UNIVERSAL ALGEBRA COMPLEXES

UNIVERSAL ALGEBRA COMPLEXES :
EXTENSIONS AND INTEGRAL ELEMENTS

By

IN YOUNG CHUNG, B.S.

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AUTHOR: In Young Chung, B.S.(Seoul National University)

SUPERVISOR: Professor B. Banaschewski

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SCOPE AND CONTENTS: Two topics are studied in this thesis. The first topic is concerned with the relation between the categories of complexes over two algebras when there is a unitary algebra homomorphism from one to the other. The second topic deals with differential forms. A certain finiteness theorem for the module of integral differential forms is studied.

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INTRODUCTION

This thesis is concerned with problems which arise out of E. Kähler's paper [7] published in 1953. In this paper, a construction of a universal algebra complex over an algebra was first given. Since a universal algebra complex over an algebra is uniquely determined by the algebra up to unique complex isomorphisms, it is natural to observe the relation between universal algebra complexes over two algebras. Some results are obtained in [9]. In Chapter I, we show that an algebra homomorphism from an algebra A into an algebra B determines a natural covariant functor from the category of complexes over A and A-complex homomorphisms into the category of complexes over B and B-complex homomorphisms which sends a universal complex over A to a universal complex over B. Explicit constructions of this functor in some special cases are given and as a consequence of this, explicit relations between universal complexes over two algebras in these cases can be obtained.

Again, in [7], Kähler defines his differential forms as elements of a certain submodule of his infinitesimal algebra. In Chapter II, we establish first that the module of Kähler's differential forms of degree k is isomorphic to the module of homogeneous differential forms of degree k as defined in Chapter 0, when the universal derivation module of the algebra is finitely generated and projective. We then introduce integral differential forms in a manner analagous to, but more general than Kähler's definition of integral differential forms and show that the set of all homogeneous differential forms of degree k is, in certain special cases, finitely generated over the ground ring.

Chapter 0 is essentially a collection of all the basic definitions and results concerning modules and derivations which are used in the ensuing chapters. Some of the results we believe to be new.

CHAPTER 0

Preliminaries

This chapter is essentially a collection of all the basic definitions and results concerning modules and derivations which will be needed in the ensuing chapters.

§1. Kronecker and Grassman algebras.

Let R be a commutative ring with unit, M and N R-modules, and M* and N* the dual modules of M and N respectively. For $\varphi \in M^*$, $\psi \in N^*$, $\varphi * \psi \in (M \otimes N)^*$ is defined by $\varphi * \psi(a \otimes b) = \varphi(a)\psi(b)$, $a \in A$, $b \in B$. The product $\varphi * \psi$ is called the Kronecker Product of φ and ψ.

Let T(M) be a tensor algebra over an R-module M, and $K_n(M) = \{\varphi \in T(M)^* \mid \varphi | T_k(M) = 0 \text{ for all } k \neq n\}$, then clearly $K_n(M)$ is a submodule of $T(M)^*$. Let $K(M) = \sum_{n=0}^{\infty} K_n(M)$, then the sum is a direct sum. Let

$$\tau_{n,m} : T_{n+m}(M) \longrightarrow T_n(M) \otimes T_m(M)$$

be the canonical isomorphism defined by $x\,y \rightsquigarrow x \otimes y$, where x is an element of degree n, y an element of degree m, and

$$\tau^*_{n,m} : (T_n(M) \otimes T_m(M))^* \longrightarrow T_{n+m}(M)^*$$

be the dual homomorphism of $\tau_{n,m}$.

For $\varphi \in K_n(M)$, $\psi \in K_m(M)$, the product $\varphi . \psi$ is defined by

$$\varphi . \psi = \tau^*_{n,m}((\varphi \circ j_n)*(\psi \circ j_m)) \circ p_{n+m},$$

where $j_n : T_n(M) \longrightarrow T(M)$, $j_m : T_m(M) \longrightarrow T(M)$ are the natural injections and $p_{n+m} : T(M) \longrightarrow T_{n+m}(M)$ is the $(n + m)$th projection. Then $\varphi . \psi \in K_{n+m}$, since $\varphi \circ j_n \in T_n(M)^*$, $\psi \circ j_m \in T_m(M)^*$, hence $(\varphi \circ j_n)*(\psi \circ j_m) \in (T_n(M) \otimes T_m(M))^*$, thus $\tau^*_{n,m}((\varphi \circ j_n)*(\psi \circ j_m)) \in T_{n+m}(M)^*$, and finally $\varphi . \psi = \tau^*_{n,m}((\varphi \circ j_n)*(\psi \circ j_m)) \circ p_{n+m} \in T(M)^*$.
Moreover, $\varphi . \psi | T_k(M) = 0$ for $k \neq n + m$, since $p_{n+m} | T_k(M) = 0$ for $k \neq n + m$. Thus we have seen that $\varphi . \psi \in K_{n+m}(M)$.
Now for arbitrary φ, $\psi \in K(M)$, let

$$\varphi . \psi = \sum_{n,m} \tau^*_{n,m}((\varphi \circ j_n)*(\psi \circ j_m) \circ p_{n+m};$$

then it is known that K(M) with this law of composition is an associative, regularly graded algebra.

Definition 1. K(M) is called the Kronecker algebra over the module M.

Let E(M) be an exterior algebra over a module M. In [4], the dual module E(M)* of E(M) with the Grassman product "$\wedge$" as its law of composition is called the Grassman algebra for the module M. But in this context, the following algebra G(M) will be called the Grassman algebra for M:

Let $G_n(M) = \{ \varphi \in E(M)^* | \ \varphi | E_k(M) = 0$ for all $k \neq n \}$, then $G_n(M)$ is a submodule of E(M)*. Let $G(M) = \sum_{n=0}^{\infty} G_n(M)$, then the sum is a direct sum, and it is proved in [4] that G(M) is a subalgebra of

$E(M)^*$ and is a regularly graded anti-commutative algebra. It is worth noting that $G(M) = E(M)^*$ when M is a finitely generated module.

Definition 2: $G(M)$ is called the Grassman algebra over the module M.

Let $p_1 : E(M) \longrightarrow E_1(M) = M$ be the 1st projection, and $p_1^* : M^* \longrightarrow E(M)^*$ be the dual homomorphism of p_1. $p_1^*(M^*) \subseteq G_1(M)$, since for any $\varphi \in M^*$, $p_1^*(\varphi) = \varphi \circ p_1 \in G_1(M)$. Moreover $(p_1^*(\varphi))^2 = 0$, since $G(M)$ is known to be an anticommunicative algebra [4]. Hence $p_1^* : M^* \longrightarrow G_1(M) (\subseteq G(M))$ extends uniquely to a graded algebra homomorphism $g : E(M^*) \longrightarrow G(M)$.

Similarly, let $q_1 : T(M) \longrightarrow T_1(M) = M$ be the 1st projection, and $q_1^* : M^* \longrightarrow T(M)^*$ be the dual homomorphism of q_1. $q_1(M^*) \subseteq K_1(M)$, since for any $\varphi \in M^*$, $q_1^*(\varphi) = \varphi \circ q_1 \in K_1(M)$. Hence $q_1^* : M^* \longrightarrow K_1(M) \subseteq K(M)$ extends uniquely to a graded algebra homomorphism $h : T(M^*) \longrightarrow K(M)$.

Remark 1. 1) Since $M^* \cong G_1(M)$ and $g : E(M^*) \longrightarrow G(M)$ is a graded algebra homomorphism, one trivial observation is that $G(M)$ is generated by $G_1(M)$ if and only if g is onto.

2) Similarly, $K(M)$ is generated by $K_1(M)$ if and only if h is onto.

Proposition 1: $g : E(M^*) \longrightarrow G(M)$ and $h : T(M^*) \longrightarrow K(M)$ are isomorphisms if M is a finitely generated projective module.

For any $x \in M$, let $\hat{x} : T(M^*) \longrightarrow R$ be the R-module homomorphism defined by $\hat{x}|T_k(M^*) = 0$ for $k \neq 1$, $\hat{x}(\varphi) = \varphi(x)$ for $\varphi \in T_1(M^*)$ (in fact, $T_1(M^*) = M^*$), then $\hat{x} \in K_1(M^*)$. Define a mapping $\alpha_o : M \longrightarrow K(M^*)$ by $\alpha_o(x) = \hat{x}$, then clearly α_o is an R-module homomorphism. Hence there exists a unique algebra homomorphism $\alpha : T(M) \longrightarrow K(M^*)$ extending α_o.

Similarly, for any $x \in M$, let $\bar{x} : E(M^*) \longrightarrow R$ be the R-module homomorphism defined by $\bar{x}|E_k(M^*) = 0$ for $k \neq 1$, $\bar{x}(\psi) = \psi(x)$ for $\psi \in E_1(M^*)$ (in fact $E_1(M^*) = M^*$), then $\bar{x} \in K_1(M^*)$. Define a mapping $\beta_o : M \longrightarrow G(M^*)$ by $\beta_o(x) = \bar{x}$, then β_o is an R-module homomorphism such that $(\beta_o(x))^2 = 0$. Hence there exists a unique algebra homomorphism $\beta : E(M) \longrightarrow G(M^*)$ extending β_o.

Proposition 2: $\alpha : T(M) \longrightarrow K(M^*)$ and $\beta : E(M) \longrightarrow G(M^*)$ are isomorphisms if M is a finitely generated projective module.

The natural module monomorphism $j_k : G_k(M) \longrightarrow K_k(M)$ will be discussed.

Let $\nu : T(M) \longrightarrow E(M)$ be the natural homomorphism. For $\varphi \in G_n(M)$, $\varphi \circ \nu \in T(M)^*$ obviously. Moreover, $\varphi \circ \nu(T_k(M)) = \varphi(E_k(M)) = 0$ for $k \neq n$. Hence $\varphi \circ \nu \in K_n(M)$ for $\varphi \in G_n(M)$.

Proposition 3: Let $j_k : G_k(M) \longrightarrow K_k(M)$ be the mapping defined by $j_k(\varphi) = \varphi \circ \nu$ for $\varphi \in G_k(M)$, then j_k is a module monomorphism.

Proof: j_k is obviously a module homomorphism. To show j_k is one-to-one, suppose $j_k(\varphi) = \varphi \circ \nu = 0$ for $\varphi \in G_k(M)$. Then $\varphi = 0$, since ν is onto. Thus j_k is a monomorphism.

Remark 2: $j_1 : G_1(M) \longrightarrow K_1(M)$ is an isomorphism.

Remark 3: For $\varphi_1, \ldots, \varphi_k \in G_1(M)$, $x_1, \ldots, x_n \in E_1(M)$

$$(\varphi_1 \wedge \ldots \wedge \varphi_k)(x_1 \ldots x_k) = \sum_{\pi} \varepsilon(\pi)\, \varphi_{\pi(1)}(x_1) \cdots \varphi_{\pi(k)}(x_k)$$

where π is a permutation of $\{1, \ldots, k\}$ and $\varepsilon(\pi) = +1$ or -1 according to the permutation π is even or odd.

Also, for $\psi_1, \ldots, \psi_k \in K_1(M)$, $y_1, \ldots, y_n \in T_1(M)$,

$$(\psi_1 \ldots \psi_k)(y_1 \ldots y_k) = \psi_1(y_1) \ldots \psi_k(y_k).$$

Proposition 4: Let $\varphi_1, \ldots, \varphi_k \in G_1(M)$ and $\psi_1, \ldots, \psi_k \in K_1(M)$ with the property that $\psi_i = \varphi_i \circ \nu$ for $i = 1, 2, \ldots, k$. Then

$$j_k(\varphi_1 \wedge \cdots \wedge \varphi_k) = \sum_{\pi} \varepsilon(\pi)\, \psi_{\pi(1)} \cdots \psi_{\pi(k)}$$

where π is a permutation of $\{1, \ldots, k\}$ and $\varepsilon(\pi) = +1$ or -1 according as π is even or odd.

Proof: Both $j_k(\varphi_1 \wedge \cdots \wedge \varphi_k)$ and $\sum_{\pi} \varepsilon(\pi)\, \psi_{\pi(1)} \cdots \psi_{\pi(k)}$ are in $K_k(M)$. This means that $j_k(\varphi_1 \wedge \cdots \wedge \varphi_k) | T_m(M) = 0$ and $\sum_{\pi} \varepsilon(\pi)\, \psi_{\pi(1)} \cdots \psi_{\pi(k)} | T_m(M) = 0$ for $m \neq k$. Hence it is sufficient to show that they coincide on $T_k(M)$. We recall that any element in $T_k(M)$ can be expressed as a sum of elements of the form $x_1 \ldots x_k$ where $x_1, \ldots, x_k \in M$.

$j_k(\varphi_1 \wedge \cdots \wedge \varphi_k)(x_1 \cdots x_k)$

$= (\varphi_1 \wedge \cdots \wedge \varphi_k) \circ \nu \ (x_1 \cdots x_k)$ (by the definition of j_k)

$= (\varphi_1 \wedge \cdots \wedge \varphi_k)\ (\nu(x_1) \cdots \nu(x_k))$

$= \sum_\pi \varepsilon(\pi)(\varphi_{\pi(1)} \circ \nu\ (x_1)) \cdots (\varphi_{\pi(k)} \circ \nu\ (x_k))$ (by Remark 3)

$= \sum_\pi \varepsilon(\pi)\ \varphi_{\pi(1)}(x_1) \cdots \varphi_{\pi(k)}(x_k)$

$= \sum_\pi \varepsilon(\pi)\ \varphi_{\pi(1)} \cdots \varphi_{\pi(k)}\ (x_1 \cdots x_k)$ (by Remark 3)

$= (\sum_\pi \varepsilon(\pi)\ \varphi_{\pi(1)} \cdots \varphi_{\pi(k)})(x_1 \cdots x_k)$.

Thus, $j_k(\varphi_1 \wedge \cdots \wedge \varphi_k) = \sum_\pi \varepsilon(\pi)\ \varphi_{\pi(1)} \cdots \varphi_{\pi(k)}$.

<u>Corollary 1</u>: The natural monomorphism $j_k : G_k(M) \longrightarrow K_k(M)$ is entirely determined by the mapping $\varphi_1 \wedge \cdots \wedge \varphi_k \rightsquigarrow \sum_\pi \varepsilon(\pi)\ \varphi_{\pi(1)} \cdots \varphi_{\pi(k)}$ if $G(M)$ is generated by $G_1(M)$.

Finally, we will study the skewsymmetric elements of $K_k(M)$.

<u>Definition 5</u>: An element $\tau \in K_k(M)$ is said to be <u>skewsymmetric</u> of degree k if there exist $\tau_{i_1}, \ldots, \tau_{i_k} \in K_1(M)$ such that

$$\tau = \sum_i \sum_{\pi_i} \varepsilon(\pi_i)\ \tau_{\pi_i(i_1)} \cdots \tau_{\pi_i(i_k)}.$$

<u>Proposition 5:</u> Suppose M to be a module such that G(M) is generated by $G_1(M)$. Let $S_k(M)$ be the set of all skewsymmetric elements of degree k, then the natural injection j_k maps $G_k(M)$ onto $S_k(M)$ or equivalently $j_k : G_k(M) \longrightarrow S_k(M)$ is an isomorphism.

<u>Proof</u>: Since G(M) is generated by $G_1(M)$, any element in $G_k(M)$ is of the form $\sum_i \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$, where $\varphi_{i_j} \in G_1(M)$ for each $j = 1,2,\ldots,k$. $j_k(G_k(M)) \subseteq S_k(M)$, since $j_k(\sum_i \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}) = \sum_i \sum_{\pi_i} \varepsilon(\pi_i)\ \tau_{\pi_i(i_1)} \cdots \tau_{\pi_i(i_k)}$, where $\tau_{i_j} = \varphi_{i_j} \circ \nu$. Conversely, for any $\varphi \in S_k(M)$, there exist

$\psi_{i_1}, \ldots, \psi_{i_k} \in K_1(M)$ such that $\psi = \sum_i \sum_{\pi_i} \varepsilon(\pi_i)\, \psi_{\pi_i(i_1)} \cdots \psi_{\pi_i(i_k)}$.

Since $j_1 : G_1(M) \longrightarrow K_1(M)$ is an isomorphism (Remark 2), there exist $\varphi_{i_1}, \ldots, \varphi_{i_k} \in G_1(M)$ such that $\varphi_{i_j} \circ \nu = j_1(\varphi_{i_j}) = \psi_{i_j}$ for $j = 1,2,\ldots,k$ and for all i. $j_k(\Sigma\, \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}) = \sum_i \sum_{\pi_i} \varepsilon(\pi_i)\, \psi_{\pi_i(i_1)} \cdots \psi_{\pi_i(i_k)}$.

Hence $j_k(G_k(M)) \supseteq S_k(M)$.

Thus $j_k(G_k(M)) = S_k(M)$.

§ 2. Derivations and Derivation modules.

Let R be a commutative ring with unit, A a unitary commutative R-algebra, and M an A-module.

Definition: An R-linear mapping $d : A \longrightarrow M$ is called a derivation from A (as R-algebra) into M if and only if $d(ab) = adb + bda$ for $a, b \in A$, and a derivation $d : A \longrightarrow A$ is called a derivation on A (as R-algebra).

Definition: A couple (M,d) is called a derivation module of A (as R-algebra) if and only if M isan A-module and d is a derivation from A (as R-algebra) into M.

Definition: Let (M,d) and (N, δ) be derivation modules of A, then a module homomorphism $f : M \longrightarrow N$ is called a derivation module homomorphism if and only if $f \circ d = \delta$. A derivation module homomorphism which is one-to-one and onto is called a derivation module isomorphism.

Definition: A derivation module (U,d) of A is said to be universal if and only if for any derivation module (M, δ), there exists a unique derivation module homomorphism $f: (U,d) \longrightarrow (M, \delta)$.

Theorem 1: For any unitary commutative R-algebra A, there exists a universal derivation module of A and it is unique up to unique derivation module isomorphisms.

Remark: A universal derivation module (U,d) of A can be constructed in the following way: Let $U = A \otimes_R A/J$ where J is the A-submodule of $A \otimes_R A$ generated by all $1 \otimes ab - a \otimes b - b \otimes a$, $a,b \in A$, and define $d : A \longrightarrow U$ by $d(a) = \nu(1 \otimes a)$, $a \in A$ where $\nu : A \otimes_R A \longrightarrow U$ is the natural module homomorphism. Then (U,d) is a universal derivation module.

Let A and B be unitary commutative R-algebras, $\varphi : A \longrightarrow B$ unitary epimorphism.

Definition: Let M be an A-module and N a B-module. A mapping $f : M \longrightarrow N$ is called a φ-homomorphism if and only if $f(x + y) = f(x) + f(y)$ and $f(ax) = \varphi(a)f(x)$ for $a \in A$, $x,y \in M$.

Theorem 2. Let (U,d) be a universal derivation module of A as R-algebra and (V,δ) a universal derivation module of B as R-algebra, then there exists a unique φ-homomorphism $f : (U,d) \longrightarrow (V,\delta)$ such that $f \circ d = \delta \circ \varphi$, and $\ker f = \ker \varphi \cdot dA + A \cdot d \ker \varphi$.

Theorem 3. Let S be a unitary subring of R, and (U(A/R), d), (U(A/S), δ) universal derivation modules of A as R-algebra and as S-algebra respectively. Then

$$(U(A/R),d) \cong (U(A/S)/A\delta R, \quad \nu\circ\delta)$$

where $\nu : U(A/S) \longrightarrow U(A/S)/A\delta R$ is the natural homomorphism.

<u>Proof</u>: The mapping $\nu \circ \delta : A \longrightarrow U(A/S)/A\,\delta R$ is clearly S-linear. $\nu \circ \delta$ is also R-linear, since $\nu \circ \delta(ra) = \nu(r\delta a + a\,\delta r) = \nu(r\delta a) = r\,\nu(\delta a), r \in R, \quad a \in A.$ Product rule for $\nu \circ \delta$ holds trivially and hence $(U(A/S)/A\delta R, \nu \circ \delta)$ is a derivation module of A as an R-algebra. Since $S \subseteq R$, $U(A/R)$ can be considered as an S-algebra and $d : A \longrightarrow U(A/R)$ as a derivation of A as an S-algebra, $(U(A/R), d)$ is a derivation module of A as an S-algebra. Hence there exists a unique A-module homomorphism $f : U(A/S) \longrightarrow U(A/R)$ such that $f \circ \delta = d$. On the other hand, $f(A\,\delta R) = A(f \circ \delta)R = A\,dR = 0$. Hence there exists an A-module homomorphism $f' : U(A/S)/A\,\delta R \longrightarrow U(A/R)$ such that $f' \circ \nu = f$. f' is a derivation module homomorphism, since $f' \circ (\nu \circ \delta) = f \circ \delta = d$.

To show $(U(A/S)/A\,\delta R, \nu \circ \delta)$ is a universal derivation module of A as an R-algebra, let (M, ∂) be an arbitrary derivation module of A as an R-algebra. Then there exists a derivation module homomorphism $g : U(A/R) \longrightarrow (M, \partial)$ and hence $g \circ f' : U(A/S)/A\,\delta R \longrightarrow (M, \partial)$ is also a derivation module homomorphism. $g \circ f'$ is unique, since $U(A/S)/A\,\delta R$ is generated by $A(\nu \circ \delta)A$ as an A-module. Thus $(U(A/S)/A\,\delta R, \nu \circ \delta)$ is a universal derivation module of A as an R-algebra and

$$(U(A/R), d) \cong (U(A/S)/A\,\delta R, \nu \circ \delta).$$

Let D be the A-module of all derivations on A, D^* the dual module of D. If we define $\delta : A \longrightarrow D^*$ by $\delta(a)(\partial) = \partial a$, $\partial \in D$, $a \in A$, then (D^*, δ) is a derivation module of A. Let (U, d) be a universal derivation module of A as an R-algebra.

Theorem 4. If U is a finitely generated projective A-module, $(U,d) \cong (D^*, \delta)$.

Definition: The Kronecker algebra K(D) of D, essentially the algebra of multilinear forms on D, is called the algebra of differential forms on A, an element $\varphi \in K(D)$ a differential form on A, and an element $\varphi \in K_n(D)$ a homogeneous differential form of degree n.

Similarly,

Definition: The Grassman algebra G(D) of D, essentially the algebra of alternating multilinear forms on D, is called the algebra of alternating differential forms on A, an element $\varphi \in G(D)$ an alternating differential form on A, and an element $\varphi \in G_n(D)$ a homogeneous alternating differential form of degree n.

Theorem 5. Let A be a unitary commutative R-algebra such that the universal derivation module (U,d) of A is finitely generated projective, then

1) $E(U) \cong G(D)$, where E(U) is an exterior algebra of U. More explicitly, let $f : (U,d) \longrightarrow (D^*,\delta)$ be the unique derivation module homomorphism and $\bar{f} : E(U) \longrightarrow E(D^*)$ be the unique extension algebra homomorphism of f to E(U), then the mapping $g' : E(U) \longrightarrow G(D)$ defined by $g' = g \circ \bar{f}$($g : E(D^*) \longrightarrow G(D)$ defined as in §1) is a graded algebra isomorphism.

2) $T(U) \cong K(D)$. More explicitly, let $f' : T(U) \longrightarrow T(D^*)$ be the

unique extension algebra homomorphism of f to T(U), then the mapping $h' : T(U) \longrightarrow K(D)$ defined by $h' = h \circ f'$ (h' defined in § 1) is a graded algebra isomorphism.

<u>Proof</u>: Immediate consequence of Proposition 1, § 1, and Theorem 4, § 2.

Let R be a commutative ring with unit, and A a unitary commutative R-algebra.

<u>Definition</u>: A pair (C,d) is called an <u>A-complex</u> or a <u>complex over A</u> if and only if C is an anti-commutative graded R-algebra such that the module C_0 of homogeneous elements of degree zero is A and $d : C \longrightarrow C$ is a homogeneous derivation of degree 1 with $d \circ d = 0$ [4].

<u>Definition</u>: Let (C,d), (D,δ) be A-complexes. A graded algebra homomorphism $\varphi : C \longrightarrow D$ is called an <u>A-complex homomorphism</u> if and only if $\varphi|A$ is the identity mapping on A and $\varphi \circ d = \delta \circ \varphi$. It is denoted by $\varphi : (C,d) \longrightarrow (D,\delta)$. An <u>A-complex isomorphism</u> is a complex homomorphism which is one-to-one and onto.

<u>Definition</u>: An A-complex (U,d) is said to be <u>universal</u> if and only if for any A-complex (C,δ) there exists a unique A-complex homomorphism from (U,d) into (C,δ).

<u>Notational Remark</u>: When no confusion arises, both a universal A-complex and a universal derivation module of A are denoted by (U,d) or

$(U(A/R),d)$. But in case it is necessary to distinguish them, (U_1,d_1) or $(U_1(A/R),d_1)$ will denote a universal derivation module of A as an R-algebra.

Definition : An A-complex (C,d) is said to be simple if and only if C is generated by $A \cup d(A)$ as an R-algebra.

Theorem 6: For any unitary commutative R-algebra A, there exists a universal A-complex and it is unique up to unique complex isomorphisms.

Remark: A universal complex is simple.

Remark: Any complex homomorphism from a simple complex over A into an arbitrary complex over A is unique.

Let $((X_\alpha,d_\alpha))_{\alpha \in I}$ be a family of A-complexes, and each X_α graded by $X_\alpha = \sum_{n \geq 0} X_{\alpha,n}$ (direct). Consider the subalgebra $\bar{A} + \sum_{n \geq 1} \pi_\alpha X_{\alpha,n}$ of the cartesian product $\pi_\alpha X_\alpha$ where $\bar{A} = \{(a_\alpha)_\alpha | a_\alpha \in X_{\alpha,o}, a_\alpha = a$ for all $\alpha \in I\}$. Clearly $\bar{A}$ is isomorphic to A. Define $d : \bar{A} + \sum_{n \geq 1} \pi_\alpha X_{\alpha,n} \longrightarrow \bar{A} + \sum_{n \geq 1} \pi_\alpha X_{\alpha,n}$ by

$$d((a_\alpha)_\alpha + \sum_{n \geq 1} (x_{\alpha,n})_\alpha) = (d_\alpha a_\alpha)_\alpha + \sum_{n \geq 1} (d_\alpha(x_{\alpha,n}))_\alpha, \; a_\alpha \in X_{\alpha,o},$$

$x_{\alpha,n} \in X_{\alpha,n}$, then d is a homogeneous derivation of degree 1 with $d \circ d = o$ [9]. Hence $(\bar{A} + \sum_{n \geq 1} \pi_\alpha X_{\alpha,n}, d)$ is an A-complex.

Definition: Let $((X_\alpha,d_\alpha))_{\alpha \in 1}$ be a family of A-complexes.

Then the complex $(\bar{A} + \sum_{n \geq 1} \pi_{\alpha} X_{\alpha,n}, d)$ is called the product of the A-complexes (X_α, d_α), $\alpha \in I$.

Consider $p_\alpha : \bar{A} + \sum_{n \geq 1} \pi_{\alpha} X_{\alpha,n} \longrightarrow X_\alpha$ defined by

$p_\alpha((a_\alpha)_\alpha + \sum_{n \geq 1} \pi_{\alpha} (x_{\alpha,n})) = a_\alpha + \sum_{n \geq 1} x_{\alpha,n}$, then

$p_\alpha : (A + \sum_{n \geq 1} \pi_{\alpha} X_{\alpha,n}, d) \longrightarrow (X_\alpha, d_\alpha)$ is a complex homomorphism.

<u>Definition</u>: p_α is called the projection of the product of a family of A-complexes $((X_\alpha, d_\alpha))_{\alpha \in I}$ with respect to $\alpha \in I$.

<u>Remark</u>: Let $((X_\alpha, d_\alpha))_{\alpha \in I}$ be a family of A-complexes. For any complex (C, δ) and any family of A-complex homomorphism $f_\alpha : (C, \delta) \longrightarrow (X_\alpha, d_\alpha)$ for $\alpha \in I$, there exists a unique complex homomorphism $f : (C, \delta) \longrightarrow (\bar{A} + \sum_{n \geq 1} \pi_{\alpha} X_{\alpha,n}, d)$ such that $p_\alpha \circ f = f_\alpha$ for each α. Hence in fact, $(\bar{A} + \sum_{n \geq 1} \pi_{\alpha} X_{\alpha,n}, d)$ is a categorical product in the category of all A-complexes and A-complex homomorphisms.

§ 3. Differents.

Let R be a commutative ring with unit and suppose A to be a unitary commutative R-algebra such that a universal derivation mudule (U,d) of A as R-algebra is a finitely generated A-module, say $U = Aw_1 + \dots + Aw_n$. Let $\mathfrak{A}$ be the collection of all sequences $(a_1, \dots, a_n)$ of n elements in A satisfying $a_1w_1 + \dots + a_nw_n = 0$, and $\mathfrak{M}$ the collection of all n x n matrices $||a_{ij}||$ where each row $(a_{i1}, \dots, a_{in})$ belongs to $\mathfrak{A}$. Clearly, $\mathfrak{A}$ is a submodule of A^n.

It is known [7] that the ideal of A generated by the determinants of all $(n - \nu) \times (n - \nu)$ submatrices of all matrices in $\mathfrak{M}$ is uniquely determined by R and A, and will be denoted by $\mathfrak{D}_\nu(A/R)$ or simply $\mathfrak{D}_\nu$ when no confusion arises.

<u>Definition</u>: $\mathfrak{D}_\nu(A/R)$ is called the <u>νth different of A over R</u>. We will simply write $\mathfrak{D}$ for $\mathfrak{D}_o$ and call $\mathfrak{D}$ the <u>different of A over R</u>.

<u>Proposition 1</u>: Let $\mathfrak{B}$ be any set of generators of $\mathfrak{A}$ as submodule of A^n and $\mathfrak{N}$ the collection of all n x n matrices $||b_{ij}||$ where each sequence $b_{i1}, \dots, b_{in}$ belongs to $\mathfrak{B}$. Then the ideal of A generated by the determinants of all $(n - \nu) \times (n - \nu)$ submatrices of all n x n matrices in $\mathfrak{N}$ is $\mathfrak{D}_\nu(A/R)$.

The following is a well known theorem concerning finitely generated modules over a Euclidian domain.

Theorem: Let A be a Euclidian domain, F a free A-module with a finite basis, S a submodule of F. Then there exists a basis $\{\tau_1, \ldots, \tau_n\}$ of F and elements $c_1, \ldots, c_n$ in A such that $c_i | c_{i+1}$ (c_i divides c_{i+1}) for $i = 1, 2, \ldots n - 1$ and $\{c_1\tau_1, \ldots, c_n\tau_n\}$ generates S.

Corollary: Let A be a Euclidian domain, M a finitely generated A-module, N a submodule of M. Then there exists a set of generators $\{w_1, \ldots, w_n\}$ of M and elements $c_1, \ldots, c_n$ in A such that $\{c_1w_1, \ldots, c_nw_n\}$ generates N, $c_i | c_{i+1}$ for $i = 1, 2, \ldots, n - 1$, and if $a_1w_1 + \ldots + a_nw_n \in N$ there exist elements $b_1, \ldots, b_n \in A$ such that $a_i = b_ic_i$ for each $i = 1, 2, \ldots, n$.

Proof: M is finitely generated and hence a homomorphic image of a free module F with a finite basis. Let $\varphi : F \longrightarrow M$ be an epimorphism and put $S = \varphi^{-1}(N)$, then S is a submodule of F and by the previous theorem, there exists a basis $\{\tau_1, \ldots, \tau_n\}$ of F and elements $c_1, \ldots, c_n$ in A such that $c_i | c_{i+1}$ for $i = 1, 2, \ldots, n - 1$ and $\{c_1\tau_1, \ldots, c_n\tau_n\}$ generates S. Let us put $w_i = \varphi(\tau_i)$ for each i, then clearly $\{w_1, \ldots, w_n\}$ is a set of generators of M and $\{c_1w_1, \ldots, c_nw_n\}$ generates N.

If $a_1w_1 + \ldots + a_nw_n \in N$, then $a_1\tau_1 + \ldots + a_n\tau_n \in S$ and hence $a_1\tau_1 + \ldots + a_n\tau_n = b_1c_1\tau_1 + \ldots + b_nc_n\tau_n$. Since F is free with $\{\tau_1, \ldots, \tau_n\}$ as its basis, each $a_i = b_ic_i$ for $i = 1, 2, \ldots, n$.

Proposition 2: Let A be a Euclidian domain whose universal derivation module as R-algebra is finitely generated over A. Let $d : A \longrightarrow M$ be a derivation of A as R-algebra and B a unitary subalgebra of A.

Then

$\mathfrak{D}(A/B)\ dA \subseteq \mathfrak{D}_1(A/B)\ dB.$

<u>Proof:</u> Let (U,δ) be a universal derivation module of A as R-algebra, then $U = A\delta A$ and $A\delta B$ is a submodule of U. By the previous Corollary, there exists a set of generators $\{w_1,\dots,w_n\}$ of U and elements $c_1,\dots,c_n$ in A such that $c_i | c_{i+1}$ for $i = 1, 2,\dots, n-1$, $\{c_1w_1,\dots,c_nw_n\}$ generates $A\delta B$, and if $a_1w_1 + \dots + a_nw_n \in A\delta B$ there exist elements $b_1,\dots,b_n \in A$ with $a_i = b_ic_i$ for each i. Recall that $U(A/B) \cong U/A\delta B$ where $(U(A/B), \partial)$ is a universal derivation module of A as B-algebra (cf. Theorem 3 ,§2 , Chapter 0) and let $\nu : U \longrightarrow U(A/B)$ be the natural homomorphism and put $\sigma_i = \nu(w_i)$ for each i, then $\{\sigma_1,\dots,\sigma_n\}$ generates $U(A/B)$.
Let $a_1\sigma_1 + \dots + a_n\sigma_n = 0$ in $U(A/B)$, then $a_1w_1 + \dots + a_nw_n \in A\delta B$ and hence $a_i = b_ic_i$ for each $i = 1, 2,\dots, n$.

On the other hand, $c_i\sigma_i = 0$ for each i, since $c_iw_i \in A\delta B$. Let $\mathcal{L} = \{(r_{i1},\dots,r_{in}) | r_{ii} = c_i,\ r_{ij} = 0 \text{ for } j \neq i,\ i = 1, 2,\dots, n\}$, then for any $(a_1,\dots,a_n)$ in $\mathfrak{A} = \{(a_1,\dots,a_n) | a_1\sigma_1 + \dots + a_n\sigma_n = 0,\ a_i \in A\}$, $a_j = \sum_i r_{ij}\, b_j$. Hence by Proposition 1, $\mathfrak{D}(A/B) = c_1\cdots c_n \cdot A$ and $\mathfrak{D}_1(A/B) = c_1\cdots c_{n-1} \cdot A$. Now let $\varphi : U \longrightarrow M$ be the derivation module homomorphism. Then $\varphi(A\delta B) = AdB$, and hence $AdB = \Sigma Ac_i\,\varphi(w_i)$. Now, each c_i divides c_n, i.e. $c_n = b_ic_i$ with suitable b_i, and hence $c_n\varphi(w_i) = b_ic_i\,\varphi(w_i) \in AdB$. It follows that $c_nAdA \subseteq AdB$, and therefore $\mathfrak{D}(A/B)dA = c_1\cdots c_nA \cdot dA \subseteq c_1\cdots c_{n-1}AdB = \mathfrak{D}_1(A/B)\ dB$. This completes the proof.

<u>Proposition 3</u>: Let A be a commutative ring with unit, A_T

a ring of quotients of A with respect to a multiplicatively closed set T in A, S a unitary subring of A, and $S_{T \cap S} = \{ \frac{s}{t} \mid s \in S, t \in S \cap T \}$. Then

$$\mathcal{D}(A_T/S') = \mathcal{D}(A/S)\, A_T$$

where S' is any subring of A_T such that $S \subseteq S' \subseteq S_{T \cap S}$.

Proof: Available in [2].

Proposition 4: Let R be a commutative ring with unit, $P = R[x_1, \ldots, x_n]$ a polynomial ring over R with $x_1, \ldots, x_n$ as indeterminates, A an R-algebra, $\varphi : P \longrightarrow A$ an R-algebra epimorphism, and $\mathcal{G}$ a set of polynomials which generates Ker φ. Then $\mathcal{D}(A/R)$ is the ideal of A generated by all $\varphi\left(\frac{\partial(g_1, \ldots, g_n)}{\partial(x_1, \ldots, x_n)}\right)$ where $g_1, \ldots, g_n \in \mathcal{G}$, and $\frac{\partial(g_1, \ldots, g_n)}{\partial(x_1, \ldots, x_n)}$ is the usual Jacobian determinant.

Proof: Let (V, δ) and (U, d) be universal derivation modules of P and A as R-algebras respectively, then $(U, d) \cong (V/J, d')$ where $J = \text{Ker}\,\varphi \cdot \delta P + P\delta\,\text{Ker}\,\varphi$ [cf. §2 , Chapter 0]. Let $\nu : V \longrightarrow U$ be the natural φ-homomorphism, then U is generated by $w_1, \ldots, w_n$, $w_i = \nu(\delta x_i) = d\,\varphi(x_i)$ for each i [cf. §2 , Chapter 0].
Suppose $g \in \text{Ker}\,\varphi$, then $\nu(\delta g) = \nu\left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \delta x_i\right) = \sum_{i=1}^{n} \varphi\left(\frac{\partial g}{\partial x_i}\right) w_i$.

On the other hand, $\nu(\delta g) = d \circ \varphi(g) = 0$.
Hence $\sum_{i=1}^{n} \varphi\left(\frac{\partial g}{\partial x_i}\right) w_i = 0$.
Let $\mathcal{L}$ be the collection of all sequences $\left(\varphi\left(\frac{\partial g}{\partial x_1}\right), \ldots, \varphi\left(\frac{\partial g}{\partial x_n}\right)\right)$, $g \in \mathcal{G}$, and for any sequence $(a_1, \ldots, a_n)$ with $a_1 w_1 + \ldots + a_n w_n = 0$, let $f_1, \ldots, f_n \in P$ with $\varphi(f_i) = a_i$ for each i. Then

$f_1\delta x_1 + \dots + f_n\delta x_n \in \text{Ker}\,\wp \cdot \delta P + P\delta\,\text{Ker}\,\wp$. Hence

$$f_1\delta x_1 + \dots + f_n\delta x_n = \sum_{i=1}^{n} h_i\delta x_i + \sum_{j=1}^{m} f_j\delta g_j \quad (\text{for some } h_i \in \text{Ker}\,\wp,$$

$f_j \in P,\ g_j \in \mathfrak{G}$)

$$= \sum_{i=1}^{n} h_i\delta x_i + \sum_{i=1}^{n} \left(\sum_{j=1}^{m} f_j \frac{\partial g_j}{\partial x_i} \right) \delta x_i.$$

Since V is known to be a free P-module with $\{\delta x_1,\dots,dx_n\}$ as its basis, each $f_i = h_i + \sum_{j=1}^{m} f_j \frac{\partial g_j}{\partial x_i}$.

Acting $\wp$ on both hand sides, we get

$$a_i = \sum_{j=1}^{m} \wp(f_j)\ \wp\left(\frac{\partial g_j}{\partial x_i}\right).$$

Hence by Proposition 1, $\mathfrak{D}(A/R)$ is the ideal of A generated by all

$$\wp\left(\frac{\partial(g_1,\dots,g_n)}{\partial(x_1,\dots,x_n)}\right).$$

•

<u>Proposition 5</u>: Let A be an integrally closed domain, F a field of quotients of A, K a finitely generated separable algebraic extension field of F, $\bar{A}$ the integral closure of A in K, and x($\in \bar{A}$) separable algebraic over F such that K = F[x]. Then $\bar{A}\mathfrak{D}(A[x]/A) \subseteq A[x]$.

<u>Proof</u>: By [P. 21 and P. 40, [2]], $\mathfrak{D}(A[x]/A)$ is equal to the classical Dedekind different of A[x] over A defined by traces [11]. From this it follows that $\bar{A}\mathfrak{D}(A[x]/A) \subseteq A[x]$ [P. 304, [11]].

<u>Proposition 6</u>: In addition to the assumptions in Proposition 5, assume that A be a Dedekind domain. Let f be the minimal polynomial of x over F, then $f'(x) \in \mathfrak{D}(\bar{A}/A)$.

<u>Proof</u>: By [P.P. 32 - 35, [2]], $\mathfrak{D}(\bar{A}/A)$ is equal to the classical Dedekind different of $\bar{A}$ over A defined by traces [11]. It is known [P. 303, [11]] that $f'(x)$ belongs to the classical Dedekind different of $\bar{A}$ over A, and hence $f'(x) \in \mathfrak{D}(\bar{A}/A)$.

§ 4. Free joins of Algebras.

Throughout this section, let R be a commutative ring with unit, M an R-module, $(M_\alpha)_{\alpha \in I}$ a family of submodules of M such that $M = \sum_{\alpha \in I} M_\alpha$ (direct). All algebras are assumed to be unitary algebras, subalgebras unitary subalgebras, and algebra homomorphisms unitary algebra homomorphisms.

This section is devoted to a partial answer to the following

Question: Let A be an algebra containing M. When is the submodule $M_{\alpha_1} \ldots M_{\alpha_k}$ ($\alpha_1, \ldots, \alpha_k$ all different) of A canonically isomorphic to $M_{\alpha_1} \otimes \ldots \otimes M_{\alpha_k}$?

Definition: An (a commutative) R-algebra A is called a free (commutative) join of a family $(A_\alpha)_{\alpha \in I}$ of its subalgebras if and only if for any (commutative) algebra C and any algebra homomorphism $f_\alpha : A_\alpha \longrightarrow C$ for each $\alpha \in I$, there exists a unique algebra homomorphism $f : A \longrightarrow C$ extending each f_α.

Example of a free join: It is well known that a tensor algebra of a direct summand of a module can be imbedded into a tensor algebra of the module, and hence without loss of generality we can assume $T(M_\alpha) \subseteq T(M)$, where $T(M)$ and $T(M_\alpha)$ are tensor algebras of M and M_α respectively. It is well known that T(M) is a free join of a family $(T(M_\alpha))_{\alpha \in I}$ of its subalgebras.

<u>Lemma 1</u>: Let A be an algebra containing M, A_α the subalgebra of A generated by M_α for each $\alpha \in I$. If A is the free join of the family $(A_\alpha)_{\alpha \in I}$ then A is generated by M.

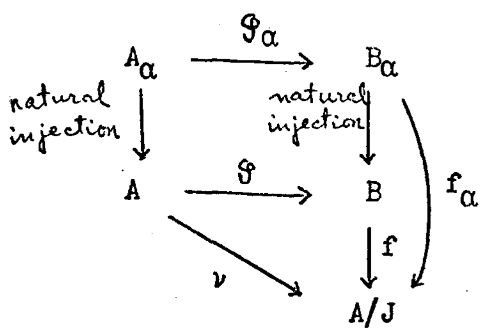
<u>Proof</u>: Let A' be the subalgebra of A generated by M, then $A_\alpha \subseteq A'$ for each α. Since A is the free join of $(A_\alpha)_{\alpha \in I}$, for the family of natural injections $(f_\alpha)_{\alpha \in I}$, $f_\alpha : A_\alpha \longrightarrow A'$, there exists a unique algebra homomorphism extending each f_α. But f may be considered as an algebra homomorphism from A to A extending each f_α. On the other hand, the identity mapping on A is also a homomorphism extending each f_α, and by the uniqueness of such homomorphisms, f is the identity on A. Thus A' = A or A is generated by M as an algebra.

A, B, C, ... will denote algebras containing M, and subalgebras $A_\alpha, B_\alpha, C_\alpha, \ldots$ subalgebras generated by M_α in A, B, C,... respectively, for each $\alpha \in I$.

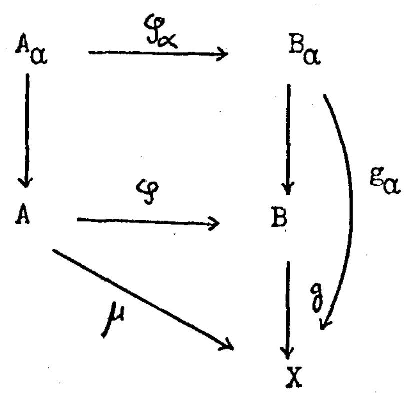
<u>Proposition 1</u>: Let A be a free join of $(A_\alpha)_{\alpha \in I}$, $\varphi : A \longrightarrow B$ an algebra homomorphism such that $\varphi|M$ is the identity on M, and $\varphi_\alpha : A_\alpha \longrightarrow B_\alpha$ be defined by $\varphi_\alpha = \varphi|A_\alpha$ for each α. Then B is a free join of $(B_\alpha)_{\alpha \in I}$ if and only if φ is onto and Ker φ is the ideal of A generated by Σ Ker φ_α.

<u>Proof</u>: Suppose B is a free join of $(B_\alpha)_{\alpha \in I}$. By Lemma

1, B is generated by M, and hence $\wp$ is onto. We will show Ker $\wp$ = J, the ideal of A generated by Σ Ker $\wp_\alpha$. Ker $\wp \supseteq J$ always, since Ker $\wp \supseteq$ Ker $\wp_\alpha$ for each α. On the other hand, since Ker $\wp_\alpha \subseteq J$, $a - a' \in$ Ker $\wp_\alpha$ for $a, a' \in A_\alpha$ implies $a - a' \in J$. Hence we can consider the mapping $f_\alpha : B_\alpha \longrightarrow A/J$ defined by $f_\alpha(b) = a + J$, $b \in B_\alpha$, $a \in \wp_\alpha^{-1}(b)$.



f_α is an algebra homomorphism such that $f_\alpha \circ \wp_\alpha = \nu | A_\alpha$, where ν : A $\longrightarrow$ A/J is the natural homomorphism. Since B is a free join of $(B_\alpha)_{\alpha \in I}$, there exists a unique algebra homomorphism $f : B \longrightarrow A/J$ extending each f_α. Here $f \circ \wp = \nu$, since $f \circ \wp$ and ν are both algebra homomorphisms from A into A/J extending $f_\alpha \circ \wp_\alpha$ for each α. Hence Ker $\wp \subseteq$ Ker ν = J. Thus Ker $\wp$ = J. Conversely, suppose B is generated by M as an algebra and Ker $\wp$ = J. Let X be any algebra and $g_\alpha : B_\alpha \longrightarrow X$ any algebra homomorphism for each $\alpha \in I$. Then there exists a unique algebra homomorphism $\mu : A \longrightarrow X$ extending each $g_\alpha \circ \wp_\alpha$.



Here $\mu(\text{Ker } \varphi) = 0$, since $\mu(\text{Ker } \varphi_\alpha) = g_\alpha \circ \varphi_\alpha(\text{Ker } \varphi_\alpha) = 0$ and Ker φ is assumed to be the ideal generated by Σ Ker φ_α. φ is assumed to be onto, hence there exists an algebra homomorphism $g : B \longrightarrow X$ such that $g \circ \varphi = \mu$. Now for any $b \in B_\alpha$, let $a \in \varphi_\alpha^{-1}(b)$, then $g(b) = g \circ \varphi(a) = \mu(a) = g_\alpha \circ \varphi_\alpha(a) = g_\alpha(b)$. Hence $g|B_\alpha = g_\alpha$ for each α. g is a unique algebra homomorphism extending g_α, since B is generated by ΣB_α. This completes the proof.

Corollary 1. Let $T(M)$ and $T(M_\alpha)$ be tensor algebras of M an M_α respectively. Let $\varphi : T(M) \longrightarrow A$ be the algebra homomorphism determined by the identity mapping $M \longrightarrow A$, and $\varphi_\alpha : T(M_\alpha) \longrightarrow A_\alpha$ the algebra homomorphism determined by the identity mapping $M_\alpha \longrightarrow A_\alpha$ on M_α for each α. Then A is a free join of $(A_\alpha)_{\alpha \in I}$ if and only if φ is onto and Ker φ is the ideal of $T(M)$ generated by Σ Ker φ_α.

Proof: $T(M)$ is a free join of $(T(M_\alpha))_{\alpha \in I}$, and hence Corollary 1 is a trivial consequence of Theorem 1.

Corollary 2: Let A be a free join of $(A_\alpha)_{\alpha \in I}$. Then $A = \sum_n A_n$ (direct), $n = 0, 1, \ldots$, is a graded algebra with $A_1 = M$ if and only if each $A_\alpha = \sum_n A_{\alpha,n}$ (direct) is a graded algebra with $A_{\alpha,1} = M_\alpha$ for each α.

Proof: Let $\varphi : T(M) \longrightarrow A$ and $\varphi_\alpha : T(M_\alpha) \longrightarrow A_\alpha$ be the algebra homomorphisms determined by the identity mappings on M and M_α respectively. Then it is known that $\varphi | T(M_\alpha) = \varphi_\alpha$. Hence $\operatorname{Ker} \varphi \cap T(M_\alpha) = \operatorname{Ker} \varphi_\alpha$.

If A is a graded algebra with $A_1 = M$, φ is a graded algebra homomorphism and hence $\operatorname{Ker} \varphi$ is a homogeneous two-sided ideal of T(M). Hence $\operatorname{Ker} \varphi \cap T(M_\alpha)$ is a homogeneous two sided ideal of $T(M_\alpha)$. Thus $\operatorname{Ker} \varphi_\alpha$ is a homogeneous two sided ideal of $T(M_\alpha)$ and hence the grading of $T(M_\alpha)$ determines one on A_α such that A_α is a graded algebra with $A_{\alpha,1} = M_\alpha$.

Conversely, suppose A_α is a graded algebra with $A_{\alpha,1} = M_\alpha$ for each α; then $\operatorname{Ker} \varphi_\alpha$ is a homogeneous two sided ideal of $T(M_\alpha)$ for each α. Since $\operatorname{Ker} \varphi$ is the two sided ideal generated by $\sum \operatorname{Ker} \varphi_\alpha$ (by Corollary 1), $\operatorname{Ker} \varphi$ is generated by homogeneous elements as an ideal and hence a homogeneous ideal. Thus φ induces a grading on A such that A becomes a graded algebra with $A_1 = M$.

Remark: As a consequence of Corollary 2, we can say the following: Let $(A_\alpha)_{\alpha \in I}$ be a family of graded algebras where each A_α is graded by $A_\alpha = \sum_{n \geq 0} A_{\alpha,n}$ and A_α is generated by $A_{\alpha,1}$. Then in the catagory of all graded algebras and graded algebra homomorphisms, there exists a free join A of (A_α), given by the ordinary free join, graded as described above, provided the latter exists.

Lemma 2: Let $\varphi : T(M) \longrightarrow A$, $\varphi_\alpha : T(M_\alpha) \longrightarrow A_\alpha$ be the same algebra homomorphisms as in Corollary 2. Then

(1) $\mathrm{Ker}\,\varphi_\alpha \cap R = \mathrm{Ker}\,\varphi \cap R$ for each α.

(2) $(\mathrm{Ker}\,\varphi_\alpha \cap R)M = 0$ for each α.

Proof: (1) Trivial (2) It is sufficient to show $(\mathrm{Ker}\,\varphi \cap R)M = 0$. For any $r \in \mathrm{Ker}\,\varphi \cap R$ and $x \in M$, $rx \in M \cap \mathrm{Ker}\,\varphi$. Since $\varphi|M$ is the identity mapping on M, $rx = \varphi(rx) = 0$. Hence $(\mathrm{Ker}\,\varphi \cap R)M = 0$.

Theorem 1: Let A be a free join of a family $(A_\alpha)_{\alpha \in I}$ of graded algebras with $A_{\alpha,1} = M_\alpha$ for each α. Then for any finite sequence $\alpha_1, \ldots, \alpha_k$ (all different), the linearization $f : M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_k} \longrightarrow M_{\alpha_1} \cdots M_{\alpha_k}$ (in A) of the multilinear mapping $f_o : M_{\alpha_1} \times \cdots \times M_{\alpha_k} \longrightarrow M_{\alpha_1} \cdots M_{\alpha_k}$ (in A) defined by $f_o(x_{\alpha_1}, \ldots, x_{\alpha_k}) = x_{\alpha_1} \cdots x_{\alpha_k}$, $x_{\alpha_i} \in M_{\alpha_i}$, is an isomorphism.

Proof: Let $\varphi : T(M) \longrightarrow A$ and $\varphi_\alpha : T(M_\alpha) \longrightarrow A_\alpha$ be the same algebra homomorphisms as in Corollary 2. We first show that $M_{\alpha_1} \cdots M_{\alpha_k}$ (in $T(M)$) $\cap \mathrm{Ker}\,\varphi = 0$. Put $J = \mathrm{Ker}\,\varphi$ and $J_k = T_k(M) \cap J$, then it is sufficient to show $M_{\alpha_1} \cdots M_{\alpha_k}$ (in $T(M)$) $\cap J_k = 0$. For any $u \in \mathrm{Ker}\,\varphi_\alpha$, $u = \sum_n u_n$, $u_n \in T_n(M_\alpha)$, $\varphi_\alpha(u) = \Sigma\,\varphi_\alpha(u_n) = 0$. Since φ_α is a graded algebra homomorphism, $\varphi_\alpha(u_n) = 0$ for all n, and hence

$\varphi_\alpha(u_1) = 0$. Since $\operatorname{Ker} \varphi_\alpha \cap M_\alpha = 0$, $u_1 = 0$. Thus $\operatorname{Ker} \varphi_\alpha \subseteq \sum_{\substack{n \geq 0 \\ n \neq 1}} T_n(M_\alpha)$.

Since J is the ideal of T(M) generated by $\Sigma \operatorname{Ker} \varphi_\alpha$ (by Corollary 1), and since $\operatorname{Ker} \varphi_\alpha \subseteq \sum_{n \neq 1} T_n(M_\alpha)$, $(\operatorname{Ker} \varphi \cap R)\, M = 0$ (by Lemma 2), $J = \sum_n J_n$ (direct) by J being homogeneous ideal, any element of J_k is expressed as $\sum_{\beta_1,\ldots,\beta_k} \sum_{i_1,\ldots,i_k} x_{i_1\beta_1} \cdots x_{i_k\beta_k}$, $x_{i_j\beta_j} \in M_{\beta_j}$ for each j, where $\beta_i = \beta_j$ for some i, j with $i \neq j$.

Suppose $x \in M_{\alpha_1} \cdots M_{\alpha_k}$ (in T(M)) $\cap J_k$ and assume $x \neq 0$; then x can be expressed in two different ways, namely,

$$x = \sum_{i_1,\ldots,i_k} x_{i_1\alpha_1} \cdots x_{i_k\alpha_k} = \sum_{\beta_1,\ldots,\beta_k} \sum_{i_1,\ldots,i_k} x_{i_1\beta_1} \cdots x_{i_k\beta_k},$$

where $x_{i_j\alpha_j} \in M_{\alpha_j}$, $x_{i_j\beta_j} \in M_{\beta_j}$ for each j and $\beta_i = \beta_j$ for some i, j with $i \neq j$. Let $\varepsilon_{\alpha_j} : T(M) \longrightarrow T(M)$ be the endomorphism determined by the mapping $\psi_{\alpha_j} : M \longrightarrow T(M)$ defined by $\psi_{\alpha_j}|M_{\alpha_j}$ = identity on M_{α_j}, $\psi_{\alpha_j}|M_\alpha = 0$ for all $\alpha \neq \alpha_j$. If we act ε_{α_1} on x, then

$$0 = \sum_{\beta_1,\ldots,\beta_k} \sum_{i_1,\ldots,i_k} x_{i_1\beta_1} \cdots x_{i_k\beta_k}, \quad \alpha_1 \notin \{\beta_1,\ldots,\beta_k\}.$$

Hence $$x = \sum_{\beta_1,\ldots,\beta_k} \sum_{i_1,\ldots,i_k} x_{i_1\beta_1} \cdots x_{i_k\beta_k}, \quad \alpha_1 \in \{\beta_1,\ldots,\beta_k\}.$$

Continuing this process for $\varepsilon_{\alpha_2},\ldots,\varepsilon_{\alpha_k}$, we get

$$x = \sum_{i_1,\ldots,i_k} x_{i_1\beta_1} \cdots x_{i_k\beta_k}, \quad \alpha_1,\ldots,\alpha_k \in \{\beta_1,\ldots,\beta_k\}.$$

But this is impossible, since the α_j are distinct but the β_j are not.

Thus $x = 0$. This proves $M_{\alpha_1} \dots M_{\alpha_k}$ (in $T(M)) \cap J = 0$. Hence φ maps $M_{\alpha_1} \dots M_{\alpha_k}$ (in $T(M)$) one-to-one and onto $M_{\alpha_1} \dots M_{\alpha_k}$ (in A). Let $g : M_{\alpha_1} \otimes \dots \otimes M_{\alpha_k} \longrightarrow M_{\alpha_1} \dots M_{\alpha_k}$ (in $T(M)$) be the canonical injection, then $f = \varphi \circ g$ is an isomorphism. It is obvious that f is the linearization of f_o.

Theorem 2: Let A be a free join of $(A_\alpha)_{\alpha \in I}$, J the ideal of A generated by $xy - yx$ for all $x, y \in M$, and J_α the ideal of A_α generated by $xy - yx$ for all $x, y \in M_\alpha$, for each α. If $Re \cap M_\alpha = 0$, e the unit of A, for each α, then A/J is a free commutative join of $(A_\alpha/J_\alpha)_{\alpha \in I}$.

Proof: Since $M = \sum_\alpha M_\alpha$ (direct), J is the ideal of A generated by $xy - yx$ for all $x \in M_\alpha$, $y \in M_\beta$, for all α, β. We will first show that $J_\alpha = J \cap A_\alpha$. Suppose $x\ (\in A_\alpha)$ is in J, then

$$x = \sum_{\alpha_1, \dots, \alpha_k} \sum{}^{*} x_{\alpha_1} \dots x_{\alpha_{i-1}} (x_{\alpha_i} y_{\alpha_{i+1}} - y_{\alpha_{i+1}} x_{\alpha_i}) x_{\alpha_{i+2}} \dots x_{\alpha_{k_n}},$$

$x_{\alpha_j}, y_{\alpha_j} \in M_{\alpha_j}$. Consider the algebra homomorphisms $f_\alpha : A_\alpha \longrightarrow A$, identity on A_α, $f_\beta : A_\beta \longrightarrow A$ defined by $f_\beta | Re$ = identity on Re, $f_\beta | M_\beta = 0$ for each $\beta \neq \alpha$. Let $f : A \longrightarrow A$ be the unique algebra homomorphism extending f_α and each f_β and act f on x. Then

$$x = \sum_i x_{i1} \dots x_{i\,j-1} (x_{ij} y_{ij} - y_{ij} x_{ij}) x_{ij+2} \dots x_{ik_i}, \text{ where all}$$

$x_{i_n} \in M_\alpha$. Hence $x \in J_\alpha$. This implies $J_\alpha \supseteq J \cap A_\alpha$ and since $J_\alpha \subseteq J \cap A_\alpha$ always, $J_\alpha = J \cap A_\alpha$.

Hence A_α/J_α can be imbedded into A/J by $a_\alpha + J_\alpha \rightsquigarrow a_\alpha + J$, $a_\alpha \in A_\alpha$. Now we will show that A/J is a free join of $(A_\alpha/J_\alpha)_{\alpha \in I}$.

For any commutative algebra X, and any algebra homomorphism $f_\alpha : A_\alpha/J_\alpha \longrightarrow X$, $f_\alpha \circ \nu_\alpha : A_\alpha \longrightarrow X$ is also an algebra homomorphism, for each α. Hence there exists a unique algebra homomorphism $\mu : A \longrightarrow X$ extending each $f_\alpha \circ \nu_\alpha$. $\mu(J) = 0$, since $\mu(xy - yx) = \mu(x)\mu(y) - \mu(y)\mu(x) = 0$, because X is commutative. Hence there exists an algebra homomorphism $f : A/J \longrightarrow X$ such that $\mu = f \circ \nu$. This algebra homomorphism f extends each f_α, since $f(a + J) = f \circ \nu(a) = \mu(a) = f_\alpha \circ \nu_\alpha(a) = f_\alpha(a + J_\alpha)$, $a \in A_\alpha$. Finally f is unique, since A/J is generated by $\Sigma A_\alpha/J_\alpha$. This completes the proof.

Corollary 3: Let A be a free join of $(A_\alpha)_{\alpha \in I}$, J the ideal of A generated by $xy - yx$ for all $x, y \in M$, and J_α the ideal of A_α generated by $xy - yx$ for all $x, y \in M_\alpha$, for each α. If A is a graded algebra with $M = A_1$ then A/J is also a graded algebra and is a free commutative join of $(A_\alpha/J_\alpha)_{\alpha \in I}$.

Proof: If A is a graded algebra, i.e. $A = \Sigma A_n$ (direct), then $Re \subseteq A_o$, $M = A_1$, and hence $Re \cap M = 0$. Moreover, $Re \cap M_\alpha = 0$ obviously. Thus Corollary 3 is a trivial consequence of Theorem 2.

Corollary 4: Let $S(M)$ and $S(M_\alpha)$ be symmetric algebras of M and M_α respectively. Then $S(M)$ is a free commutative join of $(S(M_\alpha))_{\alpha \in I}$.

Proof: Put $A = T(M)$ in Corollary 3.

Theorem 3. Let C and D be commutative algebras containing M. Let C be a free commutative join of $(C_\alpha)_{\alpha \in I}$, $\varphi : C \longrightarrow D$ an algebra homomorphism such that $\varphi|M$ is the identity mapping on M, and $\varphi_\alpha : C_\alpha \longrightarrow D_\alpha$ be defined by $\varphi_\alpha = \varphi|C_\alpha$ for each α. Then D is a free commutative join of $(D_\alpha)_{\alpha \in I}$ if and only if φ is onto and $\text{Ker}\,\varphi$ is the ideal of C generated by $\Sigma\, \text{Ker}\,\varphi_\alpha$.

Proof: Similar to the proof of Proposition 1.

Corollary 5: Let a commutative algebra C be a free commutative join of $(C_\alpha)_{\alpha \in I}$. Then $C = \sum_n C_n$ (direct), $n = 0, 1, 2, \ldots,$ is a graded algebra with $C_1 = M$ if and only if each $C_\alpha = \sum_n C_{\alpha,n}$ (direct), $n = 0, 1, 2, \ldots,$ is a graded algebra with $C_{\alpha,1} = M_\alpha$ for each α.

Proof: Similar to the proof of Corollary 2.

Theorem 4: Let a commutative algebra C be a free commutative join of a family $(C_\alpha)_{\alpha \in I}$ of graded algebras with $c_{\alpha,1} = M_\alpha$ for each α. Then for any finite sequence $\alpha_1,\ldots,\alpha_k$ (all different), $M_{\alpha_1}\ldots M_{\alpha_k}$ (in S(M)) $\cong M_{\alpha_1}\ldots M_{\alpha_k}$ (in C).

Proof: Similar to the proof of Theorem 1.

Theorem 5: Let a commutative algebra C be a free commutative join of a family $(C_\alpha)_{\alpha \in I}$ of graded algebras with $C_{\alpha,1} = M_\alpha$ for each α. Then for any α_1, α_2 $(\alpha_1 \neq \alpha_2)$, the linaarization $f : M_{\alpha_1} \otimes M_{\alpha_2} \longrightarrow M_{\alpha_1} M_{\alpha_2}$ (in C) of the bilinear mapping $f_o : M_{\alpha_1} \times M_{\alpha_2} \longrightarrow M_{\alpha_1} M_{\alpha_2}$ (in C) defined by $f_o(x_1, x_2) = x_1 x_2$, $x_i \in M_{\alpha_i}$, is an isomorphism.

Proof: Let T(M) and S(M) be a tensor algebra and a symmetric algebra of M respectively. We will show that for any $\alpha_1, \alpha_2 (\alpha_1 \neq \alpha_2)$,

$M_{\alpha_1} M_{\alpha_2}$ (in T(M)) $\cong M_{\alpha_1} M_{\alpha_2}$ (in S(M)). Let J be the ideal of T(M) generated by all xy - yx, x, y $\in$ M. Since $M = \sum_{\alpha} M_\alpha$ (direct), J is the ideal of T(M) generated by all xy - yx, $x \in M_\alpha$, $y \in M_\beta$, for all α, β. Our claim is $M_{\alpha_1} M_{\alpha_2}$ (in T(M)) $\cap J = 0$. Since T(M) is a regularly graded algebra generated by M, it is sufficient to show $M_{\alpha_1} M_{\alpha_2} \cap J_2 = 0$ where J_2 is the submodule of all homogeneous elements of degree 2 in J.

Suppose $x \in M_{\alpha_1} M_{\alpha_2} \cap J_2$, then

$x = \sum_x x_{\alpha_1} x_{\alpha_2} = \sum_{\beta_1, \beta_2} \sum_y y_{\beta_1} y_{\beta_2} - y_{\beta_2} y_{\beta_1}$. Consider the endomorphism

$\varepsilon : T(M) \longrightarrow T(M)$ determined by the module homomorphism $\varepsilon_o : M \longrightarrow M$ defined by $\varepsilon_o | M_{\alpha_i}$ = the identity in M_{α_i} for i = 1, 2, and $\varepsilon_o | M_\beta = 0$ for all $\beta \notin \{\alpha_1, \alpha_2\}$. If we act ε on x, then $\sum_x x_{\alpha_1} x_{\alpha_2} =$

$\sum_y (y_{\alpha_1} y_{\alpha_2} - y_{\alpha_2} y_{\alpha_1})$. Hence $\sum_x x_{\alpha_1} x_{\alpha_2} - \sum_y y_{\alpha_1} y_{\alpha_2} = -\sum_y y_{\alpha_2} y_{\alpha_1} \in M_{\alpha_1} M_{\alpha_2} \cap M_{\alpha_2} M_{\alpha_1}$.

On the other hand, it is well known that in T(M), $M_{\alpha_1} M_{\alpha_2} \cap M_{\alpha_2} M_{\alpha_1} = 0$,

and hence $\sum_y y_{\alpha_2} y_{\alpha_1} = 0$ and $\sum_x x_{\alpha_1} x_{\alpha_2} = \sum_y y_{\alpha_1} y_{\alpha_2}$.

It is also known that in T(M), $M_{\alpha_1} M_{\alpha_2} \cong M_{\alpha_2} M_{\alpha_1}$ by the mapping

$\Sigma x_{\alpha_1} x_{\alpha_2} \rightsquigarrow \Sigma x_{\alpha_2} x_{\alpha_1}$, and since $\sum_y y_{\alpha_2} y_{\alpha_1} = 0$ implies $\sum_y y_{\alpha_1} y_{\alpha_2} = 0$.

Hence $x = \sum_x x_{\alpha_1} x_{\alpha_2} = \sum_y y_{\alpha_1} y_{\alpha_2} = 0$, and thus $M_{\alpha_1} M_{\alpha_2} \cap J = 0$.

This means that the natural homomorphism $\nu : T(M) \longrightarrow S(M)$ maps $M_{\alpha_1} M_{\alpha_2}$ (in T(M)) one-to-one and onto $M_{\alpha_1} M_{\alpha_2}$ (in S(M)) and hence

$M_{\alpha_1} M_{\alpha_2}$ (in T(M)) $\cong M_{\alpha_1} M_{\alpha_2}$ (in S(M)) as module.

$M_{\alpha_1} \otimes M_{\alpha_2} \cong M_{\alpha_1} M_{\alpha_2}$ (in T(M)) $\cong M_{\alpha_1} M_{\alpha_2}$ (in S(M)) $\cong M_{\alpha_1} M_{\alpha_2}$ (in C) by the previous theorems and corollaries. This isomorphism is obviously given by the bilinear mapping f_o.

Proposition 2: Let a commutative algebra C be a free commutative join of a family $(C_\alpha)_{\alpha \in I}$ of graded algebras with $C_{\alpha,1} = M_\alpha$ for each α. If each M_α is a free module, then the linearization $f : M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_k} \longrightarrow M_{\alpha_1} \cdots M_{\alpha_k}$ (in C) of the multilinear mapping $f_o : M_{\alpha_1} \times \cdots \times M_{\alpha_k} \longrightarrow M_{\alpha_1} \cdots M_{\alpha_k}$ (in C) defined by $f_o(a_1, \ldots, a_k) = a_1 \cdots a_k, a_i \in M_{\alpha_i}$, is an isomorphism.

Proof: Let X_α be a basis of M_α for each $\alpha \in I$. Then it is well known that T(M) is a free algebra freely generated by the set $\bigcup_\alpha X_\alpha$ and S(M) is a free commutative algebra freely generated by the set $\bigcup_\alpha X_\alpha$ (or a polynomial ring with $\bigcup_\alpha X_\alpha$ as a set of indeterminates). Hence $M_{\alpha_1} \cdots M_{\alpha_k}$ (in T(M)) is a free module as $\{x_{\alpha_1} \cdots x_{\alpha_k} | x_{\alpha_i} \in X_{\alpha_i}\}$ as basis and $M_{\alpha_1} \cdots M_{\alpha_k}$ (in S(M)) is also a free module as $\{x_{\alpha_1} \cdots x_{\alpha_k} | x_{\alpha_i} \in X_{\alpha_i}\}$ as basis. Hence $M_{\alpha_1} \cdots M_{\alpha_k}$ (in T(M)) $\cong M_{\alpha_1} \cdots M_{\alpha_k}$ (in S(M)) where the isomorphism is given by the identity mapping on $\{x_{\alpha_1} \cdots x_{\alpha_k} | x_{\alpha_i} \in X_{\alpha_i}\}$. Since $M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_k} \cong M_{\alpha_1} \cdots M_{\alpha_k}$ (in T(M)) and $M_{\alpha_1} \cdots M_{\alpha_k}$ (in S(M)) $\cong$ $M_{\alpha_1} \cdots M_{\alpha_k}$ (in C) by Theorem 4, $M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_k} \cong M_{\alpha_1} \cdots M_{\alpha_k}$ (in C) where the isomorphism f maps $a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_k} \rightsquigarrow a_{\alpha_1} \cdots a_{\alpha_k}$. Hence f is the linearization of the multilinear mapping $f_o : M_{\alpha_1} \times \cdots \times M_{\alpha_k} \longrightarrow M_{\alpha_1} \cdots M_{\alpha_k}$

defined by $f_o(a_{\alpha_1},\dots,a_{\alpha_k}) = a_{\alpha_1}\dots a_{\alpha_k}$.

Theorem 6: Let a commutative algebra C be a free commutative join of a family $(C_\alpha)_{\alpha \in I}$ of graded algebras with $C_{\alpha,1} = M_\alpha$. If each M_α is projective, then the linearization $g : M_{\alpha_1} \otimes \dots \otimes M_{\alpha_k} \longrightarrow M_{\alpha_1}\dots M_{\alpha_k}$ (in C) of the multilinear mapping $g_o : M_{\alpha_1} \times \dots \times M_{\alpha_k} \longrightarrow M_{\alpha_1}\dots M_{\alpha_k}$ defined by $g_o(a_{\alpha_1},\dots,a_{\alpha_k}) = a_{\alpha_1}\dots a_{\alpha_k}$, $a_{\alpha_i} \in M_{\alpha_i}$, is an isomorphism.

Proof: If each M_α is projective, then there exists a free module F_α with $F_\alpha \cong M_\alpha + N_\alpha$ (direct) for some submodule N_α for each α and $F = \sum_\alpha F_\alpha$ (direct). In fact, M_α is projective, then M_α is a direct summand of a free module G_α, and if we consider $F = \bigoplus_\alpha G_\alpha$, external sum of G_α, $\alpha \in I$, and put $F_\alpha = \{ x | x \in F,\ x(\beta) = 0 \text{ for all } \beta \neq \alpha \}$, then $F = \Sigma F_\alpha$ (direct) and $F_\alpha \cong G_\alpha$ for each α, and hence each M_α can be imbedded into F_α and may be regarded as a direct summand of F_α.
By Proposition 2, the canonical mapping $f : F_{\alpha_1} \otimes \dots \otimes F_{\alpha_k} \longrightarrow F_{\alpha_1}\dots F_{\alpha_k}$ (in S(F)) is an isomorphism. Let $i : M_{\alpha_1} \otimes \dots \otimes M_{\alpha_k} \longrightarrow F_{\alpha_1} \otimes \dots \otimes F_{\alpha_k}$ and $j : M_{\alpha_1}\dots M_{\alpha_k}$ (in S(M)) $\longrightarrow F_{\alpha_1}\dots F_{\alpha_k}$ (in S(F)) be the canonical homomorphisms, then i and j are monomorphisms, since each M_{α_i} is a direct summand of F_{α_i} [cf. [4], p. 78 and p. 216].
Define a mapping $g : M_{\alpha_1} \otimes \dots \otimes M_{\alpha_k} \longrightarrow M_{\alpha_1}\dots M_{\alpha_k}$ by

$g = (i | M_{\alpha_1}\dots M_{\alpha_k})^{-1} \circ f \circ j$, then g is an isomorphism and the linearization of the multilinear mapping g_o.

<u>Remark</u>: So far we have been discussing free joins in the category $\mathcal{C}(R)$ of all unitary R-algebras and unitary algebra homomorphisms. Now let us consider the subcategory $\mathcal{C}'(R)$ of $\mathcal{C}(R)$ consisting of all unitary R-algebras A such that $a \rightsquigarrow ae$, $a \in R$ and e the unit of A, is a monomorphism, and all unitary algebra homomorphisms between these. Naturally, in $\mathcal{C}'(R)$, an R-algebra A is called a <u>free (commutative) join</u> of a family $(A_\alpha)_{\alpha \in I}$ of subalgebras if and only if for any (commutative) algebra C belonging to $\mathcal{C}'(R)$ and any family of unitary algebra homomorphisms $f_\alpha : A_\alpha \longrightarrow C$, $\alpha \in I$, there exists a unique algebra homomorphism $f : A \longrightarrow C$ extending each f_α. Now suppose A belongs to $\mathcal{C}'(R)$; then it can be easily shown that A is a free join of $(A_\alpha)_{\alpha \in I}$ in $\mathcal{C}(R)$ if and only if A is a free join of $(A_\alpha)_{\alpha \in I}$ in $\mathcal{C}'(R)$.

§ 5. Valuation rings and integral closure.

Let K be a field.

Definition 1: A place of K is a non-zero homomorphism p of a subring S of K into a field Δ with the property that $x \in K$ and $x \notin S$ implies $x^{-1} \in S$ and $p(x^{-1}) = 0$. The ring S is called the valuation ring of the place p.

Proposition 1: Ker p (usually denoted by $\mathfrak{m}$) is the only maximal ideal of S.

Proposition 2: Every valuation ring of a place of K is integrally closed.

Proposition 3: If $a_1, \ldots, a_m$ are elements of K, not all zero, then for at least one integer j, $1 \leq j \leq m$, it is true that $a_i / a_j \in S$ for $i = 1, 2, \ldots, m$, $a_j \neq 0$.

Theorem 1: Let R be an integral domain, K a field containing R. The intersection of all the valuation rings S in K with $S \supseteq R$ is the integral closure of R in K.

Let K_o be a field and K be an overfield of K_o. We say that a valuation ring S in K is an <u>extension</u> of a valuation ring S_o in K_o if $S \cap K_o = S_o$.

<u>Proposition 4</u>: Let $\mathfrak{M}$ and $\mathfrak{M}_o$ be the maximal ideals of S and S_o respectively. S is an extension of S_o or $S \cap K_o = S_o$ implies $\mathfrak{M} \cap S_o = \mathfrak{M}_o$ and is equivalent to "$S \supseteq S_o$ and $\mathfrak{M} \supseteq \mathfrak{M}_o$".

<u>Proposition 5</u>: If S is an extension of S_o, then $S_o + \mathfrak{M}/\mathfrak{M}$ is a subset of $S/\mathfrak{M}$. Moreover, if K is a finite algebraic extension of K_o, then $S/\mathfrak{M}$ is also a finite algebraic extension of $S_o + \mathfrak{M}/\mathfrak{M}$ and we have $[\, S/\mathfrak{M} : S_o + \mathfrak{M}/\mathfrak{M} \,] \leq [\, K : K_o \,]$.

<u>Proposition 6</u>: The number of valuation rings in K which are extensions of S_o is not greater than the degree of separability $[\, K : K_o \,]_S$.

<u>Proposition 7</u>: Let R be a subring of a field K, $\mathfrak{P}$ and $\mathfrak{Q}$ two prime ideals in R such that $\mathfrak{P} \subseteq \mathfrak{Q}$. Suppose S is a valuation ring in K such that $\mathfrak{M} \cap R = \mathfrak{P}$, where $\mathfrak{M}$ is the maximal ideal of S. Then there exists a valuation ring S' in K such that $S' \supseteq R$, $\mathfrak{M}' \cap R = \mathfrak{Q}$ and $S' \subseteq S$ where $\mathfrak{M}'$ is the maximal ideal of S'.

<u>Theorem 2</u>: Let K be an algebraic extension of a field K_o, S a valuation ring in K which is an extension of a valuation ring S_o of K_o and $\bar{S}_o$ the integral closure of S_o in K. Then S is a ring of quotients of $\bar{S}_o$ with respect to the prime ideal $\mathfrak{M} \cap \bar{S}_o$ of $\bar{S}_o$, where

$\mathfrak{M}$ is the maximal ideal of S.

Theorem 3: Let R be an integrally closed domain, Q its field of quotients, K a finite separable algebraic extension of Q, and $\bar{R}$ the integral closure of R in K.

1) There exists a basis $\{x_1, \ldots, x_n\}$ of K over Q such that $\bar{R}$ is contained in the R-module $\sum_i Rx_i$.

2) If R is noetherian, then $\bar{R}$ is a finite R-module and is a noetherian ring.

3) If R is a principal ideal domain, then there exists a basis $\{y_1, \ldots, y_n\}$ of K over Q such that $\bar{R} = \sum_i Ry_i$.

Proposition 8: Let R be an integrally closed domain, K a field of quotients of R, and $\mathfrak{P}$ be a prime ideal in R. If an element $x \in K$ satisfies an equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

where the coefficients a_j are in R but not all in $\mathfrak{P}$, then either x or x^{-1} belongs to the ring of quotients $R_{\mathfrak{P}}$ of R with respect to $\mathfrak{P}$.

CHAPTER I

Extensions of complexes

§ 1. Natural Functors given by algebra homomorphisms.

Let R be a commutative ring with unit, A and B unitary commutative R-algebras, $\Phi : A \longrightarrow B$ unitary algebra homomorphism, (C,d) an A-complex, and (X,δ) a B-complex.

Definition: A graded R-algebra homomorphism $\varphi : C \longrightarrow X$ is called a Φ-complex homomorphism if and only if $\varphi | A = \Phi$ and $\varphi \circ d = \delta \circ \varphi$, and will be denoted by $\varphi : (C,d) \longrightarrow (X,\delta)$. In this case, (X,δ) is said to be φ-simple if and only if X is generated by $B \cup \delta(E) \cup \varphi(C)$ as an R-algebra.

Lemma 1: Let (C,d) be an A-complex. Then for any B-complex (X,δ) and any Φ - complex homomorphism $\varphi : (C,d) \longrightarrow (X,\delta)$, there exists a φ^*-simple B-complex (X^*,δ^*) and a complex monomorphism $j : (X^*,\delta^*) \longrightarrow (X,\delta)$ over B such that $j \circ \varphi^* = \varphi$.

Proof: Let X* be the R-subalgebra of X generated by $B \cup \delta(B) \cup \varphi(C)$. C is a graded algebra and hence C is an algebra generated by its homogeneous elements, and since φ is a graded algebra homomorphism, $\varphi(C)$ is also generated by homogeneous elements of X. Hence X* is generated by $B \cup \delta(B) \cup \{ \varphi(c) \mid c \in C$ homogeneous$\}$ and hence is a graded algebra and also obviously an anti-commutative graded algebra $\delta(X^*) \subseteq X^*$, since for an arbitrary element

$$u = \sum_{i=1}^{n} b_i \varphi(c_i)\delta(b_{i_1})\ldots\delta(b_{i_{m_i}})$$

in X^*, where $b_j, b_{i_1}, \ldots, b_{i_{m_i}} \in B$, $c_i \in C$,

$$\delta(u) = \sum_{i=1}^{n} (\delta(b_i)\varphi(c_i)\delta(b_{i_1})\ldots\delta(b_{i_{m_i}}) + b_i \delta\circ\varphi(c_i)\delta(b_{i_1})\ldots\delta(b_{i_{m_i}}))$$

(since $\delta^2 = 0$)

$$\in X^*,$$

since $\delta\circ\varphi(c_i) = \varphi\circ d(c_i) \in X^*$.

Let $\delta^* = \delta|X^*$ and $\varphi^* : (C,d) \longrightarrow (X^*,\delta^*)$ be defined by $\varphi = \varphi^*$, then $\delta^* : X^* \longrightarrow X^*$ is an R-derivation, homogeneous of degree 1 with $\delta^*\circ\delta^* = 0$, and hence (X^*,δ^*) is a B-complex, φ^* is a Φ-complex homomorphism and (X^*,δ^*) is a φ^*-simple. Let $j : X^* \longrightarrow X$ be the natural injection, then j is a complex monomorphism over B and satisfies $j\circ\varphi^* = \varphi$.

<u>Theorem 1</u>: For any A-complex (C,d), there exists a B-complex (C',d') and a Φ-complex homomorphism $\pi_C : (C,d) \longrightarrow (C',d')$ such that for any B-complex (X,δ) and any Φ-complex homomorphism $\varphi : (C,d) \longrightarrow (X,\delta)$, there exists a unique complex homomorphism $\varphi' : (C',d') \longrightarrow (X,\delta)$ over A with $\varphi'\circ\pi_C = \varphi$. Moreover, (C',d') and π_C are unique in the sense that if a B-complex $(\bar{C},\bar{d})$ and a Φ-complex homomorphism $\tau_C : (C,d) \longrightarrow (\bar{C},\bar{d})$ are another such, then there exists a complex isomorphism $i : (C',d') \longrightarrow (\bar{C},\bar{d})$ over B such that $i\circ\pi_C = \tau_C$.

<u>Proof</u>: For any φ-simple B-complex (S,∂) where $\varphi : (C,d) \longrightarrow (S,\partial)$

is any Φ-complex homomorphism, $|X| \leq |B||C|\aleph_0$ holds. Hence there exists a family $((S_\alpha, \partial_\alpha))_{\alpha \in I}$ of φ_α-simple B-complexes, indexed by a set I such that for any φ-simple B-complex (S, ∂), there exists a complex isomorphism $i_\alpha : (S_\alpha, \partial_\alpha) \longrightarrow (S, \partial)$ over B with $i_\alpha \circ \varphi_\alpha = \varphi$. Here $I \neq \emptyset$, since a trivial B-complex (B, δ) is φ-simple where $\delta = 0$ and $\varphi : (C,d) \longrightarrow (B, \delta)$ is the Φ-complex homomorphism defined by $\varphi | A = \Phi$, $\varphi | C_n = 0$ for $n \geq 1$.

Let us take the product $(\bar{B} + \sum_{n \geq 1} \prod_\alpha S_{\alpha,n}, \partial)$ of a representative family $((S_\alpha, \partial_\alpha))_{\alpha \in I}$ of φ_α-simple B-complexes [cf. § 2, Chapter 0], and $\pi : (C,d) \longrightarrow (\bar{B} + \sum_{n \geq 1} \prod_\alpha S_{\alpha,n}, \partial)$ be the Φ-complex homomorphism defined by $\pi(\sum_{n \geq 0} c_n) = (\varphi_\alpha(c_0))_\alpha + \sum_{n \geq 1} (\varphi_\alpha(c_n))_\alpha$ where $c_n \in C_n$. Let C' be the subalgebra of $\bar{B} + \sum_{n \geq 1} \prod_\alpha S_{\alpha,n}$ generated by $\bar{B} \cup \partial(\bar{B}) \cup \pi(C)$. Clearly $\partial(C') \subseteq C'$ and hence if we put $d' = \partial | C'$, then (C',d') is a B-complex. Let $\pi_C : (C,d) \longrightarrow (C',d')$ be the Φ-complex homomorphism defined by $\pi_C = \pi$, then (C',d') is a π-simple B-complex.

Now, for any B-complex (X, δ) and any Φ-complex homomorphism $\varphi : (C,d) \longrightarrow (X, \delta)$, there exists φ^*-simple B-complex (X^*, δ^*) and a complex monomorphism $j : (X^*, \delta^*) \longrightarrow (X, \delta)$ over B such that $j \circ \varphi^* = \varphi$. Hence we can choose a φ_β-simple B-complex $(S_\beta, \partial_\beta)$, $\beta \in I$, from the representative family $((S_\alpha, \partial_\alpha))_{\alpha \in I}$ such that there exists a complex isomorphism $i_\beta : (S_\beta, \partial_\beta) \longrightarrow (x^*, \delta^*)$ over B with

$i_\beta \circ \varphi_\beta = \varphi^*$. Let $p_\beta : (C',d') \longrightarrow (S_\beta, \partial_\beta)$ be the projection, then $p_\beta \circ \pi_C = \varphi_\beta$ clearly. Let us put $\varphi' = j \circ i_\beta \circ p_\beta$, then $\varphi' \circ \pi_C = j \circ i_{\alpha_o} \circ p_{\alpha_o} \circ \pi_C = \varphi$. Therefore, there exists a complex homomorphism $\varphi' : (C',d') \longrightarrow (X,\delta)$ over B such that $\varphi' \circ \pi_C = \varphi$.

The uniqueness of φ' is clear, since (C',d') is π_C-simple from the definition of (C',d').

Finally, to show the uniqueness of (C',d') and π_C, let a B-complex $(\bar{C},\bar{d})$ and a Φ-complex homomorphism $\tau_C : (C,d) \longrightarrow (\bar{C},\bar{d})$ be another such, then there exists a unique complex homomorphisms $\tau'_C : (C',d') \longrightarrow (\bar{C},\bar{d})$ over B with $\tau'_C \circ \pi_C = \tau_C$, and $\pi'_C : (\bar{C},\bar{d}) \longrightarrow (C',d')$ with $\pi'_C \circ \tau_C = \pi_C$. Now $\pi'_C \circ \tau'_C : (C',d') \longrightarrow (C',d')$ is a complex homomorphism over B such that $(\pi'_C \circ \tau_C) \circ \pi_C = \pi_C$. But the identity mapping $i_{C'}$ on C' is also such a complex homomorphism over B and hence by the uniqueness, $\pi'_C \circ \tau'_C = i_{C'}$. In the same way, $\tau'_C \circ \pi'_C = i_{\bar{C}}$. Hence $\tau'_C : (C',d') \longrightarrow (\bar{C},\bar{d})$ is a complex isomorphism over B such that $\tau'_C \circ \pi_C = \tau_C$. This completes the proof.

<u>Corollary 1</u>: Let (C,d) and (D,δ) be A-complexes and (C',d') and (D',δ') be the corresponding B-complexes. Then for any A-complex homomorphism $\psi : (C,d) \longrightarrow (D,\delta)$ there exists a unique B-complex homomorphism $\psi' : (C',d') \longrightarrow (D',\delta')$ such that $\psi' \circ \pi_C = \pi_D \circ \psi$.

<u>Proof</u>: Put $\varphi = \pi_D \circ \psi$ in Theorem 1.

Let $\mathcal{C}(A)$ be the category consisting of all A-complexes and all complex homomorphisms over A, and $\mathcal{C}(B)$ be the category of all B-complexes and all complex homomorphisms over B. Let $T_{\Phi} : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$ be the mapping defined by $T_{\Phi}((C,d)) = (C',d')$ for all A-complexes (C,d) and $T_{\Phi}(\psi) = \psi'$ for all complex homomorphisms over A.

Let (C,d), (D,δ) and (G,∂) be A-complexes and let $\varphi : (C,d) \longrightarrow (D,\delta)$ and $\psi : (D,\delta) \longrightarrow (G,\partial)$ be A-complex homomorphisms. Then $\pi_G \circ \psi \circ \varphi = T_{\Phi}(\psi) \circ \pi_D \circ \varphi = T_{\Phi}(\psi) \circ T_{\Phi}(\varphi) \circ \pi_C$. Also, $\pi_G \circ \psi \circ \varphi = T_{\Phi}(\psi \circ \varphi) \circ \pi_C$. Hence by the uniqueness of a B-complex homomorphism $f : T_{\Phi}((C,d)) \longrightarrow T_{\Phi}((G,\partial))$ such that $\pi_G \circ (\psi \circ \varphi) = f \circ \pi_C$, $T_{\Phi}(\psi \circ \varphi) = T_{\Phi}(\psi) \circ T_{\Phi}(\varphi)$. It is clear that if $\varphi : (C,d) \longrightarrow (C,d)$ is the identity mapping on C then $T_{\Phi}(\varphi)$ is also the identity mapping on $T_{\Phi}((C,d))$. This shows that T_{Φ} is a covariant functor.* We shall now investigate some properties of this functor T_{Φ}.

<u>Lemma 2</u>: Let (U,d) be a universal A-complex, (X,δ) any B-complex. Then there exists a unique Φ-complex homomorphism $\varphi : (U,d) \longrightarrow (X,\delta)$.

<u>Proof</u>: Let us put $Y = A \oplus \Sigma_{n \geq 1} X_n$, and define $(a,x)(b,y) = (ab, \Phi(a)y + \Phi(b)x + xy)$, $a,b \in A$, $x,y \in \Sigma_{n \geq 1} X_n$ then $Y = \Sigma_{n \geq 0} Y_n$ (direct) is an anti-commutative graded algebra where $Y_o = \{(a,0) | a \in A\}$ (in fact $Y_o \cong A$) and $Y_n = \{(0,x_n) | x_n \in X_n\}$ for $n \geq 1$. Define $\partial : Y \longrightarrow Y$ by $\partial(a,x) = (0, \delta \circ \Phi(a) + \delta(x))$, then a straight forward calculation shows that ∂ is an R-derivative, homogeneous of

degree 1. Hence (Y, ∂) is an A-complex. Hence there exists a unique A-complex homomorphism $\varphi_0 : (U,d) \longrightarrow (Y, \partial)$.

Let us define a mapping $\varphi_1 : Y \longrightarrow X$ by $\varphi_1(a,x) = \Phi(a) + x$, then $\varphi_1 \circ \partial(a,x) = \varphi_1(0, \delta \circ \Phi(a) + \delta(x)) = \delta(\Phi(a) + x) = \delta \circ \varphi_1(a,x)$ and hence clearly φ_1 is a graded algebra homomorphism with $\varphi_1|A = \Phi$ and $\varphi_1 \circ \partial = \delta \circ \varphi_1$ i.e. φ_1 is a Φ-complex homomorphism.

Define $\varphi = \varphi_1 \circ \varphi_0$, then φ is a Φ-complex homomorphism from (U,d) to (X,δ) and the uniqueness of such φ is clear. Hence Lemma 2.

<u>Theorem 2</u>: The functor T_Φ sends a universal A-complex to a universal B-complex.

<u>Proof</u>: Let (U,d) be a universal A-complex, (X,δ) an arbitrary B-complex. By Lemma 2, there exists a unique Φ-complex homomorphism $\varphi : (U,d) \longrightarrow (X,\delta)$. Hence by Theorem 1, there exists a complex homomorphism $\varphi' : (U',d') \longrightarrow (X,\delta)$ (in fact, $(U',d') = T_\Phi((U,d))$) such that $\varphi = \varphi' \circ \pi_U$. To show the uniqueness of φ', it is sufficient to show that (U',d') is simple. We know that (U',d') is π_U-simple i.e. U' is generated by $B \cup d'(B) \cup \pi_U(U)$. Since (U,d) is universal, U is simple, i.e U is generated by $A \cup dA$, and hence $\pi_U(U)$ is generated by $\Phi(A)$ and $d'(\Phi(A))$. $\Phi(A) \subseteq B$, $d'(\Phi(A)) \subseteq d'(B)$. Hence U' is generated by B and $d'(B)$. Hence (U',d') is simple.

<u>Theorem 3</u>: Let A, B, E be unitary commutative R-algebras, $\Phi : A \longrightarrow B$, $\Psi : B \longrightarrow E$ be unitary algebra homomorphisms, and $T_\Phi : \mathfrak{C}(A) \longrightarrow \mathfrak{C}(B)$, $T_\Psi : \mathfrak{C}(B) \longrightarrow \mathfrak{C}(E)$, $T_{\Psi \circ \Phi} : \mathfrak{C}(A) \longrightarrow \mathfrak{C}(E)$

are the covariant functors defined as previously, then $T_{\Psi\circ\Phi}$ and $T_{\Psi}\circ T_{\Phi}$ are naturally equaivalent.

<u>Proof</u>: Let (C,d) be an A-complex, (X,δ) an arbitrary E-complex, $\varphi : (C,d) \longrightarrow (X,\delta)$ a $\Psi\circ\Phi$-complex homomorphism. Let $X^* = B \oplus \sum_{n \geq 1} X_n$ and define $(a,x)(b,y) = (ab, \Psi(a)y + \Psi(b)x + xy)$, $a,b \in B$, $x,y \in \sum_{n \geq 1} X_n$, and $\delta^* : X^* \longrightarrow X^*$ be the homogeneous derivation of degree 1 defined by $\delta^*(b,x) = (0, \delta\circ\Psi(b) + \delta x)$, $b \in B$, $x \in \sum_{n \geq 1} X_n$, then as in the proof of Lemma 2, (X^*,δ^*) is a B-complex. Define $\varphi_1 : (C,d) \longrightarrow (X^*,\delta^*)$ by $\varphi_1(\sum_{n \geq 0} c_n) = (\Phi(c_o), \varphi(\sum_{n \geq 1} c_n))$, $c_n \in C_n$, then φ_1 is a Φ-complex homomorphism. Hence by Theorem 1, there exists a unique B-complex homomorphism $\varphi_2 : T_{\Phi}((C,d)) \longrightarrow (X^*,\delta^*)$ such that $\varphi_2\circ\pi_C = \varphi_1$. Let $j : (X^*,\delta^*) \longrightarrow (X,\delta)$ be the mapping defined by $j((b,x)) = \Psi(b) + x$, $b \in B$, $x \in \sum_{n \geq 1} X_n$, then j is also Ψ-complex homomorphism. Hence $j\circ\varphi_2 : T_{\Phi}((C,d)) \longrightarrow (X,\delta)$ is a Ψ-complex homomorphism and hence again by Theorem 1, there exists a unique E-complex homomorphism $\varphi_3 : T_{\Psi}\circ T_{\Phi}((C,d)) \longrightarrow (X,\delta)$ such that $\varphi_3\circ\pi_{C'} = j\circ\varphi_2$. $j\circ\varphi_1 = \varphi$, since $j\circ\varphi_1(\sum c_n) = j(\Phi(c_o), \varphi(\sum_{n \geq 1} c_n)) = \Psi\circ\Phi(c_o) + \varphi(\sum_{n \geq 1} c_n) = \varphi(\sum_{n \geq 1} c_n)$, $c_n \in C_n$. Now we show that $\varphi_3(\pi_{C'}\circ\pi_C) = \varphi$. In fact, $\pi_3\circ\pi_{C'}\circ\pi_C = j\circ\varphi_2\circ\pi_C = j\circ\varphi_1 = \varphi$. The uniqueness of φ_3 is clear, since $T_{\Psi}\circ T_{\Phi}((C,d))$ is $(\pi_{C'}\circ\pi_C)$-simple. On the other hand, let $\tau_C : (C,d) \longrightarrow T_{\Psi\circ\Phi}((C,d))$ be the natural $\Psi\circ\Phi$-complex homomorphism, then $T_{\Psi\circ\Phi}(\varphi) : T_{\Psi\circ\Phi}((C,d)) \longrightarrow (X,\delta)$ is a unique E-complex homomorphism such that $T_{\Psi\circ\Psi}(\varphi)\circ\tau_C = \varphi$. Hence by the uniqueness of such E-complexes and E-complex homomorphisms

[cf. Theorem 1], there exists an E-complex isomorphism $i : T_{\Psi\circ\Phi}((C,d)) \longrightarrow T_{\Psi}\circ T_{\Phi}((C,d))$ such that $i\circ\pi_{C'}\circ\pi_C = \tau_C$. This shows that $T_{\Psi\circ\Phi}$ and $T_{\Psi}\circ T_{\Phi}$ are naturally equivalent.

§ 2. Extensions of A by indeterminates.

If B is an arbitrary unitary commutative R-algebra extension of A and $\Phi : A \to B$ the natural injection, then for an A-complex (C,d) and a B-complex (X,δ), a Φ-complex homomorphism $\varphi : (C,d) \longrightarrow (X,\delta)$ maps the elements of A identically, and hence Φ-complex homomorphisms in this case will be called complex homomorphisms over A. Also, $B \otimes_A C$ is an anit-commutative graded algebra, graded by $B \otimes_A C = \sum_{n=0}^{\infty} B \otimes_A C_n$ (direct). $\alpha_C : C \longrightarrow B \otimes C$ will denote the canonical mapping defined by $\alpha_C(c) = 1 \otimes c$ for $c \in C$.

Proposition 1: Let $d_o = d|A$ and $\bar{d}_o : B \longrightarrow B \otimes_A C_1$ be an R-derivation such that $\alpha_C \circ d_o = \bar{d}_o|A$. Then there exists a unique homogeneous R-derivation $\bar{d} : B \otimes_A C \longrightarrow B \otimes_A C$ of degree 1 such that $\alpha_C \circ d = \bar{d} \circ \alpha_C$ and $\bar{d}|B = \bar{d}_o$.

Moreover, $\bar{d} \circ \bar{d}_o = 0$ if and only if $(B \otimes_A C, \bar{d})$ is a B-complex.

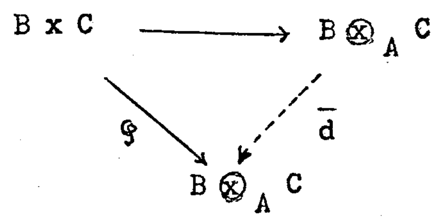
Proof: Define $\varphi : B \times C \longrightarrow B \otimes_A C$ by

$$\varphi(f,c) = \bar{d}_o f(1 \otimes c) + f \otimes dc, \quad f \in B, \; c \in C.$$

Then φ is clearly biadditive and

$$\begin{aligned}
\varphi(af,c) &= \bar{d}_o(af)(1 \otimes c) + af \otimes dc \qquad (a \in A) \\
&= ((\bar{d}_o a)f + a(\bar{d}_o f))(1 \otimes c) + af \otimes dc \\
&= ((1 \otimes da)f + a(\bar{d}_o f))(1 \otimes c) + af \otimes dc \\
&= f \otimes da + \bar{d}_o f(a \otimes c) + f \otimes adc \\
&= \bar{d}_o f(1 \otimes ac) + f \otimes d(ac) \\
&= \varphi(f,ac).
\end{aligned}$$

Thus φ is A-balanced.



Therefore there exists an additive group homomorphism $\bar{d} : B \otimes_A C \longrightarrow B \otimes_A C$ such that $\bar{d}(f \otimes c) = \bar{d}_o f(1 \otimes c) + f \otimes dc$. Clearly $\bar{d}$ is an R-linear mapping, and moreover, for $f \otimes c$, $f' \otimes c'$ where $f, f' \in B$, $c, c' \in C$, c homogeneous of degree n,

$$\bar{d}((f \otimes c)(f' \otimes c')) = \bar{d}(ff' \otimes cc')$$
$$= \bar{d}_o(ff')(1 \otimes cc') + ff' \otimes d(cc')$$
$$= ((\bar{d}_o f)f' + f(\bar{d}_o f'))(1 \otimes cc') + ff' \otimes ((dc)c' + (-1)^n c(dc'))$$
$$= (\bar{d}_o f)(1 \otimes c)(f' \otimes c') + (-1)^n (f \otimes c)(\bar{d}_o f')(1 \otimes c')$$
$$+ (f \otimes dc)(f' \otimes c') + (-1)^n (f \otimes c)(f' \otimes dc')$$
$$= (\bar{d}_o f(1 \otimes c) + f \otimes dc)(f' \otimes c') + (-1)^n (f \otimes c)(\bar{d}_o f'(1 \otimes c) + f' \otimes dc')$$
$$= (\bar{d}(f \otimes c))(f' \otimes c') + (-1)^n (f \otimes c)(d(f' \otimes c')).$$

By the definition of $\bar{d}$, $\bar{d}(B \otimes_A C_n) \subseteq B \otimes C_{n+1}$, and thus we proved that $\bar{d}$ is a homogeneous R-derivation of degree 1. For the uniqueness of $\bar{d}$, let $\delta : B \otimes_A C \longrightarrow B \otimes_A C$ be a homogeneous R-derivation of degree 1 such that $\alpha_C \circ d = \delta \circ \alpha_C$ and $\delta | B = \bar{d}_o$, then

$$\delta(f \otimes c) = \delta(f(1 \otimes c)) = \delta f\ (1 \otimes c) + f\delta(1 \otimes c)$$
$$= \bar{d}_o f(1 \otimes c) + f(1 \otimes dc)$$
$$= \bar{d}(f \otimes c).$$

Thus $\bar{d}$ is unique such that $\alpha_C \circ d = \delta \circ \alpha_C$ and $\delta | B = \bar{d}_o$.

If $\bar{d}\circ\bar{d}_o = 0$, then

$$\bar{d}\circ\bar{d}(f \otimes c) = \bar{d}(\bar{d}_o f(1 \otimes c) + f \otimes dc)$$

$$= (\bar{d}\circ\bar{d}_o f)(1 \otimes c) - (\bar{d}_o f)\, \bar{d}(1 \otimes c) + \bar{d}_o f(1 \otimes dc) + f \otimes ddc$$

$$= (\bar{d}\circ\bar{d}_o f)(1 \otimes c) - (\bar{d}_o f)(1 \otimes dc) + \bar{d}_o f(1 \otimes dc) + f \otimes ddc.$$

$$= 0.$$

Hence $(B \otimes_A C, \bar{d})$ is a B-complex.

Conversely, if $(B \otimes_A C, \bar{d})$ is a B-complex, $\bar{d}\circ\bar{d}_o f = \bar{d}\circ\bar{d} f = 0$. This completes the proof of Proposition 1.

From now on, let us consider the case when $B = A[X]$, a polynomial ring with X as a set of indeterminates.

<u>Lemma 1</u>: Let M be an A-module and $d : A \longrightarrow M$ be an R-derivation. For an arbitrary element of $A[X]$,

$$f = \sum_{\nu_1,\dots,\nu_n} a_{\nu_1\dots\nu_n} x_1^{\nu_1}\dots x_n^{\nu_n}, \quad a_{\nu_1\dots\nu_n} \in A,\ x_1,\dots,x_n \in X$$

where $\nu_1,\dots,\nu_n$ are positive integers, let

$$f_d = \sum_{\nu_1,\dots,\nu_n} x_1^{\nu_1}\dots x_n^{\nu_n} \otimes da_{\nu_1\dots\nu_n} \in A[X] \otimes_A M.$$

Then

(1) $(r_1 f + r_2 g)_d = r_1 f_d + r_2 g_d, \quad r_1, r_2 \in R, \quad f, g \in A[X].$

(2) $(fg)_d = f_d\, g + f\, g_d, \quad f, g \in A[X]$

(3) $a_d = 1 \otimes da, \quad a \in A$

(4) $x_d = 0.$

<u>Proof</u>: (1), (3) and (4) are immediate consequences of the definition.

Since $(f + g)_d = f_d + g_d$ by (1), to show (2), it is sufficient to show for $f = ax_1^{\nu_1}\dots x_n^{\nu_n}$, $g = b\, x_1^{\mu_1}\dots x_n^{\mu_n} \in A[\,X\,]$,

$$\begin{aligned}(fg)_d &= (abx_1^{\nu_1+\mu_1}\dots x_n^{\nu_n+\mu_n}) \\ &= x_1^{\nu_1+\mu_1}\dots x_n^{\nu_n+\mu_n} \otimes d(ab) \\ &= x_1^{\nu_1+\mu_1}\dots x_n^{\nu_n+\mu_n} \otimes ((da)b + a(db)) \\ &= (x_1^{\nu_1}\dots x_n^{\nu_n} \otimes da)(bx_1^{\mu_1}\dots x_n^{\mu_n}) + (ax_1^{\nu_1}\dots x_n^{\nu_n})(x_1^{\mu_1}\dots x_n^{\mu_n} \otimes db) \\ &= f_d\, g + f\, g_d.\end{aligned}$$

<u>Proposition 2:</u> Let M be an A-module and $d : A \longrightarrow M$ be an R-derivation. With every element $x \in X$, associate any element $U_x \in A[\,X\,] \otimes_A M$. Then there exists a unique R-derivation $\bar{d} : A[\,X\,] \longrightarrow A[\,X\,] \otimes_A M$ such that $\bar{d}|A = \alpha_M \circ d$ and $\bar{d}(x) = U_x$, $x \in X$.

<u>Proof</u>: For any element $f \in A[\,X\,]$, define $\bar{d} : A[\,X\,] \longrightarrow A[\,X\,] \otimes_A M$ by $\bar{d}(f) = f_d + \sum_x \frac{\partial}{\partial x} f\, U_x$. Then

(1) $\quad d(r_1 f + r_2 g) \qquad r_1, r_2 \in R, \quad f, g \in A[\,X\,]$

$$\begin{aligned}&= (r_1 f + r_2 g)_d + \sum_x \left(\frac{\partial}{\partial x} (r_1 f + r_2 g)\, U_x\right) \\ &= r_1 f_d + r_2 g_d + r_1 \sum_x \frac{\partial}{\partial x} f\, U_x + r_2 \sum_x \frac{\partial}{\partial x} g\, U_x \\ &= r_1 (f_d + \sum_x \frac{\partial}{\partial x} f\, U_x) + r_2 (g_d + \sum_x \frac{\partial}{\partial x} g\, U_x) \\ &= r_1 \bar{d} f + r_2 \bar{d} g.\end{aligned}$$

(2) $\quad d(fg) \qquad f, g \in A[\,X\,]$

$$\begin{aligned}&= (fg)_d + \sum_x \frac{\partial}{\partial x} fg\, U_x \\ &= f_d\, g + f\, g_d + \sum_x \left(\left(\frac{\partial}{\partial x} f \right) g + f\left(\frac{\partial}{\partial x} g \right)\right) U_x \\ &= (f_d + \sum_x \frac{\partial}{\partial x} f\, U_x) g + f(g_d + \sum_x \frac{\partial}{\partial x} g\, U_x) \\ &= (\bar{d} f)\, g + f\, (\bar{d} g)\end{aligned}$$

(3) $\quad \bar{d}(a) = a_d + \sum_x \frac{\partial a}{\partial x} U_x = a_d = 1 \otimes da = \alpha_M \circ d(a)$ i.e. $\bar{d}|A = \alpha_M \circ d$.

(4) $\bar{d}(x) = x_d + \frac{d}{dx}x\, U_x = U_x$

Hence $\bar{d} : A[X] \longrightarrow A[X] \otimes_A M$ so defined is an R-derivation such that $\bar{d}|A = \alpha_M \circ d$ and $\bar{d}(x) = U_x$, $x \in X$.

Finally to show the uniqueness of $\bar{d}$, let $\delta : A[X] \longrightarrow A[X] \otimes_A M$ be an R-derivation such that $\delta|A = \alpha_M \circ d$ and $\delta(x) = U_x$, $x \in X$, then

$$\delta\Big(\sum_{\nu_1,\ldots,\nu_n} a_{\nu_1\ldots\nu_n} x_1^{\nu_1}\ldots x_n^{\nu_n}\Big) = \sum_{\nu_1,\ldots,\nu_n}\Big(\delta a_{\nu_1\ldots\nu_n} x_1^{\nu_1}\ldots x_n^{\nu_n} + a_{\nu_1\ldots\nu_n}\sum_x \frac{\partial}{\partial x} x_1^{\nu_1}\ldots x_n^{\nu_n} U_x\Big)$$

$$= \sum_{\nu_1,\ldots,\nu_n}\Big(x_1^{\nu_1}\ldots x_n^{\nu_n} \otimes da_{\nu_1\ldots\nu_n} + \sum_x \frac{\partial}{\partial x} a_{\nu_1\ldots\nu_n} x_1^{\nu_1}\ldots x_n^{\nu_n} U_x \Big)$$

$$= \Big(\sum a_{\nu_1\ldots\nu_n} x_1^{\nu_1}\ldots x_n^{\nu_n}\Big)_d + \sum_x \frac{\partial}{\partial x}\Big(\sum a_{\nu_1\ldots\nu_n} x_1^{\nu_1}\ldots x_n^{\nu_n}\Big) U_x$$

$$= \bar{d}\Big(\sum_{\nu_1,\ldots,\nu_n} x_1^{\nu_1}\ldots x_n^{\nu_n}\Big).$$

Hence $\bar{d}$ is unique.

<u>Proposition</u> 3: Let (C,d) be an A-complex. With every element $x \in X$, associate any element $U_x \in A[X] \otimes_A C_1$. Then there exists a unique homogeneous R-derivation $\bar{d} : A[X] \otimes_A C \longrightarrow A[X] \otimes_A C$ of degree 1 such that $\alpha_C \circ d = \bar{d} \circ \alpha_C$ and $\bar{d}(x) = U_x$, $x \in X$. Moreover, $(A[X] \otimes_A C, \bar{d})$ is an $A[X]$-complex if and only if $\bar{d}U_x = 0$ for all $x \in X$.

<u>Proof</u>: Let $d_o = d|A$, then by Proposition 2, there exists a unique R-derivation $\bar{d}_o : A[X] \longrightarrow A[X] \otimes_A C_1$ such that $\bar{d}_o|A = \alpha_C \circ d_o$ and $\bar{d}_o(x) = U_x$, $x \in X$. By Prroposition 1, there exists a unique homogeneous R-derivation $\bar{d} : A[X] \otimes_A C \longrightarrow A[X] \otimes_A C$ of

degree 1 such that $\alpha_C \circ d = \bar{d} \circ \alpha_C$ and $\bar{d}|A[X] = \bar{d}_o$. But under the condition $\alpha_C \circ d = \bar{d}\alpha_C$, $\bar{d}|A[X] = \bar{d}_o$ if and only if $\bar{d}(x) = U_x$.
$\bar{d}|A[X] = \bar{d}_o$ implies $\bar{d}(x) = \bar{d}_o(x) = U_x$. Conversely, if $\bar{d}(x) = U_x$,

$$\bar{d}(\sum a_{\nu_1 \dots \nu_n} x_1^{\nu_1} \dots x_n^{\nu_n})$$

$$= \sum \bar{d}(a_{\nu_1 \dots \nu_n} x_1^{\nu_1} \dots x_n^{\nu_n})$$

$$= \sum \bar{d}(a_{\nu_1 \dots \nu_n}) x_1^{\nu_1} \dots x_n^{\nu_n} + \sum (\sum_k \frac{\partial}{\partial x_k} a_{\nu_1 \dots \nu_n} x_1^{\nu_1} \dots x_n^{\nu_n} \bar{d}x_k)$$

$$= \sum x_1^{\nu_1} \dots x_n^{\nu_n} \otimes da_{\nu_1 \dots \nu_n} + \sum (\sum_k \frac{\partial}{\partial x_k} a_{\nu_1 \dots \nu_n} x_1^{\nu_1} \dots x_n^{\nu_n} U_{x_k})$$

$$= \bar{d}_o(\sum a_{\nu_1 \dots \nu_n} x_1^{\nu_1} \dots x_n^{\nu_n}).$$

Thus we have shown that $\bar{d}$ is a unique homogeneous R-derivation of degree 1 such that $\alpha_C \circ d = \bar{d} \circ \alpha_C$ and $\bar{d}(x) = U_x$, $x \in X$.
Now, if $(A[X] \otimes_A C, \bar{d})$ is an $A[X]$-complex, then $\bar{d}U_x = \bar{d} \circ \bar{d}x = 0$.
Conversely, suppose $\bar{d}U_x = 0$, $x \in X$ and show $\bar{d} \circ \bar{d}_o = 0$, then by Proposition 1, we know that $(A[X] \otimes_A C, \bar{d})$ is an $A[X]$-complex. To show $\bar{d} \circ \bar{d}_o = 0$, it is sufficient to consider $ax_1^{\nu_1} \dots x_n^{\nu_n} \in A[X]$, since $\bar{d} \circ \bar{d}_o$ is linear.

$$\bar{d} \circ \bar{d}_o(ax_1^{\nu_1} \dots x_n^{\nu_n})$$

$$= \bar{d}(x_1^{\nu_1} \dots x_n^{\nu_n} \otimes da + \sum_k \frac{\partial}{\partial x_k} ax_1^{\nu_1} \dots x_n^{\nu_n} U_{x_k})$$

$$= \bar{d}_o(x_1^{\nu_1} \dots x_n^{\nu_n})(1 \otimes da) + x_1^{\nu_1} \dots x_n^{\nu_n} \otimes dda$$

$$+ \sum_k (\bar{d} (\frac{\partial}{\partial x_k} ax_1^{\nu_1} \dots x_n^{\nu_n}) U_x + \frac{\partial}{\partial x_k} ax_1^{\nu_1} \dots x_n^{\nu_n} \bar{d}U_{x_k})$$

$$= \bar{d}_o(x_1^{\nu_1} \dots x_n^{\nu_n})(1 \otimes da) + \sum_k \bar{d}(\frac{\partial}{\partial x_k} ax_1^{\nu_1} \dots x_n^{\nu_n}) U_{x_k}$$

(since $dda = 0$, $\bar{d}U_{x_k} = 0$)

$$= (\sum_k \frac{\partial}{\partial x_k} x_1^{\nu_1} \dots x_n^{\nu_n} U_{x_k})(1 \otimes da)$$

$$+ \sum_k \left(\left(\frac{\partial}{\partial x} x_1^{\nu_1} \cdots x_n^{\nu_n} \right) \otimes da + \sum_{k,m} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_k} x_1^{\nu_1} \cdots x_n^{\nu_n} U_{x_m} \right) U_{x_k}$$

$$= \left(\sum_k \frac{\partial}{\partial x_k} x_1^{\nu_1} \cdots x_n^{\nu_n} U_{x_k} \right)(1 \otimes da) + \sum_k \left(\left(\frac{\partial}{\partial x} x_1^{\nu_1} \cdots x_n^{\nu_n} \right) \otimes da \right) U_{x_k}$$

(since $U_x^2 = 0$ and $U_{x_m} U_{x_k} = - U_{x_k} U_{x_m}$ in anti- commutative graded algebra).

$= 0$

(since $U_{x_k} (1 \otimes da) = - (1 \otimes da)\, U_{x_k}$).

Hence $(A[X] \otimes_A C, \bar{d})$ is an $A[X]$-complex.

<u>Corollary 1</u>: Let (C,d) be an A-complex, then there exists a unique homogeneous R-derivation $\bar{d} : A[X] \otimes_A C \longrightarrow A[X] \otimes_A C$ of degree 1 such that $\alpha_C \circ d = \bar{d} \circ \alpha_C$ and $\bar{d}(x) = 0$, $x \in X$. Moreover, $(A[X] \otimes_A C, \bar{d})$ is an $A[X]$-complex.

<u>Proof</u>: Take $U_x = 0$ for all $x \in X$ in Proposition 3, then $\bar{d}U_x = \bar{d}0 = 0$. Thus it is just the special case of Proposition 3.

Now, we are in a position to give an explicit construction of the functor T_Φ in the present case.

Let $W = \{ w_x | x \in X \}$ such that $w_x = w_y$ implies $x = y$. Let F be a free $A[X]$-module with W as its basis. Then the following is well known.

(1) The exterior algebra E(F) of F is an anti-commututative graded algebra. Let W be totally ordered by $\leq$, and put $t_\sigma = w_{x_1} \cdots w_{x_n}$ for any finite $\sigma \subseteq W$, $\sigma = \{ w_{x_1}, \ldots, w_{x_n} \}$ with $w_{x_1} < w_{x_2} < \cdots < w_{x_n}$. Then

$$= d'(ft_\sigma \otimes c)\ (f't_\tau \otimes c') + (-1)^{p+m}(ft_\sigma \otimes c)\ d'(f't_\tau \otimes c').$$

Hence the product rule holds.

By the definition of d', it is clear that $d'(C'_n) \subseteq C'_{n+1}$

Moreover,

$$d' \circ d(ax_1^{\nu_1} \ldots x_n^{\nu_n}\ t_\sigma \otimes c) \quad (|\sigma| = m)$$

$$= d'(\partial(ax_1^{\nu_1} \ldots x_n^{\nu_n}\ t_\sigma) \otimes c + (-1)^m x_1^{\nu_1} \ldots x_n^{\nu_n}\ t_\sigma \otimes d(ac))$$

$$= \partial \circ \partial(ax_1^{\nu_1} \ldots x_n^{\nu_n}\ t_\sigma) \otimes c + (-1)^{m+1}\ (x_1^{\nu_1} \ldots x_n^{\nu_n}\ t_\sigma) \otimes d(ac)$$

$$+ (-1)^m \partial(x_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma) \otimes d(ac) + (-1)^{2m} x_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma \otimes dd(ac)$$

$$= 0$$

Therefore d' is a homogeneous R-derivation of degree 1 with $d' \circ d' = 0$.

for the uniqueness of d', let $\delta : C' \longrightarrow C'$ be a homogeneous R-derivation of degree 1 with $\delta \circ \delta = 0$ such that $\delta \circ \pi_C = \pi_C \circ d$ and $\delta(x) = w_x \otimes 1$, then

$$\delta(ax_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma \otimes c)$$

$$= \delta((x_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma \otimes 1)(a \otimes c))$$

$$= \sum_k \frac{\partial}{\partial x_k} x_1^{\nu_1} \ldots x_n^{\nu_n}\ w_{x_k} t_\sigma \otimes 1)(a \otimes c)$$

$$+(-1)^m (x_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma \otimes 1)(1 \otimes d(ac))$$

$$= \partial(ax_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma) \otimes c + (-1)^m x_1^{\nu_1} \ldots x_n^{\nu_n} \otimes d(ac)$$

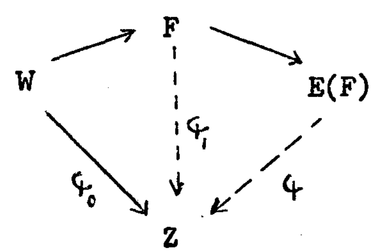
$$= d'(ax_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma \otimes c).$$

Hence $\delta = d'$ and thus d' is unique such that $d' \circ \pi_C = \pi_C \circ d$, and $d'(x) = w_x \otimes 1$. Here we have proved that C' is an anti-commutative graded algebra with $C'_o = A[\,X\,]$ and d' is a homogeneous R-derivation of degree 1 with $d' \circ d' = 0$. Therefore (C', d') is an $A[\,X\,]$-complex.

Lemma 2: Let (Z, δ) be an arbitrary $A[\,X\,]$-complex. Then there exists a unique graded algebra homomorphism $\varphi : E(F) \longrightarrow Z$ over $A[\,X\,]$ (i.e. $\varphi | A[\,X\,]$ = identity) such that $\varphi(w_x) = \delta x$. Moreover, $\varphi \circ \partial(a x_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma) = \sigma\delta(x_1^{\nu_1} \ldots x_n^{\nu_n})\, \varphi(t_\sigma)$.

Proof: Define $\varphi_o : W \longrightarrow Z$ by $\varphi_o(w_x) = \delta x$, then there exists a unique $A[\,X\,]$-homomorphism $\varphi_1 : F \longrightarrow Z$ extending φ_o. Moreover, $(\varphi_1(\Sigma f_i w_{x_i}))^2 = (\Sigma f_i \delta_{x_i})^2 = 0$,

since $\delta_{x_i}\delta_{x_j} = 0$ when $i = j$, $\delta_{x_i}\, \delta_{x_j} = -\,\delta_{x_j}\, \delta_{x_i}$ when $i = j$. Since $E(F)$ is an exterior algebra of F, φ_1 extends uniquely to an algebra homomorphism $\varphi : E(F) \longrightarrow Z$ over $A[\,X\,]$.



Thus φ is a unique algebra homomorphism over $A[\,X\,]$ such that $\varphi(w_x) = \delta x$. φ is graded by definition. Now, $\varphi \circ \partial(a x_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma) = a\, \varphi \circ \partial(x_1^{\nu_1} \ldots x_n^{\nu_n} t_\sigma)$

$$= a\, \varphi\Big(\sum_k \frac{\partial}{\partial x_k} x_1^{\nu_1} \ldots x_n^{\nu_n} w_{x_K} t_\sigma\Big)$$

$$= a\left(\sum_k \frac{\partial}{\partial x_k} x_1^{\nu_1}\cdots x_n^{\nu_n} \psi(w_{x_k})\psi(t_\sigma)\right)$$

$$= a\left(\sum_k \frac{\partial}{\partial x_k} x_1^{\nu_1}\cdots x_n^{\nu_n} \delta x_k\right)\psi(t_\sigma)$$

$$= a\delta(x_1^{\nu_1}\cdots x_n^{\nu_n})\psi(t_\sigma).$$

Proposition 5: Let (C,d) be an A-complex. For any $A[X]$-complex (Z,δ) and any complex homomorphism $\varphi : (C,d) \longrightarrow (Z,\delta)$ over A, there exists a unique complex homomorphism $\varphi' : (C',d') \longrightarrow (Z,\delta)$ such that $\varphi = \varphi' \circ \pi_C$.

Proof: Define $\varphi_o : E(F) \times C \longrightarrow Z$ by $\varphi_o(\Sigma f_\sigma t_\sigma, C) = \psi(\Sigma f_\sigma t_\sigma)\varphi(c)$. φ_o is A-bilinear, since ψ and φ are A-linear. Therefore, there exists an A-linear mapping $\varphi' : E(F) \otimes_A C \longrightarrow Z$ such that $\varphi'(\Sigma f_\sigma t_\sigma \otimes c) = \psi(\Sigma f_\sigma t_\sigma)\varphi(c)$.

We will show that φ' is what we want.

1) φ' is an algebra homomorphism, since

$$\varphi'((ft_\sigma \otimes c)(f't_\tau \otimes c')) = \varphi'((-1)^{pn}(ft_\sigma)(f't_\sigma) \otimes c_p c_q')$$

$$= (-1)^{pn}\psi(ft_\sigma)\ \psi(f't_\tau)\ \varphi(c_p)\ \varphi(c_q')$$

$$= \psi(ft_\sigma)\varphi(c_p)\ \psi(f't_\tau)\varphi(c_q')$$

$$= \varphi'(ft_\sigma \otimes c_p)\ \varphi'(f't_\tau \otimes c_q)$$

2) φ' is graded by definition.

3) $\varphi'|A[X]$ = identity on $A[X]$, since $\varphi'(f) = \varphi'(f \otimes 1) = \psi(f)\varphi(1) = \psi(f) = f$.

4) $\varphi' d' = \delta\varphi'$, since

$$\varphi' d'(a x_1^{\nu_1}\cdots x_n^{\nu_n} t_\sigma \otimes c)$$

$$= \varphi'(\partial(a x_1^{\nu_1}\cdots x_n^{\nu_n} t_\sigma) \otimes c + (-1)^m x_1^{\nu_1}\cdots x_n^{\nu_n} t_\sigma \otimes d(ac)$$

$$= \psi\circ\partial(a x_1^{\nu_1}\cdots x_n^{\nu_n} t_\sigma)\varphi(c) + (-1)^m \psi(x_1^{\nu_1}\cdots x_n^{\nu_n} t_\sigma)\varphi\circ d(ac)$$

$$= a\delta(x_1^{\nu_1}\ldots x_n^{\nu_n})\,\psi(t_\sigma)\,\varphi(c) + (-1)^m x_1^{\nu_1}\ldots x_n^{\nu_n}\psi(t_\sigma)\delta\circ\varphi(ac)$$

Also,

$$\delta\circ\varphi'(ax_1^{\nu_1}\ldots x_n^{\nu_n}t_\sigma \otimes c)$$
$$= \delta(\psi(ax_1^{\nu_1}\ldots x_n^{\nu_n}t_\sigma)\,\varphi(c))$$
$$= \delta(x_1^{\nu_1}\ldots x_n^{\nu_n}\psi(t_\sigma)\,\varphi(ac))$$
$$= \delta(x_1^{\nu_1}\ldots x_n^{\nu_n}\psi(t_\sigma))\ \varphi(ac) + (-1)^m x_1^{\nu_1}\ldots x_n^{\nu_n}\psi(t_\sigma)\delta\circ\varphi(ac)$$
$$= a\delta(x_1^{\nu_1}\ldots x_n^{\nu_n}\psi(t_\sigma))\ \varphi(c) + (-1)^m x_1^{\nu_1}\ldots x_n^{\nu_n}\psi(t_\sigma)\delta\,\varphi(ac).$$

Hence $\varphi'\delta' = \delta\varphi'$

5) $\varphi = \varphi'\circ\pi_C$, since $\varphi'\circ\pi_C(c) = \varphi'(1\otimes c) = \psi(1)\,\varphi(c) = \varphi(c)$

From (1) to (5) it follows that $\varphi' : (C', d') \longrightarrow (Z, \delta)$ is a complex homomorphism such that $\varphi = \varphi'\circ\pi_C$.

6) For the uniqueness of φ', let $\bar{\varphi} : (C', d') \longrightarrow (Z, \delta)$ be a complex homomorphism such that $\varphi = \bar{\varphi}\circ\pi_C$, then

$$\bar{\varphi}(fw_{x_1}\ldots w_{x_n}\otimes c) = \bar{\varphi}(f(w_{x_1}\otimes 1)(w_{x_2}\otimes 1)\ldots(w_{x_n}\otimes 1)(1\otimes c))$$

$$= \bar{\varphi}(fd'(x_1)d'(x_2)\ldots d'(x_n)(1\otimes c))$$

$$= f\delta\,\bar{\varphi}(x_1)\delta\,\bar{\varphi}(x_2)\ldots\delta\,\bar{\varphi}(x_n)\ \bar{\varphi}\,\pi_C(c)$$

$$= f\delta(x_1)\delta(x_2)\ldots\delta(x_n)\,\varphi(c)$$

$$= f\,\psi(w_{x_1}\ldots w_{x_n})\,\varphi(c)$$

$$= \varphi'(fw_{x_1}\ldots w_{x_n}\otimes c)$$

Hence φ' is unique such that $\varphi = \varphi'\circ\pi_C$.

This completes the proof.

Let $U(A/B)$ denote a universal derivation module of A as a B-algebra.

<u>Corollary 2</u>: $U(A[X]/R) \cong U(A[X]/A) \otimes_A U(A/R)$.

The above now prove:

<u>Theorem 1:</u> For the natural injection $\Phi : A \longrightarrow A[X]$, the functor T_Φ is explicitly given by $T_\Phi((C,d)) = (E(F) \otimes_A C, d'), d'$ as defined in Proposition 4, and $T_\Phi(\varphi) = i_{E(F)} \otimes \varphi$ for any A-complex homomorphism $\varphi : (C,d) \longrightarrow (D,\delta)$.

§ 3. Fractional extensions of A.

Let A be a commutative R-algebra with unit and E be a commutative unitary extension R-algebra of A.

Definition: An ideal D of A is called E-dense if and only if $ED = E$.

Definition: E is a fractional extension of A if and only if for each $q \in E$, there exists an E-dense ideal D such that $Dq \subseteq A$.

Definition: A module M (over A) is called E-torsion free if and only if for any E-dense ideal D and for any $x \in M$, $Dx = 0$ implies $x = 0$.

From now on, let E be a fractional extension of A, if not mentioned specially otherwise.

The following are well known facts.

(1) If D, D' are E-dense ideals, $D \cap D'$ is also E-dense.

(2) $q^{-1}A = (b \in A \mid qb \in A)$ is E-dense for any $q \in E$.

(3) An injective hull of an E-torsion free module is again E-torsion free.

(4) Any E-module is E-torsion free.

(5) Any E-torsion free module can be imbedded into an E-module i.e. if

M is E-torsion free, $M \longrightarrow E \otimes_A M$ by $x \rightsquigarrow 1 \otimes x$ is one-to-one.

Proposition 1: (1) If M, N are E-modules, any A-linear mapping $\varphi : M \longrightarrow N$ is automatically E-linear.

(2) If M is E-module, $M \cong E \otimes_A M$.

Proof: (1) For any $b \in q^{-1}A$, $b\varphi(qx) = \varphi(bqx) = b(q\varphi(x))$, $q \in E$, $x \in M$. Hence $(q^{-1}A)(\varphi(qx) - q\varphi(x)) = 0$. But N is an E-module and hence E-torsion free, and thus $\varphi(qx) = q\varphi(x)$, i.e. φ is E-linear.

(2) M is an E-module and hence E-torsion free, and thus the natural A-homomorphism $\pi_M : M \longrightarrow E \otimes_A M$ is one-to-one. By (1), π_M is automatically E-linear. Now, for any $q \otimes x \in E \otimes_A M$, $q \otimes x = 1 \otimes qx$, since $b(q \otimes x) = 1 \otimes bqx = b(1 \otimes qx)$ for all $b \in q^{-1}A$. Now, for any $q \otimes x \in E \otimes_A M$, $\pi_M(qx) = 1 \otimes qx = q \otimes x$. Hence π_M is onto and thus π_M is an isomorphism.

Lemma 1: Let D be an E-dense ideal, M be any E-module, then any A-homomorphism $\varphi : D \longrightarrow M$ has a unique extension to an A-homomorphism $\bar{\varphi} : A \longrightarrow M$.

Proof: Let H be an A-injective hull of M, then $\varphi : D \longrightarrow M \subseteq H$ has an extension $\bar{\varphi} : A \longrightarrow H$. Here $\bar{\varphi}$ is a unique extension, since if $\bar{\varphi}$ and φ^* are extensions of φ, $D\bar{\varphi}(1) = \bar{\varphi}(D) = \varphi^*(D) = D\varphi^*(1)$ implies $\bar{\varphi}(1) = \varphi^*(1)$, since H is E-torsion free. (M is an E-module and hence E-torsionfree, and H is E-torsion free, H is an injective hull of an E-torsion free module). $\bar{\varphi}$ and φ^* are A-homomorphisms with $\bar{\varphi}(1) = \varphi^*(1)$ and hence $\bar{\varphi} = \varphi^*$. Now, we will show that $\bar{\varphi}(A) \subseteq M$. $\bar{\varphi}(D) \subseteq M$ and hence $E\tilde{\varphi}(D) \subseteq M$. Thus $\tilde{\varphi}(A) = A\tilde{\varphi}(1) \subseteq E\tilde{\varphi}(1) = E\,D\tilde{\varphi}(1) = E\tilde{\varphi}(D) \subseteq M$. This completes the proof.

<u>Lemma 2</u>: Let E be a fractional extension and N be any E-module. If two derivations $d, d' : E \longrightarrow N$ coincide on A, then $d = d'$.

<u>Proof</u>: For any $q \in E$, $q^{-1}A = \{b \in A \mid qb \in A\}$ is an E-dense ideal. $(d - d')(qb) = q(d - d')(b) + b(d - d')(q)$, for each $b \in q^{-1}A$. $(d - d')(qb) = 0$ and $(d - d')(b) = 0$ imply $b(d - d')(q) = 0$ for each $b \in q^{-1}A$, i.e. $(q^{-1}A)(d(q) - d'(q)) = 0$. N is an E-module and hence E torsin free. Thus $d(q) = d'(q)$, <u>i.e.</u> $d = d'$.

Let $\pi_M : M \longrightarrow E \otimes_A M$ be the natural A-homomorphism defined by $\pi_M(x) = 1 \otimes x$, $x \in M$.

<u>Proposition 2</u>: Let M be an A-module and $d : A \longrightarrow M$ be an R-derivation, then d induces a unique derivation $\bar{d} : E \longrightarrow E \otimes_A M$ such that $\bar{d}|A = \pi_M \cdot d$.

<u>Proof</u>: For any $q \in E$, $q^{-1}A = \{b \in A \mid qb \in A\}$ is an E-dense ideal. Consider for each $q \in E$, $\varphi_q : q^{-1}A \longrightarrow E \otimes_A M$ by $\varphi_q(b) = 1 \otimes d(qb) - q \otimes db$.

Then

$$\varphi_q(b_1 + b_2) = \varphi_q(b_1) + \varphi_q(b_2), \quad b_1 b_2 \in q^{-1}A.$$

and

$$\begin{aligned}\varphi_q(ab) &= 1 \otimes d(qab) - q \otimes d(ab) \qquad a \in A,\ b \in q^{-1}A.\\ &= a \otimes d(qb) + qb \otimes da - qa \otimes db - qb \otimes da.\\ &= a(1 \otimes d(qb) - q \otimes db) = q\,\varphi_q(b).\end{aligned}$$

Hence φ_q is an A-homomorphism.

By Lemma 1, $\varphi_q : q^{-1}A \longrightarrow E \otimes_A M$ has a unique extension to an A-homomorphism $\bar{\varphi}_q : A \longrightarrow E \otimes_A M$.

Let us define $\bar{d} : E \longrightarrow E \otimes_A M$ by $\bar{d}(q) = \bar{\varphi}_q(1)$, and we will show that $\bar{d}$ is an R-derivation.

(1) R-linearity of $\bar{d}$: For any $b \in q_1^{-1} A \cap q_2^{-1} A$ $(q_1, q_2 \in E)$, $r_1q_1b + r_2q_2b \in A$, $(r_1, r_2 \in R)$ implies $b \in (r_1q_1 + r_2q_2)^{-1}A$.

$$\varphi_{r_1q_1 + r_2q_2}(b) = 1 \otimes d((r_1q_1 + r_2q_2)b) - (r_1q_1 + r_2q_2) \otimes db$$

$$= r_1(1 \otimes d(q_1b) - q_1 \otimes db) + r_2(1 \otimes d(q_2b) - q_2 \otimes db)$$

$$= r_1 \varphi_{q_1}(b) + r_2 \varphi_{q_2}(b).$$

Hence

$$b \bar{\varphi}_{r_1q_1 + r_2q_2}(1) = b(r_1\bar{\varphi}(1) + r_2 \bar{\varphi}(1)) \quad \text{for all } b \in q_1^{-1}A \cap q_2^{-1}A.$$

Since $q_1^{-1} A \cap q_2^{-1}A$ is E-dense and $E \otimes_A M$ is E-torsin free

$$\bar{\varphi}_{r_1q_1 + r_2q_2}(1) = r_1 \bar{\varphi}(1) + r_2 \bar{\varphi}(1)$$

i.e. $\bar{d}(r_1q_1 + r_2q_2) = r_1\bar{d}q_1 + r_2\bar{d}q_2$.

(2) Pruduct rule of $\bar{d}$: For $b \in (q_1q_2)^{-1}A \cap q_1^{-1}A \cap q_2^{-1}A$

$$\varphi_{q_1q_2}(b) = 1 \otimes d(q_1q_2b) - (q_1q_2) \otimes db$$

$$= 1 \otimes d(q_1q_2b) - q_1 \otimes d(q_2b) + q_1 \otimes d(q_2b) - (q_1q_2) \otimes db$$

$$= \varphi_{q_1}(q_2b) + q_1 \varphi_{q_2}(b).$$

But $\varphi_{q_1}(q_2b) = q_2 \varphi_{q_1}(b)$, since for each $c \in q_2^{-1}A$,

$$c\,\varphi_{q_1}(q_2b) = \varphi_{q_1}(cq_2b) = cq_2\,\varphi_{q_1}(b)$$

and hence

$$\varphi_{q_1}(q_2b) = q_2\,\varphi_{q_1}(b).$$

Therefore, $\varphi_{q_1q_2}(b) = q_1\,\varphi_{q_2}(b) + q_2\,\varphi_{q_1}(b)$ for each $b \in (q_1q_2)^{-1}A \cap q_1^{-1}A \cap q_2^{-1}A$.

Hence $b\,\bar{\varphi}_{q_1q_2}(1) = b(q_1\,\bar{\varphi}_{q_2}(1) + q_2\,\bar{\varphi}_{q_1}(1))$

Since $(q_1q_2)^{-1}A \cap q_1^{-1}A \cap q_2^{-1}A$ is E-dense ideal and $E \otimes_A M$ is E-torsin free,

$$\bar{\varphi}_{q_1q_2}(1) = q_1\,\bar{\varphi}_{q_2}(1) + q_2\,\bar{\varphi}_{q_1}(1).$$

i.e. $\bar{d}(q_1q_2) = q_1\bar{d}(q_2) + q_2\bar{d}(q_1)$.

For any $a \in A$, $\bar{d}(a) = \bar{\varphi}_a(1) = \varphi_a(1) = 1 \otimes da - a \otimes d(1) = 1 \otimes da = \pi_M \circ d(a)$.

Hence $\bar{d}|A = \pi_M \circ d$.

Finally, the uniqueness of $\bar{d}$ follows from Lemma 2.

<u>Corollary 1</u>: Any R-derivation $d : A \longrightarrow A$ can be uniquely extended to an R-derivation $\bar{d} : E \longrightarrow E$ where E is a fractional extension of A.

<u>Proof</u>: $E \otimes_A A \cong E$ by $f \otimes a \rightsquigarrow af$. Then this is a special case of Proposition 2.

<u>Proposition 3</u>: Let (C,d) be an A-complex. Then there exists a unique homogeneous R-derivation $\bar{d} : E \otimes_A C \longrightarrow E \otimes_A C$ of degree 1 such that $\pi_C \circ d = \bar{d} \circ \pi_C$. Moreover $(E \otimes_A C, \bar{d})$ is an E-complex.

<u>Proof</u>: Let $d_o = d|A$. By Proposition 2, $d_o : A \longrightarrow C_1$ in-

duces a unique derivation $\bar{d}_o : E \longrightarrow E \otimes_A C_1$ such that $\bar{d}_o | A = \pi_C \circ d_o$. By Proposition 1, §2, there exists a unique homogeneous R-derivation $\bar{d} : E \otimes_A C \longrightarrow E \otimes_A C$ of degree 1 such that $\pi_C \circ d = \bar{d} \circ \pi_C$, and $\bar{d}|E = \bar{d}_o$. But the condition $\bar{d}|E = \bar{d}_o$ can be omitted, since $\pi_C \circ d = \bar{d} \circ \pi_C$ implies $\bar{d}|E = \bar{d}_o$. Actually, for any $a \in A$, $\bar{d}a = \bar{d}(1 \otimes a) = \bar{d} \circ \pi_C(a) = \pi_C \circ d(a)$ and hence $d|A = \pi_C \circ d$ and by the uniqueness of such derivation $\bar{d}|E = \bar{d}_o$. Finally, to show that $(E \otimes_A C, \bar{d})$ is an E-complex, by Proposition 1, it is sufficient to show that $\bar{d} \circ \bar{d}_o = 0$. For any $q \in E$ and for any $b \in q^{-1}A$

$$\bar{d}(b(\bar{d}_o q)) = (\bar{d}b)(\bar{d}_o q) + b\bar{d}(\bar{d}_o q),$$

hence

$$b\,\bar{d}(\bar{d}_o q) = \bar{d}(b(\bar{d}_o q)) - (\bar{d}b)(\bar{d}_o q),$$

but

$$b(\bar{d}_o q) = b\,\bar{\varphi}_q(1) = \bar{\varphi}_q(b) = 1 \otimes d(qb) - q \otimes db.$$

Thus

$$\begin{aligned} b\,\bar{d}(\bar{d}_o q) &= \bar{d}(1 \otimes d(qb) - q \otimes db) - (\bar{d}b)(\bar{d}_o q) \\ &= 1 \otimes dd(qb) - (\bar{d}_o q)(1 \otimes db) - q \otimes ddb - (\bar{d}b)(\bar{d}_o q) \\ &= 0, \end{aligned}$$

since $(\bar{d}_o q)(1 \otimes db) = (\bar{d}_o q)(\bar{d}b) = -(\bar{d}b)(\bar{d}_o q)$.
This completes the proof.

Let us write $C' = E \otimes_A C$ and $\bar{d} \doteq d'$.

<u>Proposition 4</u>: For any E-complex (X, δ) and any complex homomorphism $\varphi : (C,d) \longrightarrow (X,\delta)$ over A, there exists a unique complex homomorphism $\varphi' : (C',d') \longrightarrow (X,\delta)$ such that $\varphi' \circ \pi_C = \varphi$.

<u>Proof</u>: Define $\varphi_0 : E \times C \longrightarrow X$ by $\varphi_0(q,c) = q\varphi(c)$, then φ_0 is A-bilinear, since $\varphi|A$ = identity. Hence there exists an A-homomorphism $\varphi' : E \otimes_A C \longrightarrow E \otimes_A C$ such that $\varphi'(q \otimes c) = q\varphi(c)$. We will show that φ' is what we want.

1) φ' is an algebra homomorphism, since

$$\begin{aligned}\varphi'((q_1 \otimes c_1)(q_2 \otimes c_2)) &= \varphi'(q_1q_2 \otimes c_1c_2)\\ &= q_1q_2\varphi(c_1c_2) = q_1q_2\varphi(c_1)\varphi(c_2) = (q_1\varphi(c_1))(q_2\varphi(c_2))\\ &= \varphi'(q_1 \otimes c_1)\ \varphi'(q_2 \otimes c_2)\end{aligned}$$

2) φ' is graded, since φ is graded.

3) $\varphi'|E$ = identity, since $\varphi'(q) = \varphi'(q \otimes 1) = q\varphi(1) = q$.

4) $\varphi' \circ d' = \delta \circ \varphi'$, since

$$\begin{aligned}\varphi' \circ d'(q \otimes c) &= \varphi'(\bar{d}_0 q(1 \otimes c) + q \otimes dc)\\ &= \varphi'(\bar{d}_0 q)\ \varphi'(1 \otimes c) + \varphi'(q \otimes dc)\\ &= \varphi'(\bar{d}_0 q)\ \varphi(c) + q\varphi(dc)\\ &= \varphi'(\bar{d}_0 q)\varphi(c) + q\delta\varphi(c).\end{aligned}$$

But $\bar{d}_0 q = \bar{\varphi}_q(1)$ and hence for any $b \in q^{-1}A$, $b(\bar{d}_0 q) = b\bar{\varphi}(1) = \bar{\varphi}_q(b) = 1 \otimes d(qb) - q \otimes db$.

Hence $b\ \varphi'(\bar{d}_0 q) = \varphi d(qb) - q\varphi \circ db = \delta(qb) - q\delta b = b\delta q$.

X is an E-module, and hence E-torsin free. Thus $\varphi'(\bar{d}_0 q) = \delta q$.

$$\varphi' d'(q \otimes c) = \delta q\ \varphi(c) + q\delta\varphi(c) = \delta(q\varphi(c)) = \delta\ \varphi'(q \otimes c)$$

i.e. $\varphi' d' = \delta \circ \varphi'$.

5) $\varphi = \varphi' \circ \pi_C$, since $\varphi' \circ \pi_C(c) = \varphi'(1 \otimes c) = \varphi(c)$.

By (1) ~ (5), $\varphi' : (C', d') \longrightarrow (X, \delta)$ is a complex homomorphism such that $\varphi = \varphi' \circ \pi_C$.

6) For the uniqueness of φ', let $\bar{\varphi} : (C', d') \longrightarrow (X, \delta)$ be an arbitrary complex homomorphism such that $\varphi = \bar{\varphi} \circ \pi_C$, then

$\bar{\varphi}(q \otimes c) = q\,\bar{\varphi}(1 \otimes c) = q\,\varphi(c) = \varphi'(q \otimes c)$

Hence φ' is unique such that $\varphi = \varphi' \circ \pi_C$.

This completes the proof.

Corollary 1: Let E be a fractional extension of A. Then $U(E/R) \cong E \otimes_A U(A/R)$

Corollary 2: Let E be a fractional extension of R. Then the universal E-complex $(U(E/R),d)$ is trivial.

Proof: By Corollary 1, $U(E/R) \cong E \otimes_R U(R/R)$. But $U(R/R) \cong R$ and hence $U(E/R) \cong E \otimes_R R \cong E$. Thus $U(E/R)$ is trivial.

Corollary 3. Let E be a fractional extension of A. Then $U(E/R) = U(E/A) \otimes_A U(A/R)$.

Proof: Since E is a fractional extension of A, $U(E/A) \cong E$ by Corollary 2. Then the proof is immediate from Corollary 1.

The above now proves:

Theorem 1: For the natural injection $\Phi : A \longrightarrow E$, the functor T_Φ is explicitly given by $T_\Phi((C,d)) = (E \otimes_A C, d')$, d' as defined in Proposition 3 ($d' = \tilde{d}$), and $T_\Phi(\varphi) = i_E \otimes \varphi$ for any A-complex homomorphism $\varphi : (C,d) \longrightarrow (D,\delta)$.

In case $E = A[X]$, it is very easy to see that the covariant functor $T_\Phi : \mathcal{C}(A) \longrightarrow \mathcal{C}(A[X])$ where $\Phi : A \longrightarrow A[X]$ the natural injection is <u>not</u> onto. But when E is a fractional extension of A, we have

<u>Theorem 2:</u> $T_{\Phi} : \mathcal{C}_c(A) \longrightarrow \mathcal{C}(E)$ is onto.

<u>Proof</u>: For any E-complex (X,δ), let $C = A + \sum_{n \geq 1} X_n$ and $d = \delta|C$. Then C is an anit-commutative graded algebra with $C_o = A$, and $d : C \longrightarrow C$ is a homogeneous R-derivation of degree 1 with $d \circ d = 0$. Hence (C,d) is an A-complex.

Now, consider (C',d') and show that $(C',d') = (X,\delta)$.

$C' = E \otimes_A C = E \otimes_A A + \sum_{n \geq 1} E \otimes_A X_n = E + \sum_{n \geq 1} X_n$, since X_n are E-modules and hence by Proposition 1, $X_n \cong E \otimes_A X_n$. Hence $C' = X$.

$d'|A = d_o = \delta|A$, and hence $d_o' = \delta|E$ by Proposition 2, and $d' = \delta$ by Proposition 3. Thus, $T \quad ((C,d)) = (x,\delta)$.

Finally, for any complex homomorphism $\varphi: (X,\delta) \longrightarrow (Y,\partial)$ where (Y,∂) also is an E-complex. Then $\varphi|C : A + \sum_{n \geq 1} X_n \longrightarrow A + \sum_{n \geq 1} Y_n$ is again a complex homomorphism and $T_{\Phi}(\varphi|C) = \varphi$.

Therefore T_{Φ} is onto.

CHAPTER II

Integral differential Forms

In this chapter we establish that in the context considered here, the module of Kähler's differential forms of degree k [7] is isomorphic to the module of homogeneous differential forms of degree k as defined in Chapter 0. We then introduce integral differential forms in a manner analogous to, but more general than Kähler's definition of integral differential forms in [7] and show that the set of all homogeneous integral differential forms of degree k is, in certain special cases, a finitely generated module over the ground ring.

§ 1. Preliminaries.

This section deals with the rather special results concerning valuation rings which are needed in section 3.

Proposition 1: If a valuation ring S in a field K is noetherian, then (1) the only maximal ideal $\mathfrak{m}$ of S is a principal ideal, (2) any non-zero ideal of S is a power of $\mathfrak{m}$. (Convention : $\mathfrak{m}^0 = S$).

Proof: (1) Suppose $\mathfrak{m}$ is not a principal ideal i.e.

$\mathfrak{M} = Sx_1 + \dots + Sx_m$ for $x_1, \dots, x_m \in S$ and m is the smallest possible positive integer for which this holds. Let us consider $x_1 x_2^{-1}$ and $x_2\, x_1^{-1}$, then by the definition of valuation rings, at least one of them belongs to S, say $x_1\, x_2^{-1} \in S$, and $Sx_1 = S(x_1 x_2^{-1})x_2 \subseteq Sx_2$. Hence $\mathfrak{M} = Sx_2 + \dots + Sx_m$ which contradicts the choice of m. Thus $\mathfrak{M}$ is a principal ideal.

(2) Let $\mathfrak{A}$ be a non-zero ideal of S different from S, then $\mathfrak{A} \subseteq \mathfrak{M}$. It is well known that $\bigcap_{n=1}^{\infty} \mathfrak{M}^n = 0$. And since $\mathfrak{A} \neq 0$, there exists a natural number α such that $\mathfrak{A} \subseteq \mathfrak{M}^\alpha$ but $\mathfrak{A} \nsubseteq \mathfrak{M}^{\alpha+1}$. Our claim is $\mathfrak{A} = \mathfrak{M}^\alpha$. By (1), $\mathfrak{M} = S\,x$ for some $x \in S$ and hence $\mathfrak{M}^\alpha = S\,x^\alpha$. Since $\mathfrak{A} \subseteq \mathfrak{M}^\alpha = Sx^\alpha$, $\mathfrak{A}x^{-\alpha} \subseteq S$ or more explicitly $\mathfrak{A}x^{-\alpha}$ is an ideal of S. But there exists $a \in \mathfrak{A}$ with $a\,x^{-\alpha} \notin \mathfrak{M}$, for otherwise $\mathfrak{A} \subseteq \mathfrak{M}^{\alpha+1}$. Since $\mathfrak{M}$ is the only maximal ideal and $\mathfrak{A}\,x^{-\alpha}$ is an ideal which is not contained in $\mathfrak{M}$, $\mathfrak{A}\,x^{-\alpha} = S$. Thus $\mathfrak{A} = Sx^{-\alpha} = \mathfrak{M}^\alpha$. This completes the proof.

<u>Proposition 2</u>: Let K_0 be a field, K an overfield of K_0, S_0 a valuation ring in K_0 and N be the set of all valuation rings S in K which are extensions of S_0, then the integral closure $\overline{S}_0$ of S_0 in K is $\bigcap_{S \in N} S$.

<u>Proof</u>: $\overline{S}_0 \subseteq \bigcap_{S \in N} S$ is clear from Theorem 1, § 2, Ch. 0. Hence it is sufficient to show that every valuation ring V in K which contains S_0 contains some member of N as a subset. Let $\mathfrak{N}$ be the maximal ideal in V, and let $\mathfrak{P} = \mathfrak{N} \cap S_0$, then $\mathfrak{P} \subseteq \mathfrak{M}_0$, since $\mathfrak{N} \not\ni 1$ and $\mathfrak{P}$ is an ideal of S_0. By Proposition 6, § 5, Ch. 0, there exists a valuation ring S in K such that $S \supseteq S_0$, $\mathfrak{M} \cap S = \mathfrak{M}_0$, and

$S \subseteq V$ where $\mathfrak{m}$ is the maximal ideal in S. Thus S is an extension of S_o, and hence a member of N contained in V. This completes the proof.

<u>Proposition 3</u>: Let G be an integral domain. K_o its field of quotients, K an algebraic extension of K_o, and S_o a valuation ring of K_o such that it is a ring of quotients of G. Then any valuation ring S of K which is an extension of S_o is a ring of quotients of the integral closure $\overline{G}$ of G in K with respect to $\mathfrak{m} \cap \overline{G}$, where $\mathfrak{m}$ is the maximal ideal of S.

<u>Proof</u>: It is clear that the ring of quotients in question is contained in S_o. Now let α be any non-zero element of S and let $a_o\alpha_n + a_1\alpha^{n-1} + \dots + a_n = 0$, $a_1 \in K_o$, $a_o \neq 0$, be the maximal equation of α over K_o. By Proposition 3, § 5 Ch. 0, for at least one j, $1 \leq j \leq m$, $a_i a_j^{-1} \in S_o$ for $i = 1, 2, \dots n$. Pick one of these j's, and if we set $b_i \doteq a_i a_j^{-1}$ for each i, then we have $b_o\alpha^n + b_1\alpha^{n-1} + \dots + b_n = 0$, $b_j \in S_o$ for $i = 1, 2, \dots n$. Since S_o is a ring of quotients of G, there exists an element $b \in G$ with $b \notin \mathfrak{m}$ such that $b\, b_i \in G$ for $i = 1, 2, \dots, n$. If we set $c_i = b\, b_i$ then we have $c_o\alpha^n + c_1\alpha^{n-1} + \dots + c_n = 0$, $c_i \in G \subseteq \overline{G}$. To apply Proposition 8, §5, Ch. 0, we notice that $\overline{G}$ is an integrally closed domain, K a field of quotients of $\overline{G}$, $\mathfrak{m} \cap G$ a prime ideal of $\overline{G}$ and $c_i \in \overline{G}$ for $i = 1, 2, \dots, n$ with $c_j = b \notin \mathfrak{m} \cap \overline{G}$. By Proposition 8, §5, Ch. 0, α or α^{-1} belongs to Q, the ring of quotients in question. To show $Q \supseteq S_o$, suppose $\alpha \notin Q$, then $\alpha^{-1} \in Q$ and is a non-unit in Q which implies $\alpha^{-1} \in \mathfrak{m}$ and hence $\alpha \notin S$. This is impossible, since our assumption was $\alpha \in S$. Thus $\alpha \in Q$. This completes the proof.

Let G be an integral domain, K_0 its field of quotients, and $\mathfrak{P}$ a prime ideal of G such that (1) all powers $\mathfrak{P}, \mathfrak{P}^2, \ldots$ are distinct, and $\bigcap_{\alpha=1}^{\infty} \mathfrak{P}^{\alpha} = 0$, (2) if an element a is divisible exactly by $\mathfrak{P}^{\alpha}$, i.e. $a \in \mathfrak{P}^{\alpha}$ but $a \notin \mathfrak{P}^{\alpha+1}$, and similarly, if b is divisible exactly by $\mathfrak{P}^{\beta}$, then ab is divisible exactly by $\mathfrak{P}^{\alpha+\beta}$.

By the condition (1), for any non- zero element $a \in G$, there exists a non-negative integer α such that $a \in \mathfrak{P}^{\alpha}$ but $a \notin \mathfrak{P}^{\alpha+1}$ (Convention : $\mathfrak{P}^0 = G$). Let us define a mapping $\varphi : G \longrightarrow Z$ (the ring of integers) by $\varphi(0) = 0$, $\varphi(a) = \alpha$ for non-zero $a \in G$. Then for non-zero $a,b \in G$, $\varphi(ab) = \varphi(a) + \varphi(b)$ by the condition (2).

Define a mapping $\psi : K_0 \longrightarrow Z$ by $\psi(0) = 0$, $\psi(a/b) = \varphi(a) - \varphi(b)$ for non-zero element $a/b \in K_0$, $a,b \in G$, then ψ is well defined, since $a/b = c/d$ implies $ad = bc$ and hence $\varphi(a) + \varphi(d) = \varphi(b) + \varphi(c)$ or $\varphi(a) - \varphi(b) = \varphi(c) - \varphi(d)$, and thus $\psi(a/b) = \psi(c/d)$. It is clear that ψ is an extension of φ to K_0, since $\varphi(1) = 0$.

Let $S = \{x \mid x \in K_0,\ \psi(x) \geq 0\}$, then S is a valuation ring in K_0, since for $x \in K_0 \setminus S$, $x = a/b$, $a,b \in G$ with $\psi(x) = \varphi(a) - \varphi(b) < 0$, and $\psi(x^{-1}) = \psi(b/a) = \varphi(b) - \varphi(a) > 0$, and hence $x^{-1} \in S$.

<u>Proposition 4</u>: If $\mathfrak{P}$ is a principal ideal, then $S = G_{\mathfrak{P}}$, the ring of quotients of G with respect to $\mathfrak{P}$. Hence $G_{\mathfrak{P}}$ is a valuation ring in K_0.

<u>Proof</u>: Let $x \in G_{\mathfrak{P}}$, then $x = a/b$, $a,b \in G$, $b \notin \mathfrak{P}$ (or $\varphi(b) = 0$). Hence $\psi(x) = \varphi(a) - \varphi(b) \geq 0$ or $x \in S$. Thus $S \supseteq G$.

Conversely, let $x \in S$, then $x = a/b$ with $\varphi(a) - \varphi(b) \geq 0$. Since $\mathfrak{P}$ is a principal ideal, $\mathfrak{P} = Gp$ for some element $p \in G$, and hence $a = cp^{\alpha}$, $b = dp^{\beta}$ for $c,d \in G$, with $p \nmid c$, $p \nmid d$ and $\alpha - \beta \geq 0$. Hence $x = a/b = (c/d)p^{\alpha-\beta} \in G_{\mathfrak{P}}$, since $c/d \in G$ and $p^{\alpha-\beta} \in G \subseteq G$. Thus $S \subseteq G$.

Corollary 1: Let G be a unique factorization domain or a noetherian domain, then each prime element p in G determines a valuation ring S_p in K_0 which is a ring of quotient of G and $\mathfrak{m} = S_p \cdot p$. (By a prime element p, one means an element such that $p|ab$ implies $p|a$ or $p|b$, for elements $a,b \in G$).

Proof: Let $\mathfrak{p} = Gp$, then $\mathfrak{p}$ is a prime ideal. Clearly $\mathfrak{p}$ satisfies the conditions (1) and (2). Since $\mathfrak{p}$ is a principal ideal, by Proposition 4, $G_{\mathfrak{p}}$ is a valuation ring in K_0. Put $S_p = G_p$, then obviously the only maximal ideal $\mathfrak{m}$ of S_p is $S_p \cdot p$.

Proposition 5: Let G be a unique factorization domain, K_0 a field of quotients of G, and S_p the valuation ring in K_0 such that S_p is the ring of quotients of G with respect to the prime ideal G_p of G, for each prime element p of G. Then G is the intersection of all S_p.

Proof: For any $x \in S_p$, let $x = \frac{u}{v}$ where $u,v \in G$ and u,v are relatively prime. Then $p \nmid v$. Hence for any element $x \in \bigcap_{p \text{ prime}} S_p$, if $x = \frac{u}{v}$ where $u,v \in G$ and u,v are relatively prime, then there is no prime element of G which divides v, i.e. v is an invertible element in G. This implies $x \in G$ and hence $G \supseteq \bigcap S_p$. Since each S_p is a ring of quotients of G, $G \subseteq \bigcap S_p$. Thus $G = \bigcap_{p \text{ prime}} S_p$.

Corollary 2: (1) A unique factorization domain is integrally closed. (Well known fact). (2) Let G be a unique factorization domain, and K_0 a field of quotients of G. Then G is the intersection of all valuation rings in K_0 containing G which are rings of quotients of G.

Proof: (1) Immediate consequence of Theorem 1, §5, Chapter 0 and Proposition 5. (2) Immediate consequence of Theorem 1, §5, Chapter 0 Proposition 5 and (1).

§ 2. Kähler's Differential forms.

This section is mainly to give the definition of Kähler's differential forms [7] and to explain the relation between these and the algebra of differential forms [Ch. 0] in this context. Kähler's differential forms are defined to be a certain subset of his universal infinitesimal ring [7], hence we will begin this section with the study of a slightly generalized form of Kähler's infinitesimal rings, which will be called infinitesimal algebras.

Let R be a commutative ring with unit, and A a unitary commutative R-algebra.

Definition 1: A couple (I,σ) is called an infinitesimal algebra over A if I is a commutative R-algebra containing A as a unitary subalgebra and $\sigma = (\sigma_i)_{i \in N}$, $N = \{1, 2, \ldots\}$, where each $\sigma_i : A \longrightarrow I$ is an R-algebra homomorphism such that $(\sigma_i(a) - a)(\sigma_i(b) - b) = 0$, $a,b \in A$.

The following remark tells exactly what are Kähler's infinitesimal rings.

Remark 1: Kähler's infinitesimal ring is an infinitesimal algebra in the preceding sense if 1) R is the ring of integers,

2) each $\sigma_i : A \longrightarrow I$ is an R-algebra monomorphism such that $(\sigma_i(a) - a)(\sigma_i(b) - b) = 0$, $a,b \in A$, for each $i \in N$, and 3) I is generated by $\sum_{i \in N} \sigma_i(A)$ over A.

Proposition 1: Let (I,σ) be an infinitesimal algebra over A. If we define a mapping $d_i : A \longrightarrow I$ by $d_i = \sigma_i - i_A$, where i_A is the identity mapping on A, then d_i is an R-derivation such that $d_i a\, d_i b = 0$, $a,b \in A$, for each $i \in N$.

Proof: $d_i a\, d_i b = (\sigma_i(a) - a)(\sigma_i(b) - b) = 0$, $a,b \in A$, from the definition of infinitesimal algebras.

d_i is R-linear, since $d_i(ra + sb) = \sigma_i(ra + sb) - (ra + sb) = r(\sigma_i(a) - a) + s(\sigma_i(b) - b) = rd_i a - sd_i b$, $r,s \in R$, $a,b \in A$.

d_i satisfies the product rule, since $d_i(ab) = \sigma_i(ab) - ab = \sigma_i(a)\,\sigma_i(b) - ab = a(\sigma_i(b) - b) + b(\sigma_i(a) - a) + (\sigma_i(a) - a)(\sigma_i(b) - b) = ad_i(b) + bd_i(a)$.

Proposition 2: Suppose I is a unitary commutative R-algebra containing A as a unitary subalgebra such that there exists R-derivations $d_i : A \longrightarrow I$ with $d_i a\, d_i b = 0$ for each $i \in N$. Let $\sigma_i : A \longrightarrow I$ be the mapping defined by $\sigma_i = d_i + i_A$, where i_A is the identity on A. Then (I,σ), $\sigma = (\sigma_i)_{i \in N}$, is an infinitesimal algebra over A.

Proof: Each σ_i is an R-algebra homomorphism such that $(\sigma_i(a) - a)(\sigma_i(b) - b) = 0$. For, σ_i is R-linear, since so is d_i and i_A, and $\sigma_i(ab) = d_i(ab) + ab = ad_i b + bd_i a + ab = d_i a\, d_i b + ad_i b + bd_i a + ab$ $(d_i a\, d_i b = 0) = (d_i a + a)(d_i b + b) = \sigma_i(a)\,\sigma_i(b)$, $a,b \in A$, and hence σ_i is an R-algebra homomorphism for each $i \in N$.

$(\sigma_i(a) - a)(\sigma_i(b) - b) = d_i a\, d_i b = 0, \quad a,b \in A.$

Thus, (I,σ) is an infinitesimal algebra over A.

Examples of infinitesimal algebras over A.

1) (A,σ), $\sigma = (\sigma_i)_{i \in N}$ where $\sigma_i = i_A$ for each $i \in N$.

2) Let M be an A-module, $\delta : A \longrightarrow M$ is a derivation, A as an R-algebra. Consider the external sum $A \oplus M$ and define the multiplication by $(a,x)(b,y) = (ab, ay + bx)$, $a,b \in A$, $x,y \in M$. The mapping $A \longrightarrow A \oplus M$ defined by $a \rightsquigarrow (a,o)$, $a \in A$ is an R-algebra monomorphism and hence we can put $(a,o) = a$, $a \in A$. Then $A \oplus M$ is an associative R-algebra containing A. Let $\delta_i : A \longrightarrow A \oplus M$ be a mapping defined by $\delta_i(a) = (o,o)$ for each i, $i \neq j$, $d_j(a) = (o,\delta a)$ for a fixed $j \in N$, then each δ_i, $i \in N$ is an R-derivation with $\delta_i a\, d_i b = 0$. Put $\sigma_i = \delta_i + i_A$, then $(A \oplus M, \sigma)$, $\sigma = (\sigma_i)_{i \in N}$ is an infinitesimal algebra over A.

Definition 2: Let (I,σ) and (J,τ) be two infinitesimal algebras. An R-algebra homomorphism $f : I \longrightarrow J$ is called an infinitesimal algebra homomorphism if $f|A$ is the identity on A and $f\,\sigma_i = \tau_i$ for all $i \in N$. An infinitesimal algebra homomorphism is denoted by $f : (I,\sigma) \longrightarrow (J,\tau)$.

An infinitesimal algebra homomorphism which is one-to-one and onto is called an infinitesimal algebra isomorphism.

Definition 3: An infinitesimal algebra (W,θ) over A is called a universal infinitesimal algebra if for any infinitesimal

(I,σ) over A, there exists a unique infinitesimal algebra homomorphism $f : (W,\theta) \longrightarrow (I,\sigma)$.

Remark 2: It can be shown in the usual way that universal infinitesimal algebras over A if they exist are uniquely determined up to infinitesimal algebra isomorphisms.

Proposition 3: Let (I,σ) be an infinitesimal algebra over A such that I is generated by $\sum_{i \in N} \sigma_i(A)$ over A. If there exists an infinitesimal algebra homomorphism of (I,σ) into an arbitrary infinitesimal algebra over A, then it is unique.

Proof: Let (J,τ) be an arbitrary infinitesimal algebra over A, and $f,g : (I,\sigma) \longrightarrow (J,\tau)$ be two infinitesimal algebra homomorphisms. $f|A$ = identity on $A = g|A$, and $f\circ\sigma_i = \tau_i = g\circ\sigma_i$. This means that f and g coincide on the algebra A and the set of generators of I, and hence f and g coincide on the whole I.

Proposition 4: Let (I,σ) be an infinitesimal algebra over A, and d_i is the derivation defined by $d_i = \sigma_i - i_A$ for each $i \in A$. Then

$$[\, A,\sigma_1(A),\sigma_2(A),\ldots\,] = [\, A,d_1A,d_2A,\ldots\,]$$

where the left hand side and the right hand side are the subalgebra of I generated by $\sigma_1(A),\sigma_2(A),\ldots$, and $d_1(A),d_2(A),\ldots$, over A.

Proof: Trivial, since $d_ia = \sigma_i(a) - a \in [\, A_1\sigma_1(A),\sigma_2(A),\ldots\,]$ and $\sigma_i(a) = d_ia + a \in [\, A,d_1A,d_2A,\ldots\,]$, $a \in A$.

Proposition 5: Let (W,θ) be a universal infinitesimal algebra over A. Then W is generated by $\sum_{i \in N} \theta_i(A)$ over A.

<u>Proof</u>: Let W' be the subalgebra of W generated by $\sum_{i \in N} \theta_i(A)$ over A, then (W', θ) is also an infinitesimal algebra over A. Since (W, θ) is universal, there exists a unique infinitesimal algebra homomorphism $f : (W, \theta) \longrightarrow (W', \theta)$. We may consider f as an infinitesimal algebra homomorphism of (W, θ) into itself. But the identity mapping of W into itself is also a such mapping, and hence by the uniqueness of the infinitesinal algebra homomorphism of a universal infinitesimal algebra, $(W', \theta) = (W, \theta)$. Thus $W = W'$ and W is generated by $\sum_{i \in N} \theta_i(A)$ over A.

The following is an internal characterization of a universal infinitesimal algebra over A.

<u>Theorem 1</u>: Suppose (W, θ) is an infinitesimal algebra over A. For each $i \in N$, put $d_i = \theta_i - i_A$ and $U_i = Ad_iA$. Then (W, θ) is universal if and only if

1) each (U_i, d_i) is a universal derivation module of A as an R-algebra.

2) for any commutative R-algebra C containing A as a unitary subalgebra and any family $\{f_i | f_i : U_i \longrightarrow C, i \in N, \text{ with } (f_i(U_i))^2 = 0\}$ of A-module homomorphisms, there exists a unique algebra homomorphism $f : W \longrightarrow C$ such that $f|A$ is the identity on A and f extends each module homomorphism f_i.

<u>Proof</u>: Suppose (W, θ) is universal. To show each (U_i, d_i), $i \in N$ is a universal derivation module of A as an R-algebra, let

(M,δ) be an arbitrary derivation module of A. As in Example 2), for a fixed j in N, construct an infinitesimal algebra $(A \oplus M, \sigma)$, $\sigma = (\sigma_i)_{i \in N}$, $\sigma_i = \delta_i + i_A$, $\delta_i(a) = 0$ for all $i \neq j$ and $d_j(a) = (o, \delta a)$, $a \in A$. Since (W,θ) is universal, there exists a unique algebra homomorphism $f : W \longrightarrow A \oplus M$ over A such that $f \circ \theta_i = \sigma_i$ for each $i \in N$. Put $f_j = f|U_j$, then f_j is naturally an A-module homomorphism. Let $p : A \oplus M \longrightarrow M$ be the 2nd projection, and put $g_j = p \circ f_j$ then $g_j : U_j \longrightarrow M$ is clearly an A-module homomorphism. Moreover, $g_j \circ d_j = p \circ f_j \circ d_j = p \circ f \circ (\theta - i_A) = p \circ f \circ \theta_j + p \circ f \circ i_A = p \circ \sigma_j + p \circ i_A = p(\sigma_j - i_A) = p \circ \delta_j = \delta$.

Hence $g_j : (U_j, d_j) \longrightarrow (M, \delta)$ is a derivation module homomorphism. The uniqueness of g_j can be easily checked. Thus $(U_i d_i)$ is a universal derivation module of A as an R-algebra. Next, we will show 2). Put $\sigma_i = f_i \circ d_i + i_A$ for each $i \in N$, then $(\sigma_i(a) - a)(\sigma_i(b) - b) = (f_i \circ d_i(a))(f_i \circ d_i(b)) \in (f_i(U_i))^2 = 0$. Hence $(\sigma_i(a) - a)(\sigma_i(b) - b) = 0$. Moreover, $\sigma_i : A \longrightarrow C$ is an algebra homomorphism, since $\sigma_i(ab) = f_i \circ d_i(ab) + ab = f_i(ad_i b + bd_i a) + ab = af_i \circ d_i(b) + bf_i \circ d_i(a) + ab = f_i \circ d_i(a)\ f_i \circ d_i(b) + af_i \circ d_i b + bf_i \circ d_i(a) + ab$ (since $f_i\ d_i(a)\ f_i \circ d_i(b) = 0$) $= (f_i \circ d_i(a) + a)(f_i \circ d_i(b) + b) = \sigma_i(a)\sigma_i(b)$.

Thus (C,σ) is an infiniteseimal algebra over A.

Since (W,θ) is universal, there exists a unique algebra homomorphism $f : W \longrightarrow C$ such that $f|A = i_A$ and $f \circ \theta_i = \sigma_i$ for each $i \in N$. This f is an extension of each $f_i, i \in N$, since for an arbitrary element $\Sigma\, ad_i b \in U_i$, $f(\Sigma\, ad_i b) = \Sigma\, af \circ d_i(b) = \Sigma\, af(\theta_i(b) - b) = \Sigma\, a(\sigma_i(b) - b) = \Sigma\, af_i \circ d_i(b) = f_i(\Sigma\, ad_i b)$ for each $i \in N$.

f is unique, since W is generated by $\sum_{i \in N} U_i$ over A. Conversely, suppose

(W,θ) is an infinitesimal algebra over A with the properties 1) and 2). To show the universality of (W,θ), let (I,σ) be an arbitrary infinitesimal algebra over A. Put for each $i \in N$, $\delta_i = \sigma_i - i_A$, then $\delta_i : A \longrightarrow I$ is a derivation. Since (U_i,δ_i), by 1), is a universal derivation module of A, there exists a unique A-module homomorphism $f_i : U_i \longrightarrow I$ such that $f_i \circ d_i = \delta_i$. Moreover, $(f_i(Ad_iA))^2 = A(f_i \circ d_i(A))^2 = A(\delta_i A)^2 = 0$. By 2), there exists a unique algebra homomorphism $f : W \longrightarrow C$ such that $f|A$ is the identity mapping on A and f extends each $f_i, i \in N$. The algebra homomorphism $f : W \longrightarrow C$ is an infinitiesimal algebra homomorphism, since $f \circ \theta_i = f \circ (d_i + i_A) = \delta_i + i_A = \sigma_i$. To show the uniqueness of $f : (W,\theta) \longrightarrow (I,\sigma)$, we will first show that W is generated by $\sum_{i \in N} U_i$ over A. For, let W' be the subalgebra of W generated by $\sum_{i \in N} U_i$. Then considering the identity mapping $i_{U_i} : U_i \longrightarrow W'$ for each $i \in N$, by 2) we see that for the family $(i_{U_i})_{i \in N}$ of A-module homomorphisms, there exists a unique algebra homomorphism $g : W \longrightarrow W' \subseteq W$ such that $g|W = i_A$ and g extends each i_{U_i}, $i \in N$. But the identity mapping on W is also such a mapping, hence by the uniqueness of such algebra homomorphism $W = W'$. Hence W is generated by $\sum_{i \in N} U_i$ over A. f is a unique infinitesimal algebra homomorphism by Propositions 3 and 4. This completes the proof.

<u>Corollary 1</u>: Let W be a commutative R-algebra containing A as a unitary subalgebra such that

1) there exists an R-derivation $d_i : A \longrightarrow W$ for each $i \in N$ with $U_i^2 = 0$, where $U_i = Ad_iA$.

2) each $(U_i d_i)$ is a universal derivation module of A as an R-algebra, and

3) for any commutative R-algebra C containing A as a unitary subalgebra and any family $\{f_i | f_i : U_i \longrightarrow C, i \in N \text{ with } (f_i(U_i))^2 = 0\}$ of A-module homomorphisms, there exists a unique algebra homomorphism $f : W \longrightarrow C$ such that $f|A$ is the identity in A, and f extends each module homomorphism f_i.

Then $(W, \theta), \theta = (\theta_i)_{i \in N}$ where each $\theta_i = d_i + i_A$, is a universal infinitesimal algebra over A.

<u>Proof</u>: Immediate consequence of Proposition 2 and Theorem 1.

<u>Construction</u> of a universal infinitesimal algebra over A:

Suppose (U,d) is a universal derivation module of A as an R-algebra. Consider $V = \bigoplus_{i \in N} V_i$, external sum of V_i, where $V_i = U$ for all $i \in N$. Put $U_i = \{v | v \in V, v(j) = 0 \text{ for all } j, j \neq i\}$, then $U_i \cong V_i$ and $V = \Sigma U_i$ (direct). Let S(V) be a symmetric algebra of V and put $W = S(V)/J$ where J is the ideal of S(V) generated by ΣU_i^2. $A \cap J = 0$ and $V \cap J = 0$ clearly, and this implies that A and V can be imbedded into W by the natural homomorphism $\nu : S(V) \longrightarrow W$. Hence we may consider that W contains A and V by indentifying $a = \nu(a)$, $a \in A$, and $v = \nu(v)$, $v \in V$. Each U_i is a submodule of V and by the above identification, $W_i \subseteq W$ for each $i \in N$.

Let us define $d_i : A \longrightarrow U_i (U_i \subseteq V \subseteq W)$, for each $i \in N$, by

$$d_i(a)(j) = \begin{cases} da & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$$

then d_i is a derivation and $(U_i, d_i) \cong (U,d)$ as a derivation module.

Let $\theta_i = d_i + i_A$ for each $i \in N$ and $\theta = (\theta_i)_{i \in N}$.

<u>Theorem 2</u>: (W, θ), thus obtained, is a universal infinitesimal algebra over A.

<u>Proof</u>: It is sufficient to prove those conditions 1), 2), 3) in Corollary 1. It is clear from the preceding construction that each (U_i, d_i) is a universal derivation module of A as an R-algebra and $U_i = Ad_iA$. $U_i^2 = 0$ in W, since U_i^2 considered to be in S(V) is contained in the ideal J. Thus 1) and 2) are proved. To prove 3), let C be an arbitrary commutative R-algebra containing A as a unitary subalgebra and $f_i : U_i \longrightarrow C$ be any A-module homomorphism such that $(f_i(U_i))^2 = 0$, for each $i \in N$. Since $V = \Sigma U_i$ (direct), there exists a unique homomorphism $g : V \longrightarrow C$ which is an extension of each f_i, $i \in N$. By the property of a symmetric algebra, there exists a unique algebra homomorphism $h : S(V) \longrightarrow C$ such that $g = h|V$. In this case $h(J) = 0$, since $h(U_i^2) = (h(U_i))^2 = (f_i(U_i))^2 = 0$ for each $i \in N$. This implies that there exists a unique algebra homomorphism $f : W \longrightarrow C$ such that $f \circ \nu = h$. The algebra homomorphism $f : W \longrightarrow C$ is an extension of each f_i, since $f|U_i = f \circ \nu|U_i = h|U_i = g|U_i = f_i$. Similarly, $f|A$ is the identity on A. The uniqueness of such algebra homomorphisms as f is clear, since W is generated by ΣU_i as an algebra over A. Hence 3). This completes the proof of the theorem.

We now apply the results obtained in this section, together with results of §4 in Chapter 0, to Kähler's differential forms.

Let (W, θ) be a universal infinitesimal algebra over A. From the construction of a universal infinitesimal algebra, we can easily see that W is an R-algebra generated by $\bigcup_{i \in N} d_i A$ over A, where $d_i : A \longrightarrow W$ is a derivation defined by $d_i = \theta_i - i_A$ for each $i \in N$.

Consider the A-submodule $Ad_{\nu_1}A \ldots d_{\nu_k}A$, $\nu_i \in N$, of W. ($Ad_{\nu_1}A \ldots d_{\nu_k}A = U_{\nu_1} \ldots U_{\nu_k}$ trivially).

Definition 4: An element in $Ad_{\nu_1}A \ldots d_{\nu_k}A$, is called a homogeneous infinitesimal of type $(\nu_1, \ldots, \nu_k)$ and of degree k.

Definitino 5: Kähler's differential forms of degree k are the homogeneous infinitesimals of type (1, 2, ... , k) and of degree k, or equivalently, the elements of $Ad_1A \ldots d_kA$.

Remark 3: The preceding Definition 5 is also a generalized definition of Kähler's differential forms in [7]. Kähler considered only the case when R is the ring of integers or a prime field of characteristic $p \neq 0$, and A is a finitely generated separable extension field of the field of quotients of R.

Recall that (U,d) denotes a universal derivation module of A, $T_k(U)$ the A-module of all homogeneous elements of degree k of a tensor algebra T(U) of U.

Theorem 3: If U is a projective A-module, then $T_k(U) \cong Ad_1A \ldots d_kA$.

Proof: Put $U_i = Ad_iA$ and $V = \sum_{i \in N} U_i$, then by the construction of a universal infinitesimal algebra, $V = \Sigma U_i$ (direct). Let W_i be the subalgebra of W generated by U_i for each i, then for any commutative algebra C containing A, any algebra homomorphism $g_i : W_i \longrightarrow C$ over A has the property $(g_i(U_i))^2 = 0$, since $(g_i(U_i))^2 = g_i(U_i^2) = 0$. (recall $U_i^2 = 0$ in W_i). Let $(g_i)_{i \in N}$ be a family of algebra homomorphisms over A where $g_i : W_i \longrightarrow C$ for each i, and put $f_i = g_i | U_i$, then $(f_i)_{i \in N}$ is a family of A-module homomorphisms where $f_i : U_i \longrightarrow C$, for each i such that $(f_i(U_i))^2 = (g_i(U_i))^2 = 0$. By Theorem 1, there exists a unique algebra homomorphism $f : W \longrightarrow C$ over A extending each A-module homomorphism f_i. Moreover $f|W_i = g_i$, since f and g_i coincide on U_i and U_i generates W_i. Hence W is a free commutative join of the family $(W_i)_{i \in N}$ of subalgebras. Since each U_i, $i \in N$, is a projective A-module and $V = \Sigma U_i$ (direct), $U_1 \otimes \ldots \otimes U_k = U_1 \ldots U_k$(in W) $= Ad_1A \ldots d_kA$ (cf. Theorem 6, 4, Chapter 0). By Theorem 1, we know that each $U_i = U$ and hence $T_k(U) = U_1 \times \ldots \times U_k$. Thus $T_k(U) = Ad_1A \ldots d_kA$.

Theorem 4: If U is a finitely generated projective A-module, then the A-module of all homogeneous differential forms of degree k is isomorphic to the A-module of all Kähler's differential forms of degree k.

Proof: Let K(D) be the algebra of differential forms of A, and $K_k(D)$ the A-module of all homogeneous differential forms of degree k [cf. §2, Chapter 0]. Since U is a finitely generated projective A-module, $K_k(D) \cong T_k(U)$ [cf. Theorem 5, §2, Chapter 0]. Hence by the previous theorem, $K_k(D) = Ad_1A \ldots d_kA$.

§ 3. Integral Differential Forms.

Let R be a commutative ring with unit, K a unitary commutative R-algebra, D the K-module of all derivations of K, K(D) the algebra of all multilinear forms on D. K(D) is a regularly graded algebra and hence $K(D) = \sum_n K_n(D)$ (direct). An element in K(D) is called a <u>differential form</u> and an element in $K_k(D)$ a <u>homogeneous differential form</u> of degree k. Let $D^* = \mathrm{Hom}_k(D,K)$ and we can put $D^* = K_1(D)$, since D^* is naturally imbedded into K(D), onto $K_1(D)$. If we define $d : K \longrightarrow D^*$ by $d(a)(\delta) = \delta a$ for all $a \in K$, $\delta \in D$, then d is also a derivation (cf. § 2, Chapter 0).

<u>Definition</u>: If R is an integral domain, K a field containing R, an element $x \in K(D)$ is called an <u>integral differential form</u> if and only if $x \in \sum_n S(dS)^n$ for all valuation rings S in K such that $S \supseteq R$, and an integral differential form in $K_k(D)$ is called a <u>homogeneous integral differential form</u> of degree k.

<u>Remark 1</u>: The homogeneous integral differential forms of degree zero are the elements of K integral over R. In fact, the set of all homogeneous differential forms of degree zero is the intersection of all valuation rings in K containing R by the preceding definition, and this is the integral closure of R in K (cf. Theorem 1, § 2, Chapter 0). Thus the integral differential forms are, in a sense, a generalization

of the integral elements in K over R.

The purpose of this section is to show that the R-module of all homogeneous integral differential forms of degree k is finitely generated if R is a noetherian unique factorization domain and K a finitely generated separable extension field of a field of quotients of R.

<u>Convention</u>: Unless otherwise specified,

(1) R will denote a noetherian domain, Q a field of quotients of R, $K_o = Q(x_1,\ldots,x_n)$ a purely transcendental extension of Q with transcendence degree n over Q, and $K = K_o(x_o)$ a separable algebraic extension of K_o with the minimal polynomial $f = t^m + a_1t^{m-1} + \ldots + a_m$ ($f \in K_o[\,t\,]$, polynomial ring over K_o with t as indeterminate) of x_o with respect to K_o. If f' is the usual derivative of f in $K_o[\,t\,]$, then $f'(x_o) \neq 0$, since x_o is separable algebraic over K_o.

(2) Let us put $G = R[\,x_1,\ldots,x_n\,]$, the subring of K_o generated by $x_1,\ldots,x_n$ over R, S_o dentoes a valuation ring in K_o containing G, S a valuation ring in K which is an extension of S_o, and let $\bar{G}$ and $\bar{S}_o$ be the integral closures of G and S_o in K respectively. $\mathfrak{m}_o$ and $\mathfrak{m}$ will denote the maximal ideals of S_o and S respectively.

(3) Finally $(U(S/R,\delta)$, $(U(\bar{G}/G),\partial)$,... etc, denotes a universal derivation module of S as R-algebra, a universal derivation module of $\bar{G}$ as G-algebra,...etc.

If (U,δ) is a universal derivation module of K as R-algebra,

$U = K\delta x_1 + \ldots + K\delta x_n$ (direct). Since U is a free K-module with a finite basis $\{\delta x_1, \ldots, \delta x_n\}$, the K-module $K_k(D)$ of all homogeneous differential forms of degree k is isomorphic to the K-module $T_k(U)$ of all homogeneous elements of degree k in a tensor algebra T(U) of U (cf. Theorem 5, §2, Chapter 0). Also, $T_k(U)$ is a free K-module with $\{\delta x_{i_1} \ldots \delta x_{i_k} | 1 \leq i_1, \ldots i_k \leq n\}$ as its basis [4]. We know that $(U,\delta) \cong (D^*,d)$ [cf. §2, Chapter 0], and since we put $D^* = K_1(D)$, any homogeneous differential form x of degree k is uniquely expressed in the following form:

$$x = \sum_{i_1,\ldots,i_k} a_{i_1 \ldots i_k} dx_{i_1} \ldots dx_{i_k}$$

where $a_{i_1 \ldots i_k} \in K$, the multiplication carried out is the Kronecker algebra of D.

The following Lemmas are needed to prove Proposition 1.

<u>Lemma 1</u>: Let S_o be a valuation ring in K_o such that S_o is a ring of quotients of G. Then the universal derivation modules $U(\bar{G}/R)$, $U(S/R)$, $U(\bar{G}/G)$ and $U(S/S_o)$ are all finitely generated modules.

<u>Proof</u>: Since R is a noetherian domain and G a finitely generated ring over R, G is also a moetherian domain. The integral closure $\bar{G}$ of G in K is a finitely generated G-module (cf. Theorem 3, § 5, Chapter 0), say $\bar{G} = Gw_1 + \ldots + Gw_m$, $w_i \in \bar{G}$ for $i = 1, 2, \ldots, m$. If $(U(\bar{G}/R),\delta)$ dentoes the universal derivation module of $\bar{G}$ as R-algebra,

$$\delta\bar{G} \subseteq \delta G \cdot w_1 + \ldots + \delta G \cdot w_m + G\delta w_1 + \ldots + G\delta w_m$$

$$\subseteq \bar{G}\delta x_1 + \dots + \bar{G}\delta x_n + \bar{G}\delta w_1 + \dots + \bar{G}\delta w_n.$$

Hence $\bar{G}\delta\bar{G} \subseteq \bar{G}\delta x_1 + \dots + \bar{G}\delta x_n + \bar{G}\delta w_1 + \dots + \bar{G}\delta w_m$.

On the other hand, $\bar{G}\delta\bar{G}$ contains the right hand side of the above inclusion, since the right hand side is a $\bar{G}$-module generated by elements in $\bar{G}\delta\bar{G}$.

Thus

$$U(\bar{G}/R) = \bar{G}\delta\bar{G}$$
$$= \bar{G}\delta x_1 + \dots + \bar{G}\delta x_n + \bar{G}\delta w_1 + \dots + \bar{G}\delta w_m.$$

Hence $U(\bar{G}/R)$ is a finitely generated $\bar{G}$-module.

Next, for $U(S/R)$, since S is, by Proposition 3, §1, Chapter II, a ring of quotients of $\bar{G}$,

$$U(S/R) = S \otimes_{\bar{G}} U(\bar{G}/R)$$
$$= S \otimes \delta x_1 + \dots + S \otimes \delta x_n + S \otimes \delta w_1 + \dots + S \otimes \delta w_m.$$

[cf. §3, Chapter I. Notice rings of quotients are fractional extensions].

Thus $U(S/R)$ is finitely generated.

Finally, $U(\bar{G}/G)$ and $U(S/S_o)$ are homomorphic images of $U(\bar{G}/R)$ and $U(S/R)$ respectively [cf. Theorem 2, §2, Chapter 0]. Homomorphic images of finitely generated modules are finitely generated.

<u>Corollary 1</u>: Let $\partial : S \longrightarrow M$ and $\delta : \bar{G} \longrightarrow N$ be arbitrary derivations of S and $\bar{G}$ as R-algebras respectively, where M and N are S-module and G-module respectively. Then the submodules SdS of M and G δG of N are also finitely generated modules.

<u>Proof</u>: S dS and G δG are homomorphic images of the universal derivation modules $U(S/R)$ and $U(G/R)$ respectively. Since $U(S/R)$ and $U(G/R)$ are finitely generated by Lemma 1, S dS and G δG are finitely generated.

Lemma 2: Let S_o be a valuation ring in K_o such that S_o is a ring of quotients of the subring G of K_o. If S is a valuation ring in K which is an extension of S_o, then

$$\mathfrak{d}^k S(dS)^k \subseteq \sum_{i_1,\ldots,i_k} Sdx_{i_1}\ldots dx_{i_k}$$

where $\mathfrak{d}$ is the different of S over S_o.

(Notice that the differents exist, since $U(S/S_o)$ is finitely generated).

Proof: We will first show that

(a) $SdS_o = Sdx_1 + \ldots + Sdx_n$.

It is well known that

$$U(G/R) = G\delta x_1 + \ldots + G\delta x_n$$

where $(U(G/R),\delta)$ is a universal derivation module of G as R-module. Since S_o is a ring of quotients of G,

$$U(S_o/R) = S_o \otimes_G U(G/R) = S_o \partial x_1 + \ldots + S_o \partial x_n$$

[cf. §3, Chapter I] where $(U(S_o/R), \partial)$ is a universal derivation module of S_o as R-algebra. The subset $S_o dS_o$ of U is a derivation module homomorphic image of $U(S_o/R)$ and hence

$$S_o dS_o = S_o dx_1 + \ldots + S_o dx_n .$$

Hence, $SdS_o = Sdx_1 + \ldots + Sdx_n$.

Next, notice that S is a Euclidean domain [cf. Proposition 1, §1, Chapter II], and $U(S/R)$ is finitely generated. [Lemma 1].

$$\mathfrak{d}\, dS \subseteq \mathfrak{d}_1 dS_o \quad [\text{ cf. Proposition 2, §3, Chapter 0 }]$$

Hence $\mathfrak{d}^k S(dS)^k = \mathfrak{d}^k (dS)^k \subseteq \mathfrak{d}_1^k (dS_o)^k \subseteq S(dS_o)^k$

By (a), $S(dS_o)^k = \sum_{i_1,\ldots,i_k} Sdx_{i_1}\ldots dx_{i_k}$.

Thus, $\mathfrak{d}^k S(dS)^k \subseteq \sum_{i_1,\ldots,i_k} Sdx_{i_1}\ldots dx_{i_k}$.

<u>Proposition 1</u>: Let S_o be a valuation ring in K_o with the property that S_o is a ring of quotients of G and the coefficients of the minimal polynomial f of x_o over K_o all be contained in S_o. Then

$$(f'(x_o))^{k+1}\left(\bigcap_{S \in \mathcal{S}_1} S(dS)^k\right) \subseteq \sum_{i_1,\ldots,i_k} S_o[\, x_o\,]dx_{i_1}\ldots dx_{i_k},$$

where $\mathcal{S}_1$ is the set of all valuation rings S in K which are extensions of S_o.

<u>Proof</u>: By Proposition 6, §3, Chapter 0, $f'(x_o) \in \mathcal{D}(\bar{S}_o/S_o)$. However, we know that S is a ring of quotients of $\bar{S}_o$ and hence $\mathcal{D}(S/S_o) = \mathcal{D}(\bar{S}_o/S_o)\cdot S$ [Proposition 3, §3, Chapter 0]. Thus $f'(x_o) \in \mathcal{D}(S/S_o)$. Now for $x \in \bigcap_{S \in \mathcal{S}_1} S(dS)^k$, let $x = \Sigma a_{i_1\ldots i_k}dx_{i_1}\ldots dx_{i_k}$, $a_{i_1\ldots i_k} \in K$. Then by Lemma 2, $(f'(x_o))^k x \in \Sigma S dx_{i_1}\ldots dx_{i_k}$ for all $S \in \mathcal{S}_1$, hence $f'(x_o)^k a_{i_1\ldots i_k} \in S$ for all $S \in \mathcal{S}_1$. Thus $(f'(x_o))^k a_{i_1\ldots i_k} \in \bar{S}_o$. On the other hand, $f'(x_o) \in \mathcal{D}(S_o[\, x_o\,]/S_o)$, since $U(S_o[\, x_o\,]/S_o)$ is generated by dx_o and $f(x_o) = 0$ implies $f'(x_o)dx_o = 0$. Hence $(f'(x_o))^{k+1}a_{i_1\ldots i_k} = (f'(x_o))^k a_{i_1\ldots i_k}\cdot f'(x_o) \in \bar{S}_o\mathcal{D}(S_o[\, x_o\,]/S_o)$. However, by Proposition 5, §3, Chapter 0, $\bar{S}_o\mathcal{D}(S_o[\, x_o\,]/S_o) \subseteq S_o[\, x_o\,]$. Thus $(f'(x_o))^{k+1}a_{i_1\ldots i_k} \in S_o[\, x_o\,]$. This shows that

$$(f'(x_o))^{k+1}x \in \Sigma S_o[\, x_o\,]dx_{i_1}\ldots dx_{i_k} \quad \text{for all } x \in \bigcap_{S \in \mathcal{S}_1} S(dS)^k.$$

<u>Proposition 2</u>: Let R be a noetherian unique factorization domain and the coefficients of the minimal polynomial f of x_o over K_o all be contained in G, then,

$$(f'(x_o))^{k+1} I_k \subseteq \sum_{i_1,\ldots,i_k} R[x_o, x_1, \ldots, x_n] dx_{i_1} \ldots dx_{i_k},$$

where I_k is the R-module of all homogeneous integral differential forms of degree k and $R[x_o, x_1, \ldots, x_n]$ is the subring of K generated by $x_o, x_1, \ldots, x_n$ over R.

Proof: Let $\mathcal{S}_o$ be the set of all valuation rings in K_o which are rings of quotients of G and $\mathcal{S}$ the set of all valuation rings in K which are extensions of a member of $\mathcal{S}_o$. Then

$$(f'(x_o))^{k+1} I_k \subseteq (f'(x_o))^{k+1} \bigcap_{S \in \mathcal{S}} S(dS)^k$$ (by the definition of homogeneous differential forms of degree k)

$$\subseteq \sum_{i_1,\ldots,i_k} \bigcap_{S_o \in \mathcal{S}_o} S_o[x_o] dx_{i_1} \ldots dx_{i_k}$$ (by Proposition 1)

$$= \sum_{i_1,\ldots,i_k} G[x_o] dx_{i_1} \ldots dx_{i_k}$$ (since G is a unique factorization domain and hence $G = \bigcap_{S_o \in \mathcal{S}} S_o$ by Corollary 2, § 1, Chapter II).

$$= \sum_{i_1,\ldots,i_k} R[x_o, x_1, \ldots, x_k] dx_{i_1} \ldots dx_{i_k}.$$

Proposition 3: Under the assumptions as in Proposition 2, in fact, there exists a natural number ν such that

$$(f'(x_o))^{k+1} I_k \subseteq \sum_{i_1,\ldots,i_k} T dx_{i_1} \ldots dx_{i_k}$$

where $T = \sum_{k_i \leq \nu} R x_o^{k_o} x_1^{k_1} \ldots x_n^{k_n}.$

Proof: Let (A) be the following statement: There exists a natural number ν_i for each x_i, $i = 0, 1, 2, \ldots, n$ such that

$$(f'(x_o))^{k+1} I_k \subseteq \sum_{i_1,\ldots,i_k} T_i dx_{i_1} \ldots dx_{i_k}, \quad T_i = \sum_{k_i \leq \nu_i} R[x_o, x_1, \ldots, \hat{x}_i, \ldots, x_n] x_i$$

where $\hat{x}_i$ denotes the omission of x_i.

If (A) holds, $(f'(x_o))^{k+1} I_k \subseteq \sum_{i_1,\dots,i_k} (\bigcap_{i=0}^{m} T_i) dx_{i_1} \dots dx_{i_k}$.

Put $\nu = \max\{\nu_o,\dots,\nu_n\}$, then $\bigcap_{i=0}^{n} T_i \subseteq T$, and hence

$(f'(x_o))^{k+1} I_k \subseteq \sum_{i_1,\dots,i_k} T dx_{i_1} \dots dx_{i_k}$. Hence it is sufficient

to show (A).

(1) Proof of (A) for x_o : Let $x \in (f'(x_o))^{k+1} I_k$, then

$x = \sum_{i_1,\dots,i_k} c_{i_1 \dots i_k} dx_{i_1} \dots dx_{i_k}$, $c_{i_1 \dots i_k} \in R[x_o, x_1, \dots, x_n]$ by

Proposition 2. Moreover, $K = K_o[x_o]$ is a simple algebraic extension of K_o and the leading coefficient of the minimal polynomial f of x_o over K_o is 1 and all coefficients of f are in $R[x_1,\dots,x_n]$ and hence $c_{i_1 \dots i_k} = \sum_{i=1}^{m} c_{i i_1 \dots i_k} x_o^{m-1}$, $c_{i i_1 \dots i_k} \in R[x_1,\dots,x_n]$.

Put $\nu_o = m - 1$, then

$x \in \sum_{i_1,\dots,i_k} \sum_{k_o \leq \nu_o} R[x_1,\dots,x_n] x_o^{k_o} dx_{i_1} \dots dx_{i_k}$. Thus (A) holds

for x_o.

(2) Proof of (A) for x_1 : Let us consider the subring $R[x_1^{-1}, x_2, \dots x_n]$ of K_o. Then $R[x_1^{-1}, x_2, \dots, x_n]$ is also a polynomial ring in $x_1^{-1}, x_2, \dots, x_n$

as indeterminates. Let S_o be the valuation ring in K_o determined by the irreducible element $x_1^{-1} \in R[x_1^{-1}, x_2, \dots, x_n]$ [cf. Corollary 1, §1, Chapter II]. Put $x_o x_1^{-h} = y_o$ for some positive integer h, $x_1^{-1} = y_1$, and $x_i = y_i$ for $i = 2, 3, \dots, n$. Then $Q(y_1,\dots,y_n) = K_o$ and is a purely transcendental extension of Q. Next,

$$f(x_o) = x_o^m + a_1 x_o^{m-1} + \dots + a_m$$

$$= x_1^{hm}((x_0x_1^{-h})^m + a_1x_1^{-h}(x_0x_1^{-h})^{m-1} + \dots + a_mx_1^{-hm}).$$

By replacing $x_0x_1^{-h}$ by y_0 and putting $b_i = a_ix_i^{-hi}$ for $i = 1, 2, \dots, m$,

$$f(x_0) = x_1^{hm}(y_0^m + b_1y_0^{m-1} + \dots + b_m).$$

Let us put $g = t^m + b_1t^{m-1} + \dots + b_m$; this is a polynomial over K_0 with t as indeterminate, and $g(y_0) = 0$, $K_0(y_0) = K$, and g is the polynomial of y_0 over K_0. Now we can put h sufficiently large so that

$b_i = a_ix_1^{-hi} \in S_0$.

Hence by applying Proposition 1, we have:

$$(g'(y_0))^{k+1}\left(\bigcap_{S \in \mathcal{S}_1} S(dS)^k\right) \subseteq \sum_{i_1,\dots,i_k} S_0[\,y_0\,]dy_{i_1}\dots dy_{i_k}.$$

Since $dy_1 = (-x_1^{-2})dx_1$ and $dy_i = dx_i$ for $i = 2, \dots, n$,

$$(g'(y_0))^{k+1}\left(\bigcap_{S \in \mathcal{S}} S(dS)^k\right) \subseteq \sum_{i_1,\dots,i_k} S_0[\,y_0\,](-x_1^{-2})^{q_{i_1\dots i_k}}dx_{i_1}\dots dx_i$$

where $q_{i_1\dots i_k}$ is the number of y_1 among $y_{i_1},\dots,y_{i_k}$.

Now notice that $f'(x_0) = x_1^{h(m-1)}g'(y_0)$.

Hence,

$$(f'(x_0))^{k+1}I_k \subseteq (f'(x_0))^{k+1}\left(\bigcap_{S \in \mathcal{S}_1} S(dS)^k\right)$$

$$= x_1^{h(m-1)(k+1)}(g'(y_0))^{k+1}\left(\bigcap_{S \in \mathcal{S}_1} S(dS)^k\right)$$

$$\subseteq x_1^{h(m-1)(k+1)} \sum_{i_1,\dots,i_k} S_0[\,y_0\,](-x_1^{-2})^{q_{i_1\dots i_k}}dx_{i_1}\dots dx_{i_k}$$

$$\subseteq x_1^{h(m-1)(k+1)} \sum_{i_1,\dots,i_k} S_0[\,x_0\,]dx_{i_1}\dots dx_{i_k}$$

(Since $S_0[\,y_0\,](-x_1^{-2})^{q_{i_1\dots i_k}} \subseteq S_0[\,x_0\,]$.)

$$= \sum_{i_1,\dots,i_k} x_1^{\nu_1}S_0[\,x_0\,]dx_{i_1}\dots dx_{i_k},$$

putting $\nu_1 = h(m - 1)(k + 1)$.

On the other hand, by Proposition 2,

$$(f'(x_o))^{k+1} I_k \subseteq \sum_{i_1,\ldots,i_k} R[x_o, x_1, \ldots, x_n] dx_{i_1} \ldots dx_{i_k},$$

Hence

$$f'(x_o)^{k+1} I_k \subseteq \sum_{i_1,\ldots,i_k} T_1 dx_{i_1} \ldots dx_{i_k},$$

since $x_1^{\nu_1} S_o[x_o] \cap R[x_o, x_1, \ldots, x_n] \subseteq T_1$.

(3) Proof of (A) for x_i, $i = 2, 3, \ldots, m$, is exactly the same proof with 1 replaced by i.

This completes the proof.

Theorem 1: (Main theorem) Let R be a noetherian unique factorization domain, and K a finitely generated separable extension field of a field of quotients of R. Then the R-module of all homogeneous integral differential forms of degree k is finitely generated.

Proof: It is known [11] that if K is finitely generated separable extension field of Q, then there exist elements $z, x_1, \ldots, x_n \in K$ such that $\{x_1, \ldots, x_n\}$ is a set of algebraically independent elements and if we put $K_o = Q(x_1, \ldots, x_n)$, $K = K_o(z)$ and K is a separable algebraic extension of K_o. Let $g = t^m + b_1 t^{m-1} + \ldots + b_m$, $b_i \in K_o$ for $i = 1, 2, \ldots, m$ be the minimal polynomial of z over K_o. From the fact that K_o is a field of quotients of $R[x_1, \ldots, x_n]$, there exists an element $b \in R[x_1, \ldots, x_n]$ such that $bb_i \in R[x_1, \ldots, x_n]$. Then $b^m g(z) = (bz)^m + bb_1(bz)^{m-1} + \ldots + b^m b_m$.

Let us put $bz = x_o$, $b^i b_i = a_i (\in R[x_1, \ldots, x_n])$, and

$f = t^m + a_1 t^{m-1} + \dots + a_m \in K_o[t]$; then $K = K_o(x_o)$ and f is the minimal polynomial of x_o over K_o.

Moreover, we notice that all the coefficients $a_i \in R[x_1, \dots, x_n]$.

Hence by Proposition 3,

$$f'(x_o)^{k+1} I_k \subseteq \sum_{i_1,\dots,i_k} T dx_{i_1} \dots dx_{i_k}$$

where $T = \sum_{k_i \le \nu} R x_o^{k_o} x_1^{k_1} \dots x_n^{k_n}$ for some ν.

Hence $I_k \subseteq \sum_{i_1,\dots,i_k} T \frac{1}{(f'(x_o))^{k+1}} dx_{i_1} \dots dx_{i_k}$.

Thus I_k is a submodule of a finitely generated R-module, and since R is a noetherian domain, I_k itself is a finitely generated R-module.

As a special case we note:

Corollary : If R is a noetherian unique factorization domain and K a finitely generated separable extension of the field of quotients Q of R then the integral closure of R in K is a finitely generated R- module.

For the more general class of noetherian integrally closed R this is known [11] for the case of finitely generated separable *algebraic* extensions K of Q, and hence we have a partial generalization of this latter result.

Definition: Let R be an integral domain, K a field containing R. A homogeneous alternating differential form $x \in G_k(D)$ of degree k is called *integral* if and only if $j_k(x)(\in K_k(D))$ is a homogeneous integral differential form of degree k, where $j_k : G_k(D) \longrightarrow K_k(D)$ is the natural monomorphism [cf. 4, Chapter 0].

Theorem 2: Let R be a noetherian unique factorization domain and K a finitely generated separable extension field of the field of quotients of R. Then the R-module of all homogeneous integral alternating differential forms of degree k is finitely generated.

Proof: By the definition of the integral alternating differential form, the R-module of all homogeneous integral alternating differential forms of degree k can be imbedded into the R-module of all homogeneous integral differential forms of degree k. By Theorem 1, the latter is finitely generated, and since R is noetherian domain, the former is also a finitely generated R-module.

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