EXACT ANALYSIS OF EXPONENTIAL TWO-COMPONENT SYSTEM FAILURE DATA

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By

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Abstract

A survival distribution is developed for exponential two-component systems that can survive as long as at least one of the two components in the system function. It is assumed that the two components are initially independent and non-identical. If one of the two components fail (repair is impossible), the surviving component is subject to a different failure rate due to the stress caused by the failure of the other.

In this paper, we consider such an exponential two-component system failure model when the observed failure time data are (1) complete, (2) Type-I censored, (3) Type-I censored with partial information on component failures, (4) Type-II censored and (5) Type-II censored with partial information on component failures. In these situations, we discuss the maximum likelihood estimates (MLEs) of the parameters by assuming the lifetimes to be exponentially distributed. The exact distributions (whenever possible) of the MLEs of the parameters are then derived by using the conditional moment generating function approach. Construction of confidence intervals for the model parameters are discussed by using the exact conditional distributions (when available), asymptotic distributions, and two parametric bootstrap methods. The performance of these four confidence intervals, in terms of coverage probabilities are then assessed through Monte Carlo simulation studies. Finally, some examples are presented to

illustrate all the methods of inference developed here.

In the case of Type-I and Type-II censored data, since there are no closed-form expressions for the MLEs, we present an iterative maximum likelihood estimation procedure for the determination of the MLEs of all the model parameters. We also carry out a Monte Carlo simulation study to examine the bias and variance of the MLEs.

In the case of Type-II censored data, since the exact distributions of the MLEs depend on the data, we discuss the exact conditional confidence intervals and asymptotic confidence intervals for the unknown parameters by conditioning on the data observed.

Keywords: Two-component system model; maximum likelihood estimation; bootstrap method; conditional moment generating function; exponential distribution; confidence intervals; coverage probabilities; Type-I censoring; Type-II censoring; Type-I censoring with partial information on component failures; Type-II censoring with partial information on component failures.

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Chapter 1

Introduction

1.1 Historical Background

The reliability analysis of multi-component system models with different failures has been discussed in the reliability literature. Interest typically lies in estimating the parameters of a survival distribution in a system with several components. Goel and Gupta (1983) assumed a system consisting of n independent components arranged in a series configuration and the failed component can be identified and replaced by a standby one with a constant replacement rate. Sarhan and EI-Bassiouny (2003) discussed estimations in the case of a parallel system consisting of independent but non-identical components having complementary exponential lifetime distributions with different parameters. Lin, Usher and Guess (1993), Usher (1996) and Sarhan (2003) derived the maximum likelihood estimates of the parameters for the case of a 2- or 3-component series system when the lifetimes of the system components have Weibull or exponential distribution in the case of masked system life data. Miyakawa (1984) derived closed-form expressions for the maximum likelihood estimates of the parameters in the two-component series system of exponential components. The model he considered involves only the time to first failure. This kind of data and associated inferential problems can also be encountered in competing risks data analysis; see Crowder (2001). Most of these works deal with the analysis under the assumption that the components are independently arranged in a parallel system or the failed components are not necessarily identified.

However, assumption that the components are dependent is more realistic. For example, in a multi-component system, the failure of one component may hasten the failure of the remaining components in the system; or the failure of one component may alter the failure rates of the remaining ones. In the first case, Murthy and Wilson (1994) analyzed such failure in two-component as well as multi-component systems, and termed it as *failure interaction.* They classified the interaction into two types - natural and induced, with the former being the cause of the latter. In the second case, Gross, Clark and Liu (1971) proposed a two-component system model in which an individual survives as long as at least one of the two components functions. The two components were assumed to be identical with constant failure rate λ_0. If one component fails, the failure rate of the other one changes to λ_1. They then assumed a complete sample of observations and discussed the estimation of the parameters λ_0 and λ_1.

We also note that, if the failed components can not be identified within an experiment, the inference of the unknown parameters may become difficult if not impossible. For example, in a parametric analysis involving masked systems, although it is thought to be more economical if one does not have to bother identifying the failed component within a specified system, it is clear that inference will suffer from the amount of uncertainty in the data. Similarly, for a competing risks model, full information can not be obtained for developing inference if the system can not be observed in operation after the failure of the first component.

We consider here a model which assesses the lifetimes of a multi-component system assuming that the components are dependent and the failed components can be observed within a test. We assume that the system with J components can survive until the last failure of its components. The components within a specified system are initially independent and non-identical with mean life times θ_j $(j = 1, 2, \ldots, J)$. However, failure of one component alters the subsequent lifetimes of all others, in a way that the mean lifetime of surviving components change from θ_j to θ'_j. The bivariate distribution function corresponding to such failure mechanism in a two-component system was proposed by Freund (1961). Some key references in the area of multivariate survival models are Block and Savits (1981), Hougaard (1987), Slud (1984) and Gumbel (1960).

1.2 Bivariate Extension of the Exponential Distribution

For simplicity, let us consider here that there are only two components in the individual system. All the methods presented in this thesis can, of course, be extended to the case of multi-component systems with $J > 2$, but the mathematical expressions will be for more complicated.

We assume that X and Y are random variables representing the lifetimes of Components 1 and 2, respectively, in a two-component system. Further, we assume that X and Y are initially independent exponential random variables with densities

$$f_X(x) = \frac{1}{\theta_1}\exp\left(-\frac{1}{\theta_1}x\right), \quad x > 0, \quad \theta_1 > 0,$$
$$f_Y(y) = \frac{1}{\theta_2}\exp\left(-\frac{1}{\theta_2}y\right), \quad y > 0, \quad \theta_2 > 0,$$

respectively, where θ_1 and θ_2 represent the mean lifetimes of Components 1 and 2, respectively. We assume that a simultaneous failure of both components is not possible. However, failure of one component alters the subsequent lifetime of the other component. Specifically, the mean lifetime of the surviving component changes from θ_j to θ_j' $(j = 1, 2)$. It follows that the joint density of X and Y in such a case is [see Freund (1961)]

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\theta_1\theta_2'}\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)x - \frac{1}{\theta_2'}y\right\}, & 0 < x < y, \\ & 0 < \theta_1, \theta_2, \theta_2' < \infty, \\ \frac{1}{\theta_2\theta_1'}\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)y - \frac{1}{\theta_1'}x\right\}, & 0 < y < x, \\ & 0 < \theta_1, \theta_2, \theta_1' < \infty. \end{cases} \tag{1.1}$$

All the studies in this thesis is for such a two-component system having the bivariate distribution in (1.1). Some of the basic properties of the bivariate distribution in (1.1) are as follows.

The joint survival function of X and Y is given by

$$\bar{F}(x,y) = \left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)^{-1}\left\{\frac{1}{\theta_1}\mathrm{e}^{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)x-\frac{1}{\theta_2'}y} + \left(\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)\mathrm{e}^{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)y}\right\}, \tag{1.2}$$

for $0 < x < y$, assuming that $\frac{1}{\theta_1} + \frac{1}{\theta_2} \neq \frac{1}{\theta_2'}$; correspondingly,

$$\bar{F}(x,y) = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-1} \left\{\frac{1}{\theta_2} e^{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)y - \frac{1}{\theta_1'}x} + \left(\frac{1}{\theta_1} - \frac{1}{\theta_1'}\right) e^{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)x}\right\}, \tag{1.3}$$

for $0 < y < x$, assuming that $\frac{1}{\theta_1} + \frac{1}{\theta_2} \neq \frac{1}{\theta_1'}$.

Setting $x = 0$ in (1.2) and $y = 0$ in (1.3), we obtain the marginal survival functions as

$$\bar{F}(x) = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-1} \left\{\frac{1}{\theta_2} e^{-\frac{1}{\theta_1'}x} + \left(\frac{1}{\theta_1} - \frac{1}{\theta_1'}\right) e^{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)x}\right\}, \tag{1.4}$$

$$\bar{F}(y) = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-1} \left\{\frac{1}{\theta_1} e^{-\frac{1}{\theta_2'}y} + \left(\frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) e^{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)y}\right\}. \tag{1.5}$$

These are mixture of two exponentials, rather than a single one. Independence of X and Y exists if and only if $\frac{1}{\theta_1} = \frac{1}{\theta_1'}$ and $\frac{1}{\theta_2} = \frac{1}{\theta_2'}$.

The expected values and variance of X and Y can be shown to be

$$\mathrm{E}(X) = \frac{\frac{1}{\theta_1'} + \frac{1}{\theta_2}}{\frac{1}{\theta_1'}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)}, \qquad \mathrm{E}(Y) = \frac{\frac{1}{\theta_2'} + \frac{1}{\theta_1}}{\frac{1}{\theta_2'}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)},$$

$$\mathrm{var}(X) = \frac{\frac{1}{\theta_1'^2} + \frac{2}{\theta_1\theta_2} + \frac{1}{\theta_2^2}}{\frac{1}{\theta_1'^2}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)^2}, \qquad \mathrm{var}(Y) = \frac{\frac{1}{\theta_2'^2} + \frac{2}{\theta_1\theta_2} + \frac{1}{\theta_1^2}}{\frac{1}{\theta_2'^2}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)^2},$$

$$\mathrm{cov}(X,Y) = \frac{\frac{1}{\theta_1'\theta_2'} - \frac{1}{\theta_1\theta_2}}{\frac{1}{\theta_1'\theta_2'}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)^2}.$$

The correlation coefficient (ρ) of X and Y can be obtained as the ratio of $\mathrm{cov}(X,Y)$ and $\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}$. It is of interest to note that in general $-\frac{1}{3} < \rho < 1$. The correlation coefficient approaches 1 when $\frac{1}{\theta_1'} \to \infty$ and $\frac{1}{\theta_2'} \to \infty$; physically speaking, this corresponds to the case when the two-component system cannot function if either component fails. The correlation coefficient approaches $-\frac{1}{3}$ when $\frac{1}{\theta_1} = \frac{1}{\theta_2}$ and $\frac{1}{\theta_1'} \to 0$ and $\frac{1}{\theta_2'} \to 0$; physically speaking, this corresponds to the case when either component becomes "almost infallible" as soon as the other one fails. This would not be a very realistic situation.

1.3 Asymptotic Normality of the Maximum Likelihood Estimator

Let $X_1, \ldots, X_n$ be *i.i.d.* with PDF $f(x, \theta)$, $\theta \in \Omega$. Suppose $f(x, \theta)$ has common support and is differentiable in θ. Then the log likelihood is

$$l(\theta) = \ln L(\theta) = \sum_{i=1}^{n} \ln f(x_i, \theta),$$

and the first derivative of the log likelihood is

$$l'(\theta) = \frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial \ln f(x_i, \theta)}{\partial \theta},$$

which is called the score. The maximum likelihood estimator $\hat{\theta}$ can be found by solving $l'(\theta) = 0$. That is,

$$l'(\hat{\theta}) = 0.$$

We can approximate the left-hand side of this equation in a Taylor series expanding about θ_0, namely

$$l'(\hat{\theta}) = l'(\theta_0) + l''(\theta_0)(\hat{\theta} - \theta_0) + \cdots = 0 \tag{1.6}$$

where we are going to ignore the higher-order terms under the regularity conditions: (1) For every $x \in X$, the density $f(x, \theta)$ is three times differentiable with respect to θ, the third derivative is continuous in θ, and $\int f(x,\theta)dx$ can be differentiated three times under the integral sign; (2) For every $\theta_0 \in \Omega$, there exists a positive number c and a function $M(x)$ (both of which may depend on θ_0) such that

$$\left|\frac{\partial^3 \ln f(X,\theta)}{\partial \theta^3}\right| \leq M(x)$$

for all $x \in X$ and $\theta_0 - c < \theta < \theta_0 + c$ with $E_{\theta_0}[M(X)] < \infty$.

The asymptotic of $l'(\theta)$ and $l''(\theta)$ are given by the Central Limit Theorem and the Law of Large Numbers. Since we can differentiate under the integral sign under the assumption, we see that the score $l'(\theta)$ is the sum of n independent random variables each with mean zero, and, consequently, with variance $I(\theta) = E\{[\frac{\partial \ln f(X,\theta)}{\partial \theta}]^2\}$. The function $I(\theta)$ is the Fisher Information based on one observation X. So, based on a random sample $X_1, \ldots, X_n$, we will have $\frac{l'(\theta)}{\sqrt{nI(\theta)}}$ to have the limiting distribution $N(0,1)$ by the Central Limit Theorem. Moreover, $-\frac{1}{n}l''(\theta)$ converges in probability to its expected value $I(\theta)$ by the Law of Large Numbers. This implies that $-\frac{1}{nI(\theta)}l''(\theta)$ converges in probability to 1.

If we rearrange the expansion (1.6) we get

$$\frac{\hat{\theta} - \theta}{\sqrt{\frac{1}{nI(\theta)}}} = \frac{\frac{1}{\sqrt{nI(\theta)}}l'(\theta)}{-\frac{1}{nI(\theta)}l''(\theta)}$$

which has the limiting distribution $N(0,1)$ by Slutsky's Theorem. Hence, we can say that $\hat{\theta}$ has an approximate normal distribution with mean θ and variance $\frac{1}{nI(\theta)}$, i.e., $I_n^{-1}(\theta)$.

The asymptotic of multi-parameter maximum likelihood is like the one parameter case. For large n and under similar regular conditions, $\hat{\theta} = \left(\hat{\theta}_1, \ldots, \hat{\theta}_n\right)$ has an approximate normal distribution with mean $\theta = (\theta_1, \ldots, \theta_n)$ and variance $I_n^{-1}(\theta_1, \ldots, \theta_n)$.

1.4 Types of Data in the Study

Censoring is frequently encountered in reliability and life-testing experiments because the experimenter has to terminate the experiment before all items have failed due to time or cost considerations. The two most common censoring schemes are as *Type-I* and *Type-II* censoring schemes. Some key references dealing with inference under Type-I and Type-II censoring for different parametric families of distributions include Lawless (1982), Nelson (1982), Cohen and Whitten (1988), Cohen (1991), and Balakrishnan and Cohen (1991).

1.4.1 Conventional Type-I and Type-II Censored Data

Consider a life-testing experiment in which we are testing n (non-repairable) identical units taken randomly from a population. In the typical test scenario, we have a pre-fixed time W to run the units to see if they survive or fail. The observed failure data obtained from such an experiment are called *Type-I censored data.* The termination point W of the experiment is assumed to be independent of the failure times. Another way to test is to decide in advance that you want to see exactly r $(r \leq n)$ failure times and then test until they occur. The observed failure data obtained in this way are called *Type-II censored data.*

Comparing Type-I and Type-II censoring, there are advantages as well as disadvantages with both of them. In the case of Type-I censoring, the advantage is that the duration of the experiment can be controlled by the experimenter. However, the number of observed failure times is random. If the unknown mean lifetime is not small compared to W, few failures (even no failure) may occur before time W. This may result in an adverse effect on the efficiency of inferential procedures based on Type-I censoring. In the case of Type-II censoring, the advantage is that r failures will be observed exactly which result in efficient inferential procedures. However, the termination time is unknown to the experimenter. If the unknown mean lifetime is not small, the experimentation would result in a longer life-test.

1.4.2 Type-I and Type-II Censored Data with Partial Information on Component Failures

Suppose there are n identical systems placed on a life test and that each system has two components. Assume that the experiment continues up to a pre-fixed time W. Before the time W, there are D observed failed systems. Under the conventional Type-I censoring, failures that occur after W are not observed. However, at the end of the experiment, besides D systems with complete destruction, we may observe additional D' (say) systems which have only one failed component. In order to obtain more information on lifetimes and get more accurate estimation of the parameters, we need to consider those D' failure times as well in the data. We call this type of data as "Type-I censored data with partial information on component failures". The number of observed failure times before W is random, and actually equals $2D + D'$.

Similarly, under the conventional Type-II censoring, the experiment continues until a total of (pre-fixed) d ($d \leq n$) systems fail. Assume that X and Y are random variables representing the lifetimes of Components 1 and 2, respectively, in a two-component system. If $Z_i = \max(X_i, Y_i)$ ($i = 1, \ldots, n$), the i-th system fails at time Z_i, and $Z_{1:n} < \cdots < Z_{d:n}$ are the corresponding ordered failure times. Failures that occur after $Z_{d:n}$ are not observed. However, at time $Z_{d:n}$, in addition to the d systems with complete destruction, we may observe additional d' (say) systems which have only one failed component. In order to get more accurate estimation of the parameters, we need to consider those d' failure times in the data. We call this type of data as "Type-II censored data with partial information on component failures". The number of observed failure times before $Z_{d:n}$ is random now, and actually equals $2d + d'$.

1.5 Scope of the Thesis

A survival distribution is developed for two-component systems that survive as long as at least one of the two components functions. The main goal of the thesis is to develop inference for such two-component system failure models under the conventional Type-I and Type-II censoring schemes and Type-I and Type-II censoring with partial information on component failures, respectively.

In Chapter 2, we discuss the exact inference for a two-component system failure model in the case of complete data assuming the lifetimes of the components to be exponentially distributed. In Section 2.2, we describe the model and present the MLEs of the model parameters. The exact conditional distributions of the MLEs are derived in Section 2.3. Using these exact distributions of the MLEs, we obtain in Section 2.4 the exact conditional confidence intervals for the unknown parameters. We also discuss the asymptotic distributions of the MLEs and the corresponding asymptotic confidence intervals, as well as two confidence intervals based on the parametric bootstrap method. In Section 2.5, we carry out a Monte Carlo simulation study to evaluate the performance of these confidence intervals in terms of coverage probabilities. We also present an example to illustrate all the methods of inference discussed here.

In Chapter 3, we discuss the exact inference for a two-component system failure model in the case of Type-II censored data assuming the lifetimes of the components to be exponentially distributed. In Section 3.2, we first describe the model and then discuss the likelihood estimation of the model parameters. Since there are no closed-form expressions for the MLEs, we present an iterative maximum likelihood estimation procedure to determine the MLEs of the parameters. Next, in Section 3.3, we obtain the asymptotic distributions of the MLEs and the corresponding asymptotic confidence intervals, as well as two confidence intervals based on parametric bootstrap methods. In Section 3.4, we carry out a Monte Carlo simulation study to examine the bias and variance of the MLEs and also to evaluate the performance of the three confidence intervals in terms of coverage probabilities. Numerical examples are also presented in this section to illustrate all the methods of inference discussed here.

In Chapter 4, we discuss the exact inference for a two-component system failure model in the case of Type-II censored data assuming the lifetimes of the components to be exponentially distributed. The information of the censored systems which have only one component failed at the end of the experiment is incorporated as well. In Section 4.2, we first describe the model and present the MLEs of the model parameters.

Relative risks are discussed in this section as well. The exact conditional distributions of the MLEs are derived in Section 4.3. Since the distributions of the MLEs depend on the observed data, we obtain in Section 4.4 the exact conditional confidence intervals and asymptotic confidence intervals, by conditioning on the data, for the unknown parameters. We also discuss two confidence intervals based on the parametric bootstrap methods. In Section 4.5, we carry out a Monte Carlo simulation study to examine the relative risks and also to evaluate the performance of the two parametric bootstrap confidence intervals in terms of coverage probabilities. Numerical examples are also presented in this section to illustrate all the methods of inference discussed here.

In Chapter 5, we discuss the exact inference for a two-component system failure model in the case of Type-I censored data assuming the lifetimes of the components to be exponentially distributed. In Section 5.2, we describe the model and present an iterative maximum likelihood estimation procedure to determine the MLEs of the parameters. Next, in Section 5.3, we obtain the asymptotic distributions of the MLEs and the corresponding asymptotic confidence intervals, as well as two confidence intervals based on parametric bootstrap methods. In Section 5.4, we carry out a Monte Carlo simulation study to examine the bias and variance of the MLEs and also to evaluate the performance of the three confidence intervals in terms of coverage probabilities. Numerical examples are also presented in this section to illustrate all the methods of inference discussed here.

In Chapter 6, we discuss the exact inference for a two-component system failure model in the case of Type-I censored data assuming the lifetimes of the components to be exponentially distributed. The information of the censored systems which have only one component failed at the end of the experiment is incorporated as well. In

Section 6.2, we describe the model and present the MLEs of the model parameters. Relative risks are discussed in this section as well. The exact conditional distributions of the MLEs are derived in Section 6.3. Using these exact distributions of the MLEs, we obtain in Section 6.4 the exact conditional confidence intervals for the unknown parameters. We also discuss the asymptotic distributions of the MLEs and the corresponding asymptotic confidence intervals, as well as two confidence intervals based on the parametric bootstrap method. In Section 6.5, we carry out a Monte Carlo simulation study to evaluate the performance of these confidence intervals in terms of coverage probabilities. Numerical examples are also presented in this section to illustrate all the methods of inference discussed here.

Finally, in Chapter 7, we present some concluding remarks based on the work carried out in this thesis. Some possible directions for future research are also outlined in this chapter.

Chapter 2

Exact Analysis in Complete Data

2.1 Introduction

In this chapter, we consider a two-component system failure model in the case of complete data. We then derive in Section 2.2 the maximum likelihood estimates (MLEs) of the parameters by assuming the lifetimes to be exponentially distributed. In Section 2.3, the exact distributions of the MLEs of the parameters are then derived by using the conditional moment generating function approach. Construction of confidence intervals for the model parameters are discussed in Section 2.4 by using the exact conditional distributions, asymptotic distributions, and two parametric bootstrap methods. In Section 2.5, the performance of these four confidence intervals, in terms of coverage probabilities, are assessed through a Monte Carlo simulation study. Examples are also presented in this section to illustrate all the methods of inference developed here.

2.2 Model Description and MLEs

Consider the following simple system failure model: n identical systems are placed in a life-test and each system has two components. The experiment continues until the failure of all n systems are observed. We assume that X_i and Y_i $(i = 1, \ldots, n)$ are random variables representing the lifetimes of Components 1 and 2, respectively, in the i-th system. Among the n systems, suppose Component 1 fails first n_1 times and Component 2 fails first n_2 times, with $n_1+n_2 = n$. Let $Z_i = \max(X_i, Y_i)$ $(i = 1, \ldots, n)$. Thus, the i-th system fails at time Z_i, and $Z_{1:n} < \cdots < Z_{n:n}$ are the corresponding ordered failure times of the n systems under test. The data arising from such a two-component system is as follows:

$$(T_1,\ \delta_1';\ \ Z_{1:n},\ \delta_1''), \ldots, (T_n,\ \delta_n';\ \ Z_{n:n},\ \delta_n''), \tag{2.1}$$

where $T_1, \ldots, T_n$ denote the first observed failure times in the systems, $Z_{1:n} < \cdots < Z_{n:n}$ denote the final observed failure times of the systems, and δ' and δ'' are indicators denoting the component of the first and second observed failures within the system, respectively.

If we let

$$I_1 = \{i \in (1, 2, \ldots, n) : \ \text{Component 1 fails first}\},$$

$$I_2 = \{i \in (1, 2, \ldots, n) : \ \text{Component 2 fails first}\},$$

the likelihood function of the observed data in (2.1) is

$$
\begin{aligned}
&L(\theta_1, \theta_2, \theta_1', \theta_2') = \\
&\quad (2n)! \left(\frac{1}{\theta_1\theta_2'}\right)^{n_1} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)\sum_{i\in I_1} x_i - \frac{1}{\theta_2'}\sum_{i\in I_1} z_i\right\} \\
&\quad \times \left(\frac{1}{\theta_2\theta_1'}\right)^{n_2} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)\sum_{i\in I_2} y_i - \frac{1}{\theta_1'}\sum_{i\in I_2} z_i\right\},
\end{aligned}
\tag{2.2}
$$

where $0 < x_i < z_{i:n}$ for $i \in I_1$, $0 < y_i < z_{i:n}$ for $i \in I_2$, and $0 < z_{1:n} < \cdots < z_{n:n} < \infty$.

From the likelihood function in (2.2), it is immediate that, on the condition that $1 \le n_1 \le n-1$, $\left(n_1, \sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i|, \sum_{i\in I_2}(z_i - y_i)|, \sum_{i\in I_1}(z_i - x_i)|\right)$ form a jointly complete sufficient statistic for $(\theta_1, \theta_2, \theta_1', \theta_2')$. It is also evident that the MLE of θ_2' does not exist if $n_1 = 0$ or $n_2 = n$, and the MLE of θ_1' does not exist if $n_1 = n$ or $n_2 = 0$. Therefore, the MLEs of θ_1, θ_2, θ_1' and θ_2' exist only when $1 \le n_1 \le n-1$ and may be obtained by maximizing the corresponding likelihood function in (2.2). The MLEs thus obtained are given by

$$
\begin{aligned}
\hat\theta_1 &= \frac{\sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i}{n_1}, \\
\hat\theta_2 &= \frac{\sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i}{n_2} = \frac{n_1}{n - n_1}\hat\theta_1, \\
\hat\theta_1' &= \frac{\sum_{i\in I_2}(z_i - y_i)}{n - n_1}, \\
\hat\theta_2' &= \frac{\sum_{i\in I_1}(z_i - x_i)}{n_1}.
\end{aligned}
$$

The estimates $\hat\theta_1$, $\hat\theta_2$, $\hat\theta_1'$ and $\hat\theta_2'$ presented above are conditional MLEs of θ_1, θ_2, θ_1' and θ_2', conditional on $1 \le n_1 \le n-1$.

2.3 Exact Conditional Distribution of the MLEs

We will now derive the exact marginal (conditional) distribution of the MLEs. The derivation will require the inversion of the conditional moment generating function (CMGF). To obtain the CMGF, we need to determine the distribution of random variables n_1, $\sum_{i\in I_1} X_i + \sum_{i\in I_2} Y_i$, $\sum_{i\in I_2}(Z_i - Y_i)$ and $\sum_{i\in I_1}(Z_i - X_i)$ separately.

The distribution of n_1 is established and presented in Lemma 2.3.1.

Theorem 2.3.1. *The relative risk that Component 1 fails first within a two-component system is*

$$\pi_1 = Pr(X < Y) = \int_0^\infty \frac{1}{\theta_1}\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)x\right\}dx = \frac{\theta_2}{\theta_1+\theta_2}, \quad 0 < \theta_1, \theta_2 < \infty.$$

Proof: The proof follows easily by straight forward integration.

Lemma 2.3.1. *The number of system failures due to Component 1 failing first, viz., n_1, is a non-negative random variable with binomial probability mass function*

$$p_i = Pr(n_1 = i) = \binom{n}{i}\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^i\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{n-i}, \quad i = 0, 1, \ldots, n.$$

Proof: The result follows immediately from Theorem 2.3.1.

From Lemma 2.3.1, we then have

$$Pr(n_1 = i | 1 \le n_1 \le n-1) = \frac{p_i}{\sum_{j=1}^{n-1} p_j}. \tag{2.3}$$

Next, with $S_i = \min(X_i, Y_i)$ $(i = 1, \ldots, n)$, since the minimum of two independent exponential random variables is also distributed as exponential, $\sum_{i=1}^n S_i = \sum_{i\in I_1} X_i + \sum_{i\in I_2} Y_i$ is readily seen to be distributed as a Gamma$(n, \frac{1}{\theta_1}+\frac{1}{\theta_2})$ random variable.

From Section 2.2, it can be easily seen that $\sum_{i \in I_2}(Z_i - Y_i)$ covers the situations where Component 2 fails before Component 1, while $\sum_{i \in I_1}(Z_i - X_i)$ covers the situations where Component 1 fails first. Since $Z_i - Y_i$ (for $i \in I_2$) is assumed to have an exponential distribution

$$\frac{1}{\theta_1'}\exp\left\{-\frac{1}{\theta_1'}w_i\right\}, \qquad 0 < w_i, \;\; 0 < \theta_1',$$

and similarly $Z_i - X_i$ (for $i \in I_1$) is assumed to have an exponential distribution

$$\frac{1}{\theta_2'}\exp\left\{-\frac{1}{\theta_2'}w_i\right\}, \qquad 0 < w_i, \;\; 0 < \theta_2',$$

we readily have $\sum_{i \in I_2}(Z_i - Y_i)$ to be distributed as Gamma$(n - n_1, \frac{1}{\theta_1'})$ and $\sum_{i \in I_1}(Z_i - X_i)$ to be distributed as Gamma$(n_1, \frac{1}{\theta_2'})$.

We can then establish the following two theorems.

Theorem 2.3.2. *Conditional on $1 \le n_1 \le n-1$, the CMGFs of the MLEs are given by*

$$M_{\hat{\theta}_1}(t) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \left\{1 - \left[i\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)\right]^{-1} t\right\}^{-n},$$

$$M_{\hat{\theta}_2}(t) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \left\{1 - \left[(n-i)\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)\right]^{-1} t\right\}^{-n},$$

$$M_{\hat{\theta}_1'}(t) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \left\{1 - \frac{\theta_1'}{(n-i)}t\right\}^{-(n-i)},$$

$$M_{\hat{\theta}_2'}(t) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \left\{1 - \frac{\theta_2'}{i}t\right\}^{-i}.$$

Proof: Let us first consider the CMGF of $\hat{\theta}_1$ given by

$$
\begin{aligned}
M_{\hat{\theta}_1}(t) &= E\left(e^{t\hat{\theta}_1} | 1 \le n_1 \le n-1\right) \\
&= E\left(e^{t\frac{\sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i}{n_1}} | 1 \le n_1 \le n-1\right) \\
&= \sum_{i=1}^{n-1} E\left(e^{t\frac{\sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i}{n_1}} | n_1 = i\right) Pr\left(n_1 = i | 1 \le n_1 \le n-1\right) \\
&= \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \left\{1 - \left[i\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)\right]^{-1} t\right\}^{-n}.
\end{aligned}
\tag{2.4}
$$

The proofs for the other three cases proceed analogously.

From Theorem 2.3.2, upon inverting the conditional moment generating functions, we readily derive the conditional PDFs of the MLEs, conditioned on $1 \le n_1 \le n-1$, to be as presented below in Theorem 2.3.3.

Theorem 2.3.3. *Conditional on $1 \le n_1 \le n-1$, the conditional PDFs of the MLEs are given by*

$$
f_{\hat{\theta}_1}(x) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} g\left(x; n, i\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)\right),
$$

$$
f_{\hat{\theta}_2}(x) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} g\left(x; n, (n-i)\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)\right),
$$

$$
f_{\hat{\theta}'_1}(x) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} g\left(x; n-i, \frac{(n-i)}{\theta'_1}\right),
$$

$$
f_{\hat{\theta}'_2}(x) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} g\left(x; i, \frac{i}{\theta'_2}\right),
$$

where,

$$
g(y; \alpha, \lambda) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}, & y > 0 \\ 0, & o.w. \end{cases}
$$

is the PDF of a gamma random variable.

It is of interest to note here that the conditional PDFs of the MLEs are indeed mixtures of gamma densities.

2.3.1 Properties of the MLEs

From Theorems 2.3.2 and 2.3.3, we can derive some simple distributional properties of the MLEs.

Theorem 2.3.4. *The first two moments of the MLEs are given by*

$$E(\hat{\theta}_1) = \frac{n\theta_1\theta_2}{(\theta_1+\theta_2)\sum_{j=1}^{n-1} p_j}\sum_{i=1}^{n-1}\frac{p_i}{i},$$

$$E(\hat{\theta}_1^2) = \frac{n(n+1)\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^2\sum_{j=1}^{n-1} p_j}\sum_{i=1}^{n-1}\frac{p_i}{i^2},$$

$$E(\hat{\theta}_2) = \frac{n\theta_1\theta_2}{(\theta_1+\theta_2)\sum_{j=1}^{n-1} p_j}\sum_{i=1}^{n-1}\frac{p_i}{(n-i)},$$

$$E(\hat{\theta}_2^2) = \frac{n(n+1)\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^2\sum_{j=1}^{n-1} p_j}\sum_{i=1}^{n-1}\frac{p_i}{(n-i)^2},$$

$$E(\hat{\theta}_1') = \sum_{i=1}^{n-1}\frac{p_i}{\sum_{j=1}^{n-1} p_j}\theta_1' = \theta_1',$$

$$E(\hat{\theta}_1'^2) = \frac{\theta_1'^2}{\sum_{j=1}^{n-1} p_j}\sum_{i=1}^{n-1}\frac{(n-i+1)}{(n-i)}p_i,$$

$$E(\hat{\theta}_2') = \sum_{i=1}^{n-1}\frac{p_i}{\sum_{j=1}^{n-1} p_j}\theta_2' = \theta_2',$$

$$E(\hat{\theta}_2'^2) = \frac{\theta_2'^2}{\sum_{j=1}^{n-1} p_j}\sum_{i=1}^{n-1}\frac{(i+1)}{i}p_i.$$

The above expressions for the expected values clearly reveal that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are biased estimators, while both $\hat{\theta}'_1$ and $\hat{\theta}'_2$ are unbiased estimators. The expressions for the second moments can be used to find standard errors of the MLEs. Note that, in the expressions above, the quantities within the summation sign denote the inverse moments of positive binomial random variables. Since exact expressions are not available, we may use the tabulated values of positive binomial random variables presented, for example, by Edwin and Savage (1954). Since the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are clearly biased, tabulated values of the bias given by Kundu and Basu (2000) may be used for bias correction, for example.

We can also obtain expressions for the tail probabilities from Theorem 2.3.3. These expressions, presented below in Theorem 2.3.5, will be used to construct exact confidence intervals later in Section 2.4.

Theorem 2.3.5. *The tail probabilities of the MLEs are*

$$P_{\theta_1}\left(\hat{\theta}_1 \geq b\right) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \Gamma\left(n, i\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) b\right),$$

$$P_{\theta_2}\left(\hat{\theta}_2 \geq b\right) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \Gamma\left(n, (n-i)\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) b\right),$$

$$P_{\theta'_1}\left(\hat{\theta}'_1 \geq b\right) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \Gamma\left(n-i, \frac{(n-i)}{\theta'_1} b\right),$$

$$P_{\theta'_1}\left(\hat{\theta}'_2 \geq b\right) = \sum_{i=1}^{n-1} \frac{p_i}{\sum_{j=1}^{n-1} p_j} \Gamma\left(i, \frac{i}{\theta'_2} b\right),$$

where $\Gamma(\alpha, z) = \frac{1}{\Gamma(\alpha)} \int_z^\infty y^{\alpha-1} e^{-y} dy$ $(0 < z < \infty)$ *is the incomplete gamma ratio.*

2.4 Confidence Intervals

In this section, we present different methods of constructing confidence intervals (CIs) for the unknown parameters θ_1, θ_2, θ_1' and θ_2'. The exact CIs are based on the exact conditional distributions of the MLEs presented in Theorems 2.3.3 and 2.3.5. Since the exact conditional PDFs of the MLEs are computationally intensive, we may use the asymptotic distributions of the MLEs to obtain approximate CIs for θ_1, θ_2, θ_1' and θ_2' in the case of large sample sizes. Finally, we use the parametric bootstrap methods to construct the CIs for the parameters.

2.4.1 Exact Confidence Intervals

In order to illustrate how to construct the exact confidence intervals of the parameters, we take θ_1 as an example. Determine two increasing functions of parameter θ_1, say $c(\theta_1)$ and $d(\theta_1)$, such that for each value of θ_1 we have the probability

$$P_{\theta_1}\left(\hat{\theta}_1 \geq d(\theta_1)\right) = 1 - \frac{\alpha}{2}, \tag{2.5}$$

$$P_{\theta_1}\left(\hat{\theta}_1 \geq c(\theta_1)\right) = \frac{\alpha}{2}, \tag{2.6}$$

exactly. With $c(\theta_1)$ and $d(\theta_1)$ assumed to be increasing functions, they have single-valued inverses, say $c^{-1}(\hat{\theta}_1)$ and $d^{-1}(\hat{\theta}_1)$, respectively. Thus, the events $\hat{\theta}_1 \geq d(\theta_1)$ and $\theta_1 \leq d^{-1}(\hat{\theta}_1)$, $\hat{\theta}_1 \geq c(\theta_1)$ and $\theta_1 \leq c^{-1}(\hat{\theta}_1)$ are equivalent, respectively, and so we have

$$P_{\theta_1}\left(\theta_1 \geq d^{-1}(\hat{\theta}_1)\right) = \frac{\alpha}{2}, \tag{2.7}$$

$$P_{\theta_1}\left(\theta_1 \geq c^{-1}(\hat{\theta}_1)\right) = 1 - \frac{\alpha}{2}. \tag{2.8}$$

Therefore, $\theta_{1L} = c^{-1}(\hat{\theta}_1)$ is the lower bound and $\theta_{1U} = d^{-1}(\hat{\theta}_1)$ is the upper bound for the $100(1-\alpha)\%$ confidence interval for θ_1.

Confidence Interval for θ_1

Using equations (2.5)-(2.8), a two-sided $100(1-\alpha)\%$ CI for θ_1, denoted by $(\theta_{1L}, \theta_{1U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{i=1}^{n-1} \frac{p_i(\theta_{1L}, \hat{\theta}_2)}{\sum_{j=1}^{n-1} p_j(\theta_{1L}, \hat{\theta}_2)} \Gamma\left(n, i\left(\frac{1}{\theta_{1L}} + \frac{1}{\hat{\theta}_2}\right)\hat{\theta}_1\right), \tag{2.9}$$

$$1 - \frac{\alpha}{2} = \sum_{i=1}^{n-1} \frac{p_i(\theta_{1U}, \hat{\theta}_2)}{\sum_{j=1}^{n-1} p_j(\theta_{1U}, \hat{\theta}_2)} \Gamma\left(n, i\left(\frac{1}{\theta_{1U}} + \frac{1}{\hat{\theta}_2}\right)\hat{\theta}_1\right), \tag{2.10}$$

where

$$p_i(\theta_{1L(U)}, \hat{\theta}_2) = \binom{n}{i}\left(\frac{\hat{\theta}_2}{\theta_{1L(U)} + \hat{\theta}_2}\right)^i \left(\frac{\theta_{1L(U)}}{\theta_{1L(U)} + \hat{\theta}_2}\right)^{n-i}.$$

Confidence Interval for θ_2

Similarly, a two-sided $100(1-\alpha)\%$ CI for θ_2, denoted by $(\theta_{2L}, \theta_{2U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{i=1}^{n-1} \frac{p_i(\hat{\theta}_1, \theta_{2L})}{\sum_{j=1}^{n-1} p_j(\hat{\theta}_1, \theta_{2L})} \Gamma\left(n, (n-i)\left(\frac{1}{\hat{\theta}_1} + \frac{1}{\theta_{2L}}\right)\hat{\theta}_2\right), \tag{2.11}$$

$$1 - \frac{\alpha}{2} = \sum_{i=1}^{n-1} \frac{p_i(\hat{\theta}_1, \theta_{2U})}{\sum_{j=1}^{n-1} p_j(\hat{\theta}_1, \theta_{2U})} \Gamma\left(n, (n-i)\left(\frac{1}{\hat{\theta}_1} + \frac{1}{\theta_{2U}}\right)\hat{\theta}_2\right), \tag{2.12}$$

where

$$p_i(\hat{\theta}_1, \theta_{2L(U)}) = \binom{n}{i}\left(\frac{\theta_{2L(U)}}{\hat{\theta}_1 + \theta_{2L(U)}}\right)^i \left(\frac{\hat{\theta}_1}{\hat{\theta}_1 + \theta_{2L(U)}}\right)^{n-i}.$$

Confidence Interval for θ_1'

A two-sided $100(1-\alpha)\%$ CI for θ_1', denoted by $(\theta_{1L}', \theta_{1U}')$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{i=1}^{n-1} \frac{p_i(\hat{\theta}_1, \hat{\theta}_2)}{\sum_{j=1}^{n-1} p_j(\hat{\theta}_1, \hat{\theta}_2)} \Gamma\left(n-i, \frac{(n-i)}{\theta_{1L}'}\hat{\theta}_1'\right), \tag{2.13}$$

$$1 - \frac{\alpha}{2} = \sum_{i=1}^{n-1} \frac{p_i(\hat{\theta}_1, \hat{\theta}_2)}{\sum_{j=1}^{n-1} p_j(\hat{\theta}_1, \hat{\theta}_2)} \Gamma\left(n - i, \frac{(n-i)}{\theta'_{1U}}\hat{\theta}'_1\right), \tag{2.14}$$

where

$$p_i(\hat{\theta}_1, \hat{\theta}_2) = \binom{n}{i} \left(\frac{\hat{\theta}_2}{\hat{\theta}_1 + \hat{\theta}_2}\right)^i \left(\frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}\right)^{n-i}.$$

Confidence Interval for θ'_2

A two-sided $100(1-\alpha)\%$ CI for θ'_2, denoted by $(\theta'_{2L}, \theta'_{2U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{i=1}^{n-1} \frac{p_i(\hat{\theta}_1, \hat{\theta}_2)}{\sum_{j=1}^{n-1} p_j(\hat{\theta}_1, \hat{\theta}_2)} \Gamma\left(i, \frac{i}{\theta'_{2L}}\hat{\theta}'_2\right), \tag{2.15}$$

$$1 - \frac{\alpha}{2} = \sum_{i=1}^{n-1} \frac{p_i(\hat{\theta}_1, \hat{\theta}_2)}{\sum_{j=1}^{n-1} p_j(\hat{\theta}_1, \hat{\theta}_2)} \Gamma\left(i, \frac{i}{\theta'_{2U}}\hat{\theta}'_2\right). \tag{2.16}$$

Lacking a closed-form solution, we have to apply an iterative root-finding technique in the determination of θ_{iL}, θ'_{iL}, θ_{iU} and θ'_{iU} for $i = 1, 2$; the Newton-Raphson iteration method, for instance, can be used.

It is important to mention here that our construction of the exact confidence interval is based on the assumption that $c(\theta_i)$, $c(\theta'_i)$, $d(\theta_i)$ and $d(\theta'_i)$ are increasing functions of θ_i, θ'_i, $i = 1, 2$. This assumption guarantees the invertibility of the pivotal quantities. Several authors including Chen and Bhattacharyya (1988), Gupta and Kundu (1998), Kundu and Basu (2000), and Childs *et al.* (2003) have all used this approach to construct exact CIs in different contexts. This assumption implies, for $\theta_1 < \tilde{\theta}_1$, for example, we have

$$P_{\theta_1}\left(\hat{\theta}_1 \geq c(\tilde{\theta}_1)\right) \leq P_{\theta_1}\left(\hat{\theta}_1 \geq c(\theta_1)\right) = P_{\tilde{\theta}_1}\left(\hat{\theta}_1 \geq c(\tilde{\theta}_1)\right) = \frac{\alpha}{2}. \tag{2.17}$$

Therefore, $P_{\theta_i}\left(\hat{\theta}_i \geq b\right)$ and $P_{\theta_i'}\left(\hat{\theta}_i' \geq b\right)$ are increasing functions of θ_i and θ_i', $i = 1, 2$, respectively. Values of $P_{\theta_i}\left(\hat{\theta}_i \geq b\right)$ and $P_{\theta_i'}\left(\hat{\theta}_i' \geq b\right)$ for various θ_i, θ_i' $(i = 1, 2)$ and b are presented in Tables 2.1 - 2.4 which support this monotonicity. As the concerned tail probabilities are all mixtures of gamma tail probabilities, their monotonicity can also be established using the recent results of Balakrishnan and Iliopoulos (2008).

2.4.2 Asymptotic Confidence Intervals

Using the asymptotic normality of the MLEs, we are able to construct asymptotic confidence intervals for θ_1, θ_2, θ_1' and θ_2' based on the Fisher information matrix.

Let $I(\theta_1, \theta_2, \theta_1', \theta_2') = (I_{ij}(\theta_1, \theta_2, \theta_1', \theta_2'))$, $i, j = 1, 2, 3, 4$, denote the Fisher information matrix for the parameter $(\theta_1, \theta_2, \theta_1', \theta_2')$, where

$$I_{ij}(\theta_1, \theta_2, \theta_1', \theta_2') = -\mathrm{E}\left(\frac{\partial^2 \ln\left(\mathrm{L}(\theta_1, \theta_2, \theta_1', \theta_2')\right)}{\partial\theta_i^{(')}\partial\theta_j^{(')}}\right)\Bigg|_{\theta_1=\hat{\theta}_1, \theta_2=\hat{\theta}_2, \theta_1'=\hat{\theta}_1', \theta_2'=\hat{\theta}_2'}.$$

For large n_1 and n_2 and under suitable regularity conditions, the asymptotic distributions of the pivotal quantities $\frac{\hat{\theta}_1 - E(\hat{\theta}_1)}{\sqrt{V_{11}}}$, $\frac{\hat{\theta}_2 - E(\hat{\theta}_2)}{\sqrt{V_{22}}}$, $\frac{\hat{\theta}_1' - E(\hat{\theta}_1')}{\sqrt{V_{33}}}$ and $\frac{\hat{\theta}_2' - E(\hat{\theta}_2')}{\sqrt{V_{44}}}$ are all $N(0, 1)$. Here, $V_{ii} = I_{ii}^{-1}$, $i = 1, 2, 3, 4$, and $E(\hat{\theta}_j)$, $E(\hat{\theta}_j')$, $j = 1, 2$ are all as given in Theorem 2.3.4. Then, the $100(1-\alpha)\%$ approximate CIs for $(\theta_1, \theta_2, \theta_1', \theta_2')$ can be obtained from the following expressions:

$$P\left(z_{\frac{\alpha}{2}} < \frac{\hat{\theta}_1 - \mathrm{E}(\hat{\theta}_1)}{\sqrt{V_{11}}} < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha, \tag{2.18}$$

$$P\left(z_{\frac{\alpha}{2}} < \frac{\hat{\theta}_2 - \mathrm{E}(\hat{\theta}_2)}{\sqrt{V_{22}}} < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha, \tag{2.19}$$

$$P\left(z_{\frac{\alpha}{2}} < \frac{\hat{\theta}_1' - \mathrm{E}(\hat{\theta}_1')}{\sqrt{V_{33}}} < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha, \tag{2.20}$$

$$P\left(z_{\frac{\alpha}{2}} < \frac{\hat{\theta}'_2 - \mathrm{E}(\hat{\theta}'_2)}{\sqrt{V_{44}}} < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha, \tag{2.21}$$

respectively, where z_q is the q-th upper percentile of the standard normal distribution.

From Eq. (2.2), we find

$$I_{11}(\theta_1, \theta_2, \theta'_1, \theta'_2) = -\mathrm{E}\left(\frac{n_1}{\hat{\theta}_1^2} - \frac{2S_1}{\hat{\theta}_1^3}\right) = \frac{n_1}{\hat{\theta}_1^2}, \tag{2.22}$$

$$I_{22}(\theta_1, \theta_2, \theta'_1, \theta'_2) = -\mathrm{E}\left(\frac{n_2}{\hat{\theta}_2^2} - \frac{2S_1}{\hat{\theta}_2^3}\right) = \frac{n_2}{\hat{\theta}_2^2}, \tag{2.23}$$

$$I_{33}(\theta_1, \theta_2, \theta'_1, \theta'_2) = -\mathrm{E}\left(\frac{n_2}{\hat{\theta}_1'^2} - \frac{2S_2}{\hat{\theta}_1'^3}\right) = \frac{n_2}{\hat{\theta}_1'^2}, \tag{2.24}$$

$$I_{44}(\theta_1, \theta_2, \theta'_1, \theta'_2) = -\mathrm{E}\left(\frac{n_1}{\hat{\theta}_2'^2} - \frac{2S_3}{\hat{\theta}_2'^3}\right) = \frac{n_1}{\hat{\theta}_2'^2}, \tag{2.25}$$

$$I_{12} = I_{13} = I_{14} = I_{21} = I_{23} = I_{24} = I_{31} = I_{32} = I_{34} = I_{41} = I_{42} = I_{43} = 0, \tag{2.26}$$

where

$$S_1 = \sum_{i \in I_1} x_i + \sum_{i \in I_2} y_i,$$

$$S_2 = \sum_{i \in I_2} (z_i - y_i),$$

$$S_3 = \sum_{i \in I_1} (z_i - x_i).$$

Thus, the Fisher information matrix is given by

$$\begin{bmatrix} \frac{n_1}{\hat{\theta}_1^2} & 0 & 0 & 0 \\ 0 & \frac{n_2}{\hat{\theta}_2^2} & 0 & 0 \\ 0 & 0 & \frac{n_2}{\hat{\theta}_1'^2} & 0 \\ 0 & 0 & 0 & \frac{n_1}{\hat{\theta}_2'^2} \end{bmatrix}.$$

This implies that the MLEs are asymptotically mutually independent. The asymptotic unconditional variance of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$ can be obtained from the Fisher information matrix as

$$V_{11} = \frac{\hat{\theta}_1^2}{n_1}, \quad V_{22} = \frac{\hat{\theta}_2^2}{n_2}, \quad V_{33} = \frac{\hat{\theta}_1'^2}{n_2}, \quad V_{44} = \frac{\hat{\theta}_2'^2}{n_1}.$$

Then, the $100(1-\alpha)\%$ approximate CIs for θ_1, θ_2, θ'_1 and θ'_2 are obtained accordingly.

2.4.3 Bootstrap Confidence Intervals

In this subsection, we present two methods to construct confidence intervals for θ_1, θ_2, θ'_1 and θ'_2, viz., percentile interval and the biased-corrected and accelerated (BC_α) interval. See Efron (1982), Hall (1988), and Efron and Tibshirani (1998) for pertinent details. To obtain these intervals, we use the following algorithm.

Percentile Interval

(1) Determine $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$.

(2) Generate a complete two-component system failure data set using the $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$. For this data, compute the bootstrap estimates of θ_1, θ_2, θ'_1 and θ'_2, namely, $\hat{\theta}_1^*$, $\hat{\theta}_2^*$, $\hat{\theta}_1'^*$ and $\hat{\theta}_2'^*$, by using the expressions of the MLEs presented in Section 2.2.

(3) Repeat Step 2 R times. This gives R estimates for each of the parameters θ_1, θ_2, θ'_1 and θ'_2.

(4) Arrange the R $\hat{\theta}_1^*$'s, $\hat{\theta}_2^*$'s, $\hat{\theta}_1'^*$'s and $\hat{\theta}_2'^*$'s in ascending order and take the $(R\alpha/2)$-th and $R(1-\alpha/2)$-th values. Then, a $100(1-\alpha)\%$ confidence interval for θ_1 is given by $\left(\hat{\theta}_1^{*[R\alpha/2]}, \hat{\theta}_1^{*[R(1-\alpha/2)]}\right)$. Similarly, a $100(1-\alpha)\%$ confidence interval for θ_2 is given

by $\left(\hat{\theta}_2^{*[R\alpha/2]}, \hat{\theta}_2^{*[R(1-\alpha/2)]}\right)$, $\left(\hat{\theta}_1'^{*[R\alpha/2]}, \hat{\theta}_1'^{*[R(1-\alpha/2)]}\right)$ for θ_1', $\left(\hat{\theta}_2'^{*[R\alpha/2]}, \hat{\theta}_2'^{*[R(1-\alpha/2)]}\right)$ for θ_2'.

BC_α Percentile Interval

The BC_α interval is similar to the percentile interval except that it is corrected for bias and for the rate of change of the SE of MLE $\hat{\theta}$ (say) with respect to the true parameter value θ (say); see Efron and Tibshirani (1998). The standard normal approximation assumes that the SE of $\hat{\theta}$ is the same for all values of θ, but this assumption is not correct. The BC_α interval corrects for this.

Repeat the first three steps as described for Percentile Interval. In step 4, arrange the R $\hat{\theta}_1^*$'s, $\hat{\theta}_2^*$'s, $\hat{\theta}_1'^*$'s and $\hat{\theta}_2'^*$'s in ascending order. A two-sided $100(1-\alpha)\%$ BC_α bootstrap confidence interval of $\theta_i^{(\prime)}$ is then given by

$$\left(\hat{\theta}_{iL}^{(\prime)*}, \hat{\theta}_{iU}^{(\prime)*}\right) = \left(\hat{\theta}_i^{(\prime)*[R\alpha_{1i}^{(\prime)}]}, \hat{\theta}_i^{(\prime)*[R\alpha_{2i}^{(\prime)}]}\right), \quad i = 1, 2,$$

where

$$\alpha_{1i}^{(\prime)} = \Phi\left(\hat{z}_{0i}^{(\prime)} + \frac{\hat{z}_{0i}^{(\prime)} + z_{\frac{\alpha}{2}}}{1 - \hat{\alpha}_i^{(\prime)}\left(\hat{z}_{0i}^{(\prime)} + z_{\frac{\alpha}{2}}\right)}\right) \quad \alpha_{2i}^{(\prime)} = \Phi\left(\hat{z}_{0i}^{(\prime)} + \frac{\hat{z}_{0i}^{(\prime)} + z_{1-\frac{\alpha}{2}}}{1 - \hat{\alpha}_i^{(\prime)}\left(\hat{z}_{0i}^{(\prime)} + z_{1-\frac{\alpha}{2}}\right)}\right),$$

here, $\Phi(\cdot)$ is the standard normal cumulative distribution function. z_α is the 100αth percentile point of the standard normal distribution. $\hat{z}_{0i}$ is the bias-correction and can be obtained directly from the proportion of bootstrap replications less than the original estimate $\hat{\theta}_i^{(\prime)}$,

$$\hat{z}_{0i}^{(\prime)} = \Phi^{-1}\left(\frac{\text{number of } \hat{\theta}_i^{(\prime)*} < \hat{\theta}_i^{(\prime)}}{R}\right), \quad i = 1, 2,$$

with $\Phi^{-1}(\cdot)$ denoting the inverse of the standard normal cumulative distribution function. $\hat{z}_{0i}^{(\prime)}$ is actually the median bias of $\hat{\theta}_i^{(\prime)*}$, and $\hat{z}_{0i}^{(\prime)} = 0$ if exactly half of the $\hat{\theta}_i^{(\prime)*}$ values are less than or equal to $\hat{\theta}_i^{(\prime)}$.

We calculate the acceleration value $\hat{\alpha}_i^{(')}$ by using the jackknife approach:

$$\hat{\alpha}_i^{(')} = \frac{\sum_{j=1}^{n}\left(\hat{\theta}_{i(\cdot)}^{(')} - \hat{\theta}_{i(j)}^{(')}\right)^3}{6\left[\sum_{j=1}^{n}\left(\hat{\theta}_{i(\cdot)}^{(')} - \hat{\theta}_{i(j)}^{(')}\right)^2\right]^{\frac{3}{2}}}, \quad i = 1, 2,$$

where $\hat{\theta}_{i(j)}^{(')}$ is the MLE of $\theta_i^{(')}$ based on the original sample with the j-th observation deleted, $j = 1, 2, \ldots, n$, and $\hat{\theta}_{i(\cdot)}^{(')} = \frac{\sum_{j=1}^{n}\hat{\theta}_{i(j)}^{(')}}{n}$.

2.5 Illustrations

2.5.1 Simulation Study

To compare the performance of all the confidence intervals described in Section 2.4, we carried out a Monte Carlo simulation study. We chose the values of the parameters to be $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$. We then determined the true coverage probabilities of the 90%, 95% and 99% confidence intervals for the parameters with different sample sizes by all the methods described in Section 2.4. The results for $n = 40, 20, 10$ are presented in Table 2.7. The values are based on an average over 999 bootstrap replications.

From the table, we observe that the exact method of constructing confidence intervals (based on the exact conditional distributions of the MLEs derived in Section 2.3) has its coverage probability to be quite close to the pre-fixed nominal level in all cases. As expected, the coverage probabilities of the approximate method (based on asymptotic normality of the MLEs) are most often smaller than the nominal level. This indicates that the confidence intervals obtained by this method will often be unduly narrower. Between the two bootstrap methods of constructing confidence intervals, the BC_α percentile interval seems to have coverage probabilities closer to the nominal level and hence may be used in case of large sample sizes when the computation of the exact confidence interval becomes difficult.

We notice that, when n is small, there are fewer failures observed and so inference for the parameters are not quite precise. For the approximate method, as n increases, the coverage probability for any parameter gets closer to the nominal value. This is because, when n is small, fewer failures occur during the experiment time while as n increases, the number of failures increases thus resulting in a better large-sample approximation for the distribution of MLEs. It is important to observe that for all the nominal levels considered, the coverage probabilities of the approximate method are almost always lower for small sample size n. This means that we require a much larger sample size to use the asymptotic normality of the MLEs, and in fact even for $n = 40$, the approximate method does not provide close results.

Thus, based on this simulation study, we recommend the use of the exact method for any sample size as it provides coverage probabilities quite close to the nominal levels. The use of the parametric BC_α bootstrap method can be supported for n at least moderately large. The approximate method can be used when n is large for its computational ease as well as for having its coverage probability close to the nominal level when n is large (preferably over 50).

2.5.2 Numerical Examples

In this subsection, we consider two data sets when $n = 35$ and $n = 15$. The parameters were chosen to be $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$. The data are as follows.

Data Set 1: $n = 35$

(0.76,2; 1.98,1)	(1.41,1; 2.02,2)	(1.51,2; 3.38,1)	(1.69,2; 3.99,1)	(5.14,2; 5.20,1)
(7.40,2; 8.87,1)	(10.22,1; 10.99,2)	(4.37,2; 11.12,1)	(6.32,1; 13.57,2)	(1.56,1; 15.25,2)
(7.13,1; 16.42,2)	(6.76,1; 17.52,2)	(17.10,2; 18.07,1)	(6.53,1; 18.26,2)	(8.25,2; 18.66,1)
(10.19,1; 19.66,2)	(11.45,2; 20.06,1)	(2.19,1; 21.78,2)	(20.92,2; 21.80,1)	(14.66,1; 24.79,2)
(22.67,1; 27.12,2)	(24.11,2; 28.35,1)	(1.15,1; 28.42,2)	(4.08,1; 30.03,2)	(4.34,1; 30.72,2)
(0.40,2; 30.89,1)	(30.78,1; 31.94,2)	(9.62,1; 33.07,2)	(1.30,2; 33.61,1)	(9.69,1; 33.93,2)
(11.18,2; 36.72,1)	(18.67,1; 38.32,2)	(34.16,1; 43.46,2)	(31.27,1; 43.54,2)	(34.76,1; 76.11,2)

Data Set 2: $n = 15$

(2.35,2; 5.59,1)	(0.34,2; 6.76,1)	(0.76,2; 8.93,1)	(4.84,1; 9.60,2)	(4.70,1; 10.40,2)
(5.35,1; 17.82,2)	(15.39,2; 18.60,1)	(1.87,1; 20.98,2)	(22.74,2; 23.68,1)	(3.55,2; 25.58,1)
(18.77,2; 29.30,1)	(1.97,1; 37.25,2)	(10.37,1; 41.11,2)	(24.37,1; 60.40,2)	(63.54,1; 64.98,2)

For the example when $n = 35$, we have $n_1 = 21$ and $n_2 = 14$. Using the expressions presented in Section 2.2, we find the MLEs of θ_1, θ_2, θ_1' and θ_2' to be $\hat{\theta}_1 = 18.27$, $\hat{\theta}_2 = 27.41$, $\hat{\theta}_1' = 9.08$ and $\hat{\theta}_2' = 14.70$. Similarly, for the example when $n = 15$, we have $n_1 = 8$ and $n_2 = 7$, and the MLEs to be $\hat{\theta}_1 = 22.61$, $\hat{\theta}_2 = 25.84$, $\hat{\theta}_1' = 7.79$ and $\hat{\theta}_2' = 18.19$. We then constructed the 90%, 95% and 99% confidence intervals for the four parameters by using the four methods discussed in Section 2.4, and they are presented in Tables 2.5 and 2.6, respectively.

From these results, it is seen that the exact confidence intervals are wider in general than the other intervals. It is also seen that the approximate method always provides narrower confidence intervals since, as mentioned earlier, the coverage probability for the approximate method is always lower than the nominal level. Furthermore, the two bootstrap intervals for the parameters are close to the exact confidence intervals when $n = 35$, while these intervals are not so satisfactory compared to the exact confidence intervals when $n = 15$.

Table 2.1: Values of $P_{\theta_1}\left(\hat{\theta}_1 \geq b\right)$ with $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$

θ_1	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.1340	0.0000	0.0000	0.0000
9	0.9825	0.1300	0.0009	0.0000
13	0.9999	0.7943	0.1383	0.0076
17	1.0000	0.9819	0.6058	0.1488
21	1.0000	0.9987	0.8951	0.4911
25	1.0000	0.9999	0.9779	0.7784
29	1.0000	1.0000	0.9956	0.9195
33	1.0000	1.0000	0.9991	0.9730

Table 2.2: Values of $P_{\theta_2}\left(\hat{\theta}_2 \geq b\right)$ with $\theta_1 = 20$, $\theta_1' = 9$ and $\theta_2' = 14$

θ_2	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.1399	0.0000	0.0000	0.0000
9	0.9803	0.1394	0.0014	0.0000
13	0.9999	0.7865	0.1503	0.0108
17	1.0000	0.9784	0.6024	0.1629
21	1.0000	0.9982	0.8850	0.4931
25	1.0000	0.9998	0.9729	0.7674
29	1.0000	1.0000	0.9939	0.9085
33	1.0000	1.0000	0.9986	0.9663

Table 2.3: Values of $P_{\theta_1'}\left(\hat{\theta}_1' \geq b\right)$ with $\theta_1 = 20$, $\theta_2 = 25$ and $\theta_2' = 14$

θ_1'	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.1930	0.0001	0.0000	0.0000
9	0.9326	0.1715	0.0038	0.0000
13	0.9965	0.7257	0.1637	0.0130
17	0.9997	0.9453	0.5680	0.1597
21	1.0000	0.9895	0.8425	0.4681
25	1.0000	0.9977	0.9494	0.7352
29	1.0000	0.9994	0.9838	0.8840
33	1.0000	0.9998	0.9946	0.9515

Table 2.4: Values of $P_{\theta_2'}\left(\hat{\theta}_2' \geq b\right)$ with $\theta_1 = 20$, $\theta_2 = 25$ and $\theta_1' = 9$

θ_2'	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.1704	0.0000	0.0000	0.0000
9	0.9552	0.1482	0.0014	0.0000
13	0.9989	0.7551	0.1403	0.0065
17	1.0000	0.9653	0.5833	0.1363
21	1.0000	0.9955	0.8740	0.4716
25	1.0000	0.9994	0.9685	0.7651
29	1.0000	0.9999	0.9924	0.9129
33	1.0000	1.0000	0.9981	0.9701

Table 2.5: Interval estimation for θ_1, θ_2, θ'_1 and θ'_2 in Example ($n = 35$)

C.I. for θ_1			
Method	90%	95%	99%
Exact C.I.	(13.08, 27.01)	(12.30, 29.26)	(10.93, 34.48)
Approx C.I.	(11.72, 24.83)	(10.46, 26.09)	(8.00, 28.55)
Boot-p C.I.	(12.53, 25.84)	(11.49, 27.42)	(10.17, 31.61)
BC_α C.I.	(12.83, 25.90)	(11.59, 27.58)	(10.17, 31.05)

C.I. for θ_2			
Method	90%	95%	99%
Exact C.I.	(18.19, 44.29)	(16.90, 49.04)	(14.68, 60.56)
Approx C.I.	(15.36, 39.46)	(13.05, 41.77)	(8.54, 46.28)
Boot-p C.I.	(17.84, 44.20)	(16.35, 47.62)	(13.52, 66.14)
BC_α C.I.	(17.71, 43.61)	(16.17, 46.90)	(13.46, 62.19)

C.I. for θ'_1			
Method	90%	95%	99%
Exact C.I.	(6.11, 15.23)	(5.66, 16.98)	(4.87, 21.44)
Approx C.I.	(5.09, 13.07)	(4.32, 13.84)	(2.83, 15.33)
Boot-p C.I.	(5.48, 13.46)	(4.94, 14.37)	(4.09, 16.51)
BC_α C.I.	(6.08, 14.31)	(5.47, 15.51)	(4.66, 17.10)

C.I. for θ'_2			
Method	90%	95%	99%
Exact C.I.	(10.60, 22.04)	(9,96, 23.92)	(8.83, 28.32)
Approx C.I.	(9.42, 19.98)	(8.41, 20.99)	(6.44, 22.96)
Boot-p C.I.	(9.61, 20.18)	(8.95, 21.27)	(7.89, 23.58)
BC_α C.I.	(9,95, 20.84)	(9.37, 21.63)	(8.28, 23.91)

Table 2.6: Interval estimation for θ_1, θ_2, θ_1' and θ_2' in Example ($n = 15$)

C.I. for θ_1			
Method	90%	95%	99%
Exact C.I.	(13.36, 44.05)	(12.14, 50.98)	(10.13, 69.68)
Approx C.I.	(9.46, 35.76)	(6.94, 38.28)	(2.02, 43.21)
Boot-p C.I.	(12.53, 42.99)	(11.16, 49.13)	(9.42, 67.33)
BC_α C.I.	(12.76, 43.51)	(11.33, 50.32)	(9.69, 71.17)

C.I. for θ_2			
Method	90%	95%	99%
Exact C.I.	(14.71, 52.83)	(13.30, 62.00)	(10.98, 87.71)
Approx C.I.	(9.78, 41.91)	(6.70, 44.99)	(0.68, 51.01)
Boot-p C.I.	(13.24, 51.22)	(11.07, 61.89)	(9.67, 107.24)
BC_α C.I.	(14.24, 58.09)	(12.90, 71.03)	(10.17, 118.59)

C.I. for θ_1'			
Method	90%	95%	99%
Exact C.I.	(4.54, 17.37)	(4.07, 20.89)	(3.29, 32.31)
Approx C.I.	(2.95, 12.63)	(2.02, 13.56)	(0.21, 15.37)
Boot-p C.I.	(3.43, 13.68)	(2.79, 14.91)	(1.58, 18.25)
BC_α C.I.	(4.25, 15.66)	(3.71, 18.06)	(2.84, 19.78)

C.I. for θ_2'			
Method	90%	95%	99%
Exact C.I.	(10.95, 37.70)	(9.92, 44.29)	(8.16, 63.78)
Approx C.I.	(7.61, 28.78)	(5.59, 30.80)	(1.62, 34.76)
Boot-p C.I.	(9.05, 29.97)	(7.65, 33.87)	(5.17, 40.02)
BC_α C.I.	(9.74, 32.08)	(8.66, 35.48)	(6.20, 42.10)

Table 2.7: Estimated coverage probabilities based on 999 simulations with $\theta_1 = 20$, $\theta_2 = 25$, $\theta'_1 = 9$, $\theta'_2 = 14$

$n = 40$	90% C.I.				95% C.I.				99% C.I.			
parameters	Exact	Approx	Boot-p	BC_α	Exact	Approx	Boot-p	BC_α	Exact	Approx	Boot-p	BC_α
θ_1	90.4	90.8	90.1	91.0	94.7	95.5	94.1	94.5	98.5	98.0	98.6	98.4
θ_2	89.6	89.9	88.7	90.7	95.5	94.2	94.0	95.4	98.9	97.2	98.6	98.8
θ'_1	90.1	88.7	90.5	91.2	96.0	92.7	94.7	95.7	98.9	96.6	98.4	98.3
θ'_2	89.3	88.1	88.8	89.0	94.5	92.4	93.0	93.5	99.1	96.9	98.2	98.6

$n = 20$	90% C.I.				95% C.I.				99% C.I.			
parameter	Exact	Approx	Boot-p	BC_α	Exact	Approx	Boot-p	BC_α	Exact	Approx	Boot-p	BC_α
θ_1	90.2	90.6	88.8	90.3	94.3	94.6	93.4	94.7	99.2	97.3	98.5	99.1
θ_2	91.6	91.4	90.6	91.8	95.5	94.2	94.7	96.2	99.4	97.4	99.3	99.4
θ'_1	90.8	84.8	86.4	89.9	96.1	90.6	93.5	94.9	99.3	93.6	97.3	98.4
θ'_2	89.0	88.4	89.6	90.2	956	89.5	92.6	94.3	99.6	95.2	97.4	98.3

$n = 10$	90% C.I.				95% C.I.				99% C.I.			
parameter	Exact	Approx	Boot-p	BC_α	Exact	Approx	Boot-p	BC_α	Exact	Approx	Boot-p	BC_α
θ_1	89.6	88.7	87.8	98.5	94.8	91.3	93.2	95.1	98.8	94.9	96.1	96.3
θ_2	91.5	88.7	90.1	90.9	94.1	91.1	92.5	93.9	99.0	92.9	96.2	96.8
θ'_1	88.9	81.9	83.9	87.8	95.3	82.5	88.3	92.4	98.3	87.1	92.4	95.5
θ'_2	89.0	87.8	84.1	86.9	94.5	85.1	88.9	92.4	98.1	89.7	94.1	94.9

Chapter 3

Exact Analysis under Type-II Censoring

3.1 Introduction

In this Chapter, we consider a two-component system failure model in the case of Type-II censored data. We then present an iterative maximum likelihood estimation procedure to determine the MLEs of the parameters assuming the lifetimes to be exponentially distributed. The asymptotic distributions of the MLEs are also obtained. Construction of confidence intervals for the model parameters are discussed by using the asymptotic distributions and two parametric bootstrap methods. The bias and variance of the estimates as well as the performance of the three confidence intervals in terms of coverage probabilities are assessed through Monte Carlo simulation studies. Finally, examples are presented to illustrate all the methods of inference discussed here.

3.2 Model Description and MLEs

Consider the following simple system failure model: n identical systems are placed on a life-test and each system has two components. The experiment continues until a total of d ($d \leq n$) systems fail. We assume that X_i and Y_i ($i = 1, \ldots, n$) are random variables representing the lifetimes of Components 1 and 2, respectively, in the i-th system. Among the d observations, suppose Component 1 fails first d_1 times and Component 2 fails first d_2 times, with $d_1 + d_2 = d$. Let $Z_i = \max(X_i, Y_i)$ ($i = 1, \ldots, n$). Thus, the i-th system fails at time Z_i, and $Z_{1:n} < \cdots < Z_{d:n}$ are the corresponding ordered failure times obtained from a Type-II censored sample from the n systems under test. The data arising from such a two-component system is as follows:

$$(T_1,\ \delta_1';\ \ Z_{1:n},\ \delta_1''), \ldots, (T_d,\ \delta_d';\ \ Z_{d:n},\ \delta_d''), (*, *), \ldots, \tag{3.1}$$

where $T_1, \ldots, T_d$ denote the first observed failure times in the systems, $Z_{1:n} < \cdots < Z_{d:n}$ denote the final observed failure times of the systems, and δ' denotes the component of the first observed failure within the system and δ'' denotes the component of the second observed failure within the system. We use " $*$ " to denote the censored data.

If we let

$$I_1 = \{i \in (1, 2, \ldots, d) : \text{ Component 1 fails first}\},$$

$$I_2 = \{i \in (1, 2, \ldots, d) : \text{ Component 2 fails first}\},$$

the likelihood function of the observed data in (3.1) is

$$
\begin{aligned}
&L(\theta_1, \theta_2, \theta_1', \theta_2') \\
&= \frac{n!}{(n-d)!} \prod_{i=1}^{d} f(x_i, y_i) \prod_{i=d+1}^{n} Pr\left(max(X_i, Y_i) \geq z_{d:n}\right) \\
&= \frac{n!}{(n-d)!} \times \left(\frac{1}{\theta_1 \theta_2'}\right)^{d_1} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) \sum_{i \in I_1} x_i - \frac{1}{\theta_2'} \sum_{i \in I_1} z_i\right\} \\
&\quad \times \left(\frac{1}{\theta_2 \theta_1'}\right)^{d_2} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) \sum_{i \in I_2} y_i - \frac{1}{\theta_1'} \sum_{i \in I_2} z_i\right\} \times P_{z_{d:n}}^{n-d}, \qquad (3.2)
\end{aligned}
$$

where $0 < x_i < z_{i:n}$ for $i \in I_1$, $0 < y_i < z_{i:n}$ for $i \in I_2$, $0 < z_{1:n} < \cdots < z_{d:n} < \infty$, and

$$
P_{z_{d:n}} = Pr\left(max(X_i, Y_i) \geq z_{d:n}\right) = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-1} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-1} \times \tilde{P}, \quad (3.3)
$$

where

$$
\begin{aligned}
\tilde{P} = &\frac{1}{\theta_1}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) \exp\left(-\frac{1}{\theta_2'} z_{d:n}\right) + \frac{1}{\theta_2}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) \exp\left(-\frac{1}{\theta_1'} z_{d:n}\right) \\
&- \left(\frac{1}{\theta_1 \theta_2'} - \frac{1}{\theta_1' \theta_2'} + \frac{1}{\theta_2 \theta_1'}\right) \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) z_{d:n}\right\}. \qquad (3.4)
\end{aligned}
$$

The exact derivation of $P_{z_{d:n}}$ is presented later in Lemma 3.2.2.

The maximum likelihood estimate $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_1', \hat{\theta}_2')$ of $(\theta_1, \theta_2, \theta_1', \theta_2')$ is the value that globally maximizes the likelihood function in (3.2). Taking logarithm in Eq. (3.2), we obtain the log-likelihood function to be

$$
\begin{aligned}
\ln L = &-d_1 \ln \theta_1 - d_1 \ln \theta_2' - \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) \sum_{i \in I_1} x_i - \frac{1}{\theta_2'} \sum_{i \in I_1} z_i \\
&-d_2 \ln \theta_2 - d_2 \ln \theta_1' - \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) \sum_{i \in I_2} y_i - \frac{1}{\theta_1'} \sum_{i \in I_2} z_i \\
&-(n-d) \ln\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) - (n-d) \ln\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) + (n-d) \ln \tilde{P}. \quad (3.5)
\end{aligned}
$$

Unfortunately, from (3.5), we observe that no closed-form expressions for the MLEs exist. We need to determine the MLEs by numerically maximizing the log-likelihood function in (3.5). In the next subsection, an iterative procedure for the computation of the MLEs is described.

3.2.1 Computation of the MLEs

Most iterative procedures proposed in literature strongly depend on the initial value. The well-known linear estimates from ordinary linear regression or the estimation using the method of moments is quite difficult in this context. Fortunately, since θ_i $(i = 1, 2)$ are the mean times of the first failed Component i $(i = 1, 2)$ and θ'_i $(i = 1, 2)$ are the mean times of the surviving Component i $(i = 1, 2)$ starting from the time of the first failure, the initial values for the iterative procedure can be given to be

$$\theta_1^{(0)} = \frac{\sum_{i \in I_1} x_i + \sum_{i \in I_2} y_i}{d_1}, \qquad \theta_2^{(0)} = \frac{\sum_{i \in I_1} x_i + \sum_{i \in I_2} y_i}{d_2},$$
$$\theta_1'^{(0)} = \frac{\sum_{i \in I_2} (z_i - y_i)}{d_2}, \qquad \theta_2'^{(0)} = \frac{\sum_{i \in I_1} (z_i - x_i)}{d_1}.$$

Note that these estimates do not use all the information available in the sample, but they do provide good starting values.

With these initial estimates, we could begin an iterative procedure to obtain the MLEs, by the Newton-Raphson method, for example. Let $(\hat{\theta}_1^{(0)}, \hat{\theta}_2^{(0)}, \hat{\theta}_1'^{(0)}, \hat{\theta}_2'^{(0)})$ be an initial estimate; since $\ln L$ in Eq. (3.5) is a continuous twice differentiable function, the Newton-Raphson method updates this estimate to $(\hat{\theta}_1^{(1)}, \hat{\theta}_2^{(1)}, \hat{\theta}_1'^{(1)}, \hat{\theta}_2'^{(1)})$, then this second one is updated to $(\hat{\theta}_1^{(2)}, \hat{\theta}_2^{(2)}, \hat{\theta}_1'^{(2)}, \hat{\theta}_2'^{(2)})$, and so on, through the iterative formula

$$\hat{\theta}^{(k+1)} = \hat{\theta}^{(k)} + J^{-1}(\hat{\theta}^{(k)}) U(\hat{\theta}^{(k)}), \qquad k = 0, 1, 2, \ldots,$$

where

$$\hat{\theta}^{(j)} = \begin{pmatrix} \hat{\theta}_1^{(j)} & \hat{\theta}_2^{(j)} & \hat{\theta}_1^{\prime(j)} & \hat{\theta}_2^{\prime(j)} \end{pmatrix}^T;$$

U is the score vector and is given by

$$U = \begin{pmatrix} \frac{\partial \ln L}{\partial \theta_1} & \frac{\partial \ln L}{\partial \theta_2} & \frac{\partial \ln L}{\partial \theta_1'} & \frac{\partial \ln L}{\partial \theta_2'} \end{pmatrix}^T,$$

and J is the observed information matrix given by

$$J(\theta_1, \theta_2, \theta_1', \theta_2') = (J_{ij}(\theta_1, \theta_2, \theta_1', \theta_2')), \qquad i, j = 1, 2,$$

where

$$J_{ij}(\theta_1, \theta_2, \theta_1', \theta_2') = -\left(\frac{\partial^2 \ln(\mathrm{L}(\theta_1, \theta_2, \theta_1', \theta_2'))}{\partial \theta_i^{(\prime)} \partial \theta_j^{(\prime)}} \right).$$

This iterative algorithm can be terminated by examining the convergence for each parameter separately. The convergence criterion we applied is

$$\max \left| \hat{\theta}_i^{(k+1)} - \hat{\theta}_i^{(k)} \right| < \varepsilon, \qquad \max \left| \hat{\theta}_i^{\prime(k+1)} - \hat{\theta}_i^{\prime(k)} \right| < \varepsilon, \qquad i = 1, 2,$$

with ε chosen to be 10^{-6}.

We know that a continuous twice differentiable function of one variable is convex (concave) on an interval if and only if its second derivative is non-negative (non-positive) there. If its second derivative is positive (negative) then it is strictly convex (concave). A strictly convex (concave) function will have at most one global minimum (maximum). More generally, a continuous twice differentiable function of several variables is convex (concave) on a convex (concave) set if and only if its Hessian matrix (the square matrix of second-order partial derivatives of a function) is positive (negative) semidefinite on the interior of the convex (concave) set. If its Hessian matrix (H) is positive (negative) definite then the function is strictly convex (concave) and will have global minimums (maximums) for variables.

For this iterative method, since $J = -H$, the convergence of the iterative algorithm to the MLE is dependent on the positive definiteness of the observed information matrix. If the observed information matrix is positive definite, then it is invertible. The Newton iterative formula is valid and the iterative algorithm converges to the MLE. It is clear that in some of the situations considered here in this thesis, the Fisher information matrix turns out to be diagonal (such as on Page 26, 90 and 148) in which case positive definiteness is immediately evident. In other cases, there are several ways to ensure that the observed information matrix is positive definite; one, for example, is by checking that all its eigenvalues are positive. This can be done in our cases as we are only dealing with information matrices of dimension 4, and so given the observed data, the eigen values of the information matrix can be all computed and checked for positivity.

3.2.2 Relative Risks

Relative risk is of interest in survival analysis. In this subsection, the relative risk is obtained and is presented in Theorem 3.2.1.

Lemma 3.2.1. *We have*

$$\frac{1}{\theta_1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)^{-1}\left[1-\frac{1}{\theta_2'}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)^{-1}\right]$$
$$+\frac{1}{\theta_2}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)^{-1}\left[1-\frac{1}{\theta_1'}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)^{-1}\right]=1. \tag{3.6}$$

Proof: The proof is straightforward as the identity is easily checked.

Lemma 3.2.2. *We have*

$$Pr\left(\max(X,Y)\geq a\right)=\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)^{-1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)^{-1}\times\tilde{P}_a, \tag{3.7}$$

where

$$\begin{aligned}\tilde{P}_a=&\frac{1}{\theta_1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)\exp\left(-\frac{1}{\theta_2'}a\right)+\frac{1}{\theta_2}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)\exp\left(-\frac{1}{\theta_1'}a\right)\\&-\left(\frac{1}{\theta_1\theta_2'}-\frac{1}{\theta_1'\theta_2'}+\frac{1}{\theta_2\theta_1'}\right)\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)a\right\}.\end{aligned}\tag{3.8}$$

Proof: We can express

$$\begin{aligned}&Pr\left(\max(X,Y)\geq a\right)\\&\quad=1-Pr\left(\max(X,Y)\leq a\right)\\&\quad=1-\int_0^a\int_0^y\frac{1}{\theta_1\theta_2'}\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)x-\frac{1}{\theta_2'}y\right\}dxdy\\&\qquad-\int_0^a\int_0^x\frac{1}{\theta_2\theta_1'}\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)y-\frac{1}{\theta_1'}x\right\}dydx.\end{aligned}\tag{3.9}$$

Then, the result follows by carrying out the required integration and then by using the identity in Lemma 3.2.1.

Lemma 3.2.3. *The PDF of $Z_{d:n}$ is*

$$\begin{aligned}f_{Z_{d:n}}(a)=&\sum_{i=0}^{d-1}\sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i}C_{i,j_1,j_2,j_3}\times E\\&\times\left\{I_1\exp\left(-\frac{1}{\theta_2'}a\right)+I_2\exp\left(-\frac{1}{\theta_1'}a\right)-I_3\exp\left[-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)a\right]\right\},\end{aligned}\tag{3.10}$$

where

$$C_{i,j_1,j_2,j_3}=(-1)^{i+j_3}d\binom{n}{d}\binom{d-1}{i}\binom{n-d+i}{j_1,j_2,j_3},$$

$$E=\exp\left\{-\left[\frac{j_1}{\theta_2'}+\frac{j_2}{\theta_1'}+j_3\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\right]a\right\},$$

$$I_1 = \left(\frac{1}{\theta_1}\right)^{j_1+1} \left(\frac{1}{\theta_2}\right)^{j_2} \left(\frac{1}{\theta_2'}\right) \left(\frac{1}{\theta_1\theta_2'} - \frac{1}{\theta_1'\theta_2'} + \frac{1}{\theta_2\theta_1'}\right)^{j_3}$$
$$\times \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-(j_1+j_3+1)} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-(j_2+j_3)},$$
$$I_2 = \left(\frac{1}{\theta_1}\right)^{j_1} \left(\frac{1}{\theta_2}\right)^{j_2+1} \left(\frac{1}{\theta_1'}\right) \left(\frac{1}{\theta_1\theta_2'} - \frac{1}{\theta_1'\theta_2'} + \frac{1}{\theta_2\theta_1'}\right)^{j_3}$$
$$\times \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-(j_1+j_3)} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-(j_2+j_3+1)},$$
$$I_3 = \left(\frac{1}{\theta_1}\right)^{j_1} \left(\frac{1}{\theta_2}\right)^{j_2} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) \left(\frac{1}{\theta_1\theta_2'} - \frac{1}{\theta_1'\theta_2'} + \frac{1}{\theta_2\theta_1'}\right)^{j_3+1}$$
$$\times \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-(j_1+j_3+1)} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-(j_2+j_3+1)}.$$

Proof: We can express [see Arnold, Balakrishnan and Nagaraja (1992)]

$$\begin{aligned} f_{Z_{d:n}}(a) &= \frac{n!}{(d-1)!(n-d)!} \{F_Z(a)\}^{d-1} \{1 - F_Z(a)\}^{n-d} f_Z(a) \\ &= \frac{n!}{(d-1)!(n-d)!} \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} \{1 - F_Z(a)\}^{n-d+i} f_Z(a), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} & 1 - F_Z(a) = 1 - Pr(Z \leq a) = 1 - Pr(max(X,Y) \leq a) = Pr(max(X,Y) \geq a) \\ &= \left\{ \frac{1}{\theta_1} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-1} \mathrm{e}^{-\frac{1}{\theta_2'}a} \right\} + \left\{ \frac{1}{\theta_2} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-1} \mathrm{e}^{-\frac{1}{\theta_1'}a} \right\} \\ &\quad - \left\{ \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-1} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-1} \left(\frac{1}{\theta_1\theta_2'} - \frac{1}{\theta_1'\theta_2'} + \frac{1}{\theta_2\theta_1'}\right) \mathrm{e}^{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)a} \right\} \\ &= A_1 + A_2 - A_3, \end{aligned} \tag{3.12}$$

and

$$f_Z(a) = -\frac{\partial(1 - F_Z(a))}{\partial a} = \frac{1}{\theta_2'}A_1 + \frac{1}{\theta_1'}A_2 - \left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) A_3. \tag{3.13}$$

Then, we obtain

$$
\begin{aligned}
f_{Z_{d:n}}(a) &= \frac{n!}{(d-1)!(n-d)!} \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} \{A_1 + A_2 - A_3\}^{n-d+i} f_Z(a) \\
&= \frac{n!}{(d-1)!(n-d)!} \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} \sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} \binom{n-d+i}{j_1, j_2, j_3} \\
&\quad \times A_1^{j_1} A_2^{j_2} (-1)^{j_3} A_3^{j_3} \left\{ \frac{1}{\theta_2'} A_1 + \frac{1}{\theta_1'} A_2 - \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) A_3 \right\}.
\end{aligned} \tag{3.14}
$$

from which the result follows by expanding Eq. (3.14).

Lemma 3.2.4. *We have*

$$
\begin{aligned}
P_1 &= Pr\left(X < Y < z_{d:n}\right) \\
&= \frac{1}{\theta_1} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'} \right)^{-1} \left\{ \left[1 - \frac{1}{\theta_2'} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right)^{-1} \right] \right. \\
&\quad - \sum_{i=0}^{d-1} \sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} C_{i,j_1,j_2,j_3} \left(M_1 + M_2 - M_3 \right) \\
&\quad \left. + \frac{1}{\theta_2'} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right)^{-1} \sum_{i=0}^{d-1} \sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} C_{i,j_1,j_2,j_3} \left(M_1' + M_2' - M_3' \right) \right\},
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
M_1 &= I_1 \left[\frac{j_1+2}{\theta_2'} + \frac{j_2}{\theta_1'} + j_3 \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \right]^{-1}, \\
M_2 &= I_2 \left[\frac{j_1+1}{\theta_2'} + \frac{j_2+1}{\theta_1'} + j_3 \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \right]^{-1}, \\
M_3 &= I_3 \left[\frac{j_1+1}{\theta_2'} + \frac{j_2}{\theta_1'} + (j_3+1) \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \right]^{-1}, \\
M_1' &= I_1 \left[\frac{j_1+1}{\theta_2'} + \frac{j_2}{\theta_1'} + (j_3+1) \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \right]^{-1},
\end{aligned}
$$

$$M_2' = I_2 \left[\frac{j_1}{\theta_2'} + \frac{j_2+1}{\theta_1'} + (j_3+1)\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) \right]^{-1},$$

$$M_3' = I_3 \left[\frac{j_1}{\theta_2'} + \frac{j_2}{\theta_1'} + (j_3+2)\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) \right]^{-1}.$$

Proof: We can express

$$\begin{aligned} P_1 = Pr\left(X < Y < z_{d:n}\right) &= \int_0^\infty Pr\left(X < Y < a\right) f_{Z_{d:n}}(a)da \\ &= \int_0^\infty f_{Z_{d:n}}(a) \int_0^a \int_0^y \frac{1}{\theta_1\theta_2'} \exp\left\{ -\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) x - \frac{1}{\theta_2'} y \right\} dx dy da. \end{aligned} \tag{3.16}$$

Then, the result follows by carrying out the required integration.

Lemma 3.2.5. *We have*

$$\begin{aligned} P_2 &= Pr\left(\max(X,Y) \geq z_{d:n}\right) \\ &= K_1 \sum_{i=0}^{d-1} \sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} C_{i,j_1,j_2,j_3}\left(M_1 + M_2 - M_3\right) \\ &\quad + K_2 \sum_{i=0}^{d-1} \sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} C_{i,j_1,j_2,j_3}\left(M_1'' + M_2'' - M_3''\right) \\ &\quad - K_3 \sum_{i=0}^{d-1} \sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} C_{i,j_1,j_2,j_3}\left(M_1' + M_2' - M_3'\right), \end{aligned} \tag{3.17}$$

where

$$K_1 = \frac{1}{\theta_1}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-1},$$

$$K_2 = \frac{1}{\theta_2}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-1},$$

$$K_3 = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-1}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-1}\left(\frac{1}{\theta_1\theta_2'} - \frac{1}{\theta_1'\theta_2'} + \frac{1}{\theta_2\theta_1'}\right),$$

$$M_1'' = I_1 \left[\frac{j_1+1}{\theta_2'} + \frac{j_2+1}{\theta_1'} + j_3\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) \right]^{-1},$$

$$M_2'' = I_2\left[\frac{j_1}{\theta_2'} + \frac{j_2+2}{\theta_1'} + j_3\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\right]^{-1},$$

$$M_3'' = I_3\left[\frac{j_1}{\theta_2'} + \frac{j_2+1}{\theta_1'} + j_3\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\right]^{-1},$$

and M_i, M_i', $(i = 1, 2, 3)$ are defined in Lemma 3.2.4.

Proof: The proof is similar to the one in Lemma 3.2.4.

Theorem 3.2.1. *The relative risk that Component 1 fails first within a two-component system on the condition that the system fails at the end of a Type-II censored experiment is*

$$\pi_2 = Pr(X < Y|\max(X,Y) \leq z_{d:n}) = \frac{P_1}{1-P_2}.$$

Proof: The result follows immediately from Lemmas 3.2.4 and 3.2.5.

Lemma 3.2.6. *In a Type-II censored experiment, among the d $(d \leq n)$ systems with complete destruction, the number of failures due to Component 1 failing first, viz., d_1, is a non-negative random variable with binomial probability mass function*

$$Pr(d_1 = j) = \binom{d}{j}\left(\frac{P_1}{1-P_2}\right)^j\left(1-\frac{P_1}{1-P_2}\right)^{d-j}, \quad j = 0, 1, \ldots, d.$$

Proof: The result follows immediately from Theorem 3.2.1.

3.3 Confidence Intervals

In this section, we present two different methods of constructing confidence intervals (CIs) for the unknown parameters θ_1, θ_2, θ_1' and θ_2'. First, we use the asymptotic distributions of the MLEs to obtain approximate CIs for the parameters in case of large sample sizes. Then, we use the parametric bootstrap method to construct CIs for the parameters.

3.3.1 Approximate Confidence Intervals

In the last section, we noted that closed-form expressions for the MLEs do not exist. However, we can use the asymptotic normality of the MLEs to construct approximate confidence intervals for the parameters.

The computation of the approximate confidence intervals is based on the observed Fisher information matrix, which is obtained by taking negative of the second derivatives of the log-likelihood function in (3.5) and then evaluating them at the MLEs. Specifically, we have

$$I_{\text{obs}} = - \left. \begin{pmatrix} \frac{\partial^2 \ln L}{\partial \theta_1^2} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_1'} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2'} \\ \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_2^2} & \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_1'} & \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_2'} \\ \frac{\partial^2 \ln L}{\partial \theta_1' \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_1' \partial \theta_2} & \frac{\partial^2 \ln L}{\partial \theta_1'^2} & \frac{\partial^2 \ln L}{\partial \theta_1' \partial \theta_2'} \\ \frac{\partial^2 \ln L}{\partial \theta_2' \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_2' \partial \theta_2} & \frac{\partial^2 \ln L}{\partial \theta_2' \partial \theta_1'} & \frac{\partial^2 \ln L}{\partial \theta_2'^2} \end{pmatrix} \right|_{\theta_1=\hat{\theta}_1, \theta_2=\hat{\theta}_2, \theta_1'=\hat{\theta}_1', \theta_2'=\hat{\theta}_2'}, \tag{3.18}$$

and the inverse of this observed Fisher information matrix in (3.18) gives an estimate of the variance-covariance matrix of the MLEs, which in turn can be used to construct approximate confidence intervals for the parameters. We shall make use of the asymptotic normality of the MLEs to obtain these confidence intervals.

Thus, if

$$V(\theta_1, \theta_2, \theta_1', \theta_2') = I_{\text{obs}}^{-1} = (v_{ij}(\theta_1, \theta_2, \theta_1', \theta_2')), \quad i, j = 1, 2, 3, 4,$$

is the variance-covariance matrix, the $100(1-\alpha)\%$ confidence intervals for θ_1, θ_2, θ_1', θ_2' are given by

$$\begin{aligned} &\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{11}}, \\ &\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{22}}, \\ &\hat{\theta}_1' \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{33}}, \\ &\hat{\theta}_2' \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{44}}, \end{aligned} \tag{3.19}$$

where z_q is the q-th upper percentile of the standard normal distribution. This method may work satisfactorily when n is large, but may not be satisfactory for small sample sizes.

3.3.2 Bootstrap Confidence Intervals

The bootstrap methods of percentile interval and the biased-corrected and accelerated (BC_α) interval are similar to those described in Section 2.4.3, but with a Type-II censored two-component system failure sample generated instead. The acceleration $\hat{\alpha}_i^{(')}$ in the BC_α interval should be changed to

$$\hat{\alpha}_i^{(')} = \frac{\sum_{j=1}^{d} \left(\hat{\theta}_{i(\cdot)}^{(')} - \hat{\theta}_{i(j)}^{(')} \right)^3}{6 \left[\sum_{j=1}^{d} \left(\hat{\theta}_{i(\cdot)}^{(')} - \hat{\theta}_{i(j)}^{(')} \right)^2 \right]^{\frac{3}{2}}}, \quad i = 1, 2,$$

where $\hat{\theta}_{i(j)}^{(')}$ is the MLE of $\theta_i^{(')}$ based on the original sample with the j-th observation deleted, $j = 1, 2, \ldots, d$, and $\hat{\theta}_{i(\cdot)}^{(')} = \frac{\sum_{j=1}^{d} \hat{\theta}_{i(j)}^{(')}}{d}$.

3.4 Simulation Study

In this section, a Monte Carlo simulation study based on 999 replications was carried out to examine the bias, variance and relative risks (Section 3.4.1), to evaluate the performance of the three confidence intervals in terms of coverage probabilities for different sample sizes (Section 3.4.2), and to present numerical examples to illustrate all the inferential methods discussed here (Section 3.4.3).

3.4.1 Bias, Variance and MSE of the MLEs

It is desirable to examine the bias and variance of the MLEs as they are not explicit estimators. For this purpose, we carried out a simulation study to evaluate the bias, mean squared error (MSE), mean and variances of the MLEs, and also the average of the asymptotic variance of the estimators computed from the observed information matrix. These results for different n and d are presented in Tables 3.1 - 3.3.

From the tables, we observe that, as n increases, the bias of MLEs decrease, as one would expect, with the bias tending to zero as n becomes large. Similarly, for the same sample size n, as d decreases, the bias increases. The change in d has more effect on the bias of the MLEs. The same behavior is also observed in MSE of the MLEs. This is so because when d is small, there will be fewer failures observed and so inference for θ_1, θ_2, θ_1' and θ_2' is not quite precise.

The means and variances of the estimates of the parameters over 999 were computed as well. We observe that, for large sample sizes, the means of the MLEs of the parameters θ_1, θ_2, θ_1' and θ_2' are quite close to the true values, viz., 20, 25, 9, 14, respectively. However, this is not true for smaller sample sizes. The variances of the MLEs can also be compared with the average approximate variance computed from the observed information. Once again, the variance and the average approximate variance are closer for large values of n and d, but not close for smaller sample sizes.

The theoretical values of π_1 and π_2, presented in Table 3.4, were computed from the formulas in Theorems 2.3.1 and 3.2.1, respectively. Both π_1 and π_2 are the probabilities that Component 1 fails first within a system. But, π_2 is conditional on the complete destruction of the system. $\pi_1 = 0.5556$ implies that the first failure of a system is more likely due to Component 1. However, $\pi_2 < \pi_1$ in all the cases. It reflects that the probability that Component 1 fails first within a system is weakened on the condition that the system has a complete destruction in a Type-II censoring test.

From Table 3.4, we observe that π_2 is more affected by the change of n and d. As n or d increases, the relative risk increases. This is because, when n or d is small, fewer failures occur during the experiment time. As n or d increases, the number of failures increases thus resulting in larger relative risks.

In order to check whether π_1 and π_2 can be estimated by $\frac{n_1}{n}$ and $\frac{d_1}{d}$, respectively, the results of the average of 999 replications are presented in Table 3.4 as well. We observe that the table values get closer to the corresponding theoretical values when n or d take different values. This indicates that $\frac{n_1}{n}$ and $\frac{d_1}{d}$ are good estimators of π_1 and π_2, respectively.

3.4.2 Coverage Probabilities and the Performance of the Confidence Intervals

To compare the performance of different confidence intervals described in Section 3.3, we conducted a Monte Carlo simulation study. We once again chose the values of the parameters to be $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$. We then determined the true coverage probabilities of the 90%, 95% and 99% confidence intervals for the parameters for different sample sizes by all three methods described earlier in Section 3.3. The results for $n = 40, 20$ are presented in Tables 3.5 and 3.6, and they are based on an average over 999 bootstrap replications.

From the tables, we observe that, among the three methods, the parametric BC_α bootstrap method of constructing confidence intervals has its coverage probabilities to be closer to the nominal level and is therefore recommended for large sample sizes.

As expected, the coverage probabilities of the approximate method based on asymptotic normality of the MLEs is most often smaller than the nominal level. Even for $n = 40$ and $d = 30$, the approximate method does not provide close results. This indicates that the confidence intervals obtained by this method will often be unduly narrower. We do observe that, for all the nominal levels considered, the coverage probabilities of the approximate method are lower for small sample size n or d in almost all cases. This is because, when n or d is small, there are fewer failures observed and so inference for the parameters is not precise. As n increases, the number of failures increases thus resulting in a better large-sample approximation for the distribution of MLEs. This means that we need a much larger sample size to use the asymptotic normality of the MLEs. We also observe that when n is small, even the parametric

BC_α bootstrap method does not have satisfactory coverage probabilities, but is seen to be better than the approximate method as well as percentile bootstrap method.

3.4.3 Numerical Examples

In this subsection, we consider two data sets with $n = 35$, $d = 15$, $d = 25$, and $n = 15$, $d = 7$, $d = 10$. The parameters were chosen to be $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$. The data are as follows:

Data Set 1: $n = 35$

$d = 15$				
(1.45,2; 4.73,1)	(1.46,1; 6.55,2)	(0.48,2; 7.87,1)	(0.02,1; 8.31,2)	(8.56,2; 9.18,1)
(2.88,1; 12.21,2)	(10.48,2; 12.58,1)	(8.66,2; 12.78,1)	(2.39,1; 12.99,2)	(3.39,1; 13.24,2)
(3.69,2; 14.06,1)	(9.65,1; 14.48,2)	(10.40,1; 14.85,2)	(13.31,2; 15.68,1)	(13.48,1; 16.14,2)
(*,*)				

$d = 25$				
(1.45,2; 4.73,1)	(1.46,1; 6.55,2)	(0.48,2; 7.87,1)	(0.02,1; 8.31,2)	(8.56,2; 9.18,1)
(2.88,1; 12.21,2)	(10.48,2; 12.58,1)	(8.66,2; 12.78,1)	(2.39,1; 12.99,2)	(3.39,1; 13.24,2)
(3.69,2; 14.06,1)	(9.65,1; 14.48,2)	(10.40,1; 14.85,2)	(13.31,2; 15.68,1)	(13.48,1; 16.14,2)
(3.59,1; 16.43,2)	(5.59,2; 17.01,1)	(3.44,1; 19.46,2)	(13.14,2; 20.61,1)	(7.97,1; 20.83,2)
(11.51,1; 23.02,2)	(21.45,2; 23.84,1)	(7.99,1; 24.83,2)	(6.49,2; 27.92,1)	(19.92,2; 28.20,1)
(*,*)				

Data Set 2: $n = 15$

$d = 7$				
(2.55,2; 3.87,1)	(0.94,1; 10.06,2)	(4.50,2; 11.13,1)	(8.71,2; 13.16,1)	(14.87,1; 17.89,2)
(13.64,1; 20.56,2)	(19.41,2; 20.89,1)	(*,*)		
$d = 10$				
(2.55,2; 3.87,1)	(0.94,1; 10.06,2)	(4.50,2; 11.13,1)	(8.71,2; 13.16,1)	(14.87,1; 17.89,2)
(13.64,1; 20.56,2)	(19.41,2; 20.89,1)	(7.42,1; 22.47,2)	(6.25,2; 28.58,1)	(17.18,1; 31.80,2)
(*,*)				

In the example when $n = 35$, $d = 15$, we have $d_1 = 8$ and $d_2 = 7$. Using the formulas presented in Section 3.2, the MLEs of θ_1, θ_2, θ'_1 and θ'_2 are $\hat{\theta}_1 = 19.9086$, $\hat{\theta}_2 = 33.7003$, $\hat{\theta}'_1 = 6.8415$ and $\hat{\theta}'_2 = 15.6537$.

In the example when $n = 35$, $d = 25$, we have $d_1 = 13$ and $d_2 = 12$. The MLEs are $\hat{\theta}_1 = 19.2071$, $\hat{\theta}_2 = 25.5931$, $\hat{\theta}'_1 = 8.8554$ and $\hat{\theta}'_2 = 16.6016$.

In the example when $n = 15$, $d = 7$, we have $d_1 = 3$ and $d_2 = 4$. The MLEs are $\hat{\theta}_1 = 45.4329$, $\hat{\theta}_2 = 46.2074$, $\hat{\theta}'_1 = 4.2178$ and $\hat{\theta}'_2 = 11.3774$.

In the example when $n = 15$, $d = 10$, we have $d_1 = 5$ and $d_2 = 5$. The MLEs are $\hat{\theta}_1 = 28.7220$, $\hat{\theta}_2 = 34.6628$, $\hat{\theta}'_1 = 9.6713$ and $\hat{\theta}'_2 = 16.8839$.

To assess the performance of these estimates, we constructed 90%, 95% and 99% confidence intervals using the methods outlined in Section 3.3. These results are presented in Tables 3.7-3.10.

From the results corresponding to the two examples, it is seen that, for the same sample size, as d increases, we have more accurate estimates for the parameters. We also note that the approximate method always provide narrower confidence intervals in most cases. This is because the coverage probability for the approximate method is significantly lower than the nominal level.

Table 3.1: Bias, MSE, Mean and Variance based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 40$

d	Parameters	Bias	MSE	Mean	Variance	Approximate Variance
15	θ_1	9.26	537.41	29.26	452.09	463.99
	θ_2	5.14	341.30	30.14	315.23	323.73
	θ_1'	0.57	30.53	9.57	30.42	35.28
	θ_2'	-3.31	49.58	10.69	38.62	44.95
20	θ_1	4.28	137.64	24.28	119.42	129.40
	θ_2	3.21	175.00	28.21	164.84	171.14
	θ_1'	0.44	21.14	9.44	21.01	27.38
	θ_2'	-1.58	31.88	12.42	34.08	40.40
25	θ_1	1.94	80.26	21.94	76.58	81.13
	θ_2	1.14	101.80	26.14	100.59	106.82
	θ_1'	0.38	20.02	9.38	19.84	24.33
	θ_2'	-0.14	34.07	13.86	29.42	38.31
30	θ_1	1.13	49.73	21.13	48.50	50.60
	θ_2	0.49	72.24	25.49	72.07	75.13
	θ_1'	0.38	14.77	9.38	14.45	17.18
	θ_2'	-0.15	24.29	13.85	24.30	28.59

Table 3.2: Bias, MSE, Mean and Variance based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta'_1 = 9$, $\theta'_2 = 14$ and $n = 20$

d	Parameters	Bias	MSE	Mean	Variance	Approximate Variance
10	θ_1	8.45	465.59	28.45	394.62	552.64
	θ_2	6.55	498.91	31.55	456.43	473.19
	θ'_1	0.84	39.20	9.84	38.94	51.76
	θ'_2	-2.42	57.08	11.58	52.86	92.04
12	θ_1	4.72	297.06	24.72	275.05	360.78
	θ_2	3.01	373.38	28.01	364.69	391.43
	θ'_1	0.72	35.82	9.72	35.33	46.52
	θ'_2	-0.62	46.20	13.38	46.25	81.14
14	θ_1	2.56	137.97	22.56	131.55	186.34
	θ_2	2.53	206.46	27.53	200.24	249.19
	θ'_1	0.71	28.81	9.71	28.33	42.94
	θ'_2	-0.69	45.37	13.31	44.93	67.16
16	θ_1	1.67	87.69	21.67	85.00	103.93
	θ_2	1.62	144.32	26.62	141.83	188.25
	θ'_1	0.49	22.49	9.49	22.27	34.20
	θ'_2	0.19	40.35	14.19	40.36	49.77

Table 3.3: Bias, MSE, Mean and Variance based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 10$

d	Parameters	Bias	MSE	Mean	Variance	Approximate Variance
5	θ_1	10.11	744.13	30.11	642.53	1151.42
	θ_2	7.86	794.78	32.86	733.76	1435.75
	θ_1'	1.07	59.28	10.07	57.97	295.65
	θ_2'	-2.47	97.23	11.53	91.10	173.61
6	θ_1	7.28	536.14	27.28	483.67	874.53
	θ_2	6.70	705.97	31.70	661.77	1207.96
	θ_1'	0.79	57.82	9.79	57.30	105.49
	θ_2'	-0.82	76.68	13.18	76.21	147.62
7	θ_1	6.09	436.33	26.09	399.67	764.16
	θ_2	6.01	663.56	31.01	628.13	1097.30
	θ_1'	0.76	51.14	9.76	50.05	103.59
	θ_2'	-0.72	75.42	13.28	74.98	144.94
8	θ_1	4.09	347.10	24.09	330.65	535.12
	θ_2	4.70	508.29	29.70	486.67	887.69
	θ_1'	0.61	44.11	9.61	43.54	73.51
	θ_2'	-0.37	67.67	13.63	67.60	107.01

Table 3.4: Relative risks based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$

n	d	n_1	n_2	$\frac{n_1}{n}$	π_1	d_1	d_2	$\frac{d_1}{d}$	π_2
40	30	22.18	17.82	0.55	0.56	15.57	14.43	0.52	0.52
	25	22.41	17.59	0.56	0.56	12.93	12.07	0.52	0.51
	20	22.19	17.81	0.55	0.56	9.97	10.03	0.50	0.50
	15	21.98	18.02	0.55	0.56	7.24	7.76	0.48	0.49
20	16	11.09	8.91	0.55	0.56	8.45	7.55	0.53	0.53
	14	11.20	8.80	0.56	0.56	7.32	6.68	0.52	0.52
	12	11.10	8.90	0.56	0.56	6.09	5.91	0.51	0.51
	10	11.01	8.99	0.55	0.56	4.98	5.02	0.50	0.50
10	8	5.59	4.41	0.56	0.56	4.28	3.72	0.54	0.52
	7	5.58	4.42	0.56	0.56	3.69	3.31	0.53	0.51
	6	5.53	4.47	0.55	0.56	3.03	2.97	0.51	0.51
	5	5.60	4.40	0.56	0.56	2.54	2.46	0.51	0.50

Table 3.5: Estimated coverage probabilities based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 40$

C.I. of θ_1	90% C.I.			95% C.I.			99% C.I.		
d	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
15	83.78	89.69	91.89	89.38	92.59	94.29	92.49	98.40	98.90
20	85.79	88.99	90.59	90.49	93.99	94.89	94.69	99.20	99.30
25	86.59	88.19	89.59	91.29	93.99	94.69	95.40	98.60	98.80
30	87.89	90.29	90.99	92.70	93.89	95.00	96.10	99.00	99.40
C.I. of θ_2	90% C.I.			95% C.I.			99% C.I.		
d	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
15	83.88	90.09	91.59	90.10	94.49	95.40	92.70	98.50	98.50
20	86.20	89.59	91.29	90.69	94.39	95.40	94.88	99.30	99.50
25	86.10	88.29	89.59	91.00	93.79	95.10	95.10	98.20	98.70
30	87.20	88.59	88.69	92.10	94.29	94.69	96.20	98.20	98.50
C.I. of θ_1'	90% C.I.			95% C.I.			99% C.I.		
d	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
15	82.10	86.89	88.29	87.80	91.79	92.89	89.80	95.10	95.60
20	84.80	87.59	89.09	88.20	92.19	93.69	90.59	95.50	96.30
25	85.90	88.99	90.79	90.80	92.79	94.09	93.10	96.50	97.40
30	86.80	89.89	91.69	91.79	93.99	96.20	94.20	98.90	99.30
C.I. of θ_2'	90% C.I.			95% C.I.			99% C.I.		
d	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
15	83.10	89.09	90.49	87.10	91.19	92.30	90.20	96.60	97.30
20	84.30	88.29	90.09	87.70	90.99	92.89	90.70	95.50	96.20
25	85.10	88.29	89.89	90.20	92.59	94.29	92.89	96.60	97.50
30	87.70	88.39	89.89	91.10	92.89	94.19	93.20	97.70	98.20

Table 3.6: Estimated coverage probabilities based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 20$

C.I. of θ_1	90% C.I.			95% C.I.			99% C.I.		
d	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	84.89	90.29	90.99	87.19	91.59	91.79	91.89	96.60	96.70
12	85.10	89.49	90.39	88.69	94.29	95.10	93.30	98.50	98.80
14	85.90	88.59	90.49	90.29	95.10	96.70	93.70	98.30	98.80
16	88.10	89.29	90.49	91.10	95.00	96.10	95.10	97.90	98.90
C.I. of θ_2	90% C.I.			95% C.I.			99% C.I.		
d	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	84.99	88.99	90.19	88.29	93.89	91.79	90.39	96.50	96.80
12	85.20	89.19	90.39	88.10	94.89	96.10	92.59	98.40	98.60
14	85.80	90.49	92.09	89.90	95.10	96.30	92.70	98.00	98.20
16	86.20	88.59	89.79	90.89	94.69	95.80	94.10	98.50	98.70
C.I. of θ_1'	90% C.I.			95% C.I.			99% C.I.		
d	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	76.18	84.28	85.89	80.08	86.19	88.09	85.89	91.49	91.79
12	78.78	87.59	90.59	83.68	88.49	90.99	87.79	93.19	93.99
14	77.40	87.29	89.99	82.79	90.49	93.29	88.30	95.80	96.50
16	80.20	86.89	90.59	86.09	92.89	95.30	90.60	95.90	97.00
C.I. of θ_2'	90% C.I.			95% C.I.			99% C.I.		
d	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	73.29	84.38	85.69	79.38	89.29	90.39	85.10	91.99	92.69
12	76.87	86.09	87.59	82.99	89.09	91.09	86.20	93.39	94.09
14	77.78	85.79	85.99	81.79	90.49	92.99	89.00	94.79	95.20
16	81.70	86.89	88.89	85.20	91.39	92.89	90.89	95.70	96.80

Table 3.7: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' in Example when $n = 35$, $d = 15$

C.I. for θ_1			
Method	90%	95%	99%
Approx C.I.	(2.57, 37.24)	(0*, 40.56)	(0*, 47.05)
Boot-p C.I.	(6.44, 52.84)	(5.77, 62.17)	(4.75, 85.99)
BC_α C.I.	(7.18, 57.76)	(6.28, 69.25)	(4.90, 85.99)

C.I. for θ_2			
Method	90%	95%	99%
Approx C.I.	(7.49, 59.91)	(2.47, 64.93)	(0*, 74.74)
Boot-p C.I.	(7.73, 66.08)	(6.63, 78.14)	(5.35, 95.93)
BC_α C.I.	(12.73, 86.33)	(9.61, 92.23)	(6.46, 103.13)

C.I. for θ_1'			
Method	90%	95%	99%
Approx C.I.	(0*, 13.92)	(0*, 15.28)	(0*, 17.93)
Boot-p C.I.	(2.25, 20.63)	(1.88, 26.06)	(1.27, 39.91)
BC_α C.I.	(2.79, 27.02)	(2.35, 33.64)	(1.71, 45.76)

C.I. for θ_2'			
Method	90%	95%	99%
Approx C.I.	(1.14, 30.17)	(0*, 32.95)	(0*, 38.38)
Boot-p C.I.	(3.55, 30.72)	(3.05, 36.61)	(2.11, 45.66)
BC_α C.I.	(6.72, 42.66)	(5.46, 48.66)	(3.63, 55.62)

0* stands for a non-positive number

Table 3.8: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' in Example when $n = 35$, $d = 25$

C.I. for θ_1			
Method	90%	95%	99%
Approx C.I.	(7.98, 30.44)	(5.82, 32.59)	(1.62, 36.79)
Boot-p C.I.	(11.09, 35.59)	(9.76, 42.08)	(8.29, 49.88)
BC_α C.I.	(11.19, 35.85)	(9.84, 42.56)	(8.30, 49.56)

C.I. for θ_2			
Method	90%	95%	99%
Approx C.I.	(11.74, 39.44)	(9.09, 42.10)	(3.91, 47.28)
Boot-p C.I.	(13.88, 43.37)	(12.12, 48.63)	(10.84, 63.33)
BC_α C.I.	(14.70, 44.86)	(12.60, 49.20)	(10.84, 61.34)

C.I. for θ_1'			
Method	90%	95%	99%
Approx C.I.	(2.92, 14.79)	(1.78, 15.93)	(0*, 18.15)
Boot-p C.I.	(4.70, 18.61)	(4.05, 21.40)	(3.12, 29.32)
BC_α C.I.	(4.93, 20.46)	(4.51, 23.37)	(3.66, 31.71)

C.I. for θ_2'			
Method	90%	95%	99%
Approx C.I.	(6.07, 27.14)	(4.05, 29.15)	(0.10, 33.10)
Boot-p C.I.	(7.26, 27.47)	(6.18, 29.84)	(4.09, 35.24)
BC_α C.I.	(8.32, 29.26)	(6.84, 31.76)	(4.48, 35.24)

0* stands for a non-positive number

Table 3.9: Confidence intervals for θ_1, θ_2, θ'_1 and θ'_2 in Example when $n = 15$, $d = 7$

C.I. for θ_1			
Method	90%	95%	99%
Approx C.I.	(0*, 102.98)	(0*, 114.00)	(0*, 135.55)
Boot-pC.I.	(9.64, 111.85)	(7.72, 127.39)	(6.08, 145.72)
BC_α C.I.	(9.86, 113.21)	(7.42, 124.69)	(3.71, 138.97)
C.I. for θ_2			
Method	90%	95%	99%
Approx C.I.	(0*, 86.30)	(0*, 93.98)	(0*, 108.99)
Boot-p C.I.	(8.78, 104.94)	(6.87, 120.10)	(4.10, 144.94)
BC_α C.I.	(14.91, 134.67)	(11.16, 142.88)	(6.87, 148.41)
C.I. for θ'_1			
Method	90%	95%	99%
Approx C.I.	(0*, 8.65)	(0*, 9.50)	(0*, 11.16)
Boot-p C.I.	(1.09, 13.68)	(0.84, 18.44)	(0.31, 31.14)
BC_α C.I.	(1.55, 18.44)	(1.10, 22.33)	(0.63, 38.85)
C.I. for θ'_2			
Method	90%	95%	99%
Approx C.I.	(0*, 29.68)	(0*, 33.18)	(0*, 40.03)
Boot-p C.I.	(1.45, 34.74)	(0.86, 43.14)	(0.06, 60.83)
BC_α C.I.	(2.71, 47.59)	(1.74, 52.71)	(0.30, 61.81)

0* stands for a non-positive number

Table 3.10: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' in Example when $n = 15$, $d = 10$

C.I. for θ_1			
Method	90%	95%	99%
Approx C.I.	(0.39, 57.05)	(0*, 62.48)	(0*, 73.09)
Boot-p C.I.	(11.16, 72.21)	(9.60, 89.40)	(7.33, 136.25)
BC_α C.I.	(11.28, 73.02)	(9.60, 89.37)	(7.02, 126.89)
C.I. for θ_2			
Method	90%	95%	99%
Approx C.I.	(5.41, 63.91)	(0*, 69.52)	(0*, 80.47)
Boot-p C.I.	(12.39, 86.46)	(10.80, 108.50)	(8.17, 134.05)
BC_α C.I.	(13.94, 95.28)	(11.19, 114.15)	(8.17, 134.00)
C.I. for θ_1'			
Method	90%	95%	99%
Approx C.I.	(0*, 19.95)	(0*, 21.92)	(0*, 25.77)
Boot-p C.I.	(2.96, 25.79)	(2.17, 31.87)	(1.14, 44.84)
BC_α C.I.	(3.62, 31.87)	(2.97, 39.53)	(1.97, 50.52)
C.I. for θ_2'			
Method	90%	95%	99%
Approx C.I.	(0*, 36.24)	(0*, 39.94)	(0*, 47.19)
Boot-p C.I.	(4.41, 36.68)	(3.37, 40.92)	(1.72, 56.21)
BC_α C.I.	(5.94, 41.03)	(4.51, 47.31)	(2.29, 57.69)

0* stands for a non-positive number

Chapter 4

Exact Analysis under Type-II Censoring with Partial Information on Component Failures

4.1 Introduction

In this Chapter, we consider such a two-component system failure model in the case of Type-II censored data. The information of the censored systems which have only one component failed at the end of the experiment is incorporated as well. We then obtain the MLEs of the parameters assuming the lifetimes to be exponentially distributed. The exact distributions of the MLEs of the parameters, conditioned on the data, are then derived by using the conditional moment generating function approach. Construction of confidence intervals for the model parameters are discussed by using the exact conditional distributions, asymptotic distributions, and two parametric bootstrap methods. The performance of the two parametric bootstrap confidence intervals in terms of coverage probabilities are assessed through a Monte Carlo simulation study. Finally, examples are presented to illustrate all the methods of inference discussed here.

4.2 Model Description and MLEs

Consider the following simple system failure model: n identical systems are placed on a life-test and each system has two components. The experiment continues until a total of d ($d \leq n$) systems fail. We assume that X_i and Y_i ($i = 1, \ldots, n$) are random variables representing the lifetimes of Components 1 and 2, respectively, in the i-th system. Let $Z_i = \max(X_i, Y_i)$ ($i = 1, \ldots, n$). Thus, the i-th system fails at time Z_i, and $Z_{1:n} < \cdots < Z_{d:n}$ are the corresponding ordered failure times. At the end of the experiment, we observe d systems with complete destruction, d' systems with only one failed component and $n - d - d'$ systems with no failed components. Among the d systems, there are d_1 systems in which Component 1 failed first and d_2 systems in which Component 2 failed first, with $d_1 + d_2 = d$. Among the d' systems, there are d'_1 systems of which only Component 1 failed and d'_2 systems in which only Component 2 failed, with $d'_1 + d'_2 = d'$. The data from the two-component system sample under Type-II censoring with partial information is then as follows:

$$(T_1,\ \delta'_1;\ Z_{1:n},\ \delta''_1), \ldots, (T_d,\ \delta'_d;\ Z_{d:n},\ \delta''_d), (T_{d+1},\ \delta'_{d+1}; *), \ldots, (T_{d+d'},\ \delta'_{d+d'}; *)(*, *), \tag{4.1}$$

where $T_1, \ldots, T_d$ denote the first observed failure times in the systems, $Z_{1:n} < \cdots < Z_{d:n}$ denote the final observed failure times of the systems, and δ' denotes the component of the first observed failure within the system and δ'' denotes the component of the second observed failure within the system. We use " $*$ " to denote the censored data.

If we let

$$I_1 = \{i \in (1, 2, \ldots, d) : \text{ Component 1 failed first within a failed system}\},$$

$$I_2 = \{i \in (1, 2, \ldots, d) : \text{ Component 2 failed first within a failed system}\},$$

$$I_1' = \{i \in (1, 2, \ldots, d') : \text{ only Component 1 failed within a system}\},$$

$$I_2' = \{i \in (1, 2, \ldots, d') : \text{ only Component 2 failed within a system}\},$$

the likelihood function of the observed data in (4.1) is given by

$$\begin{aligned}
L(\theta_1, \theta_2, \theta_1', \theta_2') &= \frac{(2n)!}{(2n-2d-d')!} \\
&\times \left(\frac{1}{\theta_1\theta_2'}\right)^{d_1} \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)\sum_{i\in I_1} x_i - \frac{1}{\theta_2'}\sum_{i\in I_1} z_i\right\} \\
&\times \left(\frac{1}{\theta_2\theta_1'}\right)^{d_2} \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)\sum_{i\in I_2} y_i - \frac{1}{\theta_1'}\sum_{i\in I_2} z_i\right\} \\
&\times \left(\frac{1}{\theta_1}\right)^{d_1'} \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)\sum_{i\in I_1'} x_i - \frac{1}{\theta_2'} d_1' z_{d:n}\right\} \\
&\times \left(\frac{1}{\theta_2}\right)^{d_2'} \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)\sum_{i\in I_2'} y_i - \frac{1}{\theta_1'} d_2' z_{d:n}\right\} \\
&\times \exp\left\{-(n-d-d')\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right) z_{d:n}\right\},
\end{aligned} \tag{4.2}$$

where,

$$0 < x_i < z_i < z_{d:n}, \quad \text{for } i \in I_1; \qquad 0 < y_i < z_i < z_{d:n}, \quad \text{for } i \in I_2;$$

$$0 < x_i < z_{d:n}, \quad \text{for } i \in I_1'; \quad 0 < y_i < z_{d:n}, \quad \text{for } i \in I_2'; \quad 0 < z_{1:d} < \cdots < z_{d:n} < \infty.$$

Since we need to integrate Eq. (4.2) term by term to obtain the exact conditional distribution of the MLEs in Section 4.3, details of the support have to be given. The support can be expressed as follows:

(1) for $i \in I_1 = \{i_{11}, i_{12}, \ldots, i_{1d_1}\}$,

$$0 < x_{i_{11}} < z_{i_{11}}, 0 < x_{i_{12}} < z_{i_{12}}, \ldots, 0 < x_{i_{1d_1}} < z_{i_{1d_1}};$$

(2) for $i \in I_2 = \{i_{21}, i_{22}, \ldots, i_{2d_2}\}$,

$$0 < y_{i_{21}} < z_{i_{21}}, 0 < y_{i_{22}} < z_{i_{22}}, \ldots, 0 < y_{i_{2d_2}} < z_{i_{2d_2}};$$

(3) for $i \in I_1' = \{i_{11}', i_{12}', \ldots, i_{1d_1'}'\}$,

$$0 < x_{i_{11}'} < z_{d:n} < y_{i_{11}'}, 0 < x_{i_{12}'} < z_{d:n} < y_{i_{12}'}, \ldots, 0 < x_{i_{1d_1'}'} < z_{d:n} < y_{i_{1d_1'}'};$$

(4) for $i \in I_B' = \{i_{21}', i_{22}', \ldots, i_{2d_2'}'\}$,

$$0 < y_{i_{21}'} < z_{d:n} < x_{i_{21}'}, 0 < y_{i_{22}'} < z_{d:n} < x_{i_{22}'}, \ldots, 0 < y_{i_{2d_2'}'} < z_{d:n} < x_{i_{2d_2'}'};$$

(5) $0 < z_{1:n} < z_{2:n} < \ldots < z_{d:n} < \infty$.

The maximum likelihood estimate $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_1', \hat{\theta}_2')$ of $(\theta_1, \theta_2, \theta_1', \theta_2')$ is the value that globally maximizes the likelihood function in (4.2). After some calculation, the corresponding conditional maximum likelihood estimates of the mean life times θ_1, θ_2, θ_1', θ_2' are obtained to be

$$\hat{\theta}_1 = \frac{\sum_{i \in I_1} x_i + \sum_{i \in I_2} y_i + \sum_{i \in I_1'} x_i + \sum_{i \in I_2'} y_i + (n - d - d')z_{d:n}}{d_1 + d_1'},$$

$$\hat{\theta}_2 = \frac{d_1 + d_1'}{(d - d_1) + (d' - d_1')}\hat{\theta}_1,$$

$$\hat{\theta}_1' = \frac{\sum_{i\in I_2}(z_i - y_i) + (d_2' z_{d:n} - \sum_{i\in I_2'} y_i)}{d - d_1},$$

$$\hat{\theta}_2' = \frac{\sum_{i\in I_1}(z_i - x_i) + (d_1' z_{d:n} - \sum_{i\in I_1'} x_i)}{d_1},$$

conditional on $1 \leq d_1 \leq d-1$, $0 \leq d_1' \leq d'$ and $0 \leq d' \leq n-d$.

4.2.1 Relative Risks

Based on the results of Section 3.2.2, in this subsection, two additional relative risks are derived and are presented in Theorems 4.2.1 and 4.2.2.

Lemma 4.2.1. *We have*

$$P_3 = Pr\left(X < z_{d:n} < Y\right) = \frac{1}{\theta_1}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-1}\left[\sum_{i=0}^{d-1}\sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} C_{i,j_1,j_2,j_3}\left(M_1 + M_2 - M_3\right) - \sum_{i=0}^{d-1}\sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} C_{i,j_1,j_2,j_3}\left(M_1' + M_2' - M_3'\right)\right], \tag{4.3}$$

where M_i and M_i', $i = 1, 2, 3$, are as defined in Lemma 3.2.4.

Proof: The proof is similar to that of Lemma 3.2.4.

Lemma 4.2.2. *We have*

$$P_4 = Pr(\min(X,Y) \geq z_{d:n}) = \sum_{i=0}^{d-1}\sum_{(j_1,j_2,j_3):j_1+j_2+j_3=n-d+i} C_{i,j_1,j_2,j_3}\left\{M_1' + M_2' - M_3'\right\}, \tag{4.4}$$

where M_i', $i = 1, 2, 3$, are as defined in Lemma 3.2.4.

Proof: We have

$$P_4 = Pr(\min(X,Y) \geq z_{d:n}) = \int_0^\infty Pr(\min(X,Y) \geq a) f_{Z_{d:n}}(a) da$$
$$= \int_0^\infty \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) a\right\} f_{Z_{d:n}}(a) da. \tag{4.5}$$

Then, the result follows by carrying out the required integration.

Theorem 4.2.1. *The relative risk that Component 1 fails first within a two-component system, under the condition that the system has only one failed component at the end of a Type-II censored experiment, is*

$$\pi_3 = Pr(X < Y | \min(X,Y) \leq z_{d:n}, \max(X,Y) \geq z_{d:n}) = \frac{P_3}{P_2 - P_4},$$

where P_2 is defined in Lemma 3.2.5.

Proof: The result follows immediately from Lemmas 3.2.5, 4.2.1 and 4.2.2.

Theorem 4.2.2. *The relative risk that only one component fails within a two-component system, under the condition that the system does not fail at the end of a Type-II censored experiment, is*

$$\pi_4 = Pr(\min(X,Y) \leq z_{d:n} | \max(X,Y) \geq z_{d:n}) = \frac{P_2 - P_4}{P_2},$$

where P_2 is defined in Lemma 3.2.5.

Proof: The result follows immediately from Lemmas 3.2.5 and 4.2.2.

Lemma 4.2.3. *In a Type-II censored experiment with partial information on component failures, among the d' $(0 \le d' \le n-d)$ systems with only one failed component at the end of experiment, the number of systems due to Component 1 failing first, viz., d'_1, is a non-negative random variable with binomial probability mass function given by*

$$Pr(d'_1 = j) = \binom{d'}{j}\left(\frac{P_3}{P_2 - P_4}\right)^j \left(1 - \frac{P_3}{P_2 - P_4}\right)^{d'-j}, \quad j = 0, 1, \ldots, d',$$

where P_2 is defined in Lemma 3.2.5.

Proof: The result follows immediately from Theorem 4.2.1.

Lemma 4.2.4. *In a Type-II censored experiment with partial information on component failures, among the $n-d$ systems which do not fail at the end of experiment, the number of systems of which only one component fails, viz., d', is a non-negative random variable with binomial probability mass function given by*

$$Pr(d' = j) = \binom{n-d}{j}\left(1 - \frac{P_4}{P_2}\right)^j \left(\frac{P_4}{P_2}\right)^{n-d-j}, \quad j = 0, 1, \ldots, n-d.$$

Proof: The result follows immediately from Theorem 4.2.2.

4.3 Exact Conditional Distributions of the MLEs

We will now derive the exact marginal (conditional) distribution of the MLEs. The derivation will require the inversion of the conditional moment generating function (CMGF). To obtain the CMGF, we need to find the joint PDF of X_i's, Y_i's and Z_i's first.

4.3.1 The Joint PDF of X_i's, Y_i's and Z_i's

The joint PDF of X_i's, Y_i's and Z_i's is proportional to the likelihood function $L(\theta_1, \theta_2, \theta_1', \theta_2')$ and can be obtained by integrating Eq. (4.2). However, the integration can not be performed unless we know the exact order of the observed failure times. Different order of the observations results in different forms of integration. Therefore, in this subsection, we only discuss the joint PDF of X_i's, Y_i's and Z_i's in the general case. In Section 4.3.4., a Type-II censored sample will be generated to illustrate the method discussed here.

The general process of finding the joint PDF can be done by using the following steps:

(1) Generate a Type-II censored two-component system failure data with partial information on component failures. We observe $2d + d'$ $(\leq n)$ component failure times at the end of the experiment.

(2) Rank the $2d + d'$ observed failure times in descending order: from the largest observation $(Z_{d:n})$ to the smallest one.

(3) Rewrite the likelihood function (4.2) as

$$C \times \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)\sum_{i\in I_1} x_i\right\} \times \exp\left\{-\frac{1}{\theta_2'}\sum_{i\in I_1} z_i\right\}$$
$$\times \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)\sum_{i\in I_2} y_i\right\} \times \exp\left\{-\frac{1}{\theta_1'}\sum_{i\in I_2} z_i\right\}$$
$$\times \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)\sum_{i\in I_1'} x_i\right\} \times \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)\sum_{i\in I_2'} y_i\right\}$$
$$\times \exp\left\{-\left[(n-d-d')\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)+\frac{1}{\theta_2'}d_1'+\frac{1}{\theta_1'}d_2'\right] z_{d:n}\right\}, \tag{4.6}$$

here, C is some constant given in Eq. (4.2).

(4) Take the integration of Eq. (4.6) in the corresponding order as described in Step (2). This integration results in a product consisting of $2d + d'$ multipliers, viz., $C \times M = C \times \prod_{j=1}^{2d+d'} M_i$.

(5) Step (4) implies that the general form of the joint PDF of X_i's, Y_i's and Z_i's can be expressed as $\frac{1}{C \times M} L(\theta_1, \theta_2, \theta'_1, \theta'_2)$.

We find that the critical part of this process is to find the general form of M_j ($j = 1, \ldots, 2d + d'$). Let $A = (n - d - d')(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_2} d'_1 + \frac{1}{\theta'_1} d'_2$; then, the form of M_j can be expressed as follows:

(i) When $j = 1$, $M_1 = \left[A + \frac{1}{\theta'_1}\right]^{-1}$, if the largest observation ($Z_{d:n}$) is the failure time of Component 1 within a system; otherwise, $M_1 = \left[A + \frac{1}{\theta'_2}\right]^{-1}$.

(ii) When $j = 2, \ldots, 2d + d'$, $M_j = \left[\frac{1}{M_{j-1}} + \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta'_2}\right)\right]^{-1}$, if X_i ($i \in I_1$ or $i \in I'_1$) is the j-th largest observation; $M_j = \left[\frac{1}{M_{j-1}} + \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta'_1}\right)\right]^{-1}$, if Y_i ($i \in I_2$ or $i \in I'_2$) is the j-th largest observation; $M_j = \left[\frac{1}{M_{j-1}} + \frac{1}{\theta'_2}\right]^{-1}$, if Z_i ($i \in I_1$) is the j-th largest observation; $M_j = \left[\frac{1}{M_{j-1}} + \frac{1}{\theta'_1}\right]^{-1}$, if Z_i ($i \in I_2$) is the j-th largest observation.

4.3.2 Exact Conditional Distributions of the MLEs

Based on the joint PDF of X_i's, Y_i's and Z_i's, the CMGF of the MLEs are obtained in this subsection and are presented in Theorem 4.3.1. Using the inversion theorem of the CMGF, the conditional PDFs of the MLEs are computed and are presented in Theorem 4.3.2.

Lemma 4.3.1. *We have*

$$Pr(D_1 = k, D' = m, D'_1 = q | 1 \le D_1 \le d-1, 0 \le D' \le n-d, 0 \le D'_1 \le d')$$

$$= \frac{P_{k,m,q}}{\sum_{f=1}^{d-1}\sum_{g=0}^{n-d}\sum_{s=0}^{g} P_{f,g,s}},$$

where

$$P_{k,m,q} = \binom{d}{k}\binom{m}{q}\binom{n-d}{m}\frac{P_1^k P_2^q P_3^{n-d-m}(1-P_1-P_4)^{d-k}(P_4-P_3-P_2)^{m-q}}{P_4^{n-d}(1-P_4)^d},$$

and P_1, P_2, P_3 and P_4 are as given in Lemmas 3.2.4, 3.2.5, 4.2.1 and 4.2.2, respectively.

Proof: We can express

$$\begin{aligned} P_{k,m,q} &= Pr(D_1 = k, D' = m, D'_1 = q) = Pr(D_1 = k)Pr(D'_1 = q|D' = m)Pr(D' = m) \\ &= \binom{d}{k}\left(\frac{P_1}{1-P_4}\right)^k\left(1-\frac{P_1}{1-P_4}\right)^{d-k}\binom{m}{q}\left(\frac{P_2}{P_4-P_3}\right)^q\left(1-\frac{P_2}{P_4-P_3}\right)^{m-q} \\ &\quad \times\binom{n-d}{m}\left(1-\frac{P_3}{P_4}\right)^m\left(\frac{P_3}{P_4}\right)^{n-d-m} \\ &= \binom{d}{k}\binom{m}{q}\binom{n-d}{m}\frac{P_1^k P_2^q P_3^{n-d-m}(1-P_1-P_4)^{d-k}(P_4-P_3-P_2)^{m-q}}{P_4^{n-d}(1-P_4)^d}. \end{aligned} \tag{4.7}$$

Lemma 4.3.2. *The PDF and CDF of the sum of n independent but non-identical exponential random variables with failure rates λ_i (i=1,2,...,n) are*

$$f_Y(y) = \sum_{i=1}^{n}\prod_{j=1,j\ne i}^{n}\frac{\lambda_j}{(\lambda_j-\lambda_i)}\lambda_i e^{-\lambda_i y}, \quad 0 < y < \infty, \quad \lambda_i > 0, \; i = 1,2,\ldots,n,$$

$$F_Y(y) = \sum_{i=1}^{n}\prod_{j=1,j\ne i}^{n}\frac{\lambda_j}{(\lambda_j-\lambda_i)}\left(1-e^{-\lambda_i y}\right), \quad 0 < y < \infty, \quad \lambda_i > 0, \; i = 1,2,\ldots,n,$$

respectively, with

$$\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \frac{\lambda_j}{(\lambda_j - \lambda_i)} = 1.$$

Proof: Suppose that the random variables $X_1, \ldots, X_n$ are independent and are exponentially distributed with failure rates λ_i (> 0), $i = 1, 2, \ldots, n$. Then, the joint density function of $X_1, \ldots, X_n$ is

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \lambda_1 \cdots \lambda_n e^{-\sum_{i=1}^{n} \lambda_i x_i}, \quad 0 < x_1, x_2, \ldots, x_n < \infty.$$

Let

$$Y_1 = \sum_{i=1}^{n} X_i, \;\; Y_2 = X_2, \;\; \ldots, \;\; Y_n = X_n.$$

The Jacobian of this transformation is 1. So, the joint PDF of $Y_1, \ldots, Y_n$ is

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \lambda_1 \cdots \lambda_n e^{-[\lambda_1 y_1 + (\lambda_2 - \lambda_1) y_2 + \cdots + (\lambda_n - \lambda_1) y_n]},$$

$$0 < y_2, \ldots, y_n < \infty, \quad 0 < y_1 - y_2 - \cdots - y_n < \infty.$$

Integrating the joint PDF of $Y_1, \ldots, Y_n$, we get the marginal density of Y_1 as

$$\begin{aligned}
f_{Y_1}(y_1) &= \int \cdots \int f(y_1, \ldots, y_n) dy_n dy_{n-1} \cdots dy_2 \\
&= \int_0^{y_1} \int_0^{y_1 - y_2} \cdots \int_0^{y_1 - y_2 - \cdots - y_{n-1}} \lambda_1 \cdots \lambda_n e^{-[\lambda_1 y_1 + (\lambda_2 - \lambda_1) y_2 + \cdots + (\lambda_n - \lambda_1) y_n]} \\
&\quad dy_n dy_{n-1} \cdots dy_2 \\
&\;\; \vdots \\
&= \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \frac{\lambda_j}{(\lambda_j - \lambda_i)} \lambda_i e^{-\lambda_i y_1}, \qquad (4.8)
\end{aligned}$$

which is the required result.

Lemma 4.3.3. *The moment generating functions of the random variables*

$$S_1 = \sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i + \sum_{i\in I_1'} x_i + \sum_{i\in I_2'} y_i + (n-d-d')z_{d:n},$$

$$S_2 = \sum_{i\in I_2}(z_i - y_i) + (d_2' z_{d:n} - \sum_{i\in I_2'} y_i), \qquad S_3 = \sum_{i\in I_1}(z_i - x_i) + (d_1' z_{d:n} - \sum_{i\in I_1'} x_i)$$

can be expressed as

$$M_{S_1}(t) = \prod_{i=1}^{2d+d'} (1-\alpha_i t M_i)^{-1}, \qquad M_{S_2}(t) = \prod_{i=1}^{2d+d'} (1-\alpha_i' t M_i)^{-1},$$

$$M_{S_3}(t) = \prod_{i=1}^{2d+d'} (1-\alpha_i'' t M_i)^{-1},$$

respectively. Here, M_i ($i = 1, \dots, 2d+d'$) are as defined in Section 4.3.1, and α_i, α_i' and α_i'' are some coefficients.

Proof: Let us take S_1 as an example. The derivation of the moment-generating functions for the other two random variables is quite similar.

Let $f(x_1, \dots, x_{d_1+d_1'}, y_1, \dots, y_{d_2+d_2'}, z_1, \dots, z_d)$ be the joint PDF of X_i's, Y_i's and Z_i's. Then, we have,

$$\begin{aligned} M_{S_1}(t) &= E(e^{tS_1}) \\ &= \int \cdots \int e^{tS_1} f(x_1, \dots, x_{d_1+d_1'}, y_1, \dots, y_{d_2+d_2'}, z_1, \dots, z_d) \\ &\quad dx_1 \cdots dx_{d_1+d_1'} dy_1 \cdots dy_{d_2+d_2'} dz_1 \cdots dz_d \\ &= \frac{M_t}{M} = \prod_{i=1}^{2d+d'} \frac{(M_t)_i}{M_i} = \prod_{i=1}^{2d+d'} \frac{(M_i^{-1} - \alpha_i t)^{-1}}{M_i} = \prod_{i=1}^{2d+d'} (1-\alpha_i t M_i)^{-1}; \qquad (4.9) \end{aligned}$$

here, M_t is a product consisting of $2d+d'$ multipliers and is defined as $(M_t)_i = M_i^{-1} - \alpha_i t$, $i = 1, 2, \dots, 2d+d'$.

Theorem 4.3.1. *The conditional moment generating functions of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$, conditional on $1 \le D_1 \le d-1$, $0 \le D' \le n-d$ and $0 \le D'_1 \le D'$ are given by*

$$M_{\hat{\theta}_1}(t) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \prod_{i=1}^{2d+m} \left(\frac{1}{1-\frac{\alpha_i}{k+q}M_i t}\right),$$

$$M_{\hat{\theta}_2}(t) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \prod_{i=1}^{2d+m} \left(\frac{1}{1-\frac{\alpha_i(k+q)}{(d-k)+(m-q)}M_i t}\right),$$

$$M_{\hat{\theta}'_1}(t) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \prod_{i=1}^{2d+m} \left(\frac{1}{1-\frac{\alpha'_i}{d-k}M_i t}\right),$$

$$M_{\hat{\theta}'_2}(t) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \prod_{i=1}^{2d+m} \left(\frac{1}{1-\frac{\alpha''_i}{k}M_i t}\right),$$

where

$$C_{k,m,q} = \frac{P_{k,m,q}}{\sum_{f=1}^{d-1}\sum_{g=0}^{n-d}\sum_{s=0}^{g} P_{f,g,s}},$$

α_i, α'_i and α''_i are some coefficients, and $A = (n-d-m)\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)+\frac{q}{\theta'_2}+\frac{m-q}{\theta'_1}$.

Proof: We can express

$$\begin{aligned}
&M_{\hat{\theta}_1}(t)\\
&\quad = E(e^{t\hat{\theta}_1} | 1 \le D_1 \le d-1, 0 \le D' \le n-d, 0 \le D'_1 \le D')\\
&\quad = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} E(e^{t\hat{\theta}_1} | D_1 = k, D' = m, D'_1 = q)\\
&\qquad \times Pr(D_1 = k, D' = m, D'_1 = q | 1 \le D_1 \le d-1, 0 \le D' \le n-d, 0 \le D'_1 \le D')\\
&\quad = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} E(e^{\frac{tS_1}{D_1+D'_1}} | D_1 = k, D' = m, D'_1 = q)\\
&\qquad \times Pr(D_1 = k, D' = m, D'_1 = q | 1 \le D_1 \le d-1, 0 \le D' \le n-d, 0 \le D'_1 \le D')\\
&\quad \vdots
\end{aligned} \tag{4.10}$$

Then, the result follows immediately by using Lemma 4.3.3, and the derivations of the CMGFs for the other three MLEs are quite similar.

Theorem 4.3.2. *The PDFs of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$, conditional on $1 \leq D_1 \leq d-1$, $0 \leq D' \leq n-d$ and $0 \leq D'_1 \leq D'$, are given by*

$$f_{\hat{\theta}_1}(x) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \times g\left(x, \left[\frac{\alpha_i}{k+q}M_i\right]^{-1}, 2d+m\right),$$

$$f_{\hat{\theta}_2}(x) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \times g\left(x, \left[\frac{\alpha_i(k+q)}{(d-k)+(m-q)}M_i\right]^{-1}, 2d+m\right),$$

$$f_{\hat{\theta}'_1}(x) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \times g\left(x, \left[\frac{\alpha'_i}{d-k}M_i\right]^{-1}, 2d+m\right),$$

$$f_{\hat{\theta}'_2}(x) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \times g\left(x, \left[\frac{\alpha''_i}{k}M_i\right]^{-1}, 2d+m\right),$$

here, $g(x, \lambda_i, n)$ is the PDF of the sum of n independent but non-identical exponential random variables with failure rates λ_i, and

$$g(x, \lambda_i, n) = \sum_{i=1}^{n}\prod_{j=1,j\neq i}^{n} \frac{\lambda_j}{(\lambda_j - \lambda_i)}\lambda_i e^{-\lambda_i y}, \;\; 0 < x < \infty, \;\; \lambda_i > 0, \; i = 1, 2, \ldots, n.$$

Proof: The conditional PDFs of the MLEs are computed from the inversion theorem of the moment generating functions. The results follow immediately from Theorem 4.3.1.

4.3.3 Properties of the MLEs

From the two theorems in the previous subsection, we can derive some simple distributional properties of the MLEs.

Theorem 4.3.3. *The first two moments of the MLEs are*

$$E(\hat{\theta}_1) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m}\sum_{i=1}^{2d+m} C_{k,m,q} \times \frac{\alpha_i}{k+q}M_i,$$

$$E(\hat{\theta}_1^2) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m}\sum_{i=1}^{2d+m} C_{k,m,q}\left[2\left(\frac{\alpha_i}{k+q}M_i\right)^2 + \frac{\alpha_i}{k+q}M_i \sum_{j=1,j\neq i}^{2d+m}\frac{\alpha_j}{k+q}M_j\right],$$

$$E(\hat{\theta}_2) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m}\sum_{i=1}^{2d+m} C_{k,m,q} \times \frac{\alpha_i(k+q)}{(d-k)+(m-q)}M_i,$$

$$E(\hat{\theta}_2^2) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m}\sum_{i=1}^{2d+m} C_{k,m,q}\left[2\left(\frac{\alpha_i(k+q)}{(d-k)+(m-q)}M_i\right)^2 \right.$$
$$\left. + \frac{\alpha_i(k+q)}{(d-k)+(m-q)}M_i \sum_{j=1,j\neq i}^{2d+m}\frac{\alpha_j(k+q)}{(d-k)+(m-q)}M_j\right],$$

$$E(\hat{\theta}_1') = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m}\sum_{i=1}^{2d+m} C_{k,m,q} \times \frac{\alpha_i'}{d-k}M_i,$$

$$E(\hat{\theta}_1'^2) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m}\sum_{i=1}^{2d+m} C_{k,m,q}\left[2\left(\frac{\alpha_i'}{d-k}M_i\right)^2 + \frac{\alpha_i'}{d-k}M_i \sum_{j=1,j\neq i}^{2d+m}\frac{\alpha_j'}{d-k}M_j\right],$$

$$E(\hat{\theta}_2') = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m}\sum_{i=1}^{2d+m} C_{k,m,q} \times \frac{\alpha_i''}{k}M_i,$$

$$E(\hat{\theta}_2'^2) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m}\sum_{i=1}^{2d+m} C_{k,m,q}\left[2\left(\frac{\alpha_i''}{k}M_i\right)^2 + \frac{\alpha_i''}{k}M_i \sum_{j=1,j\neq i}^{2d+m}\frac{\alpha_j''}{k}M_j\right].$$

The expressions for the expected values reveal that $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$ are biased estimators of θ_1, θ_2, θ_1' and θ_2', respectively. The expressions for the second moments can be used for finding standard errors of the estimates.

We can also obtain expressions for the tail probabilities by integrating the PDFs in Theorem 4.3.2. These expressions will be used to construct exact confidence intervals later in Section 4.4.1.

Theorem 4.3.4. *The tail probabilities of* $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ *and* $\hat{\theta}'_2$, *conditional on* $1 \le D_1 \le d-1$, $0 \le D' \le n-d$ *and* $0 \le D'_1 \le D'$, *are given by*

$$P_{\theta_1}(\hat{\theta}_1 \ge b) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \times \bar{G}(b, \left[\frac{\alpha_i}{k+q} M_i\right]^{-1}, 2d+m),$$

$$P_{\theta_2}(\hat{\theta}_2 \ge b) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \times \bar{G}(b, \left[\frac{\alpha_i(k+q)}{(d-k)+(m-q)} M_i\right]^{-1}, 2d+m),$$

$$P_{\theta'_1}(\hat{\theta}'_1 \ge b) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \times \bar{G}(b, \left[\frac{\alpha'_i}{d-k} M_i\right]^{-1}, 2d+m),$$

$$P_{\theta'_2}(\hat{\theta}'_2 \ge b) = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q} \times \bar{G}(b, \left[\frac{\alpha''_i}{k} M_i\right]^{-1}, 2d+m),$$

here, $\bar{G}(b, \lambda_i, n)$ *is the survival function of the sum of* n *independent but non-identical exponential random variables with failure rates* λ_i, *and*

$$\bar{G}(b, \lambda_i, n) = \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \frac{\lambda_i}{(\lambda_j - \lambda_i)} e^{-\lambda_i b}.$$

4.3.4 Exact Conditional Distributions of the MLEs Based on the Given Data

In this subsection, a two-component system failure data is generated under Type-II censoring, with parameters $\theta_1 = 20$, $\theta_2 = 25$, $\theta'_1 = 9$ and $\theta'_2 = 14$. Since the exact conditional distributions of the MLEs depend on the data, we will illustrate all the methods and properties presented in Section 4.3.1 - 4.3.3 by using this generated data and the results are presented in Lemmas 4.3.7-4.3.9.

***Description of the Data*:**

There are 10 systems in this sample. Each system has 2 components. The experiment continues until a total of 5 systems fail. The termination time of the experiment is $Z_{d:n} = Z_{5:10} = 21.33$. Among the 5 failed systems, we have $d_1 = 3$ and $d_2 = 2$. Among the 5 non-failed systems, we have $d'_1 = d'_2 = 2$ and 1 system in which both components did not fail before $Z_{d:n}$.

The data set is as follows:

(4.15, 1; 7.40, 2), (0.10, 1; 9.14, 2), (13.89, 2; 15.44, 1), (18.47, 1; 20.76, 2), (14.47, 2; 21.33, 1),
(10.51, 2; *), (17.80, 1; *), (* ; *), (3.77, 2; *), (8.86, 1; *)

If we rank the observed failure times in ascending order, we find that the 1^{st}, 2^{nd} and 4^{th} observations belong to set I_1; the 3^{rd} and 5^{th} observations belong to set I_2, etc. That is,

$$I_1 = \{i_{11}, i_{12}, i_{13}\} = \{1, 2, 4\}, \quad I_2 = \{i_{21}, i_{22}\} = \{3, 5\},$$

$$I'_1 = \{i'_{11}, i'_{12}\} = \{7, 10\}, \quad I'_2 = \{i'_{21}, i'_{22}\} = \{6, 9\}.$$

Next, the order of the observations are:

$$0 < X_{i_{12}} < Y_{i'_{22}} < X_{i_{11}} < Z_{i_{11}} < X_{i'_{12}} < Z_{i_{12}} < Y_{i'_{21}} < Y_{i_{21}} <$$

$$Y_{i_{22}} < Z_{i_{21}} < X_{i'_{11}} < X_{i_{13}} < Z_{i_{13}} < Z_{i_{22}} (= Z_{d:n}) < \infty.$$

Integrating the likelihood function Eq. (4.6) in the descending way, we have the result as $C \times M$, with

$$C = \left(\frac{1}{\theta_1}\right)^5 \left(\frac{1}{\theta_2}\right)^4 \left(\frac{1}{\theta'_1}\right)^2 \left(\frac{1}{\theta'_2}\right)^3,$$

$$A = (n - d - d')(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_2} d'_1 + \frac{1}{\theta'_1} d'_2 = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) + \frac{2}{\theta'_1} + \frac{2}{\theta'_2},$$

and $M = \prod_{i=1}^{14} M_i$, with

$M_1 = \frac{1}{(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{2}{\theta_2'}}$	$M_2 = \frac{1}{(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{3}{\theta_2'}}$	$M_3 = \frac{1}{2(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{2}{\theta_2'}}$
$M_4 = \frac{1}{3(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{1}{\theta_2'}}$	$M_5 = \frac{1}{3(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{4}{\theta_1'}+\frac{1}{\theta_2'}}$	$M_6 = \frac{1}{4(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{1}{\theta_2'}}$
$M_7 = \frac{1}{5(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{2}{\theta_1'}+\frac{1}{\theta_2'}}$	$M_8 = \frac{1}{6(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{1}{\theta_2'}}$	$M_9 = \frac{1}{6(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{2}{\theta_2'}}$
$M_{10} = \frac{1}{7(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{1}{\theta_2'}}$	$M_{11} = \frac{1}{7(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{2}{\theta_2'}}$	$M_{12} = \frac{1}{8(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{1}{\theta_2'}}$
$M_{13} = \frac{1}{9(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_2'}}$	$M_{14} = \frac{1}{10(\frac{1}{\theta_1+\theta_2})}$	

Then, based on this given data, the moment generating functions of the random variables S_1, S_2 and S_3 are obtained and presented in Lemmas 4.3.4-4.3.6.

Lemma 4.3.4. *The moment generating function of*

$$S_1 = \sum_{i \in I_1} x_i + \sum_{i \in I_2} y_i + \sum_{i \in I_1'} x_i + \sum_{i \in I_2'} y_i + (n-d-d')z_{d:n},$$

conditional on the given data, has the form of $\prod_{i=1}^{14}\left(1-\frac{t}{\lambda_i}\right)^{-1}$, *with*

$\lambda_1 = (\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{2}{\theta_2'}$	$\lambda_2 = (\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{3}{\theta_2'}$
$\lambda_3 = \frac{1}{2}\left[2(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{2}{\theta_2'}\right]$	$\lambda_4 = \frac{1}{3}\left[3(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{1}{\theta_2'}\right]$
$\lambda_5 = \frac{1}{3}\left[3(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{4}{\theta_1'}+\frac{1}{\theta_2'}\right]$	$\lambda_6 = \frac{1}{4}\left[4(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{3}{\theta_1'}+\frac{1}{\theta_2'}\right]$
$\lambda_7 = \frac{1}{5}\left[5(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{2}{\theta_1'}+\frac{1}{\theta_2'}\right]$	$\lambda_8 = \frac{1}{6}\left[6(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{1}{\theta_2'}\right]$
$\lambda_9 = \frac{1}{6}\left[6(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{2}{\theta_2'}\right]$	$\lambda_{10} = \frac{1}{7}\left[7(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{1}{\theta_2'}\right]$
$\lambda_{11} = \frac{1}{7}\left[7(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{2}{\theta_2'}\right]$	$\lambda_{12} = \frac{1}{8}\left[8(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_1'}+\frac{1}{\theta_2'}\right]$
$\lambda_{13} = \frac{1}{9}\left[9(\frac{1}{\theta_1}+\frac{1}{\theta_2})+\frac{1}{\theta_2'}\right]$	$\lambda_{14} = (\frac{1}{\theta_1}+\frac{1}{\theta_2})$

Lemma 4.3.5. *The moment generating function of*

$$S_2 = \sum_{i \in I_2} (z_i - y_i) + (d'_2 z_{d:n} - \sum_{i \in I'_2} y_i),$$

conditional on the given data, has the form of $\prod_{i=1}^{12} \left(1 - \frac{t}{\lambda'_i}\right)^{-1}$, *with*

$\lambda'_1 = \frac{1}{3}\left[(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{2}{\theta'_2}\right]$	$\lambda'_2 = \frac{1}{3}\left[(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{3}{\theta'_2}\right]$
$\lambda'_3 = \frac{1}{3}\left[2(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{2}{\theta'_2}\right]$	$\lambda'_4 = \frac{1}{3}\left[3(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{1}{\theta'_2}\right]$
$\lambda'_5 = \frac{1}{4}\left[3(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{4}{\theta'_1} + \frac{1}{\theta'_2}\right]$	$\lambda'_6 = \frac{1}{3}\left[4(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{1}{\theta'_2}\right]$
$\lambda'_7 = \frac{1}{2}\left[5(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{2}{\theta'_1} + \frac{1}{\theta'_2}\right]$	$\lambda'_8 = 6(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{1}{\theta'_2}$
$\lambda'_9 = 6(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{2}{\theta'_2}$	$\lambda'_{10} = 7(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{1}{\theta'_2}$
$\lambda'_{11} = 7(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{2}{\theta'_2}$	$\lambda'_{12} = 8(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{1}{\theta'_2}$

Lemma 4.3.6. *The moment generating function of*

$$S_3 = \sum_{i \in I_1} (z_i - x_i) + (d'_1 z_{d:n} - \sum_{i \in I'_1} x_i),$$

conditional on the given data, has the form of $\prod_{i=1}^{13} \left(1 - \frac{t}{\lambda_i"}\right)^{-1}$, *with*

$\lambda''_1 = \frac{1}{2}\left[(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{2}{\theta'_2}\right]$	$\lambda''_2 = \frac{1}{3}\left[(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{3}{\theta'_2}\right]$
$\lambda''_3 = \frac{1}{2}\left[2(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{2}{\theta'_2}\right]$	$\lambda''_4 = 3(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{1}{\theta'_2}$
$\lambda''_5 = 3(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{4}{\theta'_1} + \frac{1}{\theta'_2}$	$\lambda''_6 = 4(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{3}{\theta'_1} + \frac{1}{\theta'_2}$
$\lambda''_7 = 5(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{2}{\theta'_1} + \frac{1}{\theta'_2}$	$\lambda''_8 = 6(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{1}{\theta'_2}$
$\lambda''_9 = \frac{1}{2}\left[6(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{2}{\theta'_2}\right]$	$\lambda''_{10} = 7(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{1}{\theta'_2}$
$\lambda''_{11} = \frac{1}{2}\left[7(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{2}{\theta'_2}\right]$	$\lambda''_{12} = 8(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_1} + \frac{1}{\theta'_2}$
$\lambda''_{13} = 9(\frac{1}{\theta_1} + \frac{1}{\theta_2}) + \frac{1}{\theta'_2}$	

Lemma 4.3.7. *The PDFs of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$, conditional on the given data, are given by*

$$f_{\hat{\theta}_1}(x) = 5\sum_{i=1}^{14}\prod_{j=1,j\neq i}^{14}\frac{\lambda_j}{(\lambda_j-\lambda_i)}\lambda_i e^{-5\lambda_i x},$$

$$f_{\hat{\theta}_2}(x) = 4\sum_{i=1}^{14}\prod_{j=1,j\neq i}^{14}\frac{\lambda_j}{(\lambda_j-\lambda_i)}\lambda_i e^{-4\lambda_i x},$$

$$f_{\hat{\theta}'_1}(x) = 2\sum_{i=1}^{12}\prod_{j=1,j\neq i}^{12}\frac{\lambda'_j}{(\lambda'_j-\lambda'_i)}\lambda'_i e^{-2\lambda'_i x},$$

$$f_{\hat{\theta}'_2}(x) = 3\sum_{i=1}^{13}\prod_{j=1,j\neq i}^{13}\frac{\lambda''_j}{(\lambda''_j-\lambda''_i)}\lambda''_i e^{-3\lambda''_i x},$$

here, λ_i $(i = 1,\ldots,14)$ are as defined in Lemma 4.3.4, λ'_i $(i = 1,\ldots,12)$ are as defined in Lemma 4.3.5, and λ''_i $(i = 1,\ldots,13)$ are as defined in Lemma 4.3.6.

Lemma 4.3.8. *The tail probability of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$, conditional on the given data, are given by*

$$P_{\theta_1}(\hat{\theta}_1 \geq b) = \sum_{i=1}^{14}\prod_{j=1,j\neq i}^{14}\frac{\lambda_j}{(\lambda_j-\lambda_i)}e^{-5\lambda_i b},$$

$$P_{\theta_1}(\hat{\theta}_2 \geq b) = \sum_{i=1}^{14}\prod_{j=1,j\neq i}^{14}\frac{\lambda_j}{(\lambda_j-\lambda_i)}e^{-4\lambda_i b},$$

$$P_{\theta'_1}(\hat{\theta}'_1 \geq b) = \sum_{i=1}^{12}\prod_{j=1,j\neq i}^{12}\frac{\lambda'_j}{(\lambda'_j-\lambda'_i)}e^{-2\lambda'_i b},$$

$$P_{\theta'_2}(\hat{\theta}'_2 \geq b) = \sum_{i=1}^{13}\prod_{j=1,j\neq i}^{13}\frac{\lambda''_j}{(\lambda''_j-\lambda''_i)}e^{-3\lambda''_i b},$$

here, λ_i $(i = 1,\ldots,14)$ are as defined in Lemma 4.3.4, λ'_i $(i = 1,\ldots,12)$ are as defined in Lemma 4.3.5, and λ''_i $(i = 1,\ldots,13)$ are as defined in Lemma 4.3.6.

Lemma 4.3.9. *The first two moments of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$, conditional on the given data, are given by*

$$E(\hat{\theta}_1) = \sum_{i=1}^{14} \prod_{j=1,j\neq i}^{14} \frac{\lambda_j}{(\lambda_j - \lambda_i)} \frac{1}{5\lambda_i} = \sum_{i=1}^{14} \frac{1}{5\lambda_i},$$

$$E(\hat{\theta}_1^2) = \sum_{i=1}^{14} \prod_{j=1,j\neq i}^{14} \frac{\lambda_j}{(\lambda_j - \lambda_i)} \frac{2}{(5\lambda_i)^2} = \sum_{i=1}^{14} \left[\frac{2}{(5\lambda_i)^2} + \frac{1}{5\lambda_i} \sum_{j=1,j\neq i}^{14} \frac{1}{5\lambda_j} \right],$$

$$E(\hat{\theta}_2) = \sum_{i=1}^{14} \prod_{j=1,j\neq i}^{14} \frac{\lambda_j}{(\lambda_j - \lambda_i)} \frac{1}{4\lambda_i} = \sum_{i=1}^{14} \frac{1}{4\lambda_i},$$

$$E(\hat{\theta}_2^2) = \sum_{i=1}^{14} \prod_{j=1,j\neq i}^{14} \frac{\lambda_j}{(\lambda_j - \lambda_i)} \frac{2}{(4\lambda_i)^2} = \sum_{i=1}^{14} \left[\frac{2}{(4\lambda_i)^2} + \frac{1}{4\lambda_i} \sum_{j=1,j\neq i}^{14} \frac{1}{4\lambda_j} \right],$$

$$E(\hat{\theta}'_1) = \sum_{i=1}^{12} \prod_{j=1,j\neq i}^{12} \frac{\lambda'_j}{(\lambda'_j - \lambda'_i)} \frac{1}{2\lambda'_i} = \sum_{i=1}^{12} \frac{1}{2\lambda'_i},$$

$$E(\hat{\theta}_1'^2) = \sum_{i=1}^{12} \prod_{j=1,j\neq i}^{12} \frac{\lambda'_j}{(\lambda'_j - \lambda'_i)} \frac{2}{(2\lambda'_i)^2} = \sum_{i=1}^{12} \left[\frac{2}{(2\lambda'_i)^2} + \frac{1}{2\lambda'_i} \sum_{j=1,j\neq i}^{12} \frac{1}{2\lambda'_j} \right],$$

$$E(\hat{\theta}'_2) = \sum_{i=1}^{13} \prod_{j=1,j\neq i}^{13} \frac{\lambda''_j}{(\lambda''_j - \lambda''_i)} \frac{1}{3\lambda''_i} = \sum_{i=1}^{13} \frac{1}{3\lambda''_i},$$

$$E(\hat{\theta}_2'^2) = \sum_{i=1}^{13} \prod_{j=1,j\neq i}^{13} \frac{\lambda''_j}{(\lambda''_j - \lambda''_i)} \frac{2}{(3\lambda''_i)^2} = \sum_{i=1}^{13} \left[\frac{2}{(3\lambda''_i)^2} + \frac{1}{3\lambda''_i} \sum_{j=1,j\neq i}^{13} \frac{1}{3\lambda''_j} \right],$$

here, λ_i $(i = 1, \ldots, 14)$ are as defined in Lemma 4.3.4, λ'_i $(i = 1, \ldots, 12)$ are as defined in Lemma 4.3.5, and λ''_i $(i = 1, \ldots, 13)$ are as defined in Lemma 4.3.6.

4.4 Confidence Intervals

In this section, we present different methods of constructing confidence intervals (CIs) for the unknown parameters θ_1, θ_2, θ'_1 and θ'_2. The exact CIs are based on the exact conditional distributions of the MLEs presented in Theorems 4.3.2 and 4.3.4. Since the exact conditional PDFs of the MLEs are computationally intensive, we may use the asymptotic distributions of the MLEs to obtain approximate CIs for θ_1, θ_2, θ_1' and θ_2' for large sample size. Finally, we use the parametric bootstrap method to construct the CIs for the parameters.

4.4.1 Exact Confidence Intervals

The same method in Section 2.4.1 is used to construct the exact CIs for parameters θ_1, θ_2, θ_1' and θ_2'. To guarantee the invertibility for the parameters, we assume once again that the tail probabilities of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$ presented in Theorem 4.3.4 are increasing functions of θ_1, θ_2, θ_1' and θ_2', respectively. Values of the tail probabilities $P_{\theta_i^{(\prime)}}\left(\hat{\theta}_i^{(\prime)} \geq b\right)$ for various $\theta_i^{(\prime)} (i = 1, 2)$ and b are presented in Tables 4.1 - 4.4 to support this monotonicity assumption. Since the tail probabilities depend on the data, the form of $P_{\theta_i^{(\prime)}}\left(\hat{\theta}_i^{(\prime)} \geq b\right)$ is taken as in Lemma 4.3.8.

Confidence Interval for θ_1

A two-sided $100(1-\alpha)\%$ CI for θ_1, denoted by $(\theta_{1L}, \theta_{1U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q}(\theta_{1L}) \times \bar{G}(\hat{\theta}_1, \left[\frac{\alpha_i}{k+q} M_i(\theta_{1L})\right]^{-1}, 2d+m), \tag{4.11}$$

$$1-\frac{\alpha}{2} = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q}(\theta_{1U}) \times \bar{G}(\hat{\theta}_1, \left[\frac{\alpha_i}{k+q} M_i(\theta_{1U})\right]^{-1}, 2d+m), \tag{4.12}$$

where $C_{k,m,q}(\theta_{1L(U)})$ can be obtained from the expression $C_{k,m,q}$ given in Theorem 4.3.1

and $M_i(\theta_{1L(U)})$ can be obtained from the expression M_i given in Section 4.3.1. But, in both cases, we replace $(\theta_1,\ \theta_2,\ \theta_1',\ \theta_2')$ by $(\theta_{1L(U)},\ \hat{\theta}_2,\ \hat{\theta}_1',\ \hat{\theta}_2')$.

Confidence Interval for θ_2

A two-sided $100(1-\alpha)\%$ CI for θ_2, denoted by $(\theta_{2L}, \theta_{2U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q}(\theta_{2L}) \times \bar{G}(\hat{\theta}_2, \left[\frac{\alpha_i(k+q)}{(d-k)+(m-q)} M_i(\theta_{2L})\right]^{-1}, 2d+m), \quad (4.13)$$

$$1-\frac{\alpha}{2} = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q}(\theta_{2U}) \times \bar{G}(\hat{\theta}_2, \left[\frac{\alpha_i(k+q)}{(d-k)+(m-q)} M_i(\theta_{2U})\right]^{-1}, 2d+m), \quad (4.14)$$

where $C_{k,m,q}(\theta_{2L(U)})$ can be obtained from the expression $C_{k,m,q}$ given in Theorem 4.3.1 and $M_i(\theta_{2L(U)})$ can be obtained from the expression M_i given in Section 4.3.1. But, in both cases, we replace $(\theta_1,\ \theta_2,\ \theta_1',\ \theta_2')$ by $(\hat{\theta}_1,\ \theta_{2L(U)},\ \hat{\theta}_1',\ \hat{\theta}_2')$.

Confidence Interval for θ_1'

A two-sided $100(1-\alpha)\%$ CI for θ_1', denoted by $(\theta_{1L}', \theta_{1U}')$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q}(\theta_{1L}') \times \bar{G}(\hat{\theta}_1', \left[\frac{\alpha_i'}{d-k} M_i(\theta_{1L}')\right]^{-1}, 2d+m), \quad (4.15)$$

$$1-\frac{\alpha}{2} = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q}(\theta_{1U}') \times \bar{G}(\hat{\theta}_1', \left[\frac{\alpha_i'}{d-k} M_i(\theta_{1U}')\right]^{-1}, 2d+m), \quad (4.16)$$

where $C_{k,m,q}(\theta_{1L(U)}')$ can be obtained from the expression $C_{k,m,q}$ given in Theorem 4.3.1

and $M_i(\theta'_{1L(U)})$ can be obtained from the expression M_i given in Section 4.3.1. But, in both cases, we replace $(\theta_1,\ \theta_2,\ \theta'_1,\ \theta'_2)$ by $(\hat{\theta}_1,\ \hat{\theta}_2,\ \theta'_{1L(U)},\ \hat{\theta}'_2)$.

Confidence Interval for θ'_2

A two-sided $100(1-\alpha)\%$ CI for θ'_2, denoted by $(\theta'_{2L}, \theta'_{2U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q}(\theta'_{2L}) \times \bar{G}(\hat{\theta}'_2, \left[\frac{\alpha_i"}{k} M_i(\theta'_{2L})\right]^{-1}, 2d+m), \tag{4.17}$$

$$1-\frac{\alpha}{2} = \sum_{k=1}^{d-1}\sum_{m=0}^{n-d}\sum_{q=0}^{m} C_{k,m,q}(\theta'_{2U}) \times \bar{G}(\hat{\theta}'_2, \left[\frac{\alpha_i"}{k} M_i(\theta'_{2U})\right]^{-1}, 2d+m), \tag{4.18}$$

where $C_{k,m,q}(\theta'_{2L(U)})$ can be obtained from the expression $C_{k,m,q}$ given in Theorem 4.3.1 and $M_i(\theta'_{2L(U)})$ can be obtained from the expression M_i given in Section 4.3.1. But, in both cases, we replace $(\theta_1,\ \theta_2,\ \theta'_1,\ \theta'_2)$ by $(\hat{\theta}_1,\ \hat{\theta}_2,\ \hat{\theta}'_1,\ \theta'_{2L(U)})$.

Lacking a closed-form solution, we have to apply an iterative root-finding technique in the determination of θ_{iL}, θ'_{iL}, θ_{iU} and θ'_{iU}, for $i = 1, 2$; the Newton-Raphson iteration method, for instance, was used here for this purpose.

4.4.2 Approximate Confidence Intervals

Using the asymptotic normality of the MLEs, we are able to construct approximate confidence intervals for θ_1, θ_2, θ'_1 and θ'_2 using the Fisher information matrix.

Let $I(\theta_1, \theta_2, \theta'_1, \theta'_2) = (I_{ij}(\theta_1, \theta_2, \theta'_1, \theta'_2))$, $i, j = 1, 2, 3, 4$, denote the Fisher information matrix for the parameter $(\theta_1,\ \theta_2,\ \theta_1',\ \theta_2')$. From Eq. (4.2), we have

$$I_{11}\left(\theta_1,\theta_2,\theta_1',\theta_2'\right) = -\mathrm{E}\left(\frac{d_1+d_1'}{\hat{\theta}_1^2} - \frac{2S_1}{\hat{\theta}_1^3}\right) = \frac{d_1+d_1'}{\hat{\theta}_1^2}, \tag{4.19}$$

$$I_{22}\left(\theta_1,\theta_2,\theta_1',\theta_2'\right) = -\mathrm{E}\left(\frac{d_2+d_2'}{\hat{\theta}_2^2} - \frac{2S_1}{\hat{\theta}_2^3}\right) = \frac{d_2+d_2'}{\hat{\theta}_2^2}, \tag{4.20}$$

$$I_{33}\left(\theta_1,\theta_2,\theta_1',\theta_2'\right) = -\mathrm{E}\left(\frac{d_2}{\hat{\theta}_1'^2} - \frac{2S_2}{\hat{\theta}_1'^3}\right) = \frac{d_2}{\hat{\theta}_1'^2}, \tag{4.21}$$

$$I_{44}\left(\theta_1,\theta_2,\theta_1',\theta_2'\right) = -\mathrm{E}\left(\frac{d_1}{\hat{\theta}_2'^2} - \frac{2S_3}{\hat{\theta}_2'^3}\right) = \frac{d_1}{\hat{\theta}_2'^2}, \tag{4.22}$$

$$I_{12} = I_{13} = I_{14} = I_{21} = I_{23} = I_{24} = I_{31} = I_{32} = I_{34} = I_{41} = I_{42} = I_{43} = 0, \tag{4.23}$$

where S_1, S_2 and S_3 are as defined in Lemma 4.3.3. Thus, the Fisher information matrix is given by

$$\begin{bmatrix} \frac{d_1+d_1'}{\hat{\theta}_1^2} & 0 & 0 & 0 \\ 0 & \frac{d_2+d_2'}{\hat{\theta}_2^2} & 0 & 0 \\ 0 & 0 & \frac{d_2}{\hat{\theta}_1'^2} & 0 \\ 0 & 0 & 0 & \frac{d_1}{\hat{\theta}_2'^2} \end{bmatrix}.$$

This implies that the MLEs are asymptotically mutually independent. The asymptotic unconditional variance of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$ can be obtained from the Fisher information matrix as

$$V_1 = \frac{\hat{\theta}_1^2}{d_1+d_1'},\ \ V_2 = \frac{\hat{\theta}_2^2}{d_2+d_2'}, V_3 = \frac{\hat{\theta}_1'^2}{d_2},\ \ V_4 = \frac{\hat{\theta}_2'^2}{d_1}.$$

Then, the $100(1-\alpha)\%$ approximate CIs for θ_1, θ_2, θ_1' and θ_2' are obtained by using the same method as described in Section 2.4.2.

4.4.3 Bootstrap Confidence Intervals

The bootstrap methods of percentile interval and the biased-corrected and accelerated (BC_α) interval are similar to those described in Section 2.4.3, but with a Type-II censored two-component system failure sample instead. The acceleration $\hat{\alpha}_i^{(\prime)}$ in the BC_α interval should be changed to

$$\hat{\alpha}_i^{(\prime)} = \frac{\sum_{j=1}^{d}\left(\hat{\theta}_{i(\cdot)}^{(\prime)} - \hat{\theta}_{i(j)}^{(\prime)}\right)^3}{6\left[\sum_{j=1}^{d}\left(\hat{\theta}_{i(\cdot)}^{(\prime)} - \hat{\theta}_{i(j)}^{(\prime)}\right)^2\right]^{\frac{3}{2}}}, \quad i = 1, 2,$$

where $\hat{\theta}_{i(j)}^{(\prime)}$ is the MLE of $\theta_i^{(\prime)}$ based on the original sample with the j-th observation deleted, $j = 1, 2, \ldots, d$, and $\hat{\theta}_{i(\cdot)}^{(\prime)} = \frac{\sum_{j=1}^{d}\hat{\theta}_{i(j)}^{(\prime)}}{d}$.

4.5 Simulation Study

In this section, a Monte Carlo simulation study based on 999 replications was carried out to examine the relative risks (Section 4.5.1), to evaluate the performance of the two bootstrap confidence intervals in terms of coverage probabilities for different sample sizes (Section 4.5.2). We also present numerical examples in Section 4.5.3 to illustrate all the inferential methods discussed here.

4.5.1 Relative Risks

The theoretical values of π_1, π_2, π_3 and π_4 with $\theta_1 = 20$, $\theta_2 = 25$ $\theta_1' = 9$, $\theta_2' = 14$ are presented in Table 4.12 when n and d take on different values. The results were calculated by using the equations presented in Theorems 2.3.1, 3.2.1, 4.2.1 and 4.2.2.

All π_1, π_2 and π_3 are the probabilities that Component 1 fails first within a system. But π_2 is conditional on the complete destruction of the systems and π_3 is conditional on the incomplete destruction of the systems. $\pi_1 = 0.5556$ implies that the first failure of a system is more likely due to Component 1. However, in most cases, $\pi_2 < \pi_1 < \pi_3$. It reflects that a system is more likely to survive if its Component 1 fails first in a Type-II censoring experiment.

From Table 4.12, we observe that π_2, π_3 and π_4 are more affected by the change of n and d. As n or d increases, the three relative risks increase. This is because when n or d is small, fewer failures occur during the experiment time. As n or d increases, the number of failures increases thus resulting in larger relative risks.

In order to examine whether π_1, π_2, π_3 and π_4 can be estimated by $\frac{n_1}{n}$, $\frac{d_1}{d}$, $\frac{d_1'}{d'}$ and $\frac{d'}{n-d}$, respectively, the results of the average of 999 replications are presented in Table 4.13. We observe that the tabled values get closer to the corresponding theoretical values when n or d take on different values. This indicates that $\frac{n_1}{n}$, $\frac{d_1}{d}$, $\frac{d_1'}{d'}$ and $\frac{d'}{n-d}$ are good estimators of π_1, π_2, π_3 and π_4, respectively.

4.5.2 Coverage Probabilities and the Performance of the Confidence Intervals

Since the exact confidence intervals and asymptotic confidence intervals depend on the data, we carry out a Monte Carlo simulation study to compare only the performance of the two bootstrap confidence intervals described in Section 4.4.3. We once again chose the values of the parameters to be $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$.

We then determined the true coverage probabilities of the 90%, 95% and 99% confidence intervals for the parameters for different sample size by these two methods. The results for $n = 40$, 20 and 10 are presented in Tables 4.5-4.7, and they are based on average over 999 bootstrap replications.

From Tables 4.5-4.7, we observe that the parametric BC_α bootstrap method has the coverage probability to be comparatively closer to the nominal level for all the parameters in most cases. In Tables 4.6 and 4.7, when the sample size n is small, the coverage probabilities of the bootstrap percentile method and the parametric BC_α bootstrap method are most often smaller than the nominal level. But, the parametric BC_α bootstrap method is seen to be better than the percentile bootstrap method, even though it does not have satisfactory coverage probabilities.

4.5.3 Numerical Examples

In this subsection, we consider two data sets when $n = 35$ with $d = 15$ and $d = 25$ and $n = 15$ with $d = 7$ and $d = 10$. The parameters were chosen to be $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$. The data are as follows:

Data Set 1: $n = 35$

$d = 15$				
(0.54,1; 2.45,2)	(1.70,1; 4.08,2)	(4.27,2; 4.87,1)	(2.62,1; 5.53,2)	(4.01,2; 6.32,1)
(2.78,2; 7.44,1)	(5.29,1; 7.56,2)	(1.30,2; 7.65,1)	(1.32,2; 7.70,1)	(6.14,2; 8.12,1)
(5.29,1; 10.81,2)	(11.35,2; 11.79,1)	(4.07,1; 12.54,2)	(10.98,1; 15.24,2)	(13.76,1; 16.31,2)
(3.60,2; *)	(2.03,2; *)	(7.37,1; *)	(* ; *)	(1.29,1; *)
(* ; *)	(5.56,1; *)	(8.55,1; *)	(* ; *)	(* ; *)
(0.73,1; *)	(* ; *)	(* ; *)	(* ; *)	(1.61, 1; *)
(14.09,1; *)	(14.83, 1; *)	(* ; *)	(* ; *)	(* ; *)

$d = 25$				
(0.54,1; 2.45,2)	(1.70,1; 4.08,2)	(4.27,2; 4.87,1)	(2.62,1; 5.53,2)	(4.01,2; 6.32,1)
(2.78,2; 7.44,1)	(5.29,1; 7.56,2)	(1.30,2; 7.65,1)	(1.32,2; 7.70,1)	(6.14,2; 8.12,1)
(5.29,1; 10.81,2)	(11.35,2; 11.79,1)	(4.07,1; 12.54,2)	(10.98,1; 15.24,2)	(13.76,1; 16.31,2)
(3.60,2; 18.01,1)	(2.03,2; 19.61,1)	(7.37,1; 19.81,2)	(19.00,2; 21.30,1)	(1.29,1; 21.57,2)
(19.91,2; 22.12,1)	(5.56,1; 22.70,2)	(8.55,1; 25.03,2)	(25.39,1; 27.16,2)	(22.66,2; 27.74,1)
(0.73,1; *)	(23.22,1; *)	(17.60,2; *)	(27.55,1; *)	(1.61, 1; *)
(14.09,1; *)	(14.83, 1; *)	(* ; *)	(19.69,1 ; *)	(* ; *)

Data Set 2: $n = 15$

$d = 7$				
(1.21,2; 6.16,1)	(7.21,1; 9.53,2)	(9.06,2; 13.00,1)	(7.95,1; 13.98,2)	(5.97,1; 14.19,2)
(8.45,2; 17.90,1)	(6.73,1; 17.94,2)	(5.28,1; *)	(* ; *)	(* ; *)
(0.92,2; *)	(10.24,1; *)	(* ; *)	(16.95,1; *)	(* ; *)

$d = 10$				
(1.21,2; 6.16,1)	(7.21,1; 9.53,2)	(9.06,2; 13.00,1)	(7.95,1; 13.98,2)	(5.97,1; 14.19,2)
(8.45,2; 17.90,1)	(6.73,1; 17.94,2)	(5.28,1; 21.59,2)	(19.06,2; 24.45,1)	(21.43,2; 26.13,1)
(0.92,2; *)	(10.24,1; *)	(* ; *)	(16.95,1; *)	(* ; *)

In the example when $n = 35$, $d = 15$, we have $d_1 = 8$, $d_2 = 7$, $d'_1 = 8$ and $d'_2 = 2$. Using the expressions presented in Section 4.2, the MLEs of θ_1, θ_2, θ'_1 and θ'_2 are $\hat{\theta}_1 = 18.6379$, $\hat{\theta}_2 = 33.1340$, $\hat{\theta}'_1 = 7.1030$ and $\hat{\theta}'_2 = 13.3399$.

In the example when $n = 35$, $d = 25$, we have $d_1 = 13$, $d_2 = 12$, $d'_1 = 7$ and $d'_2 = 1$.

The MLEs are $\hat{\theta}_1 = 18.2793$, $\hat{\theta}_2 = 28.1220$, $\hat{\theta}'_1 = 6.2055$ and $\hat{\theta}'_2 = 14.6813$.

In the example when $n = 15$, $d = 7$, we have $d_1 = 4$, $d_2 = 3$, $d'_1 = 3$ and $d'_2 = 1$. The MLEs are $\hat{\theta}_1 = 21.6739$, $\hat{\theta}_2 = 37.9294$, $\hat{\theta}'_1 = 11.7864$ and $\hat{\theta}'_2 = 12.2768$.

In the example when $n = 15$, $d = 10$, we have $d_1 = 5$, $d_2 = 5$, $d'_1 = 2$ and $d'_2 = 1$. The MLEs are $\hat{\theta}_1 = 24.6722$, $\hat{\theta}_2 = 28.7843$, $\hat{\theta}'_1 = 10.7288$ and $\hat{\theta}'_2 = 13.8272$.

To assess the performance of these estimates, we constructed 90%, 95% and 99% confidence intervals using the methods outlined in Section 4.4. The results are presented in Tables 4.8-4.11.

From these results, it is seen that the exact confidence intervals are wider in general than the other three intervals. It is also seen that the approximate method always provide narrower confidence intervals. This is because the coverage probabilities for the approximate method are significantly lower than the nominal levels.

Table 4.1: Values of $P_{\theta_1}\left(\hat{\theta}_1 \geq b\right)$ with $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$

θ_1	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.8660	0.1502	0.0050	0.0001
9	0.9850	0.6059	0.1409	0.0158
13	0.9962	0.8109	0.3535	0.0848
17	0.9985	0.8935	0.5157	0.1802
21	0.9992	0.9317	0.6241	0.2715
25	0.9995	0.9519	0.6968	0.3488
29	1.0000	0.9997	0.7471	0.4119
33	0.9998	0.9712	0.7832	0.4630

Table 4.2: Values of $P_{\theta_2}\left(\hat{\theta}_2 \geq b\right)$ with $\theta_1 = 20$, $\theta_1' = 9$ and $\theta_2' = 14$

θ_2	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.9564	0.3745	0.0382	0.0016
9	0.9962	0.8029	0.3329	0.0728
13	0.9991	0.9189	0.5753	0.2212
17	0.9996	0.9573	0.7107	0.3577
21	0.9998	0.9734	0.7876	0.4613
25	0.9999	0.9816	0.8344	0.5374
29	0.9999	0.9862	0.8648	0.5939
33	0.9999	0.9891	0.8857	0.6366

Table 4.3: Values of $P_{\theta_1'}\left(\hat{\theta}_1' \geq b\right)$ with $\theta_1 = 20$, $\theta_2 = 25$ and $\theta_2' = 14$

θ_1'	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.1442	0.0002	0.0000	0.0000
5	0.9874	0.7123	0.2711	0.0624
9	0.9981	0.9120	0.6172	0.2986
13	0.9993	0.9569	0.7617	0.4717
17	0.9996	0.9730	0.8295	0.5770
21	0.9997	0.9806	0.8665	0.6437
25	0.9998	0.9848	0.8891	0.6886
29	0.9999	0.9874	0.9041	0.7204
33	0.9999	0.9891	0.9147	0.7440

Table 4.4: Values of $P_{\theta_2'}\left(\hat{\theta}_2' \geq b\right)$ with $\theta_1 = 20$, $\theta_2 = 25$ and $\theta_1' = 9$

θ_2'	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0012	0.0000	0.0000	0.0000
5	0.5944	0.0286	0.0004	0.0000
9	0.7828	0.1129	0.0051	0.0001
13	0.8445	0.1797	0.0132	0.0006
17	0.8734	0.2261	0.0214	0.0014
21	0.8897	0.2590	0.0286	0.0021
25	0.9002	0.2832	0.0348	0.0029
29	0.9074	0.3018	0.0400	0.0036
33	0.9127	0.3163	0.0444	0.0043

Table 4.5: Estimated coverage probabilities based on 999 simulations with $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 40$

C.I. of θ_1	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
15	90.49	90.79	94.09	95.39	98.60	99.20
20	89.69	89.99	95.10	95.99	98.80	99.20
25	88.19	89.69	93.79	95.19	98.69	98.79
30	89.29	89.98	94.79	95.20	99.29	99.40

C.I. of θ_2	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
15	89.89	91.19	94.89	95.99	98.60	99.20
20	89.99	91.49	94.50	95.50	98.30	98.80
25	88.69	88.79	93.09	94.09	98.80	99.00
30	88.69	90.29	95.50	96.69	98.59	99.20

C.I. of θ_1'	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
15	89.99	91.69	93.89	94.79	98.60	99.20
20	87.89	89.09	95.50	96.50	99.50	99.60
25	89.39	91.91	94.09	96.09	98.39	98.89
30	90.49	92.49	95.69	96.39	98.39	98.69

C.I. of θ_2'	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
15	89.09	89.59	94.19	94.39	98.60	99.20
20	89.09	89.79	93.99	94.89	98.79	98.70
25	88.29	88.69	93.69	94.80	98.99	99.30
30	90.39	91.39	94.89	95.49	97.79	98.89

Table 4.6: Estimated coverage probabilities based on 999 simulations with $\theta_1 = 20$, $\theta_2 = 25$, $\theta'_1 = 9$, $\theta'_2 = 14$ and $n = 20$

C.I. of θ_1	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
10	86.49	87.39	92.89	93.79	97.50	97.70
12	88.49	90.49	94.69	95.99	97.99	98.10
14	89.19	91.69	94.09	94.69	98.20	98.99
16	88.29	90.29	94.39	95.80	98.80	99.50
C.I. of θ_2	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
10	89.89	91.59	93.69	94.79	97.50	97.70
12	89.80	91.69	94.19	95.60	97.80	98.30
14	89.79	91.79	94.29	96.40	98.70	99.10
16	90.19	91.39	93.69	94.69	98.20	99.00
C.I. of θ'_1	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
10	86.99	89.79	90.49	92.99	95.89	96.30
12	88.09	90.99	93.09	95.20	96.60	96.80
14	87.39	89.90	93.39	95.20	97.40	98.40
16	86.89	90.79	93.09	94.79	96.60	97.40
C.I. of θ'_2	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
10	88.89	88.69	92.09	93.19	97.60	97.60
12	87.79	89.79	92.49	93.39	96.20	96.10
14	88.89	90.89	93.19	96.20	97.20	97.80
16	89.69	91.89	92.29	93.89	97.60	98.00

Table 4.7: Estimated coverage probabilities based on 999 simulations with $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 10$

C.I. of θ_1	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
5	63.16	62.76	69.37	68.77	65.97	65.67
6	76.48	76.48	78.38	78.78	83.28	83.18
7	82.08	81.68	87.09	86.79	88.39	88.29
8	86.79	86.59	89.99	90.99	92.49	92.49
C.I. of θ_2	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
5	64.86	63.36	69.47	68.87	66.17	65.97
6	77.68	76.78	79.68	78.88	82.88	82.88
7	82.08	80.98	86.29	86.69	88.89	88.99
8	86.89	87.99	90.79	90.89	93.59	93.59
C.I. of θ_1'	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
5	57.96	60.86	64.36	66.57	63.76	63.56
6	70.37	73.57	72.37	75.48	80.18	80.78
7	75.48	80.48	81.48	84.18	84.68	85.59
8	81.88	85.99	83.78	87.79	89.19	90.39
C.I. of θ_2'	90% C.I.		95% C.I.		99% C.I.	
d	Boot-p	BC_α	Boot-p	BC_α	Boot-p	BC_α
5	57.06	58.56	66.37	67.57	63.86	64.06
6	70.87	72.77	75.38	77.78	80.38	80.78
7	76.88	80.28	81.78	83.98	86.89	87.29
8	82.08	84.28	85.49	87.39	91.69	92.29

Table 4.8: Interval estimation for the simulated sample with $n = 35$, $d = 25$

90% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(12.99, 27.49)	(19.16, 53.74)	(4.03, 11.57)	(9.56, 23.85)
Approx.	(11.56, 25.00)	(15.29, 40.95)	(3.26, 9.15)	(7.98, 21.38)
Boot-p	(12.60, 27.09)	(18.02, 47.17)	(3.43, 9.63)	(9.11, 22.82)
BC_α	(12.57, 26.98)	(18.95, 52.52)	(3.84, 10.67)	(9.34, 23.40)
95% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(12.18, 29.53)	(17.59, 60.80)	(3.55, 13.38)	(8.46, 26.99)
Approx.	(10.27, 26.29)	(12.84, 43.41)	(2.69, 9.72)	(6.70, 22.66)
Boot-p	(11.77, 28.63)	(16.61, 55.87)	(3.15, 10.68)	(8.17, 25.66)
BC_α	(11.77, 28.62)	(17.19, 58.81)	(3.48, 11.84)	(8.38, 26.02)
99% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(10.80, 34.74)	(14.84, 75.66)	(3.09, 17.59)	(6.16, 30.85)
Approx.	(7.75, 28.81)	(8.03, 48.21)	(1.59, 10.82)	(4.19, 25.17)
Boot-p	(10.54, 33.42)	(14.05, 69.44)	(2.41, 13.29)	(6.37, 29.94)
BC_α	(10.54, 33.00)	(14.30, 69.44)	(3.03, 14.11)	(6.37, 29.65)

Table 4.9: Interval estimation for the simulated sample with $n = 35$, $d = 15$

90% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(13.01, 29.54)	(19.73, 59.72)	(3.70, 14.74)	(7.96, 24.40)
Approx.	(10.97, 26.30)	(14.97, 51.30)	(2.69, 11.52)	(5.58, 21.10)
Boot-p	(12.86, 28.61)	(19.73, 59.16)	(3.36, 13.65)	(6.82, 22.63)
BC_α	(12.88, 28.61)	(19.63, 58.08)	(3.41, 13.72)	(7.58, 23.60)
95% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(12.45, 32.13)	(18.88, 68.72)	(3.17, 17.74)	(6.88, 27.80)
Approx.	(9.51, 27.77)	(11.49, 54.78)	(1.84, 12.36)	(4.10, 22.58)
Boot-p	(11.98, 31.10)	(18.36, 67.13)	(2.81, 15.77)	(5.85, 24.67)
BC_α	(11.98, 31.01)	(18.32, 66.23)	(2.81, 15.48)	(6.61, 26.20)
99% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(10.41, 41.09)	(16.26, 88.59)	(2.04, 22.92)	(5.37, 34.40)
Approx.	(6.64, 30.64)	(4.68, 61.58)	(0.19, 14.02)	(1.19, 25.49)
Boot-p	(9.89, 37.42)	(15.56, 86.66)	(2.16, 20.71)	(4.24, 31.34)
BC_α	(9.89, 36.53)	(15.25, 82.56)	(1.39, 19.23)	(4.81, 31.71)

Table 4.10: Interval estimation for the simulated sample with $n = 15$, $d = 10$

90% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(12.28, 48.80)	(14.22, 61.37)	(4.79, 21.10)	(5.72, 26.49)
Approx.	(9.33, 40.01)	(9.46, 48.11)	(2.84, 18.62)	(3.66, 23.99)
Boot-p	(13.24, 50.04)	(15.25, 62.23)	(4.06, 21.17)	(5.67, 26.88)
BC_α	(12.77, 47.69)	(14.90, 59.50)	(4.28, 21.79)	(5.93, 27.84)
95% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(10.87, 57.10)	(11.60, 70.43)	(4.15, 25.50)	(4.98, 31.79)
Approx.	(6.40, 42.95)	(5.75, 51.82)	(1.32, 20.13)	(1.71, 25.95)
Boot-p	(11.74, 59.87)	(13.79, 74.61)	(3.25, 24.27)	(4.74, 33.40)
BC_α	(11.54, 57.46)	(13.06, 69.49)	(3.46, 24.51)	(4.74, 33.40)
99% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(8.78, 79.11)	(9.25, 95.35)	(2.24, 31.26)	(3.33, 53.17)
Approx.	(0.65, 48.69)	(0*, 59.05)	(0*, 23.09)	(0*, 29.76)
Boot-p	(10.15, 94.13)	(10.88, 113.28)	(1.71, 30.85)	(3.35, 55.19)
BC_α	(9.89, 80.10)	(9.14, 93.85)	(1.71, 29.66)	(3.13, 50.93)

0* stands for a non-positive number

Table 4.11: Interval estimation for the simulated sample with $n = 15$, $d = 7$

90% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(11.27, 43.69)	(16.49, 88.11)	(3.41, 36.40)	(3.85, 25.77)
Approx.	(8.29, 35.15)	(6.74, 69.12)	(0.59, 22.98)	(2.18, 22.37)
Boot-p	(11.76, 42.47)	(17.76, 96.40)	(3.07, 32.58)	(5.05, 24.31)
BC_α	(11.68, 40.87)	(17.93, 96.40)	(3.57, 34.43)	(5.21, 25.22)
95% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(10.02, 52.34)	(15.50, 117.23)	(2.67, 43.54)	(3.43, 33.21)
Approx.	(5.62, 37.73)	(0.76, 75.09)	(0*, 25.12)	(0.25, 24.31)
Boot-p	(10.65, 48.68)	(15.98, 132.89)	(2.08, 39.21)	(4.21, 32.88)
BC_α	(10.64, 48.13)	(15.88, 130.29)	(2.48, 40.51)	(4.20, 32.52)
99% C.I.				
Method	θ_1	θ_2	θ_1'	θ_2'
Exact	(8.15, 77.45)	(13.71, 168.33)	(1.72, 58.17)	(2.79, 47.67)
Approx.	(0.57, 42.78)	(0*, 86.79)	(0*, 29.31)	(0*, 28.09)
Boot-p	(8.37, 70.65)	(13.39, 196.47)	(0.31, 62.51)	(2.31, 45.95)
BC_α	(8.37, 70.18)	(13.25, 195.85)	(0.27, 54.47)	(2.17, 43.98)

0* stands for a non-positive number

Table 4.12: Theoretical values of relative risks with $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$

n	d	π_1	π_2	π_3	π_4
40	30	0.5556	0.5215	0.6939	0.7412
	25	0.5556	0.5097	0.6661	0.6871
	20	0.5556	0.4992	0.6445	0.6312
	15	0.5556	0.4894	0.6256	0.5672
20	16	0.5556	0.5274	0.7043	0.7555
	14	0.5556	0.5176	0.6803	0.7126
	12	0.5556	0.5083	0.6609	0.6697
	10	0.5556	0.5006	0.6437	0.6237
10	8	0.5556	0.5214	0.6832	0.7123
	7	0.5556	0.5132	0.6654	0.6738
	6	0.5556	0.5057	0.6499	0.6336
	5	0.5556	0.4985	0.6361	0.5899

Table 4.13: Relative risks based on 999 simulations with $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$

n	d	n_1	n_2	$\frac{n_1}{n}$	d_1	d_2	$\frac{d_1}{d}$	d_1'	d_2'	$\frac{d_1'}{d'}$	$\frac{d'}{n-d}$
40	30	22.2242	17.7758	0.5556	15.6016	14.3984	0.5201	5.1762	2.2793	0.6943	0.7455
	25	22.2643	17.7358	0.5566	12.7397	12.2603	0.5096	6.8759	3.4394	0.6666	0.6877
	20	22.1331	17.8669	0.5533	9.9199	10.0801	0.4960	8.0571	4.4735	0.6430	0.6265
	15	22.2593	17.7407	0.5565	7.3193	7.6807	0.4880	8.9159	5.3133	0.6266	0.5692
20	16	11.0771	8.9229	0.5539	8.3914	7.6086	0.5245	2.1381	0.8769	0.7092	0.7538
	14	11.0931	8.9069	0.5466	7.1522	6.8478	0.5109	2.9219	1.3003	0.6920	0.7037
	12	11.0320	8.9680	0.5516	6.0440	5.9560	0.5037	3.5516	1.8308	0.6599	0.6728
	10	11.1972	8.8028	0.5599	5.0851	4.9149	0.5085	4.0150	2.1722	0.6489	0.6187
10	8	5.5285	4.4715	0.5529	4.1932	3.8068	0.5242	1.0440	0.4434	0.7018	0.7438
	7	5.6466	4.3534	0.5647	3.6967	3.3033	0.5281	1.4515	0.6386	0.6945	0.6970
	6	5.5335	4.4665	0.5534	3.0280	2.9720	0.5047	1.7287	0.8829	0.6619	0.6529
	5	5.5155	4.4845	0.5516	2.5325	2.4675	0.5065	1.8959	1.1391	0.6247	0.6070

Chapter 5

Exact Analysis under Type-I Censoring

5.1 Introduction

In this Chapter, we consider a two-component system failure model in the case of Type-I censored data. We then present an iterative maximum likelihood estimation procedure to determine MLEs of the parameters assuming the lifetimes to be exponentially distributed. The asymptotic distributions of the MLEs are also obtained. Construction of confidence intervals for the model parameters are discussed by using the asymptotic distributions and two parametric bootstrap methods. The bias and variance of the estimates as well as the performance of the three confidence intervals in terms of coverage probabilities are assessed through a Monte Carlo simulation study. Finally, examples are presented to illustrate all the methods of inference discussed here.

5.2 Model Description and MLEs

Consider the following simple system failure model: n identical units are placed on a life test and each system has two components. Assume that the experiment continues up to a pre-fixed time W. Before the time W there are D observed failed systems. Failures that occur after W are not observed. The termination point W of the experiment is assumed to be independent of the failure times. Among the D failed systems, there are D_1 systems in which Component 1 failed first and D_2 systems in which Component 2 failed first, with $D_1 + D_2 = D$. We assume that X_i and Y_i $(i = 1, \ldots, n)$ are random variables representing the lifetimes of Components 1 and 2, respectively, in the i-th system. Let $Z_i = \max(X_i, Y_i)$ $(i = 1, \ldots, n)$. Thus, the i-th system fails at time Z_i, and $Z_{1:n} < \cdots < Z_{D:n}$ are the corresponding ordered failure times obtained from a Type-I censored sample from the n systems under test. The data arising from such a two-component system is as follows:

$$(T_1,\ \delta_1';\ \ Z_{1:n},\ \delta_1''), \ldots, (T_D,\ \delta_D';\ \ Z_{D:n},\ \delta_D''), (*, *), \ldots, \tag{5.1}$$

where $T_1, \ldots, T_D$ denote the first observed failure times in the systems, $Z_{1:n} < \cdots < Z_{D:n}$ denote the final observed failure times of the systems, δ' denotes the component of the first observed failure within the system, and δ'' denotes the component of the second observed failure within the system. We use " $*$ " to denote the censored data.

If we let

$$I_1 = \{i \in (1, 2, \ldots, D) : \ \text{Component 1 failed first}\},$$

$$I_2 = \{i \in (1, 2, \ldots, D) : \ \text{Component 2 failed first}\},$$

the likelihood function of the observed data in (5.1) is

$$
\begin{aligned}
&L(\theta_1, \theta_2, \theta_1', \theta_2') \\
&= \frac{n!}{(n-D)!} \prod_{i=1}^{D} f(x_i, y_i) \prod_{i=D+1}^{n} Pr\left(max(X_i, Y_i) \geq W\right) \\
&= \frac{n!}{(n-D)!} \times \left(\frac{1}{\theta_1 \theta_2'}\right)^{D_1} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) \sum_{i \in I_1} x_i - \frac{1}{\theta_2'} \sum_{i \in I_1} z_i \right\} \\
&\times \left(\frac{1}{\theta_2 \theta_1'}\right)^{D_2} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) \sum_{i \in I_2} y_i - \frac{1}{\theta_1'} \sum_{i \in I_2} z_i \right\} \times P_W^{n-D}, \qquad (5.2)
\end{aligned}
$$

where $0 < x_i < z_{i:n}$ for $i \in I_1$, $0 < y_i < z_{i:n}$ for $i \in I_2$, $0 < z_{1:n} < \cdots < z_{D:n} < W$, and

$$
P_W = Pr\left(max(X_i, Y_i) \geq W\right) = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right)^{-1} \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right)^{-1} \times \tilde{P}, \qquad (5.3)
$$

where

$$
\begin{aligned}
\tilde{P} = &\frac{1}{\theta_1}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) \exp\left(-\frac{1}{\theta_2'} W\right) + \frac{1}{\theta_2}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) \exp\left(-\frac{1}{\theta_1'} W\right) \\
&- \left(\frac{1}{\theta_1 \theta_2'} - \frac{1}{\theta_1' \theta_2'} + \frac{1}{\theta_2 \theta_1'}\right) \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) W\right\}. \qquad (5.4)
\end{aligned}
$$

The exact derivation of P_W is presented later in Lemma 5.2.2.

The maximum likelihood estimate ($\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$, $\hat{\theta}_2'$) of (θ_1, θ_2, θ_1', θ_2') is the value that globally maximizes the likelihood function in (5.2). Taking logarithm in Eq. (5.2), we obtain the log-likelihood function to be

$$
\begin{aligned}
\ln L = &-d_1 \ln \theta_1 - d_1 \ln \theta_2' - \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) \sum_{i \in I_1} x_i - \frac{1}{\theta_2'} \sum_{i \in I_1} z_i \\
&- d_2 \ln \theta_2 - d_2 \ln \theta_1' - \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) \sum_{i \in I_2} y_i - \frac{1}{\theta_1'} \sum_{i \in I_2} z_i \\
&- (n-D) \ln\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) - (n-D) \ln\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) + (n-D) \ln \tilde{P}. \qquad (5.5)
\end{aligned}
$$

Unfortunately, from (5.5), we observe that no closed-form for the MLEs as a function of the sample exist. We need to determine the MLEs by numerically maximizing the log-likelihood function in (5.5). Once again, an iterative procedure for the computation of the MLEs is needed. This procedure is the same as the one described in Section 3.2.1., except for changing the initial values to be as

$$\theta_1^{(0)} = \frac{\sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i}{D_1}, \qquad \theta_2^{(0)} = \frac{\sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i}{D_2},$$
$$\theta_1'^{(0)} = \frac{\sum_{i\in I_2}(z_i - y_i)}{D_2}, \qquad \theta_2'^{(0)} = \frac{\sum_{i\in I_1}(z_i - x_i)}{D_1}.$$

Note that these estimates do not use all the information available in the sample, but they do provide good starting values, for the iterative procedure.

5.2.1 Relative Risks

Based on the results of Section 3.2.2 and Section 4.2.1, in this subsection, one additional relative risk is derived and is presented in Theorem 5.2.1.

Lemma 5.2.1. *We have*

$$\begin{aligned} P_5 = Pr(X < Y < W) = \\ \frac{1}{\theta_1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)^{-1}\left[1-\frac{1}{\theta_2'}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)^{-1}\right] - \frac{1}{\theta_1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)^{-1} \\ \times\left\{\exp\left(-\frac{1}{\theta_2'}W\right) - \frac{1}{\theta_2'}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)^{-1}\exp\left[-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)W\right]\right\}. \end{aligned} \tag{5.6}$$

Proof: We can express

$$P_5 = Pr(X < Y < W) = \int_0^W \int_0^y \frac{1}{\theta_1\theta_2'}\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)x - \frac{1}{\theta_2'}y\right\}dxdy.$$

Then, the result follows by carrying out the required integration.

Lemma 5.2.2.

$$P_6 = Pr\left(\max(X,Y) \geq W\right) =$$
$$\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)^{-1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)^{-1}\left[\frac{1}{\theta_1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)\exp\left(-\frac{1}{\theta_2'}W\right)\right.$$
$$+\frac{1}{\theta_2}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)\exp\left(-\frac{1}{\theta_1'}W\right)$$
$$\left.-\left(\frac{1}{\theta_1\theta_2'}-\frac{1}{\theta_1'\theta_2'}+\frac{1}{\theta_2\theta_1'}\right)\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)W\right\}\right]. \tag{5.7}$$

Proof: We have

$$P_6 = Pr\left(\max(X,Y) \geq W\right) = 1 - Pr\left(max(X_i,Y_i) \leq W\right)$$
$$= 1 - \int_0^W \int_0^y \frac{1}{\theta_1\theta_2'}\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_2'}\right)x - \frac{1}{\theta_2'}y\right\}dxdy$$
$$-\int_0^W \int_0^x \frac{1}{\theta_2\theta_1'}\exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta_1'}\right)y - \frac{1}{\theta_1'}x\right\}dydx. \tag{5.8}$$

Then, the result follows by carrying out the required integration and by using the identity in Lemma 3.2.1.

Theorem 5.2.1. *The relative risk that Component 1 fails first within a two-component system, under the condition that the system fails by time W, is*

$$\pi_5 = Pr(X < Y|\max(X,Y) \leq W) = \frac{P_5}{1-P_6}.$$

Proof: The proof is straightforward.

Lemma 5.2.3. *In a Type-I censored experiment, there are D ($D \leq n$) systems with complete destruction observed before the pre-fixed time. Among the D systems, the* *number of failures due to Component 1 failing first, viz.,* D_1*, is a non-negative random variable with probability mass function given by*

$$Pr(D_1 = j) = \binom{D}{j}\left(\frac{P_5}{1-P_6}\right)^j\left(1-\frac{P_5}{1-P_6}\right)^{D-j}, \quad j = 0, 1, \ldots, D.$$

Proof: The result follows immediately from Theorem 5.2.1.

5.3 Confidence Intervals

In this section, we present two different methods of constructing confidence intervals (CIs) for the unknown parameters θ_1, θ_2, θ_1' and θ_2'. First, we use the asymptotic distributions of the MLEs to obtain approximate CIs for the parameters in case of large sample sizes. Next, we use the parametric bootstrap method to construct CIs for the parameters.

5.3.1 Approximate Confidence Intervals

In the last section, we noted that closed-form expressions for the MLEs do not exist. However, we can use the asymptotic normality of the MLEs to construct approximate confidence intervals for the parameters. The computation of the approximate confidence intervals is based on the observed Fisher information matrix, by taking negative of the second derivatives of the log-likelihood function in (5.5) and then evaluating them at the MLEs. Specifically, we have

$$I_{\text{obs}} = - \begin{pmatrix} \frac{\partial^2 \ln L}{\partial \theta_1^2} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_1'} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2'} \\ \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_2^2} & \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_1'} & \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_2'} \\ \frac{\partial^2 \ln L}{\partial \theta_1' \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_1' \partial \theta_2} & \frac{\partial^2 \ln L}{\partial \theta_1'^2} & \frac{\partial^2 \ln L}{\partial \theta_1' \partial \theta_2'} \\ \frac{\partial^2 \ln L}{\partial \theta_2' \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_2' \partial \theta_2} & \frac{\partial^2 \ln L}{\partial \theta_2' \partial \theta_1'} & \frac{\partial^2 \ln L}{\partial \theta_2'^2} \end{pmatrix}_{\theta_1=\hat{\theta}_1, \theta_2=\hat{\theta}_2, \theta_1'=\hat{\theta}_1', \theta_2'=\hat{\theta}_2'}, \tag{5.9}$$

and the inverse of this observed Fisher information matrix in (5.9) gives an estimate of the variance-covariance matrix of the MLEs, which in turn can be used to construct approximate confidence intervals for the parameters. We shall make use of the asymptotic normality of the MLEs to obtain these confidence intervals.

Thus, if

$$V(\theta_1, \theta_2, \theta_1', \theta_2') = I_{\text{obs}}^{-1} = (V_{ij}(\theta_1, \theta_2, \theta_1', \theta_2')), \quad i, j = 1, 2, 3, 4,$$

is the variance-covariance matrix, the $100(1-\alpha)\%$ confidence intervals for θ_1, θ_2, θ_1', θ_2' are given by

$$\begin{aligned} &\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{11}}, \\ &\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{22}}, \\ &\hat{\theta}_1' \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{33}}, \\ &\hat{\theta}_2' \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{44}}, \end{aligned} \tag{5.10}$$

where z_q is the q-th upper percentile of the standard normal distribution. This method may work satisfactorily when n is large, but may not be satisfactory for small sample sizes.

5.3.2 Bootstrap Confidence Intervals

The bootstrap methods of percentile interval and the biased-corrected and accelerated (BC_α) interval are similar to those described in Section 2.4.3, but with a Type-I censored two-component system failure sample instead. The acceleration $\hat{\alpha}_i^{(')}$ in the BC_α interval should be changed to

$$\hat{\alpha}_i^{(')} = \frac{\sum_{j=1}^{D}\left(\hat{\theta}_{i(\cdot)}^{(')} - \hat{\theta}_{i(j)}^{(')}\right)^3}{6\left[\sum_{j=1}^{D}\left(\hat{\theta}_{i(\cdot)}^{(')} - \hat{\theta}_{i(j)}^{(')}\right)^2\right]^{\frac{3}{2}}}, \quad i = 1, 2,$$

where $\hat{\theta}_{i(j)}^{(')}$ is the MLE of $\theta_i^{(')}$ based on the original sample with the j-th observation deleted, $j = 1, 2, \ldots, D$, and $\hat{\theta}_{i(\cdot)}^{(')} = \frac{\sum_{j=1}^{D}\hat{\theta}_{i(j)}^{(')}}{D}$.

5.4 Simulation Study

In this section, a Monte Carlo simulation study based on 999 replications was carried out to examine the bias, variance and relative risks (Section 5.4.1), and to evaluate the performance of the three confidence intervals in terms of coverage probabilities for different sample sizes (Section 5.4.2). We also present numerical examples in Section 5.4.3 to illustrate all the inferential methods discussed here.

5.4.1 Bias, Variance and MSE of the MLEs

It is desirable to examine the bias and variance of the MLEs as they are not explicit estimators. For this purpose, we carried out a simulation study to evaluate the bias, mean squared error (MSE), means and variances of the MLEs, and also the average of the asymptotic variance of the estimators computed from the observed information matrix. These results for $n = 40, 20, 10$ and $W = 40, 35, 30, 25, 20, 15, 10$ for each choice of n are presented in Tables 5.1-5.3.

We observe from Tables 5.1-5.3 that, as sample size n increases, the bias of MLEs decrease, with the bias tending to zero as n becomes large. Similarly, for the same sample size n, as the pre-fixed experimental time W decreases, the bias of MLEs increase. Similar behavior is also observed in MSE of the MLEs. As the sample size n and pre-fixed experimental time W increase, the MSE of MLEs are nearly identical to their corresponding variances. This indicates that MLEs are unbiased estimators for large sample with long experimental time.

The MSE of an estimator is one of many ways to quantify the amount by which an estimator differs from the true value of the quantity being estimated. In our case, as n or W decreases, big difference occurs between the true value and the estimator. The primary reason causing the big difference is the fewer failures that are observed in a small sample with short experiment time. This is so because when n or W is small, there will be fewer failures observed and so inference for θ_1, θ_2, θ_1' and θ_2' is not quite precise.

We observe a negative bias for the parameter θ_2' for most cases. This implies the underestimation of θ_2'.

We also determined the means and variances of the estimates of the parameters over 999 simulations. We observe that, for large sample sizes, the means of the MLEs of the parameters θ_1, θ_2, θ_1' and θ_2' are quite close to the true values, viz., 20, 25, 9, 14, respectively. However, this is not true for smaller sample sizes. The variances of the MLEs can also be compared with the average approximate variance computed from the observed information. Once again, the variance and the average approximate variance are closer for large values of n and W, but not so for smaller sample sizes. The reason for this is the same as the one given earlier for the bias and MSE of the MLEs.

The behaviors of relative risks π_1 and π_5 are checked through the average of 999 replications. The results are presented in Table 5.4. The values of π_1 and π_5 are computed from the expressions in Theorems 2.3.1 and 5.2.1, respectively. From the expression we find that π_1 is fixed for all sample sizes n and length of experiment time W; π_5 is only affected by the length of experiment time W. $\pi_1 = 0.5556$ implies that the first failure occurring within a specified system is more likely due to Component 1, with the true parameters $\theta_1 = 20$ and $\theta_2 = 25$. π_5 increases as the experiment time W becomes longer. $\pi_5 = 0.5$ when $W = 20$. This implies, if the pre-fixed termination time is 20, Component 1 and Component 2 have the equal chance to fail first within a system with complete destruction. However, when the experiment time W is longer than 20, Component 1 is more likely to fail first; otherwise, Component 2 is more likely to fail first. $\pi_5 < \pi_1$ in all the cases. This implies that the probability that Component 1 fails first within a system is weakened on the condition that the system has a complete destruction in a Type-I censoring test.

We observe that as W increases, the value of $\frac{n_1}{n}$ gets closer to π_1; the value of $\frac{D_1}{D}$ gets closer to π_5. This indicates that $\frac{n_1}{n}$ and $\frac{D_1}{D}$ are good estimators of π_1 and π_5, respectively.

5.4.2 Coverage Probabilities and the Performance of the Confidence Intervals

The purpose of this subsection is to carry out a Monte Carlo simulation study based on Type-I censored sample to compare the performance of different confidence intervals described in Section 5.3. We once again chose the values of the parameters to be $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$. We then determined the true coverage probabilities of the 90%, 95% and 99% confidence intervals for the parameters for different sample sizes by all three methods described earlier in Section 5.3. These values, based on 999 Monte Carlo simulations and $R = 999$ bootstrap replications, are presented in Tables 5.5 and 5.6.

From these tables, we observe that, among the three methods, the parametric BC_α bootstrap method of constructing confidence intervals has its coverage probabilities to be closer to the nominal level and is therefore recommended for large sample sizes.

As expected, the approximate method based on the asymptotic normality of the MLEs has its true coverage probabilities to be always less than the nominal level. Though the coverage probability improves for larger sample sizes, we still find it to be unsatisfactory even for n as large as 40 particularly when the pre-fixed termination time is $W = 40$. This indicates that the confidence intervals obtained by this method will often be unduly narrower. We do observe that for all the nominal levels considered, the coverage probabilities of the approximate method are lower for small sample size n or short length of pre-fixed experiment time W in almost all cases. This is because, when n or W is small, there are fewer failures observed and so inference for the parameters is not precise. As n increases, the number of failures increases thus resulting in a better large-sample approximation for the distribution of MLEs. This means that we need a much larger sample size to use the asymptotic normality of the MLEs. We also observe that when n is small, even the parametric BC_α bootstrap method does not have satisfactory coverage probabilities, but is seen to be better than the approximate methods as well as percentile bootstrap method.

5.4.3 Illustrative Examples

In this subsection, we consider two data sets by using small and moderately large sample sizes to illustrate all the methods of inference developed in the preceding sections.

Data Set 1: $n = 35$

W	Failure Times				
15	(2.28,2; 2.49,1)	(2.12,2; 2.95,1)	(2.60,2; 3.12,1)	(3.08,1; 3.70,2)	(2.92,1; 4.87,2)
	(6.41,2; 6.41,1)	(2.76,1; 6.50,2)	(4.81,2; 6.54,1)	(0.19,1; 7.09,2)	(4.55,1; 8.49,2)
	(8.79,1; 9.40,2)	(4.68,1; 10.39,2)	(6.58,2; 11.15,1)	(8.67,1; 11.49,2)	(0.64,1; 12.90,2)
	(12.98,1; 13.21,2)	(2.10,2; 13.72,1)	(13.27,1; 14.76,2)	(12.91,1; 14.90,2)	(*, *)
25	(2.28,2; 2.49,1)	(2.12,2; 2.95,1)	(2.60,2; 3.12,1)	(3.08,1; 3.70,2)	(2.92,1; 4.87,2)
	(6.41,2; 6.41,1)	(2.76,1; 6.50,2)	(4.81,2; 6.54,1)	(0.19,1; 7.09,2)	(4.55,1; 8.49,2)
	(8.79,1; 9.40,2)	(4.68,1; 10.39,2)	(6.58,2; 11.15,1)	(8.67,1; 11.49,2)	(0.64,1; 12.90,2)
	(12.98,1; 13.21,2)	(2.10,2; 13.72,1)	(13.27,1; 14.76,2)	(12.91,1; 14.89,2)	(1.83,2; 15.91,1)
	(10.42,2; 20.67)	(20.19,2; 20.71,1)	(3.28,1; 22.45,2)	(19.64,2; 24.45,1)	(*, *)

In the case when $n = 35$, had we fixed $W = 15$, we would have $D_1 = 12$ and $D_2 = 7$, and we would obtain the MLEs of θ_1, θ_2, θ_1' and θ_2' by using the methods presented in Section 5.2 to be

$$\hat{\theta}_1 = 19.6243, \qquad \hat{\theta}_2 = 37.7553, \qquad \hat{\theta}_1' = 3.5774, \qquad \hat{\theta}_2' = 5.3832.$$

Instead, had we fixed $W = 25$, we would have $D_1 = 13$ and $D_2 = 11$, and we would obtain the MLEs to be

$$\hat{\theta}_1 = 25.1019, \quad \hat{\theta}_2 = 30.4837, \quad \hat{\theta}'_1 = 5.8473, \quad \hat{\theta}'_2 = 6.5068.$$

Data Set 2: $n = 15$

W	Failure Times				
15	(0.11,2; 1.67,1)	(3.93,1; 8.89,2)	(5.63,2; 9.26,1)	(7.28,1; 9.75,2)	(4.15,1; 12.19,2)
	(7.92,2; 12.99,1)	(10.25,1; 14.54,2)	(*, *)		
25	(0.11,2; 1.67,1)	(3.93,1; 8.89,2)	(5.63,2; 9.26,1)	(7.28,1; 9.75,2)	(4.15,1; 12.19,2)
	(7.92,2; 12.99,1)	(10.25,1; 14.55,2)	(7.51,1; 17.70,2)	(13.33,1; 18.62,2)	(9.94,2; 21.14,1)
	(*, *)				

In the case when $n = 15$, had we fixed $W = 15$, we would have $D_1 = 4$ and $D_2 = 3$, and we would obtain the MLEs of θ_1, θ_2, θ'_1 and θ'_2 by using the methods presented in Section 5.2 to be

$$\hat{\theta}_1 = 19.8219, \quad \hat{\theta}_2 = 35.2041, \quad \hat{\theta}'_1 = 4.8815, \quad \hat{\theta}'_2 = 9.9186.$$

Instead, had we fixed $W = 25$, we would have $D_1 = 6$ and $D_2 = 4$, and we would obtain the MLEs to be

$$\hat{\theta}_1 = 21.3098, \quad \hat{\theta}_2 = 34.0602, \quad \hat{\theta}'_1 = 7.5193, \quad \hat{\theta}'_2 = 9.1564.$$

To assess the performance of these estimates, we constructed 90%, 95% and 99% confidence intervals using the methods outlined in Section 5.3. The results are presented in Tables 5.7-5.10. Notice that the approximate method always provide narrower confidence intervals in most cases. This is because the coverage probabilities for the approximate method are significantly lower than the nominal levels.

Table 5.1: Bias, MSE, Mean and Variance based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 40$

W	Parameters	Bias	MSE	Mean	Variance	Approximate Variance
10	θ_1	7.51	346.14	27.51	289.74	359.74
	θ_2	3.22	337.95	28.22	327.59	417.34
	θ_1'	1.21	36.42	10.21	34.96	50.00
	θ_2'	-2.69	52.83	11.31	45.60	66.46
15	θ_1	4.70	192.14	24.70	170.05	198.77
	θ_2	2.53	229.73	27.53	223.32	294.96
	θ_1'	0.92	29.99	9.93	29.15	40.88
	θ_2'	-1.65	41.01	12.35	38.29	58.81
20	θ_1	3.37	107.73	23.37	96.49	109.61
	θ_2	2.16	118.65	27.16	114.11	130.27
	θ_1'	0.71	22.02	9.71	21.66	25.17
	θ_2'	-0.95	34.41	13.05	33.52	50.25
25	θ_1	2.25	89.70	22.25	84.75	92.65
	θ_2	1.55	95.18	26.55	92.86	102.06
	θ_1'	0.59	18.02	9.59	17.69	19.50
	θ_2'	-0.43	29.79	13.57	29.63	40.06
30	θ_1	1.42	65.00	21.42	63.06	69.66
	θ_2	1.20	89.54	26.20	88.19	91.15
	θ_1'	0.50	16.11	9.50	15.92	16.71
	θ_2'	-0.04	27.60	13.96	27.62	35.59
35	θ_1	0.91	41.91	20.91	41.12	42.89
	θ_2	1.08	67.57	26.08	66.24	67.34
	θ_1'	0.46	14.49	9.46	14.16	15.72
	θ_2'	-0.04	26.38	13.96	26.40	29.02
40	θ_1	0.66	35.42	20.66	35.14	37.88
	θ_2	0.62	61.40	25.62	61.30	62.31
	θ_1'	0.29	11.01	9.29	10.94	11.70
	θ_2'	0.18	21.04	14.18	21.03	22.20

Table 5.2: Bias, MSE, Mean and Variance based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 20$

W	Parameters	Bias	MSE	Mean	Variance	Approximate Variance
10	θ_1	10.25	713.85	30.25	608.17	793.94
	θ_2	9.62	791.88	34.62	698.65	833.70
	θ_1'	2.01	53.78	11.01	49.79	71.39
	θ_2'	-3.24	102.32	10.76	91.84	198.66
15	θ_1	8.35	547.67	28.35	477.57	615.03
	θ_2	7.29	592.82	32.29	539.51	671.73
	θ_1'	1.71	43.33	10.71	40.26	59.98
	θ_2'	-2.27	77.45	11.73	72.35	97.16
20	θ_1	6.34	396.39	26.34	356.62	498.48
	θ_2	5.02	423.17	30.02	398.65	540.16
	θ_1'	1.25	35.51	10.25	33.99	45.20
	θ_2'	-1.35	58.70	12.65	56.92	74.92
25	θ_1	4.90	264.33	24.90	240.32	359.67
	θ_2	4.23	323.00	29.23	305.53	421.80
	θ_1'	0.81	32.19	9.81	31.37	40.22
	θ_2'	-0.58	50.82	13.42	50.52	60.23
30	θ_1	3.71	164.82	23.71	151.21	267.95
	θ_2	3.27	248.720	28.27	238.24	356.72
	θ_1'	0.72	31.40	9.72	31.09	34.16
	θ_2'	-0.48	48.45	13.52	48.26	56.52
35	θ_1	2.26	115.68	22.26	110.69	145.60
	θ_2	1.92	189.84	26.92	186.26	289.90
	θ_1'	0.69	29.53	9.69	29.05	30.39
	θ_2'	-0.24	47.86	14.24	47.85	53.33
40	θ_1	1.89	104.42	21.69	100.68	114.21
	θ_2	1.80	154.76	26.80	151.75	246.82
	θ_1'	0.56	21.71	9.56	21.42	26.06
	θ_2'	0.23	46.60	14.43	46.46	48.28

Table 5.3: Bias, MSE, Mean and Variance based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 10$

W	Parameters	Bias	MSE	Mean	Variance	Approximate Variance
10	θ_1	13.75	981.41	33.75	792.74	1067.97
	θ_2	11.77	1059.98	36.77	920.71	1261.65
	θ_1'	2.79	127.94	11.79	119.35	121.25
	θ_2'	-4.51	154.82	9.49	134.50	179.71
15	θ_1	10.85	791.62	30.85	673.49	863.33
	θ_2	9.83	917.71	34.83	820.65	1049.24
	θ_1'	2.16	86.79	11.16	81.81	102.48
	θ_2'	-3.30	111.90	10.70	101.03	133.08
20	θ_1	9.54	680.47	29.54	589.17	696.02
	θ_2	8.01	820.25	33.01	756.81	882.81
	θ_1'	1.73	72.66	10.73	69.19	91.01
	θ_2'	-1.80	93.32	12.20	90.16	126.41
25	θ_1	8.80	558.09	28.80	481.12	512.70
	θ_2	7.35	674.07	32.35	620.65	725.00
	θ_1'	1.07	60.02	10.07	59.15	82.77
	θ_2'	-1.57	87.23	12.43	84.82	101.30
30	θ_1	7.85	419.91	27.85	357.88	398.57
	θ_2	5.89	551.34	30.89	517.18	594.81
	θ_1'	0.97	51.51	9.97	56.57	74.81
	θ_2'	-0.73	83.90	13.87	82.96	90.05
35	θ_1	5.04	339.55	25.04	314.59	324.44
	θ_2	4.99	471.92	29.99	446.92	504.21
	θ_1'	0.76	53.97	9.76	52.79	68.23
	θ_2'	0.22	80.46	13.78	80.54	84.19
40	θ_1	4.45	322.34	24.45	302.88	307.92
	θ_2	4.83	423.80	29.83	400.71	430.96
	θ_1'	0.70	51.10	9.70	50.66	65.04
	θ_2'	0.04	78.49	14.04	78.49	80.66

Table 5.4: Relative risks based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$

n	W	n_1	n_2	$\frac{n_1}{n}$	π_1	D_1	D_2	$\frac{D_1}{D}$	π_5
40	40	22.36	17.64	0.56	0.56	18.55	15.97	0.54	0.53
	35	22.33	17.67	0.56	0.56	17.19	15.25	0.53	0.53
	30	22.10	17.90	0.55	0.56	15.18	14.31	0.51	0.52
	25	22.27	17.73	0.56	0.56	13.26	12.61	0.51	0.51
	20	22.38	17.62	0.56	0.56	10.71	10.44	0.51	0.50
	15	22.36	17.64	0.56	0.56	7.57	7.79	0.49	0.49
	10	22.31	17.69	0.56	0.56	4.34	4.69	0.48	0.48
20	40	11.13	8.87	0.56	0.56	9.31	8.04	0.54	0.53
	35	11.14	8.86	0.56	0.56	8.53	7.64	0.53	0.53
	30	11.03	8.97	0.55	0.56	7.60	7.19	0.51	0.52
	25	11.06	8.94	0.55	0.56	6.53	6.38	0.51	0.51
	20	11.09	8.91	0.55	0.56	5.33	5.25	0.50	0.50
	15	11.28	8.72	0.56	0.56	3.74	3.84	0.49	0.49
	10	11.16	8.84	0.56	0.56	2.32	2.50	0.48	0.48
10	40	5.56	4.44	0.56	0.56	4.59	4.02	0.53	0.53
	35	5.47	4.53	0.55	0.56	4.22	3.91	0.52	0.53
	30	5.51	4.49	0.55	0.56	3.85	3.60	0.52	0.52
	25	5.58	4.42	0.56	0.56	3.35	3.17	0.51	0.51
	20	5.55	4.45	0.56	0.56	2.68	2.68	0.50	0.50
	15	5.48	4.52	0.55	0.56	2.09	2.17	0.49	0.49
	10	5.45	4.55	0.55	0.56	1.50	1.60	0.48	0.48

Table 5.5: Estimated coverage probabilities based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 40$

C.I. of θ_1	90% C.I.			95% C.I.			99% C.I.		
W	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	81.49	89.69	90.20	88.10	94.69	95.89	90.49	98.69	99.20
20	83.10	88.30	89.50	89.69	94.10	95.40	93.69	98.20	99.30
30	86.99	89.19	89.69	91.49	93.79	94.89	95.10	98.59	98.80
40	88.19	90.09	90.19	92.60	94.39	95.00	96.20	98.89	99.10
C.I. of θ_2	90% C.I.			95% C.I.			99% C.I.		
W	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	81.88	89.09	91.19	87.80	94.49	95.99	89.70	98.89	99.60
20	84.09	88.59	90.29	89.20	94.29	95.49	93.10	98.80	99.10
30	86.20	88.70	89.10	91.10	93.89	94.59	94.80	98.60	99.00
40	87.39	88.69	89.30	92.09	94.89	95.69	95.90	98.39	98.80
C.I. of θ_1'	90% C.I.			95% C.I.			99% C.I.		
W	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	81.20	89.39	91.29	87.40	93.69	94.79	89.80	96.90	97.90
20	83.89	87.89	89.59	88.10	95.19	96.39	92.89	97.50	98.10
30	85.99	88.99	90.30	90.79	94.09	95.70	94.20	98.39	98.60
40	87.20	89.29	91.19	91.89	95.29	96.10	94.39	98.10	98.70
C.I. of θ_2'	90% C.I.			95% C.I.			99% C.I.		
W	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	82.19	89.10	90.19	87.90	91.79	93.39	89.80	96.70	97.89
20	83.30	88.49	90.09	88.70	93.49	94.89	92.70	97.10	98.09
30	86.70	88.49	89.70	90.89	93.19	94.50	94.30	98.10	99.20
40	87.89	89.89	90.39	92.10	94.89	95.20	95.89	97.80	98.89

Table 5.6: Estimated coverage probabilities based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 20$

C.I. of θ_1	90% C.I.			95% C.I.			99% C.I.		
W	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	80.89	86.20	87.69	85.19	92.20	92.79	89.89	97.05	97.70
20	82.30	88.39	90.20	87.10	94.49	95.55	91.10	98.20	98.49
30	83.10	89.10	90.69	89.29	94.59	95.69	93.20	98.20	98.89
40	87.10	88.19	90.10	90.89	94.69	95.20	95.10	98.30	99.20
C.I. of θ_2	90% C.I.			95% C.I.			99% C.I.		
W	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	78.49	89.40	91.19	84.10	93.29	93.79	88.39	7.10	97.30
20	80.79	89.40	91.30	86.89	94.59	95.80	90.69	98.10	98.49
30	81.80	89.79	91.80	89.70	94.69	96.30	92.70	98.30	98.60
40	85.70	89.19	90.79	90.20	94.19	95.29	94.30	98.50	98.89
C.I. of θ_1'	90% C.I.			95% C.I.			99% C.I.		
W	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	77.39	85.28	86.89	81.39	88.39	90.50	86.49	93.69	94.09
20	78.89	87.79	90.10	85.10	90.79	93.09	88.70	94.80	95.39
30	80.19	87.10	89.90	85.79	91.90	94.29	91.30	96.60	97.40
40	82.30	87.89	90.59	86.19	92.99	95.09	93.60	96.20	97.20
C.I. of θ_2'	90% C.I.			95% C.I.			99% C.I.		
W	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α	Approx.	Boot-p	BC_α
10	79.39	86.69	87.19	82.39	90.69	91.79	85.19	94.79	95.19
20	79.88	86.99	88.69	84.99	90.79	92.20	87.20	93.79	95.09
30	81.79	87.39	88.10	86.39	91.79	94.59	90.89	95.89	96.50
40	84.20	88.29	90.40	88.20	91.80	93.389	92.89	96.60	97.40

Table 5.7: Confidence intervals for θ_1, θ_2, θ'_1 and θ'_2 when $n = 35$, $W = 15$

C.I. for θ_1			
Method	90%	95%	99%
Approx C.I.	(8.01, 31.24)	(5.78, 33.47)	(1.43, 37.82)
Boot-p C.I.	(6.32, 32.61)	(5.50, 37.60)	(4.29, 47.78)
BC_α C.I.	(8.38, 39.49)	(6.62, 42.38)	(4.93, 47.92)

C.I. for θ_2			
Method	90%	95%	99%
Approx C.I.	(12.55, 62.96)	(7.72, 67.79)	(0*, 77.23)
Boot-p C.I.	(10.06, 78.90)	(8.64, 95.92)	(5.80, 132.60)
BC_α C.I.	(14.81, 99.59)	(10.64, 111.90)	(7.08, 139.44)

C.I. for θ'_1			
Method	90%	95%	99%
Approx C.I.	(0.48, 6.67)	(0*, 7.26)	(0*, 8.42)
Boot-p C.I.	(1.22, 10.70)	(1.02, 15.50)	(0.45, 33.03)
BC_α C.I.	(1.71, 21.15)	(1.44, 33.03)	(1.05, 43.26)

C.I. for θ'_2			
Method	90%	95%	99%
Approx C.I.	(1.23, 9.53)	(0.44, 10.33)	(0*, 11.88)
Boot-p C.I.	(2.29, 12.52)	(2.10, 15.78)	(1.53, 26.21)
BC_α C.I.	(2.83, 19.97)	(2.50, 26.01)	(2.10, 31.16)

0* stands for a non-positive number

Table 5.8: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' when $n = 35$, $W = 25$

C.I. for θ_1			
Method	90%	95%	99%
Approx C.I.	(11.59, 38.61)	(9.00, 41.20)	(3.95, 46.26)
Boot-p C.I.	(13.49, 43.09)	(11.52, 48.03)	(8.35, 66.01)
BC_α C.I.	(15.22, 46.31)	(13.06, 51.13)	(8.92, 66.01)

C.I. for θ_2			
Method	90%	95%	99%
Approx C.I.	(13.59, 47.38)	(10.35, 50.62)	(4.03, 56.94)
Boot-p C.I.	(16.05, 52.63)	(13.59, 61.15)	(8.95, 84.38)
BC_α C.I.	(17.12, 55.05)	(14.55, 62.74)	(8.95, 82.59)

C.I. for θ_1'			
Method	90%	95%	99%
Approx C.I.	(1.55, 10.15)	(0.72, 10.97)	(0*, 12.58)
Boot-p C.I.	(2.95, 12.97)	(2.39, 16.83)	(1.96, 26.56)
BC_α C.I.	(3.18, 15.94)	(2.84, 19.19)	(2.16, 31.55)

C.I. for θ_2'			
Method	90%	95%	99%
Approx C.I.	(1.82, 11.20)	(0.92, 12.10)	(0*, 13.85)
Boot-p C.I.	(3.45, 12.62)	(3.19, 15.14)	(2.58, 20.62)
BC_α C.I.	(3.94, 15.87)	(3.53, 19.50)	(3.14, 23.75)

0* stands for a non-positive number

Table 5.9: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' when $n = 15$, $W = 15$

C.I. for θ_1			
Method	90%	95%	99%
Approx C.I.	(0*, 44.08)	(0*, 48.72)	(0*, 57.81)
Boot-pC.I.	(4.16, 60.72)	(3.67, 72.11)	(2.80, 134.65)
BC_α C.I.	(6.49, 87.37)	(5.25, 132.06)	(3.75, 143.93)
C.I. for θ_2			
Method	90%	95%	99%
Approx C.I.	(0*, 73.93)	(0*, 81.35)	(0*, 95.85)
Boot-p C.I.	(5.23, 79.81)	(4.32, 103.05)	(2.70, 138.73)
BC_α C.I.	(15.86, 146.44)	(12.12, 146.44)	(7.52, 146.44)
C.I. for θ_1'			
Method	90%	95%	99%
Approx C.I.	(0*, 12.06)	(0*, 13.44)	(0*, 16.12)
Boot-p C.I.	(0.81, 23.15)	(0.53, 28.96)	(0.19, 64.40)
BC_α C.I.	(1.24, 28.96)	(0.84, 38.68)	(0.28, 71.03)
C.I. for θ_2'			
Method	90%	95%	99%
Approx C.I.	(0*, 24.06)	(0*, 26.77)	(0*, 32.06)
Boot-p C.I.	(1.26, 25.21)	(0.87, 33.52)	(0.25, 47.14)
BC_α C.I.	(3.28, 46.80)	(2.53, 61.58)	(1.33, 81.85)

0* stands for a non-positive number

Table 5.10: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' when $n = 15$, $W = 25$

C.I. for θ_1			
Method	90%	95%	99%
Approx C.I.	(2.26, 40.36)	(0*, 44.01)	(0*, 51.15)
Boot-p C.I.	(7.62, 48.08)	(6.42, 62.07)	(4.23, 102.36)
BC_α C.I.	(8.95, 61.91)	(7.66, 69.76)	(5.07, 105.49)

C.I. for θ_2			
Method	90%	95%	99%
Approx C.I.	(1.16, 66.96)	(0*, 73.27)	(0*, 85.59)
Boot-p C.I.	(10.72, 84.26)	(7.99, 97.84)	(5.60, 139.78)
BC_α C.I.	(14.30, 103.11)	(11.49, 120.44)	(6.89, 143.95)

C.I. for θ_1'			
Method	90%	95%	99%
Approx C.I.	(0*, 17.37)	(0*, 19.26)	(0*, 22.95)
Boot-p C.I.	(1.67, 24.84)	(0.97, 31.50)	(0.28, 67.70)
BC_α C.I.	(2.81, 41.66)	(2.11, 60.01)	(0.91, 85.88)

C.I. for θ_2'			
Method	90%	95%	99%
Approx C.I.	(0*, 19.54)	(0*, 21.53)	(0*, 25.41)
Boot-p C.I.	(2.84, 22.17)	(2.32, 26.46)	(0.96, 39.83)
BC_α C.I.	(3.97, 28.50)	(3.32, 36.20)	(2.25, 49.26)

0* stands for a non-positive number

Chapter 6

Exact Analysis under Type I Censoring with Partial Information on Component Failures

6.1 Introduction

In this Chapter, we consider a two-component system failure model in the case of Type-I censored data. The information of the censored systems which have only one component failed at the end of the experiment is incorporated as well. We then obtain the MLEs of the parameters assuming the lifetimes to be exponentially distributed. The exact distributions of the MLEs of the parameters are derived by using the conditional moment generating function approach. Construction of confidence intervals for the model parameters are discussed by using the exact conditional distributions, asymptotic distributions, and two parametric bootstrap methods. The performance of these four confidence intervals, in terms of coverage probabilities are assessed through Monte Carlo simulation studies. Finally, examples are presented to illustrate all the methods of inference discussed here.

6.2 Model Description and MLEs

Consider the following simple system failure model: n identical systems are placed in a life test and each system has two components. Assume that the experiment continues up to a pre-fixed time W. Before the time W, a total of D ($D \leq n$) systems fail. Let X_i and Y_i ($i = 1, \ldots, n$) represent the lifetimes of Components 1 and 2, respectively. If $Z_i = \max(X_i, Y_i)$ ($i = 1, \ldots, n$), the system i fails at time Z_i. Let $Z_{1:n} < \cdots < Z_{D:n} < W$ be the corresponding ordered failure times. By time W, we observe D systems with complete destruction, D' systems with only one failed component and $n - D - D'$ systems which are completely censored. Among the D systems, there are D_1 systems in which Component 1 failed first and D_2 systems in which Component 2 failed first, with $D_1 + D_2 = D$. Among the D' systems, there are D'_1 systems in which only Component 1 failed and D'_2 systems in which only Component 2 failed, with $D'_1 + D'_2 = D'$. The data from the two-component series system sample under Type-I censoring with partial information is as follows:

$$(T_1,\ \delta'_1;\ Z_{1:n},\ \delta''_1), \ldots, (T_D,\ \delta'_D;\ Z_{D:n},\ \delta''_D), (T_{D+1},\ \delta'_{D+1}; *), \ldots, (T_{D+D'},\ \delta'_{D+D'}; *), (*, *) \tag{6.1}$$

where $T_1, \ldots, T_D$ denote the first observed failure times of the systems, $Z_{1:n} < \cdots < Z_{D:n}$ denote the second observed failure times of the systems, and δ's are the indicator variables which denote the failed components, δ' stands for the first failed component, and δ'' stands for the second failed component. We denote the completely censored systems with a " $*$ ".

If we let

$$I_1 = \{i \in (1, 2, \ldots, D): \text{ Component 1 failed first within a failed system}\},$$

$$I_2 = \{i \in (1, 2, \ldots, D): \text{ Component 2 failed first within a failed system}\},$$

$$I_1' = \{i \in (1, 2, \ldots, D'): \text{ only Component 1 failed within a system}\},$$

$$I_2' = \{i \in (1, 2, \ldots, D'): \text{ only Component 2 failed within a system}\},$$

then the likelihood function of the observed data in (6.1) is

$$\begin{aligned}
L(\theta_1, \theta_2, \theta_1', \theta_2') = & \frac{(2n)!}{(2n - 2D - D')!} \\
& \times \left(\frac{1}{\theta_1 \theta_2'}\right)^{D_1} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) \sum_{i \in I_1} x_i - \frac{1}{\theta_2'} \sum_{i \in I_1} z_i\right\} \\
& \times \left(\frac{1}{\theta_2 \theta_1'}\right)^{D_2} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) \sum_{i \in I_2} y_i - \frac{1}{\theta_1'} \sum_{i \in I_2} z_i\right\} \\
& \times \left(\frac{1}{\theta_1}\right)^{D_1'} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_2'}\right) \sum_{i \in I_1'} x_i - \frac{1}{\theta_2'} D_1' W\right\} \\
& \times \left(\frac{1}{\theta_2}\right)^{D_2'} \exp\left\{-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\theta_1'}\right) \sum_{i \in I_2'} y_i - \frac{1}{\theta_1'} D_2' W\right\} \\
& \times \exp\left\{-(n - D - D')\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) W\right\},
\end{aligned} \tag{6.2}$$

where

$$0 < x_i < z_i, \quad \text{for } \; i \in I_1; \qquad 0 < y_i < z_i, \quad \text{for } \; i \in I_2;$$

$$0 < x_i < W, \quad \text{for } \; i \in I_1'; \quad 0 < y_i < W, \quad \text{for } \; i \in I_2'; \quad 0 < z_{1:D} < \cdots < z_{D:n} < W.$$

The maximum likelihood estimator $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}'_1, \hat{\theta}'_2)$ of θ_1, θ_2, θ'_1 and θ'_2 is the value that globally maximizes (6.2), and can be obtained by taking the logarithm of (6.2) and equating the partial derivatives to zero. After some calculation, the corresponding conditional maximum likelihood estimators of the mean life times θ_1, θ_2, θ'_1, θ'_2 are found to be as follows

$$\hat{\theta}_1 = \frac{\sum_{i\in I_1} x_i + \sum_{i\in I_2} y_i + \sum_{i\in I'_1} x_i + \sum_{i\in I'_2} y_i + (n-D-D')W}{D_1 + D'_1},$$

$$\hat{\theta}_2 = \frac{D_1 + D'_1}{(D-D_1)+(D'-D'_1)}\hat{\theta}_1,$$

$$\hat{\theta}'_1 = \frac{\sum_{i\in I_2}(z_i - y_i) + \sum_{i\in I'_2}(W - y_i)}{D - D_1},$$

$$\hat{\theta}'_2 = \frac{\sum_{i\in I_1}(z_i - x_i) + \sum_{i\in I'_1}(W - x_i)}{D_1},$$

conditional on $1 \le D_1 \le D-1$, $2 \le D \le n$, $0 \le D'_1 \le D'$, $0 \le D' \le n-D$.

6.2.1 The Relative Risks

Based on the results of Section 3.2.2, in this subsection, three additional relative risks are derived and are presented in Theorems 6.2.1, 6.2.2 and 6.2.3.

Lemma 6.2.1. *We have*

$$\begin{aligned} P_7 &= Pr(X < W < Y) \\ &= \frac{1}{\theta_1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}-\frac{1}{\theta'_2}\right)^{-1}\left[\exp\left(-\frac{1}{\theta'_2}W\right) - \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)W\right\}\right]. \end{aligned} \tag{6.3}$$

Proof: The proof is similar to that of Lemma 5.2.1.

Lemma 6.2.2. *We have*

$$P_8 = Pr(\min(X,Y) \geq W) = \exp\left\{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)W\right\}. \tag{6.4}$$

Proof: The proof is straightforward.

Theorem 6.2.1. *The relative risk that Component 1 fails first within a two-component system, under the condition that the system has only one failed component by time W, is given by*

$$\pi_6 = Pr(X < Y|\min(X,Y) \leq W, \max(X,Y) \geq W) = \frac{P_7}{P_6 - P_8}.$$

Proof: The result follows immediately from Lemmas 5.2.2, 6.2.1 and 6.2.2.

Theorem 6.2.2. *The relative risk that only one component fails within a two-component system, under the condition that the system does not fail by the time W, is*

$$\pi_7 = pr(\min(X,Y) \leq W|\max(X,Y) \geq W) = \frac{P_6 - P_8}{P_6}.$$

Proof: The result follows immediately from Lemmas 5.2.2 and 6.2.2.

Theorem 6.2.3. *The relative risk that the system fails by time W is then*

$$\pi_8 = Pr(\max(X,Y) \leq W) = 1 - P_6.$$

Proof: The result follows immediately from Lemma 5.2.2.

Lemma 6.2.3. *In a Type-I censored experiment, among the D' $(0 \le D' \le n - D)$ systems with only one failed component by W, the number of systems due to Component 1 fails first, viz., D'_1, is a non-negative random variable with probability mass function*

$$Pr(D'_1 = j) = \binom{D'}{j} \left(\frac{P_7}{P_6 - P_8}\right)^j \left(1 - \frac{P_7}{P_6 - P_8}\right)^{D'-j}, \quad j = 0, 1, \ldots, D'.$$

Proof: The proof is simply based on Theorem 6.2.1.

Lemma 6.2.4. *In a Type-I censored experiment, among the $n - D$ systems which do not fail by W, the number of systems in which only one component failed, viz., D', is a non-negative random variable with probability mass function*

$$Pr(D' = j) = \binom{n - D}{j} \left(1 - \frac{P_8}{P_6}\right)^j \left(\frac{P_8}{P_6}\right)^{n-D-j}, \quad j = 0, 1, \ldots, n - D.$$

Proof: The proof is based on Theorem 6.2.2.

Lemma 6.2.5. *In a Type-I censored experiment, the number of systems with complete destruction, viz., D, is a non-negative random variable with probability mass function*

$$Pr(D = j) = \binom{n}{j} (1 - P_6)^j \, P_6^{N-j}, \quad j = 0, 1, \ldots, n.$$

Proof: The proof is based on Theorem 6.2.3.

6.3 Conditional Distribution of the MLEs

Using the forms of the estimators given earlier in Section 6.2, we now derive the exact distribution of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$ conditional on $1 \le D_1 \le D - 1$, $D \ge 2$,

$0 \leq D_1' \leq D'$ and $0 \leq D' \leq n - D$. These distributions are useful in constructing exact confidence intervals. The derivations once again require the inversion of the conditional moment generating function (CMGF).

Lemma 6.3.1. *Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$ and CDF $F(x)$. Then, the conditional distribution of $X_{i:n}$, given that $X_{j:n} = x_j$ for $i < j$, is the same as the distribution of the* i*-th order statistic in a sample of size $j-1$ from a population whose distribution is obtained by truncating the distribution $F(x)$ on the right at x_j, with PDF $h(x_i) = \frac{f(x_i)}{F(x_j)}$ and CDF $H(x_i) = \frac{F(x_i)}{F(x_j)}$, for $0 < x_i < x_j$.*

Proof: See Arnold, Balakrishnan, and Nagaraja (1992, pp. 23-24).

Lemma 6.3.2. *Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$ and CDF $F(x)$. Let D denote the number of $X_{i:n}$'s that are less than or equal to some pre-fixed number W. The conditional joint PDF of $X_{1:n} < \cdots < X_{n:n}$, given $D = d$, is the same as the joint PDF of all order statistics of a random sample of size d from the right truncated distribution with PDF $h(x_i) = \frac{f(x_i)}{F(W)}$ and CDF $H(x_i) = \frac{F(x_i)}{F(W)}$, for $0 < x_i < W$, i.e.*

$$f_{1,2,\ldots,d:n}(x_1, \ldots, x_d | D = d) = d! \prod_{i=1}^{d} \frac{f(x_i)}{F(W)}, \qquad 0 \leq x_{1:n} < \ldots < x_{d:n} \leq W. \quad (6.5)$$

Proof: The proof is straightforward.

Lemma 6.3.3. *For $d \geq 2$, $\int_0^{x_d} \cdots \int_0^{x_2} \exp\left(-a \sum_{i=1}^{d-1} x_i\right) dx_1 \cdots dx_{d-1} = \frac{[1-\exp(-ax_d)]^{d-1}}{a^{d-1}(d-1)!}$.*

Proof: The first two steps immediately yield

$$\int_0^{x_2} \exp\left(-ax_1\right) dx_1 = \frac{[1-\exp(-ax_2)]}{a \times 1!},$$

$$\int_0^{x_3} \frac{1}{a}\left[1-\exp(-ax_2)\right] \exp\left(-ax_2\right) dx_2 = \frac{[1-\exp(-ax_2)]^2}{a^2 \times 2!};$$

repeating this procedure, we obtain the required expression.

Lemma 6.3.4. *Let $X_1, \ldots, X_n$ be a random sample of size n from the exponential distribution with the mean θ, and $X_{1:n} < \cdots < X_{n:n}$ be the corresponding order statistics. Let D denote the number of $X_{i:n}$'s that are less than or equal to some pre-fixed number W. Then the moment-generating function (MGF) of $Z = \sum_{i=1}^{D} X_i$, given $D = d$, is*

$$M_Z(t) = E\left(e^{tZ}|D=d\right) = \frac{\left\{1-\left[\exp\left(-\frac{W}{\theta}\right)\right]^{(1-t\theta)}\right\}^d}{\left\{\left[1-\exp\left(-\frac{W}{\theta}\right)\right]\left[1-t\theta\right]\right\}^d}. \tag{6.6}$$

Proof: Since $X_{i:n}$'s are ordered iid $\exp(\theta)$, by Lemma 6.3.2, the conditional joint PDF of $X_{1:n} < \cdots < X_{d:n}$, given $D = d$, is,

$$f_{1,2,\ldots,d:n}\left(x_1, \ldots, x_d|D=d\right) = \frac{d!}{\theta^d\left[1-\exp\left(-\frac{W}{\theta}\right)\right]^d} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{d} x_i\right),$$

$$0 \leq x_{1:n} < \ldots < x_{d:n} \leq W. \tag{6.7}$$

Therefore, the MGF of $Z = \sum_{i=1}^{D} X_i$, given $D = d$, is given by

$$\begin{aligned}
M_Z(t) &= \mathrm{E}\left(e^{tZ}|D=d\right) = \mathrm{E}\left[\exp\left(t\sum_{i=1}^{D} X_i\right)|D=d\right] \\
&= \int_0^W \int_0^{x_d} \cdots \int_0^{x_2} \exp\left(t\sum_{i=1}^{d} x_i\right) f_{X_1,\ldots,X_D}\left(x_1,\ldots,x_d|D=d\right) dx_1 \cdots dx_d \\
&= \frac{d!}{\theta^d\left[1-\exp\left(-\frac{W}{\theta}\right)\right]^d} \\
&\quad \times \int_0^W \int_0^{x_d} \cdots \int_0^{x_2} \exp\left[\left(t-\frac{1}{\theta}\right)\left(\sum_{i=1}^{d-1} x_i + x_d\right)\right] dx_1 \cdots dx_d \\
&= \frac{d}{\theta\left[1-\exp\left(-\frac{W}{\theta}\right)\right]^d (1-t\theta)^{d-1}} \\
&\quad \times \int_0^W \left[1-\exp\left(t-\frac{1}{\theta}\right)x_d\right]^{d-1} \left[\exp\left(t-\frac{1}{\theta}\right)x_d\right] dx_d \\
&= \frac{\left\{1-\left[\exp\left(-\frac{W}{\theta}\right)\right]^{(1-t\theta)}\right\}^d}{\left\{\left[1-\exp\left(-\frac{W}{\theta}\right)\right]\left[1-t\theta\right]\right\}^d}. \qquad (6.8)
\end{aligned}$$

Lemma 6.3.5. *We have*

$$Pr(D_1' = d_1', D_1 = d_1, D' = d', D = d|G) = \frac{P_{d_1',d_1,d',d}}{\sum_{s=2}^{N}\sum_{g=0}^{N-s}\sum_{f=1}^{s-1}\sum_{c=0}^{g} P_{c,f,g,s}},$$

where

$$G = 0 \le D_1' \le D',\ 1 \le D_1 \le D-1,\ 0 \le D' \le N-D,\ 2 \le D \le N,$$

and

$$\begin{aligned}
P_{d_1',d_1,d',d} &= \binom{N}{d_1, d-d_1, d_1', d'-d_1', N-d-d'} \\
&\quad \times P_5^{d_1}(1-P_6-P_5)^{d-d_1} P_7^{d_1'}(P_6-P_8-P_7)^{d'-d_1'} P_8^{N-d-d'}.
\end{aligned}$$

Proof: We have

$$\begin{aligned}
P_{d_1',d_1,d',d} &= Pr(D_1' = d_1', D_1 = d_1, D' = d', D = d) \\
&= Pr(D_1' = d_1'|D_1 = d_1, D = d, D' = d')Pr(D_1 = d_1|D = d, D' = d') \\
&\quad \times Pr(D' = d'|D = d)Pr(D = d) \\
&= \binom{d'}{d_1'}\pi_6^{d_1'}(1-\pi_6)^{d'-d_1'}\binom{d}{d_1}\pi_5^{d_1}(1-\pi_5)^{d-d_1} \\
&\quad \times \binom{N-d}{d'}\pi_7^{d'}(1-\pi_7)^{N-d-d'}\binom{N}{d}\pi_8^{d}(1-\pi_8)^{N-d} \qquad (6.9)
\end{aligned}$$

from which the result follows immediately.

Lemma 6.3.6. *For the random variable Y with PDF*

$$g(y;\alpha,\lambda) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1}e^{-\lambda y}, & y > 0 \\ 0, & \textit{otherwise}, \end{cases}$$

the MGF of $Y + A$ is

$$M_{Y+A} = E\left(e^{t(Y+A)}\right) = e^{tA}E\left(e^{tY}\right) = e^{tA}\left(1 - \frac{t}{\lambda}\right)^{-\alpha}. \qquad (6.10)$$

Proof: The proof follows from the definition of the moment-generating function of the gamma distribution.

Using Lemmas 6.3.4 - 6.3.6, we can derive the CMGF of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$ and these results are presented in Lemmas 6.3.7 - 6.3.9.

Lemma 6.3.7. *The joint CMGF of $\hat{\theta}_1$ and $\hat{\theta}_2$, conditional on $1 \leq D_1 \leq D - 1$,*

$2 \leq D \leq N$, $0 \leq D' \leq N - D$ *and* $0 \leq D_1' \leq D'$, *is given by*

$$\sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l} \times \exp\left[\frac{W(N-r-m+l)}{(k+q)}t_1 + \frac{W(N-r-m+l)}{(r-k)+(m-q)}t_2\right]$$
$$\times\left(1 - \frac{\theta}{k+q}t_1 - \frac{\theta}{(r-k)+(m-q)}t_2\right)^{-r-m}, \tag{6.11}$$

where

$$C_{k,r,q,m,l} = \frac{(-1)^l}{\sum_{c=2}^{N}\sum_{f=1}^{c-1}\sum_{g=0}^{n-c}\sum_{s=0}^{g} P_{c,f,g,s}}$$
$$\times\binom{r+m}{l}\exp\left(-\frac{W}{\theta}l\right)\left[1-\exp\left(-\frac{W}{\theta}\right)\right]^{-r-m} P_{k,r,m,q}, \tag{6.12}$$

and $\theta = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)^{-1}$.

Proof: The proof is similar to that of Theorem 2.3.2.

Lemma 6.3.8. *The CMGF of* $\hat{\theta}_1'$, *conditional on* $1 \leq D_1 \leq D - 1$, $2 \leq D \leq N$, $0 \leq D' \leq N - D$ *and* $0 \leq D_1' \leq D'$, *is given by*

$$\sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r-k} \tilde{C}_{k,r,q,m,l} \times \exp\left(\frac{Wl}{(r-k)}t\right) \times \left(1 - \frac{\theta_1'}{r-k}t\right)^{-(r-k)-(m-q)}, \tag{6.13}$$

where

$$\tilde{C}_{k,r,q,m,l} = \frac{(-1)^l}{\sum_{c=2}^{N}\sum_{f=1}^{c-1}\sum_{g=0}^{n-c}\sum_{s=0}^{g} P_{c,f,g,s}}$$
$$\times\binom{r-k}{l}\exp\left(-\frac{W}{\theta_1'}l\right)\left[1-\exp\left(-\frac{W}{\theta_1'}\right)\right]^{-(r-k)} P_{k,r,m,q}. \tag{6.14}$$

Proof: The proof is similar to that of Theorem 2.3.2.

Lemma 6.3.9. *The CMGF of $\hat{\theta}_2'$, conditional on $1 \le D_1 \le D-1$, $2 \le D \le N$, $0 \le D' \le N-D$ and $0 \le D_1' \le D'$, is given by*

$$\sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k} \tilde{\tilde{C}}_{k,r,q,m,l} \times \exp\left(\frac{Wl}{k}t\right) \times \left(1 - \frac{\theta_2'}{k}t\right)^{-k-q}, \tag{6.15}$$

where

$$\tilde{\tilde{C}}_{k,r,q,m,l} = \frac{(-1)^l}{\sum_{c=2}^{N}\sum_{f=1}^{c-1}\sum_{g=0}^{n-c}\sum_{s=0}^{g} P_{c,f,g,s}} \times \binom{k}{l} \exp\left(-\frac{W}{\theta_2'}l\right)\left[1 - \exp\left(-\frac{W}{\theta_2'}\right)\right]^{-k} P_{k,r,m,q}. \tag{6.16}$$

Proof: The proof once again similar to that of Theorem 2.3.2.

Theorem 6.3.1. *The PDFs of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$, conditional on $1 \le D_1 \le D-1$, $2 \le D \le N$, $0 \le D' \le N-D$ and $0 \le D_1' \le D'$, are given by*

$$f_{\hat{\theta}_1}(x) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l} \times g(x - \tau_{k,r,q,m,l}; r+m, \frac{k+q}{\theta}), \tag{6.17}$$

$$f_{\hat{\theta}_2}(x) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l} \times g(x - \tilde{\tau}_{k,r,q,m,l}; r+m, \frac{(r-k)+(m-q)}{\theta}), \tag{6.18}$$

$$f_{\hat{\theta}_1'}(x) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r-k} \tilde{C}_{k,r,q,m,l} \times g(x - \tau_{k,r,l}; (r-k)+(m-q), \frac{r-k}{\theta_1'}), \tag{6.19}$$

$$f_{\hat{\theta}_2'}(x) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k} \tilde{\tilde{C}}_{k,r,q,m,l} \times g(x - \tau_{k,l}; k+q, \frac{k}{\theta_2'}), \tag{6.20}$$

where,

$$\tau_{k,r,q,m,l} = \frac{W(N-r-m+l)}{(k+q)}, \quad \tilde{\tau}_{k,r,q,m,l} = \frac{W(N-r-m+l)}{(r-k)+(m-q)},$$

$$\tau_{k,r,l} = \frac{Wl}{r-k}, \quad \tau_{k,l} = \frac{Wl}{k}.$$

Proof: From Lemmas 6.3.7-6.3.9, using the inversion theorem of a moment-generating function, we obtain the conditional PDFs of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$.

It is of interest to note here that the conditional PDFs of the MLEs are all mixtures of gamma densities.

6.3.1 Properties of the MLEs

From Theorem 6.3.1. we can derive some simple distributional properties of the MLEs as presented in the following theorems.

Theorem 6.3.2. *The first two moments of* $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ *and* $\hat{\theta}'_2$ *are as follows:*

$$\mathrm{E}(\hat{\theta}_1) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k} C_{k,r,q,m,l}\left(\tau_{k,r,q,m,l} + \frac{r+m}{k+q}\theta\right),$$

$$\mathrm{E}(\hat{\theta}_1^2) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k} C_{k,r,q,m,l}\left(\frac{(r+m)(r+m+1)}{(k+q)^2}\theta^2 + 2\frac{r+m}{k+q}\theta\tau_{k,r,q,m,l} + \tau^2_{k,r,q,m,l}\right),$$

$$\mathrm{E}(\hat{\theta}_2) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k} C_{k,r,q,m,l}\left(\tilde{\tau}_{k,r,q,m,l} + \frac{r+m}{(r-k)+(m-q)}\theta\right),$$

$$\mathrm{E}(\hat{\theta}_2^2) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k} C_{k,r,q,m,l}\left(\frac{(r+m)(r+m+1)}{[(r-k)+(m-q)]^2}\theta^2 + 2\frac{r+m}{(r-k)+(m-q)}\theta\tilde{\tau}_{k,r,q,m,l} + \tilde{\tau}^2_{k,r,q,m,l}\right),$$

$$\mathrm{E}(\hat{\theta}'_1) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k} \tilde{C}_{k,r,q,m,l}\left(\tau_{k,r,l} + \frac{(r-k)+(m-q)}{r-k}\theta'_1\right),$$

$$\mathrm{E}(\hat{\theta}_1'^2) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k}\tilde{C}_{k,r,q,m,l}\left(\frac{[(r-k)+(m-q)][(r-k)+(m-q)+1]}{(r-k)^2}\theta_1'^2 + 2\frac{(r-k)+(m-q)}{r-k}\theta_1'\tau_{k,r,l} + \tau_{k,r,l}^2\right),$$

$$\mathrm{E}(\hat{\theta}_2') = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k}\tilde{\tilde{C}}_{k,r,q,m,l}\left(\tau_{k,l} + \frac{r+q}{k}\theta_2'\right),$$

$$\mathrm{E}(\hat{\theta}_2'^2) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{k}\tilde{\tilde{C}}_{k,r,q,m,l}\left(\frac{(k+q)(k+q+1)}{k^2}\theta_2'^2 + 2\frac{k+q}{k}\theta_2'\tau_{k,l} + \tau_{k,l}^2\right).$$

The expressions for the expected values clearly reveal that $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$ are all biased estimators of θ_1, θ_2, θ_1' and θ_2', respectively. The expressions for the second moments can be used for finding standard errors of the estimates.

We can also obtain expressions for the tail probabilities from Theorem 6.3.1. These expressions will be used to construct exact confidence intervals for the relevant parameters later in Section 6.4.

Theorem 6.3.3. *The tail probability of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$ are given by*

$$P_{\theta_1}\left(\hat{\theta}_1 \geq a\right) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l}\times\Gamma\left(r+m, \frac{k+q}{\theta} < a - \tau_{k,r,q,m,l} >\right), \tag{6.21}$$

$$P_{\theta_2}\left(\hat{\theta}_2 \geq a\right) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l}\times\Gamma\left(r+m, \frac{(r-k)+(m-q)}{\theta} < a - \tilde{\tau}_{k,r,q,m,l} >\right), \tag{6.22}$$

$$P_{\theta_1'}\left(\hat{\theta}_1' \geq a\right) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} \tilde{C}_{k,r,q,m,l}\times\Gamma\left((r-k)+(m-q), \frac{r-k}{\theta_1'} < a - \tau_{k,r,l} >\right), \tag{6.23}$$

$$P_{\theta'_2}\left(\hat{\theta}'_2 \geq a\right) = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} \tilde{\tilde{C}}_{k,r,q,m,l} \times \Gamma\left(k+q, \frac{k}{\theta'_2} < a - \tau_{k,l} >\right), \quad (6.24)$$

where, $\Gamma(\alpha, z) = \frac{1}{\Gamma(\alpha)}\int_z^{\infty} y^{\alpha-1}e^{-y}dy$ *is the incomplete gamma ratio and* $< x >= \max\{x, 0\}$.

6.4 Confidence Intervals

In this section, we present different methods of constructing confidence intervals (CIs) for the unknown parameters θ_1, θ_2, θ'_1 and θ'_2. The exact CIs are based on the exact conditional distributions of the MLEs presented in Theorems 6.3.3. Since the exact conditional PDFs of the MLEs are computationally intensive, we may use the asymptotic distributions of the MLEs to obtain approximate CIs for θ_1, θ_2, θ'_1 and θ'_2 for large sample sizes. Finally, we also use the parametric bootstrap method to construct CIs for the parameters.

6.4.1 Exact Confidence Intervals

The same method, as described in Section 2.4.1, is used to construct exact CIs for the parameters θ_1, θ_2, θ'_1 and θ'_2. To guarantee the invertibility for the parameters, we assume once again that the tail probabilities of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$ and $\hat{\theta}'_2$ presented in Theorem 6.3.3 are increasing functions of θ_1, θ_2, θ'_1 and θ'_2, respectively. Values of the tail probabilities $P_{\theta_i^{(\prime)}}\left(\hat{\theta}_i^{(\prime)} \geq b\right)$ for various $\theta_i^{(\prime)} (i = 1, 2)$ and b are presented in Tables 6.1

- 6.4 to support this monotonicity assumption.

Confidence Interval for θ_1

A two-sided $100(1-\alpha)\%$ CI for θ_1, denoted by $(\theta_{1L}, \theta_{1U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l}(\theta_{1L}, \hat{\theta}_2, \hat{\theta}_1', \hat{\theta}_2') \times \Gamma\left(r+m, \frac{k+q}{\theta_L} < \hat{\theta}_1 - \tau_{k,r,q,m,l} >\right), \tag{6.25}$$

$$1-\frac{\alpha}{2} = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l}(\theta_{1U}, \hat{\theta}_2, \hat{\theta}_1', \hat{\theta}_2') \times \Gamma\left(r+m, \frac{k+q}{\theta_U} < \hat{\theta}_1 - \tau_{k,r,q,m,l} >\right), \tag{6.26}$$

where

$$\theta_{L(U)} = \left(\frac{1}{\theta_{1L(U)}} + \frac{1}{\hat{\theta}_2}\right)^{-1},$$

and $C_{k,r,q,m,l}(\theta_{1L(U)}, \hat{\theta}_2, \hat{\theta}_1', \hat{\theta}_2')$ is same as defined in Lemma 6.3.7, but with (θ_1, θ_2, θ_1', θ_2') replaced by ($\theta_{1L(U)}$, $\hat{\theta}_2$, $\hat{\theta}_1'$, $\hat{\theta}_2'$).

Confidence Interval for θ_2

A two-sided $100(1-\alpha)\%$ CI for θ_2, denoted by $(\theta_{2L}, \theta_{2U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l}(\hat{\theta}_1, \theta_{2L}, \hat{\theta}_1', \hat{\theta}_2') \times\Gamma\left(r+m, \frac{(r-k)+(m-q)}{\tilde{\theta}_L} < \hat{\theta}_2 - \tilde{\tau}_{k,r,q,m,l} >\right), \tag{6.27}$$

$$1-\frac{\alpha}{2}=\sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} C_{k,r,q,m,l}(\hat{\theta}_1,\theta_{2U},\hat{\theta}'_1,\hat{\theta}'_2)$$

$$\times\Gamma\left(r+m,\frac{(r-k)+(m-q)}{\tilde{\theta}_U}<\hat{\theta}_2-\tilde{\tau}_{k,r,q,m,l}>\right), \quad (6.28)$$

where

$$\tilde{\theta}_{L(U)}=\left(\frac{1}{\hat{\theta}_1}+\frac{1}{\theta_{2L(U)}}\right)^{-1},$$

and $C_{k,r,q,m,l}(\hat{\theta}_1,\theta_{2L(U)},\hat{\theta}'_1,\hat{\theta}'_2)$ is same as defined in Lemma 6.3.7, but with (θ_1, θ_2, θ'_1, θ'_2) replaced by ($\hat{\theta}_1$, $\theta_{2L(U)}$, $\hat{\theta}'_1$, $\hat{\theta}'_2$).

Confidence Interval for θ'_1

A two-sided $100(1-\alpha)\%$ CI for θ'_1, denoted by $(\theta'_{1L},\theta'_{1U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2}=\sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} \tilde{C}_{k,r,q,m,l}(\hat{\theta}_1,\hat{\theta}_2,\theta'_{1L},\hat{\theta}'_2)$$

$$\times\Gamma\left((r-k)+(m-q),\frac{r-k}{\theta'_{1L}}<\hat{\theta}'_1-\tau_{k,r,l}>\right), \quad (6.29)$$

$$1-\frac{\alpha}{2}=\sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m} \tilde{C}_{k,r,q,m,l}(\hat{\theta}_1,\hat{\theta}_2,\theta'_{1U},\hat{\theta}'_2)$$

$$\times\Gamma\left((r-k)+(m-q),\frac{r-k}{\theta'_{1U}}<\hat{\theta}'_1-\tau_{k,r,l}>\right), \quad (6.30)$$

where, $\tilde{C}_{k,r,q,m,l}(\hat{\theta}_1,\hat{\theta}_2,\theta'_{1L(U)},\hat{\theta}'_2)$ is same as defined in Lemma 6.3.8, but with (θ_1, θ_2, θ'_1, θ'_2) replaced by ($\hat{\theta}_1$, $\hat{\theta}_2$, $\theta'_{1L(U)}$, $\hat{\theta}'_2$).

Confidence Interval for ${\theta_2}'$

A two-sided $100(1-\alpha)\%$ CI for θ'_2, denoted by $(\theta'_{2L}, \theta'_{2U})$, can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m}\tilde{\tilde{C}}_{k,r,q,m,l}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}'_1, \theta'_{2L}) \times \Gamma\left(k+q, \frac{k}{\theta'_{2L}} < \hat{\theta}'_2 - \tau_{k,l} >\right), \tag{6.31}$$

$$1-\frac{\alpha}{2} = \sum_{r=2}^{N}\sum_{k=1}^{r-1}\sum_{m=0}^{N-r}\sum_{q=0}^{m}\sum_{l=0}^{r+m}\tilde{\tilde{C}}_{k,r,q,m,l}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}'_1, \theta'_{2U}) \times \Gamma\left(k+q, \frac{k}{\theta'_{2U}} < \hat{\theta}'_2 - \tau_{k,l} >\right), \tag{6.32}$$

where $\tilde{\tilde{C}}_{k,r,q,m,l}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}'_1, \theta'_{2L(U)})$ is same as defined in Lemma 6.3.9, but with (θ_1, θ_2, θ'_1, θ'_2) replaced by ($\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}'_1$, $\theta'_{2L(U)}$).

Lacking a closed-form solution, we have to apply an iterative root-finding technique in the determination of θ_{iL}, θ'_{iL}, θ_{iU} and θ'_{iU}, for $i = 1, 2$; the Newton-Raphson iteration method, for instance, was used in our study.

6.4.2 Asymptotic Confidence Intervals

Using the asymptotic normality of the MLEs, we can construct approximate confidence intervals for θ_1, θ_2, θ'_1 and θ'_2, using the Fisher information matrix.

Let $I(\theta_1, \theta_2, \theta'_1, \theta'_2) = (I_{ij}(\theta_1, \theta_2, \theta'_1, \theta'_2))$, $i, j = 1, 2, 3, 4$, denote the Fisher information matrix for the parameter (θ_1, θ_2, θ'_1, θ'_2). From Eq. (6.2), we have

$$I_{11}(\theta_1, \theta_2, \theta'_1, \theta'_2) = -\mathrm{E}\left(\frac{D_1 + D'_1}{\hat{\theta}_1^2} - \frac{2S_1}{\hat{\theta}_1^3}\right) = \frac{D_1 + D'_1}{\hat{\theta}_1^2}, \tag{6.33}$$

$$I_{22}(\theta_1, \theta_2, \theta'_1, \theta'_2) = -\mathrm{E}\left(\frac{D_2 + D'_2}{\hat{\theta}_2^2} - \frac{2S_1}{\hat{\theta}_2^3}\right) = \frac{D_2 + D'_2}{\hat{\theta}_2^2}, \tag{6.34}$$

$$I_{33}(\theta_1, \theta_2, \theta'_1, \theta'_2) = -\mathrm{E}\left(\frac{D_2}{\hat{\theta}_1'^2} - \frac{2S_2}{\hat{\theta}_1'^3}\right) = \frac{D_2}{\hat{\theta}_1'^2}, \tag{6.35}$$

$$I_{44}(\theta_1, \theta_2, \theta_1', \theta_2') = -\mathrm{E}\left(\frac{D_1}{\hat{\theta}_2'^2} - \frac{2S_3}{\hat{\theta}_2'^3}\right) = \frac{D_1}{\hat{\theta}_2'^2}, \tag{6.36}$$

$$I_{12} = I_{13} = I_{14} = I_{21} = I_{23} = I_{24} = I_{31} = I_{32} = I_{34} = I_{41} = I_{42} = I_{43} = 0, \tag{6.37}$$

where

$$S_1 = \sum_{i \in I_A} x_i + \sum_{i \in I_B} y_i + \sum_{i \in I_A'} x_i + \sum_{i \in I_B'} y_i + (N - D - D')W,$$

$$S_2 = \sum_{i \in I_B} (z_i - y_i) + \sum_{i \in I_B'} (W - y_i),$$

$$S_3 = \sum_{i \in I_A} (z_i - x_i) + \sum_{i \in I_A'} (W - x_i).$$

Thus, the Fisher information matrix is given by

$$\begin{bmatrix} \frac{D_1 + D_1'}{\hat{\theta}_1^2} & 0 & 0 & 0 \\ 0 & \frac{D_2 + D_2'}{\hat{\theta}_2^2} & 0 & 0 \\ 0 & 0 & \frac{D_2}{\hat{\theta}_1'^2} & 0 \\ 0 & 0 & 0 & \frac{D_1}{\hat{\theta}_2'^2} \end{bmatrix}.$$

This implies that the MLEs are asymptotically mutually independent. The asymptotic unconditional variance of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$ can be obtained readily from the Fisher information matrix as

$$V_1 = \frac{\hat{\theta}_1^2}{n_1}, \; V_2 = \frac{\hat{\theta}_2^2}{n_2}, V_3 = \frac{\hat{\theta}_1'^2}{n_2}, \; V_4 = \frac{\hat{\theta}_2'^2}{n_1}.$$

Then, the $100(1-\alpha)\%$ approximate CIs for θ_1, θ_2, θ_1' and θ_2' are obtained accordingly by using the same method as described in Section 2.4.2.

6.4.3 Bootstrap Confidence Intervals

The bootstrap methods of percentile interval and the biased-corrected and accelerated (BC_α) interval are similar to those described earlier in Section 2.4.3, but with a

Type-I censored two-component system failure sample instead. The acceleration $\hat{\alpha}_i^{(')}$ in the BC_α Percentile Interval should be changed accordingly to

$$\hat{\alpha}_i^{(')} = \frac{\sum_{j=1}^{D}\left(\hat{\theta}_{i(\cdot)}^{(')} - \hat{\theta}_{i(j)}^{(')}\right)^3}{6\left[\sum_{j=1}^{D}\left(\hat{\theta}_{i(\cdot)}^{(')} - \hat{\theta}_{i(j)}^{(')}\right)^2\right]^{\frac{3}{2}}}, \quad i = 1, 2,$$

where $\hat{\theta}_{i(j)}^{(')}$ is the MLE of $\theta_i^{(')}$ based on the original sample with the j-th observation deleted, $j = 1, 2, \ldots, D$, and $\hat{\theta}_{i(\cdot)}^{(')} = \frac{\sum_{j=1}^{D}\hat{\theta}_{i(j)}^{(')}}{D}$.

6.5 Simulation Study

In this section, a Monte Carlo simulation study based on 999 replications was carried out in order to examine the relative risks (Section 6.5.1), and to evaluate the performance of the four confidence intervals in terms of coverage probabilities for different sample sizes (Section 6.5.2). We also present numerical examples in Section 6.5.3 to illustrate all the inferential methods discussed here.

6.5.1 Relative Risks

The theoretical values of π_1, π_5, π_6, π_7 and π_8 with $\theta_1 = 20$, $\theta_2 = 25$ $\theta_1' = 9$, $\theta_2' = 14$ are presented in Table 6.5 when $n = 40, 20, 10$ and $W = 40, 35, 30, 25, 20, 15, 10$. The results were calculated by using the expressions presented in Theorems 2.3.1, 5.2.1, 6.2.1, 6.2.2 and 6.2.3, respectively.

π_1 is the probability that Component 1 fails first within a system. $\pi_1 = 0.5556$ implies that the first failure within a system is more likely due to Component 1. π_5

and π_6 are also the probabilities that Component 1 fails first within a system, but π_5 is conditional on the complete destruction of the system and π_6 is conditional on the incomplete destruction of the system. In most cases, $\pi_5 < \pi_1 < \pi_6$. It reflects that a system is more likely to survive if its Component 1 fails first in a Type-I censoring experiment.

As expected, π_1 is fixed for all sample sizes n and length of the experiment time W. π_5, π_6, π_7 and π_8 are only affected by the length of the experiment time W; they all increase as the experiment time W becomes longer. $\pi_5 = 0.5$ when $W = 20$. This means that if the pre-fixed termination time is 20, Component 1 and Component 2 have equal chance to fail first within a system with complete destruction. However, when the experiment time W is longer than 20, Component 1 is more likely to fail first within a system with complete destruction.

We observe that as W increases, the values of $\frac{n_1}{n}$, $\frac{D_1}{D}$, $\frac{D_1'}{D'}$, $\frac{D'}{n-D}$ and $\frac{D}{n}$ get closer to the corresponding theoretical values when n or W take on different values. This indicates that $\frac{n_1}{n}$, $\frac{D_1}{D}$, $\frac{D_1'}{D'}$, $\frac{D'}{n-D}$ and $\frac{D}{n}$ are good estimators of π_1, π_5, π_6, π_7 and π_8, respectively.

6.5.2 Coverage Probabilities and the Performance of the Confidence Intervals

The purpose of this subsection is to carry out a Monte Carlo simulation study based on Type-I censored sample to compare the performance of different confidence intervals described in Section 6.4. We once again chose the values of the parameters to be $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$. We then determined the true coverage probabilities of the 90%, 95% and 99% confidence intervals for all the parameters for different sample size ($n = 40, 20,\ W = 10, 20, 30, 40$) by all four methods described earlier in Section 6.4. These values, based on 999 Monte Carlo simulations and $R = 999$ bootstrap replications, are presented in Tables 6.10 and 6.11.

From these tables, we observe that, among the four methods, the exact method of constructing confidence intervals (based on the exact conditional distributions of the MLEs derived in Section 6.3) has its coverage probability to be quite close to the nominal level in all cases. Between the two bootstrap methods of constructing confidence intervals, the parametric BC_α method seems to have coverage probabilities to be closer to the nominal level and is therefore recommended for large sample sizes.

As expected, the approximate method based on the asymptotic normality of the MLEs has its true coverage probabilities to be always less than the nominal level. Though the coverage probability improves for larger sample sizes, we still find it to be unsatisfactory even for $n = 40$ when the pre-fixed termination time is $W = 40$. This indicates that the confidence intervals obtained by this method will often be unduly narrower. We do observe that, for all the nominal levels considered, the coverage probabilities of the approximate method are lower for small n or W in almost all cases. This is because when n or W is small, there are fewer failures observed and so inference for the parameters is not precise. As n increases, the number of failures increases thus resulting in a better large-sample approximation for the distribution of MLEs. This means that we need a much larger sample size to rely on the asymptotic normality of the MLEs. We also observe that when n is small, even the parametric BC_α bootstrap method does not have satisfactory coverage probabilities, but is seen to be better than the approximate method as well as the percentile bootstrap method.

6.5.3 Illustrative Examples

In this subsection, we consider two data sets with small and moderately large sample sizes to illustrate all the methods of inference developed in the preceding sections.

Data Set 1: $n = 35$

W	Failure Times				
15	(0.42,2; 0.58,1)	(2.10,2; 2.76,1)	(3.35,2; 3.66,1)	(5.59,1; 5.893,2)	(5.09,2; 6.39,1)
	(8.22,1; 8.36,2)	(3.57,1; 8.56,2)	(6.66,1; 9.25,2)	(3.57,1; 9.37,2)	(3.78,2; 12.37,1)
	(2.36,2; 12.54,1)	(7.81,2; 13.44,1)	(8.48,1; 14.12,2)	(1.12,2; 14.43,1)	(9.87,2; $*$)
	(9.52,2; $*$)	(12.02,1; $*$)	(5.99,1; $*$)	($*$; $*$)	($*$; $*$)
	(3.38,1; $*$)	(1.36,1; $*$)	($*$; $*$)	($*$; $*$)	(7.76,1; $*$)
	($*$; $*$)	(5.41,1; $*$)	($*$; $*$)	($*$; $*$)	(9.52,1; $*$)
	(9.22,1; $*$)	($*$; $*$)	($*$; $*$)	($*$; $*$)	($*$; $*$)
25	(0.42,2; 0.58,1)	(2.10,2; 2.76,1)	(3.35,2; 3.66,1)	(5.59,1; 5.89,2)	(5.09,2; 6.39,1)
	(8.22,1; 8.36,2)	(3.57,1; 8.56,2)	(6.66,1; 9.25,2)	(3.57,1; 9.37,2)	(3.78,2; 12.37,1)
	(2.36,2; 12.54,1)	(7.81,2; 13.44,1)	(8.48,1; 14.12,2)	(1.12,2; 14.43,1)	(9.87,2; 15.04,1)
	(9.52,2; 18.20,1)	(12.02,1; 19.47,2)	(5.99,1; 20.41,2)	(18.71,1; 21.35,2)	(21.93,2;22.00,1)
	(3.38,1; 23.23,2)	(1.36,1; 23.86,2)	(16.83,1; 24.06,2)	(17.60,2; 24.12,1)	(7.76,1; 24.27,2)
	($*$; $*$)	(5.41,1; $*$)	($*$; $*$)	(17.32,2; $*$)	(9.52,1; $*$)
	(9.22,1; $*$)	($*$; $*$)	($*$; $*$)	($*$; $*$)	($*$; $*$)

In the case when $n = 35$, had we fixed $W = 15$, we would have $D_1 = 6$, $D_2 = 8$, $D_1' = 8$ and $D_2' = 2$, and we would obtain the MLEs of θ_1, θ_2, θ_1' and θ_2' by using the methods presented in Section 6.2 to be

$$\hat{\theta}_1 = 21.5316, \qquad \hat{\theta}_2 = 30.1443, \qquad \hat{\theta}_1' = 6.3100, \qquad \hat{\theta}_2' = 14.1301.$$

Instead, had we fixed $W = 25$, we would have $D_1 = 13$, $D_2 = 12$, $D_1' = 3$ and $D_2' = 1$, and we would obtain the MLEs to be

$$\hat{\theta}_1 = 23.6774, \qquad \hat{\theta}_2 = 29.1414, \qquad \hat{\theta}_1' = 5.6654, \qquad \hat{\theta}_2' = 12.3760.$$

Data Set 2: $n = 15$

W	Failure Times				
15	(2.73,2; 3.50,1)	(2.68,1; 4.00,2)	(4.84,1; 8.04,2)	(2.32,1; 8.55,2)	(1.60,2; 12.48,1)
	(0.02,2; 13.32,1)	(10.33,2; 13.67,1)	(11.98,2; 14.71,1)	(* ; *)	(1.48,1; *)
	(* ; *)	(12.68,1; *)	(* ; *)	(* ; *)	(10.17,1; *)
25	(2.73,2; 3.50,1)	(2.68,1; 4.00,2)	(4.84,1; 8.04,2)	(2.32,1; 8.55,2)	(1.60,2; 12.48,1)
	(0.02,2; 13.32,1)	(10.33,2; 13.67,1)	(11.98,2; 14.71,1)	(17.15,1; 18.74,2)	(1.48,1; 20.44,2)
	(20.16,1; *)	(12.68,1; *)	(* ; *)	(* ; *)	(10.17,1; *)

In the case when $n = 15$, had we fixed $W = 15$, we would have $D_1 = 3$, $D_2 = 5$, $D_1' = 3$ and $D_2' = 0$, and we would obtain the MLEs of θ_1, θ_2, θ_1' and θ_2' by using the expressions presented in Section 6.2 to be

$$\hat{\theta}_1 = 20.1380, \qquad \hat{\theta}_2 = 24.1656, \qquad \hat{\theta}_1' = 6.2030, \qquad \hat{\theta}_2' = 10.4730.$$

Instead, had we fixed $W = 25$, we would have $D_1 = 5$, $D_2 = 5$, $D_1' = 3$ and $D_2' = 0$, and we would obtain the MLEs to be

$$\hat{\theta}_1 = 18.5170, \qquad \hat{\theta}_2 = 29.6273, \qquad \hat{\theta}_1' = 6.2030, \qquad \hat{\theta}_2' = 12.6577.$$

To assess the performance of these estimates, we constructed 90%, 95% and 99% confidence intervals by using the methods described in Section 6.4. These results are presented in Tables 6.5-6.9. Notice that the approximate method always provide narrower confidence intervals in most cases. This is because the coverage probabilities for the approximate method are significantly lower than the nominal levels.

Table 6.1: Values of $P_{\theta_1}\left(\hat{\theta}_1 \geq b\right)$ with $\theta_2 = 25$, $\theta_1' = 9$ and $\theta_2' = 14$

θ_1	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.2576	0.0028	0.0000	0.0000
9	0.8997	0.2548	0.0368	0.0059
13	0.9894	0.6861	0.2690	0.0887
17	0.9985	0.8914	0.5626	0.2833
21	0.9997	0.9621	0.7637	0.4964
25	0.9999	0.9858	0.8749	0.6648
29	1.0000	0.9942	0.9327	0.7807
33	1.0000	0.9975	0.9626	0.8561

Table 6.2: Values of $P_{\theta_2}\left(\hat{\theta}_2 \geq b\right)$ with $\theta_1 = 20$, $\theta_1' = 9$ and $\theta_2' = 14$

θ_2	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.2633	0.0036	0.0001	0.0000
9	0.8953	0.2625	0.0424	0.0077
13	0.9880	0.6827	0.2778	0.0980
17	0.9982	0.8855	0.5633	0.2934
21	0.9996	0.9581	0.7590	0.5010
25	0.9999	0.9835	0.8690	0.6641
29	1.0000	0.9930	0.9275	0.7769
33	1.0000	0.9968	0.9586	0.8512

Table 6.3: Values of $P_{\theta_1'}\left(\hat{\theta}_1' \geq b\right)$ with $\theta_1 = 20$, $\theta_2 = 25$ and $\theta_2' = 14$

θ_1'	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.3773	0.0477	0.0102	0.0034
9	0.8257	0.3953	0.1723	0.0846
13	0.9388	0.6980	0.4599	0.3035
17	0.9716	0.8451	0.6765	0.5297
21	0.9840	0.9127	0.8033	0.6914
25	0.9897	0.9460	0.8744	0.7941
29	0.9927	0.9639	0.9153	0.8579
33	0.9945	0.9742	0.9399	0.8982

Table 6.4: Values of $P_{\theta_2'}\left(\hat{\theta}_2' \geq b\right)$ with $\theta_1 = 20$, $\theta_2 = 25$ and $\theta_1' = 9$

θ_2'	$b = 6$	$b = 11$	$b = 16$	$b = 21$
1	0.0000	0.0000	0.0000	0.0000
5	0.3726	0.0327	0.0052	0.0015
9	0.8610	0.3974	0.1550	0.0681
13	0.9626	0.7349	0.4755	0.3012
17	0.9862	0.8841	0.7176	0.5577
21	0.9935	0.9443	0.8476	0.7355
25	0.9964	0.9703	0.9141	0.8409
29	0.9978	0.9825	0.9486	0.9011
33	0.9985	0.9889	0.9673	0.9360

Table 6.5: Relative risks based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta'_1 = 9$, $\theta'_2 = 14$

n	W	n_1	$\frac{n_1}{n}$	π_1	D_1	$\frac{D_1}{D}$	π_5	D'_1	$\frac{D'_1}{D'}$	π_6	$\frac{D'}{n-D}$	π_7	$\frac{D}{n}$	π_8
40	40	22.26	0.56	0.56	18.59	0.54	0.53	3.13	0.73	0.73	0.81	0.80	0.87	0.86
	35	22.21	0.56	0.56	16.98	0.52	0.53	4.24	0.72	0.71	0.77	0.78	0.81	0.81
	30	22.25	0.56	0.56	15.36	0.52	0.52	5.41	0.70	0.69	0.74	0.74	0.74	0.74
	25	22.40	0.56	0.56	13.20	0.51	0.51	6.85	0.67	0.67	0.71	0.70	0.64	0.65
	20	22.46	0.56	0.56	10.67	0.50	0.50	8.02	0.66	0.65	0.65	0.65	0.53	0.53
	15	22.14	0.55	0.56	7.53	0.49	0.49	9.82	0.63	0.63	0.58	0.58	0.39	0.38
	10	22.38	0.56	0.56	4.33	0.48	0.48	9.06	0.61	0.60	0.48	0.48	0.23	0.22
20	40	11.04	0.55	0.56	9.13	0.53	0.53	1.65	0.73	0.73	0.81	0.80	0.86	0.86
	35	11.09	0.55	0.56	8.53	0.53	0.53	2.09	0.71	0.71	0.78	0.78	0.81	0.81
	30	11.28	0.56	0.56	7.73	0.53	0.52	2.77	0.70	0.69	0.74	0.74	0.73	0.74
	25	11.27	0.56	0.56	6.69	0.52	0.51	3.43	0.68	0.67	0.70	0.70	0.64	0.65
	20	11.07	0.55	0.56	5.22	0.50	0.50	4.06	0.66	0.65	0.65	0.65	0.53	0.53
	15	11.21	0.56	0.56	3.80	0.49	0.49	4.50	0.64	0.63	0.58	0.58	0.39	0.38
	10	11.01	0.55	0.56	2.33	0.48	0.48	4.30	0.60	0.60	0.47	0.48	0.24	0.22
10	40	5.51	0.55	0.56	4.57	0.53	0.53	0.80	0.74	0.73	0.80	0.80	0.87	0.86
	35	5.55	0.55	0.56	4.23	0.52	0.53	1.06	0.72	0.71	0.77	0.78	0.81	0.81
	30	5.57	0.56	0.56	3.87	0.52	0.52	1.33	0.70	0.69	0.74	0.74	0.74	0.74
	25	5.57	0.56	0.56	3.32	0.51	0.51	1.64	0.68	0.67	0.69	0.70	0.65	0.65
	20	5.53	0.55	0.56	2.77	0.51	0.50	1.85	0.64	0.65	0.64	0.65	0.55	0.53
	15	5.50	0.55	0.56	2.05	0.49	0.49	2.09	0.62	0.63	0.59	0.58	0.42	0.38
	10	5.52	0.55	0.56	1.57	0.50	0.48	1.96	0.59	0.60	0.48	0.48	0.32	0.22

Table 6.6: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' when $n = 35$, $W = 15$

C.I. for θ_1			
Methods	90%	95%	99%
Exact C.I.	(14.82, 35.32)	(13.05, 37.89)	(11.04, 51.44)
Approx C.I.	(12.07, 31.00)	(10.25, 32.81)	(6.71, 36.35)
Boot-p C.I.	(14.45, 34.02)	(13.17, 35.78)	(11.34, 47.54)
BC_α C.I.	(14.31, 34.63)	(12.82, 38.06)	(10.89, 49.50)

C.I. for θ_2			
Methods	90%	95%	99%
Exact C.I.	(18.63, 56.36)	(17.48, 65.23)	(14.33, 87.22)
Approx C.I.	(14.46, 45.82)	(11.46, 48.83)	(5.59, 54.70)
Boot-p C.I.	(18.84, 54.23)	(17.50, 62.12)	(14.39, 89.66)
BC_α C.I.	(18.71, 54.89)	(17.42, 63.52)	(14.39, 91.65)

C.I. for θ_1'			
Methods	90%	95%	99%
Exact C.I.	(3.35, 12.88)	(2.87, 15.44)	(2.43, 41.72)
Approx C.I.	(2.64, 9.98)	(1.94, 10.68)	(0.56, 12.06)
Boot-p C.I.	(3.11, 12.29)	(2.73, 14.97)	(2.00, 26.16)
BC_α C.I.	(3.46, 14.04)	(3.06, 16.98)	(2.22, 37.59)

C.I. for θ_2'			
Methods	90%	95%	99%
Exact C.I.	(8.06, 28.89)	(7.03, 29.63)	(5.35, 57.33)
Approx C.I.	(4.64, 23.62)	(2.82, 25.44)	(0*, 28.99)
Boot-p C.I.	(7.78, 30.03)	(7.06, 35.97)	(5.36, 52.38)
BC_α C.I.	(7.69, 30.78)	(6.98, 36.87)	(4.82, 58.59)

0* stands for a non-positive number

Table 6.7: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' when $n = 35$, $W = 25$

C.I. for θ_1			
Methods	90%	95%	99%
Exact C.I.	(16.78, 35.63)	(16.31, 39.42)	(13.21, 48.23)
Approx C.I.	(13.94, 33.41)	(12.08, 35.28)	(8.43, 38.92)
Boot-p C.I.	(16.51, 35.34)	(15.45, 37.25)	(12.71, 41.39)
BC_α C.I.	(16.52, 35.57)	(15.52, 38.75)	(12.71, 43.69)
C.I. for θ_2			
Methods	90%	95%	99%
Exact C.I.	(18.33, 45.36)	(17.89, 50.43)	(15.89, 68.35)
Approx C.I.	(15.85, 42.44)	(13.30, 44.98)	(8.32, 49.96)
Boot-p C.I.	(19.18, 43.89)	(17.77, 47.31)	(15.32, 65.30)
BC_α C.I.	(18.41, 44.13)	(17.21, 49.93)	(14.50, 67.92)
C.I. for θ_1'			
Methods	90%	95%	99%
Exact C.I.	(3.84, 9.11)	(3.22, 9.88)	(2.67, 16.78)
Approx C.I.	(2.98, 8.36)	(2.46, 8.87)	(1.45, 9.88)
Boot-p C.I.	(3.35, 9.07)	(3.09, 9.78)	(2.44, 13.38)
BC_α C.I.	(3.47, 9.26)	(3.14, 10.18)	(2.56, 13.88)
C.I. for θ_2'			
Methods	90%	95%	99%
Exact C.I.	(7.89, 20.54)	(6.33, 22.23)	(5.21, 30.13)
Approx C.I.	(6.73, 18.02)	(5.65, 19.10)	(3.53, 21.22)
Boot-p C.I.	(7.33, 20.58)	(6.71, 23.23)	(5.30, 30.76)
BC_α C.I.	(7.81, 22.37)	(7.12, 24.16)	(5.66, 32.50)

0* stands for a non-positive number

Table 6.8: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' when $n = 15$, $W = 15$

C.I. for θ_1			
Methods	90%	95%	99%
Exact C.I.	(10.99, 44.13)	(9.91, 52.82)	(8.16, 77.97)
Approx C.I.	(6.62, 33.66)	(4.02, 36.25)	(0*, 41.31)
Boot-pC.I.	(10.79, 40.05)	(9.58, 49.70)	(8.42, 72.28)
BC_α C.I.	(10.47, 42.48)	(9.28, 51.29)	(7.85, 76.05)
C.I. for θ_2			
Methods	90%	95%	99%
Exact C.I.	(12.44, 57.49)	(11.13, 70.65)	(9.05, 111.34)
Approx C.I.	(6.39, 41.94)	(2.98, 45.35)	(0*, 52.00)
Boot-p C.I.	(12.56, 56.22)	(11.19, 67.75)	(9.64, 120.37)
BC_α C.I.	(12.56, 56.47)	(11.19, 69.11)	(9.64, 125.37)
C.I. for θ_1'			
Methods	90%	95%	99%
Exact C.I.	(2.78, 15.31)	(2.42, 20.46)	(1.82, 51.53)
Approx C.I.	(1.64, 10.77)	(0.77, 11.64)	(0*, 13.35)
Boot-p C.I.	(2.08, 15.32)	(1.61, 19.10)	(0.82, 32.30)
BC_α C.I.	(2.85, 19.10)	(2.18, 26.69)	(1.36, 36.24)
C.I. for θ_2'			
Methods	90%	95%	99%
Exact C.I.	4.48, 21.14)	(3.94, 26.75)	(3.02, 50.92)
Approx C.I.	(0.53, 20.42)	(0*, 22.32)	(0*, 26.05)
Boot-p C.I.	(3.78, 27.00)	(3.11, 35.51)	(1.98, 52.14)
BC_α C.I.	(3.90, 28.24)	(3.28, 35.51)	(1.81, 52.90)

0* stands for a non-positive number

Table 6.9: Confidence intervals for θ_1, θ_2, θ_1' and θ_2' when $n = 15$, $W = 25$

C.I. for θ_1			
Methods	90%	95%	99%
Exact C.I.	(10.98, 35.41)	(10.01, 40.83)	(8.41, 55.27)
Approx C.I.	(7.75, 29.29)	(5.69, 31.35)	(1.65, 35.38)
Boot-p C.I.	(10.28, 35.24)	(9.02, 39.20)	(7.55, 46.94)
BC_α C.I.	(10.11, 35.37)	(9.06, 39.53)	(7.55, 47.05)

C.I. for θ_2			
Methods	90%	95%	99%
Exact	(15.36, 69.37)	(13.73, 84.98)	(11.12, 133.76)
Approx C.I.	(7.83, 51.42)	(3.66, 55.60)	(0*, 63.76)
Boot-p C.I.	(15.41, 66.73)	(13.63, 80.30)	(10.59, 169.98)
BC_α C.I.	(14.72, 67.77)	(12.75, 83.34)	(9.89, 174.06)

C.I. for θ_1'			
Methods	90%	95%	99%
Exact C.I.	(3.08, 15.63)	(2.70, 20.45)	(2.06, 44.36)
Approx C.I.	(1.64, 10.77)	(0.77, 11.64)	(0*, 13.35)
Boot-p C.I.	(2.13, 13.72)	(1.45, 17.35)	(0.66, 24.38)
BC_α C.I.	(2.95, 18.09)	(2.35, 22.48)	(1.29, 34.23)

C.I. for θ_2'			
Methods	90%	95%	99%
Exact C.I.	(6.77, 23.56)	(6.10, 27.64)	(4.97, 39.71)
Approx C.I.	(3.35, 21.97)	(1.56, 23.75)	(0*, 27.24)
Boot-p C.I.	(5.89, 28.89)	(5.01, 32.34)	(3.57, 54.41)
BC_α C.I.	(6.16, 29.48)	(5.30, 33.40)	(3.75, 54.85)

0* stands for a non-positive number

Table 6.10: Estimated coverage probabilities based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 40$

C.I. of θ_1	90% C.I.				95% C.I.				99% C.I.			
W	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α
10	88.10	79.28	84.18	84.68	93.99	84.10	88.09	90.89	97.89	88.10	94.09	95.89
20	91.20	81.38	91.29	92.59	94.59	88.20	93.29	96.00	98.30	89.09	97.90	98.70
30	90.69	85.59	89.09	89.69	94.89	89.09	95.50	95.40	98.89	91.00	98.40	99.00
40	90.59	86.19	89.89	90.29	95.10	90.99	93.79	94.59	99.20	92.09	99.10	99.20

C.I. of θ_2	90% C.I.				95% C.I.				99% C.I.			
W	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α
10	89.10	80.68	84.58	84.98	94.30	86.59	91.49	93.99	98.20	88.59	94.89	95.99
20	89.99	83.39	88.10	90.99	94.89	90.39	94.89	96.10	98.90	90.90	98.20	99.30
30	89.90	85.09	88.19	89.39	95.60	90.10	93.39	94.29	99.20	92.00	98.60	98.90
40	90.39	86.79	88.59	90.79	95.10	91.59	93.39	95.10	99.00	93.70	99.10	99.10

C.I. of θ_1'	90% C.I.				95% C.I.				99% C.I.			
W	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α
10	88.60	79.58	82.58	83.48	94.40	87.40	92.69	94.79	98.20	87.88	92.29	93.39
20	89.70	84.79	88.89	91.59	95.60	91.39	93.39	95.60	98.10	91.10	98.80	99.30
30	90.10	85.79	89.29	92.59	95.79	91.89	93.89	95.70	99.10	93.00	98.70	99.20
40	89.90	86.09	89.99	90.89	96.00	90.99	93.79	95.20	98.80	94.00	97.60	98.30

C.I. of θ_2'	90% C.I.				95% C.I.				99% C.I.			
W	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α
10	88.90	80.39	84.38	84.38	94.30	87.90	91.79	93.39	98.80	87.59	90.99	91.79
20	89.10	85.10	89.99	90.79	96.00	91.29	94.69	95.70	98.79	95.30	98.70	99.10
30	89.79	86.00	90.09	91.99	95.40	92.79	94.59	95.80	99.20	95.60	99.40	99.70
40	89.09	87.10	87.99	89.89	95.90	92.69	94.19	95.80	99.00	95.30	98.50	98.90

Table 6.11: Estimated coverage probabilities based on 999 simulations when $\theta_1 = 20$, $\theta_2 = 25$, $\theta_1' = 9$, $\theta_2' = 14$ and $n = 20$

C.I. of θ_1	90% C.I.				95% C.I.				99% C.I.			
W	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α
10	88.10	76.79	81.18	81.69	94.10	82.19	89.10	90.30	98.40	85.59	88.09	90.29
20	88.79	81.19	86.49	87.59	96.00	89.50	92.79	93.59	98.79	91.29	97.00	97.00
30	89.99	85.10	90.69	91.99	95.79	90.39	93.89	95.40	99.20	92.70	98.30	98.80
40	90.89	86.10	89.79	90.99	95.20	91.99	93.49	95.30	99.30	93.59	97.70	98.40
C.I. of θ_2	90% C.I.				95% C.I.				99% C.I.			
W	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α
10	88.30	77.00	82.19	82.79	94.20	83.80	86.49	87.00	98.79	86.59	88.89	89.19
20	91.20	85.59	87.19	88.69	96.19	89.79	91.69	92.79	99.60	90.09	97.50	97.60
30	90.30	88.79	89.39	91.59	95.70	91.19	93.89	95.60	99.10	92.10	98.40	98.80
40	90.10	87.99	88.49	90.59	95.19	91.29	94.29	95.60	99.00	93.60	98.60	99.40
C.I. of θ_1'	90% C.I.				95% C.I.				99% C.I.			
W	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α
10	88.90	74.59	80.19	81.49	95.19	81.30	86.20	88.10	97.10	83.88	88.29	88.39
20	91.40	83.78	87.29	89.69	95.90	85.39	89.59	92.49	99.80	91.09	96.40	96.60
30	90.19	85.29	87.79	91.69	96.20	89.19	93.29	96.10	99.20	93.59	97.10	97.80
40	90.30	86.09	89.69	91.99	95.10	89.59	93.29	96.40	99.06	94.49	98.10	98.90
C.I. of θ_2'	90% C.I.				95% C.I.				99% C.I.			
W	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α	Exact	Approx.	Boot-p	BC_α
10	88.70	76.20	81.19	82.20	94.79	81.79	87.10	88.79	97.79	84.59	88.79	88.99
20	91.10	84.49	87.49	89.59	95.30	86.09	92.19	93.49	99.60	90.29	96.20	96.80
30	90.30	86.49	88.49	91.59	95.70	91.29	93.29	95.39	99.10	93.49	98.50	99.00
40	89.19	86.09	89.69	91.99	95.40	89.89	92.89	96.20	98.79	94.89	98.10	98.60

Chapter 7

Conclusions and Future Work

In this thesis, we have considered the two-component system failure model when the observed failure time data are complete in Chapter 2, Type-II censored in Chapter 3, Type-II censored with partial information on component failures in Chapter 4, Type-I censored in Chapter 5, and finally Type-I censored with partial information on component failures in Chapter 6. For each situation, we have obtained the MLEs of the model parameters θ_1, θ_2, θ_1', θ_2', and have derived their exact conditional distributions (when possible). Several different procedures for constructing confidence intervals have been discussed. Simulation studies and numerical examples have been presented to assess the performance of these confidence intervals and also to illustrate the methods developed in this thesis.

From our simulation studies, we have observed that the exact method of constructing confidence intervals (based on the exact conditional distributions of the MLEs $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_1'$ and $\hat{\theta}_2'$) always maintains its coverage probability at the nominal level, even in the case of small sample sizes. The approximate method of constructing confidence intervals (based on the asymptotic normality of the MLEs) are almost always unsatisfactory in terms of coverage probabilities. Between the two bootstrap methods of constructing confidence intervals, the adjusted percentile and biased-corrected bootstrap method (BC_α method) seems to have coverage probabilities closer to the nominal level in case of larger sample sizes. Hence, we recommend the use of the exact method (when available) whenever possible, especially in the case of small sample sizes; the adjusted percentile method is recommended for at least moderately large sample sizes when the computation of the exact confidence interval becomes difficult; and the approximate method is recommended only for large sample size because of its computational ease as well as for having its coverage probability close to the nominal level when n is large (preferably over 50).

7.1 Future Work

Several problems that are worthy of consideration for further studies are as follows.

In this thesis, we have considered the two-component system failure model under Type-I and Type-II censoring schemes with or without partial information on component failures, assuming the lifetimes of the components to be exponentially distributed. One possible extension of interest will be the exact analysis of the two-component system failure under different censoring schemes such as: (1) progressive Type-I or progressive Type-II censoring, and (2) hybrid Type-I or hybrid Type-II censoring. Another possible extension of interest will be to consider different lifetime distributions for the components such as (1) Gamma and (2) Weibull. Therefore, in each of these situations, we may develop the corresponding models and discuss the determination of MLEs of the unknown parameters. We can also discuss the construction of confidence intervals for the parameters and evaluate their performance by means of Monte Carlo simulations and illustrative examples.

We can also extend all the methods presented in this thesis to the case of k-out-of-n systems, consisting of n non-identical and dependent components. A system having k-out-of-n structure can survive if at least k of its n components are operating; it fails if $n-k+1$ or more components fail. Two important special cases of this model are: (1) $k=1$ corresponding to a parallel system, and (2) $k=n$ corresponding to a series system. We also need to note that, since all components start working at the same time, this approach may lead to a kind of redundancy called active redundancy of $n-k$ components.

In the models considered in this thesis, we only assumed that failure of one component forces a change in the surviving component in that the mean lifetime changes from θ_j to θ'_j. We do not assume any relationship between θ_j and θ'_j. There may, however, be some situations where in X and Y, the lifetimes of the two components, are independent, i.e., $\theta_1 = \theta'_1$ and $\theta_2 = \theta'_2$. We could, therefore, develop hypothesis tests for $H_0 : \theta_1 = \theta'_1,\ \theta_2 = \theta'_2$ using the likelihood ratio method. The hypothesis H_0 is equivalent to testing whether we have a constant hazard rate if the lifetimes have exponential distributions. The development of suitable tests and a study of their power properties will certain be of interest.

Bibliography

[1] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1992), *A First Course in Order Statistics*, John Wiley & Sons, New York.

[2] Balakrishnan, N. and Cohen, A. C. (1991), *Order Statistics and Inference: Estimation Methods*, Academic Press, San Diego.

[3] Balakrishnan, N. and Iliopoulos, G. (2008), Stochastic monotonicity of the MLE of exponential mean under different censoring schemes, *Annals of the Institute of Statistical Mathematics* (to appear).

[4] Block, H. W. and Savits, T. H. (1981), Multivariate classes of life distributions in reliability theory, *Mathematics of Operations Research*, vol. 6, 453-461.

[5] Childs, A., Chandrasekar, B., Balakrishnan, N. and Kundu, D. (2003), Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution, *Annals of the Institute of Statistical Mathematics*, vol. 55, 319-330.

[6] Chen, S.M. and Bhattacharya, G.K. (1988), Exact confidence bound for an exponential parameter under hybrid censoring, *Communications in Statistics - Theory and Methods*, vol. 16, 1857-1870.

[7] Cohen, A. C. (1991), *Truncated and Censored Samples: Theory and Applications*, Marcel Dekker, New York.

[8] Cohen, A. C. and Whitten, B. J. (1988), *Parameter Estimation in Reliability and Life Span Models*, Marcel Dekker, New York.

[9] Crowder, M.J. (2001), *Classical Competing Risks*, Chapman & Hall, New York.

[10] Edwin, G.L. and Savage, R.I. (1954), Tables of expected value of 1/X for positive Bernoulli and Poisson variables, *Journal of the American Statistical Association*, vol. 49, 169-177.

[11] Efron, B. (1982), The Jackknife, the Bootstrap and other re-sampling plans, *CBMS/NSF Regional Conference Series in Applied Mathematics*, vol. 38, SIAM, Philadephia, PA.

[12] Efron, B. and Tibshirani, R. (1998), *An Introduction to the Bootstrap*, Chapman & Hall, New York.

[13] Freund, J.E. (1961), A bivariate extension of the exponential distribution, *Journal of the American Statistical Association*, vol. 56, 971-977.

[14] Goel, L.R. and Gupta, R. (1983), A multi-standby multi-failure mode system with repair and replacement policy, *Microelectronics and Reliability*, vol. 23, 809-812.

[15] Gross, A.J., Clark, V. A. and Liu, V. (1971), Estimation of survival parameters when one of two organs must function for survival, *Biometrics*, vol. 27, 369-377.

[16] Gumbel, E. J. (1960), Bivariate exponential distributions, *Journal of the American Statistical Association*, vol. 55, 698-707.

[17] Gupta, R.D. and Kundu, D. (1998), Hybrid censoring schemes with exponential failure distribution, *Communications in Statistics - Theory and Methods*, vol. 27, 3065-3083.

[18] Hall, P. (1988), Theoretical comparison of bootstrap confidence intervals, *Annals of Statistics*, vol. 16, 927-953.

[19] Hougaard, P. (1987), Modeling multivariate survival, *Scandinavian Journal of Statistics*, vol. 14, 291-304.

[20] Kundu, D. and Basu, S. (2000), Analysis of incomplete data in presence of competing risks, *Journal of Statistical Planning and Inference*, vol. 87, 221-239.

[21] Lawless, J. F. (1982), *Statistical Models and Methods for Lifetime Data*, John Wiley & Sons, New York.

[22] Lin, D.K., Usher, J.S. and Guess, F.M. (1993), Exact maximum likelihood estimation using masked system data, *IEEE Transactions on Reliability*, vol. 42, 631-635.

[23] Miyakawa, M. (1984), Analysis of incomplete data in competing risks model, *IEEE Transactions on Reliability*, vol. 33, 293-296.

[24] Murthy, D. N. P. and Wilson, R. J. (1994), Parameter estimation in multi-component systems with failure interaction, *Applied Stochastic Models and Data Analysis*, vol. 10, 47-60.

[25] Nelson, W. (1982), *Applied Life Data Analysis*, John Wiley & Sons, New York.

[26] Sarhan, A.M. (2003), Estimation of system components reliabilities using masked data, *Applied Mathematics and Computation*, vol. 136, 79-92.

[27] Sarhan, A.M., El-Bassiouny, A.H. (2003), Estimation of components reliability in a parallel system using masked system life data, *Applied Mathematics and Computation*, vol. 138, 61-75.

[28] Slud, E. (1984), Multivariate dependent renewal processes, *Advances in Applied Probability*, vol. 16, 347-362.

[29] Usher, J.S. (1996), Weibull component reliability-prediction in the presence of masked data, *IEEE Transactions on Reliability*, vol. 45, 229-232.