11}{2}\int_0^1 \left(\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x\right) dx = \frac{1.3.11}{2.4.12}.$$

– – – – – – – – – – – – – – – – – – – – – – etc.

and substituting these values in 4.18 we get

$$u = \frac{1}{2} + \frac{3}{4} + P_1(\cos\theta) - \frac{1.7}{2.8} P_3(\cos\theta) + \frac{1.3.11}{2.4.12} + {}^5 P_5(\cos\theta)$$

– – – – – – – – – – – – etc.

which is the required solution to our problem.

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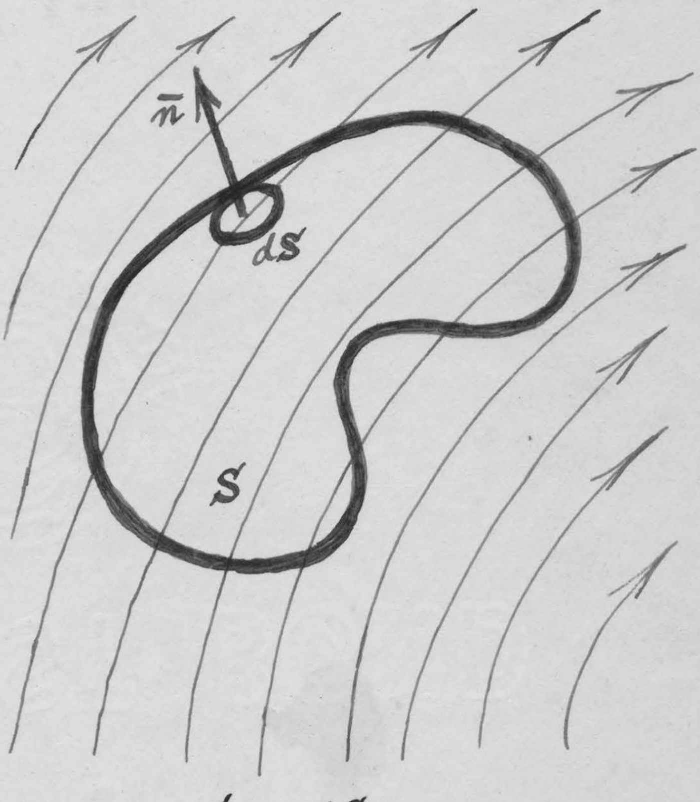
## APPENDIX A

### The Divergence Theorem.

Consider any closed surface $S$ lying in the field of a vector $\bar{q}$, which represents the maximum rate of flow of heat at any point within $S$. The excess of heat that flows out of $S$ over that which flows inwards may be measured in two different ways,

(i) by finding the total outward normal flux across $S$, and

(ii) by summing the sources and sinks or divergences for every infinitesimal volume element contained within $S$.




Equating (i) and (ii) we have

$$\int_S q_n \, dS = \int_V \operatorname{div} \bar{q} \, dV$$

or

$$\int_S \bar{n} \cdot \bar{q} \, dS = \int_V \nabla \cdot \bar{q} \, dV$$

where $\bar{n}$ is the outward-drawn unit normal to $S$, $dS$ is the element of surface, $dV$ the element of volume and $q_n$ or $\bar{n} \cdot \bar{q}$ is the outward-drawn component of $\bar{q}$ normal to $S$.

Thus in a vector field the surface integral of the normal component of the flux over any closed surface $S$ equals the volume integral of the divergence taken throughout the volume enclosed by $S$.

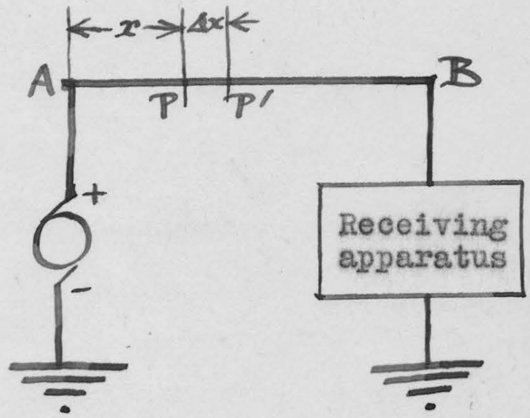
For a formal proof of this theorem see A.G.Webster's " Dynamics " or R.Gans's " Einführung in die Vektoranalysis ".

## CHAPTER 5

### Simultaneous Partial Differential Equations.

Flow of electricity in a long imperfectly insulated cable.

Consider a long imperfectly insulated cable throughout the whole length of which current leaks to earth. Suppose that the cable $AB$ is of length $\ell$ miles.




The end $A$ is attached to the positive pole of a generator, the negative pole of which is earthed. When the circuit is closed the cable carries an electric current which flows from $A$ through a receiving apparatus at $B$ ( e.g. a telegraphic key ) to earth.

Consider a point $P$ distant $x$ miles from $A$. Both the voltage, $V$ volts, and the current, $I$ amps., at $P$ are continuous functions of the time and the distance $x$ of $P$ from $A$.

I shall assume that

(i) the resistance of the cable is constant throughout its length and is $R$ ohms per mile.

(ii) the conductance (inverse of resistance) from the insulating sheathing of the cable to earth is constant throughout the length of the cable and is $G$ mhos per mile.

(iii) the cable acts as an electrostatic condenser and that the capacity of the cable and its inductance are each constant throughout the length of the cable, the capacity being $C$ Farads per mile and the inductance $H$ Henrys per mile.

Consider an element $PP'$ of length $\Delta x$ , with e.m.f. at $P$ and $V + \Delta V$ at $P'$ . The change in voltage across $PP'$ is caused by the resistance of the wire and decreased inductance. We have

$$\Delta V = -\left(IR\,\Delta x + \frac{\partial I}{\partial t} H\,\Delta x\right)$$

$$\therefore \frac{\Delta V}{\Delta x} = -IR - \frac{\partial I}{\partial t} H$$

and $$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \frac{\partial V}{\partial x} = -IR - H\frac{\partial I}{\partial t} \qquad 5.I$$

The decrease in current across $PP'$ is due to leakage and the action of the cable as a condenser. We have

$$\Delta I = -VG\,\Delta x - \frac{\partial V}{\partial t} C\,\Delta x$$

Hence $$\frac{\partial I}{\partial x} = -VG - C\frac{\partial V}{\partial t} \qquad 5.2$$

Equations 5.I and 5.2 are simultaneous differential equations to determine $V$ and $I$ .

In normal telegraphic practice the conductance $G$ and the inductance $H$ are small and it is reasonable to neglect them, but this would not be true for high frequency transmission. The capacity $C$ of the cable cannot be neglected.

I shall consider the problem of low frequency transmission and neglect $G$ and $H$ , so that 5.I and 5.2 become

$$\frac{\partial V}{\partial x} = -IR \qquad 5.3$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \qquad 5.4$$

Differentiating 5.3 with respect to $t$ and 5.4 with respect to $x$ gives

$$\frac{\partial^2 V}{\partial x \partial t} = -\frac{\partial I}{\partial t} R$$

$$\frac{\partial^2 I}{\partial x^2} = -C\frac{\partial^2 V}{\partial x \partial t} = RC\frac{\partial I}{\partial t}$$

Again, differentiating 5.3 with respect to $x$ we have

$$\frac{\partial^2 V}{\partial x^2} = -R\frac{\partial I}{\partial x} = RC\frac{\partial V}{\partial t}$$

Thus we have two simultaneous equations to determine and ,

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t} \qquad 5.5$$

$$\frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t} \qquad 5.6$$

I shall suppose that before the receiving end of the cable at is earthed there exists a steady state of voltage distribution in the cable and that the voltage at $A$ is $V_A$ and the voltage at $B$ is $V_B$. The voltage $V$ at $P$ will be a function of $x$ alone.

At time $t=0$ the recieving end at $B$ is earthed and hence the voltage at $B$ becomes zero. Suppose that the voltage at $A$ is maintained at a voltage $V_A$. Thus for $t=0$ 5.5 becomes

$$\frac{d^2V}{dx^2} = 0$$

with $V = V_A$ for $x=0$, $V = V_B$ for $x = l$.

The voltage at $P$ at time $t=0$ is

$$V = \frac{x}{l}(V_B - V_A) + V_A \qquad 5.7$$

To find the voltage at $P$ at time $t > 0$ we have to solve[I]

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}, \quad t > 0 \tag{5.8}$$

subject to the conditions that

(a) When $t = 0$, $V = \frac{x}{l}(V_B - V_A) + V_A$

(b) When $t > 0$, $V = V_A$ for $x = 0$
and $V = 0$ for $x = l$ 5.9

The voltage $V(x,t)$ for $t > 0$ can be considered as made up of a steady state distribution under the new conditions, say $V_S(x)$ since the steady state is a function of $x$ alone, and a transient voltage, $V_T(x,t)$, which decreases rapidly with the time. Hence

$$V(x,t) = V_S(x) + V_T(x,t)$$

After a short time the transient effects will become negligible and a steady state will be established where the voltage at $A$ is $V_A$ and the voltage at $B$ is zero. Hence, similarly to 5.7, we have

$$V_S(x) = -\frac{x}{l}V_A + V_A \tag{5.10}$$

and

$$V(x,t) = V_A\left(1 - \frac{x}{l}\right) + V_T(x,t) \tag{5.11}$$

When $x = 0$ we have from 5.9 for $t \geqslant 0$, $V(0,t) = V_A$, and from 5.11

$$V(0,t) = V_A + V_T(0,t)$$

$$\therefore V_T(0,t) = 0, \quad t \geqslant 0. \tag{5.12}$$

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I. The method of solution used will be similar to that used in the first part of Chapter 2, but the equation may also be solved by operational methods ( See O.M. in A.M., Chapter IX, paras. 84-86, and in particular Example 4, para. 84, which is similar to the problem here treated.)

Again when $t>0$ we have from 5.9 for $x=l$, $V(l,t)=0$, and from 5.II

$$V(l,t) = V_T(l,t)$$

$$\therefore \quad V_T(l,t) = 0 \ , \quad t>0. \qquad 5.I3$$

From 5.9 we have $V(x,0) = \frac{x}{l}(V_B - V_A) + V_A$ when $t=0$, whilst from 5.II

$$V(x,0) = V_A\left(1 - \frac{x}{l}\right) + V_T(x,0)$$

$$\therefore \quad V_T(x,0) = \frac{x}{l} V_B \qquad 5.I4$$

Now $V_S(x)$ satisfies 5.5 and so $V_T(x,t)$ must satisfy 5.5.

$V_T(x,t)$ is therefore a solution of 5.5 which satisfies the conditions 5.I2, 5.I3 and 5.I4 .

Writing $a^2 = \frac{1}{RC}$ equation 5.5 becomes

$$\frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial x^2} \ , \quad t>0 \qquad 5.I5$$

Assume a solution of 5.I5 as a product of two functions , one of which is a function of $x$ alone and the other a function of $t$ alone. i.e.

$$V_T(x,t) = X(x).T(t) \qquad 5.I6$$

Substituting 5.I6 in 5.I5 we get

$$X \frac{dT}{dt} = a^2 T \frac{d^2X}{dx^2}$$

$$\therefore \quad \frac{1}{a^2 T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

The left-hand side of the last equation is a function of $t$ alone and the right-hand side is a function of $x$ alone. Hence the equation will hold in general only if the members on either side are equal to the same constant , say $-\beta^2$ . Then we have

$$\frac{dT}{dt} + a^2\beta^2 T = 0. \tag{5.17}$$

and
$$\frac{d^2X}{dx^2} + \beta^2 X = 0. \tag{5.18}$$

Solutions of 5.17 and 5.18 are

$$T = k_1 e^{-a^2\beta^2 t}$$

and
$$X = k_2 \cos\beta x + k_3 \sin\beta x$$

where $k_1$, $k_2$ and $k_3$ are constants. The required solution of 5.15 is therefore

$$V_T(x,t) = e^{-a^2\beta^2 t}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$$

where $c_1$ and $c_2$ are constants determined by 5.12, 5.13 and 5.14.

From 5.12 $\quad V_T(0,t) = 0 \;, t > 0. \qquad \therefore c_1 = 0$

From 5.13 $\quad V_T(l,t) = 0 \;, t > 0. \qquad \therefore c_2 \sin\beta x = 0$

But $c_2 \neq 0$ since this would lead to no solution. Therefore

$$\sin\beta x = 0 \qquad \therefore \beta = \frac{n\pi}{l},\ (n = 1,2,3,\cdots\cdots)$$

Hence
$$V_T(x,t) = c\, e^{-a^2\left(\frac{n\pi}{l}\right)^2 t} \sin\frac{n\pi}{l}x \tag{5.19}$$

The condition 5.14 remains to be satisfied. Since 5.19 satisfies 5.15 for $n = 1,2,3,\cdots\cdots$ then

$$V_T(x,t) = \sum_{n=1}^{\infty} c_n e^{-a^2\left(\frac{n\pi}{l}\right)^2 t} \sin\frac{n\pi}{l}x \tag{5.20}$$

is also a solution of 5.15. When $t = 0$ 5.20 becomes

$$V_T(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}x$$
$$= \frac{x}{l} V_B \quad \text{from 5.14}$$

Write
$$\frac{x}{l} V_B = \sum_{s=1}^{\infty} c_s \sin\frac{s\pi}{l}x$$

$$\frac{x}{l} V_B = \sum_{s=1}^{\infty} C_s \sin \frac{s\pi x}{l}$$

Multiply each side of this equation by $\sin \frac{n\pi x}{l} dx$ and integrate between the limits $x = 0$ to $x = l$. We have

$$\frac{V_B}{l} \int_0^l x \sin \frac{n\pi x}{l} dx = \sum_{s=1}^{\infty} \int_0^l C_s \sin \frac{s\pi x}{l} \sin \frac{n\pi x}{l} dx$$

The integral on the left-hand side is

$$= \left[ -x \frac{l}{n\pi} \cos \frac{n\pi x}{l} + \left(\frac{l}{n\pi}\right)^2 \sin \frac{n\pi x}{l} \right]_0^l = -\frac{l^2}{n\pi} \cos n\pi$$

and on the right-hand side the term containing $C_n$ is

$$= C_n \int_0^l \left(\sin \frac{n\pi x}{l}\right)^2 dx = \frac{C_n}{2} \left[ x - \frac{l}{2n\pi} \sin \frac{2n\pi x}{l} \right]_0^l = \frac{l}{2} C_n .$$

and a term containing any other coefficient $C_s$ is, for $s \neq n$,

$$= C_s \int_0^l \sin \frac{s\pi x}{l} \sin \frac{n\pi x}{l} dx = \frac{C_s}{2} \int_0^l \left\{ \cos \frac{(n-s)\pi x}{l} - \cos \frac{(n+s)\pi x}{l} \right\} dx$$

$$= \frac{C_s}{2} \left[ \frac{l}{(n-s)\pi} \sin \frac{(n-s)\pi x}{l} - \frac{l}{(n+s)\pi} \sin \frac{(n+s)\pi x}{l} \right]_0^l$$

$= 0$ since $n$ and $s$ are integers.

$$\therefore \quad \frac{l}{2} C_n = -\frac{V_B l}{n\pi} \cos n\pi$$

$$C_n = -\frac{2 V_B}{n\pi} \cos n\pi$$

Hence the required solution is

$$V_T(x,t) = -\frac{2V_B}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-a^2 \left(\frac{n\pi}{l}\right)^2 t} \cos n\pi \sin \frac{n\pi x}{l}$$

and

$$V(x,t) = V_A\left(1 - \frac{x}{l}\right) - \frac{2V_B}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-a^2 \left(\frac{n\pi}{l}\right)^2 t} \cos n\pi \sin \frac{n\pi x}{l}$$

i.e.

$$V(x,t) = V_A\left(1-\frac{x}{l}\right) - \frac{2V_B}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{n^2\pi^2 t}{RCl^2}}\cos n\pi \sin\frac{n\pi x}{l} \qquad 5.2I$$

From 5.2I it is seen that as $t$ increases the contribution of $V_T$ to $V$ rapidly decreases and quickly becomes negligible when compared with $V_A\left(1-\frac{x}{l}\right)$. Thus if the circuit remains closed ( i.e. if the receiving end at $B$ is earthed ) for a sufficiently long period a steady state of voltage distribution is attained finally with

$$V(x,t) \doteq V_A\left(1-\frac{x}{l}\right) \qquad 5.22$$

The magnitude of the current $I$ at a point $P$ in the cable at time $t$ can be calculated by similar working from equation 5.6 and no useful purpose is served in doing so here.

The above method can also be applied to the problem of variable heat flow in a rod of small uniform cross section which is composed of a substance of uniform density, specific heat and thermal conductivity; where the surface of the rod is impervious to heat and no transfer of heat takes place across the surface, the ends of the rod are kept at constant temperature and the initial temperature at any point in the rod is thus a function of the distance of the point along the rod. The temperature $u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

where $a^2 = k/c\rho$, $k$ = thermal conductivity, $c$ = specific heat and $\rho$ = density

This equation will be subject to the following initial and boundary conditions

(i) $u = f(x)$ when $t=0$.

(ii) $u = u_1$ when $x=0$, $t \geq 0$.

(iii) $u = u_2$ when $x=l$, $t \geq 0$.