A Class Of New Continuous Covering Problems

FORMULATIONS AND EXACT SOLUTION METHODS FOR A CLASS OF NEW CONTINUOUS COVERING PROBLEMS

By

OZAN ÇAKIR, MBA., B.ENG.

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Author	Ozan ÇAKIR
Thesis supervisor	Dr. George O. WESOLOWSKY
Supervisory committee	Dr. George O. WESOLOWSKY, Dr. George STEINER, Dr. Elkafi HASSINI McMaster University, Hamilton, Ontario, Canada Dr. Jack BRIMBERG Royal Military College, Kingston, Ontario, Canada
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Ozan ÇAKIR DeGroote School of Business

Abstract

This thesis is devoted to introducing new problem formulations and exact solution methods for a class of continuous covering location models. The manuscript includes three self-contained studies which are organized as in the following.

In the first study, we introduce the *planar expropriation problem with non-rigid rectangular facilities* which has many applications in regional planning and undesirable facility location domains. This model is proposed for determining the locations and formations of two-dimensional rectangular facilities. Based on the geometric properties of such facilities, we developed a new formulation which does not require employing distance measures. The resulting model is a mixed integer nonlinear program. For solving this new model, we derived a continuous branch-and-bound framework utilizing linear approximations for the tradeoff curve associated with the facility formation alternatives. Further, we developed new problem generation and bounding strategies suitable for this particular branch-and-bound procedure. We designed a computational study where we compared this algorithm with two well-known mixed integer nonlinear programming solvers. Computational experience showed that the branch-and-bound procedure we developed performs better than BARON and SBB both in terms of processing time and size of the branching tree.

The second study is referred to as the *planar maximal covering problem with single convex polygonal shapes* and it has ample applications in transmitter location, inspection of geometric shapes and directional antenna location. In this study, we investigated maximal point containment by any convex polygonal shape in the

Euclidean plane. Using a fundamental separation property of convex sets, we derived a mixed integer linear formulation for this problem. We were able to identify two types of special cuts based on the geometric properties of the shapes under study, which were later employed for developing a branch-and-cut procedure for solving this particular location model. We also evaluated the resultant bound quality after employing the afore-mentioned cuts.

In the third study, we discuss the *dynamic planar expropriation problem with single convex polygonal shapes*. We showed how the basic problem formulations discussed in the first two studies extend to their diametric opposites, and further to models in higher dimensions. Subsequently, we allowed a dynamic setting where the shape under study is expected to function over a finite planning horizon and the system parameters such as the fixed point locations and expropriation costs are subject to change. The shape was permitted to relocate at the beginning of each time period according to particular relocation costs. We showed that this dynamic problem structure can be decomposed into a set of static problems under a particular vector of relocations. We discussed the solution of this model by two enumeration procedures. Subsequently, we derived an incomplete dynamic programming procedure which is suitable for this distinct problem structure. In this method, there is no need to evaluate all the branches of the branching tree and one proceeds with keeping the minimum stage cost. The evaluation of a branch is postponed until relocation takes place in the lower-level problems. With this postponing structure, the procedure turned out to be superior to the two enumeration procedures in terms of tree size.

Acknowledgements

First and foremost, I would like to express my sincere gratitude to my supervisor, Dr. George O. Wesolowsky, for his continuous guidance and kind tolerance throughout this research. I must also pay tribute to his high-quality supervision and academic instruction. During my five-year research period with Dr. Wesolowsky, I have always found his approach to location theory very professional and elegant. I strongly believe that I acquired many great values from him -not necessarily all academic- during this period. I thereby consider myself indeed fortunate for having the opportunity of being his student. Also, I would like to thank him for his generous financial support which stretched beyond the expected graduation period of four years.

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Chapter 1
Preliminaries

1.1 Preface

Location Science consists of siting single or multiple facilities within a feasible space while ensuring that a particular objective function is optimized. Problems in location science are set apart from those in layout research, as in the location-theoretic view, the facilities to be sited are usually relatively small compared to the feasible location space. The location space may vary with regards to the problem structure. Some examples of location spaces are: lines, Euclidean planes, multi-dimensional spaces, routes, networks, spherical surfaces, network interiors and grids. The reach of location science is far beyond the operations research/management science discipline and is related to many domains such as: industrial engineering, computational geometry, geography, mathematics, economics, pattern recognition, marketing, regional planning and socio-economic planning. Therefore, location science may be regarded as the discipline of finding the best locations for any type of entities that are of particular interest in the above-stated domains.

1.2 Location space

One method of categorizing location problems is by the type of space on which facilities are located. If the location space is continuous, the location of a facility can be specified by continuous variables denoting the coordinates. Therefore, if it is feasible to locate on a space where the coordinates of the candidate sites vary continuously, we call this a *continuous location problem.* If the candidate sites are pre-determined in terms of a discrete set of feasible location points, they can be modeled by employing discrete variables. Then, the resultant location model will be a *discrete location problem.* If the candidate sites are restricted to the vertex set of a given network, then the resulting location model is a *discrete network location problem.* If the facility can be positioned along the edges of the network, as well as on its vertices, then the corresponding location model is an *absolute network location problem.* These location spaces are illustrated in Figure 1.

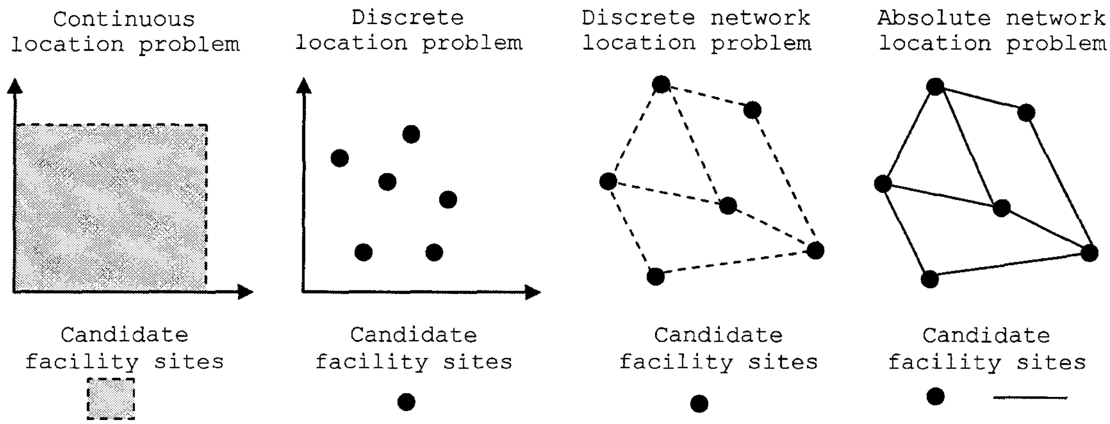

Figure 1. Location spaces and corresponding problems

1.3 Distance metrics

In location science, the proximity relationship, or the actual travel distance between the points on a space, is measured by employing a *distance metric.* This is a function which returns a real value, the distance, between each pair of points. Let (P_1, P_2) be a point pair located on a space $\wp$, $d(P_1, P_2)$ be the distance between these points and $P_3 \in \wp$ be a third point. A distance metric defined on $\wp$ will essentially satisfy the following properties (Love et al., 1988):

(1) $d(P_1, P_2) \geq 0, \quad \forall P_1, P_2 \in \wp$ (non-negativity property);

(2) $d(P_1, P_2) = 0 \Longleftrightarrow P_1 = P_2, \quad \forall P_1, P_2 \in \wp$ (identity property);

(3) $d(P_1, P_2) = d(P_2, P_1), \quad \forall P_1, P_2 \in \wp$ (symmetry property);

(4) $d(P_1, P_2) \leq d(P_1, P_3) + d(P_2, P_3), \quad \forall P_1, P_2, P_3 \in \wp$ (triangle inequality).

To employ the principles of convex programming, it is required that $d(P_1, P_2)$ is a convex function $\forall P_1, P_2 \in \wp$. The real travel distances are usually approximated with closed form analytical expressions assuming the properties (1-4) as well as convexity (Love and Morris, 1972, 1979; Brimberg, 1989; Brimberg and Love, 1991, 1992, 1995).

Consider a compact set $\mathcal{U}$ containing the origin O and the points P_i, such that $d(O, P_i) \leq 1, \forall P_i \in \mathcal{U}$ as illustrated in Figure 2. Then, $\mathcal{U}$ is said to be the *unit ball* of the distance metric d. The distance between the origin and a fixed point $P_j, j \notin \mathcal{U}$ can be calculated by expanding the unit ball $\mathcal{U}$ by r times (i.e. $r \times \mathcal{U}$: $r \in \Re^+$) until the unit ball boundary $\partial\mathcal{U}$ touches the point P_j. Then, distance between the origin and the point P_j, measured employing the metric d, is said to be r, which is given by the Minkovski relation $d(O, P_j) = \min\{r \in \Re^+ : P_j \in r \times \mathcal{U}\}$.

Excluding the satisfaction of symmetry property (3), the distance metric d can be derived from a *gauge* g which is a real-valued function defined on $\wp$ satisfying the homogeneity property $g(O, P_j) = r \cdot g(O, \partial\mathcal{U})$ and completely defined by its unit ball

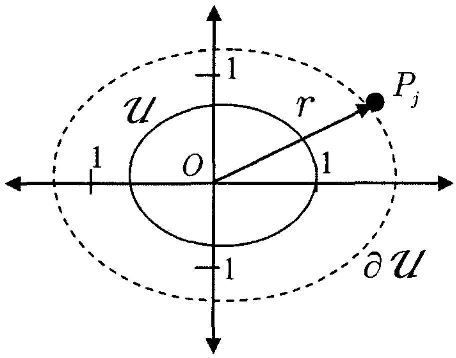

Figure 2. Calculating the distance using the unit ball of a metric

$\mathcal{U}$. The unit ball $\mathcal{U}$ illustrated in Figure 2 is an example for gauge distance unit balls since it violates the symmetry property. When the symmetry property is satisfied by a gauge g, its unit ball is symmetric around the origin O. A gauge satisfying the properties (1-4) and the symmetry property is said to define a *distance norm.*

In mathematics, a *norm* is a function that assigns a non-negative *length* to a vector in a vector space. In location science, the set of all distance norms which can be defined by a single parameter p and which satisfy the triangle inequality are referred to as *Minkovski distances of order p.* These distance measures are all convex functions and they are equivalently named ℓ_p *norms.* Let the points P_1, P_2 be located at the coordinates (a_{11}, a_{21}) and (a_{12}, a_{22}) in the plane, respectively. Then the ℓ_p distance between these two points is given by the term:

$$\ell_p(P_1, P_2) = \left(|a_{11} - a_{12}|^p + |a_{21} - a_{22}|^p\right)^{1/p}, \quad 1 \le p \le \infty.$$

For $p = 1$, ℓ_1 is called the *rectangular, rectilinear, Manhattan* or sometimes the *Hamming* distance. This distance measure is usually employed when movements parallel to the axis system are allowed for one direction at a time (i.e. traveling simultaneously in all directions is forbidden). For example, consider an automatic guided robot used to handle material between machining centers in a shop floor. In flexible manufacturing environments, the job orders are usually transmitted to those type of automatic guided robots electronically, via cables engraved in the floor forming a grid-like path. Hence, movements may be allowed only along paths aligned with an orthogonal axis system. The appropriate distance measure on the movements of this robot is then the rectangular distance norm:

$$\ell_1(P_1, P_2) = |a_{11} - a_{12}| + |a_{21} - a_{22}|.$$

As opposed to the movements in one direction at a time, now assume that simultaneous movements along the directions of an axis system are allowed. In this case, we have the *Chebyshev, maximum* or ℓ_∞ distance:

$$\ell_\infty(P_1, P_2) = \max\{|a_{11} - a_{12}|, |a_{21} - a_{22}|\}.$$

If movements are allowed along all the directions homogeneously, the distance between two points can be measured by the length of the straight line segment joining these two points. This distance measure is called the *straight-line, Euclidean,* or ℓ_2 norm between two points:

$$\ell_2(P_1, P_2) = \sqrt{(a_{11} - a_{12})^2 + (a_{21} - a_{22})^2}.$$

Distance norms can also be classified according to their unit balls. If the unit ball of a norm is composed of linear segments, it is said to be a member of the distance measures called the *block norms.* The subset of gauge distances having this linearity property are referred to as the *polyhedral gauges.* Unit balls of these gauges are convex polyhedrons. This corresponds to the situation where movement is permitted only in a finite set of directions. These directions are defined by the half-lines from the origin to the vertices of the unit ball. Symmetric polyhedral gauges are equivalent to block norms. *Round norms* are those distance measures which have no linear segments on their unit balls. Examples of the unit balls corresponding to some distance measures used in location science are illustrated in Figure 3.

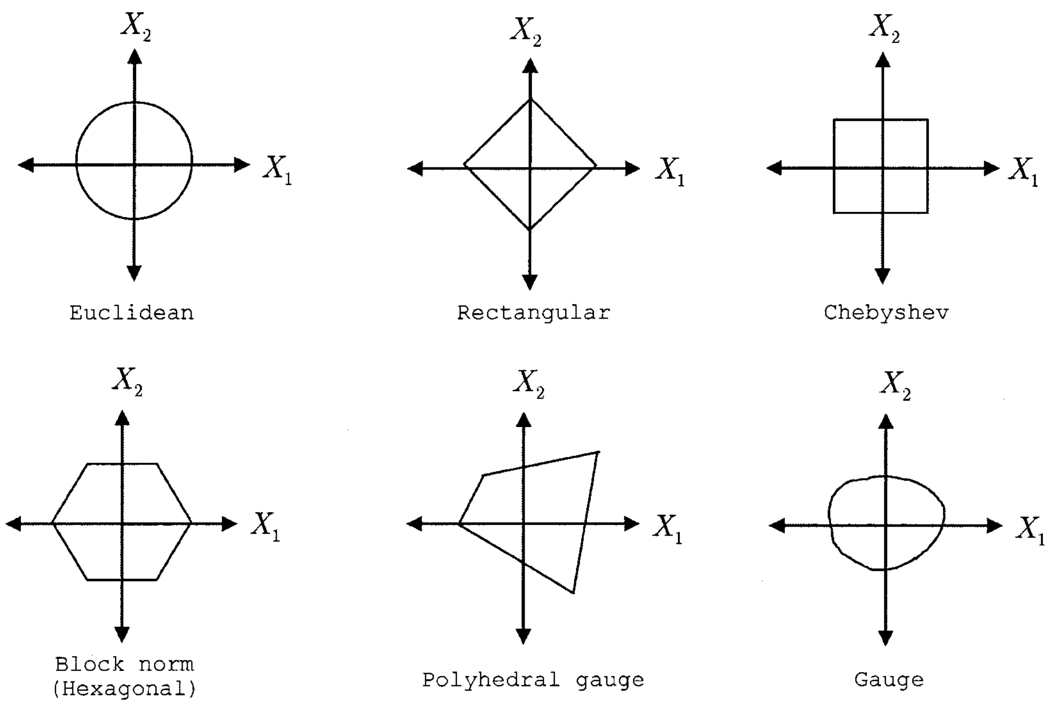

Figure 3. Unit balls of some distance measures

1.4 Objective functions

Apart from having different feasible location spaces, location problems may vary according to their objective functions. There are two predominant objective functions in location science: *mini-sum* and *mini-max*. Location problems with these two objective functions are called *median* and *center* problems, respectively. The mini-sum objective function corresponds to minimizing the sum of the (weighted or un-weighted) distances between the fixed points and the facility(ies). The mini-max objective leads to finding the facility location(s) such that the maximum distance between the facility(ies) and the farthest point to the facility(ies) is minimized.

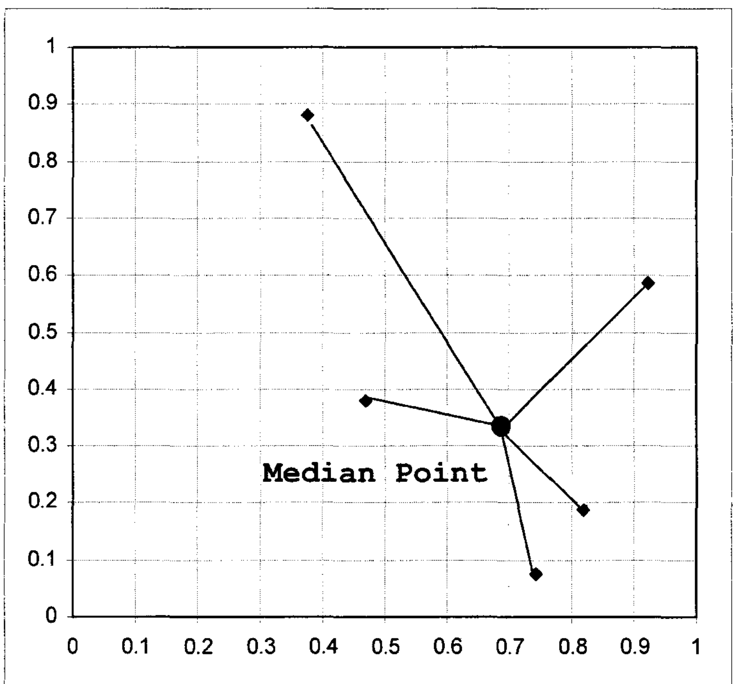

Figure 4. Solution for the mini-sum problem: *Median point*

To illustrate these objectives, we generated a small example with five fixed points located in a unit square where the point locations are randomly generated from uniform distribution $U \sim [0,1]$ as illustrated in Figures 4-5. The single facility mini-sum (median) and mini-max (center) problem solutions for this example were (0.694, 0.334) and (0.560, 0.473), respectively.

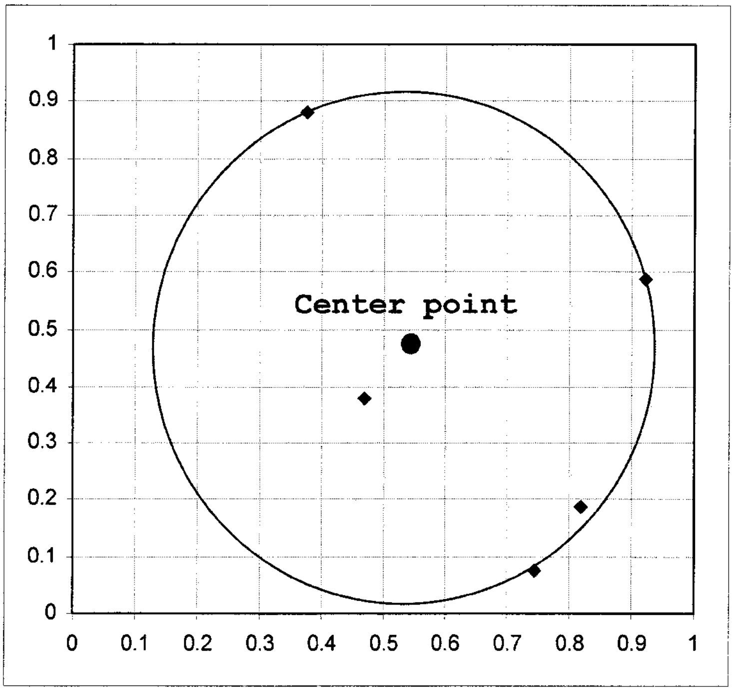

Figure 5. Solution for the mini-max problem: *Center point*

Many extensions and hybrids of these two objective functions have also been studied in location science literature. A summary of some important objective functions is provided below.

Anti-median or maxian (Church and Garfinkel, 1978). This objective corresponds to finding a facility location that maximizes the sum of the fixed point-to-facility distances within a bounded feasible space (see, Figure 6).

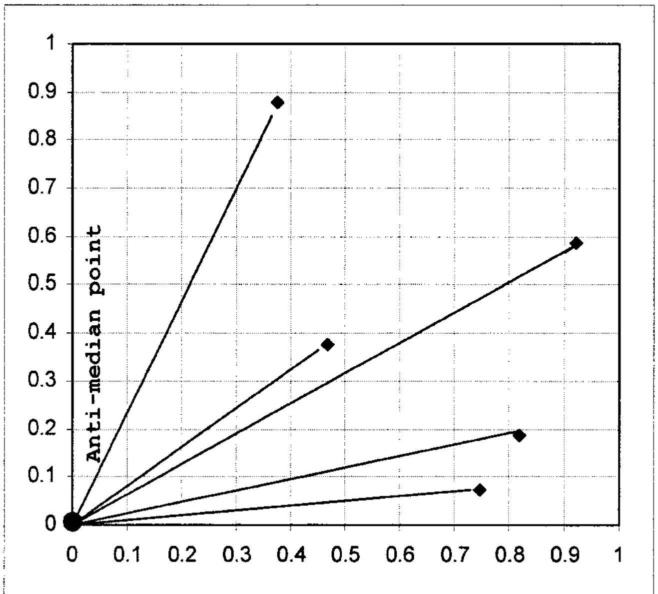

Figure 6. Solution for the maxi-sum problem: *Anti-median point*

Anti-center or maxi-min (Dasarathy and White, 1980). This objective corresponds to finding a facility location that maximizes the closest fixed point-facility distance within a bounded feasible space (see, Figure 7).

Cent-dian (Halpern, 1976). This objective function is the convex combination of center and median objectives.

Max-sum-min (Moon and Chaudry, 1984). This objective function corresponds to finding a set of facility locations that maximize the sum of closest distances between the facilities and the fixed points which patronize them.

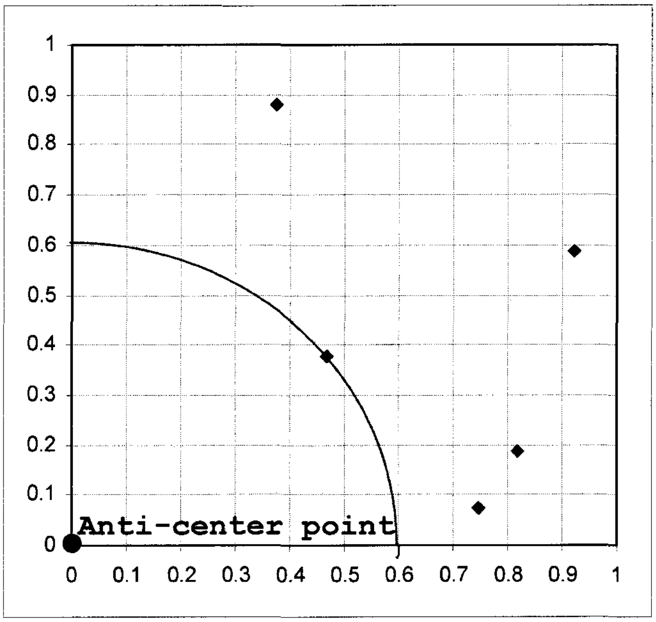

Figure 7. Solution for the maxi-min problem: *Anti-center point*

Max-min-min dispersion and maxi-sum dispersion (Kuby, 1987). These models are employed for dispersing a set of facilities within a bounded feasible space; thus there are no fixed points in these models. The first objective function maximizes the closest distances between a set of facilities, whereas the latter one maximizes the sum of the distances between the facilities (see, Figure 8).

Medi-center (Khumawala, 1973). This objective function minimizes the average distance such that every distance between facilities and fixed points is smaller than a threshold.

For illustration purposes, we solved the anti-median and anti-center problems on the same small problem illustrated in Figures 4-5. The optimal solution found for

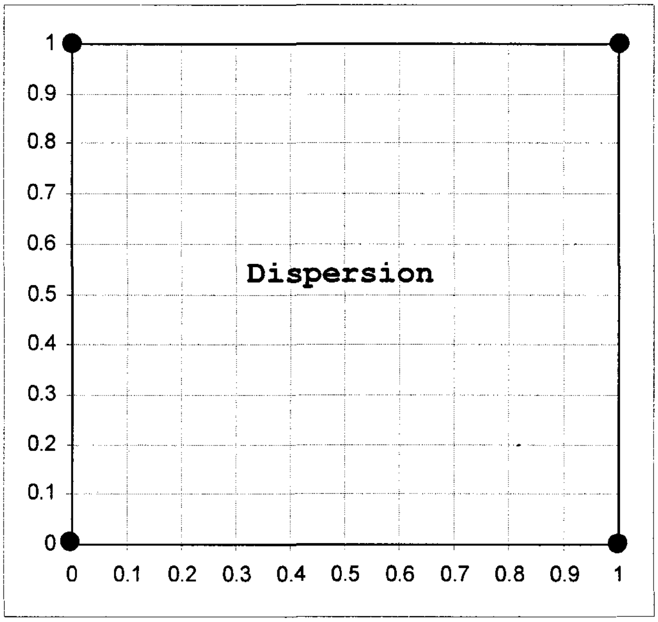

Figure 8. Solution for the *max-min-min* and *maxi-sum dispersion* problems

both problems was (0,0) as illustrated in Figures 6-7. We also solved two dispersion problems with four facilities in a unit square employing different objective functions. The solutions for max-min-min and maxi-sum dispersion problems were the same and it pointed out that the four facilities should be located at the following locations: (0,0), (0,1), (1,0) and (1,1) as illustrated in Figure 8.

A detailed analysis and discussion of the objective functions used in location science can be found in Eiselt and Laporte (1995).

1.5 Other classifications of location problems

In addition to the differences in feasible location space, the distance metric used, and the objective function, location problems may be classified according to the following: (1) the number of facilities to be located (i.e. single facility vs. multi-facility); (2) the nature of system parameters (i.e. deterministic vs. stochastic, static vs. dynamic etc.); (3) facility type (i.e. capacitated vs. uncapacitated, desirable vs. undesirable etc.); and (4) the categorization of demand (i.e. continuous vs. discrete, deterministic vs. stochastic).

An extensive taxonomy for distinguishing location problems can be found in Brandeau and Chiu (1989). For other overviews that classify location problems, one can refer to Hale and Moberg (2003) and ReVelle and Eiselt (2005).

1.6 Evolution of location science

The foundations of location research date back to 17th century and the first location problem is variously attributed to the scientists Pierre Fermat, Evangelista Torricelli and Battista Cavallieri. They independently studied the *spatial 1-median problem* and proposed solution methods based on geometric principles. The popularization of location science starts with Alfred Weber's observation (see, Weber, 1909) that the spatial 1-median problem has an industrial application and essentially solves the problem of locating a facility to minimize the sum of transportation costs from that facility to a given set of fixed points. Since then, the spatial 1-median problem has usually been named after Weber. Today, there exists a huge body of knowledge related to the problem, its extensions, solution approaches, and applications. A complete history of the Weber problem can be found in Wesolowsky (1993). Another extensive but earlier historical perspective on the Weber problem is due to Kuhn

(1973). The properties and variations of the problem are discussed in detail by Love et al. (1988).

Discrete location theory evolved after Hakimi's (1964) seminal paper which contains the first results regarding the location problems on network spaces. The main result in this paper is known as the *Hakimi property* today. This is the concavity proof of the mini-sum objective function, for the case where the fixed demand points are located at the vertices of a graph. Moreover, he showed that the minimum of this concave function is attained at one of these vertices. This is a ground-breaking result because it allows the decision maker to restrict the candidate facility sites only to the vertices of the graph and leads to a very effective solution method. If the shortest path between each vertex pair is known, Hakimi's (1964) result guarantees that the 1-median problem can be solved in $\mathcal{O}\left(|V|^2\right)$ computation time on general graphs $G = (V, E)$. Later, Hakimi (1965) showed that this property also holds for multiple facilities. More information about the literature concerning discrete location problems can be found in Tansel et al. (1983a; 1983b), Mirchandani and Francis (1990), Daskin (1995) and the references therein.

Apart from Weber's (1909) and Hakimi's (1964) influential work, there are other milestone studies in the evolution of location science. These can be summarized as: Sylvester's (1857) proposition and Chrystal's (1885) solution method regarding the *minimum covering circle problem*, which led to the development of *center problems* in location science. Hotelling's (1929) paper initiating a branch of location problems which are known today as *competitive location problems*. Weiszfeld's (1937) algorithm for efficiently solving the Weber problem is, known today as the *Weiszfeld procedure*.

Cooper's (1963) paper gave rise to another family of location problems referred to as *location-allocation problems* and Toregas et al.'s (1971) *location set covering problem* led to the establishment of another stream of location research called *covering location problems.*

More detailed information on the variety of problems in location science can be found in books by Love et al. (1988), Francis et al. (1992), Drezner (Ed., 1995), Sule (2001), Drezner and Hamacher (Eds., 2002) and in survey articles by Francis and Goldstein (1974), Brandeau and Chiu (1989), Chhajed et al. (1993), Owen and Daskin (1998), Hale and Moberg (2003), ReVelle and Eiselt (2005) and Melo et al. (2009). The computational aspects of location problems are investigated, in detail, in closely related books by Eilon et al. (1971), Scott (1971a), Christofides (1975), Daskin (1995), and in articles by Megiddo and Supowit (1984), Guha and Khuller (1999), Arya et al. (2004), Gabor and van Ommeren (2006) and Xu and Xu (2009).

Chapter 2
Scope of the thesis and related literature

2.1 Scope of the thesis

Apart from the two predominant objective functions mentioned in the previous chapter, another important objective in location science is the *covering objective.* This will be the particular concentration of this thesis. Covering problems may be categorized into four groups of problems: (1) the *set covering location problem* ($\mathcal{SCLP}$); (2) the *maximal covering location problem* ($\mathcal{MXCLP}$); (3) the *minimal covering location problem* ($\mathcal{MNCLP}$); and (4) the *maximum expected coverage location problem* ($\mathcal{MECLP}$).

In general, a fixed point is deemed as being *covered* by a facility if the distance between the fixed point and the facility is smaller than a pre-determined threshold value. The first problem, $\mathcal{SCLP}$ (Toregas et al., 1971), is employed for locating a minimum number of facilities within a feasible space such that every fixed point is covered by a facility. As it stands, $\mathcal{SCLP}$ is very restrictive and it is often not financially viable to apply this model because the spread of fixed points is important in covering location problems. For example, if the spread of the fixed demand points is not condensed in particular locations, the $\mathcal{SCLP}$ solution will require too many facilities and this may be far beyond the budgetary restrictions as illustrated in Figure 9.

Church and ReVelle (1974) and White and Case (1974) independently pro-

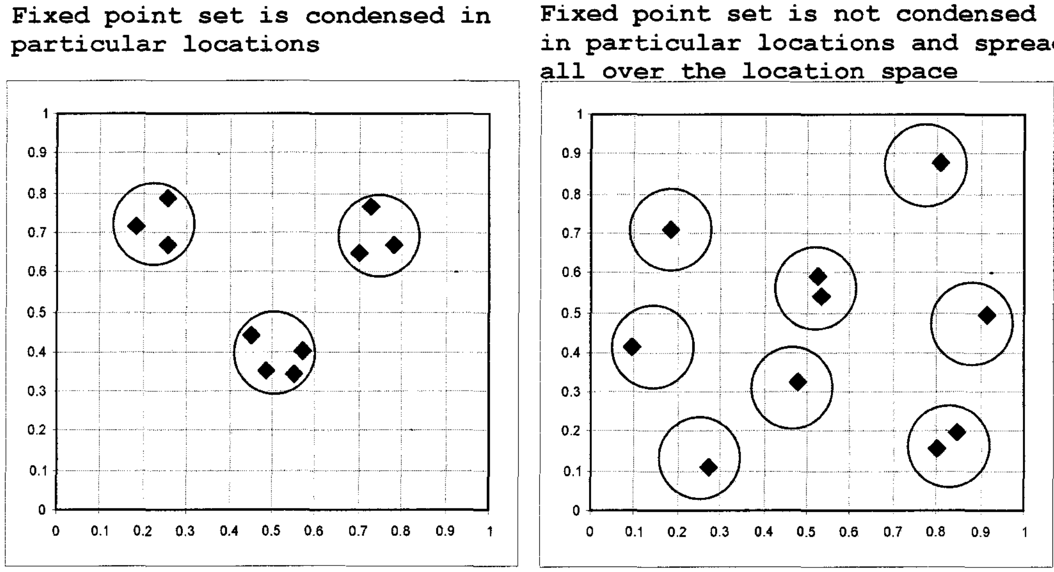

Figure 9. Spread of the fixed point set

posed the $\mathcal{MXCLP}$, which relaxed the obligatory coverage requirement. The objective of $\mathcal{MXCLP}$ was to solve for the maximum number of points covered by analyzing a set of possible facility site alternatives.

The diametric opposite of the $\mathcal{MXCLP}$ objective also exists and was first studied by Drezner and Wesolowsky (1994). This problem solves for the minimal point coverage by a circular disk or a rectangle and is referred to as $\mathcal{MNCLP}$. Another approach for partially covering the fixed point set is the $\mathcal{MECLP}$ (Daskin, 1983). In this objective, a demand originating from a fixed point may not be covered because the facilities may be serving some other fixed points. This corresponds to the *busy state* of the facilities in the system. Daskin (1983) proposed and solved the $\mathcal{MECLP}$ with the objective of maximizing the expected coverage under the assumption that the probability regarding the busy state of the facilities in the system is known as a priori.

This thesis is particularly devoted to introducing new formulations and exact solution methods for a group of new *continuous minimal and maximal covering location problems.* Unlike the facilities considered in existing covering location models, the facilities employed in this text are *dimensional facilities.* That is, we study covering location problems of facilities having particular shapes rather than being analytical solution points in the feasible location space. The facilities may assume many diverse dimensional forms according to the application area. For example in material cutting, the *facility* is nothing but a shape that has to be placed on the material with covering minimum fault-points. A practical example of this application is the problem of cutting polygonal pieces from a bigger flat section of leather in the leather industry. Since we study the covering location problems of different dimensional objects in various chapters, we will generally use the terms *shape* and *facility* interchangeably, throughout this thesis.

All the proposed problems in this thesis solve for the location of shapes in a continuous feasible location space while ensuring that the shapes are aligned with the orthogonal axis system. That is, the shapes are restricted to move only via translations, and rotational moves are not permitted. The translations allowed for the shapes are affine transformations which do not permit dilations and distortions of the shape. In other words, the shapes preserve their interior angles and the points on the edges of the shape preserve their co-linearity. Problems considering rotational aspects of the shapes, which are beyond the scope of this text, can be tackled either algorithmically or by iteratively rotating the axis system and employing the models discussed in this manuscript using modified fixed point locations.

We also discuss the implementation of continuous covering model under a dynamic setting. In this scenario, the facilities are expected to serve for multiple time periods and system parameters such as fixed point locations and costs are subject to change.

Prior to a formal statement of problems to be studied, it is essential to identify the gaps in the literature regarding this group of covering models. In this respect, we review the relevant literature on *continuous covering location problems*, *dimensional facility location problems* and *dynamic facility location problems* in the following section. In the next chapter, we summarize the gaps identified in the literature and state the problems which are the particular subject of this thesis.

2.2 Literature review

In this section, we provide a detailed literature review on continuous covering problems, dimensional facility location problems and dynamic facility location problems which are closely related to the thesis problems.

2.2.1 Continuous covering location problems

The group of maximal and minimal covering problems which will be discussed in this thesis fall into the realm of continuous covering location models. In continuous location problems, the location space can be described by continuous variables, denoting the coordinates. Unlike in discrete location problems, a finite set of candidate sites for the facilities can not be specified in continuous location problems. Therefore, instead of a pre-specified set of candidate sites, we consider a feasible location space. Usually, continuous location problems are regarded as *site-generating* (Love et al., 1988) models in the sense that the facility location can be anywhere within the feasible location space.

In general, a metric is employed as a mathematical description of the distance or closeness relation between the fixed points and the facilities. The metrics which are members of the Minkovski distances of order p are all convex functions for $p \geq 1$ (Love et al., 1988), hence their mixed norms (i.e. convex combinations) also have this property. Therefore, continuous covering problems usually call for the utilization of convex optimization and global optimization methods (Hirirart-Urruty and Lemarechal, 1993; Hansen et al., 1995).

The distance threshold below which the fixed points are deemed as *covered* is called the *impact radius* (Carrizosa and Plastria, 1995). Since we study covering location problems of dimensional facilities in this thesis, it is worthwhile to clearly state the distinction between a dimensional facility and an impact radius. An impact radius applies to an analytical solution point and sets the threshold for the covering decision, whereas a dimensional facility has its coverage area already incorporated within.

In this section, we further categorize the covering models introduced in Section 2.1 into four classes: (1) *obligatory* (or sometimes referred to as *full*) *covering* *models*, where the objective is to cover the fixed point set with the minimum possible impact radius; (2) *maximal covering models*, where the objective is to cover the largest number (or weight) of fixed points with a fixed impact radius; (3) *empty covering models*, where the objective is not to cover any fixed points but to maximize the possible impact radius; and (4) *minimal covering models*, where the objective is to cover least possible number (or weight) of fixed points with a fixed impact radius.

2.2.1.1 Obligatory (or full) covering models

The minimal radius full covering problem with a fixed point set can be solved by finding the minimum impact radius of a circular disk (in case of Euclidean distance) enclosing the fixed point set. This is equivalent to a renowned problem in location science, namely, the *minimum covering circle problem* proposed by Sylvester (1857). An early method for this problem was proposed by Elzinga and Hearn (1972). Their method is based on iteratively expanding the radius of the circle either touching two fixed points which define the diameter of the circle or three fixed points on the circle forming an acute triangle, until the circle covers the whole fixed point set. The method can be summarized as follows.

Procedure: Elzinga-Hearn

Step 1. Pick 2 demand points P_1 and P_2 from the fixed point set.

Step 2. Construct the circle C such that the two points on hand define its diameter.

Step 3. If C covers the fixed point set, it is the minimum covering circle, STOP. Otherwise, add a fixed point P_3 lying outside C.

Step 4. Check the triangle T defined by the three points on hand. If there is an obtuse angle in T, drop the point at this angle, return to **Step 2** and continue with the two points on hand.

Step 5. If all the angles of the triangle T are acute, compute the new circle C passing through these three points. If C covers the fixed point set, it is the minimum covering circle, STOP.

Step 6. Otherwise, add a point P' outside C according to procedure ADD-DROP, return to **Step 4** and continue with these three points.

Procedure: ADD-DROP

Step 1. Let, P^T be the farthest point of T from P' as in Figure 10, and L be the line passing through P^T and the centre of C.

Step 2. L separates the two remaining vertices of T (see, Figure 10). Then, drop the point which is on the same half-plane as P'. ♣

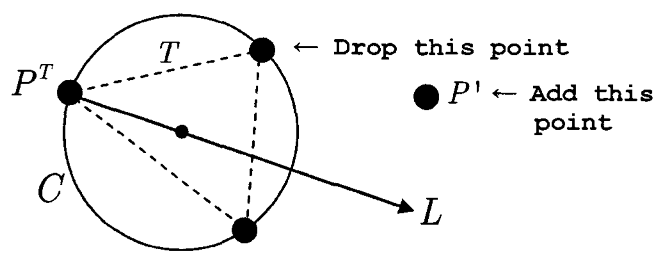

Figure 10. Adding and dropping points in Elzinga-Hearn method

The average complexity of Elzinga-Hearn method is still not known on a randomly generated fixed point set. Preparata and Shamos (1985), reported that the method has an $\mathcal{O}(N^2)$ worst case complexity, where N is the number of fixed points. Later, Drezner and Shelah (1987), contributed with an example where the method actually takes $\mathcal{O}(N^2)$ steps.

The minimum covering circle problem can be tackled by computing a tessellation of the plane, finding a non-dominated set of circle locations and finally searching the resultant set of location alternatives. This procedure requires employment of *farthest point Voronoi diagrams* (Okabe et al., 1992; Suzuki and Okabe, 1995), a basic tool of computational geometry. Many efficient methods are readily available for obtaining Voronoi diagrams: see, for example, the studies of Ohya et al. (1984) and Fortune (1987). The first step, computing the non-dominated set of location alternatives (i.e. the farthest point Voronoi diagram), will essentially require $\mathcal{O}(N \log N)$ time (see, Aurenhammer, 1991; Okabe et al., 1992). The resulting structure is a tree network. Then, the solution of the minimum covering circle problem corresponds to finding the circle location minimizing the impact radius on the absolute tree network (i.e. on the vertices and the continuum of the edges), that covers the fixed point set.

Welzl (1991) proposed another efficient method for the full covering problem, which works with $\mathcal{O}(N)$ complexity, by adding the fixed points one by one to the incumbent solution. According to Welzl's (1991) procedure, the new points added to the solution either lie inside the current minimum covering circle or on the new minimum covering circle.

Ohsawa and Imai (1997) studied some interesting properties of the problem. They considered a fixed radius larger than the impact radius and found the set of all full covering circles. Then, they tried to define a parametric function for computing the cardinality of this set by means of the fixed radius (i.e. the parameter).

Hearn and Vijay (1982) and Charalambous (1982) independently studied the weighted version of the basic problem and extended Elzinga and Hearn's (1972) method. Follow-up work is due to Megiddo (1983), where an $\mathcal{O}(N)$ algorithm was proposed.

Another variant of the problem is to consider locational constraints. Suppose that the location (i.e. center) of the minimal covering circle is restricted to lie within a polygonal region. In this case, the best known solution method is reported by Woeginger (1998), which runs in $\mathcal{O}(N \log N)$ time.

The only study considering full covering in $\Re^3$ is due to Späth (1978). The rectangular distance metric was employed for this problem. Thus, the covering shape of Späth's problem is a regular octahedron.

The full covering problem under other distance measures was also studied. Drezner and Wesolowsky (1980a) studied the mini-max single facility center problem with general ℓ_p norms, Beer and Pai (1990) investigated the Chebyshev norm center problems, and Nickel (1998) solved a restricted center problem under polyhedral gauges.

Drezner and Wesolowsky (1983a), proposed another variant of the full covering problem, the full covering problem on a sphere, by considering spherical distances. Later, this problem was re-addressed by Patel (1995) who considered the Euclidean norm. Sarkar and Chaudhuri (1996) analyzed the problem on a hemisphere using equal fixed point weights. Das et al. (1999) proposed a polynomial time algorithm for the mini-max location problem on the hemisphere.

The multi-facility extension of the full covering problem is called the *p-centre problem.* This model is employed for solving the location problem of p circular disks of equal radius (for the Euclidean distance) that fully cover the fixed point set, while minimizing the impact radius. This problem is known to be $\mathcal{NP}$-hard (Fowler et al., 1981; Masuyama et al., 1981; Megiddo and Supowit, 1984). If the rectangular distance metric is used, the circular disks are replaced by squares tilted 45°. For the rectangular case, the p-centre problem also remains $\mathcal{NP}$-hard (Hsu and Nemhauser, 1979; Megiddo and Supowit, 1984). However, there exist some instances where such problems are polynomially solvable. These instances call for a special arrangement of the fixed point set. See, for example, the one dimensional p-centre problems discussed in Megiddo et al. (1981) and Frederickson and Johnson (1983).

The p-centre problem has algorithmic solution approaches calling for $\mathcal{O}\left(N^{3p}\right)$ computational complexity. At the beginning, the minimum number of potential shape locations that cover the whole fixed point set within a fixed impact radius must be found. Let this fixed impact radius be R. First, circular disks of radius R are located on all the fixed points and the intersections of these disks are found. Then, the minimum number of these intersections which ensure that the fixed point set is fully covered are determined as *potential* shape locations. This procedure requires $\mathcal{O}\left(N^{3}\right)$ computational effort (Ryzhkov, 1973; Vijay, 1985). The next step is to search for the minimal impact radius R, ensuring that the number of potential shape locations is not greater than p. This yield an $\mathcal{O}\left(N^{3p}\right)$ algorithm overall. The rectangular distance case of this problem was also studied by Drezner (1987) and later by Ko et al. (1990), who referred the problem as the *rectilinear m-center problem.*

Several other algorithmic procedures and heuristics were proposed and studied for this problem. Exact algorithms (Drezner, 1984), simple heuristics (Dyer and Frieze, 1985), interactive graphics procedures (Ezra et al., 1994), simulated annealing procedures (Maffioli and Righini, 1994), seed point heuristics (Pelegrin and Canovas, 1998) and fixed point aggregation methods (Rayco et al., 1999) were successfully applied.

2.2.1.2 Maximal covering models

The first basic covering location model was proposed by Toregas et al. (1971). This is used to determine the locations of a fixed number of *servers* to fulfill an obligatory coverage requirement of all fixed points. As mentioned in Section 2.1, this model is very restrictive and not suitable for situations where the decision maker can not choose the impact radius or the radius threshold turns out to be insufficient for covering the whole fixed point set. An extension of this type of covering model is to consider a partial coverage of the fixed point set. This idea led location scientists to consider *maximal coverage* instead of an obligatory coverage requirement for all fixed points. The *maximal covering location problem* ($\mathcal{MXCLP}$) was introduced independently by Church and ReVelle (1974) and White and Case (1974). These authors discussed covering a maximum number of fixed demand points by analyzing a set of possible site alternatives. The models introduced by these authors were discrete problems rather than being continuous covering models, since the feasible locations were pre-specified. Soon after, $\mathcal{MXCLP}$ was extended to continuous covering models where the facility location could be anywhere on the entire plane (Mehrez, 1983; Church, 1984).

Multi-facility version of $\mathcal{MXCLP}$ was studied under different names and heuristic solution procedures were proposed. Among them, Watson-Gandy (1982) named his problem the *m-partial covering problem* and solved it for Euclidean distances, whereas Drezner (1986) discussed the problem of maximal covering with rectangular distances and referred to the resulting model as a *p-cover problem.*

One possible objective is to investigate how far the impact radius can be reduced while still guaranteeing a desired coverage level. This calls for allowing a varying impact radius and searching for the location of a facility, and its minimal impact radius, which do not violate the threshold coverage level. This problem was proposed by Carrizosa and Plastria (1995) and named the *minimal-quantile location problem.*

It is interesting to note that one decade before the formal introduction of $\mathcal{MXCLP}$, the following group of authors studied a hierarchical version of this problem in the context of emergency service facility planning: Schultz (1970), Calvo and Marks (1973), Dökmeci (1973) and Narula and Ogbu (1979). Moore and ReVelle's (1982) two-level model suggested a *hierarchical maximal covering problem* that can be characterized by two types of facilities: the lower level facilities providing a type-1 service and higher level facilities providing a type-2 service. Then, the problem is to maximize the population covered with maintaining specified service distance standards. Recent studies on this branch of $\mathcal{MXCLP}$ are due to Boffey and Narula (1997) and Espejo et al. (2003).

In the $\mathcal{MXCLP}$, a particular fixed point is either covered or left uncovered. That is, the facilities are fully covered up to a distance R (i.e. the impact radius), yet the coverage reduces to zero immediately beyond R, say at a distance $R + \varepsilon$, where ε is a very small number. In some application areas such as marketing and retail location, this problem characteristic has not been regarded as realistic. Hence, the coverage level was modeled as a decreasing function of the distance from the fixed points to the facility location. Examples are the *spatial interaction models* (Huff, 1964), related to competitive facility location. One may refer to Ghosh et al. (1995) for a detailed review of this body of knowledge.

This approach was adapted to the $\mathcal{MXCLP}$ by Berman and Krass (2002) and Drezner et al. (2004), but using different decay functions. Berman and Krass (2002) studied a generalization of the $\mathcal{MXCLP}$ by modeling the level of coverage as a decreasing step function of the distance between the fixed points and their nearest facilities. Drezner et al. (2004) investigated a somewhat different problem by considering three types of coverage levels: (1) up to some distance standard the fixed points are fully covered; (2) beyond some other distance standard the fixed points are not covered; and (3) between these two distance standards, the coverage level is linear according to the distance of the fixed point to the facility. They refer to this problem as the *gradual covering problem*, and proposed a branch-and-bound method for its solution. Earlier models discussing the coverage over the impact radius can be found in Church and Roberts (1983) and Pirkul and Schilling (1991).

Further extensions of the $\mathcal{MXCLP}$ include multiple, excess, backup and expected coverage (Daskin et al., 1988) and stochastic models (Sherali et al., 1991).

Effects of demand aggregation on the fixed points and the corresponding errors were also investigated by Daskin et al. (1989) and Current and Schilling (1990).

There are many application areas of the $\mathcal{MXCLP}$. Some interesting reports on diverse applications include: control monitor location (Hougland and Stephens, 1976), ambulance deployment (Eaton et al., 1986), locating emergency warning sirens (Current and O'Kelly, 1992), and branch location in the private sector (Pastor, 1994). An early review of the applications of $\mathcal{MXCLP}$ can be found in Chung (1986). A detailed review of covering problems in general, is given by Schilling et al. (1993).

It is noticeable that the planar maximal covering problem literature is sparse on the subject of formulations where the facilities are defined by an area. A recent study discussing the location of area-type facilities using a classical objective function is due to Carrizosa et al. (1998). In this article, the fixed point set is assumed to be distributed over a region according to a probability measure. The location of a rectangular facility minimizing the expected distance to this fixed point set is discussed. The studies considering the area-type facilities using covering objectives are very limited. The only two recent studies are due to Younies and Wesolowsky (2004; 2007). In the first article, the problem of locating inclined parallelograms for the planar maximal covering problem is proposed. A mixed integer linear program (MILP) and an alternative algorithm are presented. In the latter one, the problem is extended to general block-norm distances. Borrowing concepts from graph theory, Younies and Wesolowsky (2007) converted the anticipated maximal covering problem to an equivalent maximal clique partitioning problem on a multi-interval graph. Further, the equivalent maximal clique partitioning problem was represented as a binary unconstrained quadratic problem. Two problems were then tackled via a genetic algorithm and computational examples were demonstrated. Other than these two recent studies, there is no earlier work considering area-type facilities for planar maximal covering problems.

2.2.1.3 Empty covering models

The maximal radius empty covering problem for a fixed demand point set using Euclidean distances can be solved by inspecting the maximum impact radius of a circular disk that does not enclose any of these fixed points. This is the opposite of full covering models, and is equivalent to the *largest empty circle problem* of computational geometry. For a detailed discussion of the largest empty circle problem one may refer to Toussaint (1983) and Preparata and Shamos (1985).

The minimum enclosing circle problem and the largest empty circle problem have almost similar solution procedures, as discussed in Section 2.2.1.1, which compute a tessellation of the plane and investigate the resultant non-dominated feasible locations. However, as opposed to the procedure discussed in Section 2.2.1.1, for solving the largest empty circle problem, one should compute a *closest point Voronoi diagram.* Methods of constructing closest point Voronoi diagrams can be found in Toussaint (1983), Okabe et al. (1992) and Suzuki and Okabe (1995). Similarly to the case with farthest point Voronoi diagrams, this procedure requires $\mathcal{O}(N \log N)$ computational effort. For fixed points having different weights, the Voronoi diagram has to be adapted accordingly. Auranhammer and Edelsbrunner (1984) proposed a method of constructing a weighted extension of the basic Voronoi diagram.

The largest empty circle problem is equivalent to the maxi-min or anti-center location problems, named after their objective functions. These objective functions imply that the main goal is to choose a facility location which is as far as possible from the closest fixed point to the facility. In real-life applications, this corresponds to minimizing the undesired impact of the facility by locating it to the farthest possible site from the community. Therefore, these models are regarded as *noxious* (i.e. *undesirable* or sometimes called *obnoxious*) *facility location problems.*

The Euclidean distance maxi-min problem was studied by Dasarathy and White (1980) and Melachrinoudis and Cullinane (1985). Extensions of the basic problem include: the maxi-min problem in a workroom environment (Melachrinoudis, 1985), the maxi-min problem within a polygonal region (Melachrinoudis and Cullinane, 1986), the parametric maxi-min problem (Erkut and Öncü, 1991), and the weighted maxi-min problem (Carrizosa et al., 1994). The problem with rectangular distances was studied by Drezner and Wesolowsky (1983b), Mehrez et al. (1986), and later by Appa and Giannikos (1994).

In noxious facility location problems, undesirable nature of a facility produces a *push effect* away from the demand points. However, in some applications, there may be maximum distance constraints on the facility which *pull* the facility location closer to the community. Consider a landfill to be located within a region by a municipality. The pollution effect of the landfill site will force the management to consider a location as far as possible from the community; however, in that case, transportation costs will rise with the distance. Also, factors such as the geographical boundaries of the municipality will restrict the landfill location. A problem of this type (i.e. with maximum distance constraints) was first proposed and solved by Drezner and Wesolowsky (1980b).

Drezner and Wesolowsky (1980b, and later in (Love et al., 1988)) discussed a solution method for the distance-constrained problem which is known today as the *Drezner-Wesolowsky algorithm.* The algorithm first constructs circles on the fixed points and then gradually expands them. Since the non-dominated maxi-min candidate locations lie outside the union of these circles, the circles can only be expanded up to some extent, as otherwise the solution will occur outside the feasible area defined by obligatory distance constraints. Therefore, the solution of Drezner-Wesolowsky algorithm is the last point lying outside the union but intersecting with the feasible area defined by obligatory distance constraints. Later, Mehrez et al. (1986) proposed an enhancement of this algorithm for rectangular distances. Recent studies on distance-constrained location problems are due to Berman and Huang (2008) and Williams (2008).

The only study that considered the maxi-min objective in higher dimensions is Goetze et al.'s (1990) article investigating a *maximum empty box location problem.* A closely related higher dimensional location problem was discussed in Nguyen and Strodiot (1992) under the name *design center problem.* This involved solving for the location of a maximal convex polyhedron within a polyhedral region.

2.2.1.4 Minimal covering models

In minimal covering location models ($\mathcal{MNCLP}$), the objective is to minimize the number (or total weight) of the fixed points covered. The minimal covering problem was first studied by Drezner and Wesolowsky (1994). The objective was to find a circle or rectangle containing the minimum weight of points in the plane. Brimberg and ReVelle (1999) studied the multi-facility version of this problem by allowing a partial satisfaction of demand at the fixed points. Muñoz-Pérez and Saameño-Rodríguez (1999) discussed minimal covering location within a polygonal region that include forbidden zones.

Plastria and Carrizosa (1999) associated an additional objective with minimal covering models and referred to this as the *maximal radius problem.* They investigated the tradeoffs between a set of impact radii and a set of threshold coverage levels. That is, they solved the basic minimal covering problem considering a set of alternative impact radii and the maximal radius problem for a set of threshold coverage levels. This approach requires analyzing bi-criterion formulations.

A group of authors discussed pull-push type problems which are closely related to distance constrained maxi-min noxious (i.e. undesirable) facility location problems discussed in Section 2.2.1.3. This group of covering problems was named *semi-obnoxious* (Carrizosa and Plastria, 1999a; Ohsawa et al., 2006) *location problems* and it integrates both the attractive and undesirable features of the facilities. The solution methods are usually based on bi-criterion models (Carrizosa and Plastria, 1999b; Ohsawa, 2000) or some algorithmic approaches integrating the farthest and closest point Voronoi diagrams (Ohsawa and Tamura, 2003).

For general reviews on noxious facility location problems, one may refer to Erkut and Neumann (1989) and Kleindorfer and Kunreuther (1994).

2.2.2 Dimensional and extensive facility location problems

Location problems studied to date generally consider facility locations as being analytical solution points. However, there is now a growing interest on locating dimensional structures such as lines, segments, routes, hyper-planes and spheres. This branch of location science is referred to as *extensive facility location problems* and it is a relatively new area compared to the models discussed in the previous sections. The problems that are of particular interest in this area were still studied up to a limited degree and there exist several forms of dimensional facilities according to specific applications. Therefore, new results in location science may especially come from this area.

One extensive facility location model that is well known in all scientific disciplines is the *simple linear regression.* This problem is essentially a mini-sum line location problem where the distances to the fixed points are given in terms of squared residuals.

Some interesting results in locating a *line* facility with rectangular and Euclidean distances were given by Wesolowsky (1972; 1975), respectively. One was the theorem proving the existence of an optimal *median line* passing through at least two points from the fixed point set. Later, the same result was re-addressed by Morris and Norback (1980) and Love et al. (1988). Subsequently, this theorem was proven to be valid for the block norms as well (Schöbel, 1996).

Similarly, *center line* problems were studied under rectangular and Euclidean norms (Morris and Norback, 1980), and under distances derived from norms (Schöbel,

1998) and under gauges (Schöbel, 1999).

Solution procedures for median line location problems include: Morris and Norback's (1980) $\mathcal{O}(N^3)$ enumeration algorithm, which was later reduced to a complexity of $\mathcal{O}(N^2 \log N)$ by Megiddo and Tamir (1983). For problems under rectangular distance, polynomial time algorithms can be generated based on solving two independent line location problems (see, for example, Imai et al., 1989).

The *noxious line problem* was tackled in the context of computational geometry. An equivalent form of this problem in computational geometry literature is called *largest empty corridor problem* (or sometimes called computing the *width* of a set). Houle and Toussaint (1988) discussed this problem in detail and proposed an $\mathcal{O}(N \log N)$ algorithm. For the problem with unequal-weight fixed point sets Lee and Wu (1986) developed an $\mathcal{O}(N^2 \log N)$ algorithm.

Another important line location problem is the *noxious route location problem* which is applicable to hazardous materials logistics planning. This problem was proposed by Drezner and Wesolowsky (1989) using classical distance metrics, and was later extended to polyhedral norm distances by Hinojosa and Puerto (1999).

The *full covering line location problem* is equivalent to siting multiple lines in the plane ensuring that the fixed point set is a subset of the union of these lines. Alternatively, this problem may be named as the *line cover problem.* Megiddo and Tamir (1982) showed that this problem is $\mathcal{NP}$-hard.

One interesting extension of basic line location problems is the *line transversal problem.* This problem investigates the existence of a line passing through a set which consists of N convex polygonal sub-sets. Avis et al. (1989) solved this problem in $\mathcal{O}(N \log N)$ time by using Hershberger's (1989) $\mathcal{O}(N \log N)$ algorithm. Efficient methods for sets having the same orientation in space do exist. For example, Edelsbrunner (1985) computed the line transversal of N identically oriented rectangles in $\mathcal{O}(N)$ time.

Another extensively studied dimensional facility form is the *hyperplane*. Usually, the approaches listed above for line location problems are generalized for *hyperplane location problems*. These problems are concerned with finding a hyperplane cutting the convex hull of N points in $\Re^n$, such that a certain objective function is minimized. Hyperplane location problems were studied under the classical objective functions, which led to the *median-hyperplane* and *center-hyperplane* problems. Related results can be found in Houle et al. (1993), Korneenko and Martini (1993), Martini and Schöbel (1998), and Schöbel (1999).

An extension of this problem is to find a k dimensional subspace closest to a set of N points in $\Re^n$, using median and center objectives. This problem was studied by Martini (1994) and referred to as *k-flat problem.*

Finding the closest circle to a set of points is an essential problem as it has some applications in roundness inspection which is an important topic in precision engineering. The *closest circle problem* was studied by many authors in several application areas such as: data approximation (Rivlin, 1972; Boffey et al., 1988; Garcia-López et al. 1998), in production engineering under the name of *roundness inspection* (Yeralan and Ventura, 1988; Ventura and Yeralan, 1989) and in location science (Drezner et al., 2002).

Location problems regarding polygonal facilities are studied in the context of best approximation to a fixed point set. These problems have applications in robotic path design, pattern recognition and graphic design fields. A group of authors discussed approximating a polygonal curve with another simple one. This stream includes the works of Imai and Iri (1988), Melkman and O'Rourke (1988), Chin et al. (1992) and Chan and Chin (1996).

Some specific problems in this branch are concerned with ellipse and hyperbola fitting (Späth, 1997), finding double-ray center of a fixed point set (Glozman et al., 1999), locating a convex polygon within another polygon with the classic maximin objective (Imai et al., 1999) and approximating the fixed point set by a 1-bend polygonal curve (Díaz-Báñez et al., 2000).

Savaş et al. (2002) argued that the "infinitesimal facility" assumption of many location models is only valid when the physical aspects of the facilities are negligible with respect to those of the location space. That is, the analytical solution point generated by such location models can be used as a facility location only if the area and dimensions of the facility are remarkably small, when compared to the size of the location space. They considered the problem of locating a single, finite-area facility under rectilinear metric and barriers to travel. In their approach, there exists a set of fixed points distributed over a planar region, and the facility is serving to this set from a "server" located on its boundary. They analyzed candidate facility locations under the median objective and proposed a heuristic procedure for the solution of this location problem. In a similar study, Sarkar et al. (2007) discussed locating a finite-area facility under the same setting but with using the center objective.

Covering location problems with dimensional and extensive facilities (except

Younies and Wesolowsky, 2004; 2007) are limited. They include finding an annulus of minimum width covering a set of fixed points (De Berg et al., 1997), and finding a largest empty annulus (Díaz-Báñez et al., 2002).

An interesting open problem of extensive facility location is the problem of finding the location and orientation of two co-centric cylinders, such that the smallest possible open space between the cylinders fully contains a given set of fixed points in $\Re^3$. Díaz-Báñez et al. (2004) report that there is no solution described for this problem up to now, other than brute-force attack algorithms.

2.2.3 Dynamic facility location problems

In most of the applications, facilities may be expected to function over a period of time throughout which the system parameters such as demand at the fixed points and transportation costs are subject to change. For those cases, most of the static location problems were extended to accommodate such changes in the system parameters and were called *dynamic location problems.*

The establishment of dynamic facility location research dates back to the pioneering works of Klein and Klimpel (1967) who studied the *fixed charge problem* with economies of scale, Ballou (1968) who analyzed dynamic location of warehouses from a marketing point of view, and Scott (1971b) who discussed facilities entering the system one-by-one without relocation. Later, studies dealt with parameters such as costs evolving through multiple periods. Changes over time in destination locations and product volumes were analyzed in the location science context by Wesolowsky

(1973) and Wesolowsky and Truscott (1975). *Dynamic facility phase in-phase out location problems* were discussed by Roodman and Schwarz (1975; 1977).

Other extensions of basic models include: the *dynamic uncapacitated simple plant location problem* (Van Roy and Erlenkotter, 1982; Chardaire et al., 1996), dynamic extension of p-median problem (Galvão and Santibañez-Gonzalez, 1992), *dynamic warehouse location models* (Khumawala and Whybark, 1976; Kelly and Maruckeck, 1984).

Several versions of problems dealing with facility capacities were discussed. For example, demand pattern analysis and capacity adjustment of a given set of facilities through multiple periods was proposed independently by Sweeney and Tatham (1976), Erlenkotter (1981), and Fong and Srinivasan (1981; 1986). Lee and Luss (1987) analyzed *capacity expansion* and discussed some solution algorithms. Antunes and Peeters (2001) proposed a simulated annealing procedure for the capacity expansion problem. Shulman (1991) analyzed the case with discrete expansion sizes and contributed an algorithmic solution procedure. Melachrinoudis et al. (1995) studied the noxious facility case and proposed a multi-objective programming approach. Wang et al. (2003) considered budget constraints while analyzing opening-closing policies, and Melo et al. (2005) proposed a dynamic multi-commodity framework for strategic supply chain modeling.

Further generalizations of the dynamic facility location problem do exist. These can be summarized as follows: the problem of locating multi-echelon intermediate facilities (Canel et al., 2001), *relocation problems* of hybrid manufacturing-distribution facilities (Min and Melachrinoudis, 1999), the dynamic two-echelon *relo-*

cation and phase-out problem (Melachrinoudis and Min, 2000), and the *multi-period, multi-commodity and multi-echelon location problem* (Hinojosa et al., 2000).

Chapter 3
Thesis problems and organization

3.1 Gaps identified in the literature

Review of the subject literature revealed some gaps in the current body of knowledge regarding continuous covering location problems. These gaps can be summarized as follows:

Gap 1. In the present literature, continuous covering problems have been tackled only by using distance measures. Other than Younies and Wesolowsky (2004), we did not encounter any study attempting to formulate a covering problem without employing any distance measure but using the geometrical properties of the facility. Moreover, facility locations generated by employing the present planar continuous covering models are analytical solution points in the plane. Notice that these models can not account for dimensional facilities.

Gap 2. It is possible that the lateral dimensions of the facilities are themselves decision variables. These type of facilities may be characterized as *non-rigid* facilities. There is no work discussing location problems of non-rigid 2-dimensional facilities in the literature.

Gap 3. There is only one study (Späth, 1978) discussing a covering problem in $\Re^3$ using the classic mini-max objective function. It is evident from the literature that continuous covering problems in $\Re^3$ considering 3-dimensional shapes have not been studied.

Gap 4. If facilities are expected to function over multiple time periods and the system parameters are likely to change, then we deal with dynamic location problems. The dynamic versions of some possible new problems that will fill Gaps 1-3 will indeed be new location problems.

This thesis attempts to formulate and study a group of problems that will fill these gaps in location science research. To achieve this objective, we introduce a class of continuous covering location problems. It was also important to propose alternative exact solution approaches to these new models, hence for each problem, we also introduced a purpose-built solution procedure. A statement of the problems studied in this thesis are provided in the next section.

3.2 Problem statements

3.2.1 The planar expropriation problem with non-rigid rectangular facilities

In Chapter 4, we study the planar expropriation problem with non-rigid rectangular facilities ($\mathcal{PENR}$). This problem is intended to address Gaps 1 and 2 identified in the continuous covering problem literature.

Formal statement of the problem $\mathcal{PENR}$. *Given N fixed points indexed by i and the expropriation costs $c_i \in \Re^+$ associated with these points in the Euclidean plane $\Re^2$, find the location and formation of a (the K) non-rigid rectangular facility(ies), such that the total cost of the expropriated points is minimum.* ♣

The *expropriation location problem* is intimately related to minimal covering and single noxious (i.e. undesirable) facility maxi-min location problems. Noxious facility location problems have the classical *push objectives* (Eiselt and Laporte, 1995), in the sense that the more distant the facilities located are to the fixed points, the better the value of the objective function is. A logical objective function in such problems is to find a location maximizing the minimum distance between the facility and any fixed point (Dasarathy and White, 1980; Melachrinoudis and Cullinane, 1985). As mentioned in Section 2.2.1.3, this problem is equivalent to one of the fundamental problems in the computational geometry domain, namely, the largest empty circle problem which is discussed in detail by Toussaint (1983) and Preparata and Shamos (1985). However, when the largest empty disks found by such an approach are smaller than the impact radius of the noxious facility, we formally switch to the minimal covering problem in order to *minimize the undesirable effect* of the facility. This corresponds to finding the location of a disk of given radius (in Euclidean case) within a bounded region, covering the minimum possible weight of fixed points. What is different in the expropriation location problem is that it associates specific expropriation costs with each fixed point and the points that are covered have to be bought in developer's expense. Therefore, the expropriation location problem is applicable when locating noxious facilities and coverage up to some degree is inevitable due to configuration of the fixed point set.

The expropriation idea originated from studying point coverage (Drezner and Wesolowsky, 1994) in a planar setting using Euclidean and rectangular norms. Therefore, the expropriation area is a circular disk in the Euclidean norm case, whereas it is a diamond shaped area when a rectangular norm is adopted. The expropriation location problem was first proposed by Berman et al. (2003), considering a new noxious facility serving a particular proportion of demand. In their original work, the authors proposed two formulations for the problem using the Euclidean distance norm. Given a predetermined budget for expropriations, the first formulation finds the location of the facility such that the fixed demand points which fall into the expropriation area are *expropriated* according to this budget, and the *non-expropriated* fixed points are located as far as possible from the facility. In the second formulation, there is a predetermined impact radius such that the points that are closer than this distance to the facility have to be expropriated. In this scenario, the facility location minimizing the total expropriation cost was investigated. Recent studies on the expropriation location problem include Berman and Wang (2007; 2008) and Berman et al. (2008). One of the motivating conclusions drawn in Berman et al. (2003) is that further investigation of such a problem may consider the facility having an area of some particular shape rather than being a mathematical solution point.

In Chapter 4, we extend the expropriation problem to rectangular non-rigid facilities which can assume any rectangular formation between reasonable limitations imposed on their structure. The facility is non-rigid in the sense that we allow the formation of the facility to be a decision issue and formulate the problem in such a way that both location and formation decisions are made simultaneously with minimizing the total expropriation cost.

Possible application areas of the $\mathcal{PENR}$ exist in industrial siting applications. Some examples of such applications are: (1) Locating a facility where it is inevitable that some current points will be covered. As mentioned above, these have to be bought or expropriated at the developer's expense. In this case, the expropriation costs will reflect land prices. (2) Locating a facility which is undesirable due to hazardous materials processed or due to contamination effects (i.e. a landfill). (3) Locating a facility operation of which may pose risks to the residents (i.e. a high voltage electricity distribution station). (4) Locating a facility where some points are forbidden. In this case a very large expropriation cost, say M, may be assigned to each forbidden point.

3.2.2 The planar maximal covering problem with single convex polygonal shapes

In Chapter 5, we study the planar maximal covering problem with single convex polygonal shapes ($\mathcal{PMCS}$). This problem is intended to address Gap 1 identified in the continuous covering problem literature.

Formal statement of the problem $\mathcal{PMCS}$. *Given N fixed points indexed by i and the weights $\omega_i \in \Re^+$ associated with these points in the Euclidean plane $\Re^2$, find the location of a single convex polygonal shape, such that the total weight covered is maximum.* ♣

The $\mathcal{PMCS}$ is equivalent to investigating maximal point containment by a coverage area in the Euclidean plane $\Re^2$. While some algorithmic approaches (see, Barequet et al., 1997; Dickerson and Scharstein, 1998) investigating planar translations of convex polygonal shapes to facilitate maximal point containment exist in the computational geometry literature, there are only two (see, Section 2.2.1.2) studies on the exact methods for general m-sided regular convex polygonal shapes. In Chapter 5, we illustrate a general methodology that can be used to generate the formulations of this problem for any m-sided convex polygonal shape, and an exact procedure suitable for solving these problems.

For generating the formulations of the planar maximal covering problems by various single convex polygonal shapes, we use a simple methodology which defines a convex shape to be the intersection of a set of half-planes. Consider a bounded feasible location plane $\mathcal{B}$, and let i and j be two points with known locations (x_{1i}, x_{2i}) and (x_{1j}, x_{2j}) in this plane, respectively. A line λ_{ij} passing through these two points is a set of points (x'_1, x'_2) such that,

$$\lambda_{ij} \equiv \left\{ (x'_1, x'_2) \in \mathcal{B} : x'_2 - x_{2i} - \frac{x_{2j} - x_{2i}}{x_{1j} - x_{1i}} \cdot (x'_1 - x_{1i}) \ = 0 \right\}.$$

Such a line defines two closed half-planes. Denote these half-planes with π^+_{ij} and π^-_{ij}. These half-planes are given by:

$$\pi^+_{ij} \equiv \left\{ (x'_1, x'_2) \in \mathcal{B} : x'_2 - x_{2i} - \frac{x_{2j} - x_{2i}}{x_{1j} - x_{1i}} \cdot (x'_1 - x_{1i}) \ \geq 0 \right\}$$

and

$$\pi^-_{ij} \equiv \left\{ (x'_1, x'_2) \in \mathcal{B} : x'_2 - x_{2i} - \frac{x_{2j} - x_{2i}}{x_{1j} - x_{1i}} \cdot (x'_1 - x_{1i}) \ \leq 0 \right\}.$$

Next, consider a convex polygonal shape $\mathcal{S}$, which is to be located on the plane

$\mathcal{B}$. The coverage area of the shape can be represented as a closed convex subset of $\mathcal{B}$. To illustrate this, consider the following lemma regarding the separation of a convex set.

Lemma 1 *Let $\mathcal{S}$ be a closed convex set in $\Re^n$. Then $\mathcal{S}$ is the intersection of all half-spaces containing $\mathcal{S}$.*

PROOF See, Bazaraa and Shetty (1979). ■

For the formulation of $\mathcal{PMCS}$, we use the property illustrated by Lemma 1. We represent the shape $\mathcal{S}$ as a closed convex subset of the feasible location plane $\mathcal{B}$. We exemplify an m-sided convex polygonal shape by the intersection of m half-planes. These half-planes are defined by means of a collection of line inequalities which are constructed by using the extreme points (i.e. vertices) of the convex shape. Let λ_{ij} be the line equalities which are constructed by using the vertices i and j of the convex shape. We convert the line equalities λ_{ij} into line inequalities which define the half-plane containing the coverage area of the shape. We control the covering decision by specifying if a fixed point falls to the interior of the intersection of these half planes (i.e. into the coverage area). We employ $(m + 1)$ binary variables to produce these types of covering constraints for each fixed point. This strategy results in mixed integer linear formulations for the anticipated $\mathcal{PMCS}$ problems regarding various convex polygonal shapes.

These types of covering problems have several real-life applications such as location of convex area facilities, transmitter location, inspection of geometric shapes and locating directional antennas whose coverage areas extend in particular directions (Younies and Wesolowsky, 2004). The formulations are applicable to triangles, rectangles, symmetric quadrilaterals, parallelograms, rhombi and regular polygons.

3.2.3 The dynamic planar expropriation problem with single convex polygonal shapes

In Chapter 6, we study the *dynamic planar expropriation problem with single convex polygonal shapes* ($\mathcal{DPECS}$). This problem is intended to address Gaps 1 and 4 identified in the continuous covering problem literature.

Formal statement of the problem $\mathcal{DPECS}$. *Given N fixed points indexed by i, the expropriation costs $c_{it} \in \Re^+$ associated with these points in the Euclidean plane $\Re^2$ and the relocation costs $v_t \in \Re^+$ associated with multiple time periods indexed by t, find the successive locations of a single convex polygonal shape through multiple time periods, such that the total cost of the expropriated points and relocation costs through this planning horizon is minimum.* ♣

When the shape is expected to function over multiple time periods, it is true that the system parameters such as expropriation costs and fixed point locations are subject to change. In such a dynamic setting, relocations of the shape may be allowed within the planning horizon to minimize the total expropriation cost. However, in reality, such relocations can not be carried out without incurring relocation costs. Therefore, when single-period continuous covering problems are extended to multiple time periods, the planner must ensure a balance between the total expropriation cost reductions achieved by relocating the shape and additional costs due to such relocation decisions.

To study the above-stated problem, we first formulate its single-period variant, namely, the *single-period planar expropriation problem with single convex polygonal shapes* ($\mathcal{SPECS}$). A formal statement of this problem is as follows.

Formal statement of the problem $\mathcal{SPECS}$. *Given N fixed points indexed by i and the expropriation costs $c_i \in \Re^+$ associated with these points in the Euclidean plane $\Re^2$, find the location of a single convex polygonal shape, such that the total cost of the expropriated points is minimum.* ♣

The *diametric opposite* of an optimization problem arises when the optimization direction for its objective function is reversed (i.e. from minimization to maximization or from maximization to minimization). Throughout this thesis, we will use this term to indicate such related models. Under this definition, one may now observe that the $\mathcal{SPECS}$ is the diametric opposite of the *planar maximal covering problem with single convex shapes* ($\mathcal{PMCS}$) introduced in Section 3.2.2.

Therefore, in Chapter 6, we first show how the formulation for *planar maximal covering problem with single convex shapes* ($\mathcal{PMCS}$) extends to its diametric opposite $\mathcal{SPECS}$. Subsequently, allowing a dynamic setting where the system parameters evolve through multiple periods, we show how the formulation for $\mathcal{SPECS}$ extends to its multi-period variant, namely, the *dynamic planar expropriation problem with single convex polygonal shapes* ($\mathcal{DPECS}$). We also illustrate an appropriate dynamic programming procedure for the solution of this problem.

Wesolowsky (1973) introduced two assumptions for the dynamic location models, both of which also apply to our particular problem formulation. These are summarized as follows.

Assumption 1 (Wesolowsky, 1973). *Each relocation cost v_t is independent from the distance the shape is moved, and is also independent from the number of time periods it will remain at its new location.*

This assumption establishes that the dynamic model we discuss in Chapter 6 is a fixed-charge location model where the relocation costs are associated only with the relocation decision.

Assumption 2 (Wesolowsky, 1973). *Each expropriation cost c_{it} and relocation cost v_t, is adjusted to represent its present value at $t = 1$.*

This assumption establishes that the cost parameters are forecasted for the planning horizon and appropriately discounted to reflect the time value at $t = 1$.

In addition, we would like to introduce the following two assumptions, in that, the solutions generated by $\mathcal{DPECS}$ are evocative. That is, the facility planner should be able to interpret a meaningful location-relocation strategy from this solution.

Assumption 3. *Phasing-in occurs at the beginning of the first period.*

This assumption establishes that the shape is located into the feasible space for the first time at the beginning of the first period.

Assumption 4. *Phasing-out is not allowed.*

This assumption establishes that the shape is present in the feasible space throughout the planning horizon.

3.3 Illustration of the models

All the problems are formulated as mixed integer programs (MIPs) and illustrated through examples and computational studies. Since we study a group of new problems in this thesis, it is worthwhile to note that sample benchmark problems regarding the models discussed in this thesis actually do not exist in the literature. Therefore, we choose to illustrate all the proposed formulations and procedures on randomly generated problem instances. All the MIP programs and proposed solution procedures were coded using the General Algebraic Modeling Systems (GAMS). Details of implementation for each study are provided in the corresponding sections.

3.4 Organization of the thesis

The remainder of this thesis is organized into four chapters and four appendices whose contents are as follows.

In Chapter 4, we study the *planar expropriation problem with non-rigid rectangular facilities.* In Section 4.2, we provide a mixed integer nonlinear (MINLP) problem formulation for a single facility with four sets of constraints. In Section 4.3, we discuss a mixed integer linear problem for approximating the MINLP formulation. Section 4.4 is devoted to describe the extension of the problems to multiple facilities case by considering special non-overlapping constraints. In Section 4.5, we develop three types of strengthening cuts based on the geometric properties of the problems introduced in the earlier sections. Section 4.6 is reserved for introducing a branch-and-bound algorithm we have developed for the solutions of MINLP problems. In particular, we discuss new problem generation and bounding strategies for this specific problem structure and provide an outline of the branch-and-bound algorithm. In Section 4.7, we present two numerical examples for the branch-and-bound algorithm by considering single and multiple faciliy cases, respectively. In the last section we report our computational experience with the branch-and-bound algorithm. In particular, we discuss many implementation details including preliminary experiments, performance measure, experimental design, resource settings, random problem generation, problem sizes, experiment termination statistics and results.

In Chapter 5, we study the *planar maximal covering problem with single convex polygonal shapes.* In Section 5.2, we provide a five-step methodology for formulating the planar maximal covering problem regarding any convex polygonal shape. Section 5.3 is alotted to illustrating two types of cuts which can be identified based on the geometric properties of the shapes under study. In Section 5.4, we provide a numerical example for the problem formulations. Section 5.5 is organized in three sub-sections to introduce the branch-and-cut algorithm we developed for the solution of planar maximal covering problems with single convex polygonal shapes. In particular, we introduce notations and calculation of upper and lower bounds. Then, we provide an outline of the branch-and-cut procedure. In Section 5.6, we present a numerical example for the branch-and-cut algorithm. In the last section, we demonstrate a small computational study where we evaluate the bound qualities attained by utilizing cuts provided in Section 5.3.

In Chapter 6, we study the *dynamic planar expropriation problem with single convex polygonal shapes.* In Section 6.2 we formulate the single-period planar expropriation problem with single convex polygonal shapes. Section 6.3 is devoted to explaining the extension of this problem to multiple time periods. Thus, we formulate the dynamic planar expropriation problem with single convex polygonal shapes in this section. In Section 6.4 we discuss the decomposition of the dynamic problem structure to a set of static expropriation problems where the shape is present at the same location through multiple time periods. Section 6.5 is reserved for discussing two basic solution procedures for the dynamic planar expropriation problem with single convex polygonal shapes. In Section 6.6 we demonstrate an incomplete dynamic programming procedure we developed for the solution of this problem. Section 6.7 is alotted to analyzing the growth of the branching trees for three solution procedures discussed in Sections 6.5 and 6.6. In the last section, we present a numerical example for the incomplete dynamic programming procedure.

In Appendix A, we provide two small computational studies to evaluate the cuts introduced in Section 4.5 in terms of their effect on solution times of expropriation problem formulations. We also present a convergence analysis for the branch-and-bound algorithm developed in Section 4.6.

In Appendix B, we provide a series of formulations for illustrative planar maximal covering problems. In particular, we formulate the *planar maximal covering problems with triangles and quadrilaterals.*

In Appendix C, we provide a formulation for the *three-dimensional expropriation problem with single convex polyhedral shapes.*

In Appendix D, we provide a formulation for the *planar maximal covering problem with non-rigid rectangular facilities.*

Chapter 4
Planar expropriation problem with non-rigid rectangular facilities

4.1 Introduction

In this chapter, we study the *planar expropriation problem with non-rigid rectangular facilities* ($\mathcal{PENR}$). We extend the scope of the original expropriation problem (Berman et al., 2003) to rectangular *non-rigid* facilities which can assume any rectangular formation between reasonable limitations imposed on their structure. The facility is non-rigid in the sense that the formation of a facility is a decision subject. That is, the lateral dimensions of the facility are decision variables. The problem requires both location and formation decisions to be made simultaneously with minimizing the total expropriation cost. Formation requirements are introduced in the model by limiting the aspect ratio of the length and width of the facility. The decision of whether a fixed point falls into the expropriation area or not is managed by employing binary variables.

Let the area requirement for such a rectangular facility be A, and let its length and width be l and w, respectively. Since l and w are also decision variables, the area restriction $A = l \cdot w$ calls for a mixed integer non-linear program (MINLP) formulation and the formation decision requires an extensive exploration of the $l(w) = A/w$ tradeoff curve. To overcome this non-linearity, we substitute an approximate mixed integer linear (MILP) problem for the non-linear formulation. We also propose a continuous branch-and-bound framework which utilizes linear approximations for the tradeoff curve associated with the facility formation alternatives. The branch-and-bound procedure exploits the curve structure by investigating each non-fathomed curve segment by further partitioning it according to reasonable aspect ratio steps, forming approximation lines over the curve segments, solving the corresponding approximate expropriation problems on the approximation lines, and bounding these segments.

When working with noxious facility location models the feasible location space should be bounded, as otherwise the solution will occur at infinity. Therefore, we define lower and upper limits for location on both directions to describe such a bounded feasible location space (i.e. we use a rectangular feasible location space).

4.2 Problem formulation for a single facility

The $\mathcal{PENR}$ problem requires construction of four types of constraints: expropriation constraints, inclusion constraints, area restriction, and integrality and non-negativity constraints. In this section, we explain how each type of constraint is constructed and we provide the MINLP formulation of the problem.

4.2.1 Expropriation constraints

Consider a rectangular facility with an area requirement of A. Let (x_1, x_2) denote the location of the center of the facility, and let (a_{1i}, a_{2i}) be the locations of N fixed points indexed by i. Define an envelope Ω_i^P as a rectangle of the same formation and dimensions as the facility, but centered at a fixed point (a_{1i}, a_{2i}) as illustrated in Figure 11.

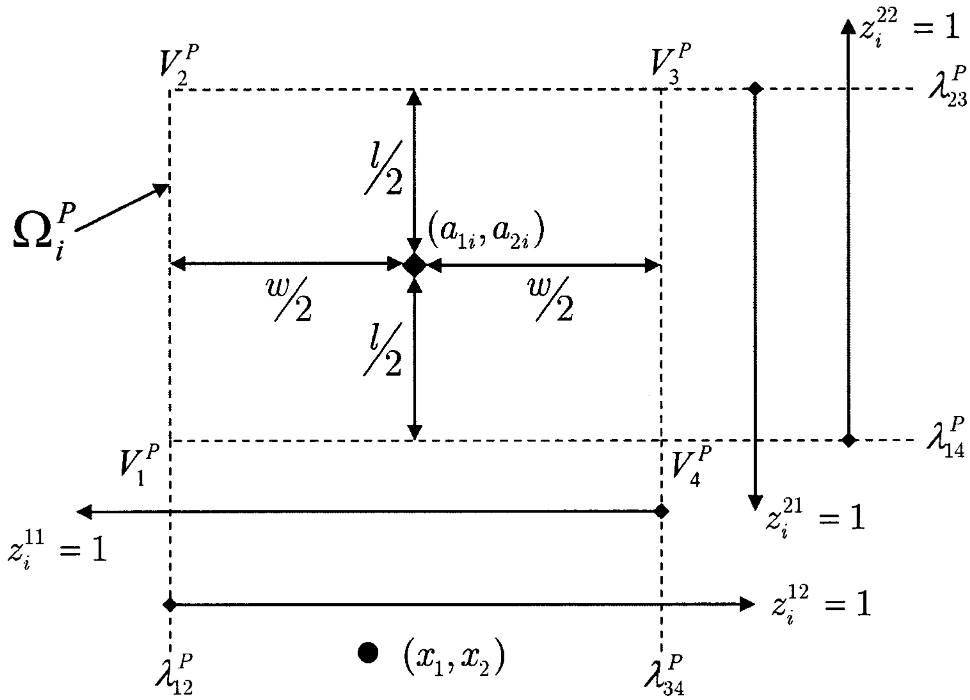

Figure 11. Binary definitions for expropriation

Let $\lambda_{12}^P, \lambda_{23}^P, \lambda_{34}^P$ and λ_{14}^P be four gridlines passing through the vertices V_1^P, V_2^P, V_3^P and V_4^P of this envelope, shown as dashed contours. Note that a fixed demand point i is expropriated if and only if there exists a facility location (x_1, x_2) in its envelope Ω_i^P. The explanation of the binary definitions given by Figure 11 is as follows. Let $z_i^{11} = 1$ if x_1 falls to the left of the right gridline λ_{34}^P and 0 otherwise; $z_i^{12} = 1$ if x_1 falls to the right of the left gridline λ_{12}^P and 0 otherwise; $z_i^{21} = 1$ if x_2 falls below the upper gridline λ_{23}^P and 0 otherwise, and finally, $z_i^{22} = 1$ if x_2 falls above the bottom gridline

λ_{14}^P and 0 otherwise.

These definitions imply that a fixed point i is covered along the X_1 dimension if following two constraints hold:

$$(x_1 - a_{1i}) + z_i^{11} \cdot M \geq \frac{w}{2} \tag{1}$$

$$(x_1 - a_{1i}) - z_i^{12} \cdot M \leq \frac{-w}{2} \tag{2}$$

where M is a very large number. Using the same logic as in the X_1 dimension case, we conclude that a fixed point i is covered along the X_2 dimension if the following two constraints hold:

$$(x_2 - a_{2i}) + z_i^{21} \cdot M \geq \frac{l}{2} \tag{3}$$

$$(x_2 - a_{2i}) - z_i^{22} \cdot M \leq \frac{-l}{2}. \tag{4}$$

Recall that, a fixed point i is expropriated if and only if a facility is located within the envelope Ω_i^P. To state it more formally in terms of constraints (1-4), a fixed point i is expropriated if and only if it is covered both along the X_1 and X_2 dimensions.

To illustrate this, assume that there exists a facility located at (x_1, x_2) as shown in Figure 11. For the fixed point i, we have $z_i^{11} = 1, z_i^{12} = 1, z_i^{21} = 1$ but $z_i^{22} = 0$. In this case, we conclude that the facility covers point i along the X_1 dimension. However, it does not cover the point along the X_2 dimension. Therefore, for the expropriation to be necessary we should have all four binary variables with values equal to 1. To introduce this into the formulation, we have to add the constraint:

$$z_i^{11} + z_i^{12} + z_i^{21} + z_i^{22} \leq 3 + z_i \tag{5}$$

where $z_i = 1$ if fixed point i is expropriated, and 0 otherwise.

4.2.2 Inclusion constraints

For getting sensible solutions to the noxious facility location problems, the location area should be bounded. Let, L^{X1} and U^{X1} be the lower and upper bounds along the X_1 dimension, respectively. Similarly, let L^{X2} and U^{X2} be the corresponding lower and upper bounds along the X_2 dimension. The feasible location space is then bounded by four lines $x_1 = L^{X1}$, $x_1 = U^{X1}$, $x_2 = L^{X2}$ and $x_2 = U^{X2}$. We can then construct the inclusion constraints that ensure the facility will be located within the feasible space as follows:

$$L^{X1} + \frac{w}{2} \leq x_1 \tag{6}$$

$$U^{X1} - \frac{w}{2} \geq x_1 \tag{7}$$

$$L^{X2} + \frac{l}{2} \leq x_2 \tag{8}$$

$$U^{X2} - \frac{l}{2} \geq x_2. \tag{9}$$

4.2.3 Area restriction, integrality and non-negativity constraints

These constraints are straightforward and given by (10-12)

$$A = l \cdot w \tag{10}$$

$$x_1, x_2, l, w \in \Re^+ \tag{11}$$

$$z_i^{11}, z_i^{12}, z_i^{21}, z_i^{22}, z_i \in \{0, 1\}. \tag{12}$$

4.2.4 The MINLP formulation

Let the expropriation cost for a fixed point i be c_i. The MINLP formulation of the $\mathcal{PENR}$ problem can be stated as follows:

$$(\mathcal{PENR}) \qquad \min \left\{ \sum_{i=1}^{N} c_i \cdot z_i \ : \ \text{s.t.} \ (1-5, \ 12) \ \forall i; \ (6-11) \right\}. \tag{13}$$

4.3 The approximate problem

The MINLP formulation (13) can be approximated by a linear approximate problem, denoted by $\mathcal{PENRA}$, by constructing an approximation line over the trade-off curve, and hence, substituting the area restriction (10) by a linear relationship between l and w as shown in Figure 12.

The facility is a rectangle and the planner should be able to impose some restrictions on its lateral dimensions to obtain a reasonable formation. Let $\alpha = l/w$ denote the *aspect ratio* of the length and width of the facility. We let the facility assume any aspect ratio between the limits $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$. The constants $\alpha_{\min}$ and $\alpha_{\max}$ define the extreme values of the aspect ratio within which the formation is considered acceptable by the planner. The aspect ratio constants now define the limits on the length and width of the facility as follows:

$$w \leq \sqrt{\frac{A}{\alpha_{\min}}} = w_{\max} \tag{14}$$

$$w_{\min} = \sqrt{\frac{A}{\alpha_{\max}}} \leq w \tag{15}$$

$$l_{\min} = \sqrt{A \cdot \alpha_{\min}} \leq l \tag{16}$$

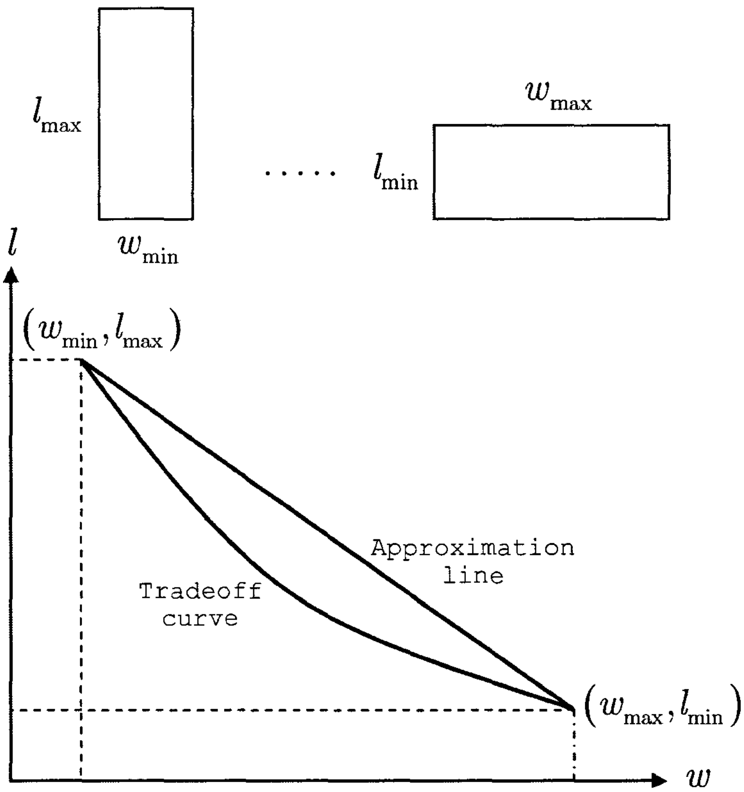

Figure 12. Approximation to the tradeoff curve

$$l \leq \sqrt{A \cdot \alpha_{\max}} = l_{\max}. \tag{17}$$

These limits are illustrated in Figure 12. The approximation line for the tradeoff curve defined by the formation limits may now be written as:

$$l = l_{\max} + \left(\frac{l_{\min} - l_{\max}}{w_{\max} - w_{\min}} \right) \cdot (w - w_{\min}). \tag{18}$$

Along with the approximation line (18), it suffices to write either (14-15) or (16-17) to complete the formation restrictions, as a linear relationship between l and w is already constructed by the approximation line. The approximate problem is then given by the following MILP:

$$(\mathcal{PENRA}) \quad \min \quad \left\{ \sum_{i=1}^{N} c_i \cdot z_i \ : \ \text{s.t.} \ (1-5,\ 12)\ \forall i;\ (6-9,\ 11,\ 16-18) \right\}. \quad (19)$$

4.4 Extension to multiple facilities

A logical extension of the single facility problem is to consider multiple facilities having particular area requirements. In this case, the objective is to locate multiple facilities in such a way that: (1) the area requirements of each facility is approximately satisfied; (2) the total expropriation cost is minimized; (3) overlapping between facilities is avoided; and (4) formations of the facilities are decided.

In addition to the constraints considered in the single facility case, we have to construct special constraints to enforce non-overlapping between multiple facilities. In what follows, we show how the constraints of the single facility formulation (19) are extended to the multiple facilities case and how the non-overlapping constraints can be constructed. Finally, we state the *planar expropriation problem with multiple non-rigid facilities* ($\mathcal{PENRM}$).

4.4.1 Extensions of the expropriation, inclusion and formation constraints

Consider K rectangular facilities indexed by k, with corresponding area requirements A_k. Let the length and width of such facilities be l_k and w_k, respectively.

Further, let (x_{1k}, x_{2k}) denote the locations of the centers of the facilities. Consider the same binary definitions as those in Figure 11, with the exception that this time the binary variables are additionally indexed for each facility. This is because we consider K envelopes Ω_{ik}^{P} utilized for the same purpose as the Ω_{i}^{P} used in the single facility case. These particular envelopes have the same formation and dimensions as their corresponding facility. Therefore, the envelope Ω_{ik}^{P} of facility k has its length and width equal to l_k and w_k, respectively. Thus, the expropriation constraints for the multiple facility case are given by:

$$(x_{1k} - a_{1i}) + z_{ik}^{11} \cdot M \geq \frac{w_k}{2} \tag{20}$$

$$(x_{1k} - a_{1i}) - z_{ik}^{12} \cdot M \leq \frac{-w_k}{2} \tag{21}$$

$$(x_{2k} - a_{2i}) + z_{ik}^{21} \cdot M \geq \frac{l_k}{2} \tag{22}$$

$$(x_{2k} - a_{2i}) - z_{ik}^{22} \cdot M \leq \frac{-l_k}{2} \tag{23}$$

$$z_{ik}^{11} + z_{ik}^{12} + z_{ik}^{21} + z_{ik}^{22} \leq 3 + z_{ik} \tag{24}$$

where $z_{ik} = 1$ if fixed point i is expropriated by facility k, and 0 otherwise.

Let the feasible location space be bounded similarly to what was done in Section 4.2.2. We can construct the inclusion constraint set similarly to (6-9) as follows:

$$L^{X1} + \frac{w_k}{2} \leq x_{1k} \tag{25}$$

$$U^{X1} - \frac{w_k}{2} \geq x_{1k} \tag{26}$$

$$L^{X2} + \frac{l_k}{2} \le x_{2k} \tag{27}$$

$$U^{X2} - \frac{l_k}{2} \ge x_{2k}. \tag{28}$$

As in the single facility case, we allow formation limitations imposed by the planner on facility k. Let α_k denote the aspect ratio of facility k, then we let the facility assume any formation between the predetermined limits $\alpha_{k,\min} \le \alpha_k \le \alpha_{k,\max}$, where $\alpha_{k,\min}$ and $\alpha_{k,\max}$ define the extreme values of the aspect ratio within which the formation of facility k is considered acceptable by the planner. Therefore, we generate the formation restrictions and the linear relationship between l_k and w_k as follows:

$$w_k \le \sqrt{\frac{A_k}{\alpha_{k,\min}}} \tag{29}$$

$$\sqrt{\frac{A_k}{\alpha_{k,\max}}} \le w_k \tag{30}$$

$$\sqrt{A_k \cdot \alpha_{k,\min}} \le l_k \tag{31}$$

$$l_k \le \sqrt{A_k \cdot \alpha_{k,\max}} \tag{32}$$

$$l_k = l_{k,\max} + \left(\frac{l_{k,\min} - l_{k,\max}}{w_{k,\max} - w_{k,\min}} \right) \cdot (w_k - w_{k,\min}) \tag{33}$$

where $l_{k,\max} = \sqrt{A_k \cdot \alpha_{k,\max}}$, $l_{k,\min} = \sqrt{A_k \cdot \alpha_{k,\min}}$, $w_{k,\min} = \sqrt{A_k / \alpha_{k,\max}}$ and $w_{k,\max} = \sqrt{A_k / \alpha_{k,\min}}$ are known parameters.

4.4.2 Non-overlapping constraints

Non-overlapping constraints for facilities can be constructed by modifying the expropriation scheme used for fixed points as discussed in Section 4.2.1. In the expropriation scheme, we form a rectangular envelope defined by length and width of the facility around each fixed point by constructing four imaginary gridlines. We then ask the following question: "Is there a facility located within this envelope?". If the answer is "yes" (indicated by binary variables), then we conclude that the fixed point is expropriated (i.e. the point inevitably falls into the expropriation area of the facility).

For deriving logical non-overlapping constraints between two facilities j and k, first we define a rectangular envelope around facility j, centered at (x_{1j}, x_{2j}) as illustrated in Figure 13. This envelope is denoted by Ω^F_{jk} and we construct it by using the dimensions of both facilities j and k (i.e. it is a rectangular envelope whose length and width are $(l_j + l_k)$ and $(w_j + w_k)$, respectively).

Let $\lambda^F_{12}, \lambda^F_{23}, \lambda^F_{34}$ and λ^F_{14} be four gridlines passing through the vertices of this envelope, shown as dashed contours. The explanation of binary definitions given by Figure 13 is as follows. Let $\vartheta^{11}_{jk} = 1$ if x_{1k} falls to the left of the right gridline λ^F_{34}, and 0 otherwise. $\vartheta^{12}_{jk} = 1$ if x_{1k} falls to the right of the left gridline λ^F_{12}, and 0 otherwise. $\vartheta^{21}_{jk} = 1$ if x_{2k} falls below the upper gridline λ^F_{23}, and 0 otherwise. $\vartheta^{22}_{jk} = 1$ if x_{2k} falls above the bottom gridline λ^F_{14}, and 0 otherwise. We can now consider how to enforce non-overlapping: "Is there a facility k located within this envelope Ω^F_{jk}?". If the answer is "yes" (indicated by the above binary variable definitions), then we conclude that the facilities j and k are overlapping. Thus two facilities j and k overlap along the X_1 dimension if following two constraints hold:

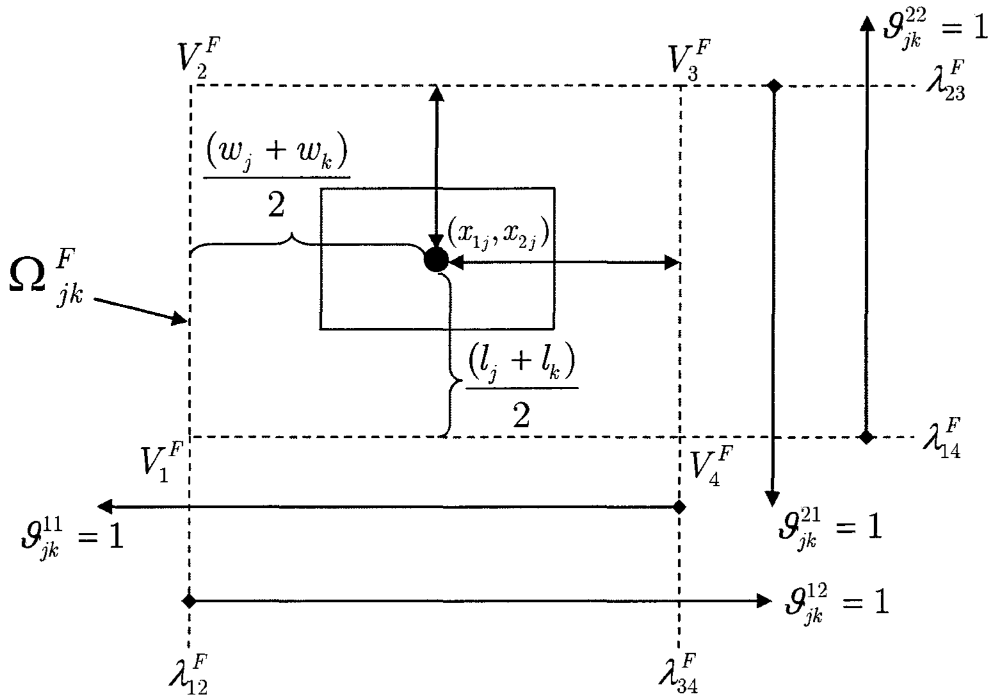

Figure 13. Binary definitions for non-overlapping

$$(x_{1k} - x_{1j}) + \vartheta_{jk}^{11} \cdot M \geq \frac{(w_j + w_k)}{2} \tag{34}$$

$$(x_{1k} - x_{1j}) - \vartheta_{jk}^{12} \cdot M \leq \frac{-(w_j + w_k)}{2}. \tag{35}$$

Similarly, two facilities j and k overlap along the X_2 dimension if following two constraints hold:

$$(x_{2k} - x_{2j}) + \vartheta_{jk}^{21} \cdot M \geq \frac{(l_j + l_k)}{2} \tag{36}$$

$$(x_{2k} - x_{2j}) - \vartheta_{jk}^{22} \cdot M \leq \frac{-(l_j + l_k)}{2}. \tag{37}$$

Note the important fact that two facilities overlap if and only if they overlap both along X_1 and X_2 dimensions. To illustrate this, consider the case where we have $\vartheta_{jk}^{11} = 1, \vartheta_{jk}^{12} = 1, \vartheta_{jk}^{21} = 1$ but $\vartheta_{jk}^{22} = 0$. In this case we see that the facilities j and k overlap along the X_1 dimension; however they do not overlap along the X_2 dimension. It follows that facility k lies below facility j because there is a vertical distance more than $(l_j + l_k)/2$ between their centers. Hence, to avoid overlapping, binary variables $\vartheta_{jk}^{11}, \vartheta_{jk}^{12}, \vartheta_{jk}^{21}$ and ϑ_{jk}^{22} should never be equal to 1 simultaneously. To introduce this into the formulation, we have to add the constraint:

$$\vartheta_{jk}^{11} + \vartheta_{jk}^{12} + \vartheta_{jk}^{21} + \vartheta_{jk}^{22} \leq 3. \tag{38}$$

Whereas we discussed the constraints between two facilities, the same type of constraints are required for all pairs of facilities (i.e. none of them should overlap). Therefore, in the formulation, we include the non-overlapping constraints between all facility pairs $\{(j,k) : \ j \neq k\}$. Hence, the expropriation problem with multiple non-rigid facilities $\mathcal{PENRM}$ can be stated as follows:

$$(\mathcal{PENRM}) \quad \min \left\{ \sum_{k=1}^{K} \sum_{i=1}^{N} c_i \cdot z_{ik} \ : \ \text{s.t. } (20-24), \left(z_{ik}^{11}, z_{ik}^{12}, z_{ik}^{21}, z_{ik}^{22}, \right.\right.$$

$$z_{ik} \in \{0,1\}\big) \ \forall i,k; \ (25-30,33), \ \left(x_{1k}, x_{2k}, l_k, w_k \in \Re^+\right) \ \forall k;$$

$$\left. (34-38,), \ \left(\vartheta_{jk}^{11}, \vartheta_{jk}^{12}, \vartheta_{jk}^{21}, \vartheta_{jk}^{22} \in \{0,1\}\right) \ \ \forall j,k, \ j \neq k \right\}. \tag{39}$$

4.5 Strengthening cuts

In this section, we illustrate three types of strengthening cuts based on the geometric properties of the problems discussed in the earlier sections of this chapter. We refer to the first group of cuts as *global cuts.* These are suitable for all the problems discussed in this chapter. The second group of cuts are *approximate problem-specific cuts*, which are intended to strengthen particular approximate problem formulations. Finally, we discuss a third group of cuts which are useful for the multiple facilities case.

Lemma 2 *(Global cuts) The inequalities:*

$$(a) \quad z_i^{11} + z_i^{12} \geq 1 \quad \forall i \tag{40}$$

$$(b) \quad z_i^{21} + z_i^{22} \geq 1 \quad \forall i \tag{41}$$

hold for all the planar expropriation problem formulations.

PROOF (a) From the binary definitions in Figure 11, one can observe that the binary variables z_i^{11} and z_i^{12} can never be simultaneously equal to zero in a feasible integer configuration. (b) The same argument as in part (a) applies. ■

For the *approximate-problem specific cuts*, consider an example approximate problem p defined on a curve segment as illustrated in Figure 14. Let the limits on the formation of the facility for this specific approximate problem be $(w^p_{\min}, l^p_{\max})$ and $(w^p_{\max}, l^p_{\min})$.

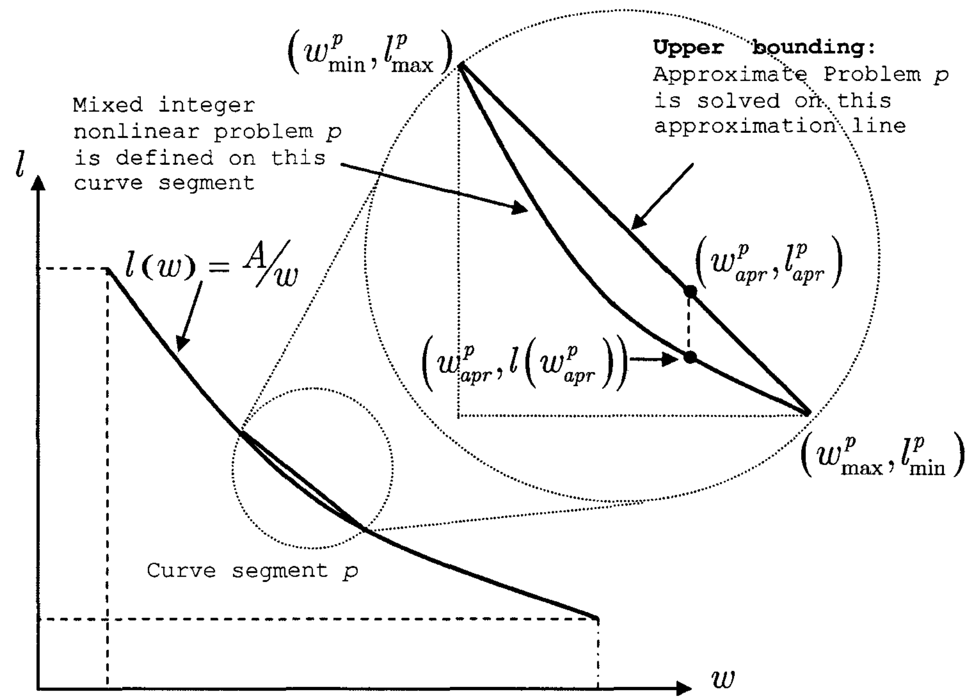

Figure 14. An approximate problem on a curve segment

Lemma 3 *(Approximate problem-specific cuts) If one of the arguments (a)* $|a_{1i} - a_{1h}| > w^p_{max}$; *(b)* $|a_{2i} - a_{2h}| > l^p_{max}$ *is true for the points* i *and* h, $i \neq h$, *then the inequality:*

$$z_i + z_h \leq 1 \tag{42}$$

holds for the approximate expropriation problem p.

PROOF (a) Notice that, the term $|a_{1i} - a_{1h}|$ is the rectangular distance between points i and h along X_1 dimension. If this distance is bigger than the maximum width allowed in the approximate problem p, the points i and h can not be expropriated at the same time. (b) Similar to the argument in part (a), therefore omitted. ■

Lemma 4 *(Cuts for the multiple facilities case) The inequalities:*

$$\sum_{k=1}^{K} z_{ik} \leq 1 \quad \forall i \tag{43}$$

hold for the $\mathcal{PENRM}$ formulations.

PROOF Since the overlapping between facilities is forbidden, a fixed point i can only be expropriated by one facility. ■

In Appendix A, we illustrate two computational studies where we evaluate the effect of these cuts on solution times of expropriation problem instances.

4.6 A branch-and-bound procedure

In this section, we discuss a branch-and-bound procedure which iteratively partitions the $l(w) = A/w$ tradeoff curve to segments and bounds the objective function value of the mixed integer nonlinear expropriation problems of type (13) defined on these segments. For this purpose, we differentiate between the following problems defined on a curve segment:

(1) Mixed integer nonlinear expropriation problem p (MINLP-p). This is the expropriation problem of type (13) defined on curve segment p (see, Figures 14-15).

(2) Approximate expropriation problem p. This is the approximate expropriation problem of type (19) defined on curve segment p (see, Figure 14). The objective function value of this problem is used as an upper bound on the objective function value of the MINLP-p.

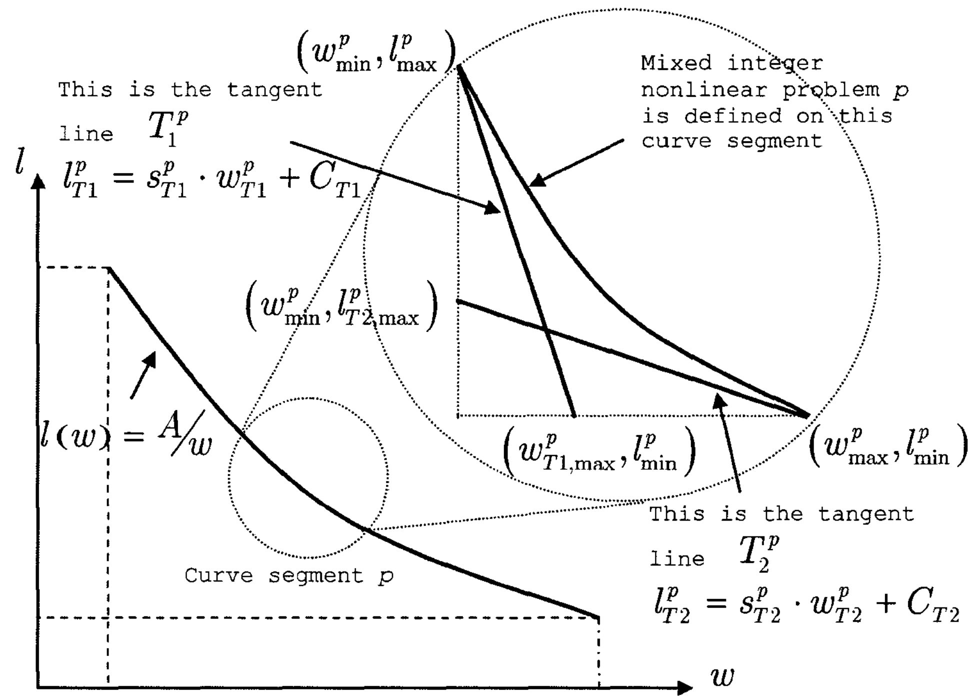

Figure 15. Two auxiliary expropriation problems

(3) Auxiliary expropriation problems. These are two auxiliary expropriation problems defined on the tangent lines T_1^p and T_2^p as illustrated in Figure 15. These two problems are used to compute a lower bound on the objective function value of the MINLP-p.

The tangent lines can be found as follows. Let the slopes of these two tangent lines be s^p_{T1} and s^p_{T2}, respectively. We can find the slope s^p_{T1}, with equalizing it to the first derivative of the curve function evaluated at the formation $(w^p_{\min}, l^p_{\max})$. That is, $dl/dw = -A/\left(w^p_{\min}\right)^2 = s^p_{T1}$. Similarly, the slope s^p_{T2} can be found with equalizing it to the first derivative of the curve function evaluated at the formation $(w^p_{\max}, l^p_{\min})$. Thus, $dl/dw = -A/\left(w^p_{\max}\right)^2 = s^p_{T2}$. Let, the possible formations on tangent lines T^p_1 and T^p_2 be (w^p_{T1}, l^p_{T1}) and (w^p_{T2}, l^p_{T2}), respectively. Then, the tangent lines T^p_1 and T^p_2 are given by the equations $l^p_{T1} = s^p_{T1} \cdot w^p_{T1} + C_{T1}$ and $l^p_{T2} = s^p_{T2} \cdot w^p_{T2} + C_{T2}$ where C_{T1} and C_{T2} are constant terms. Let, $w^p_{T1,\max}$ be the largest width allowed on tangent line T^p_1 and $l^p_{T2,\max}$ be the largest length allowed on tangent line T^p_2, as shown in Figure 15. By the definition of slope, we have $s^p_{T1} = (l^p_{\min} - l^p_{\max}) / \left(w^p_{T1,\max} - w^p_{\min}\right)$ and $s^p_{T2} = \left(l^p_{\min} - l^p_{T2,\max}\right) / (w^p_{\max} - w^p_{\min})$. From these two equalities, $w^p_{T1,\max}$ and $l^p_{T2,\max}$ can be found. Then, using the formations $\left(w^p_{T1,\max}, l^p_{\min}\right)$ and $\left(w^p_{\min}, l^p_{T2,\max}\right)$, the constant terms C_{T1} and C_{T2} can be derived as follows: $C_{T1} = l^p_{\min} - s^p_{T1} \cdot w^p_{T1,\max}$ and $C_{T2} = l^p_{T2,\max} - s^p_{T2} \cdot w^p_{\min}$.

Note that, the approximate expropriation problem and two auxiliary expropriation problems are simple mixed integer linear problems. In these problems, the area restriction $A = l/w$ is substituted with a linear relationship between l and w.

Each segment p on the tradeoff curve $l(w) = A/w$ is defined by an aspect ratio interval $[\alpha^p_{\min}, \alpha^p_{\max}]$, where $\alpha^p_{\min} = l^p_{\min}/w^p_{\max}$ and $\alpha^p_{\max} = l^p_{\max}/w^p_{\min}$ (see, Figure 14). Problem generation is done by a *cutoff* approach over aspect ratio intervals; hence, if it is decided that a segment p will be investigated further, the corresponding aspect ratio interval $[\alpha^p_{\min}, \alpha^p_{\max}]$ is divided into n equal intervals. We thus create n new curve segments; hence n new expropriation problems need to be investigated. Node selection from the branching tree is done by a *best node first* strategy. Thus, among the candidate problems, we select the problem with the lowest objective function (i.e. upper bound) value.

4.6.1 Notations

p : index for the segment problems;

f^p : optimal objective function value of the MINLP-p;

f^p_{apr} : optimal objective function value of the approximate expropriation problem p;

$\left(w^p_{apr}, l^p_{apr}\right)$: optimal formation (i.e. width and length of the facility) found by the approximate expropriation problem p;

f^p_{T1} and f^p_{T2} : optimal objective function values of the auxiliary expropriation problems defined on the tangent lines T^p_1 and T^p_2, respectively;

n : a number pre-determined by the user denoting the number of problems to be generated;

Ψ : a list keeping track of the active (i.e. non-fathomed) problems;

Ψ^T : a list where n new problems are kept temporarily;

$UB(p)$: upper bound on the objective function value of the MINLP-p;

$LB(p)$: lower bound on the objective function value of the MINLP-p;

UB^* : best upper bound found.

4.6.2 Upper and lower bounds

The objective function value of the approximate problem p can be used as an upper bound on the objective function value of the mixed integer nonlinear expropriation problem p. This is given by the following result.

Lemma 5 *(a)* $A \leq w_{apr}^{p} \cdot l_{apr}^{p}$; *(b)* $f^{p} \leq f_{apr}^{p}$.

PROOF (a) Due to convexity of the tradeoff curve $l(w) = A/w$. Since $w \geq 0$, the second derivative of curve function, $d^2l/dw^2 = 2A/w^3$ is always positive, hence the tradeoff curve is convex. Consider any convex combination $w_{apr}^{p} = \delta \cdot w_{\min}^{p} + (1-\delta) \cdot w_{\max}^{p}$, $\delta \in [0,1]$ of the limits on the width in curve segment p. By the definition of convexity,

$$l\left(\delta \cdot w_{\min}^{p} + (1-\delta) \cdot w_{\max}^{p}\right) \leq \delta \cdot l\left(w_{\min}^{p}\right) + (1-\delta) \cdot l\left(w_{\max}^{p}\right) \tag{44}$$

is true $\forall w_{apr}^{p} \in \left[w_{\min}^{p}, w_{\max}^{p}\right]$. One may observe that, the left and right hand side of (44) is equal to $l\left(w_{apr}^{p}\right)$ and l_{apr}^{p}, respectively, as shown in Figure 14. Thus, $l\left(w_{apr}^{p}\right) \leq l_{apr}^{p}$, $\forall w_{apr}^{p} \in \left[w_{\min}^{p}, w_{\max}^{p}\right]$ and hence $A = l\left(w_{apr}^{p}\right) \cdot w_{apr}^{p} \leq l_{apr}^{p} \cdot w_{apr}^{p}$.

(b) By contradiction. Assume there exists an optimal approximate problem formation $\left(w_{apr}^{p}, l_{apr}^{p}\right)$ and an optimal MINLP-p formation (w^{p}, l^{p}) at the same curve segment with corresponding objective functions f_{apr}^{p} and f^{p}, such that $f^{p} > f_{apr}^{p}$ as shown in Figure 16. Observe that there always exists a MINLP-p formation (w^{p*}, l^{p*}) on the same curve segment which can fit into the coverage area of the approximate problem formation $\left(w_{apr}^{p}, l_{apr}^{p}\right)$, and hence, covering at most the same number of fixed points. Thus we have $f^{p} > f_{apr}^{p} \geq f^{p*}$ which contradicts the optimality of f^{p}. ■

Let $f_{aux}^{p} = \min\left\{f_{T1}^{p}, f_{T2}^{p}\right\}$, and $\left(w_{aux}^{p}, l_{aux}^{p}\right)$ be the optimal formation corre-

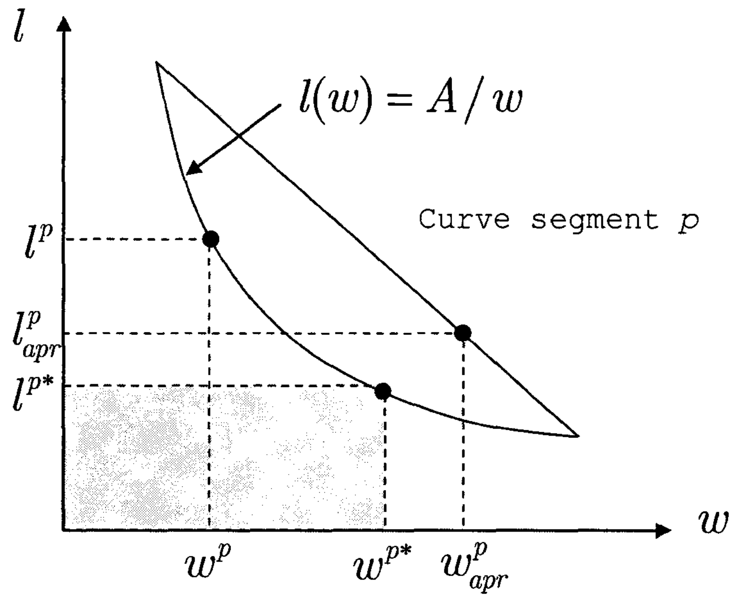

Figure 16. Illustration of Lemma 5(b)

sponding to f^p_{aux}. The minimum of the objective function values of two auxiliary expropriation problems can be used as a lower bound on the objective function value of the mixed integer nonlinear expropriation problem p. This is given by the following result.

Lemma 6 $f^p_{aux} \leq f^p$.

PROOF Let the unknown optimal formation for the MINLP-p be (w^p, l^p) as shown in Figure 17. Recall from the notations that, the objective function value regarding this formation is f^p. It is easy to observe that one can always find an optimal formation (w^p_{aux}, l^p_{aux}) for an auxiliary expropriation problem on one of the tangent lines which can fit into the coverage area of the optimal MINLP-p formation (w^p, l^p), and hence, covers at most the same number of fixed points as (w^p, l^p), as shown in Figure 17. Therefore, $f^p_{aux} \leq f^p$. ∎

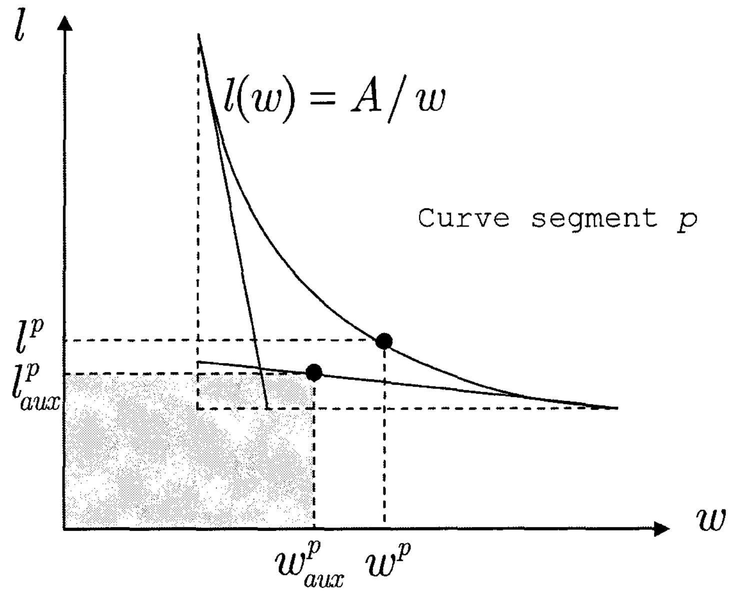

Figure 17. Illustration of Lemma 6

4.6.3 Outline of the branch-and-bound algorithm

Step 1 (Initialization).

Define n; the list Ψ only includes the first problem corresponding to the aspect ratio interval $[\alpha_{\min}, \alpha_{\max}]$; $UB(1) = UB^* = \infty,\ LB(1) = 0$.

Step 2 (Problem generation).

(2.1) From the list Ψ, select the problem p' such that $UB(p') = \min_{p \in \Psi} \{UB(p)\}$, remove p' from Ψ. Partition the aspect ratio interval of problem p' into n equal intervals and create n new problems, add these problems to Ψ^T.

(2.2) $\forall p \in \Psi^T$: **(2.2.1)** Solve the approximate expropriation problems and set the upper bounds $UB(p) = f^p_{apr}$. **(2.2.2)** Solve the auxiliary expropriation problems and set the lower bounds $LB(p) = f^p_{aux}$.

Step 3 (Bounding).

(3.1) If $\min_{p \in \Psi^T} \{UB(p)\} < UB^*$, update the upper bound $UB^* \longleftarrow \min_{p \in \Psi^T} \{UB(p)\}$.

(3.2) Remove the problems from Ψ^T and add them to Ψ.

Step 4 (Pruning).

$\forall p \in \Psi$: **(4.1-*Prune by bound*)** If $LB(p) \geq UB^*$; **(4.2-*Prune by optimality*)** Else if $LB(p) = UB(p)$; fathom p and remove from Ψ.

Step 5 (Check the list).

If $\Psi \neq \varnothing$, return to **Step 2**; else stop with optimal solution UB^*. ♣

A convergence analysis of this procedure is provided in Appendix A.

4.7 Numerical examples for the branch-and-bound algorithm

4.7.1 Single facility $\mathcal{PENR}$ example: Example 1

We constructed a 50-point example problem (Example 1 hereafter) in a bounded square having a dimension of 10×10 units, where the fixed point locations in each dimension are randomly generated from uniform distribution $U \sim [0, 10]$. Each fixed point has a unit expropriation cost of 1 and the area requirement for the single facility

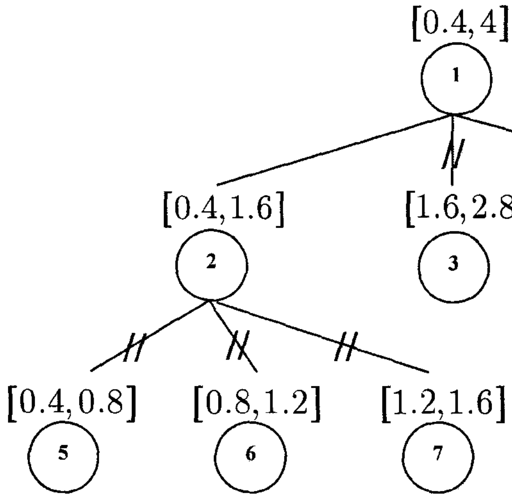

Figure 18. Branching tree for Example 1

is 20 unit squares. The formation of the single shape is considered to be acceptable in the aspect ratio interval $[\alpha_{\min}, \alpha_{\max}] = [0.4, 4]$. The number of new problems to be generated after each branching decision is $n = 3$. The data for this example problem is given in Table 1.

The branching tree for this example is illustrated in Figure 18. Square brackets above each node of the branching tree indicate the aspect ratio interval of the corresponding problem. Solutions attained at each node in the branching tree are summarized in Table 2.

According to the results presented at Figure 18, there are two alternative optimal solutions at the nodes (5) and (6) of the branching tree with an objective function value of 3. The facility location found by the approximate expropriation problem at node (5) is $(x_1, x_2) = (7.087, 2.855)$ with lateral dimensions $(w^5, l^5) =$

Point	a_{1i}	a_{2i}	**Point**	a_{1i}	a_{2i}
1	2.624264	2.672071	**26**	5.323392	4.831004
2	7.214577	0.420937	**27**	2.401478	9.221698
3	5.030869	5.04965	**28**	2.73379	9.967937
4	9.979267	7.057403	**29**	7.190492	5.200507
5	4.401408	3.390079	**30**	5.838773	8.300443
6	1.534631	1.179493	**31**	2.064844	7.012389
7	1.992227	1.182514	**32**	3.911611	4.015819
8	4.543277	2.179437	**33**	6.303549	9.195663
9	2.484197	4.44614	**34**	8.963257	9.337799
10	2.290888	5.327973	**35**	0.03591	7.248852
11	9.756471	6.981464	**36**	7.688158	2.438863
12	2.975805	9.533832	**37**	0.394231	1.292512
13	8.617978	6.952832	**38**	9.300951	0.123071
14	5.320528	6.530289	**39**	3.682208	4.262025
15	2.141231	9.946365	**40**	7.05713	0.628475
16	3.341322	5.738205	**41**	5.062653	5.671356
17	8.628258	3.854502	**42**	5.184912	0.229162
18	4.20512	8.106408	**43**	1.108956	4.518587
19	2.458259	6.719806	**44**	2.082176	8.453704
20	7.080638	2.574753	**45**	0.382973	0.623229
21	6.831599	5.754518	**46**	7.955007	6.259559
22	1.85656	8.229517	**47**	2.24412	9.241891
23	3.355435	0.599625	**48**	5.787337	7.942322
24	7.513105	5.897806	**49**	6.696141	9.111632
25	3.810221	1.478399	**50**	6.906478	0.879958

Table 1. Data for Example 1

p	x_1	x_2	w^p	l^p	$UB(p)$	$LB(p)$	Pruned by
1	-	-	-	-	∞	0	-
2	7.22	2.73	5.353	4.203	4	2	-
3	8.585	6.058	2.789	7.237	6	5	Bound $UB^* = 4$
4	8.834	4.801	2.291	8.760	4	4	Optimality
5	7.087	2.855	5.087	3.95	3	3	Optimality $UB^* = 3$
6	7.662	3.042	4.677	4.317	3	3	Optimality
7	7.772	3.156	3.969	5.056	5	5	Optimality
p	Area	% Area gap	α	Points covered			
1	-	-	-	-			
2	22.49	12.45	0.785	{17,20,36,50}			
3	20.183	0.915	2.5948	{11,13,17,24,34,46}			
4	20.069	0.345	3.8236	{11,13,17,46}			
5	20.0936	0.468	0.7764	{17,20,36}			
6	20.19	0.95	0.923	{17,20,36}			
7	20.0672	0.336	1.2738	{17,20,29,36,50}			

Table 2. Summary of the solutions for Example 1

(5.087, 3.951) and a resulting area of $A^5 = 20.098$. The facility location found by the approximate expropriation problem at node (6) is $(x_1, x_2) = (7.662, 3.042)$ with lateral dimensions $(w^6, l^6) = (4.677, 4.317)$ and a resulting area of $A^6 = 20.19$. These solutions are illustrated in Figures 19 and 20, respectively.

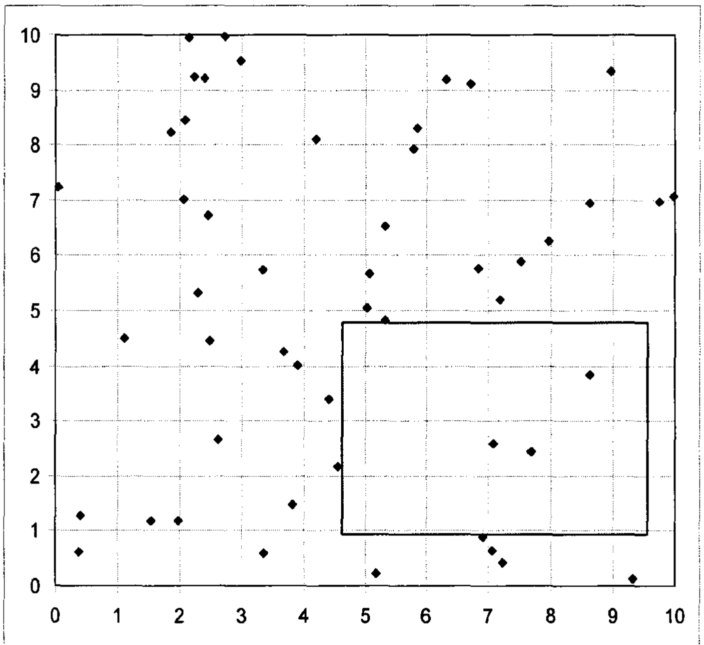

Figure 19. Optimal location for the single facility found at node (5)

4.7.2 Multiple facilities $\mathcal{PENRM}$ example: Example 2

To illustrate the case of multiple facilities, consider the same set of fixed points and bounded region used in the single facility case. Two facilities each having an area requirement of 15 unit squares are going to be located within this region. The for-

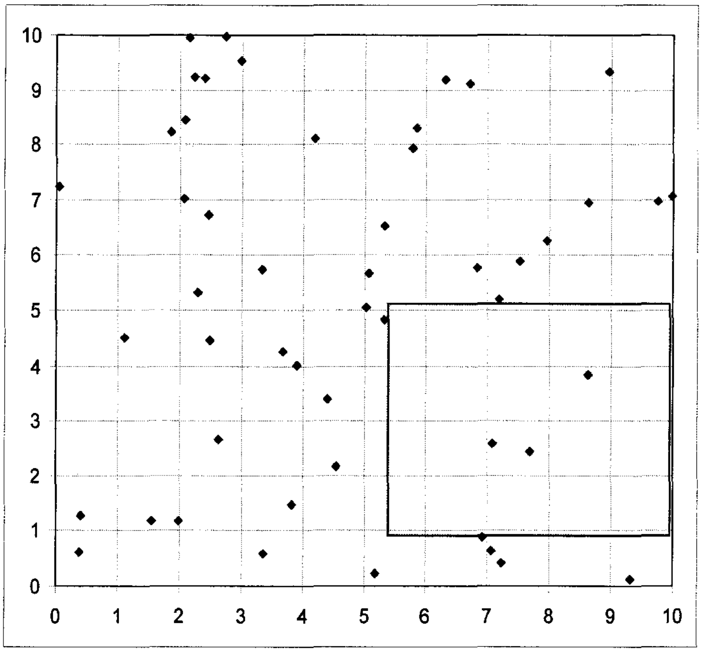

Figure 20. Optimal location for the single facility found at node (6)

mation limitations imposed on the facilities are given by the aspect ratio intervals $[\alpha_{1,\min}, \alpha_{1,\max}] = [0.5, 3]$ and $[\alpha_{2,\min}, \alpha_{2,\max}] = [0.6, 2.8]$. Number of problems to be generated after each branching decision is taken as $n = 4$. The branching tree regarding this example problem (Example 2 hereafter) is illustrated in Figure 21 and solutions attained at each node of the branching tree are summarized in Tables 3-5.

According to the results presented in Figure 21, the optimal solution for the multiple facilities example is attained at node (25) of the branching tree with an objective function value of 4. Optimal facility locations found by the approximate expropriation problem at node (25) are: (1) $(x_{11}, x_{21}) = (1.154, 4.875)$ for facility 1; and (2) $(x_{12}, x_{22}) = (8.735, 3.191)$ for facility 2. Optimal lateral dimensions found for

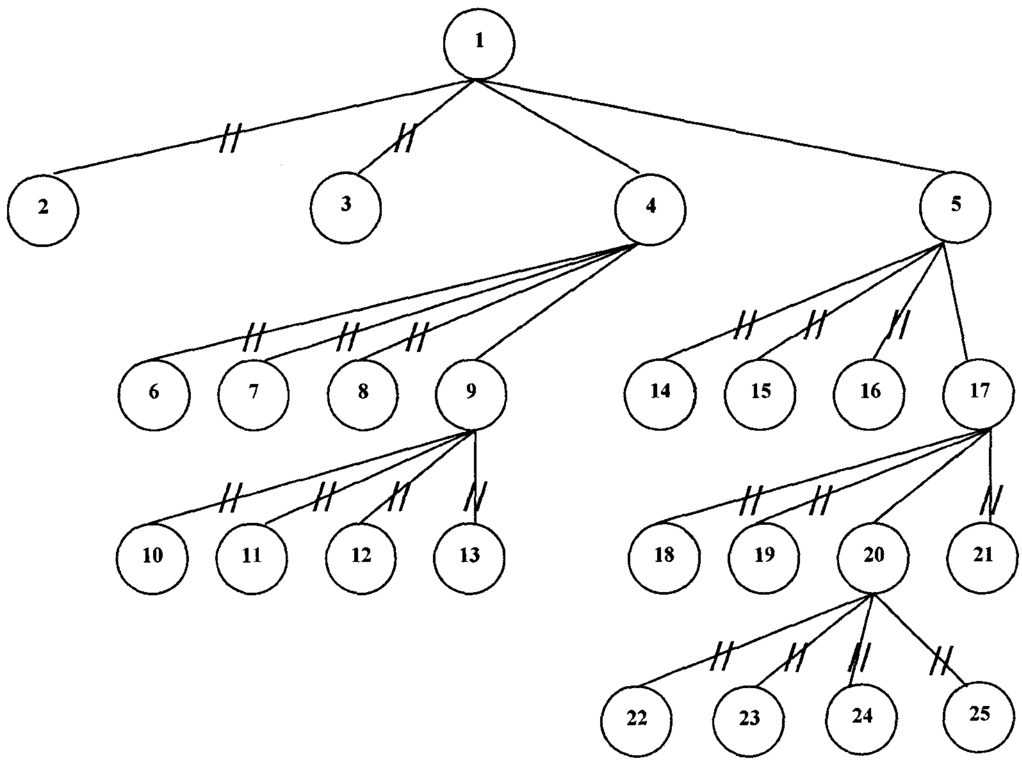

Figure 21. Branching tree for Example 2

these facilities are: (1) $(w_1^{25}, l_1^{25}) = (2.236, 6.708)$ with a resulting area of $A_1^{25} = 15$ for facility 1; and (2) $(w_2^{25}, l_2^{25}) = (2.445, 6.136)$ with a resulting area of $A_2^{25} = 15.0025$ for facility 2. This solution is illustrated in Figure 22.

4.8 Computational study

In this section, we report our computational experience with the branch-and-bound algorithm we developed for the solution of $\mathcal{PENR}$. In particular, we compared this algorithm with two conventional MINLP algorithms, namely, BARON's *branch-and-reduce* algorithm and SBB's *branch-and-bound* algorithm.

p	$[\alpha_{1,\min}, \alpha_{1,\max}]$	x_{11}	x_{21}	w_1^p	l_1^p	A_1^p	% Area gap	α_1	Points covered
1	[0.5, 3]	-	-	-	-	-	-	-	-
2	[0.5, 1.75]	6.944	8.618	5.477	2.739	15.0015	0.01	0.5	{30,33,34,48,49}
3	[0.5,1.75]	5.714	8.618	5.477	2.739	15.0015	0.01	0.5	{18,30,33,48,49}
4	[1.75,3]	8.597	2.872	2.765	5.497	15.1992	1.328	1.988	{17,36}
5	[1.75,3]	8.585	3.142	2.789	5.442	15.1777	1.184666	1.9512	{17,36}
6	[1.75,2.375]	5.389	5.061	2.614	5.763	15.0644	0.429333	2.2046	{3,5,14,26,41}
7	[1.75,2.375]	8.597	2.851	2.765	5.456	15.0858	0.572	1.9732	{17,36}
8	[2.375,3]	8.844	3.376	2.312	6.506	15.0418	0.278666	2.814	{17,46}
9	[2.375,3]	1.163	4.621	2.255	6.658	15.0137	0.091333	2.9525	{31,43}
10	[2.375,2.6875]	8.746	3.167	2.466	6.087	15.0105	0.07	2.4683	{17,36}
11	[2.375,2.6875]	3.741	6.374	2.513	5.969	15	0	2.375	{16,18,32,39}
12	[2.6875,3]	1.163	4.62	2.255	6.654	15.0047	0.031333	2.9512	{31,43}
13	[2.6875,3]	1.163	4.806	2.255	6.655	15.007	0.046666	2.9512	{31,43}
14	[1.75,2.375]	8.528	3.191	2.926	5.126	15	0	1.7518	{17,20,29,36,50}
15	[1.75,2.375]	8.597	2.957	2.765	5.456	15.0858	0.572	1.9732	{17,36}
16	[2.375,3]	1.122	4.636	2.244	6.686	15.0033	0.022	2.9799	{31,35,43}
17	[2.375,3]	1.122	4.636	2.244	6.687	15.0056	0.037333	2.9799	{31,35,43}
18	[2.375,2.6875]	8.757	3.141	2.487	6.035	15.009	0.06	2.4266	{17,36}
19	[2.375,2.6875]	8.746	3.167	2.466	6.087	15.0105	0.07	2.4683	{17,36}
20	[2.6875,3]	1.122	4.821	2.244	6.685	15.011	0.073333	2.979	{31,35,43}
21	[2.6875,3]	1.163	4.806	2.254	6.655	15.0003	0.002	2.9512	{31,43}
22	[2.6875,2.84375]	8.844	3.474	2.312	6.489	15.0025	0.016666	2.8066	{17,46}
23	[2.6875,2.84375]	1.217	4.653	2.362	6.349	15	0	2.6879	{10,31,43}
24	[2.84375,3]	1.163	4.619	2.255	6.653	15.0025	0.016666	2.9503	{31,43}
25	[2.84375,3]	1.154	4.875	2.236	6.708	15	0	3	{31,43}

Table 3. Summary of the solutions for Facility 1 - Example 2

p	$[\alpha_{2,\min}, \alpha_{2,\max}]$	x_{12}	x_{22}	w_2^p	l_2^p	A_2^p	% Area gap	α_2	Points covered
1	[0.6,2.8]	-	-	-	-	-	-	-	-
2	[0.6,1.7]	4.449	1.772	4.914	3.087	15.1695	1.13	0.6282	{1,8,23,25}
3	[1.7,2.8]	8.607	2.85	2.785	5.423	15.1281	0.854	1.9472	{17,36}
4	[0.6,1.7]	1.488	4.001	2.976	5.044	15.0109	0.072666	1.6948	{1,9,10,43}
5	[1.7,2.8]	3.828	4.618	2.407	6.28	15.1159	0.772666	2.609	{5,8,16,32,39}
6	[0.6,1.15]	2.025	3.403	3.977	3.85	15.3114	2.076	0.968	{1,9,32,39,43}
7	[1.15,1.7]	5.019	4.436	3.355	4.512	15.1377	0.918	1.3448	{3,5,14,26,32,39,41}
8	[0.6,1.15]	2.536	7.238	5	3	15	0	0.6	{18,19,22,31,44}
9	[1.15,1.7]	8.453	3.067	3.094	4.878	15.0925	0.616666	1.5765	{17,20,29,36}
10	[1.15,1.425]	1.689	6.791	3.305	4.545	15.0212	0.141333	1.3751	{10,19,22,31,44}
11	[1.425,1.7]	6.946	2.733	3.244	4.623	15	0	1.425	{20,36,40,50}
12	[1.15,1.425]	8.109	2.957	3.612	4.153	15	0	1.15	{17,20,36}
13	[1.425,1.7]	8.318	3.192	3.244	4.623	15	0	1.425	{17,20,29,36}
14	[1.7,2.25]	5.733	3.705	2.663	5.65	15.0459	0.306	2.1216	{3,8,21,26,41}
15	[2.25,2.8]	3.828	6.515	2.407	6.25	15.0437	0.291333	2.5965	{12,16,18,32,39}
16	[1.7,2.25]	8.607	2.829	2.784	5.412	15.067	0.446666	1.9432	{17,36}
17	[2.25,2.8]	8.746	3.173	2.466	6.1	15.0426	0.284	2.4736	{17,36,46}
18	[2.25,2.525]	1.327	4.087	2.582	5.809	15	0	2.25	{9,10,19,37,43}
19	[2.525,2.8]	1.247	4.575	2.422	6.194	15.0018	0.012	2.5573	{10,31,43}
20	[2.25,2.525]	8.746	3.166	2.466	6.085	15.0056	0.037333	2.4679	{17,36}
21	[2.525,2.8]	8.732	3.077	2.437	6.154	15	0	2.5252	{17,36,38}
22	[2.25,2.3875]	3.758	6.961	2.547	5.891	15.0043	0.028666	2.3129	{12,16,18,39}
23	[2.3875,2.525]	8.746	3.165	2.466	6.083	15.0006	0.004	2.4667	{17,36}
24	[2.25,2.3875]	8.506	2.905	2.582	5.810	15.0014	0.009333	2.2501	{17,36,38}
25	[2.3875,2.525]	8.735	3.191	2.445	6.136	15.0025	0.016666	2.5096	{17,36}

Table 4. Summary of the solutions for Facility 2 - Example 2

p	$UB(p)$	$LB(p)$	Pruned by
1	∞	0	-
2	9	7	Bound $UB^* = UB(4) = 6$
3	7	6	Bound $UB^* = UB(4) = 6$
4	6	4	-
5	7	4	-
6	10	6	Bound $UB^* = UB(9) = 6$
7	7	7	Optimality
8	7	7	Optimality
9	6	5	-
10	7	6	Bound $UB^* = UB(12) = 5$
11	8	6	Bound $UB^* = UB(12) = 5$
12	5	5	Optimality
13	6	5	Bound $UB^* = UB(12) = 5$
14	10	8	Bound $UB^* = UB(12) = 5$
15	7	5	Bound $UB^* = UB(12) = 5$
16	5	5	Optimality
17	6	4	-
18	7	7	Optimality
19	5	5	Optimality
20	5	4	-
21	5	5	Optimality
22	6	6	Optimality
23	5	5	Optimality
24	5	5	Optimality
25	4	4	Optimality $UB^* = UB(25) = 4$

Table 5. Summary of the bounds for Example 2

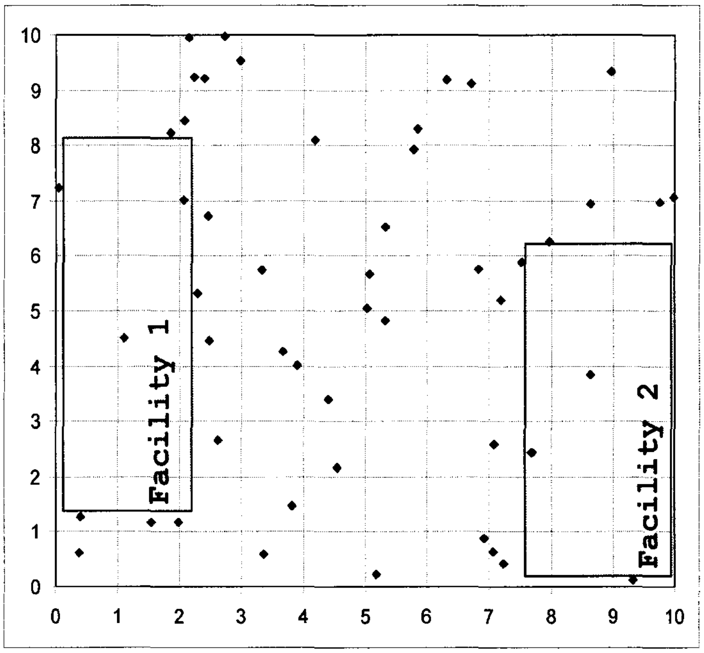

Figure 22. Illustration of the optimal solution for Example 2

4.8.1 Experiments and basis for measuring performance

In our experiments, we found that many conventional MINLP solvers are not suitable for our particular $\mathcal{PENR}$ problem. The objective function of $\mathcal{PENR}$ is neither convex nor differentiable. Hence well-known MINLP solvers such as MOSEK, KNITRO, and FILTER did not work for this problem because these solvers make use of the convexity or second-derivative information. We also tried some other solvers such as CONOPT, DICOPT and LOQO which are suitable for non-convex optimization problems. However, it is well-known that when models with non-differentiable and non-convex functions are submitted to these solvers, they become less reliable and may terminate at solutions which are not local optima. Our experience with these solvers showed that their performance in solving $\mathcal{PENR}$ is unsatisfactory. Most of the time, these procedures stopped prematurely and returned relaxed MINLP solutions or terminated after facing local infeasibility for a subproblem in the branching tree. Hence, we did not find the solutions generated by these solvers comparable to our branch-and-bound scheme. The results we obtained by using BARON and SBB were much more meaningful and comparable to our branch-and-bound algorithm than those gererated by other solvers. Hence, to provide a basis for illustrating the performance of our algorithm, we choose to compare it with BARON's branch-and-reduce and SBB's branch-and-bound algorithms.

4.8.2 Design of experiments and resource settings

We created a set of test problems considering two factors: (1) number of fixed points N; and (2) number of non-rigid facilities K. Each combination of the sets $N = \{5, 10, 20, 50, 75, 100, 200, 500\}$ and $K = \{1, 2, 3\}$ resulted in an instance of the $\mathcal{PENR}$ problem. The fixed point coordinates a_{1i} and a_{2i} in each dimension were randomly generated from a uniform distribution $U \sim [0, 10]$ within a 10×10 square. Finally, we allowed each fixed point to assume a unit expropriation cost $c_i = 1$. The properties and optimal solutions of the test problems under study are summarized in Table 6, where #Discrete and #Constraints stands for the number of discrete variables, and number of constraints, respectively.

The area requirement for each facility considered in this experiment is taken as 15 unit squares and the dimensional formation of each facility is considered to be

N	K	#Discrete	#Constraints	Optimal
5	1	25	59	0
	2	54	127	0
	3	87	204	0
10	1	50	109	0
	2	104	227	0
	3	162	354	0
20	1	100	209	0
	2	204	427	0
	3	312	654	0
50	1	250	509	1
	2	504	1027	2
	3	762	1554	5
75	1	375	759	4
	2	754	1527	8
	3	1137	2304	13
100	1	500	1009	6
	2	1004	2027	13
	3	1512	3054	20
200	1	1000	2009	15
	2	2004	4027	33
	3	3012	6054	56
500	1	2500	5009	44
	2	5004	10027	102
	3	7512	15054	171

Table 6. Properties and optimal solutions of the problems under study

acceptable within the aspect ratio limits $[\alpha_{\min}, \alpha_{\max}] = [0.5, 1.5]$. We selected the number of new problems to be generated after a branching decision as $n = 2$ for small problems $N = \{5, 10, 20\}$, $n = 3$ for moderate problems $N = \{50, 75, 100\}$, and $n = 4$ for big problems $N = \{200, 500\}$.

The algorithm was implemented in the GAMS modeling system and all experiments were conducted on a 1.6 GHz desktop computer with 512 MB RAM. We utilized CPLEX 10 as the MILP solver for tackling the approximate expropriation problems generated by our algorithm. For the subproblems generated by BARON and SBB, we utilized MINOS as the NLP solver and CPLEX 10 as the LP solver. Moreover, it is worthwhile to note that our branch-and-bound algorithm has no preprocessing capabilities whereas most of the conventional solvers do have such enhancements to reduce solution times. For example, BARON and SBB are known to implement *preprocessing* and perform multi-start local search algorithms on the root problem.

For evaluating the performance of these three algorithms by equal standards, we also imposed some resource restrictions by using the option files provided by the GAMS system. First, we imposed a 3 hour limitation on the CPU time by adding the command `option reslim=10800`. Second, we used the command `option iterlim=100000` for imposing a limit of 100,000 on the number of iterations. Finally, we ensured that the branching tree for each procedure is not bigger than 100,000 nodes by utilizing the command `expropriation.nodlim=100000` within our code.

4.8.3 Problem size

Problem size depends on two factors: N and K. A single facility $\mathcal{PENR}$ problem is composed of $5N$ discrete variables and $10N + 9$ constraints. The multi-facility problem calls for $5NK + 2K(K-1)$ discrete variables and $10NK + 9K(K-1)/2 + 9K$ constraints. These numbers point out that the number of discrete variables and constraints are more sensitive to the number of facilities considered. Clearly, an increase in the number of facilities has a critical effect on increasing problem size.

4.8.4 Experiment results

The results of our experiments on the test problems with three algorithms are summarized in Table 7. The legend for reading this table and average gap formula (45) is as follows.

Legend.

GT : generation time (in seconds);

SR : solution returned;

CPU : total CPU time elapsed (in `h:mm:ss` format);

SBT : size of the branching tree;

† : procedure reached 3 hour CPU time limit;

‡ : procedure terminated prematurely by returning a feasible integer solution. This corresponds to case 8 of GAMS's `SOLVE SUMMARY` report. In this case, at the time of termination `MODEL STATUS` will show that the solution obtained is a feasible integer solution, but it is not necessarily optimal;

		BARON Branch & Reduce				SBB Branch & Bound				Algorithm			
N	K	GT	SR	SBT	CPU	GT	SR	SBT	CPU	GT	SR	SBT	CPU
5	1	0.08	0	5	0:00:02	0.04	0	30	0:00:09	0.04	0	3	0:00:04
	2	0.08	0	13	0:00:03	0.11	0	64	0:00:19	0.06	0	4	0:00:04
	3	0.12	0	56	0:00:06	0.13	0	1901	0:04:46	0.06	0	5	0:00:08
10	1	0.07	0	7	0:00:03	0.07	0	113	0:00:35	0.06	0	3	0:00:04
	2	0.07	0	18	0:00:04	0.11	0	2136	0:09:38	0.07	0	4	0:00:04
	3	0.12	0	178	0:00:19	0.14	0	5886	0:25:15	0.07	0	5	0:00:08
20	1	0.07	0	33	0:00:10	0.07	0	2828	0:15:27	0.05	0	3	0:00:07
	2	0.09	0	235	0:00:28	0.08	0	3026	0:16:14	0.07	0	4	0:00:18
	3	0.13	0	13576	0:33:41	0.13	7*	14988	1:42:08‡	0.08	0	5	0:00:18
50	1	0.1	1	4259	0:08:46	0.08	13*	21590	2:02:14‡	0.17	1	4	0:00:24
	2	0.17	7*	13937	3:00:00†	0.25	NA	7578	1:04:38	0.11	2	7	0:03:41
	3	0.19	13*	4483	3:00:00†	0.32	NA	9778	1:46:51	0.11	5	39	0:49:34
75	1	0.1	5*	47442	3:00:00†	0.21	12*	11176	1:13:41‡	0.11	4	10	0:02:05
	2	0.19	12*	5317	3:00:00†	0.25	24*	16119	1:34:10‡	0.15	8	16	0:45:12
	3	0.88	24*	1787	3:00:00†	0.22	NA	6460	1:40:35	0.15	13	21	2:05:56
100	1	0.14	7*	25596	3:00:00†	0.21	9*	11472	1:39:15‡	0.11	6	10	0:07:20
	2	0.32	20*	3074	3:00:00†	0.43	NA	10790	2:14:04	0.11	13*	9	3:00:00†
	3	0.21	38*	859	3:00:00†	0.28	NA	10204	2:52:01	0.14	20*	10	3:00:00†
200	1	0.21	21*	7187	3:00:00†	0.23	30*	10080	2:07:42‡	0.26	15	9	1:25:54
	2	0.27	53*	617	3:00:00†	0.23	NA	3648	1:36:27	0.31	33*	7	3:00:00†
	3	1.83	82*	229	3:00:00†	0.36	NA	612	0:20:38	0.39	57*	6	3:00:00†
500	1	0.49	55*	825	3:00:00†	0.27	65*	6068	2:46:56‡	0.46	44*	4	3:00:00†
	2	0.5	153*	93	3:00:00†	0.42	NA	1632	1:44:14	0.58	108*	4	3:00:00†
	3	0.57	NA	18	2:54:47	0.76	NA	979	1:13:51	1.03	177*	4	3:00:00†

Table 7. Comparison of the algorithm with MINLP solvers

Terminated by	BARON		SBB		Algorithm	
	Count	Average Gap	Count	Average Gap	Count	Average Gap
Optimality	10	-	8	-	17	-
CPU time limit	13	73.24 %	-	-	7	1.57 %
Integer solution	-	-	7	300 %	-	-
Local infeasibility	1	-	9	-	-	-

Table 8. Termination summary

NA : procedure terminated after facing local infeasibility in branching tree by returning no solution;

$*$: solution returned is not proven to be optimal;

OPT : optimal solution of the instance;

SAI : set of all instances;

NI : number of instances. ♣

It is evident from Table 7 that the results for our branch-and-bound procedure compete favorably with two alternatives. Among the solutions presented in Table 7, those generated by SBB were of inferior quality. SBB was able to solve only small problems with $N = \{5, 10, 20\}$ to optimality. For a single facility moderate $N = \{50, 75, 100\}$, and big $N = \{200, 500\}$ problems, SBB terminated prematurely by returning integer feasible solutions. It was evident that multi-facility problems become unsolvable for SBB very quickly.

The termination statistics regarding the three algorithms under study are summarized in Table 8. Of those not solved by SBB, 7 instances were terminated prematurely by returning integer solutions, and 9 instances returned no solution by terminating due to local infeasibility in the branching tree. The average gap values in this table are calculated as follows:

$$\text{Average Gap (\%)} = \left(\sum_{\text{SAI}} \frac{\text{SR} - \text{OPT}}{\text{OPT}} \right) / \text{ NI} \tag{45}$$

where the optimal solutions for instances were computed before the experiment without imposing resource restrictions. These values were summarized in Table 6.

For SBB, the average gap between the solutions returned and the optimal solution after terminating with integer solution was as high as 300%. As far as our particular problem structure is concerned, the performance of BARON was better than SBB. It was able to solve small problems $N = \{5, 10, 20\}$ to optimality, yet the numerical evidence in Table 7 shows that moderate $N = \{50, 75, 100\}$ and big $N = \{200, 500\}$ problems are not solvable by BARON within the time constraint. For those 13 instances, solution times quickly reached the 3 hour CPU time limit and BARON terminated the procedure by returning best integer solutions. The average gap after terminating by CPU time limit was 73.24% which is quite high. The largest problem, with $N = 500$, $K = 3$, was not solvable with BARON and the procedure terminated by local infeasibility.

Among the three algorithms under study, our branch-and-bound procedure was superior to BARON and SBB. We were able to compute optimal solutions for 70% of the instances within CPU time limit by using our algorithm. The solution times for these problems were reasonable. Figure 23 is a depiction of the resultant solution times on a logarithmic scale for each test problem. It is evident that our algorithm, in general, resulted in better solution times. As seen from Table 8, our algorithm terminated due to the CPU time limit in 7 instances. Of those 7 runs, our algorithm was able to detect the optimal solution in 4 cases. That is, at the time of termination our algorithm was able to find the optimal solution but it could not verify its optimality. The average gap after terminating by CPU time limit was merely 1.57%. Moreover, the growth of the branching tree was significantly smaller than with BARON or SBB. It is also worthwhile to note that the subproblems in our algorithm are hard MILP problems. It was evident from our computational experience that using our branch-and-bound procedure for solving $\mathcal{PENR}$ and $\mathcal{PENRM}$ is much better than using BARON or SBB.

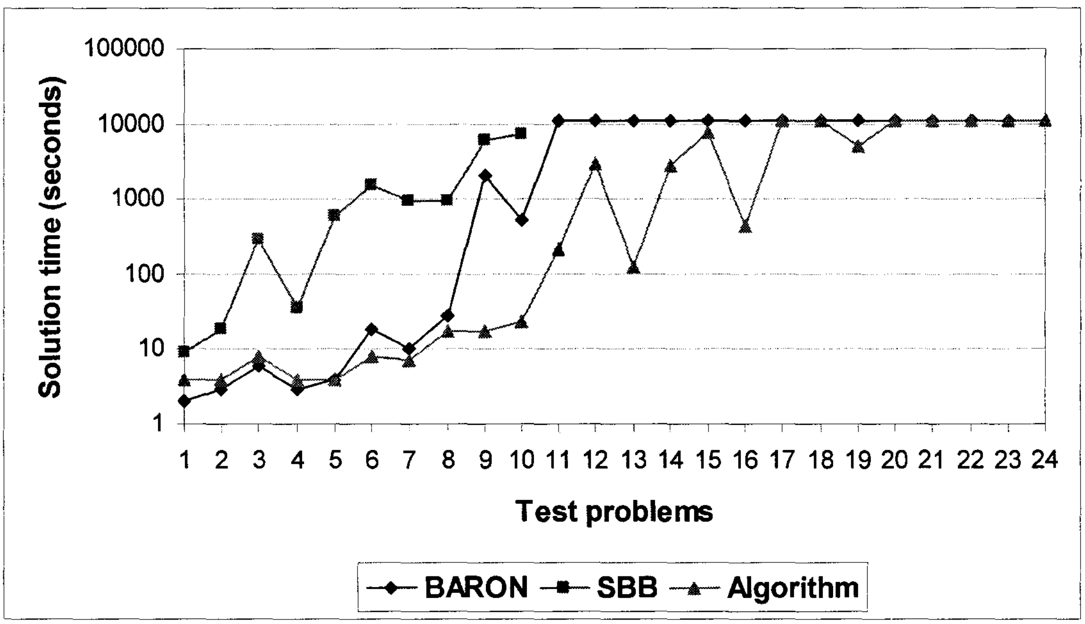

Figure 23. Illustration of solution times

Chapter 5
Planar maximal covering problem with single convex polygonal shapes

5.1 Introduction

In this chapter, we study the *planar maximal covering problem with single convex polygonal shapes* ($\mathcal{PMCS}$). As discussed in Section 3.2.2, this problem has real-life applications in location of convex area facilities, transmitter location, material cutting, inspection of geometric shapes and directional antenna location where the coverage areas of the antennas extend in particular directions (Younies and Wesolowsky, 2004).

Current maximal covering literature is sparse on the subject of formulations where the facilities have particular area requirements and specific formations (such as convex polygonal shapes), rather than being analytical solution points in the plane. As discussed in Sections 2.2.1.2, 3.1 and 3.2.2, there are only two studies on the exact methods for the maximal covering problems of regular convex polygonal shapes: (1) the *planar maximal covering problem for inclined parallelograms* (Younies and Wesolowsky, 2004), and (2) the *planar maximal covering problem under block-norm distance measure* (Younies and Wesolowsky, 2007). The $\mathcal{PMCS}$ problem discussed in this chapter is a more general formulation and encompasses the planar maximal covering problems that are solvable by Younies and Wesolowsky (2004; 2007) approaches, as special cases. To illustrate this, we provide a comparative analysis of

Shape	Y&W (2004)★	Y&W (2007)★	$\mathcal{PMCS}$
Triangles	×	×	✓
Rectangles	✓	✓	✓
Rhombi	✓	✓	✓
Quadrilaterals	✓	✓	✓
Regular polygons-symmetric (hexagons, octagons etc.)	×	✓	✓
Regular polygons-non-symmetric (pentagons, heptagons etc.)	×	×	✓
★ Younies and Wesolowsky (2004; 2007)			

Table 9. A comparative analysis of three formulations

these three formulations based on the convex polygonal shapes that can be handled by these formulations in Table 9.

5.2 Problem formulation

In this section, we discuss how to formulate $\mathcal{PMCS}$ using a simple five-step process. We'll only demonstrate the construction for the formulation of a regular hexagon; however, one can generate the formulation for any particular convex polygonal shape by following the same procedure. For illustration purposes, we provide the $\mathcal{PMCS}$ formulations for the triangles and quadrilaterals in the Appendix B.

Consider a regular hexagon with an area requirement of A. Denote the six vertices of this hexagon by V_1, V_2, V_3, V_4, V_5 and V_6. Let the location of the hexagon be (x_1, x_2) and its half-length be l as illustrated in Figure 24. Since an interior angle

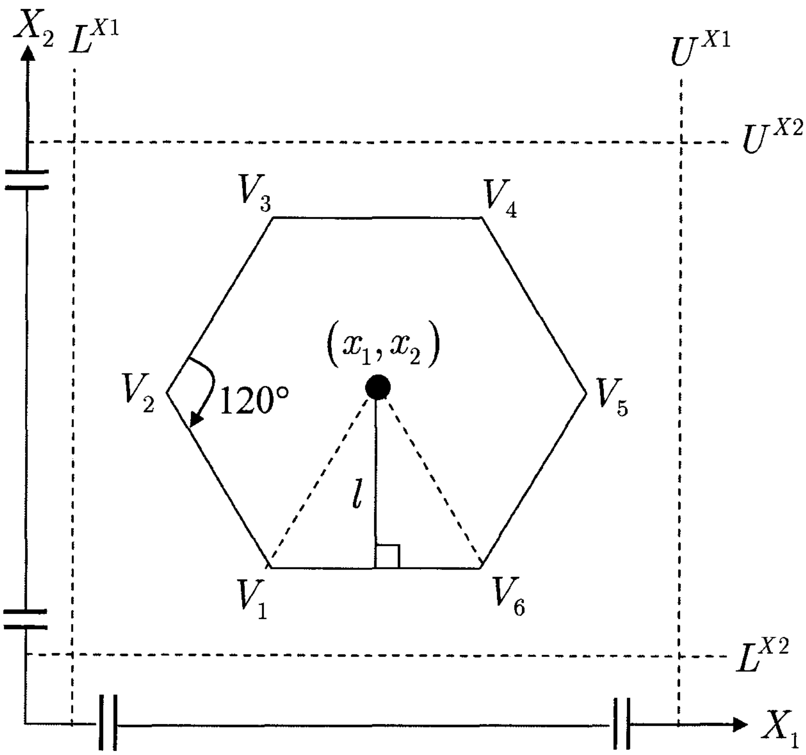

Figure 24. A hexagon and its vertices

equals to 120° for the hexagon and the area requirement is known, the half-length of the hexagon is given by $l = \sqrt{A\sqrt{3}/6}$, which is a known parameter.

Step 1. *Write the relative locations of the vertices of the convex polygonal shape in terms of shape location.*

For the hexagon in Figure 24, these are given as follows: $V_1 = (x_1 - l/\sqrt{3}, x_2 - l)$, $V_2 = (x_1 - 2 \cdot l/\sqrt{3}, x_2)$, $V_3 = (x_1 - l/\sqrt{3}, x_2 + l)$, $V_4 = (x_1 + l/\sqrt{3}, x_2 + l)$, $V_5 = (x_1 + 2 \cdot l/\sqrt{3}, x_2)$ and $V_6 = (x_1 + l/\sqrt{3}, x_2 - l)$.

Step 2. *Using the relative locations found in Step 1, generate the underlying line* *equalities for the lines passing through the edges of the shape.*

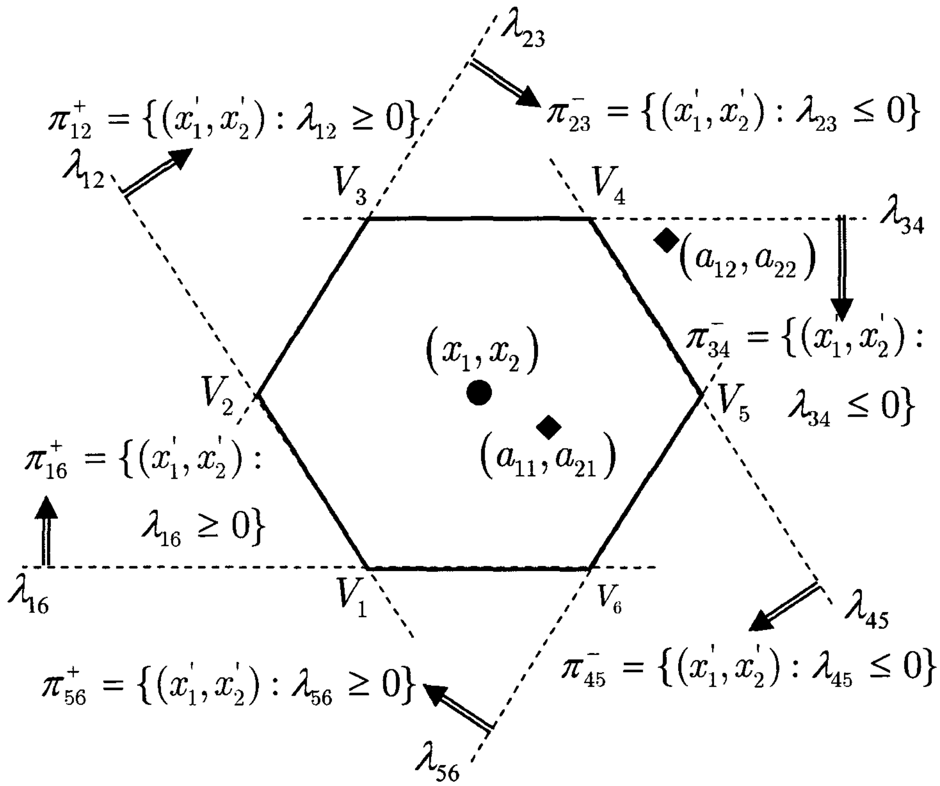

Figure 25. Underlying half-planes of the hexagon

Let (x_1', x_2') denote an arbitrary point on the corresponding line equality. These lines are shown as dashed contours in Figure 25. We start with λ_{12} and write

$$\lambda_{12} \equiv x_2' - (x_2 - l) - \frac{l - (x_2 - l)}{(x_1 - 2 \cdot l/\sqrt{3}) - (x_1 - l/\sqrt{3})} \cdot \left(x_1' - \left(x_1 - l/\sqrt{3}\right)\right) = 0.$$

After further simplification, the line equality λ_{12} is given by

$$\lambda_{12} \equiv x_2' + \sqrt{3} \cdot x_1' - x_2 - \sqrt{3} \cdot x_1 + 2 \cdot l = 0. \tag{46}$$

Continuing with the same procedure, we can define the other line equalities as follows:

$$\lambda_{23} \equiv x_2' - \sqrt{3} \cdot x_1' - x_2 + \sqrt{3} \cdot x_1 - 2 \cdot l = 0 \tag{47}$$

$$\lambda_{34} \equiv x_2' - x_2 - l = 0 \tag{48}$$

$$\lambda_{45} \equiv x_2' + \sqrt{3} \cdot x_1' - x_2 - \sqrt{3} \cdot x_1 - 2 \cdot l = 0 \tag{49}$$

$$\lambda_{56} \equiv x_2' - \sqrt{3} \cdot x_1' - x_2 + \sqrt{3} \cdot x_1 + 2 \cdot l = 0 \tag{50}$$

$$\lambda_{16} \equiv x_2' - x_2 + l = 0. \tag{51}$$

Step 3. *Using the underlying line equalities found in Step 2, generate the logical inequalities for the coverage decision. This is done by selecting the half-plane containing the coverage area of the shape.*

Let (a_{1i}, a_{2i}) be the locations of N fixed points indexed by i. Consider as a first example the fixed point located at (a_{11}, a_{21}) shown in Figure 25. If this point is covered by the hexagon, when we substitute the point location (a_{11}, a_{21}) for (x_1', x_2') in line equalities (46-51), the following inequalities should hold:

$$\lambda_{12} \geq 0 \tag{52}$$

$$\lambda_{23} \leq 0 \tag{53}$$

$$\lambda_{34} \leq 0 \tag{54}$$

$$\lambda_{45} \leq 0 \tag{55}$$

$$\lambda_{56} \geq 0 \tag{56}$$

$$\lambda_{16} \geq 0. \tag{57}$$

To illustrate this, consider the half-planes: $\pi_{12}^{+} = \{(x_1', x_2') : \lambda_{12} \geq 0\}$, $\pi_{23}^{-} = \{(x_1', x_2') : \lambda_{23} \leq 0\}$, $\pi_{34}^{-} = \{(x_1', x_2') : \lambda_{34} \leq 0\}$, $\pi_{45}^{-} = \{(x_1', x_2') : \lambda_{45} \leq 0\}$, $\pi_{56}^{+} = \{(x_1', x_2') : \lambda_{56} \geq 0\}$, and $\pi_{16}^{+} = \{(x_1', x_2') : \lambda_{16} \geq 0\}$ illustrated in Figure 25. Coverage area of the hexagon is the intersection of these six half-planes. As an example of non-coverage, consider the fixed point located at (a_{12}, a_{22}) shown in Figure 25. This point falls into the half-plane π_{45}^{+}, therefore it can not be in the intersection. We conclude that this fixed point does not fall into the coverage area.

Step 4. *Define the binary variables assisting the covering decision for a fixed point and generate the coverage constraints using line inequalities (*52-57*) found in Step 3.*

Consider the following binary definitions. Let, $y_i^1 = 1$ if (52) holds for point i, and 0 otherwise; $y_i^2 = 1$ if (53) holds for point i, and 0 otherwise; $y_i^3 = 1$ if (54) holds for point i, and 0 otherwise; $y_i^4 = 1$ if (55) holds for point i, and 0 otherwise; $y_i^5 = 1$ if (56) holds for point i, and 0 otherwise; $y_i^6 = 1$ if (57) holds for point i, and 0 otherwise. Further, let M be a very large number employed for controlling the active-inactive status of each constraint. Using the above binary definitions together with the line inequalities (52-57), we define the covering constraints for a fixed point i as follows:

$$a_{2i} + \sqrt{3} \cdot a_{1i} - x_2 - \sqrt{3} \cdot x_1 + 2 \cdot l \geq \left(1 - y_i^1\right) \cdot (-M) \tag{58}$$

$$a_{2i} - \sqrt{3} \cdot a_{1i} - x_2 + \sqrt{3} \cdot x_1 - 2 \cdot l \leq \left(1 - y_i^2\right) \cdot M \tag{59}$$

$$a_{2i} - x_2 - l \leq \left(1 - y_i^3\right) \cdot M \tag{60}$$

$$a_{2i} + \sqrt{3} \cdot a_{1i} - x_2 - \sqrt{3} \cdot x_1 - 2 \cdot l \leq \left(1 - y_i^4\right) \cdot M \tag{61}$$

$$a_{2i} - \sqrt{3} \cdot a_{1i} - x_2 + \sqrt{3} \cdot x_1 + 2 \cdot l \geq \left(1 - y_i^5\right) \cdot (-M) \tag{62}$$

$$a_{2i} - x_2 + l \geq \left(1 - y_i^6\right) \cdot (-M). \tag{63}$$

Recall from Step 3 that if a point is covered, constraints (58-63) should all hold for that point. That is, if the point falls into intersection of half-planes π_{12}^{+}, π_{23}^{-}, π_{34}^{-}, π_{45}^{-},

π_{56}^{+} and π_{16}^{+}, then the binary variables y_i^1, y_i^2, y_i^3, y_i^4, y_i^5 and y_i^6 should all be equal to 1. Therefore, in order to implement coverage for a point by the hexagon, we have to add the constraint

$$\frac{1}{6} \cdot \left(y_i^1 + y_i^2 + y_i^3 + y_i^4 + y_i^5 + y_i^6\right) \geq y_i \tag{64}$$

where $y_i = 1$ if the fixed point is covered by the hexagon, and 0 otherwise.

Step 5. *(a) Generate the inclusion constraints to ensure that the shape is located in the feasible location area. (b) Add the integrality and non-negativity constraints.*

(a) Let, L^{X1} and U^{X1} be the lower and upper bounds for feasible shape locations along the X_1 dimension, and let L^{X2} and U^{X2} be the corresponding lower and upper bounds along the X_2 dimension as shown in Figure 24. We can then construct the inclusion constraints that ensure the shape will be located within the feasible space as follows:

$$x_1 - 2 \cdot l/\sqrt{3} \geq L^{X1} \tag{65}$$

$$x_1 + 2 \cdot l/\sqrt{3} \leq U^{X1} \tag{66}$$

$$x_2 - l \geq L^{X2} \tag{67}$$

$$x_2 + l \leq U^{X2}. \tag{68}$$

(b) These constraints are straightforward and given by

$$y_i^1,\ y_i^2,\ y_i^3,\ y_i^4,\ y_i^5,\ y_i^6,\ y_i \in \{0, 1\} \tag{69}$$

$$x_1, x_2 \in \Re^+. \tag{70}$$

The MILP formulation of $\mathcal{PMCS}$ can now be stated as follows:

$$(\mathcal{PMCS}) \quad \max \quad \left\{ \sum_{i=1}^{N} \omega_i \cdot y_i \ : \ \text{s.t.}\ (58 - 64,\ 69)\ \forall i;\ (65 - 68,\ 70) \right\} \tag{71}$$

where the objective function maximizes the total weight covered, the constraints (58–64) are the covering constraints, the constraints (65-68) are the inclusion constraints, (69) and (70) are the integrality and non-negativity constraints, respectively.

5.3 Cuts

In this section, we illustrate two types of cuts which later will be useful for developing a *branch-and-cut* algorithm for the solution of $\mathcal{PMCS}$. These cuts can be identified based on the geometric properties of the convex polygonal shapes under study. The first type of cuts are suitable for the $\mathcal{PMCS}$ problem formulations regarding regular even-sided polygons that have their opposite edges parallel, namely, the *parallel-sided polygons* (Younies, 2004). Let, m be the number of the edges of a parallel-sided polygon. Assume the vertices $V_1, ..., V_m$ are indexed clock-wise as shown in Figure 26.

Lemma 7 *The inequalities:*

$$y_i^r + y_i^{r+(m/2)} \geq 1 \quad \forall i, \ r = 1, ..., m/2 \tag{72}$$

hold for all $\mathcal{PMCS}$ problem formulations regarding parallel-sided polygons.

PROOF Similar argument to those introduced in the proof of Lemma 2. Observe from the binary definitions discussed in Section 5.2 that the binary variables y_i^r and $y_i^{r+(m/2)}$, where $r = 1, ..., m/2$, can never be simultaneously equal to zero in a feasible integer configuration. ■

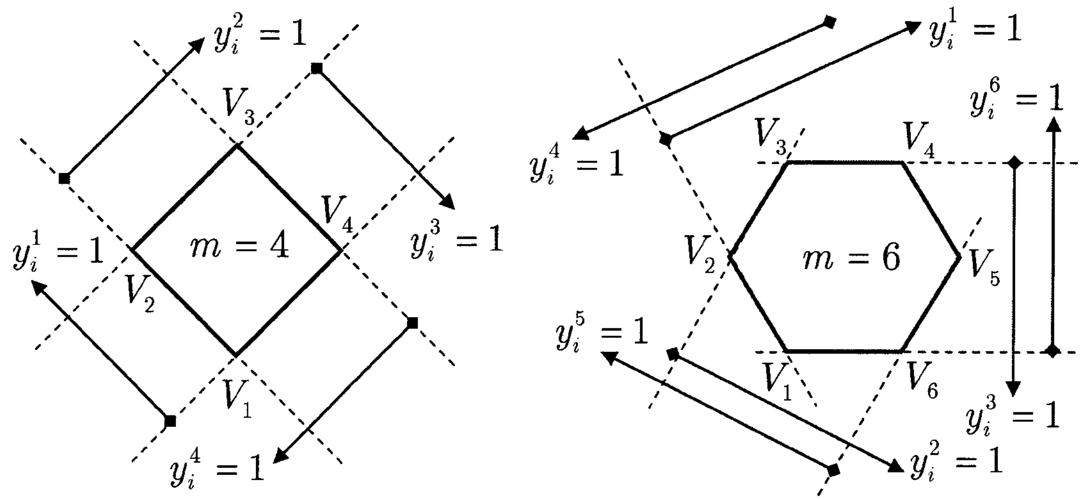

Figure 26. Illustration of Lemma 7

This is illustrated for the cases of rhombus and hexagon in Figure 26. The second group of cuts is related to the binary variables y_i utilized for the covering decision. These cuts are suitable for all regular convex polygonal shapes. The cuts belonging to this group can be found by using the dimensions of the *rectilinear convex hull* of the shape. Denote the rectilinear convex hull of a shape by $\mathcal{H}$. Let the width of $\mathcal{H}$ along X_1 dimension be $\mathcal{W}(\mathcal{H})$. Similarly, let the length of $\mathcal{H}$ along X_2 dimension be $\mathcal{L}(\mathcal{H})$. These are illustrated for some example shapes in Figure 27 where the rectilinear convex hulls are shown as double lines.

Lemma 8 *If one of the arguments: (a)* $|a_{1i} - a_{1j}| > \mathcal{W}(\mathcal{H})$*; (b)* $|a_{2i} - a_{2j}| > \mathcal{L}(\mathcal{H})$ *is* *true for the points i and j, $i \neq j$, then the inequality:*

$$y_i + y_j \leq 1 \tag{73}$$

holds for all $\mathcal{PMCS}$ problem formulations.

PROOF Same argument as the one used in the proof of Lemma 3 applies, therefore omitted. ■

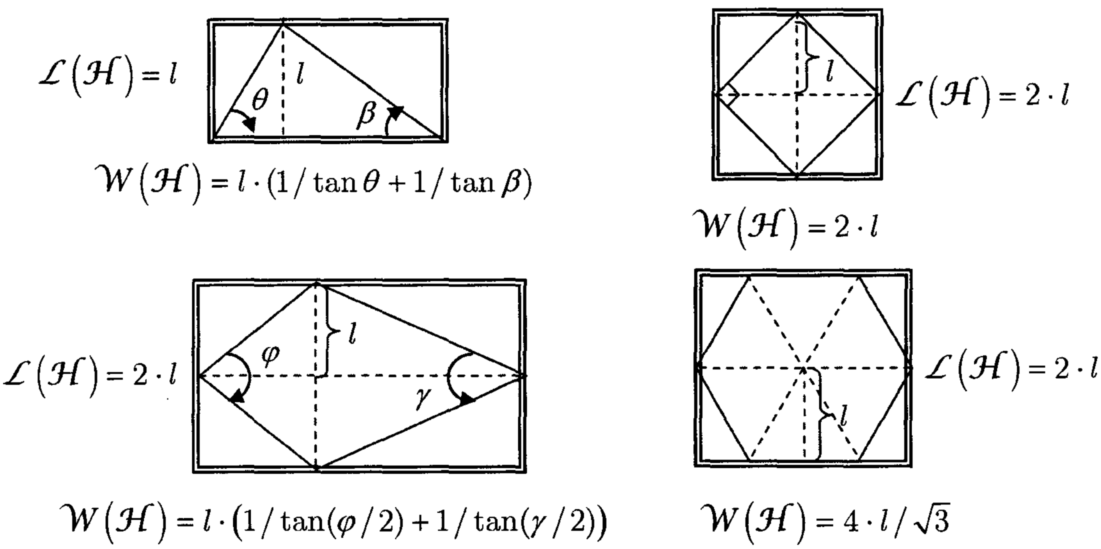

Figure 27. Dimensions of some rectilinear convex hulls

5.4 Numerical example for $\mathcal{PMCS}$ formulations: Example 3

In this section, we illustrate an example planar maximal covering problem (Example 3 hereafter) by considering four different shapes each having an area requirement of 10 unit squares. We used the fixed point set illustrated in Section 4.7.1

Shape	Obj. func.	x_1	x_2	Points covered
Triangle	9	0.905	0.42	{1,2,6,7,8,23,25,40,50}
Rhombus	9	3.662	4.836	{3,5,9,10,16,26,32,39,41}
Quadrilateral	10	4.378	5.426	{3,10,14,16,21,26,29,32,39,41}
Hexagon	10	2.835	8.271	{12,15,18,19,22,27,28,31,44,47}

Table 10. Summary of the solutions for Example 3

with allowing a unit weight $\omega_i = 1$ for each fixed point. The shapes under study are as follows.

1. a triangle with interior angles $\theta = 60°$ and $\beta = 30°$ (see, Figure 27);

2. a regular rhombus;

3. a quadrilateral with interior angles $\varphi = 80°$ and $\gamma = 60°$ (see, Figure 27);

4. a regular hexagon.

We solved the resultant $\mathcal{PMCS}$ problems for these shapes. Corresponding optimal shape locations are shown in Figures 28-29, whereas the solutions are summarized in Table 10 for this small example.

5.5 A branch-and-cut procedure

In this section, we describe a *branch-and-cut* procedure we have developed to solve the $\mathcal{PMCS}$ problems. A typical branch-and-cut algorithm is a combination of *cutting plane* and *branch-and-bound* methods. The method works with generating a series of LP relaxations of the original MILP problem and dynamically adding the cutting planes, whenever they are violated, throughout the branching tree. Then,

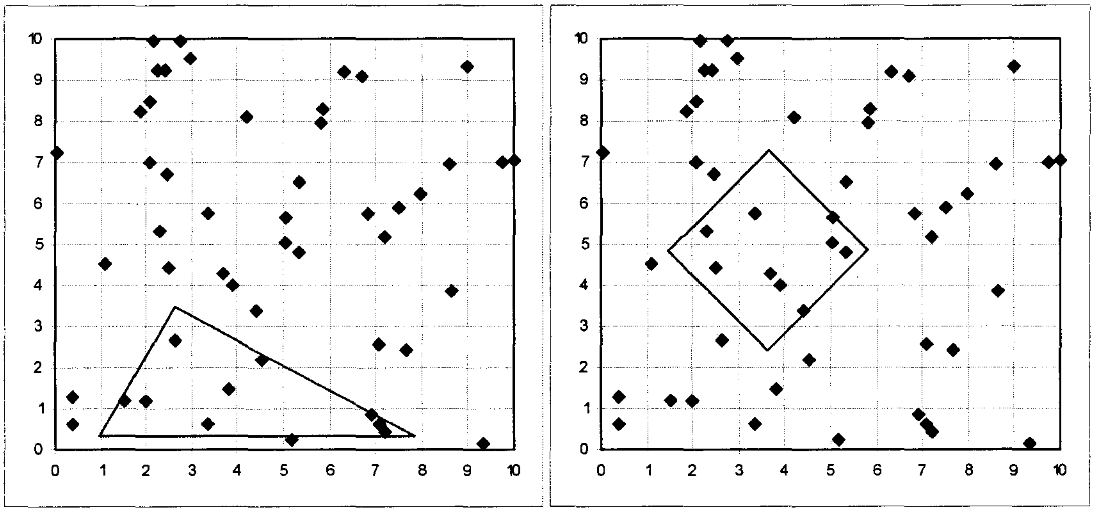

Figure 28. $\mathcal{PMCS}$ solutions for the triangle and rhombus in Example 3

re-optimization of these restricted LP subproblems will potentially provide better bounds than those obtained by a conventional branch-and-bound algorithm.

There are two important issues in the implementation of branch-and-cut method: (1) how to branch, and (2) how to generate the cutting planes. When a restricted LP subproblem is solved, the user has two choices: (1) to improve the LP relaxation with adding a cutting plane, or (2) to split the problem into two new subproblems with branching. In some applications determining the cutting planes that are valid for the original MILP but violated by the upper-level nodes in that branch may require too much computational effort. Thus, in those applications one may choose to continue branching without adding cuts at that level. Another option is to search for cutting planes after a pre-determined number of levels in the branching tree, say, at every four levels of depth. One variant of the branch-and-cut method is referred to as the

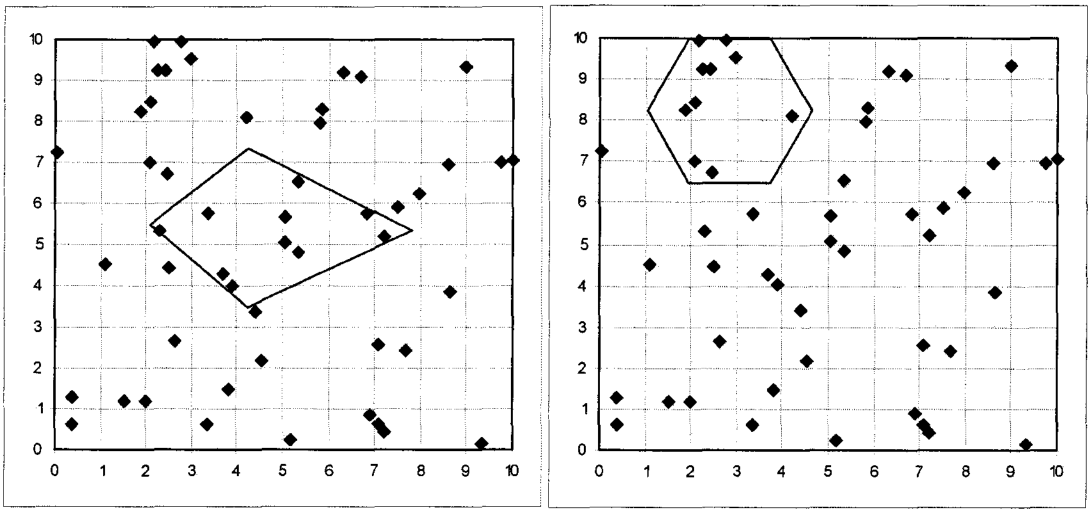

Figure 29. $\mathcal{PMCS}$ solutions for the quadrilateral and hexagon in Example 3

cut-and-branch technique, where the user adds all the cutting planes once at the first node of the branching tree and then proceeds with a branch-and-bound algorithm.

The branch-and-cut procedure we developed may be regarded as a hybrid of branch-and-cut and cut-and-branch approaches. First, we relax all the integrality requirements of a $\mathcal{PMCS}$ problem and obtain its LP relaxation. This problem is denoted by $\mathcal{LP}$. Then, all cuts of type (72) illustrated in Lemma 7 are added to $\mathcal{LP}$ to obtain the first problem of the branching tree. Node selection from the branching tree is done by a *best node first* strategy. That is, among the active problems, we select the one whose optimal value is as large as possible. When a problem is decided to be split into two new problems, we branch for one of the binary variables y_i utilized for the covering decision. The problem is split into two new problems, one with the additional constraint $y_i \geq 1$ (i.e. the point i is covered), and the other having the additional constraint $y_i \leq 0$ (i.e. the point i is left uncovered). Then, all cuts of type (73) illustrated in Lemma 8 are determined by inspecting the fixed point set $\{1, \ldots, N\} \setminus \{i\}$ and then added to both problems.

5.5.1 Notations

p : index for the problems;

f^p : optimal objective function value of the restricted LP problem p;

Ψ : a list keeping track of the active (i.e. non-fathomed) problems;

Ψ^T : a list where two new problems are kept temporarily;

$\psi(p)$: lists keeping track of the fixed points, for which a branching decision was not made at the preceding upper-level problems of that branch (i.e. the branch backtracking from the problem p to the first problem);

$UB(p)$: upper bounds;

$LB(p)$: lower bounds;

LB^* : best lower bound found.

5.5.2 Upper and lower bounds

Let a restricted LP problem p in the branching tree be split into two problems p' including the additional constraint $y_i \geq 1$ and p'' including the additional constraint $y_i \leq 0$. The optimal objective function values of these new problems will constitute their upper bounds. Lower bounds are obtained as shown in Figure 30.

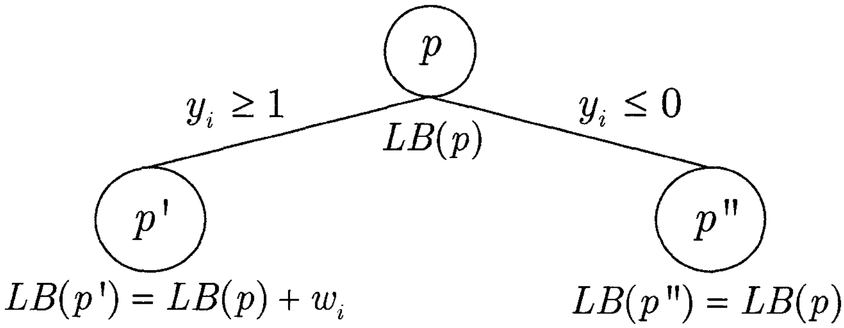

Figure 30. Lower bounding in the branch-and-cut algorithm

5.5.3 Outline of the branch-and-cut algorithm

Step 1 (Initialization).

Given a $\mathcal{PMCS}$ problem to maximize, relax the integrality requirements and obtain the problem $\mathcal{LP}$. Add all cuts of type (72) illustrated in Lemma 7 to $\mathcal{LP}$ and obtain the first problem of the branching tree. The list Ψ only includes the first problem, the list $\psi(1)$ includes all the fixed points. Solve this simple LP problem and set $UB(1) = f^1$, $LB(1) = LB^* = 0$.

Step 2 (Problem generation).

(2.1) From the list Ψ, select the problem p' such that $UB(p') = \max_{p\in\Psi}\{UB(p)\}$, remove p' from Ψ. From the list $\psi(p')$, select the point i^* such that $\omega_{i^*} = \max_{i\in\psi(p')}\{\omega_i\}$, remove i^* from $\psi(p')$.

(2.2) Create two new problems, one with the additional constraint $y_{i^*} \geq 1$, and the other having the additional constraint $y_{i^*} \leq 0$, add these problems to Ψ^T.

Step 3 (Re-optimization and bounding).

(3.1) Add the cuts of type (73) illustrated in Lemma 8 to both problems in Ψ^T.

(3.2) $\forall p \in \Psi^T$, solve the restricted LP problems and set the upper bounds $UB(p) = f^p$, and set the lower bounds as shown in Figure 30.

(3.3) If $\max_{p \in \Psi^T} \{LB(p)\} > LB^*$, update the lower bound $LB^* \longleftarrow \max_{p \in \Psi^T} \{LB(p)\}$.

(3.4) Remove the problems from Ψ^T and add them to Ψ.

Step 4 (Pruning).

$\forall p \in \Psi$: **(4.1-***Prune by infeasibility***)** If p is infeasible; **(4.2-***Prune by bound***)** Else if $UB(p) < LB(p')$, $p' \in \Psi \backslash \{p\}$; **(4.3-***Prune by optimality***)** Else if $LB(p) = UB(p)$; **(4.4-***Prune by integrality***)** Else if p has a feasible integral solution, update the lower bound $LB(p) = UB(p)$ (and LB^* if necessary); fathom p and remove from Ψ.

Step 5 (Check the list).

If $\Psi \neq \varnothing$, return to **Step 2**; else stop with optimal solution LB^*. ♣

Lemma 9 *After each branching decision, identifying the valid cuts of type (73) can be performed in $\mathcal{O}(N)$ computation time.*

PROOF This can be realized by checking for the conditions (a) and (b) of Lemma 8 between the fixed point i^* and each point in the set $\{1, \ldots, N\} \setminus \{i^*\}$. Clearly, it will require $\mathcal{O}(N)$ time. For each point pair (i^*, j), $j \in \{1, \ldots, N\} \setminus \{i^*\}$, checking the conditions (a) and (b) can be performed in linear time which does not add to the overall complexity. ■

Point	a_{1i}	a_{2i}	ω_i
1	4.100095	9.105754	2
2	5.13224	6.291908	7
3	7.572742	3.531016	3
4	1.520224	9.46276	4
5	8.228584	9.77409	6
6	6.916508	1.124133	1
7	9.866785	9.103011	5
8	9.486212	2.295648	9
9	2.705124	1.280016	4
10	5.05719	6.703921	3

Table 11. Data for Example 4

5.6 Numerical example for the branch-and-cut algorithm: Example 4

In this section, we show how the branch-and-cut algorithm works step-by-step on a small example problem (Example 4 hereafter). We considered a 10-point small example problem in a bounded square having a dimension of 10×10 units, where the fixed point locations in each dimension are randomly generated from uniform distribution $U \sim [0, 10]$. Each fixed point i has a weight ω_i. For convenience, we allowed the weights ω_i to be integer values. Thus, the weights were randomly generated from uniform distribution $U \sim [0, 10]$ and rounded to the nearest integer value. The single shape to be located into the afore-mentioned bounded region is a hexagon having an area requirement of 15 unit squares. The data for this example problem is given

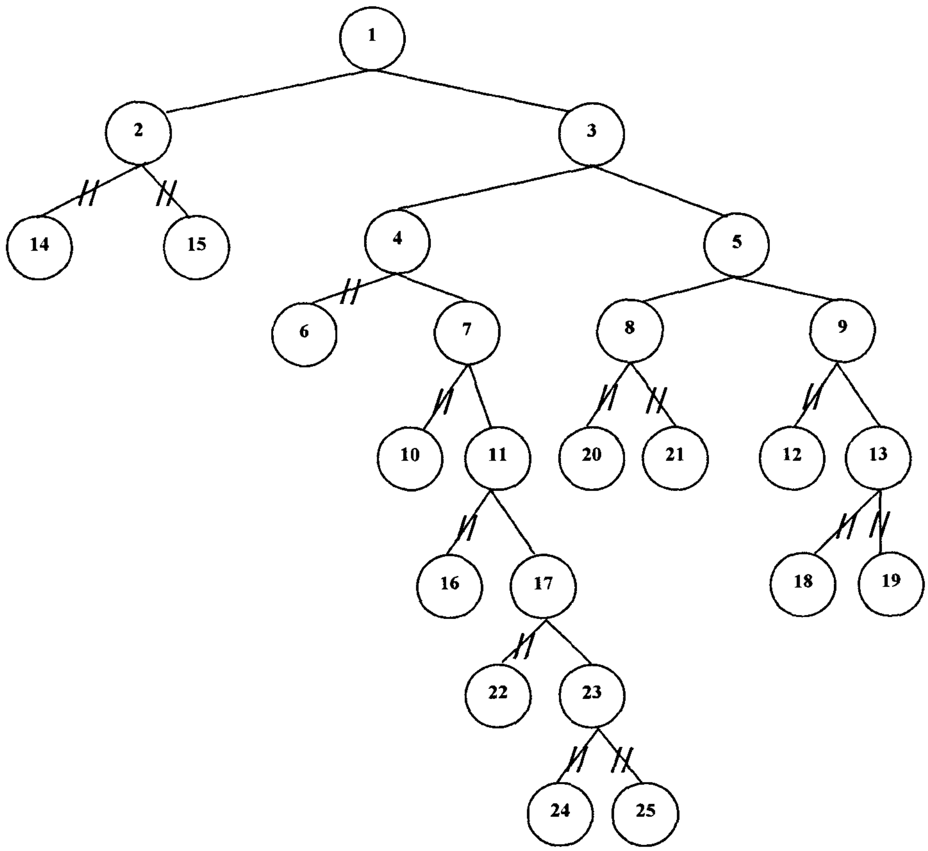

Figure 31. Branching tree for Example 4

in Table 11.

The branching tree for this example is shown in Figure 31, whereas the solutions attained at each node of the branching tree are summarized in Table 12.

According to the results presented in Figure 31 and Table 12, there exist two alternative optimal solutions at the nodes (15) and (24) of the branching tree with an objective function of 13. The solution found at node (15) points out that the hexagon should be located at $(x_1, x_2) = (7.597, 3.186)$ and covering the points {3,6,8}. The

p	Branch	x_1	x_2	$UB(p)$	$LB(p)$	LB^*	Pruned by
1	-	-	-	43.9671	0	0	-
2	$y_8 \geq 1$	7.597	3.186	19.9951	9	9	-
3	$y_8 \leq 0$	6.65	7.693	34.9778	0	9	-
4	$y_2 \geq 1$	6.688	7.759	29.9884	7	9	-
5	$y_2 \leq 0$	6.65	7.693	27.9778	0	9	-
6	$y_5 \geq 1$	-	-	-	-	9	Infeasibility
7	$y_5 \leq 0$	6.47	7.382	23.9890	7	9	-
8	$y_5 \geq 1$	7.188	7.919	15.9975	6	9	-
9	$y_5 \leq 0$	5.797	6.216	21.9805	0	9	-
10	$y_7 \geq 1$	-	-	-	-	9	Infeasibility
11	$y_7 \leq 0$	3.359	7.382	18.9949	7	9	-
12	$y_7 \geq 1$	-	-	-	-	9	Infeasibility
13	$y_7 \leq 0$	3.715	5.612	16.9893	0	9	-
14	$y_2 \geq 1$	-	-	-	-	9	Infeasibility
15	$y_2 \leq 0$	7.597	3.186	13	9	13	Integrality, new $LB(15) = LB^* = 13$
16	$y_4 \geq 1$	-	-	-	-	13	Infeasibility
17	$y_4 \leq 0$	6.371	5.612	14.9984	7	13	-
18	$y_4 \geq 1$	3.032	7.919	8.9997	4	13	Bound, $UB(18) < LB^*$
19	$y_4 \leq 0$	5.072	3.361	12.9953	0	13	Bound, $UB(19) < LB^*$
20	$y_7 \geq 1$	-	-	-	-	13	Infeasibility
21	$y_7 \leq 0$	6.897	7.919	10.9993	6	13	Bound, $UB(21) < LB^*$
22	$y_9 \geq 1$	-	-	-	-	13	Infeasibility
23	$y_9 \leq 0$	6.371	5.612	14.9984	7	13	-
24	$y_3 \geq 1$	6.371	5.612	13	10	13	Integrality, new $LB(24) = 13$
25	$y_3 \leq 0$	3.153	7.025	12	7	13	Bound, $UB(25) < LB^*$

Table 12. Summary of the solutions for Example 4

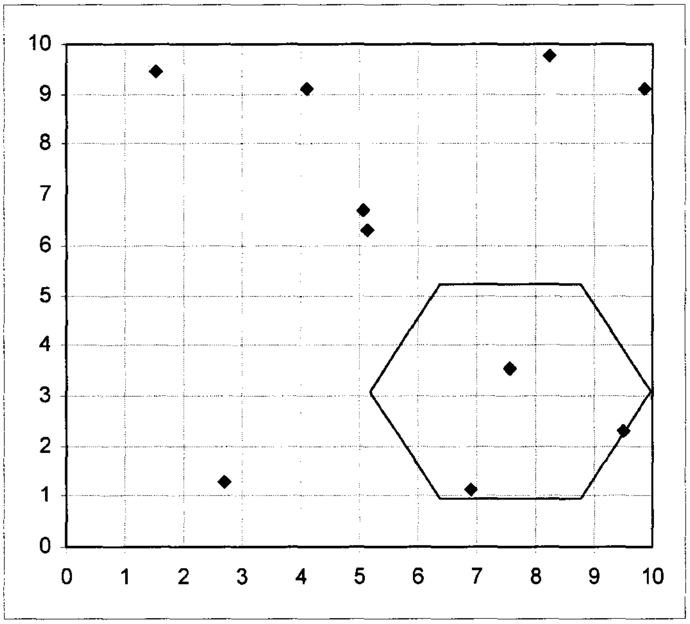

Figure 32. Branch-and-cut algorithm solution found at node (15)

alternative solution found at node (24) points out that the hexagon should be located at $(x_1, x_2) = (6.371, 5.612)$ and covering the points {2,3,10}. These solutions are shown in Figures 32-33, respectively.

5.7 Evaluation of bound quality with cuts

In this section, we illustrate the results of a small computational experiment we conducted to evaluate the bound qualities attained by utilizing the cuts provided in Section 5.3.

For this purpose, we created a set of test problems by using the same setting introduced in Section 5.6 and generated instances with $N = \{3, 5, 10, 20, 50, 75, 100, 200,$

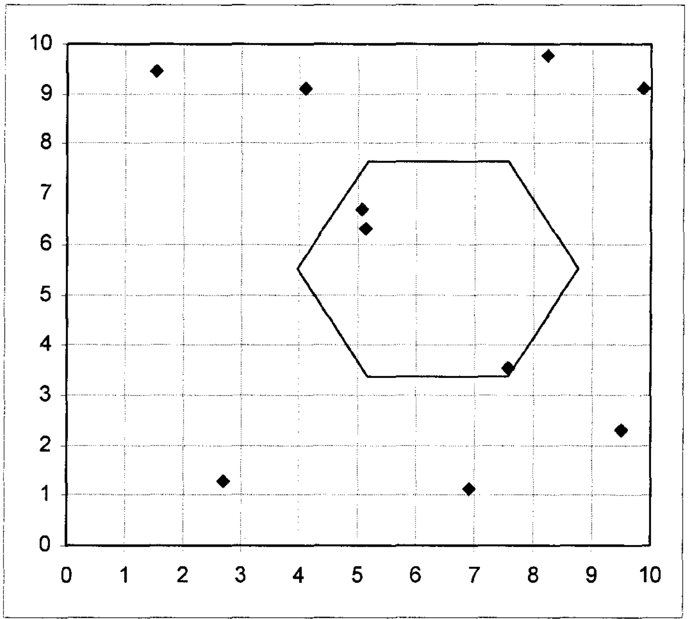

Figure 33. Branch-and-cut algorithm solution found at node (24)

500} fixed points. The coding was implemented in the GAMS modeling system. CPLEX 10 was utilized as the MILP solver and BDMLP was utilized as the LP solver. The results are summarized in Table 13. In this table, MILP stands for the optimal solution of the mixed-integer linear problem, LP stands for the optimal solution of its LP relaxation, and RLP denotes the optimal solution of the restricted LP obtained by adding cuts discussed in Section 5.3 to the LP. The last column is reserved for PI value, the percentage improvement on bounds, obtained by utilizing the cuts.

It is evident from this table that the effect of the cuts on bound quality is very positive, justified by the percentage improvement on bounds given at the last

N	MILP	LP	RLP	PI
3	7	8.9963	7	100%
5	12	22.9881	12	100%
10	24	54.9619	27.99	87.08%
20	33	93.9393	47	77.02%
50	72	263.8102	132	68.72%
75	90	1011.7463	631	41.3%
100	118	1136.6788	805.934	32.46%
200	259	1668.1534	1387.4254	19.92%
500	479	2577.2796	2375.055	9.64%

Table 13. Summary of bound quality

column. The percentage improvement values in this table are calculated as follows:

$$\text{PI} = \frac{\text{LP} - \text{RLP}}{\text{LP} - \text{MILP}}. \tag{74}$$

The performance of the cuts were significant for small problems, yet having a diminishing percentage improvement on bounds as the problem size gets larger. However, even for the largest problem considered here, having $N = 500$ with 3500 discrete variables and 7006 constraints, one can obtain an improvement of 9.64% on bounds, by employing the cuts provided in Section 5.3. The effect of these cuts is illustrated in Figure 34.

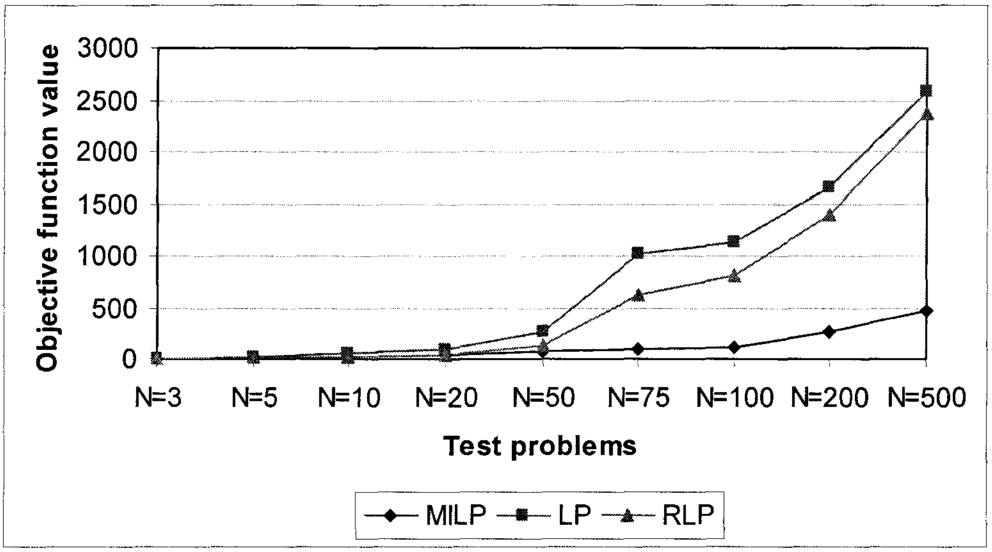

Figure 34. Improvement of bounds by the cuts

Chapter 6
Dynamic planar expropriation problem with single convex polygonal shapes

6.1 Introduction

In this chapter, we study the *dynamic planar expropriation problem with single convex polygonal shapes* ($\mathcal{DPECS}$). In this problem, the shape is expected to function through a planning horizon composed of T time periods indexed by t. The fixed point locations and expropriation costs are allowed to change for each time period. Moreover, the shape is allowed to move at the *beginning* of each time period at a relocation cost of v_t. Hence, reduction in the total expropriation costs is possible by relocating the shape to new locations at some time periods within the planning horizon. However, in this setting, there is an inherent tradeoff associated with the reductions in the total expropriation costs and additional relocation costs. The objective is to find a *location-relocation* strategy, given by the vector of relocations, and corresponding successive locations of the shape through T time periods while minimizing the total expropriation and relocation costs.

A complementary objective of this chapter is to illustrate how thesis problems extend to their diametric opposites as defined in Section 3.2.3, and further to three dimensions. Recall from Section 3.2.3 that the single-period variant of the above-stated problem, the *single-period planar expropriation problem with single convex polygonal shapes* ($\mathcal{SPECS}$), is the diametric opposite of the *planar maximal covering* *problem with single convex polygonal shapes* ($\mathcal{PMCS}$) discussed in Chapter 5. Hence, for studying the *dynamic planar expropriation problem with single convex polygonal shapes* ($\mathcal{DPECS}$), we first illustrate how the $\mathcal{PMCS}$ formulation extends to $\mathcal{SPECS}$.

Moreover, to show how the problem formulation extends to three dimensions, we also formulate another problem, namely the *three-dimensional expropriation problem with single convex polyhedral shapes* ($\mathcal{TDECS}$) in Appendix C. For illustration purposes, the diametric opposite of the *planar expropriation problem with rectangular non-rigid facilities* ($\mathcal{PENR}$) discussed in Chapter 4, the *planar maximal covering problem with rectangular non-rigid facilities* ($\mathcal{PMNR}$), is also formulated in Appendix D.

In the remainder of this chapter, we first formulate $\mathcal{SPECS}$. Then we illustrate how the multi-period extension of $\mathcal{SPECS}$, namely $\mathcal{DPECS}$, decomposes into a series of static problems under a particular vector of relocations. Subsequently, we provide an incomplete dynamic programming procedure for the solution of $\mathcal{DPECS}$. We illustrate the whole procedure on a numerical example.

6.2 Single-period planar expropriation problem with single convex polygonal shapes

For convenience, we introduce the following notation for the constraint set of a $\mathcal{PMCS}$ problem:

$\mathcal{C}$: the set of covering constraints, for example, the constraint set (58-64);

$\mathcal{IC}$: the set of inclusion constraints, for example, the constraint set (65-68);

$\mathcal{IN}$: the set of integrality constraints, for example, the constraint set (69);

$\mathcal{N}$: the set of non-negativity constraints, for example, the constraint set (70).

Then, the $\mathcal{PMCS}$ is given by the following simplified formulation:

$$(\mathcal{PMCS}) \quad \max \left\{ \sum_{i=1}^{N} \omega_i \cdot y_i \ : \ \text{s.t.}\ \mathcal{C},\ \mathcal{IN}\ \ \forall i;\ \ \mathcal{IC},\ \mathcal{N} \right\}. \tag{75}$$

The problem formulation for $\mathcal{SPECS}$ can be generated by substituting the objective function of the $\mathcal{PMCS}$ defined for a particular m-sided convex polygonal shape, with its diametric opposite:

$$\min \sum_{i=1}^{N} c_i \cdot y_i \tag{76}$$

and replacing the individual covering constraint set

$$\frac{1}{m}\left(\sum_{r=1}^{m} y_i^r\right) \geq y_i \qquad \forall i \tag{77}$$

with the following constraint set:

$$\sum_{r=1}^{m} y_i^r \leq (m-1) + y_i \qquad \forall i \tag{78}$$

whereas the other remaining restrictions should be generated according to the particular shape, as shown in Appendix B. Observe that, when the objective function of $\mathcal{PMCS}$ is substituted with its diametric opposite (76), the constraint set (77) becomes redundant. To illustrate this, consider the vector $\mathbf{y} = (y_i)^T = (0, ..., 0)^T$. Clearly, this vector is a feasible solution with an objective function value of 0, because the optimization direction is to *minimize* in (76). In this case, note that constraint set (77) is not binding because, for any possible values of y_i^r, we have:

$$\frac{1}{m}\left(\sum_{r=1}^{m} y_i^r\right) \geq 0 \qquad \forall i. \tag{79}$$

Therefore, regardless of the input, if we use the constraint set (77) we will obtain the solution 0. Recall from Section 5.2 that, the rule for implementing coverage is given by the following:

$$\textbf{If} \quad y_i^r = 1 \ \ \forall r, \quad \textbf{then} \quad y_i = 1. \tag{80}$$

Therefore, for applying this rule in $\mathcal{SPECS}$, we have to replace the constraint set (77) with (78). Now, suppose that the constraint set (78) is substituted for (77) in $\mathcal{C}$ to obtain the covering constraints $\mathcal{C}'$. Then, the $\mathcal{SPECS}$ problem is given by the following program:

$$(\mathcal{SPECS}) \qquad \min \quad \left\{ \sum_{i=1}^{N} c_i \cdot y_i \ : \ \text{s.t. } \mathcal{C}',\ \mathcal{IN} \ \ \forall i; \ \ \mathcal{IC},\ \mathcal{N} \right\}. \tag{81}$$

6.3 Dynamic planar expropriation problem with single convex polygonal shapes

Consider the dynamic setting introduced in Sections 3.2.3 and 6.1, and let the constraints of the $\mathcal{SPECS}$ problem formulation (81) be additionally indexed for the time periods t. Denote the location of the shape at time period t by $\mathbf{x}_t = (x_{1t}, x_{2t})$, and let z_t be a binary variable taking a value of 1 if the shape is relocated at the beginning of period t, and 0 otherwise. That is, we define:

$$z_t = \begin{cases} 1, & \text{if } \ \mathbf{x}_t \neq \mathbf{x}_{t-1}, \\ 0, & \text{if } \ \mathbf{x}_t = \mathbf{x}_{t-1}. \end{cases} \tag{82}$$

Let the constraint set $z_t \in \{0, 1\}$ be denoted by $\mathcal{Z}$. Moreover, for using this binary definition, we define the following constraint set denoted by $\mathcal{U}$:

$$\frac{|x_{1t} - x_{1,t-1}|}{U^{X1} - L^{X1}} \leq z_t, \tag{83}$$

$$\frac{|x_{2t} - x_{2,t-1}|}{U^{X2} - L^{X2}} \leq z_t. \tag{84}$$

This constraint set will ensure the utilization of relocation costs in the objective function. According to this set, if the location of the shape is different than its location in the preceding period (i.e. at least one of the statements $|x_{1t} - x_{1,t-1}| > 0$ or $|x_{2t} - x_{2,t-1}| > 0$ holds) corresponding binary variable is forced to be one. The denominators of (83-84) are the width and length of the feasible location space, respectively. These will ensure that the LHS values of (83-84) are smaller than one (i.e. can not be equal to one because of the inclusion constraints).

Finally, let the planning horizon be composed of T periods. Then, the *dynamic planar expropriation problem with single convex polygonal shapes* is given by the following program:

$$(\mathcal{DPECS}) \qquad \min \quad \left\{ \; g(1) \equiv \sum_{t=1}^{T} \sum_{i=1}^{N} c_{it} \cdot y_{it} \; + v_t \cdot z_t \; : \; \text{s.t.} \right.$$

$$\left. \mathcal{C}', \, \mathcal{IN} \;\; \forall i,t; \quad \mathcal{IC}, \, \mathcal{N}, \, \mathcal{U}, \, \mathcal{Z} \;\; \forall t \right\}. \tag{85}$$

For now, we leave the discussion of $g(1)$ appearing in the above formulation to Section 6.5.2.

6.4 Decomposition of $\mathcal{DPECS}$ to static problems

In this section, we illustrate that the $\mathcal{DPECS}$ problem can be decomposed into a set of *static problems* where the shape functions through multiple periods at the same location. Let $\mathbf{z} = (z_t)^T$ be the vector of relocations. Recall from Assumption 3 that, we allow any $\mathbf{z}$ vector to start with a one in this study. That is, the shape is located into the feasible space for the first time at the beginning of the first period.

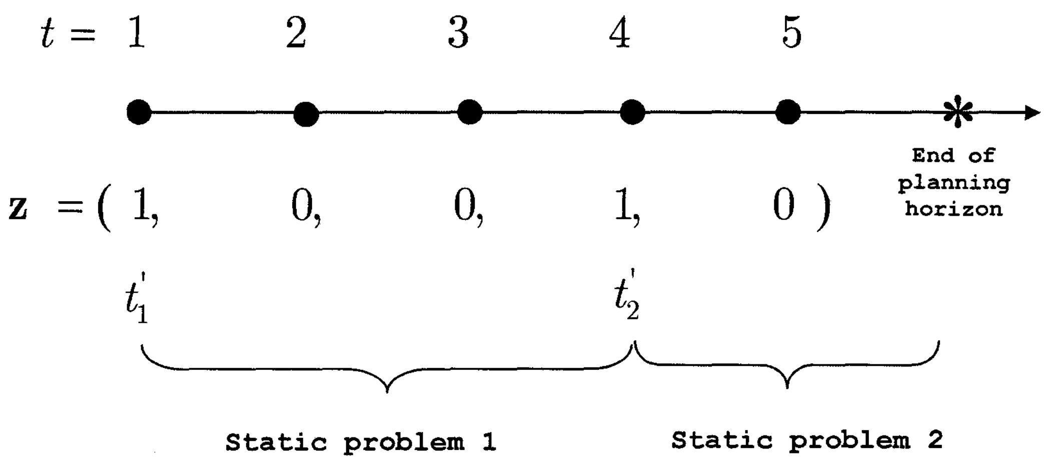

Figure 35. Illustration of the decomposition of $\mathcal{DPECS}$ to static problems

Consider the small example with five time periods illustrated in Figure 35. Let Q be the number of ones in the $\mathbf{z}$ vector. Then, it is easy to see that the number of different locations for the shape throughout the planning horizon also equals to Q. Hence, under a particular $\mathbf{z}$, problem (85) is composed of a set of static problems where the shape functions through multiple periods at the same location. Let q be the index for such problems. Every static problem q starts with the beginning of a time period t'_q and ends with the beginning of a time period t'_{q+1} as shown in Figure 35. Denote the location of the shape during the planning period $[t'_q, t'_{q+1})$ of the static problem q by $\mathbf{x}_q = (x_{1q}, x_{2q})$ and let the shape locations $\mathbf{x}_t = (x_{1t}, x_{2t})$ in the formulation (85) be replaced by $\mathbf{x}_q = (x_{1q}, x_{2q})$. Then the static problems $\mathcal{SP}(q)$ are given by the following formulation:

$$(\mathcal{SP}(q)) \qquad \min \left\{ f_q \equiv v_{t'_q} + \sum_{t=t'_q}^{(t'_{q+1})-1} \sum_{i=1}^{N} c_{it} \cdot y_{it} \ : \ \text{s.t. } \mathcal{C}',\ \mathcal{IN}\ \ \forall i, \right.$$

$$\left. \forall t \in [t'_q, t'_{q+1});\ \ \mathcal{IC},\ \mathcal{N},\ \ \forall t \in [t'_q, t'_{q+1}) \right\}. \qquad (86)$$

Therefore, under a particular $\mathbf{z}$ vector, the solutions of Q static problems define the optimal shape locations $\mathbf{x}_q = (x_{1q}, x_{2q})$ through the planning horizon. Moreover, the optimal objective function value of the $\mathcal{DPECS}$ is given by the following term:

$$g(1) = \sum_{q=1}^{Q} f_q. \qquad (87)$$

6.5 Two basic solution methods

The decomposition scheme discussed in Section 6.4 leads to the following important observation.

Remark 1 (Wesolowsky, 1973). *The optimal solution of $\mathcal{DPECS}$ can be identified by evaluating the total costs associated with every possible $\mathbf{z}$ vector.* ♣

In light of this observation, we now discuss the solution of $\mathcal{DPECS}$ with two simple procedures, namely, *complete enumeration* and *enumeration with postponement.* Subsequently, we explain an *incomplete dynamic programming procedure* which is superior to these methods in terms of *tree size.*

6.5.1 Complete enumeration

In this scenario one may compute the total costs of expropriations and relocations associated with every possible $\mathbf{z}$ vector and simply select the minimum cost. In that case, there exist 2^{T-1} such vectors to evaluate. Wesolowsky (1973) pointed out that each candidate $\mathbf{z}$ vector calls for the solution of $1 + (T-1)/2$ static problems, *on average.* Hence, the total number of static problems that have to be solved by using complete enumeration is given by the following tree size:

$$TS(\mathcal{CE}) = 2^{T-1}\left(1 + \frac{T-1}{2}\right). \tag{88}$$

6.5.2 Enumeration with postponement

In this section we illustrate a better enumeration method where one starts from the last period and calculates the optimal solution of the dynamic problem by working backwards to the first period. In this method the time periods represent the *stages.* At each stage t, the open nodes in the branching tree are divided to two new nodes with labels 0 and 1, representing the two possible values of the binary variable

z_t. But, the nodes labeled with 0 can not be evaluated at the stages they appear. Hence, the evaluation of such nodes are postponed to a lower stage, where the shape is relocated at the beginning, as illustrated in Figure 36. Before going through the calculations, we introduce the following notation.

$f[t', t'')$: the total expropriation cost associated with maintaining the shape at a particular location from the beginning of time period t' to the beginning of time period t'';

$g(t')$: the minimum total cost incurred from the beginning of time period t' to the end of the planning horizon, given that the shape is relocated at the beginning of time period t'.

Hence, according to the above notation, $g(1)$ is the optimal objective function value of $\mathcal{DPECS}$. The first cost term $f[t', t'')$ can be found by solving the modified static problem (89).

$$(\mathcal{MSP}) \quad \min \quad \left\{ f[t', t'') \equiv \sum_{t=t'}^{t''-1} \sum_{i=1}^{N} c_{it} \cdot y_{it} \ : \ \text{s.t. } \mathcal{C}', \ \mathcal{IN} \ \forall i, \right.$$

$$\left. \forall t \in [t', t''); \ \mathcal{IC}, \ \mathcal{N}, \ \forall t \in [t', t'') \right\}. \tag{89}$$

Consider the enumeration scenario illustrated in Figure 36. The numbers inside the brackets appearing below the labels are used for indexing the nodes that have to be evaluated at corresponding stages. If a relocation is specified by the $\mathbf{z}$ vector for stage t', the total cost $g(t')$ can be calculated by summing the cost of relocation

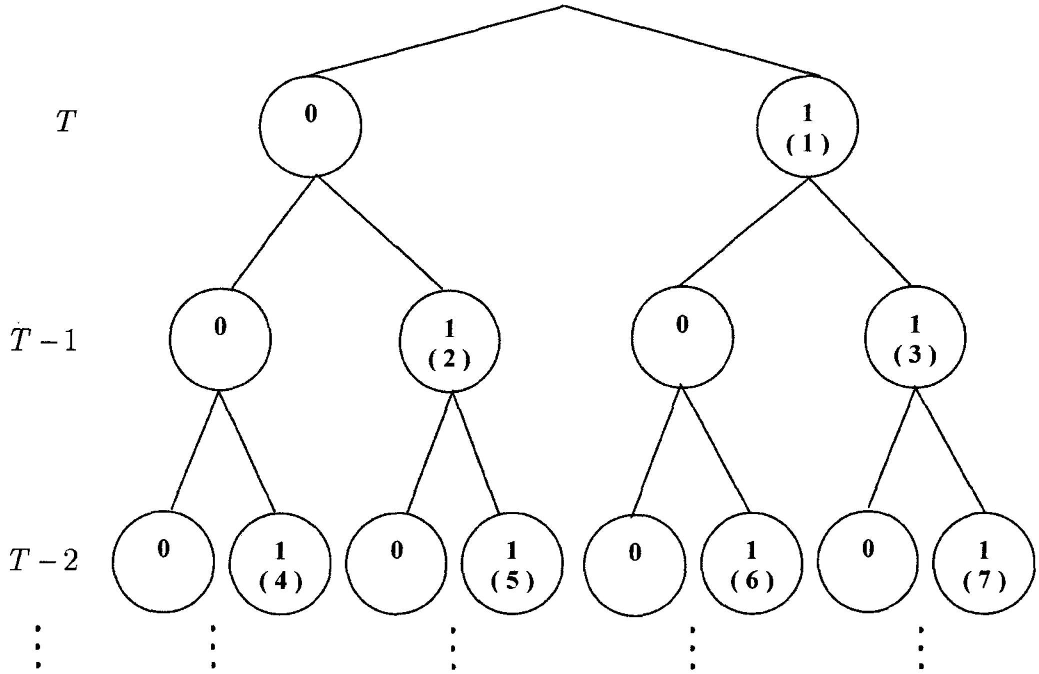

Figure 36. Branching tree for the enumeration with postponement

$v_{t'}$ at the beginning of this stage, the total expropriation cost $f[t', t'')$ associated with maintaining the shape at the same location for $(t'' - t')$ time periods, and the total cost $g(t'')$ of stage t''. Since the planning horizon is composed of T stages only, one may use notation $f[t', *)$ for the total expropriation costs associated with maintaining the shape at a particular location from the beginning of time period t' to the end of planning horizon, and assume $g(*) = 0$ for initiating the calculations. For illustration purposes, calculation of costs for the last three stages are summarized in Table 14, where the evaluation of the nodes labeled with 0 are postponed to lower stages and $g(*)$ were ignored.

It is recognizable from Figure 36 that the enumeration procedure with post-

Node	**Total cost**
1	$g(T) = f[T, *) + v_T$
2	$g(T-1) = f[T-1, *) + v_{T-1}$
3	$g(T-1) = f[T-1, T) + v_{T-1} + g(T)$
4	$g(T-2) = f[T-2, *) + v_{T-2}$
5	$g(T-2) = f[T-2, T-1) + v_{T-2} + g(T-1)$
6	$g(T-2) = f[T-2, T) + v_{T-2} + g(T)$
7	$g(T-2) = f[T-2, T-1) + v_{T-2} + g(T-1)$
⋮	⋮ ⋮

Table 14. Calculations for the enumeration with postponement

ponement leads to evaluation of 2^{T-t} nodes at each stage t. Hence, the total number of modified static problems that have to be solved by using enumeration with postponement is given by the following tree size:

$$TS(\mathcal{EWP}) = \sum_{t=1}^{T} 2^{T-t}. \tag{90}$$

6.6 An incomplete dynamic programming procedure

In this section, we formulate an incomplete dynamic programming procedure which leads to significant reduction in the tree size. This procedure is based on the following important observation regarding dynamic location problems having a similar structure to $\mathcal{DPECS}$.

Remark 2 (Wesolowsky, 1973). *Once a relocation is specified by the* $\mathbf{z}$ *vector, the subsequent locations of the shape are independent from the previous ones. As a con-*

clusion, many of the modified static problems evaluated through the branching tree are identical. ♣

To illustrate this, consider the nodes (5) and (7) of the branching tree illustrated in Figure 36. It is easy to see from Table 14 that the calculation of total costs $g(T-2)$ for these nodes is exactly the same, and hence need not to be duplicated. This is because the modified static problems evaluated at the nodes (5) and (7) are identical.

This observation leads to an incomplete dynamic programming procedure having the following properties:

Property 1. At each stage, evaluation of all nodes is not necessary. By Remark 2, one can simply proceed from the last stage to the first stage by keeping the minimum stage cost $g(t)$ and the corresponding node. The other nodes labeled with 1 at the same stage are pruned.

Property 2. The nodes labeled with 0 can not be evaluated at the stages they appear and their evaluation are postponed to lower stages. Hence, the dynamic programming procedure is deemed as *incomplete.*

Property 3. The ties between costs at any stage are broken arbitrarily. ♣

By using Properties 1-3 and the definitions given in Section 6.5.2, the *backward recursion* for the incomplete dynamic program can be formulated as follows, where the star in the superscript denotes the minimum stage cost.

$$g^{\star}(t') = \min_{t',t''} \left\{ f[t',t'') + v_{t'} + g^{\star}(t'') \right\}. \tag{91}$$

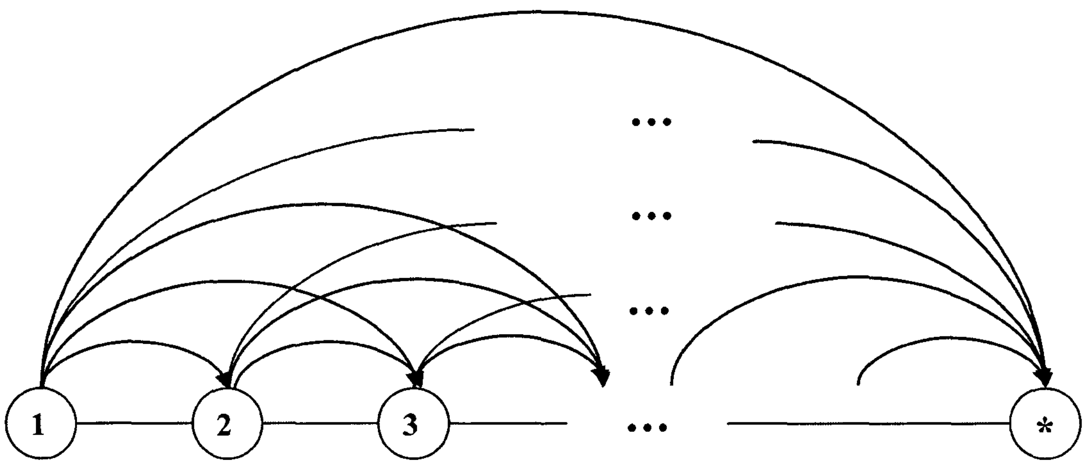

Figure 37. Network representation of the dynamic recursion (91)

A network representation of this dynamic recursion is illustrated in Figure 37. For illustration purposes, we provide an example branching tree for the incomplete dynamic programming procedure in Figure 38. It is recognizable from this figure that the incomplete dynamic programming procedure leads to evaluation of $(T - t + 1)$ nodes at each stage t.

Therefore, the total number of modified static problems that have to be solved by using the incomplete dynamic programming procedure is given by the following tree size:

$$TS(\mathcal{IDP}) = \sum_{t=1}^{T} T - t + 1. \tag{92}$$

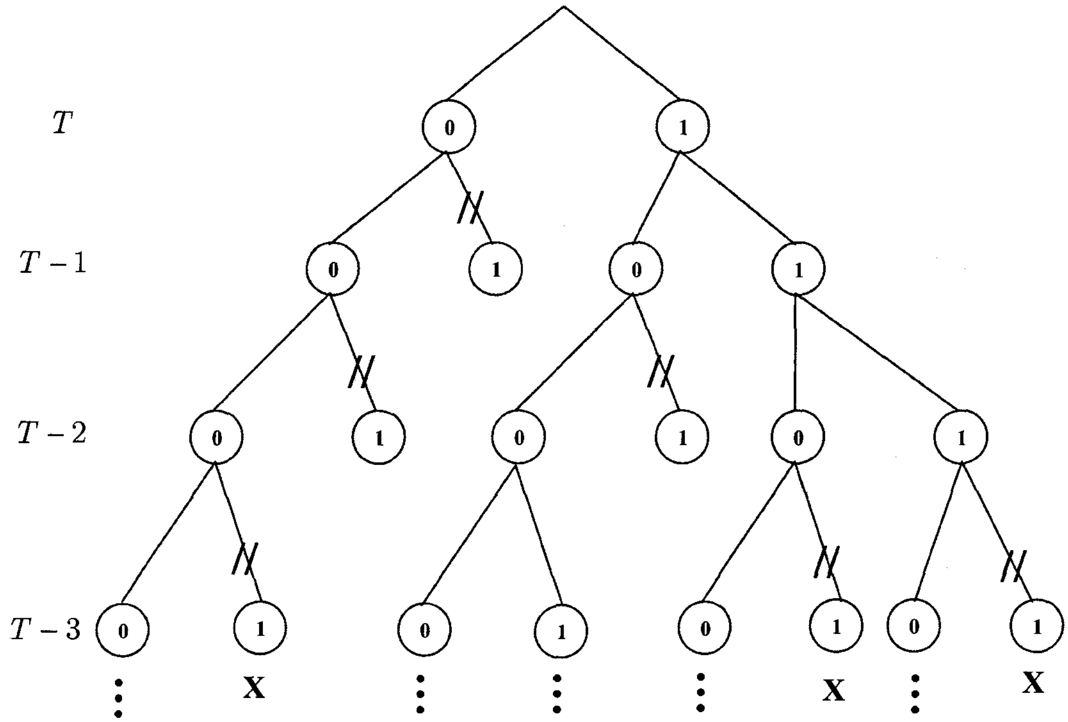

Figure 38. Branching tree for the incomplete dynamic programming procedure

6.7 Growth of the branching trees

In this section, we provide a simple simulation to illustrate the superiority of the incomplete dynamic programming to two procedures introduced in Sections 6.5.1 and 6.5.2, respectively. The criterion we may use for this comparison is the growth of resultant branching trees.

For this purpose, by using the formulas (88, 90 and 92), we created the resultant tree sizes for each method, where the number of periods ranging from 1 to 15 were considered. The results are summarized in Table 15.

T	$TS(\mathcal{CE})$	$TS(\mathcal{EWP})$	$TS(\mathcal{IDP})$
1	1	1	1
2	3	3	3
3	8	7	6
4	20	15	10
5	48	31	15
6	112	63	21
7	256	127	28
8	576	255	36
9	1280	511	45
10	2816	1023	55
11	6144	2047	66
12	13312	4095	78
13	28672	8191	91
14	61440	16383	105
15	131072	32767	120

Table 15. Growth of the tree sizes

This table shows that when one uses the incomplete dynamic programming for the solution of $\mathcal{DPECS}$, the growth of the branching tree is very slow compared to those resulting from using complete enumeration and enumeration with postponement.

6.8 Numerical example for the incomplete dynamic programming procedure: Example 5

In this section we show step-by-step how the incomplete dynamic programming procedure works on an illustrative example problem (Example 5 hereafter). We considered $T = 5$ time periods, $N = 50$ fixed points. The feasible location space for this problem is a bounded square having a dimension of 10×10 units. The single shape to be located into this region is a square having an area requirement of 16 unit squares. For each time period, the fixed point locations in each dimension were randomly generated from a uniform distribution $U \sim [0, 10]$. Each fixed point i has an expropriation cost c_{it}, and each time period has a relocation cost v_t. For convenience, we allowed the cost parameters to assume integer values. Thus, the expropriation costs were randomly generated from uniform distribution $U \sim [1, 10]$, the relocation costs were randomly generated from uniform distribution $U \sim [10, 20]$ and rounded to the nearest integer values. The resultant relocation costs were $\mathbf{v} = (18, 13, 15, 14, 17)$, whereas the remaining data are summarized in Tables 16-20.

The calculations for stage costs are detailed as in the following, whereas the solutions for modified static problems and corresponding shape locations are summarized in Table 21. Moreover, the branching tree for this example is illustrated in Figure 39. Similarly to the branching tree presented in Figure 36, the numbers appearing in the brackets at each node are used for indexing the static problems that have to be evaluated.

Cost calculation for Stage 5.

$g^\star(5) = \min \ \{(1): \ g(5) = f[5, *) + v_5 = 13 + 17 = 40$

Hence, the minimum stage cost is $g^\star(5) = 40$.

Point	a_{1it}	a_{2it}	c_{it}	**Point**	a_{1it}	a_{2it}	c_{it}
1	5.3131	2.1249	2	**26**	4.2652	3.8075	9
2	2.3131	7.4028	6	**27**	0.0337	9.6059	9
3	2.4730	0.9106	10	**28**	9.6194	7.9781	4
4	0.4086	1.0068	8	**29**	6.6348	1.1239	4
5	5.5931	9.2531	5	**30**	2.2893	6.9720	2
6	4.2585	9.2595	9	**31**	1.1792	9.0421	8
7	6.2287	6.5887	8	**32**	2.6719	7.3063	7
8	0.0462	4.6570	1	**33**	3.7869	4.4114	4
9	4.0417	3.0214	5	**34**	9.3372	3.1089	6
10	0.6118	7.3829	4	**35**	6.5898	1.1067	7
11	6.8407	2.6123	3	**36**	5.7052	6.3358	9
12	3.9861	5.7496	1	**37**	6.7344	8.2831	10
13	5.7070	0.4412	4	**38**	4.3088	5.1449	2
14	7.7411	8.7576	5	**39**	7.3151	2.3579	4
15	7.0338	3.6039	4	**40**	9.0863	5.0667	8
16	2.9477	0.5435	3	**41**	1.7781	2.2098	5
17	2.4738	0.5412	4	**42**	9.4691	4.6461	1
18	5.4241	3.3759	1	**43**	9.6341	8.8944	2
19	4.0217	1.8569	2	**44**	7.1215	7.6343	8
20	4.6022	0.7175	2	**45**	6.0638	2.6975	9
21	6.8804	2.5118	5	**46**	6.8298	9.2066	7
22	0.5608	8.8659	7	**47**	8.6332	1.4285	2
23	4.3284	8.6787	7	**48**	3.3125	6.0280	6
24	2.8194	8.1749	8	**49**	7.1409	4.3300	9
25	3.3317	3.3587	3	**50**	9.8905	8.3660	6

Table 16. Data for t=1 in Example 5

Point	a_{1it}	a_{2it}	c_{it}	**Point**	a_{1it}	a_{2it}	c_{it}
1	0.7229	2.9931	9	**26**	1.3180	0.9827	9
2	8.5938	5.6570	2	**27**	0.3808	5.1371	9
3	4.1685	8.5107	2	**28**	5.7280	6.6686	4
4	7.0171	1.9008	8	**29**	1.7166	1.7053	9
5	9.7823	5.8519	6	**30**	6.6596	4.1346	1
6	9.7887	8.9421	3	**31**	5.1922	4.2614	9
7	7.6065	1.8952	5	**32**	5.0416	5.9943	9
8	7.2161	5.5154	6	**33**	1.4903	3.9423	3
9	0.1835	3.2950	7	**34**	9.3368	3.9885	4
10	2.2009	6.4509	3	**35**	4.6783	7.9240	5
11	1.0060	5.0481	1	**36**	3.5225	8.8277	2
12	5.4374	5.5775	8	**37**	7.4105	8.7820	7
13	2.3022	3.3937	4	**38**	2.2777	6.1545	4
14	3.8120	1.7377	6	**39**	1.4539	9.8601	3
15	4.3132	0.4178	8	**40**	5.7436	3.6288	7
16	3.9090	5.3606	8	**41**	6.6565	6.9485	9
17	1.8865	9.3676	3	**42**	2.6008	8.3729	7
18	3.4288	1.3588	7	**43**	6.4088	1.3820	5
19	0.5948	9.4144	3	**44**	7.9421	7.7133	1
20	6.8792	3.9242	3	**45**	8.2669	9.5273	10
21	1.8644	7.7153	3	**46**	3.0472	5.5929	7
22	9.0988	3.7740	6	**47**	3.7243	1.2895	4
23	8.9576	3.0338	1	**48**	4.9564	9.0639	5
24	9.2399	9.4691	8	**49**	6.8889	0.6967	7
25	0.4115	9.7737	8	**50**	3.0842	7.8151	1

Table 17. Data for t=2 in Example 5

Point	a_{1it}	a_{2it}	c_{it}	**Point**	a_{1it}	a_{2it}	c_{it}
1	8.3973	3.5052	1	**26**	6.6013	4.4643	10
2	3.6705	0.3809	1	**27**	0.6364	8.3702	6
3	1.7119	0.0964	3	**28**	8.8341	9.7870	3
4	1.4696	0.3142	6	**29**	0.9471	3.3306	4
5	2.6998	6.4781	2	**30**	9.8089	8.0990	8
6	5.4400	4.9912	8	**31**	1.2959	8.0681	5
7	6.6448	5.2124	3	**32**	8.5793	8.2772	8
8	8.8598	3.6934	2	**33**	7.1196	6.3286	9
9	1.4874	9.9703	10	**34**	4.1123	2.9296	9
10	9.8204	4.9373	2	**35**	4.5094	8.1850	3
11	8.9136	7.1783	6	**36**	9.6174	3.8328	1
12	2.6272	7.8685	1	**37**	2.8576	4.5421	5
13	9.1916	6.0451	10	**38**	2.2912	6.8460	1
14	5.0074	2.4305	2	**39**	2.8019	4.5449	10
15	8.3111	8.4816	5	**40**	5.4616	4.3011	1
16	7.8978	6.0293	10	**41**	5.8856	7.5435	4
17	8.0338	7.0316	4	**42**	8.7177	5.0731	7
18	2.9073	4.8028	1	**43**	6.5270	5.1143	5
19	6.8024	3.5004	1	**44**	0.8505	7.5984	7
20	6.6215	0.3929	10	**45**	1.7207	9.0334	3
21	8.3503	1.1876	1	**46**	1.7515	7.3203	1
22	3.6646	7.4903	5	**47**	6.2536	5.7099	3
23	8.8884	7.1872	1	**48**	6.1589	3.2499	1
24	6.7075	2.0375	2	**49**	8.6599	2.6919	3
25	9.6579	0.7950	6	**50**	3.0162	7.1591	3

Table 18. Data for t=3 in Example 5

Point	a_{1it}	a_{2it}	c_{it}	**Point**	a_{1it}	a_{2it}	c_{it}
1	0.9205	7.2253	7	**26**	2.2454	9.6024	2
2	3.2645	2.8705	9	**27**	9.0400	0.4854	2
3	1.7204	1.4834	2	**28**	0.4988	4.8629	5
4	4.4962	3.7819	3	**29**	0.9635	1.8681	2
5	3.1373	0.1854	3	**30**	7.4487	6.3984	6
6	5.4758	5.5741	5	**31**	3.4519	6.5403	6
7	9.3239	7.7727	4	**32**	0.1204	5.8647	2
8	4.4472	4.9049	9	**33**	6.8356	5.2839	6
9	9.5073	3.3872	6	**34**	9.0512	6.1188	10
10	7.8733	1.7481	5	**35**	1.4886	0.8003	4
11	8.5877	0.3683	2	**36**	8.2898	1.5819	5
12	2.5419	3.8793	2	**37**	7.9190	0.0656	1
13	0.3002	9.0869	2	**38**	2.5848	3.5863	4
14	4.5547	9.3823	9	**39**	7.9380	5.1603	2
15	0.1690	6.9516	7	**40**	7.7527	0.8437	6
16	9.8388	5.6006	2	**41**	9.0232	4.3810	4
17	2.0594	2.0325	7	**42**	3.2531	0.6750	1
18	1.1682	3.2210	7	**43**	5.7634	0.0143	3
19	0.9739	1.3922	8	**44**	1.8080	6.4271	1
20	0.8458	9.8848	9	**45**	2.2489	0.1706	5
21	7.9128	6.7092	2	**46**	0.4603	1.5178	8
22	5.7312	8.7731	3	**47**	1.1402	6.8447	9
23	2.1400	1.1579	4	**48**	3.5805	7.3916	7
24	4.5868	5.8127	9	**49**	6.5248	7.7540	1
25	2.4055	2.0883	10	**50**	6.4074	2.0094	5

Table 19. Data for t=4 in Example 5

Point	a_{1it}	a_{2it}	c_{it}	Point	a_{1it}	a_{2it}	c_{it}
1	2.1494	1.3188	2	**26**	9.2521	9.2642	2
2	6.6299	6.9851	2	**27**	7.0735	9.3104	3
3	5.8876	0.4804	6	**28**	7.8821	3.4189	5
4	4.7741	9.5794	1	**29**	2.2196	3.3809	9
5	3.4921	8.8859	9	**30**	2.2417	3.1621	1
6	9.0517	8.9086	7	**31**	8.2585	5.6151	3
7	6.5069	5.3752	6	**32**	5.6196	8.0794	2
8	5.1838	7.7574	3	**33**	0.9247	1.9611	3
9	4.2920	5.9613	4	**34**	1.9327	1.4752	3
10	8.7851	6.6734	5	**35**	5.9226	8.1122	1
11	1.7137	7.9524	4	**36**	5.0347	1.6783	6
12	9.4090	5.4619	1	**37**	4.9742	0.3660	5
13	4.5441	7.0788	1	**38**	4.1373	5.9378	8
14	7.5190	8.1686	1	**39**	7.3355	2.0089	6
15	8.0748	0.5752	8	**40**	8.8085	2.8719	3
16	2.4401	2.3993	9	**41**	4.0171	3.2395	10
17	7.6454	5.0886	4	**42**	9.3335	6.1922	7
18	7.1876	0.3813	5	**43**	7.5718	5.2781	3
19	4.2254	1.9125	7	**44**	8.4537	1.8901	9
20	8.0015	3.9024	10	**45**	2.4243	4.0314	3
21	2.0361	2.1039	6	**46**	5.9333	4.0061	6
22	0.1139	3.4882	5	**47**	5.7059	6.5739	10
23	2.5545	4.9787	5	**48**	4.6113	6.3931	10
24	4.2787	1.6817	9	**49**	9.0177	3.6135	8
25	3.2078	5.4074	5	**50**	4.1507	6.3275	3

Table 20. Data for t=5 in Example 5

q	**Solution**	x_{1q}	x_{2q}
1	$f[5,*) = 13$	2.000	5.448
2	$f[4,*) = 34$	6.774	7.813
3	$f[4,5) = 8$	5.580	2.800
4	$f[3,*) = 63$	6.279	2.844
5	$f[3,4) = 4$	2.000	2.381
6	$f[3,5) = 24$	5.264	2.393
7	$f[2,*) = 101$	6.279	2.844
8	$f[2,4) = 36$	4.627	7.994
9	$f[2,3) = 18$	7.744	6.135
10	$f[2,5) = 62$	4.627	7.994
11	$f[1,*) = 136$	2.120	5.381
12	$f[1,4) = 75$	3.006	5.021
13	$f[1,2) = 13$	2.000	3.007
14	$f[1,3) = 51$	3.006	3.857
15	$f[1,5) = 108$	2.184	5.221

Table 21. Summary of the solutions for modified static problems

Cost calculations for Stage 4.

$$g^{\star}(4) = \min \begin{cases} (2): \; g(4) = f[4,*) + v_4 = 34 + 14 = 48 \\ (3): \; g(4) = f[4,5) + v_4 + g(5) = 8 + 14 + 40 = 62 \end{cases}$$

Hence, the minimum stage cost is $g^{\star}(4) = 48$. Node (3) is pruned at this stage.

Cost calculations for Stage 3.

$$g^{\star}(3) = \min \begin{cases} (4): \; g(3) = f[3,*) + v_3 = 63 + 15 = 78 \\ (5): \; g(3) = f[3,4) + v_3 + g(4) = 4 + 15 + 48 = 67 \\ (6): \; g(3) = f[3,5) + v_3 + g(5) = 24 + 15 + 40 = 79 \end{cases}$$

Hence, the minimum stage cost is $g^{\star}(3) = 67$. Nodes (4) and (6) are pruned at this stage.

Cost calculations for Stage 2.

$$g^{\star}(2) = \min \begin{cases} (7): \; g(2) = f[2,*) + v_2 = 101 + 13 = 114 \\ (8): \; g(2) = f[2,4) + v_2 + g(4) = 36 + 13 + 48 = 97 \\ (9): \; g(2) = f[2,3) + v_2 + g(3) = 18 + 13 + 67 = 98 \\ (10): \; g(2) = f[2,5) + v_2 + g(5) = 62 + 13 + 40 = 115 \end{cases}$$

Hence, the minimum stage cost is $g^{\star}(3) = 97$. Nodes (7), (9) and (10) are pruned at this stage.

Cost calculations for Stage 1.

$$g^{\star}(1) = \min \begin{cases} (11): \; g(1) = f[1,*) + v_1 = 136 + 18 = 154 \\ (12): \; g(1) = f[1,4) + v_1 + g(4) = 75 + 18 + 48 = 141 \\ (13): \; g(1) = f[1,2) + v_1 + g(2) = 13 + 18 + 97 = 128 \\ (14): \; g(1) = f[1,3) + v_1 + g(3) = 51 + 18 + 67 = 136 \\ (15): \; g(1) = f[1,5) + v_1 + g(5) = 108 + 18 + 40 = 166 \end{cases}$$

Hence, the minimum stage cost and the optimal solution of the dynamic program is $g^{\star}(1) = 128$. Nodes (11), (12), (14), (15), and those labeled with 0 are pruned at this stage. Backtracking to the root of the branching tree gives the relocation vector $\mathbf{z} = (1, 1, 0, 1, 0)$. According to this vector, the shape should be phased-in at the beginning of the planning horizon and relocated at the beginning of the second time period. It should be maintained at this location until another relocation taking place at the beginning of the fourth period. Finally, it should be maintained at this location to the end of the planning horizod. These successive locations are presented in Figures 40-42.

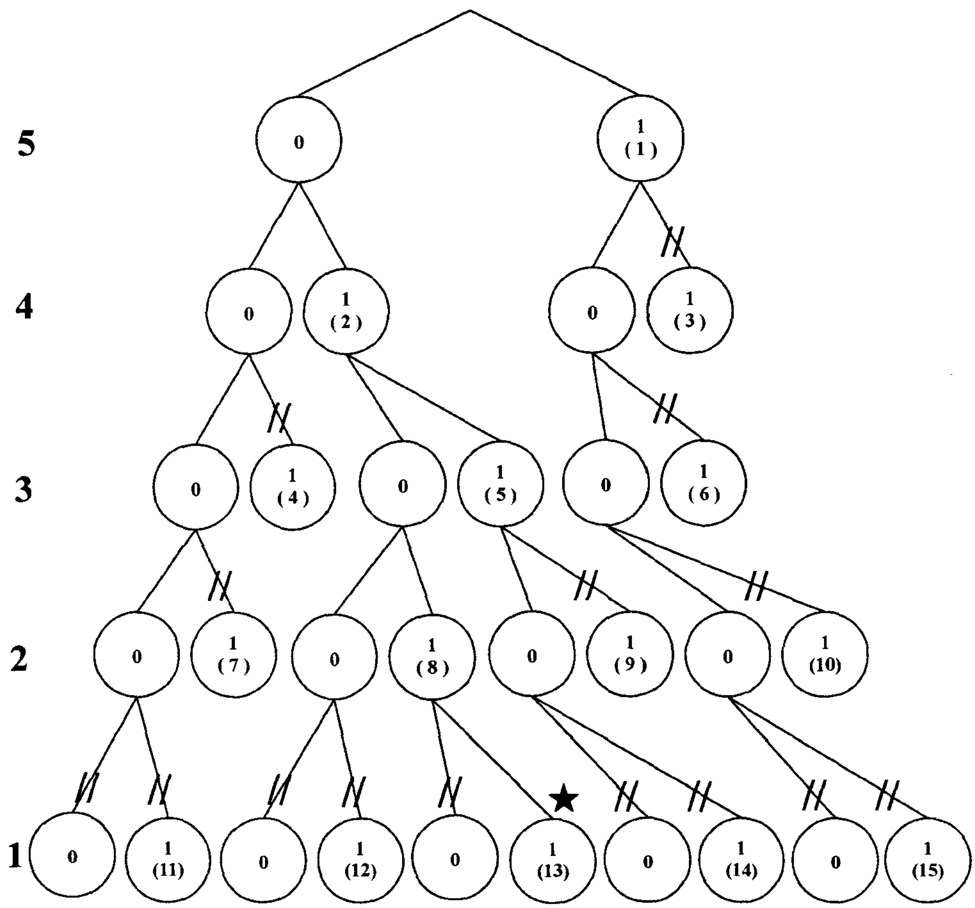

Figure 39. Branching tree for Example 5

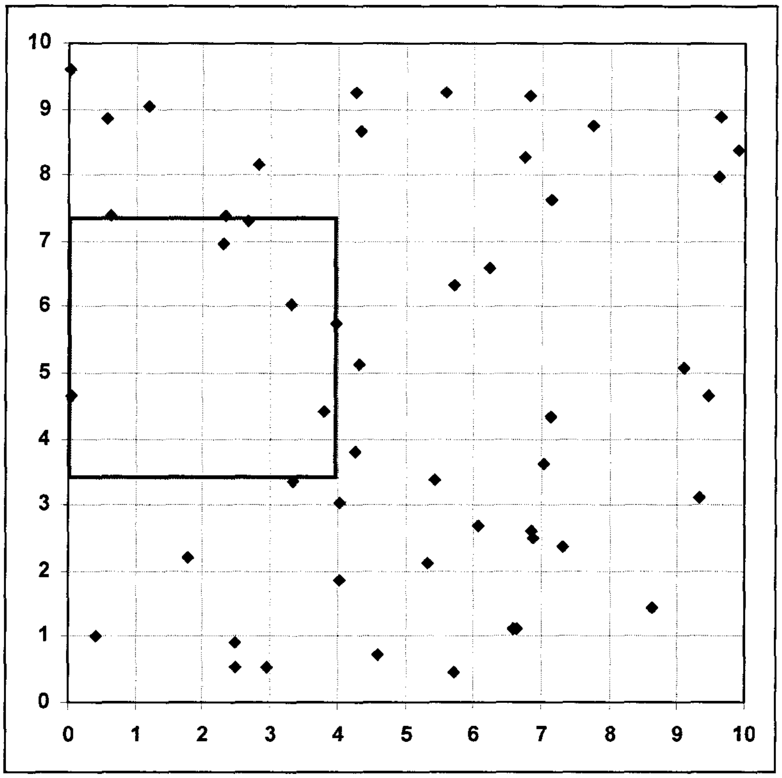

Figure 40. Shape location for period 1 illustrated at $t = 1$

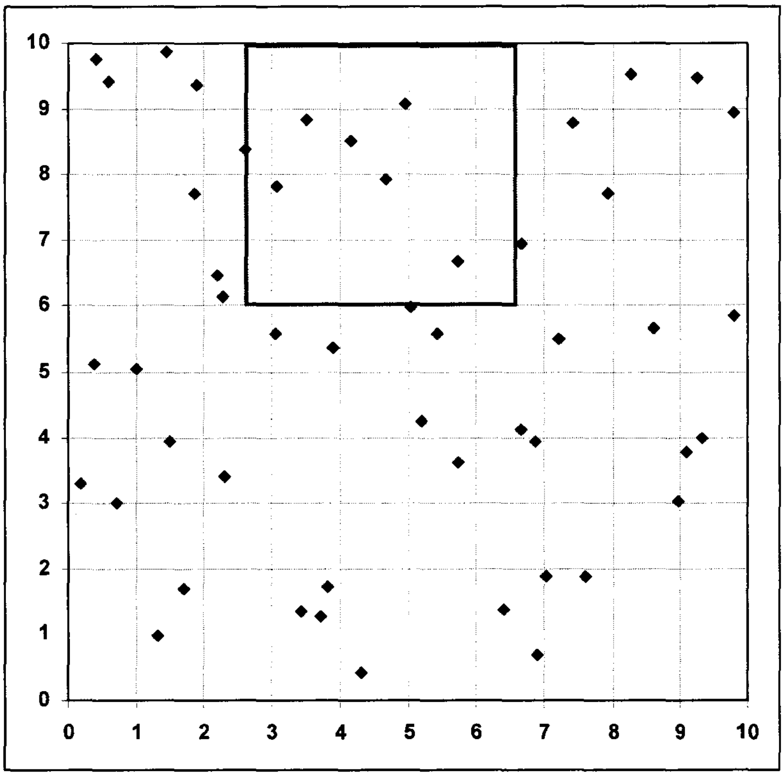

Figure 41. Shape location for periods 2 and 3 illustrated at $t = 2$

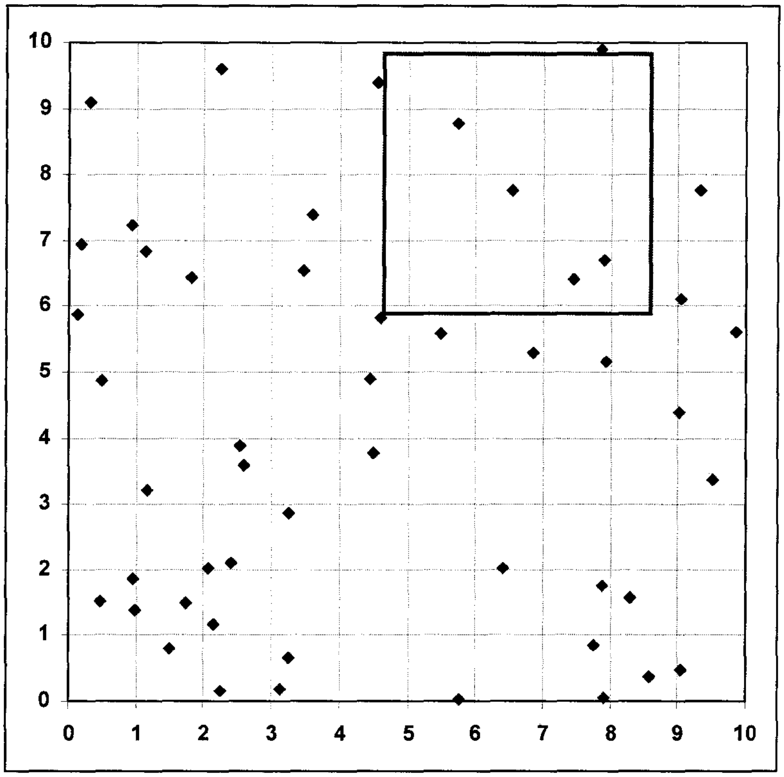

Figure 42. Shape location for periods 4 and 5 illustrated at $t = 4$

Chapter 7
Conclusions and future research

7.1 Summary and concluding remarks

In Chapter 4, we studied the *planar expropriation problem with non-rigid rectangular facilities.* We considered two-dimensional facilities of rectangular shape and allowed the facility dimensions to be decision variables. Hence we introduced the concept of *non-rigid* facilities. Other than this chapter, there is no early work in location theory literature for finding the locations and formations of facilities simultaneously. It is recognized that the current continuous covering models in the location theory literature can not deal with *dimensional facilities.* This is because such models essentially make use of well-known distance measures. As a result, the solutions are analytical points in the plane. In this study, based on the concept of non-rigid facilities, we developed a new continuous covering model that can solve for dimensional facilities without employing distance measures. In addition to site generation, the formations of the facilities are simultaneously optimized within the same model.

Given area requirements A_k, the area restrictions $A_k = l_k \cdot w_k$ for each facility k form convex nonlinear tradeoff curves associated with facility formations defined by all alternative 2-tuples (w_k, l_k). Hence, the resulting covering models are hard mixed integer nonlinear programs. For the solution of these models, we developed a continuous branch-and-bound framework which utilizes linear approximations for such tradeoff curves. By introducing an approximate problem and two auxiliary problems, we were able to bound the objective function value of the mixed integer nonlinear program. We also developed a new problem generation scheme based on the concept of *aspect ratio* of the length and width of the facilities. Computational experience showed that the branch-and-bound procedure we developed is more effective than the conventional mixed integer nonlinear solvers BARON and SBB for solving this particular location model.

One limitation of this approach is the exponential growth of the branching trees in multi-facility planar expropriation problems. It can be recognized from Example 2 that the facilities are not necessarily symmetrical. That is, for a given problem p in the branching tree, facilities do not necessarily assume the same aspect ratio interval. Therefore, the facility planner should consider investigating combinations of aspect ratio intervals for multiple facilities. This requirement is the most important obstacle for implementing the proposed branch-and-bound algorithm for the problems where the number of facilities is large.

In Chapter 5, we studied the *planar maximal covering problem with single convex polygonal shapes.* This problem is equivalent to investigating maximal point containment by a convex polygonal shape in the Euclidean plane. Until 2004, the problem was studied in the computational geometry domain. Some algorithmic approaches based on the translations of the shapes were derived. The investigation of the problem in a location-theoretic application was initiated by Younies and Wesolowsky (2004) who considered inclined parallelograms. This study was the only optimization approach to the planar maximal covering problems with dimensional shapes that allowed modeling parallelograms, rhombi and rectangles. Later, this work was extended to solving planar maximal covering problems using *block norm distances* (Younies and Wesolowsky, 2007). These extended models were equivalent to planar maximal covering problems with *even-sided polygons.* In Chapter 5, we extended the scope of the planar maximal covering problem to account for any *convex polygonal shape.* We did not employ any distance measure and used the geometric properties of the shape when modeling the corresponding planar maximal covering problems. The advantage of our approach is that the shape need not be the replica of any block norm contour. Therefore, many different convex polygonal shapes can be modeled by this methodology. Note that we also solved the planar maximal covering problems with *polyhedral gauges* in this way.

We also provided two types of cuts by using the geometric properties of the shapes under study and derived a branch-and-cut algorithm for solving this new location model. By this approach we created a series of LP relaxations of the original planar maximal covering problem and dynamically added the cuts, whenever they were violated, throughout the branching tree. A small computational study revealed the effect of these cuts on the *bound qualities* of these planar maximal covering problems. As shown in this experiment, the bound qualities were significantly improved after the utilization of such cuts.

One limitation of this approach is that we were only able to solve for shape locations where the shapes are aligned with an orthogonal axis system. Hence, rotations of the shapes were not analyzed. However, it is worthwhile to note that this is not an easy task. As of 2008, there is one solution method for planar maximal covering problems involving *rotations* of shapes (see, Barequet et al., 2008). This is an algorithmic approach for computing a special type of diagram that involves the point-containment information. Given a shape and a contact point from the fixed point set, this diagram parameterizes the translations and rotations of shape for generating the point-containment information for all fixed points. Barequet et al. (2008) showed how to use this diagram for solving the planar maximal covering problem involving rotations. However, the related problem family is open for novel optimization approaches. A list of these open problems can be found in Dickerson and Scharstein (1998).

In Chapter 6, we studied the *dynamic planar expropriation problem with single convex polygonal shapes.* We showed how to locate and relocate the shape in location space during a planning horizon where the fixed point locations and expropriation costs are subject to change. The dynamic model we developed was solved for successive locations of the shape through multiple time periods while minimizing the total expropriation and relocation costs. For any particular vector of relocations, we showed that the dynamic problem structure can be decomposed into a set of static expropriation problems. Therefore, the problem can be solved by inspecting all possible relocation vectors. We discussed two enumeration procedures to undertake this task.

The modeling principles of dynamic location problems were first established in Wesolowsky (1973). According to those principles, the problem we discussed in this chapter can be solved much more efficiently without duplicating the evaluation of many identical static problems in the branching tree. Moreover, the evaluation of expropriation costs in some stages where the shape is not relocated may be postponed to lower stages. Based on these observations, we developed an incomplete dynamic programming procedure for the solution of the dynamic planar expropriation problem with single convex polygonal shapes.

We provided formulas for calculating corresponding tree sizes for all the solution procedures discussed in this study. One advantage of the incomplete dynamic programming procedure we developed is the slow growth of the corresponding branching trees. It was shown that the non-duplicative problem evaluation and postponement strategies lead to significant reductions in the number of static problems to be evaluated through the algorithm; as a result, corresponding tree sizes are diminished.

7.2 Future research

The models studied in this thesis are versatile in the sense that they can be modified to solve various problems in other domains of natural sciences such as *urban and regional planning*, *information processing*, *robotic task planning*, *graphic design* and *computational geometry.*

For example, the first study reported in this thesis has applications in *regional planning.* Consider a particular region where a municipality is planning to site new facilities in its interior. Each facility is planned to have a certain area requirement. However, the properties that fall into such planned areas have to be expropriated at the municipality's expense. In this case, one can use the planar expropriation model discussed in Chapter 4 for siting such facilities while minimizing the total expropriation cost. From a social objective point-of-view, the same model will reduce to the problem of locating undesirable facilities (i.e. a landfill, a power-station etc.) within a bounded geographical region, while minimizing the population centers affected by the facility.

Next, we may consider a *nesting problem* arising in the *production management* domain. In leather and garment manufacturing, an operator has to cut out a set of given pieces from a flat section of raw material. It is generally true that the pieces in one batch are not necessary identical, hence they may assume particular geometric shapes. It is also true that the raw material is not flawless, and there may be some undesirable fault points on its surface. Nevertheless, in material cutting practice, trim-loss is only permitted up to a reasonable degree. Therefore, locating such pieces on the flat section of raw material is an important problem that has to be solved. In this case, one may assign a large penalty cost, say M, for the fault points covered, and use the single-period planar expropriation model illustrated in Chapter 6 for locating such pieces on the flat section of raw material.

As another real-life examples, we may consider *clustering problems* (see, for example Brusco and Köhn 2008; Tsai and Chiu 2008) arising in *pattern recognition* and *information processing* domains. Suppose that there is a set of points with specific weights assigned to them, indicating the *score* of a point with respect to a common *criterion.* We would like to *cluster* a subset of these points and set them apart from the remaining set, such that the distance between any two points in this cluster is not larger than a threshold, and the collective score of the points in the cluster is maximum. One may use the planar maximal covering model introduced in Chapter 5 for solving such clustering problems.

One of the major challenges in continuous facility location is related to siting dimensional structures such as lines, segments, routes and hyperplanes in high-dimensional spaces. In this context, the modeling approaches presented in this thesis may be extended for studying some interesting problems.

Consider the following problem raised by Barequet et al. (2008), which originates in the *computational geometry* domain. Suppose that we have a geometric description of an object that is planned to be manufactured. We also have a set of points which comes from sampling by using a coordinate-measuring apparatus that inspects the surface of an actual physical prototype of this object. We would like to check if the point set sampled from the actual physical prototype matches with our geometric description. It is easy to see that, one method of checking this is to try to cover the sampled points with the shape outlined by our geometric description. This type of covering problems may be essential for determining whether a manufactured object prototype meets the tolerances specified by related production standarts, before the actual production takes place. Also, one may be interested in finding the optimum scaling that the model must undergo in order to cover the sampled points within some tolerance limits. The covering problems we studied in this thesis may serve as practical alternative approaches for solving such geometric problems. This can be achieved by fixing the number of covered points to the number of sampled points. Observe that, in our models, there is no restriction on the number of covered points. However, for solving the above-described problems, we may fix this number and allow the area requirement to vary, for checking the optimal scaling of the shapes under study.

Consider the following problem which has its roots in the *precision engineering* and *robotic task planning* fields. Suppose that we are given an object that has to be tested for circularity. Then, we have to take a sample of points from the surface of this object and measure the circularity of this point set. If the circularity is good enough, we accept the object, and otherwise we reject the object. This problem is referred to as the *roundness inspection problem* in *precision engineering.* In *robotic task planning,* the problem is equivalent to siting a circular route in the convex hull of a set of points such that the maximum distance from the circular route to any point in this set is minimum. Hence, we may consider it as a *min-max circular route location problem.* In location theory, the problem was first investigated by Drezner et al. (2002).

We note that the above problem can be solved by our approach by considering an equivalent *annulus location problem.* An annulus is the open region between two co-centric circles. Hence, the min-max circular route location problem can be modeled as an equivalent problem of computing the *annulus* of smallest width covering the whole point set. Observe that, there exists a circumference whose distance to inner-circle is equal to its distance to outer-circle of the annulus. This circumference is the solution of the max-min circular route location problem described above. To solve the equivalent annulus location problem, one may use the planar maximal covering models we introduced in this thesis, by allowing the facility to be an annulus. In this case, the binary definitions should be adapted accordingly. For example, one binary variable may be dictated to assume a value of one if a point falls outside the inner-circle, and another binary variable may be dictated to assume a value of one if a point falls inside the outer-circle. Then, a third binary variable should assume a value of one, if both of the above two binary variables are one. These binary definitions will allow us to implement coverage by an annulus within our modeling framework introduced in this thesis. Then, one may fix the number of covered points to the cardinality of point set, and allow varying radii both for the inner- and outer-circles of the annulus.

It is evident from these examples that the research we initiated in this thesis have potential to be useful in many domains. It is worthwhile to note that above-stated problems can be formulated as mixed-integer nonlinear problems. Moreover, their objective functions may assume neither convexity nor differentiability. Thus, it is expected that non-convex optimization methods will be useful for developing appropriate solution approaches for such extended problems.

AppendixA
Analysis of strengthening cuts and convergence of the branch-and-bound algorithm

In this Appendix we present two small computational studies to illustrate the effect of strengthening cuts on solution times of expropriation problem formulations, and a convergence analysis for the branch-and-bound algorithm.

A.1 Analysis of MINLP formulation

In this section, we analyze single and multiple facility MINLP formulations of $\mathcal{PENR}$. Recall from Section 4.5 that *approximate problem-specific cuts* (APC) are not applicable to MINLP formulations. Hence, we investigate the effect of *global cuts* (GC) and *cuts for the multiple facilities case* (CMF) on the solution times of randomly generated MINLP instances. Also note that the CMF are not applicable to single facility instances, hence they were only appended to multi facility problems.

We created a set of test problems by considering the combinations of the sets $N = \{5, 10, 20, 50, 75\}$ and $K = \{1, 2, 3\}$. The fixed point coordinates in each dimension were randomly generated from a uniform distribution $U \sim [0, 10]$ within a 10×10 square. We allowed each fixed point i to assume an expropriation cost $c_i \in \mathbb{Z}^+$. Thus, the expropriation costs were randomly generated from a uniform distribution $U \sim [0, 10]$ and rounded to the nearest integer. The area requirement for each facility under study is taken as 20 unit squares. The coding was implemented in GAMS modeling system where BARON was utilized as the MINLP solver. Moreover, we

		CPU (h:mm:ss)	
N	K	MINLP	MINLP+GC+CMF
5	1	0:00:03	0:00:03
	2	0:00:03	0:00:03
	3	0:00:08	0:00:07
10	1	0:00:05	0:00:04
	2	0:00:11	0:00:08
	3	0:00:39	0:00:32
20	1	0:00:37	0:00:32
	2	0:01:41	0:01:24
	3	0:52:19	0:35:51
50	1	0:13:55	0:12:11
	2	1:54:22	1:30:27
	3	2:57:34	2:23:34
75	1	3:00:00	2:51:17
	2	3:00:00	3:00:00
	3	3:00:00	3:00:00

Table 22. Summary of solution times for MINLP problems

imposed a resource limit of 3 hours on the CPU time. Finally, it is worthwhile to note that this experiment was conducted on a 1.6 GHz desktop computer with 512 MB RAM.

The results of this study are summarized in Table 22 and Figure 43. When appended to MINLP formulations, GC and CMF resulted in a slight reduction in solution times. Percentage reduction in solution times ranged from 0% to 31.47%, with an average reduction of 12.8%. In Figure 43, the improvement on solution times seems marginal but observe that they are on a logarithmic scale.

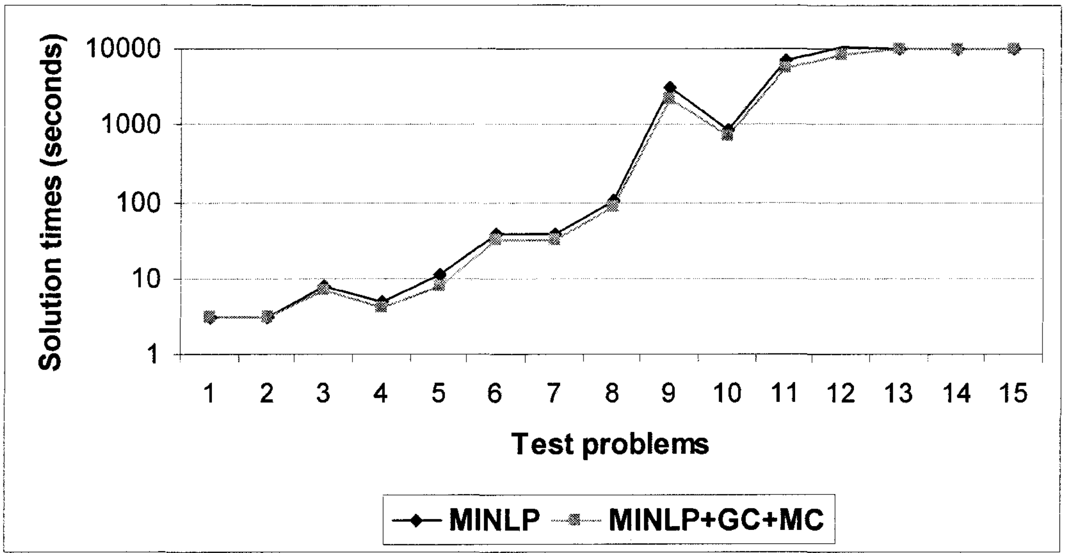

Figure 43. Illustration of solution times for MINLP problems

A.2 Analysis of the approximate problem

In this section, we analyze single and multiple facility approximate problem formulations $\mathcal{PENRA}$ and $\mathcal{PENRM}$. In particular, we investigate the effect of GC, APC and CMF on the solution times of randomly generated approximate expropriation problem instances where CMF were only appended to multi facility problems. We used the same problem setting, resource limitation and computing environment introduced in Section A.1. However, $\mathcal{PENRA}$ and $\mathcal{PENRM}$ formulations are essentially MILP models which are easy to solve when the number of fixed points is small. Thereby, in this study we increased the number of fixed points for each instance and considered the combinations of the sets $N = \{25, 50, 75, 100, 200\}$ and

N	K	CPU (h:mm:ss) MILP	MILP+GC+APC+CMF
25	1	0:00:04	0:00:03
	2	0:00:11	0:00:08
	3	0:00:17	0:00:12
50	1	0:00:06	0:00:05
	2	0:00:42	0:00:29
	3	0:03:18	0:02:02
75	1	0:00:29	0:00:22
	2	0:03:04	0:02:21
	3	0:09:52	0:06:26
100	1	0:00:54	0:00:42
	2	0:18:13	0:11:03
	3	0:38:56	0:21:18
200	1	0:17:21	0:12:34
	2	0:49:14	0:31:15
	3	1:37:18	0:56:12

Table 23. Summary of solution times for approximate problems

$K = \{1, 2, 3\}$. Moreover, we utilized CPLEX 10 as the MILP solver. Finally, the formations of the facilities are considered to be acceptable in the aspect ratio interval $[\alpha_{\min},\ \alpha_{\max}] = [0.4,\ 4]$.

The results of this study are summarized in Table 23 and Figure 44. When we append GC, APC and CMF to MILP formulations, we were able to obtain considerable reductions in solution times. Percentage reduction in solution times ranged from 16.66% to 45.29%, with an average reduction of 31.14%. Similarly to Figure 43, we employed a logarithmic scale to illustrate the resultant solution times in Figure 44.

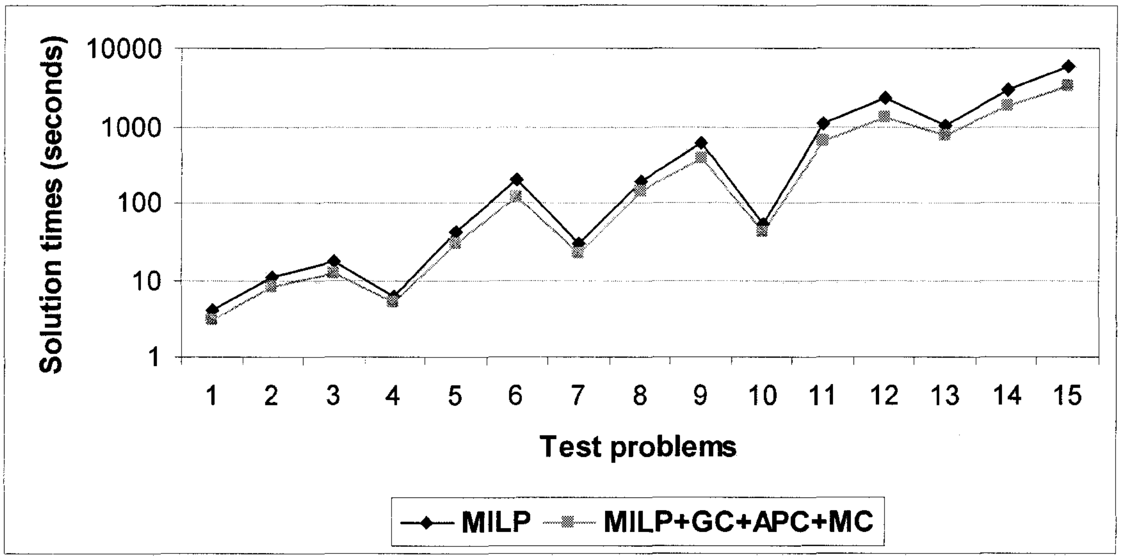

Figure 44. Illustration of solution times for approximate problems

A.3 Convergence analysis of the branch-and-bound algorithm

To prove the convergence of the branch-and-bound algorithm, let us now introduce an index q denoting the levels of the branching tree. Consider a segment problem in level q where we create n new problems from it. Let the optimal objective function values associated with its approximate and auxiliary problems be f^q_{apr} and f^q_{aux}, respectively. For convenience, in this section we do not illustrate indices p utilized for numbering segment problems.

Lemma 10 $\min\left\{f^{q+1}_{apr}\right\} \leq f^q_{apr}$.

PROOF Let $\min\left\{f^{q+1}_{apr}\right\} = f^{q+1*}_{apr}$ and assume there exists an approximate problem with a formation (w^*, l^*) and objective function f_{apr}^{q+1*} at level $q+1$ such that $f_{apr}^{q+1*} > f_{apr}^{q}$. But, observe that there also exists another approximate problem at the same level with objective function f_{apr}^{q+1**}, and formation (w^{**}, l^{**}) which can fit into the coverage area of the formation (w^q, l^q), and hence, covering at most the same number of fixed points as shown in Figure 45. Thereby, we have $f_{apr}^{q+1*} > f_{apr}^{q} \geq f_{apr}^{q+1**}$ which contradicts $\min\left\{f_{apr}^{q+1}\right\} = f_{apr}^{q+1*}$. ■

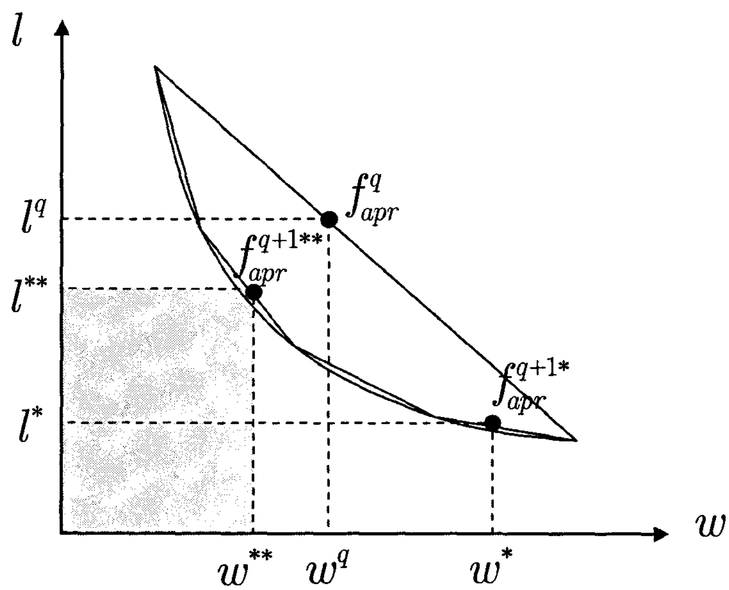

Figure 45. Illustration of Lemma 10

Lemma 11 $\max\{f_{aux}^{q+1}\} \geq f_{aux}^{q}$.

PROOF Let $\max\{f_{aux}^{q+1}\} = f_{aux}^{q+1*}$ and assume there exists an auxiliary problem with a formation (w^*, l^*) and objective function f_{aux}^{q+1*} at level $q+1$ such that $f_{aux}^{q+1*} <$

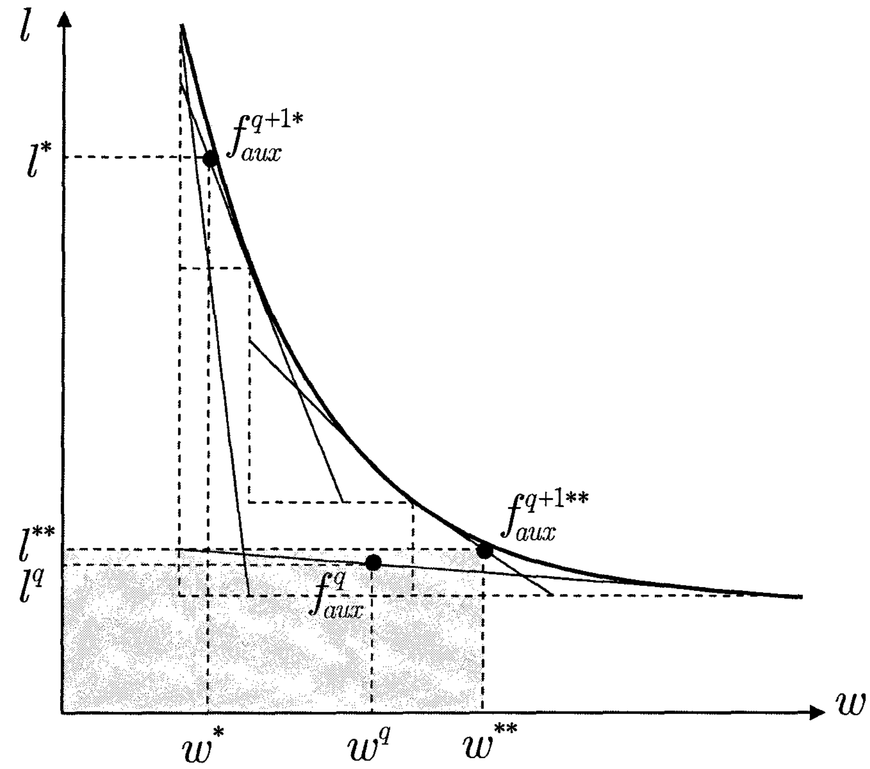

Figure 46. Illustration of Lemma 11

f_{aux}^q. But, observe that there also exists another auxiliary problem at the same level with objective function f_{aux}^{q+1**}, and formation (w^{**}, l^{**}) which is an envelope over the coverage area of the formation (w^q, l^q), and hence, covering at least the same number of fixed points as shown in Figure 46. Thereby, we have $f_{apr}^{q+1**} \geq f_{apr}^q > f_{apr}^{q+1*}$ which contradicts $\max\{f_{aux}^{q+1}\} = f_{aux}^{q+1*}$. ■

Lemma 12 *The branch-and-bound algorithm introduced in Section* 4.6 *is convergent.*

PROOF Let the optimal objective function value of the MINLP problem at the root node be f. From Lemmas 5 and 6 we know that $f_{aux}^q \leq f \leq f_{apr}^q$. Thus, by Lemmas 10 and 11 we have

$$f_{aux}^q \leq f_{aux}^{q+1*} \leq f \leq f_{apr}^{q+1*} \leq f_{apr}^q. \tag{93}$$

Then, it follows that

$$\max\{f_{aux}^q\} \leq \max\left\{f_{aux}^{q+1*}\right\} \leq f \leq \min\left\{f_{apr}^{q+1*}\right\} \leq \min\left\{f_{apr}^q\right\} \tag{94}$$

is true for all levels q. Let $\max\{f_{aux}^q\} = f_L^q$, $\max\{f_{aux}^{q+1*}\} = f_L^{q+1}, \min\left\{f_{apr}^{q+1*}\right\} = f_U^{q+1}$ and $\min\left\{f_{apr}^q\right\} = f_U^q$. Then, (94) can be written as

$$f_L^q \leq f_L^{q+1} \leq f \leq f_U^{q+1} \leq f_U^q \tag{95}$$

which shows that the sequence $\{f_L^q\}_{q=0}^{\infty}$ is non-decreasing and the sequence $\{f_U^q\}_{q=0}^{\infty}$ is non-increasing. Consider now, the sequence $\{f_U^q - f_L^q\}_{q=0}^{\infty}$. Since $\{f_L^q\}_{q=0}^{\infty}$ is non-decreasing and $\{f_U^q\}_{q=0}^{\infty}$ is non-increasing, it follows that $\{f_U^q - f_L^q\}_{q=0}^{\infty}$ is also non-increasing. This shows that if the tradeoff curve structure is sufficiently inspected as $q \to \infty$, the limit of the sequence $\{f_U^q - f_L^q\}_{q=0}^{\infty}$ exists at

$$\lim_{q\to\infty} (f_U^q - f_L^q) < \varepsilon \tag{96}$$

where ε is a very small number. Observe that, at optimality we have $f_L^q = f = f_U^q$ where f_L^q is equal to lower bound of the optimal node and f_U^q is equal to UB^*.

■

AppendixB
Example $\mathcal{PMCS}$ problem formulations

B.1 $\mathcal{PMCS}$ with an acute triangle

Consider the acute triangle shown in Figure 47 with an area requirement of A. Denote the two bottom interior angles of this triangle by θ and β . Further, denote the location and length of the triangle with $V_1 = (x_1, x_2)$ and l, respectively. The length of this triangle is given by $l = \sqrt{2 \cdot A / (1/\tan\theta + 1/\tan\beta)}$, which is a known parameter.

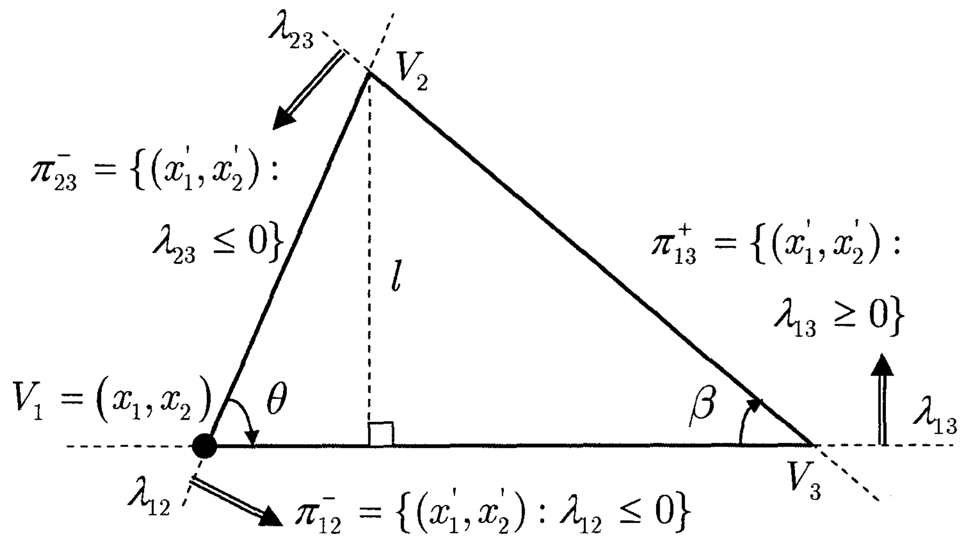

Figure 47. An acute triangle and its underlying half-planes

Step 1. The relative locations of the vertices are given by the following:

$V_1 = (x_1, x_2), V_2 = (x_1 + l/\tan\theta, x_2 + l), V_3 = (x_1 + l \cdot (1/\tan\theta + 1/\tan\beta), x_2)$.

Step 2. The lines passing through the edges of the triangle can be written as:

$$\lambda_{12} \equiv x'_2 - \tan\theta \cdot x'_1 - x_2 + \tan\theta \cdot x_1 = 0 \tag{97}$$

$$\lambda_{23} \equiv x'_2 + \tan\beta \cdot x'_1 - x_2 - \tan\beta \cdot x_1 - l \cdot (1 + \tan\beta/\tan\theta) = 0 \tag{98}$$

$$\lambda_{13} \equiv x'_2 - x_2 = 0. \tag{99}$$

Step 3. The following line inequalities should hold for a point that is covered:

$$\lambda_{12} \leq 0 \tag{100}$$

$$\lambda_{23} \leq 0 \tag{101}$$

$$\lambda_{13} \geq 0. \tag{102}$$

Step 4. The covering constraints can be generated as follows:

$$a_{2i} - \tan\theta \cdot a_{1i} - x_2 + \tan\theta \cdot x_1 \leq (1 - y_i^1) \cdot M \tag{103}$$

$$a_{2i} + \tan\beta \cdot a_{1i} - x_2 - \tan\beta \cdot x_1 - l \cdot (1 + \tan\beta/\tan\theta) \leq (1 - y_i^2) \cdot M \tag{104}$$

$$a_{2i} - x_2 \geq (1 - y_i^3) \cdot (-M) \tag{105}$$

$$\frac{1}{3} \cdot \left(y_i^1 + y_i^2 + y_i^3\right) \geq y_i. \tag{106}$$

Step 5. The inclusion constraints can be written as follows:

$$x_1 \geq L^{X1} \tag{107}$$

$$x_1 + l \cdot (1/\tan\theta + 1/\tan\beta) \leq U^{X1} \tag{108}$$

$$x_2 \geq L^{X2} \tag{109}$$

$$x_2 + l \leq U^{X2}. \tag{110}$$

Integrality and non-negativity constraints can be written as follows:

$$y_i^1,\ y_i^2,\ y_i^3,\ y_i \in \{0, 1\} \tag{111}$$

$$x_1, x_2 \in \Re^+. \tag{112}$$

The *planar maximal covering problem with an acute triangle* is given by the following MILP:

$$\max \quad \left\{ \sum_{i=1}^{N} \omega_i \cdot y_i : \text{ s.t. } (103-106,\ 111)\ \forall i;\ (107-110,\ 112) \right\}. \tag{113}$$

B.2 $\mathcal{PMCS}$ with an obtuse triangle

Planar maximal covering problem formulation for obtuse triangles can be derived with minor modifications on the formulation (113). Consider the obtuse triangle illustrated in Figure 48. The length of this triangle is $l = \sqrt{2 \cdot A / \left(\frac{1}{\tan\beta} - \frac{1}{\tan(180-\theta)}\right)}$, which is a known parameter.

Step 1. The relative locations of the vertices are given by the following: $V_1 = (x_1, x_2)$, $V_2 = (x_1 - l/\tan(180-\theta), x_2 + l)$, $V_3 = \left(x_1 + l \cdot (\frac{1}{\tan\beta} - \frac{1}{\tan(180-\theta)}), x_2\right)$.

Step 2. The lines passing through the edges of the triangle can be written as:

$$\lambda_{12} \equiv\ x_2' + \tan(180-\theta) \cdot x_1' - x_2 + \tan(180-\theta) \cdot x_1 = 0 \tag{114}$$

$$\lambda_{23} \equiv\ x_2' + \tan\beta \cdot x_1' - x_2 - \tan\beta \cdot x_1 + l \cdot (\tan\beta / \tan(180-\theta)) = 0 \tag{115}$$

$$\lambda_{13} \equiv\ x_2' - x_2 = 0. \tag{116}$$

Step 3. The following line inequalities should hold for a point that is covered:

$$\lambda_{12} \geq 0 \tag{117}$$

$$\lambda_{23} \leq 0 \tag{118}$$

$$\lambda_{13} \geq 0. \tag{119}$$

Step 4. The covering constraints can be generated as follows:

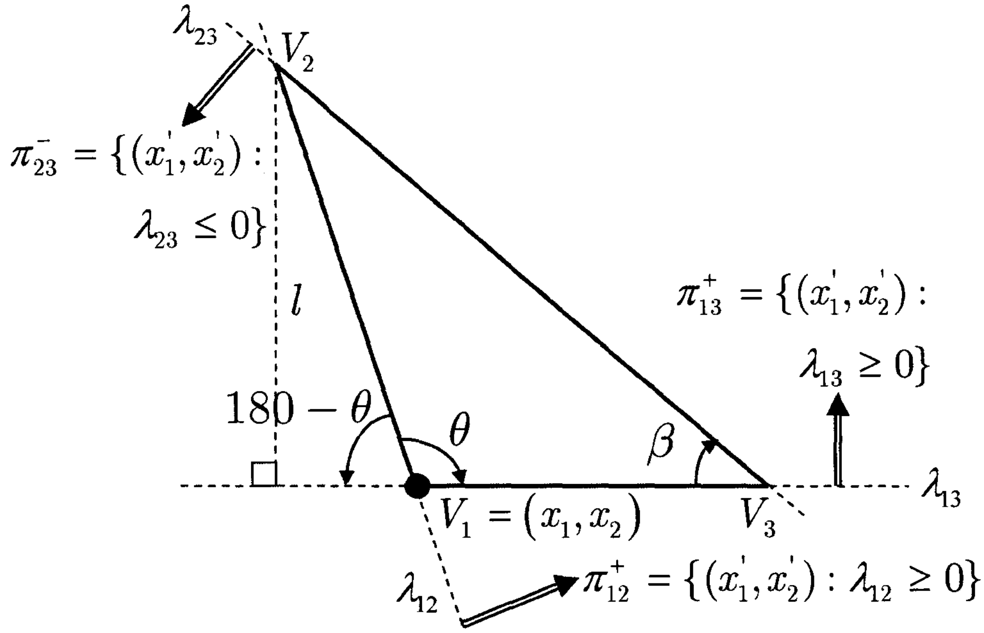

Figure 48. An obtuse triangle and its underlying half-planes

$$a_{2i} + \tan(180 - \theta) \cdot a_{1i} - x_2 - \tan(180 - \theta) \cdot x_1 \geq (1 - y_i^1) \cdot (-M) \tag{120}$$

$$a_{2i} + \tan\beta \cdot a_{1i} - x_2 - \tan\beta \cdot x_1 + l \cdot (\tan\beta / \tan(180 - \theta)) \leq (1 - y_i^2) \cdot M \tag{121}$$

$$a_{2i} - x_2 \geq (1 - y_i^3) \cdot (-M) \tag{122}$$

$$\frac{1}{3} \cdot \left(y_i^1 + y_i^2 + y_i^3\right) \geq y_i. \tag{123}$$

Step 5. The inclusion constraints can be written as follows:

$$x_1 - l / \tan(180 - \theta) \geq L^{X1} \tag{124}$$

$$x_1 + l \cdot \left(\frac{1}{\tan\beta} - \frac{1}{\tan(180 - \theta)}\right) \leq U^{X1} \tag{125}$$

$$x_2 \geq L^{X2} \tag{126}$$

$$x_2 + l \leq U^{X2}. \tag{127}$$

Integrality and non-negativity constraints are the same with the problem formulation for an acute triangle and given by (111, 112). The *planar maximal covering problem with an obtuse triangle* is given by the following MILP:

$$\max \quad \left\{ \sum_{i=1}^{N} \omega_i \cdot y_i : \text{ s.t. } (120-123,\ 111)\ \forall i;\ (124-127,\ 112) \right\}. \tag{128}$$

B.3 $\mathcal{PMCS}$ with a quadrilateral

Consider the symmetric quadrilateral shown in Figure 49 with an area requirement of A. Let the two non-neighboring interior angles of this quadrilateral be φ and γ. Further, denote the location and the half-length of this quadrilateral by (x_1, x_2) and l, respectively. The half-length of this quadrilateral is given by $l = \sqrt{A/\left(\frac{1}{\tan(\varphi/2)} + \frac{1}{\tan(\gamma/2)}\right)}$, which is a known parameter.

Step 1. The relative locations of the vertices are given by the following:
$V_1 = (x_1, x_2 - l)$, $V_2 = \left(x_1 - \frac{l}{\tan(\varphi/2)}, x_2\right)$, $V_3 = (x_1, x_2 + l)$, $V_4 = \left(x_1 + \frac{l}{\tan(\gamma/2)}, x_2\right)$.

Step 2. The lines passing through the edges of the quadrilateral can be written as follows:

$$\lambda_{12} \equiv x_2' + \tan(\varphi/2) \cdot x_1' - x_2 - \tan(\varphi/2) \cdot x_1 + l = 0 \tag{129}$$

$$\lambda_{23} \equiv x_2' - \tan(\varphi/2) \cdot x_1' - x_2 + \tan(\varphi/2) \cdot x_1 - l = 0 \tag{130}$$

$$\lambda_{34} \equiv x_2' + \tan(\gamma/2) \cdot x_1' - x_2 - \tan(\gamma/2) \cdot x_1 - l = 0 \tag{131}$$

$$\lambda_{14} \equiv x_2' - \tan(\gamma/2) \cdot x_1' - x_2 + \tan(\gamma/2) \cdot x_1 + l = 0. \tag{132}$$

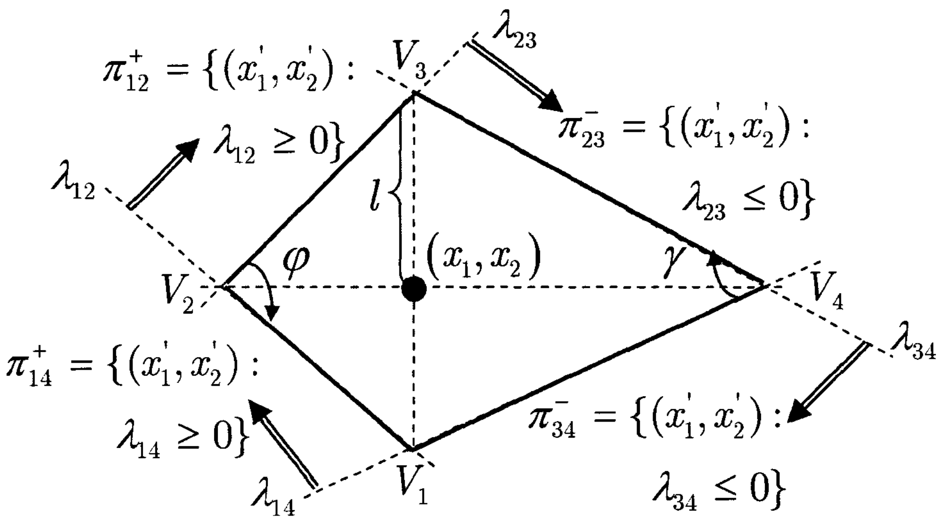

Figure 49. A quadrilateral and its underlying half-planes

Step 3. The following line inequalities should hold for a point that is covered:

$$\lambda_{12} \geq 0 \tag{133}$$

$$\lambda_{23} \leq 0 \tag{134}$$

$$\lambda_{34} \leq 0 \tag{135}$$

$$\lambda_{14} \geq 0. \tag{136}$$

Step 4. The covering constraints can be generated as follows:

$$a_{2i} + \tan(\varphi/2) \cdot a_{1i} - x_2 - \tan(\varphi/2) \cdot x_1 + l \geq (1 - y_i^1) \cdot (-M) \tag{137}$$

$$a_{2i} - \tan(\varphi/2) \cdot a_{1i} - x_2 + \tan(\varphi/2) \cdot x_1 - l \leq (1 - y_i^2) \cdot M \tag{138}$$

$$a_{2i} + \tan(\varphi/2) \cdot a_{1i} - x_2 - \tan(\varphi/2) \cdot x_1 - l \leq (1 - y_i^3) \cdot M \tag{139}$$

$$a_{2i} - \tan(\varphi/2) \cdot a_{1i} - x_2 + \tan(\varphi/2) \cdot x_1 + l \geq (1 - y_i^4) \cdot (-M) \tag{140}$$

$$\frac{1}{4} \cdot \left(y_i^1 + y_i^2 + y_i^3 + y_i^4\right) \geq y_i. \tag{141}$$

Step 5. The inclusion constraints can be written as follows:

$$x_1 - l/\tan(\varphi/2) \geq L^{X1} \tag{142}$$

$$x_1 + l/\tan(\gamma/2) \leq U^{X1} \tag{143}$$

$$x_2 - l \geq L^{X2} \tag{144}$$

$$x_2 + l \leq U^{X2}. \tag{145}$$

Integrality and non-negativity constraints are as follows:

$$y_i^1,\ y_i^2,\ y_i^3,\ y_i^4,\ y_i \in \{0, 1\} \tag{146}$$

$$x_1, x_2 \in \Re^+. \tag{147}$$

The *planar maximal covering problem with a quadrilateral* is given by the following MILP:

$$\max \quad \left\{ \sum_{i=1}^{N} \omega_i \cdot y_i : \text{ s.t. } (137 - 141,\ 146)\ \forall i;\ (142 - 145,\ 147) \right\}. \tag{148}$$

Remark 3. *This formulation is also applicable to inclined parallelograms and rhombi, as these are nothing but special cases of symmetric quadrilaterals. If the quadrilateral is not symmetric over at least one of its axes, the corresponding problem will be equivalent to solving a planar maximal covering problem with using a specific polyhedral gauge. In this case, a complete description of the unit ball is required (i.e. all the interior angles must be known).* ♣

AppendixC
Three-dimensional expropriation problem with single convex polyhedral shapes

In this Appendix, we provide the formulation for the *three-dimensional expropriation problem with single convex polyhedral shapes* ($\mathcal{TDECS}$). This problem is the extension of *single-period planar expropriation problem with single convex polygonal shapes* ($\mathcal{SPECS}$) to three dimensions. A formal statement of this problem is as follows.

Formal statement of the problem $\mathcal{TDECS}$. *Given N fixed points indexed by i and the expropriation costs $c_i \in \Re^+$ associated with these points in the Euclidean three-dimensional space $\Re^3$, find the location of a three-dimensional single convex polyhedral shape, such that the total cost of the expropriated points is minimum.* ♣

This problem is directly applicable in material cutting systems. In non-destructive material inspection, a solid material can be investigated by using a probe transmitting sound waves into the material and accumulating these waves back. Then, the locations of fault points such as cracks and air chinks in the material can be spotted in terms of point coordinates in $\Re^3$. Thus, the $\mathcal{TDECS}$ problem is applicable to finding the location of a regular convex polyhedral shape that has to be cut out from a solid material with a minimum number of fault points.

In particular, the formulation of $\mathcal{TDECS}$ calls for a generalization of the procedure described for $\mathcal{SPECS}$ in Section 6.2 to three dimensions. For generating the formulations of $\mathcal{TDECS}$, we define the convex polyhedral shape as an intersection of a set of half-spaces. Consider a bounded three-dimensional feasible location space $\Im$, and let i, j and k be three points with known locations (x_{1i}, x_{2i}, x_{3i}), (x_{1j}, x_{2j}, x_{3j}) and (x_{1k}, x_{2k}, x_{3k}) in this space, respectively. A *plane* passing through these three points is a set of points (x'_1, x'_2, x'_3) satisfying the following determinant equation:

$$\begin{vmatrix} x'_1 - x_{1i} & x'_2 - x_{2i} & x'_3 - x_{3i} \\ x_{1j} - x_{1i} & x_{2j} - x_{2i} & x_{3j} - x_{3i} \\ x_{1k} - x_{1i} & x_{2k} - x_{2i} & x_{3k} - x_{3i} \end{vmatrix} = 0. \tag{149}$$

Such a plane will define two closed half-spaces denoted by ς^{+}_{ijk} and ς^{-}_{ijk}. These half-spaces are given as follows:

$$\varsigma^{+}_{ijk} = \left\{ (x'_1, x'_2, x'_3) \in \Im : \begin{vmatrix} x'_1 - x_{1i} & x'_2 - x_{2i} & x'_3 - x_{3i} \\ x_{1j} - x_{1i} & x_{2j} - x_{2i} & x_{3j} - x_{3i} \\ x_{1k} - x_{1i} & x_{2k} - x_{2i} & x_{3k} - x_{3i} \end{vmatrix} \geq 0 \right\} \tag{150}$$

and

$$\varsigma^{-}_{ijk} = \left\{ (x'_1, x'_2, x'_3) \in \Im : \begin{vmatrix} x'_1 - x_{1i} & x'_2 - x_{2i} & x'_3 - x_{3i} \\ x_{1j} - x_{1i} & x_{2j} - x_{2i} & x_{3j} - x_{3i} \\ x_{1k} - x_{1i} & x_{2k} - x_{2i} & x_{3k} - x_{3i} \end{vmatrix} \leq 0 \right\}. \tag{151}$$

Therefore, using the separation property illustrated in Lemma 1, a convex polyhedral shape can be represented as a closed convex subset of $\Im$. Similarly to the procedure described in Section 5.2, we define a convex polyhedral shape having m faces as an intersection of m half-spaces. These half-spaces are given by means of a collection of plane inequalities which are constructed by using the vertices of the convex polyhedral shape. Now we demonstrate the construction for the problem formulation of a regular tetrahedron by using a simple five-step precedure similar to the one introduced in Section 5.2. The problem formulation for any particular convex polyhedral shape can be generated by following the same procedure.

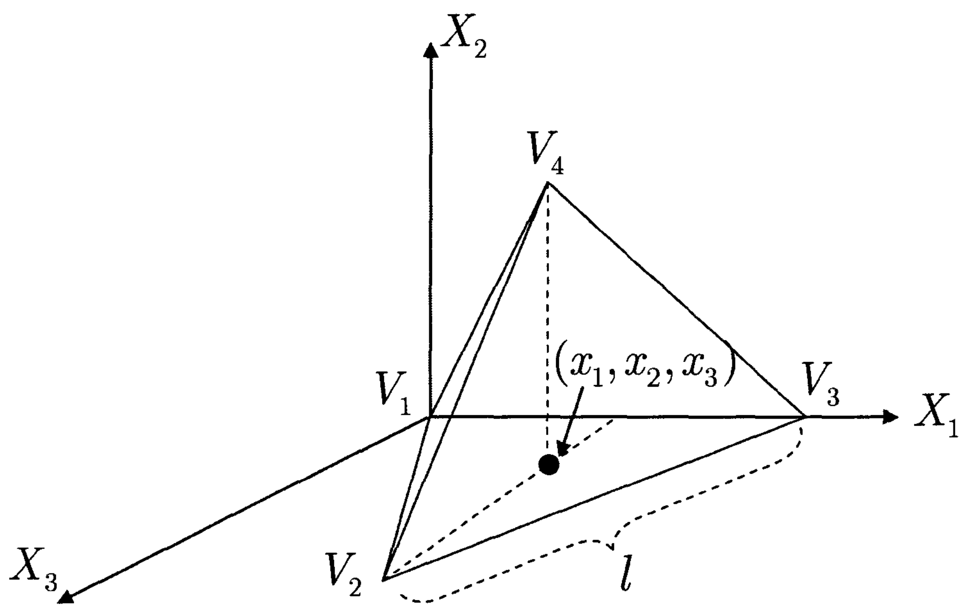

Figure 50. A tetrahedron and its vertices

Step 1. (*Finding the relative locations of the vertices of the shape*)

Consider the tetrahedron illustrated in Figure 50. Let the location of this tetrahedron be (x_1, x_2, x_3), and let the length of its edges be l. We can write the relative locations of the vertices V_1, V_2, V_3 and V_4 of this tetrahedron, in terms of shape location as follows: $V_1 = (x_1 - l/2, x_2, x_3 - l\sqrt{3}/6)$, $V_2 = (x_1, x_2, x_3 + l\sqrt{3}/3)$, $V_3 = (x_1 + l/2, x_2, x_3 - l\sqrt{3}/6)$, and $V_4 = (x_1, x_2 + l\sqrt{2}/\sqrt{3}, x_3)$.

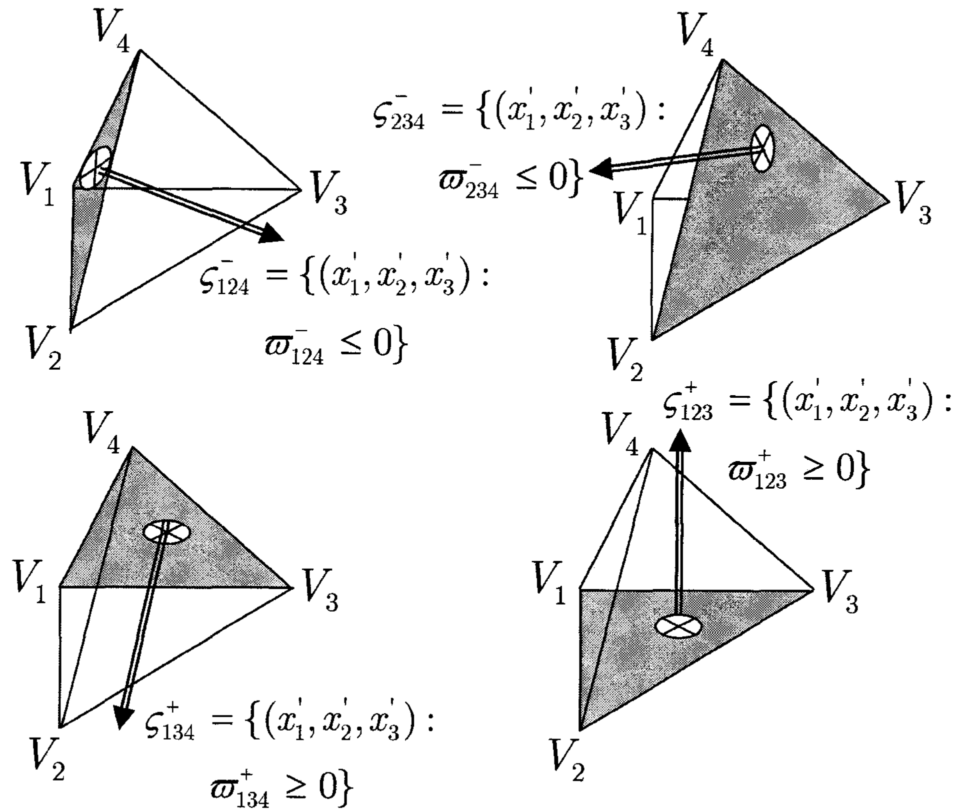

Figure 51. Underlying half-spaces of the tetrahedron

Step 2. (*Generating the plane equations*)

Using the relative locations of the vertices, we now generate the equations for the planes passing through the edges of the shape. These planes are shown as shaded areas in Figure 51. Denote the equations of these planes by ϖ_{ijk}, and let (x'_1, x'_2, x'_3) denote an arbitrary point on the corresponding plane equation. To start with ϖ_{124}, we write:

$$\varpi_{124} \equiv \begin{vmatrix} x'_1 - (x_1 - l/2) & x'_2 - x_2 & x'_3 - (x_3 - l\sqrt{3}/6) \\ x_1 - (x_1 - l/2) & x_2 - x_2 & (x_3 + l\sqrt{3}/3) - (x_3 - l\sqrt{3}/6) \\ x_1 - (x_1 - l/2) & (x_2 + l\sqrt{2}/\sqrt{3}) - x_2 & x_3 - (x_3 - l\sqrt{3}/6) \end{vmatrix} = 0,$$

which is equal to

$$\varpi_{124} \equiv \begin{vmatrix} x'_1 - x_1 + l/2 & x'_2 - x_2 & x'_3 - x_3 + l\sqrt{3}/6 \\ l/2 & 0 & l\sqrt{3}/2 \\ l/2 & l\sqrt{2}/\sqrt{3} & l\sqrt{3}/6 \end{vmatrix} = 0.$$

Upon opening the determinant we get $\varpi_{124} \equiv (x'_1 - x_1 + l/2) \cdot [(0 \cdot l\sqrt{3}/6) - (l\sqrt{2}/\sqrt{3} \cdot l\sqrt{3}/2)] - (x'_2 - x_2) \cdot [(l/2 \cdot l\sqrt{3}/6) - (l/2 \cdot l\sqrt{3}/2)] + (x'_3 - x_3 + l\sqrt{3}/6) \cdot [(l/2 \cdot l\sqrt{2}/\sqrt{3}) - (l/2 \cdot 0)] = 0$.

After further simplification, the plane equation ϖ_{124} is given by

$$\varpi_{124} \equiv \frac{-l^2}{\sqrt{2}}(x'_1 - x_1) + \frac{l^2}{\sqrt{2}}(x'_2 - x_2) + \frac{l^2}{\sqrt{6}}(x'_3 - x_3) - \frac{l^3}{\sqrt{3}} = 0. \tag{152}$$

Continuing with the same procedure, we define remaining plane equations as follows:

$$\varpi_{234} \equiv \frac{l^2}{\sqrt{2}}\left(x_1' - x_1\right) + \frac{l^2}{\sqrt{2}}\left(x_2' - x_2\right) + \frac{l^2}{\sqrt{6}}\left(x_3' - x_3\right) - \frac{l^3}{\sqrt{3}} = 0 \tag{153}$$

$$\varpi_{134} \equiv \frac{-l^2}{\sqrt{2}}\left(x_2' - x_2\right) + \frac{l^2}{\sqrt{6}}\left(x_3' - x_3\right) + \frac{l^3}{\sqrt{3}} = 0 \tag{154}$$

$$\varpi_{123} \equiv x_2' - x_2 = 0. \tag{155}$$

Step 3. (*Generating the inequalities assisting the coverage decision*)

If a fixed point i located at (a_{1i}, a_{2i}, a_{3i}) is covered by the tetrahedron, when we substitute the point location (a_{1i}, a_{2i}, a_{3i}) for (x_1', x_2', x_3') in plane equations (152-155), the following inequalities should hold:

$$\varpi_{124} \leq 0 \tag{156}$$

$$\varpi_{234} \leq 0 \tag{157}$$

$$\varpi_{134} \geq 0 \tag{158}$$

$$\varpi_{123} \geq 0. \tag{159}$$

This is because the coverage space of the tetrahedron is the intersection of half-spaces $\varsigma_{124}^- = \{(x_1', x_2', x_3') : \varpi_{124} \leq 0\}$, $\varsigma_{234}^- = \{(x_1', x_2', x_3') : \varpi_{234} \leq 0\}$, $\varsigma_{134}^+ = \{(x_1', x_2', x_3') : \varpi_{134} \geq 0\}$, and $\varsigma_{123}^+ = \{(x_1', x_2', x_3') : \varpi_{123} \geq 0\}$, as shown in Figure 51.

Step 4. (*Generating the covering constraints*)

First we define the binary variables assisting the covering decision. Let, $y_i^1 = 1$ if (156) holds for point i, and 0 otherwise; $y_i^2 = 1$ if (157) holds for point i, and 0 otherwise; $y_i^3 = 1$ if (158) holds for point i, and 0 otherwise; $y_i^4 = 1$ if (159) holds for point i, and 0 otherwise. Then, we define the covering constraints as follows:

$$\frac{-l^2}{\sqrt{2}}(a_{1i} - x_1) + \frac{l^2}{\sqrt{2}}(a_{2i} - x_2) + \frac{l^2}{\sqrt{6}}(a_{3i} - x_3) - \frac{l^3}{\sqrt{3}} \leq \left(1 - y_i^1\right) \cdot M \tag{160}$$

$$\frac{l^2}{\sqrt{2}}(a_{1i} - x_1) + \frac{l^2}{\sqrt{2}}(a_{2i} - x_2) + \frac{l^2}{\sqrt{6}}(a_{3i} - x_3) - \frac{l^3}{\sqrt{3}} \leq \left(1 - y_i^2\right) \cdot M \tag{161}$$

$$\frac{-l^2}{\sqrt{2}}(a_{2i} - x_2) + \frac{l^2}{\sqrt{6}}(a_{3i} - x_3) + \frac{l^3}{\sqrt{3}} \geq \left(1 - y_i^3\right) \cdot (-M) \tag{162}$$

$$a_{2i} - x_2 \geq \left(1 - y_i^4\right) \cdot (-M). \tag{163}$$

When a fixed point i falls into the intersection of half-spaces ς_{124}^-, ς_{234}^-, ς_{134}^+ and ς_{123}^+, then the binary variables y_i^1, y_i^2, y_i^3 and y_i^4 should all be equal to 1. Therefore, we add the constraint

$$y_i^1 + y_i^2 + y_i^3 + y_i^4 \leq 3 + y_i \tag{164}$$

where $y_i = 1$ if the fixed point i is covered by the tetrahedron, and 0 otherwise.

Step 5. (*Generating the inlusion, integrality and non-negativity constraints*)

In addition to the lower and upper bounds introduced for the planar case in Section 5.2, we let L^{X3} and U^{X3} be the upper and lower bounds, respectively, for feasible shape locations along the X_3 dimension. It follows that the location space is bounded with six planes $x_1 = L^{X1}$, $x_1 = U^{X1}$, $x_2 = L^{X2}$, $x_2 = U^{X2}$, $x_3 = L^{X3}$ and $x_3 = U^{X3}$. Thus, the inclusion constraint set can be constructed as follows:

$$x_1 - l/2 \geq L^{X1} \tag{165}$$

$$x_1 + l/2 \leq U^{X1} \tag{166}$$

$$x_2 \geq L^{X2} \tag{167}$$

$$x_2 + l\sqrt{2}/\sqrt{3} \leq U^{X2} \tag{168}$$

$$x_3 - l\sqrt{3}/6 \geq L^{X3} \tag{169}$$

$$x_3 - l\sqrt{3}/3 \leq U^{X3}. \tag{170}$$

The integrality and non-negativity constraints are straightforward and given by the following:

$$y_i^1,\ y_i^2,\ y_i^3,\ y_i^4,\ y_i\ \in \{0,1\} \tag{171}$$

$$x_1, x_2, x_3\ \in \Re^+. \tag{172}$$

The formulation of $\mathcal{TDECS}$ can now be stated as follows:

$$(\mathcal{TDECS}) \quad \min \quad \left\{ \sum_{i=1}^{N} c_i \cdot y_i :\ \text{s.t. } (160-164,\ 171)\ \forall i;\ (165-170,\ 172) \right\}. \tag{173}$$

AppendixD
Planar maximal covering problem with non-rigid rectangular facilities

In this appendix, we provide a formulation for the *planar maximal covering problem with non-rigid rectangular facilities.* A formal statement of this problem is as follows.

Formal statement of the problem $\mathcal{PMNR}$. *Given N fixed points indexed by i and the weights $\omega_i \in \Re^+$ associated with these points in the Euclidean plane $\Re^2$, find the location of a (the K) non-rigid rectangular facility(ies), such that the total weight covered is maximum.* ♣

Similarly to the procedure described in Section 6.2, we can generate the formulation for $\mathcal{PMNR}$ by substituting the objective function of the program (13) with its diametric opposite

$$\max \sum_{i=1}^{N} \omega_i \cdot z_i, \tag{174}$$

and replacing the constraint set (5) with the following set of constraints:

$$\frac{1}{4} \cdot \left(z_i^{11} + z_i^{12} + z_i^{21} + z_i^{22}\right) \geq z_i \qquad \forall i. \tag{175}$$

Observe that, when the objective function of $\mathcal{PENR}$ is substituted with its diametric opposite (174), the constraint set (5) becomes redundant. To illustrate this, consider the vector $\mathbf{z} = (z_i)^T = (1, ..., 1)^T$. Clearly, this vector is a feasible solution with an objective function value of $\sum_{i=1}^{N} \omega_i$, because the optimization direction is to *maximize* in (174). In this case, note that constraint set (5) is not binding because, for any possible values of z_i^{11}, z_i^{12}, z_i^{21} and z_i^{22} we have:

$$z_i^{11} + z_i^{12} + z_i^{21} + z_i^{22} \leq 4 \qquad \forall i. \tag{176}$$

Therefore, regardless the input, if we use the constraint set (5) we will obtain the solution $\sum_{i=1}^{N} \omega_i$. Recall from Section 4.2.1 that, the rule for implementing coverage is given by the following:

$$\textbf{If} \quad z_i^{11} = 1,\ z_i^{12} = 1,\ z_i^{21} = 1,\ \textit{and}\ z_i^{22} = 1, \quad \textbf{then} \quad z_i = 1. \tag{177}$$

Accordingly, the $\mathcal{PMNR}$ formulation can be stated as follows:

$$(\mathcal{PMNR}) \qquad \max \quad \left\{ \sum_{i=1}^{N} \omega_i \cdot z_i \ : \ \text{s.t. } (1-4, 175\ , 12)\ \forall i;\ (6-11) \right\}. \tag{178}$$

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