ON THE STABILIZATION OF CERTAIN NONLINEAR CONTROL SYSTEMS

ON THE STABILIZATION OF NONLINEAR CONTROL SYSTEMS SUBJECT TO STOCHASTIC DISTURBANCES AND INPUT CONSTRAINTS

by

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree
Master of Applied Science

McMaster University

MASTER OF APPLIED SCIENCE (2015)	McMaster University
(Chemical Engineering)	Hamilton, Ontario, Canada

TITLE:	On the stability of nonlinear control systems subject to stochastic disturbances and input constraints
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NUMBER OF PAGES:	ix, 58

ABSTRACT

This thesis investigates the broad theme of guaranteeing the stability of nonlinear control systems. In the first section, we describe the application of discrete controller for the stabilization of certain nonlinear stochastic control systems subject to unavailability of state measurements. In the second section, we consider input constrained nonlinear systems and characterize the region from which stabilization to the origin is possible. We then use this information to design a controller which stabilizes everywhere in this set.

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my supervisor, Dr. Prashant Mhaskar. I could not have asked for a wiser or more patient teacher. Much is still owed for the opportunity he has given me.

I would also like to thank the Department of Chemical Engineering, the McMaster Advanced Control Consortium, and the National Science and Engineering Research Council for their funding.

I am also grateful to the all of the faculty, staff, and fellow graduate students who helped me along the way. The list is too long to enumerate each of you.

Foremost, however, I would like to thank my identical twin brother, Tom. I would not have achieved anything without his support and encouragement.

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Chapter 1

Introduction

This thesis is, in a broad sense, is an investigation of mathematical control systems. In this way, all we have studied is how to influence dynamical systems to provoke desirable properties. However, this thesis is also as much as a study of control systems engineering, where we design control systems to accomplish engineering objectives.

1.1 Motivation and Goals

This research spans several themes, the most prominent of which is the design of controllers which recognize the nonlinearities exhibited by many systems. In chemical engineering applications, particularly the dynamics of chemical reactors, we face the challenge of designing control systems to stabilize highly nonlinear reaction kinetics. We are also challenged to control systems with input constraints. For example, all reactors have a maximum heating and cooling rate. We consider both this saturation nonlinearity and other nonlinearities in the system dynamics.

In many other applications, we must design control systems which are robust to disturbances to their dynamics. For instance, the feed composition into a reactor may experience wide variations. Controllers (and observers) which address the uncertainty caused by disturbances is another theme of this research.

In yet another set of themes, we employ controllers with the objective of guaranteeing stability. For stability analysis, we make use of Lyapunov's much celebrated direct method. To stabilize systems, we frequently propose the technology of model predictive control. This is again motivated by chemical engineering applications where predictive controllers are often used to steer complex dynamics close to their setpoints.

Hence, this research is motivated by the challenges which afflict modern control systems. The goal of this research is to design control systems which improve upon the performance of current designs.

1.2 Key Contributions

This thesis is an exposition of work on two topics.

The first topic is a study of the properties of a model predictive controller for systems with stochastic disturbances. In this topic, covered in two papers, we investigate how, and in what sense, stability can be imparted by a discrete controller operating on a continuous random dynamical system. In that work, we address the additional complication where the system has states which cannot be measured and must be reconstructed by an observer. A further introduction can be found in Chapter 2.

The second topic considers the problem of characterizing the null controllability region of nonlinear control systems. We use the machinery of optimal control theory to trace the boundary of this set. We then use the definition of this set to define a control law which guarantees stability from the largest possible set of initial conditions. This research has not yet formed any published or submitted document. Again, further introductions are contained in Chapter 3.

Both works focus on the stabilization of nonlinear systems with input constraints. Because of this, both works begin with a characterization of the system's region of attraction. Secondly, both works use Lyapunov analysis to show stability of the feedback systems and employ Lyapunov-based predictive controllers.

1.3 Preliminaries

What follows is a very fast introduction to a few basic technical prerequisites in this thesis, however, it would be intractable for this document to be self-contained. As such, the reader should consider the material in the texts: H. Khalil "Nonlinear Systems", and C. Chen "Linear Systems Theory and Design", to be essential. Less is assumed from the material in L. Evans "An Introduction to Mathematical Optimal Control Theory" and L. Evans "An Introduction to Stochastic Differential Equations", but these two manuscripts should also be considered prerequisite.

In this thesis, we consider the problem of controlling a generalized dynamical system to some desirable setpoint. However, we are more concerned about the states of the system converging to the setpoint (i.e. the origin) than we about the controller's dynamic performance. Likewise, the prevention of closed-loop instability has important safety and reliability implications. This is why, in our formulations, we make a special effort to confer stability guarantees upon the controller. If there is a region of state-space from which the system necessarily converges to the origin, we say that it is locally asymptotically stable. For example, observe that the system $\dot{x} = -2x$ is everywhere stable.

It is well known the linear autonomous system are asymptotically stable if the closed-loop system has all eigenvalues with negative real parts. We use Lyapunov's second method to analyze the stability of nonlinear systems. In this technique, we define a positive-definite 'energy' associated with the deviations of the states from the origin. Importantly, if the control action always causes the energy to decrease, then this shows that the system is stable. To make this more precise let $\dot{x} = f(x, u)$ be a system dynamics and let $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a functional which admits $V(0) = 0$ and $V(x) > 0$ whenever $x \neq 0$. Then, if there exists a control policy $\phi(x)$ such that $\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x, \phi(x)) < 0$ for all $x \neq 0$, and $\dot{V}(0) = 0$, then the dynamics are asymptotically stable under the control $\phi(x)$.

For example, it is known that a controllable linear system $\dot{x} = Ax + Bu$ can be stabilized with an appropriate choice of the gain K, where $u = Kx$. Thus, nothing is stopping us from suggesting a Lyapunov function of the form $V(x) = x^T Px$. With the appropriate manipulations, we can show that its derivative along the flow of the trajectory $x(t)$ is just $\dot{V}(x) = -x^T Qx$, where $Q > 0$ is determined from the dynamics. Thus, this system is asymptotically stable. Conversely, if a system is asymptotically stable, then there exists a $V(x)$ which verifies this fact.

Moreover, we make frequent use of model predictive control. This controller often takes the form of implementing the sequence of feasible control actions which are predicted to minimize the states' distance from the setpoint. The controller then regularly re-computes the optimal control plan as new information becomes available. MPC can be augmented by other logic, such as to obey state constraints or minimize control effort. Notice that because the control objective is arbitrary in general, we must employ additional conditions explicitly to guarantee stability of the closed-loop system.

In Chapter 2, we consider the control of stochastic dynamical systems. These systems have dynamics that evolve randomly in time but where possible realizations have well-characterized probabilities. A stochastic system (in the sense of Ito) is shown below:

$$\mathrm{d}X_t = f(X_t, u_t)\mathrm{d}t + g(X_t)\mathrm{d}W_t \tag{1.1}$$

where W_t is the 'Brownian motion' random variable. Notice that the system is written as an infinitesimal because stochastic processes are non-differentiable. The usual rules of calculus do not apply to these systems, so quantities such as the derivative of the Lyapunov function have their own special definitions in this setting. For instance, Ito's Lemma giving expansion of a function $p(X_t)$ of a scalar stochastic process X_t is shown below:

$$\mathrm{d}p(X_t) = \left(\frac{\partial p(X_t)}{\partial t} + f(X_t, u_t)\frac{\partial p(X_t)}{\partial x} + \frac{1}{2}g(X_t)^2 \frac{\partial^2 p(X_t)}{\partial x^2} \right) \mathrm{d}t + g(X_t)\frac{\partial p(X_t)}{\partial x}\mathrm{d}W_t \tag{1.2}$$

Terms in the dynamics that are purely deterministic are called 'drift' (e.g. $f(\cdot)$), otherwise they are called 'diffusion' (e.g. $g(\cdot)$). In this chapter, we use the idea of the diffusion process to model state-dependent disturbances. Unlike deterministic disturbances, random disturbances have unbounded magnitudes, so stability is achieved in a probabilistic sense.

In many applications, some states are not measured directly by instruments and so they must be reconstructed indirectly by observing other system behavior. In general, the estimated states are adjusted until the dynamics of the measured states are corroborated by what is observed in the plant. To show convergence of the estimator, we show asymptotic stability of its error dynamics. To this end, consider the following typical Luenberger observer for a linear control system:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{1.3}$$

Where $y = Cx$ is a measurable state (an output), and $\hat{y}$ is its estimate using the observer. Next, notice that the dynamics of the error $e = x - \hat{x}$ is simply $\dot{e} = (A - LC)e$. It is well known that if $A - LC$ has eigenvalues with all negative real parts, then the error dynamics are asymptotically stable.

In Chapter 3, we make extensive use of results from optimal control. These results are a re-statement of the famed Euler-Lagrange equation of the calculus of variations, which can be used to determine a whole path which minimizes some objective.

Finally, a system is said to be controllable if there exists a sequence of control actions which controls the system from any state to any other state. That is, there exists a $u(t)$ and T such that $x_2 = x_1 + \int_0^T Ax(t) + Bu(t)dt$ for all x_1 and x_2. In this formulation, u can take on any value, but in practice, the presence of input constraints will limit the set of states which can be controlled to the origin. Criteria for unconstrained nonlinear controllability can be found in Slotine "Applied Nonlinear Control". The question of constrained controllability is considered in Chapter 3.

1.4 Thesis Overview

This thesis presents the two topics in individual chapters and then concludes. It reproduces verbatim the text of their corresponding journal paper drafts.

Chapter 2

Output Feedback Model Predictive Control of Stochastic Nonlinear Systems

In this chapter we present our results on stochastic model predictive control. This work was first published as "Output-Feedback Model Predictive Control of Stochastic Nonlinear Systems" in the Proceedings of the American Control Conference 2015. It was then greatly expanded and submitted as "Output-Feedback Lyapunov-based Predictive Control of Nonlinear Systems" to a prominent control theory journal. At the time of writing, we are awaiting a verdict from the reviewers.

This chapter has two related facets. The first contribution of this chapter concerns the design of stochastic observers. There we discuss the existing results on observers for stochastic nonlinear systems that can be transformed exactly into a linear system and design an observer for an example system. We also go on to show closed loop stability of this system. For more on feedback linearization, see the end of this chapter. We then create a novel observer design for nonlinear stochastic systems and prove that it has error dynamics which are asymptotically stable in probability under certain conditions. We use a Lyapunov method (see Chapter 1) for this purpose and show that the observer is the first that is applicable to stochastic chemical reactor systems.

The second contribution in this chapter concerns the design of discrete controllers for stochastic dynamical systems. The first step is to define the finite region of state space (the region of attraction) where the control action is strong enough to stabilize the system to the origin in expectation if continuous implementation of the controller were possible. In that analysis, the fact that the estimated states are corrupted by error is used to shrink this region appropriately. Next, using a particular hold time of the control, our results give the probability that the trajectory will adopt stabilizing behaviour over the whole hold period. The proof of this result is analogous to calculating a maximum escape speed of the states. A second result gives the probability that the system remains within a certain neighbourhood of the origin for a period of time, even in view of the random disturbances.

We then design a model predictive controller that inherits the described properties by satisfying a constraint that it must choose a control action which decreases the energy of the expected dynamics. We simulate a stochastic chemical reactor system (with unmeasured reactant concentration) and show that realizations have stabilizing behaviour with the anticipated likelihoods. Lastly, because the MPC is only feasible inside the region of attraction, we show that the disturbance rejection probability can be re-stated to calculate the probability of successive feasibility of the predictive controller.

The rest of this chapter follows the journal paper submission verbatim, except in the final section on stochastic feedback linearization which was removed to condense the paper.

2.1 Abstract

In this chapter, the problem of output-feedback control of stochastic nonlinear systems is considered. A predictive controller is designed for which stability and feasibility (in a probabilistic sense) are guaranteed from an explicitly characterized region of attraction. The controller's performance is characterized by its risk of allowing destabilizing system

behavior. Since the controller design relies on certain convergence properties of the state observer, two compatible observer designs are presented: an existing design using stochastic feedback linearization, and a novel generalization of a nonlinear observer to the stochastic setting. Simulation results illustrating the efficacy of these designs are presented.

2.2 Introduction

Model predictive control (MPC) is an effective solution for controlling complex plants characterized by nonlinearities, constraints, and uncertainties [1]. The prevalence of these challenges has driven a large number of research contributions and industrial implementations of nonlinear MPC.

For systems with stochastic disturbances, MPC designs which recognize their inherent stochastic nature can out-perform designs based on conservative bounded disturbance models. As such, research on model predictive control of stochastic systems has diverged into two main approaches. One method is to attenuate the largest sequence of disturbances [2; 3]. In this direction, robust Lyapunov-based MPC (LMPC) designs have been proposed that guarantee robust stability from an explicitly characterized stability region (see e.g. [3] and [4] for results in the absence of uncertainty). These formulations constrain the search space to only trajectories that decrease a given Lyapunov function.

Another solution is to mitigate the disturbances with a quantifiable probability. Along these lines, MPC formulations that use stochastic dynamic programming provide disturbance rejection but are often too computationally burdensome [5]. Another solution, perhaps more popular, has been to compute the control action by minimizing the cost of the expected trajectory [6].

Drawing on stochastic Lyapunov theory [7; 8], LMPC has recently been adapted to stochastic nonlinear systems [9] under the assumption of full state feedback and in the absence of measurement noise. To address the issue of lack of state measurements, work on the stochastic nonlinear output feedback problem has produced results for controlling systems that can be transformed into observer canonical [10] or feedback linearized form [11; 12; 13]. However, nonlinear model predictive control designs that simultaneously address the problems of lack of state measurements and stochastic uncertainty have not been developed.

Motivated by the above considerations, in this chapter, we consider the problem of model predictive control of nonlinear stochastic systems subject to limited availability of measurements and design an output feedback Lyapunov-based MPC. The design, applicable to a class of nonlinear systems with additive stochastic noise, assumes the existence of an observer which has a sufficiently strong convergence rate and the design allows an explicit characterization of the stability (in probability) region.

The rest of the manuscript is organized as follows: we begin by covering some preliminaries in Section 2.3, including a brief review of the Stochastic LMPC results in [9]. We use Section 2.4 to present two compatible observer designs. The first observer is an existing design for systems that are stochastic feedback-linearizable. The second stochastic observer is a novel generalization of a design for a class of deterministic nonlinear systems [14]. In Section 2.5, we present the LMPC design, and characterize the region in which the controller has both initial feasibility and stability in probability. Then, further properties of this controller are characterized, including the risk of destabilizing control action, the probability of remaining close to the origin before escaping the stability region. Lastly, in Section 2.6, the efficacy of the proposed results are illustrated using two simulation examples.

2.3 Preliminaries

2.3.1 Notation

We assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and say that $X : \Omega \to \mathbb{R}$ is a *random variable* if it is $\mathcal{F}$-measurable on every Borel subset of $\mathcal{F}$. For each $\omega \in \Omega$, we say that $x_\omega(t)$ is a *realization* or *trajectory* of the solution $x(t)$. We use $\mathbb{N}(n)$ to refer to the set $\{1, ..., n\}$. We use the p-norm $|\cdot|_p$, the Euclidean norm $|\cdot|$, the weighted norm $|\cdot|_Q = x^T Q x$ and the supremum norm $|\cdot|_\infty$. By a *ball*, we mean an open neighbourhood of the origin defined by $B_d^Q = \{x \in \mathbb{R}^n : |x|_Q < d\}$. We further use ∂A, A°, A^c, and $\overline{A}$ to indicate the boundary, interior, complement and closure of set A, whereas $A \backslash B$ indicates the relative complement. Finally, we refer to the Lie derivative of functional V and vector field f as $L_f V$. The *hitting time* of a set Ω with respect to a trajectory $x(t; x_0 \in \Omega^\circ)$ and process time $t > t_0$ is defined as $\tau_\Omega(t) = \min\{t, \inf_{t>t_0}\{t : x(t) \notin \Omega\}\}$. Wherever we refer to a state observer, we use $\hat{x}$ for the estimate of state x and $\tilde{x}$ for its observation error, and then $x = \hat{x} + \tilde{x}$.

2.3.2 System Description

We consider autonomous nonlinear stochastic systems with affine input and additive disturbances described by:

$$\begin{aligned} \mathrm{d}x_i &= f_i(x)\mathrm{d}t + g_i(x)u_t\mathrm{d}t + h_i(y)\mathrm{d}W_t \\ \mathrm{d}x_{n+1} &= m(x)\mathrm{d}t + r(x)u_t\mathrm{d}t + \psi(y)\mathrm{d}W_t \end{aligned} \tag{2.1}$$

where y is the scalar output and $x \in \mathbb{R}^{n+1}$ is the vector of states. The first n states are unmeasured but can be observed from the output $y = x_{n+1}$. $W \in \mathbb{R}^w$ is the vector of independent Brownian motions associated with the probability triplet $(\Omega, \mathcal{F}, \mathbb{P})$, $u \in \mathbb{R}^k$ is the vector of control inputs and $f(\cdot)$, $m(\cdot)$, $g(\cdot)$, $r(\cdot)$, and $H(\cdot) = [h_1, \ldots, h_n, \psi]$ are suitably sized arrays. The convex set of admissible control inputs is $\mathcal{U} = \{u \in \mathbb{R}^k : u_{min} \leq u \leq u_{max}\} \neq \emptyset$, where $u_{min}, u_{max} \in \mathbb{R}^k$. For all $x(t_0) = x_0$, to give us a unique solution almost surely, we assume that on $t \in [0, \infty)$ that all terms in Eq. (2.1) are locally Lipschitz continuous and also that they are C^2 causal processes so that the Itô integral is well defined. Lastly, we assume that the disturbance is vanishing at the origin (i.e. $H(0) = 0$) and that the origin is an equilibrium point of the unperturbed system (i.e. $f(0) = 0$ and $m(0) = 0$).

2.3.3 Stochastic Stability

We start by reviewing a definition of stability and some concepts in Lyapunov analysis. In what follows, we assume the existence of a C^2 stochastic control Lyapunov functional (SCLF) $V : \mathbb{R}^{n+1} \to \mathbb{R}_0^+$.

Definition 1 [15]: *Given a C^2 scalar Lyapunov function $V(x_t)$ and the system in Eq. (2.1), the operator $\mathcal{L}$ is the infinitesimal generator, defined as:*

$$\mathcal{L}V(x) = L_f V + L_g V u(t) + \frac{1}{2}\text{tr}\left\{ H^T \frac{\partial^2 V}{\partial x^2} H \right\} \tag{2.2}$$

Proposition 1 [16]: *Given a Lyapunov function V associated to the system in Eq. (2.1), we have:*

$$\mathbb{E}(V(x_t)) = V(x_0) + \int_0^t \mathcal{L}V(x_t)\text{d}t \tag{2.3}$$

Clearly if $\mathcal{L}V$ is negative definite, then $\mathbb{E}(V(x_t)) < V(x_0)$.

Definition 2 [10]: *The equilibrium $x = 0$ of a system is said to be globally asymptotically stable in probability if for any real numbers $\varepsilon > 0$ and $t_0 > 0$, the system admits*

$$\begin{gathered} \mathbb{P}\left(\lim_{t\to\infty} x(t) = 0\right) = 1 \\ \lim_{x(t_0)\to 0} \mathbb{P}\left(\sup_{t>t_0} |x(t)| > \varepsilon\right) = 0 \end{gathered} \tag{2.4}$$

Proposition 2 [8]: *If $\mathcal{L}V(x) \leq -\Theta(|x|)$ for some class $\mathcal{K}_\infty$ function $\Theta(\cdot)$, then the system in Eq. (2.1) is globally asymptotically stable in probability.*

2.3.4 Stochastic Lyapunov-based MPC

In this section, we review stochastic Lyapunov MPC results applicable under state feedback systems, to enable us to clearly see the impact of the lack of state measurements on the probabilistic stability guarantees. To this end, consider Eq. (2.1) and assume x is measurable. The presentation here is kept very brief. For details, see [9].

First, suppose we have some SCLF from which a stabilizing in probability Lyapunov-based control policy $u = \phi(x)$ can be designed for the constrained-input system that yields $\mathcal{L}V(x) \leq -\rho V(x)$ for some $\rho > 0$. Then, we can define the set of states Π as the largest closed set containing the origin for which negative definiteness of $\mathcal{L}V$ can be achieved as:

$$\Pi = \{x \in \mathbb{R}^{n+1} : \inf_{u\in\mathcal{U}} \mathcal{L}V(x) + \rho V(x) \leq 0\} \tag{2.5}$$

We define $\mho_c$ as the maximal Lyapunov level set that is contained within Π:

$$\mho_c = \sup_{c\in\mathbb{R}_0^+} \{x \in \mathbb{R}^{n+1} : x \in \Pi,\ V(x) \leq c\} \tag{2.6}$$

By comparison of Eq. (2.5) with Eq. (2.3), it results that if the system is initialized in $\mho_c$, then under continuous implementation of the control action, the expected realization will be stabilizing. If $\mho_c$ is the whole state-space, then the controller readily satisfies *Proposition 2*.

We assume without loss of generality that $\mho_c$ corresponds with $c = 1$ and, in the rest of the manuscript, drop the subscript, and write it as $\mho$. Next, consider a discrete (sample and hold) state feedback controller implementation $u(t) = u(j\Delta) = \phi(x(j\Delta))$, $j \in \mathbb{N}$, for all $j\Delta \leq t \leq (j+1)\Delta$. It has been shown in [9] that, provided the system is initialized within $\mho$, there exists a sufficiently small hold period Δ such that the system will realize stabilizing behavior with any given probability λ. This is formalized in *Proposition 3* below:

Proposition 3 [9]: *For the system in Eq. (2.1) under state feedback and some SCLF V, given any probability $\lambda \in [0,1)$, there exists positive real numbers $\Delta^* := \Delta^*(\lambda)$, and $\delta < \delta' < 1$, such that if $\Delta \in (0, \Delta^*]$, then*

$$\begin{aligned} &\text{(i)}\ \ \mathbb{P}\left(\sup\nolimits_{t\in[0,\ \tau_{\mho\backslash\mho_\delta^\circ}(\Delta)]} \mathcal{L}V(x_t) < 0\right) \geq 1-\lambda, && x_0 \in \mho\backslash\mho_\delta^\circ \\ &\text{(ii)}\ \mathbb{P}\left(\sup\nolimits_{t\in[0,\ \tau_{\mho\backslash\Omega_\delta}(\Delta)]} V(x_t) \leq \delta'\right) \geq 1-\lambda, && x_0 \in \mho_\delta \end{aligned} \tag{2.7}$$

Next, these results were used to quantify the stability in probability of the system under Lyapunov-based feedback control subject to discrete control action.

Proposition 4 [9]: *For the system in Eq. (2.1) under the Lyapunov-based controller $u = \phi(x[t])$ designed using some SCLF V, given any positive real number d and probability $\lambda \in [0,1)$, there exists positive real numbers $\Delta^* := \Delta^*(\lambda)$, $\delta < \alpha$ and probabilities α, $\beta \in [0,1)$ such that if $\Delta \in (0, \Delta^*]$, then the following will hold for the closed–loop system:*

$$\begin{aligned} &\text{(i)}\ \ \mathbb{P}\left(\sup\nolimits_{t\in[0,\Delta]} \|x_t\| \leq d\right) \geq (1-\beta)(1-\lambda), \qquad x_0 \in \mho_\delta \\ &\text{(ii)}\ \mathbb{P}\left(\sup\nolimits_{t\in[0,\Delta]} V(x_t) < 1,\ \sup\nolimits_{t\in[0,\Delta]} \|x(t+\tau_{\mathbb{R}^n\backslash\mho_\delta^\circ})\| < d\right) \\ &\qquad \geq (1-\alpha)(1-\beta)(1-\lambda)^2, \qquad x_0 \in \mho_\alpha\backslash\mho_\delta^\circ \end{aligned} \tag{2.8}$$

In *Proposition 4*, (i) defines the likelihood that the states will remain bounded to within a given ball during each hold period, whereas (ii) gives the probability that the system's states can be stabilized to within a target neighborhood of the origin (and remain there) while never escaping the stability region.

2.4 Observer Design

In this section, we present two examples of observer designs that can satisfy the convergence properties required in our output feedback controller design. These designs rely on the existence of a stabilizing feedback control law, which we will subsequently present and characterize in Section 2.5.

We use the Lyapunov functional $V_1 = \frac{1}{4}\sum_1^{n+1}\hat{x}^4 + \frac{1}{4}\sum_1^{n+1}\tilde{x}^4$ to show stability of the following observers and to design the Lyapunov-based output-feedback controller $\phi(\hat{x})$.

2.4.1 Stochastic Feedback Linearizing Observer

It has been shown [11] that observers can be designed for SISO systems of the form of Eq. (2.1) if it can be transformed exactly using some diffiomorphism $z_t = \chi(x_t)$ into the integrator chain form:

$$\begin{aligned} \mathrm{d}z_i &= z_{i+1}\mathrm{d}t + \varphi_i(y)\mathrm{d}W \qquad && \forall\ i \in \mathbb{N}(n) \\ \mathrm{d}z_{n+1} &= \nu\mathrm{d}t + \varphi_{n+1}(y)\mathrm{d}W && y = z_1 \end{aligned} \tag{2.9}$$

Recall that a stochastic dynamical system is called stochastic feedback linearizable if it can be transformed into Eq. (2.9). For an excellent explanation of stochastic feedback linearization, including existence and solution methods for various special forms, see [13].

Then, a full-order Luenberger observer [11] for the transformed system Eq. (2.9) takes the form:

$$\begin{aligned} \dot{\hat{z}}_i &= \hat{z}_{i+1} + \ell_i(y - \hat{z}_1) \qquad && \forall\ i \in \mathbb{N}(n) \\ \dot{\hat{z}}_{n+1} &= \nu + \ell_{n+1}(y - \hat{z}_1) && y = z_1 \\ \mathrm{d}\tilde{z} &= A_0\tilde{z} + \varphi(y)\mathrm{d}W_t \end{aligned} \tag{2.10}$$

where $\varphi = [\varphi_i \dots \varphi_{n+1}]^T$ and where $\ell \in \mathbb{R}^{n+1}$ is the vector of observer gains. The matrix A_0 is given by

$$A_0 = \begin{bmatrix} \ell_1 & 1 & \cdots & 0 \\ \vdots & \vdots & I & \vdots \\ \vdots & 0 & \cdots & 1 \\ \ell_{n+1} & 0 & \cdots & 0 \end{bmatrix} \tag{2.11}$$

Once $\hat{z}$ is obtained, $\hat{x}$ can be recovered from $\hat{x} = \chi^{-1}(\hat{z})$. It has been shown that, when employed with some stabilizing controller $\phi(\hat{x})$, this observer can be used to globally asymptotically stabilize a system (in probability) provided that ℓ is chosen so that the matrix A_0 is Hurwitz (e.g. [11; 17]).

2.4.2 Nonlinear Stochastic Observer

In this section, we generalize the deterministic observer from [14] to a stochastic setting. To begin, consider the nonlinear system defined in Eq. (2.1) and assume that the drift dynamics are observable and controllable. We first recall the estimator design from [14].

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + L[\dot{y} - m(\hat{x}) - r(\hat{x})u_t] \tag{2.12}$$

where L is an n-vector of observer gains.

Subsequently, the change of variables $z = \hat{x} - Ly$ is introduced, which results in:

$$\dot{z} = \dot{\hat{x}} - L\dot{y} = f(\hat{x}) + g(\hat{x})u - L[m(\hat{x}) + r(\hat{x})u_t] \tag{2.13}$$

With this design, $z(t)$ is calculated at each time and $\hat{x}$ is recovered through $\hat{x} = z + Ly$. Note that this presentation of the observer has no explicit dependency on $\dot{y}$ and is a reduced order observer because no estimate of y is maintained. Thus, for this observer, $\tilde{x}_{n+1} \equiv 0$ and $\hat{x} = [\hat{x}_1, \dots, \hat{x}_n, y]$. Note that to reduce noise amplification from the observer, the observer could be augmented by a filtering technique.

In this section, we consider the application of this observer to a subset of the stochastic nonlinear system in Eq. (2.1) and then characterize the error convergence properties in

Theorem 1. To better motivate the required assumptions, we first consider the following error dynamics of the system in Eq. (2.1), using the estimator in (2.12).

$$\begin{aligned} \mathrm{d}\tilde{x} = &\Big[f(x) - f(x-\tilde{x}) + g(x)u_t - g(x-\tilde{x})u_t\Big]\mathrm{d}t \\ &+L\Big[m(x) - m(x-\tilde{x})\Big]\mathrm{d}t + [h_1 \cdots h_n, -L\psi]\mathrm{d}W_t \end{aligned} \tag{2.14}$$

where $\mathrm{d}W = [\mathrm{d}W^{x_1} \cdots \mathrm{d}W^{x_n}, \mathrm{d}W^{y}]^T$ are independent Brownian motions and where we would like the $r(\cdot)$ terms to not appear. For clarity, we label the above as

$$\mathrm{d}\tilde{x}_i = \gamma_i(\tilde{x}, x, u)\mathrm{d}t + \tilde{H}_i(y)\mathrm{d}W_t \quad \forall i \in \mathbb{N}(n) \tag{2.15}$$

Then, the assumptions required for the observer are:

Assumption 1: (i) *The state drift terms* $f_i(\cdot)$ *satisfies* $x_i f_i(\cdot) < 0$ *and error dynamics drift terms* $\gamma_i(\cdot)$ *satisfy* $\tilde{x}_i\gamma_i(\cdot) < 0$ *for some* $L^* \in R^n$, $\forall \tilde{x}, x, u \neq 0$, $i \in \mathbb{N}(n)$; (ii) *the drift term* $m(\cdot)$ *is globally Lipschitz continuous in* $x_1, \ldots, x_n$; (iii) $r(\cdot)$ *is a function of only* y; (iv) $\mathrm{rank}\{[g_i(\hat{x}), \cdots, g_n(\hat{x}), r(y)]\} \leq k \ \forall \hat{x} \neq 0$; (v) *for all* $\hat{x}$, $\frac{\partial \dot{y}}{\partial \hat{x}_i} \neq 0 \ \forall i \in \mathbb{N}(n)$.

Part (i) of *Assumption 1* can be seen as a requirement that the zero-input drift dynamics be inherently strongly stable. Part (ii) is satisfied by bounded or linear output dynamics and part (iii) is satisfied similarly. Part (iv) is equivalent to $[g_1(\cdot), \ldots, g_n(\cdot), r(\cdot)]u = Gu \leq b$ being a consistent linear inequality, that is, has a solution u for all b, which is a type of reachability condition. It is trivially satisfied by linear G of sufficient rank. Note that condition (iv) is only a sufficient condition, and (v) assures observability. *Assumption 1* allows the result of *Theorem 1* (for all the proofs, see the Appendix).

Theorem 1: *Consider the system Eq. (2.1) subject to Assumption 1, and the state observer of Eq. (2.13). Then there exists an* $L^* \in R^n$ *and a stabilizing controller* $u = \phi(\hat{x})$ *such that if* $\mathrm{sign}(L_i^*)L_i \geq |L_i^*| \ \forall \ i \in \mathbb{N}(n)$, *the observer Eq. (2.13) achieves global asymptotic stability in probability.*

2.5 Controller Design

In this section, we assume the existence of an appropriate state estimation design. We first consider a Lyapunov-based controller and characterize its stability region subject to constrained input and then give its stability properties, which are the main results of the chapter. Then, we present an output feedback LMPC design that has the same properties, within a defined feasibility set.

2.5.1 Stability Region for State Estimates

Firstly, to infer the position of the states from the state estimates, we need to assume $\tilde{x}$ is bounded. That is,

Assumption 2: *For any finite time $t \geq t_0$, all sample paths have observation errors that admit $|\tilde{x}|_\infty \leq E$ almost surely.*

We note that no proper Brownian motion would have a bounded distribution in the limit of infinite time. To construct the following sets, we will make use of $V_2 = \frac{1}{4}\sum_1^{n+1} \hat{x}^4$ (for a justification, see Remark 2 . We define the set $\tilde{\mho}'_{c'}$ such that if the state estimates are in $\tilde{\mho}'_{c'}$, utilizing the assumed bound on the error, we can infer that the states are in $\mho_c$. With $\mho_{c'} = \{\hat{x} : V_2(\hat{x}) \leq c'\}$, we have:

$$\tilde{\mho}'_{c'} = \sup_{c' \in \mathbb{R}_0^+} \{\hat{x} : \hat{x} \in \mho_{c'} \Rightarrow \hat{x} + \tilde{x} \in \mho_c,\ \forall |\tilde{x}|_\infty \leq E\} \tag{2.16}$$

Similar to before, we assume without loss of generality that the largest level set is $c' = 1$ (resulting in $c > 1$) and denote this space by $\tilde{\mho}'_{c'} = \tilde{\mho}'_1 \equiv \tilde{\mho}'$.

Finally, we also define $\mho_N = \{\hat{x} : V_2(\hat{x}) \leq \frac{K_7}{\rho} E\}$, with K_7 being a constant to be defined later and $\tilde{\mho} = \tilde{\mho}' \backslash \mho_N$. Figure 2.1(A) shows a pictorial representation of the sets $\tilde{\mho}$, $\tilde{\mho}'$, $\mho_N$, and Π.

Remark 1: Note that $\tilde{\mho}$, $\tilde{\mho}'$, and $\mho_N$ are images in $\mathbb{R}^{n+1}$. Using this construction of $\tilde{\mho}'$, $\hat{x} \in \tilde{\mho}$ and $\tilde{x} \in \{\tilde{x} : |\tilde{x}|_\infty \leq E\}$ can vary independently, while guaranteeing state x_t is positively invariant with respect to $\mho$.

Remark 2: V_2 was chosen because it takes all observable arguments so that it is possible to evaluate it and use as a constraint in a controller implementation. We refer the reader to *Appendix 2.10* where *Lemma 1* justifies our claim that it is sufficient to use V_2 and the constraint $V_2(\hat{x})+\mathcal{L}V_2(\hat{x})|_{\tilde{x}=0} \leq 0$ if the observer gain L is large enough. We note that the claim is resolved by prescribing a rate of error convergence, and that both observers presented can have these desired convergence properties for compatibility with our controller.

Remark 3: In our formulation, we use the specific Lyapunov function V_2 to construct the stability region, to show certain stability properties, and to compute the associated risk margins. Note that this specific choice of the Lyapunov function is not unique. Other suitable Lyapunov functions could readily be used, and would results in the same properties in general, albeit with the corresponding values of the constants and risk margins.

2.5.2 Discrete Implementation of the Output Feedback Lyapunov-based Controller

In this section, we will characterize the stability in probability properties of the output feedback LMPC. Recall that in *Proposition 3*, we quantified the probability of a discrete state-feedback system realizing stabilizing behavior. To adapt this to an output feedback setting, we have to address the additional risk associated with the observation error.

Lemma 2: *Consider the system in Eq. (2.1) under the discrete output feedback controller* $\phi(\hat{x}[t])$ *and subject to Assumption 2. Then, given any probabilities* $\zeta, \lambda \in [0,1)$*, there exists positive real numbers* $\tilde{\Delta}^* \leq \Delta^*$*, where* Δ^* *was defined in Proposition 3 and* $\delta < \delta' < 1$ *such that if* $\tilde{\Delta} \in (0, \tilde{\Delta}^*]$*, then*

$$\begin{aligned} &\text{(i)} \ \mathbb{P}\left(\sup\nolimits_{t\in[0,\tau_{\tilde{\mho}\backslash\mho_\delta^\circ}(\tilde{\Delta})]} \mathcal{L}V_2(x_t) < 0\right) \geq (1-\lambda)(1-\zeta), \quad \hat{x}_0 \in \tilde{\mho}\backslash\mho_\delta^\circ \\ &\text{(ii)} \ \mathbb{P}\left(\sup\nolimits_{t\in[0,\tau_{\tilde{\mho}\backslash\Omega_\delta}(\tilde{\Delta})]} V_2(x_t) \leq \delta'\right) \geq (1-\lambda)(1-\zeta), \qquad \hat{x}_0 \in \mho_\delta \end{aligned} \tag{2.17}$$

Remark 4: At $t=0$, we know that $\hat{x}_0 \in \tilde{\mho}$ implies $\mathcal{L}V_2(\hat{x}_0) \leq -\rho V_2(\hat{x}_0)$. As the system evolves within $t \in [0, \tilde{\Delta}$, the states are not confined to $\mho$ as a result of the discrete control action, but instead obey *Lemma 2*. This stabilization in probability property is only valid outside of a given small target set near the origin, inside of which the hold time is too large to guarantee *Lemma 2* (i).

The proof technique is to calculate a maximum hold time $\tilde{\Delta}^*$. It begins with determining the maximum 'speed' of the state dynamics implied by the risk λ. The rest of the proof is analogous to calculating, given a worst-case 'position' in state-space and the maximum 'speed', the minimum time until exiting the given set. The risk margin ζ for which the observation error meets a required bound is determined by solving the error SDE and finding its variance.

Next, the stability and boundedness of the states that results from a output feedback discrete time implementation of the controller is characterized. Similar to *Proposition 4*, we have the following analogue for an output feedback controller.

Theorem 2: *For system Eq. (2.1) with discrete control $\phi(\hat{x}[t])$, for any given positive number d and some λ, ζ, and $\tilde{\Delta}$ defined in Lemma 2, there exists probabilities $\tilde{\beta} \in (\beta, 1]$ and $\tilde{\alpha} \in (\alpha, 1]$ such that the system is characterized by*

$$\begin{aligned}
&\text{(i) } \mathbb{P}\left(\sup\nolimits_{t\in[0,\tilde{\Delta}]} \|x_t\|_Q \leq d\right) \geq (1-\tilde{\beta})(1-\lambda)(1-\zeta), && \hat{x}_0 \in \mho_\delta \\
&\text{(ii) } \mathbb{P}\left(\sup\nolimits_{t\in[0,\tilde{\Delta}]} V_2(x_t) < 1\right) \geq (1-\tilde{\alpha})(1-\lambda)(1-\zeta), && \hat{x}_0 \in \mho_\alpha \backslash \mho_\delta^\circ \\
&\text{(iii) } \mathbb{P}\left(\sup\nolimits_{t\in[0,\tilde{\Delta}]} V_2(x_t) < 1, \sup\nolimits_{t\in[0,\tilde{\Delta}]} \|x(t+\tau_{\mathbb{R}^n\backslash\mho_\delta^\circ})\| < d\right) \\
&\qquad \geq (1-\tilde{\alpha})(1-\tilde{\beta})(1-\lambda)^2(1-\zeta)^2, && \hat{x}_0 \in \mho_\alpha \backslash \mho_\delta^\circ
\end{aligned} \tag{2.18}$$

This result has an identical interpretation to *Proposition 4*. The proof follows immediately from some specific results in stochastic Lyapunov analysis from [16].

Figure 2.1(B) shows a pictorial representation of the sets $\mho_\alpha$ and $\mho_\delta$ indicated in *Lemma 2* and *Theorem 2*.

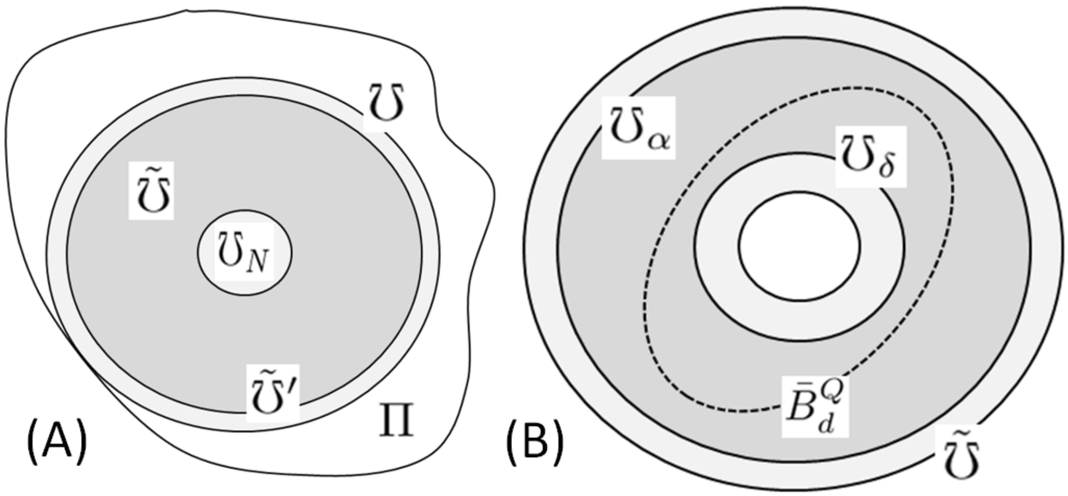

Figure 2.1: Pictorial representation of (A) the sets $\tilde{\mho}$, $\tilde{\mho}'$, $\mho_N$, and Π; (B) the sets $\mho_\alpha$ and $\mho_\delta$ used in *Theorem 2*.

2.5.3 Design of the Model Predictive Controller

The proposed Lyapunov-based model predictive controller takes the following form:

$$
\begin{aligned}
u_{MPC}(\bar{x}) &= \arg\min\{J(\bar{x},t,u) : u[t] \in \mathcal{U}\} \\
s.t.\ \bar{x}_i(0) &= \hat{x}_i(t)\ \forall i \in \mathbb{N}(n+1) \\
\tfrac{\mathrm{d}\bar{x}_i}{\mathrm{d}\bar{t}} &= f_i(\bar{x}_{\bar{t}}) + g_i(\bar{x}_{\bar{t}})u(\bar{t}) \\
\tfrac{\mathrm{d}\bar{x}_{n+1}}{\mathrm{d}\bar{t}} &= m(\bar{x}_{\bar{t}}) + r(\bar{x}_{\bar{t}})u(\bar{t}) \\
&\mathcal{L}V_2(\bar{x}_{\bar{t}}) + \rho V_2(\bar{x}_{\bar{t}})) \leq 0
\end{aligned}
\tag{2.19}
$$

where u_{MPC} is the sequence of optimal control inputs, $N = T/\Delta$ is the number of hold periods, V_2 is a SCLF that yields a stabilizing $\phi(\hat{x})$, and the Lyapunov constraint has observable arguments. The cost-to-go is given below.

$$
J(\bar{x},t,u) = \textstyle\int_0^T |\bar{x}^u(s;\bar{x},t)|_{Q_w} + |u(s)|_{R_w}\ \mathrm{d}s \tag{2.20}
$$

At each hold period, there is an observer that first updates the state estimates, and then passes them to the prediction model in Eq. (2.19). The observer updates independently as new measurements become available, much more quickly than the actuation changes.

The algorithm implements only the first control action $u[t]$ in the plant, then re-solves the optimization problem at $t + \Delta$ based on the updated state estimates.

As proven previously for state feedback in [9], it is clear that the output feedback LMPC implementation in Eq. (2.19) inherits the properties of the Lyapunov-based controller in *Theorem 2*. Stability (in probability) of the observer error follows from *Lemma 1* and the asymptotic stability of the observer presented in Section 2.4.

Lastly, we discuss the feasibility of the MPC problem. Rather than using soft constrains to ensure feasibility, we give the probability that the LMPC is feasible (i) initially, and (ii) for $\bar{N}$ hold periods. Note that in a probabilistic setting, we cannot guarantee that our problem is feasible for an infinite amount of time.

Theorem 3: *(i) If $\hat{x}(t_0) \in \tilde{\mho}$, then the LMPC in Eq. (2.19) is initially feasible. (ii) If $\hat{x}(t_0) \in \mho_\alpha \subset \tilde{\mho}$, then the event A_F that the LMPC is successively feasible for times $t \in [0, \bar{N}\Delta]$ admits the probability*

$$\mathbb{P}(A_F) \geq (1-\lambda)^{\bar{N}}(1-\zeta)^{\bar{N}}(1-\tilde{\alpha})^{\bar{N}} \tag{2.21}$$

We can recover the successive feasibility result applicable to the state feedback case simply by setting $\zeta = 0$ and $\tilde{\alpha} = \alpha$.

2.6 Simulation Results

We present two examples: the first shows stabilization using the observer from Section 2.4.1, whereas the second is a chemical reactor example using the state estimator in Section 2.4.2. This second system is used to show the experimental probabilities for the events from *Theorem 2*.

2.6.1 Example using Stochastic Feedback Linearization

We consider a stochastic nonlinear system, using an example modified from [13]. The Itô SDE of this system is given by

$$\begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1^2 \\ \Sigma^2 - 2x_1^3 - x_2 + 2x_1x_2 - x_1^2x_2 \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ 1 + x_1^2 \end{bmatrix} u\mathrm{d}t + \Sigma \begin{bmatrix} 1 \\ 1 + 2x_1 \end{bmatrix} \mathrm{d}W_t \tag{2.22}$$

with an output of $y = -x_1$ and where Σ is a noise scale factor. It turns out that this SISO system is globally feedback linearizable to Eq. (2.9) using the following diffiomorphism.

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_1^2 - x_2 \end{bmatrix} \tag{2.23}$$

Using Eq. (2.23), it is easy to verify that in the z domain the Itô system becomes simply

$$\begin{bmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \nu\mathrm{d}t + \begin{bmatrix} -\Sigma \\ -\Sigma \end{bmatrix} \mathrm{d}W_t \tag{2.24}$$

where the output in this space is just $y = z_1$ and the physical input u is related to the virtual input ν by $u = x_2 - (1 + x_1^2)^{-1}\nu$. The observer in Section 2.4.1 is implemented as $\dot{\hat{z}}_i = z_{i+1} + L_i(y - \hat{z}_1)$ with $z_3 = \nu$, $L = [20, 20]^T$. The output-feedback LMPC as described in Section 2.5.3 is designed with a Lyapunov function of the form of $V_2(\cdot)$, given by $V = 100\hat{x}_1^4 + \hat{x}_2^4$, $J = 100x_1^2 + x_2^2 + 5 \times 10^{-6}u^2$, $\Sigma = 0.2$, $\rho = 0.001$, $\Delta = 0.1$, $T_p = 2\Delta$, $T_f = 3$ and subject to the input constraint $|u| \leq 200$.

To verify the design, a thousand sample paths were simulated with an initial condition of $\hat{x}_0 = x_0 = [-5, -5]^T$, and then another one thousand were run from random initializations in the domain $x_0 \in [-5, 5] \times [-5, 5]$. These realizations are shown in Figure 2.6.1 and Figure 2.6.1. In both cases, the closed-loop system stabilizes at the unstable equilibrium point. Because the noise is non-vanishing in this example, we achieve only practical stability in probability (instead of asymptotic stability).

Although not shown, if Δ is too large, then the system is frequently unstable.

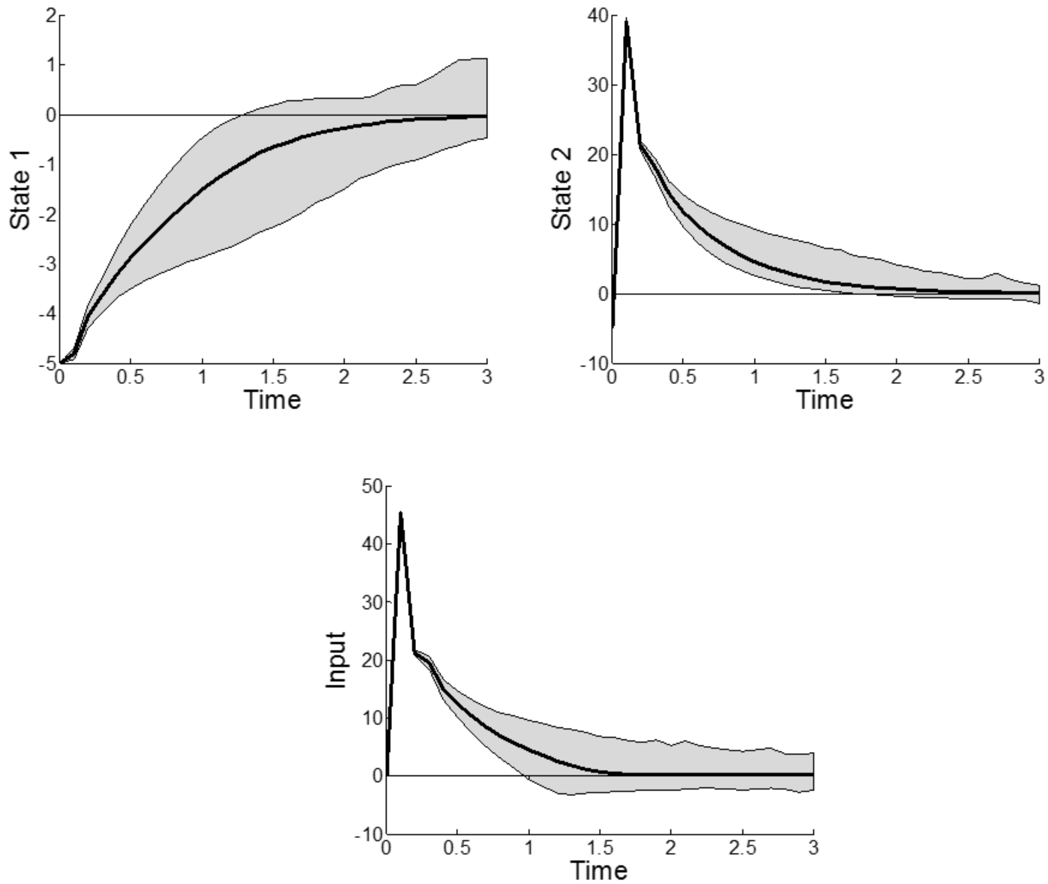

Figure 2.2: (A) Realizations of the system Eq. (2.22) under LMPC action with static initializations at $x_0 = [-5, 5]^T$. Bold line represents the median of all realizations, whereas the upper and lower lines are the 5^{th} and 95^{th} percentiles.

2.6.2 Application to a Chemical Process Example

Consider a constant volume continuous stirred tank reactor (CSTR) where an exothermic reaction $A \rightarrow B$ takes place. The overall dynamics are given by:

$$\begin{aligned} \mathrm{d}C_A &= \left(\frac{F}{V_R}(C_A^{in} - C_A) - k_0 C_A e^{\frac{-E}{RT}}\right)\mathrm{d}t + \sigma_{C_A}(T_R - T_R^S)\mathrm{d}W_{C_A} \\ \mathrm{d}T_R &= \left(\frac{F}{V_R}(T_R^{in} - T_R) + \frac{(-\Delta H)}{\rho_d c_p} k_0 C_A e^{\frac{-E}{RT}} + \frac{Q_R}{\rho_d c_p V_R}\right)\mathrm{d}t + \sigma_{T_R}(T_R - T_R^S)\mathrm{d}W_{T_R} \end{aligned} \tag{2.25}$$

where T_R and C_A are the reactor's prevailing temperature and the concentration of species A, and T_R^{in} and C_A^{in} are the temperature and concentration at the reactor inlet. Other

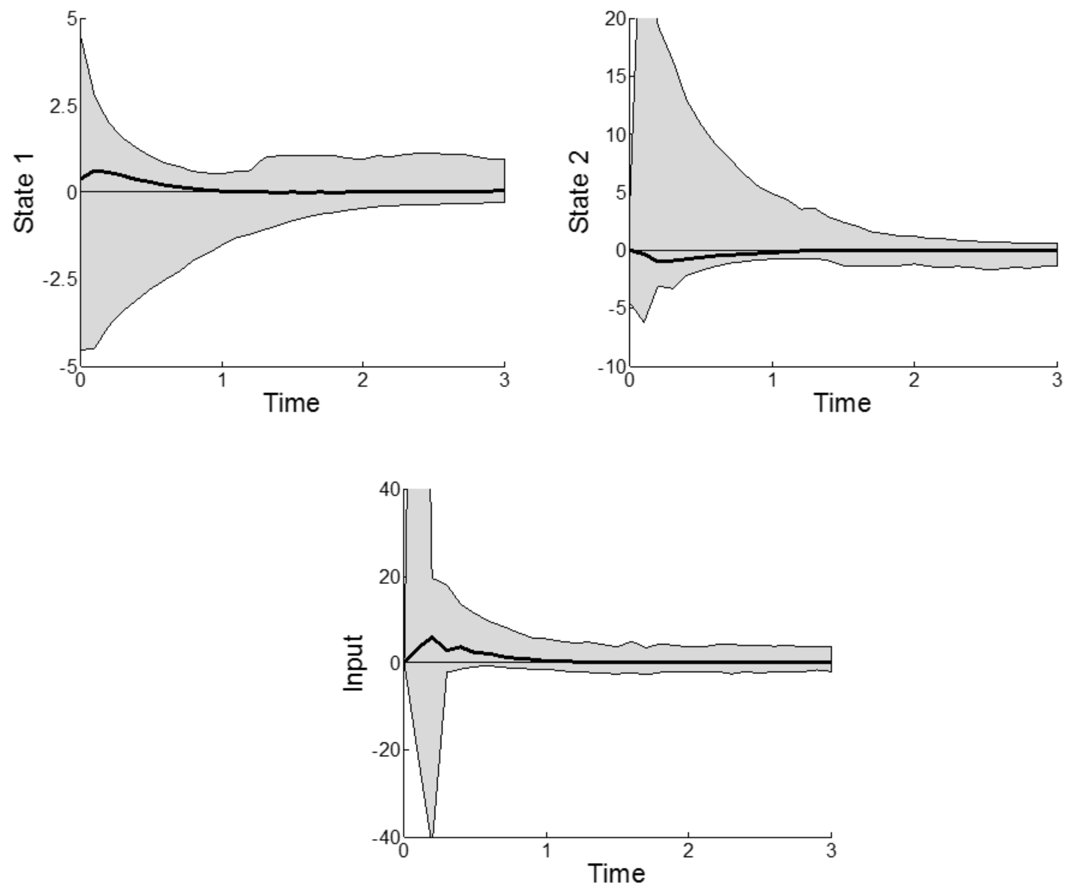

Figure 2.3: Realizations of the system Eq. (2.22) under LMPC action using the variable domain $x_0 \in [-5, 5] \times [5, 5]$. Bold line represents the median of all realizations, whereas the upper and lower lines are the 5^{th} and 95^{th} percentiles.

parameters are defined in Table 1. The manipulated inputs are the heater duty Q_R subject to $|Q_R| \leq 90$ kJ/min, and the feedstock concentration C_{A0} subject to $0 \leq C_{A0} \leq 2$ kmol/m^3. The output-dependent stochastic component impacts both the concentration and temperature.

Remark 5: Note that the specific form of the diffusion terms in Eq. (2.25) (in particular, the vanishing nature, and dependence only on the observable variable) was chosen to illustrate the key idea of the proposed results. The proposed control design, however, when applied to more physically realistic problems (e.g., with non-vanishing diffusion), would yield practical stability in probability (instead of asymptotic stability).

Table 2.1: Chemical reactor parameters and steady–state values.

Parameter	Value	Units
F	100×10^{-3}	$\mathrm{m^3/min}$
V_R	0.1	$\mathrm{m^3}$
ΔH	-4.78×10^4	kJ/kmol
k_0	72×10^9	$\mathrm{min^{-1}}$
E	8.314×10^4	kJ/kmol
R	8.314	kJ/kmol · K
ρ_d	1000.0	$\mathrm{kg/m^3}$
c_p	0.239	kJ/kg · K
C_{A0}	1.0	$\mathrm{kmol/m^3}$
T_{R0}	350.0	K
T_R^s	388.48	K
C_A^s	0.8076	$\mathrm{kmol/m^3}$
σ_{C_A}	5.7×10^{-3}	–
σ_{T_R}	2	–

To estimate C_A, we implement the stochastic nonlinear observer from Section 2.4.2. It can be shown that C_A is observable from T_R whenever $T_R > 0$ and $\Delta H \neq 0$. Using Eq. (2.13) and the coordinate transform $z = C_A - LT_R$, and again referencing [14], our observer takes the following form:

$$\dot{z} = \frac{F}{V_R}(C_A^{in} - LT_R^{in}) - \frac{F}{V_R}z - \left[1 + \frac{L(-\Delta H)}{\rho_d c_p}\right](z + LT_R)k_0 e^{\frac{-E}{RT}} - \frac{LQ_R}{\rho_d c_p V_R} \tag{2.26}$$

where $z(0) = C_A(0) + LT_R(0)$. It is easy to show that this gives the following error dynamics.

$$\mathrm{d}\tilde{z}_t = -\left(\tfrac{F}{V_R} + \left[1 + \tfrac{L(-\Delta H)}{\rho_d c_p}\right] k_0 e^{\frac{-E}{RT}}\right)\tilde{z}_t\,\mathrm{d}t + [\sigma_{C_A}(T_R - T_R^S), -L\sigma_{T_R}(T_R - T_R^S)]\mathrm{d}W \tag{2.27}$$

which is stable whenever $L(-\Delta H) > 0$. It meets the globally Lipschitz requirement of *Assumption 1* because of physical limitations which ensure $T_R > 0$. The control objective is that of stabilization at the unstable equilibrium point (C_A^S, T_R^S).

All SDEs were simulated using *MATLAB*'s *Econometrics Toolbox* and function *SDE.simByEuler* with a step size of $h = 0.001$ min, whereas the optimizations were carried out with the function *fmincon*. We used a hold period of $\Delta = 0.02$ min, constraint parameter $\rho = 0.001$, error bound $E_\zeta = 0.2$, a prediction horizon of $T_p = 2\Delta$, and ran for a total time of $T_f = 2$ min. Lastly, we used the Lyapunov functional of the form $V = (\hat{x}^T P \hat{x})^2$, where $P = 0.05\begin{bmatrix} 0.3333 & 0.0215 \\ 0.0215 & 0.0024 \end{bmatrix}$, while $R_w = 10^{-6}\begin{bmatrix} 0.1 & 0 \\ 0 & 100 \end{bmatrix}$ and $Q_w = \begin{bmatrix} 100 & 0 \\ 0 & 0.1 \end{bmatrix}$ were used for the objective function.

Table 2.2: Observed event frequencies for different values of α and β.

$\tilde{\alpha}$	$\sup_{x_0 \in \mho_\alpha \setminus \mho_\delta^\circ} \mathbb{P}(A_\alpha)$	$\tilde{\beta}$	$\sup_{x_0 \in \partial\mho_{\delta'}} \mathbb{P}(A_\delta)$
0.5	0.103	0.5	0.033
0.6	0.228	0.6	0.082
0.7	0.340	0.7	0.161
0.8	0.490	0.8	0.266
0.9	0.682	0.9	0.386

Similar to [9], we consider the events $A_\alpha = \{\tau_{\mathbb{R}\setminus\mho_\delta^\circ} > \tau_\mho, x_0 \in \mho_\alpha \setminus \mho_\delta^\circ\}$ and $A_\delta = \{\exists t : |x(t)| > d, x_0 \in \mho_\delta\}$. Note that these events imply those in *Theorem 2*.

The simulator 'observed' the occurrence of these events from the simulated trajectories during a single hold period of the controller. The results are reported in Table 2. We discretized the initial sets $\partial\mho_\alpha$ and $\partial\mho_\delta$ into 18 locations and simulated 1000 realizations from each one. In this case, we used $\delta = V_2([C_A^0, T_R^0])$ and calculated an appropriate sequence of disturbances. Since the computed frequencies are less than theoretical maximal probabilities, the simulation corroborates the results presented in *Theorem 2*.

2.7 Conclusion

We presented an output feedback LMPC design for stochastic nonlinear systems. To this end, we first reviewed a compatible observer design and presented a generalization of an existing nonlinear observer in the stochastic setting. We then presented a controller design and defined the stability in probability properties of the closed-loop system and illustrated them using two simulation examples.

2.8 Acknowledgements

Financial support from the National Science and Engineering Research Council and the McMaster Advanced Control Consortium is gratefully acknowledged.

2.9 Proof of Theorem 1

Our strategy will be for the observer-controller feedback system to satisfy *Proposition 2* and hence prove convergence of the observer.

It is clear in view of Eq. (2.14) and vanishing $\tilde{H}$, that asymptotic convergence of the error occurs only at the origin. Elsewhere, the estimates are corrupted by the diffusion term.

Let $V_1 = \frac{1}{4}\sum_i^n \hat{x}_i^4 + \frac{1}{4}\sum_i^n \tilde{x}_i^4 + \frac{1}{4}y^4$. Then we have

$$\begin{aligned}\mathcal{L}V_1(x, y, \tilde{x}) = \sum_i^n \hat{x}_i^3[f_i(\hat{x}) + g_i(\hat{x})u_t + L_i\dot{\tilde{y}}] + \; y^3[m(x) + r(y)u_t] \\ + \sum_i^n \tilde{x}_i^3\gamma_i(\tilde{x}, x, u) + \; \frac{3}{2}\mathrm{tr}\{\tilde{H}(y)\mathrm{diag}[\tilde{x}_1^2 \cdots \tilde{x}_n^2, y^2]\tilde{H}(y)^T\}\end{aligned} \tag{2.28}$$

where $\dot{\tilde{y}} = \dot{y} - m(\hat{x}) - r(y)u_t$.

From here, we invoke *Assumption 1* (iv). Then, we can select some $u(\hat{x}, \dot{y})$, where $\dot{y}$ is a known signal, such that $[g_i(\hat{x}), \cdots, g_n(\hat{x}), r(y)]^T u \leq [-L_1\dot{\tilde{y}}, \cdots, -L_n\dot{\tilde{y}}, -\eta y - m(\hat{x})]^T$, where η is to be determined. We can solve any $(n+1 \times n+1)$ block subsystem and set $k - n - 1$

controls to zero.

Moreover, from *Assumption 1* (ii), we know $m(\cdot)$ is globally Lipschitz in x, and therefore we can infer that $|m(x) - m(\hat{x})| \leq M|x - \hat{x}| \leq M|\tilde{x}|$.

Next, we examine $f(\cdot)$ and $\gamma(\cdot)$. It is a premise of the observer (i.e. *Assumption 1* (i)) that the error dynamics admit $\tilde{x}_i\gamma_i(\cdot) < 0$. Because they are vanishing, by the mean value theorem, we can write each component as $f_i(x) = -x_i\theta_i(x)$ and $\gamma_i(\tilde{x}, x, u, L) = -\tilde{x}_i\sigma_i(\tilde{x}, x, u, L)$, where both $\theta(\cdot), \sigma(\cdot) \geq 0$. By exploiting this, we get

$$\begin{aligned}\mathcal{L}V_1(\hat{x}, \tilde{x}, L) \leq & -\textstyle\sum_i^n \hat{x}_i^4\theta_i(\hat{x}) - \sum_i^n \tilde{x}_i^4\sigma_i(\tilde{x}, x, u, L) \\ & + \tfrac{3}{2}\mathrm{tr}\{H(y)\mathrm{diag}[\tilde{x}_1^2 \cdots \tilde{x}_n^2, y^2]H(y)^T\} + My^3|\tilde{x}| - \eta y^4\end{aligned} \tag{2.29}$$

To proceed, we use Young's Inequality [18], which states that for any $a, b \in \mathbb{R}$, $k > 0$, that

$$ab \leq \frac{k^p|a|^p}{p} + \frac{|b|^q}{qk^q} \tag{2.30}$$

With it, the last term of (2.29) can be written as

$$My^3|\tilde{x}| \leq \frac{3}{4}M\varepsilon_y^{\frac{3}{4}}|y|^4 + \frac{1}{4}\varepsilon_y^{-4}|\tilde{x}|^4 \tag{2.31}$$

For the next step, we use a technique similar to [11]. Using Young's Inequality, properties of norms, the definition of the trace, a standard polynomial inequality, and the mean value theorem, the second last term of (2.29) can be written as

$$\begin{aligned}&\mathrm{tr}\{H(y)\mathrm{diag}[\tilde{x}_1^2 \cdots \tilde{x}_n^2, y^2]H(y)^T\} \\ &\quad \leq n|H(y)H(y)^T\mathrm{diag}[\tilde{x}_1^2 \cdots \tilde{x}_n^2, y^2]|_\infty \\ &\quad \leq n\sqrt{n}|H(y)H(y)^T||\mathrm{diag}[\tilde{x}_1^2 \cdots \tilde{x}_n^2, y^2]|_\infty \\ &\quad \leq n\sqrt{n}|H(y)|^2(|\tilde{x}^2|_\infty + |y^2|_\infty) \\ &\quad \leq n\sqrt{n}|H(y)|^2(|\tilde{x}|_\infty^2 + \sqrt{n}|y|^2) \\ &\quad \leq \tfrac{1}{2}n\sqrt{n}(\varepsilon^{-2}|H(y)|^4 + \varepsilon^2(|\tilde{x}|_\infty^2 + \sqrt{n}|y|^2)^2) \\ &\quad \leq \tfrac{1}{2}n\sqrt{n}(\varepsilon^{-2}|H(y)|^4 + 2\varepsilon^2(|\tilde{x}|_\infty^4 + \sqrt{n}|y|^4)) \\ &\quad \leq \tfrac{1}{2}n\sqrt{n}(\varepsilon^{-2}|\Psi(y)|^4|y|^4 + 2\varepsilon^2(|\tilde{x}|_\infty^4 + \sqrt{n}|y|^4))\end{aligned} \tag{2.32}$$

Now we can choose η as

$$\eta = \tfrac{3}{4}M\varepsilon_y^{\frac{3}{4}} + \tfrac{3}{4}n\sqrt{n}(\varepsilon^{-2}|\Psi(y)|^4 + 2\sqrt{n}\varepsilon^2) \tag{2.33}$$

Lastly, we only have to choose constants such that the effect of the error terms on $\mathcal{L}V$ is overall negative. That is,

$$\textstyle\sum_i^n \tilde{x}_i^4\sigma_i(\tilde{x},x,u) - (n\sqrt{n}\varepsilon^2 + \frac{1}{4}\sqrt{n}\varepsilon_y^{-4})|\tilde{x}|_\infty^4 > 0 \tag{2.34}$$

In view of Eq. (2.14), we can assert that there exists some L such that $\sigma(\cdot) \geq \kappa_{i1}$ and hence $\sigma(\cdot)$ has the property that $0 < \kappa_{i1} \leq \sigma_i$. Then, using the fact that $\sum_i^n \tilde{x}_i^4 = |\tilde{x}^4|_1 \geq |\tilde{x}^4|_\infty = |\tilde{x}|_\infty^4$, we only have to choose ε, ε_y and L such that $p > \rho > 0$.

$$p = \min_i\{\kappa_{i1}\} - (n\sqrt{n}\varepsilon^2 + \tfrac{1}{4}\sqrt{n}\varepsilon_y^{-4}) \tag{2.35}$$

Substituting Eq. (2.33), Eq. (2.35) and Eq. (2.32) into (2.29) leaves us with

$$\mathcal{L}V_1(x,\tilde{x}) \leq -\textstyle\sum_i^n \hat{x}_i^4\theta_i(\hat{x}) - p|\tilde{x}|_\infty^4 \tag{2.36}$$

This proves global asymptotic stability in probability for the state estimator, and also the system as a whole. Given this result, z is also stabilized. ■

2.10 Lemma 1 and Proof

Consider the Lyapunov functionals V_1 and V_2 and the sets $\Pi(V_1,\rho) = \{x \in \mathbb{R}^{n+1} : \inf_{u\in\mathcal{U}} \mathcal{L}V_1(\hat{x},\tilde{x}) + \rho V_1(\hat{x},\tilde{x}) \leq 0,\ \forall\ \tilde{x} \in \mathbb{R}^n\}$ and $\Pi(V_2,\rho) = \{x \in \mathbb{R}^{n+1} : \inf_{u\in\mathcal{U}} \mathcal{L}V_2(\hat{x},\tilde{x}) + \rho V_2(\hat{x},\tilde{x}) \leq 0,\ \forall\ \tilde{x} \in \mathbb{R}^n\}$.

In *Theorem 1*, V_1 was used to show stability of the overall system by accounting for the evolution of the unobservable error, whereas V_2 does not incorporate the error nor does its use imply stabilization. To use the observable V_2 as a proxy for V_1 in our LMPC design, we need to show that for both observers, under certain conditions, satisfying $x \in \Pi(V_2,\bar{\rho})$ implies $x \in \Pi(V_1,\rho)$, with $\bar{\rho}$ to be determined.

Lemma 1: *For the system Eq. (2.1), there exists an observer gain vector* L^* *such that if each* L_i *verifies* $\mathrm{sign}(L_i^*)L_i \geq |L_i^*|\ \forall\ i \in \mathbb{N}(n)$, *then* $\Pi(V_2,\bar{\rho}) \subseteq \Pi(V_1,\rho)$. *(i) For the observer* *in Eq. (2.10) of Sections 2.4.1,* $\bar{\rho} = \rho$*; (ii) for the observer in Eq. (2.13) of Section 2.4.2,* $\bar{\rho} > \rho$*.*

Proof of Lemma 1: Let $\mathcal{C}(V(x)) = \mathcal{L}V(x) + \rho V(x)$, the value of the Lyapunov constraint in Eq. (2.19). Eq. (21) in [11] shows that for the observer in Section 2.4.1:

$$\mathcal{L}V(\hat{z}, \tilde{x}) \leq -\sum_i^{n+1} c_i \hat{z}_i^4 - p \sum_i^{n+1} \tilde{x}_i^4 \tag{2.37}$$

for some constants c and virtual variable $z := z(\hat{x})$ diffiomorphic to x. It is analogous to Eq. (2.36) in this chapter for the observer in Section 2.4.2. We see that for either observer:

$$\begin{aligned} \mathcal{C}(V_1(x)) &= \mathcal{C}(V_2(x)) + \mathcal{C}(V_1(x)) - \mathcal{C}(V_2(x)) \\ &\geq \mathcal{C}(V_2(x)) + \eta y^4 - p|\tilde{x}|_\infty^4 + \tfrac{1}{4}\rho|\tilde{x}|_\infty^4 \\ &\geq \mathcal{C}(V_2(x)) + 4\eta V_2(x) + (\tfrac{1}{4}\rho - p)|\tilde{x}|_\infty^4 \\ &\geq \mathcal{L}V_2(x) + (\bar{\rho} + 4\eta)V_2(x) + (\tfrac{1}{4}\rho - p)|\tilde{x}|_\infty^4 \end{aligned} \tag{2.38}$$

In [11] Eq. (17) and similarly in Eq. (2.35) of this chapter (i.e. for both observers), it was shown that $p := p(L)$ can be made arbitrarily large by choice of observer gain vector L (or equivalently, ℓ). By choosing $p > \frac{1}{4}\rho$, the last term in (2.38) is negative definite.

For part (i), we have $\eta = 0$ and hence (2.38) proves that $\Pi(V_2, \rho) \subseteq \Pi(V_1, \rho)$. For part (ii), we have $\mathcal{C}(V_2(x), \bar{\rho}) \leq 0$ for some $\bar{\rho} = \rho + 4\eta$, where $\eta := \eta(\varepsilon_y)$ can be regarded as a design parameter. Hence, we have $\mathcal{C}(V_1(x), \rho) \leq 0$ which gives $\Pi(V_2, \bar{\rho}) \subseteq \Pi(V_1, \rho)$, as desired. ■

2.11 Proof of Lemma 2

We will first show part (i), following along the lines of the Proof of Lemma 1 in [9].

Consider the collection of realizations $A_B = \{\omega : \sup_{t\in[0,\Delta]} |W_t| < B, B \in \mathbb{R}^+\}$ where the vector of Brownian motions is bounded. It is a standard result that there exists a probability λ such that $\mathbb{P}(A_B) = (1 - \lambda)$ [19].

We also make use of the fact that trajectories $\omega \in A_B$ are Holder continuous, meaning that $|x_t - x_0|_\infty \leq K_1(t-t_0)^\gamma$ for any $\gamma < \frac{1}{2}$, where $K_1 := K_1(\lambda, \Delta)$ for all $t - t_0 \leq \Delta$.

We have also assumed that $\mathcal{L}V_2(x_t)$ from Eq. (2.2) is composed of terms that are all locally Lipschitz continuous on a sufficiently large compact rectangle, so let K_2, K_3, and K_4 be their respective Lipschitz constants. Then for any z_2, z_1, we get

$$\begin{aligned} \mathcal{L}V_2(z_2) - \mathcal{L}V_2(z_1) &= (L_fV_2(z_2) - L_fV_2(z_1)) + (L_gV_2(z_2) - L_gV_2(z_1))u_0 \\ &+ \tfrac{1}{2}\left(\mathrm{tr}\left\{h(z_2)^T \tfrac{\partial^2 V_2(z_2)}{\partial x^2} h(z_2)\right\} - \mathrm{tr}\left\{h(z_1)^T \tfrac{\partial^2 V_2(z_1)}{\partial x^2} h(z_1)\right\}\right) \\ &\leq (K_2 + K_3 + K_4)|z_2 - z_1|_\infty \equiv K_7|z_2 - z_1|_\infty \end{aligned} \tag{2.39}$$

To employ this result, we begin by separating $\mathcal{L}V_2$ into

$$\mathcal{L}V_2(x_t) = \mathcal{L}V_2(x_0) + \left[\mathcal{L}V_2(x_t) - \mathcal{L}V_2(x_0)\right] \tag{2.40}$$

However, recall whenever $x_0 \in \tilde{\mho}$ and $u = \phi(x[t])$, the system will satisfy $\mathcal{L}V_2(\hat{x}) \leq -\rho V_2(\hat{x})$ at $t = t_0$. To proceed, we note that local Lipschitz continuity of $\mathcal{L}V$ implies that

$$\begin{aligned} \mathcal{L}V_2(x_t) &= \mathcal{L}V_2(\hat{x}_0) + \left[\mathcal{L}V_2(x_0) - \mathcal{L}V_2(\hat{x}_0)\right] + \left[\mathcal{L}V_2(x_t) - \mathcal{L}V_2(x_0)\right] \\ &\leq \mathcal{L}V_2(\hat{x}_0) + K_7|x_0 - \hat{x}_0|_\infty + K_7|x_t - x_0|_\infty \end{aligned} \tag{2.41}$$

Next, in view of *Assumption 2*, we claim that there is an event A_ζ such that the observation errors satisfy $|\tilde{x}_0|_\infty \leq E_\zeta < E$ with probability $\mathbb{P}(A_\zeta) = (1-\zeta)$. Using this, the Holder continuity established previously, and the Lyapunov constraint using the ρ defined in (2.38), we can write Eq. (2.41) as

$$\begin{aligned} \mathcal{L}V_2(x) &= -\rho V_2(\hat{x}) + K_7|\tilde{x}| + K_7K_1\Delta^\gamma \\ &\leq -\rho\tilde{\delta} + K_7K_1\Delta^\gamma \end{aligned} \tag{2.42}$$

where $\tilde{\delta} = \delta - \frac{K_7}{\rho}E_\zeta$. Note that Eq. (2.42) applies to $\hat{x}_0 \in \tilde{\mho}\backslash\mho_\delta$ because we used $\hat{x}_0 \in \partial\mho_\delta$, which is the smallest level set. Remember also that by by construction, $\tilde{\delta} > 0$, since $\mho_N \subseteq \mho_\delta \subset \tilde{\mho}$.

In view of Eq. (2.42), to satisfy $\mathcal{L}V_2(x_t) \leq -\epsilon$, where $0 < \epsilon < \rho\tilde{\delta}$ is an arbitrarily small positive number, we can define

$$\Delta_1 = \left(\frac{\rho\tilde{\delta} - \epsilon}{K_1K_7}\right)^{\frac{1}{\gamma}} \tag{2.43}$$

which would imply that $\mathcal{L}V_2(x) < 0$ for all $t \in [0, \tau_{\tilde{\mho}\backslash\mho_\delta}(\Delta_1)]$ whenever $\omega \in A_B \cup A_\zeta$. The time domain given is justified by the fact that we cannot guarantee constraint satisfaction in $\tilde{\mho}^c$, and because the hold time is too large to succeed in $\mho_\delta^\circ$.

For part (ii), consider again $\omega \in A_B \cup A_\zeta$. By compactness, we have $|V_2(x_2) - V_2(x_1)| \leq K_5|x_2 - x_1|_\infty$. Since $\hat{x}_0 \in \mho_\delta$, we know that $V_2(\hat{x}_0) \leq \delta$. Then, we can write $V_2(x_t)$ as

$$\begin{aligned} V_2(x_t) &= [V_2(x_t) - V_2(x_0)] + [V_2(x_0) - V_2(\hat{x}_0)] + V_2(\hat{x}_0) \\ &\leq K_5 K_1 \Delta^\gamma + K_5|\tilde{x}| + \delta \leq K_5 K_1 \Delta^\gamma + \breve{\delta} \end{aligned} \tag{2.44}$$

where we have labeled $\breve{\delta} = K_5 E_\zeta + \delta$.

Next, consider some excursion of magnitude $|x|_Q = d$, and let $\mathcal{Z} = \mathbb{R}^n \backslash \bar{B}_d^Q = (\bar{B}_d^Q)^c$. Then, we can use continuity of V_2 to impose the existence of a δ' that satisfies $\delta' = \inf_{y \in \mathcal{Z}} V_2(y) = \inf_{|y|=d} V_2(y)$. Using this, provided that $\delta' - \breve{\delta} > 0$, we can solve (2.44) for a hold time

$$\Delta_2 \leq \left(\frac{\delta' - \breve{\delta}}{K_1 K_5} \right)^{\frac{1}{\gamma}} \tag{2.45}$$

This is the maximal hold time such that realizations starting on $\hat{x} \in \partial\mho_\delta$ still cannot reach $\mho_{\delta'}$. Thus, finally, choosing $\tilde{\Delta}^* \leq \min\{\Delta_1, \Delta_2\}$ proves the Lemma.

To prove our claim that $|\tilde{x}|$ can be made to obey $|\tilde{x}|_\infty \leq E_\zeta$, we start by solving the error SDE. Recall that observer updates independently from the control action, and much more frequently. The event A_B restricts the trajectory to a compact set, so, referring to *Theorem 1*, we can define maximal diffusion $\Phi = |\sup_{x,y} \tilde{H}_i(x,y)|_\infty$ and drift $A|\tilde{x}|_\infty = |\sup_{x,y} \sigma_i(x,y)|_\infty |\tilde{x}|_\infty$. Assuming for simplicity that $\tilde{x}(0) = \mathbb{E}(\tilde{x}(0)) = 0$, we can write the error SDE as the scalar process $\mathrm{d}|\tilde{x}|_\infty \leq A|\tilde{x}|_\infty \mathrm{d}t + M\mathrm{d}W_t$. This equation has the integral definition $|\tilde{x}|_\infty = \Phi \int_0^t e^{A(s-t)} dW_s$ [20]. Using Itô's isometry, we can easily compute its variance.

$$\begin{aligned} \mathrm{Var}(|\tilde{x}|_\infty) &= \mathbb{E}(|\tilde{x}|_\infty^2) - \mathbb{E}(|\tilde{x}|_\infty)^2 \\ &= \mathbb{E}\left\{ (\Phi \textstyle\int_0^t e^{A(s-t)} dW_s)^2 \right\} \\ &= \Phi^2 \textstyle\int_0^t e^{2A(s-t)} ds = \frac{\Phi^2}{2A}(1 - e^{-2At}) \end{aligned} \tag{2.46}$$

Clearly, Eq. (2.46) eventually approaches $|\tilde{x}|_\infty \sim \mathcal{N}(0, \frac{\Phi^2}{2A})$. Hence, by considering only some subset of realizations A_ζ, we can retrieve from standard probability tables a probability $\mathbb{P}(A_\zeta) = (1-\zeta) \in [0,1]$ such that $|\tilde{x}|_\infty \leq E_\zeta$. ■

2.12 Proof of Theorem 2

The reader is advised that our proof draws on some definitions stated in the proof of *Lemma 2*.

In what follows, we denote by the superscript $*$ any probability that is conditional upon the event $A_B \cup A_\zeta$.

To show part (i) for all $\hat{x}_0 \in \mho_\delta$, it suffices to show that it holds for $\hat{x}_0 \in \partial\mho_\delta$. As such, we can be assured that $V_2(x_0) \leq \breve{\delta}$, where $\breve{\delta} \geq \delta$. To approach part (i), we use an argument from [16] as follows.

$$\begin{aligned} \mathbb{E}(V(x_t)) &= \int V(y)\mathrm{d}\mathbb{P}(y) \geq \int_{\mathcal{Z}} V(y)\mathrm{d}\mathbb{P}(y) \\ &\geq \int_{\mathcal{Z}} \inf_{y\in\mathcal{Z}} V(y)\mathrm{d}\mathbb{P}(y) \geq \inf_{y\in\mathcal{Z}} V(y)\cdot\mathbb{P}(\mathcal{Z}) \end{aligned} \tag{2.47}$$

Using Eq. (2.3) and *Lemma 2* (i) which together imply the supermartingale property of $\mathbb{E}(V_t) < V(x_0)$, we get

$$\mathbb{P}^*\left(|x_t| > d \text{ for some } t > 0\right) \leq \frac{V_2(x_0)}{\inf_{y\in Z} V_2(y)} \tag{2.48}$$

Next, recall that $\delta < \breve{\delta} < \delta'$. Then, we can define the probability $\tilde{\beta} \in (\beta, 1)$ as $\tilde{\beta} = \breve{\delta}/\delta' > \beta = \delta/\delta'$. Taking complements and the supremum of both sides, we get

$$\sup_{\hat{x}_0\in\partial\mho_\delta} \mathbb{P}^*\left(|x_t| \leq d,\ \forall t > 0\right) \geq (1-\tilde{\beta}) \tag{2.49}$$

Lastly, using Bayes' formula, part (i) is complete.

For (ii), we want to quantify the probability of the event $A_T = \{\omega : \tau_{\mathbb{R}^n\setminus\mho^\circ_{\breve{\delta}}} > \tau_{\tilde{\mho}}\}$. With a reminder that we are strictly confined to the time domain $t \in [0, \Delta]$ defined previously, we begin by recalling from [9] that these hitting times occur in finite time almost surely (i.e. $\tau_{\tilde{\mho}\backslash\mho_\delta} < \infty$).

Now, similar to the state feedback configuration in *Proposition 4*, by realizing that $A_T \subseteq \{\omega : V_2(x(\tau_{\tilde{\mho}\backslash\mho_\delta})) \geq 1\}$, we can state that

$$\mathbb{P}^*(A_T) \leq \mathbb{P}^* \left(V_2(x(\tau_{\tilde{\mho}\backslash\mho_\delta})) \geq 1 \right) \tag{2.50}$$

Next, consider (2.47) and the inequality in Eq. (2.48). To apply this to Eq. (2.50), we observe that because $\inf\{V(\cdot) : V_t(\cdot) \geq 1\} = 1$, we can write simply that

$$\mathbb{P}^* \left(V_2(x(\tau_{\tilde{\mho}\backslash\mho_\delta})) \geq 1 \right) \leq \mathbb{E}^* \left(V_2(x(\tau_{\tilde{\mho}\backslash\mho_\delta})) \right) \tag{2.51}$$

Then, taking the supremum over x_0 on both sides and again using $\mathbb{E}(V_t) < V(x_0)$ for any $t > t_0$, we get

$$\sup_{x_0 \in \mho_\alpha \backslash \mho_\delta^\circ} \mathbb{P}^*(\tau_{\mathbb{R}^n \backslash \mho_\delta^\circ} > \tau_\mho) < \sup_{x_0 \in \mho_\alpha \backslash \mho_\delta^\circ} V_2(x_0) \tag{2.52}$$

To convert this statement to one about estimated states, it is enough to evaluate $V_2(x_0)$ over the largest space of initial process conditions implied by both the given set of estimates and constraints on $\tilde{x}$. To this end, we can write (2.52) as

$$\begin{aligned} \sup_{\hat{x}_0 \in \mho_\alpha \backslash \mho_\delta^\circ} \mathbb{P}^*(A_T) &\leq \sup_{\hat{x}_0 \in \mho_\alpha \backslash \mho_\delta^\circ} V_2(x_0) \\ &= \sup_{\hat{x}_0 \in \mho_\alpha \backslash \mho_\delta^\circ} \{V_2(x_0) - V_2(\hat{x}_0) + V_2(\hat{x}_0)\} \\ &\leq K_5 E_\zeta + \alpha \equiv \tilde{\alpha} \end{aligned} \tag{2.53}$$

where to separate $V_2(x_0)$, we used a procedure similar to (2.44).

Then, it follows that, if E_ζ and $\mho_\alpha$ are chosen such that $\tilde{\alpha} < 1$, we can recover $\mathbb{P}^*(\tau_{\mathbb{R}^n \backslash \mho_\delta^\circ} < \tau_\mho)$ by taking the complement of (2.53) and recognizing that continuity implies $\tau_{\mathbb{R}^n \backslash \mho_\delta^\circ} \neq \tau_{\tilde{\mho}}$ almost surely.

For part (iii), we see that it remains only to write the probability that the event in part (i) occurs after A_T occurs. Thus, by applying Bayes' law again, the proof is complete. ■

2.13 Proof of Theorem 3

For part (i), we use the property that $\hat{x}_0 \in \tilde{\mho}$ by construction implies $\hat{x}_0 \in \Pi$. When subject to $u[t] = \phi(x_0)$, for the first input in the sequence, we will have $\mathcal{L}V_2(\hat{x}_0) < -\rho V_2(\hat{x}_0)$. If, assuming N prediction intervals, the other $N-1$ control actions in the prediction horizon were set to zero, then this sequence of controls would give us initial feasibility of the optimization problem.

For part (ii), we use *Theorem 2* (ii), which gives the probability that the system trajectory is confined to $\{V_2(\hat{x}_t) < 1\}$ for all $t \in [0, \Delta]$. Because this implies that $\hat{x}(\Delta) \in \tilde{\mho}$, using part (i) of this Theorem, it also implies that the problem is feasible. To show successive feasibility for $\bar{N}$ periods, we see that we can recursively apply *Theorem 2* (ii). The probability can then be written as:

$$\begin{aligned} &\mathbb{P}(x(0) \in \tilde{\mho} \cup \cdots \cup x(\bar{N}\Delta) \in \tilde{\mho}) \\ &\quad \geq (1-\tilde{\alpha}_1)(\cdots)(1-\tilde{\alpha}_{\bar{N}})(1-\lambda)^{\bar{N}}(1-\zeta)^{\bar{N}} \end{aligned} \tag{2.54}$$

To get the risk margins $\tilde{\alpha}_k$ at each of the $k \in \mathbb{N}(\bar{N})$ periods, note that from Eq. (2.51) and (2.53), we can quantify it as $\tilde{\alpha}_k \geq \mathbb{E}(V_2(\tau_{\tilde{\mho}})|V_2(x[(k-1)\Delta]) = \tilde{\alpha}_{k-1}) + K_5 E_\zeta$. However, $V_2(x[(k-1)\Delta])$ is unknown at $t = 0$ (except for $V_2(x_0)$). By observing that repeated occurrence of $A_B \cup A_\zeta$ implies that $\mathbb{E}(V_2(\bar{N}\Delta)) < \cdots < \mathbb{E}(V_2(\Delta)) < V_2(x_0) \leq \alpha_1$, we can use $\tilde{\alpha}_k = \tilde{\alpha}$ for all k. ■

2.14 Stochastic Feedback Linearization

This summary was omitted from the submitted papers for brevity, although it explains how Example 1 was constructed. It is reproduced below.

The following concepts enabled us to construct the observer in Section 2.4.1 for the system in (2.1). For another design, see Section 2.4.2.

One class of systems for which a suitable observer can be designed are those that can be transformed exactly using some $z_t = \chi(x_t)$ into the integrator chain form:

$$\begin{aligned} \mathrm{d}z_i &= z_{i+1}\mathrm{d}t + \varphi_i(y)\mathrm{d}W_i \quad && \forall\ i \in \mathbb{N}(n-1) \\ \mathrm{d}z_n &= \nu\mathrm{d}t + \varphi_i(y)\mathrm{d}W_i \quad && y = z_1 \end{aligned} \tag{2.55}$$

This leads to the following definition:

Definition 3 [13]: A system as in (2.1) is called feedback linearizable if it can be transformed into (2.55).

Next, recall (e.g. [20]) that the dynamics in (2.1) is equivalent to its Stratonovich integral representation. Because the Stratonovich integral will be desired in the ensuing formulation, we will state the conversion from an Itô integral to a Stratonovich integral.

Proposition 5 [13]: Given the Itô dynamical system (2.1), there is a transformation that gives the system as a Stratonovich SDE $\mathrm{d}x_t = f(x_t)\mathrm{d}t + g(x_t)u_t\mathrm{d}t + H(x_t) \circ \mathrm{d}W_t$ through the correction

$$f(x_t) = a(x_t) - \frac{1}{2}\frac{\mathrm{d}h(x_t)}{\mathrm{d}x}h(x_t) \tag{2.56}$$

One fortunate property of the Stratonovich representation is that it permits the same linearizing diffiomorphism as the analogous deterministic system in. This is formalized below.

Proposition 6 [13]: Given the dynamics of a Stratonovich stochastic process, if the corresponding deterministic system can be feedback linearized using $\chi(x)$, then the stochastic system is also feedback linearized by the same $\chi(x)$.

The inverse correction of (2.56) is then used to convert the linearized system back to an Itô integral.

It has been shown that the transformation $\chi(x_t)$ exists if

Proposition 7 [13]: The system in (2.1) is feedback linearizable if and only if it verifies the following

$$\begin{aligned} &\mathrm{rank}[g, \mathrm{ad}_f g \ldots \mathrm{ad}_f^{n-2} g] = n-1 \\ &\mathrm{rank}[h_i, \mathrm{ad}_f h_i \ldots \mathrm{ad}_f^{n-2} h_i] = n-1 \quad \forall\ i \in \mathbb{N}(n) \end{aligned} \tag{2.57}$$

The follows from standard feedback linearization theory that the diffiomorphism is found from

Proposition 8 [12]: The diffiomorphism, if it exists, is found by solving the partial differential equations

$$\begin{aligned} &\langle \mathrm{d}\tau, \mathrm{ad}_f^i \rangle = 0, \qquad \forall\ i \in \mathbb{N}(n-2) \\ &\langle \mathrm{d}\tau, \mathrm{ad}_f^{n-1} \rangle \neq 0 \end{aligned} \tag{2.58}$$

Then the components of the linearizing diffiomorphism $z_t = \chi(x_t)$ and the control $u = \alpha(z)\nu + \beta(z)$ are determined from

$$\begin{aligned} &z_j = L_f^{j-1}\tau, \quad \forall\ j \in \mathbb{N}(n) \\ &\alpha = \tfrac{-L_f^n \tau}{L_g L_f^{n-1}\tau} \qquad \beta = \tfrac{1}{L_g L_f^{n-1}\tau} \end{aligned} \tag{2.59}$$

Chapter 3

Constructing Constrained Control Lyapunov Functions for Control-Affine Nonlinear Systems

This chapter encompasses the ongoing investigation into the null controllable region (NCR) of nonlinear control systems. Although the paper has been mostly been prepared individually by the author, it is being co-authored with Maaz Mahmoud and Prashant Mhaskar. At the time of writing, this work has not formed any submitted paper.

In this work, we first endeavor to define the boundary of the region from which the states can be controlled to the origin (i.e. the NCR). We have found from recent results [21] that this problem is unexpectedly related to the time-optimal control of the same system. We prove that the boundary is defined by a set of trajectories with saturated control action (i.e. $u = \pm 1$). The proof technique for this is to show that there is a quantity (the Hamiltonian) that must be maximized along this trajectory, and the only way to accomplish that is to apply saturated control. We next show that for many systems, the equilibrium points of the system under saturated control must be on the boundary of the NCR. This is shown by noticing that under certain conditions, which for now we have assumed, it would take infinite time to reach these equilibrium points in reverse time, beginning at the origin. Lastly, we show that by computing time-optimal controls from one forced-system equilibrium point to the other, we traverse the boundary of the NCR for planar systems. We propose a numerical method which uses an indirect optimization method to exactly solve for the controller. This method, the solution of the costate ODE, is described later. The results are general enough to be applicable to higher dimensional systems, although these results are not yet complete.

The second contribution in this chapter is to describe a control law which stabilizes nonlinear systems with constrained inputs. Our results show that if a controller has the property that it forces the system into successively smaller shells of the NCR, then it conveys asymptotic stability. Identical to [22], we then show that the MPC formulation has these necessary properties, and hence stabilizes from everywhere in the NCR.

The rest of this chapter is formed by a reproduction of this draft paper.

3.1 Abstract

In this work we consider control affine nonlinear systems, and present a construction method for constrained control Lyapunov functions, and an illustrative control design that achieves stabilization from the null controllable region (NCR). To this end, we first propose a construction of the NCR for control affine nonlinear systems subject to input constraints. Then, the characterization is utilized in the construction of a constrained control Lyapunov function. An illustrative control design is presented that enables stabilization from all initial conditions in the NCR.

3.2 Introduction

Systems exhibiting nonlinearity, constraints, and uncertainty are ubiquitous in practice and can pose significant challenges. Previous results have shown that control designs which do not acknowledge these features can under-perform more advanced techniques or even result in closed-loop instability [23]. In many applications, particularly those systems with input constraints, control designs are sought that guarantee stabilization to the origin from the largest possible set of initial conditions. This set has been termed the *null controllable region* (NCR) [24].

Constrained Control Lyapunov Functions (CCLF) are Lyapunov functions designed so that the control laws which guarantee their decay are admissible and stabilizing for the constrained system with a region of attraction equal to the NCR [22]. Clearly, control designs using control Lyapunov functions that are not CCLFs only guarantee stabilization within subsets of the NCR [25]. There currently do not exist results that identify the NCR of a general nonlinear system or give controls laws for its stabilization everywhere within the NCR.

Motivated by the above, in this work we consider the problem of determining the NCR of control-affine nonlinear unstable systems with constrained controls.

Our construction of the nonlinear NCR is a generalization of the results for linear systems [23], where it was shown that for single-input linear systems of arbitrary dimension, the NCR of the system was covered by extremal trajectories of the reverse-time system, i.e. its reachable set. Via convex analysis, it was shown that these trajectories are induced solely by all bang-bang controls with a known number of switches.

To determine the NCR for nonlinear systems, we will exploit the fact that time-optimal trajectories traverse the boundary of the NCR [21; 26]. Nonlinear time-optimal controls were prominently investigated in [27] (and later works) where conditions were given for solutions to be strictly bang-bang. These characterizations, which give rise to feedback control laws, also have an analogue in the nonlinear setting.

To study time-optimal trajectories in the nonlinear setting, we will make use of Pontryagin's Maximum Principle and the Hamilton-Jacobi-Bellman equation (for more details, see [28]). However, because of the various complexities of the nonlinear case, we use a direct optimization approach in conjunction with some simplifications.

We will show that the NCR is a CCLF and that decay of the trajectory into successive concentric sub-NCRs, corresponding to proportionally smaller input requirements, is sufficient to result in stabilization to the origin. To illustrate the proposed approach, we will employ the CCLF in a model predictive controller that guarantees stability everywhere by constraining the control action to those which force the trajectory into successive shells of the NCR.

The rest of the chapter is organized as follows: In Section 3.3, we will introduce some preliminaries. In Section 3.4, we will first justify our method of constructing the NCR, and then outline a numerical method for the most general and then more simplified cases. Then we will employ this NCR as a CCLF and show that it results in a stabilizing control law. Lastly, in Section 3.5, we will illustrate the application using simulation examples.

3.3 Preliminaries

We consider nonlinear, single-input, affine-in-the-control dynamical systems of the form:

$$\dot{x}(t) = f(x) + g(x)u \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, so that $f(\cdot), g(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$.

We further assume that all controls are constrained to the *control set* $u \in \mathcal{U} = [-1, 1]$ so that $u(t) : \mathbb{R}^+ \to \mathcal{U}$ and call such maps *admissible controls*. We also assume that $f(\cdot), g(\cdot)$ are locally Lipschitz continuous.

We also declare the following assumption on (3.1) that will be useful later. Using (iterated) Lie bracket notation, we need that:

Assumption 1: For any vector $\lambda \in \mathbb{R}^n$, and $x \in \mathbb{R}\backslash\{0\}$ there exists a $k := k(x) \in \mathbb{N}$ such that $\lambda \cdot \mathrm{ad}_g^k f \neq 0$.

Remark 1: For planar systems with $[f(x), g(x)] \equiv \alpha(x)f(x) + \beta(x)g(x)$, it is sufficient for $\alpha(\cdot)$ to be sign definite [26]. Situations where $k \geq 3$ are quite rare [26].

We will denote the solution of (3.1) given $x(0) = x_0$ and applying $u(t)$ by $x(t, u(t); x_0)$. We assume throughout that $x = 0$ is globally asymptotically unstable for the unforced system, i.e. $\lim_{t\to\infty} |x(t, 0; \xi_0)| = \infty$ whenever $\xi_0 \neq 0$. For simplicity of presentation, we assume that the unconstrained system is globally controllable [29].

Remark 2: Note that the condition requiring unstable states is strict. It is a future avenue of research to determine the NCR of systems which have one or more open-loop stable states.

Furthermore, if $V(\cdot) : \mathbb{R}^n \to R^+$ is a Lyapunov functional for (3.1), and $\dot{V}$ is its derivative along the solution $x(t)$, then:

$$\begin{aligned} &\Pi \equiv \{x : \inf_{u\in\mathcal{U}} \dot{V}(x, u) \leq 0\} \\ &\mho_c \equiv \{x : x \in \Pi, V(x) \leq c\} \end{aligned} \tag{3.2}$$

As discussed earlier, for an arbitrary choice of $V(\cdot)$, $\mho = \sup_c \mho_c$ is not necessarily the largest set of ξ_0 such that $x(t, u; \xi_0) \to 0$ as $t \to \infty$.

Towards finding a largest such set, we define the NCR of the system in (3.1) as:

$$\mathcal{C} = \{\xi_0 : \exists\, \tilde{T} : \ s.t.\ x(\tilde{T}, u(t); \xi_0) = 0, u \in \mathcal{U}\} \tag{3.3}$$

Note that $\mathcal{C} \neq \emptyset$ and $\mho \subseteq \mathcal{C} \subset \mathbb{R}^n$. It is known that $\mathcal{C}$ is compact and connected, but generally not convex or symmetric.

Note that $\mathcal{C}$ is equivalent to the *reachable set* of the reverse time system given by:

$$\dot{z}(t) = -f(z) - g(z)v \tag{3.4}$$

with $v(t) = u(T - t)$ on any $t \in [0, T]$. $v(t)$ can thus be thought of as being the reverse plan to $u(t)$ on the interval $[0, T]$. Recall that trajectories of $z(t)$ trace the same paths as $x(t)$, but in reverse. Then the reachable set $\mathcal{R}_0$ (for the original system) is given simply by:

$$\mathcal{R}_0 = \{z : z(t, v(t); 0), v \in \mathcal{U}\} \tag{3.5}$$

We will now review several useful properties of Pontryagin's Maximum Principle. Let us define a value function as:

$$J = -\int_0^T L(x, u)\mathrm{d}t \tag{3.6}$$

and define u^* as the control trajectory that maximizes J and $x*$ as the corresponding state trajectory. This value function, along the flow of the optimal solution (x^*, u^*), is known to satisfy the Hamilton-Jacobi-Bellman equation. That is:

$$J_t + J_x \frac{\mathrm{d}x}{\mathrm{d}t}(x^*, u^*) + L(x^*, u^*) = 0 \tag{3.7}$$

where J_a is a partial derivative with respect to parameter a. It is known that there exists a unique *viscosity solution* to this equation whenever the dynamics are Lipschitz continuous [30].

It is useful now to recall the general free-time time-optimal control problem. In this case, the value function is just the time taken to reach some target manifold from a specified initial point. Hence, $J = -t_f$ and $L(\cdot) = 1$.

Notice that the definition of $\mathcal{C}$ implies that any $x \in \mathbb{R}^n$ for which there exists a solution to the free-time time-optimal control problem that also belongs to $\mathcal{C}$.

We also define the so-called costate $p(t) \in \mathbb{R}^n$ and the *Hamiltonian* of the system as:

$$H(x, u, p) = \lambda_0 L(x, u) + p(t)[f(x) + g(x)u(t)]^T \tag{3.8}$$

It is well known that the control $u^*(t)$ that maximizes the value function (3.6) also maximizes the Hamiltonian (3.8). That is:

$$H^*(t, x^*, u^*) = \max_{u(t)} \{H(x, u, p)\} \tag{3.9}$$

For problems with free end-times, the Hamiltonian is actually everywhere zero along the flow of $x^*(t)$, that is:

$$H^*(t, x^*, u^*, p^*) = 0 \tag{3.10}$$

and the solution evolves as a set of coupled ODEs governed by:

$$\begin{aligned} \dot{p}^*(t) = &\quad -H_x(x^*, u^*, p^*) \\ \dot{x}^*(t) = &\quad H_p(x^*, u^*, p^*) \end{aligned} \tag{3.11}$$

We say that $u(t)$ is a *bang-bang* control if it is only at the extreme points of $\mathcal{U}$, i.e. $u(t) \in \{-1, 1\}$ almost everywhere.

The next proposition is essential to our NCR construction procedure:

Proposition 1: The boundary of the reachable set is covered by time-optimal trajectories.

Proof: See [21], Lemma 6.2, Page 71. ■

3.4 Main Results

3.4.1 Support for the NCR Construction Procedure

We will now present a construction procedure for the null controllable region for (3.1). The next lemma further characterizes free time optimal solutions for system 3.1:

Lemma 1: Time-optimal trajectories of the system (3.1) are bang-bang controls.

Proof: It turns out that, without loss of generality, we can assume that $\lambda_0 = 1$. In general, either $\lambda_0 = 0$ (called abnormal) or $\lambda_0 = 1$ (called normal) [28]. However, one can verify that the abnormal case implies there are no switches [26], which would obviously fail to cover the reachable set when $n \geq 2$.

Next, we know that the optimum solution must maximize its Hamiltonian. For time optimal control problems, we get that:

$$H^*(t) = -1 + (p^*)^T f(x^*) + \max_{u(t)}\{(p^*)^T g(x^*)u\} \tag{3.12}$$

Hence it is clear that to maximize H, we need only that:

$$u^*(t) = \Phi(t) = \text{sign}\{p^*(t) \cdot g(x^*)\} \tag{3.13}$$

This shows if an affine-input system has $\Phi \neq 0$ almost everywhere, then its time-optimal solutions are realized by bang-bang controls.

If $\Phi(t) = 0$ over some open interval $t \in I$, then the solution is said to contain *singular arcs*. In these intervals, $u(t)$ is not given by the maximum principle and takes on values in the interior of the control set. While singular arcs are well-defined and not necessarily 'uncommon',(see [26; 21]), their characterization adds complexity that we will avoid here. To rule them out, observe that if $x(t)$ were singular, then $\dot{\Phi}$ and all other derivatives must also vanish identically on I. That is:

$$\Phi^{(k)}(t) = p(t) \cdot \mathrm{ad}_f^k g = 0 \tag{3.14}$$

Hence, it is sufficient to invoke Assumption 1 to guarantee that there cannot be any such singular arcs. ■

Next we show that to cover the reachable set with simulated trajectories, it is first required to identify a suitable location on the boundary to initialize on.

We define x_e^+ (x_e^-) as an external forced equilibrium point of the subsystem if it solves $f(x) + g(x)u = 0$ with $u = 1$ ($u = -1$). Let $x_e^\pm = \{x_e^+, x_e^-\}$. We next make use of the following assumption.

Assumption 2: (i) We assume that elements of x_e are reachable, that is (with some abuse of notation) that there exists a control $\theta(t)$ such that $z(\breve{t}, \theta(t); 0) = x_e$ for some $\breve{t}$, possibly infinite. (ii) We also assume that $z(t, \theta(t), 0)$ does not contain a limit cycle for any time optimal control $\theta(t)$. (iii) We also assume there is only one external forced equilibrium point for each control $u = 1$ and $u = -1$. (iv) We further assume the control $\theta^+(t)$ that reaches x_e^+ admits that $\theta^+(t) = 1$ for all t sufficiently large, and that x_e^- is reached similarly such that $\theta^-(t) = -1$ for all large enough t.

The above assumptions are restrictive. We note that an objective of future research is to identify the class of systems which satisfy the above assumptions. For many systems, we expect (iv) to be implied by (i) and (ii), although we have not proved this.

Lemma 2: Trajectories of the reverse-time system emanating from the external forced equilibrium points are a finite cover of the reachable set.

Proof: We show that it is sufficient to consider only those trajectories that emanate from the external forced equilibrium points. To prove this, we will first show that the external forced equilibrium points reside on the boundary of the reachable set.

Because $f(\cdot)$ is globally asymptotically unstable, we know that (3.1) has a finite compact reachable set [21]. We know that trajectories are confined in some compact set D and, by invoking Assumption 1 (i), $x_e \subset D$.

Next, consider the autonomous system $\dot{z}|_{v=\kappa_1}$, where κ_1 is a constant. Appealing to LaSalle's Invariance Principle, we know that solutions $z_1(t) = z(t, \kappa_1; 0)$ must tend towards some maximal invariant set in D. In particular, because D contains only isolated equilibrium points via Assumption 1 (ii), then this trajectory approaches one of them. As such, consider the set of all such points $L(\kappa_1) = \{z : \dot{z}|_{v=\kappa_1} = 0\}$. Then:

$$\lim_{t\to\infty} z(t, \kappa_1; 0) = \ell_1 \in L(\kappa_1) \tag{3.15}$$

It is now clear that trajectories of the reverse time system under bang-bang control flow from the origin to an equilibrium point of the forced system.

Next, consider another trajectory of (3.4), $z_2(t) = z(t, \kappa_2; \ell_1)$, where $\kappa_2 \neq \kappa_1$ is another constant and where $z_2(\tilde{T}) = \tilde{Z}$ for some $\tilde{T} < \infty$. Thus, in forward time, there exists a trajectory of (3.1), $x(t, u_\kappa(t); \tilde{Z})$, where:

$$u_\kappa(t) = \begin{cases} \kappa_2, & t \leq \tilde{T} + t_0 \\ \kappa_1, & t > \tilde{T} + t_0 \end{cases} \tag{3.16}$$

Hence, the concatenation of the trajectories $z_1(t)$ and $z_2(t)$ is feasible for the reverse time system as well.

Lastly, to see that $\ell_1 \in \partial\mathcal{R}_0$, notice that Assumption 1 (iv) implies that, in a neighborhood of x_e, the dynamics 3.4 (which are under constant control) are Lipschitz continuous and hence $x \to x_e$ only in the limit when $t \to \infty$ (e.g. see [31]). Hence, ℓ_1 is reachable with bang-bang controls but only in infinite time. To conclude, we note that the boundary of the reachable set must at least contains all the points not reachable in finite time. That is, if $\mathcal{R}_0^\infty = \inf\{\mathcal{R}_0 \backslash \mathcal{R}_0(T) : T < \infty\}$, then since $\partial\mathcal{R}_0 \supseteq \mathcal{R}_0^\infty$, we know that $\pm x_e \in \mathcal{R}_0^\infty$.

Finally, we know that time-optimal trajectories emanating from these equilibrium points are also on $\partial\mathcal{R}_0$ since [21] asserts that optimal trajectories on $\partial\mathcal{R}_0$ remain on $\partial\mathcal{R}_0$ for all time. ■

Lemma 3: Trajectories that cover $\partial\mathcal{R}_0$ have at most $n-1$ arbitrarily designated switching times.

Proof: To see why we cannot assert n or more switches, observe that the trajectory is necessarily traced by its corresponding state-costate pair, $(\psi(t), \tilde{p}(t))$. For $\tilde{p}(t)$ to corroborate $\psi(t)$, there must be an initial co-state such that $g(\psi(t_i))\tilde{p}^T(t_i) = 0$. Along with the the Hamiltonian at $\psi(t_0) = 0$, this imposes n constraints on p_0. To see this, consider the following inner product formed with the costate trajectory along $\psi(t)$:

$$\begin{aligned}
p^T(t_i) \quad & = p^T(t_0) + \textstyle\int_{t_0}^{t_i} \dot{p}^T(\tau)\mathrm{d}\tau \\
g(t_i)p^T(t_i) \quad & = g(t_i)p^T(t_0) + g(t_i) \textstyle\int_{t_0}^{t_i} \dot{p}(\tau)\mathrm{d}\tau \\
0 \quad & = g(t_i)p^T(t_0) + g(t_i) \textstyle\int_{t_0}^{t_i} \dot{p}^T(\tau)\mathrm{d}\tau
\end{aligned} \tag{3.17}$$

Next, if we write this down for all switching times and include the Hamiltonian constraint, we get that p_0 must satisfy:

$$\begin{bmatrix} g(t_1) \\ \vdots \\ g(t_{n-1}) \\ f(x_0) + g(x_0)\Phi(t_0) \end{bmatrix} p_0 = \begin{bmatrix} g(t_1)(p_1 - p_0)^T \\ \vdots \\ g(t_{n-1})(p_{n-1} - p_0)^T \\ -1 \end{bmatrix} \tag{3.18}$$

Hence we cannot guarantee existence if we require more than $n-1$ arbitrary switches because otherwise (3.18) may not have a initial co-state solution. ■

3.4.2 The NCR Construction Procedure

The equations in (3.11) suggest that, given any x_0, we could produce an optimal trajectory if only we knew the corresponding optimal initial costate p_0^*. Then, we could integrate the coupled ODEs using the control law in (3.13).

We first consider planar systems. We will take the approach of finding the optimal initial costate p_0^*, since, in contrast with finding the switching times, there are only n components to p_0^*, and an unknown number of switches. We will use the shooting method to find an initial costate p_0 such that the Hamiltonian vanishes (i.e. (3.10)) at the known points x_0 and x_f. In this construction, we use elements of the set x_e initial and final points. This produces trajectories wholly on the boundary of the NCR because of the fact that they time-optimally connect two points also on the boundary.

To make this precise, let us denote $\hat{p}^k$ as the kth iteration of an estimate of the costate and let $p^k(t, u(t); x_0, p_0)$ be the $p(t)$ subspace of a trajectory $(x(t), p(t))$ integrated from (x_0, p_0) . Then, if:

$$h_f^k(\hat{p}_0^k) = \lim_{t \to \infty} \hat{p}^k(t, \Phi(t); x_0, \hat{p}_0^k) \cdot \dot{x}_f - 1 \tag{3.19}$$

where $\dot{x}_f = f(x_f) + g(x_f)u_f$. It is clear that h^k is an approximation of the Hamiltonian. Hence, $h^k \to 0$ as $p_0^k \to p_0^*$. Let $p_0 = [p_{0,1}, p_{0,2}]$. To reduce the dimensionality of this search space (to one), we solve the following equation for $p_{0,2}^k$ using a guess of $p_{0,1}^k$.

$$h_0^k(\hat{p}_0^k) = \hat{p}_0^k \cdot (f(x_0) + g(x_0)u_0) - 1 \tag{3.20}$$

The search of $p_{0,1}$ can use standard techniques. We see that the solution for planar NCRs is completely determined by x_0, x_f, and the switching law Φ.

The picture in higher dimensions is more complex. We want to rely on transversality conditions to define $p^*(T)$, but because the sub-manifold defining the target set is a point, the typical conditions do not apply. Unfortunately, the boundary value problem defining the flow of (x, p) is degenerate in the sense that we don't have uniqueness from the boundary conditions at the initial and final states. They give rise to a family of admissible costate trajectories. The correct one is the unique path such that $H(t) = 0$ along the solution. Hence, the solution method takes the form of sifting through the (large) space of admissible initial co-states. Methods for solving this intractable problem are not discussed further.

3.4.3 The Constrained Control Lyapunov Function

Here we seek an asymptotically stabilizing control policy that makes use of our knowledge of the NCR. We will exploit the fact that, in a neighborhood of the origin, (3.1) has the property that the control action necessary to stabilize the system, λ, tends monotonically to zero as $x \to 0$.

If denote $\mathcal{C}^{\lambda}$ as the null controllable region that results from the control set $|u(t)| \leq \lambda \leq 1$, we will assert that $\mathcal{C}^{\lambda_1} \subset \mathcal{C}^{\lambda_2}$ whenever $\lambda_2 > \lambda_1$ and also that $\mathcal{C}^0 = \{0\}$. Let $\mathcal{C}^1 = \mathcal{C}$. We can now state the following theorem:

Theorem 1: Any control law which guarantees that $\dot{\lambda}(t) < 0$ everywhere in some region $E \supset \{0\}$ is asymptotically stable in E.

Proof: Follows immediately from $V(x) = \lambda(x)$. Clearly, $\dot{\lambda}(t) < 0$ implies $\dot{V} < 0$. Then, $V(x) \to 0$, which implies $x \to \mathcal{C}^0$.

To determine $\lambda(x)$, it is sufficient to compute the level sets $\mathcal{C}^{\lambda}$ for numerous λ, and interpolate between them. We claim that if $x \in \mathcal{C}$, then $\Gamma(\lambda)x \in \mathcal{C}^{\lambda}$, where $\Gamma(\cdot)$ is a class Kappa scalar function (see [29]).

Remark 3: It is common that the value function inspires a choice of a Lyapunov function. In view of this, even the time to stabilization t^* is a (somewhat trivial) CCLF for a controller which admits $\dot{t}^* < 0$.

For example, one controller which satisfies this requirement is a Lyapunov-based model predictive controller. This controller could take the following form:

$$
\begin{aligned}
& u(t) = \arg\min J(u) \\
& s.t.\ \tfrac{\mathrm{d}\hat{x}}{\mathrm{d}t} = f(\hat{x}) + g(\hat{x})u(t) \\
& \hat{x}(\bar{t}_0) = x(t) \\
& \hat{x} \in \mathcal{C}, u \in \mathcal{U} \\
& \tfrac{\mathrm{d}\lambda}{\mathrm{d}t} < 0
\end{aligned}
\tag{3.21}
$$

where $J(x,u,t) = \int_t^{t+\Delta} x(\bar{\tau})^T P x(\bar{\tau}) + R u(\bar{\tau})^2 \mathrm{d}\bar{\tau}$ and Δ is the time horizon being considered. More details of such an MPC implementation can be found in [22].

3.5 Examples

Here we present just a few examples of reachable set boundaries for nonlinear systems. Notice that in view of [T. Hu], obviously the reachable set for linear systems can also be produced by our construction.

Example 1 Consider the planar system:

$$\begin{aligned}\dot{x}_1 &= x_1(\sin(x_2) + 1) + x_2 + u \\ \dot{x}_2 &= x_2(\sin(x_1) + 1) + u\end{aligned} \tag{3.22}$$

Figure 3.5 shows the null controllable region of this system. As discussed, it is produced by integrating the dynamics to form bang-bang arcs that emanate from each forced system equilibrium point.

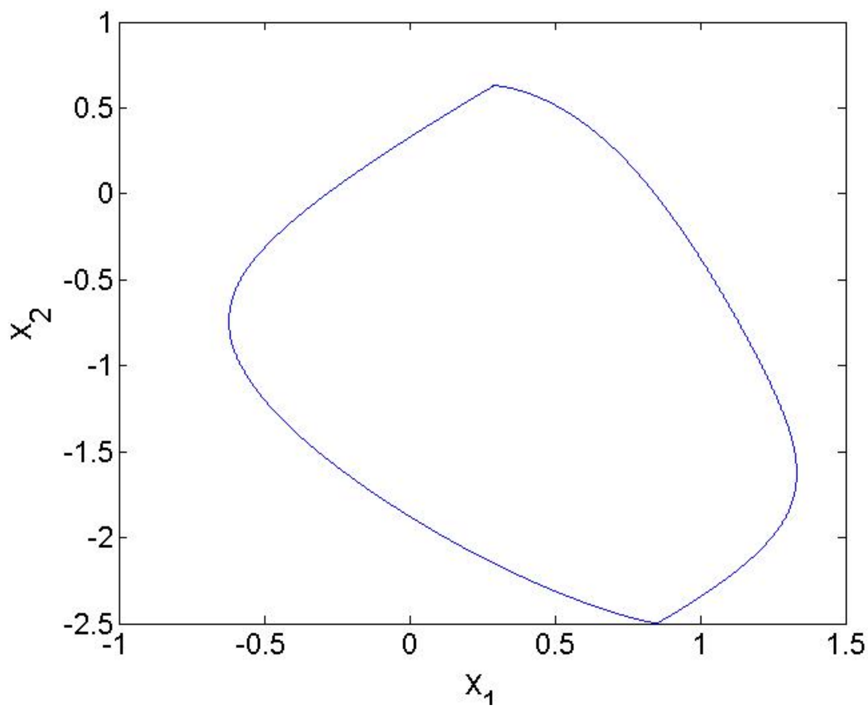

Figure 3.1: The boundary of the reachable set in Example 1.

Example 2 Consider the planar system:

$$\begin{aligned}\dot{x}_1 &= k_1(x_1 - k_2) - k_3 x_1 \exp(\tfrac{-k_4}{x_2+k_5}) \\ \dot{x}_2 &= k_1(x_2 - k_6) - k_7 x_1 \exp(\tfrac{-k_4}{x_2+k_5}) - k_8 u\end{aligned} \tag{3.23}$$

where $[k_1, ..., k_8] > 0$. For certain values of the parameters, this system has the form of a time-reversed continuously stirred tank reactor. Figure 3.5 shows the null controllable region of this system, where we have translated the state space so that its equilibrium point corresponds with $x = 0$.

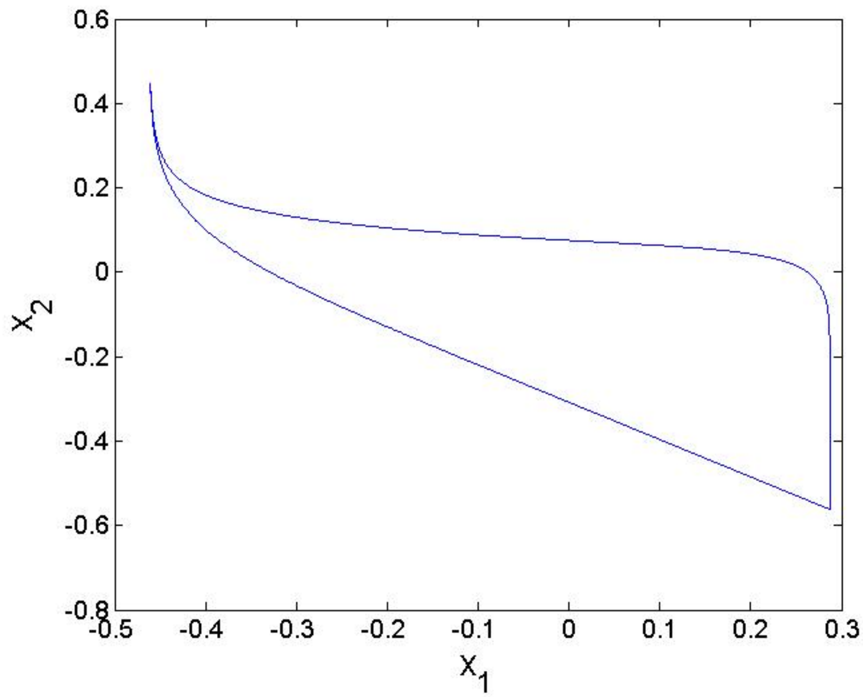

Figure 3.2: The boundary of the reachable set in Example 2.

3.6 Conclusion

In conclusion, we have demonstrated a procedure for constructing the null controllable region of unstable nonlinear systems with affine constrained controls. A future avenue of investigation will be to construct the null controllable region for semi-stable systems. Yet another will be to determine the control law implied by these time-optimal solutions.

Chapter 4

Conclusion

4.1 Conclusions

In this thesis, we have focused on the stability of nonlinear systems. In particular, we have designed controllers that guarantee stabilization of systems with stochastic disturbances and also of deterministic systems with input constraints. In a reflection of the theme of this work, we have proposed solutions to control systems afflicted by the challenges of nonlinearity, uncertainty, and input constraints.

4.2 Recommendations for Further Work

There are significant opportunities for future investigation. The most direct opportunity is in completing our characterization of the null controllable region. We will endeavour prove the hypotheses laid out in the third chapter which will show that the proposed construction procedure is rigorous.

Following this, there are many opportunities to extend the results given. The first opportunity is to extend the validity of the construction procedure to semi-stable systems. That is, to systems with semi-unbounded null controllable regions (NCRs). The second opportunity is to investigate the NCR of multiple-input systems. Here we will consider both linear and nonlinear systems, since the path that would trace the boundary of the NCR is not well-understood in either case. The third opportunity is to describe the switching manifolds of the system under bang-bang control, which would result in a feedback controller. The fourth opportunity is to implement the resulting stabilizing control design into an industrial system. The fifth opportunity would be to extend the results to stochastic systems using the stochastic analogues of key optimal control results.

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