INFERENCE FOR BIRNBAUM-SAUNDERS, LAPLACE AND SOME RELATED DISTRIBUTIONS UNDER CENSORED DATA

INFERENCE FOR BIRNBAUM-SAUNDERS, LAPLACE AND SOME RELATED DISTRIBUTIONS UNDER CENSORED DATA

By

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ABSTRACT: The Birnbaum-Saunders (BS) distribution is a positively skewed distribution and is a popular model for analyzing lifetime data. In this thesis, we first develop an improved method of estimation for the BS distribution and the corresponding inference. Compared to the maximum likelihood estimators (MLEs) and the modified moment estimators (MMEs), the proposed method results in estimators with smaller bias, but having the same mean squared errors (MSEs) as these two estimators. Next, the existence and uniqueness of the MLEs of the parameters of BS distribution are discussed based on Type-I, Type-II and hybrid censored samples. In the case of five-parameter bivariate Birnbaum-Saunders (BVBS) distribution, we use the distributional relationship between the bivariate normal and BVBS distributions to propose a simple and efficient method of estimation based on Type-II censored samples. Regression analysis is commonly used in the analysis of life-test data when some covariates are involved. For this reason, we consider the regression problem based on BS and BVBS distributions and develop the associated inferential methods.

One may generalize the BS distribution by using Laplace kernel in place of the normal kernel, referred to as the Laplace BS (LBS) distribution, and it is one of the generalized Birnbaum-Saunders (GBS) distributions. Since the LBS distribution has a close relationship with the Laplace distribution, it becomes necessary to first carry out a detailed study of inference for the Laplace distribution before studying the LBS distribution. Several inferential results have been developed in the literature for the Laplace distribution based on complete samples. However, research on Type-II

censored samples is somewhat scarce and in fact there is no work on Type-I censoring. For this reason, we first start with MLEs of the location and scale parameters of Laplace distribution based on Type-II and Type-I censored samples. In the case of Type-II censoring, we derive the exact joint and marginal moment generating functions (MGF) of the MLEs. Then, using these expressions, we derive the exact conditional marginal and joint density functions of the MLEs and utilize them to develop exact confidence intervals (CIs) for some life parameters of interest. In the case of Type-I censoring, we first derive explicit expressions for the MLEs of the parameters, and then derive the exact conditional joint and marginal MGFs and use them to derive the exact conditional marginal and joint density functions of the MLEs. These densities are used in turn to develop marginal and joint CIs for some quantities of interest.

Finally, we consider the LBS distribution and formally show the different kinds of shapes of the probability density function (PDF) and the hazard function. We then derive the MLEs of the parameters and prove that they always exist and are unique. Next, we propose the MMEs, which can be used as initial values in the numerical computation of the MLEs. We also discuss the interval estimation of parameters.

KEY WORDS: BS distribution; BVBS distribution; Cumulative hazard; GBS distribution; Hazard function; Kaplan-Meier curve; Kolmogorov-Smirnov test; Laplace distribution; LBS distribution; Mixture distribution; P-P plot; Q-Q plot.

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Co-authorship and inclusion of previously published material

This thesis is based on the following papers:

1. Balakrishnan, N. and Zhu, X. (2014). An improved method of estimation for the parameters of the Birnbaum-Saunders distribution. *Journal of Statistical Computation and Simulation*, **10**, 2285-2294 (work in Chapter 2).

2. Balakrishnan, N. and Zhu, X. (2014). On the existence and uniqueness of the maximum likelihood estimates of the parameters of Birnbaum-Saunders distribution based on Type-I, Type-II and hybrid censored samples. *Statistics*, **48**, 1013-1032 (work in Chapter 3).

3. Balakrishnan, N. and Zhu, X. (2013). A simple and efficient method of estimation of the parameters of a bivariate Birnbaum-Saunders distribution based on Type-II censored samples. In: *Multivariate Statistics: Theory and Applications* (Ed., T. Kollo), pp, 34-47, World Scientific Publishers, Singapore (work in Chapter 4).

4. Balakrishnan, N. and Zhu, X. (2014). Inference for the Birnbaum-Saunders lifetime regression model with applications. *Communications in Statistics-Simulation and Computation*, to appear (work in Chapter 5).

5. Balakrishnan, N. and Zhu, X. (2014). Inference for the bivariate Birnbaum-Saunders lifetime regression model and associated inference. *Metrika*, to appear (work in Chapter 6).

6. Balakrishnan, N. and Zhu, X. (2014). Exact likelihood-based point and interval estimation for Laplace distribution based on Type-II right censored samples. *Journal of Statistical Computation and Simulation*, to appear (part of work in Chapter 7).

7. Zhu, X. and Balakrishnan, N. (2014). Exact inference for Laplace quantile, reliability and cumulative hazard functions based on Type-II censored data. *IEEE Transactions on Reliability*, to appear (part of work in Chapter 7).

8. Zhu, X. and Balakrishnan, N. (2014). Exact likelihood-based point and interval estimation for lifetime characteristics of Laplace distribution based on a time-constrained life-testing experiment. Under review (work in Chapter 8).

9. Zhu, X. and Balakrishnan, N. (2014). Birnbaum-Saunders distribution based on Laplace kernel and some properties and inferential issues. *Statistics & Probability Letters*, to appear (work in Chapter 9).

The different research works that are used in the thesis are joint papers with my supervisor Professor N. Balakrishnan. The co-authorship of my supervisor in these research publications are due to his role in two ways. In the beginning of my research work, he suggested possible research problems to undertake which was important for me at that stage. However, all the problems were solved by me fully and all the corresponding numerical work and simulations were all carried out by me alone. Then, when I prepared a research paper based on my findings, he again helped by

going over what I had written and offering some suggestions and corrections in the language and presentation, and with this I could finalize the paper and publish it successfully. This was his primary role and that is why he is present as a co-author in these papers. As I progressed with my research work, I also could construct some research problems on my own and worked on them with occasional discussions with my supervisor, and that was the nature of our collaboration in the latter parts. I felt that it was necessary and appropriate to include him as a co-author in the work due to this role he played and also for his help in going over the prepared manuscript and offering once again some valuable suggestions and corrections.

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Chapter 1

Introduction

1.1 Birnbaum-Saunders distribution

The Birnbaum-Saunders distribution, written shortly as BS distribution, proposed by Birnbaum and Saunders (1969a), is a model that has become quite useful in the analysis of reliability data. The cumulative distribution function (CDF) of a two-parameter BS random variable T is given by

$$F(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right], \qquad t > 0,\ \alpha > 0,\ \beta > 0, \tag{1.1.1}$$

where $\Phi(\cdot)$ is the standard normal CDF, and β and α are the scale and shape parameters, respectively.

The BS distribution has found applications in a wide array of problems. Birnbaum and Saunders (1969b) fitted this distribution to several data sets on the fatigue life

of 6061-T6 aluminum coupons. Desmond (1985) extended the model for failures in random environments and investigated the fatigue damage at the root of the cantilever beam. Chang and Tang (1993) applied the model to active repair time of an airborne communication transceiver.

The corresponding PDF is

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right\}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{\beta}{t}+\frac{t}{\beta}-2\right)\right],$$
$$t>0, \alpha>0, \beta>0. \qquad (1.1.2)$$

The following interesting properties of the BS distribution in (1.1.1) are well-known; see, for example, Birnbaum and Saunders (1969a).

Property 1.1.1 *Suppose $T \sim BS(\alpha,\beta)$ as defined in (1.1.1). Then:*

(1) $\frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim N(0,1)$;

(2) $cT \sim BS(\alpha, c\beta)$;

(3) $\frac{1}{T} \sim BS(\alpha, \frac{1}{\beta})$.

By using Result (1) in Property 1.1.1, the expected value and variance of T can be readily obtained as

$$E(T) = \beta\left(1+\frac{1}{2}\alpha^2\right), \qquad (1.1.3)$$
$$Var(T) = (\alpha\beta)^2\left(1+\frac{5}{4}\alpha^2\right). \qquad (1.1.4)$$

Similarly, by using Result (3) in Property 1.1.1, we readily have

$$E(T^{-1}) = \beta^{-1}\left(1+\frac{1}{2}\alpha^2\right), \tag{1.1.5}$$

$$Var(T^{-1}) = \alpha^2\beta^{-2}\left(1+\frac{5}{4}\alpha^2\right). \tag{1.1.6}$$

1.1.1 Bivariate Birnbaum-Saunders distribution

Recently, through a transformation of the bivariate normal distribution, Kundu et al. (2010) derived the bivariate Birnbaum-Saunders distribution, written shortly as BVBS distribution. The bivariate random vector (T_1, T_2) is said to have a BVBS if it has the joint CDF as

$$P(T_1 \le t_1, T_2 \le t_2) = \Phi_2\left[\frac{1}{\alpha_1}\left(\sqrt{\frac{t_1}{\beta_1}}-\sqrt{\frac{\beta_1}{t_1}}\right), \frac{1}{\alpha_2}\left(\sqrt{\frac{t_2}{\beta_2}}-\sqrt{\frac{\beta_2}{t_2}}\right); \rho\right], \quad t_1 > 0, t_2 > 0, \tag{1.1.7}$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ are the shape parameters, $\beta_1 > 0$ and $\beta_2 > 0$ are the scale parameters, $-1 < \rho < 1$ is the dependence parameter, and $\Phi_2(z_1, z_2; \rho)$ is the joint CDF of a standard bivariate normal vector (Z_1, Z_2) with correlation coefficient ρ.

Then, the corresponding joint PDF of (T_1, T_2) is given by

$$\begin{aligned} f_{T_1,T_2}(t_1,t_2) &= \phi_2\left[\frac{1}{\alpha_1}\left(\sqrt{\frac{t_1}{\beta_1}}-\sqrt{\frac{\beta_1}{t_1}}\right), \frac{1}{\alpha_2}\left(\sqrt{\frac{t_2}{\beta_2}}-\sqrt{\frac{\beta_2}{t_2}}\right); \rho\right] \\ &\quad \times \frac{1}{2\alpha_1\beta_1}\left\{\left(\frac{\beta_1}{t_1}\right)^{\frac{1}{2}}+\left(\frac{\beta_1}{t_1}\right)^{\frac{3}{2}}\right\}\frac{1}{2\alpha_2\beta_2}\left\{\left(\frac{\beta_2}{t_2}\right)^{\frac{1}{2}}+\left(\frac{\beta_2}{t_2}\right)^{\frac{3}{2}}\right\}, \\ &\qquad t_1>0,\ t_2>0, \end{aligned} \tag{1.1.8}$$

where $\phi_2(z_1, z_2; \rho)$ is the joint PDF of Z_1 and Z_2 given by

$$\phi_2(z_1,z_2,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}(z_1^2+z_2^2-2\rho z_1 z_2)\right].$$

Then, the following interesting properties of the BVBS in (1.1.7) are well-known; see, for example, Kundu et al. (2010).

Property 1.1.2 *If $(T_1, T_2) \sim BVBS(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$ as defined in (1.1.7), then:*

(1) $\left(\frac{1}{\alpha_1}\left(\sqrt{\frac{T_1}{\beta_1}}-\sqrt{\frac{\beta_1}{T_1}}\right), \frac{1}{\alpha_2}\left(\sqrt{\frac{T_2}{\beta_2}}-\sqrt{\frac{\beta_2}{T_2}}\right)\right) \sim n\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right]$;

(2) $T_i \sim BS(\alpha_i, \beta_i)$, $i = 1, 2$;

(3) $(T_1^{-1}, T_2^{-1}) \sim BVBS(\alpha_1, \beta_1^{-1}, \alpha_2, \beta_2^{-1}, \rho)$;

(4) $(T_1^{-1}, T_2) \sim BVBS(\alpha_1, \beta_1^{-1}, \alpha_2, \beta_2, -\rho)$;

(5) $(T_1, T_2^{-1}) \sim BVBS(\alpha_1, \beta_1, \alpha_2, \beta_2^{-1}, -\rho)$.

Kundu et al. (2010) also further showed that

$$E\left[T_1T_2\right] = \beta_1\beta_2\left[1+\frac{1}{2}(\alpha_1^2+\alpha_2^2)+\frac{1}{4}\alpha_1^2\alpha_2^2(1+\rho^2)+\alpha_1\alpha_2 I_1\right], \tag{1.1.9}$$

where $I_1 = E\left[Z_1Z_2\sqrt{\left(\frac{1}{2}\alpha_1 Z_1\right)^2+1}\sqrt{\left(\frac{1}{2}\alpha_2 Z_2\right)^2+1}\right]$. For non-negative integers m and n, it is known from Kotz et al. (2000), for example, that

$$\begin{aligned} a_{m,n} &= E(Z_1^{2m+1}Z_2^{2n+1}) \\ &= \frac{(2m+1)!(2n+1)!}{2^{m+n}}\sum_{i=0}^{\min[m,n]}\frac{(2\rho)^{2i+1}}{(m-i)!(n-i)!(2i+1)!}. \end{aligned} \tag{1.1.10}$$

Then, we have

$$\begin{aligned} I_1 &= a_{0,0}+\frac{1}{2^3}a_{0,1}(\alpha_1^2+\alpha_2^2)+\frac{1}{2^6}\alpha_1^2\alpha_2^2 a_{1,1} \\ &\quad+\sum_{i=2}^{\infty}(-1)^{i-1}\frac{1\cdot 3\cdots(2i-3)}{2^{3i}i!}a_{0,i}(\alpha_1^{2i}+\alpha_2^{2i}) \\ &\quad+\sum_{i=2}^{\infty}(-1)^{i-1}\frac{1\cdot 3\cdots(2i-3)}{2^{3i+3}i!}a_{1,i}(\alpha_1^2\alpha_2^{2i}+\alpha_2^2\alpha_1^{2i}) \\ &\quad+\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(-1)^{i+j}\frac{1\cdot 3\cdots(2i-3)}{2^{3i}i!}\times\frac{1\cdot 3\cdots(2j-3)}{2^{3j}j!}\alpha_1^{2i}\alpha_2^{2j}a_{i,j}. \end{aligned} \tag{1.1.11}$$

Similarly, we have

$$E\left[\sqrt{T_1T_2}\right] = \sqrt{\beta_1\beta_2}\left\{\frac{1}{4}\alpha_1\alpha_2\rho+I_2\right\}, \tag{1.1.12}$$

where $I_2 = E\left[\left(\sqrt{\left(\frac{1}{2}\alpha_1 Z_1\right)^2+1}\right)\left(\sqrt{\left(\frac{1}{2}\alpha_2 Z_2\right)^2+1}\right)\right]$. It can be shown that

$$\begin{aligned} I_2 &= 1+\frac{1}{2^3}(\alpha_1^2+\alpha_2^2)+\frac{1}{2^6}\alpha_1^2\alpha_2^2(1+2\rho^2) \\ &\quad +\sum_{i=2}^{\infty}(-1)^{i-1}\frac{1\cdot 3\cdots(2i-3)}{2^{3i}i!}b_{0,i}(\alpha_1^{2i}+\alpha_2^{2i}) \\ &\quad +\sum_{i=2}^{\infty}(-1)^{i-1}\frac{1\cdot 3\cdots(2i-3)}{2^{3i+3}i!}b_{1,i}(\alpha_1^2\alpha_2^{2i}+\alpha_2^2\alpha_1^{2i}) \\ &\quad +\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(-1)^{i+j}\frac{1\cdot 3\cdots(2i-3)}{2^{3i}i!}\frac{1\cdot 3\cdots(2j-3)}{2^{3j}j!}\alpha_1^{2i}\alpha_2^{2i}b_{i,j}, \end{aligned} \quad (1.1.13)$$

where $b_{m,n} = E\left(Z_1^{2m}Z_2^{2n}\right) = \frac{(2m)!(2n)!}{2^{m+n}}\sum_{i=0}^{\min(m,n)}\frac{(2\rho)^{2i}}{(m-i)!(n-i)!(2i)!}$; see Kotz et al. (2000). By using these results, we readily obtain the following properties.

Property 1.1.3 *If* $(T_1, T_2) \sim BVBS(\alpha_1, \alpha_2, \beta_1, \beta_2, \rho)$, *then:*

(1) $E\left[\left(\sqrt{\frac{T_1}{\beta_1}}-\sqrt{\frac{\beta_1}{T_1}}\right)\left(\sqrt{\frac{T_2}{\beta_2}}-\sqrt{\frac{\beta_2}{T_2}}\right)\right] = \alpha_1\alpha_2\rho;$

(2) $E\left[\left(\sqrt{\frac{T_1}{\beta_1}}-\sqrt{\frac{\beta_1}{T_1}}\right)\left(\sqrt{\frac{T_2}{\beta_2}}+\sqrt{\frac{\beta_2}{T_2}}\right)\right] = E\left[\left(\sqrt{\frac{T_1}{\beta_1}}+\sqrt{\frac{\beta_1}{T_1}}\right)\left(\sqrt{\frac{T_2}{\beta_2}}-\sqrt{\frac{\beta_2}{T_2}}\right)\right] = 0;$

(3) $E\left[\left(\sqrt{\frac{T_1}{\beta_1}}+\sqrt{\frac{\beta_1}{T_1}}\right)\left(\sqrt{\frac{T_2}{\beta_2}}+\sqrt{\frac{\beta_2}{T_2}}\right)\right] = 4I_2,$

where I_2 is as given in (1.1.13).

1.1.2 Generalized Birnbaum-Saunders distribution

The generalized Birnbaum-Saunders distribution, written shortly as GBS distribution, was proposed by Díaz-García and Leiva-Sánchez (2005, 2007) and Díaz -García

and Domínguez-Molina (2006) by using other symmetric distributions in place the normal kernel, such as Cauchy, Pearson type VII, t, Bessel, Laplace and logistic.

The CDF of a two-parameter GBS random variable T is given by

$$F(t;\alpha,\beta) = G\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right], \qquad t > 0,\ \alpha > 0,\ \beta > 0, \tag{1.1.14}$$

where $G(.)$ is the CDF of any symmetric distributions, and here again α and β are the shape and scale parameters, respectively.

1.2 Laplace distribution

The Laplace distribution, also known as double exponential distribution, is a symmetric distribution, having its PDF as

$$f(x) = \frac{1}{2\sigma} e^{-\frac{|\mu - x|}{\sigma}}, \qquad -\infty < x < \infty, \tag{1.2.1}$$

where μ and σ are the location and scale parameters, respectively. The CDF corresponding to (1.2.1) is

$$F(x) = \begin{cases} \frac{1}{2} e^{-\frac{\mu - x}{\sigma}}, & x \leq \mu, \\ 1 - \frac{1}{2} e^{-\frac{x - \mu}{\sigma}}, & x \geq \mu. \end{cases} \tag{1.2.2}$$

Several inferential results have been developed for Laplace distribution based on

complete and censored samples. Interested readers may refer to Johnson et al. (1995) and Kotz et al. (2001) for detailed overviews of all these developments. Based on a complete sample, Bain and Engelhardt (1973) constructed approximate CIs. Kappenman (1975, 1977) subsequently derived conditional CIs and tolerance intervals. Balakrishnan and Cutler (1995) derived the MLEs based on general Type-II censored samples in an explicit form. Childs and Balakrishnan (1996, 1997, 2000) used these closed-form expressions of the MLEs to develop conditional inference procedures based on Type-II and progressively Type-II censored samples. With a similar motivation, linear estimation methods based on order statistics have also been developed by using the means, variances and covariances of Laplace order statistics derived by Govindarajulu (1963, 1966). For example, Balakrishnan and Chandramouleeswaran (1996) derived the best linear unbiased estimators (BLUEs) of μ and σ based on Type-II censored samples and used them to develop estimators of the reliability function and tolerance limits. Similarly, Balakrishnan et al. (1996) constructed CIs for μ and σ using BLUEs based on Type-II censored samples.

1.3 Common Censoring Schemes

We introduce here different forms of censored data that are discussed in this thesis. First, let us assume n independent units are placed simultaneously on a life-test and their lifetimes are observed. Let $t_{1:n} < t_{2:n} < \cdots < t_{n:n}$ denote the ordered lifetimes of the n units. Suppose the experimenter decides to observe only the first k failures and

then terminate the experiment. Then, the data so observed will be $(t_{1:n}, \cdots, t_{k:n})$ with the largest $n-k$ censored, which are referred to as *Type-II censored data*. Instead, if the experimenter chooses to conduct the life-test for a pre-fixed time T and then terminate the experiment, then the data observed in this manner are said to be *Type-I censored data*, and will be in the form $(t_{1:n}, \cdots, t_{D:n})$, where D is the random number of units that fail before termination time T. Inferential procedures have been discussed for many lifetime distributions based on Type-I and Type-II censored data; see, for example, Balakrishnan and Cohen (1991).

As a compromise between Type-I and Type-II censoring schemes, Epstein (1954) considered the *Type-I hybrid censoring scheme* (Type-I HCS) in which the life-test would be terminated at the k-th failure if it were to occur before time T, and otherwise the termination would occur at time T; that is, the termination time is $\min\{t_{k:n}, T\}$. Since such a Type-I HCS may result in few failures or no failures at all, Childs et al. (2003) proposed the *Type-II hybrid censoring scheme* (Type-II HCS) with termination time as $\max\{t_{k:n}, T\}$. Several inferential procedures have been developed based on these two forms of HCS for many different lifetime distributions, and one may refer to Balakrishnan and Kundu (2012) for a comprehensive review on this topic.

1.4 Some methods used in point and interval estimation

1.4.1 Jackknifing method

From a sample of size n, after dropping the i-th observation (t_i) from the sample, find the corresponding $\hat{\boldsymbol{\theta}}^*_{(\boldsymbol{i})} = (\hat{\theta}^*_{1(i)}, \cdots, \hat{\theta}^*_{p(i)})$ as an estimator of $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)$, from the sample of remaining $(n-1)$ observations by using MLE or any other estimation method. Then, from the set of n estimators $\hat{\boldsymbol{\theta}}^*_{(\mathbf{1})}, \cdots, \hat{\boldsymbol{\theta}}^*_{(\boldsymbol{n})}$ so obtained, we can estimate the variance of the estimator $\hat{\theta}_j$ (for $j = 1, \cdots, p$) as

$$\widehat{Var(\hat{\theta}_j)} = \frac{n}{n-1} \sum_{i=1}^{n} (\hat{\theta}^*_{j(i)} - \bar{\hat{\theta}}^*_j)^2, \tag{1.4.1}$$

where $\bar{\hat{\theta}}^*_j = \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}^*_{j(i)}$; see Efron (1970) for details.

1.4.2 Bootstrap confidence interval

The bootstrap approach, developed by Efron (1970), provides another way of obtaining CIs. Suppose $\hat{\boldsymbol{\theta}}$ is the MLE or any other estimator of $\boldsymbol{\theta}$ obtained from the original sample. We then draw samples

$$t^*_1, t^*_2, \cdots, t^*_n \sim d(\hat{\boldsymbol{\theta}}),$$

where $d(\hat{\boldsymbol{\theta}})$ is the original distribution. If we take B such samples, we can then obtain a $100(1-\alpha)\%$ bootstrap CI for parameter θ_j, for example, as

$$(\hat{\theta}_{j(\frac{\alpha B}{2})}, \hat{\theta}_{j((1-\frac{\alpha B}{2})B)}),$$

where $\hat{\theta}_{j(l)}$ denotes the l-th ordered value of $\hat{\theta}_j$ from the B bootstrap simulations. Furthermore, the bootstrap approach can also be applied to estimate the corresponding standard error (SE) of estimate $\hat{\theta}_j$; see Lehmann (1999) for details.

1.5 Scope of the Thesis

The aim of this thesis is to develop inference for the BS, Laplace and some associated distributions.

In Chapter 2, we propose a simple method of estimation for the parameters of the two-parameter BS distribution by making use of some key properties of the distribution. Compared to the MLEs and the MMEs, the proposed method has smaller bias, but having the same MSEs as these two estimators. We also discuss some methods of construction of CIs. The performance of the estimators are then assessed by means of Monte Carlo simulations. Finally, an example is used to illustrate the method of estimation developed here.

In Chapter 3, we discuss the existence and uniqueness of the MLEs of the parameters of BS distribution based on Type-I, Type-II and hybrid censored samples.

The line of proof is based on the monotonicity property of the likelihood function. We then describe the numerical iterative procedure for determining the MLEs of the parameters, and point out briefly some recently developed simple methods of estimation in the case of Type-II censoring. Some graphical illustrations of the approach are given for three data sets from the reliability literature. Finally, for illustrative purpose, we also present a simulated example in which the MLEs do not exist.

In Chapter 4, we propose a method of estimation for the parameters of a BVBS distribution based on Type-II censored samples. The distributional relationship between the bivariate normal and BVBS distributions is used for the development of these estimators. The performance of the estimators are then assessed by means of Monte Carlo simulations. Finally, an example is used to illustrate the method of estimation developed here.

In Chapter 5, we consider the regression problem based on BS model and discuss the MLEs of the model parameters as well as associated inference. We discuss the likelihood-ratio tests for some hypotheses of interest as well as some interval estimation methods. A Monte Carlo simulation study is then carried out to examine the performance of the proposed estimators and the interval estimation methods. Finally, some numerical data analyses are performed for illustrating all the inferential methods developed here.

In Chapter 6, we extend the regression model to the BVBS distribution. We derive the MLEs of the model parameters and then develop associated inference. Next, we

briefly describe likelihood-ratio tests for some hypotheses of interest as well as some interval estimation methods. Monte Carlo simulations are then carried out to examine the performance of the estimators as well as the interval estimation methods. Finally, a numerical data analysis is performed for illustrating all the inferential methods developed here.

In Chapter 7, we first present explicit expressions for the MLEs of the location and scale parameters of the Laplace distribution based on a Type-II right censored sample under different cases. Next, we derive the exact marginal densities of the MLEs from the MGFs and utilize them to develop exact CIs for the parameters. Then, we derive the same results based on the spacing property of the Laplace distribution and utilize it to develop exact CIs for the quantile of the distribution. Along the same lines, we derive explicit expressions for the MLEs of reliability and cumulative hazard functions and also use them to construct exact CIs for these functions. We then present analogous results based on the BLUEs. Finally, we present two examples to illustrate all the point and interval estimation methods developed here.

In Chapter 8, we first present explicit expressions for the MLEs of the parameters of the Laplace distribution based on a Type-I right censored sample by considering different cases. We derive the conditional joint MGF of these MLEs and use them to derive the bias and MSEs of the MLEs for all the cases. We then derive the exact conditional marginal and joint density functions of the MLEs and utilize them to develop exact marginal and joint conditional CIs for the parameters. Next, a Monte

Carlo simulation study is carried out to evaluate the performance of the developed inferential results. Finally, some examples are presented to illustrate the point and interval estimation methods developed here under Type-I censoring.

In Chapter 9, we introduce the BS distribution based on Laplace kernel and then discuss several properties of this LBS distribution. We first show that the PDF has two kinds of shape and the hazard function has three different kinds of shape. We then derive the MLEs of the model parameters and develop associated inferential methods. We also show that the MLEs always exist and are unique. Simple and explicit MMEs are developed which can be used effectively as initial estimates for the numerical computation of the MLEs. The performance of these estimators are then assessed by means of Monte Carlo simulations. Finally, some examples are presented to illustrate the methods of estimation developed here.

Finally, in Chapter 10, we make some concluding remarks and also indicate some directions for possible future research.

Chapter 2

Improved estimation for the parameters of BS distribution

2.1 Introduction

The MLEs of the parameters α and β were derived originally by Birnbaum and Saunders (1969b) and their asymptotic distributions were obtained by Engelhardt et al. (1981). Ng et al. (2003) derived the MMEs based on complete samples. Subsequently, Ng et al. (2006) and Wang et al. (2006) extended the MMEs to the case of Type-II right censored samples. Rieck (1995) discussed the estimation problem based on symmetrically censored samples. Here, we propose another simple explicit estimation method and show it has a smaller bias compared to the MMEs and the MLEs, especially in the case of small samples.

The rest of this chapter proceeds as follows. In Section 2.2, we describe briefly the MLEs and the MMEs and the corresponding inferential results. In Section 2.3, we present the proposed method of estimation and show that the estimators always exist uniquely. In Section 2.4, we show that the proposed estimator of the shape parameter has a negative bias, and that the bias is smaller than that of the MME. In Section 2.5, we describe the interval estimation of parameters based on the delta method. A Monte Carlo simulation study is carried out in Section 2.6 to examine the bias and MSEs of the proposed estimators, and to compare their performance with those of the MMEs and the MLEs. Finally, in Section 2.7, we illustrate the approach by using a data set from the reliability literature.

2.2 MLEs and MMEs

2.2.1 MLEs

Let $(t_1, t_2, \cdots, t_n)$ be a random sample of size n from the BS distribution with PDF as given in (1.1.2). Then, the MLE of β (denoted by $\hat{\beta}$) can be obtained from the equation

$$\beta^2 - \beta[2r + K(\beta)] + r[s + K(\beta)] = 0, \tag{2.2.1}$$

where $s = \frac{1}{n}\sum_{i=1}^{n} t_i$, $r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}$, and $K(x) = \left[\frac{1}{n}\sum_{i=1}^{n}(x + t_i)^{-1}\right]^{-1}$ for $x \geq 0$. Since this is a non-linear equation, one may have to use either the Newton-Raphson

algorithm or some other numerical method. Once $\hat{\beta}$ is obtained, the MLE of α (denoted by $\hat{\alpha}$) can be obtained explicitly as

$$\hat{\alpha} = \left[\frac{s}{\hat{\beta}} + \frac{\hat{\beta}}{r} - 2\right]^{\frac{1}{2}}. \tag{2.2.2}$$

Note that this estimator always exists since $\frac{s}{\hat{\beta}} + \frac{\hat{\beta}}{r} \geq 2\sqrt{\frac{s}{r}} \geq 2$.

Engelhardt et al. (1981) showed that the asymptotic joint distribution of $\hat{\alpha}$ and $\hat{\beta}$ is bivariate normal given by

$$\sqrt{n}\begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\alpha^2}{2} & 0 \\ 0 & \frac{\beta^2}{0.25+\alpha^{-2}+I(\alpha)} \end{pmatrix}\right], \tag{2.2.3}$$

where $I(\alpha) = 2\int_0^\infty \{[1 + g(\alpha x)]^{-1} - 0.5\}^2 d\Phi(x)$ and $g(y) = 1 + \frac{y^2}{2} + y\left(1 + \frac{y^2}{4}\right)^{\frac{1}{2}}$. Based on the results of an extensive Monte Carlo simulation study, Ng et al. (2003) observed that

$$Bias(\hat{\alpha}) \approx -\frac{\alpha}{n}, \tag{2.2.4}$$

$$Bias(\hat{\beta}) \approx \frac{\alpha^2}{4n}. \tag{2.2.5}$$

2.2.2 MMEs

Ng et al. (2003) proposed the MMEs from Eqs. (1.1.3) and (1.1.5). In this case, the unique MMEs for α and β, denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, are given explicitly by

$$\tilde{\alpha} = \left\{2\left[\left(\frac{s}{r}\right)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{2}}, \tag{2.2.6}$$

$$\tilde{\beta} = (sr)^{\frac{1}{2}}. \tag{2.2.7}$$

The asymptotic joint distribution of $\tilde{\alpha}$ and $\tilde{\beta}$ has been shown to be bivariate normal given by

$$\sqrt{n}\begin{pmatrix} \tilde{\alpha} - \alpha \\ \tilde{\beta} - \beta \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\alpha^2}{2} & 0 \\ 0 & (\alpha\beta)^2\left(\frac{1+\frac{3}{4}\alpha^2}{(1+\frac{1}{2}\alpha^2)^2}\right) \end{pmatrix}\right]. \tag{2.2.8}$$

Based on the results of an extensive Monte Carlo simulation study, Ng et al. (2003) also observed that the MLEs and the MMEs performed very similarly in terms of both bias and MSE, especially for small values of α. Upon inspecting the pattern of the bias of the MMEs, they found that the same formulae in (2.2.4) and (2.2.5) for the bias also apply to these estimators.

2.3 Proposed estimators

Let $T \sim BS(\alpha, \beta)$ as defined in (1.1.1), and $(T_1, \cdots, T_n)$ be a complete sample of size n. Then, let us define

$$Z_{ij} = T_i \frac{1}{T_j}, \qquad for \quad 1 \leq i \neq j \leq n. \tag{2.3.1}$$

It is evident that $Z_{ij} = \frac{1}{Z_{ji}}$, and we thus have $\binom{n}{2}$ pairs (Z_{ij}, Z_{ji}).

By exploiting the fact that $\frac{1}{T} \sim BS(\alpha, \frac{1}{\beta})$ (see Result 3 in Property 1.1.1) and the independence of T_i and T_j, we immediately find

$$E\left(Z_{ij}\right) = E\left(T_i \frac{1}{T_j}\right) = E\left(T_i\right) E\left(\frac{1}{T_j}\right) = \left(1 + \frac{1}{2}\alpha^2\right)^2. \tag{2.3.2}$$

Then, the sample mean of z_{ij} (observed value of Z_{ij}), calculated as

$$\bar{z} = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} z_{ij}, \tag{2.3.3}$$

may be equated to $E(Z_{ij}) = (1 + \frac{1}{2}\alpha^2)^2$ and solved for α to obtain the estimator

$$\hat{\alpha}^* = \left[2\left(\sqrt{\bar{z}} - 1\right)\right]^{\frac{1}{2}}. \tag{2.3.4}$$

Also, since

$$E(\bar{T}) = E\left(\frac{1}{n}\sum_{i=1}^{n} T_i\right) = \beta\left(1 + \frac{1}{2}\alpha^2\right), \tag{2.3.5}$$

we can readily get an estimator of β (denoted by $\hat{\beta}_1^*$) as

$$\hat{\beta}_1^* = \frac{2s}{(\hat{\alpha}^*)^2 + 2} = \frac{s}{\sqrt{\bar{z}}}, \tag{2.3.6}$$

where $s = \frac{1}{n}\sum_{i=1}^{n} t_i$. Moreover, since

$$E\left[\overline{\left(\frac{1}{T}\right)}\right] = \frac{1}{n}\sum_{i=1}^{n} E\left(\frac{1}{T_i}\right) = \frac{1}{\beta}\left(1 + \frac{1}{2\alpha^2}\right), \tag{2.3.7}$$

we can also get another estimator of β (denoted by $\hat{\beta}_2^*$) as

$$\hat{\beta}_2^* = r\left(1 + \frac{1}{2}(\hat{\alpha}^*)^2\right) = r\sqrt{\bar{z}}, \tag{2.3.8}$$

where $r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}$. Now, these two estimators of β can be combined to obtain an estimator of β as

$$\hat{\beta}^* = (\hat{\beta}_1^*\hat{\beta}_2^*)^{\frac{1}{2}} = \left(\frac{s}{\sqrt{\bar{z}}} r\sqrt{\bar{z}}\right)^{\frac{1}{2}} = (sr)^{\frac{1}{2}}, \tag{2.3.9}$$

which is interestingly the same as the MME $\tilde{\beta}$ given in Eq. (2.2.7).

Property 2.3.1 *The proposed estimators always exist uniquely.*

Proof It is equivalent to showing that $\hat{\alpha}^*$ in Eq. (2.3.4) is always non-negative. For this purpose, we note that

$$\begin{aligned}
\hat{\alpha} &= \left[2\left(\sqrt{\bar{z}}-1\right)\right]^{\frac{1}{2}} \\
&= \left[2\left(\sqrt{\frac{1}{n(n-1)}\sum_{1\le i<j\ne n}\left(z_{ij}+\frac{1}{z_{ij}}\right)}-1\right)\right]^{\frac{1}{2}} \\
&\ge \left[2\left(\sqrt{\frac{n(n-1)}{n(n-1)}}-1\right)\right]^{\frac{1}{2}} \\
&= 0,
\end{aligned}$$

as required. ■

2.4 Comparison with MMEs

Ng et al. (2003) observed that the performance of the MMEs is quite similar to that of the MLEs. While the MLEs are obtained by solving a non-linear equation, the MMEs have simple explicit expressions. But, they noted that both estimators are somewhat biased, and especially so in case of small sample sizes. In this section, we examine some properties of the proposed estimate $\hat{\alpha}^*$ in (2.3.4) and compare it with the MME $\tilde{\alpha}$ in (2.2.6).

Property 2.4.1 *Based on a sample $t_1, \cdots, t_n$, we have $\hat{\alpha}^* > \tilde{\alpha}$.*

Proof Proving this result is equivalent to showing that $\bar{z} \geq \frac{s}{r}$, where $\bar{z}$, s and r are as in (2.3.3) and (2.2.1). By applying Cauchy-Schwarz inequality, we find

$$\bar{z} = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} t_i \frac{1}{t_j} = \frac{1}{n^2 - n} \sum_{1 \leq i < j \leq n} \left(t_i \frac{1}{t_j} + t_j \frac{1}{t_i} \right) \geq 1.$$

Now, by utilizing the fact that $\frac{x}{y} > \frac{x+c}{y+c}$, when $x \geq y \geq 0$ and $c > 0$, we obtain

$$\begin{aligned}
\bar{z} &= \frac{1}{n^2 - n} \sum_{1 \leq i < j \leq n} \left(t_i \frac{1}{t_j} + t_j \frac{1}{t_i} \right) \\
&\geq \frac{\sum_{1 \leq i < j \leq n} \left(t_i \frac{1}{t_j} + t_j \frac{1}{t_i} \right) + n}{n^2 - n + n} \\
&= \frac{\sum_{1 \leq i, j \leq n} \left(t_i \frac{1}{t_j} + t_j \frac{1}{t_i} \right)}{n^2} \\
&= \frac{1}{n^2} \sum_{i=1}^{n} t_i \sum_{i=1}^{n} \frac{1}{t_i} \\
&= \frac{s}{r}.
\end{aligned}$$

Hence, the result. ■

Property 2.4.2 $\left(\frac{(\hat{\alpha}^*)^2 + 2}{2} \right)^2$ *is an unbiased estimator of* $\left(\frac{\alpha^2 + 2}{2} \right)^2$.

Proof Evidently, we have

$$E\left[\left(\frac{(\hat{\alpha}^*)^2 + 2}{2} \right)^2 \right] = E(\bar{Z}) = E(T_i) E\left(\frac{1}{T_j} \right) = \left(\frac{\alpha^2 + 2}{2} \right)^2.$$

■

Property 2.4.3 $\hat{\alpha}^*$ *is a negatively biased estimator of* α.

Proof Let $g(x) = \left(\frac{x^2+2}{2}\right)^2$. Then, $g(x)$ is clearly a convex function which is monotone increasing for $x \geq 0$. So, by using Jensen's inequality, we immediately have

$$g[E(\hat{\alpha}^*)] \leq E[g(\hat{\alpha}^*)] = \left(\frac{\alpha^2+2}{2}\right)^2,$$

from which we obtain that $[E(\hat{\alpha}^*)] \leq \alpha$ by the monotonicity of $g(.)$. ■

Property 2.4.4 *The MME $\tilde{\alpha}$ in (2.2.6) is a negatively biased estimator of α, with its bias being greater than that of $\hat{\alpha}^*$ in (2.3.4), i.e., $Bias(\tilde{\alpha}) < Bias(\hat{\alpha}^*) < 0$.*

Proof This can be readily proved by using Properties 2.4.1 and 2.4.3. ■

Property 2.4.4 immediately reveals that the proposed estimator of α has less bias than the MME (and the MLE) of α. This can also be seen in the simulation results presented in Table 2.6.1.

2.5 Interval estimation of parameters

One may use the jackknifing and bootstrap methods described in Chapter 1 for this purpose. In this section, we will construct CIs by using the asymptotic distribution of the estimators. Since $\hat{\beta}^* = \tilde{\beta}$, we have the same asymptotic distribution as

$$\sqrt{n}\left(\hat{\beta}^* - \beta\right) \sim N\left[0, (\alpha\beta)^2\left(\frac{1+\frac{3}{4}\alpha^2}{(1+\frac{1}{2}\alpha^2)^2}\right)\right], \tag{2.5.1}$$

which can be used to construct an asymptotic CI for β.

Next, for deriving the asymptotic distribution of $\bar{Z}$, we first of all observe

$$E(\bar{Z}) = \frac{1}{n(n-1)} \sum_{1 \le i \ne j \le n} E(Z_{ij}) = \left(1 + \frac{1}{2}\alpha^2\right)^2$$

and

$$\begin{aligned} E\left[(\bar{Z})^2\right] &= \frac{1}{n^2(n-1)^2} E\Bigg[\sum_{1 \le i \ne j \ne k \ne l \le n} \frac{t_i t_j}{t_k t_l} + \sum_{1 \le i \ne j \ne k \le n} \frac{t_i^2}{t_j t_k} + \sum_{1 \le i \ne j \ne k \le n} \frac{t_j t_k}{t_i^2} \\ &\quad + 2 \sum_{1 \le i \ne j \ne k \le n} \frac{t_i t_k}{t_k t_j} + \sum_{1 \le i \ne j \le n} \frac{t_i^2}{t_j^2} + n(n-1) \Bigg] \\ &= \left(1 + \frac{1}{2}\alpha^2\right)^2 + \frac{2\alpha^4 \left(1 + \frac{1}{2}\alpha^2\right)^2}{n-1} + \frac{\frac{7}{16}\alpha^8 - \frac{5}{2}\alpha^6 - 6\alpha^4 - 4\alpha^2 - 1}{n(n-1)}. \end{aligned} \tag{2.5.2}$$

These expressions yield

$$Var(\bar{Z}) = \frac{2\alpha^4 \left(1 + \frac{1}{2}\alpha^2\right)^2}{n-1} + \frac{\frac{7}{16}\alpha^8 - \frac{5}{2}\alpha^6 - 6\alpha^4 - 4\alpha^2 - 1}{n(n-1)}. \tag{2.5.3}$$

Note that

$$\bar{Z} = \frac{n}{n-1}\left(\frac{s}{r}\right) - \frac{1}{n-1}. \tag{2.5.4}$$

Since Ng et al. (2003) proved that S and R are asymptotically distributed as bivariate normal, upon using their result and delta method, we obtain from (2.5.4) that

$$\sqrt{n}\left[\bar{Z}-\left(1+\frac{1}{2}\alpha^2\right)^2\right]\underset{n\to\infty}{\longrightarrow}N\left(0,2\alpha^4\left(1+\frac{1}{2}\alpha^2\right)^2\right). \tag{2.5.5}$$

Now, for obtaining the asymptotic distribution of $\hat{\alpha}^*$, upon using Taylor series expansion, we obtain

$$\hat{\alpha}^*=\sqrt{2(\sqrt{\bar{z}}-1)}=g(\bar{z})=g(a)+(\bar{z}-a)g'(a)+\frac{(\bar{z}-a)^2}{2}g''(a)+\cdots, \tag{2.5.6}$$

where $a=\left(1+\frac{1}{2}\alpha^2\right)^2$, and $g'(.)$ and $g''(.)$ are the first and second derivatives of the function $g(.)$. Thus, the asymptotic distribution of $\sqrt{n}(\hat{\alpha}^*-\alpha)$ is $N\left(0,2\alpha^2\right)$. We also have

$$Bias(\hat{\alpha}^*)\approx-\frac{\alpha(2+3\alpha^2)}{4n(2+\alpha^2)}, \tag{2.5.7}$$

which can be used to correct the bias of $\hat{\alpha}^*$ if need be.

2.6 Simulation Study

We carried out an extensive Monte Carlo simulation study for different choices of n and α by setting $\beta=1$, without loss of any generality. For the cases when the sample size n equals 10 and 50, and the values of α are 0.10, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50 and 2.00, the empirical values of the means and MSEs of the proposed estimator

(PE) are presented in Table 2.6.1, along with the corresponding results for the MLEs and the MMEs. These empirical results were determined from 10,000 Monte Carlo simulations. The values of the shape parameter α have been chosen so as to examine the performance of the proposed estimation method under low, moderate and high skewness. The required BS samples were generated from standard normal samples and then using the inverse relationship between the BS and normal variates stated in Property 1.1.1. As observed earlier by Ng et al. (2003), we also see from our empirical results in Table 2.6.1 that the MLEs and MMEs are quite close both in terms of bias and MSE, and that the corresponding estimates of α are quite biased for small sample sizes especially for large values of α. However, a comparison of these results with those of the proposed method in Table 2.6.1 reveals that the proposed method of estimation of α possesses lower bias and similar MSE. It is clear that the proposed estimator of α performs better than the MLE and MME of α in terms of bias, especially in the case of small sample size. Moreover, we observe that the proposed estimators of α and β have nearly the same MSEs as the MLEs and the MMEs.

2.7 An illustrative example

In this section, we illustrate the results established in the preceding sections with a data taken from the reliability literature.

Example 2.7.1 *The data presented in Table A.1.1, due to Birnbaum and Saunders*

Table 2.6.1: Simulated values of means and MSEs (within brackets) of the proposed estimator (PE) in comparison with those of MLEs and MMEs.

		PE	MLEs		MMEs	
n	α	$\hat{\alpha}^*$	$\hat{\alpha}$	$\hat{\beta}$	$\tilde{\alpha}$	$\tilde{\beta}$
10	0.10	0.0976	0.0926	1.0003	0.0926	1.0003
		(0.0006)	(0.0005)	(0.0010)	(0.0005)	(0.0010)
	0.25	0.2438	0.2315	1.0025	0.2315	1.0025
		(0.0034)	(0.0034)	(0.0062)	(0.0034)	(0.0062)
	0.50	0.4855	0.4620	1.0107	0.4620	1.0107
		(0.0136)	(0.0137)	(0.0242)	(0.0137)	(0.0242)
	0.75	0.7242	0.6911	1.0232	0.6911	1.0231
		(0.0303)	(0.0309)	(0.0528)	(0.0309)	(0.0528)
	1.00	0.9600	0.9188	1.0384	0.9186	1.0384
		(0.0538)	(0.0555)	(0.0899)	(0.0554)	(0.0900)
	1.25	1.1934	1.1455	1.0548	1.1449	1.0549
		(0.0844)	(0.0877)	(0.1332)	(0.0875)	(0.1337)
	1.50	1.4251	1.3715	1.0712	1.3702	1.0716
		(0.1224)	(0.1280)	(0.1805)	(0.1274)	(0.1820)
	2.00	1.8850	1.8228	1.1013	1.8186	1.1032
		(0.2220)	(0.2335)	(0.2800)	(0.2317)	(0.2864)
50	0.10	0.0995	0.0985	1.0001	0.0985	1.0001
		(0.0001)	(0.0001)	(0.0002)	(0.0001)	(0.0002)
	0.25	0.2486	0.2462	1.0005	0.2462	1.0005
		(0.0006)	(0.0006)	(0.0012)	(0.0006)	(0.0012)
	0.50	0.4969	0.4922	1.0021	0.4922	1.0021
		(0.0025)	(0.0025)	(0.0048)	(0.0025)	(0.0048)
	0.75	0.7445	0.7378	1.0044	0.7378	1.0045
		(0.0057)	(0.0057)	(0.0100)	(0.0057)	(0.0100)
	1.00	0.9915	0.9833	1.0071	0.9832	1.0072
		(0.0102)	(0.0102)	(0.0161)	(0.0102)	(0.0161)
	1.25	1.2381	1.2285	1.0098	1.2284	1.0101
		(0.0158)	(0.0160)	(0.0223)	(0.0160)	(0.0225)
	1.50	1.4844	1.4737	1.0123	1.4734	1.0127
		(0.0228)	(0.0231)	(0.0281)	(0.0231)	(0.0285)
	2.00	1.9764	1.9641	1.0161	1.9631	1.0172
		(0.0408)	(0.0413)	(0.0374)	(0.0412)	(0.0389)

The PE of β, $\hat{\beta}^*$, is identical to the MME of β, $\tilde{\beta}$.

Table 2.7.1: Estimates of the parameters (based on the PEs, MLEs and MMEs) and the corresponding 95% CIs based on data in Table A.1.1.

	$\hat{\alpha}$	$\hat{\beta}$
PEs	0.1712	131.8193
Jackknife 95% CI	(0.1396,0.2028)	(127.4088,136.2297)
Bootstrap 95% CI	(0.1473,0.1949)	(127.4794,136.2791)
Asymptotic 95% CI	(0.1476,0.1948)	(127.4330,136.2046)
MLEs	0.1704	131.8188
Jackknife 95% CI	(0.1390,0.2018)	(127.4080,136.2296)
Bootstrap 95% CI	(0.1480,0.1923)	(127.5290,135.9603)
Asymptotic 95% CI	(0.1469,0.1939)	(127.4544,136.1832)
MMEs	0.1704	131.8193
Jackknife 95% CI	(0.1390,0.2018)	(127.4088,136.2297)
Bootstrap 95% CI	(0.1480,0.1923)	(127.4794,136.2791)
Asymptotic 95% CI	(0.1469,0.1939)	(127.4330,136.2046)

(1969b), give the fatigue lifetimes of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The complete sample contains 101 observations with maximum stress per cycle as 31,000 psi.

Ng et al. (2003) analyzed these data by using both MLEs and MMEs. Here, we estimate the parameters by the proposed method, and then use the Jackknife, Bootstrap and asymptotic methods to construct 95% *CIs for the parameters* α *and* β*. All these results, presented in Table 2.7.1, are also compared with the corresponding results based on the MLEs and MMEs. We observe in this case that all the obtained estimates and the CIs are quite similar. Finally, the Kolmogorov-Smirnov (KS) test is carried out to verify the assumption of the BS model for this data, and the P-values of the KS test presented in Table 2.7.2 do not reject the BS model assumption made in our analysis.*

Table 2.7.2: KS distances and the corresponding P-values based on the PEs, MLEs and MMEs.

	PEs	MLEs	MMEs
KS-Distance	0.0853	0.0850	0.0850
P-value	0.4544	0.4594	0.4595

Chapter 3

MLEs of BS distribution based on censored samples

3.1 Introduction

Even though the existence and uniqueness of the MLEs have been discussed in the literature for the case of complete samples [see, for example, Birnbaum and Saunders (1969b) and Johnson et al. (1995)], this issue has not been addressed for censored samples. For this reason, motivated by the work of Ng et al. (2006), we present here a formal proof for the result that the MLEs of the parameters α and β of the BS distribution in (1.1.1) based on Type-II censored data are unique when they exist. The method of proof is based on the monotonicity property of the likelihood function given the relationship between the MLEs $\hat{\alpha}$ and $\hat{\beta}$. We also show that the MLEs may

not always exist. When the MLEs do exist, then the uniqueness of the MLE of β does imply the uniqueness of the MLE of (α, β) in turn. Using similar arguments, we also discuss the existence and uniqueness of the MLEs of the parameters based on Type-I and hybrid censored samples.

The rest of this chapter is organized as follows. In Section 3.2, we consider the case of Type-II censoring and present the corresponding likelihood function and a relationship between the MLEs $\hat{\alpha}$ and $\hat{\beta}$. In Section 3.3, we discuss the existence and uniqueness of the MLEs based on Type-II censored samples. In Section 3.4, we present results for the case of Type-I censoring, while the corresponding results for two forms of hybrid censoring are presented in Section 3.5. In Section 3.6, we describe the numerical iterative procedure for determining the MLEs of the parameters, and point out briefly some recently developed simple methods of estimation in the case of Type-II censoring. Finally in Section 3.7, we illustrate the approach by using three data sets from the reliability literature and one simulated data.

3.2 Case of Type-II censoring

For the case of Type-II right censored data $(t_{1:n}, \cdots, t_{k:n})$ from the BS distribution in (1.1.1), we have the likelihood function as

$$L = \frac{n!}{(n-k)!} \left[\prod_{i=1}^{k} f(t_{i:n}) \right] [1 - F(t_{k:n})]^{n-k}, \qquad 0 < t_{1:n} < \cdots < t_{k:n}, \tag{3.2.1}$$

where $F(\cdot)$ and $f(\cdot)$ are the CDF and PDF of the BS distribution given in (1.1.1) and (1.1.2), respectively. From (3.2.1), we obtain the log-likelihood function as

$$\begin{aligned} lnL &= C - k\log\alpha - k\log\beta + \sum_{i=1}^{k}\log\left[\left(\frac{\beta}{t_{i:n}}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t_{i:n}}\right)^{\frac{3}{2}}\right] \\ &\quad -\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left(\frac{t_{i:n}}{\beta} + \frac{\beta}{t_{i:n}} - 2\right) + (n-k)\log\left[1 - F(t_{k:n})\right]. \end{aligned} \tag{3.2.2}$$

From (3.2.2), we get the first derivatives with respect to α and β as

$$\frac{\partial lnL}{\partial\alpha} = -\frac{k}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{k}\left(\frac{t_{i:n}}{\beta} + \frac{\beta}{t_{i:n}} - 2\right) + \frac{(n-k)A\eta}{\alpha}, \tag{3.2.3}$$

$$\begin{aligned} \frac{\partial lnL}{\partial\beta} &= -\frac{k}{\beta} + \sum_{i=1}^{k}\frac{t_{i:n} + 3\beta}{2\beta\left(t_{i:n} + \beta\right)} - \frac{1}{2\alpha^2}\sum_{i=1}^{k}\left(\frac{1}{t_{i:n}} - \frac{t_{i:n}}{\beta^2}\right) \\ &\quad + \frac{(n-k)A}{2\alpha t_{k:n}}\left[\left(\frac{t_{k:n}}{\beta}\right)^{\frac{1}{2}} + \left(\frac{t_{k:n}}{\beta}\right)^{\frac{3}{2}}\right], \end{aligned} \tag{3.2.4}$$

where $A = \frac{\phi(\eta)}{1-\Phi(\eta)}$, $\phi(\cdot)$ is the PDF of the standard normal distribution, and $\eta = \frac{1}{\alpha}\left(\sqrt{\frac{t_{k:n}}{\beta}} - \sqrt{\frac{\beta}{t_{k:n}}}\right)$. Equating (3.2.3) and (3.2.4) to zero, we obtain the likelihood equations, after some simplification, to be

$$-k\alpha^2 + \sum_{i=1}^{k}\left(\frac{t_{i:n}}{\beta} + \frac{\beta}{t_{i:n}} - 2\right) = (n-k)\alpha\left(\sqrt{\frac{\beta}{t_{k:n}}} - \sqrt{\frac{t_{k:n}}{\beta}}\right)A, \tag{3.2.5}$$

$$\alpha^2\sum_{i=1}^{k}\frac{\beta - t_{i:n}}{\beta + t_{i:n}} - \sum_{i=1}^{k}\left(\frac{\beta}{t_{i:n}} - \frac{t_{i:n}}{\beta}\right) = -(n-k)\alpha\left(\sqrt{\frac{\beta}{t_{k:n}}} + \sqrt{\frac{t_{k:n}}{\beta}}\right)A. \tag{3.2.6}$$

From (3.2.5) and (3.2.6), we observe that if the MLEs exist, then A could be expressed as

$$(n-k)A = \frac{\alpha^2 \sum_{i=1}^{k} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{k} \left(1 - \frac{t_{i:n}}{\beta}\right)}{\alpha\sqrt{\frac{t_{k:n}}{\beta}}} \tag{3.2.7}$$

or

$$(n-k)A = \frac{-\alpha^2 \sum_{i=1}^{k} \frac{\beta}{\beta+t_{i:n}} + \sum_{i=1}^{k} \left(\frac{\beta}{t_{i:n}} - 1\right)}{\alpha\sqrt{\frac{\beta}{t_{k:n}}}}. \tag{3.2.8}$$

These two equations together imply that

$$\frac{\alpha^2 \sum_{i=1}^{k} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{k} \left(1 - \frac{t_{i:n}}{\beta}\right)}{(n-k)\alpha\sqrt{\frac{t_{k:n}}{\beta}}} = \frac{-\alpha^2 \sum_{i=1}^{k} \frac{\beta}{\beta+t_{i:n}} + \sum_{i=1}^{k} \left(\frac{\beta}{t_{i:n}} - 1\right)}{(n-k)\alpha\sqrt{\frac{\beta}{t_{k:n}}}}, \tag{3.2.9}$$

which yields a relationship between the MLEs of the parameters $\hat{\alpha}$ and $\hat{\beta}$ as

$$\hat{\alpha} = \left[\frac{\sum_{i=1}^{k} (t_{k:n} - t_{i:n}) \left(\frac{1}{t_{i:n}} - \frac{1}{\hat{\beta}}\right)}{\sum_{i=1}^{k} \frac{t_{k:n}+t_{i:n}}{t_{i:n}+\hat{\beta}}}\right]^{\frac{1}{2}}. \tag{3.2.10}$$

Remark 3.2.1 *As in the case of complete samples, we have the MLE of α in (3.2.10) to be an explicit estimator (of course, as a function of the MLE of β) and so the existence and uniqueness issue reduces to just for one parameter, namely, β.*

Lemma 3.2.1 *Suppose $\hat{\alpha}$ is the MLE of the BS shape parameter α based on a Type-II*

censored sample. Then, $\hat{\alpha}$ is an increasing function of β.

Proof Let us consider, from (3.2.10), the first derivative of $\hat{\alpha}$ with respect to β given by

$$\hat{\alpha}'(\beta) = \frac{\sum_{i=1}^{k}(t_{k:n} - t_{i:n})}{2\beta^2\hat{\alpha}(\beta)\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} + \frac{\hat{\alpha}(\beta)\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{(t_{i:n}+\beta)^2}}{2\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} > 0.$$

This readily shows that $\hat{\alpha}(\beta)$ is an increasing function of β. ■

Lemma 3.2.2 *The MLEs do not exist when $k = 1$, i.e., when only one failure is observed.*

Proof It is readily seen in this case that Eq. (3.2.9) can never be satisfied for any $\alpha > 0$. ■

3.3 Existence and uniqueness of the MLEs under Type-II censoring

Before we present a formal proof, we need the following lemmas.

Lemma 3.3.1 *Let $h(x) = \frac{\phi(x)}{1-\Phi(x)}$ be the hazard function of the standard normal distribution. Then, $h(x)$ is an increasing function of x.*

Proof This is a well-known property, and one may refer to Gupta and Gupta (2001).

■

Lemma 3.3.2 *Let* $g_1(\beta) = \frac{1}{\alpha(\beta)}\left(\sqrt{\frac{t_{k:n}}{\beta}} - \sqrt{\frac{\beta}{t_{k:n}}}\right)$, *where* $\alpha(\beta)$ *is as defined in (3.2.10). Then,* $g_1(\beta)$ *is a decreasing function of* β.

Proof See Appendix. ■

Lemma 3.3.3 *Let* $A = \frac{\phi(\eta)}{1-\Phi(\eta)}$, *where* $\eta = \frac{1}{\alpha(\beta)}\left(\sqrt{\frac{t_{k:n}}{\beta}} - \sqrt{\frac{\beta}{t_{k:n}}}\right)$ *and* $\alpha(\beta)$ *is as defined in (3.2.10). Then,* A *is a decreasing function of* β.

Proof From Lemma 3.3.2, it is known that $\eta(\beta)$ is a decreasing function of β. From Lemma 3.3.1, $\frac{\phi(\eta)}{1-\Phi(\eta)}$ is an increasing function of η. These together imply that A is a decreasing function of β. ■

Lemma 3.3.4 *The function* $g_2(\beta) = \frac{\sqrt{\beta}}{\alpha(\beta)}\sum_{i=1}^{k}\left(1 - \frac{t_{i:n}}{\beta}\right) = \frac{1}{\alpha(\beta)}\sum_{i=1}^{k}\left(\sqrt{\beta} - \frac{t_{i:n}}{\sqrt{\beta}}\right)$ *is an increasing function of* β.

Proof See Appendix. ■

Lemma 3.3.5 *The function* $g_3(\beta) = \alpha(\beta)\sqrt{\beta}\sum_{i=1}^{k}\frac{t_{i:n}}{\beta+t_{i:n}}$ *is an increasing function of* β.

Proof See Appendix. ■

We are now ready to present the main result concerning the existence and uniqueness of the MLE of β.

Theorem 3.3.1 *The MLEs of the parameters α and β, based on a Type-II censored sample with $k \geq 2$ (i.e., with at least two observed failures) exist only when*

$$\frac{1}{\sqrt{t_{k:n}}}\left[\sum_{i=1}^{k} t_{i:n}\sqrt{\frac{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}{\sum_{i=1}^{k}(t_{k:n}+t_{i:n})}}+k\sqrt{\frac{\sum_{i=1}^{k}(t_{k:n}+t_{i:n})}{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}}\right] > (n-k)\frac{\phi\left(\sqrt{\frac{\sum_{i=1}^{k}(t_{k:n}+t_{i:n})\frac{1}{t_{k:n}}}{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}}\right)}{\Phi\left(\sqrt{\frac{\sum_{i=1}^{k}(t_{k:n}+t_{i:n})\frac{1}{t_{k:n}}}{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}}\right)}, \tag{3.3.1}$$

and in this case they are unique.

Proof See Appendix. ■

Corollary 3.3.1 *For complete samples, the MLEs of α and β always exist and are unique.*

Proof In the complete sample case, i.e., $k = n$, the left hand side of (3.2.7) is 0. But, the right hand side of (3.2.7) is an increasing function of β with the left limit being less than 0 and the right limit being greater than 0 as shown in the proof of Theorem 3.3.1. Hence, the solution of (3.2.7) always exists and is unique in this case. This was first established by Birnbaum and Saunders (1969b). ■

Corollary 3.3.2 *Suppose $(t_{1:n}, \cdots, t_{k:n})$ is a Type-II right censored sample from the BS distribution and $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of α and β, respectively, based on this right censored sample. Then, $\left(\frac{1}{t_{k:n}}, \cdots, \frac{1}{t_{1:n}}\right)$ is a Type-II left censored sample from the BS distribution and the MLEs of α and $\frac{1}{\beta}$ in this case are simply $\hat{\alpha}$ and $\frac{1}{\hat{\beta}}$, respectively.*

Proof The likelihood function based on the Type-II left censored sample is

$$\begin{aligned} L &= \frac{n!}{(n-k)!}\left[\prod_{i=1}^{k} f(t_{i:n}^{-1};\alpha,\beta)\right]\left\{\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t_{k:n}^{-1}}{\beta}}-\sqrt{\frac{\beta}{t_{k:n}^{-1}}}\right)\right]\right\}^{n-k} \\ &= \frac{n!}{(n-k)!}\left[\prod_{i=1}^{k} t_{i:n}^{2} f(t_{i:n};\alpha,\frac{1}{\beta})\right]\left\{1-\Phi\left[\frac{1}{\alpha}\left(\sqrt{t_{k:n}\beta}-\frac{1}{\sqrt{t_{k:n}\beta}}\right)\right]\right\}^{n-k}. \end{aligned}$$

Thus, if $(\hat{\alpha},\hat{\beta})$ are the MLEs based on the Type-II right censored sample, then $\left(\hat{\alpha},\frac{1}{\hat{\beta}}\right)$ are the MLEs based on the Type-II left censored sample, since the equations to be solved are the same for both cases. ■

3.4 Case of Type-I censoring

In this section, we consider the MLEs of the parameters of the BS distribution based on Type-I censored samples, with U as the pre-fixed duration of the life-test, and discuss the existence and uniqueness of the MLEs.

Lemma 3.4.1 *If no failures occur before time U out of the n units under test, then the MLEs do not exist.*

Proof In this case, the likelihood function is simply

$$L=\left\{1-\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{U}{\beta}}-\sqrt{\frac{\beta}{U}}\right)\right]\right\}^{n}=\left\{\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{U}}-\sqrt{\frac{U}{\beta}}\right)\right]\right\}^{n}.$$

Clearly, this function is an increasing function of β due to the monotonicity of $\Phi(\cdot)$, and so the MLEs do not exist in this case. ■

Now, suppose during the test up to time U, we observe $(t_{1:n}, \cdots, t_{D:n})$ as the Type-I censored sample with $1 \leqslant D \leqslant n$ as the number of failures. Then, the log-likelihood function is

$$\begin{aligned} lnL &= C - D\log\alpha - D\log\beta + \sum_{i=1}^{D}\log\left[\left(\frac{\beta}{t_{i:n}}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t_{i:n}}\right)^{\frac{3}{2}}\right] \\ &\quad -\frac{1}{2\alpha^2}\sum_{i=1}^{D}\left(\frac{t_{i:n}}{\beta} + \frac{\beta}{t_{i:n}} - 2\right) + (n-D)\log\left[1 - F(U)\right]. \end{aligned} \tag{3.4.1}$$

Taking partial derivatives with respect to α and β, we obtain from (3.4.1) that

$$\frac{\partial lnL}{\partial\alpha} = -\frac{D}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{D}\left(\frac{t_{i:n}}{\beta} + \frac{\beta}{t_{i:n}} - 2\right) + \frac{(n-D)A\eta}{\alpha}, \tag{3.4.2}$$

$$\begin{aligned} \frac{\partial lnL}{\partial\beta} &= -\frac{D}{\beta} + \sum_{i=1}^{D}\frac{t_{i:n} + 3\beta}{2\beta\left(t_{i:n} + \beta\right)} - \frac{1}{2\alpha^2}\sum_{i=1}^{D}\left(\frac{1}{t_{i:n}} - \frac{t_{i:n}}{\beta^2}\right) \\ &\quad + \frac{(n-D)A}{2\alpha U}\left[\left(\frac{U}{\beta}\right)^{\frac{1}{2}} + \left(\frac{U}{\beta}\right)^{\frac{3}{2}}\right], \end{aligned} \tag{3.4.3}$$

where $A = \frac{\phi(\eta)}{1-\Phi(\eta)}$ as before, and $\eta = \frac{1}{\alpha}\left(\sqrt{\frac{U}{\beta}} - \sqrt{\frac{\beta}{U}}\right)$.

As in the case of Type-II censoring, if the MLEs of α and β exist, we can obtain

$$(n-D)A = \frac{\alpha^2\sum_{i=1}^{D}\frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{D}\left(1 - \frac{t_{i:n}}{\beta}\right)}{\alpha\sqrt{\frac{U}{\beta}}} \tag{3.4.4}$$

or

$$(n-D)A = \frac{-\alpha^2 \sum_{i=1}^{D} \frac{\beta}{\beta+t_{i:n}} + \sum_{i=1}^{D} \left(\frac{\beta}{t_{i:n}} - 1\right)}{\alpha\sqrt{\frac{\beta}{U}}}. \tag{3.4.5}$$

These two equations jointly imply that

$$\frac{\alpha^2 \sum_{i=1}^{D} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{D} \left(1 - \frac{t_{i:n}}{\beta}\right)}{\alpha\sqrt{\frac{U}{\beta}}} = \frac{-\alpha^2 \sum_{i=1}^{D} \frac{\beta}{\beta+t_{i:n}} + \sum_{i=1}^{D} \left(\frac{\beta}{t_{i:n}} - 1\right)}{\alpha\sqrt{\frac{\beta}{U}}},$$

which provides a relationship between the MLEs of the parameters α and β as

$$\hat{\alpha} = \left[\frac{\sum_{i=1}^{D} (U - t_{i:n}) \left(\frac{1}{t_{i:n}} - \frac{1}{\hat{\beta}}\right)}{\sum_{i=1}^{D} \frac{U+t_{i:n}}{t_{i:n}+\hat{\beta}}}\right]^{\frac{1}{2}}. \tag{3.4.6}$$

Theorem 3.4.1 *The MLEs of the parameters α and β, based on a Type-I censored sample with at least one observed failure, will exist only when*

$$\frac{1}{\sqrt{U}}\left[\sum_{i=1}^{D} t_{i:n}\sqrt{\frac{\sum_{i=1}^{D}(U-t_{i:n})\frac{1}{t_{i:n}}}{\sum_{i=1}^{D}(U+t_{i:n})}} + D\sqrt{\frac{\sum_{i=1}^{D}(U+t_{i:n})}{\sum_{i=1}^{D}(U-t_{i:n})\frac{1}{t_{i:n}}}}\right] > (n-D)\frac{\phi\left(\sqrt{\frac{\sum_{i=1}^{D}(U+t_{i:n})\frac{1}{U}}{\sum_{i=1}^{D}(U-t_{i:n})\frac{1}{t_{i:n}}}}\right)}{\Phi\left(\sqrt{\frac{\sum_{i=1}^{D}(U+t_{i:n})\frac{1}{U}}{\sum_{i=1}^{D}(U-t_{i:n})\frac{1}{t_{i:n}}}}\right)}, \tag{3.4.7}$$

and in this case they are unique.

Proof See Appendix. ■

3.5 Hybrid censoring cases

In this section, we discuss the existence and uniqueness of the MLEs of the parameters α and β of the BS distribution (1.1.1) based on Type-I and Type-II HCS described earlier in Chapter 1.

3.5.1 MLEs for Type-II HCS

Let the termination time of the life-test be $U_2^* = \max\{T_{k:n}, U\}$, where k $(2 \leq k < n)$ and U are pre-fixed, and D be the number of failures up to time U. Then, the likelihood function is given by

$$L = \begin{cases} \frac{n!}{(n-k)!}\left[\prod_{i=1}^{k} f(t_{i:n};\alpha,\beta)\right]\left\{1-\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t_{k:n}}{\beta}}-\sqrt{\frac{\beta}{t_{k:n}}}\right)\right]\right\}^{n-k} & \text{if } U < t_{k:n}, \\ \frac{n!}{(n-D)!}\left[\prod_{i=1}^{D} f(t_{i:n};\alpha,\beta)\right]\left\{1-\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{U}{\beta}}-\sqrt{\frac{\beta}{U}}\right)\right]\right\}^{n-D} & \text{if } U > t_{k:n}. \end{cases}$$

Following exactly the same lines as used in the last two sections, we obtain in this case that

$$\hat{\alpha} = \begin{cases} \left[\frac{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\right]^{\frac{1}{2}} & \text{if } U < t_{k:n}, \\ \left[\frac{\sum_{i=1}^{D}(U-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}{\sum_{i=1}^{D}\frac{U+t_{i:n}}{t_{i:n}+\beta}}\right]^{\frac{1}{2}} & \text{if } U > t_{k:n}. \end{cases}$$

Theorem 3.5.1 *The MLEs of α and β based on a Type-II HCS exist only when*

$$\begin{cases} (n-k)\dfrac{\phi\left(\sqrt{\dfrac{\sum_{i=1}^{k}\left(\frac{t_{i:n}}{t_{k:n}}+1\right)}{\sum_{i=1}^{k}\left(\frac{t_{k:n}}{t_{i:n}}-1\right)}}\right)}{\Phi\left(\sqrt{\dfrac{\sum_{i=1}^{k}\left(\frac{t_{i:n}}{t_{k:n}}+1\right)}{\sum_{i=1}^{k}\left(\frac{t_{k:n}}{t_{i:n}}-1\right)}}\right)} < \frac{1}{\sqrt{t_{k:n}}}\left[\sum_{i=1}^{k} t_{i:n}\sqrt{\dfrac{\sum_{i=1}^{k}\left(\frac{t_{k:n}}{t_{i:n}}-1\right)}{\sum_{i=1}^{k}(t_{k:n}+t_{i:n})}} + k\sqrt{\dfrac{\sum_{i=1}^{k}(t_{k:n}+t_{i:n})}{\sum_{i=1}^{k}\left(\frac{t_{k:n}}{t_{i:n}}-1\right)}}\right] & \text{if } U < t_{k:n}, \\ (n-D)\dfrac{\phi\left(\sqrt{\dfrac{\sum_{i=1}^{D}\left(\frac{t_{i:n}}{U}+1\right)}{\sum_{i=1}^{D}\left(\frac{U}{t_{i:n}}-1\right)}}\right)}{\Phi\left(\sqrt{\dfrac{\sum_{i=1}^{D}\left(\frac{t_{i:n}}{U}+1\right)}{\sum_{i=1}^{D}\left(\frac{U}{t_{i:n}}-1\right)}}\right)} < \frac{1}{\sqrt{U}}\left[\sum_{i=1}^{D} t_{i:n}\sqrt{\dfrac{\sum_{i=1}^{D}\left(\frac{U}{t_{i:n}}-1\right)}{\sum_{i=1}^{D}(U+t_{i:n})}} + D\sqrt{\dfrac{\sum_{i=1}^{D}(U+t_{i:n})}{\sum_{i=1}^{D}\left(\frac{U}{t_{i:n}}-1\right)}}\right] & \text{if } U > t_{k:n}, \end{cases}$$

and in this case they are unique.

Proof Proof is similar to that of Theorem 3.3.1 and is therefore omitted for the sake of brevity. ■

3.5.2 MLEs for Type-I HCS

Let the termination test of the life-test be $U_1^* = \min\{T_{k:n}, U\}$, where k $(2 \le k \le n)$ and U are pre-fixed, and once again, let D be the number of failures up to time U. Then, the log-likelihood function is given by

$$L = \begin{cases} \frac{n!}{(n-D)!}\left[\prod_{i=1}^{D} f(t_{i:n},\alpha,\beta)\right]\left\{1-\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{U}{\beta}}-\sqrt{\frac{\beta}{U}}\right)\right]\right\}^{n-D} & \text{if } U < t_{k:n}, \\ \frac{n!}{(n-k)!}\left[\prod_{i=1}^{k} f(t_{i:n},\alpha,\beta)\right]\left\{1-\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t_{k:n}}{\beta}}-\sqrt{\frac{\beta}{t_{k:n}}}\right)\right]\right\}^{n-k} & \text{if } U > t_{k:n}. \end{cases}$$

In this case, we obtain

$$\hat{\alpha} = \begin{cases} \left[\dfrac{\sum_{i=1}^{D} (U - t_{i:n}) \left(\frac{1}{t_{i:n}} - \frac{1}{\hat{\beta}} \right)}{\sum_{i=1}^{D} \frac{U + t_{i:n}}{t_{i:n} + \hat{\beta}}} \right]^{\frac{1}{2}} & \text{if } U < t_{k:n}, \\ \left[\dfrac{\sum_{i=1}^{k} (t_{k:n} - t_{i:n}) \left(\frac{1}{t_{i:n}} - \frac{1}{\hat{\beta}} \right)}{\sum_{i=1}^{k} \frac{t_{k:n} + t_{i:n}}{t_{i:n} + \hat{\beta}}} \right]^{\frac{1}{2}} & \text{if } U > t_{k:n}. \end{cases}$$

First of all, it should be noted that, as in Lemmas 3.4.1 and 3.2.2, the MLEs of α and β will not exist when D is either 0 or 1. So, in this case, the following theorem can be established which is presented here without a proof.

Theorem 3.5.2 *The MLEs of α and β based on a Type-I HCS exist only when $D \geq 2$ and*

$$\begin{cases} (n-D) \dfrac{\phi\left(\sqrt{\dfrac{\sum_{i=1}^{D} \left(\frac{t_{i:n}}{U} + 1 \right)}{\sum_{i=1}^{D} \left(\frac{U}{t_{i:n}} - 1 \right)}} \right)}{\Phi\left(\sqrt{\dfrac{\sum_{i=1}^{D} \left(\frac{t_{i:n}}{U} + 1 \right)}{\sum_{i=1}^{D} \left(\frac{U}{t_{i:n}} - 1 \right)}} \right)} < \dfrac{1}{\sqrt{U}} \left[\sum_{i=1}^{D} t_{i:n} \sqrt{\dfrac{\sum_{i=1}^{D} \left(\frac{U}{t_{i:n}} - 1 \right)}{\sum_{i=1}^{D} (U + t_{i:n})}} + D \sqrt{\dfrac{\sum_{i=1}^{D} (U + t_{i:n})}{\sum_{i=1}^{D} \left(\frac{U}{t_{i:n}} - 1 \right)}} \right], & \textit{if } U < t_{k:n} \\ (n-k) \dfrac{\phi\left(\sqrt{\dfrac{\sum_{i=1}^{k} \left(\frac{t_{i:n}}{t_{k:n}} + 1 \right)}{\sum_{i=1}^{k} \left(\frac{t_{k:n}}{t_{i:n}} - 1 \right)}} \right)}{\Phi\left(\sqrt{\dfrac{\sum_{i=1}^{k} \left(\frac{t_{i:n}}{t_{k:n}} + 1 \right)}{\sum_{i=1}^{k} \left(\frac{t_{k:n}}{t_{i:n}} - 1 \right)}} \right)} < \dfrac{1}{\sqrt{t_{k:n}}} \left[\sum_{i=1}^{k} t_{i:n} \sqrt{\dfrac{\sum_{i=1}^{k} \left(\frac{t_{k:n}}{t_{i:n}} - 1 \right)}{\sum_{i=1}^{k} (t_{k:n} + t_{i:n})}} + k \sqrt{\dfrac{\sum_{i=1}^{k} (t_{k:n} + t_{i:n})}{\sum_{i=1}^{k} \left(\frac{t_{k:n}}{t_{i:n}} - 1 \right)}} \right] & \textit{if } U > t_{k:n}, \end{cases}$$

and in this case they are unique.

3.6 Numerical procedure and other methods

In this section, we first describe the numerical procedure employed for the determination of the MLEs of α and β in our study, and then make some comparative comments with other methods of estimation.

First of all, we note that the MLE of α is explicit, albeit a function of $\hat{\beta}$, in all cases considered. So, for determining the MLEs $\hat{\alpha}$ and $\hat{\beta}$, we adopted the following iterative procedure:

1. Treat the given censored data as a pseudo-complete sample, that is, by taking all the censored observations to be $t_{k:n}$ or U corresponding to the Type-II and Type-I censored cases, respectively. Then, determine the MMEs of Ng et al. (2003) given by

$$\hat{\alpha} = \left\{ 2 \left[\left(\frac{s_1}{s_2} \right)^{\frac{1}{2}} - 1 \right] \right\}^{\frac{1}{2}}, \qquad \hat{\beta} = (s_1 s_2)^{\frac{1}{2}}, \tag{3.6.1}$$

   where $s_1 = \frac{1}{n}\left(\sum_{i=1}^{k} t_{i:n} + (n-k)t_{k:n}\right)$, $\frac{1}{s_2} = \frac{1}{n}\left(\sum_{i=1}^{k} t_{i:n}^{-1} + (n-k)t_{k:n}^{-1}\right)$ for the case of Type-II censoring and $s_1 = \frac{1}{n}\left(\sum_{i=1}^{D} t_{i:n} + (n-D)U\right)$, $\frac{1}{s_2} = \frac{1}{n}\left(\sum_{i=1}^{D} t_{i:n}^{-1} + (n-D)U^{-1}\right)$ for the case of Type-I censoring.

   Let us denote these estimates by $\hat{\alpha}^*$ and $\hat{\beta}^*$, and use them as initial values;

2. Use $\hat{\beta}^*$ in the explicit expression of $\hat{\alpha}$ in (3.2.10) for the Type-II censoring case and in (3.4.6) for the case of Type-I censoring case to obtain the estimate $\hat{\alpha}_1$;

3. Taking $\hat{\alpha}_1$ as α, and with $\hat{\beta}^*$ as an initial estimate of β, solve the likelihood equation for β by Newton-Raphson method to find $\hat{\beta}_1$ (one may also instead directly maximize the log-likelihood function with respect to this one parameter β using *maxim* or *maxnr* function in R);

4. Use $\hat{\beta}_1$ now in the explicit expression of $\hat{\alpha}$ to find $\hat{\alpha}_2$, and so on;

5. Continue until convergence to the desired level of accuracy is achieved (i.e., $|\frac{\hat{\beta}_{r+1}-\hat{\beta}_r}{\hat{\beta}_r}| < \epsilon$, where ϵ is the tolerance level, say 10^{-5}).

The estimates developed here are the MLEs and are therefore fully efficient, consistent and asymptotically normally distributed. One can therefore develop asymptotic CIs and tests of hypotheses by using the asymptotic distribution along with the Fisher information. In this regard, the following expressions for the second-order derivatives may be used at the values of the MLEs $\hat{\alpha}$ and $\hat{\beta}$ to determine the Fisher information:

$$-\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{k}{\alpha^2} + \frac{3}{\alpha^4}\sum_{i=1}^{k}\left(\frac{t_{i:n}}{\beta} + \frac{\beta}{t_{i:n}} - 2\right) + \frac{(n-k)A\eta}{\alpha^2}\left(2 - \eta + A\right), \tag{3.6.2}$$

$$-\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = \frac{1}{\alpha^3}\sum_{i=1}^{k}\left(\frac{t_{i:n}}{\beta^2} - \frac{1}{t_{i:n}}\right) + \frac{(n-k)}{2\alpha^2\beta}\left[\sqrt{\frac{\beta}{t_{k:n}}} + \sqrt{\frac{t_{k:n}}{\beta}}\right]\left(A - \eta^2 A + \eta A^2\right), \tag{3.6.3}$$

$$-\frac{\partial^2 \ln L}{\partial \beta^2} = -\sum_{i=1}^{k}\frac{t_{i:n}^2 + 2t_{i:n}\beta - \beta^2}{2\beta^2(t_{i:n}+\beta)^2} + \frac{1}{\alpha^2\beta^3}\sum_{i=1}^{k} t_{i:n} + \frac{(n-k)A}{4\alpha\beta^2}\left[\sqrt{\frac{\beta}{t_{k:n}}} + 3\sqrt{\frac{t_{k:n}}{\beta}}\right] + \frac{n-k}{4\alpha^2\beta^2}\left[\sqrt{\frac{\beta}{t_{k:n}}} + \sqrt{\frac{t_{k:n}}{\beta}}\right]^2\left(-\eta A + A^2\right). \tag{3.6.4}$$

Just in the case of Type-II censoring, some other methods of estimation have been

discussed recently in the literature. Ng et al. (2006) and Wang et al. (2006) have both discussed MMEs and evaluated their performance through Monte Carlo simulations, but their consistency and asymptotic distributions have not been established formally. Recently, Barreto et al. (2013) discussed adjustments to the profile likelihood function for the shape parameter and then obtained the adjusted profile MLEs of the parameters. However, such estimators have not been developed for other forms of censored data considered here.

3.7 Illustrative examples

We illustrate the results established in the preceding sections here with three data examples from the reliability literature. We also use another simulated sample to present a case in which the MLEs of α and β do not exist.

Example 3.7.1 *We will use the data in Example 2.7.1. Ng et al. (2006) analyzed these data as Type-II censored data introducing different censoring proportions. We shall do the same here and, in addition, analyze these data as a Type-I censored data for various choices of censoring time U. The corresponding MLEs $\hat{\alpha}$ and $\hat{\beta}$ for Type-II and Type-I censoring cases are presented in Tables 3.7.1 and 3.7.2, respectively. Figures 3.7.1 and Figure 3.7.2 graphically show the uniqueness of the MLEs for the Type-II and Type-I censoring cases, respectively.*

Example 3.7.2 *In this example, we carry out an extensive Monte Carlo simulation*

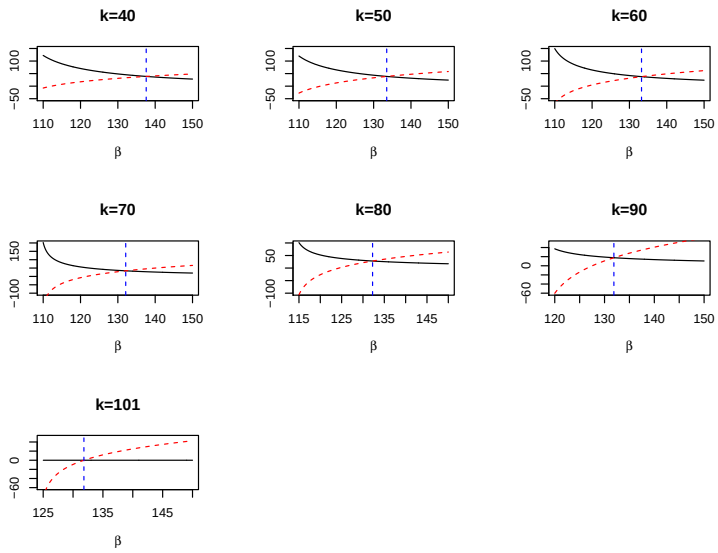

Figure 3.7.1: Graphical check for the uniqueness in the case of Type-II censoring in Example 3.7.1. Here, the solid, red broken and blue vertical lines represent $(n-k)\frac{\phi(\eta)}{1-\Phi(\eta)}$, $\frac{(\alpha(\beta))^2 \sum_{i=1}^{k} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{k}\left(1-\frac{t_{i:n}}{\beta}\right)}{\alpha(\beta)\sqrt{\frac{t_{k:n}}{\beta}}}$ and $\hat{\beta}$, respectively, where $\eta = \frac{1}{\alpha(\beta)}\left(\sqrt{\frac{t_{k:n}}{\beta}} - \sqrt{\frac{\beta}{t_{k:n}}}\right)$ and $\alpha(\beta) = \left[\frac{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\right]^{\frac{1}{2}}$.

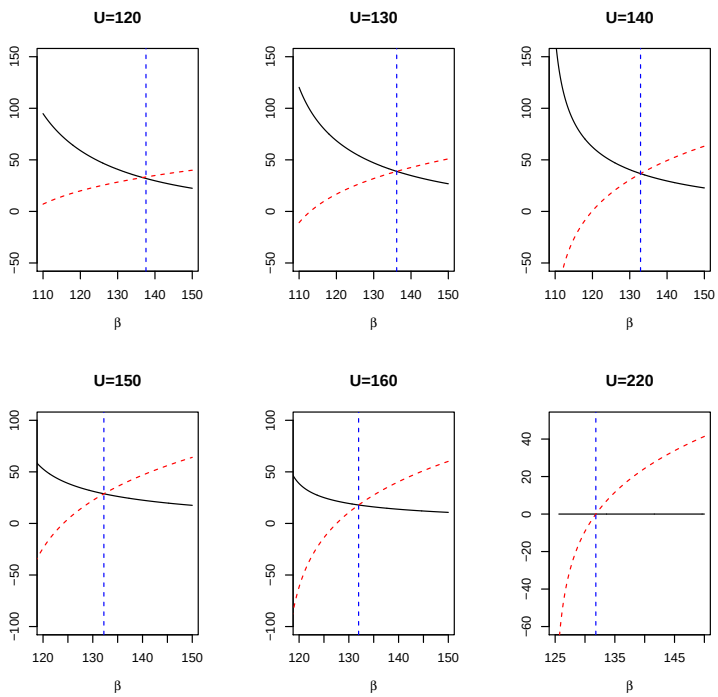

Figure 3.7.2: Graphical check for the uniqueness in the case of Type-I censoring in Example 3.7.1. Here, the solid, red broken and blue vertical lines represent $(n-D)\frac{\phi(\eta)}{1-\Phi(\eta)}$, $\frac{(\alpha(\beta))^2 \sum_{i=1}^{D} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{D}\left(1-\frac{t_{i:n}}{\beta}\right)}{\alpha(\beta)\sqrt{\frac{U}{\beta}}}$ and $\hat{\beta}$, respectively, where $\eta = \frac{1}{\alpha(\beta)}\left(\sqrt{\frac{U}{\beta}} - \sqrt{\frac{\beta}{U}}\right)$ and $\alpha(\beta) = \left[\frac{\sum_{i=1}^{D}(U-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}{\sum_{i=1}^{D}\frac{U+t_{i:n}}{t_{i:n}+\beta}}\right]^{\frac{1}{2}}$.

Table 3.7.1: MLEs of α and β based on Type-II censored sample for different choices of k in Example 3.7.1.

k	$\hat{\alpha}$	$\hat{\beta}$
101	0.1704	131.8188
90	0.1706	131.8776
80	0.1751	132.2527
70	0.1735	132.1069
60	0.1829	133.2411
50	0.1848	133.5043
40	0.2112	137.5925

Table 3.7.2: MLEs of α and β based on Type-I censored sample for different choices of U in Example 3.7.1.

U	$\hat{\alpha}$	$\hat{\beta}$
220	0.1704	131.8188
160	0.1717	131.9651
150	0.1749	132.2323
140	0.1802	132.8903
130	0.2024	136.1655

study based on the estimates of the parameters presented in Tables 3.7.1 and 3.7.2. Specifically, by taking the values of $\hat{\alpha}$ and $\hat{\beta}$ in these tables as the true values of α and β and then simulating the corresponding censored data from the BS distribution. We determine the bias, variances and MSEs of the estimates as well as the average variances determined from the observed Fisher information matrix. These results are presented in Tables 3.7.3 and 3.7.4 for the cases of Type-II and Type-I censoring, respectively. In addition, relative efficiencies of the estimates are also computed as ratios of MSEs of the estimates based on the first row of the table to that of censored data in that row of the table. From the results presented in Tables 3.7.3 and 3.7.4, we observe that the bias of the estimates are quite small even when the censoring

Table 3.7.3: Simulated values of Bias, Variances, MSEs and average variances from the observed Fisher Information, and relative efficiencies of the estimates for the case of Type-II censoring.

k	$Bias(\hat{\alpha})$	$Bias(\hat{\beta})$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$	$\overline{Var(\hat{\alpha})}$	$\overline{Var(\hat{\beta})}$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$RE(\hat{\alpha})$	$RE(\hat{\beta})$
101	-0.0011	0.0251	0.0001	5.0697	0.0001	4.9211	0.0001	5.0698	1.0000	1.0000
90	-0.0014	0.0041	0.0002	5.2174	0.0001	5.0298	0.0002	5.2168	0.8416	0.9718
80	-0.0017	-0.0233	0.0002	5.7591	0.0001	5.4690	0.0002	5.7591	0.6846	0.8803
70	-0.0020	-0.0611	0.0002	6.0384	0.0002	5.5613	0.0002	6.0416	0.5842	0.8392
60	-0.0026	-0.1204	0.0003	7.5842	0.0002	6.5973	0.0003	7.5979	0.4281	0.6673
50	-0.0032	-0.2082	0.0004	9.3086	0.0003	7.2121	0.0004	9.3511	0.3329	0.5422
40	-0.0045	-0.3971	0.0007	16.8205	0.0010	10.8970	0.0007	16.9765	0.1938	0.2986

Table 3.7.4: Simulated values of Bias, Variances, MSEs and average variances from the observed Fisher Information, and relative efficiencies of the estimates for the case of Type-I censoring.

U	$Bias(\hat{\alpha})$	$Bias(\hat{\beta})$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$	$\overline{Var(\hat{\alpha})}$	$\overline{Var(\hat{\beta})}$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$RE(\hat{\alpha})$	$RE(\hat{\beta})$
220	-0.0011	0.0262	0.0001	5.0710	0.0001	4.9215	0.0001	5.0712	1.0000	1.0000
160	-0.0008	0.0495	0.0002	5.3317	0.0001	5.1120	0.0002	5.3336	0.8066	0.9508
150	-0.0008	0.0677	0.0002	5.8644	0.0001	5.5163	0.0002	5.8684	0.6570	0.8641
140	-0.0008	0.0908	0.0003	7.3160	0.0002	6.4018	0.0003	7.3236	0.4618	0.6924
130	-0.0011	0.1577	0.0007	15.3311	0.0007	10.2128	0.0007	15.3544	0.2199	0.3303

proportion is as large as 60%. *Also, we note that the average variances determined from the observed Fisher information are quite close to the simulated variances of the estimates when the censoring proportion is small, but become underestimate as the censoring amount increases. Finally, from the values of relative efficiencies presented, we observe that the loss of information due to censoring is minimal when the censoring is light, but becomes substantial when the amount of censoring increases; moreover, the loss in efficiency is more pronounced in the estimation of the parameter* α.

Example 3.7.3 *The second censored sample, presented in Table A.1.2, is due to Dodson (2006). In this sample, twenty identical grinders were tested, with the test ending when the 12th grinder failed. Now, we assume these data to be a Type-II censored sample from the BS distribution. In addition, we will treat this sample as a Type-I censored sample by terminating the test at time* $U = 120$. *The MLEs* $\hat{\alpha}$ *and* $\hat{\beta}$ *for the Type-II censoring case turn out to be* 1.0957 *and* 122.2826, *respectively,*

while for the Type-I censoring case they are given by 1.2629 *and* 143.1036*, respectively. Figures 3.7.3 and 3.7.4 graphically display the uniqueness of the MLEs with corresponding plots of the log-likelihood function.*

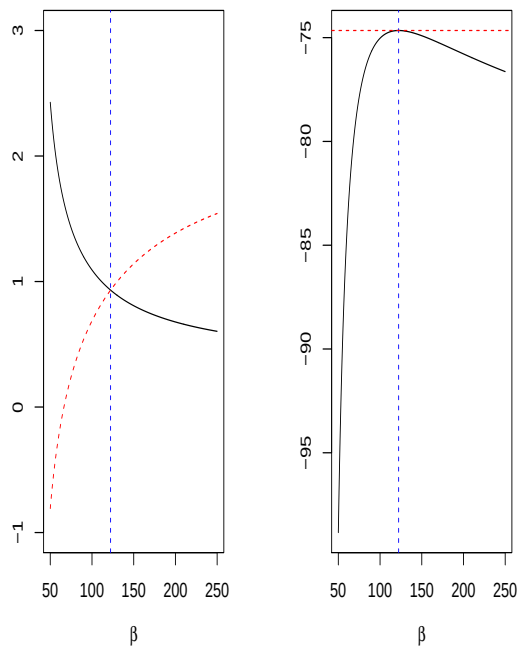

Figure 3.7.3: Graphical check for the uniqueness in the case of Type-II censoring in Example 3.7.3. For the graph on the left, the solid, red broken and blue vertical lines represent $\frac{\phi(\eta)}{1-\Phi(\eta)}$, $\frac{(\alpha(\beta))^2 \sum_{i=1}^{k} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{k}\left(1-\frac{t_{i:n}}{\beta}\right)}{(n-k)\alpha(\beta)\sqrt{\frac{t_{k:n}}{\beta}}}$ and $\hat{\beta}$, respectively, where $\eta = \frac{1}{\alpha(\beta)}\left(\sqrt{\frac{t_{k:n}}{\beta}} - \sqrt{\frac{\beta}{t_{k:n}}}\right)$ and $\alpha(\beta) = \left[\frac{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\right]^{\frac{1}{2}}$. For the graph on the right, the solid, red horizontal and blue vertical lines are $\ln L(\beta)$, the maximal value of the log-likelihood function and $\hat{\beta}$, respectively.

Example 3.7.4 *In this example, with the simulated data presented in Table 3.7.5 representing the first 4 of 50 failures, we will show that the MLEs based on Type-II*

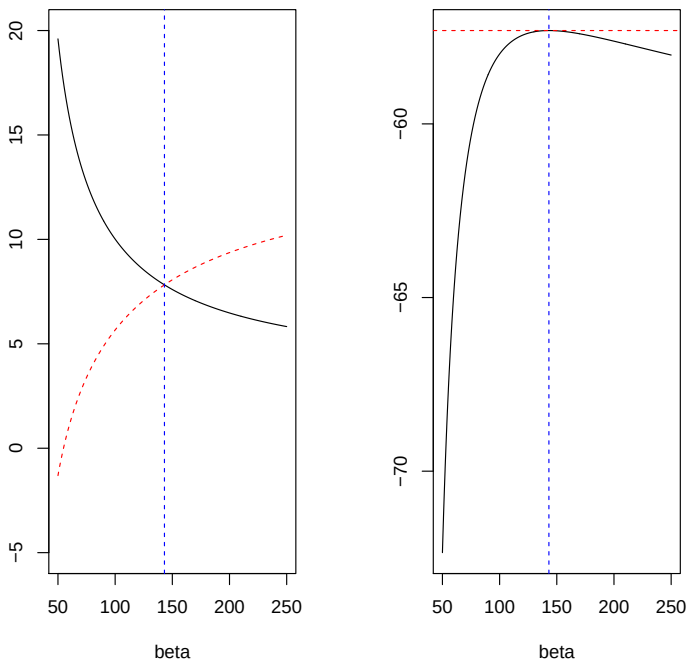

Figure 3.7.4: Graphical check for the uniqueness in the case of Type-I censoring in Example 3.7.3. For the graph on the left, the solid, red broken and blue vertical lines represent $\frac{\phi(\eta)}{1-\Phi(\eta)}$, $\frac{(\alpha(\beta))^2 \sum_{i=1}^{D} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{D}\left(1-\frac{t_{i:n}}{\beta}\right)}{(n-D)\alpha(\beta)\sqrt{\frac{U}{\beta}}}$ and $\hat{\beta}$, respectively, where $\eta = \frac{1}{\alpha(\beta)}\left(\sqrt{\frac{U}{\beta}} - \sqrt{\frac{\beta}{U}}\right)$ and $\alpha(\beta) = \left[\frac{\sum_{i=1}^{D}(U-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}{\sum_{i=1}^{D}\frac{U+t_{i:n}}{t_{i:n}+\beta}}\right]^{\frac{1}{2}}$. For the graph on the right, the solid, red horizontal and blue vertical lines are $\ln L(\beta)$, the maximal value of the log-likelihood function and $\hat{\beta}$, respectively.

Table 3.7.5: Simulated censored sample with $k = 4$ and $n = 50$.

0.5	1	1.5	2

Table 3.7.6: MLEs of α and β for Type-II HCS and Type-I HCS in Example 3.7.5.

k	Type-II hybrid censoring scheme		Type-I hybrid censoring scheme	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
9	1.5314	61.3522	1.3950	54.1846
11	1.5314	61.3522	1.6348	66.9105
13	1.4689	58.2094	1.5314	61.3522
15	1.4398	56.8102	1.5314	61.3522

and Type-I censored samples do not exist. By considering the Type-II censoring case, as well as the Type-I censoring case with test ending at time $U = 3$, we find that the MLEs do not exist in both these cases since the data violate the necessary conditions in Theorems 3.3.1 and 3.4.1, respectively. This is graphically displayed in the two plots in Figure 3.7.5.

Example 3.7.5 *For illustrating the results regarding hybrid censoring schemes,we use the data of Bartholomew (1963), presented in Table A.1.3. Childs et al. (2003) examined this sample as a Type-II HCS wherein 20 items were on a life-test with the test ending at $U = 150$. The corresponding failure times are presented in Table A.1.3. In order to discuss both Type-II and Type-I HCS, we also assume $U = 90$ and consider the situations when $k = 9$, $k = 11$, $k = 13$ and $k = 15$. The MLEs $\hat{\alpha}$ and $\hat{\beta}$ determined for these cases are all presented in Table 3.7.6, and the corresponding plots of the log-likelihood functions are presented in Figures 3.7.6 and 3.7.7.*

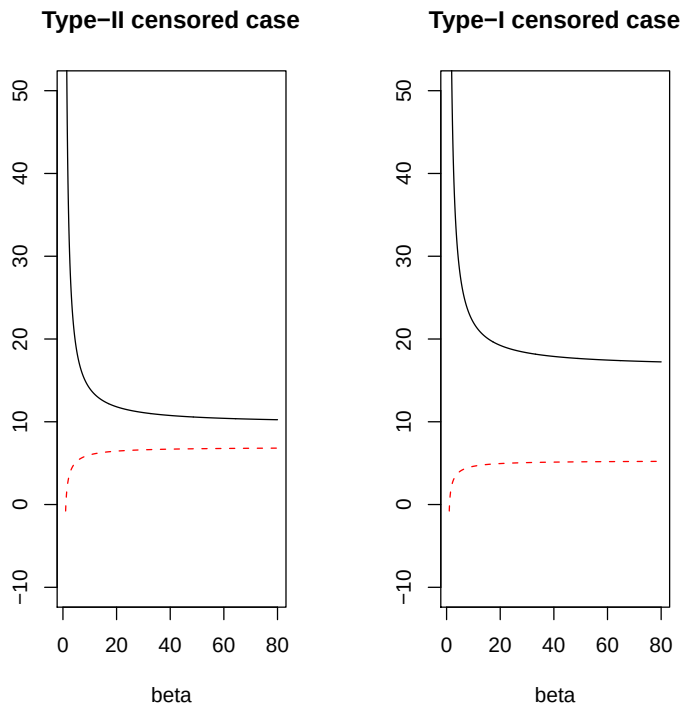

Figure 3.7.5: For the graph on the left for the Type-II censored sample in Example 3.7.4, the solid and broken lines represent $(n-k)\frac{\phi(\eta)}{1-\Phi(\eta)}$ and $\frac{(\alpha(\beta))^2 \sum_{i=1}^{k} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{k}\left(1-\frac{t_{i:n}}{\beta}\right)}{\alpha(\beta)\sqrt{\frac{t_{k:n}}{\beta}}}$, respectively, where $\eta = \frac{1}{\alpha(\beta)}\left(\sqrt{\frac{t_{k:n}}{\beta}} - \sqrt{\frac{\beta}{t_{k:n}}}\right)$ and $\alpha(\beta) = \left[\frac{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\right]^{\frac{1}{2}}$. For the graph on the right for the Type-I censored sample in Example 3.7.4, the solid and broken lines represent $(n-D)\frac{\phi(\eta)}{1-\Phi(\eta)}$ and $\frac{(\alpha(\beta))^2 \sum_{i=1}^{D} \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^{D}\left(1-\frac{t_{i:n}}{\beta}\right)}{\alpha(\beta)\sqrt{\frac{U}{\beta}}}$, respectively, where $\eta = \frac{1}{\alpha(\beta)}\left(\sqrt{\frac{U}{\beta}} - \sqrt{\frac{\beta}{U}}\right)$ and $\alpha(\beta) = \left[\frac{\sum_{i=1}^{D}(U-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}{\sum_{i=1}^{D}\frac{U+t_{i:n}}{t_{i:n}+\beta}}\right]^{\frac{1}{2}}$.

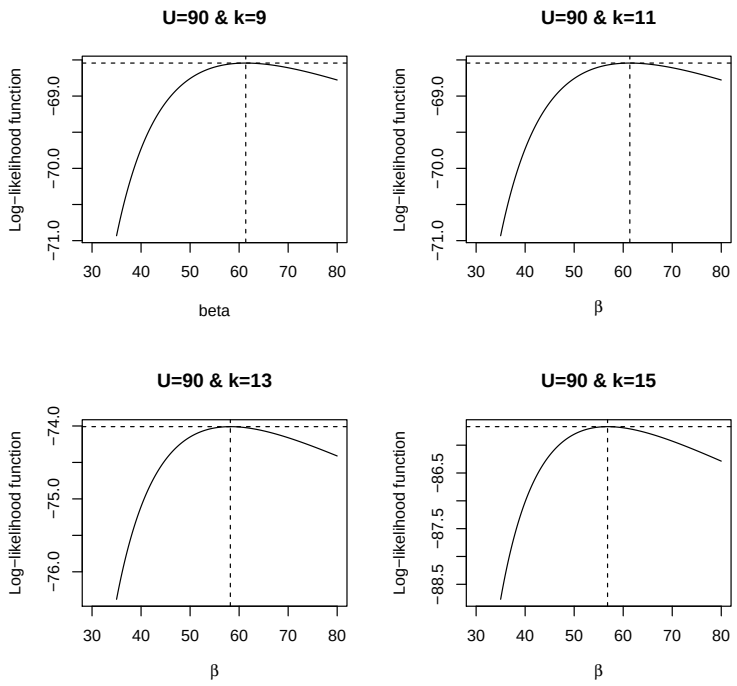

Figure 3.7.6: Log-likelihood function, $\ln L(\beta)$, for Type-II HCS in Example 3.7.5.

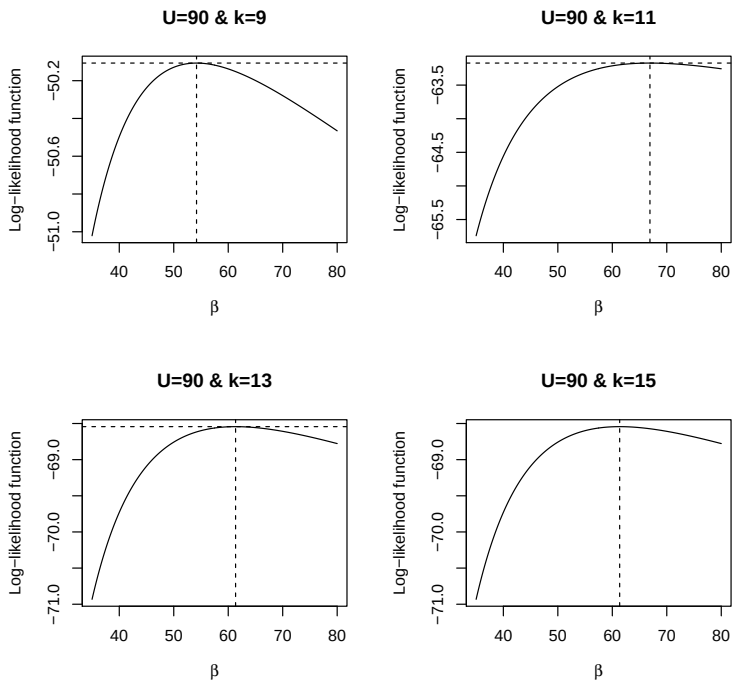

Figure 3.7.7: Log-likelihood function, $\ln L(\beta)$, for Type-I HCS in Example 3.7.5.

Chapter 4

Inference for BVBS model based on Type-II censored data

4.1 Introduction

Through a transformation of the bivariate normal distribution, Kundu et al. (2010) derived a BVBS distribution, and then discussed the MLEs and MMEs of the five parameters of the model based on complete samples. Here, we consider the situation when the available sample is Type-II censored and develop a method of estimation for the model parameters. Note that work done here is based on the BVBS developed by Kundu et al. (2010), whereas Lemonte et al (2015) have introduced a multivariate BS distribution using different construction. The method discussed in this chapter does not apply to their model, and only to the model of Kundu et al. (2013).

Suppose $(T_i^{(1)}, T_i^{(2)})$, $i = 1, \cdots, n$, denote n independent and identically distributed observations from the BVBS distribution in (1.1.7). We now assume that only a Type-II censored sample of the following form is observed instead of a complete sample. Suppose the ordered values of $T^{(2)}$ are directly observable, and that the first k order statistics of $T^{(2)}$-values, denoted by $T_{1:n}^{(2)} < T_{2:n}^{(2)} < \cdots < T_{k:n}^{(2)}$ (for $k \leqslant n$), are observed. For the $T^{(1)}$-variable, the corresponding $T^{(1)}$-values are observed, which are called the *concomitants of order statistics* [see, for example, David and Nagaraja (2003)] and are denoted by $T_{[1:n]}^{(1)}, \cdots, T_{[k:n]}^{(1)}$; that is, we have $T_{[i:n]}^{(1)} = T_j^{(1)}$ if $T_{i:n}^{(2)} = T_j^{(2)}$ for $i, j = 1, 2, \cdots, k$. We thus have the available data from the BVBS in (1.1.7) to be in the form

$$(T_{[1:n]}^{(1)}, T_{1:n}^{(2)}), (T_{[2:n]}^{(1)}, T_{2:n}^{(2)}), \cdots, (T_{[k:n]}^{(1)}, T_{k:n}^{(2)}), \tag{4.1.1}$$

where $k(\leqslant n)$ is the pre-fixed number of order statistics to be observed from $T^{(2)}$-values. Of course, when $k = n$, we will have a complete sample in an ordered form based on the $T^{(2)}$-values.

Based on the form of data in (4.1.1), we develop in this chapter a simple method of estimation for the parameters α_1, α_2, β_1, β_2 and ρ of the BVBS in (1.1.7).

The rest of this chapter proceeds as follows. In Section 4.2, we develop the method of estimation for all five parameters of the BVBS based on Type-II censored samples. In Section 4.3, a Monte Carlo simulation study is carried out to examine the bias and MSEs of the proposed estimators for different choices of the parameters, sample

sizes and different proportions of censoring. Finally, in Section 4.4, an example is presented to illustrate the proposed method of estimation. The obtained estimates are also compared with the MLEs and the MMEs in the complete sample situation.

4.2 Estimation based on Type-II censored samples

From (1.1.7), it is evident that the BVBS distribution has a close relationship with the bivariate normal distribution. Suppose (X, Y) is bivariate normal distributed, then from Harrell and Sen (1979), it is known that based on a Type-II censored data of the form (4.1.1), the MLEs of the parameters of the bivariate normal distribution are given by

$$\hat{\mu}_X = \bar{X} + (S_{XY}/S_Y^2)(\hat{\mu}_Y - \bar{Y}), \tag{4.2.1}$$

$$\hat{\sigma}_X^2 = S_X^2 + (S_{XY}^2/S_Y^2)(\hat{\sigma}_Y^2/S_Y^2 - 1), \tag{4.2.2}$$

$$\hat{\rho} = \hat{\sigma}_Y S_{XY}/(\hat{\sigma}_X S_Y^2), \tag{4.2.3}$$

where

$$\bar{X} = \frac{1}{k}\sum_{i=1}^{k} X_{[i:n]}, \qquad \bar{Y} = \frac{1}{k}\sum_{i=1}^{k} Y_{i:n}, \qquad S_X^2 = \frac{1}{k}\sum_{i=1}^{k}(X_{[i:n]} - \bar{X})^2,$$
$$S_Y^2 = \frac{1}{k}\sum_{i=1}^{k}(Y_{i:n} - \bar{Y})^2, \qquad S_{XY} = \frac{1}{k}\sum_{i=1}^{k}(X_{[i:n]} - \bar{X})(Y_{i:n} - \bar{Y}); \tag{4.2.4}$$

in the above, $\hat{\mu}_Y$ and $\hat{\sigma}_Y$ need to be obtained by solving the corresponding likelihood equations. For more details, one may refer to Harrell and Sen (1979).

Now, we use these known results for the bivariate normal case to develop a simple and efficient method of estimation for the parameters of the BVBS distribution in (1.1.7). For this, we first of all assume that the censoring present in the data is light or moderate and not heavy, that is, $\frac{n}{2} < k \leqslant n$. This is necessary for the ensuing method of estimation. We then proceed as follows.

Let us first of all consider the Type-II right censored sample on $T^{(2)}$ given by $T^{(2)}_{1:n}, T^{(2)}_{2:n}, \cdots, T^{(2)}_{k:n}$. From Property 1.1.2, it is known that

$$T^{(2)} \sim BS(\alpha_2, \beta_2) \qquad \textit{and} \qquad \frac{1}{T^{(2)}} \sim BS\left(\alpha_2, \frac{1}{\beta_2}\right). \tag{4.2.5}$$

Consequently, the order statistics from the $T^{(2)}$-values satisfy a "reciprocal property", and for utilizing it, we can only make use of the order statistics $T^{(2)}_{n-k+1:n}, T^{(2)}_{n-k+2:n}, \cdots$, $T^{(2)}_{k:n}$. Then, we propose the following intuitive estimates of β_2 and α_2:

$$\hat{\beta}_2 = \frac{\sum_{i=n-k+1}^{k} \sqrt{T^{(2)}_{i:n}}}{\sum_{i=n-k+1}^{k} \sqrt{1/T^{(2)}_{i:n}}} \tag{4.2.6}$$

and

$$\hat{\alpha}_2 = \sqrt{\frac{1}{n-1}\left[\sum_{i=1}^{k}\left(\frac{T^{(2)}_{i:n}}{\hat{\beta}_2} + \frac{\hat{\beta}_2}{T^{(2)}_{i:n}} - 2\right) + \sum_{i=1}^{n-k}\left(\frac{T^{(2)}_{i:n}}{\hat{\beta}_2} + \frac{\hat{\beta}_2}{T^{(2)}_{i:n}} - 2\right)\right]}. \tag{4.2.7}$$

Next, for the estimation of α_1, β_1 and ρ, from (1.1.7), we consider the transformation $Z_1 = \sqrt{\frac{T^{(1)}}{\beta_1}} - \sqrt{\frac{\beta_1}{T^{(1)}}}$ and $Z_2 = \sqrt{\frac{T^{(2)}}{\beta_2}} - \sqrt{\frac{\beta_2}{T^{(2)}}}$ which results in $(Z_1, Z_2) \sim N(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{\Sigma} = \begin{bmatrix} \alpha_1^2 & \rho\alpha_1\alpha_2 \\ \rho\alpha_1\alpha_2 & \alpha_2^2 \end{bmatrix}$. Now, by using the MLEs of the parameters of the bivariate normal distribution presented earlier, β_1 can be estimated from the equation

$$\bar{Z}_1 = (S_{Z_1Z_2}/S^2_{Z_2})\bar{Z}_2, \tag{4.2.8}$$

where

$$Z_1 = \sqrt{\frac{T^{(1)}}{\hat{\beta}_1}} - \sqrt{\frac{\hat{\beta}_1}{T^{(1)}}}, \qquad \bar{Z}_1 = \frac{1}{k}\sum_{i=1}^{k} Z_{1_{[i:n]}}, \qquad S^2_{Z_1} = \frac{1}{k}\sum_{i=1}^{k}(Z_{1_{[i:n]}} - \bar{Z}_1)^2,$$
$$Z_2 = \sqrt{\frac{T^{(2)}}{\hat{\beta}_2}} - \sqrt{\frac{\hat{\beta}_2}{T^{(2)}}}, \qquad \bar{Z}_2 = \frac{1}{k}\sum_{i=1}^{k} Z_{2_{i:n}}, \qquad S^2_{Z_2} = \frac{1}{k}\sum_{i=1}^{k}(Z_{2_{i:n}} - \bar{Z}_2)^2,$$
$$S_{Z_1Z_2} = \frac{1}{k}\sum_{i=1}^{k}(Z_{1_{[i:n]}} - \bar{Z}_1)(Z_{2_{i:n}} - \bar{Z}_2). \tag{4.2.9}$$

Finally, the estimates of $\hat{\alpha}_1$ and $\hat{\rho}$ can be determined as

$$\hat{\alpha}_1 = \sqrt{S^2_{Z_1} + (S^2_{Z_1Z_2}/S^2_{Z_2})(\hat{\alpha}_2^2/S^2_{Z_2} - 1)}, \tag{4.2.10}$$

$$\hat{\rho} = \hat{\alpha}_2 S_{Z_1Z_2}/(\hat{\alpha}_1 S^2_{Z_1}). \tag{4.2.11}$$

The implicit form of the solution for $\hat{\beta}_1$ in (4.2.8) makes it difficult to establish the existence and uniqueness of the solution. However, based on extensive Monte Carlo simulations, we have observed that Eq. (4.2.8) always resulted in a unique solution

for $\hat{\beta}_1$. It is important to mention here that the method is applicable in the case of complete sample as well.

4.3 Simulation Study

We carried out an extensive Monte Carlo simulation study for different sample sizes n and values of ρ, by taking $\alpha_1 = 0.25$, $\alpha_2 = 0.25, 1.00$, and $\beta_1 = \beta_2 = 1.00$. We chose sample sizes n to be 20 and 100, and the values of ρ to be 0.95, 0.50, 0.25 and 0.00. With these choices of n and all the parameters, we simulated bivariate Type-II censored data of the form in (4.1.1) with the first k order statistics on Y and the corresponding concomitants on X being observed. Then, by using the method of estimation proposed in the preceding Section, we determined the empirical values of the means and MSEs of the estimates of α_1, α_2, β_1, β_2 and ρ for various degrees of censoring, namely, 0%, 10%, 20%, 30% and 40%. These results are all presented in Tables 4.3.1-4.3.4. From these tables, we observe that the precision of the estimates do not seem to depend on the value of ρ. Moreover, we observe that all the bias values are very small even when the sample size is as small as 20 which reveals that the proposed estimates are very nearly unbiased.

Table 4.3.1: Simulated values of means and MSEs (reported within brackets) of the proposed estimates when $\alpha_1 = \alpha_2 = 0.25$, $\beta_1 = \beta_2 = 1$ and $n = 20$. Here, d.o.c. denotes degree of censoring.

ρ	d.o.c.(%)	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$
0.95	0	0.2474	0.2474	1.0014	1.0016	0.9504
		(0.0016)	(0.0016)	(0.0032)	(0.0032)	(0.0007)
	10	0.2457	0.2465	1.0012	1.0016	0.9482
		(0.0022)	(0.0022)	(0.0033)	(0.0034)	(0.0009)
	20	0.2453	0.2460	1.0013	1.0017	0.9472
		(0.0027)	(0.0026)	(0.0036)	(0.0036)	(0.0012)
	30	0.2450	0.2461	1.0014	1.0020	0.9463
		(0.0033)	(0.0030)	(0.0040)	(0.0039)	(0.0016)
	40	0.2449	0.2463	1.0018	1.0024	0.9451
		(0.0039)	(0.0035)	(0.0048)	(0.0043)	(0.0022)
0.50	0	0.2474	0.2474	1.0011	1.0019	0.4908
		(0.0016)	(0.0016)	(0.0031)	(0.0033)	(0.0321)
	10	0.2436	0.2467	1.0010	1.0020	0.4869
		(0.0020)	(0.0022)	(0.0034)	(0.0034)	(0.0497)
	20	0.2437	0.2464	1.0009	1.0023	0.4768
		(0.0025)	(0.0027)	(0.0043)	(0.0037)	(0.0663)
	30	0.2457	0.2467	1.0029	1.0030	0.4702
		(0.0032)	(0.0032)	(0.0060)	(0.0041)	(0.0873)
	40	0.2477	0.2470	1.0045	1.0034	0.4582
		(0.0040)	(0.0036)	(0.0089)	(0.0046)	(0.1173)
0.25	0	0.2473	0.2474	1.0010	1.0020	0.2449
		(0.0016)	(0.0016)	(0.0031)	(0.0033)	(0.0477)
	10	0.2435	0.2466	1.0014	1.0021	0.2440
		(0.0019)	(0.0022)	(0.0035)	(0.0034)	(0.0709)
	20	0.2446	0.2465	1.0014	1.0025	0.2349
		(0.0023)	(0.0027)	(0.0046)	(0.0037)	(0.0973)
	30	0.2460	0.2467	1.0025	1.0031	0.2296
		(0.0028)	(0.0032)	(0.0064)	(0.0041)	(0.1201)
	40	0.2491	0.2470	1.0051	1.0035	0.2269
		(0.0035)	(0.0036)	(0.0102)	(0.0046)	(0.1564)
0.00	0	0.2473	0.2474	1.0010	1.0020	0.0012
		(0.0017)	(0.0016)	(0.0031)	(0.0033)	(0.0533)
	10	0.2432	0.2466	1.0006	1.0021	-0.0022
		(0.0019)	(0.0022)	(0.0035)	(0.0034)	(0.0790)
	20	0.2444	0.2465	1.0017	1.0026	0.0011
		(0.0021)	(0.0027)	(0.0045)	(0.0037)	(0.1041)
	30	0.2475	0.2469	1.0031	1.0033	0.0026
		(0.0026)	(0.0032)	(0.0067)	(0.0041)	(0.1343)
	40	0.2505	0.2471	1.0041	1.0037	0.0016
		(0.0033)	(0.0036)	(0.0106)	(0.0046)	(0.1736)

Table 4.3.2: Simulated values of means and MSEs (reported within brackets) of the proposed estimates when $\alpha_1 = \alpha_2 = 0.25$, $\beta_1 = \beta_2 = 1$ and $n = 100$. Here, d.o.c. denotes degree of censoring.

ρ	d.o.c.(%)	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$
0.95	0	0.2500	0.2495	1.0005	1.0007	0.9496
		(0.0003)	(0.0003)	(0.0006)	(0.0006)	(0.0001)
	10	0.2497	0.2496	1.0007	1.0009	0.9499
		(0.0005)	(0.0005)	(0.0007)	(0.0007)	(0.0001)
	20	0.2499	0.2499	1.0010	1.0012	0.9497
		(0.0006)	(0.0005)	(0.0007)	(0.0007)	(0.0002)
	30	0.2499	0.2500	1.0010	1.0014	0.9496
		(0.0007)	(0.0006)	(0.0008)	(0.0008)	(0.0002)
	40	0.2501	0.2502	1.0013	1.0016	0.9495
		(0.0008)	(0.0007)	(0.0010)	(0.0009)	(0.0002)
0.50	0	0.2500	0.2491	1.0003	1.0009	0.4988
		(0.0003)	(0.0003)	(0.0006)	(0.0006)	(0.0057)
	10	0.2495	0.2493	1.0006	1.0010	0.4995
		(0.0004)	(0.0004)	(0.0007)	(0.0007)	(0.0087)
	20	0.2498	0.2496	1.0010	1.0014	0.4983
		(0.0005)	(0.0005)	(0.0008)	(0.0007)	(0.0120)
	30	0.2498	0.2498	1.0011	1.0016	0.4959
		(0.0006)	(0.0006)	(0.0012)	(0.0008)	(0.0155)
	40	0.2503	0.2499	1.0013	1.0018	0.4920
		(0.0008)	(0.0007)	(0.0018)	(0.0009)	(0.0214)
0.25	0	0.2500	0.2491	1.0002	1.0009	0.2497
		(0.0003)	(0.0003)	(0.0006)	(0.0006)	(0.0089)
	10	0.2492	0.2492	1.0004	1.0010	0.2502
		(0.0004)	(0.0004)	(0.0007)	(0.0007)	(0.0139)
	20	0.2500	0.2494	1.0012	1.0014	0.2515
		(0.0004)	(0.0005)	(0.0009)	(0.0007)	(0.0189)
	30	0.2501	0.2496	1.0011	1.0016	0.2488
		(0.0005)	(0.0006)	(0.0013)	(0.0008)	(0.0249)
	40	0.2508	0.2497	1.0015	1.0017	0.2466
		(0.0006)	(0.0007)	(0.0020)	(0.0009)	(0.0331)
0.00	0	0.2500	0.2490	1.0001	1.0009	0.0009
		(0.0003)	(0.0003)	(0.0006)	(0.0006)	(0.0100)
	10	0.2492	0.2491	1.0004	1.0010	0.0024
		(0.0003)	(0.0004)	(0.0007)	(0.0007)	(0.0160)
	20	0.2496	0.2493	1.0008	1.0013	0.0035
		(0.0004)	(0.0005)	(0.0009)	(0.0007)	(0.0217)
	30	0.2503	0.2495	1.0018	1.0015	0.0072
		(0.0004)	(0.0006)	(0.0013)	(0.0008)	(0.0285)
	40	0.2510	0.2495	1.0010	1.0016	0.0029
		(0.0005)	(0.0007)	(0.0020)	(0.0009)	(0.0378)

Table 4.3.3: Simulated values of means and MSEs (reported within brackets) of the proposed estimates when $\alpha_1 = 0.25$, $\alpha_2 = 1.00$, $\beta_1 = \beta_2 = 1$ and $n = 20$. Here, d.o.c. denotes degree of censoring.

ρ	d.o.c.(%)	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$
0.95	0	0.2474	0.9861	1.0014	1.0216	0.9472
		(0.0016)	(0.0257)	(0.0032)	(0.0457)	(0.0007)
	10	0.2458	0.9840	1.0013	1.0242	0.9481
		(0.0022)	(0.0367)	(0.0034)	(0.0521)	(0.0009)
	20	0.2453	0.9829	1.0013	1.0267	0.9471
		(0.0027)	(0.0457)	(0.0037)	(0.0584)	(0.0012)
	30	0.2449	0.9840	1.0013	1.0303	0.9462
		(0.0033)	(0.0540)	(0.0040)	(0.0651)	(0.0016)
	40	0.2447	0.9860	1.0014	1.0348	0.9451
		(0.0039)	(0.0636)	(0.0046)	(0.0740)	(0.0022)
0.50	0	0.2474	0.9861	1.0011	1.0227	0.4907
		(0.0016)	(0.0260)	(0.0031)	(0.0461)	(0.0321)
	10	0.2436	0.9848	1.0010	1.0260	0.4870
		(0.0020)	(0.0373)	(0.0035)	(0.0531)	(0.0497)
	20	0.2438	0.9849	1.0009	1.0298	0.4769
		(0.0025)	(0.0471)	(0.0043)	(0.0602)	(0.0663)
	30	0.2456	0.9872	1.0029	1.0353	0.4702
		(0.0032)	(0.0571)	(0.0060)	(0.0692)	(0.0873)
	40	0.2476	0.9893	1.0044	1.0406	0.4580
		(0.0040)	(0.0674)	(0.0089)	(0.0801)	(0.1173)
0.25	0	0.2473	0.9861	1.0010	1.0230	0.2449
		(0.0016)	(0.0262)	(0.0031)	(0.0461)	(0.0477)
	10	0.2435	0.9845	1.0014	1.0261	0.2441
		(0.0019)	(0.0375)	(0.0035)	(0.0532)	(0.0709)
	20	0.2446	0.9852	1.0015	1.0305	0.2351
		(0.0023)	(0.0476)	(0.0046)	(0.0608)	(0.0973)
	30	0.2460	0.9875	1.0025	1.0360	0.2296
		(0.0028)	(0.0577)	(0.0064)	(0.0702)	(0.1200)
	40	0.2490	0.9897	1.0051	1.0411	0.2269
		(0.0035)	(0.0679)	(0.0101)	(0.0807)	(0.1562)
0.00	0	0.2473	0.9861	1.0010	1.0232	0.0012
		(0.0017)	(0.0264)	(0.0031)	(0.0461)	(0.0532)
	10	0.2432	0.9845	1.0006	1.0263	-0.0022
		(0.0019)	(0.0376)	(0.0035)	(0.0533)	(0.0790)
	20	0.2444	0.9853	1.0017	1.0309	0.0013
		(0.0021)	(0.0479)	(0.0045)	(0.0613)	(0.1040)
	30	0.2475	0.9881	1.0031	1.0367	0.0028
		(0.0026)	(0.0579)	(0.0067)	(0.0706)	(0.1342)
	40	0.2505	0.9901	1.0042	1.0416	0.0017
		(0.0033)	(0.0679)	(0.0106)	(0.0808)	(0.1734)

Table 4.3.4: Simulated values of means and MSEs (reported within brackets) of the proposed estimates when $\alpha_1 = 0.25$, $\alpha_2 = 1.00$, $\beta_1 = \beta_2 = 1$ and $n = 100$. Here, d.o.c. denotes degree of censoring.

ρ	d.o.c.(%)	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$
0.95	0	0.2498	0.9972	1.0005	1.0055	0.9495
		(0.0003)	(0.0050)	(0.0006)	(0.0084)	(0.0001)
	10	0.2498	0.9984	1.0007	1.0070	0.9498
		(0.0005)	(0.0075)	(0.0007)	(0.0098)	(0.0001)
	20	0.2499	1.0001	1.0010	1.0090	0.9497
		(0.0006)	(0.0095)	(0.0007)	(0.0112)	(0.0002)
	30	0.2498	1.0009	1.0010	1.0103	0.9496
		(0.0007)	(0.0113)	(0.0008)	(0.0126)	(0.0002)
	40	0.2500	1.0018	1.0012	1.0116	0.9495
		(0.0008)	(0.0131)	(0.0009)	(0.0142)	(0.0002)
0.50	0	0.2500	0.9959	1.0003	1.0061	0.4987
		(0.0004)	(0.0051)	(0.0006)	(0.0086)	(0.0051)
	10	0.2495	0.9971	1.0006	1.0076	0.4995
		(0.0004)	(0.0073)	(0.0007)	(0.0101)	(0.0087)
	20	0.2498	0.9988	1.0010	1.0096	0.4982
		(0.0005)	(0.0091)	(0.0009)	(0.0115)	(0.0120)
	30	0.2498	0.9998	1.0010	1.0112	0.4958
		(0.0006)	(0.0108)	(0.0012)	(0.0130)	(0.0155)
	40	0.2502	1.0007	1.0012	1.0125	0.4918
		(0.0008)	(0.0124)	(0.0018)	(0.0145)	(0.0214)
0.25	0	0.2500	0.9956	1.0002	1.0063	0.2496
		(0.0003)	(0.0051)	(0.0006)	(0.0086)	(0.0089)
	10	0.2492	0.9966	1.0004	1.0077	0.2502
		(0.0004)	(0.0073)	(0.0007)	(0.0102)	(0.0139)
	20	0.2500	0.9981	1.0012	1.0095	0.2514
		(0.0004)	(0.0091)	(0.0009)	(0.0116)	(0.0189)
	30	0.2501	0.9993	1.0011	1.0112	0.2487
		(0.0005)	(0.0107)	(0.0013)	(0.0131)	(0.0249)
	40	0.2508	0.9997	1.0015	1.0121	0.2465
		(0.0006)	(0.0123)	(0.0020)	(0.0146)	(0.0331)
0.00	0	0.2500	0.9955	1.0001	1.0064	0.0009
		(0.0003)	(0.0051)	(0.0006)	(0.0087)	(0.0101)
	10	0.2492	0.9962	1.0004	1.0076	0.0024
		(0.0003)	(0.0073)	(0.0007)	(0.0103)	(0.0160)
	20	0.2496	0.9977	1.0008	1.0094	0.0035
		(0.0004)	(0.0091)	(0.0009)	(0.0117)	(0.0217)
	30	0.2503	0.9985	1.0018	1.0108	0.0071
		(0.0004)	(0.0107)	(0.0013)	(0.0132)	(0.0285)
	40	0.2510	0.9990	1.0010	1.0119	0.0029
		(0.0005)	(0.0125)	(0.0020)	(0.0149)	(0.0378)

4.4 Illustrative Data Analysis

In this Section, we will illustrative the proposed method of estimation by considering a data used earlier by Kundu et al. (2010).

Example 4.4.1 *These data, presented in Table A.1.4, given by Johnson and Wichern (1999), represent the bone mineral density (BMD) measured in g/cm^2 for* 24 *individuals, who had participated in an experimental study. The first figure represents the BMD of the bone dominant radius before starting the study and the second figure represents the BMD of the same bone after one year. Kundu et al. (2010) analyzed this data by assuming a BVBS distribution for the given complete bivariate data.*

First, based on the complete bivariate data in Table A.1.4, we determined the MLEs, MMEs, and the proposed estimates of the parameters α_1, α_2, β_1, β_2 and ρ. These results are all presented in Table 4.4.1 from which we immediately observe that the proposed estimates are indeed very nearly the same as the other two sets of estimates in this complete sample situation. Incidentally, the values of MLEs and MMEs in this case have also been reported earlier by Kundu et al. (2010).

Next, with the estimates obtained by these three methods, we also carried out the KS test for the goodness-of-fit of the BS distribution for the data on individual components X and Y in Table A.1.4. The computed values of the KS distances and the corresponding P-values for all three methods of estimation are presented in Table 4.4.2, and these results do not reject the model assumption made in our analysis.

Finally, for the purpose of illustrating the proposed method of estimation for Type-

Table 4.4.1: Estimates of the parameters based on the data in Table A.1.4.

Estimation	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\rho}$
MMEs	0.1491	0.8316	0.1674	0.8293	0.9343
MLEs	0.1491	0.8313	0.1674	0.8292	0.9343
Proposed estimates	0.1491	0.8321	0.1674	0.8302	0.9343

II censored data, we introduced various levels of censoring in the data presented in Table A.1.4 by taking $k = 22$, 20, 18 *and* 16. *We then determined the estimates of the parameters* α_1, α_2, β_1, β_2 *and* ρ *based on these censored data, and these are all presented in Table 4.4.3. Their SEs were determined by Monte Carlo simulations, and these values are also reported in Table 4.4.3. Upon comparing the estimates of the parameters in Table 4.4.3 with the corresponding ones in Table 4.4.1 based on the complete data, we observe that the estimates of the parameters are all close even in the case of* $k = 16$ *(i.e.,* 8 *of the* 24 *observations censored). We also note that, even though the estimates remain fairly stable when the amount of censoring increases, the SEs of the estimates do increase, as one would expect. Finally, in Figure 4.4.1, we have demonstrated the existence and uniqueness of the estimate of* β_1 *when solving the corresponding estimating equation.*

Table 4.4.2: KS distance and the corresponding P-value for the BS goodness-of-fit for the data on components X and Y in Table A.1.4.

	X		Y	
Estimation	Distance	P-value	Distance	P-value
MMEs	0.1537	0.5696	0.1650	0.4805
MLEs	0.1530	0.5757	0.1654	0.4777
Proposed estimates	0.1549	0.5599	0.1629	0.4969

Table 4.4.3: Estimates of the parameters and SEs (reported within brackets) based on Type-II censored data on Y, where k denotes the rank of the last observed order statistic from Y.

k	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\rho}$
22	0.1808	0.8438	0.2175	0.8496	0.9589
	(0.0302)	(0.0323)	(0.0364)	(0.0393)	(0.0206)
20	0.1871	0.8484	0.2242	0.8550	0.9583
	(0.0347)	(0.0347)	(0.0419)	(0.0409)	(0.0240)
18	0.1861	0.8471	0.2260	0.8569	0.9537
	(0.0375)	(0.0359)	(0.0442)	(0.0437)	(0.0303)
16	0.1891	0.8496	0.2296	0.8609	0.9464
	(0.0414)	(0.0389)	(0.0475)	(0.0463)	(0.0463)

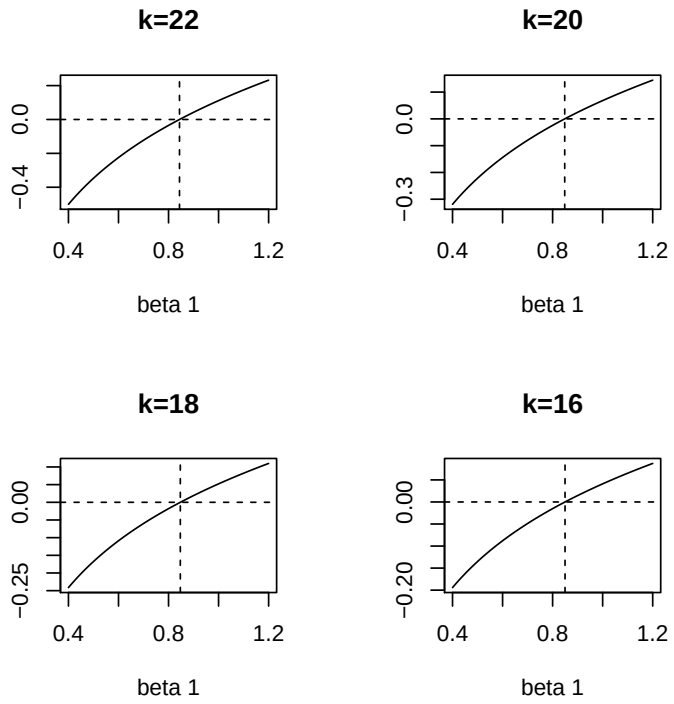

Figure 4.4.1: Uniqueness check for the estimate $\hat{\beta}_1$ for Example 4.4.1.

Chapter 5

Inference for BS regression model

5.1 Introduction

For incorporating covariates into the analysis, Rieck (1989), Rieck and Nedelman (1991) and Lemonte (2013) discussed a log-linear model based on the BS distribution. Tsionas (2001) considered Bayesian inference in this context. Galea et al. (2004) and Leiva et al. (2007) developed influence diagnostics for the log-BS regression model. The analysis of the BS regression model with current status data has been carried out by Xiao et al. (2010). Recently, Leiva et al. (2014) introduced a new approach for BS regression models based on the original scale of the data and can model non-constant variance.

Motivated by these works, we develop here some inferential methods for the BS lifetime regression model with equal and unequal shape parameters. We discuss the

MLEs of the model parameters under a log-linear link function. Interval estimation of parameters and hypotheses tests are also discussed.

The rest of this chapter proceeds as follows. In Section 5.2, after first presenting the BS regression model with log-linear link function, we discuss the MLEs of the model parameters and also explain the process of setting up the initial values required for the iterative process. In Section 5.3, we describe the hypothesis testing procedure and interval estimation methods by using the asymptotic properties of MLEs as well as the bootstrap approach. In Section 5.4, we consider the case of unequal shape parameters in the model, and then discuss the MLEs of the model parameters along with the corresponding initial values required for the iterative process. In Section 5.5, we discuss the hypothesis testing and interval estimation methods in the case of unequal shape parameters in the model. A simulation study is carried out in Section 5.6 for evaluating the performance of the proposed point estimation methods based on bias and MSE and also the CIs based on coverage probabilities. In Section 5.7, some goodness-of-fit procedures are proposed based on Q-Q plots and KS test. Finally, two data sets are analyzed in Section 5.8 for illustrating all the inferential methods developed here.

5.2 Regression model and ML estimation

5.2.1 Model

Suppose the lifetime $T \sim BS(\alpha, \theta)$ with shape parameter α and scale parameter θ. Further, suppose there are p covariates $\mathbf{x} = (x_1, \cdots, x_p)^T$ associated with the lifetimes of units under the life-test. Then, from (1.1.1), by assuming a log-linear link function for the scale parameter θ, we have the CDF as

$$F(t; \alpha, \theta) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\theta}} - \sqrt{\frac{\theta}{t}}\right)\right], \tag{5.2.1}$$

where $\theta = e^{\boldsymbol{\beta}'\mathbf{x}}$ and $\boldsymbol{\beta}'\mathbf{x} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$. The corresponding PDF is

$$f(t; \alpha, \theta) = \frac{1}{2\sqrt{2\pi}\alpha\theta}\left\{\left(\frac{\theta}{t}\right)^{1/2} + \left(\frac{\theta}{t}\right)^{3/2}\right\}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{\theta}{t} + \frac{t}{\theta} - 2\right)\right]. \tag{5.2.2}$$

5.2.2 ML estimation

Suppose we observe $n(\geqslant p+2)$ failure times, denoted by $(t_1, t_2, \cdots, t_n)$, with the covariates corresponding to t_i as $\mathbf{x}_i = (x_{1i}, x_{2i}, \cdots, x_{pi})$. Then, taking the corresponding scale parameter as $\theta_i = \exp\{\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_p x_{pi}\}$, we have the likelihood function as

$$L = \prod_{i=1}^{n} f(t_i; \alpha, \theta_i), \tag{5.2.3}$$

where $f(\cdot)$ is the PDF of the BS distribution as given in (5.2.2). From (5.2.3), we obtain the log-likelihood function (without the constant) as

$$lnL = -n\log\alpha + \sum_{i=1}^{n}\log\left[\left(\frac{t_i}{\theta_i}\right)^{\frac{1}{2}} + \left(\frac{\theta_i}{t_i}\right)^{\frac{1}{2}}\right] - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right). \quad (5.2.4)$$

From (5.2.4), it is evident that, for given $\beta_0, \beta_1, \cdots, \beta_p$, the MLE of α is

$$\hat{\alpha} = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right)\right]^{\frac{1}{2}}. \quad (5.2.5)$$

Next, the MLEs of $\beta_0, \beta_1, \cdots, \beta_p$ need to be obtained by solving the equations

$$\sum_{i=1}^{n}\frac{\theta_i - t_i}{\theta_i + t_i} - \frac{1}{(\alpha(\theta(x)))^2}\sum_{i=1}^{n}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right) = 0, \quad (5.2.6)$$

$$\sum_{i=1}^{n}\frac{x_{ji}(\theta_i - t_i)}{\theta_i + t_i} - \frac{1}{(\alpha(\theta(x)))^2}\sum_{i=1}^{n}\left[x_{ji}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right)\right] = 0, \quad j = 1, 2\cdots p. \quad (5.2.7)$$

We shall use the Newton-Raphson method for solving Eqs. (5.2.6)-(5.2.7). The question of the existence and uniqueness of the solution, however, remains as an open problem. But in the case of no covariates, this problem has been resolved earlier in Chapter 3. For the implementation of the Newton-Raphson method, it is important to provide good initial values, which is discussed next.

5.2.3 Initial values

For the $BS(\alpha, \theta)$ distribution, it is known that the mode and median are both equal to θ, and so we may consider the relationship

$$\ln t_i = \ln \theta_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi} + \epsilon_i, \qquad i = 1, \cdots, n, \tag{5.2.8}$$

to develop least-squares estimation (LSE) of the parameter $\boldsymbol{\beta}$. By minimizing the sum of squares

$$S(\boldsymbol{\beta}) = \sum_{i=1}^{n} (\ln t_i - \beta_0 - \beta_1 x_{1i} - \cdots - \beta_p x_{pi})^2, \tag{5.2.9}$$

we obtain the LSE of $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta}_0 = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \ln \mathbf{t}, \tag{5.2.10}$$

where $\mathbf{X} = \begin{pmatrix} 1 & x_{11} & x_{21} & \cdots & x_{p1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & x_{1n} & x_{2n} & \cdots & x_{pn} \end{pmatrix}$ and $\ln \mathbf{t} = \begin{pmatrix} \ln t_1 \\ \vdots \\ \ln t_n \end{pmatrix}$. The estimate $\hat{\boldsymbol{\beta}}_0$ in (5.2.10) will be used as an initial value in the Newton-Rahphson method for solving the system of equations in (5.2.6) and (5.2.7).

5.3 Hypotheses testing and interval estimation

In this section, we first discuss likelihood-ratio tests for some hypotheses of interest. Then, we discuss CIs by using the asymptotic properties of MLEs and the bootstrap approach.

5.3.1 Hypotheses testing

Suppose we are interested in testing whether the j-th covariate is significant or not, i.e., $H_0 : \beta_j = 0$. In this case, the MLEs of the unknown parameters under H_0 are as follows:

$$\tilde{\alpha} = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{t_i}{\theta(\mathbf{x_{-j}})} + \frac{\theta(\mathbf{x_{-j}})}{t_i} - 2\right)\right]^{\frac{1}{2}}, \tag{5.3.1}$$

where $\mathbf{x_{-j}}$ means the covariate vector without the j-th covariate, while the MLE of $\boldsymbol{\beta}$ in this case is obtained by maximizing the profile log-likelihood function

$$\ln L_{profile} = -n\ln\tilde{\alpha}(\theta(\mathbf{x_{-j}})) + \sum_{i=1}^{n}\ln\left[\left(\frac{t_i}{\theta(\mathbf{x_{-j}})}\right)^{\frac{1}{2}} + \left(\frac{\theta(\mathbf{x_{-j}})}{t_i}\right)^{\frac{1}{2}}\right]. \tag{5.3.2}$$

If $(\hat{\alpha}, \hat{\boldsymbol{\beta}})$ is the global MLE that maximizes (5.2.4) and $(\tilde{\alpha}, \tilde{\boldsymbol{\beta}})$ is the MLE under H_0 obtained from (5.3.1) and (5.3.2), then under H_0, for large n,

$$D = -2\left\{\ln L_{profile}(\tilde{\alpha}, \tilde{\boldsymbol{\beta}}) - \ln L(\hat{\alpha}, \hat{\boldsymbol{\beta}})\right\} \sim \chi^2_1. \tag{5.3.3}$$

Evidently, we will reject H_0 for large values of D.

5.3.2 Interval estimation

Asymptotic confidence intervals

Under the usual regularity conditions, it can be shown that the MLEs are consistent and asymptotically normally distributed, as presented in the following theorem.

Theorem 5.3.1 *With $\boldsymbol{\eta} = (\alpha, \boldsymbol{\beta})$ as the parameter vector and $\hat{\boldsymbol{\eta}} = (\hat{\alpha}, \hat{\boldsymbol{\beta}})$ as the MLE of $\boldsymbol{\eta}$, as $n \to \infty$, we have*

$$\sqrt{n}\left(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}\right) \xrightarrow{D} N_{p+2}\left(\mathbf{0}, \mathbf{J}_{\mathbf{p+2}}^{-\mathbf{1}}\right), \tag{5.3.4}$$

where $\xrightarrow{D}$ denotes convergence in distribution and $N_{p+2}\left(\mathbf{0}, \mathbf{J}_{\mathbf{p+2}}^{-\mathbf{1}}\right)$ denotes the $(p+2)$-variate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{J}_{\mathbf{p+2}}^{-\mathbf{1}}$, with the matrix $\mathbf{J}_{\mathbf{p+2}}$ being the corresponding Fisher information matrix.

Here, the Fisher information matrix is defined as

$$\boldsymbol{J}_{p+2} = E\begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta_0} & \cdots & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta_p} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta_p} & -\frac{\partial^2 \ln L}{\partial \beta_0 \partial \beta_p} & \cdots & -\frac{\partial^2 \ln L}{\partial \beta_p^2} \end{bmatrix}. \tag{5.3.5}$$

One may also construct a bootstrap CI by adopting the process described earlier in Chapter 1.

5.3.3 Fisher information matrix

From Theorem 5.3.1, we readily have an approximate CI for the parameter η_j as

$$\left(\hat{\eta}_j - z_{1-\frac{\alpha}{2}}\widehat{J_{jj}^{-1}}, \hat{\eta} + z_{1-\frac{\alpha}{2}}\widehat{J_{jj}^{-1}}\right), \tag{5.3.6}$$

where $z_{1-\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ percentage point of the standard normal distribution. The first-order derivatives of the log-likelihood function with respect to the parameters, required for the Newton-Raphson method, are given by

$$\frac{\partial \ln L}{\partial \alpha} = -\frac{n}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{n}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right), \tag{5.3.7}$$

$$\frac{\partial \ln L}{\partial \beta_0} = \frac{1}{2}\sum_{i=1}^{n}\frac{\theta_i - t_i}{\theta_i + t_i} - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \tag{5.3.8}$$

$$\frac{\partial \ln L}{\partial \beta_j} = \frac{1}{2}\sum_{i=1}^{n}x_{ji}\frac{\theta_i - t_i}{\theta_i + t_i} - \frac{1}{2\alpha^2}\sum_{i=1}^{n}x_{ji}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \qquad j = 1, \cdots, p. \tag{5.3.9}$$

Next, the negative of the second-order derivatives of the log-likelihood function with respect to the parameters are obtained from (5.3.7)-(5.3.9) as follows:

$$-\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \frac{3}{\alpha^4}\sum_{i=1}^{n}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right), \tag{5.3.10}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_0^2} = -\sum_{i=1}^{n}\frac{\theta_i t_i}{(\theta_i + t_i)^2} + \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \tag{5.3.11}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_j^2} = -\sum_{i=1}^{n}\frac{x_{ji}^2\theta_i t_i}{(\theta_i + t_i)^2} + \frac{1}{2\alpha^2}\sum_{i=1}^{n}x_{ji}^2\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \quad j = 1, \cdots, p, \tag{5.3.12}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial \beta_0} = -\frac{1}{\alpha^3}\sum_{i=1}^{n}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \tag{5.3.13}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial \beta_j} = -\frac{1}{\alpha^3}\sum_{i=1}^{n}x_{ji}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \qquad j = 1, \cdots, p, \tag{5.3.14}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_0 \partial \beta_j} = -\sum_{i=1}^{n}\frac{x_{ji}\theta_i t_i}{(\theta_i + t_i)^2} + \frac{1}{2\alpha^2}\sum_{i=1}^{n}x_{ji}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \quad j = 1, \cdots, p, \tag{5.3.15}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_j \partial \beta_k} = -\sum_{i=1}^{n}\frac{x_{ji}x_{ki}\theta_i t_i}{(\theta_i + t_i)^2} + \frac{1}{2\alpha^2}\sum_{i=1}^{n}x_{ji}x_{ki}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \; j \neq k = 1, \cdots, p. \tag{5.3.16}$$

By using the properties of BS distribution presented in Chapter (see Property 1.1.1), the expected values of the negative of the second-order derivatives of the log-likelihood

with respect to the parameters, from (5.3.10)-(5.3.16), can be shown to be as follows:

$$
\begin{aligned}
-E\left[\frac{\partial^2 \ln L}{\partial \alpha^2}\right] &= \frac{2n}{\alpha^2}, \\
-E\left[\frac{\partial^2 \ln L}{\partial \beta_0^2}\right] &= \frac{n}{\alpha^2}\left(1+\frac{1}{2}\alpha^2\right) - nE\left[\frac{V}{(1+V)^2}\right] \\
-E\left[\frac{\partial^2 \ln L}{\partial \beta_j^2}\right] &= \frac{1}{\alpha^2}\left(1+\frac{1}{2}\alpha^2\right)\sum_{i=1}^{n} x_{ji}^2 - E\left[\frac{V}{(1+V)^2}\right]\sum_{i=1}^{n} x_{ji}^2, \qquad j=1,\cdots,p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha \partial \beta_0}\right], &= 0, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha \partial \beta_j}\right] &= 0, \qquad j=1,\cdots,p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \beta_0 \partial \beta_j}\right] &= \frac{1}{\alpha^2}\left(1+\frac{1}{2}\alpha^2\right)\sum_{i=1}^{n} x_{ji} - E\left[\frac{V}{(1+V)^2}\right]\sum_{i=1}^{n} x_{ji}, \qquad j=1,\cdots,p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \beta_j \partial \beta_k}\right] &= \frac{1}{\alpha^2}\left(1+\frac{1}{2}\alpha^2\right)\sum_{i=1}^{n} x_{ji}x_{ki} - E\left[\frac{V}{(1+V)^2}\right]\sum_{i=1}^{n} x_{ji}x_{ki}, \\
&\qquad j \neq k = 1,\cdots,p, \qquad (5.3.17)
\end{aligned}
$$

where $V \sim BS(\alpha, 1)$.

While computing the above required expectations, we need the value of $E\left[\frac{V}{(1+V)^2}\right]$, which is not available in an analytic form. So, we may approximate it by Monte Carlo method (one may also instead directly use integration function in R), or we may use the approximate formula (see Kendall and Stuart (1977))

$$E\left[\frac{X}{Y}\right] \approx \frac{E[X]}{E[Y]} \qquad (5.3.18)$$

to arrive at an approximation

$$E\left[\frac{V}{(1+V)^2}\right] \approx \frac{E[V]}{E[(1+V)^2]} = \frac{1+\frac{1}{2}\alpha^2}{4+3\alpha^2+\frac{3}{2}\alpha^4}. \tag{5.3.19}$$

Instead, we may also use the binomial approximation to obtain

$$E\left[\frac{V}{(1+V)^2}\right] \approx E\left[V(1-2V+3V^2)\right] = \frac{45}{2}\alpha^6 + 24\alpha^4 + 10\alpha^2. \tag{5.3.20}$$

5.4 Model with unequal shape parameters

In this section, we will further consider the case when the shape parameter for each observation varies with the covariates $\mathbf{x}$. Then, by assuming once again a log-linear link function $\gamma_i = \exp\{\alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \cdots + \alpha_p x_{pi}\}$, and with $T_i \sim BS(\gamma_i, \theta_i)$, we have the likelihood function as

$$L = \prod_{i=1}^{n} f(t_i; \gamma_i, \theta_i), \tag{5.4.1}$$

where $f(\cdot)$ is the PDF of the BS distribution as given in (5.2.2). From (5.4.1), we obtain the log-likelihood function (without the constant) as

$$\ln L = -\sum_{i=1}^{n} \ln \gamma_i + \sum_{i=1}^{n} \ln\left[\left(\frac{t_i}{\theta_i}\right)^{\frac{1}{2}} + \left(\frac{\theta_i}{t_i}\right)^{\frac{1}{2}}\right] - \sum_{i=1}^{n} \frac{1}{2\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right). \tag{5.4.2}$$

The MLEs of the parameters $\alpha_0, \alpha_1, \cdots, \alpha_p$ and $\beta_0, \beta_1, \cdots, \beta_p$ can be obtained by solving the following system of equations:

$$\sum_{i=1}^{n} \frac{1}{\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right) - n = 0, \tag{5.4.3}$$

$$\sum_{i=1}^{n} \frac{x_{ji}}{\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right) - \sum_{i=1}^{n} x_{ji} = 0, \quad j = 1, 2, \cdots, p, \tag{5.4.4}$$

$$\sum_{i=1}^{n} \frac{\theta_i - t_i}{\theta_i + t_i} - \sum_{i=1}^{n} \frac{1}{\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right) = 0, \tag{5.4.5}$$

$$\sum_{i=1}^{n} \frac{x_{ji}(\theta_i - t_i)}{\theta_i + t_i} - \sum_{i=1}^{n} \frac{x_{ji}}{\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right) = 0, \quad j = 1, 2, \cdots, p. \tag{5.4.6}$$

These equations can be solved by employing the Newton-Raphson method as before, for which it is important to provide good initial values, which is discussed next.

5.4.1 Initial values

For the $BS(\gamma, \theta)$ distribution, it is known that the mode and median are both equal to θ and is independent of γ, as mentioned earlier. Because of this, an initial value of $\boldsymbol{\beta}$ can be derived from Eq. (5.2.10).

Now, in order to obtain the initial value of $\boldsymbol{\alpha}$, we will use the following property.

Property 5.4.1 *If $T \sim BS(\gamma, \theta)$, then $\frac{1}{\gamma}\left(\sqrt{\frac{T}{\theta}} - \sqrt{\frac{\theta}{T}}\right) \sim N(0, 1)$. So, we readily have $E\left(\frac{T}{\theta} + \frac{\theta}{T} - 2\right) = \gamma^2$.*

Let us now define $y_i = \frac{t_i}{\hat{\theta}_i} + \frac{\hat{\theta}_i}{t_i} - 2$. We may then minimize the sum of squares

$$S(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \left(\frac{1}{2} \ln y_i - \alpha_0 - \alpha_1 x_{1i} - \cdots - \alpha_p x_{pi} \right)^2, \tag{5.4.7}$$

which, by the use of LSE, yields

$$\boldsymbol{\alpha}_0 = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \ln \mathbf{y}, \tag{5.4.8}$$

where $\mathbf{X}$ is as defined in Eq. (5.2.10) and $\ln \mathbf{y} = \frac{1}{2} (\ln y_1, \cdots, \ln y_n)'$.

5.5 Hypotheses testing and interval estimation

In this section, we first discuss likelihood-ratio tests for some hypotheses of interest. Then, we discuss CIs by using the asymptotic properties of MLEs and the bootstrap approach.

5.5.1 Hypotheses testing

In this model, we may be interested in testing whether the j-th covariate affects the shape parameter, or whether it affects both shape and scale parameters, or whether the shape parameters are independent of the covariates. These hypotheses may be tested by specifying the null hypothesis H_0 as

1. $\alpha_j = 0$,

2. $\alpha_j = 0$ and $\beta_j = 0$,

3. $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$,

respectively.

In these cases, with $\ln L_0$ and $\ln L_1$ denoting the log-likelihood functions under H_0 and H_1, we have under H_0, for large n,

$$D = -2\left(\ln L_0 - \ln L_1\right) \tag{5.5.1}$$

following χ_1^2, χ_2^2 and χ_p^2, respectively. Evidently, we will reject H_0 for large values of D.

One may instead use Akaike information criterion (AIC) [see Akaike (1998)] which is defined as

$$AIC = 2k - 2\ln L, \tag{5.5.2}$$

where k is the number of model parameters. Let us denote by AIC_0 and AIC_1 the AIC values under H_0 and H_1, respectively. We may then use the statistic

$$D^* = AIC_0 - AIC_1 = -2\ln L_0 + 2\ln L_1 + 2k_0 - 2k_1, \tag{5.5.3}$$

where k_0 and k_1 are the numbers of parameters under H_0 and H_1, respectively. In this case, we will reject H_0 if $D^* > 0$.

5.5.2 Interval estimation

Asymptotic confidence intervals

For the model with unequal shape parameters and based on the usual regularity conditions, it can be shown that the MLEs are consistent and asymptotically normally distributed, as presented in the following theorem.

Theorem 5.5.1 *With $\boldsymbol{\eta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ as the parameter vector and $\hat{\boldsymbol{\eta}} = (\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}})$ as the MLE of $\boldsymbol{\eta}$, as $n \to \infty$, we have*

$$\sqrt{n}\left(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}\right) \xrightarrow{D} N_{2p+2}\left(\mathbf{0}, \mathbf{J}_{\mathbf{2p+2}}^{-\mathbf{1}}\right), \tag{5.5.4}$$

where $\xrightarrow{D}$ denotes convergence in distribution and $N_{2p+2}\left(\mathbf{0}, \mathbf{J}_{\mathbf{2p+2}}^{-\mathbf{1}}\right)$ denotes the $(2p+2)$-variate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{J}_{\mathbf{2p+2}}^{-\mathbf{1}}$, with the matrix $\mathbf{J}_{\mathbf{2p+2}}$ being the corresponding Fisher information matrix.

Here, the Fisher information matrix is defined as

$$\boldsymbol{J}_{2p+2} = E\begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha_0^2} & \cdots & -\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \alpha_p} & -\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \beta_0} & \cdots & -\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \beta_p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \beta_p} & \cdots & -\frac{\partial^2 \ln L}{\partial \alpha_p \partial \beta_p} & -\frac{\partial^2 \ln L}{\partial \beta_0 \partial \beta_p} & \cdots & -\frac{\partial^2 \ln L}{\partial \beta_p^2} \end{bmatrix}. \tag{5.5.5}$$

5.5.3 Fisher information matrix

The first-order derivatives of the log-likelihood function with respect to the model parameters, required for the Newton-Raphson method, are follows:

$$\frac{\partial \ln L}{\partial \alpha_0} = -n + \sum_{i=1}^{n} \frac{1}{\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right), \tag{5.5.6}$$

$$\frac{\partial \ln L}{\partial \alpha_j} = -\sum_{i=1}^{n} x_{ji} + \sum_{i=1}^{n} \frac{x_{ji}}{\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right), \qquad j = 1, \cdots, p, \tag{5.5.7}$$

$$\frac{\partial \ln L}{\partial \beta_0} = \frac{1}{2}\sum_{i=1}^{n} \frac{\theta_i - t_i}{\theta_i + t_i} - \sum_{i=1}^{n} \frac{1}{2\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \tag{5.5.8}$$

$$\frac{\partial \ln L}{\partial \beta_j} = \frac{1}{2}\sum_{i=1}^{n} x_{ji}\frac{\theta_i - t_i}{\theta_i + t_i} - \sum_{i=1}^{n} \frac{x_{ji}}{2\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \qquad j = 1, \cdots, p. \tag{5.5.9}$$

Next, the negative of the second-order derivatives of the log-likelihood function with respect to the parameters, obtained from (5.5.6)-(5.5.9), are as follows:

$$-\frac{\partial^2 \ln L}{\partial \alpha_0^2} = \sum_{i=1}^{n} \frac{2}{\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right), \tag{5.5.10}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_j^2} = \sum_{i=1}^{n} \frac{2x_{ji}^2}{\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right), \qquad j = 1, \cdots, p, \tag{5.5.11}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \alpha_j} = \sum_{i=1}^{n} \frac{2x_{ji}}{\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right), \qquad j = 1, \cdots, p, \tag{5.5.12}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_j \partial \alpha_k} = \sum_{i=1}^{n} \frac{2x_{ji}x_{ki}}{\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i} - 2\right), \qquad j \neq k = 1, \cdots, p, \tag{5.5.13}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_0^2} = -\sum_{i=1}^{n} \frac{\theta_i t_i}{(\theta_i + t_i)^2} + \sum_{i=1}^{n} \frac{1}{2\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \tag{5.5.14}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_j^2} = -\sum_{i=1}^{n} \frac{x_{ji}^2 \theta_i t_i}{(\theta_i + t_i)^2} + \sum_{i=1}^{n} \frac{x_{ji}^2}{2\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \quad j = 1, \cdots, p, \tag{5.5.15}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_0 \partial \beta_j} = -\sum_{i=1}^{n} \frac{x_{ji}\theta_i t_i}{(\theta_i + t_i)^2} + \sum_{i=1}^{n} \frac{x_{ji}}{2\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \quad j = 1, \cdots, p, \tag{5.5.16}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_j \partial \beta_k} = -\sum_{i=1}^{n} \frac{x_{ji}x_{ki}\theta_i t_i}{(\theta_i + t_i)^2} + \sum_{i=1}^{n} \frac{x_{ji}x_{ki}}{2\gamma_i^2}\left(\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right),$$

$$j \neq k = 1, \cdots, p, \tag{5.5.17}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \beta_0} = -\sum_{i=1}^{n} \frac{1}{\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \tag{5.5.18}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \beta_j} = -\sum_{i=1}^{n} \frac{x_{ji}}{\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \qquad j = 1, \cdots, p, \tag{5.5.19}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_0} = -\sum_{i=1}^{n} \frac{x_{ji}}{\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \qquad j = 1, \cdots, p, \tag{5.5.20}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_j} = -\sum_{i=1}^{n} \frac{x_{ji}^2}{\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \qquad j = 1, \cdots, p, \tag{5.5.21}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_k} = -\sum_{i=1}^{n} \frac{x_{ji}x_{ki}}{\gamma_i^2}\left(-\frac{t_i}{\theta_i} + \frac{\theta_i}{t_i}\right), \qquad j \neq k = 1, \cdots, p. \tag{5.5.22}$$

By using Property 1.1.1, we then obtain the expected values of the negative of the second-order derivatives of the log-likelihood with respect to the parameters, from (5.5.10)-(5.5.22), as follows:

$$
\begin{aligned}
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_0^2}\right] &= 2n, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j^2}\right] &= 2\sum_{i=1}^{n} x_{ji}^2, \qquad j = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \alpha_j}\right] &= 2\sum_{i=1}^{n} x_{ji}, \qquad j = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j \partial \alpha_k}\right] &= 2\sum_{i=1}^{n} x_{ji} x_{ki}, \qquad j \neq k = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \beta_0^2}\right] &= \sum_{i=1}^{n}\left(\frac{1}{\gamma_i^2} + \frac{1}{2}\right) - \sum_{i=1}^{n} E\left[\frac{V_i}{(1+V_i)^2}\right], \\
-E\left[\frac{\partial^2 \ln L}{\partial \beta_j^2}\right] &= \sum_{i=1}^{n}\left(\frac{1}{\gamma_i^2} + \frac{1}{2}\right) x_{ji}^2 - \sum_{i=1}^{n} E\left[\frac{V_i}{(1+V_i)^2}\right] x_{ji}^2, \qquad j = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \beta_0 \partial \beta_j}\right] &= \sum_{i=1}^{n}\left(\frac{1}{\gamma_i^2} + \frac{1}{2}\right) x_{ji} - \sum_{i=1}^{n} E\left[\frac{V_i}{(1+V_i)^2}\right] x_{ji}, \qquad j = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \beta_j \partial \beta_k}\right] &= \sum_{i=1}^{n}\left(\frac{1}{\gamma_i^2} + \frac{1}{2}\right) x_{ji} x_{ki} - \sum_{i=1}^{n} E\left[\frac{V_i}{(1+V_i)^2}\right] x_{ji} x_{ki}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \neq k = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \beta_0}\right] &= 0, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_0 \partial \beta_j}\right] &= 0, \qquad j = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_0}\right] &= 0, \qquad j = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_j}\right] &= 0, \qquad j = 1, \cdots, p, \\
-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_k}\right] &= 0, \qquad j \neq k = 1, \cdots, p.
\end{aligned}
$$

where $V_i \sim BS(\gamma_i, 1)$ as in the preceding section.

Moreover, if we use $\mathbf{A}$ and $\mathbf{B}$ to denote the variance-covariance matrices of $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$, then we observe from the above expressions that

$$Var(\hat{\boldsymbol{\eta}}) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix}. \tag{5.5.23}$$

Then, as in (5.3.5), we can construct an asymptotic $100(1-\alpha)\%$ CI for the parameter η_j by using (5.5.23).

One may also construct a bootstrap CI by adopting the process described earlier in Chapter 1.

5.6 Simulation study

We carried out an extensive Monte Carlo simulation study for different sample sizes n, number of covariates p, the shape parameter α, and scale parameters $\boldsymbol{\beta}$. In Tables 5.6.1-5.6.3, we chose the sample size n to be 20, 50 and 100, the number of covariates p to be 1, 2 and 4, the value of α to be 0.25, 0.50 and 0.75, and the covariates $\mathbf{x}$ randomly selected from uniform $U(-1, 1)$ distribution. Then, we determined the empirical values of means and MSEs of the ML estimates of the parameters and also the coverage probabilities of the 95% asymptotic CIs, using the exact Fisher information. From these tables, we observe that the MLEs of β_j's have negligible bias even for n as small as 20, while the MLEs of the shape parameter α is negatively

biased for small n but the bias becomes negligible when n gets large. We also note that the coverage probabilities are close to the nominal level of 0.95 even for n as small as 20. However, larger n is required in the case of larger number of covariates p.

For the case of unequal shape parameters, we also carried out a simulation study with different sample sizes n, number of covariates p, the shape parameters $\boldsymbol{\alpha}$, and scale parameters $\boldsymbol{\beta}$. For the case of 1 covariate, we chose the sample size n to be 20, 50 and 100, while for the case of 2 covariates, the value of n to be 50, 100, 200 and 400. For both cases, the covariates $\mathbf{x}$ were randomly selected from uniform $U(-1, 1)$ distribution. Then, we determined the empirical values of means and the MSEs of the ML estimates of the parameters and also the coverage probabilities of the 95% asymptotic CIs, using the exact Fisher information. The results so obtained are presented in Tables 5.6.4-5.6.7, from which we observe that, while the MLEs of the shape parameter α_i and β_i are biased for small n, the bias becomes negligible when n becomes large. We also note that the coverage probabilities of $\boldsymbol{\alpha}$ are close to the nominal level of 0.95 when the sample size increases. But, the coverage probabilities of $\boldsymbol{\beta}$ are a little lower than the level of 0.95, but do get closer to the nominal level when n becomes large.

Table 5.6.1: Simulated values of means and MSEs (within brackets in the second row) of the MLEs for the regression model involving 1 covariate with $\beta_0 = 1.00$ and $\beta_1 = -1.00$, and coverage probabilities of 95% CIs (within brackets in the third row).

α	n	$\hat{\alpha}$	$\hat{\beta}_0$	$\hat{\beta}_1$
0.25	20	0.2335	0.9994	-0.9982
		(0.0018)	(0.0034)	(0.0102)
		(0.9336)	(0.9384)	(0.9625)
	50	0.2435	0.9997	-1.0005
		(0.0007)	(0.0013)	(0.0038)
		(0.9449)	(0.9506)	(0.9685)
	100	0.2468	0.9997	-0.9999
		(0.0003)	(0.0006)	(0.0019)
		(0.9493)	(0.9456)	(0.9466)
0.50	20	0.4655	0.9987	-0.9979
		(0.0075)	(0.0122)	(0.0393)
		(0.9308)	(0.9466)	(0.9329)
	50	0.4860	0.9993	-1.0005
		(0.0027)	(0.0048)	(0.0144)
		(0.9460)	(0.9479)	(0.9501)
	100	0.4931	0.9995	-0.9997
		(0.0013)	(0.0024)	(0.0072)
		(0.9464)	(0.9472)	(0.9311)
0.75	20	0.6987	1.0010	-1.0142
		(0.0166)	(0.0260)	(0.0824)
		(0.9310)	(0.9428)	(0.9288)
	50	0.7289	1.0005	-0.9991
		(0.0061)	(0.0102)	(0.0308)
		(0.9422)	(0.9464)	(0.9247)
	100	0.7399	1.0011	-0.9997
		(0.0029)	(0.0048)	(0.0149)
		(0.9473)	(0.9500)	(0.9320)

Table 5.6.2: Simulated values of means and MSEs (within brackets in the second row) of the MLEs for the regression model involving 2 covariates with $\beta_0 = 1.00$, $\beta_1 = 0.75$ and $\beta_2 = 0.40$, and coverage probabilities of 95% CIs (within brackets in the third row).

α	n	$\hat{\alpha}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
0.25	20	0.2259	0.9997	0.7500	0.3989
		(0.0021)	(0.0035)	(0.0109)	(0.0107)
		(0.9101)	(0.9354)	(0.9401)	(0.9127)
	50	0.2408	1.0003	0.7508	0.3995
		(0.0007)	(0.0013)	(0.0040)	(0.0039)
		(0.9341)	(0.9465)	(0.9308)	(0.9198)
	100	0.2452	0.9999	0.7499	0.4004
		(0.0003)	(0.0006)	(0.0020)	(0.0019)
		(0.9430)	(0.9480)	(0.9545)	(0.9693)
0.50	20	0.4521	1.0007	0.7490	0.4011
		(0.0084)	(0.0132)	(0.0423)	(0.0416)
		(0.9087)	(0.9349)	(0.8997)	(0.8968)
	50	0.4819	0.9997	0.7484	0.3994
		(0.0028)	(0.0048)	(0.0152)	(0.0151)
		(0.9384)	(0.9466)	(0.9568)	(0.9281)
	100	0.4908	1.0001	0.7508	0.4003
		(0.0013)	(0.0024)	(0.0072)	(0.0073)
		(0.9414)	(0.9499)	(0.9331)	(0.9434)
0.75	20	0.6765	1.0003	0.7528	0.3988
		(0.0189)	(0.0286)	(0.0876)	(0.0879)
		(0.9138)	(0.9271)	(0.9257)	(0.9477)
	50	0.7200	1.0010	0.7517	0.4035
		(0.0065)	(0.0104)	(0.0316)	(0.0312)
		(0.9290)	(0.9444)	(0.9545)	(0.9290)
	100	0.7351	0.9997	0.7510	0.4000
		(0.0031)	(0.0048)	(0.0145)	(0.0149)
		(0.9680)	(0.9585)	(0.9565)	(0.9515)

Table 5.6.3: Simulated values of means and MSEs (within brackets in the second row) of the MLEs for the regression model involving 4 covariates with $\beta_0 = 1.00$, $\beta_1 = 0.75$, $\beta_2 = 0.50$, $\beta_3 = 0.25$ and $\beta_4 = -0.40$, and coverage probabilities of 95% CIs (within brackets in the third row).

α	n	$\hat{\alpha}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
0.25	20	0.2122	1.0001	0.7511	0.5004	0.2486	-0.3995
		(0.0030)	(0.0039)	(0.0124)	(0.0121)	(0.0120)	(0.0125)
		(0.8397)	(0.9155)	(0.9342)	(0.9230)	(0.8999)	(0.8921)
	50	0.2355	1.0004	0.7486	0.4998	0.2500	-0.3997
		(0.0008)	(0.0014)	(0.0040)	(0.0041)	(0.0041)	(0.0041)
		(0.9149)	(0.9366)	(0.9494)	(0.9560)	(0.9454)	(0.9340)
	100	0.2428	1.0002	0.7498	0.5003	0.2501	-0.4001
		(0.0004)	(0.0006)	(0.0020)	(0.0019)	(0.0019)	(0.0019)
		(0.9305)	(0.9470)	(0.9468)	(0.9442)	(0.9415)	(0.9266)
0.50	20	0.4230	0.9997	0.7506	0.4975	0.2529	-0.4029
		(0.0122)	(0.0151)	(0.0479)	(0.0476)	(0.0498)	(0.0470)
		(0.8326)	(0.9151)	(0.8770)	(0.9307)	(0.8982)	(0.8760)
	50	0.4706	1.0002	0.7494	0.5003	0.2511	-0.3991
		(0.0033)	(0.0052)	(0.0154)	(0.0155)	(0.0155)	(0.0153)
		(0.9128)	(0.9350)	(0.9482)	(0.9355)	(0.9573)	(0.9734)
	100	0.4847	1.0004	0.7501	0.4999	0.2497	-0.3984
		(0.0014)	(0.0025)	(0.0073)	(0.0073)	(0.0073)	(0.0072)
		(0.9324)	(0.9466)	(0.9520)	(0.9432)	(0.9616)	(0.9561)
0.75	20	0.6294	1.0001	0.7455	0.4971	0.2452	-0.4044
		(0.0279)	(0.0314)	(0.1022)	(0.0993)	(0.1023)	(0.0997)
		(0.8302)	(0.9154)	(0.9241)	(0.8684)	(0.8445)	(0.8708)
	50	0.7040	0.9989	0.7518	0.5001	0.2499	-0.4007
		(0.0078)	(0.0110)	(0.0325)	(0.0332)	(0.0334)	(0.0332)
		(0.9053)	(0.9367)	(0.9207)	(0.9210)	(0.9557)	(0.9305)
	100	0.7275	1.0003	0.7512	0.4995	0.2503	-0.3989
		(0.0032)	(0.0051)	(0.0156)	(0.0154)	(0.0155)	(0.0148)
		(0.9331)	(0.9477)	(0.9528)	(0.9448)	(0.9492)	(0.9296)

Table 5.6.4: Simulated values of means and MSEs (within brackets in the second row) of the MLEs for the regression model involving 1 covariate with $\alpha_0 = -1.00$, $\alpha_1 = -0.5$, $\beta_0 = 1.00$ and $\beta_1 = -1.00$, and coverage probabilities of 95% CIs (within brackets in the third row).

n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$
20	-1.1038	-0.5091	0.9996	-0.9976
	(0.2292)	(0.2620)	(0.6960)	(0.5121)
	(0.8576)	(0.8673)	(0.8650)	(0.8512)
50	-1.0413	-0.5040	0.9994	-0.9984
	(0.0129)	(0.0428)	(0.0031)	(0.0098)
	(0.9137)	(0.9186)	(0.8931)	(0.8913)
100	-1.0200	-0.5037	0.9999	-1.0002
	(0.0057)	(0.0191)	(0.0015)	(0.0047)
	(0.9350)	(0.9415)	(0.9079)	(0.9065)

Table 5.6.5: Simulated values of means and MSE (within brackets in the second row) of the MLEs for the regression model involving 1 covariate with $\alpha_0 = -1.00$, $\alpha_1 = 0.25$, $\beta_0 = 1.00$ and $\beta_1 = -1.00$, and coverage probabilities of 95% CIs (within brackets in the third row).

n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$
20	-1.1137	0.2794	1.0012	-1.0005
	(0.0685)	(0.1421)	(0.0808)	(0.0507)
	(0.8515)	(0.8693)	(0.9465)	(0.9319)
50	-1.0421	0.2611	1.0000	-0.9988
	(0.0130)	(0.0425)	(0.0026)	(0.0089)
	(0.9125)	(0.9206)	(0.9469)	(0.9440)
100	-1.0202	0.2530	1.0000	-1.0004
	(0.0057)	(0.0192)	(0.0013)	(0.0046)
	(0.9335)	(0.9389)	(0.9471)	(0.9433)

Table 5.6.6: Simulated values of means and MSEs (within brackets in the second row) of the MLEs for the regression model involving 2 covariates with $\alpha_0 = -1.00$, $\alpha_1 = -0.50$, $\alpha_2 = 0.25$, $\beta_0 = 1.00$, $\beta_1 = -1.00$ and $\beta_2 = 0.25$, and coverage probabilities of 95% CIs (within brackets in the third row).

n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
50	-1.0637	-0.5091	0.2602	0.9999	-0.9986	0.2493
	(0.0155)	(0.0475)	(0.0516)	(0.0033)	(0.0100)	(0.0090)
	(0.8849)	(0.9044)	(0.8999)	(0.8829)	(0.8785)	(0.9340)
100	-1.0299	-0.5130	0.2548	1.0001	-1.0003	0.2493
	(0.0063)	(0.0207)	(0.0201)	(0.0015)	(0.0049)	(0.0037)
	(0.9209)	(0.9286)	(0.9283)	(0.8986)	(0.9008)	(0.9390)
200	-1.0152	-0.5047	0.2508	0.9998	-0.9999	0.2495
	(0.0028)	(0.0095)	(0.0080)	(0.0008)	(0.0021)	(0.0019)
	(0.9332)	(0.9390)	(0.9458)	(0.8771)	(0.8958)	(0.9245)
400	-1.0082	-0.5035	0.2519	1.0001	-1.0008	0.2501
	(0.0014)	(0.0042)	(0.0039)	0.0004)	(0.0010)	(0.0009)
	(0.9390)	(0.9454)	(0.9456)	(0.9065)	(0.9098)	(0.9401)

Table 5.6.7: Simulated values of means and MSE (within brackets in the second row) of the MLEs for the regression model involving 2 covariates with $\alpha_0 = -1.00$, $\alpha_1 = -0.25$, $\alpha_2 = 0.50$, $\beta_0 = 1.00$, $\beta_1 = -1.00$ and $\beta_2 = 0.25$, and coverage probabilities of 95% CIs (within brackets in the third row).

n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
50	-1.0645	-0.2490	0.5189	0.9999	-0.9984	0.2494
	(0.0156)	(0.0476)	(0.0520)	(0.0032)	(0.0087)	(0.0105)
	(0.8813)	(0.9048)	(0.8999)	(0.8955)	(0.9120)	(0.9058)
100	-1.0303	-0.2582	0.5098	0.9999	-1.0000	0.2494
	(0.0063)	(0.0207)	(0.0201)	(0.0015)	(0.0045)	(0.0044)
	(0.9203)	(0.9275)	(0.9270)	(0.9070)	(0.9303)	(0.9134)
200	-1.0152	-0.2528	0.5025	0.9998	-1.0000	0.2494
	(0.0028)	(0.0095)	(0.0080)	(0.0008)	(0.0018)	(0.0022)
	(0.9337)	(0.9381)	(0.9461)	(0.8671)	(0.9274)	(0.8857)
400	-1.0082	-0.2523	0.5030	1.0000	-1.0008	0.2501
	(0.0014)	(0.0042)	(0.0039)	(0.0004)	(0.0009)	(0.0010)
	(0.9390)	(0.9453)	(0.9455)	(0.9041)	(0.9364)	(0.9116)

5.7 Model Validation

In order to test the fit of the model to the observed data, we now describe some methods based on the BS Q-Q plot, Normal Q-Q plot and the KS test.

5.7.1 BS Q-Q plot

Since the BS distribution belongs to the scale-family, with $T_i \sim BS(\alpha, \theta_i)$, we could consider the transformation $Z_i = \frac{T_i}{\theta_i}$, which would result in z_i being independent and identically distributed observations from $BS(\alpha, 1)$ distribution. For this purpose, from the data, we estimate θ by $\hat{\theta}$ and use it to determine the values of z_i's from which we find the sample quantile $\hat{q}_j = Z_{j:n}$, where $Z_{j:n}$ is the j-th smallest value obtained from the ordered values of z_i's. Then, a plot of the points with coordinates

$$[q_j, \hat{q}_j]$$

is the BS Q-Q plot, where $q_j = F^{-1}(\frac{j}{n+1})$ is the corresponding theoretical quantile of the $BS(\alpha, 1)$ distribution. If the assumed BS model is suitable for the data, then we should expect all the points to fall close to a straight line.

5.7.2 Normal Q-Q plot

The BS distribution has a close relationship with the normal distribution; in fact, if $T \sim BS(\alpha, \theta)$, then $\frac{1}{\alpha}\left(\sqrt{\frac{T}{\theta}} - \sqrt{\frac{\theta}{T}}\right) \sim N(0, 1)$. By exploiting this relationship, we

could consider the transformation

$$Z_i = \frac{1}{\alpha}\left(\sqrt{\frac{T_i}{\theta_i}} - \sqrt{\frac{\theta_i}{T_i}}\right), i = 1, \cdots, n, \tag{5.7.1}$$

which should result in z_i being independent and identically distributed observations from the standard normal distribution. Thus, from the given data, we may estimate $\boldsymbol{\theta}$ by $\hat{\boldsymbol{\theta}}$ and α by $\hat{\alpha}$ and use them to determine the z_i values from (5.7.1), from which we find the sample quantile $\hat{q}_j = Z_{j:n}$, where $Z_{j:n}$ is the j-th smallest value obtained from the ordered values of z_i's.

Then, a plot of the points with coordinates

$$[q_j, \hat{q}_j]$$

is the normal Q-Q plot, where q_j is the the corresponding theoretical standard normal quantile. Once again, if the assumed BS model is suitable for the data, then we would expect all the points to fall close to a straight line.

5.7.3 KS test

Instead of verifying by the use of Q-Q plots suggested in the preceding subsections, we can propose a formal KS test. Suppose the theoretical distribution $F(t_i)$ is completely known, and the corresponding empirical distribution $\hat{F}(t_i)$ is obtained from the MLEs or any other consistent estimates, then a test could be based on the difference between

$F(t_i)$ and $\hat{F}(t_i)$, which is the well-known KS test of the form

$$D_n = \sup |\hat{F}(t_i) - F(t_i)|. \tag{5.7.2}$$

Critical values of D_n can be obtained from the tables in D'Agostino and Stephens (1986), for example.

Since the true theoretical distribution $F(t_i)$ will be unknown in practice, we may consider the transformation

$$T_i^* = \frac{T_i}{\theta_i}, \qquad i = 1, \cdots, n, \tag{5.7.3}$$

which should result in t_i^*'s being independent and identically distributed observations from $BS(\alpha, 1)$. Now, the empirical distribution function $F(t^*)$ from n i.i.d. observations t_i^* is given by

$$F_0(t^*) = \frac{1}{n} \sum_{j=1}^{n} I_{t_j^* \leqslant t^*}, \tag{5.7.4}$$

where $I_{t_i^* \leqslant t_i}$ is equal to 1 if $t_i^* \leqslant t_i$ and 0 otherwise. However, the true value of the scale parameter is unknown, and so the MLEs may be used to find

$$F(t^*) = \frac{1}{n} \sum_{i=1}^{n} I_{\hat{t}_i^* \leqslant t^*}, \tag{5.7.5}$$

where $\hat{t}_i^* = \frac{t_i}{\hat{\theta}_i}$. Now, with the KS statistic D_n as defined in (5.7.2), if the proposed

BS distribution is a good model for the data, then we would expect a small value of D_n.

5.7.4 Unequal shape parameters

In the case of unequal shape parameters, we may use the transformation

$$Z_i = \frac{1}{\gamma_i}\left(\sqrt{\frac{T_i}{\theta_i}} - \sqrt{\frac{\theta_i}{T_i}}\right), \qquad i = 1, \cdots, n, \tag{5.7.6}$$

which should result in Z_i's being independent and identically distributed observations from the standard normal distribution. Thus, from the given data, we may estimate $\boldsymbol{\theta}$ by $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\gamma}$ by $\hat{\boldsymbol{\gamma}}$ and then use them to determine z_i values from (5.7.6), from which we may find the sample quantile $\hat{q}_j = Z_{j:n}$, where $Z_{j:n}$ is the j-th ordered value obtained from the ordered values of z_i's. Following the same procedure as in Section 5.7.2, the normal Q-Q plot can then be constructed. Moreover, the ordered values of z_i's can be also used for the KS test.

5.8 Illustrative examples

In this section, we will illustrate the proposed methods of inference by using two data sets from the literature.

Example 5.8.1 *These data, due to Birnbaum and Saunders (1969b), give the fatigue lifetimes of 6061-T6 aluminum coupons cut parallel to the direction of rolling and*

Table 5.8.1: Likelihood-ratio test and AIC values for testing the hypothesis $\alpha_1 = 0$ for Example 5.8.1.

$\ln L_0$	$\ln L_1$	$\chi^2_{1,0.95}$	P-value	AIC_0	AIC_1
77.1725	77.3202	3.8415	0.5867	-148.3449	-146.6404

oscillated at 18 cycles per second. These data contain 101 observations with maximum stress per cycle as 31,000 psi and 102 observations with maximum stress per cycle as 26,000 psi. The MLEs of α and β for the first data are found to be 0.1704 and 131.8193, respectively, while for the second data, they are found to be 0.1614 and 392.7635.

Now, we shall use the regression model with 1 *level to analyze these data. The covariate matrix* $\mathbf{X}$ *is defined as* $x_{1i} = 1$ *if the observation comes from* $26,000$ *psi level, and* $x_{1i} = 0$ *otherwise. Based on the given data and the determined MLEs, we assume they have the same shape parameter, which is not rejected by the likelihood-ratio test and AIC values, as shown in Table 5.8.1. We then determined the MLEs of the parameters* α, β_0 *and* β_1, *along with the corresponding* 95% *CIs, and these are presented in Table 5.8.2. The BS Q-Q plot and the normal Q-Q plot, presented in Figures 5.8.1 and 5.8.2, show an approximate straight line, not rejecting the assumed BS model. We also carried out the KS test, and the computed value of the KS statistic and the corresponding P-value are presented in Table 5.8.3, and these results do not lead to the rejection of the assumed BS model.*

Example 5.8.2 *In this example, we continue to use the data analyzed in the previous example, but with the addition of one more data set, containing 101 observations with*

Table 5.8.2: MLEs of the parameters and the corresponding 95% CIs for Example 5.8.1.

	$\hat{\alpha}$	$\hat{\beta}_0$	$\hat{\beta}_1$
MLEs	0.1660	4.8814	1.0918
Bootstrap CI	(0.1497,0.1804)	(4.8483,4.9125)	(1.0474,1.1378)
Asymptotic CI	(0.1498,0.1821)	(4.8248,4.9380)	(1.0352,1.1484)
Appro. CI	(0.1498,0.1821)	(4.8587,4.9042)	(1.0597,1.1239)
Bino. Appro. CI	(0.1498,0.1821)	(4.8587,4.9042)	(1.0597,1.1239)

Table 5.8.3: KS statistic and the corresponding P-value for Example 5.8.1.

KS-Statistic	P-value
0.0479	0.7395

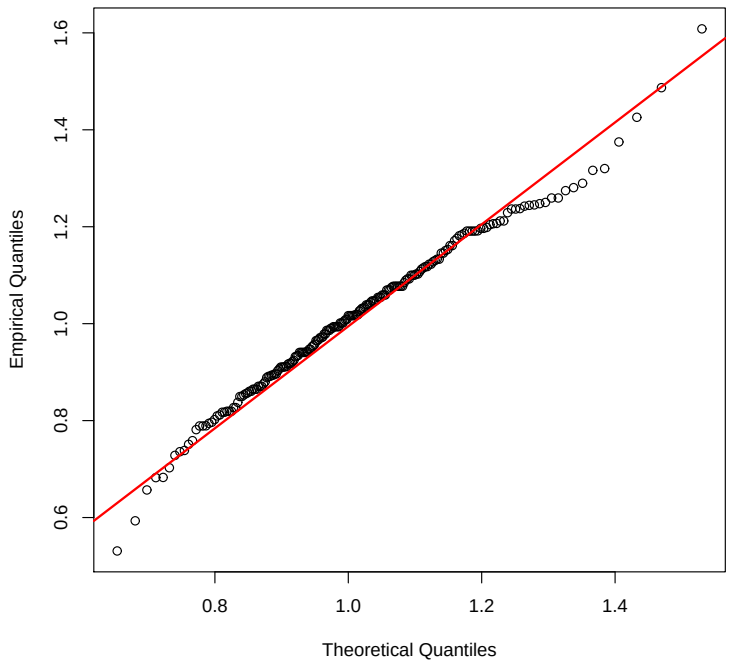

Figure 5.8.1: BS Q-Q plot for Example 5.8.1.

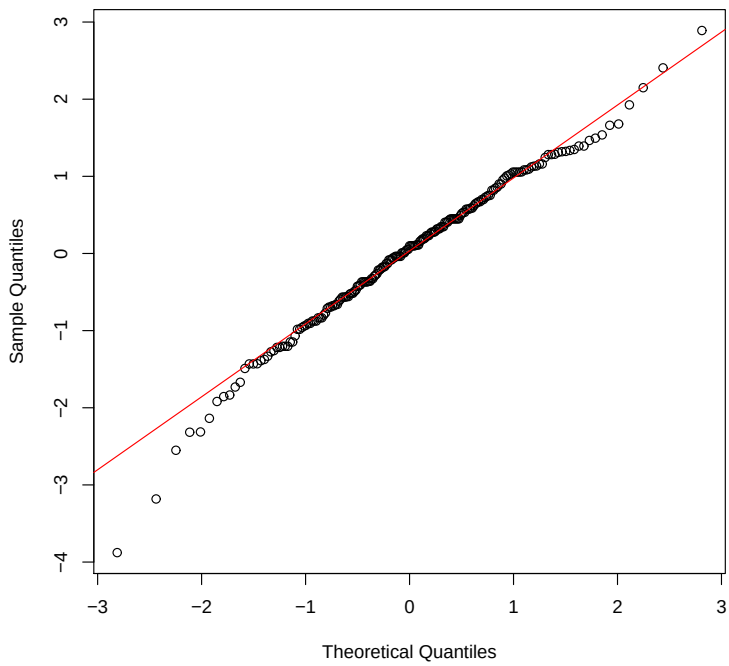

Figure 5.8.2: Normal Q-Q plot for Example 5.8.1.

Table 5.8.4: Estimates of the parameters and the corresponding 95% CIs for Example 5.8.2.

	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
MLEs	-1.770	-0.054	0.599	4.881	1.091	2.316
Bootstrap CI	(-1.926,-1.644)	(-0.254, 0.145)	(0.402,0.798)	(4.848,4.915)	(1.046,1.138)	(2.249,2.386)
Asymptotic CI	(-1.849,-1.690)	(-0.191, 0.083)	(0.461,0.737)	(4.860,4.903)	(1.061,1.123)	(2.257,2.376)
Appro. CI	(-1.849,-1.690)	(-0.191, 0.083)	(0.461,0.737)	(4.8602,4.903)	(1.061,1.123)	(2.257,2.376)
Bino. Appro. CI	(-1.849,-1.690)	(-0.191,0.083)	(0.461,0.737)	(4.860,4.903)	(1.061,1.123)	(2.254,2.379)

Table 5.8.5: KS statistic and the corresponding P-value for Example 5.8.2.

KS-Statistic	P-value
0.0465	0.5256

maximum stress per cycle as 21,000 psi. The MLEs of α *and* β *for this third data are found to be 0.3101 and 1336.5640, respectively.*

Now, we shall use the regression model with unequal shape parameters based on 2 *levels to model these data. For this purpose, we set* $x_{1i} = 1$ *if the* i*-th observation comes from 26,000 psi level, and* 0 *otherwise, and similarly,* $x_{2i} = 1$ *if the* i*-th observation comes from 21,000 psi level, and* 0 *otherwise. Based on these data, we determined the MLEs of the parameters* α_0, α_1, α_2, β_0, β_1 *and* β_2 *along with the corresponding* 95% *CIs, and these are presented in Table 5.8.4. The normal Q-Q plot in Figure 5.8.3 shows an approximate straight line, not rejecting the assumed BS model. We also carried out the KS test, and the computed values of the KS statistic and the corresponding P-value are presented in Table 5.8.5, and these results once again provide strong evidence for the assumed BS model. Finally, we performed tests for three different hypotheses, and the corresponding likelihood-ratio test and AIC results are presented in Table 5.8.6, and these results suggest that we can take* $\alpha_1 = 0$ *in the assumed model.*

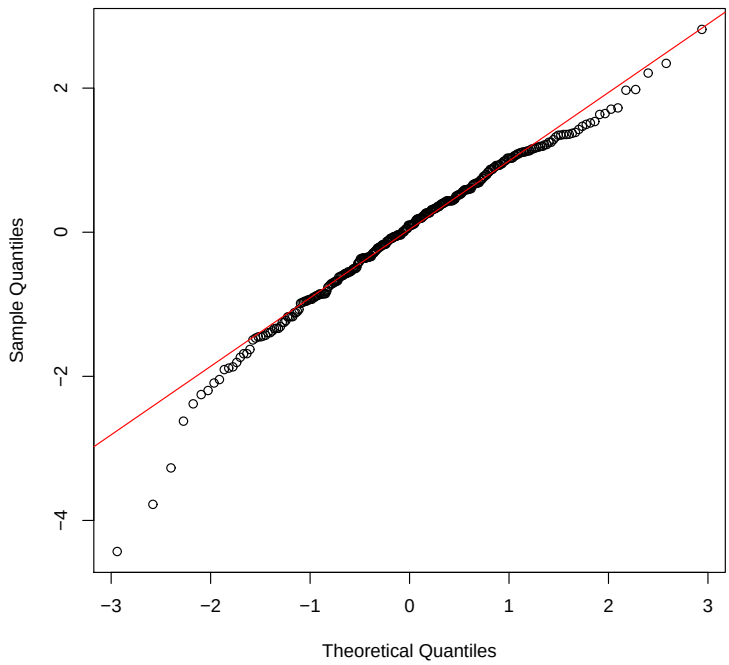

Figure 5.8.3: Normal Q-Q plot for Example 5.8.2.

Table 5.8.6: Likelihood-ratio test and AIC values for testing different hypotheses for Example 5.8.2.

Hypothesis	$\ln L_0$	$\ln L_1$	$\chi^2_{0.95}$	P-value	AIC_0	AIC_1
$\alpha_1 = 0$	52.8102	52.9574	3.8415	0.5874	-93.9148	-95.6204
$\alpha_2 = 0$	35.8343	52.9574	3.8415	$< 10^{-8}$	-93.9148	-61.6687
$\alpha_1 = \alpha_2 = 0$	24.2760	52.9574	5.9914	$< 10^{-12}$	-93.9148	-40.5520

Table 5.8.7: Estimates of the parameters and the corresponding 95% CIs for Example 5.8.3.

	$\hat{\alpha}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
MLEs	0.1587	2.2936	0.0510	0.0005
Bootstrap CI	(0.1062,0.1930)	(2.1880,2.3961)	(0.0347,0.0666)	(0.0002,0.0009)
Asymptotic CI	(0.1147,0.2027)	(2.2315, 2.3556)	(0.0454,0.0566)	(0.0004,0.0006)
Appro. CI	(0.1147,0.2027)	(2.1057, 2.2297)	(0.0454,0.0566)	(0.0004,0.0006)
Bin. Appro. CI	(0.1147,0.2027)	(2.2315, 2.3556)	(0.0454,0.0566)	(0.0004,0.0006)

Example 5.8.3 *The third data, presented in Table A.1.5, are from Montgomery et al. (2006). A soft drink bottler is interested in predicting the amount of time required by the route driver to service the vending machines in an outlet. The two most important variables affecting the delivery time (t) are the number of cases of product stocked (x_1) and the distance walked by the route driver (x_2). We will analyze these data here by fitting a BS regression model with 2 covariates. The MLEs of the model parameters are presented in Table 5.8.7 with the corresponding 95% CIs. The likelihood-ratio test results for the significance of the 2 covariates are presented in Table 5.8.8. With the obtained MLEs, we have constructed the BS Q-Q plot and the normal Q-Q plot in Figures 5.8.4 and 5.8.5, respectively. We have also carried out the KS test for the suitability of the assumed BS model, and the obtained results are presented in Table 5.8.9. The points falling close to a straight line in the two Q-Q plots as well as the lager P-value of the KS test in Table 5.8.9 provide strong evidence in favour of the assumed BS regression model.*

Table 5.8.8: Likelihood-ratio test for testing the hypotheses $\beta_1 = 0$ and $\beta_2 = 0$ for Example 5.8.3.

Hypothesis	$\ln L_0$	$\ln L_1$	$\chi^2_{1,0.95}$	P-value
$\beta_1 = 0$	-0.8300	10.6196	3.8415	$< 10^{-5}$
$\beta_2 = 0$	6.8182	10.6196	3.8415	0.0058

Table 5.8.9: KS statistic and corresponding P-value for Example 5.8.3.

KS-Statistic	P-value
0.1365	0.6904

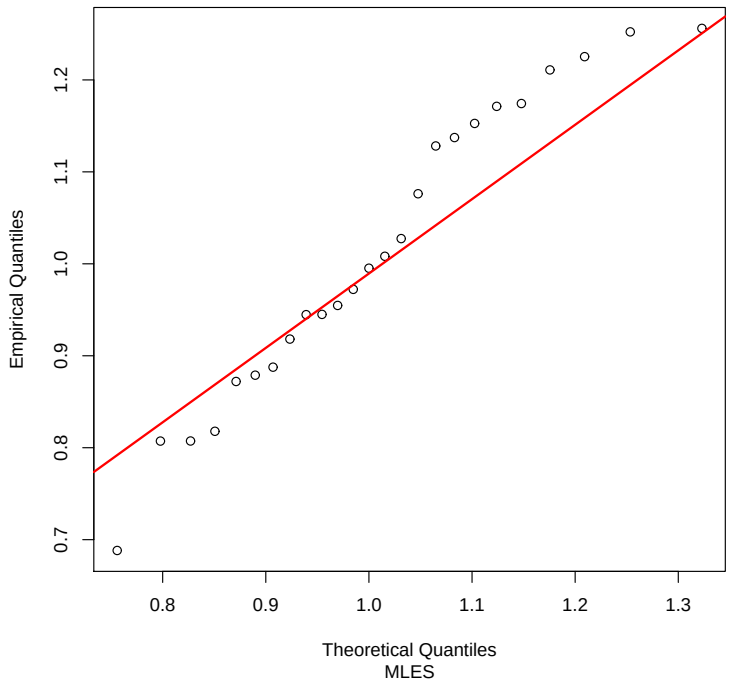

Figure 5.8.4: BS Q-Q plot for Example 5.8.3.

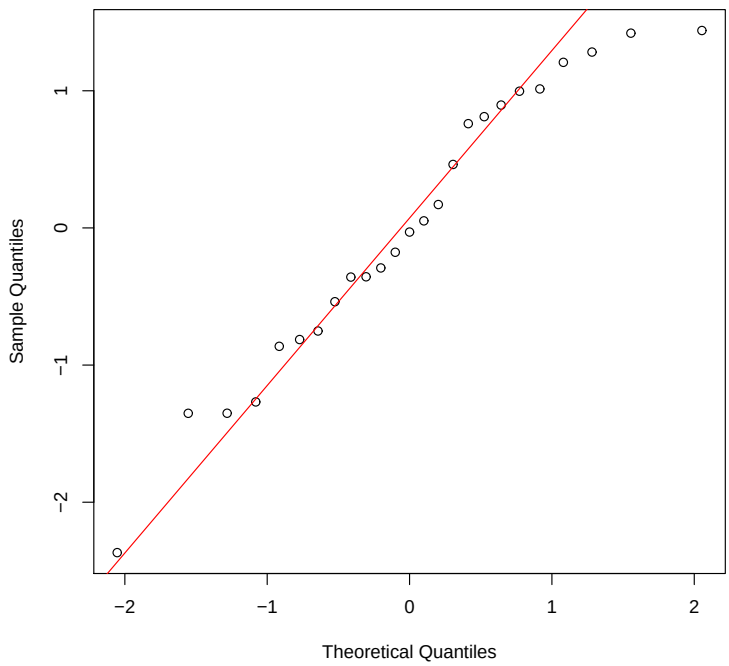

Figure 5.8.5: Normal Q-Q plot for Example 5.8.3.

Chapter 6

Inference for BVBS regression model

6.1 Introduction

Lemonte, A.J. (2013) discussed the regression based on multivariate Birnbaum-Saunders distribution by assuming independence of two sample. In this chapter, we develop some inferential results for the BVBS lifetime regression model which allows the dependence. We discuss the MLEs of the model parameters under a log-linear link function and also propose methods for the choice of initial values needed for the required numerical iterative process. Interval estimation of parameters and hypotheses tests are also discussed. Some model validation techniques are described. An extensive Monte Carlo simulation study is carried out to examine the performance of all the

inferential methods developed here, after which an illustrative example is presented.

The rest of this chapter proceeds as follows. In Section 6.2, after presenting the BVBS regression model with log-linear link function, we discuss the MLEs of the model parameters and explain the process of setting up initial values required for the iterative process by the method of least-squares. We also discuss briefly likelihood-ratio tests for some hypotheses of interest. In Section 6.3, we discuss the interval estimation methods based on asymptotic properties of the MLEs. In Section 6.4, some model validation techniques are briefly mentioned. A simulation study is carried out in Section 6.5 to evaluate the performance of the proposed estimation methods based on bias and MSE. Finally, an example is analyzed in Section 6.6 to illustrate all the inferential methods developed here.

6.2 Bivariate regression model and ML estimation

6.2.1 Model

Suppose $(T_1, T_2) \sim BS(\alpha_1, \alpha_2, \theta_1, \theta_2, \rho)$ with PDF as in (1.1.8). Now, assume that there are p and q covariates, denoted by $\boldsymbol{X}^{(1)} = (x_1^{(1)}, \cdots, x_p^{(1)})^T$ and $\boldsymbol{X}^{(2)} = (x_1^{(2)}, \cdots, x_q^{(2)})^T$, associated with the lifetimes T_1 and T_2, respectively. It should be noted that $\boldsymbol{X}^{(1)}$ could be equal to $\boldsymbol{X}^{(2)}$, but not necessary. Then, from

(1.1.7), by assuming a log-linear link function, we have the joint CDF as

$$P(T_1 \le t_1, T_2 \le t_2) = \Phi_2 \left[\frac{1}{\alpha_1} \left(\sqrt{\frac{t_1}{\theta_1}} - \sqrt{\frac{\theta_1}{t_1}} \right), \frac{1}{\alpha_2} \left(\sqrt{\frac{t_2}{\theta_2}} - \sqrt{\frac{\theta_2}{t_2}} \right); \rho \right],$$
$$t_1 > 0, t_2 > 0, \quad (6.2.1)$$

where $\theta_j = e^{\boldsymbol{\beta}_j{}'\boldsymbol{X}^{(j)}}$, $\boldsymbol{\beta}_j{}'\boldsymbol{X}^{(j)} = \beta_{j0} + \beta_{j1}x_1^{(j)} + \beta_{j2}x_2^{(j)} + \cdots + \beta_{jp(q)}x_{p(q)}^{(j)}$, for $j = 1, 2$. Also, the corresponding joint PDF is given by (1.1.8) with the above choices of θ_j, $j = 1, 2$.

6.2.2 ML estimation

Let (t_{1i}, t_{2i}), $i = 1, \cdots, n$, be the joint lifetimes of n units, with the covariates corresponding to t_{1i} as $\boldsymbol{X}_i^{(1)} = (x_{1i}^{(1)}, x_{2i}^{(1)}, \cdots, x_{pi}^{(1)})$ and t_{2i} as $\boldsymbol{X}_i^{(2)} = (x_{1i}^{(2)}, x_{2i}^{(2)}, \cdots, x_{qi}^{(2)})$, respectively. Then, by taking the corresponding scale parameters as $\theta_{ji} = \exp\left\{\beta_{j0} + \beta_{j1}x_{1i}^{(j)} + \beta_{j2}x_{2i}^{(j)} + \cdots + \beta_{jp(q)}x_{p(q)i}^{(j)}\right\}$, for $j = 1, 2$, we have the likelihood function as

$$L = \prod_{i=1}^{n} f(t_{1i}, t_{2i}; \alpha_1, \alpha_2, \theta_{1i}, \theta_{2i}, \rho), \quad (6.2.2)$$

where $f(\cdot)$ is the joint PDF of the BVBS distribution in (1.1.8). From (6.2.2), we obtain the log-likelihood function (without the constant) as

$$\begin{aligned} lnL &= -n\sum_{j=1}^{2}\ln\alpha_j - \frac{n}{2}\ln(1-\rho^2) + \sum_{j=1}^{2}\sum_{i=1}^{n}\ln\left[\left(\frac{t_{ji}}{\theta_{ji}}\right)^{\frac{1}{2}} + \left(\frac{\theta_{ji}}{t_{ji}}\right)^{\frac{1}{2}}\right] \\ &\quad -\sum_{j=1}^{2}\sum_{i=1}^{n}\frac{1}{2(1-\rho^2)\alpha_j^2}\left(\frac{t_{ji}}{\theta_{ji}} + \frac{\theta_{ji}}{t_{ji}} - 2\right) \\ &\quad +\frac{\rho}{(1-\rho^2)\alpha_1\alpha_2}\sum_{i=1}^{n}\left(\sqrt{\frac{t_{1i}}{\theta_{1i}}} - \sqrt{\frac{\theta_{1i}}{t_{1i}}}\right)\left(\sqrt{\frac{t_{2i}}{\theta_{2i}}} - \sqrt{\frac{\theta_{2i}}{t_{2i}}}\right). \end{aligned} \tag{6.2.3}$$

From (6.2.3), for given $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, the MLEs of α_1, α_2 and ρ can be expressed as

$$\hat{\alpha}_j(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{t_{ji}}{\theta_{ji}} + \frac{\theta_{ji}}{t_{ji}} - 2\right)\right]^{\frac{1}{2}}, \quad j = 1, 2, \tag{6.2.4}$$

$$\hat{\rho}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \frac{\sum_{i=1}^{n}\left(\sqrt{\frac{t_{1i}}{\theta_{1i}}} - \sqrt{\frac{\theta_{1i}}{t_{1i}}}\right)\left(\sqrt{\frac{t_{2i}}{\theta_{2i}}} - \sqrt{\frac{\theta_{2i}}{t_{2i}}}\right)}{\sqrt{\sum_{i=1}^{n}\left(\sqrt{\frac{t_{1i}}{\theta_{1i}}} - \sqrt{\frac{\theta_{1i}}{t_{1i}}}\right)^2}\sqrt{\sum_{i=1}^{n}\left(\sqrt{\frac{t_{2i}}{\theta_{2i}}} - \sqrt{\frac{\theta_{2i}}{t_{2i}}}\right)^2}}. \tag{6.2.5}$$

It is evident that $\hat{\alpha}_1(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is a function of $\boldsymbol{\beta}_1$ only, and similarly $\hat{\alpha}_2(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is a function of $\boldsymbol{\beta}_2$ only. Finally, the MLEs of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ can be obtained by maximizing the profile log-likelihood function

$$\begin{aligned} L_{profile}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) &= -n\sum_{j=1}^{2}\ln\hat{\alpha}_j(\boldsymbol{\beta}_j) - \frac{n}{2}\ln(1-\hat{\rho}^2(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)) \\ &\quad +\sum_{j=1}^{2}\sum_{i=1}^{n}\ln\left\{\left(\frac{\theta_{ji}}{t_{ji}}\right)^{\frac{1}{2}} + \left(\frac{t_{ji}}{\theta_{ji}}\right)^{\frac{1}{2}}\right\}. \end{aligned} \tag{6.2.6}$$

We shall adopt the Newton-Raphson method to maximize the profile likelihood function in (6.2.6). However, for the implementation of the Newton-Raphson method, it is important to provide good initial values, which is discussed next.

6.2.3 Initial values by least-squares method

It is known that the mode and median of the univariate $BS(\alpha, \beta)$ distribution are both equal to β. So, by treating t_{ji} as an estimator of θ_{ji}, we have the relationship

$$\ln t_{ji} = \beta_{j0} + \beta_{j1} x_{1i}^{(j)} + \cdots + \beta_{jp(q)} x_{p(q)i}^{(j)} + \epsilon_{ji}, \qquad i = 1, \cdots, n, \tag{6.2.7}$$

to develop the LSE of $\boldsymbol{\beta}_j$, $j = 1, 2$. Note ϵ_1 and ϵ_2 follows the bivariate sinh-normal distribution [see Díaz-García and Domínguez-Molina (2006), Kundu (2014a,b) and Vilca et al. (2014)]. Now, by minimizing the sum of squares

$$S(\boldsymbol{\beta}_j) = \sum_{i=1}^{n} (\ln t_{ji} - \beta_{j0} - \beta_{j1} x_{1i}^{(j)} - \cdots - \beta_{jp(q)} x_{p(q)i}^{(j)})^2, \tag{6.2.8}$$

we obtain the LSE of $\boldsymbol{\beta}_j$ as

$$\boldsymbol{\beta}_j^{(0)} = \left(\boldsymbol{X}^{(j)\prime} \boldsymbol{X}^{(j)} \right)^{-1} \boldsymbol{X}^{(j)\prime} \ln \boldsymbol{t}_j, \tag{6.2.9}$$

where $\boldsymbol{X}^{(j)} = \begin{pmatrix} 1 & x_{11}^{(j)} & x_{21}^{(j)} & \cdots & x_{p1}^{(j)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{1n}^{(j)} & x_{2n}^{(j)} & \cdots & x_{pn}^{(j)} \end{pmatrix}$ and $\ln \boldsymbol{t}_j = \begin{pmatrix} \ln t_{j1} \\ \vdots \\ \ln t_{jn} \end{pmatrix}$. Upon substituting $\hat{\boldsymbol{\beta}}_1^{(0)}$ and $\hat{\boldsymbol{\beta}}_2^{(0)}$ in Eqs. (6.2.4) and (6.2.5), we obtain the estimates of α_1, α_2 and ρ. Moreover, the estimate $(\hat{\boldsymbol{\beta}}_1^{(0)}, \hat{\boldsymbol{\beta}}_2^{(0)})$ in (6.2.9) can be used as an initial value in the Newton-Raphson method for maximizing (6.2.6).

6.2.4 Hypotheses testing

We can use likelihood-ratio tests to test some hypotheses of interest. For example, the following specific testing problems of interest can be handled readily:

- Test 1: $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$;
- Test 2: $H_0 : \alpha_1 = \alpha_2$ vs. $H_1 : \alpha_1 \neq \alpha_2$;
- Test 3: $H_0 : \beta_{ij} = 0$ vs. $H_1 : \beta_{ij} \neq 0$, for $j = 1, 2, \cdots, p(q)$, $i = 1(2)$.

If T_1 and T_2 have a common covariate, say $x_{j_1}^{(1)}$ and $x_{j_2}^{(2)}$, we can also similarly test

- Test 4: $H_0 : \beta_{1j_1} = \beta_{2j_2}$ vs. $H_1 : \beta_{1j_1} \neq \beta_{2j_2}$;
- Test 5: $H_0 : \beta_{1j_1} = \beta_{2j_2} = 0$ vs. $H_1 : \beta_{1j_1} \neq 0$ or $\beta_{2j_2} \neq 0$.

We describe below the procedure for testing the correlation coefficient, and since other tests can be developed similarly, we refrain from providing pertinent details. The following properties will be used for developing the required tests.

Property 6.2.1 *Let ρ^* be the correlation coefficient between T_1 and T_2, i.e., $\rho^* = corr(T_1, T_2)$. Then, if $\rho = 0$, ρ^* must be equal to 0.*

Proof

If $\rho = 0$, we will have $a_{m,n} = 0$ and $I_1 = 0$ in (1.1.10) and (1.1.11), respectively. We thus obtain

$$\begin{aligned} E(T_1T_2) &= \beta_1\beta_2\left[1 + \frac{1}{2}(\alpha_1^2 + \alpha_2^2) + \frac{1}{4}\alpha_1^2\alpha_2^2\right] = \beta_1\left(1 + \frac{1}{2}\alpha_1^2\right)\beta_2\left(1 + \frac{1}{2}\alpha_2^2\right) \\ &= E(T_1)E(T_2), \end{aligned}$$

implying that $\rho^* = 0$. ■

Property 6.2.2 *Z_1 and Z_2 are independent if and only if T_1 and T_2 are independent.*

Proof Let us use $f_{T_1,T_2}(t_1, t_2)$ to denote the joint PDF of T_1 and T_2, and $f_{Z_1,Z_2}(z_1, z_2)$ to denote the standard bivariate normal density function. Then, for any $(z_1, z_2) \in (-\infty, \infty) \times (-\infty, \infty)$, we have $f_{Z_i}(z_i) = f_{T_i}(\psi_i(z_i))$ and

$$f_{Z_1,Z_2}(z_1, z_2) = f_{T_1,T_2}(\psi_1(z_1), \psi_2(z_2)),$$

where $\psi_i(.)$ is a one-to-one function given by $\psi_i(z_i) = \beta_i\left[1 + \frac{1}{2}\alpha_i^2 + \alpha_i z_i\sqrt{1 + \left(\frac{1}{2}\alpha_i z_i\right)^2}\right]$, for $i = 1, 2$.

<u>**Sufficiency**</u>

If T_1 and T_2 are independent, then we have

Table 6.2.1: Values of correlation coefficient ρ^* for various choices of α_1, α_2 and ρ, by taking $\beta_1 = \beta_2 = 1$, without loss of any generality.

α_1	α_2	ρ	ρ^*	α_1	α_2	ρ	ρ^*	α_1	α_2	ρ	ρ^*
0.25	0.25	0.95	0.9485	0.25	0.50	0.95	0.9373	0.25	1.00	0.95	0.8915
		0.50	0.4925			0.50	0.4812			0.50	0.4505
		0.25	0.2448			0.25	0.2377			0.25	0.2205
		0.00	0.0007			0.00	0.0006			0.00	0.0005
0.50	0.50	0.95	0.9450	0.50	1.00	0.95	0.9241	1.00	1.00	0.95	0.9373
		0.50	0.4752			0.50	0.4515			0.50	0.4378
		0.25	0.2320			0.25	0.2169			0.25	0.2050
		0.00	0.0006			0.00	0.0004			0.00	0.0003

$f_{Z_1,Z_2}(z_1, z_2) = f_{T_1,T_2}(\psi_1(z_1), \psi_2(z_2)) = f_{T_1}(\psi_1(z_1))f_{T_2}(\psi_2(z_2)) = f_{Z_1}(z_1)f_{Z_2}(z_2) \Rightarrow Z_1$ and Z_2 are independent.

Necessity

If Z_1 and Z_2 are independent, then we have

$f_{T_1,T_2}(t_1, t_2) = f_{Z_1,Z_2}(\psi_1^{-1}(t_1), \psi_2^{-1}(t_2)) = f_{Z_1}(\psi_1^{-1}(t_1))f_{Z_2}(\psi_2^{-1}(t_2)) = f_{T_1}(t_1)f_{T_2}(t_2) \Rightarrow$ T_1 and T_2 are independent. ■

By means of a Monte Carlo study, we determined the values of ρ^* for different values of α_1, α_2 and ρ, by taking $\beta_1 = \beta_2 = 1$ without loss of any generality. These results are presented in Table 6.2.1 from which it is seen that $|\rho| \geq |\rho^*|$, while ρ^* value remains close to ρ.

From Property 6.2.2, we observe that testing independence of T_1 and T_2 is equivalent to testing the independence of Z_1 and Z_2. We can conclude that if T_1 and T_2 are independent, then we must have $\rho = 0$, though the inverse is not necessarily true. So, it will be of interest to test whether $H_0 : \rho = 0$. In this case, the MLEs of α_1 and

α_2, denoted by $\tilde{\alpha}_1(\boldsymbol{\beta}_1)$ and $\tilde{\alpha}_2(\boldsymbol{\beta}_2)$, are the same as given in (6.2.4), while the MLEs of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are obtained by maximizing the profile log-likelihood function

$$\tilde{L}_{profile}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = -n \sum_{j=1}^{2} \ln \tilde{\alpha}_j(\boldsymbol{\beta}_j) + \sum_{j=1}^{2} \sum_{i=1}^{n} \ln \left\{ \left(\frac{\theta_{ji}}{t_{ji}} \right)^{\frac{1}{2}} + \left(\frac{t_{ji}}{\theta_{ji}} \right)^{\frac{1}{2}} \right\}. \quad (6.2.10)$$

In this case, it is equivalent to obtaining $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ by separately maximizing

$$\tilde{L}_{profile1} = -n \ln \tilde{\alpha}_1(\boldsymbol{\beta}_1) + \sum_{i=1}^{n} \ln \left\{ \left(\frac{\theta_{1i}}{t_{1i}} \right)^{\frac{1}{2}} + \left(\frac{t_{1i}}{\theta_{1i}} \right)^{\frac{1}{2}} \right\}, \quad (6.2.11)$$

$$\tilde{L}_{profile2} = -n \ln \tilde{\alpha}_2(\boldsymbol{\beta}_2) + \sum_{i=1}^{n} \ln \left\{ \left(\frac{\theta_{2i}}{t_{2i}} \right)^{\frac{1}{2}} + \left(\frac{t_{2i}}{\theta_{2i}} \right)^{\frac{1}{2}} \right\}, \quad (6.2.12)$$

since $\tilde{L}_{profile1} + \tilde{L}_{profile2} = \tilde{L}_{profile}$, in (6.2.10). If $(\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2)$ and $(\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_2)$ are the global MLEs that maximize (6.2.10) and (6.2.6) under H_0 and H_1, respectively, then, for large n,

$$D = -2 \left\{ \tilde{L}_{profile}(\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2) - \hat{L}_{profile}(\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_2) \right\} \sim \chi^2_1. \quad (6.2.13)$$

We will reject H_0 for large values of D.

We may also instead use AIC to reject H_0 if $-2\tilde{L}_{profile}(\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2) + 2\hat{L}_{profile}(\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_2) - 2 > 0$.

6.3 Interval estimation

In this section, we discuss briefly the CIs based on the asymptotic properties of MLEs. One may also develop it from the bootstrap approach described earlier in Chapter 1.

6.3.1 Asymptotic confidence intervals

Under the usual regularity conditions, it can be shown that the MLEs are consistent and asymptotically normally distributed, as presented in the following theorem.

Theorem 6.3.1 *With* $\boldsymbol{\eta} = (\alpha_1, \alpha_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \rho)$ *as the parameter vector and* $\hat{\boldsymbol{\eta}} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\boldsymbol{\beta}}_{\mathbf{1}}, \hat{\boldsymbol{\beta}}_{\mathbf{2}}, \hat{\rho})$ *as its MLE, as* $n \to \infty$*, under usual regularity conditions, we have*

$$\sqrt{n}\left(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}\right) \xrightarrow{D} N_{p+q+5}\left(\mathbf{0}, \boldsymbol{J}_{p+q+5}^{-1}\right), \tag{6.3.1}$$

where $\xrightarrow{D}$ *denotes convergence in distribution and* $N_{p+q+5}\left(\mathbf{0}, \boldsymbol{J}_{p+q+5}^{-1}\right)$ *denotes the* $(p+q+5)$*-variate normal distribution with mean vector* $\mathbf{0}$ *and covariance matrix* $\boldsymbol{J}_{p+q+5}^{-1}$*, with the matrix* $\boldsymbol{J}_{p+q+5}$ *being the corresponding limiting Fisher information matrix; see Lehmann (1999).*

Here, the information matrix is defined as

$$\boldsymbol{J}_{p+q+5} = E\begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha_1^2} & -\frac{\partial^2 \ln l}{\partial \alpha_1 \partial \alpha_2} & -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \beta_{10}} & \cdots & -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \beta_{1p}} & -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \beta_{20}} & \cdots & -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \beta_{2q}} & -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \rho} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \rho} & -\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \rho} & -\frac{\partial^2 \ln L}{\partial \beta_{10} \partial \rho} & \cdots & -\frac{\partial^2 \ln L}{\partial \beta_{1p} \partial \rho} & -\frac{\partial^2 \ln L}{\partial \beta_{20} \partial \rho} & \cdots & -\frac{\partial^2 \ln L}{\partial \beta_{2q} \partial \rho} & -\frac{\partial^2 \ln L}{\partial \rho^2} \end{bmatrix}. \tag{6.3.2}$$

6.3.2 Fisher information matrix

From Theorem 6.3.1, we readily have the CI for η_j as

$$\left(\hat{\eta}_j - z_{1-\frac{\alpha}{2}}\widehat{J_{jj}^{-1}}, \hat{\eta} + z_{1-\frac{\alpha}{2}}\widehat{J_{jj}^{-1}}\right), \tag{6.3.3}$$

where $z_{1-\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ percentage point of the standard normal distribution. Let us define

$z_{ji} = \frac{1}{\alpha_j}\left(\sqrt{\frac{t_{ji}}{\theta_{ji}}} - \sqrt{\frac{\theta_{ji}}{t_{ji}}}\right)$,

$z'_{jki} = \frac{\partial z_{ji}}{\partial \beta_{jk}} = -\frac{1}{2\alpha_j}\left(\sqrt{\frac{t_{ji}}{\theta_{ji}}} + \sqrt{\frac{\theta_{ji}}{t_{ji}}}\right) x_{ki}^{(j)}$,

$z''_{jkli} = \frac{\partial^2 z_{ji}}{\partial \beta_{jk}\partial \beta_{jl}} = \frac{1}{4\alpha_j}\left(\sqrt{\frac{t_{ji}}{\theta_{ji}}} - \sqrt{\frac{\theta_{ji}}{t_{ji}}}\right) x_{ki}^{(j)} x_{li}^{(j)}$,

$(z_{jki}^2)' = \frac{\partial (z_{ji}^2)}{\partial \beta_{jk}} = -\frac{1}{\alpha_j^2}\left(\frac{t_{ji}}{\theta_{ji}} - \frac{\theta_{ji}}{t_{ji}}\right) x_{ki}^{(j)}$,

$(z_{jkli}^2)'' = \frac{\partial^2 (z_{ji}^2)}{\partial \beta_{jk}\partial \beta_{jl}} = \frac{1}{\alpha_j^2}\left(\frac{t_{ji}}{\theta_{ji}} + \frac{\theta_{ji}}{t_{ji}}\right) x_{ki}^{(j)} x_{li}^{(j)}$,

and $x_{0i}^{(0)} = x_{0i}^{(1)} = 1$ for $i = 1, \cdots, n$, $j = 1(2)$, $k = 0, \cdots, p(q)$, $l = 0, \cdots, p(q)$. Then, the first-order derivatives of the log-likelihood function with respect to the model parameters, required for the Newton-Raphson method, are given by

$$\frac{\partial \ln L}{\partial \alpha_j} = -\frac{n}{\alpha_j} + \frac{1}{(1-\rho^2)\alpha_j}\sum_{i=1}^{n} z_{ji}^2 - \frac{\rho}{(1-\rho^2)\alpha_j}\sum_{i=1}^{n} z_{ji}z_{j_1 i}, \tag{6.3.4}$$

$$\frac{\partial \ln L}{\partial \rho} = \frac{n\rho}{1-\rho^2} - \frac{\rho}{(1-\rho^2)^2}\sum_{i=1}^{n}(z_{1i}^2 + z_{2i}^2) + \frac{1+\rho^2}{(1-\rho^2)^2}\sum_{i=1}^{n} z_{1i}z_{2i}, \tag{6.3.5}$$

$$\frac{\partial \ln L}{\partial \beta_{jk}} = -\frac{1}{2}\sum_{i=1}^{n}\left(\frac{t_{ji}-\theta_{ji}}{t_{ji}+\theta_{ji}}\right) x_{ki}^{(j)} - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}(z_{jki}^2)' + \frac{\rho}{1-\rho^2}\sum_{i=1}^{n} z_{j_1 i}z'_{jki}, \tag{6.3.6}$$

for $j \neq j_1 \in \{1, 2\}$. Next, the negative of all the second-order derivatives of the log-likelihood function with respect to the model parameters are obtained from (6.3.4)-(6.3.6) as follows:

$$-\frac{\partial^2 \ln L}{\partial \alpha_j^2} = -\frac{n}{\alpha_j^2} + \frac{3}{(1-\rho^2)\alpha_j^2}\sum_{i=1}^{n} z_{ji}^2 - \frac{2\rho}{(1-\rho^2)\alpha_j^2}\sum_{i=1}^{n} z_{ji}z_{j_1 i}, \tag{6.3.7}$$

$$-\frac{\partial^2 \ln L}{\partial \rho^2} = -\frac{n(1+\rho^2)}{(1-\rho^2)^2} + \frac{1+3\rho^2}{(1-\rho^2)^3}\sum_{i=1}^{n}(z_{1i}^2 + z_{2i}^2) - \frac{2\rho(3+\rho^2)}{(1-\rho^2)^3}\sum_{i=1}^{n} z_{1i}z_{2i}, \tag{6.3.8}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_{jk}^2} = \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}(z_{jkki}^2)'' - \sum_{i=1}^{n}\frac{\theta_{ji}t_{ji}}{(t_{ji}+\theta_{ji})^2}[x_{ki}^{(j)}]^2 - \frac{\rho}{1-\rho^2}\sum_{i=1}^{n}(z_{jkki})''z_{j_1 i}, \tag{6.3.9}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \alpha_2} = -\frac{\rho}{(1-\rho^2)\alpha_1\alpha_2}\sum_{i=1}^{n} z_{ji}z_{j_1 i}, \tag{6.3.10}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_j \partial \rho} = -\frac{2\rho}{(1-\rho^2)^2\alpha_j}\sum_{i=1}^{n} z_{ji}^2 + \frac{1+\rho^2}{(1-\rho^2)^2\alpha_j}\sum_{i=1}^{n} z_{1i}z_{2i}, \tag{6.3.11}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_{jk}} = \frac{\rho}{(1-\rho^2)\alpha_j}\sum_{i=1}^{n} z'_{jki}z_{j_1 i} - \frac{1}{(1-\rho^2)\alpha_j}\sum_{i=1}^{n}[z_{jki}^2]', \tag{6.3.12}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_{j_1 k}} = \frac{\rho}{(1-\rho^2)\alpha_j}\sum_{i=1}^{n} z_{ji}z'_{j_1 ki}, \tag{6.3.13}$$

$$-\frac{\partial^2 \ln L}{\partial \rho \partial \beta_{jk}} = \frac{\rho}{(1-\rho^2)^2}\sum_{i=1}^{n}[z_{jki}^2]' - \frac{1+\rho^2}{(1-\rho^2)^2}\sum_{i=1}^{n} z'_{jki}z_{j_1 i}, \tag{6.3.14}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_{jk} \partial \beta_{jl}} = \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}[z_{jkli}^2]'' - \sum_{i=1}^{n}\frac{\theta_{ji}t_{ji}}{(t_{ji}+\theta_{ji})^2}x_{ki}^{(j)}x_{li}^{(j)} - \frac{\rho}{1-\rho^2}\sum_{i=1}^{n} z''_{jkli}z_{j_1 i}, \quad k \neq l = 0, \cdots, p(q), \tag{6.3.15}$$

$$-\frac{\partial^2 \ln L}{\partial \beta_{1k_1} \partial \beta_{2k_2}} = -\frac{\rho}{1-\rho^2}\sum_{i=1}^{n} z'_{1k_1 i}z'_{2k_2 i}, \qquad k_1 = 0, \cdots, p,\ k_2 = 0, \cdots, q. \tag{6.3.16}$$

By using the known properties of the BS distribution listed in Property 1.1.3, we then obtain the expected values of the second-order derivatives of the log-likelihood with respect to the parameters, in (6.3.7)-(6.3.16), as follows:

$$-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j^2}\right] = \frac{n(2-\rho^2)}{\alpha_j^2(1-\rho^2)}, \tag{6.3.17}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \rho^2}\right] = \frac{n(1+\rho^2)}{(1-\rho^2)^2}, \tag{6.3.18}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \beta_{jk}^2}\right] = \frac{1}{1-\rho^2}\sum_{i=1}^{n}\left(\frac{1}{\alpha_j^2}+\frac{1}{2}-\frac{\rho^2}{4}\right)[x_{ki}^{(j)}]^2 - \sum_{i=1}^{n}[x_{ki}^{(j)}]^2 E\left[\frac{V_j}{(1+V_j)^2}\right], \tag{6.3.19}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \alpha_2}\right] = -\frac{n\rho^2}{(1-\rho^2)\alpha_1\alpha_2}, \tag{6.3.20}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j \partial \rho}\right] = -\frac{n\rho}{(1-\rho^2)\alpha_j}, \tag{6.3.21}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_{jk}}\right] = 0, \tag{6.3.22}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \alpha_j \partial \beta_{j_1 k}}\right] = 0, \tag{6.3.23}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \rho \partial \beta_{jk}}\right] = 0, \tag{6.3.24}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \beta_{jk} \partial \beta_{jl}}\right] = \frac{2+\alpha_1^2}{2(1-\rho^2)\alpha_1^2}\sum_{i=1}^{n} x_{ki}^{(j)} x_{li}^{(j)} - \sum_{i=1}^{n} x_{ki}^{(j)} x_{li}^{(j)} E\left[\frac{V_j}{(1+V_j)^2}\right] - \frac{\rho^2}{4(1-\rho^2)}\sum_{i=1}^{n} x_{ki}^{(j)} x_{li}^{(j)}, \tag{6.3.25}$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \beta_{1k_1} \partial \beta_{2k_2}}\right] = -\frac{\rho I_2}{(1-\rho^2)\alpha_1\alpha_2}\sum_{i=1}^{n} x_{k_1 i}^{(1)} x_{k_2 i}^{(2)}, \tag{6.3.26}$$

where $V_j \sim BS(\alpha_j, 1)$.

The value of $E\left[\frac{V}{(1+V)^2}\right]$ is discussed in (5.3.18) and (5.3.19), respectively. Now, if we use $\mathbf{J_1}$ to denote the information matrix for α_1, α_2 and ρ, and use $\mathbf{J_2}$ to denote the

information matrix for $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, then we observe from the expressions in (6.3.17)-(6.3.26) that

$$Var(\hat{\boldsymbol{\eta}}) = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{J}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2^{-1} \end{bmatrix}, \tag{6.3.27}$$

where the last equality holds if the matrix $Var(\hat{\boldsymbol{\eta}})$ is positive definite. Then, as in (6.3.3), we can construct an asymptotic $100(1-\alpha)\%$ CI for the parameter η_j by using the expression in (6.3.27).

6.4 Model Validation

By making use of the marginal distributions, we can use either a BS Q-Q plot or a normal Q-Q plot after transforming the data. We may also use the KS test either directly on the BS data or equivalently on the transformed normal data. Another alternative is to use the joint distribution and perform a chi-square test on the transformed bivariate normal data. Since these are standard procedures and similar to the univariate case described earlier in Chapter 5, we refrain from furnishing further details here.

6.5 Simulation study

We carried out an extensive Monte Carlo simulation study for different sample sizes n, scale parameters $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, and correlation parameter ρ. We chose the sample size n to be 50 and 100, the number of covariates $p = 2$ and $q = 2$, 3, the value of α_1 to be 0.25, the value of α_2 to be 0.25 and 0.50, and the correlation ρ to be 0.95, 0.50, 0.25 and 0.00. The covariates $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ were both randomly selected from uniform $U(-1, 1)$ distribution. Then, we determined the empirical values of means and MSEs of the MLEs and LSEs of the parameters. Moreover, for a given ρ, we calculated the corresponding $\hat{\rho}^* = corr(\frac{T_1}{\hat{\theta}_1}, \frac{T_2}{\hat{\theta}_2})$ and $\tilde{\rho}^* = corr(\frac{T_1}{\tilde{\theta}_1}, \frac{T_2}{\tilde{\theta}_2})$. Since we do not know the true value for ρ^*, we calculated $E[(\hat{\rho}^* - \bar{\hat{\rho}}^*)^2]$ and $E[(\tilde{\rho}^* - \bar{\tilde{\rho}}^*)^2]$ instead of the MSE. All these results are presented in Tables 6.5.1-6.5.2. From these tables, we observe that both MLEs and LSEs of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ have negligible bias even for n as small as 50, but the MLEs of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ (exclude β_{10} and β_{20}) have smaller MSEs, compared to the LSEs if the correlation coefficient is high. While the MLEs and LSEs of the shape parameters α_1 and α_2 are negatively biased for small n, but the bias becomes negligible when n gets large. We observe that the average value of $\hat{\rho}^*$ is quite close to the true value of ρ. We also find that the MSE of all the MLEs and LSEs are small for a large ρ and small α, and that the LSEs are quite efficient compared to the MLEs. Moreover, the MLEs did converge in almost all the 10,000 simulations that were performed for each parameter setting. This may be due to the fact that the LSEs, as initial values in the iterative process, are quite close to the MLEs.

Table 6.5.1: Simulated values of means and MSEs (reported within brackets) of the MLEs and LSEs when $\alpha_1 = \alpha_2 = 0.25$, $\boldsymbol{\beta}_1 = (-1.00, 1.00)$ and $\boldsymbol{\beta}_2 = (-1.00, 1.00)$.

					MLEs				
n	ρ	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_{10}$	$\hat{\beta}_{11}$	$\hat{\beta}_{20}$	$\hat{\beta}_{21}$	$\hat{\rho}$	$\hat{\rho}^*$
50	0.95	0.2460	0.2459	-1.0000	1.0002	-1.0000	1.0004	0.9511	0.9497
		(0.0006)	(0.0006)	(0.0012)	(0.0005)	(0.0013)	(0.0004)	(0.0002)	(0.0002)
	0.50	0.2444	0.2442	-1.0003	1.0006	-0.9999	0.9997	0.5044	0.4976
		(0.0007)	(0.0007)	(0.0013)	(0.0035)	(0.0013)	(0.0029)	(0.0124)	(0.0134)
	0.25	0.2439	0.2437	-1.0000	1.0006	-0.9998	0.9995	0.2533	0.2481
		(0.0007)	(0.0007)	(0.0013)	(0.0044)	(0.0014)	(0.0036)	(0.0193)	(0.0200)
	0.00	0.2437	0.2435	-1.0001	1.0006	-0.9998	0.9993	0.0013	0.0009
		(0.0007)	(0.0007)	(0.0013)	(0.0047)	(0.0014)	(0.0038)	(0.0219)	(0.0218)
100	0.95	0.2481	0.2481	-0.9999	1.0002	-0.9999	0.9999	0.9506	0.9493
		(0.0003)	(0.0003)	(0.0006)	(0.0002)	(0.0006)	(0.0002)	(0.0001)	(0.0001)
	0.50	0.2473	0.2471	-1.0000	1.0001	-0.9999	0.9997	0.5030	0.4961
		(0.0003)	(0.0003)	(0.0006)	(0.0014)	(0.0006)	(0.0013)	(0.0058)	(0.0064)
	0.25	0.2470	0.2468	-1.0000	1.0002	-0.9999	0.9996	0.2524	0.2474
		(0.0003)	(0.0003)	(0.0006)	(0.0018)	(0.0007)	(0.0017)	(0.0091)	(0.0095)
	0.00	0.2469	0.2467	-1.0000	1.0002	-0.9999	0.9996	0.0013	0.0013
		(0.0003)	(0.0003)	(0.0006)	(0.0019)	(0.0007)	(0.0018)	(0.0104)	(0.0103)
					LSEs				
n	ρ	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$	$\tilde{\beta}_{10}$	$\tilde{\beta}_{11}$	$\tilde{\beta}_{20}$	$\tilde{\beta}_{21}$	$\tilde{\rho}$	$\tilde{\rho}^*$
50	0.95	0.2437	0.2436	-1.0001	1.0007	-0.9999	0.9986	0.9293	0.9274
		(0.0007)	(0.0007)	(0.0013)	(0.0042)	(0.0013)	(0.0034)	(0.0011)	(0.0008)
	0.50	0.2437	0.2435	-1.0001	1.0007	-0.9999	0.9985	0.4861	0.4793
		(0.0007)	(0.0007)	(0.0013)	(0.0042)	(0.0013)	(0.0033)	(0.0123)	(0.0132)
	0.25	0.2437	0.2435	-1.0001	1.0007	-0.9999	0.9985	0.2428	0.2378
		(0.0007)	(0.0007)	(0.0013)	(0.0042)	(0.0013)	(0.0033)	(0.0182)	(0.0189)
	0.00	0.2437	0.2435	-1.0002	1.0007	-0.9998	0.9986	0.0003	0.0000
		(0.0007)	(0.0007)	(0.0013)	(0.0041)	(0.0013)	(0.0033)	(0.0204)	(0.0204)
100	0.95	0.2470	0.2469	-1.0000	1.0006	-0.9999	1.0001	0.9399	0.9383
		(0.0003)	(0.0003)	(0.0006)	(0.0022)	(0.0006)	(0.0017)	(0.0003)	(0.0002)
	0.50	0.2470	0.2468	-1.0000	1.0006	-0.9999	0.9998	0.4938	0.4870
		(0.0003)	(0.0003)	(0.0006)	(0.0021)	(0.0006)	(0.0017)	(0.0058)	(0.0063)
	0.25	0.2469	0.2467	-1.0000	1.0006	-0.9999	0.9997	0.2473	0.2423
		(0.0003)	(0.0003)	(0.0006)	(0.0021)	(0.0006)	(0.0017)	(0.0088)	(0.0092)
	0.00	0.2469	0.2467	-1.0000	1.0005	-0.9999	0.9996	0.0011	0.0010
		(0.0003)	(0.0003)	(0.0006)	(0.0021)	(0.0006)	(0.0017)	(0.0100)	(0.0099)

Table 6.5.2: Simulated values of means and MSEs (reported within brackets) of the MLEs and LSEs when $\alpha_1 = 0.25$, $\alpha_2 = 0.50$, $\boldsymbol{\beta}_1 = (-1.00, 1.00)$ and $\boldsymbol{\beta}_2 = (-1.00, 1.00, -0.50)$.

					MLEs					
n	ρ	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_{10}$	$\hat{\beta}_{11}$	$\hat{\beta}_{20}$	$\hat{\beta}_{21}$	$\hat{\beta}_{22}$	$\hat{\rho}$	$\hat{\rho}^*$
50	0.95	0.2460	0.4912	-1.0000	1.0002	-1.0000	1.0001	-0.4997	0.9521	0.9406
		(0.0006)	(0.0026)	(0.0013)	(0.0005)	(0.0048)	(0.0015)	(0.0013)	(0.0002)	(0.0003)
	0.50	0.2444	0.4841	-1.0000	1.0006	-0.9998	0.9994	-0.4993	0.5082	0.4915
		(0.0007)	(0.0028)	(0.0013)	(0.0036)	(0.0051)	(0.0112)	(0.0097)	(0.0128)	(0.0147)
	0.25	0.2439	0.4820	-1.0000	1.0006	-0.9997	0.9990	-0.4993	0.2556	0.2442
		(0.0007)	(0.0029)	(0.0013)	(0.0044)	(0.0052)	(0.0138)	(0.0122)	(0.0202)	(0.0213)
	0.00	0.2437	0.4813	-1.0001	1.0006	-0.9996	0.9987	-0.4994	0.0012	0.0007
		(0.0007)	(0.0029)	(0.0013)	(0.0047)	(0.0052)	(0.0146)	(0.0130)	(0.0230)	(0.0225)
100	0.95	0.2481	0.4958	-0.9999	1.0000	-0.9999	0.9998	-0.5000	0.9511	0.9391
		(0.0003)	(0.0013)	(0.0006)	(0.0002)	(0.0024)	(0.0007)	(0.0007)	(0.0001)	(0.0002)
	0.50	0.2473	0.4920	-1.0000	1.0001	-0.9998	0.9993	-0.4993	0.5049	0.4876
		(0.0003)	(0.0013)	(0.0006)	(0.0014)	(0.0025)	(0.0051)	(0.0056)	(0.0059)	(0.0070)
	0.25	0.2470	0.4910	-1.0000	1.0002	-0.9998	0.9992	-0.4990	0.2537	0.2420
		(0.0003)	(0.0014)	(0.0006)	(0.0018)	(0.0025)	(0.0064)	(0.0070)	(0.0093)	(0.0101)
	0.00	0.2469	0.4905	-1.0000	1.0002	-0.9998	0.9992	-0.4987	0.0014	0.0014
		(0.0003)	(0.0014)	(0.0006)	(0.0019)	(0.0025)	(0.0068)	(0.0075)	(0.0106)	(0.0105)
					LSEs					
n	ρ	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$	$\tilde{\beta}_{10}$	$\tilde{\beta}_{11}$	$\tilde{\beta}_{20}$	$\tilde{\beta}_{21}$	$\tilde{\beta}_{22}$	$\tilde{\rho}$	$\tilde{\rho}^*$
50	0.95	0.2437	0.4814	-1.0001	1.0007	-0.9997	0.9971	-0.4977	0.9198	0.9074
		(0.0007)	(0.0029)	(0.0013)	(0.0042)	(0.0049)	(0.0131)	(0.0144)	(0.0018)	(0.0012)
	0.50	0.2437	0.4813	-1.0001	1.0007	-0.9997	0.9970	-0.4988	0.4811	0.4645
		(0.0007)	(0.0029)	(0.0013)	(0.0042)	(0.0049)	(0.0129)	(0.0144)	(0.0127)	(0.0143)
	0.25	0.2437	0.4812	-1.0001	1.0007	-0.9997	0.9970	-0.4993	0.2402	0.2293
		(0.0007)	(0.0029)	(0.0013)	(0.0042)	(0.0049)	(0.0129)	(0.0143)	(0.0184)	(0.0196)
	0.00	0.2437	0.4812	-1.0002	1.0007	-0.9997	0.9971	-0.4998	0.0002	-0.0001
		(0.0007)	(0.0029)	(0.0013)	(0.0041)	(0.0049)	(0.0129)	(0.0143)	(0.0205)	(0.0204)
100	0.95	0.2470	0.4910	-1.0000	1.0006	-0.9998	1.0001	-0.5000	0.9351	0.9225
		(0.0003)	(0.0013)	(0.0006)	(0.0022)	(0.0024)	(0.0067)	(0.0061)	(0.0005)	(0.0004)
	0.50	0.2470	0.4907	-1.0000	1.0006	-0.9998	0.9996	-0.5001	0.4915	0.4742
		(0.0003)	(0.0014)	(0.0006)	(0.0021)	(0.0025)	(0.0067)	(0.0062)	(0.0058)	(0.0069)
	0.25	0.2469	0.4906	-1.0000	1.0006	-0.9998	0.9994	-0.5002	0.2463	0.2348
		(0.0003)	(0.0014)	(0.0006)	(0.0021)	(0.0025)	(0.0067)	(0.0063)	(0.0088)	(0.0096)
	0.00	0.2469	0.4905	-1.0000	1.0005	-0.9998	0.9993	-0.5002	0.0014	0.0014
		(0.0003)	(0.0014)	(0.0006)	(0.0021)	(0.0025)	(0.0067)	(0.0063)	(0.0100)	(0.0099)

6.6 Illustrative example

In this section, we illustrate the proposed methods of inference by using a data from the literature.

Example 6.6.1 *These data, obtained from Johnson and Wichern (2002), represent the bone mineral density (BMD) measured in g/cm^2 for 24 individuals, who had participated in an experimental study. These data represent the BMD of the bone dominant radius before starting the study and after one year, respectively. Kundu et al. (2010) analyzed these data by fitting a BVBS model without covariates. The summary of the variables of interest are presented in Table 6.6.1.*

Now, we shall use the regression model with 5 covariates for each T to analyze these data. The covariate matrices $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ are as defined in (6.2.9). Then, we calculated the initial values based on LSEs and utilized them to determine the MLEs of the parameters α_1, α_2, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ and ρ along with the corresponding 95% and 90% CIs, and these results are presented in Table 6.6.2. The BS Q-Q plot and the normal Q-Q plot based on the marginal distributions of T_1 and T_2 and the chi-square Q-Q plot for the joint distribution of T_1 and T_2 all show an approximate straight line, not rejecting the assumed BVBS model. These plots are not presented here for conciseness. We also carried out the KS test, and the corresponding P-values are presented in Table 6.6.3; these results also do not lead to the rejection of the assumed BVBS model.

Example 6.6.2 *In this example, we continue using the data from the previous example. Now, instead of using all 5 covariates, we only use the covariates Dominant*

Table 6.6.1: Summary of the variables of interest for Example 6.6.1.

Dominant radius	Radius	Dominant humerus	Humerus	Dominant Ulna	Ulna
(t_1)	$(x_1^{(1)})$	$(x_2^{(1)})$	$(x_3^{(1)})$	$(x_4^{(1)})$	$(x_5^{(1)})$
(t_2)	$(x_1^{(2)})$	$(x_2^{(2)})$	$(x_3^{(2)})$	$(x_4^{(2)})$	$(x_5^{(2)})$

Table 6.6.2: LSEs and MLEs of the parameters, SEs and the corresponding 95% and 90% CIs from the observed information matrix for Example 6.6.1.

Parameter	LSEs	MLEs	SEs	95% CIs	90% CIs
α_1	0.0561	0.0647	0.0084	(0.0482 0.0813)	(0.0509 0.0786)
α_2	0.0857	0.0923	0.0120	(0.0688 0.1159)	(0.0726 0.1121)
β_{10}	-1.2995	-1.2360	0.0074	(-1.2505 -1.2216)	(-1.2482 -1.2239)
β_{11}	1.0824	0.7977	0.0090	(0.7801 0.8153)	(0.7828 0.8125)
β_{12}	0.4822	0.2606	0.0041	(0.2526 0.2686)	(0.2539 0.2673)
β_{13}	-0.5091	-0.2177	0.0042	(-0.2259 -0.2094)	(-0.2246 -0.2107)
β_{14}	0.2633	0.5486	0.0104	(0.5281 0.5690)	(0.5313 0.5658)
β_{15}	0.1039	-0.0988	0.0106	(-0.1196 -0.0780)	(-0.1163 -0.0813)
β_{20}	-1.1958	-1.2063	0.0104	(-1.2269 -1.1857)	(-1.2236 -1.1890)
β_{21}	1.1248	0.7047	0.0128	(0.6796 0.7299)	(0.6835 0.7259)
β_{22}	0.4123	0.2578	0.0058	(0.2464 0.2692)	(0.2482 0.2674)
β_{23}	-0.3213	-0.1179	0.0060	(-0.1297 -0.1060)	(-0.1278 -0.1079)
β_{24}	-0.1557	0.2434	0.0146	(0.2149 0.2720)	(0.2194 0.2674)
β_{25}	0.0412	0.0291	0.0151	(-0.0005 0.0587)	(0.0042 0.0540)
ρ	0.6385	0.8302	0.0488	(0.7345 0.9259)	(0.7497 0.9107)

Table 6.6.3: KS statistics and the corresponding P-values based on the marginal and joint distributions for Example 6.6.1.

	KS-Statistic	P-value
T_1	0.1451	0.6408
T_2	0.1954	0.2802
T_1 & T_2	0.1870	0.3287

humerus $(x_2^{(1)}, x_2^{(2)})$ *and Dominant Ulna* $(x_4^{(1)}, x_4^{(2)})$*. Let us use* $x_1^{(1)}$ *and* $x_1^{(2)}$ *to denote the values of Dominant humerus, and* $x_2^{(1)}$ *and* $x_2^{(2)}$ *for the values of Dominant Ulna for* T_1 *and* T_2*, respectively.*

In this case, we first calculated the LSEs as initial values to determine the MLEs of the parameters α_1*,* α_2*,* $\boldsymbol{\beta}_1$*,* $\boldsymbol{\beta}_2$ *and* ρ*, along with the corresponding* 95% *and* 90% *CIs, as presented in Table 6.6.4. The BS Q-Q plot, the normal Q-Q plot, and the chi-square Q-Q plot all show an approximate straight line (not presented here for brevity), not rejecting the assumed BVBS model. We also carried out the KS test based on both marginal and joint distributions, and the computed values of the KS statistic and the corresponding P-values are presented in Table 6.6.5, and these results once again provide strong evidence in favour of the assumed BVBS model. Finally, we performed tests for three different hypotheses, and the corresponding likelihood-ratio test and AIC results are presented in Table 6.6.6, and these results do suggest that we can take* $\beta_{12} = \beta_{22}$ *in the assumed model.*

Table 6.6.4: LSEs and MLEs of the parameters, SEs and the corresponding 95% and 90% CIs from the observed information matrix for Example 6.6.2.

Parameter	$LSEs$	MLEs	SEs	95% CIs	90% CIs
α_1	0.0887	0.0895	0.0072	(0.0753 0.1037)	(0.0776 0.1014)
α_2	0.1113	0.1127	0.0091	(0.0949 0.1306)	(0.0978 0.1277)
β_{10}	-1.0757	-1.0915	0.0282	(-1.1069 -1.0761)	(-1.1045 -1.0785)
β_{11}	0.2320	0.1919	0.0079	(0.1730 0.2107)	(0.1760 0.2077)
β_{12}	0.6852	0.8105	0.0096	(0.8020 0.8191)	(0.8034 0.8177)
β_{20}	-0.9744	-1.0623	0.0044	(-1.0818 -1.0429)	(-1.0787 -1.0459)
β_{21}	0.2923	0.2472	0.0099	(0.2234 0.2710)	(0.2271 0.2672)
β_{22}	0.3775	0.6131	0.0121	(0.6023 0.6239)	(0.6040 0.6222)
ρ	0.8903	0.9020	0.0055	(0.8467 0.9574)	(0.8554 0.9486)

Table 6.6.5: KS statistics and the corresponding P-values based on marginal and joint distributions for Example 6.6.2.

	KS-Statistic	P-value
T_1	0.0971	0.9609
T_2	0.1094	0.9068
T_1 & T_2	0.1422	0.6647

Table 6.6.6: Likelihood-ratio test and AIC values for testing different null hypotheses for Example 6.6.2.

H_0	$\ln L_0$	$\ln L_1$	$2\ln L_1 - 2\ln L_0$	p-value	$AIC_0 - AIC_1$
$\alpha_1 = \alpha_2$	59.3482	62.4210	6.1456	0.0132	4.1456
$\beta_{11} = 0$	57.7036	62.4210	9.4352	0.0021	7.4352
$\beta_{12} = 0$	51.4615	62.4210	21.9190	$< 1e-05$	19.9190
$\beta_{11} = \beta_{21}$	61.6336	62.4210	1.5748	0.2095	-0.4252

Chapter 7

Likelihood inference for Laplace model under Type-II censoring

7.1 Introduction

Childs and Balakrishnan (1996, 1997, 2000) used the closed-form expressions of the MLEs to develop conditional inference procedures based on Type-II and progressively Type-II censored samples. Iliopoulos and Balakrishnan (2011) developed exact likelihood inference and tests based on some pivotal quantities. Recently, Iliopoulos and MirMostafaee (2014) developed exact prediction intervals based on the MLEs from censored samples. Motivated by all these works, we will derive here the exact distribution of the MLEs of the location and scale parameters of the Laplace distribution based on Type-II censored samples and then utilize them to construct the exact CIs.

The rest of this chapter is organized as follows. In Section 7.2, we present closed-form expressions of the MLEs based on Type-II right censored samples for different censoring cases. In Section 7.3, we first derive the joint MGF of the MLEs, and then use it to obtain the exact density functions of the MLEs, which are then used to develop exact CIs for μ and σ. In Section 7.4, after reviewing some important properties of spacings from exponential and Laplace distributions under Type-II censoring, we derive the exact densities of the MLEs from an approach based on spacings. In Section 7.5, we first present the MLE of the quantile and derive its exact density function and use it to find the mean and variance of the estimate and also to develop an exact CI for the quantile of the distribution. In Sections 7.6 and 7.7, we derive the exact distributions of the MLEs of reliability and cumulative hazard functions, and then use them to find the mean and variance of the estimators and further to develop exact CIs for the reliability and cumulative hazard functions. Analogous results are then developed based on BLUEs in Section 7.8. In Section 7.9, we present two examples to illustrate all the methods of inference developed here.

7.2 MLEs for Type-II right censored samples

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$ denote the Type-II right censored sample from the Laplace distribution in (1.2.2). Then, we have the likelihood function as [see Balakr-

ishnan and Cohen (1991) and Arnold et al. (1992)]

$$L = C_r \prod_{i=1}^{r} f(x_{i:n}) S(x_{r:n})^{n-r}, \tag{7.2.1}$$

where $C_r = \frac{n!}{(n-r)!}$ and $S(.) = 1 - F(.)$ is the survival function.

By maximizing the likelihood function in (7.2.1), we obtain the MLEs of μ and σ, as done by Balakrishnan and Cutler (1995). According to different censoring cases, we can present explicit simple expressions for the MLEs. In all the cases, the MLE $\hat{\sigma}$ exists uniquely, and so does the MLE $\hat{\mu}$ except when n is even and $r > \frac{n}{2}$; in this case, $\hat{\mu}$ can be any value between $x_{\frac{n}{2}:n}$ and $x_{\frac{n}{2}+1:n}$ and so we choose the average of these two values as the MLE of μ which is indeed unbiased for μ. We then have the following expressions for the MLEs.

Case I: $r < \frac{n}{2}$

$$\begin{cases} \hat{\mu} = x_{r:n} + \hat{\sigma} \ln \frac{n}{2r}, \\ \hat{\sigma} = \frac{1}{r} \sum_{i=1}^{r-1} (x_{r:n} - x_{i:n}); \end{cases} \tag{7.2.2}$$

Case II: $n = 2m + 1$ and $r \geq m + 2$

$$\begin{cases} \hat{\mu} = x_{m+1:n}, \\ \hat{\sigma} = \frac{1}{r} \left[\sum_{i=m+2}^{r} x_{i:n} - \sum_{i=1}^{m} x_{i:n} + (n-r) x_{r:n} \right]; \end{cases} \tag{7.2.3}$$

Case III: $n = 2m + 1$ and $r = m + 1$

$$\begin{cases} \hat{\mu} = x_{m+1:n}, \\ \hat{\sigma} = \frac{1}{r}\left[\sum_{i=1}^{m}(x_{m+1:n} - x_{i:n})\right]; \end{cases} \tag{7.2.4}$$

Case IV: $n = 2m$ and $r > m + 1$

$$\begin{cases} \hat{\mu} = \frac{1}{2}\left(x_{m:n} + x_{m+1:n}\right), \\ \hat{\sigma} = \frac{1}{r}\left[\sum_{i=m+1}^{r} x_{i:n} - \sum_{i=1}^{m} x_{i:n} + (n-r)x_{r:n}\right]; \end{cases} \tag{7.2.5}$$

Case V: $n = 2m$ and $r = m + 1$

$$\begin{cases} \hat{\mu} = \frac{1}{2}\left(x_{m:n} + x_{m+1:n}\right), \\ \hat{\sigma} = \frac{1}{r}\left[m x_{r:n} - \sum_{i=1}^{m} x_{i:n}\right]; \end{cases} \tag{7.2.6}$$

Case VI: $n = 2m$ and $r = m$

$$\begin{cases} \hat{\mu} = x_{m:n}, \\ \hat{\sigma} = \frac{1}{r}\left[(r-1)x_{r:n} - \sum_{i=1}^{r-1} x_{i:n}\right]. \end{cases} \tag{7.2.7}$$

7.3 Exact inference based on MLEs from the MGF approach

The joint MGF of $(\hat{\mu}, \hat{\sigma})$ will be derived in this section for all the cases presented in the preceding section, and then it will be used to obtain the bias and MSEs of $\hat{\mu}$

and $\hat{\sigma}$. Next, we derive the exact marginal distributions of $\hat{\mu}$ and $\hat{\sigma}$, which are then utilized to develop exact CIs for the parameters μ and σ.

7.3.1 Exact joint MGF of MLEs

The PDF of the r-th order statistic is given by [see Balakrishnan and Cohen (1991) and Arnold et al. (1992)]

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1} f(x)[S(x)]^{n-r}, \qquad -\infty < x < \infty. \quad (7.3.1)$$

Even though it will not be known whether the order statistic $X_{r:n}$ is greater than μ or not, it is clear that its density in (7.3.1) will take on two different forms for these two cases. For this reason, we need to use the conditional approach and let J $(0 \leq J \leq r)$ denote the number of observations in the Type-II censored data that are smaller than μ. Below, we will give the detailed procedure for obtaining the joint and marginal MGFs for Case I and then just present the final results for all other cases since their derivations are quite similar.

Case I: $r < \frac{n}{2}$

The joint MGF of $(\hat{\mu}, \hat{\sigma})$ can be obtained as

$$
\begin{aligned}
E\left(\mathrm{e}^{t_1\hat{\mu}+t_2\hat{\sigma}}\right) &= \sum_{j=0}^{r-1} E\left(\mathrm{e}^{t_1\hat{\mu}+t_2\hat{\sigma}}|J=j\right) P(J=j) + E\left(\mathrm{e}^{t_1\hat{\mu}+t_2\hat{\sigma}}|J=r\right) P(J=r) \\
&= \sum_{j=0}^{r-1} E\left(\mathrm{e}^{s_1 x_{r:n} - s_2 \sum_{i=1}^{r-1} x_{i:n}}|J=j\right) P(J=j) \\
&\quad + E\left(\mathrm{e}^{s_1 x_{r:n} - s_2 \sum_{i=1}^{r-1} x_{i:n}}|J=r\right) P(J=r) \\
&= \sum_{j=0}^{r-1} \frac{n!}{j!(r-1-j)!(n-r)!} \left[\int_{-\infty}^{\mu} \mathrm{e}^{-s_2 x} f(x)dx\right]^j \\
&\quad \times \int_{\mu}^{\infty} \left[\int_{\mu}^{x_{r:n}} \mathrm{e}^{-s_2 x} f(x)dx\right]^{r-1-j} \mathrm{e}^{s_1 x_{r:n}} f(x_{r:n})[1-F(x_{r:n})]^{n-r} dx_{r:n} \\
&\quad + \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \frac{n!}{(r-1)!(n-r)!} \\
&\quad \times \int_{-\infty}^{\mu} \left[\int_{-\infty}^{x_{r:n}} \mathrm{e}^{-s_2 x} f(x)dx\right]^{r-1} \mathrm{e}^{s_1 x_{r:n}} f(x_{r:n})[F(x_{r:n})]^l dx_{r:n} \\
&= \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_1 e^{t_1\mu} (1-s_2\sigma)^{-j} (1+s_2\sigma)^{-(r-1-j)} \left[1 - \frac{(s_1 - s_2 l)\sigma}{n-r+1+l}\right]^{-1} \\
&\quad + \sum_{l=0}^{n-r} p_2 e^{t_1\mu} (1-s_2\sigma)^{-(r-1)} \left(1 + \frac{t_1\sigma}{l+r}\right)^{-1}, \qquad (7.3.2)
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= \frac{(-1)^l c_{jr}(r-1-j)!}{2^n l!(r-1-j-l)!(n-r+1+l)}, \\
p_2 &= \frac{(-1)^l c'_{jr}(n-r)!}{2^{r+l} l!(n-r-l)!(l+r)}, \\
c_{jr} &= \frac{n!}{j!(r-1-j)!(n-r)!}, \\
c'_{jr} &= \frac{n!}{(r-1)!(n-r)!}, \\
s_1 &= \left(\frac{r-1}{r}\ln\frac{n}{2r}+1\right)t_1 + \frac{r-1}{r}t_2, \\
s_2 &= \frac{1}{r}\ln\frac{n}{2r}t_1 + \frac{1}{r}t_2,
\end{aligned}
$$

Throughout Chapters 7 and 8, t_1 and t_2 are defined so as to make all the MGFs involved to be positive.

Taking derivatives with respect to t_1 and t_2 and then setting them to 0, we get

$$
\begin{aligned}
E(\hat{\mu}\hat{\sigma}) &= \left.\frac{\partial E\left(\mathrm{e}^{t_1\hat{\mu}+t_2\hat{\sigma}}\right)}{\partial t_1 \partial t_2}\right|_{t_1=0,t_2=0} \\
&= \sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1 \left\{ \left[j s_2^{(t_2)} - (r-1-j)s_2^{(t_2)} + \frac{s_1^{(t_2)} - l s_2^{(t_2)}}{n-r+l+1} \right] \mu\sigma \right. \\
&\quad + j s_2^{(t_1)} \left[(j+1)s_2^{(t_2)} - (r-1-j)s_2^{(t_2)} + \frac{s_1^{(t_2)} - l s_2^{(t_2)}}{n-r+l+1} \right] \sigma^2 \\
&\quad - (r-1-j)s_2^{(t_1)} \left[j s_2^{(t_2)} - (r-j)s_2^{(t_2)} + \frac{s_1^{(t_2)} - l s_2^{(t_2)}}{n-r+l+1} \right] \sigma^2 \\
&\quad \left. + \frac{s_1^{(t_2)} - l s_2^{(t_2)}}{n-r+l+1} \left[j s_2^{(t_2)} - (r-j-1)s_2^{(t_2)} + 2\frac{s_1^{(t_2)} - l s_2^{(t_2)}}{n-r+l+1} \right] \right\} \\
&\quad + \sum_{l=0}^{n-r} p_2 (r-1) s_2^{(t_2)} \sigma \left[\mu + r s_2^{(t_1)}\sigma - \frac{1}{l+r}\sigma \right], \qquad (7.3.3)
\end{aligned}
$$

where $s_1^{(t_1)} = \frac{\partial s_1}{\partial t_1} = \frac{r-1}{r}\ln\frac{n}{2r} + 1$, $s_2^{(t_1)} = \frac{\partial s_2}{\partial t_1} = \frac{1}{r}\ln\frac{n}{2r}$, $s_1^{(t_2)} = \frac{\partial s_1}{\partial t_2} = \frac{r-1}{r}$, and $s_2^{(t_2)} = \frac{\partial s_2}{\partial t_2} = \frac{1}{r}$. Note, here

Upon setting $t_2 = 0$ and $t_1 = 0$ in (7.3.2), we obtain the marginal MGFs of $\hat{\mu}$ and $\hat{\sigma}$, respectively, as

$$\begin{aligned} E\left(\mathrm{e}^{t\hat{\mu}}\right) &= \mathrm{e}^{\mu t}\left\{\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left(1-\frac{tb\sigma}{r}\right)^{-j}\left(1+\frac{tb\sigma}{r}\right)^{-r+j+1}\right. \\ &\quad \times\left[1-\frac{r+b(r-1-l)}{r(n-r+1+l)}\sigma t\right]^{-1} \\ &\quad \left.+\sum_{l=0}^{n-r} p_2\left(1-\frac{tb\sigma}{r}\right)^{-(r-1)}\left(1+\frac{t\sigma}{l+r}\right)^{-1}\right\}, \qquad (7.3.4) \\ E\left(\mathrm{e}^{t\hat{\sigma}}\right) &= \sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left(1-\frac{t\sigma}{r}\right)^{-j}\left(1+\frac{t\sigma}{r}\right)^{-r+j+1}\left[1-\frac{(r-1-l)\sigma t}{r(n-r+1+l)}\right]^{-1} \\ &\quad +\sum_{l=0}^{n-r} p_2\left(1-\frac{t\sigma}{r}\right)^{1-r}, \qquad (7.3.5) \end{aligned}$$

where $b = \ln\frac{n}{2r}$. From (7.3.4) and (7.3.5), we readily obtain the first two moments of $\hat{\mu}$ and $\hat{\sigma}$ as follows:

$$
\begin{aligned}
E(\hat{\mu}) &= \mu + \frac{\sigma}{r}\left\{\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left[jb-(r-j-1)b+\frac{r+b(r-1-l)}{n-r+l+1}\right]\right. \\
&\quad \left.+\sum_{l=0}^{n-r} p_2\left[(r-1)b-\frac{r}{l+r}\right]\right\}, \qquad (7.3.6)\\
E(\hat{\mu}^2) &= \mu^2+\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\frac{1}{r^2}\left\{\frac{2\mu\sigma}{r}\left[jb-(r-j-1)b+\frac{r+b(r-1-l)}{n-r+l+1}\right]\right. \\
&\quad +\frac{\sigma^2}{r^2}\left[j(j+1)b^2+(r-j-1)(r-j)b^2+2\left(\frac{r+b(r-1-l)}{n-r+l+1}\right)^2\right. \\
&\quad \left.\left.+2\frac{(r-1-l)j}{n-r+l+1}-2j(r-j-1)-2\frac{(r-1-l)(r+b(r-1-l))}{n-r+l+1}\right]\right\} \\
&\quad +\sum_{l=0}^{n-r} p_2\left\{\frac{2\mu\sigma}{r}\left[(r-1)b-\frac{r}{l+r}\right]\right. \\
&\quad \left.+\frac{\sigma^2}{r}\left[(r-1)b^2+2\frac{r}{(l+r)^2}-2\frac{b(r-1)}{l+r}\right]\right\}, \qquad (7.3.7)\\
E(\hat{\sigma}) &= \sigma\left\{\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left[\frac{2j+1-r}{r}+\frac{r-1-l}{r(n-r+l+1)}\right]+\sum_{l=0}^{n-r} p_2\frac{r-1}{r}\right\}, \qquad (7.3.8)\\
E(\hat{\sigma}^2) &= \sigma^2\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\frac{1}{r^2}\left[j(j+1)+(r-j-1)(r-j)+2\left(\frac{r-1-l}{n-r+l+1}\right)^2\right. \\
&\quad \left.-2j(r-j-1)-2\frac{(r-1-l)(r-j-1)}{n-r+l+1}+2\frac{(r-1-l)j}{n-r+l+1}\right] \\
&\quad +\sigma^2\sum_{l=0}^{n-r} p_2\frac{r-1}{r}. \qquad (7.3.9)
\end{aligned}
$$

Now, the variance, covariance and MSEs of $\hat{\mu}$ and $\hat{\sigma}$ can all be readily obtained from (7.3.3) and (7.3.6) - (7.3.9).

The following lemma will enable us to derive the exact distributions of $\hat{\mu}$ and $\hat{\sigma}$ from their MGFs in (7.3.4) and (7.3.5), respectively; see, for example, Johnson et al.

(1994).

Lemma 7.3.1 *If X follows the exponential distribution with scale parameter θ, then the MGF of X and $-X$ are $(1-t\theta)^{-1}$ and $(1+t\theta)^{-1}$, respectively. Also, if X follows the gamma distribution with shape parameter α and scale parameter β, then the MGF of X and $-X$ are $(1-t\beta)^{-\alpha}$ and $(1+t\beta)^{-\alpha}$, respectively with $t < \frac{1}{\beta}$.*

Let us now use $\Gamma(\alpha, \beta)$ to denote a random variable following the gamma distribution with shape parameter α and scale parameter β, and use $\Gamma^*(\alpha, \beta)$ to denote a random variable following the negative gamma distribution, i.e., if $X \sim \Gamma(\alpha, \beta)$, then $-X \sim \Gamma^*(\alpha, \beta)$. Similarly, we will use $E(\theta)$ and $E^*(\theta)$ to denote the exponential random variable with scale parameter θ and its negative, respectively. Then, by utilizing Lemma 7.3.1, we obtain from (7.3.4) and (7.3.5) the exact distributions of $\hat{\mu}$ and $\hat{\sigma}$ as follows:

$$\begin{aligned}
\hat{\mu} &\stackrel{d}{=} \mu + \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_1 \left[\Gamma\left(j, \frac{b\sigma}{r}\right) + \Gamma^*\left(r-1-j, \frac{b\sigma}{r}\right) + E\left(\frac{r+b(r-l-1)\sigma}{r(n-r+l+1)}\right)\right] \\
&\quad + \sum_{l=0}^{n-r} p_2 \left[\Gamma\left(r-1, \frac{b\sigma}{r}\right) + E^*\left(\frac{\sigma}{l+r}\right)\right], \qquad (7.3.10) \\
\hat{\sigma} &\stackrel{d}{=} \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_1 \left[\Gamma\left(j, \frac{\sigma}{r}\right) + \Gamma^*\left(r-1-j, \frac{\sigma}{r}\right) + E\left(\frac{(r-l-1)\sigma}{r(n-r+l+1)}\right)\right] \\
&\quad + \sum_{l=0}^{n-r} p_2 \Gamma\left(r-1, \frac{\sigma}{r}\right), \qquad (7.3.11)
\end{aligned}$$

where $\Gamma(0, \frac{\sigma}{r})$, $\Gamma^*(0, \frac{\sigma}{r})$, $E(0)$ and $E^*(0)$ are all degenerate at 0. Throughout Chapters 7 and 8, $\sum_{i=1}^{k} P_i X_i$ denotes the generalized mixture of distributions of variables $X_1, \cdots,$

X_k with probabilities $P_1, \cdots, P_k$, such that $\sum_{i=1}^{k} P_i = 1$ but P_i's being not necessarily non-negative.

Furthermore, in the special case when σ is known, we have $\hat{\mu} = x_{r:n} + \sigma \ln \frac{n}{2r}$, and the MGF of $\hat{\mu}$ in this case can be expressed as

$$\begin{aligned} E(\mathrm{e}^{t\hat{\mu}}) &= E(\mathrm{e}^{t\hat{\mu}}|\mu > x_{r:n})P(\mu > x_{r:n}) + E(\mathrm{e}^{t\hat{\mu}}|\mu < x_{r:n})P(\mu < x_{r:n}) \\ &= \sum_{l=0}^{n-r} p_1' \left(1 + \frac{t\sigma}{r+l}\right)^{-1} \mathrm{e}^{(\mu+b\sigma)t} + \sum_{l=0}^{r-1} p_2' \left(1 - \frac{t\sigma}{n-r+l+1}\right)^{-1} \mathrm{e}^{(\mu+b\sigma)t}, \end{aligned} \tag{7.3.12}$$

where

$$\begin{aligned} p_1' &= \frac{(-1)^l n!}{(r-1)!l!(n-r-l)!(r+l)}, \\ p_2' &= \frac{(-1)^l n!}{2^{n-r+l+1} l!(r-1-l)!(n-r)!(n-r+l+1)}. \end{aligned}$$

From (7.3.12), the first two moments of $\hat{\mu}$ can be readily obtained as

$$E(\hat{\mu}) = \mu + b\sigma + \sigma \left[-\sum_{l=0}^{n-r} p_1' \frac{1}{r+l} + \sum_{l=0}^{r-1} p_2' \frac{1}{n-r+l+1} \right], \tag{7.3.13}$$

$$\begin{aligned} E(\hat{\mu}^2) &= \mu^2 + b^2\sigma^2 + 2b\mu\sigma + \left[\sum_{l=0}^{n-r} p_1' \frac{2\sigma}{r+l} \left(\frac{\sigma}{r+l} - \mu - b \right) \right] \\ &\quad + \left[\sum_{l=0}^{r-1} p_2' \frac{2\sigma}{n-r+l+1} \left(\frac{\sigma}{n-r+l+1} + \mu + b \right) \right]. \end{aligned} \tag{7.3.14}$$

Moreover, it is evident that $\hat{\mu}$ in this case has the same distribution as $X_{r:n}$ from

Laplace (μ, σ) distribution with a shift of $\sigma \ln \frac{n}{2r}$.

We shall now present the analogous results for all other cases just in the final form without the corresponding derivations for the sake of brevity.

Case II: $n = 2m + 1$ and $r \geq m + 2$

$$
\begin{aligned}
E\left(\mathrm{e}^{t_1\hat{\mu}+t_2\hat{\sigma}}\right) &= \sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_1 \mathrm{e}^{\mu t_1}\left(1-\frac{\sigma t_2}{r}\right)^{m+1-r-j}\left(1+\frac{\sigma t_2}{r}\right)^{j-m} \\
&\quad \times\left[1-\frac{(m-l_1)\sigma t_2}{(m+1+l_1)r}-\frac{\sigma t_1}{m+1+l_1}\right]^{-1} \\
&\quad +\sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{r-1-j} p_2 \mathrm{e}^{\mu t_1}\left(1-\frac{\sigma t_2}{r}\right)^{j-r-m}\left(1+\frac{\sigma t_2}{r}\right)^{m+1-j} \\
&\quad \times\left[1-\frac{(m-l_1)\sigma t_2}{(m+1+l_1)r}+\frac{\sigma t_1}{m+1+l_1}\right]^{-1} \\
&\quad +\sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_3 \mathrm{e}^{\mu t_1}\left(1-\frac{\sigma t_2}{r}\right)^{-m}\left(1+\frac{\sigma t_2}{r}\right)^{m+2-r} \\
&\quad \times\left[1-\frac{(m-l_2)t_2\sigma}{(m+l_2+1)r}+\frac{t_1\sigma}{m+l_2+1}\right]^{-1}\left(1+\frac{t_1\sigma}{r+l_1}\right)^{-1}, \quad (7.3.15)
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= \frac{(-1)^{l_1+l_2}C_1(m-j)!(r-2-m)!}{2^n l_1!(m-j-l_1)!l_2!(r-2-m-l_2)!(l_1+m+1)(n-r+1+l_2)},\\
p_2 &= \frac{(-1)^{l_1+l_2}C_2(j-m-1)!(r-1-j)!}{2^n l_1!(j-m-1-l_1)!l_2!(r-1-j-l_2)!(l_1+m+1)(n-r+1+l_2)},\\
p_3 &= \frac{(-1)^{l_1+l_2}C_3(n-r)!(r-2-m)!}{2^{r+l_1} l_1!(n-r-l_1)!l_2!(r-2-m-l_2)!(r+l_1)(m+l_2+1)},\\
C_1 &= \frac{n!}{j!(m-j)!(r-2-m)!(n-r)!},\\
C_2 &= \frac{n!}{m!(j-1-m)!(r-1-j)!(n-r)!},\\
C_3 &= \frac{n!}{m!(r-2-m)!(n-r)!};
\end{aligned}
$$

$$
\begin{aligned}
E(\hat{\mu}\hat{\sigma}) &= \frac{\sigma}{r}\Bigg\{\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_1\Bigg[\mu\left(r+2j-n+\frac{m-l_1}{m+l_1+1}\right)\\
&\quad+\frac{\sigma}{m+1+l_1}\left(r+2j-n+2\frac{m-l_1}{m+l_1+1}\right)\Bigg]\\
&\quad+\sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{r-1-j} p_2\Bigg[\mu\left(r+n-2j+\frac{m-l_1}{m+l_1+1}\right)\\
&\quad+\frac{\sigma}{m+1+l_1}\left(r+n-2j+\frac{2(m-l_1)}{m+l_1+1}\right)\Bigg]\\
&\quad+\sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_3\Bigg[\left(n+1-r+\frac{m-l_2}{m+l_2+1}\right)\\
&\quad-\frac{\sigma}{r+l_1}\left(n+1-r+\frac{m-l_2}{m+l_2+1}\right)\\
&\quad-\frac{1}{m+l_2+1}\left(n+1-r+2\frac{m-l_2}{m+l_2+1}\right)\Bigg]\Bigg\};
\end{aligned}
$$

$$
\begin{aligned}
E\left(\mathrm{e}^{t\hat{\sigma}}\right) &= \sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_1\left(1-\frac{\sigma t}{r}\right)^{-(r+j-m-1)}\left(1+\frac{\sigma t}{r}\right)^{-(m-j)} \\
&\quad \times\left[1-\frac{(m-l_1)\sigma t}{(m+1+l_1)r}\right]^{-1} \\
&\quad +\sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{r-1-j} p_2\left(1-\frac{\sigma t}{r}\right)^{-(r+m-j)}\left(1+\frac{\sigma t}{r}\right)^{-(j-m-1)} \\
&\quad \times\left[1-\frac{(m-l_1)\sigma t}{(m+1+l_1)r}\right]^{-1} \\
&\quad +\sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_3\left(1-\frac{\sigma t}{r}\right)^{-m}\left(1+\frac{\sigma t}{r}\right)^{-(r-m-2)}\left[1-\frac{(m-l_2)\sigma t}{(m+l_2+1)r}\right]^{-1}; \\
E(\mathrm{e}^{t\hat{\mu}}) &= \sum_{l=0}^{m} p_\mu \mathrm{e}^{t\mu}\left[\left(1-\frac{t\sigma}{m+l+1}\right)^{-1}+\left(1+\frac{t\sigma}{m+l+1}\right)^{-1}\right],
\end{aligned}
$$

where $p_\mu = (-1)^l \frac{n!}{2^{m+l+1}m!(m-l)!l!(m+l+1)}$;

$$
\begin{aligned}
\hat{\mu} &\stackrel{d}{=} \mu+\sum_{l=0}^{m} p_\mu E\left(\frac{\sigma}{m+l+1}\right)+\sum_{l=0}^{m} p_\mu E^*\left(\frac{\sigma}{m+l+1}\right); \\
\hat{\sigma} &\stackrel{d}{=} \sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_1\left[\Gamma\left(r+j-m-1,\frac{\sigma}{r}\right)+\Gamma^*\left(m-j,\frac{\sigma}{r}\right)\right. \\
&\quad \left.+E\left(\frac{(m-l_1)\sigma}{r(m+l_1+1)}\right)\right] \\
&\quad +\sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-1-m}\sum_{l_2=0}^{r-1-j} p_2\left[\Gamma\left(r+m-j,\frac{\sigma}{r}\right)+\Gamma^*\left(j-m-1,\frac{\sigma}{r}\right)\right. \\
&\quad \left.+E\left(\frac{(m-l_1)\sigma}{r(m+l_1+1)}\right)\right] \\
&\quad +\sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_3\left[\Gamma\left(m,\frac{\sigma}{r}\right)+\Gamma^*\left(r-m-2,\frac{\sigma}{r}\right)+E\left(\frac{(m-l_2)\sigma}{r(m+l_2+1)}\right)\right];
\end{aligned}
$$

$$
\begin{aligned}
E(\hat{\mu}) &= \mu; \\
E(\hat{\mu}^2) &= \mu^2 + 4\sigma^2 \sum_{l=0}^{m} \frac{p_\mu}{(m+l+1)^2}; \\
E(\hat{\sigma}) &= \sigma + \frac{\sigma}{r} \left[\sum_{j=0}^{m} \sum_{l_1=0}^{m-j} \sum_{l_2=0}^{r-2-m} p_1 \left(2j - n + \frac{m-l_1}{n-m+l_1} \right) \right. \\
&\quad + \sum_{j=m+1}^{r-1} \sum_{l_1=0}^{j-1-m} \sum_{l_2=0}^{r-1-j} p_2 \left(n - 2j + \frac{m-l_1}{l_1+m+1} \right) \\
&\quad \left. + \sum_{l_1=0}^{n-r} \sum_{l_2=0}^{r-2-m} p_3 \left(n+1-2r + \frac{m-l_2}{m+l_2+1} \right) \right]; \\
E(\hat{\sigma}^2) &= \frac{\sigma^2}{r^2} \left\{ \sum_{j=0}^{m} \sum_{l_1=0}^{m-j} \sum_{l_2=0}^{r-2-m} p_1 \left[(r+j-m-1)(r+j-m) + (m-j)(m-j+1) \right. \right. \\
&\quad \left. +2\frac{(m-l_1)^2}{(m+1+l_1)^2} + 2\frac{(m-l_1)(r+2j-n)}{m+1+l_1} - 2(r+j-m-1)(m-j) \right] \\
&\quad + \sum_{j=m+1}^{r-1} \sum_{l_1=0}^{j-1-m} \sum_{l_2=0}^{r-1-j} p_2 \left[(r+m-j)(r+m+1-j) + (j-m)(j-m-1) \right. \\
&\quad \left. +2\frac{(m-l_1)^2}{(l_1+m+1)^2} + 2\frac{(m-l1)(r+n-2j)}{l_1+m+1} - 2(r+m-j)(j-m-1) \right] \\
&\quad + \sum_{l_1=0}^{n-r} \sum_{l_2=0}^{r-2-m} p_3 \left[m(m+1) + (r-2-m)(r-1-m) + 2\frac{(m-l_2)^2}{(m+l_2+1)^2} \right. \\
&\quad \left. \left. +2\frac{(m-l_2)(n+1-r)}{m+l_2+1} - 2m(r-2-m) \right] \right\}.
\end{aligned}
$$

Case III: $n = 2m+1$ and $r = m+1$

$$
\begin{aligned}
E\left(\mathrm{e}^{t_1\hat{\mu}+t_2\hat{\sigma}}\right) &= \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_1 \mathrm{e}^{\mu t_1} \left(1 - \frac{\sigma t_2}{r}\right)^{-j} \\
&\quad \left(1 + \frac{\sigma t_2}{r}\right)^{-(r-1-j)} \left(1 - \frac{(r-1-l)t_2\sigma}{r(r+l)} - \frac{t_1\sigma}{r+l}\right)^{-1} \\
&\quad + \sum_{l=0}^{n-r} p_2 \mathrm{e}^{\mu t_1} \left(1 - \frac{\sigma t_2}{r}\right)^{-(r-1)} \left(1 + \frac{t_1\sigma}{r+l}\right)^{-1}, \qquad (7.3.16)
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= \frac{(-1)^l n!}{2^n j! l! (r-1-j-l)!(n-r)!(r+l)}, \\
p_2 &= \frac{(-1)^l n!}{2^{r+l}(r-1)! l! (n-r-l)!(r+l)};
\end{aligned}
$$

$$
\begin{aligned}
E(\hat{\mu}\hat{\sigma}) &= \frac{\sigma}{r}\left\{\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left[(2j+1-r)\left(\mu+\frac{\sigma}{r+l}\right)+\frac{r-l-1}{r+l}\left(\mu+\frac{2\sigma}{r+l}\right)\right]\right. \\
&\quad \left.+\sum_{l=0}^{n-r} p_2(r-1)\left(\mu-\frac{\sigma}{r+l}\right)\right\}; \\
E\left(\mathrm{e}^{t\hat{\sigma}}\right) &= \sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left(1-\frac{\sigma t}{r}\right)^{-j}\left(1+\frac{\sigma t}{r}\right)^{-(r-1-j)}\left(1-\frac{(r-1-l)t\sigma}{r(r+l)}\right)^{-1} \\
&\quad +\sum_{l=0}^{n-r} p_2\left(1-\frac{\sigma t}{r}\right)^{-(r-1)}; \\
\hat{\sigma} &\stackrel{d}{=} \sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left[\Gamma\left(j,\frac{\sigma}{r}\right)+\Gamma^*\left(r-1-j,\frac{\sigma}{r}\right)+E\left(\frac{(r-l-1)\sigma}{r(r+l)}\right)\right] \\
&\quad +\sum_{l=0}^{n-r} p_2\Gamma\left(r-1,\frac{\sigma}{r}\right); \\
E(\hat{\sigma}) &= \sigma+\frac{\sigma}{r}\left\{\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left[(2j-2r+1)+\frac{r-1-l}{r+l}\right]-\sum_{l=0}^{n-r} p_2\right\}; \\
E(\hat{\sigma}^2) &= \frac{\sigma^2}{r^2}\left\{\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left[j(j+1)+(r-1-j)(r-3j)\right.\right. \\
&\quad \left.\left.+\frac{2(r-1-l)^2}{(r+l)^2}+\frac{2(r-l-1)(2j+1-r)}{(r+l)}\right]+\sum_{l=0}^{n-r} p_2 r(r-1)\right\}.
\end{aligned}
$$

In this case, $\hat{\mu}$ has the same form as in Case II leading to the same expressions for the moments and are therefore not presented again.

Case IV $n = 2m$ and $r > m+1$

$$
\begin{aligned}
E\left(\mathrm{e}^{t_1\hat{\mu}+t_2\hat{\sigma}}\right) &= \sum_{j=0}^{m-1}\sum_{l_1=0}^{m-1-j}\sum_{l_2=0}^{r-2-m} p_1\mathrm{e}^{\mu t_1}\left(1-\frac{t_2\sigma}{r}\right)^{-(r+j-m-1)}\left(1+\frac{t_2\sigma}{r}\right)^{-(m-1-j)} \\
&\quad \times\left(1-\frac{t_2\sigma}{r}-\frac{t_1\sigma}{n}\right)^{-1}\left(1-\frac{(m-l_1-1)t_2\sigma}{r(m+l_1+1)}-\frac{t_1\sigma}{m+l_1+1}\right)^{-1} \\
&\quad + \sum_{l=0}^{r-2-m} p_2\mathrm{e}^{\mu t_1}\left(1-\frac{t_2\sigma}{r}\right)^{-(r-2)}\left(1-\frac{t_2\sigma}{r}+\frac{t_1\sigma}{n}\right)^{-1} \\
&\quad \times\left(1-\frac{t_2\sigma}{r}-\frac{t_1\sigma}{n}\right)^{-1} \\
&\quad + \sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{r-j-1} p_3\mathrm{e}^{\mu t_1}\left(1-\frac{t_2\sigma}{r}\right)^{-(r+m-j-1)}\left(1+\frac{t_2\sigma}{r}\right)^{-(j-m-1)} \\
&\quad \times\left(1-\frac{t_2\sigma}{r}+\frac{t_1\sigma}{n}\right)^{-1}\left(1-\frac{(m-l_1-1)t_2\sigma}{r(m+l_1+1)}+\frac{t_1\sigma}{m+l_1+1}\right)^{-1} \\
&\quad + \sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-m-2} p_4\mathrm{e}^{\mu t_1}\left(1-\frac{t_2\sigma}{r}\right)^{-(m-1)}\left(1+\frac{t_2\sigma}{r}\right)^{-(r-2-m)} \\
&\quad \times\left(1-\frac{t_2\sigma}{r}+\frac{t_1\sigma}{n}\right)^{-1}\left(1+\frac{t_1\sigma}{r+l_1}\right)^{-1} \\
&\quad \times\left(1-\frac{(m-l_2-1)t_2\sigma}{r(m+l_2+1)}+\frac{t_1\sigma}{m+l_2+1}\right)^{-1}, \qquad (7.3.17)
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= \frac{(-1)^{l_1+l_2}C_1(m-j-1)!(r-2-m)!}{2^n l_1!(m-j-1-l_1)!l_2!(r-2-m-l_2)!m(m+l_1+1)(n-r+1+l_2)},\\
p_2 &= \frac{(-1)^{l}C_2(r-2-m)!}{2^n l!(r-2-m-l)!(n-r+1+l)m^2},\\
p_3 &= \frac{(-1)^{l_1+l_2}C_3(j-m-1)!(r-1-j)!}{2^n l_1!(j-m-1-l_1)!l_2!(r-1-j-l_2)!(l_1+m+1)(n-r+1+l_2)m},\\
p_4 &= \frac{(-1)^{l_1+l_2}C_4(n-r)!}{2^{r+l_1}l_1!(n-r-l_1)!l_2!(r-2-m-l_2)!(r+l_1)(m+1+l_2)m},\\
C_1 &= \frac{n!}{j!(m-1-j)!(r-2-m)!(n-r)!},\\
C_2 &= \frac{n!}{(m-1)!(r-2-m)!(n-r)!},\\
C_3 &= \frac{n!}{(m-1)!(j-1-m)!(r-1-j)!(n-r)!},\\
C_4 &= \frac{n!}{(m-1)!(r-2-m)!(n-r)!};
\end{aligned}
$$

$$
\begin{aligned}
E(\hat{\mu}\hat{\sigma}) \;=\; & \sum_{j=0}^{m-1}\sum_{l_1=0}^{m-1-j}\sum_{l_2=0}^{r-2-m} p_1\left\{\frac{\mu\sigma}{r}\left(r+2j-n+1+\frac{m-l_1-1}{m+l_1+1}\right)\right. \\
& +\frac{\sigma^2}{rn}\left(r+2j-n+2+\frac{m-l_1-1}{m+l_1+1}\right) \\
& \left.+\frac{\sigma^2}{r(m+l_1+1)}\left(r+2j-n+1+\frac{m-l_1-1}{m+l_1+1}\right)\right\} \\
& +\sum_{l=0}^{r-2-m} p_2\mu\sigma \\
& +\sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{r-1-j} p_3\left\{\frac{\mu\sigma}{r}\left(r-2j+n+1+\frac{m-l_1-1}{m+l_1+1}\right)\right. \\
& -\frac{\sigma^2}{nr}\left(r-2j+n+2+\frac{m-l_1-1}{m+l_1+1}\right) \\
& \left.-\frac{\sigma^2}{2r(m+l_1+1)}\left(r-2j+n+1+2\frac{m-l_1-1}{m+l_1+1}\right)\right\} \\
& \sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_4\left\{\left(\frac{\mu\sigma}{r}-\frac{\sigma^2}{r(r+l_1)}\right)\left(n-r+2+\frac{m-l_2-1}{m+l_2+1}\right)\right. \\
& -\frac{\sigma^2}{nr}\left(n-r+3+\frac{m-l_2-1}{m+l_2+1}\right) \\
& \left.-\frac{\sigma^2}{r(m+l_2+1)}\left(n-r+2+2\frac{m-l_2-1}{m+l_2+1}\right)\right\};
\end{aligned}
$$

$$
\begin{aligned}
E(\mathrm{e}^{\hat{\mu}t}) \;=\; & \sum_{l_1=0}^{m-1} p_{1\mu}\mathrm{e}^{\mu t}\left[\left(1-\frac{\sigma t}{n}\right)^{-1}\left(1-\frac{\sigma t}{l_1+m+1}\right)^{-1}\right. \\
& \left.+\left(1+\frac{\sigma t}{n}\right)^{-1}\left(1+\frac{\sigma t}{m+1+l_1}\right)^{-1}\right]+p_{2\mu}\mathrm{e}^{\mu t}\left(1+\frac{\sigma t}{n}\right)^{-1}\left(1-\frac{\sigma t}{n}\right)^{-1},
\end{aligned}
$$

where $p_{1\mu}=\frac{(-1)^{l_1}n!}{2^{l_1+1+m}m!l_1!(m-1-l_1)!(l_1+m+1)}$ and $p_{2\mu}=\frac{n!2^{-n}}{m!m!}$;

$$
\begin{aligned}
E\left(\mathrm{e}^{t\hat{\sigma}}\right) &= \sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_{1\sigma}\left(1-\frac{\sigma t}{r}\right)^{-(r+j-m-1)}\left(1+\frac{\sigma t}{r}\right)^{-(m-j)} \\
&\quad \times\left[1-\frac{(m-l_1)\sigma t}{r(m+l_1)}\right]^{-1} \\
&\quad +\sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{r-1-j} p_{2\sigma}\left(1-\frac{\sigma t}{r}\right)^{-(r+m-j)}\left(1+\frac{\sigma t}{r}\right)^{-(j-m-1)} \\
&\quad \times\left[1-\frac{(m-1-l_1)\sigma t}{(m+1+l_1)r}\right]^{-1} \\
&\quad +\sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_{3\sigma}\left(1-\frac{\sigma t}{r}\right)^{-m}\left(1+\frac{\sigma t}{r}\right)^{-(r-m-2)} \\
&\quad \times\left[1-\frac{(m-l_2-1)\sigma t}{(m+l_2+1)r}\right]^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
p_{1\sigma} &= \frac{(-1)^{l_1+l_2}C_{1\sigma}(m-j)!(r-2-m)!}{2^n l_1!(m-j-l_1)!l_2!(r-2-m-l_2)!(n-m+l_1)(n-r+1+l_2)}, \\
p_{2\sigma} &= \frac{(-1)^{l_1+l_2}C_{2\sigma}(j-m-1)!(r-1-j)!}{2^n l_1!(j-m-1-l_1)!l_2!(r-1-j-l_2)!(l_1+m+1)(n-r+1+l_2)}, \\
p_{3\sigma} &= \frac{(-1)^{l_1+l_2}C_{3\sigma}(n-r)!(r-2-m)!}{2^{r+l_1} l_1!(n-r-l_1)!l_2!(r-2-m-l_2)!(r+l_1)(m+1+l_2)}, \\
C_{1\sigma} &= \frac{n!}{j!(m-j)!(r-2-m)!(n-r)!}, \\
C_{2\sigma} &= \frac{n!}{m!(j-1-m)!(r-1-j)!(n-r)!}, \\
C_{3\sigma} &= \frac{n!}{m!(r-2-m)!(n-r)!};
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mu} &\stackrel{d}{=} \mu + \sum_{l_1=0}^{m-1} p_{1\mu} \left[E\left(\frac{\sigma}{n}\right) + E\left(\frac{\sigma}{l_1+m+1}\right)\right] \\
&\quad + \sum_{l_1=0}^{m-1} p_{1\mu} \left[E^*\left(\frac{\sigma}{n}\right) + E^*\left(\frac{\sigma}{l_1+m+1}\right)\right] + p_{2\mu}\left[E\left(\frac{\sigma}{n}\right) + E^*\left(\frac{\sigma}{n}\right)\right]; \\
\hat{\sigma} &\stackrel{d}{=} \sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_{1\sigma}\left[\Gamma\left(r+j-m-1,\frac{\sigma}{r}\right) + \Gamma^*\left(m-j,\frac{\sigma}{r}\right)\right. \\
&\quad \left. + E\left(\frac{(m-l_1)\sigma}{r(m+l_1)}\right)\right] \\
&\quad + \sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-1-m}\sum_{l_2=0}^{r-1-j} p_{2\sigma}\left[\Gamma\left(r+m-j,\frac{\sigma}{r}\right) + \Gamma^*\left(j-m-1,\frac{\sigma}{r}\right)\right. \\
&\quad \left. + E\left(\frac{(m-1-l_1)\sigma}{r(m+1+l_1)}\right)\right] \\
&\quad + \sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_{3\sigma}\left[\Gamma\left(m,\frac{\sigma}{r}\right) + \Gamma^*\left(r-m-2,\frac{\sigma}{r}\right) + E\left(\frac{(m-1-l_2)\sigma}{r(m+1+l_2)}\right)\right]; \\
E(\hat{\mu}) &= \mu; \\
E(\hat{\mu}^2) &= \mu^2 + 2\frac{\sigma^2}{n^2} + 4\sigma^2 \sum_{l_1=0}^{m-1} p_{1\mu}\left[\frac{1}{(m+l_1+1)^2} + \frac{1}{n(m+l_1+1)}\right]; \\
E(\hat{\sigma}) &= \sigma + \frac{\sigma}{r}\left[\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_{1\sigma}\left(2j-n-\frac{2l_1}{n-m+l_1}\right)\right. \\
&\quad + \sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-1-m}\sum_{l_2=0}^{r-1-j} p_{2\sigma}\left(n-2j+\frac{n}{l_1+m+1}\right) \\
&\quad \left. + \sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_{3\sigma}\left(n+2-2r+\frac{m-l_2-1}{m+l_2+1}\right)\right];
\end{aligned}
$$

$$
\begin{aligned}
E(\hat{\sigma}^2) \;=\; & \frac{\sigma^2}{r^2}\left\{\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_{1\sigma}\left[(r+j-m-1)(r+j-m)\right.\right.\\
& +(m-j)(m-j+1)+2\frac{(m-l_1)^2}{(m+l_1)^2}+2\frac{(m-l_1)(r+2j-n-1)}{m+l_1}\\
& \left.-2(r+j-m-1)(m-j)\right]\\
& +\sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-1-m}\sum_{l_2=0}^{r-1-j} p_{2\sigma}\left[(r+m-j)(r+m+1-j)\right.\\
& +(j-m)(j-m-1)+2\frac{(m-l_1-1)^2}{(l_1+m+1)^2}+2\frac{(m-l_1-1)(r+n-2j)}{l_1+m+1}\\
& \left.-2(r+m-j)(j-m-1)\right]\\
& +\sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_{3\sigma}\left[m(m+1)+(r-2-m)(r-1-m)+2\frac{(m-l_2-1)^2}{(m+l_2+1)^2}\right.\\
& \left.\left.+2\frac{(m-l_2-1)(n+1-r)}{m+l_2+1}-2m(r-2-m)\right]\right\}.
\end{aligned}
$$

Case V: $n = 2m$ and $r = m+1$

$$
\begin{aligned}
E\left(\mathrm{e}^{t_1\hat{\mu}+t_2\hat{\sigma}}\right) \;=\; & \sum_{j=0}^{m-1}\sum_{l=0}^{m-1-j} p_1\mathrm{e}^{\mu t_1}\left(1-\frac{\sigma t_2}{r}\right)^{-j}\left(1+\frac{\sigma t_2}{r}\right)^{-(m-j-1)}\\
& \times\left(1-\frac{t_2\sigma}{r}-\frac{t_1\sigma}{n}\right)^{-1}\left(1-\frac{(n-r-l)t_2\sigma}{r(n-r+l+2)}-\frac{t_1\sigma}{n-r+l+2}\right)^{-1}\\
& +p_2\mathrm{e}^{\mu t_1}\left(1-\frac{t_2\sigma}{r}\right)^{-(m-1)}\left(1-\frac{t_2\sigma}{r}+\frac{t_1\sigma}{n}\right)^{-1}\left(1-\frac{t_2\sigma}{r}-\frac{t_1\sigma}{n}\right)^{-1}\\
& +\sum_{l=0}^{n-r} p_3\mathrm{e}^{\mu t_1}\left(1-\frac{t_2\sigma}{r}\right)^{-(m-1)}\left(1-\frac{t_2\sigma}{r}+\frac{t_1\sigma}{n}\right)^{-1}\\
& \times\left(1+\frac{t_1\sigma}{m+l+1}\right)^{-1}, \qquad (7.3.18)
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= \frac{(-1)^l C_1 (m-j-1)!}{2^n l!(m-j-1-l)!(n-r+1)(n-r+l+2)}, \\
p_2 &= \frac{C_2}{2^n m(n-r+1)}, \\
p_3 &= \frac{(-1)^l C_3 (n-r)!}{2^{r+l} l!(n-r-l)! m(l+m+1)}, \\
C_1 &= \frac{n!}{j!(m-j-1)!(n-r)!}, \\
C_2 &= \frac{n!}{(m-1)!(n-r)!}, \\
C_3 &= \frac{n!}{(m-1)!(n-r)!};
\end{aligned}
$$

$$
\begin{aligned}
E(\hat{\mu}\hat{\sigma}) &= \sum_{j=0}^{m-1} \sum_{l=0}^{m-1-j} p_1 \left[\frac{\mu\sigma}{r} \left(2j - m + 2 + \frac{n-r-l}{n-r+l+2} \right) \right. \\
&\quad + \frac{\sigma^2}{rn} \left(2j - m + 3 + \frac{n-r-l}{n-r+l+2} \right) \\
&\quad \left. + \frac{\sigma^2}{r(n-r+l+2)} \left(2j - m + 2 + 2\frac{n-r-l}{n-r+l+2} \right) \right] \\
&\quad + p_2 \frac{\mu\sigma(m+1)}{r} + \sum_{l=0}^{n-r} p_3 \left[\frac{m\mu\sigma}{r} - \frac{(m+1)\sigma^2}{nr} - \frac{m\sigma^2}{r(m+l+1)} \right]; \\
E(\mathrm{e}^{t\hat{\sigma}}) &= \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_{1\sigma} \left(1 - \frac{t\sigma}{r} \right)^{-j} \left(1 + \frac{t\sigma}{r} \right)^{-(r-1-j)} \left(1 - \frac{t(m-l)\sigma}{r(m+l)} \right)^{-1} \\
&\quad + \sum_{l=0}^{n-r} p_{2\sigma} \left(1 - \frac{t\sigma}{r} \right)^{-(r-1)},
\end{aligned}
$$

where

$$
\begin{aligned}
p_{1\sigma} &= \frac{(-1)^l C_{1\sigma}(r-1-j)!}{2^n l!(r-1-j-l)!(m+l)}, \\
p_{2\sigma} &= \frac{(-1)^l C_{2\sigma}(n-r)!}{2^{r+l} l!(n-r-l)!(l+m+1)}, \\
C_{1\sigma} &= \frac{n!}{j!(r-1-j)!(n-r)!}, \\
C_{2\sigma} &= \frac{n!}{(r-1)!(n-r)!};
\end{aligned}
$$

$$
\begin{aligned}
\hat{\sigma} &\stackrel{d}{=} \sum_{j=0}^{m}\sum_{l=0}^{m-j} p_{1\sigma}\left[\Gamma\left(j,\frac{\sigma}{r}\right)+\Gamma^*\left(r-1-j,\frac{\sigma}{r}\right)+E\left(\frac{\sigma(m-l)}{r(m+l)}\right)\right] \\
&\quad +\sum_{l=0}^{n-r} p_{2\sigma}\Gamma\left(r-1,\frac{\sigma}{r}\right); \\
E(\hat{\sigma}) &= \sigma+\frac{\sigma}{r}\left[\sum_{j=0}^{m}\sum_{l=0}^{m-j} p_{1\sigma}\left(2j-2r+1+\frac{m-l}{m+l}\right)-\sum_{l=0}^{n-r} p_{2\sigma}\right]; \\
E(\hat{\sigma}^2) &= \frac{\sigma^2}{r^2}\left\{\sum_{j=0}^{m}\sum_{l=0}^{m-j} p_{1\sigma}\left[j\left(3j+3-2r+\frac{2(m-l)}{m+l}\right)\right.\right. \\
&\quad \left.\left.+(r-1-j)\left(r-j-\frac{2(m-l)}{m+l}\right)+\frac{2(m-l)^2}{(m+l)^2}\right]+\sum_{l=0}^{n-r} p_{2\sigma}2r(r-1)\right\}.
\end{aligned}
$$

In this case, $\hat{\mu}$ has the same form as in Case IV leading to the same expressions for the moments, and are therefore not presented again.

Case VI: $n = 2m$ and $r = m$

$$
\begin{aligned}
E(\mathrm{e}^{\hat{\mu}t_1+\hat{\sigma}t_2}) &= \sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1 \mathrm{e}^{\mu t_1}\left(1-\frac{t_2\sigma}{r}\right)^{-j}\left(1+\frac{t_2\sigma}{r}\right)^{-(r-1-j)} \\
&\quad \times\left[1-\frac{(r-l-1)t_2\sigma}{r(r+l+1)}-\frac{t_1\sigma}{r+l+1}\right]^{-1} \\
&\quad +\sum_{l=0}^{n-r} p_2 \mathrm{e}^{\mu t_1}\left(1-\frac{t_2\sigma}{r}\right)^{-(r-1)}\left[1+\frac{t_1\sigma}{r+l}\right]^{-1}, \qquad (7.3.19)
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= \frac{(-1)^l n!}{2^n j! l! (r-1-j-l)!(n-r)!(r+l+1)}, \\
p_2 &= \frac{(-1)^l n!}{2^{r+l}(r-1)! l! (n-r-l)!(r+l)};
\end{aligned}
$$

$$
\begin{aligned}
E(\hat{\mu}\hat{\sigma}) &= \frac{\mu\sigma}{r}\left\{\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left[2j+1-r+\frac{r-l-1}{r+l+1}\right]+(r-1)\sum_{l=0}^{n-r} p_2\right\} \\
&\quad +\frac{\sigma^2}{r}\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j}\frac{p_1}{r+l+1}\left[2j+1-r+\frac{2(r-l-1)}{r+l+1}\right] \\
&\quad -\frac{\sigma^2(r-1)}{r}\sum_{l=0}^{n-r}\frac{p_2}{r+l}; \\
E(\mathrm{e}^{t\hat{\mu}}) &= \sum_{l=0}^{r-1} p_{1\mu}\mathrm{e}^{\mu t}\left[1-\frac{\sigma t}{l+r+1}\right]^{-1}+\sum_{l=0}^{n-r} p_{2\mu}\mathrm{e}^{\mu t}\left[1+\frac{\sigma t}{r+l}\right]^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
p_{1\mu} &= \frac{(-1)^l n!}{2^{r+l+1} l!(r-1-l)!(n-r)!(r+l+1)}, \\
p_{2\mu} &= \frac{(-1)^l n!}{2^{r+l}(r-1)!l!(n-r-l)!(r+l)};
\end{aligned}
$$

$$
\begin{aligned}
E(\mathrm{e}^{\hat{\sigma}t}) &= \sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1 \left(1-\frac{t_2\sigma}{r}\right)^{-j}\left(1+\frac{t_2\sigma}{r}\right)^{-(r-1-j)}\left(1-\frac{(r-l-1)t_2\sigma}{r(r+l+1)}\right)^{-1} \\
&\quad + \sum_{l=0}^{n-r} p_2 \left(1-\frac{t_2\sigma}{r}\right)^{-(r-1)}; \\
\hat{\mu} &\stackrel{d}{=} \mu + \sum_{l=0}^{r-1} p_{1\mu} E\left(\frac{\sigma}{r+l+1}\right) + \sum_{l=0}^{n-r} p_{2\mu} E^*\left(\frac{\sigma}{r+l}\right), \\
\hat{\sigma} &\stackrel{d}{=} \sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1 \left[\Gamma\left(j,\frac{\sigma}{r}\right) + \Gamma^*\left(r-1-j,\frac{\sigma}{r}\right) + E\left(\frac{(r-l-1)\sigma}{r(r+l+1)}\right)\right] \\
&\quad + \sum_{l=0}^{n-r} p_2 \Gamma\left(r-1,\frac{\sigma}{r}\right); \\
E(\hat{\mu}) &= \mu + \sigma\left[\sum_{l=0}^{r-1}\frac{p_{1\mu}}{r+l+1} - \sum_{l=0}^{n-r}\frac{p_{2\mu}}{r+l}\right]; \\
E(\hat{\mu}^2) &= \mu^2 + \sum_{l=0}^{r-1} p_{1\mu}\left[\frac{2\mu\sigma}{r+l+1} + \frac{2\sigma^2}{(r+l+1)^2}\right] + \sum_{l=0}^{n-r} p_{2\mu}\left[\frac{2\sigma^2}{(r+l)^2} - \frac{2\mu\sigma}{r+l}\right]; \\
E(\hat{\sigma}) &= \sigma + \frac{\sigma}{r}\left[\sum_{j=0}^{r-1}\sum_{l=0}^{r-1} p_1\left(2j-2r+1+\frac{r-l-1}{r+l+1}\right) - \sum_{l=0}^{n-r} p_2\right]; \\
E(\hat{\sigma}^2) &= \frac{\sigma^2}{r^2}\left\{\sum_{j=0}^{r-1}\sum_{l=0}^{r-1-j} p_1\left[(r-1)\left(r-\frac{2(r-1-l)}{r+l+1}\right) + \frac{2(r-1-l)^2}{(r+l+1)^2}\right]\right. \\
&\quad \left. + \sum_{l=0}^{n-r} p_2 r(r-1)\right\}.
\end{aligned}
$$

7.3.2 Exact density functions of MLEs and interval estimation

In this section, we will use the results of the preceding section to derive the exact density functions of $\hat{\mu}$ and $\hat{\sigma}$ and then utilize them to develop interval estimation for the parameters μ and σ. For this purpose, we first need the following lemmas.

Lemma 7.3.2 *If $X \sim \Gamma(\alpha, \beta)$ and α is a positive integer, then the CDF of X can be expressed as*

$$F(x, \alpha, \beta) = 1 - \sum_{i=0}^{\alpha-1} \frac{1}{i!} \left(\frac{x}{\beta}\right)^i \exp\left(-\frac{x}{\beta}\right),$$

denoted by $\Gamma(x, \alpha, \beta)$, and let us denote $S_\Gamma(x, \alpha, \beta) = 1 - \Gamma(x, \alpha, \beta)$.

Lemma 7.3.3 *If j is a positive integer and $h > 0$, then we have*

$$\int_0^u x^j \exp(hx) = \sum_{i=0}^{j} \frac{(-1)^{j-i} j!}{h^{j-i+1} i!} \exp(hu) u^i - \frac{(-1)^j j!}{h^{j+1}}.$$

Lemma 7.3.4 *If $X \sim \Gamma(\alpha_1, \beta_1)$, $Y \sim \Gamma(\alpha_2, \beta_2)$, $Y^* = -Y$, and $Z \sim E(\theta)$, with X, Y and Z being independent, where α_1 and α_2 are both positive integers, then the CDF*

of $W = X + Y^* + Z$ *can be derived as follows:*

$$
\begin{aligned}
P(W \leq a) &= P(X - Y + Z \leq a) \\
&= \int_{\max(0,-a)}^{\infty} \int_0^{a+y} F_X(a+y-z) f_Z(z) f_Y(y) dz dy \\
&= \int_{\max(0,-a)}^{\infty} \int_0^{a+y} f_Z(z) f_Y(y) dz dy \\
&\quad - \int_{\max(0,-a)}^{\infty} \int_0^{a+y} \sum_{i=0}^{\alpha_1 - 1} \frac{(a+y-z)^i}{\beta_1^i i!} \exp\left(\frac{z-a-y}{\beta_1}\right) f_Z(z) f_Y(y) dz dy \\
&= S_\Gamma[\max(0,-a), \alpha_2, \beta_2] \\
&\quad - \frac{\exp\left(-\frac{a}{\theta}\right)}{\beta_2^{\alpha_2}\left(\frac{1}{\theta} + \frac{1}{\beta_2}\right)^{\alpha_2}} \left[S_\Gamma\left(\max(0,-a), \alpha_2, \left(\frac{1}{\theta} + \frac{1}{\beta_2}\right)^{-1}\right)\right] \\
&\quad - \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \sum_{k=0}^{i-j} C \int_{\max(0,-a)}^{\infty} \int_0^{a+y} z^j \exp\left[-z\left(\frac{1}{\theta} - \frac{1}{\beta_1}\right)\right] y^{\alpha_2 + k - 1} \\
&\quad \times \exp\left[-y\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)\right] dz dy, \qquad (7.3.20)
\end{aligned}
$$

where $C = \frac{(-1)^j a^{i-j-k} \exp(-\frac{a}{\beta_1})}{j!k!(i-j-k)!\beta_1^i \theta \Gamma(\alpha_2) \beta_2^{\alpha_2}}$.

We now need to distinguish three different cases depending on whether the term $\left(\frac{1}{\theta} - \frac{1}{\beta_1}\right)$ *in the exponent is* < 0, > 0 *or* $= 0$.

Case 1: $\theta > \beta_1$

In this case, we see that the last term in (7.3.20) is

$$
\begin{aligned}
&= \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{k=0}^{i-j} C\int_{\max(0,-a)}^{\infty}\left\{\sum_{l=0}^{j}\frac{(-1)^{j-l}j!}{(\frac{1}{\beta_1}-\frac{1}{\theta})^{j-l+1}l!}\exp\left[(a+y)\left(\frac{1}{\beta_1}-\frac{1}{\theta}\right)\right](a+y)^l\right.\\
&\quad\left.-\frac{(-1)^j j!}{(\frac{1}{\beta_1}-\frac{1}{\theta})^{j+1}}\right\}y^{\alpha_2+k-1}\exp\left[-y\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)\right]dy\\
&= \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{k=0}^{i-j}\sum_{l=0}^{j}\sum_{m=0}^{l}\frac{C(-1)^{j-l}j!a^{l-m}\Gamma(\alpha_2+m+k)}{\left(\frac{1}{\beta_1}-\frac{1}{\theta}\right)^{j-l+1}\left(\frac{1}{\theta}+\frac{1}{\beta_2}\right)^{\alpha_2+m+k}(l-m)!m!}\\
&\quad\times\exp\left[a\left(\frac{1}{\beta_1}-\frac{1}{\theta}\right)\right]\left[S_\Gamma\left(\max(0,-a),\alpha_2+m+k,\left(\frac{1}{\theta}+\frac{1}{\beta_2}\right)^{-1}\right)\right]\\
&\quad+\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{k=0}^{i-j}\frac{C(-1)^{j+1}j!\Gamma(\alpha_2+k)}{\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)^{\alpha_2+k}\left(\frac{1}{\beta_1}-\frac{1}{\theta}\right)^{j+1}}\\
&\quad\times\left[S_\Gamma\left(\max(0,-a),\alpha_2+k,\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)^{-1}\right)\right];
\end{aligned}
\tag{7.3.21}
$$

Case 2: $\theta < \beta_1$

In this case, we see that the last term in (7.3.20) is

$$
\begin{aligned}
&= \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{k=0}^{i-j}\frac{C\Gamma(j+1)}{\left(\frac{1}{\theta}-\frac{1}{\beta_1}\right)^{j+1}}\int_{\max(0,-a)}^{\infty}\left\{1-\sum_{l=0}^{j}\frac{(a+y)^l\left(\frac{1}{\theta}-\frac{1}{\beta_1}\right)^l}{l!}\right.\\
&\quad\left.\times\exp\left[-(a+y)\left(\frac{1}{\theta}-\frac{1}{\beta_1}\right)\right]\right\}y^{\alpha_2+k-1}\exp\left[-y\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)\right]dy\\
&= \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{k=0}^{i-j}\frac{C\Gamma(j+1)\Gamma(\alpha_2+k)}{\left(\frac{1}{\theta}-\frac{1}{\beta_1}\right)^{j+1}\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)^{\alpha_2+k}}\\
&\quad\left[S_\Gamma\left(\max(0,-a),\alpha_2+k,\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)^{-1}\right)\right]\\
&\quad-\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{k=0}^{i-j}\sum_{l=0}^{j}\sum_{m=0}^{l}\frac{C\Gamma(j+1)\Gamma(\alpha_2+k+m)a^{l-m}}{m!(l-m)!\left(\frac{1}{\theta}-\frac{1}{\beta_1}\right)^{j+1-l}\left(\frac{1}{\theta}+\frac{1}{\beta_2}\right)^{\alpha_2+k+m}}\\
&\quad\times\exp\left[-a\left(\frac{1}{\theta}-\frac{1}{\beta_1}\right)\right]\left[S_\Gamma\left(\max(0,-a),\alpha_2+k+m,\left(\frac{1}{\theta}+\frac{1}{\beta_2}\right)^{-1}\right)\right];
\end{aligned}
\tag{7.3.22}
$$

Case 3: $\theta = \beta_1$

Finally, we see in this case that $X + Z \sim \Gamma(\alpha_1 + 1, \theta)$ *and*

$$
\begin{aligned}
P(W \le a) &= P(X - Y + Z \le a)\\
&= \int_{\max(0,-a)}^{\infty} F_{X+Z}(a+y)f_Y(y)dy\\
&= S_\Gamma\left(\max(0,-a),\alpha_2,\beta_2\right) - \sum_{i=0}^{\alpha_1}\sum_{j=0}^{i}\frac{\Gamma(\alpha_2+j)a^{i-j}\exp\left(-\frac{a}{\theta}\right)}{j!(i-j)!\theta^i\Gamma(\alpha_2)\beta_2^{\alpha_2}\left(\frac{1}{\beta_2}+\frac{1}{\theta}\right)^{\alpha_2+j}}\\
&\quad\times\left[S_\Gamma\left(\max(0,-a),\alpha_2+j,\left(\frac{1}{\beta_2}+\frac{1}{\theta}\right)^{-1}\right)\right].
\end{aligned}
\tag{7.3.23}
$$

For the special case when $\alpha_1 = 0$ or $\alpha_2 = 0$, we have the following lemmas, respectively.

Lemma 7.3.5 *If $Y \sim \Gamma(\alpha_2, \beta_2)$, $Y^* = -Y$, and $Z \sim E(\theta)$, with Y and Z being independent, where α_2 is a positive integer, then the CDF of $W = Y^* + Z$ can be derived similarly as*

$$\begin{aligned} P(W \leq a) &= S_\Gamma(\max(0, -a), \alpha_2, \beta_2) \\ &\quad - \frac{\exp\left(-\frac{a}{\theta}\right)}{\beta_2^{\alpha_2}\left(\frac{1}{\theta} + \frac{1}{\beta_2}\right)^{\alpha_2}} S_\Gamma\left[\max(0, -a), \alpha_2, \left(\frac{1}{\theta} + \frac{1}{\beta_2}\right)^{-1}\right]. \end{aligned} \tag{7.3.24}$$

Lemma 7.3.6 *If $X \sim \Gamma(\alpha_1, \beta_1)$ and $Z \sim E(\theta)$, with X and Z being independent, where α_1 is a positive integer, then the CDF of $W = X + Z$ can be derived similarly as*

$$P(W \leq a) = \begin{cases} \Gamma(a, \alpha_1, \beta_1) - \frac{\exp\left(-\frac{a}{\theta}\right)}{\beta_1^{\alpha_1}\left(\frac{1}{\beta_1} - \frac{1}{\theta}\right)^{\alpha_1}} \Gamma\left[a, \alpha_1, \left(\frac{1}{\beta_1} - \frac{1}{\theta}\right)\right] & \text{if } \theta > \beta_1, \\ \Gamma(a, \alpha_1, \beta_1) + \sum_{i=0}^{\alpha_1 - 1} \frac{a^i \exp\left(-\frac{a}{\beta_1}\right)}{i!\beta_1^{\alpha_1}\left(\frac{1}{\beta_1} - \frac{1}{\theta}\right)^{\alpha_1 - i}} - \frac{\exp\left(-\frac{a}{\theta}\right)}{\beta_1^{\alpha_1}\left(\frac{1}{\beta_1} - \frac{1}{\theta}\right)^{\alpha_1}} & \text{if } \theta < \beta_1, \\ \Gamma(a, \alpha_1, \beta_1) - \frac{\exp\left(-\frac{a}{\theta}\right) a^{\alpha_1}}{\alpha_1! \beta_1^{\alpha_1}} & \text{if } \theta = \beta_1, \end{cases} \tag{7.3.25}$$

where $a > 0$ and $P(W < 0) = 0$.

Upon using the above lemmas and the expressions in (7.3.21) - (7.3.25), we obtain explicit expressions for the exact CDF of $\hat{\sigma}$ for the six cases as follows:

Case I: $r < \frac{n}{2}$

$$P(\hat{\sigma} \leq a) = \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_1 P(W \leq a) + \sum_{l=0}^{n-r} p_2 \Gamma(a, r-1, \frac{\sigma}{r}), \tag{7.3.26}$$

where $W = \Gamma\left(j, \frac{\sigma}{r}\right) + \Gamma^*\left(r-1-j, \frac{\sigma}{r}\right) + E\left(\frac{(r-l-1)\sigma}{r(n-r+l+1)}\right)$, and p_1, p_2 are as defined in (7.3.2);

Case II: $n = 2m+1$ and $r \geq m+2$

$$\begin{aligned} P(\hat{\sigma} \leq a) &= \sum_{j=0}^{m} \sum_{l_1=0}^{m-j} \sum_{l_2=0}^{r-2-m} p_1 P(W_1 \leq a) + \sum_{j=m+1}^{r-1} \sum_{l_1=0}^{j-1-m} \sum_{l_2=0}^{r-1-j} p_2 P(W_2 \leq a) \\ &\quad + \sum_{l_1=0}^{n-r} \sum_{l_2=0}^{r-2-m} p_3 P(W_3 \leq a), \end{aligned} \tag{7.3.27}$$

where

$$\begin{aligned} W_1 &= \Gamma\left(r+j-m-1, \frac{\sigma}{r}\right) + \Gamma^*\left(m-j, \frac{\sigma}{r}\right) + E\left(\frac{(m-l_1)\sigma}{r(m+l_1+1)}\right), \\ W_2 &= \Gamma\left(r+m-j, \frac{\sigma}{r}\right) + \Gamma^*\left(j-m-1, \frac{\sigma}{r}\right) + E\left(\frac{(m-l_1)\sigma}{r(m+l_1+1)}\right), \\ W_3 &= \Gamma\left(m, \frac{\sigma}{r}\right) + \Gamma^*\left(r-m-2, \frac{\sigma}{r}\right) + E\left(\frac{(m-l_2)\sigma}{r(m+l_2+1)}\right), \end{aligned}$$

and p_1, p_2, p_3 are as defined in (7.3.15);

Case III: $n = 2m+1$ and $r = m+1$

$$P(\hat{\sigma} \leq a) = \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_1 P(W \leq a) + \sum_{l=0}^{n-r} p_2 \Gamma\left(a, r-1, \frac{\sigma}{r}\right), \tag{7.3.28}$$

where $W = \Gamma\left(j, \frac{\sigma}{r}\right) + \Gamma^*\left(r-1-j, \frac{\sigma}{r}\right) + E\left(\frac{(r-l-1)\sigma}{r(r+l)}\right)$, and p_1, p_2 are as defined in (7.3.16);

Case IV: $n = 2m$ and $r > m+1$

$$\begin{aligned} P(\hat{\sigma} \leq a) &= \sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{r-2-m} p_{1\sigma}P(W_1 \leq a) + \sum_{j=m+1}^{r-1}\sum_{l_1=0}^{j-1-m}\sum_{l_2=0}^{r-1-j} p_{2\sigma}P(W_2 \leq a) \\ &\quad + \sum_{l_1=0}^{n-r}\sum_{l_2=0}^{r-2-m} p_{3\sigma}P(W_3 \leq a), \end{aligned} \tag{7.3.29}$$

where

$$\begin{aligned} W_1 &= \Gamma\left(r+j-m-1, \frac{\sigma}{r}\right) + \Gamma^*\left(m-j, \frac{\sigma}{r}\right) + E\left(\frac{(m-l_1)\sigma}{r(m+l_1)}\right), \\ W_2 &= \Gamma\left(r+m-j, \frac{\sigma}{r}\right) + \Gamma^*\left(j-m-1, \frac{\sigma}{r}\right) + E\left(\frac{(m-1-l_1)\sigma}{r(m+1+l_1)}\right), \\ W_3 &= \Gamma\left(m, \frac{\sigma}{r}\right) + \Gamma^*\left(r-m-2, \frac{\sigma}{r}\right) + E\left(\frac{(m-1-l_2)\sigma}{r(m+1+l_2)}\right), \end{aligned}$$

and $p_{1\sigma}$, $p_{2\sigma}$, $p_{3\sigma}$ are as defined in (7.3.17);

Case V: $n = 2m$ and $r = m+1$

$$P(\hat{\sigma} \leq a) = \sum_{j=0}^{m}\sum_{l=0}^{m-j} p_{1\sigma}P(W \leq a) + \sum_{l=0}^{n-r} p_{2\sigma}\Gamma\left(a, r-1, \frac{\sigma}{r}\right), \tag{7.3.30}$$

$W = \Gamma\left(j, \frac{\sigma}{r}\right) + \Gamma^*\left(r-1-j, \frac{\sigma}{r}\right) + E\left(\frac{\sigma(m-l)}{r(m+l)}\right)$, and $p_{1\sigma}$, $p_{2\sigma}$ are as defined in (7.3.18);

Case VI: $n = 2m$ and $r = m$

$$P(\hat{\sigma} \leq a) = \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_1 P(W \leq a) + \sum_{l=0}^{n-r} p_2 \Gamma\left(a, r-1, \frac{\sigma}{r}\right), \tag{7.3.31}$$

where $W = \Gamma\left(j, \frac{\sigma}{r}\right) + \Gamma^*\left(r-1-j, \frac{\sigma}{r}\right) + E\left(\frac{(r-l-1)\sigma}{r(r+l+1)}\right)$, and p_1, p_2 are as defined in (7.3.19). Note that the CDF of W is discussed based on $a \in (-\infty, \infty)$, while the support of $\sigma > 0$. For this reason, when constructing the CDF of $\hat{\sigma}$, we only consider the case when $a > 0$.

By making use of the above Lemmas, we can further derive the following explicit expressions for the exact CDF of $\hat{\mu}$:

Case I: $r < \frac{n}{2}$

$$P(\hat{\mu} \leq a + \mu) = \sum_{j=0}^{r-1} \sum_{l=0}^{r-1-j} p_1 P(W_1 \leq a) + \sum_{l=0}^{n-r} p_2 P(W_2^* \leq a), \tag{7.3.32}$$

where $W = \Gamma(j, \frac{b\sigma}{r}) + \Gamma^*(r-j-1, \frac{b\sigma}{r}) + E(\frac{r+b(r-l-1)\sigma}{r(n-r+l+1)})$, $W_2^* = \Gamma(r-1, \frac{\sigma}{r}) + E^*(\frac{\sigma}{l+r}) = -(\Gamma^*(r-1, \frac{\sigma}{r}) + E(\frac{\sigma}{l+r}))$, and p_1, p_2 are as defined in (7.3.2), while the CDF of W_2^* can be obtained from Lemma 7.3.5;

Case II: $n = 2m+1$ and $r \geq m+2$

$$P(\hat{\mu} \leq a + \mu) = \begin{cases} p_\mu \exp(\frac{a(m+l+1)}{\sigma}) & \text{if} \quad a < 0, \\ p_\mu \left[1 - \exp\left(-\frac{a(m+l+1)}{\sigma}\right)\right] + p_\mu & \text{if} \quad a \geq 0, \end{cases} \tag{7.3.33}$$

where p_μ is as defined earlier;

Case III: $n = 2m + 1$ and $r = m + 1$

In this case, we have the same form as in Case II;

Case IV: $n = 2m$ and $r > m + 1$

$$P(\hat{\mu} \leq \mu + a) = \sum_{l_1=0}^{m-1} p_{1\mu} P(W_1 \leq a) + \sum_{l_1=0}^{m-1} p_{1\mu} P(W_2^* \leq a) + p_{2\mu} P(W_3 \leq a), \quad (7.3.34)$$

where $W_1 = E\left(\frac{\sigma}{n}\right) + E\left(\frac{\sigma}{l_1+m+1}\right)$, $W_2^* = E^*\left(\frac{\sigma}{n}\right) + E^*\left(\frac{\sigma}{l_1+m+1}\right)$, $W_3 = E(\frac{\sigma}{n}) + E^*(\frac{\sigma}{n})$, and $p_{1\mu}$, $p_{2\mu}$ are as defined in (7.3.17). Note that the CDF of W_2^* can be obtained from Lemma 7.3.6 by setting $\alpha_1 = 1$;

Case V: $n = 2m$ and $r = m + 1$

In this case, we have the same form as in Case IV;

Case VI: $n = 2m$ and $r = m$

$$P(\hat{\mu} \leq a + \mu) = \begin{cases} 1 - \sum_{l=0}^{r-1} p_{1\mu} \exp\left(-\frac{a(r+l+1)}{\sigma}\right) & \text{if} \quad a \geq 0, \\ \sum_{l=0}^{n-r} p_{2\mu} \exp\left(\frac{a(r+l)}{\sigma}\right) & \text{if} \quad a < 0, \end{cases} \quad (7.3.35)$$

where $p_{1\mu}$ and $p_{2\mu}$ are as defined earlier.

Now by making use of the explicit expressions of the CDFs of $\hat{\mu}$ and $\hat{\sigma}$ in (7.3.32)-(7.3.35) and (7.3.26)-(7.3.31), respectively, exact $100(1-\alpha)\%$ CIs for μ and σ, denoted by (μ_L, μ_U) and (σ_L, σ_U), can then be determined as follows:

$$F_{\hat{\mu}}(\mu_U) = 1 - \frac{\alpha}{2} \quad \textit{and} \quad F_{\hat{\mu}}(\mu_L) = \frac{\alpha}{2}, \quad (7.3.36)$$

$$F_{\hat{\sigma}}(\sigma_U) = 1 - \frac{\alpha}{2} \quad \textit{and} \quad F_{\hat{\sigma}}(\sigma_L) = \frac{\alpha}{2}. \quad (7.3.37)$$

In the construction of CI for μ based on (7.3.36), we only use the density of $\hat{\mu}$ in (7.3.32)-(7.3.35), which depend on σ, but then σ is replaced by $\hat{\sigma}$ to form the CI.

7.4 Exact inference based on MLEs using spacings

In this section, we first present some key properties of order statistics from the standard exponential distribution and then their extensions to the case of Laplace distribution.

Theorem 7.4.1 *Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the order statistics from a sample of size n from the standard exponential population. Then, the random variables $Z_1, Z_2, \cdots, Z_n$, where*

$$Z_i = (n - i + 1)(X_{i:n} - X_{i-1:n}), \qquad i = 1, 2, \cdots, n,$$

with $X_{0:n} = 0$, are all statistically independent and identically distributed as standard exponential.

Proof See Sukhatme (1937).

Theorem 7.4.2 *Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics from a sample of size n from $L(\mu, \sigma)$, and suppose $\mu \in (X_{j:n}, X_{j+1:n})$. Then, the random variables*

$Z_1^{(j)}, Z_2^{(j)}, \cdots, Z_n^{(j)}$, *where*

$$Z_i^{(j)} = \begin{cases} i(X_{i+1:n} - X_{i:n}), & 1 \leq i \leq j-1, \\ i(\mu - X_{i:n}), & i = j, \\ (n-i+1)(X_{i:n} - \mu), & i = j+1, \\ (n-i+1)(X_{i:n} - X_{i-1:n}), & j+2 \leq i \leq n, \end{cases}$$

are all statistically independent and exponentially distributed with scale parameter σ*, i.e.,* $Z_i^{(j)} \overset{i.i.d}{\sim} E(\sigma)$.

Proof See Iliopoulos and Balakrishnan (2009).

Lemma 7.4.1 *If* $X_1, \cdots, X_n$ *is a random sample from* $L(\mu, \theta)$*, and* D *denotes the number of* X_i*'s less than* μ*, i.e.,* $D = \{\#X's < \mu\}$*, then,*

$$P(D = d) = \binom{n}{d} \left(\frac{1}{2}\right)^n, \qquad 0 \leq d \leq n. \tag{7.4.1}$$

Proof It is evident from the fact that D follows a binomial $B\left(n, \frac{1}{2}\right)$ distribution.

Lemma 7.4.2 *If* $X_i \sim E(\lambda_i)$ *with* $\lambda_i \neq \lambda_j > 0$*, for* $i, j = 1, \cdots, n$*, and they are all independent, then* $Z = \sum_{i=1}^{n} X_i$ *follows Hypoexponential distribution, denoted by* $H(\boldsymbol{\lambda})$*, with PDF*

$$f(z) = \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{\lambda_i^{-1}\lambda_j^{-1}}{\lambda_j^{-1} - \lambda_i^{-1}} \right) e^{-\frac{z}{\lambda_i}}, \quad z \geq 0. \tag{7.4.2}$$

It is of interest to mention here that the Hypoexponential distribution has found some interesting applications in reliability theory; see Scheuer (1988).

Lemma 7.4.3 *If $Z_1 \sim H(\boldsymbol{\lambda})$ and $Z_2 \sim \Gamma(\alpha, \beta)$ are independent random variables, where α is a positive integer, $\boldsymbol{\lambda} > 0$ and $\beta > 0$, then the CDF of $Z^- = Z_1 - Z_2$ and $Z^+ = Z_1 + Z_2$ are as follows:*

$$P(Z^- \leq z) = \begin{cases} 1 - \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{\lambda_j^{-1}}{\lambda_j^{-1} - \lambda_i^{-1}} \right) \frac{e^{-z\lambda_i^{-1}}}{\beta^\alpha \left(\lambda_i^{-1} + \beta^{-1}\right)^\alpha}, & z \geq 0, \\ S_\Gamma(-z, \alpha, \beta) - \sum_{i=1}^{n} \left[\prod_{j=1, j\neq i}^{n} \frac{\lambda_j^{-1}}{\lambda_j^{-1} - \lambda_i^{-1}} \right] \frac{S_\Gamma\left[-z, \alpha, \left(\lambda_i^{-1} + \beta^{-1}\right)^{-1}\right]}{e^{z\lambda_i^{-1}} \beta^\alpha \left(\lambda_i^{-1} + \beta^{-1}\right)^\alpha}, & z < 0, \end{cases} \tag{7.4.3}$$

$$\begin{aligned} P(Z^+ \leq z) &= F_\Gamma(z, \alpha, \beta) - \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{\lambda_j^{-1}}{\lambda_j^{-1} - \lambda_i^{-1}} \right) \\ &\quad \times \frac{e^{-z\lambda_i^{-1}}}{\Gamma(\alpha)\beta^\alpha} \int_0^z z_2^{\alpha-1} e^{-z_2(\beta^{-1} - \lambda_i^{-1})} dz_2, \end{aligned} \tag{7.4.4}$$

where

$$\int_0^z z_2^{\alpha-1} e^{-z_2\left(\frac{1}{\beta} - \frac{1}{\lambda_i}\right)} dz_2 = \begin{cases} \frac{\Gamma(\alpha)}{\left(\beta^{-1} - \lambda_i^{-1}\right)^\alpha} F_\Gamma\left(z, \alpha, \left(\beta^{-1} - \lambda_i^{-1}\right)^{-1}\right), & \beta < \lambda_i, \\ \frac{z^\alpha}{\alpha}, & \beta = \lambda_i, \\ \sum_{i=0}^{\alpha-1} \frac{(-1)^{\alpha-1-i}(\alpha-1)!}{(\lambda_i^{-1} - \beta^{-1})^{\alpha-i} i!} z^i e^{z(\lambda_i^{-1} - \beta^{-1})} - \frac{(-1)^{\alpha-1}(\alpha-1)!}{(\lambda_i^{-1} - \beta^{-1})^\alpha}, & \beta > \lambda_i. \end{cases} \tag{7.4.5}$$

Proof See Appendix.

Lemma 7.4.4 *In the general case when there are n distinct sums of independent exponential variables with scale parameters $\lambda_1, \cdots, \lambda_n > 0$ with $r_1, \cdots, r_n$ being the number of terms in these n sums, the CDF of this generalized Hypoexponential distribution (GH) is given by*

$$F(z) = 1 - \left(\prod_{i=1}^{n} \lambda_i^{-r_i}\right) \sum_{i=1}^{n} \sum_{l=1}^{r_i} \frac{\Psi_{i,l}(-\lambda_i^{-1}) z^{r_i - l} e^{-z\lambda_i^{-1}}}{(r_i - l)!(l-1)!}, \qquad z \geq 0, \tag{7.4.6}$$

where $\Psi_{i,l}(x) = -\frac{\partial^{l-1}}{\partial x^{l-1}} \left(\prod_{j=0, j\neq i}^{n} (\lambda_j^{-1} + x)^{-r_j} \right)$, *with* $\lambda_0 = 0$ *and* $r_0 = 1$.

It is of interest to mention here that a closed-form expression for $\Psi_{i,l}(x)$ has been provided by Amari and Misra (1997) as

$$\Psi_{i,l}(x) = (-1)^{l-1}(l-1)! \sum_{\Omega} \prod_{j=1}^{n} \begin{pmatrix} r_j + k_j - 1 \\ k_j \end{pmatrix} (\eta_j + x)^{-(r_j + k_j)}, \tag{7.4.7}$$

where $\Omega : \sum_{j=0, j\neq i}^{n} k_j = l - 1$.

Lemma 7.4.5 *Let $Z_1 \sim GH(\boldsymbol{\lambda}_1)$, with $\boldsymbol{\lambda}_1 = (\lambda_{11}, \cdots, \lambda_{1n})$ and $\mathbf{r_1} = (r_{11}, \cdots, r_{1n})$. Further, let $Z_2 \sim GH(\boldsymbol{\lambda}_2)$ be independent of Z_1, with $\boldsymbol{\lambda}_2 = (\lambda_{21}, \cdots, \lambda_{2m})$ and $\mathbf{r_2} =$*

$(r_{21}, \cdots, r_{2m})$. *Then, the CDF of* $Z = Z_1 - Z_2$ *is given by*

$$\begin{aligned} F(z) &= 1 - \left(\prod_{i=1}^{m} \lambda_{2i}^{-r_{2i}} \prod_{j=1}^{n} \lambda_{1j}^{-r_{1j}}\right) \sum_{\Omega} \frac{\Psi_{2i,l_2}(-\lambda_{2i}^{-1})\Psi_{1j,l_1}(-\lambda_{1j}^{-1}) z^{r_{1j}-l_1-k} e^{-z\lambda_{1j}^{-1}}}{(r_{2i}-l_2)!(l_2-1)!k!(r_{1j}-l_1-k)!(l_1-1)!} \\ &\quad \times \left[\frac{(r_{2i}-l_2+k)!}{\lambda_{2i}(\lambda_{1j}^{-1}+\lambda_{2i}^{-1})^{r_{2i}-l_2+k+1}} - \frac{(r_{2i}-l_2+k-1)!(r_{2i}-l_2)}{(\lambda_{1j}^{-1}+\lambda_{2i}^{-1})^{r_{2i}-l_2+k}}\right], \quad z \geq 0, \end{aligned} \tag{7.4.8}$$

where $\Omega : (i \in [1, m], l_2 \in [1, r_{2i}], j \in [1, n], l_1 \in [1, r_{1j}], k \in [0, r_{1j} - l_1])$. *For the case when* $z < 0$, *the CDF can be obtained through the identity*

$$P(Z_1 - Z_2 \leq z) = P(Z_2 - Z_1 > -z) = 1 - P(Z_2 - Z_1 \leq -z).$$

Proof See Appendix.

For the sake of completeness and also for computational use, we now present the following two lemmas.

Lemma 7.4.6 *Let* $Z_1 \sim H(\boldsymbol{\lambda}_1)$ *and* $Z_2 \sim H(\boldsymbol{\lambda}_2)$ *be independent variables, with* $\boldsymbol{\lambda}_1 = (\lambda_{11}, \cdots, \lambda_{1n})$ *and* $\boldsymbol{\lambda}_2 = (\lambda_{21}, \cdots, \lambda_{2m})$, *respectively. Then, the CDF of* $Z = Z_1 - Z_2$ *is given by*

$$F(z) = 1 - \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{C_{ij} e^{-z\lambda_{1j}^{-1}}}{\lambda_{2i}}, \quad z \geq 0, \tag{7.4.9}$$

where $C_{ij} = \left(\prod_{i=1}^{m} \lambda_{2i}^{-1} \prod_{j=1}^{n} \lambda_{1j}^{-1}\right) \frac{\Psi_{2i,1}(-\lambda_{2i}^{-1})\Psi_{1j,1}(-\lambda_{1j}^{-1})}{\lambda_{1j}^{-1}+\lambda_{2i}^{-1}}$.

Proof It is a special case of Lemma 7.4.5 when all $r_{1j} = 1$ and $r_{2i} = 1$.

Lemma 7.4.7 *Let $Z_1 \sim H(\boldsymbol{\lambda}_1)$, $Z_2 \sim H(\boldsymbol{\lambda}_2)$ and $Z_3 \sim \Gamma(\alpha, \beta)$ be independent variables, with a positive integer α, $\beta > 0$, $\boldsymbol{\lambda}_1 = (\lambda_{11}, \cdots, \lambda_{1n}) > 0$ and $\boldsymbol{\lambda}_2 = (\lambda_{21}, \cdots, \lambda_{2m}) > 0$. Then, the CDF of $Z = Z_1 - Z_2 - Z_3$ is given by*

$$F(z) = \begin{cases} 1 - \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{C_{ij} e^{-z\lambda_{1j}^{-1}}}{\lambda_{2i} \beta^{\alpha} (\lambda_{1j}^{-1} + \beta^{-1})^{\alpha}}, & z \geq 0, \\ S_{\Gamma}(-z, \alpha, \beta) + \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{C_{ij}}{\beta^{\alpha}} \left[\frac{e^{z\lambda_{2i}^{-1}}}{\Gamma(\alpha)\lambda_{1j}} f(z) - \frac{e^{-z\lambda_{1j}^{-1}} S_{\Gamma}\left(-z, \alpha, \left(\lambda_{1j}^{-1} + \beta^{-1}\right)^{-1}\right)}{\lambda_{2i} \left(\lambda_{1j}^{-1} + \beta^{-1}\right)^{\alpha}} \right], & z < 0, \end{cases} \tag{7.4.10}$$

where $f(z) = \int_0^{-z} z_3^{\alpha-1} e^{-z_3(\frac{1}{\beta} - \frac{1}{\lambda_{2i}})} dz_3$ and C_{ij} are as defined in Eq. (7.4.5) and Eq. (7.4.9).

Proof See Appendix.

By using the above Lemmas, we can get the following result which becomes useful later while developing exact inferential results for some quantities of interest.

Theorem 7.4.3 *Let $X_{1:n} < X_{2:n} < \cdots < X_{r:n}$ be a Type-II right censored sample from a sample of size n from $L(\mu, \sigma)$, and Z be any linear combination, $Z = \sum_{i=1}^{r} c_i X_{i:n}$. Then, the CDF of Z is given by*

$$P(Z \leq z + \mu \sum_{i=1}^{r} c_i) = \sum_{d=0}^{n} P(GH(\boldsymbol{\lambda}_{1d}) - GH(-\boldsymbol{\lambda}_{2d}) \leq z) P(D = d), \tag{7.4.11}$$

where $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$, *with*

$$\sigma_j = \begin{cases} \sum_{i=j}^{r} \frac{c_i \sigma}{n-j+1}, & d+1 \le j \le r \ d < r, \\ -\sum_{i=1}^{j} \frac{c_i \sigma}{j}, & 1 \le j \le d \ d < r, \\ -\sum_{i=1}^{r} \frac{c_i \sigma}{j}, & d \ge r, \end{cases}$$

and $\boldsymbol{\lambda}_{1d}$ *being the subset of* $\boldsymbol{\sigma}$ *that contains all the non-negative entries and* $\boldsymbol{\lambda}_{2d}$ *being the subset of* $\boldsymbol{\sigma}$ *that contains all the negative entries.*

Proof See Appendix.

Now, by using Theorem 7.4.3 in conjunction with the above explicit expressions of the MLEs of μ and σ, we can obtain the following two theorems.

Theorem 7.4.4 *The exact conditional distributions of* $\frac{r(\hat{\mu}-\mu)}{\sigma}$ *and* $\frac{r\hat{\sigma}}{\sigma}$ *given* $D = d$ *are as follows:*

Case I: $r < \frac{n}{2}$

$$\frac{r\hat{\sigma}}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum_{j=d+1}^{r} E\left(\frac{j-1}{n-j+1}\right) + \Gamma(d,1) & 0 \le d < r, \\ \Gamma(r-1,1) & d \ge r, \end{cases}$$

$$\frac{r(\hat{\mu}-\mu)}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum_{j=d+1}^{r} E\left(\frac{r+(j-1)\ln\frac{n}{2r}}{n-j+1}\right) + \Gamma(d, \ln\frac{n}{2r}) & 0 \le d < r, \\ \Gamma(r-1, \ln\frac{n}{2r}) + \sum_{j=r}^{d} E\left(-\frac{r}{j}\right) & d \ge r; \end{cases}$$

Case II: $n = 2m+1$ *and* $r \geq m+2$

$$\frac{r\hat{\sigma}}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum\limits_{j=d+1}^{m+1} E\left(\frac{j-1}{n-j+1}\right) + \Gamma(r-m-1+d,1) & 0 \leq d \leq m, \\ \Gamma(r-1,1) + E\left(\frac{m}{m+1}\right) & d = m+1, \\ \sum\limits_{j=m+1}^{d} E\left(\frac{n-j}{j}\right) + \Gamma(r-d+m,1) & m+1 < d < r, \\ \sum\limits_{j=m+1}^{r-1} E\left(\frac{n-j}{j}\right) + \Gamma(m,1) & d \geq r, \end{cases}$$

$$\frac{r(\hat{\mu}-\mu)}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum\limits_{j=d+1}^{m+1} E\left(\frac{r}{n-j+1}\right) & 0 \leq d < m+1, \\ \sum\limits_{j=m+1}^{d} E\left(-\frac{r}{j}\right) & d \geq m+1; \end{cases}$$

Case III: $n = 2m+1$ *and* $r = m+1$

$$\frac{r\hat{\sigma}}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum\limits_{j=d+1}^{r} E\left(\frac{j-1}{n-j+1}\right) + \Gamma(d,1) & 0 \leq d < r, \\ \Gamma(d-1,1) & d \geq r, \end{cases}$$

$$\frac{r(\hat{\mu}-\mu)}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum\limits_{j=d+1}^{m+1} E\left(\frac{r}{n-j+1}\right) & 0 \leq d < m+1, \\ \sum\limits_{j=m+1}^{d} E\left(-\frac{r}{j}\right) & d \geq m+1; \end{cases}$$

Case IV: $n = 2m$ *and* $r > m+1$

$$\frac{r\hat{\sigma}}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum_{j=d+1}^{m} E\left(\frac{j-1}{n-j+1}\right) + \Gamma(r-m+d,1) & 0 \le d < m, \\ \Gamma(r,1) & d = m, \\ \sum_{j=m+1}^{d} E\left(\frac{n-j}{j}\right) + \Gamma(r-d+m,1) & m < d < r, \\ \sum_{j=m+1}^{r-1} E\left(\frac{n-j}{j}\right) + \Gamma(m,1) & d \ge r, \end{cases}$$

$$\frac{r(\hat{\mu}-\mu)}{\sigma}|d \stackrel{d}{=} \begin{cases} E\left(\frac{r}{n}\right) + \sum_{j=d+1}^{m} E\left(\frac{r}{n-j+1}\right) & 0 \le d < m, \\ E\left(\frac{r}{n}\right) + E\left(-\frac{r}{n}\right) & d = m, \\ \sum_{j=m+1}^{r} E\left(-\frac{r}{j}\right) + E\left(-\frac{r}{n}\right) & d \ge m+1; \end{cases}$$

Case V: $n = 2m$ *and* $r = m+1$

$$\frac{r\hat{\sigma}}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum_{j=d+1}^{m} E\left(\frac{j-1}{n-j+1}\right) + \Gamma(1+d,1) & 0 \le d < r-1, \\ \Gamma(r,1) & d = r-1, \\ \Gamma(r-1,1) & d \ge r, \end{cases}$$

$$\frac{r(\hat{\mu}-\mu)}{\sigma}|d \stackrel{d}{=} \begin{cases} E\left(\frac{r}{n}\right) + \sum_{j=d+1}^{m} E\left(\frac{r}{n-j+1}\right) & 0 \le d < r-1, \\ E\left(\frac{r}{n}\right) + E\left(-\frac{r}{n}\right) & d = r-1, \\ \sum_{j=m+1}^{r} E\left(-\frac{r}{j}\right) + E\left(-\frac{r}{n}\right) & d \ge r; \end{cases}$$

Case VI: $n = 2m$ *and* $r = m$

$$\frac{r\hat{\sigma}}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum\limits_{j=d+1}^{m} E\left(\frac{j-1}{n-j+1}\right) + \Gamma(d,1) & 0 \leq d < r, \\ \Gamma(r-1,1) & d \geq r, \end{cases}$$

$$\frac{r(\hat{\mu}-\mu)}{\sigma}|d \stackrel{d}{=} \begin{cases} \sum\limits_{j=d+1}^{m} E\left(\frac{r}{n-j+1}\right) & 0 \leq d < r, \\ \sum\limits_{j=m}^{d} E\left(-\frac{r}{j}\right) & d \geq r. \end{cases}$$

Theorem 7.4.5 *The expected values of* $\hat{\mu}$ *and* $\hat{\sigma}$ *are as follows:*

Case I: $r < \frac{n}{2}$

$$\begin{aligned} E(\hat{\sigma}) &= \frac{\sigma}{r}\left[\sum_{d=0}^{r-1}\left(d + \sum_{j=d+1}^{r}\frac{j-1}{n-j+1}\right)P(D=d) + (r-1)P^*(D=r)\right], \\ E(\hat{\mu}) &= \mu + \frac{\sigma}{r}\left\{\sum_{d=0}^{r-1}\left(d\ln\frac{n}{2r} + \sum_{j=d+1}^{r}\frac{r+(j-1)\ln\frac{n}{2r}}{n-j+1}\right)P(D=d)\right. \\ &\quad \left. + \sum_{d=r}^{n}\left[(r-1)\ln\frac{n}{2r} - \sum_{j=r}^{d}\frac{r}{j}\right]P(D=d)\right\}; \end{aligned}$$

Case II: $n = 2m + 1$ *and* $r \geq m + 2$

$$
\begin{aligned}
E(\hat{\sigma}) &= \frac{\sigma}{r}\left\{\sum_{d=0}^{m}\left[r - m - 1 + d + \sum_{j=d+1}^{m+1}\frac{j-1}{n-j+1}\right]P(D=d)\right. \\
&\quad + \left(r - \frac{1}{m+1}\right)P(D = m+1) \\
&\quad + \sum_{d=m+2}^{r-1}\left[r - d + m + \sum_{j=m+1}^{d}\frac{n-j}{j}\right]P(D=d) \\
&\quad \left. + \left(m + \sum_{j=m+1}^{r-1}\frac{n-j}{j}\right)P^{*}(D=r)\right\}, \\
E(\hat{\mu}) &= \mu;
\end{aligned}
$$

Case III: $n = 2m + 1$ *and* $r = m + 1$

$$
\begin{aligned}
E(\hat{\sigma}) &= \frac{\sigma}{r}\left[\sum_{d=0}^{r-1}\left(d + \sum_{j=d+1}^{r}\frac{j-1}{n-j+1}\right)P(D=d) + (r-1)P^{*}(D=r)\right], \\
E(\hat{\mu}) &= \mu;
\end{aligned}
$$

Case IV: $n = 2m$ *and* $r > m + 1$

$$
\begin{aligned}
E(\hat{\sigma}) &= \frac{\sigma}{r}\left\{\sum_{d=0}^{m-1}\left[r - m + d + \sum_{j=d+1}^{m}\frac{j-1}{n-j+1}\right]P(D=d) + rP(D=m)\right. \\
&\quad + \sum_{d=m+1}^{r-1}\left[r - d + m + \sum_{j=m+1}^{d}\frac{n-j}{j}\right]P(D=d) \\
&\quad \left. + \left(m + \sum_{j=m+1}^{r-1}\frac{n-j}{j}\right)P^{*}(D=r)\right\}, \\
E(\hat{\mu}) &= \mu;
\end{aligned}
$$

Case V: $n = 2m$ *and* $r = m + 1$

$$\begin{aligned} E(\hat{\sigma}) &= \frac{\sigma}{r}\left\{\sum_{d=0}^{m-1}\left(1+d+\sum_{j=d+1}^{m}\frac{j-1}{n-j+1}\right)P(D=d)+(m+1)P(D=m)\right. \\ &\quad \left. +mP^*(D=r)\right\}, \\ E(\hat{\mu}) &= \mu; \end{aligned}$$

Case VI: $n = 2m$ *and* $r = m$

$$\begin{aligned} E(\hat{\sigma}) &= \frac{\sigma}{r}\left\{\sum_{d=0}^{m-1}\left(d+\sum_{j=d+1}^{m}\frac{j-1}{n-j+1}\right)P(D=d)+(m-1)P^*(D=r)\right\}, \\ E(\hat{\mu}) &= \mu+\sigma\left\{\left[\sum_{d=0}^{m-1}\left(\sum_{j=d+1}^{m}\frac{P(D=d)}{n-j+1}\right)\right]-\left[\sum_{d=m}^{n}\left(\sum_{j=m}^{d}\frac{P(D=d)}{j}\right)\right]\right\}, \end{aligned}$$

where $P(D = d)$ *is as defined in Eq. (7.4.1) for* $d < r$ *and* $P^*(D = r) = 1-\sum_{d=0}^{r-1} P(D = d)$.

From the expressions of expectations of $\hat{\sigma}$ and $\hat{\mu}$ presented in Theorem 7.4.5, it is evident that we can generally express

$$E(\hat{\mu}) = \mu + h_1(n,r)\sigma, \tag{7.4.12}$$

$$E(\hat{\sigma}) = h_2(n,r)\sigma. \tag{7.4.13}$$

Table 7.4.1: Values of $h_1(n,r)$ for $n = 2(1)20$ and $r \leq \frac{n}{2}$.

n	r	$h_1(n,r)$	n	r	$h_1(n,r)$	n	r	$h_1(n,r)$	n	r	$h_1(n,r)$	n	r	$h_1(n,r)$
2	1	-0.7500	10	1	-2.2358		5	-0.1886		5	-0.2273		9	-0.0659
3	1	-1.1250		2	-0.7773		6	-0.1221		6	-0.1614	19	1	-2.8546
4	1	-1.3854		3	-0.3930	14	1	-2.5584		7	-0.1145		2	-1.0755
	2	-0.3438		4	-0.2259		2	-0.9320		8	-0.0748		3	-0.5861
5	1	-1.5885		5	-0.1246		3	-0.4934	17	1	-2.7464		4	-0.3725
	2	-0.4596	11	1	-2.3267		4	-0.3046		2	-1.0229		5	-0.2576
6	1	-1.7563		2	-0.8208		5	-0.2030		3	-0.5521		6	-0.1877
	2	-0.5460		3	-0.4215		6	-0.1377		4	-0.3476		7	-0.1409
	3	-0.2188		4	-0.2496		7	-0.0864		5	-0.2381		8	-0.1061
7	1	-1.8995		5	-0.1512	15	1	-2.6251		6	-0.1710		9	-0.0766
	2	-0.6163	12	1	-2.4101		2	-0.9642		7	-0.1247	20	1	-2.9046
	3	-0.2785		2	-0.8607		3	-0.5142		8	-0.0876		2	-1.0999
8	1	-2.0246		3	-0.4474		4	-0.3199	18	1	-2.8020		3	-0.6019
	2	-0.6765		4	-0.2700		5	-0.2157		2	-1.0499		4	-0.3840
	3	-0.3238		5	-0.1717		6	-0.1505		3	-0.5695		5	-0.2667
	4	-0.1592		6	-0.1021		7	-0.1021		4	-0.3604		6	-0.1952
9	1	-2.1358	13	1	-2.4870	16	1	-2.6876		5	-0.2481		7	-0.1477
	2	-0.7296		2	-0.8976		2	-0.9944		6	-0.1797		8	-0.1132
	3	-0.3610		3	-0.4712		3	-0.5337		7	-0.1333		9	-0.0851
	4	-0.1969		4	-0.2881		4	-0.3342		8	-0.0978		10	-0.0588

From these expressions, we can readily correct for the bias to produce "bias-corrected MLEs" as

$$\hat{\sigma}^* = h_2^{-1}(n,r)\hat{\sigma}, \tag{7.4.14}$$

$$\hat{\mu}^* = \hat{\mu} - h_1(n,r)h_2^{-1}(n,r)\hat{\sigma}. \tag{7.4.15}$$

In Tables 7.4.1 and 7.4.2, we have presented the exact values of $h_1(n,r)$ and $h_2(n,r)$ for sample sizes n= 2(1)20 and various choices of r. For the sake of brevity, the values of $h_1(n,r)$ are not presented for $r > \frac{n}{2}$, since $\hat{\mu}$ is unbiased in these cases. Note also that $h_2(n,r)$ exists only when the MLE of $\hat{\sigma}$ exists, i.e., when $r \geq 2$.

Table 7.4.2: Values of $h_2(n,r)$ for $n = 2(1)20$ and $r \geq 2$.

n	r	$h_2(n,r)$	n	r	$h_2(n,r)$	n	r	$h_2(n,r)$	n	r	$h_2(n,r)$	n	r	$h_2(n,r)$
2	2	0.7500		4	0.7555		12	0.9504		11	0.9492		18	0.9697
3	2	0.5625		5	0.8203		13	0.9542		12	0.9542	19	2	0.5000
	3	0.7500		6	0.8913	14	2	0.5000		13	0.9579		3	0.6667
4	2	0.5208		7	0.9182		3	0.6667		14	0.9610		4	0.7500
	3	0.8056		8	0.9308		4	0.7504		15	0.9636		5	0.8001
	4	0.8646		9	0.9388		5	0.8018		16	0.9658		6	0.8337
5	2	0.5078		10	0.9449		6	0.8395	17	2	0.5000		7	0.8584
	3	0.7205	11	2	0.5000		7	0.8745		3	0.6667		8	0.8788
	4	0.8268		3	0.6672		8	0.9163		4	0.7501		9	0.8984
	5	0.8646		4	0.7528		9	0.9350		5	0.8003		10	0.9212
6	2	0.5031		5	0.8110		10	0.9443		6	0.8345		11	0.9399
	3	0.6896		6	0.8662		11	0.9500		7	0.8608		12	0.9495
	4	0.8453		7	0.9056		12	0.9543		8	0.8850		13	0.9550
	5	0.8888		8	0.9229		13	0.9578		9	0.9122		14	0.9587
	6	0.9083		9	0.9325		14	0.9608		10	0.9340		15	0.9616
7	2	0.5013		10	0.9394	15	2	0.5000		11	0.9449		16	0.9640
	3	0.6769		11	0.9449		3	0.6667		12	0.9510		17	0.9661
	4	0.7948	12	2	0.5000		4	0.7502		13	0.9552		18	0.9680
	5	0.8655		3	0.6669		5	0.8009		14	0.9585		19	0.9697
	6	0.8926		4	0.7514		6	0.8368		15	0.9613	20	2	0.5000
	7	0.9083		5	0.8059		7	0.8675		16	0.9637		3	0.6667
8	2	0.5006		6	0.8521		8	0.9008		17	0.9658		4	0.7500
	3	0.6714		7	0.9054		9	0.9267	18	2	0.5000		5	0.8000
	4	0.7719		8	0.9276		10	0.9392		3	0.6667		6	0.8335
	5	0.8722		9	0.9383		11	0.9462		4	0.7500		7	0.8579
	6	0.9057		10	0.9450		12	0.9510		5	0.8002		8	0.8773
	7	0.9211		11	0.9500		13	0.9548		6	0.8340		9	0.8950
	8	0.9311		12	0.9542		14	0.9581		7	0.8593		10	0.9140
9	2	0.5002	13	2	0.5000		15	0.9608		8	0.8811		11	0.9378
	3	0.6689		3	0.6668	16	2	0.5000		9	0.9038		12	0.9501
	4	0.7609		4	0.7507		3	0.6667		10	0.9320		13	0.9567
	5	0.8380		5	0.8032		4	0.7501		11	0.9460		14	0.9608
	6	0.8893		6	0.8440		5	0.8005		12	0.9533		15	0.9637
	7	0.9104		7	0.8860		6	0.8353		13	0.9578		16	0.9660
	8	0.9224		8	0.9175		7	0.8633		14	0.9610		17	0.9680
	9	0.9311		9	0.9321		8	0.8911		15	0.9636		18	0.9698
10	2	0.5001		10	0.9402		9	0.9250		16	0.9659		19	0.9714
	3	0.6677		11	0.9458		10	0.9410		17	0.9679		20	0.9728

7.5 MLE of the quantile

Let q denote the quantile of the standard $L(0,1)$ distribution. Then, the quantile of $L(\mu,\sigma)$ is clearly $Q = q\sigma + \mu$ and so its MLE is simply

$$\hat{Q} = q\hat{\sigma} + \hat{\mu}; \tag{7.5.1}$$

moreover, its unbiased estimate is given by

$$\tilde{Q} = qh_2^{-1}(n,r)\hat{\sigma} + \hat{\mu} - h_1(n,r)h_2^{-1}(n,r)\hat{\sigma}. \tag{7.5.2}$$

Thus, in general, the estimate can be considered in the form

$$\hat{Q}^* = (aq+b)\hat{\sigma} + \hat{\mu} = q^*\hat{\sigma} + \hat{\mu}. \tag{7.5.3}$$

For example, upon setting $a = 1$ and $b = 0$, we obtain the MLE of Q in Eq. (7.5.1) and if we set $a = h_2^{-1}(n,r)$ and $b = -h_1(n,r)h_2^{-1}(n,r)$, we obtain the unbiased estimate of Q in Eq. (7.5.2).

Theorem 7.5.1 *The exact conditional distribution of* $\frac{r(\hat{Q}^*-\mu)}{\sigma}$ *given* $D = d$ *is as follows:*

Case I: $r < \frac{n}{2}$

$$\frac{r\left(\hat{Q}^* - \mu\right)}{\sigma}|d \stackrel{d}{=} \begin{cases} \Gamma\left(d, q^* + \ln\left(\frac{n}{2r}\right)\right) + \sum\limits_{j=d+1}^{r} E\left(\frac{r+(r-1)\left(q^*+\ln\left(\frac{n}{2r}\right)\right)}{n-j+1}\right) & 0 \leq d < r, \\ \Gamma\left(r-1, q^* + \ln\left(\frac{n}{2r}\right)\right) + \sum\limits_{j=r}^{d} E\left(-\frac{r}{j}\right) & d \geq r; \end{cases}$$

Case II: $n = 2m+1$ *and* $r \geq m+2$

$$\frac{r\left(\hat{Q}^* - \mu\right)}{\sigma}|d \stackrel{d}{=} \begin{cases} \Gamma\left(r-m-1+d, q^*\right) + \sum\limits_{j=d+1}^{m+1} E\left(\frac{q^*(j-1)+r}{n-j+1}\right) & 0 \leq d < m+1, \\ \Gamma\left(r-d+m, q^*\right) + \sum\limits_{j=m+1}^{d} E\left(\frac{q^*(n-j)-r}{j}\right) & m+1 \leq d < r, \\ \Gamma\left(m, q^*\right) + \sum\limits_{j=m+1}^{r-1} E\left(\frac{q^*(n-j)-r}{j}\right) + \sum\limits_{j=r}^{d} E\left(-\frac{r}{j}\right) & d \geq r; \end{cases}$$

Case III: $n = 2m+1$ *and* $r = m+1$

$$\frac{r\left(\hat{Q}^* - \mu\right)}{\sigma}|d \stackrel{d}{=} \begin{cases} \Gamma\left(d, q^*\right) + \sum\limits_{j=d+1}^{m+1} E\left(\frac{q^*(j-1)+r}{n-j+1}\right) & 0 \leq d < r, \\ \Gamma\left(m, q^*\right) + \sum\limits_{j=r}^{d} E\left(-\frac{r}{j}\right) & d \geq r; \end{cases}$$

Case IV: $n = 2m$ *and* $r > m+1$

$$\frac{r\left(\hat{Q}^* - \mu\right)}{\sigma}|d \stackrel{d}{=} \begin{cases} \Gamma\left(r-m-1+d, q^*\right) + E\left(\frac{nq^*+r}{n}\right) + \sum\limits_{j=d+1}^{m} E\left(\frac{q^*(j-1)+r}{n-j+1}\right) & 0 \leq d < m, \\ \Gamma\left(r-2, q^*\right) + E\left(\frac{nq^*+r}{n}\right) + E\left(\frac{nq^*-r}{n}\right) & d = m, \\ \Gamma\left(r-d+m-1, q^*\right) + \sum\limits_{j=m+1}^{d} E\left(\frac{(n-j)q^*-r}{j}\right) + E\left(\frac{nq^*-r}{n}\right) & m < d < r, \\ \Gamma\left(m-1, q^*\right) + \sum\limits_{j=m+1}^{r-1} E\left(\frac{(n-j)q^*-r}{j}\right) + E\left(\frac{nq^*-r}{n}\right) + \sum\limits_{j=r}^{d} E\left(-\frac{r}{j}\right) & d \geq r; \end{cases}$$

Case V: $n = 2m$ *and* $r = m + 1$

$$\frac{r\left(\hat{Q}^* - \mu\right)}{\sigma}|d \stackrel{d}{=} \begin{cases} \Gamma\left(d, q^*\right) + E\left(\frac{nq^*+r}{n}\right) + \sum\limits_{j=d+1}^{m} E\left(\frac{q^*(j-1)+r}{n-j+1}\right) & 0 \leq d < m, \\ \Gamma\left(m-1, q^*\right) + E\left(\frac{nq^*+r}{n}\right) + E\left(\frac{nq^*-r}{n}\right) & d = m, \\ \Gamma\left(m-1, q^*\right) + E\left(\frac{nq^*-r}{n}\right) + \sum\limits_{j=r}^{d} E\left(-\frac{r}{j}\right) & d \geq r; \end{cases}$$

Case VI: $n = 2m$ *and* $r = m$

$$\frac{r\left(\hat{Q}^* - \mu\right)}{\sigma}|d \stackrel{d}{=} \begin{cases} \Gamma\left(d, q^*\right) + \sum\limits_{j=d+1}^{m} E\left(\frac{q^*(j-1)+r}{n-j+1}\right) & 0 \leq d < m, \\ \Gamma\left(m-1, q^*\right) + \sum\limits_{j=r}^{d} E\left(-\frac{r}{j}\right) & d \geq r. \end{cases}$$

By using the expressions of $E(\hat{\mu})$ and $E(\hat{\sigma})$ in Theorem 7.4.5, we readily have

$$E\left(\hat{Q}^*\right) = \left((aq + b)h_2(n, r) + h_1(n, r)\right)\sigma + \mu \tag{7.5.4}$$

from which the bias of the estimate can be easily determined.

Theorem 7.5.2 *The exact second moment of* $\hat{Q}^*$*, from which the variance and MSE can be easily determined, are as follows:*

Case I: $r < \frac{n}{2}$

$$
\begin{aligned}
E\left(\hat{Q}^{*2}\right) &= \frac{\sigma^2}{r^2}\left\{\sum_{d=0}^{r-1}\left[\sum_{j=d+1}^{r}\left(\frac{r+(r-1)\left(q^*+\ln\left(\frac{n}{2r}\right)\right)}{n-j+1}\right)^2 + d\left(q^*+\ln\left(\frac{n}{2r}\right)\right)^2\right.\right. \\
&\quad \left.+\left(\sum_{j=d+1}^{r}\frac{r+(r-1)\left(q^*+\ln\left(\frac{n}{2r}\right)\right)}{n-j+1} + d\left(q^*+\ln\left(\frac{n}{2r}\right)\right)\right)^2\right] P(D=d) \\
&\quad +\sum_{d=r}^{n}\left[\sum_{j=r}^{d}\left(\frac{r}{j}\right)^2 + (r-1)\left(q^*+\ln\left(\frac{n}{2r}\right)\right)^2\right. \\
&\quad \left.\left.+\left((r-1)\left(q^*+\ln\left(\frac{n}{2r}\right)\right) - \sum_{j=r}^{d}\frac{r}{j}\right)^2\right] P(D=d)\right\} + 2\mu E\left(\hat{Q}^*\right) \\
&\quad -\mu^2; \qquad (7.5.5)
\end{aligned}
$$

Case II: $n = 2m+1$ *and* $r \geq m+2$

$$
\begin{aligned}
E\left(\hat{Q}^{*2}\right) &= \frac{\sigma^2}{r^2}\left\{\sum_{d=0}^{m}\left[\sum_{j=d+1}^{m+1}\left(\frac{q^*(j-1)+r}{n-j+1}\right)^2 + (r-m-1+d)(q^*)^2\right.\right. \\
&\quad \left.+\left(\sum_{j=d+1}^{m+1}\frac{q^*(j-1)+r}{n-j+1} + (r-m-1+d)q^*\right)^2\right] P(D=d) \\
&\quad +\sum_{d=m+1}^{r-1}\left[\sum_{j=m+1}^{d}\left(\frac{q^*(n-j)-r}{j}\right)^2 + (r-d+m)(q^*)^2\right. \\
&\quad \left.+\left(\sum_{j=m+1}^{d}\frac{q^*(n-j)-r}{j} + (r-d+m)q^*\right)^2\right] P(D=d) \\
&\quad +\sum_{d=r}^{n}\left[\sum_{j=m+1}^{r-1}\left(\frac{q^*(n-j)-r}{j}\right)^2 + m(q^*)^2 + \sum_{j=r}^{d}\left(\frac{r}{j}\right)^2\right. \\
&\quad \left.\left.+\left(\sum_{j=m+1}^{r-1}\frac{q^*(n-j)-r}{j} + mq^* - \sum_{j=r}^{d}\frac{r}{j}\right)^2\right] P(D=d)\right\} + 2\mu E\left(\hat{Q}^*\right) \\
&\quad -\mu^2; \qquad (7.5.6)
\end{aligned}
$$

Case III: $n = 2m+1$ *and* $r = m+1$

$$
\begin{aligned}
E\left(\hat{Q}^{*2}\right) &= \frac{\sigma^2}{r^2}\left\{\sum_{d=0}^{r-1}\left[\sum_{j=d+1}^{m+1}\left(\frac{q^*(j-1)+r}{n-j+1}\right)^2 + d(q^*)^2\right.\right. \\
&\quad \left.+\left(\sum_{j=d+1}^{m+1}\frac{q^*(j-1)+r}{n-j+1} + dq^*\right)^2\right] P(D=d) \\
&\quad \left.+\sum_{d=r}^{n}\left[m(q^*)^2 + \sum_{j=r}^{d}\frac{r^2}{j^2} + \left(mq^* - \sum_{j=r}^{d}\frac{r}{j}\right)^2\right] P(D=d)\right\} \\
&\quad +2\mu E\left(\hat{Q}^*\right) - \mu^2; \qquad (7.5.7)
\end{aligned}
$$

Case IV: $n = 2m$ *and* $r > m+1$

$$
\begin{aligned}
E\left(\hat{Q}^{*2}\right) &= \frac{\sigma^2}{r^2}\left\{\sum_{d=0}^{m-1}\left[\sum_{j=d+1}^{m}\left(\frac{q^*(j-1)+r}{n-j+1}\right)^2+\left(\frac{nq^*+r}{n}\right)^2\right.\right.\\
&+(r-m-1+d)(q^*)^2\\
&+\left.\left(\sum_{j=d+1}^{m}\frac{q^*(j-1)+r}{n-j+1}+\frac{r}{n}+(r-m+d)q^*\right)^2\right]P(D=d)\\
&+\left[\left(\frac{nq^*+r}{n}\right)^2+\left(\frac{nq^*-r}{n}\right)^2+(r-2)(q^*)^2+(rq^*)^2\right]P(D=m)\\
&+\sum_{d=m+1}^{r-1}\left[\sum_{j=m+1}^{d}\left(\frac{q^*(n-j)-r}{j}\right)^2+(r-d+m-1)(q^*)^2\right.\\
&+\left(\frac{nq^*-r}{n}\right)^2\\
&+\left.\left(\sum_{j=m+1}^{d}\frac{q^*(n-j)-r}{j}+(r-d+m)q^*-\frac{r}{n}\right)^2\right]P(D=d)\\
&+\sum_{d=r}^{n}\left[\sum_{j=m+1}^{r-1}\left(\frac{q^*(n-j)-r}{j}\right)^2+(m-1)(q^*)^2+\sum_{j=r}^{d}\left(\frac{r}{j}\right)^2\right.\\
&+\left(\frac{nq^*-r}{n}\right)^2\\
&+\left.\left.\left(\sum_{j=m+1}^{r-1}\frac{q^*(n-j)-r}{j}+mq^*-\sum_{j=r}^{d}\frac{r}{j}-\frac{r}{n}\right)^2\right]P(D=d)\right\}\\
&+2\mu E\left(\hat{Q}^*\right)-\mu^2; \qquad (7.5.8)
\end{aligned}
$$

Case V: $n = 2m$ *and* $r = m + 1$

$$
\begin{aligned}
E(\hat{Q}^{*2}) \;=\; & \frac{\sigma^2}{r^2}\left\{\sum_{d=0}^{m-1}\left[\sum_{j=d+1}^{m}\left(\frac{q^*(j-1)+r}{n-j+1}\right)^2+\left(\frac{nq^*+r}{n}\right)^2+d(q^*)^2\right.\right. \\
& \left.\left(\sum_{j=d+1}^{m}\frac{q^*(j-1)+r}{n-j+1}+\frac{r}{n}+(d+1)q^*\right)^2\right]P(D=d) \\
& +\left[\left(\frac{nq^*+r}{n}\right)^2+\left(\frac{nq^*-r}{n}\right)^2+(m-1)(q^*)^2+(rq^*)^2\right]P(D=m) \\
& +\sum_{d=r}^{n}\left[(m-1)(q^*)^2+\sum_{j=r}^{d}\frac{r^2}{j^2}+\left(\frac{nq^*-r}{n}\right)^2\right. \\
& \left.\left.+\left(mq^*-\sum_{j=r}^{d}\frac{r}{j}-\frac{r}{n}\right)^2\right]\times P(D=d)\right\}+2\mu E\left(\hat{Q}^*\right)-\mu^2; \qquad (7.5.9)
\end{aligned}
$$

Case VI: $n = 2m$ *and* $r = m$

$$
\begin{aligned}
E\left(\hat{Q}^{*2}\right) \;=\; & \frac{\sigma^2}{r^2}\left\{\sum_{d=0}^{m-1}\left[\sum_{j=d+1}^{m}\left(\frac{q^*(j-1)+r}{n-j+1}\right)^2\right.\right. \\
& \left.+d(q^*)^2\left(\sum_{j=d+1}^{m}\frac{q^*(j-1)+r}{n-j+1}+dq^*\right)^2\right]P(D=d) \\
& +\sum_{d=r}^{n}\left[(m-1)(q^*)^2+\left((m-1)q^*-\sum_{j=r}^{d}\frac{r}{j}\right)^2\right. \\
& \left.\left.+\sum_{j=r}^{d}\left(\frac{r}{j}\right)^2\right]P(D=d)\right\}+2\mu E\left(\hat{Q}^*\right)-\mu^2. \qquad (7.5.10)
\end{aligned}
$$

In Tables 7.5.1 and 7.5.2, we have presented the exact bias and MSEs of the estimates $\hat{Q}_\alpha$ and $\tilde{Q}_\alpha$ for different choices of α and also the RE, defined as $RE = 100 \times \frac{MSE(\tilde{Q}_\alpha)}{MSE(\hat{Q}_\alpha)}$, for $n = 15$ and $n = 20$. These values readily reveal that the biased estimate always

Table 7.5.1: Exact bias and MSEs (in parentheses) for biased and unbiased estimates of quantile Q_α for different choices of α when $n = 15$. Here, $RE = 100 \times \frac{MSE(\tilde{Q}_\alpha)}{MSE(\hat{Q}_\alpha)}$.

	r				
	3	6	9	12	15
$\hat{Q}_{0.10}$	0.0223	0.1121	0.1180	0.0789	0.0630
	(0.4378)	(0.3993)	(0.3648)	(0.3075)	(0.2661)
$\tilde{Q}_{0.10}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.4478)	(0.4411)	(0.3939)	(0.3230)	(0.2759)
RE	1.0229	1.1047	1.0799	1.0503	1.0369
$\hat{Q}_{0.25}$	-0.2831	-0.0374	0.0508	0.0340	0.0271
	(0.4218)	(0.1526)	(0.1427)	(0.1355)	(0.1278)
$\tilde{Q}_{0.25}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.4239)	(0.1458)	(0.1478)	(0.1383)	(0.1297)
RE	1.0049	0.9554	1.0360	1.0211	1.0142
$\hat{Q}_{0.50}$	-0.5142	-0.1505	0.0000	0.0000	0.0000
	(0.7816)	(0.1537)	(0.0963)	(0.0963)	(0.0963)
$\tilde{Q}_{0.50}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.9635)	(0.1481)	(0.0963)	(0.0963)	(0.0963)
RE	1.2328	0.9634	1.0000	1.0000	1.0000
$\hat{Q}_{0.75}$	-0.7452	-0.2636	-0.0508	-0.0340	-0.0271
	(1.4616)	(0.3166)	(0.1549)	(0.1356)	(0.1278)
$\tilde{Q}_{0.75}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(1.9836)	(0.3448)	(0.1610)	(0.1385)	(0.1297)
RE	1.3571	1.0892	1.0394	1.0212	1.0142
$\hat{Q}_{0.90}$	-1.0506	-0.4131	-0.1180	-0.0789	-0.0630
	(2.8522)	(0.7800)	(0.3932)	(0.3078)	(0.2661)
$\tilde{Q}_{0.90}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(4.0695)	(0.9032)	(0.4246)	(0.3233)	(0.2759)
RE	1.4268	1.1579	1.0799	1.0503	1.0369

has a smaller MSE.

Upon using the conditional distribution of $\frac{r(\hat{Q}^* - \mu)}{\sigma}$ presented in Theorem 7.5.1, we obtain explicit expressions for the exact CDF as follows.

Theorem 7.5.3 *Let W denote the variable $\frac{r(\hat{Q}^* - \mu)}{\sigma}$ and W_d denote the variable $\frac{r(\hat{Q}^* - \mu)}{\sigma}$, conditioned on $D = d$. Then, the CDF of W can be expressed as*

$$F(w) = \sum_{d=0}^{n} P(W_d \leq w) P(D = d), \tag{7.5.11}$$

where the CDF of W_d can be obtained by the use of Lemmas 7.4.2-7.4.7.

Table 7.5.2: Exact bias and MSEs (in parentheses) for biased and unbiased estimates of quantile Q_α for different choices of α when $n = 20$. Here, $RE = 100 \times \frac{MSE(\tilde{Q}_\alpha)}{MSE(\hat{Q}_\alpha)}$.

	r				
	5	10	11	12	20
$\hat{Q}_{0.10}$	0.0552	0.0796	0.1001	0.0803	0.0438
	(0.3100)	(0.2928)	(0.2815)	(0.2728)	(0.1947)
$\tilde{Q}_{0.10}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.3279)	(0.3117)	(0.3008)	(0.2883)	(0.1999)
RE	1.0578	1.0645	1.0687	1.0567	1.0269
$\hat{Q}_{0.25}$	-0.1281	0.0008	0.0431	0.0346	0.0189
	(0.1891)	(0.1051)	(0.1020)	(0.1031)	(0.0904)
$\tilde{Q}_{0.25}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.1768)	(0.1052)	(0.1053)	(0.1059)	(0.0913)
RE	0.9350	1.0010	1.0323	1.0269	1.0107
$\hat{Q}_{0.50}$	-0.2667	-0.0588	0.0000	0.0000	0.0000
	(0.3207)	(0.0734)	(0.0666)	(0.0666)	(0.0666)
$\tilde{Q}_{0.50}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.3414)	(0.0711)	(0.0666)	(0.0666)	(0.0666)
	1.0644	0.9692	1.0000	1.0000	1.0000
$\hat{Q}_{0.75}$	-0.4053	-0.1184	-0.0431	-0.0346	-0.0189
	(0.6446)	(0.1367)	(0.1177)	(0.1092)	(0.0904)
$\tilde{Q}_{0.75}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.7462)	(0.1423)	(0.1220)	(0.1123)	(0.0913)
RE	1.1577	1.0409	1.0368	1.0284	1.0107
$\hat{Q}_{0.90}$	-0.5885	-0.1972	-0.1001	-0.0803	-0.0438
	(1.3676)	(0.3662)	(0.3179)	(0.2870)	(0.1947)
$\tilde{Q}_{0.90}$	0.0000	0.0000	0.0000	0.0000	0.0000
	(1.6501)	(0.3979)	(0.3397)	(0.3032)	(0.1999)
RE	1.2065	1.0864	1.0684	1.0565	1.0269

Suppose (W_L, W_U) is an interval such that $P(W_L \leq W \leq W_U) = 1 - \alpha$. Then, an exact $100(1-\alpha)\%$ CI for Q, denoted by (Q_L, Q_U), can be determined as follows:

$$Q_L = \frac{\sigma W_L}{r} + \mu \qquad \text{and} \qquad Q_U = \frac{\sigma W_U}{r} + \mu. \tag{7.5.12}$$

Since, the underlying parameters will be unknown and need to be estimated, we replace (μ, σ) by the unbiased estimate $(\hat{\mu}^*, \hat{\sigma}^*)$. These CIs can then be utilized to determine bounds in a Quantile-Quantile plot (Q-Q plot), for example.

In Table 7.5.3, we have presented the exact CIs based on biased and bias-corrected MLEs of quantiles when $n = 10$, by taking $\mu = 0$ and $\sigma = 1$. With the values of r taken as 4(1)7 and 10, we computed the 95% CIs for 10%, 25%, 70%, 75% and 90% quantiles. The performance of these two CIs are then compared by the RE, defined as $RE = 100 \times \frac{\text{Width of CI based on the unbiased MLEs}}{\text{Width of CI based on the biased MLEs}}$, and these values are also given in Table 7.5.3. These results reveal once again that the biased estimates in general lead to CI with smaller width, especially when r is small.

7.6 MLE of the reliability function

Balakrishnan and Chandramouleeswaran (1996) discussed the estimation of the reliability function by the use of BLUEs. Motivated by their work, we will consider here the MLE of the reliability function. Based on $(\hat{\mu}, \hat{\sigma})$ or $(\hat{\mu}^*, \hat{\sigma}^*)$, a natural estimator

Table 7.5.3: Exact 95% CIs based on biased and unbiased estimates of quantiles, and $RE = 100 \times \frac{\text{Width of CI based on } \tilde{Q}_\alpha}{\text{Width of CI based on } \hat{Q}_\alpha}$.

	r				
	4	5	6	7	10
$\hat{Q}_{0.10}$	(-3.1478, -0.2922)	(-3.1299, -0.2852)	(-3.0601, -0.3075)	(-3.0606, -0.3583)	(-2.9116, -0.4654)
$\tilde{Q}_{0.10}$	(-3.5123, -0.3568)	(-3.4648, -0.3540)	(-3.3926, -0.3882)	(-3.2964, -0.4246)	(-3.0533, -0.5186)
RE	1.1051	1.0936	1.0915	1.0627	1.0362
$\hat{Q}_{0.25}$	(-1.8474, 0.0538)	(-1.7223, 0.1176)	(-1.6076, 0.1720)	(-1.6391, 0.1620)	(-1.6021, 0.1418)
$\tilde{Q}_{0.25}$	(-1.7664, 0.0933)	(-1.7222, 0.1177)	(-1.7303, 0.1165)	(-1.7278, 0.1189)	(-1.6558, 0.1104)
RE	0.9781	1.0000	1.0377	1.0253	1.0127
$\hat{Q}_{0.50}$	(-1.2602, 0.5807)	(-0.9830, 0.6448)	(-0.7748, 0.7748)	(-0.7757, 0.7757)	(-0.7757, 0.7757)
$\tilde{Q}_{0.50}$	(-1.0870, 0.9609)	(-0.8681, 0.8030)	(-0.7748, 0.7748)	(-0.7757, 0.7757)	(-0.7757, 0.7757)
RE	1.1124	1.0266	1.0000	1.0000	1.0000
$\hat{Q}_{0.75}$	(-0.8980, 1.5580)	(-0.5388, 1.5105)	(-0.2590, 1.6793)	(-0.1825, 1.6529)	(-0.1418, 1.6021)
$\tilde{Q}_{0.75}$	(-0.6920, 2.4264)	(-0.3903, 1.9782)	(-0.2099, 1.8066)	(-0.1407, 1.7421)	(-0.1104, 1.6558)
RE	1.2697	1.1558	1.0403	1.0258	1.0127
$\hat{Q}_{0.90}$	(-0.5600, 3.1042)	(-0.1382, 2.9871)	(0.1889, 3.1575)	(0.3277, 3.0775)	(0.4653, 2.9116)
$\tilde{Q}_{0.90}$	(-0.3249, 4.5485)	(0.0382, 3.8487)	(0.2682, 3.4918)	(0.3931, 3.3133)	(0.5186, 3.0533)
RE	1.3300	1.2192	1.0859	1.0620	1.0362

of the reliability at mission time t can be proposed as

$$\hat{S}(t) = \begin{cases} 1 - \frac{1}{2}e^{-\left(\frac{\hat{\mu}+b\hat{\sigma}-t}{a\hat{\sigma}}\right)} & \text{if } t < \hat{\mu} + b\hat{\sigma}, \\ \frac{1}{2}e^{-\left(\frac{t-\hat{\mu}-b\hat{\sigma}}{a\hat{\sigma}}\right)} & \text{if } t \geq \hat{\mu} + b\hat{\sigma}, \end{cases} \tag{7.6.1}$$

where a and b are as defined in (7.5.3). The distribution function of $\hat{S}(t)$ can then be obtained as

$$P\left(\hat{S}(t) \leq s\right) = \begin{cases} P\left(\hat{\mu} + \left(a\log(2(1-s)) + b\right)\hat{\sigma} \leq t\right) & \text{if } t < \hat{\mu} + b\hat{\sigma}, s < \frac{1}{2}, \\ P\left(\hat{\mu} + (b - a\log 2s)\hat{\sigma} \leq t\right) & \text{if } t \geq \hat{\mu} + b\hat{\sigma}, s \geq \frac{1}{2}, \end{cases} \tag{7.6.2}$$

where $\hat{\mu} + \left(a\log(2(1-s)) + b\right)\hat{\sigma}$ and $\hat{\mu} + (b - a\log 2s)\hat{\sigma}$ are both linear combinations of the order statistics and so they follow the generalized Hypoexponential distribution

as shown in Theorem 7.4.3. To develop an exact equi-tailed $100(1-\alpha)\%$ CI for $\hat{S}(t)$ is equivalent to finding an s such that

$$\begin{cases} P\left(\hat{\mu} + (a\log(2(1-s)) + b)\,\hat{\sigma} \leq t\right) = \frac{\alpha}{2} & \text{if } t < \hat{\mu} + b\hat{\sigma}, s < \frac{1}{2}, \\ P\left(\hat{\mu} + (b - a\log 2s)\hat{\sigma} \leq t\right) = \frac{\alpha}{2} & \text{if } t \geq \hat{\mu} + b\hat{\sigma}, s \geq \frac{1}{2}, \end{cases} \tag{7.6.3}$$

and

$$\begin{cases} P\left(\hat{\mu} + (a\log(2(1-s)) + b)\,\hat{\sigma} \leq t\right) = 1 - \frac{\alpha}{2} & \text{if } t < \hat{\mu} + b\hat{\sigma}, s < \frac{1}{2}, \\ P\left(\hat{\mu} + (b - a\log 2s)\hat{\sigma} \leq t\right) = 1 - \frac{\alpha}{2} & \text{if } t \geq \hat{\mu} + b\hat{\sigma}, s \geq \frac{1}{2}. \end{cases} \tag{7.6.4}$$

These CIs can then be used, for example, to provide bounds for the Kaplan-Meier curve (K-M curve) or the Probability-Probability plot (P-P plot).

7.7 MLE of the cumulative hazard function

The cumulative hazard function is defined as

$$\Lambda(t) = -\ln(S(t)). \tag{7.7.1}$$

Based on $(\hat{\mu}, \hat{\sigma})$ or $(\hat{\mu}^*, \hat{\sigma}^*)$, the natural estimator for the cumulative hazard function at mission time t is then

$$\hat{\Lambda}(t) = -\ln(\hat{S}(t)), \tag{7.7.2}$$

where $\hat{S}(t)$ is as defined in Eq. (7.6.1). So, the distribution function of $\hat{\Lambda}(t)$ can be readily expressed as

$$P(\hat{\Lambda}(t) \leq h) = P(\hat{S}(t) \geq e^{-h}) = 1 - P(\hat{S}(t) < e^{-h}). \tag{7.7.3}$$

Consequently, if an exact equi-tailed $100(1-\alpha)\%$ CI for $\hat{S}(t)$ is (s_l, s_u), then an exact equi-tailed $100(1-\alpha)\%$ CI for $\hat{\Lambda}(t)$ becomes $(-\log(s_u), -\log(s_l))$.

7.8 BLUEs

Let $X_{i:n}$ denote the i-th order statistic from a sample of size n from $L(\mu, \sigma)$. Furthermore, let

$$\begin{aligned}
\boldsymbol{X} &= (X_{1:n}, X_{2:n}, \cdots, X_{r:n})^T, \\
\boldsymbol{\alpha} &= (\alpha_{1:n}, \alpha_{2:n}, \cdots, \alpha_{r:n})^T, \\
\boldsymbol{\Sigma} &= ((\sigma_{ij})), \qquad 1 \leq i, j \leq r,
\end{aligned}$$

where $\boldsymbol{\alpha}$ is the mean vector and $\boldsymbol{\Sigma}$ is the variance-covariance matrix of $\frac{\mathbf{X}-\mu\mathbf{1}}{\sigma}$. Then, the BLUEs of μ and σ are given by

$$\tilde{\mu} = \left\{ \frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}\mathbf{1}'\boldsymbol{\Sigma}^{-1} - \boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2} \right\} \mathbf{X} = \sum_{i=1}^{r} a_i X_{i:n} \tag{7.8.1}$$

and

$$\tilde{\sigma} = \left\{ \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1} - \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2} \right\} \mathbf{X} = \sum_{i=1}^{r} b_i X_{i:n}. \tag{7.8.2}$$

Moreover, the variances and covariance of these BLUEs are given by

$$\begin{aligned} Var(\tilde{\mu}) &= \sigma^2 \left\{ \frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2} \right\}, && (7.8.3) \\ Var(\tilde{\sigma}) &= \sigma^2 \left\{ \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2} \right\}, && (7.8.4) \\ Cov(\tilde{\mu}, \tilde{\sigma}) &= -\sigma^2 \left\{ \frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2} \right\}; && (7.8.5) \end{aligned}$$

for more elaborate details, one may refer to the books by Arnold et al. (1992) and Balakrishnan and Cohen (1991). Now, by applying Theorem 7.4.3, it can be shown that the BLUEs follow generalized Hypoexponential distributions. One may also present estimation for quantile, reliability function and the cumulate hazard function based on BLUEs, and develop exact CIs for all these quantities as well. Since the procedure is analogous to the one based on MLEs described above, we omit these details for the sake of brevity.

7.9 Illustrative examples

In this section, we illustrate the results established in the preceding sections with two sets of data taken from the reliability literature.

Example 7.9.1 *The data presented in Table A.1.6, given by Mann and Fertig (1973), are the lifetimes of 13 aeroplane components with the last 3 lifetimes of components having been censored. Here, we analyze these data by assuming a Laplace distribution and computed the MLEs based on Type-II censored samples with different degrees of censoring, and also the MSEs and the correlation coefficient of the MLEs based on the exact formulae derived in Section 7.3. All these results are presented in Table 7.9.1. Note that we have used bias-adjusted MLEs of* μ *and* σ*, denoted by* $\tilde{\mu}$ *and* $\tilde{\sigma}$*, in place of* μ *and* σ *to compute the* $MSE(\hat{\mu})$ *and* $MSE(\hat{\sigma})$ *values, and the CIs for* μ *and* σ*. The* 95% *CIs, constructed from the CDFs derived in Section 7.4, are presented in Table 7.9.2, and are compared with the results from Monte Carlo simulations. The figures of the exact CDF of* $\frac{\hat{\sigma}}{\sigma}$ *are presented in Figure 7.9.1. These CIs are also incorporated into the Q-Q plot, presented in Figure 7.9.2. The K-M curve, with the exact* 95% *confidence bounds, are presented in Figure 7.9.3. These figures do reveal that the Laplace model provides a good fit to these data.*

Example 7.9.2 *The data presented in Table A.1.7, given by Bain and Engelhardt (1973), are 33 values of differences in flood levels between Wrightstown and Berlin, obtained originally by Gumbel and Mustafi (1967). Bain and Engelhardt (1973) showed*

Table 7.9.1: MLEs of the parameters based on data in Table A.1.6 and their MSEs and correlation coefficient based on the exact formulae.

r	$\hat{\mu}$	$\tilde{\mu}$	$\hat{\sigma}$	$\tilde{\sigma}$	$\frac{MSE(\hat{\mu})}{\sigma^2}$	$MSE(\hat{\mu})$	$\frac{MSE(\hat{\sigma})}{\sigma^2}$	$MSE(\hat{\sigma})$	$\widehat{Corr(\hat{\mu},\hat{\sigma})}$
6	1.3664	1.4322	0.4550	0.5391	0.1402	0.0408	0.1647	0.0479	0.1366
7	1.5400	1.5400	0.5700	0.6433	0.1138	0.0471	0.1402	0.0580	0.1319
10	1.5400	1.5400	1.1010	1.1711	0.1138	0.1560	0.0975	0.1338	0.0039

Table 7.9.2: Exact and simulated 95% CIs for μ and σ based on data in Table A.1.6.

	Exact results		Simulated results	
r	μ	σ	μ	σ
6	(1.0177, 1.7880)	(0.1493, 0.9279)	(1.0147, 1.7862)	(0.1500, 0.9299)
7	(1.0976, 1.9824)	(0.2124, 1.0997)	(1.0985, 1.9794)	(0.2120, 1.0955)
10	(0.7346, 2.3454)	(0.5135, 1.9078)	(0.7362, 2.3398)	(0.5145, 1.9031)

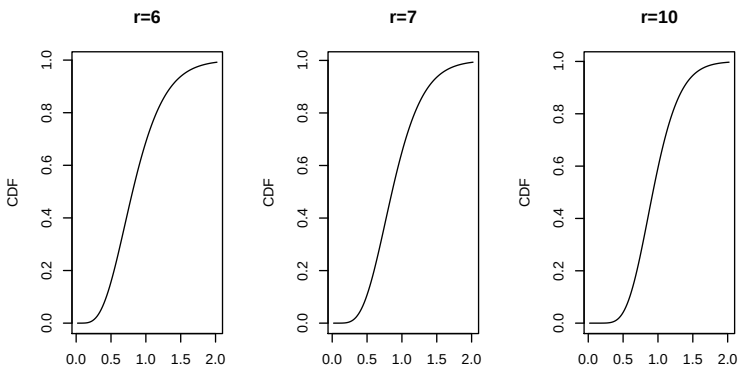

Figure 7.9.1: CDF of $\frac{\hat{\sigma}}{\sigma}$ for Example 7.9.1.

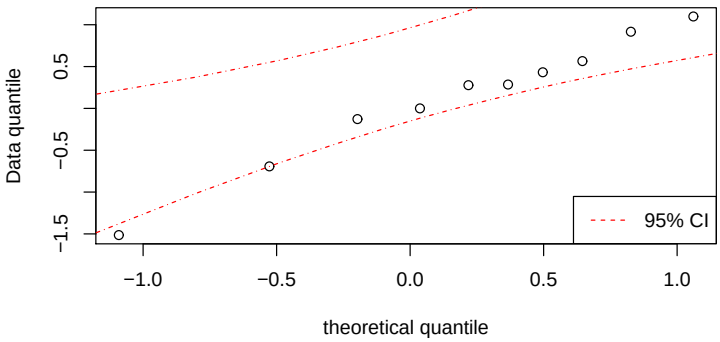

Figure 7.9.2: Q-Q plot with 95% confidence bounds for data in Table A.1.6.

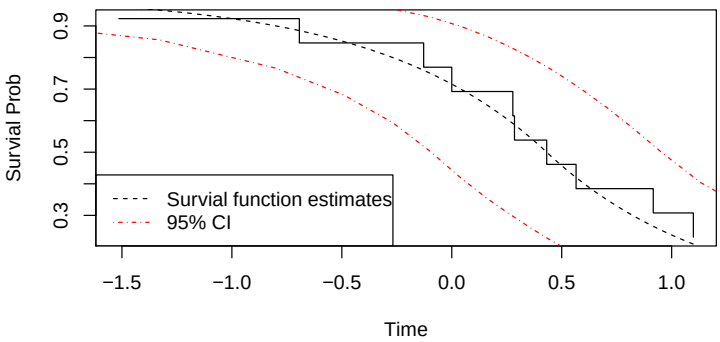

Figure 7.9.3: K-M curve and the estimated survival function with 95% confidence bounds for the data in Table A.1.6.

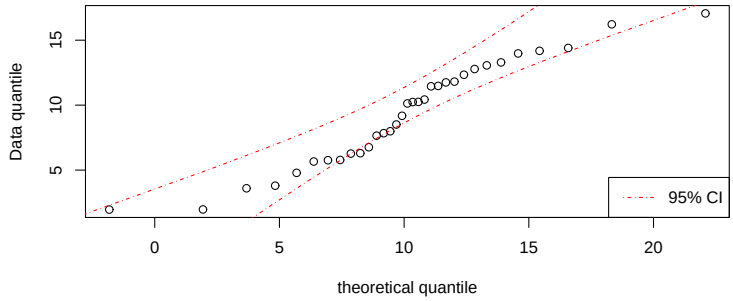

Figure 7.9.4: Q-Q plot with 95% confidence bounds for data in Table A.1.7.

that the Laplace model provides a good fit for this data set. For illustrative purpose, we analyze these data with the choice of $r = 10$, 20, *and* 33. *We obtained the MLEs and the MLEs of quantiles with their estimated bias and MSEs, and the corresponding* 95% *CIs, and these results are presented in Table 7.9.3. Here again, the bias-adjusted MLEs* $\hat{\mu}^*$ *and* $\hat{\sigma}^*$ *are used to compute the estimates of bias and MSEs. Puig and Stephens (2000) discussed a formal goodness-of-fit test for the Laplace model based on these data, and concluded that the Laplace model provides a very good fit for the complete data. Next, the Q-Q plot and K-M curve are presented in Figures 7.9.4 and 7.9.5 based on the complete sample, and the corresponding* 95% *confidence bounds. These figures also do not reject the Laplace assumption for these data.*

Table 7.9.3: MLEs of quantiles and estimates of their bias and MSEs, and 95% CIs based on the data in Table A.1.7.

r	Parameter	MLE	$\widehat{bias}$	$\widehat{MSE}$	95% CI
10	μ	7.1568	-0.2204	0.4026	(5.9169, 8.2536)
	σ	1.7110	-0.1900	0.3614	(0.7803, 3.0020)
	$Q_{0.10}$	4.4031	0.1625	0.6794	(2.6183, 5.8361)
	$Q_{0.25}$	5.9709	-0.1685	0.2923	(4.7985, 6.8929)
	$Q_{0.50}$	7.1568	-0.4190	0.4027	(5.9169, 8.2536)
	$Q_{0.75}$	8.3428	-0.6694	0.8604	(6.7355, 10.1279)
	$Q_{0.90}$	9.9106	-1.0004	1.9986	(7.6233, 12.7866)
20	μ	10.1300	0.0000	0.5790	(8.6352, 11.6250)
	σ	3.6345	-0.1085	0.6936	(2.2028, 5.4321)
	$Q_{0.10}$	4.2805	0.6533	2.3243	(0.9787, 6.9079)
	$Q_{0.25}$	7.6108	0.2814	0.8737	(5.6159, 9.2938)
	$Q_{0.50}$	10.1300	0.0000	0.5502	(8.6352, 11.6250)
	$Q_{0.75}$	12.6492	-0.2814	0.8931	(10.9306, 14.6688)
	$Q_{0.90}$	15.9795	-0.6533	2.3693	(13.2914, 19.2864)
33	μ	10.1300	0.0000	0.4829	(8.7649, 11.4953)
	σ	3.3606	-0.0574	0.3513	(2.3055, 4.6086)
	$Q_{0.10}$	4.7213	0.3159	1.3688	(2.2394, 6.8189)
	$Q_{0.25}$	7.8006	0.1361	0.6276	(6.1323, 9.2787)
	$Q_{0.50}$	10.1300	0.0000	0.4588	(8.7649, 11.4953)
	$Q_{0.75}$	12.4594	-0.1361	0.6276	(10.9823, 14.1241)
	$Q_{0.90}$	15.5387	-0.3159	1.3688	(13.4419, 18.0034)

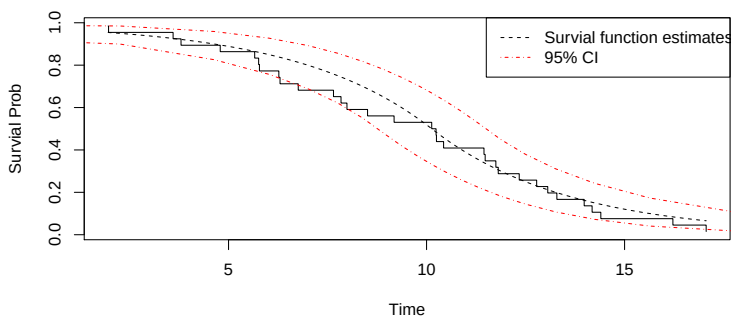

Figure 7.9.5: K-M curve and the estimated survival function with 95% confidence bounds for the data in Table A.1.7.

Chapter 8

Likelihood inference for Laplace model under Type-I censoring

8.1 Introduction

As mentioned in the last chapter, several inferential results have been developed for Laplace distribution based on complete, Type-II and progressively Type-II censored samples and no work has been developed for the case when the life-testing experiment is time-constrained resulting in a Type-I censored sample. For this reason, we will first derive here the MLEs of Laplace parameters based on Type-I censored samples. Then, we will derive conditional marginal and joint distributions of the MLEs, from which we obtain bias, MSEs, and covariance of the estimates. We also develop exact conditional CIs from the exact conditional distributions of the MLEs.

The rest of this chapter is organized as follows. In Section 8.2, we present closed-form expressions of the MLEs based on Type-I right censored samples for different censoring cases. In Section 8.3, we derive the conditional joint MGF of the MLEs, and use it to obtain the mean, variance and covariance of the MLEs. The exact conditional marginal and joint density functions of the MLEs are derived in Section 8.4, which are then used to develop exact conditional marginal and joint CIs for μ and σ. A Monte Carlo simulation study is carried out in Section 8.5 to evaluate the performance of thest point and interval estimates. Finally, in Section 8.6, we present two examples to illustrate all the methods of inference developed here. Through out this paper, we use "$E(\theta)$" to denote a more general exponential distribution, by allowing the scale parameter to be any real value. Of course, this would mean that for example, if $X \sim E(\theta)$ and $\theta < 0$, then $-X \sim E(-\theta)$. Moreover, we use "$NE(\theta)$" and "$N\Gamma(\alpha, \beta)$" with a positive scale parameter to denote the negative exponential and negative gamma distributions, i.e., if $X \sim NE(\theta)$, then $-X \sim E(\theta)$.

8.2 MLEs based on Type-I right censored samples

Let $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ denote the ordered lifetimes of n units under a life-test. Suppose the experimenter decides to conduct a life-testing experiment until a pre-fixed time T and terminate the experiment at time T. The data so observed will be $(x_{1:n} < x_{2:n} < \cdots < x_{D:n})$, where D is the random number of units that fail until time T, commonly referred to as a Type-I censored sample. Then, we have the

corresponding likelihood function as [see Balakrishnan and Cohen (1991) and Arnold et al. (1992)]

$$L = C_d \prod_{i=1}^{d} f(x_{i:n})[S(T)]^{n-d}, \qquad -\infty < x_{1:n} < x_{2:n} < \cdots < x_{d:n} < T, \tag{8.2.1}$$

where $C_d = \frac{n!}{(n-d)!}$ and $S(\cdot) = 1 - F(\cdot)$ is the survival function. First of all, it is clear that the MLEs of μ and σ exist only when $D \geq 1$, and so all subsequent results developed here are based on this condition of observing at least one failure.

Theorem 8.2.1 *By maximizing the likelihood function in (8.2.1), the MLEs of μ and σ can be derived as follows:*

$$\hat{\mu} = \begin{cases} [X_{m:n}, X_{m+1:n}], & n = 2m, d \geq m+1, \\ X_{m+1:n}, & n = 2m+1, d \geq m+1, \\ [X_{m:n}, T], & n = 2m, d = m, \\ T + \hat{\sigma}\log(\frac{n}{2d}), & d < \frac{n}{2}; \end{cases} \tag{8.2.2}$$

$$\hat{\sigma} = \begin{cases} \frac{1}{d}\left[(n-d)T + \sum\limits_{i=m+1}^{d} X_{i:n} - \sum\limits_{i=1}^{m} X_{i:n}\right], & n = 2m, d \geq m, \\ \frac{1}{d}\left[(n-d)T + \sum\limits_{i=m+2}^{d} X_{i:n} - \sum\limits_{i=1}^{m} X_{i:n}\right], & n = 2m+1, d \geq m+1, \\ \frac{1}{d}\sum\limits_{i=1}^{d}\left(T - X_{i:n}\right), & d < \frac{n}{2}. \end{cases} \tag{8.2.3}$$

Proof See Appendix. ■

We observe that in (8.2.2), in some cases, $\hat{\mu}$ can be any value in a specific interval with equal likelihood. In these cases, as done usually, we take the mid-points of the intervals to obtain

$$\hat{\mu} = \begin{cases} \frac{1}{2}(X_{m:n} + X_{m+1:n}) & n = 2m, d \geq m+1, \\ X_{m+1:n} & n = 2m+1, d \geq m+1, \\ \frac{1}{2}(X_{m:n} + T) & n = 2m, d = m, \\ T + \hat{\sigma}\log(\frac{n}{2d}) & d < \frac{n}{2}. \end{cases} \tag{8.2.4}$$

8.3 Exact conditional MGF of the MLEs

In this section, we derive the exact conditional marginal and joint MGFs of $(\hat{\mu}, \hat{\sigma})$ conditioned on observing at least one failure for all the cases presented in the preceding section, and then determine the bias and MSEs of $\hat{\mu}$ and $\hat{\sigma}$ from these expressions. We describe in detail the procedure for obtaining the conditional marginal MGF of $\hat{\sigma}$ only for the case when $n = 2m$, $d \geq m+1$, and then just present the final results for all other cases for the sake of brevity since their derivations are similar.

8.3.1 Even sample size

Due to the two forms of density function in (1.2.1), we need to consider the cases of $\mu \geq T$ and $\mu < T$ separately. In the case when $\mu < T$, we will use the conditional approach and let J $(0 \leq J \leq D)$ denote the number of observations in the Type-I

censored data that are smaller than μ.

Theorem 8.3.1 *The MGF of $\hat{\sigma}$, conditional on observing at least one failure, is as follows:*

$$\begin{aligned} E(e^{t\hat{\sigma}}|D>0) &= 1_{\{T>\mu\}}\left\{\sum_{d=1}^{m}\sum_{j=0}^{d}\sum_{l=0}^{d-j} p_1 M_{Z_1^{(1)}}(t) + \sum_{d=m+1}^{n}\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{d-m-1} p_{7\sigma} M_{Z_\sigma^{(7)}}(t)\right. \\ &\quad + \sum_{d=m+1}^{n}\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{d-m-1} p_{8\sigma} M_{Z_\sigma^{(8)}}(t) \\ &\quad \left. + \sum_{d=m+1}^{n}\sum_{j=m+1}^{d}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{d-j} p_{11} M_{Z_{1,e}^{(11)}}(t)\right\} \\ &\quad + 1_{\{T\le\mu\}}\left\{\sum_{d=1}^{m} q_1 M_{Z_1^{(12)}}(t) + \sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1} q_3 M_{Z_{1,e}^{(14)}}(t)\right\}, \end{aligned} \tag{8.3.1}$$

where $M_X(t)$ denotes the MFG of X and the coefficients and variables involved are all as presented in Appendix.

Proof See Appendix. ■

Theorem 8.3.2 *The conditional marginal MGF of $\hat{\mu}$ is given by*

$$\begin{aligned}
E(e^{s\hat{\mu}}|D>0) &= 1_{\{T>\mu\}}\left\{\sum_{d=1}^{m-1}\sum_{j=0}^{d}\sum_{l=0}^{d-j} p_1 M_{Z_2^{(1)}}(t) + \sum_{l=0}^{m-1}\left[p_{2\mu}M_{Z_2^{(2)}}(t) + p_{3\mu}M_{Z_2^{(3)}}(t)\right]\right. \\
&\quad + p_4 M_{Z_2^{(4)}}(t) + \sum_{d=m+1}^{n}\sum_{l_1=0}^{m-1}\sum_{l_2=0}^{d-m-1}\left[p_{5\mu}M_{Z_2^{(5)}}(t) + p_{6\mu}M_{Z_2^{(6)}}(t)\right. \\
&\quad \left. + p_{7\mu,e}M_{Z_{2,e}^{(7)}}(t) + p_{8\mu,e}M_{Z_{2,e}^{(8)}}(t)\right] \\
&\quad + \sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1}\left[p_9 M_{Z_2^{(9)}}(t) + p_{10}M_{Z_2^{(10)}}(t)\right] \\
&\quad \left. + \sum_{d=m+1}^{n}\sum_{l_1=0}^{d-m-1} p_{11\mu}M_{Z_{2,e}^{(11)}}(t)\right\} \\
&\quad + 1_{\{T\le\mu\}}\left\{\sum_{d=1}^{m-1} q_1 M_{Z_2^{(12)}}(t) + q_2 M_{Z_2^{(13)}}(t)\right. \\
&\quad \left. + \sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1} q_3 M_{Z_{2,e}^{(14)}}(t)\right\}. \qquad (8.3.2)
\end{aligned}$$

For obtaining the exact conditional joint distribution of the MLEs from the above conditional joint MGF, we first need the following lemma.

Lemma 8.3.1 *Let $Y_1 \sim \Gamma(\alpha_1, \beta_1)$, $Y_2 \sim N\Gamma(\alpha_2, \beta_2)$, $Z_1 \sim E(1)$, $Z_2 \sim E(1)$, with Y_1, Y_2, Z_1 and Z_2 being independent, and β_1, $\beta_2 > 0$. Further, let $W_1 = Y_1 + Y_2 + a_1^* Z_1 + a_2^* Z_2$ and $W_2 = b_1^* Z_1 + b_2^* Z_2$. Then, for any real values a_1^*, a_2^*, b_1^* and b_2^*, the joint MGF of W_1 and W_2 is given by*

$$E\left(e^{tW_1+sW_2}\right) = (1-t\beta_1)^{-\alpha_1}(1+t\beta_2)^{-\alpha_2}(1-a_1^*t-b_1^*s)^{-1}(1-a_2^*t-b_2^*s)^{-1}. \qquad (8.3.3)$$

Proof It is readily obtained from the well-known properties of exponential and gamma distributions. ■

The joint and marginal CDFs of W_1 and W_2 are presented in Appendix, while the CDF of $Y_1 + Y_2 + aZ_1$ is presented in (7.4.2)-(7.4.6).

By using Lemma 8.3.1, we readily obtain the following theorem.

Theorem 8.3.3 *The conditional joint MGF of $(\hat{\mu}, \hat{\sigma})$ is as follows:*

$$\begin{aligned}
E(e^{t\hat{\sigma}+s\hat{\mu}}|D>0) &= 1_{\{T>\mu\}}\Bigg\{\sum_{d=1}^{m-1}\sum_{j=0}^{d}\sum_{l=0}^{d-j} p_1 M_{Z_1^{(1)},Z_2^{(1)}}(t,s) \\
&+\sum_{j=0}^{m-1}\sum_{l=0}^{m-1-j}\Big[p_2 M_{Z_1^{(2)},Z_2^{(2)}}(t,s)+p_3 M_{Z_1^{(3)},Z_2^{(3)}}(t,s)\Big] \\
&+p_4 M_{Z_1^{(4)},Z_2^{(4)}}(t,s)+\sum_{d=m+1}^{n}\sum_{j=0}^{m-1}\sum_{l_1=0}^{m-1-j}\sum_{l_2=0}^{d-m-1}\Big[p_5 M_{Z_1^{(5)},Z_2^{(5)}}(t,s) \\
&+p_6 M_{Z_1^{(6)},Z_2^{(6)}}(t,s)+p_{7,e} M_{Z_{1,e}^{(7)},Z_{2,e}^{(7)}}(t,s)+p_{8,e} M_{Z_{1,e}^{(8)},Z_{2,e}^{(8)}}(t,s)\Big] \\
&+\sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1}\Big[p_9 M_{Z_1^{(9)},Z_2^{(9)}}(t,s)+p_{10} M_{Z_1^{(10)},Z_2^{(10)}}(t,s)\Big] \\
&+\sum_{d=m+1}^{n}\sum_{j=m+1}^{d}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{d-j} p_{11} M_{Z_{1,e}^{(11)},Z_{2,e}^{(11)}}(t,s)\Bigg\} \\
&+1_{\{T\le\mu\}}\Bigg\{\sum_{d=1}^{m-1} q_1 M_{Z_1^{(12)},Z_2^{(12)}}(t,s)+q_2 M_{Z_1^{(13)},Z_2^{(13)}}(t,s) \\
&+\sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1} q_3 M_{Z_{1,e}^{(14)},Z_{2,e}^{(14)}}(t,s)\Bigg\}. \qquad (8.3.4)
\end{aligned}$$

8.3.2 Odd sample size

Theorem 8.3.4 *The conditional joint MGF of $(\hat{\mu}, \hat{\sigma})$ is as follows:*

$$\begin{aligned} E(e^{t\hat{\sigma}+s\hat{\mu}}|D>0) &= 1_{\{T>\mu\}}\Bigg\{\sum_{d=1}^{m}\sum_{j=0}^{d}\sum_{l=0}^{d-j} p_1 M_{Z_1^{(1)},Z_2^{(1)}}(t,s) \\ &+ \sum_{d=m+1}^{n}\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{d-m-1}\left[p_{7,o}M_{Z_{1,o}^{(7)},Z_{2,o}^{(7)}}(t,s)+p_{8,o}M_{Z_{1,o}^{(8)},Z_{2,o}^{(8)}}(t,s)\right] \\ &+ \sum_{d=m+1}^{n}\sum_{j=m+1}^{d}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{d-j} p_{11}M_{Z_{1,o}^{(11)},Z_{2,o}^{(11)}}(t,s)\Bigg\} \\ &+1_{\{T\le\mu\}}\Bigg\{\sum_{d=1}^{m} q_1 M_{Z_1^{(12)},Z_2^{(12)}}(t,s) \\ &+ \sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1} q_3 M_{Z_{1,o}^{(14)},Z_{2,o}^{(14)}}(t,s)\Bigg\}. \end{aligned} \tag{8.3.5}$$

The conditional marginal MGFs of $\hat{\mu}$ and $\hat{\sigma}$ are deduced from (8.3.5) by setting $t = 0$ and $s = 0$, respectively. Also, $\hat{\mu}$ can be obtained by setting $t = 0$ for $D < m + 1$ and utilizing the exact distribution of $X_{m+1:n}$ for $D \geq m + 1$. In this approach, we get the following result for $\hat{\mu}$.

Theorem 8.3.5 *The conditional marginal MGF of $\hat{\mu}$ is given by*

$$\begin{aligned} E(e^{s\hat{\mu}}|D>0) &= 1_{\{T>\mu\}}\Bigg\{\sum_{d=1}^{m}\sum_{j=0}^{d}\sum_{l=0}^{d-j} p_1 M_{Z_2^{(1)}}(t) + \sum_{d=m+1}^{n}\sum_{l_1=0}^{d-m-1} p_{11\mu}M_{Z_{2,o}^{(11)}}(t) \\ &+ \sum_{d=m+1}^{n}\sum_{l_1=0}^{m}\sum_{l_2=0}^{d-m-1}\left[p_{7\mu,o}M_{Z_{2,o}^{(7)}}(t)+p_{8\mu,o}M_{Z_{2,o}^{(8)}}(t)\right]\Bigg\} \\ &+1_{\{T\le\mu\}}\Bigg\{\sum_{d=1}^{m} q_1 M_{Z_2^{12}}(t) + \sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1} q_3 M_{Z_{2,o}^{(14)}}(t)\Bigg\}. \end{aligned} \tag{8.3.6}$$

8.4 Exact conditional densities and conditional confidence intervals

From the conditional marginal MGFs of $\hat{\sigma}$ and $\hat{\mu}$, we can readily obtain the conditional density functions of MLEs as presented below.

Theorem 8.4.1 *If the sample size is even (i.e., $n = 2m$), then the MLEs $\hat{\sigma}$ and $\hat{\mu}$ are distributed as follows:*

$$\begin{aligned}\hat{\sigma}|(D>0) \stackrel{d}{=} & \; 1_{\{T>\mu\}}\left\{\sum_{d=1}^{m}\sum_{j=0}^{d}\sum_{l=0}^{d-j} p_1 Z_1^{(1)} + \sum_{d=m+1}^{n}\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{d-m-1} p_{7\sigma} Z_\sigma^{(7)}\right. \\ & \left. + \sum_{d=m+1}^{n}\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{d-m-1} p_{8\sigma} Z_\sigma^{(8)} + \sum_{d=m+1}^{n}\sum_{j=m+1}^{d}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{d-j} p_{11} Z_{1,e}^{(11)}\right\} \\ & + 1_{\{T\le\mu\}}\left\{\sum_{d=1}^{m} q_1 Z_1^{(12)} + \sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1} q_3 Z_{1,e}^{(14)}\right\}, \end{aligned} \tag{8.4.1}$$

$$\begin{aligned}\hat{\mu}|(D>0) \stackrel{d}{=} & \; 1_{\{T>\mu\}}\left\{\sum_{d=1}^{m-1}\sum_{j=0}^{d}\sum_{l=0}^{d-j} p_1 Z_2^{(1)} + \sum_{l=0}^{m-1}\left[p_{2\mu} Z_2^{(2)} + p_{3\mu} Z_2^{(3)}\right] + p_4 Z_2^{(4)}\right. \\ & + \sum_{d=m+1}^{n}\sum_{l_1=0}^{m-1}\sum_{l_2=0}^{d-m-1}\left[p_{5\mu} Z_2^{(5)} + p_{6\mu} Z_2^{(6)} + p_{7\mu,e} Z_{2,e}^{(7)} + p_{8\mu,e} Z_{2,e}^{(8)}\right] \\ & \left. + \sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1}\left[p_9 Z_2^{(9)} + p_{10} Z_2^{(10)}\right] + \sum_{d=m+1}^{n}\sum_{l_1=0}^{d-m-1} p_{11\mu} Z_{2,e}^{(11)}\right\} \\ & + 1_{\{T\le\mu\}}\left\{\sum_{d=1}^{m-1} q_1 Z_2^{(12)} + q_2 Z_2^{(13)} + \sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1} q_3 Z_{2,e}^{(14)}\right\}, \end{aligned} \tag{8.4.2}$$

where $\sum_{i=1}^{n} P_{X_i} X_i$ *denotes the generalized mixture of distributions of variables* $X_1, \cdots,$ X_n *with probabilities* $P_{X_1}, \cdots, P_{X_n}$*, such that* $\sum_{i=1}^{n} P_{X_i} = 1$ *but* P_{X_i}*'s not necessarily being non-negative.*

Theorem 8.4.2 *If the sample size is odd (i.e.,* $n = 2m+1$*), then the MLEs* $\hat{\sigma}$ *and* $\hat{\mu}$ *are distributed as follows:*

$$\begin{aligned}
\hat{\sigma}|(D>0) \stackrel{d}{=} & \; 1_{\{T>\mu\}} \left\{ \sum_{d=1}^{m} \sum_{j=0}^{d} \sum_{l=0}^{d-j} p_1 Z_1^{(1)} \right. \\
& + \sum_{d=m+1}^{n} \sum_{j=0}^{m} \sum_{l_1=0}^{m-j} \sum_{l_2=0}^{d-m-1} \left[p_{7,o} Z_{1,o}^{(7)} + p_{8,o} Z_{1,o}^{(8)} \right] \\
& \left. + \sum_{d=m+1}^{n} \sum_{j=m+1}^{d} \sum_{l_1=0}^{j-m-1} \sum_{l_2=0}^{d-j} p_{11} Z_{1,o}^{(11)} \right\} \\
& + 1_{\{T \le \mu\}} \left\{ \sum_{d=1}^{m} q_1 Z_1^{(12)} + \sum_{d=m+1}^{n} \sum_{l=0}^{d-m-1} q_3 Z_{1,o}^{(14)} \right\}, \qquad (8.4.3)
\end{aligned}$$

$$\begin{aligned}
\hat{\mu}|(D>0) \stackrel{d}{=} & \; 1_{\{T>\mu\}} \left\{ \sum_{d=1}^{m} \sum_{j=0}^{d} \sum_{l=0}^{d-j} p_1 Z_2^{(1)} + \sum_{d=m+1}^{n} \sum_{l_1=0}^{m} \sum_{l_2=0}^{d-m-1} \left[p_{7\mu,o} Z_{2,o}^{(7)} + p_{8\mu,o} Z_{2,o}^{(8)} \right] \right. \\
& \left. + \sum_{d=m+1}^{n} \sum_{l_1=0}^{d-m-1} p_{11\mu} Z_{2,o}^{(11)} \right\} \\
& + 1_{\{T \le \mu\}} \left\{ \sum_{d=1}^{m} q_1 Z_2^{(12)} + \sum_{d=m+1}^{n} \sum_{l=0}^{d-m-1} q_3 Z_{2,o}^{(14)} \right\}. \qquad (8.4.4)
\end{aligned}$$

From Theorems 8.4.1 and 8.4.2, we readily obtain the conditional moments of $\hat{\mu}$ and $\hat{\sigma}$ as well as the product moment. For example, when the sample size is odd, we

have

$$
\begin{aligned}
E(\hat{\sigma}|D>0) &= 1_{\{T>\mu\}}\left\{\sum_{d=1}^{m}\sum_{j=0}^{d}\sum_{l=0}^{d-j}\frac{p_1}{d}\left[(T-\mu)(d-l)+(2j-d)\sigma\right]\right. \\
&+\sum_{d=m+1}^{n}\sum_{j=0}^{m}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{d-m-1}\left[\frac{p_{7,o}}{d}\left((T-\mu)(m-l_2)+(2j+d-n)\sigma\right.\right. \\
&\left.+\frac{(l_2-l_1)\sigma}{l_1+l_2+1}\right) \\
&\left.+\frac{p_{8,o}}{d}\left((T-\mu)(m-l_1)+(2j+d-n)\sigma+\frac{(l_2-l_1)\sigma}{l_1+l_2+1}\right)\right] \\
&+\sum_{d=m+1}^{n}\sum_{j=m+1}^{d}\sum_{l_1=0}^{j-m-1}\sum_{l_2=0}^{d-j}\frac{p_{11}}{d}\left[(T-\mu)(n-d+l_2)+(d-2j+n)\sigma\right. \\
&\left.\left.+\frac{(m-l_1)\sigma}{m+l_1+1}\right]\right\} \\
&+1_{\{T\le\mu\}}\sigma\left\{\sum_{d=1}^{m}q_1+\sum_{d=m+1}^{n}\sum_{l=0}^{d-m-1}\frac{q_3}{d}\left[n-d+\frac{m-l}{l+m+1}\right]\right\}. \quad (8.4.5)
\end{aligned}
$$

By using Lemma 8.3.1, we can also obtain the exact marginal and joint conditional CDFs of $\hat{\mu}$ and $\hat{\sigma}$, which can then be utilized to derive the exact marginal and joint conditional CIs.

8.5 Monte Carlo simulation study

We carried out a Monte Carlo simulation study for $n = 15$, 20 by taking $\mu = 0$ and $\sigma = 1$, without loss of any generality. The values of T were chosen as $-0.10(0.05)0.10$, and then we computed the first, second and product moments of $\hat{\mu}$ and $\hat{\sigma}$ through simulations and also by the use of exact formulae derived in the preceding sections. All

Table 8.5.1: Simulated values of the first, second and product moments of $\hat{\mu}$ and $\hat{\sigma}$ when $\mu = 0$, $\sigma = 1$, with the corresponding exact values within parentheses.

n	T	$\hat{\mu}$	$\hat{\mu}^2$	$\hat{\sigma}$	$\hat{\sigma}^2$	$\hat{\mu}\hat{\sigma}$
15	-0.10	0.0259	0.1646	0.9830	1.1274	0.0780
		(0.0263)	(0.1593)	(0.9826)	(1.1271)	(0.0771)
	-0.05	0.0169	0.1405	0.9774	1.1056	0.0552
		(0.0183)	(0.1402)	(0.9773)	(1.1067)	(0.0574)
	0.00	0.0088	0.1237	0.9707	1.0831	0.0369
		(0.0104)	(0.1244)	(0.9707)	(1.0843)	(0.0396)
	0.05	0.0022	0.1113	0.9644	1.0623	0.0221
		(0.0042)	(0.1129)	(0.9648)	(1.0644)	(0.0257)
	0.10	-0.0008	0.1051	0.9611	1.0492	0.0141
		(0.0008)	(0.1054)	(0.9613)	(1.0505)	(0.0166)
20	-0.10	0.0144	0.1032	0.9892	1.0959	0.0445
		(0.0143)	(0.1014)	(0.9891)	(1.0963)	(0.0432)
	-0.05	0.0079	0.0904	0.9853	1.0811	0.0299
		(0.0077)	(0.0902)	(0.9848)	(1.0810)	(0.0292)
	0.00	0.0009	0.0803	0.9793	1.0629	0.0165
		(0.0012)	(0.0808)	(0.9792)	(1.0634)	(0.0165)
	0.05	-0.0037	0.0734	0.9742	1.0469	0.0071
		(-0.0035)	(0.0739)	(0.9743)	(1.0479)	(0.0072)
	0.10	-0.0057	0.0690	0.9715	1.0367	0.0018
		(-0.0056)	(0.0695)	(0.9716)	(1.0377)	(0.0017)

these results are presented in Table 8.5.1 from which we observe that in all cases, $\hat{\sigma}$ is negatively biased and that the bias decreases when T increases. Moreover, the second moments of both estimators decrease with increasing T. Finally, it is of interest to observe that the exact values are in close agreement with the corresponding simulated values thus validating the accuracy of the derived results.

8.6 Illustrative examples

In this section, we illustrate the results established in the preceding sections with two data sets from the reliability literature.

Example 8.6.1 *The data presented in Table A.1.8, given by Lawless (1982), is a Type-II right-censored sample. Twelve components were placed on a life-test and the experiment was terminated as soon as the eighth failure occurred. For illustrative purpose, we analyze these data here by assuming a Laplace distribution and computed the MLEs based on Type-I censored samples with different choices of* T*, and also the MSEs and the correlation coefficient of the MLEs based on the exact formulae derived in Section 8.3. These results are presented in Table 8.6.1. It needs to be mentioned here that we used the bias-adjusted MLEs of* μ *and* σ*, denoted by* $\tilde{\mu}$ *and* $\tilde{\sigma}$*, in place of* μ *and* σ *to compute* $MSE(\hat{\mu})$ *and* $MSE(\hat{\sigma})$*, the CIs for* μ *and* σ*, as well as the KS distance and P-value, where* $\tilde{\mu}$ *and* $\tilde{\sigma}$ *are the roots of* μ *and* σ *from equations* $\hat{\mu} = E(\hat{\mu})$ *and* $\hat{\sigma} = E(\hat{\sigma})$*. The* 90% *CIs, constructed from the exact conditional CDFs derived in Section 8.4, are presented in Table 8.6.2, and are compared with the results obtained from Monte Carlo simulations. It should be noted that these two sets of results are quite close. Finally, the KS-test is carried out and these results, presented in Table 8.6.3, do not reject the Laplace model assumption made in our analysis of these censored data.*

Example 8.6.2 *Once again, we will use the data presented in Example 7.6.1, by assuming a Laplace distribution and compute the MLEs based on Type-I censored sam-*

Table 8.6.1: MLEs of the parameters based on data in Table A.1.8 and their MSEs and correlation coefficient based on the exact formulas.

T	$\hat{\mu}$	$\tilde{\mu}$	$\hat{\sigma}$	$\tilde{\sigma}$	$\widehat{MSE(\hat{\mu})}$	$\widehat{MSE(\hat{\sigma})}$	$\widehat{Cor(\hat{\mu},\hat{\sigma})}$
400	446.27	438.1556	253.80	257.9485	17560.02	14813.31	0.42
480	475.00	474.86	279.83	290.16	14121.52	15128.61	0.23
580	483.50	486.75	313.71	328.52	12834.54	14979.30	0.06

Table 8.6.2: Exact and simulated 90% CIs for μ and σ based on the data in Table A.1.8.

	Exact results		Simulated results	
T	μ	σ	μ	σ
400	(291.77, 656.22)	(89.79, 472.97)	(291.05, 670.85)	(96.72, 478.09)
480	(310.95, 662.37)	(116.58, 506.36)	(309.98, 657.78)	(118.78, 507.07)
580	(301.31, 663.25)	(150.92, 524.05)	(300.38, 663.25)	(152.70, 540.21)

Table 8.6.3: KS distances and the corresponding P-values for different levels of censoring based on data in Table A.1.8.

T	KS distance	P-value
400	0.1459	0.3245
480	0.1492	0.5367
580	0.1338	0.7025

Table 8.6.4: MLEs of the parameters based on data in Table A.1.6 and their MSEs and correlation coefficient based on the exact formulas.

T	$\hat{\mu}$	$\tilde{\mu}$	$\hat{\sigma}$	$\tilde{\sigma}$	$\widehat{MSE(\hat{\mu})}$	$\widehat{MSE(\hat{\sigma})}$	$\widehat{Cor(\hat{\mu},\hat{\sigma})}$
0.75	1.2097	1.2038	0.3900	0.3900	0.2425	0.0869	0.8625
1.75	1.5400	1.5413	0.7500	0.7868	0.0713	0.0789	0.0727
2.75	1.5400	1.5400	1.1122	1.1525	0.1511	0.1263	0.0170

Table 8.6.5: Exact and simulated 90% CIs for μ and σ based on data in Table A.1.6.

	Exact results		Simulated results	
T	μ	σ	μ	σ
0.75	(0.8039, 2.1377)	(0.0529, 0.9472)	(0.8037, 2.1383)	(0.0532, 0.9386)
1.75	(1.1047, 1.9470)	(0.3715, 1.2279)	(1.1019, 1.9433)	(0.3726, 1.2644)
2.75	(0.9004, 2.1796)	(0.6260, 1.7573)	(0.8962, 2.1807)	(0.6274, 1.7530)

ples with different choices of T. We computed the MSEs and the correlation coefficient of the MLEs based on the exact formulas derived in Section 8.3, and these results are presented in Table 8.6.4. The 90% CIs, constructed from the exact conditional CDFs derived in Section 8.4 as well as those obtained by Monte Carlo simulations, are presented in Table 8.6.5. The two sets of CIs are seen to be quite close once again. Finally, the KS-test is carried out and these results, presented in Table 8.6.6, do not reject the Laplace model assumption made in our analysis of these censored data.

Table 8.6.6: KS distances and the corresponding P-values for different levels of censoring based on data in Table A.1.6.

T	KS distance	P-value
0.75	0.0716	0.5020
1.75	0.0793	0.9350
2.75	0.0903	0.9559

Chapter 9

LBS distribution and associated inferential issues

9.1 Introduction

Generalized Birnbaum-Saunders distribution, written shortly as GBS distribution, was proposed by Díaz-García and Leiva-Sánchez (2005, 2007), Díaz -García and Domínguez-Molina (2006) by using other symmetric distributions in place of the normal kernel, such as Cauchy, Pearson type VII, t, Bessel, Laplace and logistic. In a similar vein, Gómez et al. (2009) proposed the generalized slash BS distribution. Balakrishnan et al. (2007) developed acceptance sampling plans when the life-test is truncated at a pre-fixed time for the GBS distribution. Leiva et al. (2008) and Sanhueza et al. (2008) discussed several properties of this family of distributions and

developed some inferential methods. An R-package for the analysis of GBS distribution has been developed by Barros et al. (2009). In this chapter, we study in detail the GBS distribution based on the Laplace kernel, written shortly as LBS distribution.

The rest of this chapter is organized as follows. In Section 9.2, we briefly describe some basic properties of the LBS distribution. In Section 9.3, we formally show that there are two forms of shape of the PDF of the LBS distribution. In Section 9.4, we then prove that there are three different kinds of shape of the hazard function of the LBS distribution. In Section 9.5, we show that the change points of the hazard function can be determined as a solution of a non-linear equation. In Section 9.6, we discuss the MLEs and formally establish that the MLEs always exist and are unique. In Section 9.7, we develop MMEs, which have simple explicit forms. A Monte Carlo simulation study is then carried out in Section 9.8 to evaluate the performance of these estimators. Finally, in Section 9.9, we present two examples to illustrate all the methods of inference developed here. Through out this chapter, we use "inc" and "dec" to denote increasing and decreasing behaviors, respectively. We will also use BS to denote only the classical Birnbaum-Saunders distribution based on the normal kernel.

9.2 LBS distribution

The BS distribution, based on the Laplace kernel, has its PDF as

$$f(t;\alpha,\beta)=\begin{cases}\frac{1}{4\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]e^{-\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\frac{t}{\beta}}\right)}, & 0<t<\beta,\ \alpha,\beta>0,\\ \frac{1}{4\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]e^{-\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)}, & \beta\le t<\infty,\ \alpha,\beta>0.\end{cases} \tag{9.2.1}$$

The CDF corresponding to (9.2.1) is

$$F(t;\alpha,\beta)=\begin{cases}\frac{1}{2}e^{-\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\frac{t}{\beta}}\right)}, & 0<t<\beta,\ \alpha,\beta>0,\\ 1-\frac{1}{2}e^{-\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)}, & \beta\le t<\infty,\ \alpha,\beta>0.\end{cases} \tag{9.2.2}$$

Property 9.2.1 *The PDF of the LBS distribution in (9.2.1) is continuous but not differentiable at β.*

Proof This is seen easily. ■

As in the case of the BS distribution, we also readily have the following properties for the LBS distribution.

Property 9.2.2 *Suppose $T\sim LBS(\alpha,\beta)$, as defined in (9.2.2). Then:*

1. $\frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}}-\sqrt{\frac{\beta}{T}}\right)\sim Laplace(0,1)$;
2. $cT\sim LBS(\alpha,c\beta)$;
3. $\frac{1}{T}\sim LBS(\alpha,\beta^{-1})$.

Property 9.2.3 *Suppose* $X \sim$ *Laplace*$(0,1)$ *and* $T = \beta\left[\frac{1}{2}\alpha X + \sqrt{1+\left(\frac{1}{2}\alpha X\right)^2}\right]^2$. *Then,* $T \sim LBS(\alpha,\beta)$.

This transformation enables the determination of the moments of T through known results on expectations of the standard Laplace variable presented, for example, in Johnson et al. (1995). For example, we can show that

$$E(T) = \beta(1+\alpha^2), \tag{9.2.3}$$

$$V(T) = (\alpha\beta)^2(2+11\alpha^2), \tag{9.2.4}$$

$$\beta_1(T) = \frac{\alpha^2(326\alpha^2+30)^2}{(2+11\alpha^2)^3}, \tag{9.2.5}$$

$$\beta_2(T) = \frac{18789\alpha^4+1308\alpha^2+24}{(2+11\alpha^2)^2}, \tag{9.2.6}$$

$$E(T^k) = \beta^k \sum_{j=0}^{k}\binom{2k}{2j}\sum_{i=0}^{j}\binom{j}{i}E(X^{k+i-j})\left(\frac{\alpha}{2}\right)^{2(k+i-j)}, \tag{9.2.7}$$

where $X \sim$ *Laplace*$(0,1)$. By using Property 9.2.2, we also readily have

$$E\left(\frac{1}{T}\right) = \beta^{-1}(1+\alpha^2), \tag{9.2.8}$$

$$V\left(\frac{1}{T}\right) = (\alpha\beta^{-1})^2(2+11\alpha^2). \tag{9.2.9}$$

Corollary 9.2.1 *The coefficient of variation (CV) of LBS is less than* $\sqrt{11}$ *and is an inc function of* α.

From Corollary 9.2.1, we can conclude that LBS has more variability than BS, since the CV of BS distribution is known to be less than $\sqrt{5}$. The comparison of

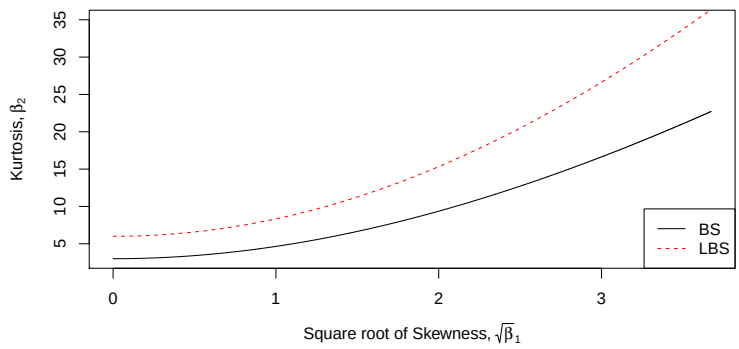

Figure 9.2.1: Comparison of $(\sqrt{\beta_1}, \beta_2)$ between BS and LBS distributions.

$(\sqrt{\beta_1}, \beta_2)$ between BS and LBS distributions displayed in Figure 9.2.1 shows that LBS possesses heavier tails than BS.

9.3 Shape of the density function of LBS

In this section, we will discuss the shape of the density function of LBS. It is easy to see that the PDF of LBS in (9.2.1) is a dec function of t for $t \geq \beta$. For $t < \beta$, let us consider the first derivative of the PDF with respect to t given by

$$\begin{aligned} f'(t) &= \frac{1}{8\alpha\beta^2}\left\{\frac{1}{\alpha}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]^2 - \left[\left(\frac{\beta}{t}\right)^{\frac{3}{2}} + 3\left(\frac{\beta}{t}\right)^{\frac{5}{2}}\right]\right\} \\ &\quad \times \exp\left[-\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}} - \sqrt{\frac{t}{\beta}}\right)\right]. \end{aligned} \tag{9.3.1}$$

The sign of $f'(t)$ depends just on the function

$$g(t) = \frac{1}{\alpha}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]^2 - \left[\left(\frac{\beta}{t}\right)^{\frac{3}{2}} + 3\left(\frac{\beta}{t}\right)^{\frac{5}{2}}\right]. \tag{9.3.2}$$

The left and right limits of $g(t)$ are seen to be $\lim_{t\to 0} g(t) = +\infty$ and $\lim_{t\to\beta} g(t) = \frac{4}{\alpha} - 4$. Now, let us set

$$g_2(t) = \left(\frac{t}{\beta}\right)^3 g(t) = \frac{1}{\alpha}\left[1 + 2\left(\frac{t}{\beta}\right) + \left(\frac{t}{\beta}\right)^2\right] - \left[\left(\frac{t}{\beta}\right)^{\frac{3}{2}} + 3\left(\frac{t}{\beta}\right)^{\frac{1}{2}}\right], \tag{9.3.3}$$

which has its first derivative as

$$g_2'(t) = \frac{2}{\alpha\beta}\left[1 + \left(\frac{t}{\beta}\right)\right] - \frac{3}{2\beta}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right]. \tag{9.3.4}$$

It is easy to see that $g_2'(t)$ is an inc function of t, with $\lim_{t\to 0} g_2'(t) = -\infty$ and $\lim_{t\to\beta} g_2'(t) = \frac{1}{\beta}\left(\frac{4}{\alpha} - 3\right)$. Since the right limits of $g_2'(t)$ and $g(t)$ depend on the value of α, we need to discuss the cases $\alpha \geq \frac{4}{3}$, $1 < \alpha < \frac{4}{3}$ and $0 < \alpha \leq 1$ separately.

Case I: $\alpha \geq \frac{4}{3}$

In the case when $\alpha > \frac{4}{3}$, I have $\lim_{t\to\beta} g_2'(t) < 0$. Since $g_2'(t)$ is an inc function with $\lim_{t\to 0} h'(t) = -\infty$, we readily obtain that $g_2'(t) < 0$ for all $t < \beta$, implying $g_2(t)$ is a dec function of t for all $t < \beta$. Moreover, in this case, $\lim_{t\to\beta} g_2(t) > 0$ and $\lim_{t\to\beta} g_2(t) < 0$, implying that there exists one and only one root of $g_2(t) = 0$ for $t \in (0, \beta)$, say t_0. Now, since $g(t)$ and $g_2(t)$ have the same sign, we know that $f(t)$ is an upside down

function for $t < \beta$. For the special case when $\alpha = \frac{4}{3}$, it can also be shown that $f(t)$ is an upside-down function in a similar way.

Case II: $1 \leq \alpha < \frac{4}{3}$

In this case, it can be shown that $g_2(t)$ is a dec-inc function of t with $\lim_{t \to 0} g_2(t) > 0$ and $\lim_{t \to \beta} g_2(t) < 0$. Then, there exists one and only one root for $g_2(t) = 0$ for $t \in (0, \beta)$. Since $g(t)$ has the same sign as $g_2(t)$, there exists one and only one root for $g(t) = 0$ for $t \in (0, \beta)$. Thus, $f(t)$ is an upside down function for $t < \beta$. In a similar way, it can be shown that for $\alpha = 1$, $f(t)$ is once again an upside down function.

Case III: $\alpha < 1$

In this case also, $g_2(t)$ is a dec-inc function of t with $\lim_{t \to 0} g_2(t) > 0$ and $\lim_{t \to \beta} g_2(t) > 0$. Since $g(t)$ has the same sign as $g_2(t)$, then there may exist none, one or two root(s) for $g(t) = 0$. In the cases of no root and one root, $f(t)$ is a non-dec function of t, while in the case of two roots, $f(t)$ is an inc-dec-inc function of t. The number of roots can be determined by checking the minimum value of $g_2(t)$. Form (9.3.4), we see that $g_2(t)$ takes the minimum value when

$$\alpha = \frac{4\left[1 + \left(\frac{t}{\beta}\right)\right]}{3\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right]} = \frac{4}{3}\left(\frac{t}{\beta}\right)^{\frac{1}{2}}, \quad 0 < t < \frac{9\beta}{16}. \tag{9.3.5}$$

Substituting it into (9.3.3), we get

$$\begin{aligned}\min(g_2(t)) &= \frac{3}{4}\left(\frac{\beta}{t}\right)^{\frac{1}{2}} - \frac{3}{2}\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \frac{1}{4}\left(\frac{t}{\beta}\right)^{\frac{3}{2}} \\ &= -\frac{3}{4}\left(\frac{\beta}{t}\right)^{\frac{1}{2}}\left[\frac{1}{3}\left(\frac{t}{\beta}\right)^{2} + 2\left(\frac{t}{\beta}\right) - 1\right], \qquad (9.3.6)\end{aligned}$$

which is positive if $\frac{t}{\beta} \in (0, 2\sqrt{3}-3)$ and negative if $\frac{t}{\beta} \in (2\sqrt{3}-3, \frac{9}{16})$. Thus, we have the result that if $\alpha < \frac{4}{3}\sqrt{2\sqrt{3}-3}$, $g_2(t)$ has no root, if $\frac{4}{3}\sqrt{2\sqrt{3}-3} < \alpha < 1$, $g_2(t)$ has two roots, and if $\alpha = \frac{4}{3}\sqrt{2\sqrt{3}-3}$, there will be only one root which is $t = (2\sqrt{3}-3)\beta$. From these facts, we see that if $\alpha \leq \frac{4}{3}\sqrt{2\sqrt{3}-3}$, $f(t)$ is an inc function of t for $t \leq \beta$ and is dec after that, and if $\frac{4}{3}\sqrt{2\sqrt{3}-3} < \alpha < 1$, $f(t)$ is an inc-dec-inc function of t for $t \leq \beta$ and is dec after that.

From the above discussion and the findings, we can state the following theorem.

Theorem 9.3.1 *If $\frac{4}{3}\sqrt{2\sqrt{3}-3} < \alpha < 1$, then the PDF in (9.2.1) is an inc-dec-inc-dec function of t; otherwise, it is an upside-down function of t.*

The PDF of the LBS distribution is plotted in Figure 9.3.1 for some values of α, when $\beta = 1$ without loss of any generality. In Figure 9.3.2, we have plotted the PDF for $\alpha = 0.90$, 0.91, 0.99 and 1.00, and a magnified view in some intervals revealing that the critical point of α is in the interval $(\frac{4}{3}\sqrt{2\sqrt{3}-3}, 1)$, thus supporting the result stated in Theorem 9.3.1.

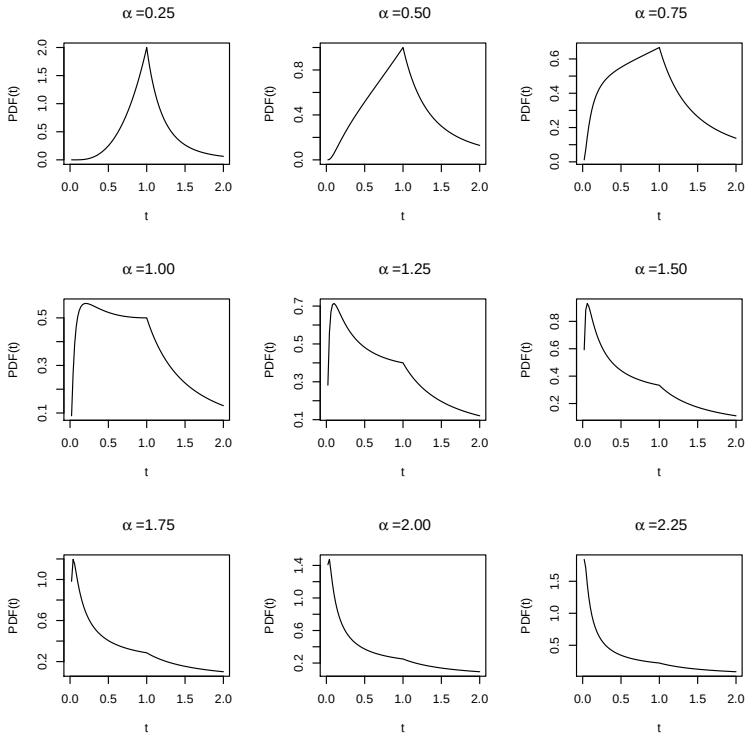

Figure 9.3.1: Plots of the PDF for various values of the shape parameter α.

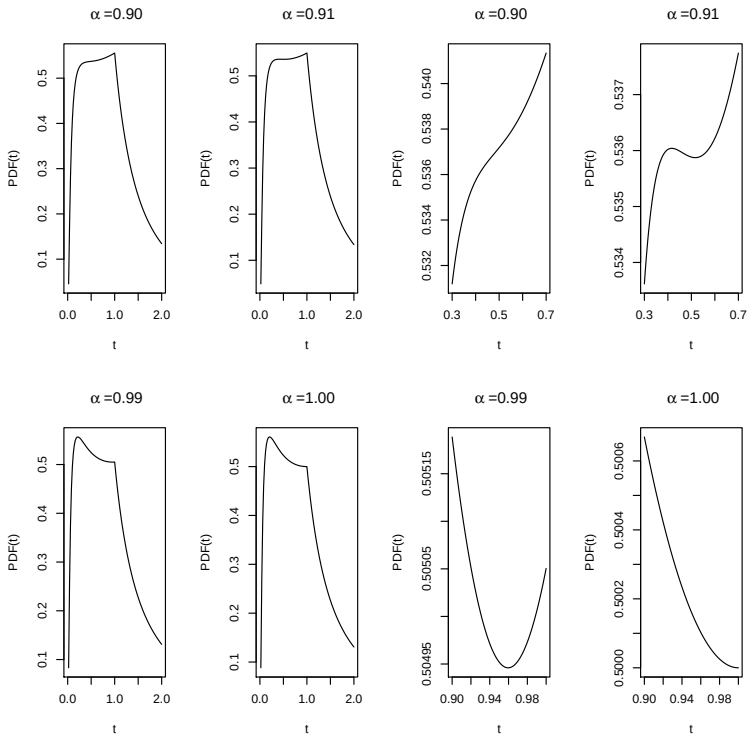

Figure 9.3.2: Plots of the PDF of LBS for the cases $\alpha = 0.90$, 0.91, 0.99 and 1.00 in some intervals.

9.4 Shape characteristics of the hazard function of LBS distribution

It is of interest to mention here that, by using the results of Glaser (1980), Kundu et al. (2008) have shown that the hazard function of the BS distribution is unimodal and discussed further inferential methods for the change point of the hazard function. One may also refer to Bebbington et al. (2008) for some additional discussion in this direction. A more general such analysis has been carried out recently on the hazard function of the BS distribution based on a t-kernel by Azevedo et al. (2012). In this section, we will study the shape of the hazard function of LBS, given by

$$h(t) = \begin{cases} \dfrac{\frac{1}{4\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]e^{-\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\frac{t}{\beta}}\right)}}{1-\frac{1}{2}e^{-\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\frac{t}{\beta}}\right)}}, & 0 < t < \beta, \alpha, \beta > 0, \\ \dfrac{\frac{1}{4\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]e^{-\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)}}{\frac{1}{2}e^{-\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)}}, & \beta \leq t < \infty, \alpha, \beta > 0. \end{cases} \tag{9.4.1}$$

Property 9.4.1 *The hazard function of the LBS is continuous, but is not differentiable at the point $t = \beta$.*

Proof It is easy to prove and is therefore omitted for brevity. ■

Since the PDF in (9.2.1) is non-differentiable, we are unable to use the theorem of Glaser (1980), who developed some basic analysis for the shape of the hazard function when the PDF is continuous and twice-differentiable. Now, to begin with, it is evident that the hazard function in (9.4.1) will take on two different forms when $t < \beta$ and

$t \geq \beta$. For this reason, we need to discuss these two cases separately.

Lemma 9.4.1 *When $t \geq \beta$, $h(t)$ is a dec function of t.*

Proof In this case,

$$h(t) = \frac{1}{2\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right], \tag{9.4.2}$$

which is readily seen to be a dec function of t. ■

To examine the shape of the hazard function for the case when $t < \beta$, let us consider

$$R(t) = \frac{1}{h(t)} = \frac{4\alpha\beta e^{\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\frac{t}{\beta}}\right)}}{\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}} - \frac{2\alpha\beta}{\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}}. \tag{9.4.3}$$

The first derivative of $R(t)$ with respect to t is obtained as

$$\begin{aligned} R'(t) &= -2e^{\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\frac{t}{\beta}}\right)} + \frac{2\alpha\left[\left(\frac{\beta}{t}\right)^{\frac{3}{2}} + 3\left(\frac{\beta}{t}\right)^{\frac{5}{2}}\right]}{\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]^2} e^{\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\frac{t}{\beta}}\right)} - \frac{\alpha\left[\left(\frac{\beta}{t}\right)^{\frac{3}{2}} + 3\left(\frac{\beta}{t}\right)^{\frac{5}{2}}\right]}{\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]^2} \\ &= \frac{2\alpha e^{\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\frac{t}{\beta}}\right)}\left[\left(\frac{\beta}{t}\right)^{\frac{3}{2}} + 3\left(\frac{\beta}{t}\right)^{\frac{5}{2}}\right]}{\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]^2} \\ &\quad \times \left\{-\frac{(1+\frac{t}{\beta})^2}{\alpha\left[\left(\frac{t}{\beta}\right)^{\frac{3}{2}} + 3\left(\frac{t}{\beta}\right)^{\frac{1}{2}}\right]} + \left[1 - \frac{1}{2}e^{\frac{1}{\alpha}\left(\sqrt{t}\beta-\sqrt{\frac{\beta}{t}}\right)}\right]\right\}, \end{aligned}$$

with

$$\lim_{t\to 0} R'(t) = -\infty, \tag{9.4.4}$$

$$\lim_{t\to \beta} R'(t) = \alpha - 2. \tag{9.4.5}$$

Further, let us define

$$M(t) = \left[1 - \frac{1}{2}e^{\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)}\right] - \frac{(1+\frac{t}{\beta})^2}{\alpha\left[\left(\frac{t}{\beta}\right)^{\frac{3}{2}} + 3\left(\frac{t}{\beta}\right)^{\frac{1}{2}}\right]}. \tag{9.4.6}$$

Then, the first derivative of $M(t)$ is given by

$$\begin{aligned} M'(t) &= -\frac{1}{4\alpha\beta}e^{\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right] \\ &\quad - \frac{\left(\frac{t}{\beta}\right)^{\frac{5}{2}} + 7\left(\frac{t}{\beta}\right)^{\frac{3}{2}} + 3\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - 3\left(\frac{t}{\beta}\right)^{-\frac{1}{2}}}{2\alpha\beta\left[\left(\frac{t}{\beta}\right)^{3} + 6\left(\frac{t}{\beta}\right)^{2} + 9\left(\frac{t}{\beta}\right)\right]} \\ &= -\frac{\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}}{4\alpha\beta}\left[e^{\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)} - \frac{24}{\left(\frac{t}{\beta}+3\right)^2} + 2\right], \end{aligned} \tag{9.4.7}$$

with

$$\lim_{t\to 0} M'(t) = +\infty, \tag{9.4.8}$$

$$\lim_{t\to \beta} M'(t) = -\frac{3}{4\alpha\beta}. \tag{9.4.9}$$

Now, let us introduce

$$N(t) = e^{\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)} - \frac{24}{\left(\frac{t}{\beta}+3\right)^2} + 2, \tag{9.4.10}$$

which is readily seen to be an inc function of t. From the results in (9.4.7)-(9.4.10), we observe that there exists only one $t_0 \in (0, \beta)$ such that $M'(t_0) = 0$, and we then have

$$M'(t) \begin{cases} > 0, & 0 < t < t_0, \\ < 0, & t_0 < t < 1, \end{cases} \tag{9.4.11}$$

implying

$$M(t) \begin{cases} \text{inc}, & 0 < t < t_0, \\ \text{dec}, & t_0 < t < \beta. \end{cases} \tag{9.4.12}$$

Moreover,

$$\lim_{t\to 0} M(t) = -\infty, \tag{9.4.13}$$

$$\lim_{t\to \beta} M(t) = \frac{1}{2} - \frac{1}{\alpha}. \tag{9.4.14}$$

Lemma 9.4.2 *If $\alpha \geq 2$, the shape of the hazard function of the LBS distribution is inc-dec for $0 < t < \infty$.*

Proof For the case when $\alpha > 2$, by using the results in (9.4.12)-(9.4.14), we observe that there will exist only one root $t_0 \in (0, \beta)$ for which $M(t)$=0, and

$$M(t) \begin{cases} < 0, & 0 < t < t_0, \\ \\ > 0, & t_0 < t < \beta. \end{cases}$$

Since $R'(t)$ has the same sign as $M(t)$, and $h'(t)$ has the opposite sign as $R'(t)$, we have $h(t)$ to be an inc-dec function for $t < \beta$. For the case when $\alpha = 2$, we have

$$M(t) < 0, \quad 0 < t < \beta.$$

Similarly, we can show that $h(t)$ is an inc function for $t < \beta$. As $h(t)$ is continuous and is a dec function for $t \geq \beta$, the lemma follows. ■

Lemma 9.4.3 *If $\alpha < 1.10$, then the shape of the hazard function is inc-dec with the change point $t = \beta$. If $1.10 < \alpha < 2$, then the shape of the hazard function is a inc-dec-inc-dec function.*

Proof In the case when $\alpha < 2$, with $M(t)$ as defined in (9.4.6), the equation $M(t) = 0$ will have no root, one root or two roots. To determine the exact number of roots is equivalent to checking the sign of the maximum value of $M(t)$. Let t_0 be the unique

root for $M'(t) = 0$ from Eq. (9.4.7). Then, we have the condition

$$e^{\frac{1}{\alpha}\left(\sqrt{\frac{t_0}{\beta}}-\sqrt{\frac{\beta}{t_0}}\right)} + 2 - \frac{24}{\left(\frac{t_0}{\beta}+3\right)^2} = 0, \tag{9.4.15}$$

or

$$\frac{1}{\sqrt{\frac{t_0}{\beta}}-\sqrt{\frac{\beta}{t_0}}} \log\left[\frac{24}{\left(\frac{t_0}{\beta}+3\right)^2} - 2\right] = \frac{1}{\alpha}, \tag{9.4.16}$$

where $t_0 \in (0, (\sqrt{12}-3)\beta)$ so as to make the log-function defined.

Now, substituting (9.4.15) or (9.4.16) into (9.4.6), we obtain

$$L(t_0) \;=\; \max(M(t)) = 2 - \frac{12}{\left(\frac{t_0}{\beta}+3\right)^2} - \frac{\left(1+\frac{t_0}{\beta}\right)^2}{\left(\frac{t_0}{\beta}\right)^2 + 2\left(\frac{t_0}{\beta}\right) - 3} \log\left(\frac{24}{\left(\frac{t_0}{\beta}+3\right)^2} - 2\right),$$

with its first derivative being given by

$$
\begin{aligned}
L'(t_0) &= \frac{24}{\beta(\frac{t_0}{\beta}+3)^3} + \frac{48\left(1+\frac{t_0}{\beta}\right)^2}{\beta\left[24\left(\frac{t_0}{\beta}+3\right)-2\left(\frac{t_0}{\beta}+3\right)^3\right]\left(\frac{t_0}{\beta}+3\right)\left(\frac{t_0}{\beta}-1\right)} \\
&\quad -\frac{2\left(1+\frac{t_0}{\beta}\right)\left[\left(\frac{t_0}{\beta}\right)^2+2\left(\frac{t_0}{\beta}\right)-3\right]-2\left(1+\frac{t_0}{\beta}\right)^3}{\beta\left[\left(\frac{t_0}{\beta}\right)^2+2\left(\frac{t_0}{\beta}\right)-3\right]^2}\log\left(\frac{24}{\left(\frac{t_0}{\beta}+3\right)^2}-2\right) \\
&= \frac{8\left(1+\frac{t_0}{\beta}\right)}{\beta\left[\left(\frac{t_0}{\beta}\right)^2+2\frac{t_0}{\beta}-3\right]^2}\log\left(\frac{24}{\left(\frac{t_0}{\beta}+3\right)^2}-2\right) \\
&\quad -\frac{384\frac{t_0}{\beta}}{\beta\left(\frac{t_0}{\beta}+3\right)^3\left(1-\frac{t_0}{\beta}\right)\left[12-\left(\frac{t_0}{\beta}+3\right)^2\right]} \\
&< 0,
\end{aligned}
$$

revealing that $L(t_0)$ is a dec function of t_0.

By numerical computation, we find the root of $L(t_0) = 0$ to be $t_0^* = 0.2561\beta$, and

$$
L(t_0)\begin{cases} > 0, & 0 < t_0 < t_0^*, \\ < 0, & t_0^* < t_0 < \beta. \end{cases} \tag{9.4.17}
$$

It is clear that the LHS of (3.16) is inc, and $\alpha = 1.10$ when $t_0 = 0.2561\beta$. In addition, if $\alpha < 1.10$, then $t_0 < t_0^*$ implying the maximum of $M(t)$ is positive, and if $\alpha > 1.10$, then the maximum value of $M(t)$ is negative. From these results, we obtain the Lemma. ■

From the above discussion and the obtained results, we can state the following theorem.

Theorem 9.4.1 *If $\alpha < 1.10$, then the hazard function of the LBS is inc first until $t = \beta$ and then dec; if $1.10 < \alpha < 2$, then the hazard function of the LBS is inc first, then dec and inc until $t = \beta$ and dec after that; and if $\alpha \geq 2$, then the hazard function of the LBS is a inc-dec function.*

The hazard function of the LBS is plotted in Fig. 9.4.1 for some values of α, taking $\beta = 1$ without loss of any generality. In Fig. 9.4.2, we have plotted the hazard function for $\alpha = 1.10$ and 1.11, and a magnified view in the interval $t \in (0.2, 0.4)$, suggesting that the critical point of α is in the interval $(1.10, 1.11)$, thus supporting the result stated in Theorem 9.4.1.

9.5 Change point(s) of the hazard function

In the preceding section, we have shown that the hazard function of the LBS distribution has three different cases. It is then of natural interest to study the change point of the hazard function. From the discussion in the last section, the following lemma is evident.

Lemma 9.5.1 *If $\alpha < 1.10$, then there exists only one change point, which is exactly equal to β. If $1.10 < \alpha < 2$, then there exists three change points, where the smallest*

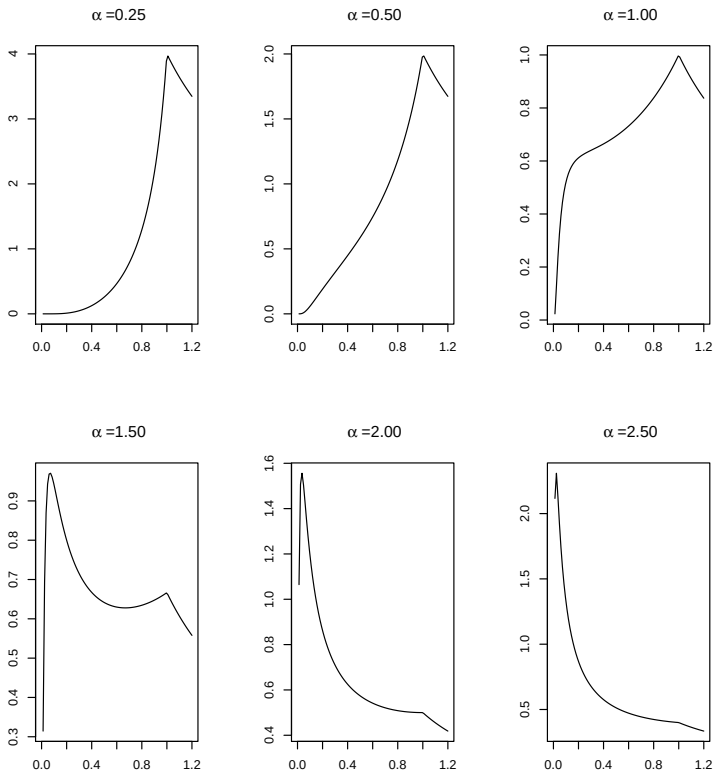

Figure 9.4.1: Plot of the hazard function for various values of the shape parameter α.

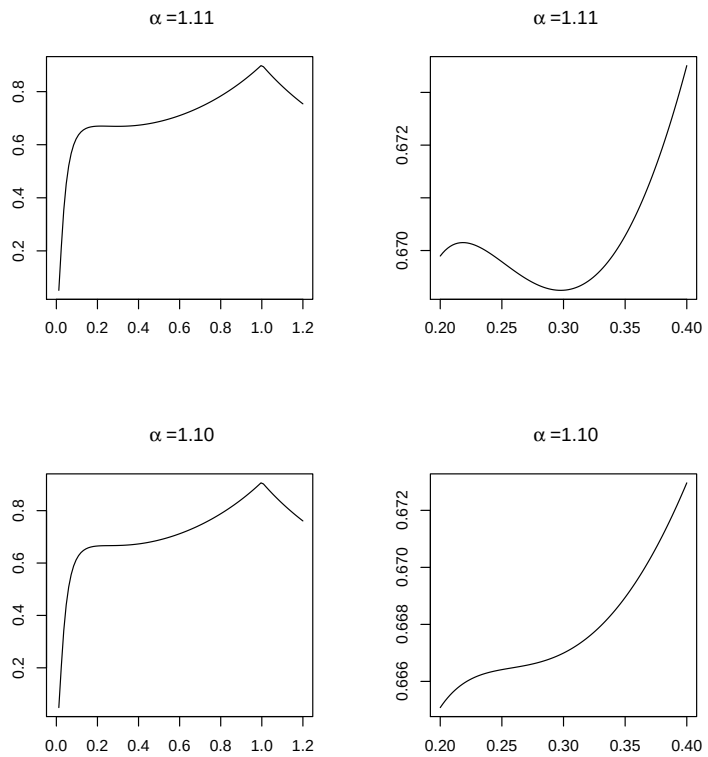

Figure 9.4.2: Comparison of the hazard function for the cases $\alpha = 1.10$ and $\alpha = 1.11$.

Table 9.5.1: Values of change points (say c_α) for different values of α.

α	c_α	α	c_α	α	c_α
1.1	1.0000	1.5	0.0673, 0.6678, 1.0000	1.9	0.0375, 0.9337, 1.0000
1.2	0.1366, 0.4301, 1.0000	1.6	0.0569, 0.7356, 1.0000	2.0	0.0333
1.3	0.1022, 0.5204, 1.0000	1.7	0.0489, 0.8019, 1.0000	2.1	0.0298
1.4	0.0815, 0.5971, 1.0000	1.8	0.0426, 0.8678, 1.0000	2.2	0.0268

two can be obtained as the solution of the non-linear equation

$$\alpha\left[\left(\frac{t}{\beta}\right)^{\frac{3}{2}}+3\left(\frac{t}{\beta}\right)^{\frac{1}{2}}\right]\left[1-\frac{1}{2}e^{\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)}\right]-\left(1+\frac{t}{\beta}\right)^{2}=0, \tag{9.5.1}$$

and the largest is exactly equal to β. If $\alpha \geq 2$, then there exists only one change point, which can be solved from the non-linear equation in (9.5.1).

There is no explicit solution for (9.5.1), but it can be numerically determined for given values of α. Let us use c_α to denote the change point of the hazard function of LBS with shape parameter α and scale parameter $\beta = 1$. Some values of the change points c_α are presented in Table 9.5.1 for different choices of α.

Plots of the first and second change points for the case $1.10 < \alpha < 2$ (with $\beta = 1$) are presented in Fig. 9.5.1, while the plots of the change point for $\alpha > 2$ (with $\beta = 1$) are presented in Fig. 9.5.2. The results of Table 9.5.1 and Figs. 9.5.1 and 9.5.2 suggest that if $1.10 < \alpha < 2$, then the smallest change point is a dec function of α, while the second one is an inc function of α and approaches β; and if $\alpha \geq 2$, then the change point is a dec function of α and approaches zero as α tends to infinity.

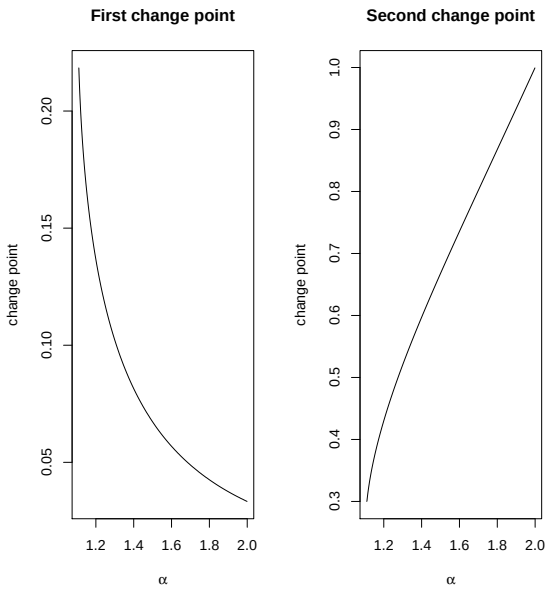

Figure 9.5.1: First and second change points for $1.10 < \alpha < 2.00$.

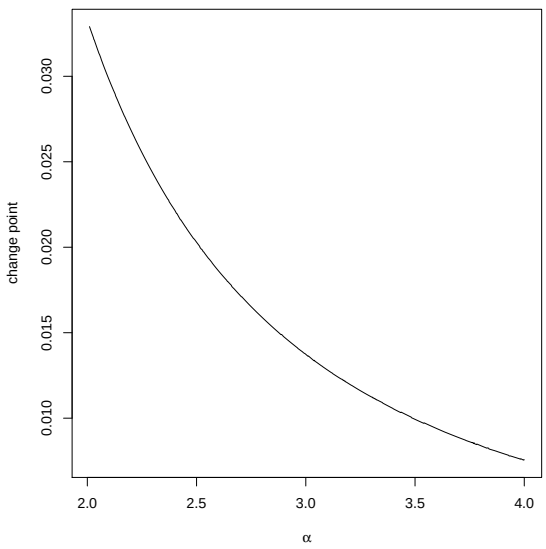

Figure 9.5.2: Change point for $2.00 < \alpha < 4.00$.

9.6 Maximum likelihood estimates

Let $(t_1, t_2, \cdots, t_n)$ be a random sample of size n from the LBS distribution with PDF in (9.2.1). Then, the log-likelihood function (without the constant) is given by,

$$\ln L = -n \log \alpha + \sum_{i=1}^{n} \log \left[\left(\frac{\beta}{t_i} \right)^{\frac{1}{2}} + \left(\frac{t_i}{\beta} \right)^{\frac{1}{2}} \right] - \frac{1}{\alpha} \sum_{i=1}^{n} \left| \left(\frac{\beta}{t_i} \right)^{\frac{1}{2}} - \left(\frac{t_i}{\beta} \right)^{\frac{1}{2}} \right|. \quad (9.6.1)$$

From (9.6.1), we obtain the first derivative with respect to α as

$$\frac{\partial \ln L}{\partial \alpha} = -\frac{n}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^{n} \left| \left(\frac{\beta}{t_i} \right)^{\frac{1}{2}} - \left(\frac{t_i}{\beta} \right)^{\frac{1}{2}} \right|. \quad (9.6.2)$$

Equating (9.6.2) to zero, we obtain a relationship between the MLEs of the parameters $\hat{\alpha}$ and $\hat{\beta}$ as

$$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} \left| \left(\frac{\hat{\beta}}{t_i} \right)^{\frac{1}{2}} - \left(\frac{t_i}{\hat{\beta}} \right)^{\frac{1}{2}} \right|. \quad (9.6.3)$$

The MLE of β is then numerically solved by maximizing the profile log-likelihood function

$$\ln L_{profile} = -n \log(\alpha(\beta)) + \sum_{i=1}^{n} \log \left[\left(\frac{\beta}{t_i} \right)^{\frac{1}{2}} + \left(\frac{t_i}{\beta} \right)^{\frac{1}{2}} \right]. \quad (9.6.4)$$

Now, we will show that the MLEs of the parameters of LBS do exist and are unique when the sample is complete.

Lemma 9.6.1 *Suppose $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of the parameters α and β of LBS. Then, there exists a $\beta_0 > 0$, such that $\hat{\alpha}$ is a dec function of $\hat{\beta}$ if $\hat{\beta} \leq \beta_0$ and is an inc function of $\hat{\beta}$ if $\hat{\beta} > \beta_0$.*

Proof It is clear that (9.6.3) is not differentiable at $\hat{\beta} = t_i$, for $i = 1, \cdots, n$. However, it is continuous, and so we may consider $\hat{\alpha}(\hat{\beta})$ as local maximum or local minimum at t if $\lim_{x \to t^+} \hat{\alpha}'(x) \lim_{x \to t^-} \hat{\alpha}'(x) < 0$. For this reason, we may define $\hat{\alpha}'(x) = 0$ if $\lim_{x \to t^+} \hat{\alpha}'(x) \lim_{x \to t^-} \hat{\alpha}'(x) < 0$. Further, let us take $t_1 \leq t_2 \leq \cdots \leq t_n$. Now, if $t_j < \hat{\beta} < t_{j+1}$, then the first derivative of $\hat{\alpha}$ with respect to $\hat{\beta}$ is given by

$$\hat{\alpha}'(\hat{\beta}) = \frac{1}{2n\hat{\beta}} \left\{ \sum_{i=1}^{j} \left[\left(\frac{\hat{\beta}}{t_i} \right)^{\frac{1}{2}} + \left(\frac{t_i}{\hat{\beta}} \right)^{\frac{1}{2}} \right] - \sum_{i=j+1}^{n} \left[\left(\frac{\hat{\beta}}{t_i} \right)^{\frac{1}{2}} + \left(\frac{t_i}{\hat{\beta}} \right)^{\frac{1}{2}} \right] \right\}.$$

Let us set

$$g(\hat{\beta}) = 2n\hat{\beta}^{\frac{3}{2}}\hat{\alpha}'(\hat{\beta}) = \hat{\beta} \left(\sum_{i=1}^{j} t_i^{\frac{1}{2}} - \sum_{i=j+1}^{n} t_i^{\frac{1}{2}} \right) + \left(\sum_{i=1}^{j} t_i^{-\frac{1}{2}} - \sum_{i=j+1}^{n} t_i^{-\frac{1}{2}} \right)$$

which is clearly an inc function of $\hat{\beta}$ for $\hat{\beta} \in (t_j, t_{j+1})$. For any $k > j$, it is readily seen that $g(\hat{\beta}_2) > g(\hat{\beta}_1)$ with $\hat{\beta}_2 \in (t_k, t_{k+1})$ and $\hat{\beta}_1 \in (t_j, t_{j+1})$. By using the facts that

$$\begin{aligned} \lim_{\hat{\beta} \to 0} g(\hat{\beta}) &= -\sum_{i=+1}^{n} t_i^{-\frac{1}{2}} < 0, \\ \lim_{\hat{\beta} \to +\infty} g(\hat{\beta}) &= \lim_{\hat{\beta} \to +\infty} g(\hat{\beta}) = \hat{\beta} \left(\sum_{i=1}^{n} t_i^{\frac{1}{2}} \right) + \sum_{i=1}^{n} t_i^{-\frac{1}{2}} = +\infty > 0, \end{aligned}$$

we can then conclude that there must exist a β_0 for which $g(\hat{\beta}) < 0$ if $\beta < \beta_0$ and

$g(\hat{\beta}) > 0$ otherwise. Since $\alpha'(\hat{\beta})$ has the same sign as $g(\hat{\beta})$ and $\alpha(\hat{\beta})$ is continuous, we have the Lemma. ■

Theorem 9.6.1 *The MLEs of α and β always exist and are unique.*

Proof It is equivalent to proving that there exists one and only one β maximizing the profile log-likelihood in Eq. (9.6.4). As discussed in Lemma 9.6.1, this function is a continuous function of β, but not differentiable at t_i for $i = 1, \cdots, n$. As in Lemma 9.6.1, we conclude $\ln' L_{profile}$ reaches the local maximum or minimum at t if $\lim_{x \to t^+} \ln' L_{profile}(x) \lim_{x \to t^-} \ln' L_{profile}(x) < 0$. Let us consider now the first derivative with respect to β in each interval (t_j, t_{j+1}), for $j = 0 \cdots, n$, given by

$$\ln' L_{profile} = -\frac{n\alpha'(\beta)}{\alpha(\beta)} + \sum_{i=1}^{n} \frac{\beta - t_i}{2\beta(\beta + t_i)} = \frac{\gamma(\beta)}{\beta^2 \alpha(\beta)}, \tag{9.6.5}$$

where $\gamma(\beta) = -\beta^2 n\alpha'(\beta) + \beta\alpha(\beta) \sum_{i=1}^{n} \frac{\beta - t_i}{2(\beta + t_i)}$.

Existence Proving the existence is equivalent to checking the sign of the limit of $\ln' L_{profile}$ in (9.6.5) given by

$$\begin{aligned}
\lim_{\beta \to 0} \ln' L_{profile} &= \lim_{\beta \to 0} \left\{ \frac{n \sum_{i=1}^{n} \left[\left(\frac{t_i}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t_i}\right)^{\frac{1}{2}} \right]}{2\beta \sum_{i=1}^{n} \left[\left(\frac{t_i}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t_i}\right)^{\frac{1}{2}} \right]} - \frac{n}{2\beta} + \sum_{i=1}^{n} \frac{1}{\beta + t_i} \right\} \\
&= \lim_{\beta \to 0} \left\{ \frac{n \sum_{i=1}^{n} \left(\frac{\beta}{t_i}\right)^{\frac{1}{2}}}{\beta \sum_{i=1}^{n} \left[\left(\frac{t_i}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t_i}\right)^{\frac{1}{2}} \right]} + \sum_{i=1}^{n} \frac{1}{\beta + t_i} \right\} > 0,
\end{aligned}$$

$$\begin{aligned}
\lim_{\beta\to+\infty} \ln' L_{profile} &= \lim_{\beta\to+\infty}\left\{-\frac{n\sum_{i=1}^{n}\left[\left(\frac{t_i}{\beta}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t_i}\right)^{\frac{1}{2}}\right]}{2\beta\sum_{i=1}^{n}\left[\left(\frac{\beta}{t_i}\right)^{\frac{1}{2}}-\left(\frac{t_i}{\beta}\right)^{\frac{1}{2}}\right]}+\frac{n}{2\beta}-\sum_{i=1}^{n}\frac{t_i}{\beta(\beta+t_i)}\right\} \\
&= \lim_{\beta\to+\infty}\left\{-\frac{n\sum_{i=1}^{n}\left(\frac{t_i}{\beta}\right)^{\frac{1}{2}}}{\beta\sum_{i=1}^{n}\left[\left(\frac{\beta}{t_i}\right)^{\frac{1}{2}}-\left(\frac{t_i}{\beta}\right)^{\frac{1}{2}}\right]}-\sum_{i=1}^{n}\frac{t_i}{\beta(\beta+t_i)}\right\}<0,
\end{aligned}$$

implying that there exists root(s) for the equation $\ln' L_{profile}(\beta)=0$.

Uniqueness Due to the existence of the MLE, there must exist at least one root β for the equation $\ln' L_{profile}(\beta)=0$. We will then prove that there will be no root for $\beta>\hat{\beta}$. To prove this, we will first show that $\gamma(\beta)$, defined in (9.6.5), is a dec function of β when $\gamma(\beta)<0$. The first derivative of $\gamma(\beta)$, with $\beta\in(t_j,t_{j+1})$, is given by

$$\begin{aligned}
\gamma'(\beta) &= -\left[n\beta\alpha'(\beta)+\frac{n\alpha(\beta)}{4}\right]+\left[(\alpha(\beta)+\beta\alpha'(\beta))\sum_{i=1}^{n}\frac{\beta-t_i}{2(\beta+t_i)}+\sum_{i=1}^{n}\frac{\beta\alpha(\beta)t_i}{(t_i+\beta)^2}\right] \\
&= -\alpha'(\beta)\beta\sum_{i=1}^{n}\frac{\beta+3t_i}{2(\beta+t_i)}+\alpha(\beta)\sum_{i=1}^{n}\frac{(\beta-t_i)(\beta+3t_i)}{4(t_i+\beta)^2} \\
&= -\alpha'(\beta)\beta\sum_{i=1}^{n}\frac{\beta+3t_i}{2(\beta+t_i)}+\alpha(\beta)\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\beta+3t_i}{2(\beta+t_i)}\sum_{i=1}^{n}\frac{\beta-t_i}{2(\beta+t_i)}\right. \\
&\quad \left.-\sum_{i=1}^{n}\left(\frac{t_i}{t_i+\beta}\right)^2+\frac{1}{n}\left(\sum_{i=1}^{n}\frac{t_i}{t_i+\beta}\right)^2\right] \\
&= \frac{\gamma(\beta)}{n\beta}\sum_{i=1}^{n}\frac{\beta+3t_i}{2(\beta+t_i)}-\frac{\alpha(\beta)}{n}\left[n\sum_{i=1}^{n}\left(\frac{t_i}{t_i+\beta}\right)^2-\left(\sum_{i=1}^{n}\frac{t_i}{t_i+\beta}\right)^2\right]<0.
\end{aligned}$$

Moreover, for any non-differentiable point t_j for $j = 1, \cdots, n$, we have

$$\lim_{\beta \to t_j^-} \ln' L_{profile} - \lim_{\beta \to t_j^+} \ln' L_{profile} = \frac{n\left(\lim_{\beta \to t_j^+} \alpha'(\beta) - \lim_{\beta \to t_j^-} \alpha'(\beta)\right)}{\alpha(t_j)} > 0. \quad (9.6.6)$$

We can thus conclude that there exists one and only one root for the equation $\ln' L_{profile}(\beta) = 0$, implying that the MLE of β is always exist and is unique. ■

9.6.1 Bias-corrected MLEs

Based on the results of an extensive Monte Carlo simulation study, we observed that

$$E(\hat{\alpha}) \approx \sqrt{\frac{n-1}{n}}\alpha, \quad (9.6.7)$$

$$E(\hat{\beta}) \approx \left(1 + \frac{\alpha^2}{2n}\right)\beta. \quad (9.6.8)$$

Hence, we can propose "bias-corrected MLEs" (UMLEs), denoted by $\tilde{\alpha}^*$ and $\tilde{\beta}^*$, as

$$\hat{\alpha}^* = \sqrt{\frac{n}{n-1}}\hat{\alpha}, \quad (9.6.9)$$

$$\hat{\beta}^* = \left(1 + \frac{\hat{\alpha}^{*2}}{2n}\right)^{-1}\hat{\beta}. \quad (9.6.10)$$

9.6.2 Moments and interval estimation of parameters

Due to the non-differentiability of the PDF in (9.2.1), we can not obtain the Fisher information matrix and then use the asymptotic normality of the MLEs to develop CIs

for the parameters. For this reason, we use the Jackknifing and Bootstrap methods detailed earlier in Chapter 1, for the construction of CIs for the model parameters.

9.7 Modified moment estimators

For the usual moment estimators in a two-parameter case, we equate the population mean and variance with the corresponding sample moments. Ng et al. (2003) proposed the MMEs for BS distribution. Motivated by their work, we develop here the MMEs for LBS by equating $E(T)$ and $E\left(\frac{1}{T}\right)$ in Eqs. (9.2.3) and (9.2.8) with the corresponding sample moments. In this process, we obtain the MMEs for α and β, denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, as

$$\tilde{\alpha} = \left[\left(\frac{s}{r}\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}, \tag{9.7.1}$$

$$\tilde{\beta} = (sr)^{\frac{1}{2}}, \tag{9.7.2}$$

where $s = \frac{1}{n}\sum_{i=1}^{n} t_i$ and $r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}$.

Lemma 9.7.1 *$\tilde{\alpha}$ is negatively biased.*

Proof We proved that the MME $\tilde{\alpha}$ is negatively biased for BS distribution in Property 2.4.4. In a similar way, it can be easily shown that the MME $\tilde{\alpha}$ is also negatively bias for LBS distribution. ■

Lemma 9.7.2 *The asymptotic joint distribution of $\tilde{\alpha}$ and $\tilde{\beta}$ is*

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{5\alpha^2}{4} & 0 \\ 0 & \frac{\alpha^2\beta^2(2+6\alpha^2)}{n(1+\alpha^2)^2} \end{pmatrix}\right]. \tag{9.7.3}$$

Proof From the strong law of large numbers, it is known that s and r converge to $E(T)$ and $E(T^{-1})$, respectively. Then, by using the central limit theorem, we have $s \sim N\left[E(T), \frac{1}{n}Var(T)\right]$ and $r^{-1} \sim N\left[E(T^{-1}), \frac{1}{n}Var(T^{-1})\right]$. Moreover,

$$Cov(s, r^{-1}) = \frac{1}{n}Cov(T, T^{-1}) = \frac{1-(1+\alpha^2)^2}{n}.$$

Therefore,

$$\sqrt{n}\begin{pmatrix} s - E(T) \\ r^{-1} - E(T^{-1}) \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{\Sigma}\right],$$

where

$$\mathbf{\Sigma} = \begin{bmatrix} (\alpha\beta)^2(2+11\alpha^2) & 1-(1+\alpha^2)^2 \\ 1-(1+\alpha^2)^2 & (\alpha\beta^{-1})^2(2+11\alpha^2) \end{bmatrix}.$$

By using Taylor expansion, we obtain

$$\begin{aligned} \tilde{\alpha} &\approx \alpha + \frac{1}{4\alpha\beta}s + \frac{\beta}{4\alpha}r^{-1}, \\ \tilde{\beta} &\approx \beta + \frac{1}{2(1+\alpha^2)}s - \frac{\beta^2}{2(1+\alpha^2)}r^{-1}. \end{aligned}$$

Now, by using the asymptotic joint distribution of (s, r^{-1}), we obtain the asymptotic joint distribution of $(\tilde{\alpha}, \tilde{\beta})$, as required. ■

By using this asymptotic distribution, we may readily obtain CIs. We may also construct CIs by using Jackknifing or Bootstrap method detailed earlier in Chapter 1.

9.7.1 Bias-corrected MMEs

Based on the results of an extensive Monte Carlo simulation study, we observed that

$$Bias(\tilde{\alpha}) \approx -\frac{1}{n}\alpha, \tag{9.7.4}$$

$$Bias(\tilde{\beta}) \approx \frac{\alpha^2}{2n}. \tag{9.7.5}$$

So, we can simply construct "bias-corrected MMEs" (UMMEs), denoted by $\tilde{\alpha}^*$ and $\tilde{\beta}^*$, as

$$\tilde{\alpha}^* = \frac{n-1}{n}\tilde{\alpha}, \tag{9.7.6}$$

$$\tilde{\beta}^* = \left(1 + \frac{\tilde{\alpha}^{*2}}{2n}\right)^{-1} \tilde{\beta}. \tag{9.7.7}$$

9.8 Simulation study

We carried out an extensive Monte Carlo simulation study for different choices of n and α by keeping $\beta = 1$, without loss of any generality. For the cases when the

Table 9.8.1: Simulated values of means and MSEs (inside parentheses) of the MLEs, UMLEs, MMEs and UMMEs.

n	α	MLEs		UMLEs		MMEs		UMMEs	
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}^*$	$\hat{\beta}^*$	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha}^*$	$\tilde{\beta}^*$
20	0.25	0.2434	1.0021	0.2498	1.0008	0.2363	1.0025	0.2488	1.0008
		(0.0030)	(0.0045)	(0.0031)	(0.0045)	(0.0034)	(0.0064)	(0.0036)	(0.0063)
	0.50	0.4862	1.0087	0.4988	1.0034	0.4702	1.0114	0.4949	1.0048
		(0.0121)	(0.0180)	(0.0125)	(0.0178)	(0.0135)	(0.0267)	(0.0140)	(0.0262)
	0.75	0.7281	1.0190	0.7470	1.0072	0.7011	1.0251	0.7380	1.0106
		(0.0272)	(0.0406)	(0.0281)	(0.0393)	(0.0301)	(0.0596)	(0.0309)	(0.0572)
	1.00	0.9693	1.0310	0.9945	1.0101	0.9298	1.0410	0.9787	1.0153
		(0.0484)	(0.0708)	(0.0500)	(0.0671)	(0.0535)	(0.0999)	(0.0543)	(0.0934)
	1.50	1.4495	1.0588	1.4871	1.0122	1.3837	1.0708	1.4565	1.0147
		(0.1093)	(0.1491)	(0.1125)	(0.1338)	(0.1217)	(0.1854)	(0.1217)	(0.1629)
	2.00	1.9283	1.0861	1.9784	1.0051	1.8358	1.0943	1.9324	0.9982
		(0.1962)	(0.2343)	(0.2016)	(0.1975)	(0.2195)	(0.2619)	(0.2179)	(0.2151)
50	0.25	0.2473	1.0005	0.2498	1.0000	0.2442	1.0009	0.2492	1.0003
		(0.0012)	(0.0015)	(0.0012)	(0.0015)	(0.0015)	(0.0026)	(0.0015)	(0.0026)
	0.50	0.4943	1.0025	0.4993	1.0005	0.4873	1.0047	0.4972	1.0022
		(0.0049)	(0.0061)	(0.0050)	(0.0061)	(0.0058)	(0.0109)	(0.0059)	(0.0108)
	0.75	0.7281	1.0190	0.7470	1.0072	0.7011	1.0251	0.7380	1.0106
		(0.0272)	(0.0406)	(0.0281)	(0.0393)	(0.0301)	(0.0596)	(0.0309)	(0.0572)
	1.00	0.9873	1.0106	0.9974	1.0025	0.9693	1.0175	0.9891	1.0074
		(0.0196)	(0.0235)	(0.0198)	(0.0231)	(0.0230)	(0.0396)	(0.0231)	(0.0385)
	1.50	1.4791	1.0217	1.4941	1.0034	1.4489	1.0296	1.4784	1.0071
		(0.0442)	(0.0476)	(0.0447)	(0.0454)	(0.0519)	(0.0685)	(0.0518)	(0.0647)
	2.00	1.9702	1.0307	1.9902	0.9986	1.9278	1.0382	1.9672	0.9988
		(0.0789)	(0.0712)	(0.0797)	(0.0660)	(0.0929)	(0.0903)	(0.0924)	(0.0823)

sample size n is 20 and 50, and the values of α are 0.25, 0.50, 0.75, 1.00, 1.50, and 2.00, we have presented the empirical values of the means and MSEs of all the proposed estimators in Table 9.8.1. These empirical results were determined from 10,000 Monte Carlo simulations. The required LBS samples were generated from standard Laplace samples and then using the inverse relationship between the BS and Laplace variates stated in Property 9.2.3.

Table 9.9.1: Point estimates based on data in Table A.1.1.

MLEs		UMLEs		MMEs		UMMEs	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}^*$	$\hat{\beta}^*$	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha}^*$	$\tilde{\beta}^*$
0.1289	133.0000	0.1296	132.9911	0.1205	131.8193	0.1217	131.8096

9.9 Illustrative Example

In this section, we illustrate the results established in the preceding sections with two data sets taken from the reliability literature.

Example 9.9.1 *Once again, we will use the data presented in Table A.1.1, but by assuming here a LBS distribution.*

The point estimates of α and β obtained from the MLEs, UMLEs, MMEs and UMMEs are all presented in Table 9.9.1. The comparison of the log-likelihood, AIC and BIC values between BS and LBS models, presented in Table 9.9.2, reveals that the LBS model provides a better fit than the BS model for these data. The bootstrap method is then utilized to estimate the SEs and also to construct 95% *CIs for α and β. These results are presented in Table 9.9.3. Finally, the KS test is carried out, and the values of the KS-statistic and the P-value, presented in Table 9.9.4, do not reject the LBS model assumption made in our analysis of these data.*

Example 9.9.2 *These data, presented in Table A.1.9, taken from McCool (1974), give failure times of ceramic ball bearings tested at the stress level of* $0.87(10^6 psi)$. *The point estimates of α and β estimated through all the methods are presented in*

Table 9.9.2: Log-likelihood, AIC and BIC values comparison of BS and LBS models for the data in Table A.1.1.

Model	Log-likelihood	AIC	BIC
BS	-457.2705	918.5411	923.7713
LBS	-456.8177	917.6355	922.8657

Table 9.9.3: Bootstrap SEs of estimates and 95% CIs based on data in Table A.1.1.

Estimator	α		β	
	$\widehat{SE}$	95% CI	$\widehat{SE}$	95% CI
MLEs	0.0126	(0.1047, 0.1537)	1.8228	(129.3711, 136.6117)
UMLEs	0.0128	(0.1057, 0.1552)	1.8316	(129.3355, 136.6111)
MMEs	0.0131	(0.0953, 0.1466)	2.2236	(127.5938, 136.2748)
UMMEs	0.0133	(0.0972, 0.1495)	2.2457	(127.5315, 136.2999)

Table 9.9.4: KS-statistics and the corresponding P-values based on MLEs, UMLEs, MMEs and UMMEs for the data in Table A.1.1.

	MLEs	UMLEs	MMEs	UMMEs
KS-statistic	0.0667	0.0657	0.0804	0.0790
P-value	0.7595	0.7752	0.5307	0.5538

Table 9.9.5: Point estimates based on data in Table A.1.9.

MLEs		UMLEs		MMEs		UMMEs	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}^*$	$\hat{\beta}^*$	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha}^*$	$\tilde{\beta}^*$
0.2050	204.7000	0.2161	204.3185	0.1997	212.0204	0.2219	211.4995

Table 9.9.6: Log-likelihood, AIC and BIC values comparison for BS and LBS models for the data in Table A.1.9.

Model	Loglikelihood	AIC	BIC
BS	-54.9718	113.9435	129.9435
LBS	-54.5051	113.0111	129.0111

Table 9.9.5. A comparison of the log-likelihood, AIC and BIC values between BS and LBS models, presented in Table 9.9.6, does reveal that the LBS model provides a better fit than the BS model. The bootstrap method is then utilized to estimate the SEs and also to construct 95% *CIs for* α *and* β*. These results are presented in Table 9.9.7. The KS-test is then carried out, and the values of the KS-statistics and the P-values, presented in Table 9.9.8, do not reject the LBS model assumption made in our analysis. Finally, the plot of* $\gamma(\beta)$ *in (9.6.5), presented in Figure 9.9.1, reveals the existence and uniqueness of the MLEs.*

Table 9.9.7: Bootstrap SEs of estimates and 95% CIs based on data in Table A.1.9.

Estimator	α		β	
	$\widehat{SE}$	95% CI	$\widehat{SD}$	95% CI
MLEs	0.0628	(0.0900, 0.3351)	17.4688	(171.8557, 243.9264)
UMLEs	0.0697	(0.1001, 0.3723)	18.3316	(169.6016, 245.2534)
MMEs	0.0609	(0.0807, 0.3193)	19.1418	(177.2483, 252.9811)
UMMEs	0.0751	(0.0996, 0.3933)	21.1883	(172.6576, 256.3829)

Table 9.9.8: KS-statistics and the corresponding P-values based on MLEs, UMLEs, MMEs and UMMEs for the data in Table A.1.9.

	MLEs	UMLEs	MMEs	UMMEs
KS-statistic	0.1783	0.1669	0.2181	0.1965
P-value	0.8549	0.9020	0.6527	0.7666

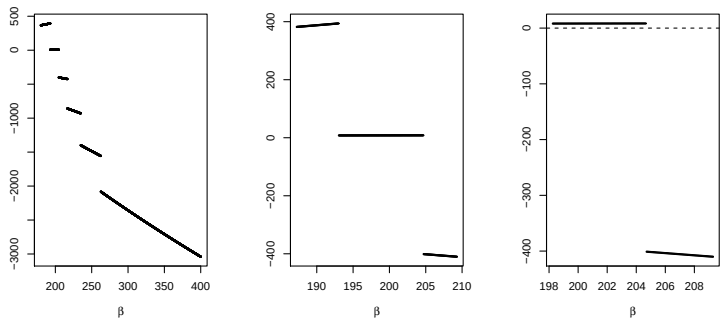

Figure 9.9.1: Plot of $\gamma(\beta)$ in (9.6.5).

Chapter 10

Summary and concluding Remarks

In this thesis, we first have developed a simple and efficient method of estimating the shape and scale parameters of the BS distribution based on complete samples. We have shown that the proposed estimator of the shape parameter α has smaller bias compared to the MME and MLE. It would be of interest to extend the proposed method of estimation to the case when the available data are censored. Also, the extension of this method to the case of BVBS and GBS distributions will be of great interest.

The MLEs of model parameters is very important in model fitting and in most cases (in this BS case as well) this would require the use of a numerical method. In Chapter 3, we have discussed the existence and uniqueness of the MLEs for the BS distribution based on Type-II, Type-I and hybrid censored samples. The extension of these results to the cases of BVBS and GBS distributions in the presence of various

forms of censoring will be of great interest. Another extension of interest will be to study similarly the MLEs of the parameters of BS distribution when the observed data are interval censored, which arises commonly in medical studies.

In Chapter 4, we have developed a simple and efficient method of estimating the five parameters of the BVBS distribution in (1.1.7) when the available data are Type-II censored and are of the form in (4.1.1). In developing this method of estimation, we have assumed that the censoring present in the data is light or moderate, with $\frac{n}{2} < k \leqslant n$. This is necessary for the method of estimation as it uses the reciprocal property of the BS distribution. This leaves some problems open for further study. First, in the Type-II censoring situation considered here, it would be of interest to develop a method of estimation when the censoring amount is heavy, i.e., $k \leqslant \frac{n}{2}$. Next, it would also be of interest to extend the proposed method of estimation to the case when the available data are Type-I censored and progressively Type-II censored; interested readers may refer to Balakrishnan and Aggarwala (2000), Balakrishnan (2007) and Balakrishnan and Cramer (2014) for detailed reviews of various developments on progressive censoring. With regard to the latter, the results of Balakrishnan and Kim (2005), extending the work of Harrell and Sen (1979) to the case of progressive Type-II censoring in the case of bivariate normal, would prove to be quite useful.

In Chapter 5, we have discussed the fitting of the regression model based on the BS distribution by the use of MLEs. We have also considered the case when the shape parameters are not equal. The extension of these results to the case of GBS

distribution will be of interest. Also, the results established here are all for complete samples, and their generalizations to the case when the available data are censored will be of practical importance.

In Chapter 6, we have discussed the fitting of the regression model based on the BVBS distribution with the use of MLEs. Leiva et al. (2008) and Kundu et al. (2013) have discussed the univariate and bivariate generalized BS distributions, respectively, and extension of the developed results to the generalized BVBS distribution along their lines will be of interest. Also, the results established here are based on complete samples, and the example provided is precisely of this form. However, in lifetime data analysis, one may often encounter censored data. In this case, we may have to develop likelihood inferential methods by considering distributions of order statistics and their vector concomitants in this set-up.

In Chapter 7, we have derived the MGFs of the MLEs of the parameters μ and σ of the Laplace distribution based on Type-II censored samples. We have then used these MGFs to obtain the exact density function, mean, variance and correlation coefficient of the MLEs $\hat{\mu}$ and $\hat{\sigma}$. The exact density functions have also been utilized to determine exact CIs for μ and σ. Similar results have been developed for the case of Type-I censored sample in Chapter 8. In Chapter 7, we have also discussed the exact distributions of MLEs of quantiles, reliability function and cumulative function as well as exact CIs for these quantities, which have then been utilized to develop exact bounds for Q-Q plot and KM curve. It will naturally be of interest to develop

analogous results when the available sample is Type-I censored and more generally hybrid censored; see Balakrishnan and Kundu (2013) for a detailed review on hybrid censoring and associated inferential results.

In Chapter 9, we have discussed several properties of the LBS distribution and some associated inferential methods. Ng et al. (2006), Lemonte and Ferrari (2011) and Barreto et al. (2013) have all discussed fitting of the BS model based on Type-II censored samples. It will be of interest to extend the method of estimation developed here for the LBS model to the case when the available data are Type-II and Type-I censored. Also, Kundu et al. (2010, 2013) constructed bivariate and multivariate BS distributions and discussed their properties and applications. It will naturally be of interest to develop a multivariate LBS distribution and discuss its properties and associated inferential methods.

Appendix A

Appendix

A.1 Data sets

Table A.1.1: Data on the fatigue lifetimes of aluminum coupons, taken from Birnbaum and Saunders (1969b).

70	90	96	97	99	100	103	104	104	105	107	108	108	108
109	109	112	112	113	114	114	114	116	119	120	120	120	121
121	123	124	124	124	124	124	128	128	129	129	130	130	130
131	131	131	131	131	132	132	132	133	134	134	134	134	134
136	136	137	138	138	138	139	139	141	141	142	142	142	142
142	142	144	144	145	146	148	148	149	151	151	152	155	156
157	157	157	157	158	159	162	163	163	164	166	166	168	170
174	196	212											

Table A.1.2: Type-II censored sample from Dodson (2006).

12.5	24.4	58.2	68.0	69.1	95.5	96.6	97.0	114.2	123.2	125.6	152.7

Table A.1.3: Failure times of units up to 150 hours from Bartholomew (1963).

3	19	23	26	27	37	38	41	45	58	84	90	99	109	138

Table A.1.4: The bone mineral density data taken from Johnson and Wichern (1999).

$T^{(1)}$	1.103	0.842	0.925	0.857	0.795	0.787
$T^{(2)}$	1.027	0.857	0.875	0.873	0.811	0.640
$T^{(1)}$	0.933	0.799	0.945	0.921	0.792	0.815
$T^{(2)}$	0.947	0.886	0.991	0.977	0.825	0.851
$T^{(1)}$	0.755	0.880	0.900	0.764	0.733	0.932
$T^{(2)}$	0.770	0.912	0.905	0.756	0.765	0.932
$T^{(1)}$	0.856	0.890	0.688	0.940	0.493	0.835
$T^{(2)}$	0.843	0.879	0.673	0.949	0.463	0.776

Table A.1.5: The delivery time data from Montgomery et al. (2006).

Observation No.	Delivery Time t (min)	No. of Cases, x_1	Distance x_2 (ft)
1	16.68	7	560
2	11.50	3	220
3	12.03	3	340
4	14.88	4	80
5	13.75	6	150
6	18.11	7	330
7	8.00	2	110
8	17.83	7	210
9	79.24	30	1460
10	21.50	5	605
11	40.33	16	688
12	21.00	10	215
13	13.50	4	255
14	19.75	6	462
15	24.00	9	448
16	29.00	10	776
17	15.35	6	200
18	19.00	7	132
19	9.50	3	36
20	35.10	17	770
21	17.90	10	140
22	52.32	26	810
23	18.75	9	450
24	19.83	8	635
25	10.75	4	150

Table A.1.6: Data from Mann and Fertig (1973).

0.22	0.50	0.88	1.00	1.32	1.33	1.54	1.76	2.50	3.00

Table A.1.7: Data from Bain and Engelhardt (1973).

1.96	1.97	3.60	3.80	4.79	5.66	5.76	5.78	6.27	6.30	6.76
7.65	7.84	7.99	8.51	9.18	10.13	10.24	10.25	10.43	11.45	11.48
11.75	11.81	12.34	12.78	13.06	13.29	13.98	14.18	14.40	16.22	17.06

Table A.1.8: Data from Lawless (1982).

31	58	157	185	300	470	497	673

Table A.1.9: Fatigue lifetime data presented by McCool (1974).

152.7	172.0	172.5	173.3	193.0	204.7	216.5	234.9	262.6	422.6

A.2 Appendix for Chapter 3

Proof of Lemma 3.3.2 We shall prove the required result by considering two cases.

Case I: $\beta \leq t_{k:n}$ From Lemma 3.2.1, it is known that $\alpha(\beta)$ in (3.2.10) is an increasing function of β and so $\frac{1}{\alpha(\beta)}$ is a decreasing function of β. Clearly, $\sqrt{\frac{t_{k:n}}{\beta}} - \sqrt{\frac{\beta}{t_{k:n}}}$ is a decreasing function of β for $\beta \leq t_{k:n}$. Hence, $g_1(\beta)$ is a decreasing function of β when $\beta \leq t_{k:n}$.

Case II: $\beta > t_{k:n}$ In this case, upon differentiating $g_1(\beta)$, we find

$$
\begin{aligned}
[g_1(\beta)]' &= -\frac{\sqrt{\frac{t_{k:n}}{\beta}} - \sqrt{\frac{\beta}{t_{k:n}}}}{\alpha^2(\beta)} \left[\frac{\frac{1}{\beta^2}\sum_{i=1}^{k}(t_{k:n}-t_{i:n})}{2\alpha(\beta)\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} + \frac{\alpha(\beta)\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{(t_{i:n}+\beta)^2}}{2\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} \right] \\
&\quad -\frac{1}{2\beta\alpha(\beta)}\left(\sqrt{\frac{t_{k:n}}{\beta}} + \sqrt{\frac{\beta}{t_{k:n}}}\right) \\
&= -\frac{1}{2\beta\alpha(\beta)}\sqrt{\frac{t_{k:n}}{\beta}} + \frac{\left(\sqrt{\frac{\beta}{t_{k:n}}} - \sqrt{\frac{t_{k:n}}{\beta}}\right)\sum_{i=1}^{k}(t_{k:n}-t_{i:n})}{2\beta^2\alpha^3(\beta)\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} - \frac{1}{2\beta\alpha(\beta)}\sqrt{\frac{\beta}{t_{k:n}}} \\
&\quad +\frac{1}{2\alpha(\beta)}\sqrt{\frac{\beta}{t_{k:n}}} + \frac{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{(t_{i:n}+\beta)^2}}{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} - \frac{1}{2\alpha(\beta)}\sqrt{\frac{t_{k:n}}{\beta}}\frac{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{(t_{i:n}+\beta)^2}}{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} \\
&< -\frac{1}{2\alpha^3(\beta)\beta^2\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\left[\sqrt{\beta t_{k:n}}\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)\right. \\
&\quad \left. -\left(\sqrt{\frac{\beta}{t_{k:n}}} - \sqrt{\frac{t_{k:n}}{\beta}}\right)\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\right] \\
&\quad -\frac{1}{2\alpha(\beta)\sqrt{\beta t_{k:n}}}\left(1 - \frac{\beta\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{(t_{i:n}+\beta)^2}}{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\right) \\
&< -\frac{\sqrt{\beta t_{k:n}}}{2\alpha^3(\beta)\beta^2\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\left[\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{t_{k:n}}\right)\right] \\
&< 0.
\end{aligned}
$$

This shows that $g_1(\beta)$ is a decreasing function of β when $\beta > t_{k:n}$. Hence, the lemma.

■

Proof of Lemma 3.3.4 Let us consider the derivative of $g_2(\beta)$ with respect to β given by

$$
\begin{aligned}
[g_2(\beta)]' &= -\frac{\sqrt{\beta}}{\alpha^2(\beta)}\sum_{i=1}^{k}\left(1-\frac{t_{i:n}}{\beta}\right)\left[\frac{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})}{2\beta^2\alpha(\beta)\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}+\frac{\alpha(\beta)\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{(t_{i:n}+\beta)^2}}{2\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\right] \\
&\quad +\frac{1}{2\beta\alpha(\beta)}\sum_{i=1}^{k}\left(\sqrt{\beta}+\frac{t_{i:n}}{\sqrt{\beta}}\right) \\
&= \frac{\sqrt{\beta}}{2\alpha^3(\beta)\beta^2\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\left[\sum_{i=1}^{k}t_{i:n}\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)\right. \\
&\quad \left.-\sum_{i=1}^{k}\left(1-\frac{t_{i:n}}{\beta}\right)\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\right] \\
&\quad +\frac{k}{2\alpha(\beta)\sqrt{\beta}}\left(1-\frac{\beta\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{(t_{i:n}+\beta)^2}}{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\right)+\frac{\sum_{i=1}^{k}t_{i:n}\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{(t_{i:n}+\beta)^2}}{2\sqrt{\beta}\alpha(\beta)\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} \\
&> \frac{\sqrt{\beta}}{2\alpha^3(\beta)\beta^2\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\left[\sum_{i=1}^{k}t_{i:n}\sum_{i=1}^{k}\left(\frac{t_{k:n}}{t_{i:n}}-1\right)-k\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\right] \\
&= \frac{t_{k:n}\sqrt{\beta}}{2\alpha^3(\beta)\beta^2\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\left[\sum_{i=1}^{k}t_{i:n}\sum_{i=1}^{k}\frac{1}{t_{i:n}}-k^2\right] \\
&> 0.
\end{aligned}
$$

Hence, the lemma.

■

Proof of Lemma 3.3.5 Consider

$$\begin{aligned} g_3(\beta) &= \sqrt{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)/\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\sqrt{\beta}\sum_{i=1}^{k}\frac{t_{i:n}}{\beta+t_{i:n}} \\ &= \sqrt{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}\sqrt{\sum_{i=1}^{k}\frac{t_{i:n}}{\frac{t_{i:n}}{\beta}+1}} \\ &\quad \times\left[\sqrt{\sum_{i=1}^{k}\frac{t_{i:n}}{\beta+t_{i:n}}}/\sqrt{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}}\right]. \end{aligned}$$

Since $\sqrt{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\left(\frac{1}{t_{i:n}}-\frac{1}{\beta}\right)}\sqrt{\sum_{i=1}^{k}\frac{t_{i:n}}{\frac{t_{i:n}}{\beta}+1}}$ increases as β increases, in order to prove that $g_3(\beta)$ is increasing in β, it is sufficient to prove that $\sqrt{\sum_{i=1}^{k}\frac{t_{i:n}}{\beta+t_{i:n}}}/\sqrt{\sum_{i=1}^{k}\frac{t_{k:n}+t_{i:n}}{t_{i:n}+\beta}} = \sqrt{\frac{1}{1+t_{k:n}g_4(\beta)}}$ is an increasing function of β, where $g_4(\beta) = \frac{\sum_{i=1}^{k}\frac{1}{\beta+t_{i:n}}}{\sum_{i=1}^{k}\frac{t_{i:n}}{\beta+t_{i:n}}}$. Let us consider the derivative of $g_4(\beta)$ given by

$$\begin{aligned} [g_4(\beta)]' &= \frac{-\sum_{i=1}^{k}\frac{1}{(\beta+t_{i:n})^2}\sum_{i=1}^{k}\frac{t_{i:n}}{\beta+t_{i:n}}+\sum_{i=1}^{k}\frac{t_{i:n}}{(\beta+t_{i:n})^2}\sum_{i=1}^{k}\frac{1}{\beta+t_{i:n}}}{\left(\sum_{i=1}^{k}\frac{t_{i:n}}{\beta+t_{i:n}}\right)^2} \\ &= \frac{-k\sum_{i=1}^{k}\left(\frac{1}{\beta+t_{i:n}}\right)^2+\sum_{i=1}^{k}\frac{1}{\beta+t_{i:n}}\sum_{i=1}^{k}\frac{1}{\beta+t_{i:n}}}{\left(\sum_{i=1}^{k}\frac{t_{i:n}}{\beta+t_{i:n}}\right)^2} \\ &< 0. \end{aligned}$$

Thus, $g_4(\beta)$ is a decreasing function of β and so $\sqrt{\frac{1}{1+t_{k:n}g_4(\beta)}}$ is an increasing function of β. Consequently, we have $g_3(\beta)$ to be an increasing function of β.

Proof of Theorem 3.3.1 We first present the proof for the existence of the estimates, and then for their uniqueness.

<u>Existence</u> To prove the existence of the MLEs, it is sufficient to check whether Eq. (3.2.7) is satisfied for some $\hat{\alpha}$ and $\hat{\beta}$, wherein $\hat{\alpha}$ and $\hat{\beta}$ in turn satisfy the relationship presented in (3.2.10). From (3.2.10), it is easy to see that $\hat{\alpha}$ exists only when exists only when $\hat{\beta} > \frac{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})}{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}$ $(= \hat{\beta}_0$, say), where $\frac{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})}{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}} < t_{k:n}$. Now, let us study the left and right limits of both sides of (3.2.7). For the left hand side of (3.2.7), we first of all note from (3.2.10) that as $\beta \to \hat{\beta}_0$, $\alpha \to 0$ and consequently $\eta \to \infty$, and so

$$\lim_{\beta\to\hat{\beta}_0}(n-k)A = \lim_{\eta\to\infty}\frac{\phi(\eta)}{1-\Phi(\eta)} = \lim_{\eta\to\infty}(n-k)\frac{\phi'(\eta)}{-\phi(\eta)} = \lim_{\eta\to\infty}(n-k)\frac{-\eta\phi(\eta)}{-\phi(\eta)} > 0,$$

by the use of L'Hospital's rule. Similarly, as $\beta \to \infty$, since α is an increasing function of β, we have $\eta \to -\frac{\sqrt{\beta}}{\alpha(\beta)\sqrt{t_{k:n}}}$, and so

$$\lim_{\beta\to\infty}(n-k)A = (n-k)\frac{\phi\left(-\frac{\sqrt{\beta}}{\alpha(\beta)\sqrt{t_{k:n}}}\right)}{1-\Phi\left(-\frac{\sqrt{\beta}}{\alpha(\beta)\sqrt{t_{k:n}}}\right)} = (n-k)\frac{\phi\left(\sqrt{\frac{\sum_{i=1}^{k}(t_{k:n}+t_{i:n})\frac{1}{t_{k:n}}}{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}}\right)}{\Phi\left(\sqrt{\frac{\sum_{i=1}^{k}(t_{k:n}+t_{i:n})\frac{1}{t_{k:n}}}{\sum_{i=1}^{k}(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}}\right)}.$$

For the right hand side of (3.2.7), we similarly find

$$
\begin{aligned}
& \lim_{\beta\to\hat{\beta}_0} \frac{\alpha^2(\beta)\sum_{i=1}^k \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^k \left(1-\frac{t_{i:n}}{\beta}\right)}{\alpha(\beta)\sqrt{\frac{t_{k:n}}{\beta}}} \\
= \quad & \lim_{\beta\to\hat{\beta}_0} \frac{\frac{\sqrt{\beta}}{\alpha(\beta)}\sum_{i=1}^k \left(1-\frac{t_{i:n}}{\beta}\right)}{\sqrt{t_{k:n}}} \\
= \quad & \lim_{\beta\to\hat{\beta}_0} \frac{\frac{\sqrt{\beta}}{\alpha(\beta)}\sum_{i=1}^k \left[1-t_{i:n}\frac{\sum_{i=1}^k\left(\frac{t_{k:n}}{t_{i:n}}-1\right)}{\sum_{i=1}^k(t_{k:n}-t_{i:n})}\right]}{\sqrt{t_{k:n}}} \\
= \quad & \lim_{\beta\to\hat{\beta}_0} \frac{\frac{\sqrt{\beta}}{\alpha(\beta)}\sum_{i=1}^k \left[\sum_{i=1}^k(t_{k:n}-t_{i:n}) - t_{i:n}\sum_{i=1}^k\left(\frac{t_{k:n}}{t_{i:n}}-1\right)\right]}{\sqrt{t_{k:n}}\sum_{i=1}^k(t_{k:n}-t_{i:n})} \\
= \quad & \lim_{\beta\to\hat{\beta}_0} \frac{\frac{\sqrt{\beta}}{\alpha(\beta)}\left[k\sum_{i=1}^k(t_{k:n}-t_{i:n}) - \sum_{i=1}^k t_{i:n}\sum_{i=1}^k\frac{1}{t_{i:n}}(t_{k:n}-t_{i:n})\right]}{\sqrt{t_{k:n}}\sum_{i=1}^k(t_{k:n}-t_{i:n})} \\
= \quad & \lim_{\beta\to\hat{\beta}_0} \frac{\sqrt{\beta t_{k:n}}\left[k^2-\sum_{i=1}^k t_{i:n}\sum_{i=1}^k\frac{1}{t_{i:n}}\right]}{\alpha(\beta)\sum_{i=1}^k(t_{k:n}-t_{i:n})} < 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \lim_{\beta\to\infty} \frac{\alpha^2(\beta)\sum_{i=1}^k \frac{t_{i:n}}{\beta+t_{i:n}} + \sum_{i=1}^k \left(1-\frac{t_{i:n}}{\beta}\right)}{\alpha(\beta)\sqrt{\frac{t_{k:n}}{\beta}}} \\
= \quad & \frac{1}{\sqrt{t_{k:n}}}\left[\sum_{i=1}^k t_{i:n}\sqrt{\frac{\sum_{i=1}^k(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}{\sum_{i=1}^k(t_{k:n}+t_{i:n})}} + k\sqrt{\frac{\sum_{i=1}^k(t_{k:n}+t_{i:n})}{\sum_{i=1}^k(t_{k:n}-t_{i:n})\frac{1}{t_{i:n}}}}\right].
\end{aligned}
$$

Since both sides of (3.2.7) are known to be monotone and continuous function of β, the MLEs will exist only when the condition in (3.3.1) is satisfied. In some cases, in fact, this condition may not be satisfied, as displayed in Example 3.7.4.

Uniqueness When the MLEs exist, then they can be shown to be unique. Since the

left hand side of (3.2.7) is a monotone decreasing function of β as proved in Lemma 3.3.3 and the right hand side of (3.2.7), being a sum of two monotone increasing functions of β as proved in Lemmas 3.3.4 and 3.3.5, is a monotone increasing function of β, they would intersect exactly once, at the MLE of β. To show that it really is the maximum and not the minimum of the log-likelihood, it is sufficient to show that the limit of the first derivative of $\ln L(\beta)$ is positive as $\beta \to \hat{\beta}_0$. In fact, by using Eq. (3.2.10) and Lemma 3.3.2, we have

$$\begin{aligned}
\lim_{\beta\to\hat{\beta}_0}[lnL(\beta)]' &= -\frac{k}{\alpha(\beta)}\alpha'(\beta) - \frac{k}{\beta} + \sum_{i=1}^{k}\frac{t_{i:n}+3\beta}{2\beta\left(t_{i:n}+\beta\right)} \\
&\quad + \frac{1}{\alpha^3(\beta)}\sum_{i=1}^{k}\left(\frac{t_{i:n}}{\beta}+\frac{\beta}{t_{i:n}}-2\right)\alpha'(\beta) \\
&\quad - \frac{1}{2\alpha(\beta)^2}\left[\sum_{i=1}^{k}\left(\frac{t_{i:n}}{\beta}+\frac{\beta}{t_{i:n}}-2\right)\right]' \\
&\quad + \left[(n-k)\log\left\{1-\Phi\left[\frac{1}{\alpha(\beta)}\left(\sqrt{\frac{t_{k:n}}{\beta}}-\sqrt{\frac{\beta}{t_{k:n}}}\right)\right]\right\}\right]' \\
&= \infty.
\end{aligned}$$

So, the log-likelihood function will increase first, and when the left hand side of (3.2.7) intersects the right hand side, the log-likelihood function will reach its maximum. Then, $[\ln L(\beta)]'$ will be negative after that point since the left hand side of (3.2.7) intersects the right hand side only once. If the left hand side of (3.2.7) dose not intersect the right hand side, then the log-likelihood function will always increase, in which case a solution does not exist.

Proof of Theorem 3.4.1 The proof follows along the lines of Theorem 3.3.1 upon using Eqs. (3.4.4) and (3.4.6). Firstly, $\hat{\alpha}$ in (3.4.6) can be shown to be an increasing function of β. Secondly, the left and right hand sides of (3.4.5) are continuous and monotone decreasing and increasing functions of β, respectively. This means that if they do intersect, they will intersect exactly once at the MLE of β. If they do not intersect, then a solution does not exist. ■

A.3 Appendix for Chapter 7

Proof of Lemma 7.4.3

Case I: $z \geq 0$

$$\begin{aligned} P(Z_1 - Z_2 \leq z) &= P(Z_1 < Z_2 + z) \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} z_2^{\alpha-1} e^{-z\beta^{-1}} \left[1 - \sum_{i=1}^n \left(\prod_{j=1, j\neq i}^n \frac{\lambda_j^{-1}}{\lambda_j^{-1} - \lambda_i^{-1}} \right) e^{-(z_2+z)\lambda_i^{-1}} \right] dz_2 \\ &= 1 - \sum_{i=1}^n \left(\prod_{j=1, j\neq i}^n \frac{\lambda_j^{-1}}{\lambda_j^{-1} - \lambda_i^{-1}} \right) \frac{1}{\beta^\alpha \left(\lambda_i^{-1} + \beta^{-1}\right)^\alpha}; \end{aligned}$$

Case II: $z < 0$

$$\begin{aligned} P(Z_1 - Z_2 \leq z) &= P(Z_1 < Z_2 + z) \\ &= \int_{-z}^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} z_2^{\alpha-1} e^{-z\beta^{-1}} \left[1 - \sum_{i=1}^n \left(\prod_{j=1, j\neq i}^n \frac{\lambda_j^{-1}}{\lambda_j^{-1} - \lambda_i^{-1}} \right) e^{-(z_2+z)\lambda_i^{-1}} \right] dz_2 \\ &= S_\Gamma(-z, \alpha, \beta) - \sum_{i=1}^n \left(\prod_{j=1, j\neq i}^n \frac{\lambda_j^{-1}}{\lambda_j^{-1} - \lambda_i^{-1}} \right) \frac{e^{-z\lambda_i^{-1}} S_\Gamma\left(-z, \alpha, \left(\lambda_i^{-1} + \beta^{-1}\right)^{-1}\right)}{\beta^\alpha \left(\lambda_i^{-1} + \beta^{-1}\right)^\alpha}; \end{aligned}$$

$$
\begin{aligned}
P(Z_1+Z_2\le z) &= P(Z_1<z-Z_2)\\
&= \int_0^z \frac{1}{\Gamma(\alpha)\beta^\alpha} z_2^{\alpha-1} e^{-z_2\beta^{-1}} \left[1-\sum_{i=1}^n \left(\prod_{j=1,j\neq i}^n \frac{\lambda_j^{-1}}{\lambda_j^{-1}-\lambda_i^{-1}}\right) e^{-(z-z_2)\lambda_i^{-1}}\right] dz_2\\
&= \Gamma(z,\alpha,\beta)-\sum_{i=1}^n \left(\prod_{j=1,j\neq i}^n \frac{\lambda_j^{-1}}{\lambda_j^{-1}-\lambda_i^{-1}}\right) \frac{e^{-z\lambda_i^{-1}}}{\Gamma(\alpha)\beta^\alpha} \int_0^z z_2^{\alpha-1} e^{-z_2(\beta^{-1}-\lambda_i^{-1})} dz_2.
\end{aligned}
$$

Using integration by parts, we can then obtain the expression for the case when $\beta > \lambda_i$.

■

Proof of Lemma 7.4.5

$$
\begin{aligned}
P(Z\le z) &= 1-P(Z_1>Z_2+z)\\
&= 1-\int_0^\infty f_{Z_2}(z_2)[1-F_{Z_1}(z_2+z)]dz_2\\
&= 1-\left(\prod_{i=1}^m \lambda_{2i}^{-r_{2i}} \prod_{j=1}^n \lambda_{1j}^{-r_{1j}}\right) \sum_{i=1}^m \sum_{l_2=1}^{r_{2i}} \sum_{j=1}^n \sum_{l_1=1}^{r_{1j}} \frac{\Psi_{2i,l_2}(-\lambda_{2i}^{-1})\Psi_{1j,l_1}(-\lambda_{1j}^{-1})}{(r_{2i}-l_2)!(l_2-1)!(r_{1j}-l_1)!(l_1-1)!}\\
&\quad \times \int_0^\infty (z_2+z)^{r_{1j}-l_1} z_2^{r_{2i}-l_2} \left(\frac{1}{\lambda_{2i}}-\frac{r_{2i}-l_2}{z_2}\right) e^{-z_2\lambda_{2i}^{-1}-(z_2+z)\lambda_{1j}^{-1}} dz_2\\
&= 1-\left(\prod_{i=1}^m \lambda_{2i}^{-r_{2i}} \prod_{j=1}^n \lambda_{1j}^{-r_{1j}}\right) \sum_{\Omega} \frac{\Psi_{2i,l_2}(-\lambda_{2i}^{-1})\Psi_{1j,l_1}(-\lambda_{1j}^{-1}) z^{r_{1j}-l_1-k} e^{-z\lambda_{1j}^{-1}}}{(r_{2i}-l_2)!(l_2-1)!k!(r_{1j}-l_1-k)!(l_1-1)!}\\
&\quad \times \int_0^\infty z_2^{r_{2i}-l_2+k} \left(\frac{1}{\lambda_{2i}}-\frac{r_{2i}-l_2}{z_2}\right) e^{-z_2(\lambda_{1j}^{-1}+\lambda_{2i}^{-1})} dz_2\\
&= 1-\left(\prod_{i=1}^m \lambda_{2i}^{-r_{2i}} \prod_{j=1}^n \lambda_{1i}^{-r_{1j}}\right) \sum_{\Omega} \frac{\Psi_{2i,l_2}(-\lambda_{2i}^{-1})\Psi_{1j,l_1}(-\lambda_{1j}^{-1}) z^{r_{1j}-l_1-k} e^{-z\lambda_{1j}^{-1}}}{(r_{2i}-l_2)!(l_2-1)!k!(r_{1j}-l_1-k)!(l_1-1)!}\\
&\quad \times \left[\frac{(r_{2i}-l_2+k)!}{\lambda_{2i}(\lambda_{1j}^{-1}+\lambda_{2i}^{-1})^{r_{2i}-l_2+k+1}} - \frac{(r_{2i}-l_2+k-1)!(r_{2i}-l_2)}{(\lambda_{1j}^{-1}+\lambda_{2i}^{-1})^{r_{2i}-l_2+k}}\right],
\end{aligned}
$$

as required. ■

Proof of Lemma 7.4.7 Let $Z = Z_1 - Z_2 - Z_3$. Then, we have the following:

Case I: $z \geq 0$

$$\begin{aligned} P(Z \leq z) &= 1 - P(Z_1 - Z_2 > Z_3 + z) \\ &= 1 - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{C_{ij}}{\lambda_{2i}\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} z_3^{\alpha-1} e^{-(z_3+z)\lambda_{1j}^{-1} - z_3\beta^{-1}} dz_3 \\ &= 1 - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{C_{ij} e^{-z\lambda_{1j}^{-1}}}{\lambda_{2i}\beta^{\alpha}(\lambda_{1j}^{-1} + \beta^{-1})^{\alpha}}; \end{aligned}$$

Case II: $z < 0$

$$\begin{aligned} P(Z \leq z) &= 1 - P(Z_1 - Z_2 > Z_3 + z \ \& \ Z_3 < -z) - P(Z_1 - Z_2 > Z_3 + z \ \& \ Z_3 \geq -z) \\ &= 1 - P(Z_3 < -z) + P(Z_2 - Z_1 > -Z_3 - z \ \& \ Z_3 < -z) \\ &\quad -P(Z_1 - Z_2 > Z_3 + z \ \& \ Z_3 \geq -z) \\ &= P(Z_3 \geq -z) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{C_{ij}}{\lambda_{1j}\Gamma(\alpha)\beta^{\alpha}} \int_0^{-z} z_3^{\alpha-1} e^{(z_3+z)\lambda_{2i}^{-1} - z_3\beta^{-1}} dz_3 \\ &\quad - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{C_{ij}}{\lambda_{2i}\Gamma(\alpha)\beta^{\alpha}} \int_{-z}^{\infty} z_3^{\alpha-1} e^{-(z_3+z)\lambda_{1j}^{-1} - z_3\beta^{-1}} dz_3 \\ &= S_{\Gamma}(-z, \alpha, \beta) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{C_{ij}}{\beta^{\alpha}} \left[\frac{e^{z\lambda_{2i}^{-1}}}{\Gamma(\alpha)\lambda_{1j}} f(z) - \frac{e^{-z\lambda_{1j}^{-1}} S_{\Gamma}\left(-z, \alpha, \left(\lambda_{1j}^{-1} + \beta^{-1}\right)^{-1}\right)}{\lambda_{2i}\left(\lambda_{1j}^{-1} + \beta^{-1}\right)^{\alpha}} \right], \end{aligned}$$

as required. ■

Proof of Theorem 7.4.3 Conditioned on $D = d$ and by using Theorem 7.4.2, we get

$$Z|_{D=d} \stackrel{d}{=} \sum_{j=1}^{r} E(\sigma_j) + \mu \sum_{i=1}^{r} c_i.$$

Upon separating $\boldsymbol{\sigma}$ into two subgroups, based on the elements being negative or

positive as described earlier in Theorem 7.4.3, we then obtain

$$Z|_{D=d} \stackrel{d}{=} GH(\boldsymbol{\lambda}_{1d}) - GH(-\boldsymbol{\lambda}_{2d}) + \mu \sum_{i=1}^{r} c_i.$$

Now unconditioning with respect to D, we obtain the distribution function of Z, as desired. ■

A.4 Appendix for Chapter 8

Proof of Theorem 8.2.1 Here, we will present the detailed proof for the derivation of the MLEs for the case when $d < \frac{n}{2}$ and abstain from presenting the proofs for all other cases for the sake of brevity since their derivations are quite similar. Even though it will not be known whether T is greater than μ or not, it is clear that the density in (8.2.1) will take on two different forms and so we will first find the MLE of μ based on the two cases $T \geq \mu$ and $T \leq \mu$. Then, finally the MLE of μ is determined by comparing the likelihoods under these two cases.

Case I: $T \geq \mu$

In this case, the likelihood function becomes

$$L_1 = \frac{C_d}{2^n \sigma^d} e^{-\frac{\sum_{i=1}^{d} |\mu - x_{i:n}| + (n-d)(T-\mu)}{\sigma}}, \qquad -\infty < x_{1:n} < \cdots < x_{d:n} < T.$$

We readily obtain the MLE of μ as $\hat{\mu}_1 = T$, with the corresponding likelihood being

$$L_1(T,\sigma) = \frac{C_d}{2^n \sigma^d} e^{-\frac{dT - \sum_{i=1}^d x_{i:n}}{\sigma}}.$$

Case II: $T \leq \mu$

In this case, the likelihood function becomes

$$L_2 = \frac{C_d}{(2\sigma)^d} e^{-\sum_{i=1}^d \frac{\mu - x_{i:n}}{\sigma}} \left(1 - \frac{1}{2} e^{-\frac{\mu - T}{\sigma}}\right)^{n-d}, \quad -\infty < x_{1:n} < \cdots < x_{d:n} < T.$$

Taking derivative with respect to μ and equating it to zero, we obtain the MLE of μ as

$$\hat{\mu}_2 = T + \sigma \log\left(\frac{n}{2d}\right).$$

Since for any σ, $L_1(T,\sigma) = L_2(T,\sigma) < L_2(\hat{\mu}_2,\sigma)$, we now obtain

$$\hat{\mu} = T + \hat{\sigma} \log\left(\frac{n}{2d}\right),$$

as given in (8.2.2). Then, the MLE of σ can be obtained from the profile likelihood function

$$L_{profile} = \frac{2dC_d}{n(2\sigma)^d} e^{-\sum_{i=1}^d \left(\frac{T - x_{i:n}}{\sigma}\right)} \left(1 - \frac{d}{n}\right)^{n-d}, \qquad -\infty < x_{1:n} < \cdots < x_{d:n} < T.$$

Taking derivative with respect to σ and equating it to zero, we obtain the MLE of σ as

$$\hat{\sigma} = \frac{1}{d}\sum_{i=1}^{d}\left(T - X_{i:n}\right),$$

as given in (8.2.3). ■

Proof of Theorem 8.3.1. We shall consider the two cases $\mu < T$ and $\mu \geq T$.

Case I: $\mu < T$. In this case, we have

$$\begin{aligned} E[e^{t\hat{\sigma}}|D>0] &= \sum_{d=1}^{m}\sum_{j=0}^{d} E\left[e^{t\hat{\sigma}}|D=d, J=j\right] P(D=d, J=j|D>0) \\ &\quad + \sum_{d=m+1}^{n}\sum_{j=0}^{d} E\left[e^{t\hat{\sigma}}|D=d, J=j\right] P(D=d, J=j|D>0). \end{aligned}$$

The joint distribution of J and D, conditional on $D > 0$, is

$$\begin{aligned} P(D=d, J=j|D>0) &= P(J=j|D=d)\frac{P(D=d, D>0)}{P(D>0)} \\ &= (1-p_0)^{-1}\binom{d}{j}\binom{n}{d}[F(T)]^d[1-F(T)]^{n-d} \\ &= \frac{n!\left(1-e^{-\frac{T-\mu}{\sigma}}\right)^{d-j} e^{-\frac{(T-\mu)(n-d)}{\sigma}}}{2^n(n-d)!(d-j)!j!(1-p_0)}. \end{aligned}$$

Now, the joint distribution of $X_{1:d}, \cdots, X_{d:d}$, conditional on $D = d$ and $J = j$, is

given by

$$f(X_{1:d},\cdots,X_{d:d}|D=d,J=j)=j!(d-j)!\prod_{i=1}^{j}\frac{1}{\sigma}e^{-\frac{\mu-x_{i:d}}{\sigma}}\prod_{i=j+1}^{d}\frac{\frac{1}{\sigma}e^{-\frac{x_{i:d}-\mu}{\sigma}}}{1-e^{-\frac{T-\mu}{\sigma}}},$$
$$x_{1:d}<\cdots<x_{j:d}<\mu<x_{j+1:d}<\cdots<x_{d:d}<T.$$

For the case when $D\leq\frac{n}{2}$, and given $J=j$, we find

$$\hat{\sigma}|_{D=d,J=j}\overset{d}{=}\sum_{i=1}^{j}E\left(\frac{\sigma}{d}\right)+\sum_{i=j+1}^{d}E_R^*\left(\frac{\sigma}{d},T-\mu\right)+T-\mu,$$

where $E^*(\theta)$ denotes the negative exponential distribution with scale parameter θ, that is, if X follows the exponential distribution, then $-X$ is negative exponentially distributed, i.e., $X\sim E(\theta)$, $-X\sim E^*(\theta)$, and $E_R^*(\sigma,T)$ denotes a negative exponential distribution right truncated at time T. Then, we have the conditional MGF of $\hat{\sigma}$ as

$$\begin{aligned}E[e^{t\hat{\sigma}}|D=d,J=j] &= e^{t(T-\mu)}\left(1-\frac{t\sigma}{d}\right)^{-j}\left\{\left(1+\frac{t\sigma}{d}\right)^{-1}\frac{1-e^{-(T-\mu)(\frac{1}{\sigma}+\frac{t}{d})}}{1-e^{-\frac{T-\mu}{\sigma}}}\right\}^{d-j}\\ &= \sum_{l=0}^{d-j}\binom{d-j}{l}(-1)^l\left(1-e^{-\frac{T-\mu}{\sigma}}\right)^{-(d-j)}e^{-\frac{(T-\mu)l}{\sigma}}\\ &\quad\times e^{(T-\mu)\frac{(d-l)t}{d}}\left(1-\frac{t\sigma}{d}\right)^{-j}\left(1+\frac{t\sigma}{d}\right)^{-(d-j)}.\end{aligned}$$

For the case when $D>m$, we need to consider $J\leq m$ and $J>m$ separately.

When $J \leq m$, similarly, we will have the first j failures as i.i.d. exponential random variables, denoted by $X_1, \cdots, X_j$. For the remaining $d-j$ failures, we may consider them as $m-j$ i.i.d. failures before $X_{m+1:d}$ and $d-m-1$ i.i.d. failures after $X_{m+1:d}$ with $\mu < X_{m+1:d} < T$, denoted by $X_{j+1}, \cdots, X_m$ and $X_{m+2}, \cdots, X_d$, respectively.

The joint pdf $\mathbf{X} = (X_1, \cdots, X_m, X_{m+1:d}, X_{m+2}, \cdots, X_d)$ is given by

$$\begin{aligned} f(\mathbf{X}|D=d, J=j) &= \frac{(d-j)!}{(m-j)!(d-m-1)!} \left(\prod_{i=1}^{j} \frac{1}{\sigma} e^{-\frac{\mu - x_i}{\sigma}} \right) \left(\prod_{i=j}^{m} \frac{\frac{1}{\sigma} e^{-\frac{x_i - \mu}{\sigma}}}{1 - e^{-\frac{T-\mu}{\sigma}}} \right) \\ &\quad \times \left(\frac{\frac{1}{\sigma} e^{-\frac{x_{m+1:d} - \mu}{\sigma}}}{1 - e^{-\frac{T-\mu}{\sigma}}} \right) \left(\prod_{i=m+2}^{d} \frac{\frac{1}{\sigma} e^{-\frac{x_i - \mu}{\sigma}}}{1 - e^{-\frac{T-\mu}{\sigma}}} \right), \\ &\quad x_1, \cdots, x_j < \mu < x_{j+1}, \cdots, x_m < x_{m+1:d} < x_{m+2}, \cdots, x_d < T. \end{aligned}$$

Then, the conditional MGF can be derived as follows:

$$
\begin{aligned}
E[e^{t\hat{\sigma}}|J=j,D=d] &= \frac{(d-j)!}{(m-j)!(d-m-1)!}\left(1-e^{-\frac{T-\mu}{\sigma}}\right)^{-(d-j)} e^{\frac{(n-d)(T-\mu)t}{d}}\left[\int_{-\infty}^{\mu}\frac{1}{\sigma}e^{-(\mu-x)(\frac{1}{\sigma}-\frac{t}{d})}dx\right]^{j} \\
&\quad \times\int_{\mu}^{T}\left[\int_{\mu}^{x_{m+1:d}}\frac{1}{\sigma}e^{-(x-\mu)(\frac{1}{\sigma}+\frac{t}{d})}dx\right]^{m-j}\left[\int_{x_{m+1:d}}^{T}\frac{1}{\sigma}e^{-(x-\mu)(\frac{1}{\sigma}-\frac{t}{d})}dx\right]^{d-m-1} \\
&\quad \times\frac{1}{\sigma}e^{-(x_{m+1:d}-\mu)(\frac{1}{\sigma}-\frac{t}{d})}dx_{m+1:d} \\
&= \frac{(d-j)!}{(m-j)!(d-m-1)!}\left(1-e^{-\frac{T-\mu}{\sigma}}\right)^{-(d-j)} e^{\frac{(n-d)(T-\mu)t}{d}}\left(1-\frac{t\sigma}{d}\right)^{-(j+d-m-1)} \\
&\quad \times\left(1+\frac{t\sigma}{d}\right)^{-(m-j)}\int_{\mu}^{T}\frac{1}{\sigma}e^{-(x_{m+1:d}-\mu)(\frac{1}{\sigma}-\frac{t}{d})}\left[1-e^{-(x_{m+1:d}-\mu)(\frac{1}{\sigma}+\frac{t}{d})}\right]^{m-j} \\
&\quad \times\left\{e^{-(x_{m+1:d}-\mu)(\frac{1}{\sigma}-\frac{t}{d})}-e^{-(T-\mu)(\frac{1}{\sigma}-\frac{t}{d})}\right\}^{d-m-1}dx_{m+1:d} \\
&= \frac{(d-j)!}{(m-j)!(d-m-1)!}\left(1-e^{-\frac{T-\mu}{\sigma}}\right)^{-(d-j)} e^{\frac{(n-d)(T-\mu)t}{d}}\left(1-\frac{t\sigma}{d}\right)^{-(j+d-m-1)} \\
&\quad \times\left(1+\frac{t\sigma}{d}\right)^{-(m-j)}\sum_{l_1=0}^{m-j}\sum_{l_2=0}^{d-m-1}(-1)^{l_1+d-m-1-l_2}\binom{m-j}{l_1}\binom{d-m-1}{l_2} \\
&\quad \times e^{-(T-\mu)(\frac{1}{\sigma}-\frac{t}{d})(d-m-1-l_2)}\int_{\mu}^{T}\frac{1}{\sigma}e^{-(x_{m+1:d}-\mu)(\frac{l_1+l_2+1}{\sigma}-\frac{t(l_2+1-l_1)}{d})}dx_{m+1:d} \\
&= \sum_{l_1=0}^{m-j}\sum_{l_2=0}^{d-m-1}(-1)^{l_1+d-m-1-l_2}\binom{m-j}{l_1}\binom{d-m-1}{l_2}\frac{(d-j)!}{(m-j)!(d-m-1)!} \\
&\quad \times(l_1+l_2+1)^{-1}\left(1-e^{-\frac{T-\mu}{\sigma}}\right)^{-(d-j)} e^{(T-\mu)\left(\frac{(m-1-l_2)t}{d}-\frac{d-m-1-l_2}{\sigma}\right)}\left(1+\frac{t\sigma}{d}\right)^{-(m-j)} \\
&\quad \times\left(1-\frac{t\sigma}{d}\right)^{-(j+d-m-1)}\left\{1-\frac{t(l_2+1-l_1)\sigma}{d(l_1+l_2+1)}\right\}^{-1}\left\{1-e^{-(T-\mu)\left[\frac{l_1+l_2+1}{\sigma}-\frac{t(l_2+1-l_1)}{d}\right]}\right\}.
\end{aligned}
$$

We can similarly derive the conditional MGF for the case of $J > m$.

Case II $\mu > T$. In this case, we have

$$
\begin{aligned}
P(D=d|D>0) &= (1-q_0)^{-1}\binom{n}{d}[F(T)]^d[1-F(T)]^{n-d} \\
&= (1-q_0)^{-1}\binom{n}{d}\left(\frac{1}{2}e^{-\frac{\mu-T}{\sigma}}\right)^d\left(1-\frac{1}{2}e^{-\frac{\mu-T}{\sigma}}\right)^{n-d}.
\end{aligned}
$$

Now, by adopting a similar procedure, we can obtain the corresponding conditional MGF. Finally, upon combining these expressions, we obtain the conditional MGF

presented in Theorem 8.3.1. ■

To derive the marginal and joint CDF of W_1 and W_2 as described in Lemma 8.3.1, we first need the following lemma.

Lemma A.4.1 *Let $Y_1 \sim \Gamma(\alpha_1, \beta_1)$ and $Y_2 \sim \Gamma(\alpha_2, \beta_2)$ be independent with integer shape parameters α_1 and α_2 and $\beta_1, \beta_2 > 0$. Let $Y = Y_1 - Y_2$. Then, the CDF of Y, denoted by $\Gamma(y, \alpha_1, \alpha_2, \beta_1, \beta_2)$, is given by*

$$\begin{aligned}
\Gamma(y, \alpha_1, \alpha_2, \beta_1, \beta_2) &= P(Y_1 \leq y + y_2) = 1 - P(Y_1 > y + y_2) \\
&= 1 - \sum_{i=0}^{\alpha_1 - 1} \int_0^\infty \frac{1}{i!} \left(\frac{y + y_2}{\beta_1} \right)^i e^{-\frac{y+y_2}{\beta_1}} \frac{y_2^{\alpha_2 - 1} e^{-\frac{y_2}{\beta_2}}}{\Gamma(\alpha_2) \beta_2^{\alpha_2}} dy_2 \\
&= 1 - \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \binom{i}{j} \frac{y^{i-j} e^{-\frac{y}{\beta_1}} \int_0^\infty y_2^{\alpha_2 + j - 1} e^{-y_2 \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right)} dy_2}{i! \beta_1^i \Gamma(\alpha_2) \beta_2^{\alpha_2} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right)^{\alpha_2 + j}} \\
&= 1 - \frac{\beta_1^{\alpha_2}}{\Gamma(\alpha_2) (\beta_1 + \beta_2)^{\alpha_2}} \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \frac{\Gamma(\alpha_2 + j) \beta_1^{j-i} \beta_2^j y^{i-j} e^{-\frac{y}{\beta_1}}}{(i-j)! j! (\beta_1 + \beta_2)^j}, \ y \geq 0;
\end{aligned}$$

$$\begin{aligned}
\Gamma(y, \alpha_1, \alpha_2, \beta_1, \beta_2) &= P(Y_2 \geq y_1 - y) \\
&= \frac{\beta_2^{\alpha_1}}{\Gamma(\alpha_1) (\beta_1 + \beta_2)^{\alpha_1}} \sum_{i=0}^{\alpha_2 - 1} \sum_{j=0}^{i} \frac{\Gamma(\alpha_1 + j) \beta_2^{j-i} \beta_1^j (-y)^{i-j} e^{\frac{y}{\beta_2}}}{(i-j)! j! (\beta_1 + \beta_2)^j}, \quad y < 0.
\end{aligned}$$

Now, the corresponding PDF is given by

$$
\begin{aligned}
f(y) &= \frac{\beta_1^{\alpha_2}}{\Gamma(\alpha_2)\left(\beta_1+\beta_2\right)^{\alpha_2}} \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i} \frac{\Gamma(\alpha_2+j)\beta_1^{j-i-1}\beta_2^j y^{i-j} e^{-\frac{y}{\beta_1}}}{(i-j)!j!\left(\beta_1+\beta_2\right)^j} \\
&\quad -\frac{\beta_1^{\alpha_2}}{\Gamma(\alpha_2)\left(\beta_1+\beta_2\right)^{\alpha_2}} \sum_{i=1}^{\alpha_1-1}\sum_{j=0}^{i-1} \frac{\Gamma(\alpha_2+j)\beta_1^{j-i}\beta_2^j y^{i-j-1} e^{-\frac{y}{\beta_1}}}{(i-j-1)!j!\left(\beta_1+\beta_2\right)^j} \\
&= \frac{\beta_1^{\alpha_2}}{\Gamma(\alpha_2)\left(\beta_1+\beta_2\right)^{\alpha_2}} \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i} \frac{\Gamma(\alpha_2+j)\beta_1^{j-i-1}\beta_2^j y^{i-j} e^{-\frac{y}{\beta_1}}}{(i-j)!j!\left(\beta_1+\beta_2\right)^j} \\
&\quad -\frac{\beta_1^{\alpha_2}}{\Gamma(\alpha_2)\left(\beta_1+\beta_2\right)^{\alpha_2}} \sum_{i=0}^{\alpha_1-2}\sum_{j=0}^{i} \frac{\Gamma(\alpha_2+j)\beta_1^{j-i-1}\beta_2^j y^{i-j} e^{-\frac{y}{\beta_1}}}{(i-j)!j!\left(\beta_1+\beta_2\right)^j} \\
&= \frac{\beta_1^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2)\left(\beta_1+\beta_2\right)^{\alpha_2}} \sum_{j=0}^{\alpha_1-1} \frac{\Gamma(\alpha_2+j)\beta_1^{j}\beta_2^j y^{\alpha_1-j-1} e^{-\frac{y}{\beta_1}}}{(\alpha_1-j-1)!j!\left(\beta_1+\beta_2\right)^j}, \qquad y \geq 0;
\end{aligned}
$$

$$
f(y) = \frac{\beta_2^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\left(\beta_1+\beta_2\right)^{\alpha_1}} \sum_{j=0}^{\alpha_2-1} \frac{\Gamma(\alpha_1+j)\beta_1^{j}\beta_2^j (-y)^{\alpha_2-j-1} e^{\frac{y}{\beta_2}}}{(\alpha_2-j-1)!j!\left(\beta_1+\beta_2\right)^j}, \qquad y < 0.
$$

The marginal distribution of W_2 is either exponential or is a linear combination of two exponential variables, and so its CDF is easy to obtain and is therefore omitted for brevity. By using Lemma A.4.1, the marginal CDF of W_1 can be readily obtained as presented in the following lemma.

Lemma A.4.2 *Let $Y_1 \sim \Gamma(\alpha_1, \beta_1)$, $Y_2 \sim N\Gamma(\alpha_2, \beta_2)$, with α_1 and α_2 being positive integer shape parameters and β_1 and $\beta_2 > 0$. Further, let $Z_1 \sim E(1)$ and $Z_2 \sim E(1)$, with $a_1 \neq a_2 \neq \beta_1$ (and β_2) > 0. If Y_1, Y_2, Z_1 and Z_2 are all independent, then the*

CDF of $W_1 = Y_1 + Y_2 + a_1 Z_1 - a_2 Z_2$ *is as follows:*

$$\begin{aligned} P(W_1 \le w_1) &= \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(w_1, \infty, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{a_2}\right) + F(w_1, \alpha_1, \alpha_2, \beta_1, \beta_2) \\ &\quad - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right) \\ &\quad - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_1}\right), \quad w_1 \ge 0; \end{aligned}$$

$$\begin{aligned} P(W_1 \le w_1) &= \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(w_1, 0, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right) \\ &\quad + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, \infty, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{a_2}\right) + \Gamma(w_1, \alpha_1, \alpha_2, \beta_1, \beta_2) \\ &\quad - \frac{a_1 e^{-\frac{w}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, w_1, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right), \quad w_1 < 0. \end{aligned}$$

The CDF of $W_1 = Y_1 + Y_2 + a_1 Z_1 + a_2 Z_2$ *is as follows:*

$$\begin{aligned} P(W_1 \le w_1) &= \Gamma(w_1, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{a_2}{a_2 - a_1} e^{-\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_2}\right) \\ &\quad - \frac{a_2 e^{-\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_2}\right) \\ &\quad + \frac{a_1}{a_2 - t a_1} e^{-\frac{w}{\theta_1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_1}\right) \\ &\quad + \frac{a_1}{a_2 - \theta_1} e^{-\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right), \quad w_1 \ge 0; \end{aligned}$$

$$\begin{aligned} P(W_1 \leq w_1) &= \Gamma(w_1, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{a_2 e^{-\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, w_1, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_2}\right) \\ &\quad + \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_2 - a_1} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, w_1, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right), \quad w_1 < 0. \end{aligned}$$

By using Lemmas A.4.1 and A.4.2, the joint CDF of W_1 and W_2 can be derived as presented in the following Lemmas. Note in these lemmas that all the coefficients a_1, a_2, b_1 and b_2 are positive, all the scale parameters β_1, β_2 are positive, and the shape parameters α_1 and α_2 are positive integers. Moreover, in these lemmas, we have used the notation

$$\begin{aligned} r_1 &= w_1 - \frac{a_1 w_2}{b_1}, \\ r_2 &= w_1, \\ r_3 &= w_1 + \frac{a_2 w_2}{b_2}, \\ r_4 &= w_1 + \frac{a_1 w_2}{b_1}, \\ r_1^* &= \max(r_1, 0), \\ r_2^* &= \max(r_2, 0), \\ r_3^* &= \max(r_3, 0), \\ r_4^* &= \max(r_4, 0), \\ S\Gamma\left(l, r, \alpha_1, \alpha_2, \beta_1, \beta_2\right) &= \Gamma\left(r, \alpha_1, \alpha_2, \beta_1, \beta_2\right) - \Gamma\left(l, \alpha_1, \alpha_2, \beta_1, \beta_2\right), \\ C_j &= \frac{\Gamma(\alpha_2 + j)\beta_1^{\alpha_2 - \alpha_1 + j}\beta_2^j}{\Gamma(\alpha_2)(\alpha_1 - j - 1)!j!\left(\beta_1 + \beta_2\right)^{\alpha_2 + j}}, \\ C_\Gamma &= \frac{1}{\Gamma(\alpha)\beta^\alpha}, \end{aligned}$$

and

$$F\left(l,r,\theta_1,\theta_2\right)=\int_l^r x^{\theta_1-1}e^{-x\theta_2}dx=\begin{cases}\frac{\Gamma(\theta_1)}{\theta_2^{\theta_1}}\left(\Gamma(r,\theta_1,\theta_2^{-1})-\Gamma(l,\theta_1,\theta_2^{-1})\right), & \theta_2>0, l>0,\\ \frac{\Gamma(\theta_1)}{(-\theta_2)^{\theta_1}}\left(\Gamma(-l,\theta_1,\theta_2^{-1})-\Gamma(-r,\theta_1,\theta_2^{-1})\right), & \theta_2>0, r<0,\\ \frac{1}{\theta_1}\left(r^{\theta_1}-l^{\theta_1}\right), & \theta_2=0,\\ -\sum\limits_{j=0}^{\theta_1-1}\frac{(\theta_1-1)!}{j!\theta_2^{\theta_1-j}}\left(r^je^{-r\theta_2}-l^je^{-l\theta_2}\right), & \theta_2<0,\end{cases}$$

where $\gamma(x,\cdot,\cdot)$ denotes the CDF of the gamma distribution.

By using these results, we can also obtain a more general result for the joint CDF of $W_1=Y_1+Y_2+a_1Z_1+a_2Z_2$ and $W_2=b_1Z_1+b_2Z_2$ by using the known results on the joint CDF of $-W_1$ and W_2 and the CDF of W_2 as

$$\begin{aligned}P(W_1\le w_1,W_2\le w_2) &= P(-W_1\ge -w_1,W_2\le w_2)\\ &= P(W_2\le w_2)-P(-W_1\le -w_1,W_2\le w_2).\end{aligned}$$

Lemma A.4.3 *Let $Y_1\sim\Gamma(\alpha_1,\beta_1)$, $Y_2\sim\Gamma(\alpha_2,\beta_2)$, with α_1 and α_2 being positive integers and β_1 and $\beta_2>0$. Further, let $Z_1\sim E(\theta_1)$ and $Z_2\sim E(\theta_2)$, with $\theta_1\neq\theta_2>0$. If Y_1, Y_2, Z_1 and Z_2 are all independent and $Y=Y_1-Y_2$, then the CDF of $W=Y+Z_1+Z_2$ is as follows:*

Case I: $w \geq 0$

$$\begin{aligned}
P(W \leq w) &= F(w, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{\theta_2}{\theta_2 - \theta_1} e^{-\frac{w}{\theta_2}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{\theta_2}\right) \\
&\quad - \frac{\theta_2}{\theta_2 - \theta_1} e^{-\frac{w}{\theta_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{\theta_2}\right) \\
&\quad + \frac{\theta_1}{\theta_2 - \theta_1} e^{-\frac{w}{\theta_1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{\theta_1}\right) \\
&\quad + \frac{\theta_1}{\theta_2 - \theta_1} e^{-\frac{w}{\theta_1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{\theta_1}\right);
\end{aligned}$$

Case II: $w < 0$

$$\begin{aligned}
P(W \leq w) &= F(w, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{\theta_2}{\theta_2 - \theta_1} e^{-\frac{w}{\theta_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, w, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{\theta_2}\right) \\
&\quad + \frac{\theta_1}{\theta_2 - \theta_1} e^{-\frac{w}{\theta_1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, w, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{\theta_1}\right).
\end{aligned}$$

Lemma A.4.4 *Let $Y_1 \sim \Gamma(\alpha_1, \beta_1)$, $Y_2 \sim \Gamma(\alpha_2, \beta_2)$, with α_1 and α_2 being positive integers and β_1 and $\beta_2 > 0$. Further let $Z_1 \sim E(\theta_1)$ and $Z_2 \sim E(\theta_2)$, with $\theta_1 \neq \theta_2 > 0$. If Y_1, Y_2, Z_1 and Z_2 are all independent and $Y = Y_1 - Y_2$, then the CDF of $W = Y + Z_1 - Z_2$ is as follows:*

Case I: $w \geq 0$

$$\begin{aligned} P(W \leq w) &= \frac{\theta_2}{\theta_1+\theta_2} e^{\frac{w}{\theta_2}} \sum_{j=0}^{\alpha_1-1} C_j F\left(w, \infty, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{\theta_2}\right) + F(w, \alpha_1, \alpha_2, \beta_1, \beta_2) \\ &\quad - \frac{\theta_1}{\theta_1+\theta_2} e^{-\frac{w}{\theta_1}} \sum_{j=0}^{\alpha_2-1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{\theta_1}\right) \\ &\quad - \frac{\theta_1}{\theta_1+\theta_2} e^{-\frac{w}{\theta_1}} \sum_{j=0}^{\alpha_1-1} C_j F\left(0, w, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{\theta_1}\right); \end{aligned}$$

Case II: $w < 0$

$$\begin{aligned} P(W \leq w) &= \frac{\theta_2}{\theta_1+\theta_2} e^{\frac{w}{\theta_2}} \sum_{j=0}^{\alpha_2-1} C_j^* F\left(w, 0, \alpha_2 - j, \frac{1}{\theta_2} - \frac{1}{\beta_2}\right) \\ &\quad + \frac{\theta_2}{\theta_1+\theta_2} e^{\frac{w}{\theta_2}} \sum_{j=0}^{\alpha_1-1} C_j F\left(0, \infty, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{\theta_2}\right) + F(w, \alpha_1, \alpha_2, \beta_1, \beta_2) \\ &\quad - \frac{\theta_1}{\theta_1+\theta_2} e^{-\frac{w}{\theta_1}} \sum_{j=0}^{\alpha_2-1} C_j^* F\left(-\infty, w, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{\theta_1}\right). \end{aligned}$$

Lemma A.4.5 *Let $Y_1 \sim \Gamma(\alpha_1, \beta_1)$, $Y_2 \sim N\Gamma(\alpha_2, \beta_2)$, $Z_1 \sim E(1)$, $Z_2 \sim E(1)$, with Y_1, Y_2, Z_1 and Z_2 being independent and β_1, $\beta_2 > 0$ and α_1 and α_2 being positive integer. Further, let $W_1 = Y_1 + Y_2 + aZ_1$ and $W_2 = b_1 Z_1 + b_2 Z_2$ with $a, b_1, b_2 > 0$. Then, the joint CDF of W_1 and W_2 is as follows:*

Case I: $b_1 \neq b_2$**,** $b_1 w_1 \geq a w_2$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(w_1 - \frac{a w_2}{b_1}, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + S\Gamma\left(w_1 - \frac{a w_2}{b_1}, w_1, \alpha_1, \alpha_2, \beta_1, \beta_2\right)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}}\right) \\
& - e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(w_1 - \frac{a w_2}{b_1}, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
& + \frac{b_2 e^{-\frac{a w_2 + (b_2 - b_1) w_1}{a b_2}}}{b_2 - b_1} \sum_{j=0}^{\alpha_1 - 1} C_j \left(w_1 - \frac{a w_2}{b_1}, w_1, \alpha_1 - j, \frac{1}{\beta_1} + \frac{b_1 - b_2}{a b_2}\right);
\end{aligned}
$$

Case II: $b_1 \neq b_2$**,** $b_1 w_1 < a w_2$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(w_1 - \frac{a w_2}{b_1}, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + S\Gamma\left(w_1 - \frac{a w_2}{b_1}, w_1, \alpha_1, \alpha_2, \beta_1, \beta_2\right)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}}\right) \\
& - e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
& - \frac{b_2}{b_1 - b_2} e^{-\frac{a w_2 + (b_2 - b_1) w_1}{a b_2}} \sum_{j=0}^{\alpha_1 - 1} C_j \left(0, w_1, \alpha_1 - j, \frac{1}{\beta_1} + \frac{b_1 - b_2}{a b_2}\right) \\
& - e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(w_1 - \frac{a w_2}{b_1}, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& - \frac{b_2 e^{-\frac{a w_2 + (b_2 - b_1) w_1}{a b_2}}}{b_1 - b_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* \left(w_1 - \frac{a w_2}{b_1}, 0, \alpha_2 - j, \frac{b_1 - b_2}{a b_2} - \frac{1}{\beta_2}\right);
\end{aligned}
$$

Case III: $b_1 = b_2 = b$**,** $bw_1 \geq aw_2$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma\left(w_1 - \frac{aw_2}{b}, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - e^{-\frac{w_2}{b}} - \frac{w_2}{b}e^{-\frac{w_2}{b}}\right) \\
&+ S\Gamma\left(w_1 - \frac{aw_2}{b_1}, w_1, \alpha_1, \alpha_2, \beta_1, \beta_2\right)\left(1 - \frac{w_1}{a}e^{-\frac{w_2}{b}}\right) \\
&- e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(w_1 - \frac{aw_2}{b_1}, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
&+ e^{-\frac{w_2}{b}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(w_1 - \frac{aw_2}{b}, w_1, \alpha_1 + 1 - j, \frac{1}{\beta_1}\right);
\end{aligned}$$

Case IV: $b_1 = b_2 = b$**,** $bw_1 < aw_2$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma\left(w_1 - \frac{aw_2}{b}, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - e^{-\frac{w_2}{b}} - \frac{w_2}{b}e^{-\frac{w_2}{b}}\right) \\
&+ S\Gamma\left(w_1 - \frac{aw_2}{b_1}, w_1, \alpha_1, \alpha_2, \beta_1, \beta_2\right)\left(1 - \frac{w_1}{a}e^{-\frac{w_2}{b}}\right) \\
&- e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
&+ e^{-\frac{w_2}{b}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w_1, \alpha_1 + 1 - j, \frac{1}{\beta_1}\right) \\
&- e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(w_1 - \frac{aw_2}{b}, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
&+ e^{-\frac{w_2}{b}} \sum_{j=0}^{\alpha_2 - 1} C_j F\left(w_1 - \frac{aw_2}{b}, 0, \alpha_2 + 1 - j, -\frac{1}{\beta_2}\right).
\end{aligned}$$

Lemma A.4.6 *Let* $Y_1 \sim \Gamma(\alpha_1, \beta_1)$*,* $Y_2 \sim N\Gamma(\alpha_2, \beta_2)$*,* $Z_1 \sim E(1)$*,* $Z_2 \sim E(1)$*, with* Y_1*,* Y_2*,* Z_1 *and* Z_2 *being independent and* β_1*,* $\beta_2 > 0$*. Further, let* $W_1 = Y_1 + Y_2 + aZ_1$ *and* $W_2 = b_1 Z_1 - b_2 Z_2$ *with* $a, b_1, b_2 > 0$*. Then, the joint CDF of* W_1 *and* W_2 *is as follows:*

Case I: $w_1 \geq 0$**,** $w_2 \geq 0$**,** $b_1 w_1 > a w_2$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(w_1 - \frac{a w_2}{b_1}, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - \frac{b_1}{b_1 + b_2} e^{-\frac{w_2}{b_1}}\right) \\
& - e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(w_1 - \frac{a w_2}{b_1}, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
& + S\Gamma\left(w_1 - \frac{a w_2}{b_1}, w_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right) \\
& - \frac{b_2}{b_1 + b_2} e^{\frac{a w_2 - (b_1 + b_2) w_1}{a b_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* \left(-\infty, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{b_1 + b_2}{a b_2}\right) \\
& - \frac{b_2}{b_1 + b_2} e^{\frac{a w_2 - (b_1 + b_2) w_1}{a b_2}} \sum_{j=0}^{\alpha_1 - 1} C_j \left(0, w_1 - \frac{a w_2}{b_1}, \alpha_1 - j, \frac{1}{\beta_1} - \frac{b_1 + b_2}{a b_2}\right);
\end{aligned}
$$

Case II: $w_1 \geq 0$**,** $w_2 \geq 0$**,** $b_1 w_1 < a w_2$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(w_1 - \frac{a w_2}{b_1}, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - \frac{b_1}{b_1 + b_2} e^{-\frac{w_2}{b_1}}\right) \\
& - e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, w_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
& - e^{-\frac{w_1}{a}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(w_1 - \frac{a w_2}{b_1}, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& + S\Gamma\left(w_1 - \frac{a w_2}{b_1}, w_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right) \\
& - \frac{b_2 e^{\frac{a w_2 - (b_1 + b_2) w_1}{a b_2}}}{b_1 + b_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* \left(-\infty, w_1 - \frac{a w_2}{b_1}, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{b_1 + b_2}{a b_2}\right);
\end{aligned}
$$

Case III: $w_1 > 0$**,** $w_2 < 0$

$$\begin{aligned} P(W_1 \le w_1, W_2 \le w_2) &= \frac{b_2 e^{\frac{w_2}{b_2}}}{b_1+b_2}\Gamma\left(w_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right) \\ &\quad -\frac{b_2 e^{\frac{aw_2-(b_1+b_2)w_1}{ab_2}}}{b_1+b_2}\sum_{j=0}^{\alpha_2-1} C_j^*\left(-\infty, 0, \alpha_2-j, -\frac{1}{\beta_2}-\frac{b_1+b_2}{ab_2}\right) \\ &\quad -\frac{b_2 e^{\frac{aw_2-(b_1+b_2)w_1}{ab_2}}}{b_1+b_2}\sum_{j=0}^{\alpha_1-1} C_j\left(0, w_1, \alpha_1-j, \frac{1}{\beta_1}-\frac{b_1+b_2}{ab_2}\right); \end{aligned}$$

Case IV: $w_1 < 0$**,** $w_2 \ge 0$

$$\begin{aligned} P(W_1 \le w_1, W_2 \le w_2) &= \Gamma\left(w_1-\frac{aw_2}{b_1}, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1-\frac{b_1}{b_1+b_2}e^{-\frac{w_2}{b_1}}\right) \\ &\quad -e^{-\frac{w_1}{a}}\sum_{j=0}^{\alpha_2-1} C_j^* F\left(w_1-\frac{aw_2}{b_1}, w_1, \alpha_2-j, -\frac{1}{\beta_2}-\frac{1}{a}\right) \\ &\quad +S\Gamma\left(w_1-\frac{aw_2}{b_1}, w_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right) \\ &\quad -\frac{b_2 e^{\frac{aw_2-(b_1+b_2)w_1}{ab_2}}}{b_1+b_2}\sum_{j=0}^{\alpha_2-1} C_j^*\left(-\infty, w_1-\frac{aw_2}{b_1}, \alpha_2-j, -\frac{1}{\beta_2}-\frac{b_1+b_2}{ab_2}\right); \end{aligned}$$

Case V: $w_1 < 0$**,** $w_2 < 0$

$$\begin{aligned} P(W_1 \le w_1, W_2 \le w_2) &= \frac{b_2 e^{\frac{w_2}{b_2}}}{b_1+b_2}\Gamma\left(w_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right) \\ &\quad -\frac{b_2 e^{\frac{aw_2-(b_1+b_2)w_1}{ab_2}}}{b_1+b_2}\sum_{j=0}^{\alpha_2-1} C_j^*\left(-\infty, w_1, \alpha_2-j, -\frac{1}{\beta_2}-\frac{b_1+b_2}{ab_2}\right). \end{aligned}$$

Lemma A.4.7 *Let* $Y_1 \sim \Gamma(\alpha_1, \beta_1)$*,* $Y_2 \sim N\Gamma(\alpha_2, \beta_2)$*,* $Z_1 \sim E(1)$*,* $Z_2 \sim E(1)$*, with* Y_1*,* Y_2*,* Z_1 *and* Z_2 *being independent and* β_1*,* $\beta_2 > 0$*. Further, let* $W_1 = Y_1 + Y_2 + a_1 Z_1 - a_2 Z_2$ *and* $W_2 = b_1 Z_1 + b_2 Z_2$ *with* $a_1, a_2, b_1, b_2 > 0$*. Then, the joint CDF of* W_1

and W_2 is as follows:

Case I: $w_1 \geq 0, b_1 \neq b_2, r_1 \geq 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \int_{-\infty}^{r_1} \int_0^{\frac{w_2}{b_2}} p\left(X_1 \leq \frac{w_2 - b_2 x_2}{b_1}\right) dx_2 dy \\
&+ \int_{r_1}^{r_3} \int_{\frac{a_1 w_2 - b_1 w_1 + b_1 y}{a_1 b_2 + a_2 b_1}}^{\frac{w_2}{b_2}} p\left(X_1 \leq \frac{w_2 - b_2 x_2}{b_1}\right) dx_2 dy \\
&+ \int_{r_1}^{r_2} \int_0^{\frac{a_1 w_2 - b_1 w_1 + b_1 y}{a_1 b_2 + a_2 b_1}} p\left(X_1 \leq \frac{w_1 - y + a_2 x_2}{a_1}\right) dx_2 dy \\
&+ \int_{r_2}^{r_3} \int_{\frac{y - w_1}{a_2}}^{\frac{a_1 w_2 - b_1 w_1 + b_1 y}{a_1 b_2 + a_2 b_1}} p\left(X_1 \leq \frac{w_1 - y + a_2 x_2}{a_1}\right) dx_2 dy \\
&= \Gamma\left(r_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right) \left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
&+ S\Gamma\left(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2\right) - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
&+ \frac{(a_1 b_2 + a_2 b_1) e^{\frac{(b_1 - b_2) w_1 - (a_1 + a_2) w_2}{a_1 b_2 + a_2 b_1}}}{(a_1 + a_2)(b_2 - b_1)} \\
&\times \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_1, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{b_1 - b_2}{a_1 b_2 + a_2 b_1}\right) \\
&- \frac{a_1}{a_1 + a_2} e^{-\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_1, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_1}\right) \\
&+ \frac{a_2}{a_1 + a_2} e^{\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{a_2}\right);
\end{aligned}
$$

Case II: $w_1 \geq 0, b_1 \neq b_2, r_1 < 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(r_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + S\Gamma\left(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2\right) - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& + \frac{(a_1 b_2 + a_2 b_1) e^{\frac{(b_1 - b_2) w_1 - (a_1 + a_2) w_2}{a_1 b_2 + a_2 b_1}}}{(a_1 + a_2)(b_2 - b_1)} \\
& \times \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 - j, -\frac{1}{\beta_2} + \frac{b_1 - b_2}{a_1 b_2 + a_2 b_1}\right) \\
& + \frac{(a_1 b_2 + a_2 b_1) e^{\frac{(b_1 - b_2) w_1 - (a_1 + a_2) w_2}{a_1 b_2 + a_2 b_1}}}{(a_1 + a_2)(b_2 - b_1)} \\
& \times \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{b_1 - b_2}{a_1 b_2 + a_2 b_1}\right) \\
& - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right) \\
& - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_1}\right) \\
& + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{a_2}\right);
\end{aligned}
$$

Case III: $w_1 < 0, b_1 \neq b_2, r_3 \geq 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(r_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + S\Gamma\left(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2\right) - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& + \frac{(a_1 b_2 + a_2 b_1) e^{\frac{(b_1 - b_2) w_1 - (a_1 + a_2) w_2}{a_1 b_2 + a_2 b_1}}}{(a_1 + a_2)(b_2 - b_1)} \\
& \times \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 - j, -\frac{1}{\beta_2} + \frac{b_1 - b_2}{a_1 b_2 + a_2 b_1}\right) \\
& + \frac{(a_1 b_2 + a_2 b_1) e^{\frac{(b_1 - b_2) w_1 - (a_1 + a_2) w_2}{a_1 b_2 + a_2 b_1}}}{(a_1 + a_2)(b_2 - b_1)} \\
& \times \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{b_1 - b_2}{a_1 b_2 + a_2 b_1}\right) \\
& - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_2, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right) \\
& + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, -\frac{1}{\beta_2} + \frac{1}{a_2}\right) \\
& + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{a_2}\right);
\end{aligned}
$$

Case IV: $w_1 < 0, b_1 \neq b_2, r_3 < 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(r_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + S\Gamma\left(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2\right) - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& + \frac{(a_1 b_2 + a_2 b_1) e^{\frac{(b_1 - b_2) w_1 - (a_1 + a_2) w_2}{a_1 b_2 + a_2 b_1}}}{(a_1 + a_2)(b_2 - b_1)} \\
& \times \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_3, \alpha_2 - j, -\frac{1}{\beta_2} + \frac{b_1 - b_2}{a_1 b_2 + a_2 b_1}\right) \\
& - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_2, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right) \\
& + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, -\frac{1}{\beta_2} + \frac{1}{a_2}\right);
\end{aligned}
$$

Case V: $w_1 \geq 0, b_1 = b_2 = b, r_1 \geq 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(r_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - e^{-\frac{w_2}{b}} - \frac{w_2}{b} e^{-\frac{w_2}{b}}\right) \\
& - \left(1 + \frac{a_2 w_2 + b w_1}{b(a_1 + a_2)}\right) e^{-\frac{w_2}{b}} S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& + \frac{e^{-\frac{w_2}{b}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_1, r_3, \alpha_1 + 1 - j, \frac{1}{\beta_1}\right) \\
& + S\Gamma\left(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& + \frac{a_1 e^{\frac{-w_2}{b}}}{a_1 + a_2} S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_1, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_1}\right) \\
& + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j \left(r_2, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{a_2}\right);
\end{aligned}
$$

Case VI: $w_1 \geq 0, b_1 = b_2 = b, r_1 < 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma\left(r_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - e^{-\frac{w_2}{b}} - \frac{w_2}{b} e^{-\frac{w_2}{b}}\right) \\
& - \left(1 + \frac{a_2 w_2 + b w_1}{b(a_1 + a_2)}\right) e^{-\frac{w_2}{b}} S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& + \frac{e^{-\frac{w_2}{b}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 + 1 - j, -\frac{1}{\beta_2}\right) \\
& + \frac{e^{-\frac{w_2}{b}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 + 1 - j, \frac{1}{\beta_1}\right) \\
& + S\Gamma\left(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& + \frac{a_1 e^{\frac{-w_2}{b}}}{a_1 + a_2} S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
& - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right) \\
& - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_1}\right) \\
& + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} \sum_{j=0}^{\alpha_1 - 1} C_j \left(r_2, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{a_2}\right);
\end{aligned}
$$

Case VII: $w_1 < 0, b_1 = b_2 = b, bw_1 + a_2w_2 \geq 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma\left(r_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - e^{-\frac{w_2}{b}} - \frac{w_2}{b}e^{-\frac{w_2}{b}}\right) \\
&- \left(1 + \frac{a_2w_2 + bw_1}{b(a_1 + a_2)}\right)e^{-\frac{w_2}{b}}S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
&+ \frac{e^{-\frac{w_2}{b}}}{a_1 + a_2}\sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 + 1 - j, -\frac{1}{\beta_2}\right) \\
&+ \frac{e^{-\frac{w_2}{b}}}{a_1 + a_2}\sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 + 1 - j, \frac{1}{\beta_1}\right) \\
&+ S\Gamma\left(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
&+ \frac{a_1 e^{\frac{-w_2}{b}}}{a_1 + a_2}S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
&- \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2}\sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_2, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2}\sum_{j=0}^{\alpha_2 - 1} C_j^* \left(r_2, 0, \alpha_2 - j, -\frac{1}{\beta_2} + \frac{1}{a_2}\right) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2}\sum_{j=0}^{\alpha_1 - 1} C_j \left(0, r_3, \alpha_1 - j, \frac{1}{\beta_1} + \frac{1}{a_2}\right);
\end{aligned}
$$

Case VIII: $w_1 < 0, b_1 = b_2 = b, bw_1 + a_2w_2 < 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma\left(r_1, \alpha_1, \beta_1, \alpha_2, \beta_2\right)\left(1 - e^{-\frac{w_2}{b}} - \frac{w_2}{b}e^{-\frac{w_2}{b}}\right) \\
&- \left(1 + \frac{a_2w_2 + bw_1}{b(a_1 + a_2)}\right)e^{-\frac{w_2}{b}}S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
&+ \frac{e^{-\frac{w_2}{b}}}{a_1 + a_2}\sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_3, \alpha_2 + 1 - j, -\frac{1}{\beta_2}\right) \\
&+ S\Gamma\left(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
&+ \frac{a_1 e^{\frac{-w_2}{b}}}{a_1 + a_2}S\Gamma\left(r_1, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2\right) \\
&- \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2}\sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_2, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a_1}\right) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2}\sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, -\frac{1}{\beta_2} + \frac{1}{a_2}\right).
\end{aligned}
$$

Lemma A.4.8 *Let $Y_1 \sim \Gamma(\alpha_1, \beta_1)$, $Y_2 \sim N\Gamma(\alpha_2, \beta_2)$, $Z_1 \sim E(1)$, $Z_2 \sim E(1)$, with Y_1, Y_2, Z_1 and Z_2 being independent and β_1, $\beta_2 > 0$. Further, let $W_1 = Y_1 + Y_2 - a_1 Z_1 - a_2 Z_2$ and $W_2 = b_1 Z_1 + b_2 Z_2$. If $a_1, a_2, b_1, b_2 > 0$, and when $a_1 b_2 \geq a_2 b_1$, then the joint CDF of W_1 and W_2 is as follows:*

Case I: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 \geq 0$**,** $a_1 b_2 > a_2 b_1$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \;=\; & \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + \frac{a_1}{a_1 - a_2} e^{\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_4, \alpha_1 - j, \frac{1}{a_1} + \frac{1}{\beta_1}\right) \\
& + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1} S\Gamma(r_2, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{e^{\frac{(b_2 - b_1) w_1 + (a_2 - a_1) w_2}{a_1 b_2 - a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} - \frac{b_1}{b_2 - b_1}\right)^{-1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_3, r_4, \alpha_1 - j, \frac{1}{\beta_1} + \frac{b_2 - b_1}{a_1 b_2 - a_2 b_1}\right) \\
& - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{a_2}{a_2 - a_1} e^{\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 - j, \frac{1}{a_2} + \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case II: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $b_2 w_1 + a_2 w_2 \geq 0$**,** $a_1 b_2 > a_2 b_1$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + \frac{a_1}{a_1 - a_2} e^{\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
& + \frac{a_1}{a_1 - a_2} e^{\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_4, \alpha_1 - j, \frac{1}{a_1} + \frac{1}{\beta_1}\right) \\
& + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1} S\Gamma(r_2, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{e^{\frac{(b_2 - b_1) w_1 + (a_2 - a_1) w_2}{a_1 b_2 - a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} - \frac{b_1}{b_2 - b_1}\right)^{-1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_3, r_4, \alpha_1 - j, \frac{1}{\beta_1} + \frac{b_2 - b_1}{a_1 b_2 - a_2 b_1}\right) \\
& - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{a_2}{a_2 - a_1} e^{\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right) \\
& + \frac{a_2}{a_2 - a_1} e^{\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{a_2} + \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case III: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $b_2 w_1 + a_2 w_2 < 0$**,** $b_1 w_1 + a_1 w_2 \geq 0$**,** $a_1 b_2 > a_2 b_1$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \;=\; & \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + \frac{a_1}{a_1 - a_2} e^{\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
& + \frac{a_1}{a_1 - a_2} e^{\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_4, \alpha_1 - j, \frac{1}{a_1} + \frac{1}{\beta_1}\right) \\
& + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1} S\Gamma(r_2, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{e^{\frac{(b_2 - b_1) w_1 + (a_2 - a_1) w_2}{a_1 b_2 - a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} - \frac{b_1}{b_2 - b_1}\right)^{-1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_3, 0, \alpha_2 - j, \frac{b_2 - b_1}{a_1 b_2 - a_2 b_1} - \frac{1}{\beta_2}\right) \\
& + \frac{e^{\frac{(b_2 - b_1) w_1 + (a_2 - a_1) w_2}{a_1 b_2 - a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} - \frac{b_1}{b_2 - b_1}\right)^{-1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_4, \alpha_1 - j, \frac{1}{\beta_1} + \frac{b_2 - b_1}{a_1 b_2 - a_2 b_1}\right) \\
& - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{a_2}{a_2 - a_1} e^{\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right);
\end{aligned}
$$

Case IV: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $b_1 w_1 + a_1 w_2 < 0$**,** $a_1 b_2 > a_2 b_1$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \;=\; & \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + \frac{a_1}{a_1 - a_2} e^{\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_4, \alpha_2 - j, \frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
& + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1} S\Gamma(r_2, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{e^{\frac{(b_2 - b_1) w_1 + (a_2 - a_1) w_2}{a_1 b_2 - a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} - \frac{b_1}{b_2 - b_1}\right)^{-1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_3, r_4, \alpha_2 - j, \frac{b_2 - b_1}{a_1 b_2 - a_2 b_1} - \frac{1}{\beta_2}\right) \\
& - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{a_2}{a_2 - a_1} e^{\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right);
\end{aligned}
$$

Case V: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 \geq 0$**,** $a_1 b_2 = a_2 b_1$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 - j, \frac{1}{a_1} + \frac{1}{\beta_1}\right) \\
&+ \frac{b_1 e^{-\frac{w_2}{b_1}} - b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 - j, \frac{1}{a_2} + \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case VI: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $b_2 w_1 + a_2 w_2 \geq 0$**,** $a_1 b_2 = a_2 b_1$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{a_1} + \frac{1}{\beta_1}\right) \\
&+ \frac{b_1 e^{-\frac{w_2}{b_1}} - b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{a_2} + \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case VII: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $b_2 w_1 + a_2 w_2 < 0$**,** $a_1 b_2 = a_2 b_1$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
&+ \frac{b_1 e^{-\frac{w_2}{b_1}} - b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right);
\end{aligned}$$

Case VIII: $a_1 = a_2 = a$**,** $b_1 \neq b_2$**,** $w_1 \geq 0$**,** $a_1 b_2 > a_2 b_1$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
&+ \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 + 1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
&- \frac{(w_1 - a) e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
&+ \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1} S\Gamma(r_2, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&- \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{(b_1 w_1 + a w_2 - a b_1) e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_3, r_4, \alpha_1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
&- \frac{b_1 e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_3, r_4, \alpha_1 + 1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right);
\end{aligned}$$

Case IX: $a_1 = a_2 = a$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $b_2 w_1 + a w_2 \geq 0$**,** $b_2 > b_1$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 + 1 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& + \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 + 1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
& - \frac{(w_1 - a) e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& - \frac{(w_1 - a) e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
& + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1} S\Gamma(r_2, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{(b_1 w_1 + a w_2 - a b_1) e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_3, r_4, \alpha_1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
& - \frac{b_1 e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_3, r_4, \alpha_1 + 1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case X: $a_1 = a_2 = a$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $b_2 w_1 + a w_2 < 0$**,** $b_1 w_1 + a w_2 \geq 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 + 1 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& - \frac{(w_1 - a) e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1} S\Gamma(r_2, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{(b_1 w_1 + a w_2 - a b_1) e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_3, 0, \alpha_2 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& + \frac{(b_1 w_1 + a w_2 - a b_1) e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_4, \alpha_1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
& - \frac{b_1 e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_3, 0, \alpha_2 + 1 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& - \frac{b_1 e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_4, \alpha_1 + 1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case XI: $a_1 = a_2 = a$**,** $b_1 \neq b_2$**,** $b_1 w_1 + a w_2 < 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \left(1 - \frac{b_2}{b_2 - b_1} e^{-\frac{w_2}{b_2}} + \frac{b_1}{b_2 - b_1} e^{-\frac{w_2}{b_1}}\right) \\
& + \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 + 1 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& - \frac{(w_1 - a) e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1} S\Gamma(r_2, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{(b_1 w_1 + a w_2 - a b_1) e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_3, r_4, \alpha_2 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
& - \frac{b_1 e^{\frac{w_1}{a}}}{a(b_2 - b_1)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_3, r_4, \alpha_2 + 1 - j, \frac{1}{a} - \frac{1}{\beta_2}\right);
\end{aligned}
$$

Case XII: $a_1 > a_2$**,** $b_1 = b_2 = b$**,** $w_1 \geq 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{b + w_2}{b} e^{-\frac{w_2}{b}} \Gamma(r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_4, \alpha_1 - j, \frac{1}{a_1} + \frac{1}{\beta_1}\right) \\
& + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 - j, \frac{1}{a_2} + \frac{1}{\beta_1}\right) \\
& - \frac{(a_1 b + a_1 w_2 + b w_1) e^{-\frac{w_2}{b}}}{b(a_1 - a_2)} S\Gamma(r_3, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& + \frac{e^{-\frac{w_2}{b}}}{a_1 - a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_3, r_4, \alpha_1 + 1 - j, \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case XIII: $a_1 > a_2$**,** $b_1 = b_2 = b$**,** $w_1 < 0$**,** $r_3 \geq 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{b + w_2}{b} e^{-\frac{w_2}{b}} \Gamma(r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_4, \alpha_1 - j, \frac{1}{a_1} + \frac{1}{\beta_1}\right) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{a_2} + \frac{1}{\beta_1}\right) \\
&- \frac{(a_1 b + a_1 w_2 + b w_1) e^{-\frac{w_2}{b}}}{b(a_1 - a_2)} S\Gamma(r_3, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{e^{-\frac{w_2}{b}}}{a_1 - a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_3, r_4, \alpha_1 + 1 - j, \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case XIIII: $a_1 \neq a_2$**,** $b_1 = b_2 = b$**,** $w_1 < 0$**,** $bw_1 + a_2 w_2 < 0$**,** $bw_1 + a_1 w_2 \geq 0$**,** $a_1 b_2 > a_2 b_1$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{b + w_2}{b} e^{-\frac{w_2}{b}} \Gamma(r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_4, \alpha_1 - j, \frac{1}{a_1} + \frac{1}{\beta_1}\right) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right) \\
&- \frac{(a_1 b + a_1 w_2 + b w_1) e^{-\frac{w_2}{b}}}{b(a_1 - a_2)} S\Gamma(r_3, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{e^{-\frac{w_2}{b}}}{a_1 - a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_3, 0, \alpha_2 + 1 - j, -\frac{1}{\beta_2}\right) \\
&+ \frac{e^{-\frac{w_2}{b}}}{a_1 - a_2} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_4, \alpha_1 + 1 - j, \frac{1}{\beta_1}\right);
\end{aligned}
$$

Case XV: $a_1 \neq a_2$**,** $b_1 = b_2 = b$**,** $bw_1 + a_1w_2 < 0$**,** $a_1b_2 > a_2b_1$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{b + w_2}{b} e^{-\frac{w_2}{b}} \Gamma(r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_4, \alpha_2 - j, \frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
&+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a_2} - \frac{1}{\beta_2}\right) \\
&- \frac{(a_1 b + a_1 w_2 + b w_1) e^{-\frac{w_2}{b}}}{b(a_1 - a_2)} S\Gamma(r_3, r_4, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{e^{-\frac{w_2}{b}}}{a_1 - a_2} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_3, r_4, \alpha_2 + 1 - j, -\frac{1}{\beta_2}\right);
\end{aligned}$$

Case XVI: $a_1 = a_2 = a$**,** $b_1 = b_2 = b$**,** $w_1 \geq 0$**,** $bw_1 + aw_2 \geq 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \Gamma\left(\frac{w_2}{b_2}, 2, 1\right) \\
&- \frac{(b + w_2) e^{-\frac{w_2}{b}}}{b} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&+ \frac{(a - w_1) e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
&+ \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_2, r_3, \alpha_1 + 1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right);
\end{aligned}$$

Case XVII: $a_1 = a_2 = a$**,** $b_1 = b_2 = b$**,** $w_1 < 0$**,** $bw_1 + aw_2 \geq 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\Gamma\left(\frac{w_2}{b_2}, 2, 1\right) \\
&\quad - \frac{(b+w_2)e^{-\frac{w_2}{b}}}{b} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&\quad + \frac{(a-w_1)e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2-1} C_j^* F\left(r_2, 0, \alpha_2 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
&\quad + \frac{(a-w_1)e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1-1} C_j F\left(0, r_3, \alpha_1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right) \\
&\quad + \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2-1} C_j^* F\left(r_2, 0, \alpha_2 + 1 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
&\quad + \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1-1} C_j F\left(0, r_3, \alpha_1 + 1 - j, \frac{1}{a} + \frac{1}{\beta_1}\right);
\end{aligned}$$

Case XVIII: $a_1 = a_2 = a$**,** $b_1 = b_2 = b$**,** $w_1 < 0$**,** $bw_1 + aw_2 < 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2)\Gamma\left(\frac{w_2}{b_2}, 2, 1\right) \\
&\quad - \frac{(b+w_2)e^{-\frac{w_2}{b}}}{b} S\Gamma(r_2, r_3, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
&\quad + \frac{(a-w_1)e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2-1} C_j^* F\left(r_2, r_3, \alpha_2 - j, \frac{1}{a} - \frac{1}{\beta_2}\right) \\
&\quad + \frac{e^{\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2-1} C_j^* F\left(r_2, r_3, \alpha_2 + 1 - j, \frac{1}{a} - \frac{1}{\beta_2}\right).
\end{aligned}$$

Lemma A.4.9 *Let* $Y_1 \sim \Gamma(\alpha_1, \beta_1)$*,* $Y_2 \sim N\Gamma(\alpha_2, \beta_2)$*,* $Z_1 \sim E(1)$*,* $Z_2 \sim E(1)$*, with* Y_1*,* Y_2*,* Z_1 *and* Z_2 *being independent and* β_1*,* $\beta_2 > 0$*. Further, let* $W_1 = Y_1 + Y_2 + a_1 Z_1 + a_2 Z_2$ *and* $W_2 = b_1 Z_1 - b_2 Z_2$ *with* $a_1, a_2, b_1, b_2 > 0$*. Then, the joint CDF of* W_1 *and* W_2 *is as follows:*

Case I: $a_1 \neq a_2$**,** $w_1, w_2 \geq 0$**,** $b_1 w_1 \geq a_1 w_2$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & S\Gamma(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& -\frac{a_2}{a_2 - a_1} e^{-\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{a_2} - \frac{1}{\beta_2}\right) \\
& -\frac{a_2}{a_2 - a_1} e^{-\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_2}\right) \\
& +\frac{a_1}{a_2 - a_1} e^{-\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_1, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_1}\right) \\
& +\frac{e^{-\frac{(b_1+b_2)w_1 + (a_2 - a_1)w_2}{a_1 b_2 + a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} + \frac{b_1}{b_1 + b_2}\right)^{-1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{b_1 + b_2}{a_1 b_2 + a_2 b_1} - \frac{1}{\beta_2}\right) \\
& +\frac{e^{-\frac{(b_1+b_2)w_1 + (a_2 - a_1)w_2}{a_1 b_2 + a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} + \frac{b_1}{b_1 + b_2}\right)^{-1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{b_1 + b_2}{a_1 b_2 + a_2 b_1}\right) \\
& +\left(1 - \frac{b_1}{b_1 + b_2} e^{-\frac{w_2}{b_1}}\right) \Gamma(r_1, \alpha_1, \alpha_2, \beta_1, \beta_2);
\end{aligned}
$$

Case II: $a_1 \neq a_2$**,** $w_1, w_2 \geq 0$**,** $b_1 w_1 < a_1 w_2$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & S\Gamma(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& -\frac{a_2}{a_2 - a_1} e^{-\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{a_2} - \frac{1}{\beta_2}\right) \\
& -\frac{a_2}{a_2 - a_1} e^{-\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_2}\right) \\
& +\frac{a_1}{a_2 - a_1} e^{-\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 - j, -\frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
& +\frac{a_1}{a_2 - a_1} e^{-\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a_1}\right) \\
& +\frac{e^{-\frac{(b_1+b_2)w_1 + (a_2 - a_1)w_2}{a_1 b_2 + a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} + \frac{b_1}{b_1 + b_2}\right)^{-1}} \\
& \times \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, r_1, \alpha_2 - j, -\frac{b_1 + b_2}{a_1 b_2 + a_2 b_1} - \frac{1}{\beta_2}\right) \\
& +\left(1 - \frac{b_1}{b_1 + b_2} e^{-\frac{w_2}{b_1}}\right) \Gamma(r_1, \alpha_1, \alpha_2, \beta_1, \beta_2);
\end{aligned}
$$

Case III: $a_1 \neq a_2$**,** $w_1 < 0, w_2 > 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & S\Gamma(r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& -\frac{a_2}{a_2 - a_1} e^{-\frac{w_1}{a_2}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, r_2, \alpha_2 - j, -\frac{1}{a_2} - \frac{1}{\beta_2}\right) \\
& +\frac{a_1}{a_2 - a_1} e^{-\frac{w_1}{a_1}} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_2, \alpha_2 - j, -\frac{1}{a_1} - \frac{1}{\beta_2}\right) \\
& +\frac{e^{-\frac{(b_1+b_2)w_1 + (a_2 - a_1)w_2}{a_1 b_2 + a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} + \frac{b_1}{b_1 + b_2}\right)^{-1}} \\
& \times \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, r_1, \alpha_2 - j, -\frac{b_1 + b_2}{a_1 b_2 + a_2 b_1} - \frac{1}{\beta_2}\right) \\
& +\left(1 - \frac{b_1}{b_1 + b_2} e^{-\frac{w_2}{b_1}}\right) \Gamma(r_1, \alpha_1, \alpha_2, \beta_1, \beta_2);
\end{aligned}
$$

Case IV: $a_1 = a_2 = a$**,** $w_1, w_2 \geq 0$**,** $b_1 w_1 \geq a w_2$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & -\frac{(ab_2 + b_2 w_1 + a w_2) e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& -\frac{(ab_2 + b_2 w_1 + a w_2) e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_1, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
& +\frac{b_2 e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, 0, \alpha_2 + 1 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& +\frac{b_2 e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_1, \alpha_1 + 1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
& +\Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{b_1}{b_1 + b_2} e^{-\frac{w_2}{b_1}} \Gamma(r_1, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& -\frac{(a + w_1) e^{-\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_1, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
& +\frac{e^{-\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(r_1, r_2, \alpha_1 + 1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right);
\end{aligned}
$$

Case V: $a_1 = a_2 = a$**,** $w_1, w_2 \geq 0$**,** $b_1 w_1 < a w_2$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & -\frac{(ab_2 + b_2 w_1 + a w_2) e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, r_1, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& + \frac{b_2 e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, r_1, \alpha_2 + 1 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& + \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{b_1}{b_1 + b_2} e^{-\frac{w_2}{b_1}} \Gamma(r_1, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& - \frac{(a + w_1) e^{-\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& - \frac{(a + w_1) e^{-\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_2, \alpha_1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right) \\
& + \frac{e^{-\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, 0, \alpha_2 + 1 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& + \frac{e^{-\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_1 - 1} C_j F\left(0, r_2, \alpha_1 + 1 - j, \frac{1}{\beta_1} - \frac{1}{a}\right);
\end{aligned}
$$

Case VI: $a_1 = a_2 = a$**,** $w_1 < 0$**,** $w_2 \geq 0$

$$
\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) \quad = \quad & -\frac{(ab_2 + b_2 w_1 + a w_2) e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, r_1, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& + \frac{b_2 e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(-\infty, r_1, \alpha_2 + 1 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& + \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{b_1}{b_1 + b_2} e^{-\frac{w_2}{b_1}} \Gamma(r_1, \alpha_1, \alpha_2, \beta_1, \beta_2) \\
& - \frac{(a + w_1) e^{-\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_2, \alpha_2 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right) \\
& + \frac{e^{-\frac{w_1}{a}}}{a} \sum_{j=0}^{\alpha_2 - 1} C_j^* F\left(r_1, r_2, \alpha_2 + 1 - j, -\frac{1}{\beta_2} - \frac{1}{a}\right).
\end{aligned}
$$

Lemma A.4.10 *Let* $Y \sim \Gamma(\alpha, \beta)$*,* $Z_1 \sim E(1)$*,* $Z_2 \sim E(1)$*, with* Y*,* Z_1 *and* Z_2 *being independent and* $\beta > 0$*. Further, let* $W_1 = Y - a_1 Z_1 - a_2 Z_2$ *and* $W_2 = b_1 Z_1 + b_2 Z_2$

with $a_1, a_2, b_1, b_2 > 0$ *and* $a_1 b_2 \geq a_2 b_1$. *Then, the joint CDF of* W_1 *and* W_2 *is as follows:*

Case I: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 \geq 0$**,** $a_1 b_2 > a_2 b_1$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1}\Gamma(r_4, \alpha, \beta) \\ &+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} C_\Gamma F\left(r_2, r_4, \alpha, \frac{1}{a_1} + \frac{1}{\beta}\right) \\ &+ \frac{e^{\frac{(b_2 - b_1)w_1 + (a_2 - a_1)w_2}{a_1 b_2 - a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} - \frac{b_1}{b_2 - b_1}\right)^{-1}} C_\Gamma F\left(r_3, r_4, \alpha, \frac{1}{\beta} + \frac{b_2 - b_1}{a_1 b_2 - a_2 b_1}\right) \\ &+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{a_2} + \frac{1}{\beta}\right); \end{aligned}$$

Case II: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $r_3 \geq 0$**,** $a_1 b_2 > a_2 b_1$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1}\Gamma(r_4, \alpha, \beta) - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) \\ &+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} C_\Gamma F\left(0, r_4, \alpha, \frac{1}{a_1} + \frac{1}{\beta}\right) \\ &+ \frac{e^{\frac{(b_2 - b_1)w_1 + (a_2 - a_1)w_2}{a_1 b_2 - a_2 b_1}}}{\left(\frac{a_1}{a_2 - a_1} - \frac{b_1}{b_2 - b_1}\right)^{-1}} C_\Gamma F\left(r_3, r_4, \alpha, \frac{1}{\beta} + \frac{b_2 - b_1}{a_1 b_2 - a_2 b_1}\right) \\ &+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{a_2} + \frac{1}{\beta}\right); \end{aligned}$$

Case III: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $r_3 < 0$**,** $r_4 \geq 0$**,** $a_1 b_2 > a_2 b_1$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1}\Gamma(r_4, \alpha, \beta) + \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} C_\Gamma F\left(0, r_4, \alpha, \frac{1}{a_1} + \frac{1}{\beta}\right) \\ &\quad + \frac{e^{\frac{(b_2-b_1)w_1 + (a_2-a_1)w_2}{a_1 b_2 - a_2 b_1}}}{\left(\frac{a_1}{a_2-a_1} - \frac{b_1}{b_2-b_1}\right)^{-1}} C_\Gamma F\left(0, r_4, \alpha, \frac{1}{\beta} + \frac{b_2 - b_1}{a_1 b_2 - a_2 b_1}\right); \end{aligned}$$

Case IV: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 \geq 0$**,** $a_1 b_2 = a_2 b_1$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) + \frac{b_1 e^{-\frac{w_2}{b_1}} - b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) \\ &\quad + \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{a_1} + \frac{1}{\beta}\right) \\ &\quad + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{a_2} + \frac{1}{\beta}\right); \end{aligned}$$

Case V: $a_1 \neq a_2$**,** $b_1 \neq b_2$**,** $w_1 < 0$**,** $r_3 \geq 0$**,** $a_1 b_2 = a_2 b_1$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \frac{b_1 e^{-\frac{w_2}{b_1}} - b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) + \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{a_1} + \frac{1}{\beta}\right) \\ &\quad + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{a_2} + \frac{1}{\beta}\right); \end{aligned}$$

Case VI: $a_1 = a_2 = a$**,** $b_2 > b_1$**,** $w_1 \geq 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1}\Gamma(r_4, \alpha, \beta) \\
&+ \frac{e^{\frac{w_1}{a}}}{a} C_\Gamma F\left(r_2, r_3, \alpha + 1, \frac{1}{a} + \frac{1}{\beta}\right) \\
&- \frac{(w_1 - a)e^{\frac{w_1}{a}}}{a} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{a} + \frac{1}{\beta}\right) \\
&+ \frac{(b_1 w_1 + a w_2 - a b_1)e^{\frac{w_1}{a}}}{a(b_2 - b_1)} C_\Gamma F\left(r_3, r_4, \alpha, \frac{1}{a} + \frac{1}{\beta}\right) \\
&- \frac{b_1 e^{\frac{w_1}{a}}}{a(b_2 - b_1)} C_\Gamma F\left(r_3, r_4, \alpha + 1, \frac{1}{a} + \frac{1}{\beta}\right);
\end{aligned}$$

Case VII: $a_1 = a_2 = a$**,** $b_2 > b_1$**,** $w_1 < 0$**,** $r_3 \geq 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1}\Gamma(r_4, \alpha, \beta) - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) \\
&+ \frac{e^{\frac{w_1}{a}}}{a} C_\Gamma F\left(0, r_3, \alpha + 1, \frac{1}{a} + \frac{1}{\beta}\right) \\
&- \frac{(w_1 - a)e^{\frac{w_1}{a}}}{a} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{a} + \frac{1}{\beta}\right) \\
&+ \frac{(b_1 w_1 + a w_2 - a b_1)e^{\frac{w_1}{a}}}{a(b_2 - b_1)} C_\Gamma F\left(r_3, r_4, \alpha, \frac{1}{a} + \frac{1}{\beta}\right) \\
&- \frac{b_1 e^{\frac{w_1}{a}}}{a(b_2 - b_1)} C_\Gamma F\left(r_3, r_4, \alpha + 1, \frac{1}{a} + \frac{1}{\beta}\right);
\end{aligned}$$

Case VIII: $a_1 = a_2 = a$**,** $b_2 > b_1$**,** $w_1 < 0$**,** $r_3 < 0$**,** $r_4 \geq 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1}\Gamma(r_4, \alpha, \beta) + \frac{(b_1 w_1 + a w_2 - a b_1)e^{\frac{w_1}{a}}}{a(b_2 - b_1)} C_\Gamma F\left(0, r_4, \alpha, \frac{1}{a} + \frac{1}{\beta}\right) \\
&- \frac{b_1 e^{\frac{w_1}{a}}}{a(b_2 - b_1)} C_\Gamma F\left(0, r_4, \alpha + 1, \frac{1}{a} + \frac{1}{\beta}\right);
\end{aligned}$$

Case IX: $a_1 > a_2$**,** $b_1 = b_2 = b$**,** $w_1 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) - \frac{b + w_2}{b} e^{-\frac{w_2}{b}} \Gamma(r_3, \alpha, \beta) \\ &+ \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} C_\Gamma F\left(r_2, r_4, \alpha, \frac{1}{a_1} + \frac{1}{\beta}\right) \\ &+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{a_2} + \frac{1}{\beta}\right) \\ &+ \frac{e^{-\frac{w_2}{b}}}{a_1 - a_2} C_\Gamma F\left(r_3, r_4, \alpha_1 + 1, \frac{1}{\beta}\right) \\ &- \frac{(a_1 b + a_1 w_2 + b w_1) e^{-\frac{w_2}{b}}}{b(a_1 - a_2)} S\Gamma(r_3, r_4, \alpha, \beta); \end{aligned}$$

Case X: $a_1 > a_2$**,** $b_1 = b_2 = b$**,** $w_1 < 0$**,** $r_3 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= -\frac{b + w_2}{b} e^{-\frac{w_2}{b}} \Gamma(r_3, \alpha, \beta) + \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} C_\Gamma F\left(0, r_4, \alpha, \frac{1}{a_1} + \frac{1}{\beta}\right) \\ &+ \frac{a_2 e^{\frac{w_1}{a_2}}}{a_2 - a_1} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{a_2} + \frac{1}{\beta}\right) + \frac{e^{-\frac{w_2}{b}}}{a_1 - a_2} C_\Gamma F\left(r_3, r_4, \alpha_1 + 1, \frac{1}{\beta}\right) \\ &- \frac{(a_1 b + a_1 w_2 + b w_1) e^{-\frac{w_2}{b}}}{b(a_1 - a_2)} S\Gamma(r_3, r_4, \alpha, \beta); \end{aligned}$$

Case XI: $a_1 > a_2$**,** $b_1 = b_2 = b$**,** $w_1 < 0$**,** $r_3 < 0$**,** $r_4 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \frac{a_1 e^{\frac{w_1}{a_1}}}{a_1 - a_2} C_\Gamma F\left(0, r_4, \alpha, \frac{1}{a_1} + \frac{1}{\beta}\right) + \frac{e^{-\frac{w_2}{b}}}{a_1 - a_2} C_\Gamma F\left(0, r_4, \alpha_1 + 1, \frac{1}{\beta}\right) \\ &- \frac{(a_1 b + a_1 w_2 + b w_1) e^{-\frac{w_2}{b}}}{b(a_1 - a_2)} S\Gamma(0, r_4, \alpha, \beta); \end{aligned}$$

Case XII: $a_1 = a_2 = a$**,** $b_1 = b_2 = b$**,** $w_1 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta)\Gamma\left(\frac{w_2}{b_2}, 2, 1\right) - \frac{(b+w_2)e^{-\frac{w_2}{b}}}{b} S\Gamma(r_2, r_3, \alpha, \beta) \\ &\quad + \frac{(a-w_1)e^{\frac{w_1}{a}}}{a} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{a} + \frac{1}{\beta}\right) \\ &\quad + \frac{e^{\frac{w_1}{a}}}{a} C_\Gamma F\left(r_2, r_3, \alpha+1, \frac{1}{a} + \frac{1}{\beta}\right); \end{aligned}$$

Case XIII: $a_1 = a_2 = a$**,** $b_1 = b_2 = b$**,** $w_1 < 0$**,** $r_3 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= -\frac{(b+w_2)e^{-\frac{w_2}{b}}}{b} S\Gamma(0, r_3, \alpha, \beta) + \frac{(a-w_1)e^{\frac{w_1}{a}}}{a} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{a} + \frac{1}{\beta}\right) \\ &\quad + \frac{e^{\frac{w_1}{a}}}{a} C_\Gamma F\left(0, r_3, \alpha+1, \frac{1}{a} + \frac{1}{\beta}\right). \end{aligned}$$

Lemma A.4.11 *Let* $Y \sim \Gamma(\alpha, \beta)$*,* $Z_1 \sim E(1)$*,* $Z_2 \sim E(1)$*, with* Y*,* Z_1 *and* Z_2 *being independent and* $\beta_1 > 0$*. Further, let* $W_1 = Y + a_1 Z_1 + a_2 Z_2$ *and* $W_2 = b_1 Z_1 - b_2 Z_2$ *with* $a_1, a_2, b_1, b_2 > 0$*. Then, the joint CDF of* W_1 *and* W_2 *is as follows:*

Case I: $a_1 \neq a_2$**,** $w_1, w_2 \geq 0$**,** $r_1 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) - \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_1 + b_2} \Gamma(r_1, \alpha, \beta) - \frac{a_2 e^{-\frac{w_1}{a_2}}}{a_2 - a_1} C_\Gamma F\left(0, r_2, \alpha, \frac{1}{\beta} - \frac{1}{a_2}\right) \\ &\quad + \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_2 - a_1} C_\Gamma F\left(r_1, r_2, \alpha, \frac{1}{\beta} - \frac{1}{a_1}\right) \\ &\quad + \frac{e^{-\frac{(b_1+b_2)w_1 + (a_2-a_1)w_2}{a_1 b_2 + a_2 b_1}}}{\left(\frac{a_1}{a_2-a_1} + \frac{b_1}{b_1+b_2}\right)^{-1}} C_\Gamma F\left(0, r_1, \alpha, \frac{1}{\beta} - \frac{b_1+b_2}{a_1 b_2 + a_2 b_1}\right); \end{aligned}$$

Case II: $a_1 \neq a_2$**,** $w_1, w_2 \geq 0$**,** $r_1 < 0$

$$P(W_1 \leq w_1, W_2 \leq w_2) = \Gamma(r_2, \alpha, \beta) + \frac{a_1 e^{-\frac{w_1}{a_1}} - a_2 e^{-\frac{w_1}{a_2}}}{a_2 - a_1} C_\Gamma F\left(r_2, \alpha, \frac{1}{\beta} - \frac{1}{a_2}\right);$$

Case III: $a_1 = a_2 = a$**,** $w_1, w_2 \geq 0$**,** $r_1 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= -\frac{(ab_2 + b_2 w_1 + a w_2) e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} C_\Gamma F\left(0, r_1, \alpha, \frac{1}{\beta} - \frac{1}{a}\right) \\ &+ \frac{b_2 e^{-\frac{w_1}{a}}}{a(b_1 + b_2)} C_\Gamma F\left(0, r_1, \alpha + 1, \frac{1}{\beta} - \frac{1}{a}\right) - \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_1 + b_2} \Gamma(r_1, \alpha, \beta) \\ &+ \Gamma(r_2, \alpha, \beta) - \frac{(a + w_1) e^{-\frac{w_1}{a}}}{a} C_\Gamma F\left(r_1, r_2, \alpha, \frac{1}{\beta} - \frac{1}{a}\right) \\ &+ \frac{e^{-\frac{w_1}{a}}}{a} C_\Gamma F\left(r_1, r_2, \alpha + 1, \frac{1}{\beta} - \frac{1}{a}\right); \end{aligned}$$

Case IV: $a_1 = a_2 = a$**,** $w_1, w_2 \geq 0$**,** $r_1 < 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha_1, \alpha_2, \beta_1, \beta_2) - \frac{(a + w_1) e^{-\frac{w_1}{a}}}{a} C_\Gamma F\left(0, r_2, \alpha, \frac{1}{\beta} - \frac{1}{a}\right) \\ &+ \frac{e^{-\frac{w_1}{a}}}{a} C_\Gamma F\left(0, r_2, \alpha + 1, \frac{1}{\beta} - \frac{1}{a}\right). \end{aligned}$$

Lemma A.4.12 *Let* $Y \sim \Gamma(\alpha, \beta)$*,* $Z_1 \sim E(1)$*,* $Z_2 \sim E(1)$*, with* Y*,* Z_1 *and* Z_2 *being independent and* $\beta > 0$*. Further, let* $W_1 = Y + a_1 Z_1 - a_2 Z_2$ *and* $W_2 = b_1 Z_1 + b_2 Z_2$ *with* $a_1, a_2, b_1, b_2 > 0$*. Then, the joint CDF of* W_1 *and* W_2 *is as follows:*

Case I: $b_1 \neq b_2$**,** $w_1, w_2 \geq 0$**,** $r_1 \geq 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) + \frac{b_1 e^{-\frac{w_2}{b_1}}}{b_2 - b_1}\Gamma(r_1, \alpha, \beta) \\
&+ \frac{e^{-\frac{(a_1+a_2)w_2+(b_2-b_1)w_1}{a_1 b_2 + b_1 a_2}}}{\left(\frac{a_1+a_2}{a_1} + \frac{b_1}{b_2-b_1}\right)^{-1}} C_\Gamma F\left(r_1, r_3, \alpha, \frac{1}{\beta} + \frac{b_1 - b_2}{a_1 b_2 + b_1 a_2}\right) \\
&- \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} C_\Gamma F\left(r_1, r_2, \alpha, \frac{1}{\beta} - \frac{1}{a_1}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_2}}}{a_1 + a_2} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{\beta} + \frac{1}{a_2}\right);
\end{aligned}$$

Case II: $b_1 \neq b_2$**,** $w_1, w_2 \geq 0$**,** $r_1 < 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) - \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) \\
&+ \frac{e^{-\frac{(a_1+a_2)w_2+(b_2-b_1)w_1}{a_1 b_2 + b_1 a_2}}}{\left(\frac{a_1+a_2}{a_1} + \frac{b_1}{b_2-b_1}\right)^{-1}} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{\beta} + \frac{b_1 - b_2}{a_1 b_2 + b_1 a_2}\right) \\
&- \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} C_\Gamma F\left(0, r_2, \alpha, \frac{1}{\beta} - \frac{1}{a_1}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_2}}}{a_1 + a_2} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{\beta} + \frac{1}{a_2}\right);
\end{aligned}$$

Case III: $b_1 \neq b_2$**,** $w_2 \geq 0$**,** $w_1 < 0$**,** $r_3 \geq 0$

$$\begin{aligned}
P(W_1 \leq w_1, W_2 \leq w_2) &= \frac{b_2 e^{-\frac{w_2}{b_2}}}{b_2 - b_1}\Gamma(r_3, \alpha, \beta) \\
&+ \frac{e^{-\frac{(a_1+a_2)w_2+(b_2-b_1)w_1}{a_1 b_2 + b_1 a_2}}}{\left(\frac{a_1+a_2}{a_1} + \frac{b_1}{b_2-b_1}\right)^{-1}} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{\beta} + \frac{b_1 - b_2}{a_1 b_2 + b_1 a_2}\right) \\
&+ \frac{a_1 e^{\frac{w_1}{a_2}}}{a_1 + a_2} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{\beta} + \frac{1}{a_2}\right);
\end{aligned}$$

Case IV: $b_1 = b_2 = b$**,** $w_1, w_2 \geq 0$**,** $r_1 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) - e^{-\frac{w_2}{b}}\Gamma(r_3, \alpha, \beta) - \frac{w_2}{b}e^{-\frac{w_2}{b}}\Gamma(r_1, \alpha, \beta) \\ &\quad - \frac{a_2 w_2 + b w_1 - a_1 b}{b(a_1 + a_2)} e^{-\frac{w_2}{b}} S\Gamma(r_1, r_3, \alpha, \beta) \\ &\quad + \frac{e^{-\frac{w_2}{b}}}{a_1 + a_2} C_\Gamma F\left(r_1, r_3, \alpha + 1, \frac{1}{\beta}\right) - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} C_\Gamma F\left(r_1, r_2, \alpha, \frac{1}{\beta} - \frac{1}{a_1}\right) \\ &\quad + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{\beta} + \frac{1}{a_2}\right); \end{aligned}$$

Case V: $b_1 = b_2 = b$**,** $w_1, w_2 \geq 0$**,** $r_1 < 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= \Gamma(r_2, \alpha, \beta) - e^{-\frac{w_2}{b}}\Gamma(r_3, \alpha, \beta) - \frac{a_2 w_2 + b w_1 - a_1 b}{b(a_1 + a_2)} e^{-\frac{w_2}{b}}\Gamma(r_3, \alpha, \beta) \\ &\quad + \frac{e^{-\frac{w_2}{b}}}{a_1 + a_2} C_\Gamma F\left(0, r_3, \alpha + 1, \frac{1}{\beta}\right) - \frac{a_1 e^{-\frac{w_1}{a_1}}}{a_1 + a_2} C_\Gamma F\left(0, r_2, \alpha, \frac{1}{\beta} - \frac{1}{a_1}\right) \\ &\quad + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} C_\Gamma F\left(r_2, r_3, \alpha, \frac{1}{\beta} + \frac{1}{a_2}\right); \end{aligned}$$

Case VI: $b_1 = b_2 = b$**,** $w_1, < 0$**,** $r_3 \geq 0$

$$\begin{aligned} P(W_1 \leq w_1, W_2 \leq w_2) &= -e^{-\frac{w_2}{b}}\Gamma(r_3, \alpha, \beta) - \frac{a_2 w_2 + b w_1 - a_1 b}{b(a_1 + a_2)} e^{-\frac{w_2}{b}}\Gamma(r_3, \alpha, \beta) \\ &\quad + \frac{a_2 e^{\frac{w_1}{a_2}}}{a_1 + a_2} C_\Gamma F\left(0, r_3, \alpha, \frac{1}{\beta} + \frac{1}{a_2}\right). \end{aligned}$$

List of notation for Chapter 8:

$$
\begin{aligned}
p_0 &= \frac{1}{2^n} e^{-\frac{n(T-\mu)}{\sigma}}, \\
q_0 &= \left(1 - \frac{1}{2} e^{-\frac{\mu - T}{\sigma}}\right)^n, \\
p_1 &= \frac{(-1)^l e^{-\frac{(T-\mu)(n+l-d)}{\sigma}}}{(1-p_0) 2^n (n-d)! j! (d-j-l)! l!}, \\
p_2 &= \frac{(-1)^l n! e^{-\frac{(T-\mu)m}{\sigma}}}{2^n j! (l+1)! (m-j-l-1)! (n-m)! (1-p_0)}, \\
p_3 &= -p_2 e^{-\frac{(T-\mu)(l+1)}{\sigma}}, \\
p_4 &= \frac{n! e^{-\frac{(T-\mu)m}{\sigma}}}{2^n m! m! (1-p_0)}, \\
p_5 &= \frac{(-1)^{l_1+d-m-1-l_2} n! e^{-\frac{(T-\mu)(m-1-l_2)}{\sigma}}}{2^n j! (m-1-j-l_1)! (d-m-1-l_2)! (l_2+1)! (n-d)! (l_1+1)! (1-p_0)}, \\
p_6 &= -p_5 e^{-\frac{(T-\mu)(l_2+1)}{\sigma}}, \\
p_{7,e} &= -\frac{p_5}{l_1 + l_2 + 2}, \\
p_{7,o} &= \frac{(-1)^{l_1+d-m-1-l_2} n! e^{-\frac{(T-\mu)(m-l_2)}{\sigma}}}{(1-p_0) 2^n (l_1+l_2+1) l_1! l_2! j! (n-d)! (m-j-l_1)! (d-m-1-l_2)!},
\end{aligned}
$$

$$
\begin{aligned}
p_{8,e} &= -p_{7,e}e^{-\frac{(T-\mu)(l_1+l_2+2)}{\sigma}}, \\
p_{8,o} &= -p_{7,o}e^{-\frac{(T-\mu)(l_1+l_2+1)}{\sigma}}, \\
p_9 &= \frac{(-1)^{d-m-l-1}n!e^{-\frac{(T-\mu)(m-l-1)}{\sigma}}}{2^n m!(d-m-l-1)!(l+1)!(n-d)!(1-p_0)}, \\
p_{10} &= -p_9 e^{-\frac{(T-\mu)(l+1)}{\sigma}}, \\
p_{11} &= \frac{(-1)^{l_1+l_2}n!e^{-\frac{(T-\mu)(n-d+l_2)}{\sigma}}}{2^n m!(j-m-l_1-1)!l_1!(d-j-l_2)!l_2!(n-d)!(m+l_1+1)(1-p_0)}, \\
p_{7\sigma} &= \frac{(-1)^{l_1+d-m-1-l_2}n!e^{-\frac{(T-\mu)(m-1-l_2)}{\sigma}}}{(1-p_0)2^n(l_1+l_2+1)l_1!l_2!j!(n-d)!(m-j-l_1)!(d-m-1-l_2)!}, \\
p_{8\sigma} &= -p_{7\sigma}e^{-\frac{(T-\mu)(l_1+l_2+1)}{\sigma}}, \\
p_{2\mu} &= \frac{(-1)^l n!e^{-\frac{(T-\mu)m}{\sigma}}}{(m-l-1)!(l+1)!m!2^{m+l+1}(1-p_0)}, \\
p_{3\mu} &= -p_{2\mu}e^{-\frac{(T-\mu)(l+1)}{\sigma}}, \\
p_{5\mu} &= \frac{n!(-1)^{l_1+d-m-1-l_2}e^{-\frac{(T-\mu)(m-l_2-1)}{\sigma}}}{2^{m+l_1+1}(m-l_1-1)!(l_1+1)!(d-m-l_2-1)!(l_2+1)!(n-d)!(1-p_0)}, \\
p_{6\mu} &= -p_{5\mu}e^{-\frac{(T-\mu)(l_2+1)}{\sigma}}, \\
p_{7\mu,e} &= -\frac{l_2+1}{l_1+l_2+2}p_{5\mu}, \\
p_{7\mu,o} &= \frac{n!(-1)^{l_1+d-m-1-l_2}e^{-\frac{(T-\mu)(m-l_2)}{\sigma}}}{2^{m+l_1+1}(m-l_1)!l_1!(d-m-1-l_2)!l_2!(n-d)!(l_1+l_2+1)(1-p_0)}, \\
p_{8\mu,e} &= -p_{7\mu,e}e^{-\frac{(T-\mu)(l_1+l_2+2)}{\sigma}}, \\
p_{8\mu,o} &= -p_{7\mu,o}e^{-\frac{(T-\mu)(l_1+l_2+1)}{\sigma}}, \\
p_{11\mu} &= \frac{(-1)^{l_1}n!e^{-\frac{(T-\mu)(n-d)}{\sigma}}\left[1-\frac{1}{2}e^{-\frac{T-\mu}{\sigma}}\right]^{d-m-1-l_1}}{2^{m+l_1+1+n-d}m!(d-m-l_1-1)!l_1!(n-d)!(m+l_1+1)(1-p_0)}, \\
q_1 &= \frac{n!}{d!(n-d)!(1-q_0)}\left(\frac{1}{2}e^{-\frac{\mu-T}{\sigma}}\right)^d\left(1-\frac{1}{2}e^{-\frac{\mu-T}{\sigma}}\right)^{n-d}, \\
q_2 &= \frac{n!}{m!m!}\left(\frac{1}{2}e^{-\frac{\mu-T}{\sigma}}\right)^m\left[1-\frac{1}{2}e^{-\frac{\mu-T}{\sigma}}\right]^m, \\
q_3 &= \frac{(-1)^l\left(1-\frac{1}{2}e^{-\frac{\mu-T}{\sigma}}\right)^{n-d}\left(\frac{1}{2}e^{-\frac{(\mu-T)}{\sigma}}\right)^d n!}{(l+m+1)m!l!(d-m-1-l)!(n-d)!(1-q_0)},
\end{aligned}
$$

$$
\begin{aligned}
Z_{\sigma}^{(7)} &\stackrel{d}{=} \Gamma\left(j+d-m-1,\frac{\sigma}{d}\right)+N\Gamma\left(m-j,\frac{\sigma}{d}\right)+E\left(\frac{(l_2-l_1+1)\sigma}{(l_2+l_1+1)d}\right)+\frac{(T-\mu)(m-1-l_2)}{d},\\
Z_{\sigma}^{(8)} &\stackrel{d}{=} \Gamma\left(j+d-m-1,\frac{\sigma}{d}\right)+N\Gamma\left(m-j,\frac{\sigma}{d}\right)+E\left(\frac{(l_2-l_1+1)\sigma}{(l_2+l_1+1)d}\right)+\frac{(T-\mu)(m-l_1)}{d},\\
Z_{1}^{(1)} &\stackrel{d}{=} \Gamma\left(j,\frac{\sigma}{d}\right)+N\Gamma\left(d-j,\frac{\sigma}{d}\right)+\frac{(T-\mu)(d-l)}{d},\\
Z_{2}^{(1)} &\stackrel{d}{=} \log\left(\frac{n}{2d}\right)Z_{1}^{(1)}+T,\\
Z_{1,e}^{(2)} &\stackrel{d}{=} \Gamma\left(j,\frac{\sigma}{m}\right)+N\Gamma\left(m-1-j,\frac{\sigma}{m}\right)-\frac{\sigma}{m}E_2+(T-\mu),\\
Z_{2,e}^{(2)} &\stackrel{d}{=} \frac{\sigma}{2(l+1)}E_2+\frac{T+\mu}{2},\\
Z_{1}^{(3)} &\stackrel{d}{=} \Gamma\left(j,\frac{\sigma}{m}\right)+N\Gamma\left(m-1-j,\frac{\sigma}{m}\right)-\frac{\sigma}{m}E_3+\frac{(T-\mu)(m-l-1)}{m},\\
Z_{2}^{(3)} &\stackrel{d}{=} \frac{\sigma}{2(l+1)}E_3+T,\\
Z_{1}^{(4)} &\stackrel{d}{=} \Gamma\left(m-1,\frac{\sigma}{m}\right)+\frac{\sigma}{m}E_4+(T-\mu),\\
Z_{2}^{(4)} &\stackrel{d}{=} -\frac{\sigma}{n}E_4+\frac{T+\mu}{2},\\
Z_{1,e}^{(5)} &\stackrel{d}{=} \Gamma\left(d-m-1+j,\frac{\sigma}{d}\right)+N\Gamma\left(m-1-j,\frac{\sigma}{d}\right)+\frac{\sigma}{d}E_{5A}-\frac{\sigma}{d}E_{5B}+\frac{(T-\mu)(m-1-l_2)}{d},\\
Z_{2,e}^{(5)} &\stackrel{d}{=} \frac{\sigma}{2(l_2+1)}E_{5A}+\frac{\sigma}{2(l_1+1)}E_{5B}+\mu,\\
Z_{1,e}^{(6)} &\stackrel{d}{=} \Gamma\left(d-m-1+j,\frac{\sigma}{d}\right)+N\Gamma\left(m-1-j,\frac{\sigma}{d}\right)+\frac{\sigma}{d}E_{6A}-\frac{\sigma}{d}E_{6B}+\frac{(T-\mu)m}{d},\\
Z_{2,e}^{(6)} &\stackrel{d}{=} \frac{\sigma}{2(l_2+1)}E_{6A}+\frac{\sigma}{2(l_1+1)}E_{6B}+\frac{T+\mu}{2},\\
Z_{1,e}^{(7)} &\stackrel{d}{=} \Gamma\left(d-m-1+j,\frac{\sigma}{d}\right)+N\Gamma\left(m-1-j,\frac{\sigma}{d}\right)\\
&\quad+\frac{(l_2-l_1)\sigma}{(l_1+l_2+2)d}E_{7A}-\frac{\sigma}{d}E_{7B}+\frac{(T-\mu)(m-l_2-1)}{d},\\
Z_{2,e}^{(7)} &\stackrel{d}{=} \frac{\sigma}{l_1+l_2+2}E_{7A}+\frac{\sigma}{2(l_1+1)}E_{7B}+\mu,\\
Z_{1,o}^{(7)} &\stackrel{d}{=} \Gamma\left(j+d-m-1,\frac{\sigma}{d}\right)+N\Gamma\left(m-j,\frac{\sigma}{d}\right)+\frac{(l_2-l_1)\sigma}{(l_2+l_1+1)d}E_7+\frac{(T-\mu)(m-l_2)}{d},\\
Z_{2,o}^{(7)} &\stackrel{d}{=} \frac{\sigma}{l_2+l_1+1}E_7+\mu,\\
Z_{1,e}^{(8)} &\stackrel{d}{=} \Gamma\left(d-m-1+j,\frac{\sigma}{d}\right)+N\Gamma\left(m-1-j,\frac{\sigma}{d}\right)\\
&\quad+\frac{(l_2-l_1)\sigma}{(l_1+l_2+2)d}E_{8A}-\frac{\sigma}{d}E_{8B}+\frac{(T-\mu)(m-l_1-1)}{d},\\
Z_{2,e}^{(8)} &\stackrel{d}{=} \frac{\sigma}{l_1+l_2+2}E_{8A}+\frac{\sigma}{2(l_1+1)}E_{8B}+T,
\end{aligned}
$$

$$
\begin{aligned}
Z_{1,o}^{(8)} &\stackrel{d}{=} \Gamma\left(j+d-m-1,\frac{\sigma}{d}\right)+N\Gamma\left(m-j,\frac{\sigma}{d}\right)+\frac{(l_2-l_1)\sigma}{(l_2+l_1+1)d}E_8+\frac{(T-\mu)(m-l_1)}{d},\\
Z_{2,o}^{(8)} &\stackrel{d}{=} \frac{\sigma}{l_2+l_1+1}E_8+T,\\
Z_1^{(9)} &\stackrel{d}{=} \Gamma\left(d-2,\frac{\sigma}{d}\right)+\frac{\sigma}{d}E_{9A}+\frac{\sigma}{d}E_{9B}+\frac{(T-\mu)(m-l-1)}{d},\\
Z_2^{(9)} &\stackrel{d}{=} -\frac{\sigma}{n}E_{9A}+\frac{\sigma}{2(l+1)}E_{9B}+\mu,\\
Z_1^{(10)} &\stackrel{d}{=} \Gamma\left(d-2,\frac{\sigma}{d}\right)+\frac{\sigma}{d}E_{10A}+\frac{\sigma}{d}E_{10B}+\frac{(T-\mu)m}{d},\\
Z_2^{(10)} &\stackrel{d}{=} -\frac{\sigma}{n}E_{10A}+\frac{\sigma}{2(l+1)}E_{10B}+\frac{T+\mu}{2},\\
Z_{1,e}^{(11)} &\stackrel{d}{=} \Gamma\left(d-j+m-1,\frac{\sigma}{d}\right)+N\Gamma\left(j-m-1,\frac{\sigma}{d}\right)+\frac{\sigma}{d}E_{11A}+\frac{(m-l_1-1)\sigma}{(m+l_1+1)d}E_{11B}\\
&\quad +\frac{(T-\mu)(n-d+l_2)}{d},\\
Z_{2,e}^{(11)} &\stackrel{d}{=} -\frac{\sigma}{n}E_{11A}-\frac{\sigma}{m+l_1+1}E_{11B}+\mu,\\
Z_{1,o}^{(11)} &\stackrel{d}{=} \Gamma\left(d-j+m,\frac{\sigma}{d}\right)+N\Gamma\left(j-m-1,\frac{\sigma}{d}\right)+\frac{(m-l_1)\sigma}{(m+l_1+1)d}E_{11}+\frac{(T-\mu)(n-d+l_2)}{d},\\
Z_{2,o}^{(11)} &\stackrel{d}{=} -\frac{\sigma}{m+l_1+1}E_{11}+\mu,\\
Z_1^{(12)} &\stackrel{d}{=} \Gamma\left(d,\frac{\sigma}{d}\right),\\
Z_2^{(12)} &\stackrel{d}{=} \log\left(\frac{n}{2d}\right)Z_1^{(12)}+T,\\
Z_1^{(13)} &\stackrel{d}{=} \Gamma\left(m-1,\frac{\sigma}{m}\right)+\frac{\sigma}{m}E_{13},\\
Z_2^{(13)} &\stackrel{d}{=} -\frac{\sigma}{n}E_{13}+T,\\
Z_{1,e}^{(14)} &\stackrel{d}{=} \Gamma\left(m-1,\frac{\sigma}{d}\right)+N\Gamma\left(d-m-1,\frac{\sigma}{d}\right)+\frac{\sigma}{d}E_{14A}+\frac{(m-l-1)\sigma}{(m+l+1)d}E_{14B},\\
Z_{2,e}^{(14)} &\stackrel{d}{=} -\frac{\sigma}{n}E_{14A}-\frac{\sigma}{m+l+1}E_{14B}+T,\\
Z_{1,o}^{(14)} &\stackrel{d}{=} \Gamma\left(m,\frac{\sigma}{d}\right)+N\Gamma\left(d-m-1,\frac{\sigma}{d}\right)+\frac{(m-l)\sigma}{(m+l+1)d}E_{14},\\
Z_{2,o}^{(14)} &\stackrel{d}{=} -\frac{\sigma}{m+l+1}E_{14}+T,\\
E. &\stackrel{d}{=} E(1).
\end{aligned}
$$

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