**Theoretical and Computational Fluid Dynamics manuscript No.**
(will be inserted by the editor)

**N. Kevlahan**

# Vortices for computing: the engines of turbulence simulation

Received: date / Accepted: date

**Abstract** Vortices have been described as the "sinews of turbulence" (9). They are also, increasingly, the computational engines driving numerical simulations of turbulence. In this paper I review some recent advances in vortex-based numerical methods for simulating high Reynolds number turbulent flows. I focus on coherent vortex simulation (CVS), where nonlinear wavelet filtering is used to identify and track the few high energy multiscale vortices that dominate the flow dynamics. This filtering drastically reduces the computational complexity for high Reynolds number simulations. It also has the advantage of decomposing the flow into two physically important components: coherent vortices and background noise. In addition to its computational efficiency, this decomposition provides a way of directly estimating how space and space–time intermittency scales with Reynolds number, $Re^{\alpha}$. Comparing $\alpha$ to its non-intermittent values gives a realistic Reynolds number upper bound for adaptive direct numerical simulation (DNS) of turbulent flows. This direct measure of intermittency also guides the development of new mathematical theories for the structure of high Reynolds number turbulence.

**Keywords** Turbulence · Vortices · Wavelets

**PACS** 42.27.De · 42.27.Ek · 42.27.Jv · 47.32.C

## 1 Introduction

Helmholtz (6) attempted to address the failure of Euler's inviscid equations to accurately describe fluid flow by introducing the concept of vorticity, and idealized singular flow structures such as vortex filaments and vortex sheets (4). Although viscosity does not appear explicitly in Helmholtz's vorticity equation, Helmholtz thought of vorticity as being introduced into the flow via internal friction at solid boundaries. In fact, he used the concept of the unsteady vortex sheet to accurately calculate the tones of an organ pipe. Helmholtz ignored the details of vorticity production, which remains a difficult theoretical problem to this day. Thus, from the very beginning, vorticity and vortices have been used to model and compute fluid flow.

Since Helmholtz's pioneering work, vorticity-based descriptions have proven to be the simplest way of understanding and computing a wide variety of compressible and incompressible flows. This is due to the power of vorticity-based theorems such as Helmholtz's theorem, Kelvin's circulation theorem and the helicity conservation theorem. For example, the force on an obstacle is the result of vortex motion,

$$\boldsymbol{F}(t) = -\frac{1}{2}\frac{d}{dt}\int_{V_\infty} \boldsymbol{x} \times \boldsymbol{\omega}\ \mathrm{d}V$$

N. Kevlahan
Department of Mathematics & Statistics, McMaster University

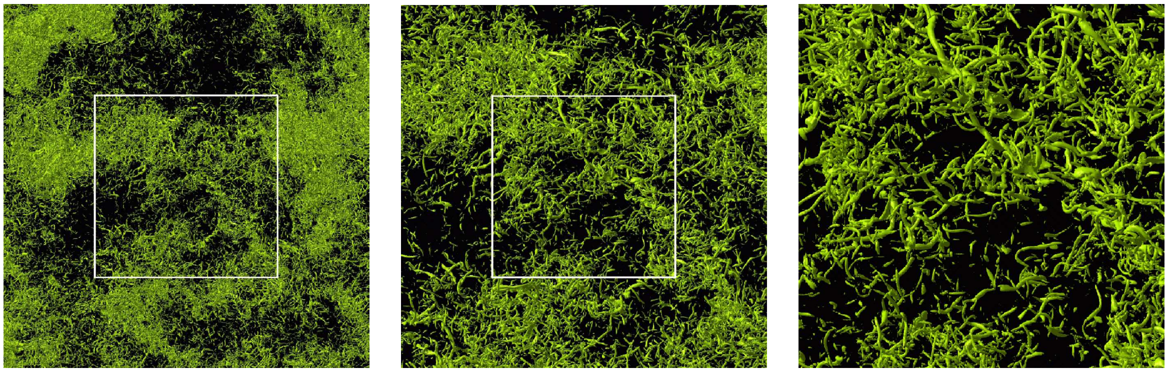

**Fig. 1** DNS of homogeneous isotropic turbulence at $\mathrm{Re}_\lambda = 1\,217$ at increasingly small scales (left to right). (From Yokokawa et al. (15).) The flow has spontaneously organized into many tube-like coherent vortices. Note the fractal-like structure of the vorticity field.

as is far-field sound emitted from a localized source, such as a jet,

$$p_F = -\frac{\varepsilon^{(1)}(t_r)}{15\pi c^2 r} - \frac{\rho_0}{c^2} Q_{ij}^{(3)}(t_r)\frac{x_i x_j}{r^3} + \frac{\rho_0}{c^3} Q_{ijk}^{(4)}(t_r)\frac{x_i x_j x_k}{r^4} + \cdots$$

where the $Q$'s are moments of the vorticity distribution. The lift generated by a wing, and other aerodynamic flows, can be explained by systems of vortex lines. Not surprisingly, rotational flow of superfluids can be described extremely well by collections of (quantized) vortex lines.

Coherent vortices are the viscous flow structures corresponding to the vortex filaments of ideal flow. In other words, they are localized regions of intense vorticity that persist for a significant period of time. Coherent vortices are most obvious in inhomogeneous flows, such as jets and mixing layers, but they are also the dominant flow structures of statistically homogeneous and isotropic turbulent flows (see figure 1). Coherent vortices are not just ubiquitous features of turbulent flows, they are also believed to control many aspects of turbulence dynamics. In the same way that Helmholtz used the vortex sheet as a simplified model of the driven flow in an organ pipe, coherent vortices could form the basis of a reduced theoretical model of turbulence.

It is important to distinguish between statistically representative coherent structures and instantaneous coherent vortices. Proper orthogonal decomposition (POD) decomposes a turbulent velocity field into a set of orthogonal eigenfunctions which are solutions of the eigenvalue problem

$$\int R_{ij}(\boldsymbol{x}, \boldsymbol{x}')\, \phi_j^{(n)}(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}' = \lambda^{(n)} \phi_i^{(n)}(\boldsymbol{x}),$$

where $R_{ij}(\boldsymbol{x}, \boldsymbol{x}') = \langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x}') \rangle$ is the two-point correlation tensor. If a flow is periodic in time, the first eigenmode few eigenmodes represent typical coherent structures. However, it is important to remember that since the POD modes correspond to eigenfunctions of energy (not enstrophy), they do not correspond precisely to vortices. In addition, if the turbulence is not periodic in time, the POD modes do not represent actual instantaneous flow structures. In fact, if the turbulence is only statistically stationary in time and homogeneous in space the POD modes are simply Fourier modes (i.e. sines and cosines), which are certainly not coherent vortices! Nevertheless, the first few POD modes provide significant insight into the flow structure, and can form the basis of a reduced dynamical model of certain flows (e.g. jets).

Coherent vortices, in contrast, are defined based on the analysis of an individual flow realization at an instant in time. Coherent vortices were introduced as a way of visualizing flows such as mixing layers (2). The criterion for identifying a coherent vortex may be based simply on vorticity thresholding, or on more robust measures, such as the eigenvalues of the symmetric tensor $\boldsymbol{S}^2 + \boldsymbol{\Omega}^2$ (7). We will focus on an alternative definition, based on de-noising the turbulent flow (i.e. coherent vortices are what is left after the turbulence has been de-noised). As in the case of POD modes, coherent vortices can also form the basis of a reduced dynamical model of the flow. It is therefore natural to try to use vorticity or coherent vortices as the basis for numerical simulations of fluid flow. This is the topic of the next section.

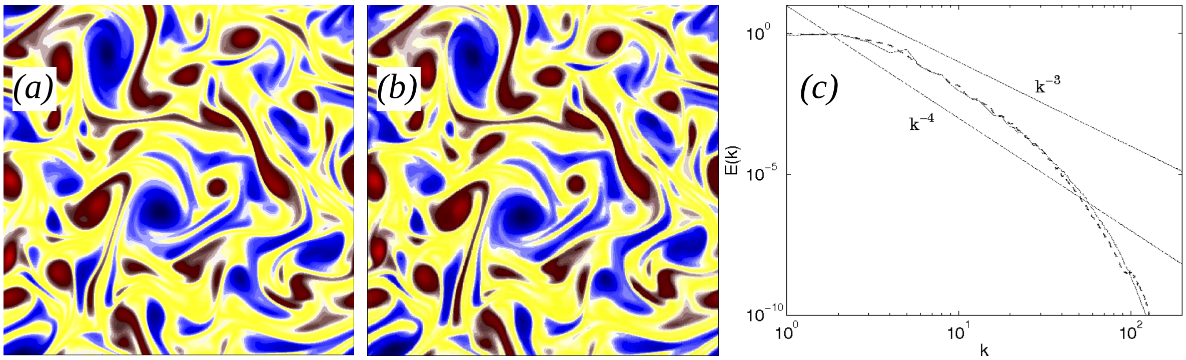


**Fig. 2** Vorticity field of two-dimensional turbulence at $Re = 40\,400$. (From Kevlahan et al. (8).) (*a*) Computed from 263 169 Fourier modes using the pseudo-spectral method, (*b*) Computed using 7 895 coherent wavelet modes, (*c*) energy spectra: - - -, wavelet, — pseudo-spectral.

## 2 Vortices for computation

Vortex methods for the numerical simulation of fluid dynamics date back over 80 years to Prager (11) and Rosenhead (12). However, they were not used commonly (or justified mathematically) until the advent of powerful electronic computers and the mathematical justification provided by the work of Chorin (3), Beale & Majda (1) and others. Although they were originally developed for two-dimensional flows, a highly accurate three-dimensional vortex method for wake flows was introduced by Winckelmans & Leonard (13). In vortex methods the circulation of the initial vorticity field is first discretized onto a set of $N$ particles (or point vortices). The particles move by Lagrangian advection due to the velocity field of all other particles. Vorticity diffusion is modelled either by adding Brownian noise, or by re-distributing circulation amongst nearby particles at the end of each time step. Vorticity production at solid boundaries is included empirically, by adding the vorticity flux necessary to cancel velocity at the boundary. This new vorticity is then advected away from the boundary. Following Helmholtz, vortex methods are used primarily for fluid–structure interaction problems.

Although the vortex particles are indeed the “engines” of the numerical simulation, they do not correspond to the coherent vortices seen in direct numerical simulations and laboratory experiments. In other words, this method does not cut away the noisy “fat” from the coherent “sinews” of turbulence.

In order to reduce the computational complexity of numerical simulations of high Reynolds number turbulence Farge et al. (5) proposed coherent vortex simulation (CVS) where the computational elements represent precisely the *coherent* vortices of the flow. To avoid the long-standing problem of how to define the coherent vortices in a turbulent flow we used the following computationally inspired ansatz: the coherent vortices are defined to be what remains once the (Gaussian) noise has been removed. Since noise is by definition incoherent, the remainder should correspond to what we intuitively think of as coherent vortices. These coherent vortices form the computational elements of the simulation: the wavelet modes are adapted at each time step to identify and track the coherent vortices. The effect of the noise is either modelled simply, or neglected entirely.

De-noising is achieved by nonlinear wavelet filtering, since this procedure optimally removes additive Gaussian noise (see figure 3). Figure 2 shows an example of this approach applied to two-dimensional turbulence. It demonstrates that *all scales* of the coherent vortices are captured by only 3% of the total possible wavelet modes. The coherent vortices represented by the significant wavelet modes are both the “sinews” of the turbulence and the “engines” of the numerical simulation (since they are the computational elements, equivalent to the point vortices of vortex methods).

## 3 Vortices and intermittency

CVS is an efficient and accurate numerical scheme for solving the Navier–Stokes equations. More importantly, it is also a coherent vortex model of turbulence. A CVS in space (i.e. coherent vortex filtering in space at each time step) estimates the number of spatial degrees of freedom, while a CVS in space–time (i.e. coherent vortex filtering in both space and time simultaneously) estimates the total

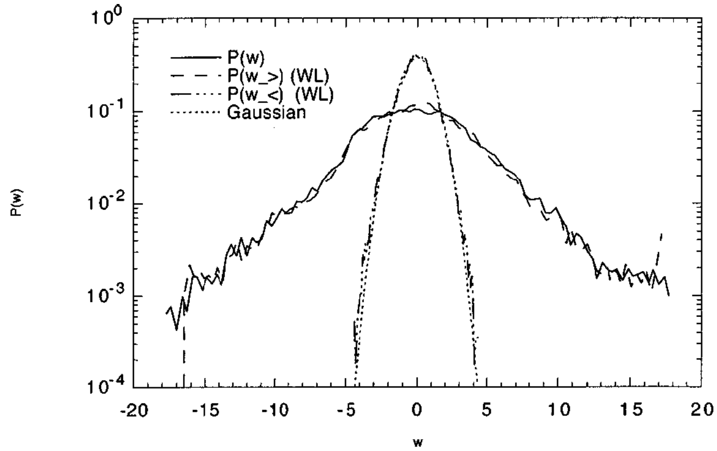


**Fig. 3** PDF of vorticity for nonlinear wavelet filtering of two-dimensional turbulence. The PDF of coherent vorticity $\omega_<$ closely matches the PDF of the total vorticity, while the PDF of incoherent vorticity $\omega_<$ is Gaussian, i.e. it is noise.

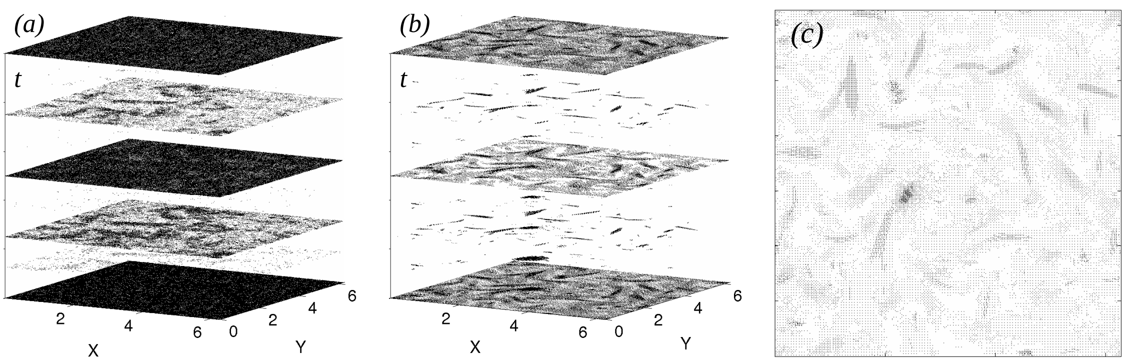


**Fig. 4** Adaptive wavelet grids at Re = 40 400 (half the time domain, i.e. 9 time levels, is shown) (From Kevlahan et al. (8).) (*a*) Space–time grid: first time interval $t \in [0, 2.1]$. (*b*) Space–time grid: final time interval $t \in [123.8, 126.0]$. (*c*) Spatial grid only at $t = 126.0$. Note the strong time intermittency of the solution: the smallest time step is strongly localized in space.

number of dynamical degrees of freedom. Note that space–time CVS ensures the vortices are truly coherent: they necessarily exist for a significant period of time.

Therefore, using CVS, the number of active wavelet modes is an upper bound on the number of degrees of freedom $\mathcal{N}$ required to represent a turbulent flow at a given Reynolds number. A sequence of CVS at different Reynolds numbers allows us to estimate the exponent $\alpha$ in the relation $\mathcal{N} \sim \mathrm{Re}^{\alpha}$. A simple non-intermittent estimate gives $\alpha = 3$ (or even $\alpha = 4$ (14)) for space–time modes and $\alpha = 9/4$ for spatial modes in three-dimensional turbulence (or 3/2 and 1, respectively, for two-dimensional turbulence). Comparing the computed $\alpha$ to their equivalent non-intermittent values directly measures turbulence intermittency. If the active regions of the flow are assumed to have a fractal structure, $\alpha$ can be used to calculate the fractal dimension (10). Note that a commonly used indicator of intermittency, the anomalous scaling of structure function exponents, measures intermittency only indirectly, and would suggest, contrary to observation, that two-dimensional turbulence is *not* intermittent. Our definition directly measures the proportion of the turbulent flow that is active in both space and time.

Kevlahan et al. (8) were able to estimate, for the first time, that the number of space–time modes of a two-dimensional turbulent flow scales like $Re^{0.9}$ (compared with the usual estimate of $Re^{1.5}$), while the number of spatial coherent modes scales like $Re^{0.7}$ (compared with the usual estimate of $Re^{1}$). These scaling exponents are a direct measure of the intermittency of the flow, and hence of the space-fillingness of the coherent vortices.

The next step in this programme is to measure the space and space–time intermittency of three-dimensional flows. It is commonly believed that three-dimensional turbulence is more intermittent than two-dimensional turbulence (perhaps due to the vortex stretching term), although the range of active scales is larger. Three-dimensional turbulence is a more challenging problem for adaptive methods like CVS, due to the complicated geometry and topology of coherent vortices in three dimensions (see figure 1). A more serious challenge is the interpretation of dynamics in the four-dimensional space–time domain. Can we find an (approximate) reduced three- or two-dimensional model of the full four-dimensional dynamics? The degree of intermittency (as measured by the exponent $\alpha$) should answer this question.

## 4 Conclusions

In this article we have briefly reviewed how vortices have been used for computation of high Reynolds number flows. In fact, Helmholtz originally introduced vorticity (and vortex lines and filaments) to improve the ability of Euler's equations to compute real flows. Since then vorticity and vortex-based methods have led to better computational schemes and deeper understanding of turbulent flows.

We propose that coherent vortex simulation (CVS) should be used not just to compute turbulent flows, but also to analyse their structure. The way the size of the CVS scales with Reynolds number provides a direct estimate of the intermittency of the flow (and hence the space-fillingness of its active regions). A space–time CVS calculation gives a numerical estimate of the size of the dynamical system governing the turbulent, and hence gives upper bounds on the dimension of a reduced dynamical model of turbulence. It is likely that the usual non-intermittent estimates of computational degrees of freedom ($Re^3$ or even $Re^4$ in three dimensions) will prove unduly pessimistic if adaptive coherent vortex methods are used.

## References

1. Beale, J., Majda, A.: Vortex method I: Convergence in three dimensions. Math. Comput. **39**, 1–27 (1982)
2. Brown, G., Roshko, A.: On density effects and large structures in turbulent mixing layers. J. Fluid Mech. **64**, 775–816 (1974)
3. Chorin, A.: Numerical study of slightly viscous flow. J. Fluid Mech. **57**, 785–796 (1973)
4. Darrigol, O.: Worlds of flow, chap. 4. Oxford (2005)
5. Farge, M., Schneider, K., Kevlahan, N.K.R.: Non-gaussianity and coherent vortex simulation for two-dimensional turbulence using an adaptive orthogonal wavelet basis. Phys. Fluids **11**, 2187–2201 (1999)
6. von Helmholtz, H.: über integrale der hydrodynamischen gleichungen, welche den wirbelbewegungen entspreschen. JRAM **55**, 25–55 (1858)
7. Jeong, J., Hussain, F.: On the identification of a vortex. J. Fluid Mech. **285**, 69–94 (1995)
8. Kevlahan, N.K.R., Alam, J., Vasilyev, O.: Scaling of space–time modes with reynolds number in two-dimensional turbulence. J. Fluid Mech. **570**, 217–226 (2007)
9. Moffatt, H., Kida, S., Ohkitani, K.: Stretched vortices - the sinews of turbulence; large-reynolds-number asymptotics. J. Fluid Mech. **259**, 241–264 (1994)
10. Paladin, G., Vulpiani, A.: Degrees of freedom of turbulence. Phys. Rev. A **35**(4), 1971–1973 (1987)
11. Prager, W.: Die druckverteilung an körpern in ebener potentialströmung. Phys. Z. **29**, 865–869 (1928)
12. Rosenhead, L.: The formation of vortices from a surface of discontinuity. Proc. R. Soc. London Ser. A **134**, 170–192 (1931)
13. Winckelmans, G., Leonard, A.: Contributions to vortex particle methods for the computation of three-dimensional incompressible unsteady flows. J. Comput. Phys. **109**, 247–273 (1993)
14. Yakhot, V., Sreenivasan, K.: Anomalous scaling of structure functions and dynamic constraints on turbulence simulations. J. Stat. Phys. **121**(5/6), 823–841 (2005)
15. Yokokawa, M., Itakura, K., Uno, A., Ishihara, T., Kaneda, Y.: 16.4-Tflops direct numerical simulation of turbulence by a fourier spectral method on the earth simulator. In: Proceedings of the Proceedings of the IEEE/ACM SC2002 Conference, p. 50. IEEE Computer Society (2002)