DEFINITE FORMS IN VALUED FIELDS

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By

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Abstract

Let $\mathcal{K} = (K, v, \dots)$ be a model of a model-complete theory, $\mathcal{T}$ of valued fields. We characterise, for certain definable subsets S of K^n, the collections of S-$\mathcal{T}$-integral definite and S-$\mathcal{T}$-infinitesimal definite rational functions. Specifically, we consider subsets S defined by both integrality and infinitesimality conditions for the theories of algebraically closed valued fields, p-adically closed fields, two model-complete theories of valued D-fields and in two model-complete theories of henselian residually valued fields.

Contents

Introduction

1. Hilbert's Seventeenth Problem

In his seminal address to the International Congress of Mathematicians in 1900, David Hilbert proposed a list of twenty-three then unsolved problems intended to promote the development of various fields of mathematics throughout the twentieth century. Work on these problems was deemed of great importance. Of the twenty-three problems, the majority have been resolved or partially resolved. (The most notable exception is Hilbert's 8th Problem, the Riemann Hypothesis, whose affirmative solution would have many practical and important number-theoretic implications.) However, aspects of even the resolved problems hold interest for mathematicians today.

In the published list of the twenty-three problems [13], Hilbert's 17th Problem is stated as follows:

> A rational integral function or form in any number of variables with real coefficient such that it becomes negative for no real values of these variables, is said to be definite. The system of all definite forms is invariant with respect to the operations of addition and multiplication, but the quotient of two definite forms, in case it should be an integral function of the variables, is also a definite form. The square of any form is evidently always a definite form. But since not every definite form can be compounded by addition from squares of forms, the question arises whether every definite form may not be expressed as a quotient of sums of squares of forms.

Hilbert's 17th Problem asks for a characterisation of the functional property of a polynomial over the real numbers being everywhere nonnegative in terms of the algebraic property of being a sum of squares of rational functions over the real numbers. This conjecture seems simplistic. It posits that the definite forms are exactly those forms that are obviously definite. The conjecture is, however, true in any real closed field. Artin provided the affirmative answer in 1927.

THEOREM 0.1 (Artin's Nichtnegativstellensatz, [1]). *Let R be any real closed field and $f \in R[X_1, \dots, X_n]$. If f is nonnegative on R^n, then f is a sum of squares in the field of rational functions $R(X_1, \dots, X_n)$.*

As Artin's methods rely on an application of Zorn's Lemma, the result is nonconstructive. We will give a model-theoretic proof of this in Section 3 of this introduction, which, too, is nonconstructive. Thus the solution leads to more questions. Given a polynomial with real coefficients, is there an algorithm to determine whether it is definite? If so, is there an algorithm to determine its representation as a sum of squares? How many squares are needed? Are there bounds on the degrees? See [7] for an account of the literature on these questions.

Hilbert's 17th Problem can be seen, also, as part of a larger problem to construct a correspondence between algebra and geometry akin to the correspondence between algebraic subsets of $\mathbb{C}^n$ and radical ideals of $\mathbb{C}[X] = \mathbb{C}[X_1, \dots, X_n]$. This theory was developed by Stengle [24] and others in the second half of the twentieth century. A lovely survey of their results can be found in [16].

Here, we prove analogues in valued fields of Artin's solution to Hilbert's 17th Problem with an eye toward developing an algebro-geometric correspondence in this setting. Our methods, too, are nonconstructive and, hence, will give rise to questions analogous to those discussed above.

2. Valued Fields

By a valued field, we shall always mean a field K together with a map

$$v : K \to \Gamma \cup \{\infty\}$$

where Γ is an ordered abelian group written additively. This map is called the valuation on K and must satisfy

$$\begin{aligned} v(0) &= \infty, \\ v(xy) &= v(x) + v(y), \\ v(x+y) &\geq \min\{v(x), v(y)\} \end{aligned}$$

for all $x, y \in K$. This third condition is the strong triangle inequality. It is important to note that this inequality becomes an equality whenever $v(x) \neq v(y)$. To see this, suppose there are $x, y \in K$ with $v(x) < v(y)$ but $v(x+y) > \min\{v(x), v(y)\} = v(x)$. Certainly,

$$v(x) = v((x+y) + (-y)) \geq \min\{v(x+y), v(-y)\}.$$

However, since $v(x+y) > v(x)$ and $v(-y) = v(y) > v(x)$, we have $\min\{v(x+y), v(-y)\} > v(x)$. This is a contradiction.

The valuation on a valued field is analogous to the ordering on the real field. Hence, we may reformulate Hilbert's 17th Problem in the valued field setting by replacing the notion of being nonnegative with the notion of having nonnegative valuation.

We fix some notation. The set of elements of nonnegative valuation,

$$\mathcal{O}_{v,K} = \{x \in K : v(x) \geq 0\},$$

is the valuation ring of (K, v). Moreover, $\mathcal{O}_v$ is a local ring whose maximal ideal is precisely the elements of positive valuation,

$$\mathcal{M}_{v,K} = \{x \in K : v(x) > 0\}.$$

Then the group of units, $\mathcal{O}_v^\times$, consists exactly of the elements of valuation zero. The quotient $\mathcal{O}_{v,K}/\mathcal{M}_{v,K}$ is the residue field of (K, v) and is denoted $k_{v,K}$. The canonical map $\mathcal{O}_{v,K} \to k_{v,K}$ is called the residue map and is denoted $\text{res}_{v,K}$. We shall often write $\mathcal{O}_v$, $\mathcal{M}_v$, k_v and res_v for the valuation ring, maximal ideal, residue field and residue map, respectively, when the valued field K is clear from context.

Then given a valued field (K, v), we shall say that a rational function $f(X) \in K(X) = K(X_1, \ldots, X_n)$ is integral at $b \in K^n$ if $f(b)$ is defined and $f(b) \in \mathcal{O}_v$. We shall say that $f(X)$ is infinitesimal at $b \in K^n$ if $f(b)$ is defined and $f(b) \in \mathcal{M}_v$. Furthermore, given a subset $S \subseteq K^n$, we say that $f(X)$ is S-integral definite if $f(X)$ is integral at each $b \in S$. Similarly, we say $f(X)$ is S-infinitesimal definite if $f(X)$ is infinitesimal at each $b \in S$.

Notice that, as $v(0) = \infty$, elements with large valuation are "close" to zero, motivating our choice of the term infinitesimal for these elements. Further, notice that these definitions explicitly require that a function $f(X)$ be defined at any point where it is integral or infinitesimal. In Chapter 3, we shall refine our notions of integrality and infinitesimality to allow certain functions to be considered integral or infinitesimal even where not defined.

The analogue of Hilbert's 17th Problem in valued fields, then, asks for an algebraic characterisation of the functional properties of being S-integral and S-infinitesimal definite. We provide this in the form of algebraic characterisations of the collections of S-integral definite and S-infinitesimal definite rational functions for certains sets S as in (1) below. We denote these sets $\mathcal{O}(S)$ and $\mathcal{M}(S)$, respectively. We will refer to these types of results as Stellensätze after the Nullstellensatz, Positivstellensatz and Nichtnegativstellensatz of real algebraic geometry.

Our method is model-theoretic in nature. Formally, we work throughout (with the exception of in the following section) in expansions of the language of valued fields $\mathcal{L}_{\text{div}} = \mathcal{L}_{\text{rings}} \cup \{\text{div}\}$ where the binary predicate div is defined by $\text{div}(x, y) \leftrightarrow v(x) \leq v(y)$. Then we may axiomatise valued fields in $\mathcal{L}_{\text{div}}$ by the axioms for fields together with

$$\forall X[\operatorname{div}(X,X)]$$
$$\forall X\forall Y[\operatorname{div}(X,Y) \vee \operatorname{div}(Y,X)]$$
$$\forall X\forall Y\forall Z[\operatorname{div}(X,Y) \wedge \operatorname{div}(Y,Z) \to \operatorname{div}(X,Z)]$$
$$\forall X\forall Y\forall Z[\operatorname{div}(X,Y) \leftrightarrow \operatorname{div}(XZ,YZ)]$$
$$\forall X\forall Y[\operatorname{div}(1,X) \wedge \operatorname{div}(1,Y) \to \operatorname{div}(1,X+Y)]$$

We denote the above $\mathcal{L}_{\text{div}}$-theory of valued fields by VF. We shall, however, often use the notation $v(X) \leq v(Y)$ and $v(X) < v(Y)$ with the understanding that these are abbreviations of the appropriate $\mathcal{L}_{\text{div}}$-formulae. The sets S we consider are the $\mathcal{L}_{\text{div}}$-definable subsets of the form

$$(1) \qquad S = \left\{ x \in K^n : \bigwedge_{i\in I} v(f_i(x)) \geq v(f(x)) \ \& \ \bigwedge_{j\in J} v(g_j(x)) > v(g(x)) \right\}$$

for some $f, f_i, g, g_j, h_k \in K[X]$.

Our model-theoretic method requires that the valued fields in which we formulate our Stellensätze be models of model-complete theories. Recall that an $\mathcal{L}$-theory $\mathcal{T}$ is model-complete if whenever $\mathcal{A}$ and $\mathcal{B}$ are models of $\mathcal{T}$ such that $\mathcal{A} \subseteq \mathcal{B}$ then for each quantifier-free $\mathcal{L}$-formula $\varphi(X,a)$ with parameters $a \in A^n$ for some $n \in \mathbb{N}$, we have

$$\mathcal{B} \models \exists X\varphi(X,a) \text{ if and only if } \mathcal{A} \models \exists X\varphi(X,a).$$

Moreover, $\mathcal{T}$ is the model-companion of $\mathcal{T}' \subset \mathcal{T}$ if $\mathcal{T}$ is model-complete and every model $\mathcal{A}$ of $\mathcal{T}'$ can be extended to a model of $\mathcal{T}$. For background on these and other model-theoretic notions, we direct the reader to [14] or [18].

In particular, if an $\mathcal{L}_{\text{div}}$-theory, $\mathcal{T}$, admits quantifier elimination in $\mathcal{L}_{\text{div}}$, then $\mathcal{T}$ is model-complete in $\mathcal{L}_{\text{div}}$. In this case, the sets S as in (1) are almost completely general as every definable subset of a model of $\mathcal{T}$ is a finite union of sets defined by formulae

$$\varphi_S(X) := \bigwedge_{i\in I} v(f_i(X)) \geq v(f(X)) \ \& \ \bigwedge_{j\in J} v(g_j(X)) > v(g(X)) \ \& \ \bigwedge_{k\in K} h_k(X) = 0$$

for some $f, f_i, g, g_j, h_k \in K[X]$. The zero conditons above introduce new complications that we will not address here. Thus, we restrict ourselves to sets S as in (1).

Kochen first studied the question of integral definite rational functions in the p-adic case. In [15], the globally integral definite functions over a p-adically closed field are characterised. Moreover, Lemma 5 of [15] is the foundation of all further results in this area. Indeed, it is an extension of this lemma (Theorem 1.2) that is critical to our results. Bélair, in [3], obtains a similar result characterising the globally integral definite functions in the case of wittian difference fields. Prestel and Roquette in [19] extend Kochen's result with a characterisation of the S-integral definite rational functions for sets S defined by integrality conditions. In [12], Haskell and Yaffe formalise a framework for proving Stellensätze in valued fields and carefully consider the question of whether a rational function might be considered integral at a point where it is not defined. In particular, they characterise the S-integral definite rational functions for sets S defined by integrality conditions in D-Henselian fields (as in [22]) and real closed valued fields. Further, in [11], Guzy introduces the notions of henselian residually p-adically closed field and characterises the globally integral definite rational functions in this setting.

The contribution of this manuscript is to extend all of these results by characterising in each case the integral definite and infinitesimal definite rational functions. Using similar model-theoretic methods, recent work of Yaffe and Lavi follows that of Dickmann [8] to obtain results on the integrality of rational functions on sets defined by positivity conditions in the real closed setting.

We shall proceed as follows. In the following section we briefly describe Artin's solution to Hilbert's 17th Problem. In Chapter 1, we prove the extension of Kochen's Lemma (Theorem 1.2) that is the foundation for the results in the remainder of the manuscript. In Chapter 2, we consider the problem of integral definite and infinitesimal definite rational functions in the pure valued field setting. Specifically, we characterise the S-integral definite and S-infinitesimal definite rational functions for subsets of algebraically closed valued fields (Theorem 2.1) and p-adically closed fields (Theorem 2.3) defined by integrality and infinitesimality conditions. In Chapter 3, we consider valued fields with additional structure. When new function or relation symbols are added to the language, we must refine our notions of integrality and infinitesimality. We use the notions of $\mathcal{T}$-integrality (see [12]) and $\mathcal{T}$-infinitesimality to obtain Stellensätze in D-henselian fields (Theorem 3.5), a model-complete theory of wittian difference fields (Theorem 3.8) and real closed valued fields (Theorem 3.10). In Chapter 4, we use the terminology of Delon in [6] to introduce the notion of a henselian residually valued field. We then obtain Stellensätze in two model-complete theories of henselian residually valued fields: henselian residually p-adically closed fields (Theorem 4.7) and henselian residually real closed valued fields (Theorem 4.9).

3. A Model-Theoretic Proof of Artin's Nichtnegativstellensatz

In order to make transparent the analogy to our own results, we provide here a model-theoretic proof of the the real Nichtnegativstellensatz that is Artin's solution to Hilbert 17th Problem.

Recall that a field is formally real if -1 is not a sum of squares. In this case, we can define an ordering $<$ on K so that $(K, <)$ is an ordered field. A formally real field is real closed if it has no formally real algebraic extensions. We work in the language of ordered rings $\mathcal{L}_< = \{+, -, \cdot, 0, 1, <\}$. It is routine to axiomatise ordered fields in $\mathcal{L}_<$. We shall denote the $\mathcal{L}_<$-theory of ordered fields by OF. We may further axiomatise real closed fields by the axioms for OF together with

$$\forall X_0 \dots \forall X_{2n} \exists Y (Y^{2n+1} + \sum_{i=0}^{2n} X_i Y^i = 0).$$

We shall denote the above $\mathcal{L}_<$-theory of real closed fields by RCF.

It is well-known that the theory RCF is model-complete as RCF admits elimination of quantifiers in the language of ordered fields $\mathcal{L}_<$. The quantifier elimination is originally due to Tarksi, but a modern treatment of this result can be found in [18]. Moreover, RCF is the model companion of OF, since every ordered field $(K, <)$ can be extended to a real closed field, its real closure.

Let $(K, <)$ be a model of RCF and let $f(X) \in K(X) = K(X_1, \dots, X_n)$. Now the property of being nonnegative is $\mathcal{L}_<$-definable by the universal $\mathcal{L}_<$-formula

$$\forall X_1, \dots, \forall X_n [f(X_1, \dots, X_n) \geq 0].$$

Thus its negation is existentially definable by

$$\exists X_1 \dots \exists X_n [f(X_1, \dots, X_n) < 0].$$

By the model-completeness of RCF, to show that $f(X)$ is not nonnegative, it will suffice to witness the above existential formula in any real closed field extending $(K, <)$. In particular, the field of rational functions $K(X)$ can be made into a model of OF, $(K(X), <')$, extending $(K, <)$. Then, as RCF is the model-companion of OF, we may further extend to some model of RCF. However, the ordering $<'$ on $K(X)$ is by no means unique. Thus some information regarding the possible orderings $<'$ on $K(X)$ is needed.

Given an ordered field $(K, <)$, let

$$P_< := \{x \in K : x \geq 0\}$$

be the positive cone of the ordering $<$. Further, let

$$\sum K^2 = \left\{ \sum_{i \in I} a_i^2 : a_i \in K, I \text{ finite} \right\}$$

be the subsemiring of sums of squares in K. Finally, let

$$q(\sum K^2) = \left\{ x \in K : ax \in \sum K^2 \text{ for some } a \in \sum K^2, a \neq 0 \right\}.$$

The following is true for any field extension L of K. We are, of course, interested in the case where $L = K(X)$.

LEMMA 0.2 (Artin's Criterion). *Let $(K, <)$ be an ordered field and let L be an extension field of K. Then $q(\sum L^2)$ is the intersection of all the positive cones $P_{<'}$ of orderings $<'$ on L that extend the ordering on K.*

The utility of Artin's Criterion comes from the contrapositive: given $a \notin q(\sum L^2)$, there is an ordering $<'$ on L extending the ordering on K such that $a <' 0$.

THEOREM 0.3 (Artin's Nichtnegativstellensatz). *Let $(K, <)$ be a real closed field. Then $f(X) \in K(X)$ is positive semidefinite if and only if $f \in q(\sum K(X)^2)$.*

A MODEL THEORETIC PROOF. The right-to-left direction is clear as every square is certainly nonnegative and the property of being nonnegative is preserved under multiplication, addition and division by positive elements. For the converse, assume for a contradicition that $f(X) = \frac{f_1(X)}{f_2(X)} \notin q(\sum K(X)^2)$. Then, as in the remarks after Lemma 0.2, there is an ordering $<'$ on $K(X)$ such that $f(X)$, as an element of the ordered field $(K(X), <')$, is negative. In particular, we have

$$(K(X), <') \models \exists X \left[(f_1(X) < 0 \vee f_2(X) < 0) \ \& \ \neg(f_1(X) < 0 \wedge f_2(X) < 0) \right].$$

Now $(K(X), <')$ is a model of OF but is not, in general, real closed. However, since RCF is the model companion of OF, we may extend $(K(X), <')$ to a real closed field, say $(L, <')$. Then, in particular, $(K, <) \subseteq (L, <')$ and

$$(L, <') \models \exists X \left[(f_1(X) < 0 \vee f_2(X) < 0) \ \& \ \neg(f_1(X) < 0 \wedge f_2(X) < 0) \right].$$

By the model-completeness of RCF, we have that

$$(K, <) \models \exists X \left[(f_1(X) < 0 \vee f_2(X) < 0) \ \& \ \neg(f_1(X) < 0 \wedge f_2(X) < 0) \right].$$

This contradicts that $f(X)$ is positive semidefinite. □

CHAPTER 1

A Key Lemma

When we work in the valued field setting, the general idea of the proof will resemble the proof in the real algebraic setting. Given a model of VF, (K, v), and $f(X) \in K(X) = K(X_1, \ldots, X_n)$, the properties of being S-integral definite and S-infinitesimal definite are $\mathcal{L}_{\text{div}}$-definable by the universal formulae

$$\forall X_1 \ldots \forall X_n [v(f(X_1, \ldots, X_n)) \geq 0]$$

and

$$\forall X_1 \ldots \forall X_n [v(f(X_1, \ldots, X_n)) > 0].$$

Thus, the properties of not being S-integral definite and of not being S-infinitesimal definite are existentially $\mathcal{L}_{\text{div}}$-definable. If $(K, v, ...)$ is, in fact, a model of a model-complete theory of valued fields, $\mathcal{T}$, then it will suffice to witness these existential formulae in any model of $\mathcal{T}$ extending $(K, v, ...)$. In particular, we will be concerned with the case where we have a model of $\mathcal{T}$ extending $(K(X), \overline{v}, \ldots)$ with $\overline{v}$ extending v. Thus, as in the real setting, we will require information regarding the possible extensions of the valuation v to $K(X)$. In particular, we require a generalisation of the following lemma of Kochen.

Lemma 1.1 (Kochen's Lemma, Lemma 5 of [15]). *Let (K, v) be a valued field with extension field L and let A be a subring of L such that $A \cap K = \mathcal{O}_v$. Let $T = \{1 + ma : m \in \mathcal{M}_v, a \in A\}$. Then the (ring-theoretic) integral closure of $A_T = \{x \in L : tx \in A$ for some $t \in T\}$ is the intersection of all valuation rings $\mathcal{O}_{\overline{v}}$ of valuations $\overline{v}$ on L extending the valuation v on K such that $A \subseteq \mathcal{O}_{\overline{v}}$.*

This is an analogue for valued fields of Artin's Criterion. Again, its utility lies in the contrapositive: given $x \in L$ not integral over A_T, Kochen's Lemma implies the existence of a valuation $\overline{v}$ on L extending v and such that $\overline{v}(a) \geq 0$ for each $a \in A$ but $\overline{v}(x) < 0$. We will generalise this lemma to include information about infinitesimal elements.

We first fix some notation. Given a subring A of a field L and a proper ideal B of A, the set $T = \{1 + b : b \in B\}$ is a multiplicative subset of A. Thus the set

$$A_T = \{x \in L : tx \in A \text{ for some } t \in T\}$$

is just the localisation of A at T and is a subring of L. Moreover, the set

$$B_T = \{x \in L : tx \in B \text{ for some } t \in T\}$$

is the image of the ideal B under the localisation. In particular, B_T is a proper ideal of A_T. Let $\sqrt[\text{int}]{A}^L$ and $\sqrt[\text{int}]{B}^L$ denote the integral closures in L of A_T and B_T, respectively. We shall omit the superscript whenever the context permits.

THEOREM 1.2. *Let (K, v) be a valued field, L a field extension of K, $A \subset L$ a subring such that $A \cap K = \mathcal{O}_v$ and B a proper ideal of A such that $B \cap K = \mathcal{M}_v$. Let $T = \{1 + b : b \in B\}$.*

(i) *The set $\sqrt[\text{int}]{A}$ is the intersection of all valuation rings $\mathcal{O}_{\overline{v}}$ of valuations $\overline{v}$ on L with maximal ideals $\mathcal{M}_{\overline{v}}$ such that $A \subseteq \mathcal{O}_{\overline{v}}$, $B \subseteq \mathcal{M}_{\overline{v}}$ and $\overline{v}$ extends v.*

(ii) *The set $\sqrt[\text{int}]{B}$ is the intersection of all maximal ideals $\mathcal{M}_{\overline{v}}$ of valuation rings $\mathcal{O}_{\overline{v}}$ of valuations $\overline{v}$ on L such that $A \subseteq \mathcal{O}_{\overline{v}}$, $B \subseteq \mathcal{M}_{\overline{v}}$ and $\overline{v}$ extends v.*

Before we prove Theorem 1.2, we will need an easy corollary to the standard valuation theoretic result known as Chevalley's Theorem.

THEOREM 1.3 (Chevalley's Theorem, Theorem 3.1.1 of [9], for example). *Let K be a field and let R be a subring of K with prime ideal p. Then there is a valuation v on K such that $R \subseteq \mathcal{O}_v$ and $\mathcal{M}_v \cap R = p$.*

Given a valued field (K, v) and an extension field L of K, the following corollary tells us when a subring of L can be extended to a valuation ring of L whose corresponding valuation extends the valuation v.

COROLLARY 1.4. *Let (K, v) be a valued field. Let L be an extension field of K, let A be a subring of L such that $A \cap K = \mathcal{O}_v$, and let I be a proper ideal of A such that $I \cap K = \mathcal{M}_v$. Then A can be extended to a valuation ring $\mathcal{O}_{\overline{v}}$ on L with maximal ideal $\mathcal{M}_{\overline{v}}$ extending I. In particular, $\overline{v}$ extends v.*

PROOF. Let $\mathcal{M}$ be any maximal ideal of A containing I. Then, in particular, $\mathcal{M}$ is prime and, by Chevalley's Theorem, there is a valuation $\overline{v}$ on L with $A \subseteq \mathcal{O}_{\overline{v}}$ and $\mathcal{M}_{\overline{v}} \cap \mathcal{O}_v = \mathcal{M}$. Then $I \subseteq \mathcal{M}_{\overline{v}}$. Moreover, $\mathcal{O}_{\overline{v}} \cap K = \mathcal{O}_v$, for $x \in \mathcal{O}_{\overline{v}} \cap K$ but $x \notin \mathcal{O}_v$ we have $x^{-1} \in \mathcal{M}_v$. Hence, $x^{-1} \in \mathcal{M} \subseteq \mathcal{M}_{\overline{v}}$. This is a contradiction as $x \in \mathcal{O}_{\overline{v}}$. Thus $\overline{v}$ extends v. □

We are now ready to proceed with the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Let $\overline{v}$ be a valuation on L such that $A \subseteq \mathcal{O}_{\overline{v}}$, $B \subseteq \mathcal{M}_{\overline{v}}$ and $\mathcal{O}_{\overline{v}} \cap K = \mathcal{O}_v$. We first show that $\sqrt[\text{int}]{A} \subseteq \mathcal{O}_{\overline{v}}$ and $\sqrt[\text{int}]{B} \subseteq \mathcal{M}_{\overline{v}}$. First, observe that each element of T has valuation zero, since $\overline{v}(b) > 0$ for each $b \in B$ so that $\overline{v}(1) < \overline{v}(b)$ and, hence, $\overline{v}(1+b) = \overline{v}(1) = 0$. Then, since for any valuation v,

$$v\left(\frac{x}{y}\right) = v(x) - v(y),$$

the properties of being integral and infinitesimal are preserved by localisation at this multiplicative set. Thus $A_T \subseteq \mathcal{O}_v$ and $B_T \subseteq \mathcal{M}_v$. Now suppose α is integral over A_T. That is, set $p(Y) = Y^n + \sum_{i=0}^{n-1} a_i Y^i$ with $a_i \in A_T$ for each $i = 0, 1, \ldots, n-1$, and suppose that $p(\alpha) = 0$. Assume, for a contradiction, that $\overline{v}(\alpha) < 0$. Now

$$\overline{v}(p(\alpha)) \geq \min\{\overline{v}(\alpha^n), \overline{v}(a_{n-1}\alpha^{n-1}), \ldots, \overline{v}(a_0)\}$$

and we have equality if the minimum value occurs uniquely. Since $\overline{v}(a_i) \geq 0$ for each i, we find that

$$\overline{v}(\alpha^n) = n\overline{v}(\alpha) < \overline{v}(a_i) + i\overline{v}(\alpha) = \overline{v}(a_i\alpha^i).$$

Thus $\overline{v}(p(\alpha)) = n\overline{v}(\alpha) < 0$. This is a contradiction as $\overline{v}(p(\alpha)) = \overline{v}(0) > 0$. Thus $\sqrt[\text{int}]{A} \subset \mathcal{O}_{\overline{v}}$.

Now suppose α is integral over B_T. Set $p(Y) = Y^n + \sum_{i=0}^{n-1} b_i Y^i$ with $b_i \in B_T$ for each $i = 0, 1, \ldots, n-1$, and suppose that $p(\alpha) = 0$. Since $b_i \in B_T$, we have $v(b_i) > 0$ for each i. Thus we may apply the residue map, $\mathrm{res}_{\overline{v}}$. In the residue field $k_{\overline{v}}$,

$$\begin{aligned}
0 = \mathrm{res}_{\overline{v}}(0) &= \mathrm{res}_{\overline{v}}(p(\alpha) \\
&= \mathrm{res}_{\overline{v}}(\alpha)^n + \sum_{i=0}^{n-1} \mathrm{res}_{\overline{v}}(b_i)\,\mathrm{res}_{\overline{v}}(\alpha)^i \\
&= \mathrm{res}_{\overline{v}}(\alpha)^n.
\end{aligned}$$

Thus $\mathrm{res}_{\overline{v}}(\alpha) = 0$ and it follows that $v(\alpha) > 0$. Thus $\sqrt[\text{int}]{B} \subseteq \mathcal{M}_{\overline{v}}$.

For the reverse inclusions, we first consider α not integral over B_T and show that there is a valuation, $\overline{v}$, on L extending v on K such that $\overline{v}(A) \geq 0$ and $\overline{v}(B) > 0$ but $\overline{v}(\alpha) < 0$.

Observe that if $\alpha \in L \backslash K$ is not integral over B_T then the ideal generated by B_T and α^{-1}, $\mathcal{M} = B_T[\alpha^{-1}]$, is a proper ideal of $A_T[\alpha^{-1}]$. To see this, suppose $\mathcal{M} = A_T[\alpha^{-1}]$. Then

$$-1 = \sum_{i=0}^{r} \beta_i \alpha^{-i}$$

for some $r \in \mathbb{N}$ and $\beta_i \in B_T$. Multiplying by α^r, we get

$$0 = (1+\beta_0)\alpha^r + \sum_{i=1}^{r} \beta_i \alpha^{r-i}.$$

Now $\beta_0 = \frac{b_1}{1+b_2}$ for some $b_1, b_2 \in B$. Thus multiplying by $1 + b_2$, we obtain

$$0 = (1+b_2+b_1)\alpha^r + \sum_{i=1}^{r} (\beta_i + b_2\beta_i)\alpha^{r-i}.$$

Since $(1 + b_2 + b_1) \in T$ we may divide by it to see that α is integral over B_T. This is a contradiction, so we must have that $\mathcal{M} = B_T[\alpha^{-1}]$ is a proper ideal.

Then, by Corollary 1.4, for α not integral over B_T, $A_T\,[\alpha^{-1}]$ can be extended to a valuation ring $\mathcal{O}_{\overline{v}}$ of L with maximal ideal $\mathcal{M}_{\overline{v}}$ extending $B_T\,[\alpha^{-1}]$ such that $\overline{v}$ extends v.

Finally, if α is not integral over A_T, then, in particular, α is not integral over B_T and the argument above yields a valuation $\overline{v}$ on L extending v on K such that $\overline{v}(A) \geq 0$ and $\overline{v}(B) > 0$ but $\overline{v}(\alpha) < 0$. □

Again, the utility of Theorem 1.2 lies in its contrapositive. Given $x \in L$ not integral over A_T, there is a valuation $\overline{v}$ on L such that $\overline{v}(a) \geq 0$ for each $a \in A$ and $\overline{v}(b) > 0$ for each $b \in B$, but $\overline{v}(x) < 0$. Similarly, for $x \in L$ not integral over B_T there is a valuation $\overline{v}$ on L such that $\overline{v}(a) \geq 0$ for each $a \in A$ and $\overline{v}(b) > 0$ for each $b \in B$, but $\overline{v}(x) \leq 0$.

Notice that we recover Kochen's Lemma as the special case of Theorem 1.2 where $B = \mathcal{M}_v \cdot A$.

CHAPTER 2

Pure Valued Fields

By a pure valued field, we mean a model of a theory of valued fields in a language $\mathcal{L}$ that is an expansion of $\mathcal{L}_{\text{div}}$ with no new function or relation symbols. As we shall see in Chapter 3, interactions between additional functions or relations and the valuation require us to re-evaluate our notions of integrality and infinitesimality. Our pure theories are likely those with which the reader is most familiar: the theories of algebraically closed valued fields and p-adically closed fields. The results in algebraically closed valued fields build on the work of Haskell and Yaffe in [12]. While the results in p-adically closed fields have been known for some time (see [15] and [19]), we have included them here for the sake of completeness and to better illustrate our method of proof.

1. Algebraically Closed Valued Fields

A valued field (K, v) is an algebraically closed valued field if the field K is algebraically closed. We can easily axiomatise algebraically closed fields in $\mathcal{L}_{\text{div}}$ by the axioms for valued fields and axioms

$$\forall X_0, \ldots, \forall X_n \exists Y \left(\sum_{i=0}^{n} X_i Y^i = 0 \right)$$

for each positive integer n. We denote the $\mathcal{L}_{\text{div}}$-theory of algebraically closed valued fields by ACVF.

That ACVF is model-complete in the language $\mathcal{L}_{\text{div}}$ is shown in [21]. In fact, ACVF admits quantifier elimination. Moreover, ACVF is the model-compnaion of the theory of valued fields, VF. To see this, one need only verify that given a valued field (K, v), the valuation can be extended to the algebraic closure of K (see [9] or [20], for example).

It follows from quantifier elimination that every definable subset of an algebraically closed valued field, (K, v), is a finite Boolean combination of sets defined by formulae

$$\varphi_S(X) := \bigwedge_{i \in I} v(f_i(X)) \geq v(f(X)) \ \& \ \bigwedge_{j \in J} v(g_j(X)) > v(g(X)) \ \& \ \bigwedge_{k \in K} h_k(X) = 0$$

for some $f, f_i, g, g_j, h_k \in K[X]$. Notice that for $g(x) \neq 0$, we have

$$v(f(x)) \geq v(g(x)) \text{ if and only if } v\left(\frac{f(x)}{g(x)}\right) = v(f(x)) - v(g(x)) \geq 0.$$

Thus we will often abuse our notation slightly and write

$$\varphi_S(X) := \bigwedge_{i \in I} v(f_i(X)) \geq 0 \ \& \ \bigwedge_{j \in J} v(g_j(X)) > 0 \ \& \ \bigwedge_{k \in K} h_k(X) = 0$$

with $f_i, g_j \in K(X)$.

Given any subset $S \subseteq K^n$, let $\mathcal{O}(S)$ denote the collection of S-integral definite rational functions. Let $\mathcal{M}(S)$ denote the collection of S-infinitesimal definite rational functions. Our goal is to characterise $\mathcal{O}(S)$ and $\mathcal{M}(S)$ for sets S of the above form. However, as remarked earlier, the zero conditons introduce new difficulties that we choose not to address here. Thus, throughout this thesis, we shall restrict ourselves to subsets defined by

$$\varphi_S(X) := \bigwedge_{i \in I} v\left(f_i(X)\right) \geq 0 \ \& \ \bigwedge_{j \in J} v\left(g_j(X)\right) > 0 \tag{2}$$

where $f_i, g_j \in K(X)$.

The approach is simple and similar to Hilbert's approach in conjecturing the Nichtnegativstellensatz. We collect all of the forms which are obviously definite and prove, with the help of Theorem 1.2, that there are no others.

Given a set as in (2), it is clear that the f_i and the g_j are S-integral definite. Moreover, we may take products and sums of these as well as multiply by elements of the ring of integral constant functions $\mathcal{O}_v$. Thus the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by the f_i and the g_j is a subset of $\mathcal{O}(S)$. Now since the product of an infinitesimal element and an integral element yields an infinitesimal element, $\mathcal{M}(S)$ is an ideal of $\mathcal{O}(S)$. Further, the g_j as well as the constant infinitesimal functions, $\mathcal{M}_v$, are clearly S-infinitesimal definite. Then the ideal generated by $\mathcal{M}_v$ and the g_j is a subset of $\mathcal{M}(S)$. Finally, as in Theorem 1.2, the properties of being S-integral definite and S-infinitesimal definite are preserved under division by elements of valuation zero and taking integral closures. This motivates the first of our main results.

THEOREM 2.1. *Suppose (K, v) is a model of ACVF and S is a nonempty subset of K^n defined by a formula φ_S as in (2). Let A be the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by $\{f_i, g_j\}_{i \in I}^{j \in J}$. Let B be the ideal of A generated by $\mathcal{M}_v$ and $\{g_j\}_{j \in J}$. Let $T = \{1 + b : b \in B\}$. Then $h \in K(X)$ is S-integral definite if and only if h is integral (in the ring-theoretic sense) over A_T, and h is S-infinitesimal definite if and only if h is integral over B_T. That is $\mathcal{O}(S) = \sqrt[\text{int}]{A}$ and $\mathcal{M}(S) = \sqrt[\text{int}]{B}$.*

Recall that A_T is just the localisation of the ring A at the multiplicative set T and B_T is the image of B under this localization.

PROOF. The right-to-left implication is clear as each element of A is S-integral definite, each element of B is S-infinitesimal definite and the properties of being S-integral and S-infinitesimal definite are preserved under localisation at T and under taking integral closures. To see this, we apply the argument in the proof of Theorem 1.2 to each point $b \in S$.

We prove the converse by contradiction. Suppose that $h(X) \in K(X)$ is S-integral definite, but that $h(X)$ is not integral over A_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X)) < 0$. In particular, $\overline{v}(f_i(X)) \geq 0$ for each $i \in I$ and $\overline{v}(g_j(X)) > 0$ for each $j \in J$. Then

$$(K(X), \overline{v}) \models \exists X \left(\bigwedge_{i \in I} v(f_i(X)) \geq 0 \ \& \ \bigwedge_{j \in J} v(g_j(X)) > 0 \ \& \ v(h(X)) < 0 \right)$$

witnessed by the element $X \in K(X)$. That is,

$$(K(X), \overline{v}) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0 \right].$$

Now $K(X)$ is not, in general, algebraically closed so we may not yet apply model-completeness. However, since ACVF is the model companion of VF, we may extend $(K(X), \overline{v})$ to an algebraically closed valued field, say (L, w). Then $(K, v) \subseteq (L, w)$ and

$$(L, w) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0 \right].$$

By the model-completeness of ACVF,

$$(K, v) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0 \right].$$

This contradicts that $h(X)$ is S-integral definite.

Similarly, suppose $h(X) \in K(X)$ is S-infinitesimal definite but $h(X)$ is not integral over B_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X)) \leq 0$. Then

$$(K(X), \overline{v}) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) \leq 0 \right].$$

Again, as ACVF is the model companion of VF, we may extend $(K(X), \overline{v})$ to a model of ACVF, (L, w), satisfying the same condition. In particular, $(K, v) \subseteq (L, w)$ and

$$(L, w) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) \leq 0 \right].$$

Then ,by the model-completeness of ACVF, (K, v) satisfies this formula as well. This contradicts that $h(X)$ is S-infinitesimal definite. □

This result extends Theorem 2.5 of [12], firstly, by considering sets S defined by both integrality and infinitesimality condiions and, secondly, by characterising both $\mathcal{O}(S)$ and $\mathcal{M}(S)$. In fact, Theorem 2.5 of [12] is the special case of Thorem 2.1 where the set J is trivial so that $B = \mathcal{M}_v \cdot A$.

2. p-adically Closed Fields

A valued field (K, v) is p-valued if $v(p)$ is minimal positive in the value group and $k_v = \mathbb{F}_p$. We can axiomatise p-valued fields in $\mathcal{L}_{\text{div}}$ by the axioms for valued fields together with

$$\neg \operatorname{div}(p, 1)$$

$$\operatorname{div}(1, p) \rightarrow \bigvee_{i=1}^{p} \operatorname{div}(p, x - i).$$

We denote this theory pVF. A p-valued field is p-adically closed if the value group, Γ_v, is a $\mathbb{Z}$-group (i.e. Γ_v is a model of $\text{Th}(\mathbb{Z})$) and Hensel's Lemma is satisfied.

Now, although we may axiomatise p-adically closed fields in $\mathcal{L}_{\text{div}}$, this $\mathcal{L}_{\text{div}}$-theory does not eliminate quantifiers. It is shown in [17] that one needs predicates P_n for the n-th powers in order to obtain quantifier elimination. Thus we work in $\mathcal{L}_{\text{mac}}$, the expansion of $\mathcal{L}_{\text{div}}$ by these predicates. The $\mathcal{L}_{\text{mac}}$-theory of p-adically closed fields admits quantifier elimination. We denote this theory by pCF. Moreover, pCF is the model-companion of the theory of p-valued fields, pVF, as any p-valued field can be extended to a p-adically closed field, namely, its p-adic closure.

Kochen first characterised the globally integral definite functions for p-adically closed fields in [15] and later Prestel and Roquette characterised the S-integral definite rational functions for sets S defined by integrality conditions [19]. Since the valuation is discrete we have

$$v(g_j(x)) > 0 \leftrightarrow v(p^{-1} g_j(x)) \geq 0.$$

Hence, we need only consider sets S defined by integrality conditions. However, we continue to work with subsets S defined as in the previous section,

$$\varphi_S(X) := \bigwedge_{i \in I} v\left(f_i(X)\right) \geq 0 \ \& \ \bigwedge_{j \in J} v\left(g_j(X)\right) > 0, \tag{3}$$

in order to better illustrate our method. Since pCF does not eliminate quantifiers in the pure language of valued fields, the Boolean combinations of sets defined by formulae of type (3) do not constitute all of the definable subsets of a p-adically closed field. Ultimately, one would want to consider sets S defined also by zero conditions and n-th power conditions. However, here we shall restrict ourselves to sets S as above.

Now we would like to proceed as in the case of algebraically closed valued fields. However, first we examine more closely a key step in the proof of Theorem 2.1. Given the set A, the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by the f_i and g_j, and the set B, the ideal of A generated by $\mathcal{M}_v$ and the g_j, we applied Theorem 1.2 to obtain a valuation $\overline{v}$ on $K(X)$ with $\overline{v}(A) \geq 0$, $\overline{v}(B) > 0$ and $\overline{v}(h) < 0$. Since any valuation w on $K(X)$ makes $(K(X), w)$ a model of VF, in particular, $(K(X), \overline{v})$ is a model of VF. Then, as ACVF is the model companion of VF, we could extend to a model of ACVF. In order to use this method to obtain Kochen's and Prestel and Roquette's results, we must ensure that the valuation obtained by applying Theorem 1.2 makes $(K(X), \overline{v})$ a model of pVF. To this end, we must introduce the Kochen operator.

Let (K, v) be any valued field with $\mathrm{char}(K) \neq p$. For any $x \in K$ we set

$$\boxed{\gamma_p(x) = \frac{1}{p}\left(\frac{x^p - x}{(x^p - x)^2 - 1}\right).}$$

PROPOSITION 2.2 (Lemma 2 of [15]). *A valued field (K, v) is p-valued if and only if $\gamma_p(K) \subseteq \mathcal{O}_v$.*

The left-to-right implication is clear as, if (K, v) is a p-valued field, then the rational function $\gamma_p(X)$ is globally integral definite. To see this let $x \in K$. Suppose $v(x) \geq 0$. Then we may apply the residue map to the $x^p - x$. Since the residue field has characteristic p, $\mathrm{res}(x^p - x) = 0$. Hence, $v(x^p - x) > 0$

$$\begin{aligned} v\left(\frac{x^p - x}{(x^p - x)^2 - 1}\right) &= v(x^p - x) - v((x^p - x) - 1) \\ &= v(x^p - x) - \min\{2v(x^p - x), 0\} \\ &= v(x^p - x) > 0. \end{aligned}$$

If $v(x) < 0$, then $v(x^p) < v(x)$. Thus $v(x^p - x) = pv(x) < 0$. Then

$$\begin{aligned} v\left(\frac{x^p - x}{(x^p - x)^2 - 1}\right) &= v(x^p - x) - v((x^p - x) - 1) \\ &= v(x^p - x) - \min\{2v(x^p - x), 0\} \\ &= v(x^p - x) - 2v(x^p - x) > 0. \end{aligned}$$

For the right-to-left implication, we evaluate $\gamma_p(X)$ at any element $x \in L$ with $v(x) > 0$. Then, as in the calculations above, $v(x^p - x) = v(x)$ and $v\left(\frac{x^p - x}{(x^p - x)^2 - 1}\right) = v(x^p -$

$x) - v((x^p - x) - 1) = v(x)$. It follows that $v(p)$ must be minimal positive as

$$v\left(\frac{1}{p}\left(\frac{x^p - x}{(x^p - x)^2 - 1}\right)\right) = v\left(\frac{1}{p}\right) + v\left(\frac{x^p - x}{(x^p - x)^2 - 1}\right)$$
$$= v\left(\frac{1}{p}\right) + v(x).$$

Then for any subset $S \subseteq K^n$ we should have $\gamma_p(K(X)) \subseteq \mathcal{O}(S)$ and $\mathcal{M}_v \cdot \gamma_p(K(X)) \subseteq \mathcal{M}(S)$.

THEOREM 2.3. *Suppose (K, v) is a model of pCF and S is a nonempty subset of K^n defined by*

$$\varphi_S(X) := \bigwedge_{i \in I} v(f_i(X)) \geq 0 \ \& \ \bigwedge_{j \in J} v(g_j(X)) > 0$$

for some $f_i, g_j \in K(X)$. Let A be the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by $\{f_i, g_j\}_{i \in I}^{j \in J}$ and $\gamma_p(K(X))$. Let B be the ideal of A generated by $\mathcal{M}_v$ and $\{g_j\}_{j \in J}$. Let $T = \{1+b : b \in B\}$. Then $h \in K(X)$ is S-integral definite if and only if h is integral over A_T, and h is S-infinitesimal definite if and only if h is integral over B_T. That is $\mathcal{O}(S) = \sqrt[\text{int}]{A}$ and $\mathcal{M}(S) = \sqrt[\text{int}]{B}$.

PROOF. As in the proof of Theorem 2.1, the right-to-left implication follows from the fact that the properties of being S-integral and S-infinitesimal definite are preserved by localisation at T and by taking integral closures. For the left-to-right implication, the proof also proceeds similarly. First, suppose $h(X) \in K(X)$ is S-integral definite but not integral over A_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$, $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X)) < 0$. In particular,

$$(K(X), \overline{v}) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right].$$

Now since $\gamma_p(K(X)) \subseteq A \subseteq \mathcal{O}_{\overline{v}}$, $(K(X), \overline{v})$ is a p-valued field and, since pCF is the model companion of pVF, we may extend $(K(X), \overline{v})$ to a p-adically closed field, (L, w). In particular, $(K, v) \subseteq (L, w)$ and

$$(L, w) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right].$$

Then by model-completeness (K, v) satisfies the same formula. This contradicts that $h(X)$ is S-integral definite.

We could repeat the same argument for the S-infinitesimal definite case, however, as the valuation is discrete, this is immediate as $\mathcal{M}(S) = \mathcal{M}_v \cdot \mathcal{O}(S)$. □

CHAPTER 3

Valued Fields with Additional Structure

We now investigate theories of valued fields in which new function or relation symbols are added to our language $\mathcal{L}$. For example, a valued D-field is a valued field (K, v) together with an additive map $D : K \to K$ satisfying the twisted Leibniz rule $D(xy) = xD(y) + yD(x) + eD(x)D(y)$ where e is a distinguished element of positive valuation. We further require a strong interaction between the valuation and the operator D in the sense that $v(D(x)) \geq v(x)$ for each $x \in K$.

Here we work in a language $\mathcal{L}_{\text{VDF}} = \mathcal{L}_{\text{div}} \cup \{D, e\}$. In $\mathcal{L}_{\text{VDF}}$, we can axiomatise valued D-fields by the axioms for valued fields together with

$$\forall X[v(DX) \geq v(X)].$$

We denote the $\mathcal{L}_{\text{VDF}}$-theory of valued D-fields by VDF.

The terms in our language $\mathcal{L}_{\text{VDF}}$ are the D-polynomials over K, $K[X]_D$. These are the polynomials in the variables $\{D^k X_i\}_{1 \leq i \leq n}^{k \geq 0}$ with coefficients in K. In particular, we will consider the ring of D-polynomials as the ring of polynomials in the variables $Y = \{Y_{i,k} = D^k X_i\}_{1 \leq i \leq n}^{k \geq 0}$ and, for any $r \times n$ matrix α, write Y^α for $\prod_{1 \leq i \leq n}^{0 \leq k \leq r} Y_{i,k}^{\alpha_{i,k}}$. In particular, for any $p \in K[X]_D$ we may write $p = \sum_\alpha p_\alpha Y^\alpha$ where the size of α will depend on the degree of p. We shall also speak of the field of D-rational functions, $K(X)_D$, whose elements are quotients of D-polynomials.

Now $\frac{DX}{X}$ is a D-rational function. For each $x \in K$, $v\left(\frac{Dx}{x}\right) = v(Dx) - v(x)$, which is always non-negative as $v(Dx) \geq v(x)$. Hence, $\frac{DX}{X}$ should be considered globally integral, even at $x = 0$. This differs from our previous notion of integrality where we required a rational function to be defined in order to be considered integral or infinitesimal. To remedy this, we will generalise our notion of integrality to that of $\mathcal{T}$-integrality as in [12]. We will define $\mathcal{T}$-infinitesimality similarly and proceed.

1. Refining Integrality

In [12], Haskell and Yaffe extend the notion of integrality to a more general notion, that of $\mathcal{T}$-integrality, which will allow us to consider some rational functions integral even where not defined. This notion takes into account any interaction between the valuation and new functions and relations in our language.

Fix a language $\mathcal{L}$ expanding the language of valued fields and an $\mathcal{L}$-theory $\mathcal{T}$ of valued fields. Let $\tilde{\mathcal{T}}$ be a model-companion of $\mathcal{T}$ and let $\mathcal{K} = (K, v, \dots)$ be a model of $\tilde{\mathcal{T}}$. Denote by $K[X]_{\mathcal{T}}$ the collection of $\mathcal{L}$-terms with parameters from K. This is the ring of $\mathcal{T}$-polynomials. The $\mathcal{T}$-rational functions are quotients of $\mathcal{T}$-polynomials. We shall denote this field by $K(X)_{\mathcal{T}}$.

Definition 3.1. With the notation above, let $\hat{v}$ be a valuation on $K(X)_{\mathcal{T}}$.

i) We say that $\hat{v}$ is a $\mathcal{T}$-valuation if there is an expansion of $(K(X)_{\mathcal{T}}, \hat{v})$ to a model of $\mathcal{T}$ extending $\mathcal{K}$.
ii) The T-valuation $\hat{v}$ is said to be given by evaluation near b or a $\mathcal{T}$-valuation near $b \in K^n$ if for any $1 \leq i \leq n$ and every $c \in K^{\times}$ we have $\hat{v}(X_i - b_i) > v(c)$.
iii) For $f \in K(X)_{\mathcal{T}}$ and $b \in K^n$, we say that f is $\mathcal{T}$-integral at b if for every $\mathcal{T}$-valuation $\hat{v}$ on $K(X)_{\mathcal{T}}$ which is given by evaluation near b we have $\hat{v}(f) \geq 0$. We say f is $\mathcal{T}$-infinitesimal at b if for every $\mathcal{T}$-valuation $\hat{v}$ near b we have $\hat{v}(f) > 0$.

Under this definition, the function $\frac{DX}{X}$ is surely globally VDF-integral as any $\mathcal{T}$-valuation near a point $b \in K$ must satisfy $\hat{v}(Db) \geq \hat{v}(b)$.

Of course, we would like to know that the notions of $\mathcal{T}$-integrality and $\mathcal{T}$-infinitesimality extend our more naive notions. That is, in order for our proofs to work, it must be the case that a $\mathcal{T}$-rational function, f, is $\mathcal{T}$-integral (or $\mathcal{T}$-infinitesimal) if and only if it is integral (or infinitesimal) wherever f is defined. This, however, is not a priori true.

Definition 3.2. We say that a theory of valued fields, $\mathcal{T}$, is conservative if whenever a function $f \in K(X)_{\mathcal{T}}$ is defined at a point $b \in K^n$ then f is $\mathcal{T}$-integral at b if and only if f is integral at b and f is $\mathcal{T}$-infinitesimal at b if and only if f is infinitesimal at b..

As in the remarks following Definition 2.15 of [12] we observe that the existence of a $\mathcal{T}$-valuation near b for every $b \in K^n$ is sufficient for the conservativity of a theory $\mathcal{T}$. This follows from the observation that for any $\mathcal{T}$-valuation near b, $\hat{v}(f - f(b)) > v(f(b))$. Thus if f is defined at b then $\hat{v}(f) = v(f(b))$ and conservativity follows.

2. D-Henselian Fields

In this section, we consider a model-complete theory of valued D-fields of equicharacteristic zero. In the formulation of Scanlon in [22], a D-Henselian field is a valued D-field satisfying certain conditions; in particular, $\text{char}(K) = \text{char}(k_v) = 0$ and D-Hensel's Lemma, a version of Hensel's Lemma for D-polynomials.

DEFINITION 3.3 (D-Hensel's Lemma). Let (K, v, D) be a valued D-field. Let $\mathcal{O}_v[X]_D$ be the subring of $K[X]_D$ consisting of D-polynomials with coefficients from $\mathcal{O}_v$. We say that (K, v, D) satisfies D-Hensel's Lemma if whenever $P \in \mathcal{O}_v(X)_D$, $a \in \mathcal{O}_v$ and $v(P(a)) > 0 = v(\frac{\partial}{\partial X_i}P(a))$ for some $i = 1, \dots, n$, then there is some $b \in K$ with $P(b) = 0$ and $v(a - b) \geq v(P(a))$.

Scanlon axiomatises D-Henselian fields in a 3-sorted language extending $\mathcal{L}_{\mathrm{VDF}}$. Moreover, Scanlon shows that this theory has quantifier elimination. In particular, the theory of D-Henselian fields is model-complete, and is the model companion of the theory of valued D-fields, VDF.

We now consider sets S of the form

$$\varphi_S(X) := \bigwedge_{i \in I} v(f_i(X)) \geq 0 \;\&\; \bigwedge_{j \in J} v(g_j(X)) > 0$$

with the $f_i, g_j \in K(X)_D$. However, as the language in which D-Henselian fields admits quantifier elimination is somewhat larger, these sets do not constitute all the definable subsets of a D-Henselian field.

Now, in order to apply our method as before, we will need to determine whether a valuation $\overline{v}$ on $K(X)$ obtained by applying Theorem 1.2 is a VDF-valuation so that we may extend $(K(X)_D, \overline{v})$ to a model of VDF. Let

$$\boxed{\mathcal{I}_D := \left\{ \frac{Dp}{p} : p \in K[X]_D, p \neq 0 \right\}}.$$

Then the following is immediate.

LEMMA 3.4. *Let (K, v, D) be a model of VDF. Then a valuation $\overline{v}$ on $K(X)_D$ is a VDF-valuation if and only if $\mathcal{I}_D \subseteq \mathcal{O}_{\overline{v}}$.*

PROOF. The forward direction is immediate from the definition of a valued D-field. If $\mathcal{I}_D \subseteq \mathcal{O}_{\overline{v}}$, then the standard interpretation of D on $K(X)_D$, where we set $D(D^k X_i) = D^{k+1} X_i)$ for each $i = 1, \dots, n$, will make $(K(X)_D, \overline{v}, D)$ a model of VDF extending (K, v, D). □

Further, we need VDF to be a conservative theory. Haskell and Yaffe show in [12] that VDF is conservative by constructing a VDF-valuation near any point as follows. Let (K, v, D) be a model of VDF with value group Γ. By translating, we may assume that $b = 0 = (0, \dots, 0)$. Fix an element $\delta > \gamma$ for each $\gamma \in \Gamma$ and let $\Gamma' = \Gamma \oplus \mathbb{Z}\delta$. Then, writing each D-polynomial $p \in K[X]_D$ as $p = \sum_\alpha p_\alpha Y^\alpha$, we define

$$\hat{v}(p) = \min\{v(p_\alpha) + |\alpha|\delta\}$$

where $|\alpha| = \sum_{1\leq i\leq n}^{0\leq k\leq r} \alpha_{i,k}$. We then extend $\hat{v}$ canonically to $K(X)_D$. This is clearly a VDF-valuation near b as each $D^k X_i$ becomes a new infinitesimal element and $\hat{v}(Dp) \geq \hat{v}(p)$ for each $p \in K(X)_D$.

We are now ready to prove our Stellensatz.

THEOREM 3.5. *Let (K, v, D) be a D-henselian valued field and S is a nonempty subset of K^n defined by*

$$\varphi_S(X) := \bigwedge_{i\in I} v\left(f_i(X)\right) \geq 0 \ \& \ \bigwedge_{j\in J} v\left(g_j(X)\right) > 0$$

with $f_i, g_j \in K(X)_D$. Let A be the $\mathcal{O}_v$-subalgebra of $K(X)_D$ generated by $\{f_i, g_j\}_{i\in I}^{j\in J}$ and $\mathcal{I}_D$. Let B be the ideal of A generated by $\mathcal{M}_v$ and $\{g_j\}_{j\in J}$. Let $T = \{1+b : b \in B\}$. Then $h \in K(X)$ is S-VDF-integral definite if and only if h is integral over A_T, and h is S-VDF-infinitesimal definite if and only if h is integral over B_T. That is, $\mathcal{O}(S) = \sqrt[\text{int}]{A}$ and $\mathcal{M}(S) = \sqrt[\text{int}]{B}$.

Since $v(Dx) \geq v(x)$ for all $x \in K$, one would expect we need require that the ring A and its ideal B be closed under the operator D. However, since $\mathcal{I}_D \subseteq A$ this is already the case. For each $f \in A$, $\frac{Df}{f} \in A$. Since A is closed under products, we get $Df \in A$. Similarly, for each $f \in B$, $\frac{Df}{f} \in A$. Since B is an ideal of A, B is closed under multiplication by elements of A. Thus we get $Df \in B$.

PROOF. For the right-to-left implication we need only verify that the properties of being VDF-integral and VDF-infinitesimal are preserved by localisation at T and by integral closure. However, this is clear as $\mathcal{O}_{\hat{v}}$ and $\mathcal{M}_{\hat{v}}$ are closed under these properties for each VDF-valuation $\hat{v}$. (See the proof of Theorem 1.2 for details.) For the left-to-right implication, the proof also proceeds much as before. First, suppose $h(X) \in K(X)$ is S-VDF-integral definite but not integral over A_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)_D$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(x)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X)) < 0$. In particular, $\overline{v}(f_i(X)) \geq 0$ for each $i \in I$ and $\overline{v}(g_j(X)) > 0$ for each $j \in J$. Then

$$(K(X)_D, \overline{v}) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right],$$

witnessed by the element $X \in K(X)_D$. Since $\mathcal{I}_D \subseteq \mathcal{O}_{\overline{v}}$, $\overline{v}$ is a VDF-valuation and interpreting D in the standard way makes $(K(X)_D, \overline{v}, D)$ a valued D-field extending (K, v, D). Then, as the theory of D-Henselian fields is the model companion of VDF, we may further extend $(K(X)_D, \overline{v}, D)$ to a D-Henselian field, (L, w, D). In particular, $(K, v, D) \subseteq (L, w, D)$ and

$$(L, w, D) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right].$$

Then, by the model-completeness of the theory of D-Henselian fields, (K, v, D) satisfies the same formula. This contradicts that $h(X)$ is S-VDF-integral definite.

Similarly, suppose $h(X)$ is S-VDF-infinitesimal definite but not integral over B_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)_D$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X)) \leq 0$. In particular, $\overline{v}(f_i(X)) \geq 0$ for each $i \in I$ and $\overline{v}(g_j(X)) > 0$ for each $j \in J$. That is,

$$(K(X)_D, \overline{v}) \models \exists X \left[\varphi_S(X) \;\&\; v(h(X)) \leq 0\right],$$

witnessed by the element $X \in K(X)_D$. Since $\mathcal{I}_D \subseteq \mathcal{O}_{\overline{v}}$, $(K(X), \overline{v}, D)$ is a valued D-field. Then we may extend $(K(X), \overline{v}, D)$ to a D-Henselian field, (L, w, D). In particular, $(K, v, D) \subseteq (L, w, D)$ and

$$(\mathcal{K}, \overline{v}) \models \exists X \left[\varphi_S(X) \;\&\; v(h(X)) < 0\right].$$

Then, by model-completeness, (K, v, D) satisfies the same formula. This contradicts that h is S-VDF-infinitesimal definite. □

This result extends that of [12] by considering sets S defined by both integrality and infinitesimality conditions and characterising both $\mathcal{O}(S)$ and $\mathcal{M}(S)$. In fact, Theorem 3.2 of [12] is the special case of Theorem 3.5 where the indexing set J is trivial so that $B = \mathcal{M}_v \cdot A$.

3. Witt Vectors

We now consider a theory of valued D-fields of mixed characteristic. We first remind the reader of the construction of Witt vectors.

A field k of characteristic p is perfect if the Frobenius endomorphism $\sigma_p : x \mapsto x^p$ is an automorphism on k. We shall say that a ring $\mathcal{O}$ is a local p-ring if $\mathcal{O}$ is a complete local ring with maximal ideal $\mathcal{M} = p\mathcal{O}$ and perfect residue field. Moreover, we shall call such a ring a strict local p-ring if $p^n \neq 0$ for each $n \in \mathbb{N}$. The rings $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{Z}_p$ are examples of a local p-ring and a strict local p-ring, respectively.

Two difficulties arise when working in a local p-ring $\mathcal{O}$. The first is that as $\mathcal{O}$ has no subfields, the residue field may not be lifted as in the characteristic zero case. Thus it is necessary to devise a canonical system of representatives for the residue field. The second difficulty is that, for example, under the standard bijection

$$\begin{aligned} \mathbb{F}_p^{\mathbb{N}} &\to \mathbb{Z}_p \\ \{x_i\}_{i \in \mathbb{N}} &\mapsto \sum_{i \in \mathbb{N}} x_i p^i \end{aligned}$$

the addition and multiplication in $\mathbb{Z}_p$ do not correspond to natural operations on $\mathbb{F}_p^{\mathbb{N}}$. One motivation for Witt vectors is to find a bijection under which addition and multiplication correspond to explicit polynomial maps $\mathbb{F}_p^{\mathbb{N}} \times \mathbb{F}_p^{\mathbb{N}} \to \mathbb{F}_p^{\mathbb{N}}$.

A canonical system of representatives always exists for a complete local ring $\mathcal{O}$ with perfect residue field of characteristic p. In this case, there is, in fact, a unique map $\tau : k \to \mathcal{O}$ satisfying $\mathrm{res}(\tau(x)) = x$ and $\tau(x^p) = \tau(x)^p$. Moreover, if $p^n \neq 0$ but $p^{n+1} = 0$ in $\mathcal{O}$, then for every $a \in \mathcal{O}$ there is a unique tuple $(x_0, \ldots, x_n) \in k^{n+1}$ such that

$$a = \sum_{i=0}^{n} \tau(x_i)p^i.$$

If $\mathcal{O}$ is a strict local p-ring, then there is a unique sequence $\{x_n\} \in k^{\mathbb{N}}$ such that

$$a = \sum_{n=0}^{\infty} \tau(x_n)p^n.$$

The sequence $\{x_n\}$ is called the Teichmüller vector of a. We may then define the Witt vector of $a \in \mathcal{O}$ as the unique sequence $\{x_n\} \in k^{\mathbb{N}}$ such that

$$a = \sum_{n=0}^{\infty} \tau(x_n^{p^{-n}})p^n.$$

The map that assigns each $a \in \mathcal{O}$ to its Witt vector is a bijection $\mathcal{O} \to k^{\mathbb{N}}$. Under this bijection, addition and multiplication in $\mathcal{O}$ correspond to explicit polynomial operations on $k^{\mathbb{N}}$. Taking these operations as addition and multiplication makes $k^{\mathbb{N}}$ a strict local p-ring with residue field isomorphic to k. Moreover, there is only one such ring, up to isomorphism. We shall denote this ring $W[k]$. Its field of fractions $W(K)$ is then a p-valued field. For more details on Witt vectors, see [23].

The model theory of Witt vectors is studied in detail in [2]. In particular, the theory WF= $\mathrm{Th}((W(\tilde{\mathbb{F}}_p), \sigma_p, v_p))$ is model-complete in the language $\mathcal{L}_{\mathrm{VDF}} \cup \{p\}$. Here $\tilde{\mathbb{F}}_p$ denotes the algebraic closure of $\mathbb{F}_p$ and σ_p denotes an extension of the Frobenius automorphism. Moreover, WF is the model companion of the theory, WVDF, of wittian valued difference fields. A wittian valued difference field is a valued difference field (K, v, σ) such that (K, v) is a p-valued field, $v(\sigma(x)) = v(x)$ for each $x \in K$ and $v(\sigma(x) - x^p) > 0$ for each $x \in K$ with $v(x) \geq 0$. We can axiomatise wittian valued difference fields in $\mathcal{L}_{\mathrm{VDF}}$ (replacing for convenience the symbol D with σ) by the axioms for p-valued fields together with

$$\forall X[v(\sigma(X)) = v(X)]$$
$$\forall X[v(\sigma(X) - X^p) > 0].$$

In [3], Bélair characterises those σ-rational functions that are integral or not defined wherever some finite collection $S \subset K(X)_\sigma$ of σ-rational functions are either integral or not defined. This essentially characterises the S-integral definite functions for sets S defined by

$$\varphi_S(X) = \bigwedge_{i \in I} v(f_i(X)) \geq 0$$

with $f_i(X) \in K(X)_\sigma$, but with no special attention paid to the undefined case. The valuation is discrete, so, as in the p-adic case, we need not consider the case where S is defined by both integrality and infinitesimality conditions. However, we shall continue to consider infinitesimality conditions for uniformity of results.

To proceed as before, we need to determine when a valuation $\overline{v}$ on $K(X)$ obtained by applying Theorem 1.2 is compatible with the automorphism σ so that $(K(X), \overline{v})$ can be extended to a model of WVDF. For this, we need a modified version of the Kochen operator, given by

$$\boxed{\gamma_\sigma(x) = \frac{1}{p}\left(\frac{\sigma(x) - x^p}{(\sigma(x) - x^p)^2 - 1}\right)}.$$

Then we have an analogue of Proposition 2.2. indeed, the proof of the following is identical.

PROPOSITION 3.6 (Lemma 2.3 of [3]). *Let (K, v, σ) be a valued difference field of characteristic zero with $v(p) > 0$. Then (K, v, σ) is wittian if and only if $\gamma_\sigma(K) \subseteq \mathcal{O}_v$.*

All that remains, before we proceed to our Stellensätze, is to show that WVDF is conservative. We do this, as before, by constructing a WVDF-valuation near each point b of a model of WVDF.

LEMMA 3.7. *The theory WVDF is conservative.*

PROOF. It suffices to show that for any model (K, v, σ) of WVDF and any $b \in K$ there is a WVDF-valuation near b on $K(X)_\sigma$. Further, we may assume $b = 0$ as we may translate to any other b. Then let Γ be the value group of (K, v, σ) and fix δ such that $\delta > \gamma$ for each $\gamma \in \Gamma$ and let $\Gamma' = \Gamma \oplus \mathbb{Z}\delta$. Then for $q(X) \in K[X]_\sigma$ write

$$q(X) = \sum q_\alpha Y^\alpha$$

where $Y^\alpha = \prod_{1 \leq i \leq n}^{k \geq 0} (\sigma^k(X_i))^{\alpha_{i,k}}$. Define

$$\hat{v}(q) = \min_\alpha \{v(q_\alpha) + |\alpha|\delta\}$$

where $|\alpha| = \sum_{i\leq n}^{k\geq 0} \alpha_{i,k}$. We extend this canonically to $K(X)_\sigma$ and claim that this is a WVDF-valuation. Extending σ to $K(X)_\sigma$ in the standard way, we observe that

$$\begin{aligned}\sigma(Y^\alpha) &= \sigma(\prod_{1\leq i\leq n}^{k\geq 0} \sigma^k(X_i)^{\alpha_{i,k}}) \\ &= \prod_{1\leq i\leq n}^{k\geq 0} \sigma^{k+1}(X_i)^{\alpha_{i,k}}.\end{aligned}$$

Hence, if $Y^\beta = \sigma(Y^\alpha)$ then $|\beta| = |\alpha|$. Thus we get

$$\begin{aligned}\hat{v}(\sigma(q)) &= \hat{v}(\sigma(\sum q_\alpha Y^\alpha)) \\ &= \hat{v}(\sum \sigma(q_\alpha)\sigma(Y^\alpha)) \\ &= \min_\alpha\{v(\sigma(q_\alpha)) + |\alpha|\delta\} \\ &= \min_\alpha\{v(q_\alpha) + |\alpha|\delta\} \\ &= \hat{v}(q).\end{aligned}$$

This shows that $(K(X), \hat{v}, \sigma)$ is a valued difference field. Then we need only verify that $(K(X), \hat{v}, \sigma)$ is wittian. Let $q \in K(X)_\sigma$ such that $\hat{v}(q) \geq 0$. Notice that since $\hat{v}(Y^\alpha) > \Gamma$ for each α, for any $q \in K(X)_\sigma$, if q has nonzero constant term then $\hat{v}(q) = v(q_0)$. Now $\sigma(q(x)) - q(x)^p$ has constant term $\sigma(q_0) + q_0^p$. If this is nonzero, then $\hat{v}(\sigma(q) - q^p) = v(\sigma(q_0) + q_0^p) > 0$ since (K, v, σ) is wittian. On the other hand, if q has trivial constant term, then so does $\sigma(q) - q^p$. In this case, $\hat{v}(\sigma(q) - q^p) > \Gamma$ and so, in particular, $\hat{v}(\sigma(q) - q^p) > 0$. Thus $(K(X)_\sigma, \overline{v}, \sigma)$ is wittian and $\hat{v}$ is a WVDF-valuation near $b = 0$. □

We are now ready to prove the main result of this section.

THEOREM 3.8. *Suppose (K, v, σ) is a model of WF and S is a nonempty subset of K^n defined by*

$$\varphi_S(X) := \bigwedge_{i\in I} v\left(f_i(X)\right) \geq 0 \;\&\; \bigwedge_{j\in J} v\left(g_j(X)\right) > 0.$$

for some $f_i, g_j \in K(X)_\sigma$. Let A be the $\mathcal{O}_v$-subalgebra of $K(X)_\sigma$ generated by $\{f_i, g_j\}_{i\in I}^{j\in J}$, $\mathcal{I}_\sigma$ and $\gamma_\sigma(K(X)_\sigma)$. Let B be the ideal of A generated by $\mathcal{M}_K$ and $\{g_j\}_{j\in J}$. Let $T = \{1+b : b \in B\}$. Then $h \in K(X)_\sigma$ is S-WVDF-integral definite if and only if h is integral over A_T, and h is S-WVDF-infinitesimal definite if and only if h is integral over B_T. That is, $\mathcal{O}(S) = \sqrt[\text{int}]{A}$ and $\mathcal{M}(S) = \sqrt[\text{int}]{B}$.

PROOF. The right-to-left implication is once again clear as the properties of being S-WVDF-integral and S-WVDF-infinitesimal definite are preserved by localisation at T and by taking integral closures. For the left-to-right implication, we proceed as before. First, suppose $h(X) \in K(X)_\sigma$ is S-WVDF-integral definite but not integral over A_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)_\sigma$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X)) < 0$. In particular, $\overline{v}(f_i(X)) \geq 0$ for each $i \in I$ and $\overline{v}(g_j(X)) > 0$ for each $j \in J$. Then

$$(K(X)_\sigma, \overline{v}) \models \exists X\, [\varphi_S(X) \;\&\; v(h(X)) < 0]$$

witnessed by the element $X \in K(X)_\sigma$. Since $\mathcal{I}_\sigma \subseteq \mathcal{O}_{\overline{v}}$, $\overline{v}$ is a VDF-valuation by Lemma 3.4 and interpreting σ in the standard way makes $(K(X)_\sigma, \overline{v}, \sigma)$ a valued difference field extending (K, v, σ). Moreover, since $\gamma_\sigma(K(X)_\sigma) \subseteq \mathcal{O}_{\overline{v}}$, $(K(X), v, \sigma)$ is a model of WVDF. Then, as WF is the model companion of WVDF, we may further extend $(K(X), \overline{v})$ to a model of WF, say, (L, w, σ).. In particular, $(K, v, \sigma) \subseteq (L, w, \sigma)$ and

$$(L, w, \sigma) \models \exists X\, [\varphi_S(X) \;\&\; v(h(X)) < 0]\,.$$

Then, by model-completeness of WF, (K, v, σ) satisfies the same formula. This contradicts that h is S-WVDF-integral definite.

We could now repeat the above argument for the S-WVDF-infinitesimal definite functions, however, as in pCF, since the valuation is discrete we need only observe that $\mathcal{M}(S) = \mathcal{M}_v \cdot \mathcal{O}(S)$. □

4. Real Closed Valued Fields

An ordered valued field is a valued field (K, v) equipped with an ordering $<$. We require that the ordering exhibit a strong interaction with the valuation in the sense that

$$0 < x < y \to v(x) \geq v(y) \tag{4}$$

for every $x, y \in K$. Then the rational function $\frac{X^2}{X^2+Y^2}$ should be considered integral even at the origin as $0 < x^2 < x^2 + y^2 \to v(x^2) \geq v(x^2 + y^2)$ for any $x, y \in K$. Thus we must frame our results within the context of OVF-integrality and OVF-infinitesimality.

An ordered valued field $(K, v, <)$ is a real closed valued field if, not surprisingly, K is real closed as a field. We then work in the language $\mathcal{L}_{\text{ovf}} = \mathcal{L}_{\text{vf}} \cup \{<\}$. In $\mathcal{L}_{\text{ovf}}$ we can axiomatise ordered valued fields by the axioms for ordered fields and the axioms for valued fields together with

$$\forall X \forall Y [0 < X < Y \to v(X) \geq v(Y)].$$

We denote this $\mathcal{L}_{\text{ovf}}$-theory by OVF. Further, we can axiomatise real closed valued fields by the axioms for real closed fields and the above axiom. We denote this $\mathcal{L}_{\text{ovf}}$-theory by RCVF.

As one might expect, RCVF is the model companion of the theory of OVF, [5]. In [12], Haskell and Yaffe characterise the S-OVF-integral definite rational functions for sets S defined by integraliity conditions. We extend these results to sets S defined by both integrality and infinitesimality conditions.

The method of proof will again be similar. Once again, we must ensure that the valuation $\overline{v}$ on $K(X)$ obtained by applying Theorem 1.2 will be compatible (in the sense of (4)) with some extension of the ordering $<$ to $K(X)$ so that $(K(X), \overline{v}, <)$ is a model of OVF.

In general, given an ordered valued field $(K, v, <)$, we shall say that a valued field extension (L, w) of (K, v) is formally real over $(K, v, <)$ if there is an extension of the ordering $<$ to (L, w) that makes $(L, w, <)$ an ordered valued field. Now for any valued field (K, v) define

$$\boxed{\mathcal{I}_{\text{ord}}(K) = \left\{\frac{1}{1+r} : r \in \sum K^2\right\}.}$$

Then the following is implicit in the remarks after Proposition 4.1 of [12].

PROPOSITION 3.9. *Let $(K, v, <)$ be a model of OVF. Let (L, w) be a valued field extension of (K, v). Then $(L, \overline{v})$ is formally real over $(K, v, <)$ if and only if $\mathcal{I}_{\text{ord}}(L) \subseteq \mathcal{O}_{\overline{v}}$.*

Let $(L, \overline{v})$ be formally real with ordering $<$. Then for each any sum of squares r in L, we have $0 < 1 < 1 + r$ so that $\overline{v}(1 + r) \leq \overline{v}(1)$. For the converse, we remark that each ordering $<$ on $(L, \overline{v})$ is induced by an ordering on l_w. If no ordering exists, then l_w is not formally real. Then there are $\overline{a}_0, \ldots, \overline{a}_k \in l_{\overline{v}}$ such that $\sum_{i=0}^{k} \overline{a}_i = -1$. Choosing $a_i \in L$ such that $\text{res}_{\overline{v}}(a_i) = \overline{a}_i$ we have $1 + \sum_{i=0}^{k} a_i = 0 \in \mathcal{M}_{\overline{v}}$. This contradicts that $\frac{1}{1 - \sum_{i=0}^{k} a_i} \in \mathcal{O}_{\overline{v}}$.

All that remains before we proceed to our theorem is to observe that RCVF is conservative. As noted at the end of Section 1, it suffices to construct an OVF-valuation on $K(X)$ near any point $b \in K^n$. This is done in Proposition 4.2 of [12]. Given an ordered valued field, $(K, v, <)$, with value group Γ, fix new elements $\delta_1, \ldots, \delta_n$ with $\delta_i > \gamma$ for each $\gamma \in \Gamma$ and let $\Gamma' = \Gamma \oplus \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_n$. Then for any polynomial $p \in K[X]$, write $p = \sum_\alpha p_\alpha (X - b)^\alpha$ where $(X - b)^\alpha = \prod_{1 \leq i \leq n} (X_i - b_i)^{\alpha_i}$. Define

$$\hat{v}(p) = \min\{v(p_\alpha) + \sum_{1 \leq i \leq n} \alpha_i \delta_i\}$$

and extend to $K(X)$. This will be an OVF-valuation if there is a compatible extension of $<$ to $K(X)$. From this definition it is clear, that every monomial in $K(X)$ has a different valuation. Then for $p \in K(X)$, let $p_\alpha(X-b)^\alpha$ be the monomial of p with least valuation. Define $p>0$ if and only if $p_\alpha > 0$. This ordering extends the ordering on K and makes $(K(X), \hat{v}, <)$ a model of OVF. Hence $\hat{v}$ is an OVF-valuation near b and OVF is, indeed, conservative.

We are ready to prove our Stellensatz.

THEOREM 3.10. *Suppose (K, v) is a model of RCVF and S is a nonempty subset of K^n defined by*

$$\varphi_S(X) := \bigwedge_{i\in I} v(f_i(X)) \geq 0 \ \& \ \bigwedge_{j\in J} v(g_j(X)) > 0$$

for some $f_i, g_j \in K(X)$. Let A be the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by $\{f_i, g_j\}_{i\in I}^{j\in J}$ and $\mathcal{I}_{ord}$. Let B be the ideal of A generated by $\mathcal{M}_v$ and $\{g_j\}_{j\in J}$. Let $T = \{1+b : b \in B\}$. Then $h \in K(X)$ is S-OVF-integral definite if and only if h is integral over A_T, and h is S-OVF-infinitesimal definite if and only if h is integral over B_T. That is $\mathcal{O}(S) = \sqrt[\text{int}]{A}$ and $\mathcal{M}(S) = \sqrt[\text{int}]{B}$.

PROOF. For the right-to-left implication is, again, clear as the properties of being S-OVF-integral definite and S-OVF-infinitesimal definite are preserved by localisation at T and by taking integral closures. For the left-to-right implication, the proof also proceeds as before. First, suppose $h(X) \in K(X)$ is S-OVF-integral definite but not integral over A_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X))) < 0$. In particular, $\overline{v}(f_i(X)) \geq 0$ for each $i \in I$ and $\overline{v}(g_j(X)) > 0$ for each $j \in J$. Then

$$(K(X), \overline{v}) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right],$$

witnessed by the element $X \in K(X)$. Since $\mathcal{I}_{\text{ord}} \subseteq \mathcal{O}_{\overline{v}}$, $(K(X), \overline{v})$ is formally real (i.e. $\overline{v}$ is an OVF-valuation) and there is an ordering $<$ on $K(X)$ such that $(K(X), \overline{v}, <)$ is an ordered valued field extending $(K, v, <)$. Then, as RCVF is the model companion of OVF, we may further extend $(K(X), \overline{v}, <)$ to a real closed valued field, $(L, w, <)$. In particular, $(K, v, <) \subseteq (L, w, <)$ and

$$(K, \overline{v}, <) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right].$$

Then, by the model-completeness of RCVF, $(K, v, <)$ satisfies the same formula. This contradicts that $h(X)$ is S-OVF-integral definite.

Similarly, suppose $h(X)$ is S-OVF-infinitesimal definite but not integral over B_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(x)) \leq 0$. In particular, $\overline{v}(f_i(X0) \geq 0$ for each $i \in I$ and $\overline{v}(g_j(X0) > 0$ for each $j \in J$. That is,

$$(K(X), \overline{v}) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) \leq 0\right],$$

witnessed by the element $X \in K(X)$. Since $\mathcal{I}_{\text{ord}} \subseteq \mathcal{O}_{\overline{v}}$, $(K(X), \overline{v})$ is formally real so there is an ordering $<$ on $K(X)$ that makes $(K(X), \overline{v}, <)$ a model of OVF. Then we may extend $(K(X), \overline{v}, <)$ to a real closed valued field, $(L, w, <)$. In particular, $(K, v, <) \subseteq (L, w, <)$ and

$$(K, \overline{v}) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right].$$

Then by model-completeness $(K, v, <)$ satisfies the same formula. This contradicts that $h(X)$ is S-OVF-infinitesimal definite. □

5. VF-integrality and pVF-integrality

The astute reader may be wondering whether our refined notions of integrality and infinitesimality will force us to rework our results from Chapter 2. Fortunately, in [12] it is shown that VF-integrality is the same as the naive notion of integrality. That is, given a pure valued field (K, v) and a rational function $f(X) \in K(X)$, $f(X)$ is VF-integral at a point $b \in K^n$ if and only if $f(X)$ is integral at b. In particular, this means that in a pure valued field no function is VF-integral at point where it is undefined. We give a similar result for p-valued fields. The proof is nearly identical to the proof of Theorem 2.16 of [12].

Proposition 3.11. *A function $h(X) \in K(X)$ is pVF-integral at $b \in K^n$ if and only if $h(X)$ is integral at b.*

Proof. First we observe that pVF is conservative. Once again, it suffices to construct a pVF-valuation near $b = 0 \in K^n$. The construction is similar. Given a p-valued field, (K, v), with value group Γ, fix an element $\delta > \gamma$ for each $\gamma \in \Gamma$ and let $\Gamma' = \Gamma \oplus \mathbb{Z}\delta$. Write $p(X) \in K(X)$ as $\sum_\alpha p_\alpha X^\alpha$. Define

$$\hat{v}(p(X)) = \min_\alpha \{v(p_\alpha) + |\alpha|\delta\}.$$

This is a p-valuation on $K(X)$ near $b = 0$, as $\hat{v}(p) = v(p) = 1$ is minimal positive and $\hat{v}(X_i) = \delta > \Gamma$ for each $i = 1, \ldots, n$. This shows that pVF is conservative.

Now it will suffice to show that whenever $h(X) \in K(X)$ is pVF-integral at b it is defined at b. As in the proof of Proposition 2.16 of [12], let A be the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by the set $\{c(X_i - b_i) : c \in K, 1 \leq i \leq n\}$. Then each $f \in A$ is defined at b. Moreover, for each $f \in \{1 + ma : m \in \mathcal{M}_v, a \in A\}$, we have $v(f(b)) = 0$ so that, in particular, $f(b) \neq 0$. Thus each $f \in \sqrt[\text{int}]{A}$ is defined at b. It will suffice to show that h is integral over A_T.

Suppose, for a contradiction, that this is not the case. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)$ extending v on K with $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$, but $v(h) < 0$. Then $\overline{v}(X_i - b_i) > \overline{v}(\frac{1}{c}) = v(\frac{1}{c})$ for each $c \in K^\times$. It follows that $\overline{v}$ is a pVF-valuation near b. Since h is pVF-integral at b, we must have $\overline{v}(h) \geq 0$. This is a contradiction. Then we have h integral over A_T and, thus, defined at b. $\square$

CHAPTER 4

Residually Valued Fields

It is well-known that the structure of a valued field (K, v) is in some sense controlled by the structure of its residue field and the structure of its value group. Nowhere is this more explicit than in the case of a henselian field of residual characteristic zero where we have the classical Ax-Kochen-Ershov Theorem.

THEOREM 4.1 (Theorem 5.4.12 of [4], for example). *Let (K, v) and (L, w) be henselian fields with* $\mathrm{char}(k_v) = \mathrm{char}(l_w) = 0$. *Then $(K, v) \equiv (L, w)$ if and only if $k_v \equiv l_v$ and $\Gamma_v \equiv \Gamma_w$.*

In particular, the Ax-Kochen-Ershov principle implies that certain properties (model-completeness, for example) can be lifted from the residue field and value group. The work of Delon concerning fields with many valuations,[6], gives us a framework for developing such theories. Let $\mathcal{L}_k$ a language extending $\{+, -, \cdot, 0, 1\}$ by adding only new relation symbols. Let $\mathcal{T}_k$ be an $\mathcal{L}_k$-theory extending the theory of fields of characteristic zero. Similarly, let $\mathcal{L}_\Gamma$ be a language extending $\{+, 0, \leq\}$ by adding only new relation symbols. Let $\mathcal{T}_\Gamma$ be an $\mathcal{L}_\Gamma$-theory extending the theory of ordered abelian groups. Let $\mathcal{L}(\mathcal{L}_k, \mathcal{L}_\Gamma) = \mathcal{L}_{\mathrm{div}} \cup \{R' : R \in \mathcal{L}_k - \{+, -, \cdot, 0, 1\}\} \cup \{R' : R \in \mathcal{L}_\Gamma - \{+, 1, \leq\}\}$. This is a one-sorted language in which we can axiomatise valued fields with residue field a model of $\mathcal{T}_k$ and value group a model of $\mathcal{T}_\Gamma$. We denote this theory $\mathcal{T}(\mathcal{T}_k, \mathcal{T}_\Gamma)$. This is axiomatised by the axioms for valued fields together with

$$R'(X_1, \ldots, X_n) \to \bigwedge_{i=1}^{n} v(x_i) \geq 0$$

$$[R'(X_1, \ldots, X_n) \,\&\, \bigwedge_{i=1}^{n} v(X_i - Y_i) > 0] \to R'(Y_1, \ldots, Y_n)$$

for each n-ary predicate R of $\mathcal{L}_k$. Then, given a model (K, v) of $\mathcal{T}(\mathcal{T}_k, \mathcal{T}_\Gamma)$, we can define an $\mathcal{L}_k$ structure on k_v by setting

$$k_v \models R(\mathrm{res}_v(X_1), \ldots, \mathrm{res}_v(X_n) \leftrightarrow (K, v) \models R'(X_1, \ldots, X_n).$$

We require that k_v be a model of $\mathcal{T}_k$. Similarly, we have axioms

$$R'(X_1, \dots, X_n) \to \bigwedge_{i=1}^{r} X_i \neq 0$$

$$[R'(X_1, \dots, X_n) \ \& \ \bigwedge_{i=1}^{n} v(X_i) = v(Y_i)] \to R'(Y_1, \dots, Y_n)$$

for each n-ary predicate R of $\mathcal{L}_\Gamma - \{\leq\}$ and we may induce an $\mathcal{L}_\Gamma$-structure on Γ_v by setting

$$\Gamma_v \models R(v(X_1), \dots, v(X_n))) \leftrightarrow (K, v) \models R'(X_1, \dots, X_n)$$

and require that Γ_v be a model of $\mathcal{T}_\Gamma$. Further, let $\mathcal{T}_h(\mathcal{T}_k, \mathcal{T}_\Gamma)$ be the theory of henselian models of $\mathcal{T}(\mathcal{T}_k, \mathcal{T}_\Gamma)$.

In this framework, we have the following strong Ax-Kochen-Ershov principle.

THEOREM 4.2 (Theorem 1 of [6]). *Using the notation above, if $\tilde{\mathcal{T}}_k$ is the model-companion of $\mathcal{T}_k$ and $\tilde{\mathcal{T}}_\Gamma$ is the model-companion of $\mathcal{T}_\Gamma$ then $\mathcal{T}_h(\tilde{\mathcal{T}}_k, \tilde{\mathcal{T}}_\Gamma)$ is the model-companion of $\mathcal{T}(\mathcal{T}_k, \mathcal{T}_\Gamma)$.*

This gives a framework for constructing model-complete theories of valued fields.

1. Henselian Residually Valued Fields

We consider first the general theory $\mathcal{T}_h(\mathrm{VF}, \mathrm{OAG})$ where OAG is the theory of ordered abelian groups in the language $\{+, 0, \leq\}$. As neither VF nor OAG is model-complete, this theory is not model-complete either. However, all our other theories shall extend this one.

We shall refer to models of this theory as henselian residually valued fields. This is motivated by the fact that any model of $\mathcal{T}_h(\mathrm{VF}, \mathrm{OAG})$ can be interpreted as a henselian valued field (K, v_h) equipped with a second nontrivial valuation v' satisfying

(5) $$v'(x) \geq v'(y) \to v_h(x) \geq v_h(y).$$

When this condition is satisfied, we shall say that v_h is compatible with v'. This notion of compatibility is known as coarsening or refinement and is well-studied in valuation theory. See [9] for more details. The valuation v' is just a valuation on K induced by a valuation v on k_{v_h}. Moreover, given a v_h compatible with v', we can recover the valuation v on k_{v_h} by setting

$$v(0) = \infty$$
$$v(x) = v'(y) \text{ for } y \in \mathcal{O}_{v_h}^{\times} \text{ such that } \mathrm{res}_{v_h}(y) = x.$$

This is well-defined as the compatibility condition (5) implies $v'(\mathcal{O}_{v_h}^{\times}) < v'(\mathcal{M}_{v_h})$ so that given $y \in \mathcal{O}_{v_h}^{\times}$, $v'(y+m) = v'(y)$ for any $m \in \mathcal{M}_{v_h}$.

In the following sections, we will assume further that the value group of the henselian valuation, Γ_{v_h}, is discrete. That is, we consider $\mathcal{T}_h(\mathrm{VF}, \mathrm{DOAG})$ where DOAG is the theory of discrete ordered abelian groups with minimal positive element α in the language $\{+, 0, \alpha, \leq\}$. Models of this theory are precisely henselian residually valued fields (K, v, v_h, a) with $v_h(a)$ minimal positive.

These are, in fact, natural objects to study as the theory has the following canonical models. Given a valued field (k, v), let (K, v', v_t, t) denote the field of Laurent series $K = k((t))$ equipped with the usual henselian t-adic valuation, v_t, and the following natural valuation on K. For $f \in k((t))$, write $f = \sum_{i \geq n} a_i t^i$ and define $v'(f) = (n, v(a_n)) \in \mathbb{Z} \times \Gamma_v$ ordered lexicographically. Then the valuation v' is compatible with v_t. In particular, this is a model of $\mathcal{T}_h(\mathrm{VF}, \mathrm{Th}(\mathbb{Z}))$. Moreover, if (k, v) is, say, a model of pCF then (K, v_t, v', t) is a model of $\mathcal{T}_h(p\mathrm{CF}, \mathrm{Th}(\mathbb{Z}))$. If (k, v) is a model of RCVF then (K, v_t, v', t) is a model of $\mathcal{T}_h(\mathrm{RCVF}, \mathrm{Th}(\mathbb{Z}))$. As the theories pCF, RCVF and $\mathrm{Th}(\mathbb{Z})$ are all model-complete, $\mathcal{T}_h(p\mathrm{CF}, \mathrm{Th}(\mathbb{Z}))$ and $\mathcal{T}_h(\mathrm{RCVF}, \mathrm{Th}(\mathbb{Z}))$ are also model-complete.

Now, in a henselian residually valued field we have access to two valuations: the henselian valuation v_h on K and the valuation v' induced by the valuation v on the residue field k_{v_h}. Thus the meanings of integrality and infinitesimality must be clarified. Throughout the following sections we shall take our notions of integrality and infinitesimality with respect to the valuation v' coming from the valuation v on the residue field. Specifically, we consider the model-complete theories $\mathcal{T}_h(p\mathrm{CF}, \mathrm{Th}(\mathbb{Z}))$. and $\mathcal{T}_h(\mathrm{RCVF}, \mathrm{Th}(\mathbb{Z}))$. For sets S defined by

$$\varphi_S(X) := \bigwedge_{i \in I} v'(f_i(X)) \geq 0 \ \& \ \bigwedge_{j \in J} v'(g_j(X)) > 0.$$

we characterise

$$\mathcal{O}(S) = \{f(x) \in K(X) : v'(f(x)) \geq 0 \text{ for all } x \in S\}$$

and

$$\mathcal{M}(S) = \{f(x) \in K(X) : v'(f(x)) > 0 \text{ for all } x \in S\}.$$

These characterisations essentially lift the results of Sections 2.2 and 3.4 from the residue field of a henselian residually valued field.

Then, as before, we need to gather information regarding extensions of the valuation v' to $K(X)$. In particular, given $(K(X), \overline{v})$ with $\overline{v}$ extending v', we will need to determine whether $(K(X), \overline{v})$ can be extended to a model of $\mathcal{T}_h(\mathrm{VF}, \mathrm{DOAG})$. We say that a valued field extension (L, w) of (K, v) is formally $\mathcal{T}_h(\mathrm{VF}, \mathrm{DOAG})$ over

(K, v, v_h, a) if there is a valuation w_h on some valued field extension $(L', \overline{w})$ of (L, w) that extends v_h and makes $(L', \overline{w}, w_h, a)$ a model of $\mathcal{T}_h(\mathrm{VF}, \mathrm{DOAG})$. This is analogous to saying that a field K is formally real if there is at least one ordering $<$ on K that makes $(K, <)$ an ordered valued field.

To obtain a result analogous to Proposition 2.2. we require the operator, γ_a, defined by

$$\boxed{\gamma_a(X) = \frac{X}{X^2 - a}} \tag{6}$$

The operator γ_a is similar to the Kochen operator γ_p in the sense that it allows us to determine for an element $a \in \mathcal{M}_v$ whether $v_h(a)$ is minimal positive.

PROPOSITION 4.3 (Lemma 2.3 of [10]). *Let (K, v) be a valued field and $a \in K^\times$. Then $v(a) = 1$ if and only if $a \in \mathcal{M}_v$ and $\gamma_a(K) \subseteq \mathcal{O}_v$.*

The left-to-right implication is clear. Suppose $v(a) = 1$ and $x \in K$. If $v(x) \leq 0$, we have $v(x^2 - a) = 2v(x)$ so that

$$\begin{aligned} v\left(\frac{x}{x^2 - a}\right) &= v(x) - v(x^2 - a) \\ &= v(x) - 2v(x) \geq 0. \end{aligned}$$

If $v(x) > 0$, then $v(x^2) > v(a) = 1$ so that $v(x^2 - a) = 1$. Then

$$\begin{aligned} v\left(\frac{x}{x^2 - a}\right) &= v(x) - v(x^2 - a) \\ &= v(x) - 1 \geq 0. \end{aligned}$$

The converse is also clear as for any $x \in K$ with $v(x) > 0$,

$$v\left(\frac{x}{x^2 - a}\right) = v(x) - v(x^2 - a) > 0$$

implies that $v(a) < 2v(x)$. In particular, if $v(x) = 1$ then we must have $v(a) = 1$.

For any valued field (K, v) and any $a \in K^\times$, let $\mathcal{I}_a(K)$ be the subring of K generated by $\gamma_a(K)$. Further, let $M_a(K) = \mathcal{M}_v \cdot \mathcal{I}_a(K)$.

PROPOSITION 4.4. *The valued field (L, w) is formally $\mathcal{T}_h(\mathrm{VF}, \mathrm{DOAG})$ over (K, v, v_h, a) if and only if $\mathcal{O}_v[M_a(L)] \subseteq \mathcal{O}_w$.*

PROOF. The left-to-right direction follows from Proposition 4.3. For the converse, suppose $\mathcal{O}_v[\mathcal{M}_{h,K} \cdot I_a] \subseteq \mathcal{O}_v$. Then the set

$$B = \{x \in L : w(c) \leq w(x) \text{ for some } c \in \mathcal{M}_{h,K} \cdot I_a\}$$

is a valuation ring of L. Let w_h be the associated valuation on L so that $\mathcal{O}_{h,L} = B$. It is clear from the definition of B that w_h is compatible with w, thus we have the valuation $\tilde{w}$ on l_h. Now $w(\mathcal{M}_{h,K} \cdot I) \geq 0$ and $a \in \mathcal{M}_{h,K}$. By the definition of B, we must have $a^{-1} \notin B$ and hence $a \in \mathcal{M}_{h,L}$. Moreover, since $\gamma_a(L) \subseteq I \subseteq B$, we have $w_h(a) = 1$. Furthermore, since $\mathcal{O}_{h,K} \subseteq \mathcal{O}_{h,L}$ and $\mathcal{M}_{h,K} = a\mathcal{O}_{h,K} \subseteq a\mathcal{O}_{h,L} = \mathcal{M}_{h,L}$, w_h extends v_h. While the valuation w_h may not itself be henselian, we may now easily extend (L, w, w_h, a) to a model of $\mathcal{T}_h(\mathrm{VF}, \mathrm{DOAG})$. □

2. Residually p-adically Closed Fields

We now turn our attention to the theory $\mathcal{T}_h(\mathrm{pVF}, \mathrm{DOAG})$. A model of this theory is a henselian residually valued field, (K, v', v_h, a), where $v' = v'_p$ is a p-valuation on K. In this case, v'_p is induced by a p-valuation $v = v_p$ on k_h. If (k_h, v_p) is, in fact, a p-adically closed field and the value group Γ_h is a $\mathbb{Z}$-group, then (K, v', v_h, a) is a model of $\mathcal{T}_h(\mathrm{pCF}, \mathrm{Th}(\mathbb{Z}))$. This theory coincides with Guzy's henselian residually p-adically closed fields of [11].

As the theory of p-adically closed fields is the model-companion of the theory of p-valued fields and the theory $\mathrm{Th}(Z)$ is the model-companion of DOAG, we get the following as a direct consequence of Theorem 4.2.

COROLLARY 4.5. *The theory* $\mathcal{T}_h(\mathrm{pCF}, \mathrm{Th}(\mathbb{Z}))$ *is the model-companion of* $\mathcal{T}_h(\mathrm{pVF}, \mathrm{DOAG}$

In proving our Stellensatze, we need to be able to extend a model (K, v'_p, v_h, a) of $\mathcal{T}_h(\mathrm{pVF}, \mathrm{Th}(\mathbb{Z}))$ to a model $(K(X), \overline{v}_p, \overline{v}_h, a)$ of $\mathcal{T}_h(\mathrm{pVF}, \mathrm{DOAG})$. In particular, we will need to determine when $(K(X), \overline{v})$, with $\overline{v}$ obtained by applying Theorem 1.2, is p-valued. Recall the Kochen operator

$$\gamma_p(X) = \frac{1}{p} \frac{X^p - X}{(X^p - X)^2 - 1}.$$

Then Proposition 2.2 asserts that a valued field (K, v) is p-valued if and only if $\gamma_p(K) \subseteq \mathcal{O}_v$. Once we have $(K(X), \overline{v})$ a p-valued field, Proposition 4.4 determines when there is a compatible henselian valuation.

Finally, before proceeding to the Stellensätze, we show that the theory $\mathcal{T}_h(\mathrm{pVF}, \mathrm{DOAG})$ is conservative..

PROPOSITION 4.6. *The theory* $\mathcal{T}_h(\mathrm{pVF}, \mathrm{DOAG})$ *is conservative.*

PROOF. Given (K, v_p, v_h, a) a model of $\mathcal{T}_h(\mathrm{pVF}, \mathrm{DOAG})$, we construct an $\mathcal{T}_h(\mathrm{pVF}, \mathrm{DOAG})$-valuation near $b = 0$ on $K(X)$. Let Γ_p be the value group $\Gamma_{v_p,K}$, fix δ such that $\delta > \gamma$ for each $\gamma \in \Gamma_p$ and let $\Gamma'_p = \Gamma_p \oplus \mathbb{Z}\delta$. Then for $q \in K[X]$ write

$$q = \sum q_\alpha X^\alpha$$

and define

$$\hat{v}_p(q) = \min_\alpha\{v_p(q_\alpha) + |\alpha|\delta\}.$$

This is the pVF-valuation on $K(X)$ from Proposition 3.11.

It remains only to show that there is a compatible henselian valuation. We may construct one in the same manner. Fix an element ϵ such that $\gamma < \epsilon$ for each $\gamma \in \Gamma_h$. Let $\Gamma_h' = \Gamma_h \oplus \mathbb{Z}\epsilon$ and define

$$\hat{v}_h(q) = \min_\alpha\{v_h(q_\alpha) + |\alpha|\epsilon\}.$$

Now, notice that if $\hat{v}_p(q) = v(q_{\alpha_0}) + |\alpha_0|\delta$, then for any α with $|\alpha| < |\alpha_0|$ we must have $q_\alpha = 0$ and, moreover, the same must hold for $\hat{v}_h(q) = v_h(q_{\beta_0}) + |\beta_0|\epsilon$. Thus we must have $|\alpha_0| = |\beta_0|$. Further, since v_p and v_h are compatible

$$v_p(q_{\alpha_0}) \leq v_p(q_\alpha) \to v_h(q_{\alpha_0}) \leq v_h(q_\alpha)$$

for each α with $|\alpha| = |\alpha_0|$. Then if $\hat{v}_p(f) = v_p(q_{\alpha_0}) + |\alpha_0|\delta$, we must have $\hat{v}_h(f) = v_h(q_{\alpha_0}) + |\alpha_0|\epsilon$ and the compatibility of $\hat{v}_p$ and $\hat{v}_h$ follows from the compatibility of v_p and v_h.

Thus $\hat{v}_p$ is an $\mathcal{T}_h$(pVF, DOAG)-valuation and the theory is, conservative. □

We are now ready to prove the main result of this section.

THEOREM 4.7. *Suppose (K, v_p, v_h, a) is a model of $\mathcal{T}_h$(pCF, Th($\mathbb{Z}$)) and S is a subset of K^n defined by*

$$\varphi_S(X) := \bigwedge_{i\in I} v_p(f_i(X)) \geq 0 \ \& \ \bigwedge_{j\in J} v_p(g_j(X)) > 0.$$

Let A be the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by $\{f_i, g_j\}_{i\in I}^{j\in J}$, $M_a(K(X))$ and $\gamma_p(K(X))$. Let B be the ideal of A generated by $\mathcal{M}_p$ and $\{g_j\}_{j\in J}$. Let $T = \{1 + b : b \in B\}$. Then $h(X) \in K(X)$ is S-$\mathcal{T}_h$(pVF, DOAG)-integral definite if and only if h is integral over A_T, and h is S-$\mathcal{T}_h$(pVF, DOAG)-infinitesimal definite if and only if h is integral over B_T. That is $\mathcal{O}(S) = \sqrt[\text{int}]{A}$ and $\mathcal{M}(S) = \sqrt[\text{int}]{B}$.

PROOF. As usual, the right-to-left implication is clear as the properties of being S-$\mathcal{T}_h$(pVF, DOAG)-integral definite and S-$\mathcal{T}_h$(pVF, DOAG)-infinitesimal definite are preserved by localisation at T and by integral closure. For the left-to-right implication, we proceed as before. First, suppose $h(X) \in K(X)$ is S-$\mathcal{T}_h$(pVF, DOAG)-integral definite but not integral over A_T. Then, by Theorem 1.2, there is a valuation $\overline{v}_p$ on $K(X)$ such that $\overline{v}_p(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}_p(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}_p(h(X)) < 0$. In particular, $\overline{v}_p(f_i(X)) \geq 0$ for each $i \in I$ and $\overline{v}_p(g_j(X)) > 0$ for each $j \in J$. That is,

$$(K(X), \overline{v}_p) \models \exists X\, [\varphi_S(X) \ \&\ v(h(X)) < 0],$$

witnessed by the element $X \in K(X)$. Since $\gamma_p(K(X)) \subseteq \mathcal{O}_{\overline{v}_p}$, $\overline{v}_p$ is a p-valuation. Moreover, since $M_a(K(X)) \subseteq \mathcal{O}_{\overline{v}_p}$, $\overline{v}_p$ is an $\mathcal{T}_h(\text{pVF}, \text{DOAG})$-valuation, so there is a henselian valuation $\overline{v}_h$ on $K(X)$ that makes $(K(X), \overline{v}_p, \overline{v}_h, a)$ a model of $\mathcal{T}_h(\text{pVF}, \text{DOAG})$. Then, as $\mathcal{T}_h(\text{pCF}, \text{Th}(\mathbb{Z}))$ is the model companion of $\mathcal{T}_h(\text{pVF}, \text{DOAG})$, we may further extend $(K(X), \overline{v}_p)$ to a model of $\mathcal{T}_h(\text{pCF}, \text{Th}(\mathbb{Z}))$, say, (L, w_p, w_h, a). In particular, $(K, v_p, v_h, a) \subseteq (L, w_p, w_h, a)$ and

$$(L, w_p, w_h, a) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right].$$

Then, by model-completeness of HrpCF, (K, v_p, v_h, a) satisfies the same formula. This contradicts that h is S-$\mathcal{T}_h(\text{pVF}, \text{DOAG})$-integral definite.

For the infinitesimal case, it suffices to observe that the valuation v_p is discrete so that $\mathcal{M}(S) = \mathcal{M}_{v_p} \cdot \mathcal{O}(S)$. □

3. Residually Real Closed Valued Fields

We now turn our attention to the theory $\mathcal{T}_h(\text{OVF}, \text{DOAG})$. A model of this theory is a model of henselian residually valued field (K, v', v_h, a) equipped also with an ordering $<'$ that makes $(K, <', v)$ an ordered valued field. In particular, this means

$$1 < x < y \rightarrow v'(x) \geq v'(y) \tag{7}$$

The ordering $<'$ is induced by the ordering on the residue field $(k_h, v, <)$ which is a model of OVF. If $(k_h, v, <)$ is, in fact, a real closed field and the value group Γ_h is a $\mathbb{Z}$-group, then $(K, v', v_h, <', a)$ is a model of $\mathcal{T}_h(\text{RCVF}, \text{Th}(\mathbb{Z}))$. These are similar to the chain-closed fields of [10], except that we have an explicit valuation v on the residue field. In particular, we get the following as a corollary to Theorem 4.2.

COROLLARY 4.8. *The theory $\mathcal{T}_h(\text{RCVF}, \text{Th}(\mathbb{Z}))$ is model-complete and is the model-companion of $\mathcal{T}_h(\text{OVF}, \text{DOAG})$.*

As before, to obtain our Stellensatz, we wish to extend a model $(K, v', v_h, <', a)$ of $\mathcal{T}_h(\text{RCVF}, \text{Th}(\mathbb{Z}))$ to a model $(K(X), \overline{v}, \overline{v}_h, <, a)$ of $\mathcal{T}_h(\text{OVF}, \text{DOAG})$. Recall from Section 3.4, for any field L we define the set

$$\mathcal{I}_{\text{ord}}(L) = \{\frac{1}{1+r} : f \text{ is a sum of squares in } L\}.$$

Proposition 3.9 states that a valued field extension (L, w) is formally real over $(K, v, <)$ if and only if $\mathcal{I}_{\text{ord}}(L) \subseteq \mathcal{O}_w$. Then Proposition 4.4 determines when there is a compatible henselian valuation.

Finally we show that $\mathcal{T}_h(\text{OVF}, \text{DOAG})$ is conservative by constructing an $\mathcal{T}_h(\text{OVF}, \text{DOAG})$-valuation near $b = 0$. In fact, we may use the same construction as in Section 3.4. Given a model of $\mathcal{T}_h(\text{OVF}, \text{DOAG})$ $(K, <, v, v_h, a)$ with value group $\Gamma = \Gamma_v$, fix new elements $\delta_1, \ldots, \delta_n$ with $\delta_i > \gamma$ for each $\gamma \in \Gamma$ and let

$\Gamma' = \Gamma \oplus \delta_1\mathbb{Z} \oplus \cdots \oplus \delta_n\mathbb{Z}$. Then for any polynomial $p \in K[X]$, write $p = \sum_\alpha p_\alpha (X-b)^\alpha$ where $(X-b)^\alpha = \prod_{1\leq i\leq n}(X_i - b_i)^{\alpha_i}$. Define

$$\hat{v}(p) = \min\{v(p_\alpha) + \sum_{1\leq i\leq n} \alpha_i\delta_i\}$$

and extend to $K(X)$. To see that this is an $\mathcal{T}_h(\mathrm{OVF}, \mathrm{DOAG})$-valuation we need only construct a compatible henselian valuation $\hat{v}_h$. We may, in fact, use the same construction. Fix elements $\epsilon_1, \ldots \epsilon_n$ such that $\gamma < \epsilon_i$ for each $\gamma \in \Gamma_h$ and each $i = 1, \ldots, n$. Let $\Gamma'_h = \Gamma_h \oplus \epsilon_1\mathbb{Z} \oplus \cdots \oplus \epsilon_n\mathbb{Z}$ and define

$$\hat{v}_h(p) = \min\{v_h(p_\alpha) + \sum_{1\leq i\leq n} \alpha_i\epsilon_i\}.$$

An argument similar to the one in the proof of Proposition 4.6 shows that $\hat{v}_h$ is, indeed, compatible with $\hat{v}$. This shows that $\mathcal{T}_h(\mathrm{OVF}, \mathrm{DOAG})$ is conservative and we are ready to prove our Stellensätze.

THEOREM 4.9. *Suppose $(K, <, v, v_h, a)$ is a model of $\mathcal{T}_h(\mathrm{RCVF}, \mathrm{Th}(\mathbb{Z}))$ and S is a nonempty subset of K^n defined by*

$$\varphi_S(X) := \bigwedge_{i\in I} v(f_i(X)) \geq 0 \;\&\; \bigwedge_{j\in J} v(g_j(X)) > 0.$$

Let A be the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by $\{f_i, g_j\}$, $M_a(K(X))$ and $\mathcal{I}_{\mathrm{ord}}$. Let B be the ideal of A generated by $\mathcal{M}_v$ and $\{g_j\}$. Let $T = \{1 + b : b \in B\}$. Then $h(X) \in K(X)$ is S-$\mathcal{T}_h(\mathrm{OVF}, \mathrm{DOAG})$-integral definite if and only if h is integral over A_T, and h is S-$\mathcal{T}_h(\mathrm{OVF}, \mathrm{DOAG})$-infinitesimal definite if and only if h is integral over B_T. That is $\mathcal{O}(S) = \sqrt[\mathrm{int}]{A}$ and $\mathcal{M}(S) = \sqrt[\mathrm{int}]{B}$.

PROOF. As usual, the right-to-left implication is clear as the properties of being S-$\mathcal{T}_h(\mathrm{OVF}, \mathrm{DOAG})$-integral definite and S-$\mathcal{T}_h(\mathrm{OVF}, \mathrm{DOAG})$-infinitesimal definite are preserved by localisation at T and by integral closure. For the left-to-right implication, we proceed as before. First, suppose $h(X) \in K(X)$ is S-$\mathcal{T}_h(\mathrm{OVF}, \mathrm{DOAG})$-integral definite but not integral over A_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)$ such that $\overline{v}(f(x)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X)) < 0$. In particular, $\overline{v}(f_i(X)) \geq 0$ for each $i \in I$ and $\overline{v}(g_j(X)) > 0$ for each $j \in J$. That is,

$$(K(X), \overline{v}) \models \exists X\, [\varphi_S(X) \;\&\; v(h(X)) < 0],$$

witnessed by the element $X \in K(X)$. Since $\mathcal{I}_{\mathrm{ord}} \subseteq \mathcal{O}_{\overline{v}}$, $\overline{v}$ is an OVF-valuation and extend the ordering $<$ to $(K(X), \overline{v})$ to get a model of OVF $(K(X), <, \overline{v})$. Moreover, since $M_a(K(X)) \subseteq \mathcal{O}_{\overline{v}}$, $\overline{v}$ is an $\mathcal{T}_h(\mathrm{OVF}, \mathrm{DOAG})$-valuation , so there is a henselian valuation $\overline{v}_h$ on $K(X)$ that makes $(K(X), <, \overline{v}, \overline{v}_h, a)$ a model of $\mathcal{T}_h(\text{OVF}, \text{DOAG})$. Then, as $\mathcal{T}_h(\text{RCVF}, \text{Th}(\mathbb{Z}))$ is the model companion of $\mathcal{T}_h(\text{OVF}, \text{DOAG})$, we may further extend $(K(X), <, \overline{v}, \overline{v}_h, a)$ to a model of $\mathcal{T}_h(\text{RCVF}, \text{Th}(\mathbb{Z}))$, say, $(L, <, w, w_h, a)$. In particular, $(K, <, v, v_h, a) \subseteq (L, <, w, w_h, a)$ and

$$(L, <, w, w_h, a) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right].$$

Then, by model-completeness of $\mathcal{T}_h(\text{RCVF}, \text{Th}(\mathbb{Z}))$, $(K, <, v, v_h, a)$ satisfies the same formula. This contradicts that h is S-v-integral definite.

Similarly, if $h(X) \in K(X)$ is S-$\mathcal{T}_h(\text{OVF}, \text{DOAG})$-infinitesimal definite but not integral over B_T. Then, by Theorem 1.2, there is a valuation $\overline{v}$ on $K(X)$ such that $\overline{v}(f(X)) \geq 0$ for each $f(X) \in A$ and $\overline{v}(g(X)) > 0$ for each $g(X) \in B$, but $\overline{v}(h(X)) \leq 0$. In particular, $\overline{v}(f_i(X)) \geq 0$ for each $i \in I$ and $\overline{v}(g_j(X)) > 0$ for each $j \in J$. Then

$$(K(X), \overline{v}) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right],$$

witnessed by the element $X \in K(X)$. Since $\mathcal{I}_{\text{ord}} \subseteq \mathcal{O}_{\overline{v}}$, $\overline{v}$ and $M_a(K(X)) \subseteq \mathcal{O}_{\overline{v}}$, $\overline{v}$ is an $\mathcal{T}_h(\text{OVF}, \text{DOAG})$-valuation, there is a henselian valuation $\overline{v}_h$ on $K(X)$ that makes $(K(X), <, \overline{v}, \overline{v}_h, a)$ a model of $\mathcal{T}_h(\text{OVF}, \text{DOAG})$. Then we extend $(K(X), <, \overline{v}, \overline{v}_h, a)$ to a model of $\mathcal{T}_h(\text{RCVF}, \text{Th}(\mathbb{Z}))$, say, $(L, <, w, w_h, a)$. In particular, $(K, <, v, v_h, a) \subseteq (L, <, w, w_h, a)$ and

$$(L, <, w, w_h, a) \models \exists X \left[\varphi_S(X) \ \& \ v(h(X)) < 0\right].$$

Then, by the model-completeness of $\mathcal{T}_h(\text{RCVF}, \text{Th}(\mathbb{Z}))$, $(K, <, v, v_h, a)$ satisfies the same formula. This contradicts that h is S-$\mathcal{T}_h(\text{OVF}, \text{DOAG})$-integral definite. $\square$

A Final Remark

We conclude by pointing out an interesting feature of our results. In each case, our algebraic characterisations of $\mathcal{O}(S)$ and $\mathcal{M}(S)$ a priori depend on the chosen definition of the set S. On the other hand, the sets $\mathcal{O}(S)$ and $\mathcal{M}(S)$ are independent of the formula chosen to define S. Thus our Stellensätze characterise also the possible definitions for S.

For example, let (K, v) be a model of ACVF. Let S be the subset of K^n defined by

$$\varphi_S(X) := \bigwedge_{i\in I} v(f_i(X)) \geq 0 \ \& \ \bigwedge_{j\in J} v(g_j(X)) > 0.$$

By Theorem 2.1, we have that $\mathcal{O}(S) = \sqrt[\text{int}]{A}$ and $\mathcal{M}(S) = \sqrt[\text{int}]{B}$ where A is the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by the f_i and g_j and B is the ideal of A generated by $\mathcal{M}_v$ and the g_j. If

$$\varphi'_S(X) := \bigwedge_{i\in I'} v(f'_i(X)) \geq 0 \ \& \ \bigwedge_{j\in J'} v(g'_j(X)) > 0.$$

is another formula defining S, then, again by Theorem 2.1, we have $\mathcal{O}(S) = \sqrt[\text{int}]{A'}$ and $\mathcal{M}(S) = \sqrt[\text{int}]{B'}$ where A' is the $\mathcal{O}_v$-subalgebra of $K(X)$ generated by the f'_i and g'_j and B' is the ideal of A generated by $\mathcal{M}_v$ and the g'_j. Then, in particular, we must have $\sqrt[\text{int}]{A} = \sqrt[\text{int}]{A'}$ and $\sqrt[\text{int}]{B} = \sqrt[\text{int}]{B'}$.

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