Stochastic differential equation models of vortex merging and reconnection

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(Received 13 October 2004; accepted 12 April 2005; published online 3 June 2005)

We show that the stochastic differential equation (SDE) model for the merger of two identical two-dimensional vortices proposed by Agullo and Verga ["Exact two vortices solution of Navier–Stokes equation," Phys. Rev. Lett. **78**, 2361 (1997)] is a special case of a more general class of SDE models for N interacting vortex filaments. These toy models include vorticity diffusion via a white noise forcing of the inviscid equations, and thus extend inviscid models to include core dynamics and topology change (e.g., merger in two dimensions and vortex reconnection in three dimensions). We demonstrate that although the $N=2$ two-dimensional model is qualitatively and quantitatively incorrect, it can be dramatically improved by accounting for self-advection. We then extend the two-dimensional SDE model to three dimensions using the semi-inviscid asymptotic approximation of Klein *et al.* ["Simplified equations for the interactions of nearly parallel vortex filaments," J. Fluid Mech. **288**, 201 (1995)] for nearly parallel vortices. This model is nonsingular and is shown to give qualitatively reasonable results until the approximation of nearly parallel vortices fails. We hope these simple toy models of vortex reconnection will eventually provide an alternative perspective on the essential physical processes involved in vortex merging and reconnection.
 [DOI: 10.1063/1.1932310]

I. INTRODUCTION

Vortex interactions are fundamental to moderate and high Reynolds number flows. The dynamics of jets, fluid-structure interaction, and mixing layers are all governed by large-scale coherent vortices. It is also believed that the dynamics of high Reynolds number turbulence is dominated by vortex interactions. In particular, most enstrophy dissipation is probably due to the vortex reconnection. Many turbulent flows are forced by vorticity production at the wall, and drag reduction techniques attempt to modify the wall vortices. Vortex-based methods are used to simulate fluid flow in both two and three dimensions.[1] It is thus clearly important to understand and analyze all stages of vortex reconnection.

Unfortunately, there is still no simple mathematical model for vortex reconnection. This is because viscous diffusion is necessary for full vortex reconnection, and viscosity renders slender vortex models, with their relatively simple Hamiltonian dynamics, inapplicable. Despite this, many semi-inviscid models have been developed that include a steady approximation for core dynamics.[2–5] These models fail once the vortices approach within a core radius, which leads to the development of a singularity in curvature (called a *hairpin* or *kink*). Various more or less *ad hoc* ways of dealing with reconnection have been proposed.[6,7] These methods use a physically based algorithm to give the end result of the reconnection. Obviously, the intermediate stages of the reconnection are not resolved.

Of course, vortex reconnection can be calculated accurately using full direct numerical simulation (DNS) of the Navier–Stokes equations.[8,9] However, DNS is computationally expensive and gives little insight into the fundamental physical mechanisms involved. In addition, DNS is limited to moderate Reynolds numbers since its space-time computational complexity scales like Re^3 (unless an adaptive method is used). For these reasons a simple analytic or semi-analytic model for interacting vorticity filaments is desirable.

Agullo and Verga[10] claimed to have produced an "exact two vortex solution of the Navier–Stokes equations," i.e., an equation for the interaction of two identical two-dimensional vortices whose solution can be approximated asymptotically. They simply took the inviscid equation for two interacting point vortices and turned it into a stochastic differential equation (SDE) by adding white noise forcing. The positions of the point vortices become random variables, and the vorticity distribution is given by the probability density function (PDF) for the positions of the point vortices. Although the equations do produce a single merged vortex, as we show in Sec. II, the dynamics and vorticity distribution are both quantitatively and qualitatively incorrect. In fact, the solution involves a severe simplification of the nonlinear term of the vorticity equation: only pairwise interactions between the point vortices are included in each realization. In the complete vorticity equations the vorticity at each point is advected by the vorticity at all other points simultaneously. It is this approximation that distinguishes Agullo and Verga's approach from numerical vortex methods, where many point vortices are included simultaneously in order to approximate a continuous vortex distribution.[1] Despite its shortcomings, Agullo and Verga's approach does suggest a general way of extending inviscid models to include diffusion, and hence topology change. We will explore this idea, and try to evaluate its usefulness, in the present paper.

The SDE toy models considered in this paper are closely related to, but distinct from, the stochastic vortex method introduced by Chorin.[11] Chorin proposed his vortex method

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as a way of efficiently solving the two-dimensional Navier–Stokes equations at high Reynolds numbers. The continuous vorticity field is divided into N vortex blobs, i.e., $\omega = \Sigma_{j=1}^{N} \omega_j$, with the corresponding stream function,

$$\psi(r) = \sum_{j=1}^{N} \Gamma_j \psi^0(r - r_j), \tag{1}$$

where the smoothed kernel $\psi^0(r)$ is given by

$$\psi^0(r) = \begin{cases} \dfrac{1}{2\pi} \ln r, & r \geqslant \sigma, \\ \dfrac{1}{2\pi\sigma} r, & r < \sigma. \end{cases} \tag{2}$$

Distant vortex blobs interact as point vortices and σ is a cutoff which regularizes the logarithmic singularity of the point vortex kernel. The motion of the vortex blobs is then calculated by a split-step particle method where the particles are first advected using the streamfunction (1) (excluding self-advection), and then perturbed by a stochastic white noise forcing (which models diffusion). Chorin saw this method as especially useful for high Reynolds number flows because it is gridless and because the error in approximating the advection term due to the random walk is $O(\mathrm{Re}^{-1/2})$. Chorin applied this simple vortex blob method to flow past a circular cylinder, where the no-slip boundary condition is enforced by the creation of M vortices (of the appropriate strength) along the cylinder boundary at each time step. M depends on the time step: as the time step decreases more vortices must be created.

In two dimensions the SDE model discussed in this paper is closely related to Chorin's model. There are, however, several important differences compared with Chorin's and other more recent vortex methods.[1,12–14]

(1) We use point vortices instead of vortex blobs.
(2) We consider only the interaction of initially well-separated physical vortices.
(3) Each physical vortex is represented by only one point vortex. In vortex methods a large number (typically tens or hundreds of thousands) of discrete vortices are used simultaneously.[13,15]
(4) The actual vorticity field is given by the ensemble average (or PDF) of many realizations.

Perhaps the most important difference, however, is our ultimate goal. We are *not* interested in the accurate numerical approximation of the vorticity equations (vortex methods[1,12–14,16–19] are already highly accurate in both two and three dimensions), but rather in the construction of simple toy models of vortex interaction which allow topology change. These models are not rigorously justified, but are intended to capture the minimal physics needed to qualitatively describe vortex interaction. In fact, the two-dimensional case is just the simplest of a class of such SDE toy models that extend semi-inviscid vortex filament equations to include vorticity diffusion by the addition of white noise forcing. This converts the inviscid partial differential equation into a stochastic differential equation, where the vorticity field is given by the PDF of its solution. These toy models should eventually give a new insight into the essential physics of vortex merger and reconnection, and can even admit analytical solution in certain cases.[10]

The paper is organized as follows. In Sec. II A we review Agullo and Verga's SDE model for the interaction of identical two-dimensional vortices, and compare its solution with a full DNS in Sec. II B. The main source of error is identified and a simple correction is proposed in Sec. II C. This corrected model is shown to be qualitatively accurate. Then in Sec. III we extend the SDE model to the case of N interacting nearly parallel vortices via a simple modification of asymptotic semi-inviscid model by Klein *et al.*[2] for nearly parallel vortices. Agullo and Verga's model is a special case of this new model when $N=2$ and vortex curvature is neglected. Because it is not a straightforward asymptotic approximation of the incompressible Navier–Stokes equations, the qualitative accuracy of this stochastic toy model is assessed by comparing it with a pseudospectral DNS of the reconnection of antiparallel vortices (i.e., the Crow instability). Finally, we make some concluding remarks and outline future research directions in Sec. IV.

II. TWO-DIMENSIONAL TWO VORTEX SDE MODEL

A. Basic model

Agullo and Verga[10] proposed the following simple model for the interaction of two identical point vortices:

$$\frac{\partial \psi_1}{\partial t} = 2i \frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'} b_1, \tag{3}$$

$$\frac{\partial \psi_2}{\partial t} = -2i \frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'} b_2, \tag{4}$$

where $\psi_j(t) = x_j(t) + i\, y_j(t)$ are the positions of the two vortices (expressed as complex numbers) and $b_j(t)$ are independent white noises (the derivative of a Wiener process). Time has been rescaled by 4π so viscosity is also rescaled, $\nu' = 4\pi\nu$. Note that the first term on the right-hand side is simply the (nonlinear) advection by the other point vortex, and the second term is a white noise forcing which represents diffusion. The vortex positions ψ_1 and ψ_2 are random variables. The vorticity field is given by the ensemble average of many realizations of $\psi_1(t)$ and $\psi_2(t)$ (which is an estimate of their PDFs), and the center of rotation of the vortices is given by the expectations $\langle \psi_1(t) \rangle$, $\langle \psi_2(t) \rangle$. In each realization the initial condition is the same: a pair of identical point vortices with circulation $\Gamma = 1$.

Before considering the solution of the basic model [(3) and (4)], let us recall the full two-dimensional vorticity equation,

$$\frac{\partial \omega}{\partial t} = -\boldsymbol{u} \cdot \boldsymbol{\nabla} \omega + \nu \Delta \omega. \tag{5}$$

Note that the vorticity is confined to the z direction, i.e., $\boldsymbol{\omega} = (0, 0, \omega)$. The velocity is a functional of the vorticity, given by the Biot–Savart law,

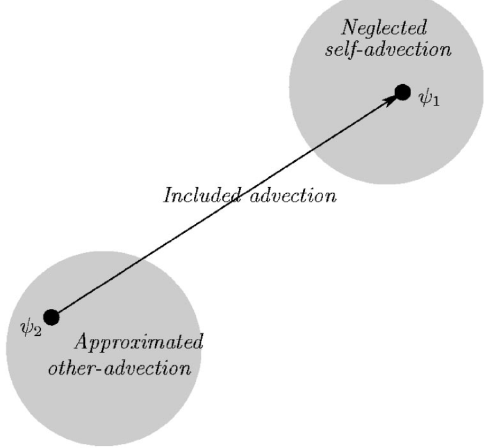

FIG. 1. Schematic illustration of the approximation to the nonlinear term in the basic SDE model. The shaded regions indicate the actual vorticity field and the black circles indicate the position of the point vortices in this particular realization. We are considering the velocity advecting vortex 1.

$$\boldsymbol{u}(\boldsymbol{x},t) = -\frac{1}{2\pi}\int \frac{(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} \times \omega(\boldsymbol{y},t)d\boldsymbol{y}, \tag{6}$$

where the integral is over the entire fluid. The interaction of two identical vortices corresponds to specifying an appropriate initial condition, e.g., with an initial separation r_0,

$$\omega(x,y;0) = \delta(-r_0/2,0) + \delta(r_0/2,0). \tag{7}$$

This is also the initial condition used in the SDE model.

Comparing (6) with the SDE model, we see immediately that the SDE model involves a drastic simplification of the advection term. In fact, only the pairwise nonlinear interactions of point vortices are included in each realization. As we will see later, it is useful to divide the neglected part of the advection into two components: the advection of the point vortex due to *self-advection* (i.e., due to the rest of the vortex) and the advection due to the vorticity of the rest of the *other vortex* (i.e., due to the fact that the actual vorticity field is distributed and not concentrated at a single point). These neglected parts of the nonlinear advection are sketched in Fig. 1. The key approximation of the SDE model is thus a pointwise approximation to the advecting velocity field, where only *one point* (which is itself a stochastic variable) is used in any given realization.

Note that the effect of self-advection is entirely neglected in the SDE model. We will see that the neglect of self-advection is by far the largest source of error in this model. Interestingly, in the case of one vortex the SDE model is exact, since the nonlinear term is zero and the dynamics are purely diffusive.

One can construct valid probabilistic approximations, or even representations, of the vorticity equations. For example, numerical vortex methods use a pointwise approximation of the vorticity field, but in this case the vorticity of a vortex is distributed over many point vortices (e.g., tens of thousands). On the other hand, Busnello *et al.*[20] have shown that one can construct an exact SDE representation of the vorticity of a three-dimensional viscous fluid on $\mathbb{R}^3$ (i.e., without solid boundaries). Again, this probabilistic model is only valid when many fluid particles are considered simultaneously. In contrast, in Agullo and Verga's model each vortex is represented by only a single point vortex in each realization.

Although the SDE model involves a rather severe approximation of the Navier–Stokes equations it does have some attractive features.

(1) Analytic solutions or approximations are possible in some cases (e.g., Agullo and Verga[10]).
(2) Numerical solutions are efficient, even at large Reynolds numbers, because the method is gridless. In addition, the SDE model reduces the dimension of the vortex merger problem from two to zero (i.e., from the continuous two-dimensional vorticity equation to point vortex interactions). A similar SDE model (see Sec. III) reduces the vortex reconnection problem from three dimensions to one dimension.
(3) Existing inviscid or semi-inviscid models can be extended easily to include topology change. This removes the finite time singularities associated with these models.
(4) The Hamiltonian structure of the original inviscid equations is retained.

For these reasons it is interesting to evaluate the SDE model, and determine how it might be improved or extended.

B. Comparison of the basic SDE model with a DNS

We now compare the basic SDE model solution with a full DNS. We use a high resolution adaptive wavelet direct numerical simulation (WDNS) of the vorticity equation (for details see Vasilyev and Kevlahan[21]). The WNDS uses an adaptive high-order explicit stiffly stable Krylov method in time.[22] The key property of WDNS is that the computational grid adapts automatically to the solution at each time step, refining or coarsening locally as necessary. One can therefore think of the WDNS as a sort of constrained vortex method, where the adapted grid points correspond to vortices. Indeed, we use the fast multipole method[23] approximation to the Biot–Savart law to find the velocity field given the vorticity at the adapted grid points. Thus, the number of points in the adapted grid gives an idea of the number of vortices required to fully represent the solution and its dynamics.

The WDNS solution for the merger of two identical vortices with circulation $\Gamma=1$ and initial separation $r_0=1$ was computed at $\mathrm{Re}=\Gamma/\nu=1000$ on a domain of $[-2.5,2.5]\times[-2.5,2.5]$ with a maximum resolution of 2048^2 grid points. The l^2 norm tolerances for grid adaptation and time integration were set to 10^{-4}. Note that the WDNS is dealiased: the actual grid used is twice as fine as the grid necessary to resolve the solution to the desired tolerance. The vorticity field at nine different times is shown in Fig. 2, while the associated adapted grid is shown in Fig. 3. Times are normalized by the initial rotation period of the pair of point vortices. Figure 4 shows that about 8000 grid points were used in the simulation, with a maximum of about 10 000 points at $t=1.37$. This suggests that roughly 10^4 point vortices would be required simultaneously for a complete repre-

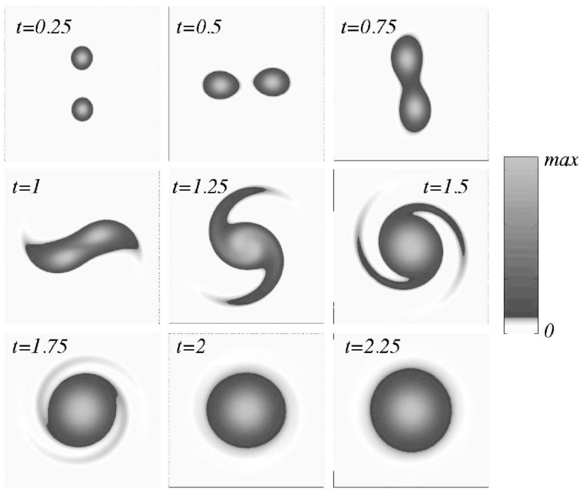

FIG. 2. Vorticity field. Vortex merging at Re=1000, full adaptive wavelet solution.

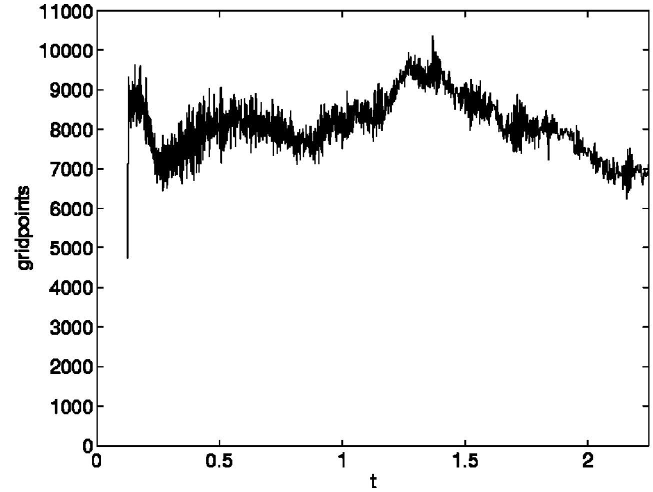

FIG. 4. Number of grid points in adaptive wavelet solution as a function of time.

sentation of the vorticity dynamics. Recall that only two-point vortices are used in any given realization of the SDE model.

The vortices are pushed together by the irrotational strain generated by the spiral vorticity filaments. This convective stage of merger is described in detail by Meunier[24] and by Cerretelli and Williamson.[25] The vortices are fully merged (i.e., the vorticity field has a single maximum) after 1.5 pair rotation periods, and a single Gaussian vortex has formed after two rotations. The asymptotic long time state is therefore a single Gaussian vortex at the center of rotation of the initial conditions. The dynamics in the final stage are purely diffusive and therefore linear.

We now solve the basic SDE model equations numerically, using the Euler–Maruyama method[26] with a small time step of $\Delta t = 10^{-4}$ to ensure accuracy. Note that since the drift coefficient $\sqrt{2\nu'}$ is constant, the Euler–Maruyama method is equivalent to the Milstein method[26] and has strong order 1. The vorticity field (which is the ensemble average of 10^5 realizations) is shown in Fig. 5. A comparison of Figs. 2 and 5 reveals that the basic SDE model is both qualitatively and quantitatively *incorrect*. Although the long-time solution is a single Gaussian vortex in both cases, the intermediate dynamics and time scale for final merging are very different. In the basic SDE model the intermediate solution is a diffusing vortex ring, and the vortices have still not completely merged by $t = 2.25$. Indeed, merging is a diffusive process in the basic SDE model, whereas the actual merging process is advection dominated, with diffusion important only in the final stages.[25] In the following section we propose a corrected model that gives a qualitatively accurate solution and identify the main source of error in the basic model.

C. Corrected model

The error in the basic SDE model is due to the approximation of the nonlinear advection term by pairwise point vortex interactions. As mentioned above, this error may be divided into self-advection error (neglected entirely) and

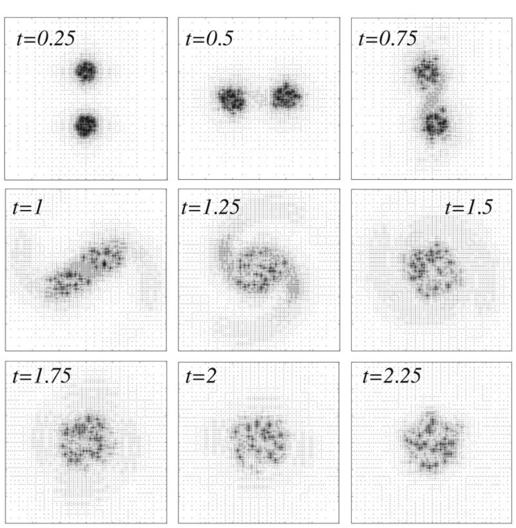

FIG. 3. Adaptive wavelet grid for vortex merging at Re=1000.

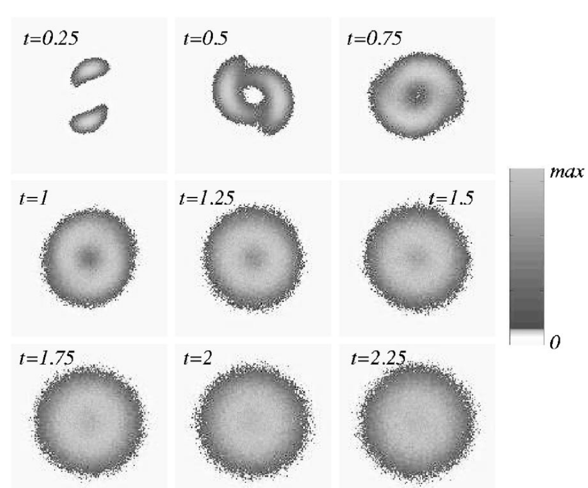

FIG. 5. Vorticity field. Vortex merging at Re=1000, basic SDE model.

TABLE I. Corrected SDE models.

Case	Advecting velocities
Self-advection	$U_1=u_1+V$, $U_2=u_2-V$
Other advection	$U_1=u_2$, $U_2=u_1$
Both	$U_1=u_1+u_2$, $U_2=u_1+u_2$

other vortex interaction error (the effect of the other vortex is approximated by concentrating all the vorticity at a single point).

A pair of inviscid point vortices (without white noise forcing) simply rotate around their center of rotation at a constant rate. We therefore propose to advect the point vortices in the SDE model by the velocity field of Gaussian vortices (of the correct age) at the positions of the inviscid point vortices. It is easy to find an analytic expression for this velocity field. Because the velocity is an integral of the vorticity, we expect it to be relatively insensitive to the precise location and form of the vortices. Besides correcting the basic SDE model, this approach also allows us to separate the effects of self-advection and other vortex advection. The corrected equations are

$$\frac{\partial \psi_1}{\partial t} = iU_1(\psi_1,t) + \sqrt{2\nu'}b_1, \tag{8}$$

$$\frac{\partial \psi_2}{\partial t} = iU_2(\psi_2,t) + \sqrt{2\nu'}b_2, \tag{9}$$

where $U_1(\boldsymbol{z},t)$ and $U_2(\boldsymbol{z},t)$ are the (complex) velocities advecting vortices 1 and 2, respectively.

Let u_1 and u_2 be the velocities of Gaussian vortices at the locations of inviscid point vortices with initial positions of vortices 1 and 2, respectively, and let $V(\psi_1,\psi_2)=2(\psi_1-\psi_2)/|\psi_1-\psi_2|^2$ be the advecting velocity in the basic SDE model. In order to identify the main source of error in the basic SDE model we consider three different cases: self-advection, other advection, and both, as summarized in Table I.

Because the distance between the inviscid point vortices never decreases, we must eventually replace the two advecting Gaussian vortices by a single vortex at their center of rotation. Cerretelli and Williamson[25] find that merging occurs when the vortex core size $\delta \approx 0.29 r_0$. We use this time (i.e., $t_c=1.07$) to switch to a single vortex. The single vortex must conserve the maximum vorticity and total circulation of the two vortices it replaces. Although this switch to a single vortex is rather *ad hoc*, it does give reasonable results. Note that for $t \geqslant t_c$ all three models described in Table I are identical.

The corrected SDE models are compared with the full WDNS solution in Fig. 6. Figures 6(c) and 6(d) show that the main source of error in the basic SDE model of Agullo and Verga is self-advection. If the effect of self-advection is approximated as described above, the solution of the SDE model is much more accurate. If the effect of the continuous vorticity of the other vortex is included as well, the SDE solution is qualitatively correct, and has approximately the right time scale. It is perhaps not surprising that self-advection is the largest source of error: it is completely neglected in the basic SDE model.

Although the goal of this paper is to develop and analyze the qualitative accuracy of simple SDE toy models for vortex

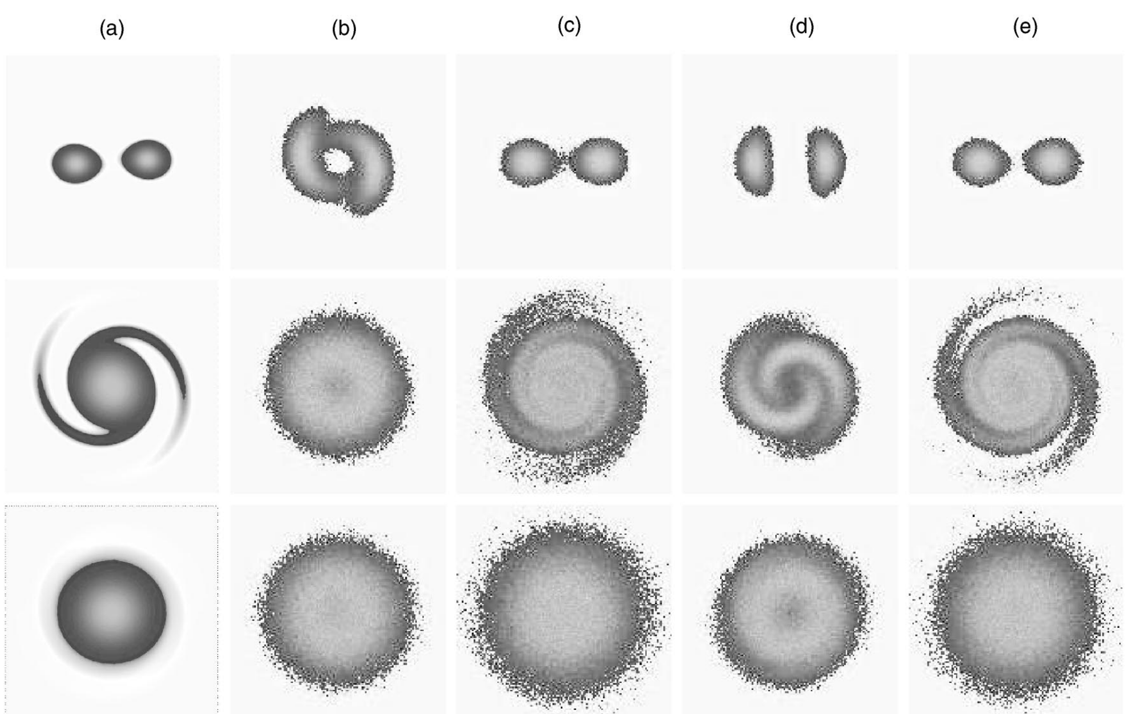

FIG. 6. Vorticity field. Two-dimensional vortex merger at times $t=0.5$, 1.25, 2. (a) Full WDNS solution. (b) Uncorrected SDE model. (c) SDE model corrected to include self-advection. (d) SDE with other vortex advection corrected. (e) SDE model with both self-vortex and other vortex corrections.

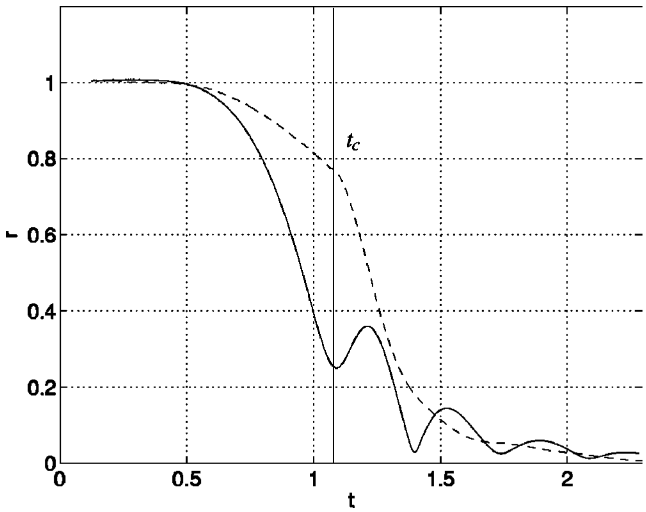

FIG. 7. The distance between the centers of rotation of the vortices as a function of time. —, exact WNDS; - - -, corrected SDE model.

interaction, it is also instructive to examine their errors quantitatively. An important quantitative measure of two-dimensional vortex merger is the distance between vortex centers as a function of time, $r(t)$. Figure 7 shows $r(t)$ for the exact WDNS solution and the corrected SDE model. Note that we have actually shown the WDNS solution of the PDE version of the corrected SDE model (i.e., a linearized form of the vorticity equation with the advecting velocity as in the SDE model). This allows us to easily measure $r(t)$ at each time step and eliminates the random noise of the numerical solution of the SDE. The PDE version of the SDE model is solved to the same l^2 tolerance used for the exact equations, i.e., 10^{-4}. The vortex centers were calculated using separate equations for each vortex, which allows the two vortices to remain identifiable for the whole simulation. The corrected model merges the vortex on approximately the correct time scale, although some details [such as the oscillations in $r(t)$ at the later stages of merging] have been lost. It is interesting to note that $r(t)$ begins to decrease well before $t=t_c$, when the two advection vortices are replaced by a single one.

A finer quantitative comparison is given by Fig. 8, which compares cuts through the vortex maxima for the exact WDNS solution and the corrected model (solved as a PDE). The agreement is reasonable until the switch to the single vortex, at which point the corrected model has much less vorticity at the center of the merging vortices. The reason is clear from Figs. 6 and 9: because the vortex maxima are still relatively far apart at $t=t_c$, the switch to a single advecting vortex leads to a rapid wind-up of vorticity accompanied by a spurious increase in enstrophy dissipation. In addition, the single advecting vortex does not move the vortex maxima together as fast as in the exact case. Despite these significant quantitative errors, the time scale for the merger and the final vortex radius are reasonable.

Finally, we would like to compare the numerical efficiencies of each method. We take a spatial tolerance for grid adaptation of 10^{-2} and a time integration tolerance of 10^{-4} for the WDNS (exact vorticity equation, and PDE version of SDE model). These WDNS are then comparable in numerical accuracy to the stochastic solution of the SDE model with a time step of 10^{-4} and 10^5 realizations. Table II shows the CPU times for each of the examples considered. Note that the WDNS require a maximum resolution of between 512^2 and 32^2 (depending on the time), and a maximum of 2586 wavelets. The WDNS of the PDE version of the SDE model is the fastest: 3.6 times faster than the exact WDNS and 49 times faster than the stochastic solution of the SDE model. The slowness of the stochastic simulation is due to the slow square root convergence of the stochastic approximation of the diffusion term (discussed below). The WDNS of the PDE version of the SDE model is faster than the WDNS of the exact equations due to three factors.

(1) There is no need to calculate the velocity from the vorticity (a time consuming part of the WDNS).
(2) The Krylov time scheme used in the WDNS is very efficient for linear equations, which allows much larger time steps.
(3) The code uses a coarser grid resolution to achieve the same tolerance (twice as coarse at many times).

These observations suggest that in two dimensions it is more numerically efficient to solve the PDE, rather than the stochastic, version of the SDE model.

Figure 10 illustrates the way the noise of the SDE model decreases with the number of realizations. From the central limit theorem, it is clear that the noise should decrease like $1/\sqrt{\mathcal{N}}$, where $\mathcal{N}$ is the number of realizations. The computational complexity of the SDE model is $O(N\mathcal{N}/\Delta t)$, i.e., it is proportional to the number of interacting vortices and the number of realizations, and inversely proportional to the time step. As in all Lagrangian methods, the time step is not limited by the Courant–Friedrichs–Lax criterion. Kloeden and Platen[26] show that the time step of the Euler–Maruyama scheme for a SDE must satisfy $\Delta t < 2/U^2|-U+1/(4\nu)|$, where U is the drift velocity.

A rough solution of the SDE can be found quickly and then improved progressively as required; this is not possible with the PDE formulation. In addition, because the realizations are independent, the method can be easily parallelized on any computer cluster and scales to an arbitrarily large number of processors. These advantages are more significant when the SDE approach is applied to vortex reconnection in three dimensions, as proposed in the following section. Note that we have chosen a very conservative time step ($\Delta t=10^{-4}$), but we also obtain reasonable results for time steps of 10^{-3} or larger. If CPU time were an issue, we could use one of the more sophisticated adaptive time step methods for SDEs, such as the one proposed recently by Burrage and Burrage.[27]

III. THREE-DIMENSIONAL NEARLY PARALLEL *N* VORTEX SDE MODEL

A. Derivation of the model

We now explain how the basic SDE model of Agullo and Verga[10] for the interaction of two identical two-dimensional vortices can be extended to N three-dimensional vortices

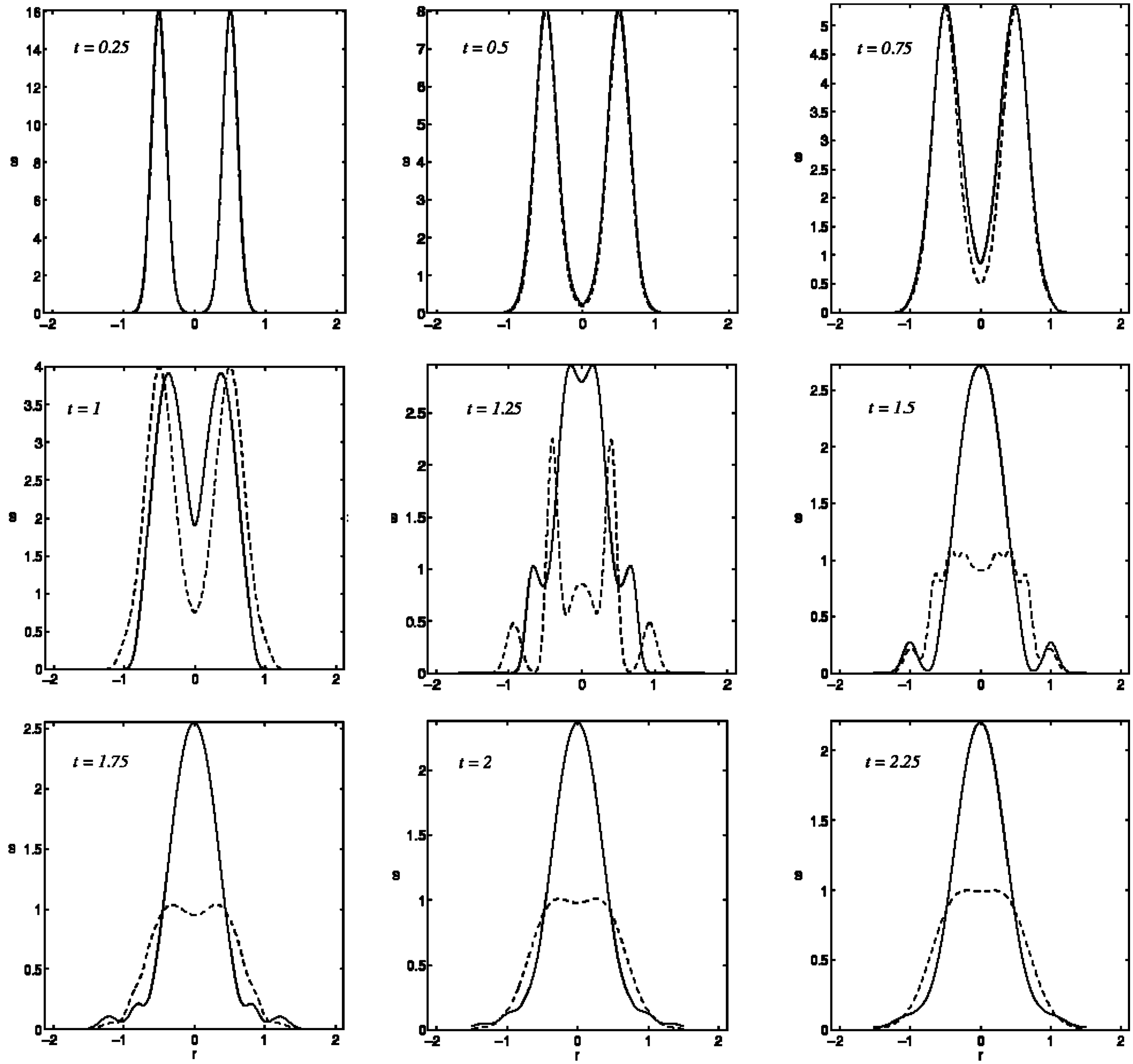

FIG. 8. Vorticity along cuts through the vortex maxima: —, exact WDNS; - - -, corrected model (WDNS solution of PDE version).

with different circulations. The simplest model for the interaction of three-dimensional vortex filaments is the semi-inviscid asymptotic theory for the interaction of N nearly parallel vortices derived by Klein *et al.*[2] They assume that the vortices are nearly aligned (e.g., with the z axis) and that the perturbation amplitudes of the vortex centerlines are much smaller than the perturbation wavelengths, which are also much larger than the core radius. As usual in such semi-inviscid theories, they also assume that the separation between vortices is much larger than the vortex core radius. With these assumptions the interaction between vortex filaments is approximated asymptotically by two-dimensional point (i.e., potential) vortex interaction in planes perpendicular to the z axis, while the self-advection is given by a geometrically simple form of the local induction approximation. Nonlocal self-stretching and mutual induction are neglected, although vortex stretching by other vortices is included. These equations are remarkably simple and are easy to solve numerically to high precision. The numerical solutions presented in Ref. 2 show that, when applied to the Crow instability of antiparallel vortices, the vortices develop a kink where the filaments are closest. At this point the theory breaks down and the solution is singular. This semi-inviscid theory is thus unable to resolve vortex reconnection or predict the final configuration of the vortices.

We propose to extend the theory of Klein *et al.* to include diffusion (and core dynamics) by adding white noise forcing to the filament equations, following the approach of Agullo and Verga.[10] This produces the following SDE toy model for the diffusive interaction of N nearly parallel vortices:

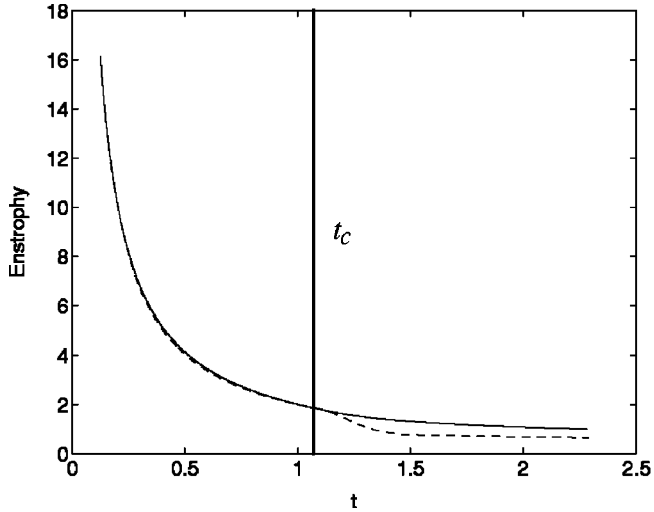

FIG. 9. Enstrophy as a function of time: —, exact WDNS solution; - - -, corrected model (WDNS solution of PDE version).

$$\frac{\partial \boldsymbol{X}_j}{\partial t} = J\left[\sum_{k\neq j}^{N} 2\Gamma_k \frac{(\boldsymbol{X}_j - \boldsymbol{X}_k)}{|\boldsymbol{X}_j - \boldsymbol{X}_k|^2}\right] + J\left[\Gamma_j \frac{\partial^2 \boldsymbol{X}_j}{\partial z^2}\right] + \sqrt{2\nu'}\boldsymbol{b}_j, \tag{10}$$

where $\boldsymbol{X}_j(z,t)=[x_j(z,t),y_j(z,t)]$, $j=1,\ldots,N$ are the coordinates of the vortex centerlines, Γ_j are their circulations,

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and $\boldsymbol{b}_j(z,t)=[b_{j1}(z,t),b_{j2}(z,t)]$ where b_{j1} and b_{j2} are independent Gaussian random variables with mean zero and variance one. Time has been rescaled by 4π so $\nu'=4\pi\nu$. The first term on the right-hand side of (10) is the point vortex interaction in z planes, the second is the local induction approximation (i.e., curvature term), and the third is the white noise forcing (which diffuses the vortices in the z planes, which are the planes approximately perpendicular to the vortex axis). The initial conditions for these equations are nonintersecting line vortices.

Formally, the only change with respect to the model of Klein *et al.* is the addition of the white noise forcing. However, the interpretation of the solution is very different. The filament's shape $\boldsymbol{X}_j(t)$ is now a random variable, and the vorticity distribution is given by its PDF. The position of the vortex centerline is the mean of $\boldsymbol{X}_j(t)$. We will see that the SDE model remains nonsingular for arbitrarily long times.

TABLE II. CPU times for different methods with the same numerical accuracy.

Case	CPU time (s)
Exact WDNS	407
Corrected SDE model (PDE, WDNS solution)	113
Corrected SDE model (stochastic, 10^5 realizations)	5500

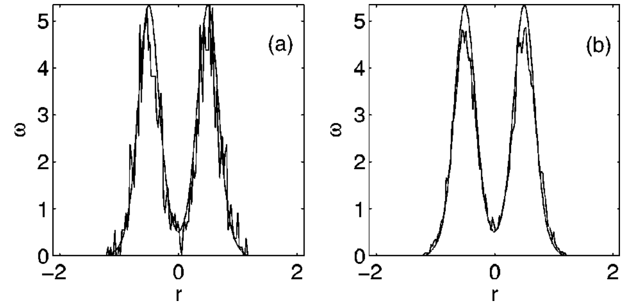

FIG. 10. The vorticity along the y axis at $t=0.75$. (a) 10^4 realizations, compared with the PDE solution. (b) 10^5 realizations, compared with the PDE solution.

This suggests that a toy model of this type could reproduce some aspects of vortex reconnection. This is likely the simplest SDE model with this property.

When $N=2$, we can use the complex notation of Ref. 2 to derive the following equations for the interaction of a pair of filaments:

$$\frac{\partial \psi_1}{\partial t} = 2i\Gamma \frac{\psi_1-\psi_2}{|\psi_1-\psi_2|^2} + i\frac{\partial^2 \psi_1}{\partial z^2} + \sqrt{2\nu'}b_1, \tag{11}$$

$$\frac{\partial \psi_2}{\partial t} = -2i \frac{\psi_1-\psi_2}{|\psi_1-\psi_2|^2} + i\frac{\partial^2 \psi_2}{\partial z^2} + \sqrt{2\nu'}b_2, \tag{12}$$

where $\psi_j=x_j(z,t)+i\,y_j(z,t)$, $b_j(z,t)=b_{j1}+i\,b_{j2}$, we have set $\Gamma_1=1$, $\Gamma=\Gamma_2/\Gamma_1$.

If we neglect the curvature term in (10) and take $z=0$ (without loss of generality), we obtain a SDE model for the interaction of N nonidentical two-dimensional vortices. If we set $\Gamma=1$, $N=2$, and $z=0$ and neglect curvature in [(11) and (12)] we recover the SDE model of Agullo and Verga [(3) and (4)]. Agullo and Verga's model equation can thus be seen as a special case of a much wider class of SDE models for vortex interaction.

The simple SDE toy model [(11) and (12)] can be solved numerically using a split-step method similar to that used by Klein *et al.*[2] We take periodic boundary conditions in z, and solve exactly for the curvature term step using a Fourier transformation in z. The grid spacing in z, Δz, should be sufficiently small to resolve the smallest filament curvature, but much larger than the amplitude of the white noise perturbations to avoid resolving spurious small scale curvature fluctuations, i.e., we require $\sqrt{2\nu'\Delta t} \ll \Delta z \ll \sqrt{\nu'}$, which can be satisfied if $\Delta t \ll 1$. The remaining two terms are solved as in the two-dimensional case.

B. Evaluation of the model: Reconnection of antiparallel vortices

This section is not intended to provide a quantitative error analysis of the SDE toy model or even to determine its range of validity. Our aim is to establish whether the SDE model is promising and merits further investigation. In order to evaluate qualitatively the SDE model for nearly parallel

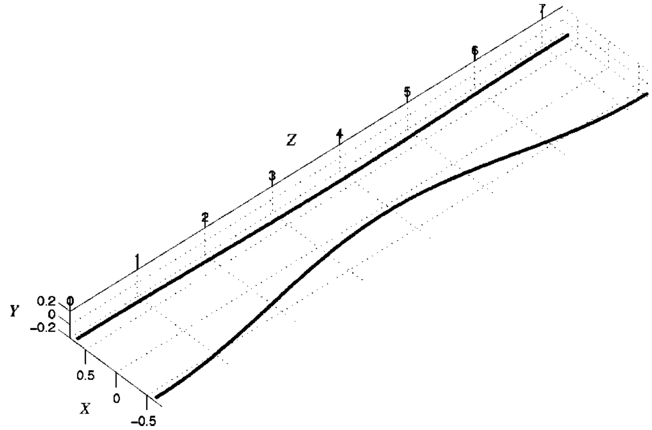

FIG. 11. Initial configuration of vortex centerlines for the antiparallel vortex reconnection simulation. The positive vortex is in the half-space $x<0$, and the initial condition is mirror symmetric about the plane $x=0$.

vortices we compare it with the semi-inviscid model of Klein *et al.* and to a full pseudospectral DNS of the reconnection of two equal strength antiparallel vortices.

In all simulations, $\Gamma=-1$ and the initial conditions for the vortex centerlines are

$$\psi_1(z,0) = -\frac{1}{2} - \frac{\epsilon}{2\sqrt{2}}(1-i)(e^{i2\pi/\lambda z} - e^{-i2\pi/\lambda z}),$$

$$\psi_2(z,0) = \frac{1}{2} - \frac{\epsilon}{2\sqrt{2}}(1+i)(e^{i2\pi/\lambda z} - e^{-i2\pi/\lambda z}),$$

where the amplitude and the wavelength of the perturbation are $\epsilon=0.2$ and $\lambda=7.3$, respectively. This initial condition corresponds to a periodic perturbation of the vortices at the most unstable mode of the Crow instability.[28] The nominal separation of the vortices is $r_0=1$. In the viscous cases (SDE and DNS) the vortex Reynolds number is $\mathrm{Re}=\Gamma/\nu=1500$, and the vortices initially have a Gaussian profile with radius 0.2.

The DNS uses the same initial conditions, grid resolution (128^3) and domain size (7.3^3) as the pseudospectral DNS of Marshall *et al.*[9] They found that these parameters give well-converged results. We use a Krylov stiffly stable adaptive time scheme[22] to ensure l^2 norm time integration accuracy of 10^{-4}. More details of this standard pseudospectral code are given in Ref. 29.

We construct a nonsingular Gaussian initial condition for the SDE model by setting to zero all terms except the stochastic term until the core has the desired thickness, i.e., 0.2. The length of the periodic computational domain in z is λ and is discretized using 128 points (as in the pseudospectral DNS). The time step is $\Delta t=10^{-3}$, and 3×10^6 realizations are used in order to obtain reasonably smooth contours of core vorticity. Note that if we were only interested in the geometry of the vortex centerlines, or the isosurfaces of vorticity, about $O(10^5)$ realizations would suffice. The initial condition for the SDE model and the model by Klein *et al.* is shown in Fig. 11.

Figure 12 compares the centerlines of the positive vortex according to the model by Klein *et al.* and the present SDE model at $t_*=0.52$, i.e., just before the vortex develops a kink

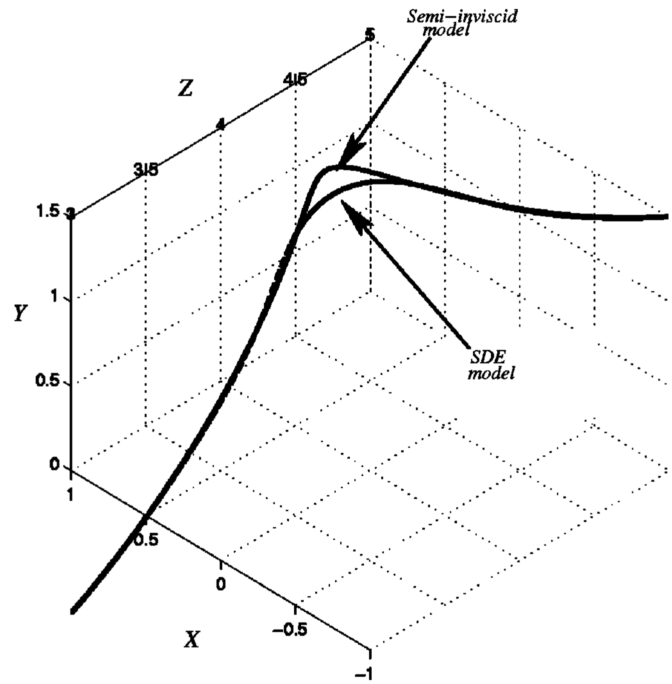

FIG. 12. Centerlines of the positive vortex near $z=\lambda/2$ at $t=0.52$ just before the semi-inviscid vortices develop kinks and touch. Note that the SDE model vortex matches the semi-inviscid vortex except near the kink, where it remains smooth.

and the model by Klein *et al.* breaks down. It is interesting to note that the two models agree, except in the region of the kink where the SDE model remains smooth. This result is perhaps not surprising, but it does show that the SDE model is physically reasonable: it remains close to the semi-inviscid model where the semi-inviscid model is valid, but avoids the unphysical kink. This result also shows how nonsteady core dynamics can control vortex curvature.

In Fig. 13 we compare the SDE model with the pseudospectral DNS at four different times. Isosurfaces of vorticity at $\omega=\Gamma/(4\pi\nu t)\exp(-1)$ (i.e., the core radius of an equivalent Gaussian vortex) are plotted. The DNS results show that the vortices interact, and eventually reconnect, at the location where they are closest (i.e., at $z=\lambda/2$). The reconnection begins at the time the semi-inviscid theory fails ($t\approx0.522$), and the reconnected vortices are still joined by thin threads at $t=1.27$. Overall the SDE toy model performs reasonably

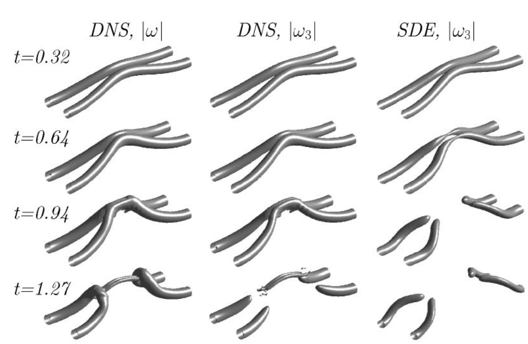

FIG. 13. Isosurfaces of vorticity: comparison of the DNS with the SDE model. Note that the semi-inviscid model of Klein *et al.* fails at $t=0.522$.

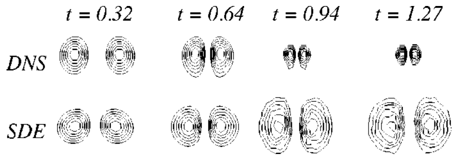

FIG. 14. Contours of vorticity in the vortex cores at $z=\lambda/2$ for the DNS and the SDE model. (At $t=0$ the DNS vortices have a finite radius $\sigma_0=0.2$.)

well: it remains nonsingular, its dynamics have the correct time scale, and the reconnection proceeds by thinning of the vorticity to create threads. Comparison of isosurfaces of $|\omega_3|$ for DNS and SDE shows that the main qualitative error is due to the lack of self-stretching in the SDE model. The threads joining the vortices rapidly become very weak in the SDE model, whereas in the DNS they strengthen due to the stretching. Of course, the SDE model includes only the z component of vorticity, and so is unable to fully reconnect the vortices (compare the $|\omega|$ and $|\omega_3|$ isosurfaces at $t=1.27$ in Fig. 13 to see the role of nonaxial vorticity in completing the reconnection). The model thus becomes increasingly inaccurate for $t>1.27$, although it remains numerically stable.

The simple three-dimensional SDE toy model performs better than the basic two-dimensional SDE model, probably because self-advection is approximated via the curvature term (i.e., the local induction approximation) instead of being totally neglected. This suggests that the curvature part of the self-advection dominates that due to core deformation.

Finally, Fig. 14 compares the core structure of the vortices in the SDE model with the DNS at $z=\lambda/2$. Seven contours are plotted between $\omega_{\max}$ and $\exp(-1)\omega_{\max}$, where $\omega_{\max}$ is the maximum vorticity at $z=\lambda/2$. The SDE model produces some elongation of the vortex core, although the deformation is smaller due to the lack of self-stretching. The lack of self-stretching also means that the SDE vortex cores are much too weak, and hence too large. Nevertheless, the fact that the core is deformed show that a suitably modified SDE model may be able to produce realistic core dynamics, as well as giving the vortex centerline geometry.

The results presented in this section suggest that, unlike the two-dimensional SDE model, the main sources of error in the three-dimensional SDE model are the assumption of nearly parallel vortices and the lack of self-stretching. These assumptions are not justified during the later stages of reconnection. The fact that the model is qualitatively correct at the early stages of reconnection is consistent with the observations of the preceding section that some self-advection must be included: it is included here via the curvature term.

IV. CONCLUSIONS

In this paper we have evaluated the accuracy of Agullo and Verga's[10] remarkable SDE model for the merger of two identical two-dimensional vortices. We demonstrated that this simple model is qualitatively and quantitatively incorrect, due to its drastic simplification of the nonlinear term of the vorticity equation. However, it can be dramatically improved by approximating vortex self-advection, which is completely neglected in their model. With this correction Agullo and Verga's model gives qualitatively accurate results when compared with a DNS.

Agullo and Verga's model may be seen as a special case of a much wider class of simple SDE toy models for vortex interaction. These models extend inviscid and semi-inviscid approximations to include vorticity diffusion and core dynamics by the addition of white noise forcing. This converts the inviscid partial differential equation into a stochastic differential equation, where the vorticity field is given by the PDF of its solution (which can be approximated as the ensemble average of many realizations). In this way, one can construct general SDE toy models for the interaction of N different two- or three-dimensional vortices. In a single realization the solution for the interaction of N vortices is geometrically simple: a set of N point vortices in two dimensions or N vortex filaments in three dimensions. Such toy models have several attractive features: they have an associated inviscid version, are computationally efficient, have nonsingular solutions, allow topology change, and retain the Hamiltonian structure of the original inviscid equations. However, since they are not derived as rigorous approximations to the vorticity equation, the accuracy and applicability of these SDE toy models is unclear.

A SDE toy model for the interaction of three-dimensional vortices has been evaluated. It is based on the semi-inviscid theory of Klein *et al.*,[2] and is therefore valid for nearly parallel vortices. We used this model to calculate the reconnection of identical antiparallel vortices, and compared the results with a pseudospectral DNS. The model avoids the finite-time curvature singularity of the semi-inviscid theory, and gives qualitatively reasonable results for intermediate times. However, it becomes unphysical at longer times, and is unable to produce complete reconnection. We conjectured that this problem is due primarily to the assumption of nearly parallel vortices and the neglect of self-stretching in the semi-inviscid model, and not to the stochastic modeling of vorticity diffusion and the associated simplification of the nonlinear terms of the vorticity equation. We will investigate a more sophisticated SDE toy model, which includes general vortex geometry and self-stretching, in future work.

The SDE toy models for vortex interaction introduced here have the potential to provide an alternative analytical or semi-analytical description of all stages of vortex connection. Because of their low computational complexity, they may also allow vortex interaction and topology change to be investigated at very high Reynolds numbers. Much work remains to be done in order to properly understand the accuracy and domain of validity of such simple SDE models for vortex interaction.

ACKNOWLEDGMENTS

The author would like to thank Oleg Vasilyev, who is a coauthor of the wavelet code used in Sec. II. Research funding from NSERC is gratefully acknowledged. This paper was written during a visit to DAMTP, University of Cambridge, and the author would like to thank them for their hospitality.

This work benefitted from conversations the author had with Tom Hurd and Sergey Nazarenko.

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