Three Essays on Credit Risk Modeling

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By

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To my parents, my sister and Xiao

Abstract

Credit risk is the risk of losses due to the failure to fulfil the obliged payment from a debtor or a counterparty. It is one of the three major components of risks that a bank faces as defined in the new Basel Accord. The credit risk literature has experienced similar rapid growth as the credit market itself. There are currently four different approaches to analyzing credit risk: structural, reduced-form, incomplete information and hybrid models. Even though there are large volumes of published research papers and books on credit risk, our understanding and management skills in this area are still very limited as evidenced by the recent crash of the subprime market. This thesis combines three working papers on credit risk modeling and aims at adding some insights and contributions to the current credit risk literature.

In the first paper, we propose to randomize the initial condition of a generalized structural model, where the solvency ratio instead of the asset value is modeled explicitly. This initial randomization assumption is motivated by the fact that market players cannot observe the solvency ratio accurately. We find that positive short spreads can be produced due to imperfect observation on the risk factor. The two models we have considered, the Randomized Merton (RM)-II and the Randomized Black-Cox (RBC)-II, both have explicit expressions for Probability of Default (PD), Loss Given Default (LGD) and Credit Spreads (CS). In the RM-II model, both PD and LGD are found to be of order of $\sqrt{T}$, as the maturity T approaches zero. It therefore

provides an example that has no well-defined default intensity but still admits positive short spreads. In the RBC-II model, the positive short spread is generated through the positive default intensity of the model. Because explicit formulas are available, these two Randomized Structure (RS) models are easily implemented and calibrated to the market data. This is illustrated by a calibration exercise on Ford Motor Corp. Credit Default Swap (CDS) spread data.

In the second paper, we introduce the inverse-CIR (iCIR) intensity model of credit risk. A multi-firm intensity-based model is constructed where negative correlations are built through the negative correlation between the Cox-Ingersoll-Ross (CIR) process and its inverse. This parsimonious setting allows us to form rich correlation structures among short spreads of different firms, while keeping nonnegative conditions for interest rates and short spreads. The bond prices are given by explicit expressions involving confluent hypogeometric functions. This model can be regarded as an extension of the Ahn & Gao (1999) one factor iCIR model on interest rates to a multi-factor framework on credit risk.

In the third paper, we derive several forms of the equity volatility as a function of the equity value, from the structural credit risk literature. We then propose a new jump to default model by taking the equity volatility to be of the form implied by the models of Leland (1994) and Leland & Toft (1996). This model involves a process we call the Dual-Jacobi process and which has explicit formulae for its moments. Gram-Charlier expansions are then applied to approximate bond and call prices. Our model generalizes Linetsky (2006) by incorporating a local volatility which is bounded below by a positive constant. This local volatility will decrease to a positive constant for increasing stock prices, making the stock process asymptotic to Geometric Brownian Motion (GBM). In this sence, our model is more realistic than Constant Elasticity of Variance (CEV) models.

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List of Acronyms

BIS Bank for International Settlements

BM Brownian Motion

BMO Bank of Montreal

bps basis points

cdf Cumulative Distribution Function

CDOs Collateralized Debt Obligations

CDS Credit Default Swap

CDX CDS Index

CEV Constant Elasticity of Variance

CIR Cox-Ingersoll-Ross

CS Credit Spreads

DD Distance to Default

EAD Exposure At Default

EKF Extended Kalman Filter

GBM Geometric Brownian Motion

GMM General Method of Moments

HJM Heath-Jarrow-Morton

iCIR inverse-CIR

IRB Internal Ratings-Based

KF Kalman Filter

MAE Mean Absolute Error

MRE Mean Relative Error

MLE Maximum Likelihood Estimation

MM Method of Moments

LGD Loss Given Default

LHS Left Hand Side

OU Ornstein-Uhlenbeck

pdf Probability Density Function

PDE Partial Differential Equation

PD Probability of Default

QMLE Quasi-Maximum Likelihood Estimation

RBC Randomized Black-Cox

RHS Right Hand Side

RM Randomized Merton

RMSE Root of Mean Square Error

RMV Recovery of Market Value

RR Recovery Rate

RS Randomized Structure

MRSR Mean-reverting Solvency Ratio

SBM Standard Brownian Motion

SDE Stochastic Differential Equation

TSCS Term Structure of Credit Spreads

Chapter 1

Introduction

1.1 Background: The 2008 Credit Crisis

The subprime mortgage market in the United States began to enter a "meltdown" in late 2006 and became a global financial crisis in July 2007, see Wikipedia (2007). Many banks, hedge funds and mortgage lenders have suffered, and continue to suffer, tremendous losses as a result of defaults from subprime mortgage borrowers. As of November 12, 2007, more than $40 billion had been lost from Wall Street top investment banks, including Citigroup Inc., Merrill Lynch & Co. and Morgan Stanley. Consequently, the CEOs of both Citigroup Inc. and Merrill Lynch & Co. were forced to resign. Numerous other companies have either filed bankcruptcy or have also suffered significant losses. Lehman Brothers, JPMorgan and some other investment banks have announced job cutting in residential mortgages and structured finance.

The U.S. Federal Reserve made a dramatic intervention in financial markets by slashing its interest rate by 50 basis points (bps) on September 18 and by 25 bps again on October 31, 2007. These decisions were designed to stimulate the market and to prevent the subprime turmoil from denting the economy, see BBCNews (2007).

The September 19 announcement made the Dow Jones industrial average increase by 2.45% on one single day. The interest rate cut in the U.S. has also strengthened the Canadian dollar, which hit parity against the U.S. dollar on September 20, 2007 for the first time in thirty-one years. However, these and further interventions have not ended the credit crisis. Recently, there have been warnings from several U.S leading banks that the subprime lending crisis will contribute to another round of losses in the fourth quarter of 2007. The already realized $40 billion losses from financial institutions is just the tip of the iceberg of much larger losses that may end up in hundreds of billions of dollars, according to Djsblack (2007). This credit crisis has become much worse than the collapse of Long Term Capital Management in 1998.

The reasons for the meltdown of subprime mortgage market are complicated and varied. However, what is clear is that global financial institutions, including leading investment banks, have exhibited poor risk management in the credit risk area. The subprime lenders made too many loans to borrowers with poor credit ratings. As of March 2007, the outstanding U.S. subprime mortgages were estimated at $1.3 trillion, and of these about 30 to 40 percent will default, according to Taub (2007). According to JPMorgan, as of August 2007, the value of global Collateralized Debt Obligations (CDOs) was estimated at $1.5 trillion of which about $500 billion to $600 billion in structured finance CDOs backed by subprime mortgages, see Anderson & Timmons (2007). Fund and portfolio managers blamed financial engineers that their models systematically underestimated the credit risk and did not prevent CDOs problems. Financial engineers defended themselves by saying that fund and portfolio managers rarely use their models correctly. The rating agencies, such as S&P, Fitch Ratings and Moody's have been criticized by investors, saying their ratings on structured finance CDOs did not reflect the real default rate.

The deficiency of credit risk management in the finance industry has long been recognized in the last century by the Basel Committee on Banking Supervision, which began the process of drafting a new Basel Accord and published the first consultation paper in June 1999. This new framework was modified in the course of ongoing discussions, based on feedbacks from banks and supervisory authorities. In June 2004, The Bank for International Settlements (BIS) released the final version of the Basel II Capital Accord, see BIS (2004). This new Basel II Accord refines risk classification and requires banks to calculate their regulatory capital based on three major components of risks: credit risk, operational risk and market risk. Banks may choose to calculate their credit risk either through the standardized approach, the foundation Internal Ratings-Based (IRB) approach or the advanced IRB approach. PD, LGD and Exposure At Default (EAD) need to be determined to calculate credit risk under the Basel II framework. As of November 2007, banks from Canada and many other countries have approved and started implementing Basel II in their risk management system. It is ironic that, on November 2, 2007, the Federal Reserve Board of United States finally approved Basel II accord too late to avert the subprime crisis.

The calculation of credit risk requires sophisticated mathematical modeling and statistical techniques. The aim of this thesis is to provide further insights into modeling methodologies on credit risk and its linkage with incomplete information, interest rates and equity derivatives. It is hoped that our results will prove useful to researchers and practitioners in credit markets. Before we move on to the detailed contributions of this thesis, let us first look at the existing modeling approaches in the credit literature.

1.2 Literature Review

Quantitative modeling of credit risk is becoming the essential tool to assess and control credit exposure for banks and other financial institutions. One can identify four different approaches to analyzing credit risk: structural, reduced form (or intensity-based), incomplete information and hybrid. The classical structural approach starts by modeling the firm asset dynamics and defines default to be the time when the asset is insufficient according to some measure. The intensity-based approach assumes the firm will default with an exogenously given hazard rate, without specifying the firm asset dynamics. The incomplete information approach generalizes the classical structural approach by incorporating the impacts of inaccurate observations. The hybrid modeling approach differs from the other approaches in that it models the pre-default equity price and combines market risk and credit risk in a unified framework. The following four subsections provide a brief review of these four different approaches respectively. For a more comprehensive review of credit risk literature, please refer to Giesecke (2004*b*), Duffie & Singleton (2003), Schönbucher (2003) and Bielecki & Rutkowski (2004).

1.2.1 Structural Modeling

The seminal work of Black & Scholes (1973) not only provided a risk-neutral pricing mechanism for European call options, but also paved the way for later development of credit risk models. The second part of Black & Scholes (1973) pointed out that corporate liabilities can be viewed as options on the asset value of the firm. Following the idea of Black & Scholes (1973), Merton (1974) put the corporate liability arguments into a more rigorous mathematical framework and studied the model implied credit spreads. He modeled the asset value to be a Geometric Brownian Motion (GBM) under the risk-neutral measure. The default time was then defined to be the maturity of the bond if the firm asset is insufficient to pay back the debt. The Modigliani-Miller theorem was found to hold in Merton's model. Merton (1974) also argued that the equity of a leveraged firm should be at least as risky as the firm as a whole. As documented in Altman, Resti & Sironi (2004), the expected recovery rate is an output of Merton's model and it increases as PD decreases.

Empirical tests show that options buyer's prices are higher than those predicted by the Black-Scholes formula and it is well documented in the literature that the Term Structure of Credit Spreads (TSCS) generated by Merton's model is too low, especially for short maturities, see Black & Cox (1976) and Giesecke & Goldberg (2004). The short spread in Merton's model is zero, which is counterfactual to empirical data, see Giesecke & Goldberg (2004).

Many researchers have since generalized Black-Scholes-Merton to take account of more empirical facts. Black & Cox (1976) used the same setting as in Merton's model, but redefined the default time to be the first time that the firm asset value passes some pre-determined barrier, to incorporate the effect of safety covenants in bond indentures. Longstaff & Schwartz (1995) extended Black & Cox (1976) to incorporate stochastic interest rates and found a strong negative correlation between credit spreads and the level of interest rates. Their result is consistent with the empirical findings in Duffee (1998). Leland (1994) and Leland & Toft (1996) introduced the idea that the default trigger is set by maximizing the equity value of a firm whose debt structure is assumed to be stationary. Leland & Toft (1996) provided a broad picture of the firm's capital structure in the context of credit risk, where tax benefits of debt, bankcrutcy cost and agency cost are all considered. It was noticed by Black & Cox (1976) that it is the ratio of asset to debt, rather than the actual values of asset or debt, that plays the major role in their analysis. This fact motivated Collin-Dufresne & Goldstein (2001) to model the log of asset over debt (solvency ratio) as a mean-reverting process, reflecting the ability of firms to adjust their capital structure. Fouque, Sircar & Solna (2006) added stochastic volatility to asset dynamics and studied its effects on credit spreads. They found that fast mean-reverting volatility significantly raises the credit spreads at short maturities.

However, none of the models mentioned above can produce positive short spreads. Zhou (2001*b*) was the first to inroduce jump diffusions to credit risk. Hilberink &

Rogers (2002) then extended the Leland-Toft model by generalizing the firm value process to be an exponential Levy with no upward jumps. Chen & Kou (2006) introduced double exponential jumps to Leland-Toft and studied their implications on credit spreads and implied volatility. These jump diffusion structural credit models successfully guarantee positive short spreads. Another advantage of these models is that the expected LGD is not a constant, but implied from the model itself. Introducing jumps, however, adds enormous mathematical and computational complexity.

Structural models are widely applied by practitioners in many ways. Based on Merton's model, Moody's KMV designed the well known Portfolio Manager software package for banks to use to manage credit risk exposures, see Crosbie & Kocagil (2003). Zhou (2001*a*) extended the first passage time model to a multi-firm setting and studied the default correlation. Li (2000) introduced copulas to the structural credit literature and applied a multi-firm model to price first-to-default swaps. Hull & White (2004) applied copula models to the pricing of CDOs and nth-to-default CDS. Baxter (2006) and Moosbrucker (2006) built multi-name correlated models by applying the time change technique used in variance-gamma processes, see Madan & Seneta (1990) and Madan, Carr & Chang (1998). These time changed models are found to fit the market CDS Index (CDX) tranche prices very well. Despite their popularity in industry, structural models are found to have very limited explanatory power for the changes of credit spreads, as suggested by Collin-Dufresne, Goldstein & Martin (2001). An empirical study by Elton, Gruber, Agrawal & Mann (2001) reported that expected losses from default can only account for 20% of the credit spread.

1.2.2 Intensity-based Modeling

Intensity-based modeling dates back to Artzner & Delbaen (1995), Jarrow & Turnbull (1995) and Madan & Unal (1998). Under the intensity-based approach, the firm asset value is not modeled, but rather the firm is assumed to default with an exogenously given hazard rate. The modeling of hazard rates then becomes the focus of the intensity-based approach on credit risk. Intensity-based models are also known as reduced-form models.

Jarrow, Lando & Turnbull (1997) extended Jarrow & Turnbull (1995) by modeling the default process as a discrete Markov chain to incorporate credit rating migrations. Lando (1998) introduced the Cox process to intensity-based modeling and generalized Jarrow et al. (1997) by allowing correlations between stochastic interest rates and the bankruptcy process. This credit migration idea was implemented by Gupton, Finger & Bhatia (1997) from CreditMetrics and further investigated by Chen & Filipovic (2005), Albanese & Chen (2005) and Hurd & Kuznetsov (2007).

Duffie & Singleton (1999) enriched the intensity-based approach by studying different recovery rate assumptions. They showed that under the recovery of market value assumption a defaultable bond can be calculated as if it is default-free using the interest rate adjusted by the hazard rate. Belanger, Shreve & Wong (2004) generalized Duffie & Singleton (1999) to include the case when default can only happen at specific times.

Reduced form models share many similarities with interest rate models. The term structure theory in interest rate modeling, see Dai & Singleton (2000), is thus widely applied to credit risk. Schönbucher (1998) represented defaultable bonds in terms of defaultable forward rates as in Heath, Jarrow & Morton (1992). Duffie, Filipovic & Schachermayer (2003) and Duffie (2005) laid out the theoretical foundation for affine processes and their applications in credit risk. Dai & Singleton (2003) gave a comprehensive survey of credit term structural models. Madan & Unal (2000) and others then extended the intensity-based approach to a multi-factor model setting. Affine models facilitate computations and hence are easily calibrated. Duffee (1999) estimated a three factor CIR model for 161 firms using an extended Kalman filter (EKF), as did Chen & Scott (2003) and Duan & Simonato (1999). The EKF is recommended by Duffee & Stanton (2004), where maximum likelihood estimation (MLE), method of moments (MM) and EKF are compared. Bakshi, Madan & Zhang (2006) empirically tested a three-factor hazard rate model and concluded that interest rate risk is of first-order importance in explaining corporate bond yields variations.

Mathematical tractability of the intensity-based approach has stimulated empirical studies in the credit market. CDS and CDOs can be priced in an efficient way under the intensity-based approach, see Hull & White (2000), Hull & White (2001), Duffie & Garleanu (2001) and Laurent & Gregory (2005). Hull, Predescu & White (2004) studied the relationship between CDS spreads and bond yields and concluded that the interest swap rate is a better benchmark of the default-free interest rate than the treasury yield. This finding was confirmed by Houweling & Vorst (2005). Longstaff, Mithal & Neis (2005) claimed that there is a non-default component in corporate bond spreads can be explained by liquidity. Using U.S. data, Blanco, Brennan & Marsh (2005) and Zhu (2006) both found that CDS spreads and corporate bond spreads move together in the long run, but not necessarily in the short run. In addition, they claimed that price discovery takes place first in the CDS market rather than the bond market. Chan-Lau & Kim (2004) looked at the emerging market of some developing countries and obtained mixed results for price discovery. The work of Norden & Weber (2004) documented the importance of taking into account of equity market information when doing credit risk analysis.

Portfolio credit reduced-form models, where the individual firm intensities are modeled first, such as Schönbucher & Schubert (2001), are termed "bottom-up models". Errais, Giesecke & Goldberg (2007) and Giesecke & Goldberg (2007), on the other hand, proposed a top down approach by modeling the aggregate credit loss directly. In their top down approach, the aggregate credit loss is modeled as a self-exciting process, whose intensity depends on the history of the process itself. This model captures the contagion effects observed in credit markets, while it avoids an ad hoc choice of copulas.

The intensity-based approach has received intensive application in industry. The well known CreditRisk+ by Credit Suisse Financial is essentially an implementation of intensity-based models. Crouhy, Galai & Mark (2000) provided a comparative analysis of existing models implemented in industry.

1.2.3 Incomplete Information Approach

The first incomplete information model was introduced by Duffie & Lando (2001). They set up their model in a structural form, but the asset value was assumed to be observed at discrete times with random noise. This noise leads to a positive short spread which makes the original structural model look like a reduced form model. Kusuoka (1999) and Nakagawa (2001) applied the filtering theory to an extention of Duffie & Lando (2001) with continuous imperfect observations of asset values. Jarrow & Protter (2004) provided an information-based perspective on relationships between structural and reduced form models. Cetin, Jarrow, Protter & Yildirim (2004) and Guo, Jarrow & Zeng (2007*a*) also obtained reduced form models by constructing an economy where only a reduction of the manager's information set is revealed in the market. Jeanblanc & Valchev (2005) studied three cases of incomplete information on asset values and found that credit spreads increase with the reductions of the information sets.

Giesecke & Goldberg (2004) on the other hand argued that investors cannot observe the default barrier, even if the asset value process can be observed accurately. They found that their incomplete information model reacts faster to new information than the classical structural models. Giesecke (2004*a*) constructed a multi-firm correlated default model with incomplete information about the default barriers. They observed contagious jumps in credit spreads of other firms when one firm goes to bankruptcy. Giesecke (2006) generalized Duffie & Lando (2001) by considering both incomplete information about firm asset value and incomplete information about the default barrier. The reduced form approach of Duffie & Singleton (1999) was generalized by Giesecke (2006) in the sense that the security pricing formulae can be expressed in terms of the accumulated intensity. This generalization was also independently introduced by Elliott, Jeanblanc & Yor (2000), Jeanblanc & Rutkowski (2000) and Bielecki & Rutkowski (2004). Based on the idea of Collin-Dufresne & Goldstein (2001), Coculescu, Geman & Jeanblanc (2006) added incomplete information to the underlying state variable triggering default. This state variable may typically be the solvency ratio of the company. Guo, Jarrow & Zeng (2007*b*) built a stochastic recovery model based on incomplete information setting in Guo et al. (2007*a*).

Largely due to its complexity, the incomplete information approach has not yet been widely used in the industry.

1.2.4 Hybrid Models

Empirical studies from both reduced form and structural models have already pointed out that the equity market information should not be ignored for credit risk analysis. Jarrow (2001) and Janosi, Jarrow & Yildirim (2003) provided a methodology for estimating default parameters using both debt and equity prices. Hull, Nelken & White (2004) proposed a calibration scheme of Merton's structural model using implied volatilities from equity options. Based on a dataset of 120,000 individual CDS quotes, they reported that the mean CDS observed spread is about 95 bps higher than the mean implied spread from the model.

The importance of jointly managing of both market risk and credit risk has been recognized by industrial practitioners. Iscoe, Kreinin & Rosen (1999) proposed a model which integrates both market risk and credit risk. It was then improved by Prisco, Iscoe, Jiang & Mausser (2007) with Monte-Carlo simulation. This joint modeling idea originated from convertible bonds literature.[1] Davis & Lischka (2002) first introduced a jump-to-default model where the stock price jumps to zero with a local intensity when the firm defaults. If the default intensity is assumed to be a negative-power of the stock price, empirical estimates of the power are in the range -2.0 to -1.2, see Muromachi (1999). Bloch & Miralles (2002) specified the local intensity function as a negative natural log function of the stock price. Arvanitis & Gregory (2001) chose a negative-exponential function of the stock price as the default intensity. Almost all the existing models on convertible bonds rely on numerical algorithms to solve the related Partial Differential Equation (PDE). A binomial lattice algorithm was proposed by Derman (1994). Andersen & Buffum (2004) provided a calibration and implementation scheme with finite difference methods. Linetsky (2006) studied the local intensity model (with constant volatility) proposed by Davis & Lischka (2002) and derived an explicit expression for the transition density of the underlying pre-default stock price. Consequently, explicit pricing formulae for both corporate bonds and equity derivatives were obtained. Carr & Linetsky (2006) extended Linetsky (2006) to incorporate constant elastisity volatility effect and also found an explicit expression for the transition density of the underlying pre-default stock price. These jump-to-default models are able to generate reasonable term structure of credit spreads and implied volatility surfaces simultaneously.

However, Carr & Wu (2006) pointed out that the changes of the stock option price and the CDS spread are perfectly correlated locally since the pre-default stock price is the only source of uncertainty. They then considered a jump-to-default model

[1]Early studies of convertible bonds include Brennan & Schwarz (1980), McConnell & Schwarz (1986), Cheung & Nelken (1994), Derman (1994), Ho & Pfeffer (1996) and Tsiveriotis & Fernandes (1998). Ayache, Forsyth & Vetzal (2003) provided a survey of existing models and their limitations on convertible bonds.

with mean-reverting stochastic volatility and stochastic hazard rate. The stochastic hazard rate was assumed to be a positive constant times the stochastic volatility, plus an independent mean-reverting factor. This specification captures the positive co-movements of CDS spreads and implied volatilities and also accommodates the fact that the credit market and the equity market are not perfectly correlated. The model was calibrated on four companies using both time series of implied volatilities and CDS spreads. However, calibration of the model by Carr & Wu (2006) does not provide satisfactory results on swap spreads: only about 50 percent of the variation in the CDS spreads on General Motors and only 30 percent on Altria Group can be explained by their model.

The classification we used here for credit risk models is general and conventional, but it by no means exhausts all existing models. Bielecki, Jeanblanc & Rutkowski (2004) applied a utility-based approach to defaultable claims, where dynamic programming techniques are used. More recent papers on utility-based credit models include Sircar & Zariphopoulou (2007*a*), Sircar & Zariphopoulou (2007*b*), Bielecki & Jang (2007) and Lakner & Liang (2007).

1.3 Contributions of This Thesis

As the size of the credit market has grown in recent years, so has the credit risk literature. Even though there are large volumes of published research papers and books on credit risk, the recent failure of the subprime market is an evidence that our understanding and management skills on credit risk are still very limited. This thesis combines three working papers on credit risk modeling and aims to add some insights and contributions to the current state of credit risk management.

Chapter 2, based on the working paper of Yi, Tchernitser & Hurd (2007), titled *"Randomized Structural Models of Credit Spreads"*, belongs to the incomplete information approach. The existing models under the incomplete information approach usually either have too many parameters or are too complicated to implement. In this paper, we propose to randomize the initial value of the solvency ratio process to take account of incomplete information. We recommend two models of very few parameters, both of which generate positive short spreads and varying realistic shapes of TSCS. Explicit formulae for PD, LGD and CS are all obtained, allowing fast calibration of the model. The RM-II model provides an example that has no well-defined default intensity but still admits positive short spreads. The RBC-II model generates positive short spreads through its positive default intensity. This randomization technique can be applied to different processes of the underlying risk factor with different assumptions on its inital value. Most of the work of this paper was done during my internship in Market Risk at BMO in summer 2006. This paper was presented at the Third International Conference on Credit and Operational Risk, April 12-13 2007, HEC Montreal.

Chapter 3, based on the working paper of Hurd & Yi (2007*b*), titled *"Inverse CIR and Semi-affine Intensity-based Modeling on Credit Risk"*, belongs to the intensity-based approach. The existing multi-factor models under the intensity-based approach are usually unable to produce rich correlation structures among the credit spreads of different firms, while preserving the nonnegativity restrictions on interest rates and credit spreads, see Duffee (1999). In this paper we propose a new multi-factor model where the iCIR process is introduced. By introducing the iCIR process as a new factor, we are able to form rich correlation structures among the short spreads of different firms, while the non-negativity conditions for interest rates and short spreads are satisfied. This paper was accepted by the 7th Annual Hawaii International Conference on Statistics, Mathematics and Related Fields, January 17-19 2008.

Chapter 4 is based on the working paper of Hurd & Yi (2007*a*), titled *"In Search of Hybrid Models for Credit Risk: from Leland-Toft to Carr-Linetsky"*. This paper belongs to hybrid models. The CEV setting for local equity volatility in Carr &

Linetsky (2006) is inappropriate for many firms which have large equity values and are also very volatile, such as internet technology companies. In a CEV setting, the equity volatility always vanishes to zero, when the stock price approaches infinity. We propose a new jump-to-default model by taking the equity volatility to be of the form implied from Leland & Toft (1996). This local volatility decreases to a positive constant with increasing stock prices, making the stock an asymptotic GBM. Therefore, our model is more realistic than CEV models. The model specification allows us to use the Gram-Charlier approximation for fast computation of the bond and call prices: our approximation scheme is more than 70 times faster than the classical finite difference method as demonstrated in the numerical examples.

The current author is the primary author of all the three working papers. All these three papers are in preparation for submission for publication.

Chapter 2

Randomized Structural Models of Credit Spreads

In this chapter, we propose to randomize the initial value of the solvency ratio process and study its implications of the term structure of credit spreads. This chapter is organized as follows. In Section 2.1, we give a brief introduction of our motivation. In Section 2.2, we briefly review Merton's model. In Section 2.3, we introduce the idea of modeling the solvency ratio. The Black-Cox model is treated in a simplified version in Section 2.4. In Section 2.5, two versions of the RM model are introduced with different assumptions on the initial distribution. In Section 2.6, two versions of the Randomized Black-Cox (RBC) model are introduced, where the default time is defined similarly as in the Black-Cox model. Section 2.7 provides a delayed information perspective on the RS model. In Section 2.8, we study another two RS models, which assume mean-reverting solvency ratio. In Section 2.9, a calibration exercise is conducted. We summarize this chapter in Section 2.10. All proofs are given in the appendix of this chapter.

2.1 Introduction

In Merton's model, the asset and debt values enter into the formula of credit spreads as a single parameter: asset over debt. Motivated by this fact, we treat the log of asset over debt (or solvency ratio) as one single risk factor and model it as a Stochastic Differential Equation (SDE), without specifying the dynamics of asset value and debt structure. The idea of directly modeling the solvency ratio is also utilized in Coculescu et al. (2006).

As pointed out by Duffie & Lando (2001), market players do not have full information of firm's capital structure in real time. Instead, the observed solvency ratio is contaminated with some random noise. We therefore propose to randomize the initial value of the solvency ratio process to take account of this imperfect information.

This randomization technique could be applied to both Merton and Black-Cox type models, where the major difference of the two models are the definition of default. Different assumptions on the distribution of the initial value will also have different effects on the short spreads of the model. In this chapter, we consider four different models: the RM-I, the RM-II, the RBC-I and the RBC-II, which cover both definition of default and different assumptions on the initial distribution. We focus on how the randomization affects the term structure of credit spreads, particularly on the short end. Another two randomized structural models with mean-reverting solvency ratio are also discussed.

2.2 Merton's Model

Merton (1974) assumed that the firm's value V_t follows a GBM under the risk-neutral measure, starting from a known constant V_0 at time zero. That is

$$dV_t = rV_t dt + \sigma V_t dW_t,$$

where r denotes constant interest rate, σ is the volatility, and W_t is a Standard Brownian Motion (SBM).

The firm is obliged to pay the debt holders a constant K at maturity T. Default happens at maturity T, when the firm has insufficient funds to pay back to the debt holders at that time, namely when $V_T < K$. Thus, the probability of default $PD(T)$, as a function of maturity T, can easily be calculated through

$$\begin{aligned} PD(T) &:= P(V_T < K) \\ &= \Phi\left(-\frac{\log\frac{V_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \end{aligned} \tag{2.1}$$

where Φ stands for the Cumulative Distribution Function (cdf) of a standard normal distribution.

Following Altman et al. (2004), the expected recovery rate $RR(T)$ (under the risk-neutral measure), as a function of maturity T, given default at maturity T, can be evaluated as[1]

$$\begin{aligned} RR(T) &:= \mathbb{E}[\frac{V_T}{K}|V_T < K] \\ &= \frac{V_0}{K}e^{rT}\frac{\Phi\left(-\frac{\log\frac{V_0}{K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)}{PD(T)}, \end{aligned}$$

where $PD(T)$ is the probability of default given in (2.1).

Expected LGD (under the risk-neutral measure), $LGD(T)$, as a function of maturity T, is defined to be $LGD(T) := 1 - RR(T)$.

Under the assumption of a constant interest rate, the TSCS in Merton's model can

[1]This recovery rate is the recovery of face value of the bond. Other recovery rate assumptions include recovery of treasury and recovery of market value, see Duffie & Singleton (1999) for a discussion.

be expressed as

$$\begin{aligned} CS(T) &:= -\frac{1}{T}\log[1 - PD(T) \times LGD(T)] \qquad (2.2) \\ &= -\frac{1}{T}\log\left(\Phi(\frac{\log\frac{V_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) + \frac{V_0}{K}e^{rT}\Phi(-\frac{\log\frac{V_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})\right), \end{aligned}$$

which is a function of maturity T, interest rate r, asset volatility σ and the initial leverage ratio $\frac{K}{V_0}$. Setting interest rate r to 0.02 and initial leverage ratio $\frac{K}{V_0}$ to 0.8, Figure 2.1 plots Merton's TSCS given by equation (2.2), for varying volatility σ.

The short spread is defined to be the right limit of CS as maturity T goes to zero.

$$CS(+0) := \lim_{T\to+0} CS(T) = \lim_{T\to+0} \frac{PD(T) * LGD(T)}{T}.$$

If $V_0 > K$, using L'Hospital's rule, one can show that Merton's short spread is always zero. If $V_0 < K$, Merton's short spread is positive infinity.

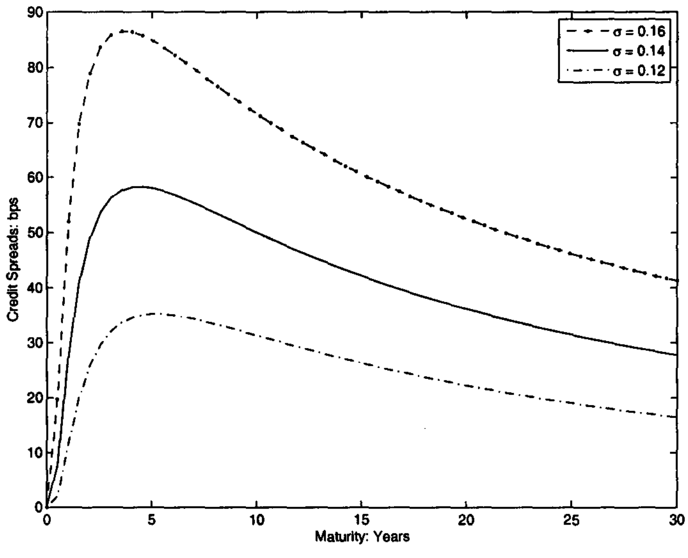


Figure 2.1: Merton's term structure of credit spreads, varying asset volatility σ. We set $\frac{K}{V_0} = 0.8$ and $r = 0.02$.

2.3 Modeling The Solvency Ratio

Merton (1974) modeled the firm's asset value V_t explicitly as a GBM and the debt K_t as a constant K. As a consequence, the solvency ratio in Merton's model, i.e. $\log(V_t/K_t)$, is a drifted Brownian Motion (BM). Instead of specifying the dynamics of the firm's asset value and the debt value separately, we can model the solvency ratio itself directly.

Assume that the solvency ratio X_t follows a drifted BM under the risk-neutral measure

$$X_t = x_0 + \mu t + \sigma W_t, \tag{2.3}$$

where x_0 is a constant. Default happens at time T if $X_T < 0$. Then, the probability of default $PD(T)$, expected recovery rate $RR(T)$ and the credit spread $CS(T)$ are given by

$$\begin{aligned}
PD(T) &:= P(X_T < 0) = \Phi\left(-\frac{x_0 + \mu T}{\sigma\sqrt{T}}\right), \\
RR(T) &:= \mathbb{E}[e^{X_T} | X_T < 0] = \frac{\Phi\left(-\frac{x_0+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right) e^{x_0+\mu T+\frac{1}{2}\sigma^2 T}}{\Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)}, \\
CS(T) &= -\frac{1}{T}\log\left(\Phi(\frac{x_0 + \mu T}{\sigma\sqrt{T}}) + \Phi(-\frac{x_0 + \mu T + \sigma^2 T}{\sigma\sqrt{T}})e^{x_0+\mu T+\frac{1}{2}\sigma^2 T}\right).
\end{aligned}$$

This is exactly Merton's model with the following parameter constraints:

$$\begin{aligned}
x_0 &= \log(V_0/K), \\
\mu &= r - \frac{1}{2}\sigma^2 > -\frac{1}{2}\sigma^2.
\end{aligned}$$

Our setting here is more general than Merton (1974), since we do not specify either asset or debt processes. The debt can be a constant, a random variable, such as Giesecke (2006) or a stochastic process, such as Collin-Dufresne & Goldstein (2001). Practically speaking, corporate restructuring is allowed in this model, but not in Merton's model. By Ito's lemma, it is easy to see that the drift of the solvency ratio μ equals the default free interest rate r minus half of the squared volatility of the solvency ratio in Merton (1974). This implies that the drift of the solvency ratio has to be larger than negative half of the squared volatility of the solvency ratio in Merton's model, considering positive interest rates. However, we do not impose any restriction on the relationship between the drift and volatility of the solvency ratio. We have a broader set of admissible parameters than in Merton's case.

2.4 The Black-Cox Model

In Merton's model, the default event can only happen at the maturity. However, in reality, defaults could happen before the maturity of an indenture. Black & Cox (1976) then proposed the well known first passage time model. Instead of describing what Black and Cox have done exactly in the 1976 paper, we give a simplified version of the model which maintains the essence of the original one.

For a given company, let its solvency ratio X_t be a drifted BM given by Equation (2.3) (under the risk-neutral measure). In addition, we impose positivity assumption on x_0 to ensure that no default has happened up to now. The risk-neutral default time τ is defined as the first time X_t crosses the zero boundary, i.e.

$$\tau = \inf\{t \geq 0; X_t = 0\}. \tag{2.4}$$

For a given future time $T > 0$, the default probability $P(\tau < T|X_0 = x_0)$ can be calculated using the reflection principle of Brownian motion and it is given by

$$P(\tau < T|X_0 = x_0) = \Phi\left(-\frac{x_0 + \mu T}{\sigma\sqrt{T}}\right) + e^{-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0 - \mu T}{\sigma\sqrt{T}}\right). \tag{2.5}$$

Detailed derivation could be found in Steele (2004). Note that the first term is exactly the Merton's default probability (the probability of default at T). The second term comes from possibilities of default before T.

Assuming constant risk-neutral $LGD = l$, then the TSCS is given by

$$CS(T) = -\frac{1}{T}\log[1 - lP(\tau < T|X_0 = x_0)]. \tag{2.6}$$

This TSCS has similar shapes as in Figure 2.1. Define the default intensity λ at time zero as

$$\lambda := \frac{\partial P(\tau < T|X_0 = x_0)}{\partial T}|_{T=0}. \tag{2.7}$$

Using L'Hospital's rule, one can show that $\lambda = 0$ in the Black-Cox model. Consequently, the short spread in this model is always zero.

2.5 Randomized Merton Model

In the classical structural models discussed previously, the solvency ratio X_t has a constant initial value $X_0 = x_0$. This means that we can fully observe the solvency ratio at current time. However, in reality, the current solvency ratio cannot be exactly observed by the market players. It is therefore reasonable to randomize the initial value X_0.

Assume that the solvency ratio X_t follows a drifted BM under the risk-neutral measure

$$X_t = X_0 + \mu t + \sigma W_t, \tag{2.8}$$

with a random initial value X_0. At time zero, we cannot observe the initial value X_0 accurately, but instead, we observe X_0 plus some random noise. We also assume that X_0 and W_t are independent for all $t > 0$. This is a reasonable assumption, since the noise should not affect the evolution of the solvency ratio process. However, it does contaminate the information observed by market players.

In the following two subsections, the solvency ratio is assumed to follow Equation (2.8). The default probability is defined as in Merton's model. The interest rate is assumed to be constant and the credit spread is calculated using Equation (2.2). Two models with different assumptions on the initial randomization are studied respectively. We focus on how the the short spread is influenced by the randomization of the initial value.

2.5.1 Randomized Merton I (RM-I)

In this RM-I model, we assume the following distribution for X_0

- RM-I Assumption on X_0: $X_0 \sim N(x_0, \sigma_0^2)$.

This is a natural assumption, since the drifted BM is normally distributed. It follows that $X_T \sim N(x_0 + \mu T, \sigma_0^2 + \sigma^2 T)$. As in Merton's model, we define the default time to be T, if $X_T < 0$. The default probability $PD(T)$, the expected recovery rate $RR(T)$ and the credit spreads $CS(T)$ are given by

$$\begin{aligned}
PD(T) &:= P(X_T < 0) = \Phi\left(-\frac{x_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right), \\
RR(T) &:= \mathbb{E}[e^{X_T} | X_T < 0] = \frac{e^{x_0 + \mu T + \frac{1}{2}(\sigma_0^2 + \sigma^2 T)} \Phi\left(-\frac{x_0 + \mu T + \sigma_0^2 + \sigma^2 T}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right)}{\Phi\left(-\frac{x_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right)}, \\
CS(T) &= -\frac{1}{T} \log\left(\Phi(\frac{x_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}) + \Phi(-\frac{x_0 + \mu T + \sigma^2 T}{\sqrt{\sigma_0^2 + \sigma^2 T}}) e^{x_0 + \mu T + \frac{1}{2}(\sigma_0^2 + \sigma^2 T)}\right).
\end{aligned}$$

When $\sigma_0 = 0$, this becomes the original Merton's model. When $T = 0$, both $PD(0)$ and $RR(0)$ are positive constants. As a result, the short spread becomes positive infinity. Therefore the RM-I model is inappriate for pricing the short spread.

2.5.2 Randomized Merton II (RM-II)

In the RM-I model, infinite short spread is due to nonzero default probability at time zero, which in turn is due to the positive probability that $X_0 < 0$. In this RM-II model, we assume the following distribution for X_0, which has no mass on $(-\infty, 0)$.

- RM-II Assumption on X_0: its Probability Density Function (pdf) $f(x_0; y_0, \sigma_0)$ is given by

$$f(x_0; y_0, \sigma_0) = \begin{cases} \phi(x_0; y_0, \sigma_0)/\Phi(y_0/\sigma_0) & if \quad x_0 \geq 0 \\ 0 & if \quad x_0 < 0. \end{cases} \tag{2.9}$$

  where function $\phi(x; \mu, \sigma)$ denotes the pdf of $N(\mu, \sigma^2)$ given by

$$\phi(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{2.10}$$

This initial randomization will ensure zero default probability at time zero, namely $P(X_0 < 0) = 0$.

Remark 2.5.1. *As time progresses from zero to $t \in (0, T)$, the solvency ratio X_t can be negative without triggering a default in Merton's model. Therefore, the nonnegative assumption on X_0 is not a consistent assumption for a dynamic model. Nevertheless, this assumption is reasonable for a static model which can be used to price the current short spread.*

The default probability $PD(T)$ and the recovery rate $RR(T)$ can be calculated by conditioning

$$PD(T) := \mathbb{E}[P(X_T < 0|X_0)], \tag{2.11}$$

$$RR(T) := \frac{\mathbb{E}[\mathbb{E}[e^{X_T} 1_{\{X_T<0\}}|X_0]]}{\mathbb{E}[P(X_T < 0|X_0)]}. \tag{2.12}$$

The following Proposition gives explicit formulas for $PD(T)$, $RR(T)$ and $CS(T)$, as well as their asymptotics when $T \to +0$.

Proposition 2.5.1. *In the RM-II model, the default probability $PD(T)$, the recovery rate $RR(T)$ and the credit spreads $CS(T)$ have the following representations*[2]

$$
\begin{aligned}
PD(T) &= \frac{A}{\Phi(y_0/\sigma_0)}, && (2.13)\\
RR(T) &= \frac{Be^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{A}, && (2.14)\\
CS(T) &= -\frac{1}{T}\log\left(\frac{\Phi(y_0/\sigma_0) - A + Be^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{\Phi(y_0/\sigma_0)}\right), && (2.15)
\end{aligned}
$$

where the function $\Phi_2(x_1, x_2, \rho)$ denotes the cdf of a bivariate normal distribution with marginal distributions being standard normal and correlation coefficent ρ and[3]

$$
\begin{aligned}
A &= \Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right),\\
B &= \Phi_2\left(-\frac{y_0+\mu T+\sigma_0^2+\sigma^2 T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}+\sigma_0, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\lim_{T\to+0}\frac{PD(T)}{\sqrt{T}} &= \frac{\sigma f(0; y_0, \sigma_0)}{\sqrt{2\pi}}, && (2.16)\\
\lim_{T\to+0}\frac{LGD(T)}{\sqrt{T}} &= \frac{\sigma\sqrt{2\pi}}{4}, && (2.17)\\
\lim_{T\to+0} CS(T) &= \frac{\sigma^2 f(0; y_0, \sigma_0)}{4}. && (2.18)
\end{aligned}
$$

From the above proposition, we can see that $PD(T)$ indeed vanishes to zero as maturity T approaches zero. When there is no random noise of the initial observation, i.e. $\sigma_0 = 0$, the RM-II model reduces to Merton's model. Both $PD(T)$ and $LGD(T)$ are found to have an order of $\sqrt{T}$, as $T \to +0$. As a result, the default intensity does not exist in the RM-II model, but it can still generate positive short spread. This positive short spread has an explicit formula given by Equation (2.18).

[2]Pykhtin (2003) obtained a similar expression for the recovery rate in his recovery risk model.

[3]A series expansion of these functions are given by Vasicek (1998). We thank Michael Gordy for pointing out this.

The short spread only depends on σ, y_0 and σ_0, and it does not depend on the drift μ. However, if we allow y_0 to be a function of μ, the short spread may depend on μ indirectly, as we can see in section 2.7. Equation (2.18) implies that the short spread increases when σ increases while holding other parameters constant. If we fix σ and σ_0, the short spread is a decreasing function of y_0/σ_0. The ratio y_0/σ_0 can be regarded as Distance to Default (DD). More uncertainty about the observed solvency ratio indicates higher risk and hence the short spread should be higher. This uncertainty should be measured by DD instead of σ_0. This result may also imply that a firm's credit spreads will fall after its annual report. This awaits empirical results from testing the model. The situation for σ_0 is more complicated. When σ_0 increases from zero, the short spread first increases to a maximum and then decreases. The maximum is reached at a σ_0^{max}, which solves the following equation

$$(y_0^2\sigma_0^2 - 1)\Phi(y_0/\sigma_0) + y_0\phi_0 = 0.$$

This equation is obtained by setting the first order derivative of $CS(+0)$ with respect to σ_0 to be zero.

Figures 2.2 and 2.3 show term structure of credit spreads for varying σ_0 and μ respectively, while holding other parameters constant. The short spread of the RM-II model is clearly above zero as seen from both figures. Figure 2.2 also shows that $CS(T)$ may decrease when σ_0 increases. Some people may argue that this is counter-intuitive, since more uncertainty about the current observation should require to pay more for the protection of default. Hence the credit spread should be higher for larger σ_0. This argument is only correct if we replace the risk measure σ_0 by DD (i.e. y_0/σ_0). The credit spread is indeed a monotone decreasing function of DD. Figure 2.3 also demonstrates that the RM-II model is capable of generating upwarding term structure of credit spreads by choosing sufficient negative μ. Merton's model cannot produce upward increasing term structure of credit spreads because of the nonnegativity restriction on the constant interest rate. In the RM-II model, however, we do not specify the dynamics of the asset value V_t. We allow $\mu + \frac{1}{2}\sigma^2$ to be negative in the RM-II model, because $\mu + \frac{1}{2}\sigma^2$ does not necessarily represent the interest rate. As a result, the RM-II model is able to generate varying shapes of term structure of credit spreads.

Figure 2.4 plots the short spread defined in Equation (2.18) as a function of y_0, σ_0 and DD. The middle picture in Figure 2.4 shows a hump-shaped curve of the short spread as a function of σ_0. The maximum short spread is achieved at $\sigma_0^{max} = 0.4167$, in the case when $\sigma = 0.12$ and $y_0 = 0.35$.

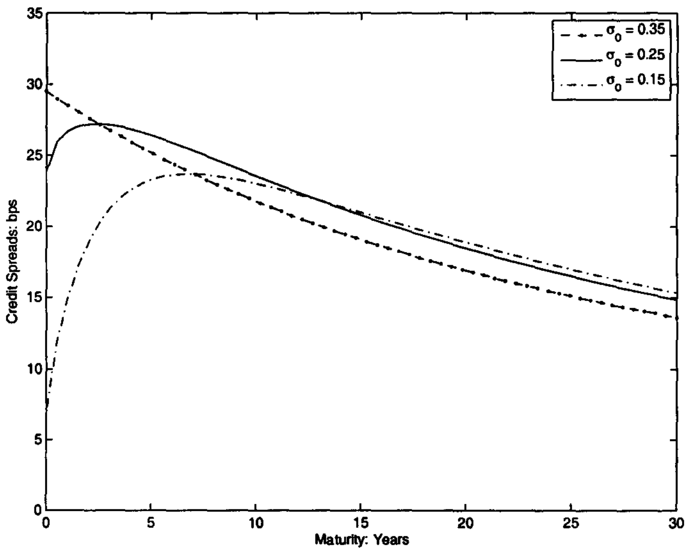


Figure 2.2: Term structure of credit spreads of RM-II model for varying σ_0: $\mu = 0.01$, $\sigma = 0.12$, $y_0 = 0.35$.

We have used the technique of randomizing the initial condition of the solvency ratio and have studied two models with Merton's definition of default. The short spread in RM-I model is infinity while the RM-II model successfully generates positive short spreads. We conclude that the RM-I model is inapproriate for modeling short spreads and the RM-II model is recommended.

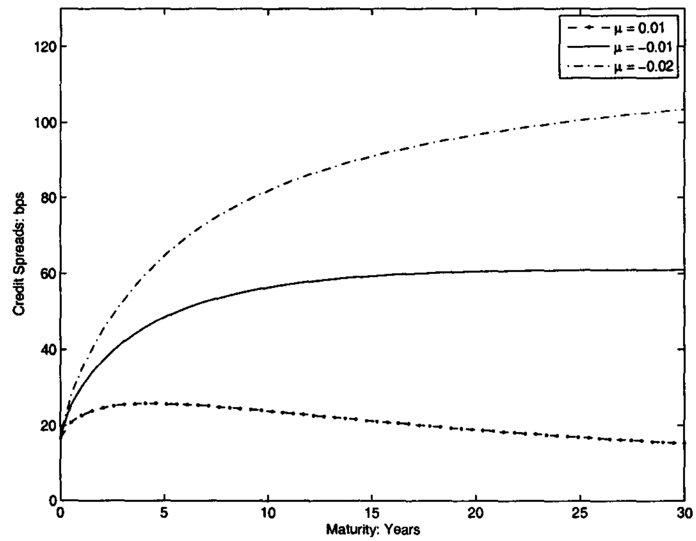


Figure 2.3: Term structure of credit spreads of RM-II model for varying μ: $\sigma = 0.12$, $y_0 = 0.35$, $\sigma_0 = 0.20$.

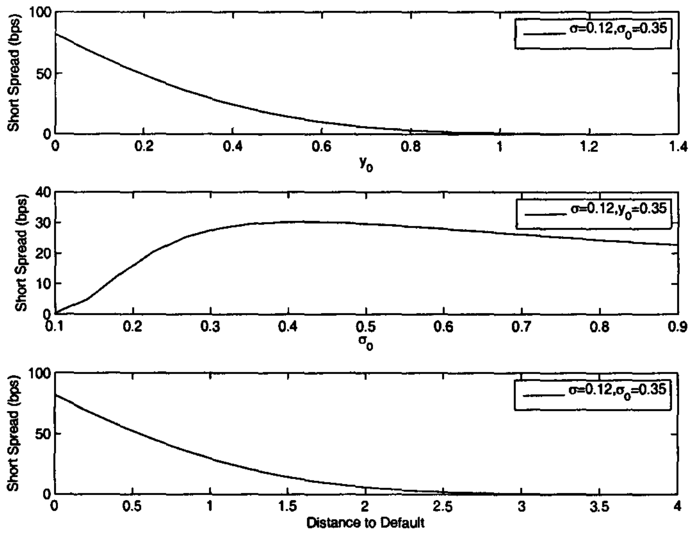


Figure 2.4: RM-II short spread as a function of y_0, σ_0 and DD.

In order to prove Proposition 2.5.1, the following two lemmas are needed.

Lemma 2.5.2. *Let (X,Y) be a bivariate normal with correlation coefficient ρ and marginals $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$. Then the following equation holds*

$$\Phi_2\left(-\frac{\mu_x}{\sigma_x}, -\frac{\mu_y}{\sigma_y}, \rho\right) = \int_0^{+\infty} \Phi\left(\frac{y\rho\sigma_x/\sigma_y + \mu_y\rho\sigma_x/\sigma_y - \mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y; -\mu_y, \sigma_y)dy. \tag{2.19}$$

Lemma 2.5.3. *As $T \to +0$ the following expansion holds*

$$\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, \frac{-\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right) = \frac{\sigma\phi(0; y_0, \sigma_0)\sqrt{T}}{\sqrt{2\pi}} + \frac{y_0\sigma^2\phi(0; y_0, \sigma_0)T}{4\sigma_0^2} + O(T^{3/2}). \tag{2.20}$$

2.6 Randomized Black-Cox (RBC) Model

As mentioned in the Remark 2.5.1, the assumption that $X_0 > 0$ is inconsistant for a dynamic model in Merton's definition of default. However, it is natural to make this assumption in the Black-Cox setting.

In this section, we apply the randomization technique to the Black-Cox model and study two different models which have different assumptions on the initial distribution. In the following two subsections, the solvency ratio is still assumed to follow Equation (2.8). The default time is assumed to be the first passage time defined by Equation (2.4). The interest rate and the expected recovery rate are assumed to be constant and the credit spreads are calculated using Equation (2.6).

2.6.1 Randomized Black-Cox I (RBC-I)

In this RBC-I model, we make the following assumption about X_0

- RBC-I Assumption on X_0: assume the pdf of X_0 is given by Equation (2.9).

The probability of default $P(\tau < T)$ can be calculated through

$$P(\tau < T) = \mathbb{E}[P(\tau < T|X_0)], \tag{2.21}$$

where $P(\tau < T|X_0 = x_0)$ is given by Equation (2.5). The following Proposition gives an explicit expression of the default probability.

Proposition 2.6.1. *For the RBC-I model, the default probability has the following expression*

$$\begin{aligned} P(\tau < T) &= \frac{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}{\Phi(y_0/\sigma_0)} \\ &+ \frac{\Phi_2\left(-\frac{y_0-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right) e^{-2\mu y_0/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}}{\Phi(y_0/\sigma_0)}. \end{aligned} \tag{2.22}$$

Moreover, we have[4]

$$\lim_{T\to+0} \frac{P(\tau < T)}{\sqrt{T}} = \frac{2\sigma f(0; y_0, \sigma_0)}{\sqrt{2\pi}}. \tag{2.23}$$

This Proposition implies that $P(\tau < T)$ has an order of $\sqrt{T}$, as $T \to +0$. For constant LGD assumption, the short spread in the RBC-I model becomes infinity. Similar to the RM-I, the RBC-I model is inappropriate for modeling the short spread. Different choices of the distribution of the initial state X_0 will yield different order of convergence for $P(\tau < T)$, as $T \to +0$. The next session provides a better alternative.

2.6.2 Randomized Black-Cox II (RBC-II)

In this RBC-II model, we propose the following distribution for X_0:

[4]The Right Hand Side (RHS) of Equation (2.23) is exactly twice of the RHS of Equation (2.16). This implies that the ratio of the first passage default probability and the Merton's default probability is 2 when $T \to +0$, see Yi (2006) for a discussion.

- RBC-II Assumption on X_0: its pdf $f(x_0; a, v_0, \sigma_0)$ is assumed to be

$$f(x_0; a, v_0, \sigma_0) = \begin{cases} \frac{\phi(x_0; a+v_0, \sigma_0) - e^{-2av_0/\sigma_0^2}\phi(x_0; v_0-a, \sigma_0)}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)} & if \quad x_0 \geq 0 \\ 0 & if \quad x_0 < 0 \end{cases} \tag{2.24}$$

where $\sigma_0 > 0$ and $a > |v_0|$.

Direct integration shows that $f(x_0; a, v_0, \sigma_0)$ is indeed a valid pdf. Figure 2.5 shows an example of the pdf.

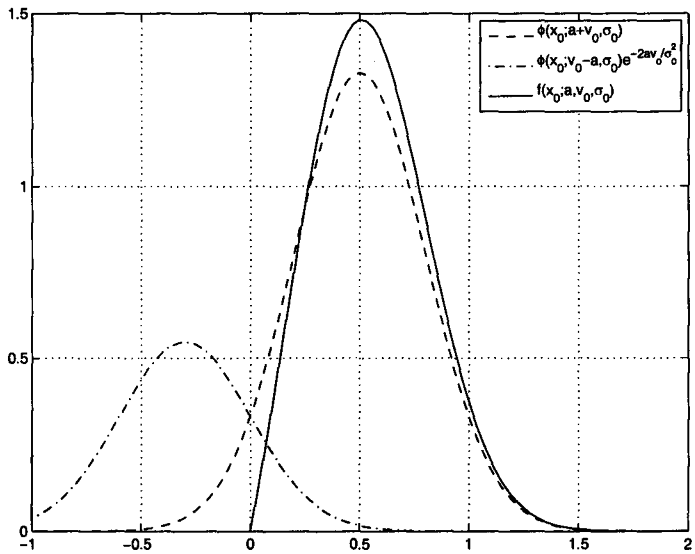


Figure 2.5: Probability Density Function $f(x_0; a, v_0, \sigma_0)$ with parameters $a = 0.4$, $v_0 = 0.1$ and $\sigma_0 = 0.3$.

The motivation of this distribution is that the conditional first passage probability $P(t < \tau < T | \tau > t)$ can be written as an integral of a pdf which has a form of $f(x_0; a, v_0, \sigma_0)$ defined above. Using Lemmas 2.5.2 and 2.5.3, the explicit formula for $P(\tau < T)$ and its asymptotics (as $T \to +0$) are given by the following Proposition.

Proposition 2.6.2. *For the RBC-II model, the default probability has the following* *expression*

$$P(\tau < T) = \frac{A + B - C - D}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},$$

where

$$\begin{aligned}
A &= \Phi_2\left(-\frac{a+v_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{a+v_0}{\sigma_0}, \rho\right), \\
B &= \Phi_2\left(-\frac{a+v_0-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{a+v_0-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, \rho\right) e^{2\mu^2\sigma_0^2/\sigma^4-2\mu(a+v_0)/\sigma^2}, \\
C &= \Phi_2\left(-\frac{v_0-a+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{v_0-a}{\sigma_0}, \rho\right) e^{-2av_0/\sigma_0^2}, \\
D &= \Phi_2\left(-\frac{v_0-a-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{v_0-a-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, \rho\right) e^{2\mu^2\sigma_0^2/\sigma^4-2av_0/\sigma_0^2-2\mu(v_0-a)/\sigma^2}, \\
\rho &= \frac{-\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\lim_{T\to+0}\frac{P(\tau<T)}{T} &= \frac{a\sigma^2\phi(0; a+v_0, \sigma_0)/\sigma_0^2}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)} \\
&= \frac{\sigma^2}{2}\frac{\partial f(x_0; a, v_0, \sigma_0)}{\partial x_0}\Big|_{x_0=0}.
\end{aligned}$$

The above Proposition ensures positive intensity for the RBC-II model. Consequently, positive short spreads are generated in the RBC-II model. Note that we have obtained an equivalent expression of the intensity as in Duffie & Lando (2001). In their model, the log of asset value follows drifted BM with a constant initial value z_0: $Z_t = z_0 + mt + \sigma W_t$. The default time τ is defined to be $\tau := \inf\{s > 0 : Z_s = 0\}$. Consider fixed time $t > 0$, assume the only information available is $\mathcal{H}_t := \{1_{\{\tau > s\}} : s \le t\}$. Conditional on $\tau > t$, Z_t has a conditional density $f(\cdot)$ which is bounded and has bounded derivative with $f(0) = 0$ and $f'(0)$ is defined from the right. Duffie & Lando (2001) stated that, the default intensity $\lambda := \lim_{h\to+0} P(t < \tau \le t+h | \tau > t)/h$, is given by $\frac{1}{2}\sigma^2 f'(0)$. However, in their model, there is perfect information at time zero and they can only establish the existence of an intensity for $t > 0$. By randomizing the initial value of the solvency ratio, the RBC-II model avoids this difficulty.

Example 2.6.1. *When $\mu/\sigma^2 = v_0/\sigma_0^2$.*
In this case, the probability of default is reduced to

$$P(\tau < T) = \frac{\Phi\left(-\frac{a+v_0+\mu T}{\sqrt{\sigma^2 T+\sigma_0^2}}\right) + e^{-2a\mu/\sigma^2}\Phi\left(-\frac{a-v_0-\mu T}{\sqrt{\sigma^2 T+\sigma_0^2}}\right) - \Phi\left(-\frac{a+v_0}{\sigma_0}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},$$

where Φ_2 functions disappear and only Φ functions are involved. The reduction of dimensionalities of the integrals can be proved either by Vasicek's expansions, or by separation of integration techniques.

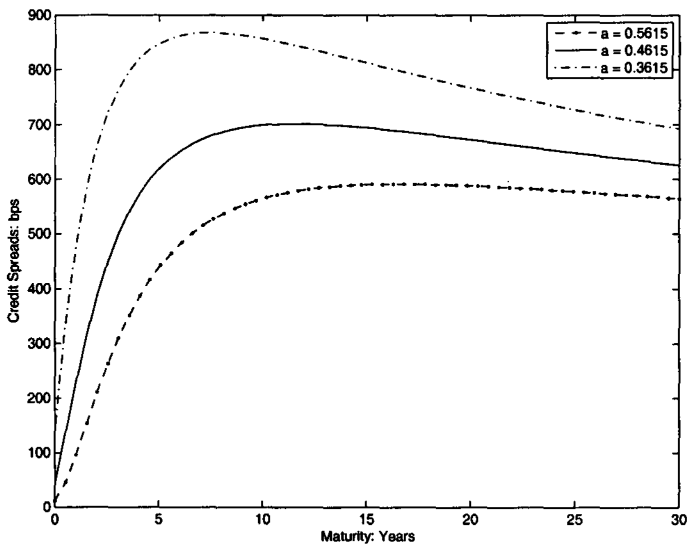


Figure 2.6: Term structure of credit spreads of RBC-II model for varying a: $\mu = -0.0417$, $\sigma = 0.2030$, $v_0 = 0.2402$, $\sigma_0 = 0.2162$ and $l = 1$.

Figures 2.6 and 2.7 show that the TSCS increases as a or v_0 decreases. Since for smaller a or v_0, the initial distribution of X_0 has more mass close to zero. It is therefore more likelly to default which in turn implies higher credit spreads. Figure 2.8 shows

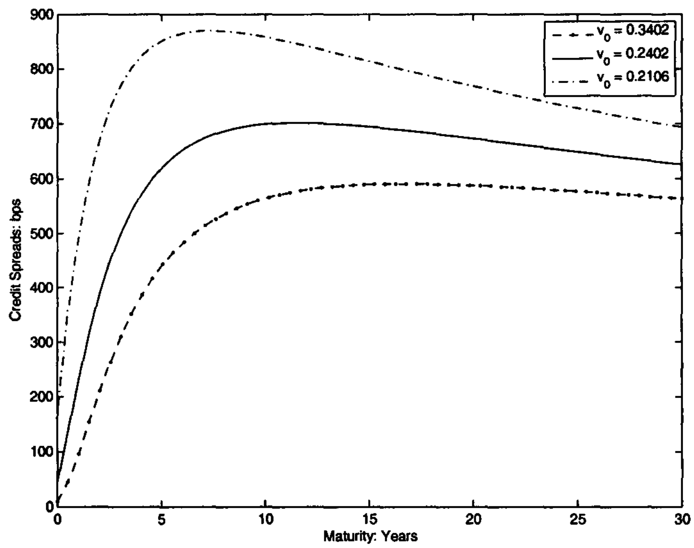


Figure 2.7: Term structure of credit spreads of RBC-II model for varying v_0: $\mu = -0.0417$, $\sigma = 0.2030$, $a = 0.4615$, $\sigma_0 = 0.2162$ and $l = 1$.

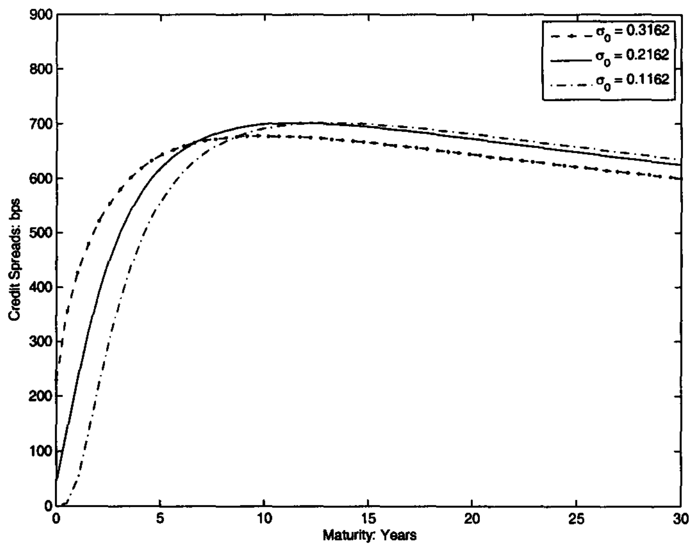


Figure 2.8: Term structure of credit spreads of RBC-II model for varying σ_0: $\mu = -0.0417$, $\sigma = 0.2030$, $v_0 = 0.2402$, $a = 0.4615$ and $l = 1$.

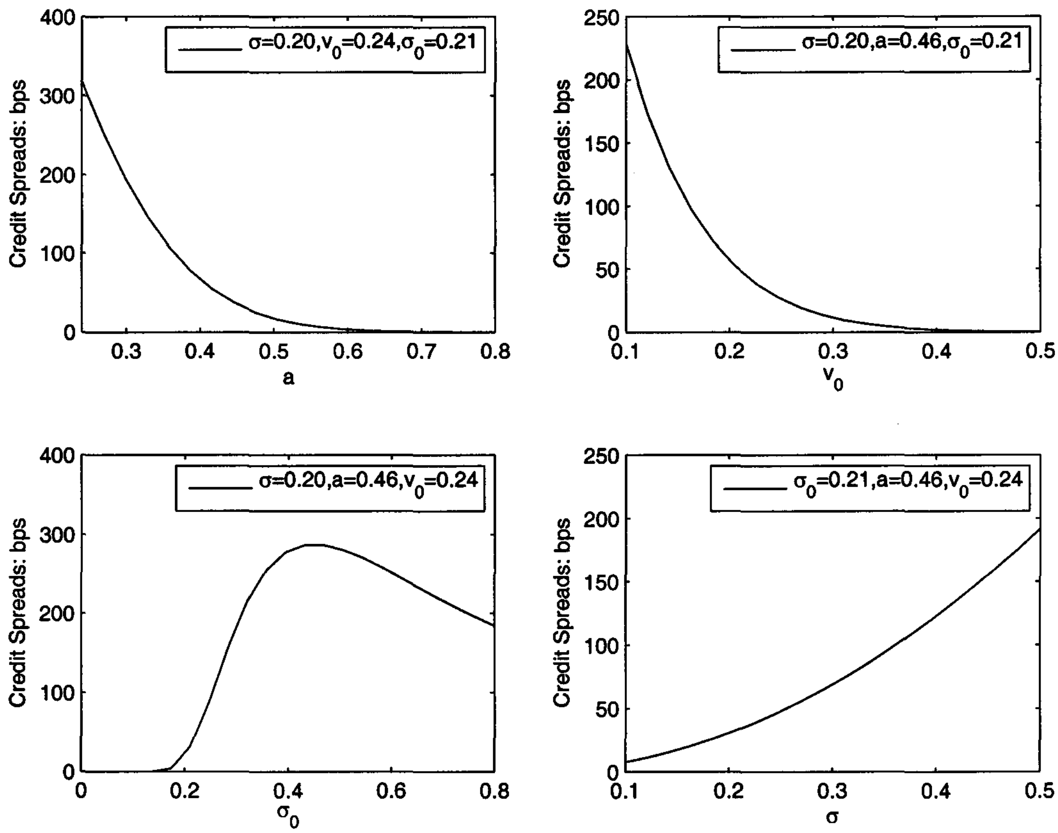


Figure 2.9: RBC-II short spread as a function of a, v_0, σ_0 and σ.

that the credit spread does not depend on σ_0 in a monotonic way. For some maturities the credit spread increases with σ_0 while it may decrease with σ_0 for other maturities. Figure 2.9 shows the short spread as a function of parameters a, v_0, σ_0 and σ. The short spread increases as a decreases, or v_0 decreases or σ increases. As similar to the RM-II model, the short spread of the RBC-II model first increases to a maximum and then decreases, as σ_0 increases from zero. This maximum achieved at some σ_0 which can be solved by setting $\partial CS(+0)/\partial\sigma_0 = 0$.

2.7 Delayed Information vs. Randomized Structure Model

Randomization of the initial value in the RS model may look awkward at first glance. This section gives a natural construction of the RS model through delayed information.

2.7.1 Delayed Information vs. RM-II

Assume the solvency ratio X_t is given by Equation (2.3) with constant initial value $X_0 = x_0$. Let time t be the current time and T be a future time. Let ϵ be a small positive number. At the current time t, we assume two sets of information available. First, we assume $X_t > 0$. Second, X_t is not observed, but the delayed solvency ratio $X_{t-\epsilon} = a$ is realized at time t. Then, conditional on $X_{t-\epsilon} = a$, we have $X_t \sim N(a + \mu\epsilon, \sigma^2\epsilon)$. This is because

$$X_t = X_{t-\epsilon} + \mu\epsilon + \sigma(W_t - W_{t-\epsilon}).$$

The conditional default probability can be calculated through

$$\begin{aligned} P(X_T < 0 | X_t > 0, X_{t-\epsilon} = a) &= \frac{P(X_T < 0, -X_t \le 0 | X_{t-\epsilon} = a)}{P(-X_t < 0 | X_{t-\epsilon} = a)} \\ &= \frac{\Phi_2\left(-\frac{a+\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}, \frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}, -\frac{\sqrt{\epsilon}}{\sqrt{T-t+\epsilon}}\right)}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right)}. \end{aligned}$$

This is equivalent to the RM-II model if we set $y_0 = a + \mu\epsilon$ and $\sigma_0 = \sigma\sqrt{\epsilon}$. This construction suggests that uncertainty about the current solvency ratio may come from the delayed realization of the solvency ratio. The longer the observation is delayed (i.e. for larger ϵ), the more uncertainty is the current solvency ratio (i.e. larger σ_0). The short spread derived from this construction becomes

$$\frac{\frac{\sigma^2}{4\sqrt{2\pi\sigma^2\epsilon}} e^{-\frac{(a+\mu\epsilon)^2}{2\sigma^2\epsilon}}}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right)}.$$

The short spread derived from this delayed information depends on the drift parameter μ through y_0. This is because y_0 is a linear function of μ. Figure 2.10 shows that the short spread increases as a or μ decreases, or σ or ϵ increases. This indicates that the longer the information is delayed, the higher the short spread.

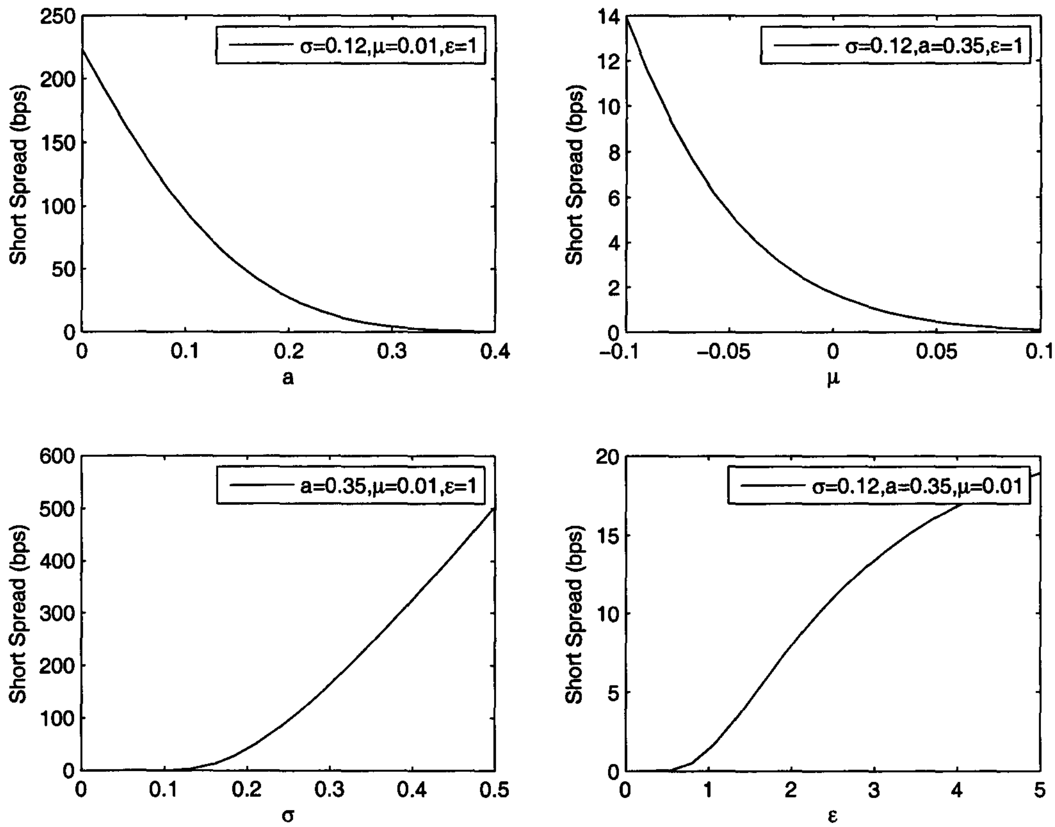


Figure 2.10: The short spread curve derived from delayed information as a function of a, μ, σ and ϵ.

2.7.2 Delayed Information vs. RBC-II

This delayed information approach can easily be extended to first passage models, where the default time τ is defined as in Equation (2.4). As before, X_t denotes the solvency ratio process which we assume has stationary increments. At the current time t, we assume the following information is available. First, default has not happened up to now, namely $\tau > t$ is known. Second, at current time t, we can only observe the path of the solvency ratio up to a previous time $t - \epsilon$, particulary $X_{t-\epsilon} = a > 0$ is realized at time t but X_t is not. Let $\mathcal{F}_{t-\epsilon}$ denote the filtration generated by the solvency ratio process up to time $t - \epsilon$. Then, the default time for a given future time $T > t$ can be calculated through

$$\begin{aligned} P(\tau < T|\mathcal{F}_{t-\epsilon}, \tau > t) &= \frac{P(t < \tau < T|\mathcal{F}_{t-\epsilon})}{P(\tau > t|\mathcal{F}_{t-\epsilon})} \\ &= \frac{P(\epsilon < \tau < T - t + \epsilon|X_0 = a)}{P(\tau > \epsilon|X_0 = a)} \\ &= \frac{P(\tau < T - t + \epsilon|X_0 = a) - P(\tau < \epsilon|X_0 = a)}{1 - P(\tau < \epsilon|X_0 = a)}. \end{aligned} \tag{2.25}$$

The above formula can be calculated explicitly if $P(\tau < \epsilon|X_0 = a)$ has an explicit expression. The default intensity is then given by

$$\lambda_t = \frac{\partial P(\tau < \eta + \epsilon|X_0 = a)/\partial\eta|_{\eta=0}}{P(\tau > \epsilon|X_0 = a)}. \tag{2.26}$$

Note that the numerator is exactly the pdf of the first passage time taking value at ϵ. We thus have obtained positive intensities through delayed information.

Consider the case when X_t is a drifted BM. We know that $P(\tau < \epsilon|X_0 = a)$ has explicit formula given by Equation (2.5). Then the default probability is given by

$$P(\tau < T|\mathcal{F}_{t-\epsilon}, \tau > t) = \frac{\Phi\left(-\frac{a+\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}\right) + e^{-2a\mu/\sigma^2}\Phi\left(-\frac{a-\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}\right) - \Phi\left(-\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon-a}{\sigma\sqrt{\epsilon}}\right)}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon-a}{\sigma\sqrt{\epsilon}}\right)}.$$

This is equivalent to the RBC-II model discussed in Example 2.6.1 with $v_0 = \mu\epsilon$ and $\sigma_0^2 = \sigma^2\epsilon$. These parameter constraints satisfy $\mu/\sigma^2 = v_0/\sigma_0^2$. The default intensity λ_t has the following elegant expression:

$$\lambda_t = \frac{ae^{-\frac{(a+\mu\epsilon)^2}{2\sigma^2\epsilon}}/\sqrt{2\pi\sigma^2\epsilon^3}}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon-a}{\sigma\sqrt{\epsilon}}\right)} \tag{2.27}$$

Note that this construction only makes sense when $t > \epsilon > 0$, since $\mathcal{F}_{-\epsilon}$ is not well-defined here. We can therefore only establish the existence of an intensity for $t > 0$ through the construction of delayed information. The Duffie-Lando approach is thus equivalent to the delayed information approach.

2.8 Mean-reverting Solvency Ratio (MRSR)

In previous models, we assumed that the solvency ratio follows a drifted BM. Many researchers propose that the solvency ratio is more likely to be mean-reverting, such as Collin-Dufresne & Goldstein (2001). Next, we consider the case when the solvency ratio is modeled by an Ornstein-Uhlenbeck (OU) process (under the risk-neutral measure) given by

$$X_t = \theta + (X_0 - \theta + \sigma\int_0^t e^{\kappa s}dW_s)e^{-\kappa t}, \tag{2.28}$$

with random initial value X_0. In the following two subsections, we assume that X_t is given by Equation (2.28).

2.8.1 RM-II with MRSR

In this section we make the following assumptions about default time and the distribution of X_0:

- Default time: the firm defaults at time T, when $X_T < 0$.

- Distribution of X_0: its pdf $f(x_0; y_0, \sigma_0)$ is given by Equation (2.9).

For fixed T, X_T is normal with mean μ_x and variance σ_x^2 given by

$$\mu_x = \theta + (y_0 - \theta)e^{-\kappa T}, \tag{2.29}$$

$$\sigma_x^2 = \sigma_0^2 e^{-2\kappa T} + \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T}). \tag{2.30}$$

Similarly, we can derive the PD, LGD, and CS. We state the following propositions without proofs.

Proposition 2.8.1. *In this RM-II with MRSR model, the $PD(T)$, $RR(T)$ and $CS(T)$ have the following representations*

$$\begin{aligned} PD(T) &= \frac{\Phi_2(-\mu_x/\sigma_x, y_0/\sigma_0, \rho)}{\Phi(y_0/\sigma_0)}, \\ RR(T) &= \frac{\Phi_2(-\mu_x/\sigma_x - \sigma_x, y_0/\sigma_0 - \rho\sigma_x, \rho)e^{\mu_x + \frac{1}{2}\sigma_x^2}}{\Phi_2(-\mu_x/\sigma_x, y_0/\sigma_0, \rho)}, \\ CS(T) &= -\frac{1}{T}\log\left(\frac{\Phi(y_0/\sigma_0) - A + Be^{\mu_x + \frac{1}{2}\sigma_x^2}}{\Phi(y_0/\sigma_0)}\right), \end{aligned}$$

where μ_x, σ_x are given in equations (2.29-2.30) and

$$\begin{aligned} \rho &= -\sigma_0 e^{-\kappa T}/\sigma_x, \\ A &= \Phi_2(-\mu_x/\sigma_x, y_0/\sigma_0, \rho), \\ B &= \Phi_2(-\mu_x/\sigma_x - \sigma_x, y_0/\sigma_0 - \rho\sigma_x, \rho). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \lim_{T \to +0} \frac{PD(T)}{\sqrt{T}} &= \frac{\sigma f(0; y_0, \sigma_0)}{\sqrt{2\pi}}, \\ \lim_{T \to +0} \frac{LGD(T)}{\sqrt{T}} &= \frac{\sigma\sqrt{2\pi}}{4}, \\ \lim_{T \to +0} CS(T) &= \frac{\sigma^2 f(0; y_0, \sigma_0)}{4}. \end{aligned}$$

Corollary 2.8.2. *In this RM-II with MRSR, we have*

$$
\begin{aligned}
\lim_{\sigma_0 \to +0} PD(T) &= \Phi\left(-\frac{y_0 e^{-\kappa T} + \theta(1-e^{-\kappa T})}{\sqrt{\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa T})}}\right), \\
\lim_{\sigma_0 \to +0} RR(T) &= \frac{\Phi\left(-\frac{y_0 e^{-\kappa T} + \theta(1-e^{-\kappa T}) + \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa T})}{\sqrt{\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa T})}}\right) e^{y_0 e^{-\kappa T} + \theta(1-e^{-\kappa T}) + \frac{\sigma^2}{4\kappa}(1-e^{-2\kappa T})}}{\Phi\left(-\frac{y_0 e^{-\kappa T} + \theta(1-e^{-\kappa T})}{\sqrt{\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa T})}}\right)}.
\end{aligned}
$$

Note that this short spread is exactly the same as in the original RM-II model. This tells us that the short spread does not directly depend on the drift of the solvency ratio. When T is sufficient large, X_T will be normal with mean approximately equal to $\theta > 0$. Hence, the default probability will converge to a positive constant which is larger than 1/2 and less than 1. As a result, $PD * LGD$ can never converge to 1 as $T \to +\infty$. This eventually leads CS to zero, as $T \to +\infty$. It therefore results in a hump-shaped TSCS.

2.8.2 RBC-II with MRSR

In this RBC-II with MRSR model, we need two assumptions from the original RBC-II model.

- Default time τ: is defined by Equation (2.4).
- Distribution of X_0: is given by Equation (2.24).

This model generalizes Collin-Dufresne & Goldstein (2001). However, we lose analytical tractability due to the complexity of the first passage time density of an OU process.

Therefore, we propose to use the delayed information construction. The probability of default and the default intensity are given by Equations (2.25) and (2.26). The pdf of the first passage time of an OU process is needed to calculate these equations explicitly. Thanks to Alili, Patie & Pedersen (2005), three semi-explicit representations of the first passage time density of an OU process are available: the series expansion representation, the integral representation and the Bessel bridge representation.[5] Collin-Dufresne & Goldstein (2001) also provided an efficient numerical scheme to calculate the first passage time probability of an OU process by utilizing Fortet's lemma.

Lemma 2.8.3. ***Collin-Dufresne and Goldstein (2001):*** *Discretize time into n equal intervals, and define date $t_j = jT/n := j\Delta t$, for $j \in (1, 2, ..., n)$. Let X_t be an OU process defined in Equation (2.28). Default time τ is defined in Equation (2.4). The default probability $P(\tau < T|X_0 = x_0)$ can be calculated through:*

$$\begin{aligned}
P(\tau < t_j|X_0 = x_0) &= \sum_{i=1}^{j} q_i, \quad j = 2, 3, ..., n \\
q_1 &= \frac{\Phi(a_1)}{\Phi(b_{(1/2)})}, \\
q_i &= \left[\Phi(a_i) - \sum_{j=1}^{i-1} q_j \Phi(b_{i-j+1/2})\right] / \Phi(b_{(1/2)}), \\
a_i &= \frac{M(i\Delta t)}{S(i\Delta t)}, \quad b_i = \frac{L(i\Delta t)}{S(i\Delta t)}, \quad i = 2, 3, ..., n \\
M(t) &= x_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \\
L(t) &= \theta(1 - e^{-\kappa t}), \\
S^2(t) &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}).
\end{aligned}$$

As $T \to +\infty$, we can see from Equation (2.6), under the assumption of a constant interest rate and constant LGD (assume less than 1), the long term spread will eventually go to zero. Although, Collin-Dufresne & Goldstein (2001) showed an upward

[5]Full explicit representation for the first passage density of an OU process is available only when the barrier coincides with the asymptotic mean of the OU process. In this case, the cdf of the first passage time admits an even simpler representation as discussed in Yi (2006).

sloping TSCS in their paper for the time span from 0 to 20 years, the TSCS is in fact hump-shaped when extending the spreads over longer maturities, such as 50 years. We conclude that the mean-reverting solvency ratio can not make the asymptotic long term spreads higher.[6]

2.9 Calibration Exercise

In this section, we demonstrate the incomplete information effects on TSCS by fitting the theoretical CS in both the RM-II model and the RBC-II model to the observed CDS spread curve. A thorough investigation of an optimal calibration procedure is beyond the scope of this paper.

In the RM-II model, the parameters of interest are μ, σ, y_0 and σ_0. In the RBC-II model, the parameters of interest are μ, σ, a, v_0 and σ_0. In this calibration exercise, we set the $LGD = 1$ for the RBC-II model and the Black-Cox model.

For comparison, we also fit the Merton's CS and the Black-Cox's CS to the market data. The parameters of interest in both the Merton and the Black-Cox models are μ, σ and y_0. However, in fact, only two parameters need to be calibrated in Merton's model because μ and σ are related through $r = \mu + 0.5\sigma^2$, where the short rate r is proxied by the 1-month U.S Treasury yield. Parameters are calibrated by minimizing the cross-sectional Mean Absolute Error (MAE).

We take the CDS curve of Ford Motor Corp. on March 16, 2007, with 0.25-, 1-, 2-, 3-, 4-, 5-, 7-, and 10-year maturities.[7] The short rate r is taken to be 5.18%, which is

[6]In practice, the longest time to maturity we are interested in is 30-year. For certain parameters, we can generate TSCS which is upward sloping in the range of 0 to 30-year. Even Merton's model can do this.

[7]The 3-month CDS is not directly available from Bloomberg. It is interpolated from its nearest two points available, namely 1-year and 2-year CDS data.

the 1-month U.S Treasury yield on March 16, 2007.

Figures 2.11 and 2.12 show the calibrated results. It can be seen from the picture that both RM-II and RBC-II models outperform the classical structural models in fitting the CDS spreads, especially for short maturities. We also find that the RBC-II model fits better than the RM-II model. In fact, the MAE for the RBC-II, the RM-II, the Black-Cox and the Merton's model are 7 bps, 15 bps, 68bps and 30 bps respectively. For the short end maturities, such as 3-month, both Merton's and Black-Cox's spreads are close to zero (0.3709 bps and 4bps respectively), but catering for incomplete information allows the RM-II and RBC-II to generate a positive value of 83.327 bps and 89 bps respectively.

The calibrated parameters are illustrated in Table 2.1. Note that $\mu + 0.5\sigma^2$ is negative (-0.1033) in the RM-II model, which does not represent the interest rate.

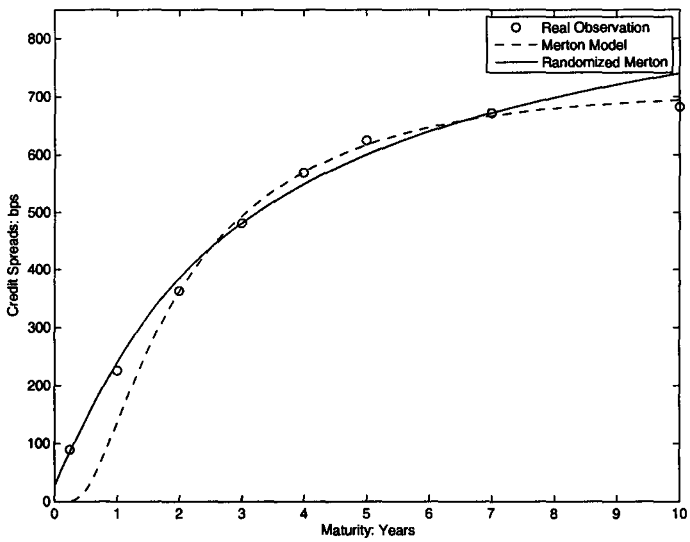


Figure 2.11: RM-II model vs. Merton model fit to Ford Motor CDS curve on March 16, 2007.

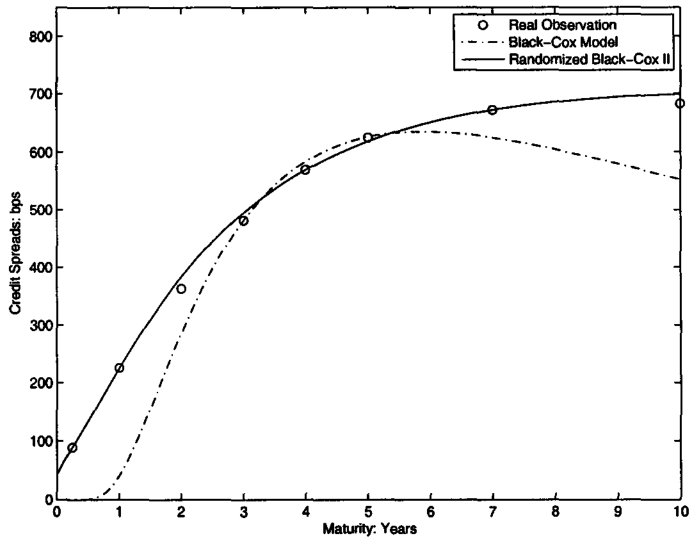


Figure 2.12: RBC-II model vs. Black-Cox model fit to Ford Motor CDS curve on March 16, 2007.

Model	μ	σ	y_0	σ_0	v_0	a	MAE (bps)
Merton	-0.2449	0.7703	1.4852				30
RM-II	-0.1432	0.2825	0.4926	0.2045			15
Black-Cox	-0.3220	0.6288	1.9588				68
RBC-II	-0.0417	0.2030		0.2162	0.2402	0.4615	7

Table 2.1: Calibrated parameters for the three models.

2.10 Summary

Motivated by Merton's model, we proposed to treat the solvency ratio directly as the risk factor. Based on the classical structural models, we then proposed to randomize the initial value of the solvency ratio to take account of imperfect information. We have mainly looked at four different RS models, which cover two types of definition of default and different assumptions on the initial randomization.

From the RM-II and the RBC-II models, we found that positive short spreads can be produced due to imperfect observation on the solvency ratio. We also found that various shapes of the term structure of credit spreads can be generated. The PD, the LGD and the CS are given in explicit formulae for both models, which have explicit expressions for their short spreads. In the RM-II model, we found that both $PD(T)/\sqrt{T}$ and $LGD(T)/\sqrt{T}$ converge to positive constants as $T \to +0$. The default intensity is thus not defined in the RM-II model, but positive short spreads can still be produced. In the RBC-II model, we found that $PD(\tau < T)/T$ converges to a positive constant. Therefore, the default intensity does exist in the RBC-II model and it generates positive short spreads under the assumption of constant LGD.

Merton's model becomes a special case of the RM-II model while the Black-Cox model is a special case of the RBC-II model. Numerical analysis and a calibration exercise illustrate that the randomized structural models outperforms the classical structural model in fitting the term structure of credit spreads, especially for short maturities. These two RS models generalize the classical structural models in two folds. First, instead of modeling the asset value and debt separately, we modeled the solvency ratio directly as a drifted BM. Second, imperfect information is considered and positive short spreads are generated. From the RM-I and the RBC-I, we also noticed that the short spread may become infinity if the random initial distribution has too much mass close to the default barrier.

We next provided a delayed information perspective on the RS models. The models constructed through delayed information are special cases in our general RS models in two ways. First, number of parameters are reduced due to some parameter constrains. Second, positive short spreads can be generated for only $t > 0$.

We finally studied two alternative models whose solvency ratio is modeled as a mean-reverting process. We found that the mean-reverting effect does not affect the short end of the TSCS. However, the long term spread of these alternative models is found to converge to zero when T tends to infinity.

2.11 Appendix I

- **Proof of Proposition 2.5.1:**

  From Equations (2.9-2.12), we have

$$
\begin{aligned}
PD(T) &= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} P(X_T < 0|X_0 = x_0)\phi(x_0; y_0, \sigma_0)dx_0 \\
&= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} \Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0, \sigma_0)dx_0, \\
&= \frac{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}{\Phi(y_0/\sigma_0)}, \\
RR(T) &= \frac{\int_0^{+\infty} e^{x_0+\mu T+\frac{1}{2}\sigma^2 T}\Phi\left(-\frac{x_0+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0, \sigma_0)dx_0}{\Phi(y_0/\sigma_0)PD(T)} \\
&= \frac{e^{y_0+\mu T+\frac{1}{2}(\sigma_0^2+\sigma^2 T)}\int_0^{+\infty}\Phi\left(-\frac{x_0+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0+\sigma_0^2, \sigma_0)dx_0}{\Phi(y_0/\sigma_0)PD(T)} \\
&= \frac{\Phi_2\left(-\frac{y_0+\mu T+\sigma_0^2+\sigma^2 T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}+\sigma_0, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)e^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}.
\end{aligned}
$$

  We have used Lemma 2.5.2 for the last steps of each calculation. Then the formula for $CS(T)$ comes in handy. For the asymptotics, Equation (2.16) is a direct result from Lemma 2.5.3. Note that $LGD(T) = 1 - RR(T)$, Equation (2.17) is obtained by applying Lemma 2.5.3 to $1 - RR(T)$. Equation (2.18) is a direct result of Equations (2.16) and (2.17).

- **Proof of Lemma 2.5.2:**

  The Left Hand Side (LHS) of Equation 2.19 is the probability that both X and Y are less than zero, i.e. $P(X < 0, Y < 0)$. We can also calculate this probability by conditioning

$$\begin{aligned}
P(X<0,Y<0) &= \int_{-\infty}^{0} P(X<0|Y=y)\phi(y;\mu_y,\sigma_y)dy \\
&= \int_{-\infty}^{0} \Phi\left(\frac{-y\rho\sigma_x/\sigma_y+\mu_y\rho\sigma_x/\sigma_y-\mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y;\mu_y,\sigma_y)dy \\
&= \int_{0}^{+\infty} \Phi\left(\frac{y\rho\sigma_x/\sigma_y+\mu_y\rho\sigma_x/\sigma_y-\mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y;-\mu_y,\sigma_y)dy
\end{aligned}$$

For the second step, we have used the fact that X is still normally distributed conditional on $Y=y$, i.e. $X|Y=y \sim N(y\rho\sigma_x/\sigma_y-\mu_y\rho\sigma_x/\sigma_y+\mu_x, \sigma_x\sqrt{1-\rho^2})$.

- **Proof of Lemma 2.5.3:**

From Lemma 2.5.2, the LHS of Equation (2.20) can be written as

$$\begin{aligned}
LHS &= \frac{1}{\sqrt{2\pi\sigma_0^2}}\int_0^{+\infty}\Phi\left(-\frac{x+\mu T}{\sigma\sqrt{T}}\right)e^{-\frac{(x-y_0)^2}{2\sigma_0^2}}dx \\
&= \frac{\sigma\sqrt{T}}{\sqrt{2\pi\sigma_0^2}}\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)e^{-\frac{(z\sigma\sqrt{T}-\mu T-y_0)^2}{2\sigma_0^2}}dz \\
&= \sigma\sqrt{T}\phi(0;y_0,\sigma_0)\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)e^{-\frac{-2y_0z\sigma\sqrt{T}+O(T)}{2\sigma_0^2}}dz \\
&= \sigma\sqrt{T}\phi(0;y_0,\sigma_0)\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)[1+y_0z\sigma\sqrt{T}/\sigma_0^2+O(T)]dz \\
&\to \frac{\sigma\phi(0;y_0,\sigma_0)\sqrt{T}}{\sqrt{2\pi}}+\frac{y_0\sigma^2\phi(0;y_0,\sigma_0)T}{4\sigma_0^2}+O(T^{3/2}).
\end{aligned}$$

For the last step, we have used the following two equalities

$$\int_0^{+\infty}\Phi(-z)dz=\frac{1}{\sqrt{2\pi}},\quad \int_0^{+\infty}z\Phi(-z)dz=\frac{1}{4}.$$

- **Proof of Proposition 2.6.1:**

Recall that $P(\tau < T|X_0 = x_0)$ is given by Equation (2.5), we have

$$\begin{aligned}
P(\tau < T) &= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} P(\tau < T|X_0 = x_0)\phi(x_0; y_0, \sigma_0)dx_0 \\
&= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} \Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0, \sigma_0)dx_0 \\
&\quad + \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} e^{-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0, \sigma_0)dx_0 \\
&= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} \Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0, \sigma_0)dx_0 \\
&\quad + \frac{e^{-2y_0\mu/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} \Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0 - 2\mu\sigma_0^2/\sigma^2, \sigma_0)dx_0 \\
&= \frac{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}{\Phi(y_0/\sigma_0)} \\
&\quad + \frac{\Phi_2\left(-\frac{y_0-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)e^{-2\mu y_0/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}}{\Phi(y_0/\sigma_0)}.
\end{aligned}$$

Lemma 2.5.2 is used for the last step. From Lemma 2.5.3, we have

$$\begin{aligned}
\lim_{T\to+0}\frac{P(\tau < T)}{\sqrt{T}} &= \frac{\sigma[\phi(0; y_0, \sigma_0) + \phi(0; y_0 - 2\mu\sigma_0^2/\sigma^2, \sigma_0)e^{-2\mu y_0/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}]}{\sqrt{2\pi}\Phi(y_0/\sigma_0)} \\
&= \frac{2\sigma\phi(0; y_0, \sigma_0)}{\sqrt{2\pi}\Phi(y_0/\sigma_0)}.
\end{aligned}$$

- **Proof of Proposition 2.6.2**

$$\begin{aligned}
P(\tau < T) &= \int_{-\infty}^{+\infty} P(\tau < T|X_0 = x_0)f(x_0; a, v_0, \sigma_0)dx_0 \\
&= \frac{A + B - C - D}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},
\end{aligned}$$

where A, B, C and D are given by

$$
\begin{aligned}
A &= \int_0^{+\infty} \Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; a+v_0, \sigma_0)dx_0,\\
B &= \int_0^{+\infty} e^{-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; a+v_0, \sigma_0)dx_0,\\
C &= \int_0^{+\infty} e^{-2av_0/\sigma_0^2}\Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; v_0-a, \sigma_0)dx_0,\\
D &= \int_0^{+\infty} e^{-2av_0/\sigma_0^2-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; v_0-a, \sigma_0)dx_0.
\end{aligned}
$$

Note that B and D can also be written as

$$
\begin{aligned}
B &= e^{-2\mu(a+v_0)/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}\\
&\quad * \int_0^{+\infty} \Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; a+v_0-2\mu\sigma_0^2/\sigma^2, \sigma_0)dx_0,\\
D &= e^{-2av_0/\sigma_0^2-2\mu(v_0-a)/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}\\
&\quad * \int_0^{+\infty} \Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; v_0-a-2\mu\sigma_0^2/\sigma^2, \sigma_0)dx_0.
\end{aligned}
$$

Explicit expressions for A, B, C and D are finally obtained by invoking Lemma 2.5.2. The asymptotic equation of $P(\tau < T)$ is obtained by using Lemma 2.5.3 and the following identities:

$$
\begin{aligned}
\phi(0; a+v_0, \sigma_0) &= \phi(0; v_0-a, \sigma_0)e^{-2av_0/\sigma_0^2}\\
&= e^{2\mu^2\sigma_0^2/\sigma^4-2\mu(a+v_0)/\sigma^2}\phi(0; a+v_0-2\mu\sigma_0^2/\sigma^2, \sigma_0)\\
&= e^{2\mu^2\sigma_0^2/\sigma^4-2av_0/\sigma_0^2-2\mu(v_0-a)/\sigma^2}\phi(0; v_0-a-2\mu\sigma_0^2/\sigma^2, \sigma_0).
\end{aligned}
$$

Chapter 3

Inverse CIR and Semi-affine Intensity-based Models

In the previous chapter, the interest rate is assumed to be a constant and credit spreads are derived by modeling the solvency ratio. In this chapter, we assume that the interest rate $r(t)$ is an iCIR process and model the short spread directly as a nonnegative process correlated with the short rate. **This section assumes constant Recovery of Market Value (RMV) under the Duffie-Singleton framework, see Duffie & Singleton (1999).** In Section 3.1, we give a brief introduction of our motivation and the model. In Section 3.2, we study the CIR and the iCIR processes and recall some elementary results from Ahn & Gao (1999). In Section 3.3, we model the interest rate as an iCIR process, and assume a two-factor model for the short spread. In Section 3.4, both default-free and defaultable zero coupon bond prices are given in explicit formulae. In Section 3.5, we focus on the correlation structure among short spreads of different firms, and the correlation structure between each firm's short spreads and the default-free interest rates. In Section 3.6, some numerical analysis is provided. Calibration issues are discussed in Section 3.7. In Section 3.8, we discuss an extension of our model. Section 3.9 summarizes this chapter. All proofs are given in the appendix of this chapter.

3.1 Introduction

The CIR process was first introduced and used to model the default-free interest rate by Cox, Ingersoll & Ross (1985). Since then, it has received great attention by both academic researchers and industrial practitioners. The CIR process has become so popular because it is the simplest nonnegative process which admits an affine term structure of bond pricing formulae. However, the empirical study by Pearson & Sun (1994) rejected CIR as a good candidate for describing the Treasury market. Duan & Simonato (1999) also provided support in favor of rejection. In order to capture the general shapes of the yield curves, two additional CIR factors may be needed as suggested by Chen & Scott (2003). Although multi-factor models are generally better than single factor models for the default-free interest rate, the expense is the addition of too many parameters.

Chan, Karolyi, Longstaff & Sanders (1992) assumed that the volatility of the short rate is a power function of the form $\sigma r(t)^{\beta}$. Their estimation of the volatility is very accurate and their results strongly suggest that $\beta = 1.5$ instead of simply 0.5 (i.e. the square root). Without specifying the parametric form of the short rate process $r(t)$, econometricians, such as Ait-Sahalia (1996*a*), Ait-Sahalia (1996*b*) and Stanton (1997), applied nonparametric techniques to estimate the drift and diffusion as a function of the short rate. Their findings suggest that the drift is non-linear in $r(t)$. Moreover, they also find that the volatility is proportional to $r(t)^{1.5}$, which is similar to that estimated by Chan et al. (1992). Inspired by these findings, Ahn & Gao (1999) proposed a non-affine model where the diffusion of the short rate is proportional to $r(t)^{1.5}$ and the drift is quadratic in $r(t)$. It turns out that their model is the same as an inverse of the

CIR process. Their empirical studies also show that the iCIR model outperforms the one-factor affine type models in fitting the default-free yield curves.

The CIR process has been widely used in the literature of credit risk as well, ever since Lando (1998) and Duffie & Singleton (1999) built the foundations for intensity-based approach to credit risk. Duffee (1999) empirically estimated a model with two CIR factors for the short rate process $r(t)$ and three CIR factors for the hazard rate process $h_j(t)$ for firm j, i.e.

$$\begin{aligned} r(t) &= -1 + X_1(t) + X_2(t), \\ h_j(t) &= \alpha_j + \beta_{1,j} X_1(t) + \beta_{2,j} X_2(t) + Y_j(t), \end{aligned}$$

where $X_i(t)$, $i = 1, 2$ and $Y_j(t)$, $j = 1, ..., n$ are independent CIR factors, and α_j, $\beta_{i,j}$ are constants. He found that most $\beta_{i,j}$ are negative. As a consequence, the interest rate and the hazard rate are found to be negatively correlated, which is consistent with most empirical findings, such as Longstaff & Schwartz (1995). However, negative coefficients are problematic. First, there is a positive probability that the hazard rate process $h_j(t)$ will go below zero, which is counterfactual. Second, if all the $\beta_{i,j}$ are negative, the correlations between any two firms' hazard rates can only be positive, which is too restrictive. In order to have the nonnegativity property of hazard rates for this model, we have to impose nonnegative coefficients. But, if all $\beta_{i,j}$ are positive, neither the negative correlation between interest rates and hazard rates is captured, nor can this model produce a rich correlation structure among hazard rates. This well known dilemma is not only for this model, but it is faced by all classical factor models, as discussed by Schönbucher (2003) and Duffie & Singleton (2003).

A good model should be able to reflect realistic properties as much as possible, while maintaining analytical tractability. For a good intensity-based factor model, we would like the following realistic properties to be satisfied.

- Nonnegativity: interest rates and short spreads should be nonnegative.

- Rich Correlation Structure: first, interest rates and short spreads should be negatively correlated in most cases as suggested in the empirical literature; second, both negative and positive correlations among different firms should be possible.
- Analytical Tractability: the pricing formulae should admit explicit expressions for both default-free and defaultable bonds.

In this paper, we propose a parsimonious model which is flexible enough to produce rich correlation structures among short spreads of differerent firms, while satisfying nonnegativity conditions. Following Ahn & Gao (1999), we model the interest rate $r(t)$ as an iCIR process. We then extend Ahn & Gao (1999) to a two-factor model for the short spread. More specifically, we model the short spread as a linear combination of the interest rate, the inverse of the interest rate and another idiosyncratic CIR factor. The coefficients of the interest rate and the short spread are imposed to be nonnegative, in order to maintain the nonnegativity property. Our model is then able to generate rich correlation structures among short spreads of different firms, while capturing the empirical fact that interest rates and short spreads are negatively correlated in most cases. This model remedies the deficiency for affine intensity-based factor models, which can only produce very restrictive correlation structures among short spreads of different firms. The pricing formulae of default-free and defaultable bonds are non-affine, but are obtained in explicit forms using the recent findings of stochastic integrals by Hurd & Kuznetsov (2006).

3.2 CIR and iCIR

We start with some simple facts about the CIR process and its inverse. Let $X(t)$ be a CIR process, starting at $X(0) = x_0 > 0$, specified as

$$dX(t) = (a - bX(t))dt + c\sqrt{X(t)}dW(t), \tag{3.1}$$

where a, b, c are positive constants and $W(t)$ is standard Brownian motion. We denote this CIR process as $CIR(a, b, c, x_0)$. The conditional and unconditional means and variances are well known and are given by

$$\begin{aligned}
\mathbb{E}[X(t)] &= \frac{a}{b} + (x_0 - \frac{a}{b})e^{-bt}, && (3.2)\\
\mathbb{E}[X] &= \frac{a}{b}, && (3.3)\\
Var[X(t)] &= \frac{ac^2}{2b^2} + \frac{c^2}{b}(x_0 - \frac{a}{b})e^{-bt} + \frac{c^2}{b}(\frac{a}{2b} - x_0)e^{-2bt}, && (3.4)\\
Var[X] &= \frac{ac^2}{2b^2}. && (3.5)
\end{aligned}$$

The CIR process $X(t)$ has unattainable boundaries if and only if $a \geq \frac{1}{2}c^2$. Given this condition, the iCIR process, $Y(t)$ will be well defined through

$$Y(t) = \frac{1}{X(t)}. \tag{3.6}$$

Ito's formula implies that $Y(t)$, starting from $Y(0) := y_0 = \frac{1}{x_0}$, has the following dynamics

$$dY(t) = [b - (a - c^2)Y(t)]Y(t)dt - cY(t)^{1.5}dW(t). \tag{3.7}$$

Ahn & Gao (1999) used this iCIR process to model the default-free interest rate. They provided the necessary and sufficient conditions for iCIR to have stationarity and unattainability of the boundaries. These conditions require that $a > c^2$ and $b > 0$, which in turn implies that the original CIR has unattainable boundaries. Ahn & Gao (1999) also derived the iCIR's conditional density as well as all the conditional and unconditional moments. We write the conditional and unconditional means and variances as follows, which will be needed later.

$$\begin{aligned}
\mathbb{E}[Y(t)] &= \frac{\zeta_t e^{-u_t}}{q} M(q, 1+q, u_t), && (3.8)\\
\mathbb{E}[Y] &= \frac{2b}{2a - c^2}, && (3.9)\\
Var[Y(t)] &= \frac{\zeta_t^2 e^{-u_t}}{q}\left[\frac{M(q-1, 1+q, u_t)}{q-1} - \frac{e^{-u_t}M(q, 1+q, u_t)^2}{q}\right], && (3.10)\\
Var[Y] &= \frac{2b^2c^2}{(a - c^2)(2a - c^2)^2}, && (3.11)
\end{aligned}$$

where

$$\begin{aligned}\zeta_t &= \frac{2b}{c^2(1-e^{-bt})},\\ u_t &= \zeta_t x_0 e^{-bt},\\ q &= \frac{2a}{c^2} - 1,\end{aligned}$$

and $M(\cdot,\cdot,\cdot)$ is the confluent hypergeometric function, which can be represented as hypergeometric series or integral form

$$\begin{aligned}M(a,b,z) &= 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \ldots\\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt.\end{aligned}$$

Following Ahn & Gao (1999), we will model the default-free interest rate as an iCIR process, as seen in next section.

3.3 Semi-affine Intensity-based Model

In this section, all the dynamics specified are under the physical measure. The default-free interest rate, $r(t)$, and the short spread $h_j(t)$, for firm $j, j = 1, ..., n$, are specified as follows:[1]

$$\begin{aligned}r(t) &= Y(t), &(3.12)\\ h_j(t) &= \alpha_j X(t) + \beta_j Y(t) + X_j(t), &(3.13)\end{aligned}$$

where α_j, β_j, $j = 1, ..., n$, are nonnegative constants, and $X_j(t)$ are independent $CIR(a_j, b_j, c_j, x_0^j)$, $j = 1, ..., n$, i.e.

$$dX_j(t) = (a_j - b_j X_j(t))dt + c_j\sqrt{X_j(t)}dW_j(t). \tag{3.14}$$

[1]Alternatively, one can assume constant recovery of the market value as in Duffie & Singleton (1999), and then model the hazard rate. Here, we model the product of loss given default and the hazard rate directly.

where a_j, b_j, c_j are nonnegative constants, and $W_j(t)$ are standard BM. We also assume independence between $X(t)$ and $X_j(t)$, for all j. Therefore, $X_j(t)$ denotes the idiosyncratic factor for firm j. Since all the coefficients are nonnegative and the CIR processes are nonnegative, this implies that the interest rate $r(t)$ and the short spread $h_j(t)$ are all nonnegative.

If $\alpha_j > 0$ and $\beta_j > 0$, from the definition of $h_j(t)$, we see that $h_j(t) \geq 2\sqrt{\alpha_j \beta_j}$, which puts a lower bound for the short spreads. Considering that the interest rates are small, say around 5%, $X(t)$ will be very big, comparatively speaking, say magnitude of 20. The short spread $h_j(t)$ is also very small in reality, say 10 basis points for a BBB company. This reality implies that α_j will be very small. For considerable β_j, the lower bound of $h_j(t)$ will be a very small positive number. Lower bounded short spreads implied from this model are not unrealistic at all. The fact that any company might default in the next infinitesimal time requires a premium in order to compensate for this risk.

Note that the short spreads specified in our model have three components, although only two CIR factors are used for each firm. The only common factor that is shared by every company is the economic systematic factor $X(t)$, which is the inverse of the interest rate process. Our model is significantly different from others in factor-model literature, because the common factors are shared in different ways. In traditional factor models, the short spreads only depend linearly on the common factor, such as Duffie & Singleton (1999), Duffee (1999), Duffie & Liu (2001), Schönbucher (2003), to mention a few. In our model, the short spreads not only depend linearly on the common factor, but also depend linearly on the inverse of the common factor. As a consequence, an increase of the common factor $X(t)$, will have two folded influences on the short spreads, given positive coefficients β_j and α_j. Increasing $X(t)$ will increase the first component of $h_j(t)$, but the second component will be decreased. This feature adds more flexibility to generate rich correlation structures among the short spreads of different firms. We will save this discussion for Section 3.5.

We list some simple facts about the mean and variances of $h_j(t)$ and $X_j(t)$ as follows for later use.

$$\begin{aligned}
\mathbb{E}[h_j(t)] &= \alpha_j\mathbb{E}[X(t)] + \beta_j\mathbb{E}[Y(t)] + \mathbb{E}[X_j(t)], && (3.15)\\
\mathbb{E}[h_j] &= \alpha_j\mathbb{E}[X] + \beta_j\mathbb{E}[Y] + \mathbb{E}[X_j], && (3.16)\\
Var[h_j(t)] &= \alpha_j^2 Var[X(t)] + 2\alpha_j\beta_j(1 - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]) && (3.17)\\
&\quad + \beta_j^2 Var[Y(t)] + Var[X_j(t)], && (3.18)\\
Var[h_j] &= \alpha_j^2 Var[X] + 2\alpha_j\beta_j(1 - \mathbb{E}[X]\mathbb{E}[Y]) + \beta_j^2 Var[Y] + Var[X_j], && (3.19)
\end{aligned}$$

where $E[X(t)], \mathbb{E}[X], Var[X(t)], Var[X], \mathbb{E}[Y(t)], \mathbb{E}[Y], Var[Y(t)], Var[Y]$ are given in equations (3.2-3.5) and equations (3.8-3.11) respectively, and $\mathbb{E}[X_j(t)]$ and $Var[X_j(t)]$ are the mean and variance of $CIR(a_j, b_j, c_j, x_0^j)$.

3.4 Bond Prices

The previous section completes our specification of a two-factor model for a multi-firm setup, under the physical measure. In this section, we provide the formulae for computing zero coupon bond prices for both default-free and defaultable bonds. In order to do so, we need to change the measure to the risk-neutral measure Q.

Assume that, under the risk-neutral measure Q, there exist some constants λ_1, λ_2, and λ_1^j, λ_2^j, $j = 1, ..., n$, such that $\widetilde{W}(t)$ and $\widetilde{W}_j(t)$, for $j = 1, ..., n$ are independent Q-Brownian motions, given by

$$\begin{aligned}
\widetilde{W}(t) &= W(t) + \int_0^t \left(\frac{\lambda_1}{c\sqrt{X(s)}} + \frac{\lambda_2\sqrt{X(s)}}{c} \right) ds,\\
\widetilde{W}_j(t) &= W_j(t) + \int_0^t \left(\frac{\lambda_1^j}{c_j\sqrt{X_j(s)}} + \frac{\lambda_2^j\sqrt{X_j(s)}}{c_j} \right) ds.
\end{aligned}$$

This implies that, under the risk-neutral measure Q, the dynamics of $X(t)$ and $X_j(t)$ can be written as

$$
\begin{aligned}
dX(t) &= [(a-\lambda_1)-(b+\lambda_2)X(t)]dt + c\sqrt{X(t)}d\widetilde{W}(t), &(3.20)\\
dX_j(t) &= [(a_j-\lambda_1^j)-(b_j+\lambda_2^j)X_j(t)]dt + c_j\sqrt{X_j(t)}d\widetilde{W}_j(t). &(3.21)
\end{aligned}
$$

In order to procure the equivalence of the two measures, the following conditions are required: $\lambda_1 < a - \frac{1}{2}c^2$, $\lambda_1^j < a_j - \frac{1}{2}c_j^2$, $\lambda_2 > -b$ and $\lambda_2^j > -b_j$. Our market price of risk specification precludes arbitrage opportunities, see Cheridito, Filipovic & Kimmel (2007) for a disscussion.

Using Ito's formula, the Q-dynamics for the iCIR process, $Y(t)$, can be written as

$$
dY(t) = [(b+\lambda_2)-(a-\lambda_1-c^2)Y(t)]Y(t)dt - cY(t)^{1.5}d\widetilde{W}(t). \qquad (3.22)
$$

In order for this to be well defined, the unattainable boundary conditions require that $\lambda_1 < a - c^2$.

Recall from Duffie & Singleton (1999), the price for a default-free zero coupon bond, $P(t,T)$, and the price for a defaultable zero coupon bond, $P_j(t,T)$, for firm j, can be calculated through

$$
\begin{aligned}
P(t,T) &= E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\right],\\
P_j(t,T) &= E_t^Q\left[\exp\left(-\int_t^T r(s)+h_j(s)ds\right)\right].
\end{aligned}
$$

The following propositions give explicit formulae for computing these bond prices.

Proposition 3.4.1. *Assume that the default-free interest rate $r(t)$ is specified as in equation (3.12). The risk-neutral dynamics for factor $X(t)$ is specified in equation (3.20), such that $a > c^2$, $b > 0$, $c > 0$, $\lambda_1 < a - c^2$ and $\lambda_2 > -b$. At time t, assume $X(t) = x_t = 1/r_t$. The time t value, $P(t,T)$, for a default-free T-bond is then given by*

$$
P(t,T) = G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t,T,x_t,0,1), \qquad (3.23)
$$

where the function $G^{CIR(a,b,c)}(t,T,x_t,l_1,l_2)$ *associated with* $X_t \sim CIR(a,b,c)$ *is given by*

$$G^{CIR(a,b,c)}(t,T,x_t,l_1,l_2) = E_t\left[\exp\left(-\int_t^T (l_1 X(s) + \frac{l_2}{X(s)})ds\right)\right]. \tag{3.24}$$

The explicit formula for the G function is given in the appendix. This formula is also derived in Ahn & Gao (1999) using the PDE approach.

Proposition 3.4.2. *Assume that the short spreads* $h_j(t)$ *for firm* j *are modeled as in equation (3.13). The risk-neutral dynamics of factors* $X(t)$, $X_j(t)$, *are specified in equations (3.20-3.21), such that* $a > c^2$, $b > 0$, $c > 0$, $a_j > 0$, $b_j > 0$, $c_j > 0$, $\lambda_1 < a - c^2$, $\lambda_2 > -b$, $\lambda_1^j < a_j - \frac{1}{2}c_j^2$ *and* $\lambda_2^j > -b_j$, $\alpha_j \geq 0$, $\beta_j \geq 0$, *for* $j = 1, ..., n$. *At time* t, *assume* $X(t) = x_t$ *and* $X_j(t) = x_t^j$. *The time* t *value,* $P_j(t,T)$, *for a defaultable* T*-bond of firm* j *is then given by*

$$\begin{aligned} P_j(t,T) &= G^{CIR(a-\lambda_1,b+\lambda_2,c)}(t,T,x_t,\alpha_j,1+\beta_j) \\ &\quad \times G^{CIR(a_j-\lambda_1^j,b_j+\lambda_2^j,c_j)}(t,T,x_t^j,1,0) \end{aligned} \tag{3.25}$$

Proposition 3.4.2 states that the defaultable bond prices can be evaluated as a product of two $G^{CIR(\cdot,\cdot,\cdot)}(\cdot,\cdot,\cdot,\cdot,\cdot)$ functions. The first one is a function of the common factor x_t and it does not depend on the idiosyncratic factor x_t^j. The second one is a function of the idiosyncratic factor x_t^j and it is independent of the common factor x_t. From the proof of Proposition 3.4.2 in the appendix, it is easy to see that the last $G^{CIR(\cdot,\cdot,\cdot)}(\cdot,\cdot,\cdot,\cdot,\cdot)$ function in equation (3.25) has a well known affine term structure in x_t^j,

$$G^{CIR(a_j-\lambda_1^j,b_j+\lambda_2^j,c_j)}(t,T,x_t^j,1,0) = A_j(t,T)e^{-B_j(t,T)x_t^j}, \tag{3.26}$$

where

$$
\begin{aligned}
A_j(t,T) &= \left(\frac{2\eta_j \exp\{(T-t)(b_j+\lambda_2^j+\eta_j)/2\}}{(b_j+\lambda_2^j+\eta_j)(\exp\{(T-t)\eta_j\}-1)+2\eta_j}\right)^{2a_j/c_j^2}, \\
B_j(t,T) &= \frac{2(\exp\{(T-t)\eta_j\}-1)}{(b_j+\lambda_2^j+\eta_j)(\exp\{(T-t)\eta_j\}-1)+2\eta_j}, \\
\eta_j &= \sqrt{(b_j+\lambda_2^j)^2+2c_j^2}.
\end{aligned}
$$

Proposition 3.4.2 provides an explicit formula for computing defaultable zero coupon bond prices. Since coupon bonds can be expressed as linear combinations of zero coupon bonds, we can also obtain an explicit formula for computing defaultable coupon bond prices. Calculating bond prices is reduced to computing $G^{CIR(\cdot,\cdot,\cdot)}(\cdot,\cdot,\cdot,\cdot,\cdot)$ functions, which involves confluent hypergeometric functions.

The yields to maturity for the default-free bond, $R(x_t,t,T)$, the yields to maturity for the defaultable bond, $R_j(x_t,x_t^j,t,T)$, and the credit spreads for the defaultable bond $CS_j(x_t,x_t^j,t,T)$, for $j=1,...,n$, are defined respectively as follows

$$
\begin{aligned}
R(x_t,t,T) &= -\frac{\log P(t,T)}{T-t}, \\
R_j(x_t,x_t^j,t,T) &= -\frac{\log P_j(t,T)}{T-t}, \\
CS_j(x_t,x_t^j,t,T) &= R_j(x_t,x_t^j,t,T) - R(x_t,t,T).
\end{aligned}
$$

Applying Propositions 3.4.1 and 3.4.2 and equation (3.26) we obtain

$$
\begin{aligned}
R(x_t,t,T) &= -\frac{1}{T-t}\left[\log G^{CIR(a-\lambda_1,b+\lambda_2,c)}(t,T,x_t,0,1)\right], &(3.27)\\
R_j(x_t,x_t^j,t,T) &= -\frac{1}{T-t}\left[\log G^{CIR(a-\lambda_1,b+\lambda_2,c)}(t,T,x_t,\alpha_j,\beta_j+1)\right. \\
&\quad \left. + \log A_j(t,T) - B_j(t,T)x_t^j\right], &(3.28)\\
CS_j(x_t,x_t^j,t,T) &= -\frac{1}{T-t}\left[\log G^{CIR(a-\lambda_1,b+\lambda_2,c)}(t,T,x_t,\alpha_j,\beta_j+1)\right. \\
&\quad - \log G^{CIR(a-\lambda_1,b+\lambda_2,c)}(t,T,x_t,0,1) \\
&\quad \left. + \log A_j(t,T) - B_j(t,T)x_t^j\right]. &(3.29)
\end{aligned}
$$

As we can see from the formula for function $G^{CIR(\cdot,\cdot,\cdot)}(\cdot,\cdot,\cdot,\cdot,\cdot)$, the yields to maturity, $R(x_t,t,T)$, for default-free bonds are non-affine in x_t. Similarly for defaultable bonds, the yields $R_j(x_t,x_t^j,t,T)$ are non-affine in x_t. However, both $R_j(x_t,x_t^j,t,T)$ and $CS_j(x_t,x_t^j,t,T)$ are affine in the idiosyncratic factors x_t^j, as indicated from the above equations. We say that this intensity-based model is *semi-affine*. This semi-affine property is very important when it comes to the calibration issue, which we will discuss in later sections.

3.5 Correlation Structure

In this section, we focus on the correlations between short spreads and interest rates, and the correlations among the short spreads of different firms. Both conditional (conditional on the initial state) and unconditional (stationary state) correlations are studied. Rich correlation structures are generated, while nonnegative interest rates and nonnegative short spreads are maintained.

3.5.1 Correlation of $r(t)$ and $h_j(t)$

Most empirical studies, such as Longstaff & Schwartz (1995), Duffee (1999) and Collin-Dufresne et al. (2001), find that the default-free interest rate $r(t)$ and the credit spread $h_j(t)$ are negatively correlated. Our model is flexible enough to allow both negative and positive correlations between $r(t)$ and $h_j(t)$. We first give a heuristic argument. Referring to equations (3.12-3.13), the correlations between $r(t)$ and $h_j(t)$ are solely determined by the coefficients α_j and β_j. Considering an extreme case when $\alpha_j = 0$ and $\beta_j > 0$, then the correlation between $r(t)$ and $h_j(t)$ is clearly positive. On the other hand, when $\beta_j = 0$ and $\alpha_j > 0$, $r(t)$ will be negatively correlated with $h_j(t)$, since $X(t)$ and $Y(t)$ are negatively correlated.

Next, let us give a quantitative analysis of the correlation structure. The covariance of $r(t)$ and $h_j(t)$ can be directly calculated through (conditional on time zero)

$$Cov[r(t), h_j(t)] = \alpha_j(1 - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]) + \beta_j Var[Y(t)], \tag{3.30}$$

where $\mathbb{E}[X(t)]$, $\mathbb{E}[Y(t)]$ and $Var[Y(t)]$ are given in equations (3.2-3.5) and equations (3.8-3.11). As time $t \to +\infty$, the covariance $Cov[r(t), h_j(t)]$ will converge asymptotically to the unconditional covariance

$$Cov[r, h_j] = \alpha_j(1 - \mathbb{E}[X]\mathbb{E}[Y]) + \beta_j Var[Y], \tag{3.31}$$

where $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $Var[Y]$ are given in equations (3.2-3.5) and equations (3.8-3.11). This can be simplified into

$$Cov[r, h_j] = \frac{c^2}{2a - c^2}\left(D_2\beta_j - \alpha_j\right). \tag{3.32}$$

where

$$D_2 = \frac{2b^2}{(a - c^2)(2a - c^2)}. \tag{3.33}$$

Recall that $a > c^2$, therefore, both $2a - c^2$ and $a - c^2$ are positive, which implies that $D_2 > 0$. The sign of the asymptotic correlation of r and h_j is hence determined by the sign of $D_2\beta_j - \alpha_j$. If $\alpha_j = \beta_j D_2$, then $Cov[r, h_j] = 0$; if $\alpha_j > \beta_j D_2$, then $Cov[r, h_j] < 0$; if $\alpha_j < \beta_j D_2$, then $Cov[r, h_j] > 0$. Since empirical findings suggest that the interest rate and short spreads are mostly negatively correlated, we expect to have most of the cases that $\alpha_j > \beta_j D_2$.

The correlation coefficient of $r(t)$ and $h_j(t)$, $\rho_j(t)$, and its asymptotic limit ρ_j, can thus be calculated by

$$\rho_j(t) := \frac{Cov[r(t), h_j(t)]}{\sqrt{Var[Y(t)]Var[h_j(t)]}}, \tag{3.34}$$

$$\rho_j := \frac{Cov[r, h_j]}{\sqrt{Var[Y]Var[h_j]}}, \tag{3.35}$$

where $Cov[r(t), h_j(t)]$ and $Cov[r, h_j]$ are given in equations (3.30) and (3.31) respectively.

3.5.2 Correlation of $h_i(t)$ and $h_j(t)$

The short spreads $h_i(t)$ and $h_j(t)$, for different firms i and j, can be positively or negatively correlated as well. As in the previous section, a similar heuristic argument can be given for the correlation between $h_i(t)$ and $h_j(t)$. Recall the following equations, for $i \neq j$,

$$h_i(t) = \alpha_i X(t) + \beta_i Y(t) + X_i(t),$$
$$h_j(t) = \alpha_j X(t) + \beta_j Y(t) + X_j(t).$$

The correlation between $h_i(t)$ and $h_j(t)$ is solely determined by α_i, α_j, β_i and β_j. Consider an extreme case when $\alpha_i = \alpha_j = 0$ and $\beta_i > 0$, $\beta_j > 0$ (or $\alpha_i > 0$, $\alpha_j > 0$ and $\beta_i = \beta_j = 0$), the correlation between $h_i(t)$ and $h_j(t)$ will be positive. On the other hand, when $\alpha_i = \beta_j = 0$ and $\beta_i > 0$, $\alpha_j > 0$ (or $\alpha_j = \beta_i = 0$ and $\beta_j > 0, \alpha_i > 0$), $h_i(t)$ will be negatively correlated with $h_j(t)$, since $X(t)$ and $Y(t)$ are negatively correlated.

The covariance of $h_i(t)$ and $h_j(t)$, $Cov[h_i(t), h_j(t)]$ can be expressed as (conditional on time zero)

$$\begin{aligned} Cov[h_i(t), h_j(t)] &= \alpha_i\alpha_j Var[X(t)] + \beta_i\beta_j Var[Y(t)] \\ &\quad +(\alpha_i\beta_j + \alpha_j\beta_i)(1 - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]), \end{aligned} \tag{3.36}$$

where $Var[X(t)]$, $Var[Y(t)]$, $\mathbb{E}[X(t)]$ and $\mathbb{E}[Y(t)]$ are given in previous sections. As $t \to +\infty$, the above equation tends to the unconditional covariance

$$Cov[h_i, h_j] = \alpha_i\alpha_j Var[X] + \beta_i\beta_j Var[Y] + (\alpha_i\beta_j + \alpha_j\beta_i)(1 - \mathbb{E}[X]\mathbb{E}[Y]). \tag{3.37}$$

which can also be written as

$$\begin{aligned} Cov[h_i, h_j] &= \alpha_i Cov[\frac{1}{r}, h_j] + \beta_i Cov[r, h_j], \\ &= \alpha_j Cov[\frac{1}{r}, h_i] + \beta_j Cov[r, h_i]. \end{aligned}$$

This expression will be very useful for calibration of α_j and β_j, if we believe that the observed time series of $r(t)$, $h_i(t)$ and $h_j(t)$ are stationary. Some simple algebra will lead to an explicit expression of $Cov[h_i, h_j]$ as follows

$$Cov[h_i, h_j] = \frac{c^2}{2a - c^2}\left[\alpha_i\left(\frac{\alpha_j}{D_1} - \beta_j\right) + \beta_i\left(D_2\beta_j - \alpha_j\right)\right], \tag{3.38}$$

where D_2 is given in equation (3.33) and D_1 is defined as

$$D_1 = \frac{2b^2}{a(2a - c^2)}. \tag{3.39}$$

It is easy to see that $D_2 > D_1 > 0$, given that $a > c^2$.

The correlation coefficient of $h_i(t)$ and $h_j(t)$, $\rho_{ij}(t)$, and its asymptotic limit, ρ_{ij} can thus be calculated by

$$\rho_{ij}(t) := \frac{Cov[h_i(t), h_j(t)]}{\sqrt{Var[h_i(t)]Var[h_j(t)]}}, \tag{3.40}$$

$$\rho_{ij} := \frac{Cov[h_i, h_j]}{\sqrt{Var[h_i]Var[h_j]}}, \tag{3.41}$$

where $Cov[h_i(t), h_j(t)]$ and $Cov[h_i, h_j]$ are given in equations (3.36) and (3.38) respectively, and $Var[h_j(t)]$ and $Var[h_j]$, for $j = 1, ..., n$, are given by

$$\begin{aligned} Var[h_j(t)] &= \alpha_j^2 Var[X(t)] + \beta_j^2 Var[Y(t)] + 2\alpha_j\beta_j(1 - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]) + Var[X_j(t)], \\ Var[h_j] &= \alpha_j^2 Var[X] + \beta_j^2 Var[Y] + 2\alpha_j\beta_j(1 - \mathbb{E}[X]\mathbb{E}[Y]) + Var[X_j]. \end{aligned}$$

All of these formulae can be written in explicit forms.

The correlations we have discussed so far are under the physical measure. Similar arguments will follow if we consider the correlations under the risk-neutral measure. The formulae for $Cov[r(t), h_j(t)]$ and $Cov[h_i(t), h_j(t)]$, under the risk-neutral measure, will be the same as in equations (3.30) and (3.36) respectively, with a replaced by $a - \lambda_1$ and b replaced by $b + \lambda_2$. The formulae for $\rho_j(t)$ and $\rho_{ij}(t)$, under the risk-neutral measure will be the same as in equations (3.34) and (3.40), with a, b, a_j and b_j replaced by $a - \lambda_1$, $b + \lambda_2$, $a_j - \lambda_1^j$ and $b_j + \lambda_2^j$ respectively.

3.6 Numerical Illustration

Figure 3.1 plots the time series of U.S. 5-year treasury yields versus the 5-year Financial Sector CDS spreads of AA-rated, which are daily market data taken from Bloomberg from the same period: March 4, 2002 to August 30, 2006. It can be seen from the graph that the 5-year treasury yield went down first from March 2002 and then went up from October 2003 and continued the upward trend to August 2006. The 5-year CDS spread behaved in an inverse pattern: it went up first and then went down. The statistical correlation coefficient in Figure 3.1 is calculated as if the two time series have reached a stationary state. The calculated number -0.5741 only serves as an estimate from the two time series. The accuracy of the estimate may be questioned, but what is more important is a strong negative correlation is presented. [2]

a	b	c	x_0	a_1	b_1	c_1	x_0^1	α_1	β_1	λ_1	λ_2	λ_1^1	λ_2^1
3.09	0.13	1.3	18	0.006	3	0.002	0.001	0.0001	0.002	0	0	0	0

Table 3.1: Base Case Parameters for Simulation.

Figure 3.2 shows simulated sample paths of the interest rate and the short spread processes using parameters given in table 3.1.[3] It is clear from the graph that the simulated interest rate process is negatively correlated with the short spread process, hence mimicking the real observed time series shown in Figure 3.1. This is not surprising, because for the base case parameters, we can easily deduce that $D_2\beta_1 = 1.0754 \times 10^{-5} < \alpha_1 = 0.0001$. To quantify this negative correlation, Figure 3.3 plots the correlation coefficient of $r(t)$ and $h_1(t)$ from equations (3.34) and (3.35). We see that the strength

[2]Treasury yield time series are available for maturities 1/12, 3/12, 6/12 1, 2, 3, 5, 7, 10, and 20-year. Financial AA CDS time series are available for maturities 1, 3, 5 and 10-year. These time series show a similar fashion as in Figure 3.1. The reason we choose 5-year data for illustration is because the the 5-year CDS has best liquidity.

[3]As recommended by Alfonsi (2005), the explicit discretization scheme was used to simulate CIR processes.

of the negative correlation diminishes gradually to the asymptotic correlation as time progresses.

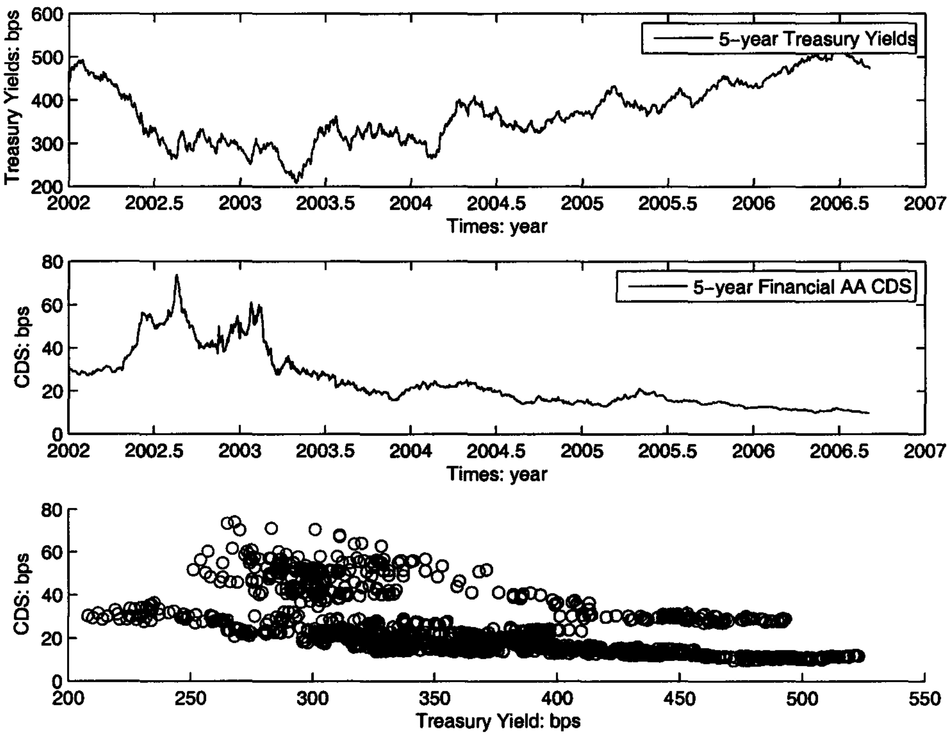


Figure 3.1: Time series of 5-year Treasury yield vs. time series of 5-year CDS spreads (daily) of Financial Sector, AA-rated. Dates: March 4, 2002 to August 30, 2006. Resources: Bloomberg. The statistical correlation coeffcient between the two time series is -0.5741.

Figure 3.4 shows the term structure of treasury yields, corporate yields and credit spreads with base case parameters. The term structure of treasury yields and corporate yields are hump-shaped for the base case parameters and the term structure of credit spreads is upward sloping. However, various shapes of these term structures could be generated using different parameters, including flat, humped, upward and downward.

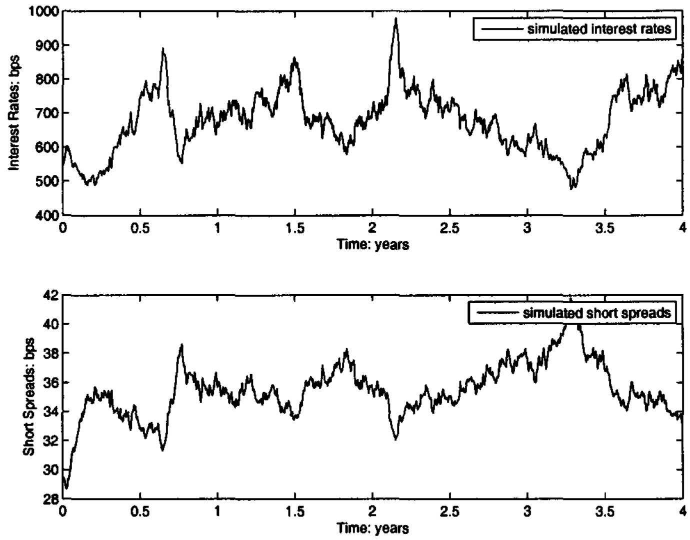


Figure 3.2: Simulated Sample Paths of the Interest Rate and the Short Spread Processes with Base Case Parameters.

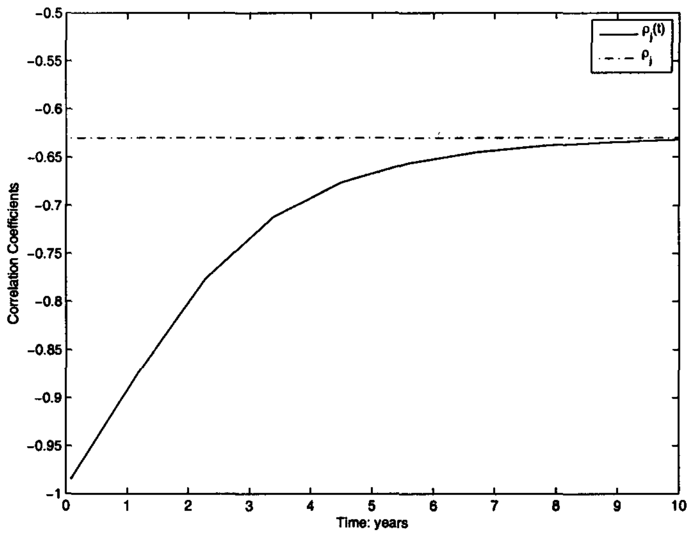


Figure 3.3: Correlation Coefficients of Interest Rates and Short Spreads with Base Case Parameters.

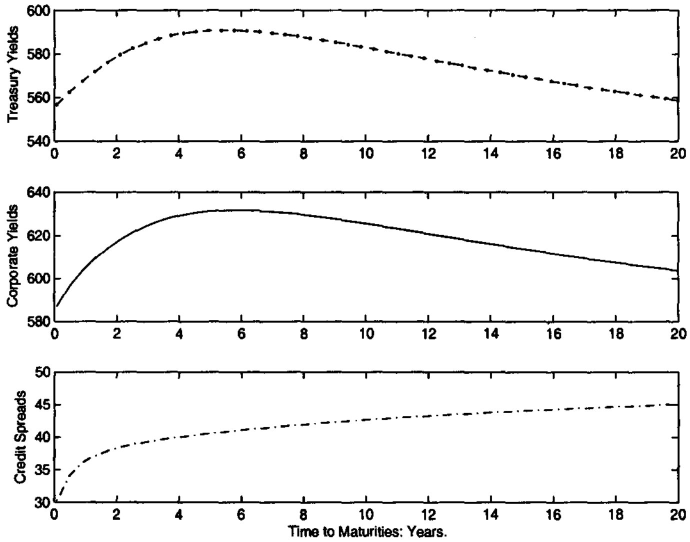


Figure 3.4: Simulated Treasury Yields Curve, Corporate Yields Curve and Credit Spreads Curve (at time 0) with Base Case Parameters.

3.7 Calibration Issues

Despite the elegant mathematics of our model, it is unfortunately the case that the calibration has proved to be difficult. In this section, we outline our proposed method and give possible reasons why we were unable to complete it.

We propose a two-step calibation of our model by using the Extended Kalman Filter (EKF) in conjunction with the Quasi-Maximum Likelihood Estimation (QMLE).[4] The first step is to calibrate the iCIR interest rate process to observed Treasury yield time series with different maturities. The *measurement equation* of the EKF is given by Equation 3.27. Since the Treasury yield is non-affine in the iCIR-factor, linearization of the measurement equation is needed for the EKF scheme. The *transition equation* for the EKF is obtained by discretizing the iCIR SDE given by Equation 3.7. Assume

[4]The readers are directed to Duffee (1999), Chen & Scott (2003), Duan & Simonato (1999) and Yi (2005) for more information on EKF and QMLE.

that we have backed out the parameters in the iCIR interest rate model and have filtered out the time series of the interest rate in the first step. Therefore, parameters a, b, c, λ_1 and λ_2, and the time series of the common factor $X(t)$ are known in step two.

In the second step, for each company j, the short spread process can be calibrated by using the Kalman Filter (KF) combined with QMLE. The measurement equation of the KF is given by Equation (3.28) or (3.29). The transition equation is obtained by discretizing the CIR SDE given by Equation 3.14. As we have discussed, both the corporate yields $R_j(x_t, x_t^j, t, T)$ and the credit spreads $CS_j(x_t, x_t^j, t, T)$ are affine in the idiosyncratic factor $X_j(t)$. Hence, the standard KF can be used. The parameters a_j, b_j, c_j, λ_1^j, λ_2^j, α_j and β_j are obtained from step two. The time series of the idiosyncratic factor $X_j(t)$ are filtered out from this step as well.

A number of difficulties were encountered when we empirically implemented the EKF and QMLE scheme for the first step. These difficulties possibly come from the following reasons. First, since the linearization of the measurement equation is a first order Taylor approximation of a very complicated function, the accuracy of the approximation should be questioned. Second, the innovations in the transition equation are non-Gaussian, where the standard KF requires Gaussian innovations. This condition is violated for both CIR and iCIR models. Duan & Simonato (1999) claimed that QMLE is still a consistent way to estimate parameters for the CIR interest rate model. However, we are not clear if this is the case for the iCIR interest rate model.

Another major obstacle is computing the hypergeometric function for different parameters for hundreds of times (in both steps). Although this function is a Matlab built-in function, that algorithm is not able to calculate for certain input parameters. A better numerical approximation of the hypergeometric function may be needed, which is beyond the scope of this paper.

3.8 Extension of the Model

We have considered a model where $r(t)$ is assumed to be a one factor iCIR process and the short spread $h_j(t)$ is assumed to be a two factor model. However, this model is not restrictive and it can be easily generalized to a model which has three or more factors.

The interest rate and the short spreads could be modeled as multi-CIR-factor, or multi-iCIR-factor, or even linear combinations of CIR and iCIR factors. Negative correlations between the interest rate and the short spread are generated through the common CIR factor (factors) and its (their) inverse(s).

It can be proved that, if $Cov[r, h_i] < 0$ and $Cov[r, h_j] < 0$, then $Cov[h_i, h_j] > 0$ for the semi-affine intensity-based model we have discussed. However, in reality, the short spreads of two firms are not necessarily positively correlated given that they are both negatively correlated with the interest rate. To the author's best knowledge, the extant factor models in the literature are not able to incorporate this feature without violating the nonnegativity property. However, our model can easily be adjusted to do so by adding one additional CIR factor to the short spread as follows:

$$\begin{aligned} r(t) &= \frac{1}{X(t)}, \\ h_i(t) &= \alpha_i X(t) + \beta_i \frac{1}{X(t)} + p_i H(t) + q_i \frac{1}{H(t)} + X_i(t), \end{aligned}$$

where p_i, q_i are nonnegative constants and $H(t)$ is an independent $CIR(a_h, b_h, 1)$ with $a_h > 1$ and $b_h > 0$. The factor $H(t)$ is thus shared by all the firms, but not by the interest rate. The correlations between $h_i(t)$ and $h_j(t)$ are thus determined not only by α_i, α_j, β_i and β_j, but also p_i, p_j, q_i and q_j. To see that $Cov[r, h_i] < 0$, $Cov[r, h_j] < 0$ and $Cov[h_i, h_j] < 0$ can coexist for the model specified above, let us consider the following example. Consider two firms i and j, assume that $\beta_i = \beta_j = p_i = q_j = 0$ and

$p_j > 0$, $q_i > 0$. Therefore

$$\begin{aligned} h_i(t) &= \alpha_i X(t) + q_i \frac{1}{H(t)} + X_i(t), \\ h_j(t) &= \alpha_j X(t) + p_j H(t) + X_j(t). \end{aligned}$$

It is obvious that both $h_i(t)$ and $h_j(t)$ are negatively correlated with the interest rate $r(t)$. In this case, the covariance of $h_i(t)$ and $h_j(t)$, $Cov[h_i(t), h_j(t)]$, can be written as

$$Cov[h_i(t), h_j(t)] = \alpha_i \alpha_j Var[X(t)] + q_i p_j Cov[\frac{1}{H(t)}, H(t)].$$

Since $Cov[\frac{1}{H(t)}, H(t)] < 0$, negative $Cov[h_i(t), h_j(t)]$ can be obtained by choosing sufficiently large product of q_i and p_j.

The present work of this paper could thus be regarded more as a framework of multi-factor models which can generate rich correlation structures and maintain nonnegativity rather than a simply the illustrative model we have discussed in previous sections.

3.9 Summary

In this chapter, we have introduced the iCIR process to the literature of intensity-based factor models on credit risk. In this framework, the short spread for each firm not only depends on the common factor, but also depends on the inverse of the common factor. By doing this, we added much flexibility of correlation structures among the interest rates and the short spreads, and among the short spreads of different firms, while both the interest rates and the short spreads remain nonnegative.

The pricing formulae of both the default-free and defaultable bonds were derived in explicit forms by utilizing the recent findings of stochastic integrals by Hurd & Kuznetsov (2006). The term structure of the credit spreads are non-affine in the common factor, but are affine in the idiosyncratic factor. This allows us to use the KF and QMLE to conduct the calibration of the short spread process, once the interest rate process is fitted. The affine intensity-based modeling is thus extended to a more general semi-affine family.

Numerical illustrations showed that our model is able to generate various shapes of the term structure of credit spreads which are consistent with the empirical findings. In addition, we were able to generate negatively correlated time series of interest rates and short spreads, without violating the nonnegativity property, mimicking the real observations as seen in Figure 3.1.

3.10 Appendix II

Lemma 3.10.1. *(Hurd & Kuznetsov (2006)) Let $X(t)$ be a CIR process given by equation (3.1). Let CIR(a,b,c) denote this process with parameters a, b, c. Assume that $X(t) = x_t$ and it has unattainable boundaries. Define function $G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2)$ associated with this CIR(a,b,c) as follows*

$$G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2) = E_t\left[\exp\left(-\int_t^T (l_1 X(s) + \frac{l_2}{X(s)})ds\right)\right],$$

where l_1, l_2 are constants. Then, the function $G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2)$ admits explicit expression given by

$$\begin{aligned} G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2) &= \left(1 + \frac{v_1}{\gamma(t,T)}\right)^{-\frac{2a}{c^2}} \exp\left\{v_1\left[x_t + a(T-t) - \frac{y(x_t,t,T)}{\gamma(t,T)}\right]\right\} \\ &\quad \times \frac{\Gamma(v_3 - v_2)}{\Gamma(v_3)} M\left(v_2, v_3, -y(x_t,t,T)\right) y(x_t,t,T)^{v_2} \end{aligned}$$

where

$$\begin{aligned}
v_1 &= \frac{b-\sqrt{b^2+2l_1c^2}}{c^2}, \\
v_2 &= \frac{\sqrt{(2a-c^2)^2+8l_2c^2}-(2a-c^2)}{2c^2}, \\
v_3 &= \frac{\sqrt{(2a-c^2)^2+8l_2c^2}+c^2}{c^2}, \\
\gamma(t,T) &= \frac{2\sqrt{b^2+2l_1c^2}}{c^2(1-\exp\{-(T-t)\sqrt{b^2+2l_1c^2}\})}, \\
y(x_t,t,T) &= \frac{x_t\gamma(t,T)^2\exp\{-(T-t)\sqrt{b^2+2l_1c^2}\}}{v_1+\gamma(t,T)}.
\end{aligned}$$

Please refer to Hurd & Kuznetsov (2006) for the proof.

- **Proof of Proposition 3.4.1:** From equation (3.23), the default-free zero coupon bond price, $P(t,T)$, can also be written as

$$\begin{aligned}
P(t,T) &= \mathbb{E}_t^Q\left[\exp\left(-\int_t^T \frac{1}{X(s)}ds\right)\right] \\
&= G^{CIR(a-\lambda_1,b+\lambda_2,c)}(t,T,x_t,0,1)
\end{aligned}$$

  where the Q-dynamics of $X(t)$ is given by equation (3.20). The second step comes from invoking the above lemma.

- **Proof of Proposition 3.4.2:** According to equation (3.23), applying equations (3.12) and (3.13), the defaultable zero coupon bond price, $P_j(t,T)$, can be calculated as

$$\begin{aligned}
P_j(t,T) &= \mathbb{E}_t^Q\left[\exp\left(-\int_t^T \left(\frac{1}{X(s)}+\alpha_jX(s)+\beta_j\frac{1}{X(s)}+X_j(s)\right)ds\right)\right] \\
&= \mathbb{E}_t^Q\left[\exp\left(-\int_t^T \left(\alpha_jX(s)+\frac{1+\beta_j}{X(s)}\right)ds\right)\right]\mathbb{E}_t^Q\left[\exp\left(-\int_t^T X_j(s)ds\right)\right] \\
&= G^{CIR(a-\lambda_1,b+\lambda_2,c)}(t,T,x_t,\alpha_j,1+\beta_j) \\
&\quad \times G^{CIR(a_j-\lambda_1^j,b_j+\lambda_2^j,c_j)}(t,T,x_t^j,1,0).
\end{aligned}$$

  For the second step, we utilized the fact that $X_j(t)$ are independent of $X(t)$, for $j=1,...,n$. The last step follows by invoking the above lemma.

Chapter 4

Hybrid Credit Risk Models

The models discussed in previous chapters are designed to manage or price credit derivatives only. In this chapter, we propose a new jump-to-default model, which can price both credit and equity derivatives in a unified framework. In Section 4.1, we give an introduction of related literature and our motivation. In Section 4.2, we derive some functional forms of equity volatility from the structural credit risk modeling literature. Our new jump-to-default model is then described in Section 4.3. In Section 4.4, we discuss pricing of both credit and equity derivatives. Section 4.5 studies the Dual-Jacobi process, which is related to the pre-default stock price. In Section 4.6, the Gram-Charlier expansion is used to approximate call and bond prices. Some numerical analyses are provided in Section 4.7. In Section 4.8, we discuss an extension of our model. Section 4.9 summarizes this chapter. All proofs are given in the appendix of this chapter.

4.1 Introduction

Classical structural models of credit risk start from modeling the dynamics of firm asset value as a discounted martingale under the risk-neutral measure, such as Merton (1974), Black & Cox (1976) and Leland (1994). The equity and defaultable bonds are then priced under this discounted martingale assumption on the asset value. In reality, common stocks rather than the firm's assets are publicly traded and hence should be considered to be a discounted martingale under the risk-neutral measure. This alternative approach, recently taken by Linetsky (2006) and Carr & Linetsky (2006), is to directly model the stock price as a risk-neutral discounted martingale. Credit risk is incorporated in this equity modeling approach by assuming that the stock price S_t at time t can jump to zero with an intensity $h(S_t)$, which is assumed to be a function of S_t.

Linetsky (2006) considered a constant volatility model with the hazard rate chosen to be a negative power function of the stock price. The pre-default stock price S_t under the risk-neutral measure is thus assumed to follow:[1]

$$\begin{aligned} dS_t &= (r + h(S_t))S_t dt + \sigma S_t dW_t, && (4.1) \\ h(S_t) &= \alpha S_t^{-p}, && (4.2) \end{aligned}$$

with initial value S_0 at time $t = 0$, where W_t denotes a standard BM. The constant r denotes the default-free interest rate, and α, p and σ are positive constants.

Let Δ denote the bankruptcy state when the firm defaults at time τ. Then we can also write the dynamics for the stock price subject to bankruptcy S_t^Δ as follows:

$$dS_t^\Delta = S_{t-}^\Delta [rdt + \sigma dW_t - dM_t],$$

[1] Linetsky (2006) and Carr & Linetsky (2006) also assumed a dividend yield q, which we set to zero here for simplicity.

where

$$M_t = 1_{\{\tau \le t\}} - \int_0^{t \wedge \tau} h(S_u)du.$$

is a martingale.

Note that the hazard rate function goes to infinity as the stock price approaches zero. Unbounded intensity kills the pre-default process almost surely by a jump-to-default instead of diffusing down to zero. The valuations of corporate liabilities and equity derivatives are then reduced to evaluation of the following risk-neutral expectations

$$F_\psi(S_0, T) := \mathbb{E}[e^{-\int_0^T r + h(S_t)dt}\psi(S_T)], \tag{4.3}$$

where ψ is any payoff function. Proposition 2.1 in Linetsky (2006) states that

$$F_\psi(S_0, T) = S_0\hat{\mathbb{E}}[S_T^{-1}\psi(S_T)], \tag{4.4}$$

where $\hat{\mathbb{E}}$ is the expectation with respect to a new probability measure $\hat{P}$ under which $\hat{W}_t := W_t - \sigma t$ is a standard BM. Under the new measure $\hat{P}$, the dynamics of the stock prices S_t has the same form as in equation (4.1), with r replaced by $r + \sigma^2$. The SDE of the stock price has an explicit transition density representation solved by a change of variables $Z := S^p$. This closed form transition density is closely related to the Bessel process as discussed in Appendix B of Linetsky (2006). The pricing formulae for defaultable bonds, European puts, and European calls can thus be derived explicitly based on the known distribution of S_t under the $\hat{P}$ measure. The firm's asset value is not of interest and is therefore not discussed in this paper.

Carr & Linetsky (2006) extended Linetsky's model by combining a jump-to-default model with CEV models. The pre-default stock price is assumed to have the following dynamics

$$dS_t = (r + h(S_t))S_t dt + \sigma(S_t)S_t dW_t, \tag{4.5}$$

$$\sigma(S_t) = aS_t^{-\beta}, \tag{4.6}$$

$$h(S_t) = b + c\sigma^2(S_t), \tag{4.7}$$

where a, b, c and β are nonnegative constants. It was found by Carr & Linetsky (2006) that the process specified in equation (4.5) can be represented as a re-scaled and time-changed power of a Bessel process. The valuations of corporate liabilities, credit quality derivatives, and equity derivatives are then reduced to calculating expectations of a known function of a standard Bessel process evaluated at a changed time.

The choices of functions $h(S_t)$ and $\sigma(S_t)$ specified in equations (4.6) and (4.7) capture a positive relationship between the default intensity and the equity volatility, and the leverage effect between volatility and stock prices. A good property of their choice is that analytical tractability of the formulae is obtained. Theoretically speaking, one can generalize Carr & Linetsky (2006) by assuming different functional forms of $h(S_t)$ and $\sigma(S_t)$. However, not all functional forms of $h(S_t)$ and $\sigma(S_t)$ will be economical appealing and mathematical attractive. A disadvantage of the CEV models is that the equity volatility will vanish to zero as the stock price approaches infinity. As a result, CEV models may not be appropriate to describe the stock dynamics of companies with high equity but low debt (i.e. low financial leverage). This fact motivates us to search for better alternatives.

4.2 Implied Equity Volatility from Structual Credit Literature

We now explore some implied functional forms of $\sigma(S_t)$ found in the structural credit literature. Particularly, we are interested in the form implied by Leland (1994) and Leland & Toft (1996). Structural modeling approach on credit risk goes back to Merton (1974). Before we move on to Leland (1994), let us first look at Merton's implied equity volatility.

4.2.1 Merton (1974)

Merton (1974) assumes that the firm's asset value V follows a GBM under the risk-neutral measure with interest rate r as its drift and volatility σ. Under Merton's assumption, the equity of the firm E can be evaluated as a call option on the asset value V, with the debt value K as the strike price and the debt maturity T as the maturity of the call option. The equity value is hence given by the celebrated Black-Scholes formula

$$E_t = V_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \tag{4.8}$$

where $\Phi(\cdot)$ stands for the cumulative distribution function (cdf) of a standard normal random variable and $d_\pm$ are given by

$$d_\pm = \frac{\log\frac{V}{K} + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The Black-Scholes-Merton equation can also be written in terms of asset, equity and debt as follows

$$E = V\frac{\partial E}{\partial V} + K\frac{\partial E}{\partial K}.$$

If we assume that equity follows the dynamics given by the following equation

$$\frac{dE}{E} = \mu_E dt + \sigma_E dW_t,$$

then the equity drift μ_E and the equity volatility σ_E can be recovered by applying Ito's lemma to equation (4.8). It is not surprising that μ_E turns out to be the default-free interest rate r. The equity volatility σ_E satisfies the following equation,

$$\sigma_E = \sigma E^{-1} V\Phi(d_+), \tag{4.9}$$

which is also derived in Hull, Nelken & White (2004). The equity volatility σ_E can thus be regarded as a function of E implicitly solved from equation (4.8) and (4.9). As seen in Figure 4.1, these two equations imply that the equity volatility σ_E goes to infinity as the equity E approaches zero, and tends to the asset volatility σ as E goes to infinity. Figure 4.1 demonstrates that equity volatility σ_E is bounded below by the asset volatility σ. The implied equity volatility is found to be a decreasing function of maturity T. This implied local volatility should not be confused with Dupire's implied local volatility. Dupire (1994) derived the famous implied local volatility equation from arbitrage arguments on pricing options on the stock without considering default risk. The local volatility obtained here is derived from Merton (1974) credit structural model, without considering pricing options on the equity.

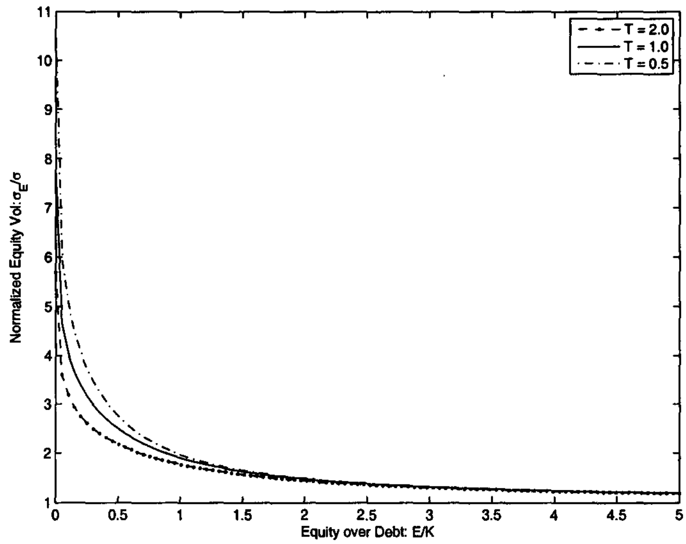


Figure 4.1: Local equity volatility implied from Merton (1974) as a function of equity. The x-axis denotes the equity-debt ratio, E/K. The y-axis denotes the normalized equity volatility, σ_E/σ.

4.2.2 Leland (1994) and Leland-Toft (1996)

This section shows that the implied equity local volatility from Leland (1994) behaves like a power decay function of equity, while bounded below by the asset volatility.

Leland (1994) introduced the concept of endogenous bankruptcy by maximizing the equity value of the firm, whose debt promises a perpetual coupon payment C. As in Merton (1974), he started with the assumption that the asset value of the firm V is a GBM with drift of interest rate r and volatility σ under the risk-neutral measure. Tax benefits of debt financing allow the firm to deduct a fraction τ of the coupon payments as long as it is solvent. The firm will suffer from a bankruptcy cost αV_B when its asset value first hits the bankruptcy level V_B. The value of the equity, E, is then derived by the formula that the equity equals the asset value V plus tax benefits minus bankruptcy cost and minus the debt.[2] This yields equation (13) in Leland (1994), which is

$$E(V) = V - b + (b - V_B)(V/V_B)^{-X}, \tag{4.10}$$

where $X = 2r/\sigma^2$, $b = (1-\tau)C/r$. The optimal bankruptcy level V_B is then determined endogenously by solving the smooth-pasting condition: $dE(V)/dV\ |_{V=V_B} = 0$, which gives

$$V_B = \frac{(1-\tau)C}{r + 0.5\sigma^2}. \tag{4.11}$$

The equity $E(V)$ as a function of the asset value is thus obtained explicitly by plugging the optimal V_B into equation (4.10), which yields

$$E(V) = V + aV^{-X} - b, \tag{4.12}$$

where a is given by

$$a = \frac{b^{X+1}X^X}{(X+1)^{X+1}}. \tag{4.13}$$

However, Leland (1994) did not address the dynamics of the equity E and left the unanswered question of whether the firm's asset is a traded asset.[3] We answer this question here by analysing the stochastic dynamics of the equity. Applying Ito's lemma

[2] We will assume constant total number of shares, N. The equity E equals total number of shares times the stock prices, i.e. $E = NS$.

[3] See footnote 11 on page 1217 of Leland (1994)

to $E(V)$, we obtain the following SDE for E

$$\frac{dE}{E} = \mu_E dt + \sigma_E dW_t,$$

where μ_E and σ_E denote the drift and volatility respectively, which solve the following equations

$$\begin{aligned} \mu_E &= r(1+\frac{b}{E}), && (4.14) \\ \hat{E}+1 &= \hat{E}\hat{\sigma}_E + (\hat{E}\hat{\sigma}_E/X + \hat{E}+1)^{-X}, && (4.15) \end{aligned}$$

where $\hat{E}$ and $\hat{\sigma}_E$ are normalized equity and normalized volatility of the equity respectively, which are defined as $\hat{E} = E/b$, $\hat{\sigma}_E := \sigma_E/\sigma$. Note that the drift μ_E is always bigger than the interest rate r unless the firm has full tax benefits, namely $\tau = 1$. It turns out that $E + b$ is a discounted martingale under the risk-neutral measure, but E is not. This contradicts the fact that stocks are commonly traded in the market. This non-martingale property mainly comes from the assumption of the constant perpetual debt services. Similarly, we can show that the equity is not a discounted martingale in Leland & Toft (1996), where they assume stationary debt structure with finite maturity.

Nevertheless, we are interested in the implied equity volatility σ_E as a function of equity itself implicitly solved from eqution (4.15). Particularly, when $X = 1$ (same as $r = 0.5\sigma^2$), we can obtain an explicit formula for σ_E given by

$$\sigma_E = \sigma\sqrt{1+\frac{2b}{E}}. \tag{4.16}$$

The above local volatility function has very broad financial implications. First, the equity volatility is always bigger than the asset volatility, which echoes the arguments in Merton (1974) that the equity of a levered firm must be at least as risky as the firm's asset. Second, the equity volatility is a decreasing function of the equity value, which is known as leverage effect discussed in Black (1976). More precisely, the equity volatility is a decreasing function of the equity/debt ratio, considering that b denotes the total debt service deducted by tax benefits if there is no default. We notice that the equity volatility specified in equation (4.16) is a power decay function of the equity value, which coincides with the power law of CEV models. For CEV models, the local volatility is specified as $\alpha E^{-\beta}$, where α and β are positive numbers. The local volatility of CEV models and the implied local volatility in Leland (1994) both go to infinity when the equity tends to zero. For small E, the equity volatility specified in equation (4.16) is $O(E^{-0.5})$. Finally, when equity E tends to infinity, the equity volatility σ_E converges to the asset volatility σ. As a consequence, the equity will behave much like GBM for large E. This is different from CEV models, because CEV local volatility will vanish to zero when equity goes to infinity. The above properties are not only true for the special case of σ_E, but are also true for all the σ_E implicitly solved from equation (4.15) as seen from numerical illustration. Figure 4.2 plots $\hat{\sigma}_E$ as a function of $\hat{E}$ for varying X. The graph demontrates that the equity volatility is an increasing function of X, when holding other parameters constant. This implies that stocks are more volatile in an economy with low interest rate than in an economy with high interest rate.

We have derived implicit equity local volatility function implied by Leland (1994) in this section. This local volatility is found to be bounded below by a positive constant which differs from the specification in Carr & Linetsky (2006).

4.3 A New Jump to Default Model

Motivated by Carr & Linetsky (2006), we have derived some implied functional forms of equity volatility from structural credit literature. In CEV models, when the equity is large, the equity volatility approaches zero. The CEV specification of the equity volatility is consequently inappropriate for many existing firms with large equity and

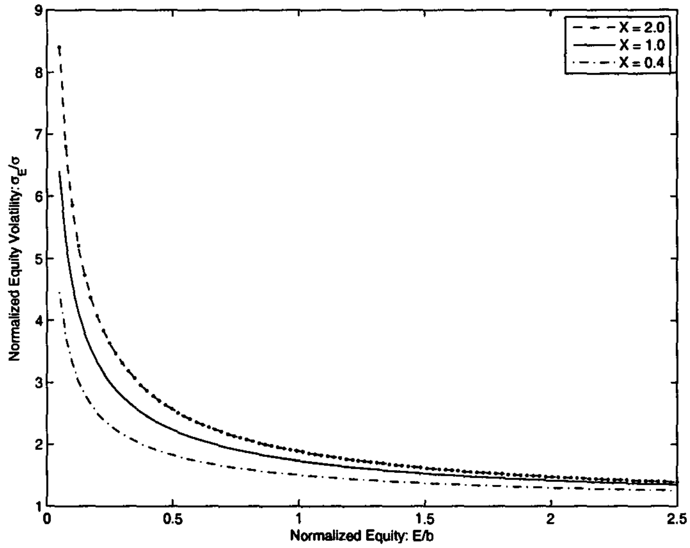


Figure 4.2: Local equity volatility implied from Leland (1994) as a function of equity. The x-axis denotes the normalized equity, E/b. The y-axis denotes the normalized equity volatility, σ_E/σ.

small debt (i.e. low financial leverage). These models predict that firms' equity values will grow linearly with almost no volatility for large equity companies. However, most high-tech firms, such as the Internet and biotechnology companies, are highly volatile, while having almost no debt as pointed out by Chen & Kou (2006). Consequently, the Carr-Linetsky model might under price deep in-the-money calls or deep out-of-money puts. The implied equity volatility function of Leland (1994) is more realistic, because it captures the fact that equity should be at least as risky as the asset value of the firm, of which the asset volatility should not be regarded as zero. In this section, we propose a new jump-to-default model with equity volatility bounded below by a positive constant, which can be regarded as the asset volatility.

Assume that the pre-default stock price S_t has the following risk-neutral dynamics

$$dS_t = (r + aS_t^{-p})S_t dt + cS_t\sqrt{1 + bS_t^{-p}}dW_t, \tag{4.17}$$

where the constant r denotes the default-free interest rate; a, c, b, and p are positive constants. This specification of the pre-default stock price has the form of equation (4.5) with volatility and hazard rate functions given by

$$\sigma(S_t) = c\sqrt{1+bS_t^{-p}}, \tag{4.18}$$

$$h(S_t) = aS_t^{-p}. \tag{4.19}$$

The volatility function $\sigma(S_t)$ specified in equation (4.18) is bounded below by a positive constant c. When S_t is large, $\sigma(S_t)$ is almost a constant, becoming asymptotically the case of constant volatility studied in Linetsky (2006). When S_t is close to zero, $\sqrt{1+bS_t^{-p}}$ can be approximated as $\sqrt{b}S_t^{-p/2}$. Consequently, our model becomes asymptotically CEV case studied in Carr & Linetsky (2006) when the company is close to default. If we allow b to be zero, then this model is reduced to a constant volatility case studied in Linetsky (2006).

The hazard rate function $h(S_t)$ is related to the volatility function $\sigma(S_t)$ through the following equation

$$h(S_t) = \frac{a}{b}(\frac{\sigma^2(S_t)}{c^2} - 1).$$

As the stock price increases, the equity volatility declines to constant c and the hazard rate approaches zero, making the stock prices asymptotically GBM. The default time is defined to be the first jump time of a Cox process with intensity given by the hazard rate $h(S_t)$. The pricing of equity derivatives can therefore be performed using standard reduced-form intensity-based credit risk framework as discussed in Duffie & Singleton (1999).

4.4 Pricing

Similarly, as in Linetsky (2006), pricing equity derivatives can be reduced to computing the expectations of the form in equation (4.3). This simplification is because of the choice of the hazard rate function. Since the hazard rate will go to infinity when the stock price vanishes to zero, the pre-default process will be almost surely killed by a jump-to-default instead of diffusing to zero. Proposition 2.1 in Linetsky (2006) can be generalized as follows:

Proposition 4.4.1. *Assume that the pre-default stock price S_t has risk-neutral dynamics*

$$dS_t = (r + h(S_t))S_t dt + \sigma(S_t)S_t dW_t,$$

starting from $S_0 > 0$ at time $t = 0$. Then for any payoff function ψ, the claim price can be written:

$$F_\psi(S_0, T) := \mathbb{E}[e^{-\int_0^T r+h(S_t)dt}\psi(S_T)] = S_0\hat{\mathbb{E}}[S_T^{-1}\psi(S_T)], \tag{4.20}$$

where $\hat{\mathbb{E}}$ is the expectation with respect to a new probability measure $\hat{P}$ under which $\hat{W}_t := W_t - \int_0^t \sigma(S_u)du$ is a standard Brownian motion and

$$dS_t = (r + h(S_t) + \sigma^2(S_t))S_t dt + \sigma(S_t)S_t d\hat{W}_t. \tag{4.21}$$

Particularly, for the equity volatility and hazard rate functions given by equations (4.18) and (4.19), the new dynamics of the pre-default stock price S_t can be written as

$$dS_t = [(r + c^2) + (a + bc^2)S_t^{-p}]S_t dt + cS_t\sqrt{1 + S_t^{-p}}d\hat{W}_t, \tag{4.22}$$

which has the same form as in the original measure, but with r replaced by $r + c^2$ and a replaced by $a + bc^2$.

Proposition 4.4.1 can be applied immediately to obtain the pricing formulae for the defaultable zero-coupon bond, European calls and European puts. For a zero-coupon bond with unit face value, let R denote the constant recovery rate paid at the maturity T. A European call option with strike price K has payoff $(S_T - K)^+$ at expiration date T if there is no default and it has no recovery if the firm defaults before T. A

European put option promises a payoff $(K - S_T)^+$ at expiration if no default happens and a recovery payment K at expiration in the event of bankruptcy before expiration date T. Then, the values of a zero coupon bond, $B_R(S_0, T)$, the call, $C_K(S_0, T)$ and the put, $P_K(S_0, T)$, take the following forms respectively

$$\begin{aligned} B_R(S_0,T) &= e^{-rT}R + (1-R)S_0\hat{\mathbb{E}}[S_T^{-1}], & (4.23)\\ C_K(S_0,T) &= S_0\hat{\mathbb{E}}[(1-KS_T^{-1})^+], & (4.24)\\ P_K(S_0,T) &= S_0\hat{\mathbb{E}}[(KS_T^{-1}-1)^+] + K(e^{-rT} - S_0\hat{\mathbb{E}}[S_T^{-1}]). & (4.25) \end{aligned}$$

Note that the put-call parity still holds here

$$C_K(S_0,T) + Ke^{-rT} = P_K(S_0,T) + S_0. \tag{4.26}$$

When $a = 0$, the bond becomes a default-free bond which only depends on the interest rate r and maturity T. When S_0 is close to zero, the company is almost in default. As a result, the call price and the zero bond with zero recovery will worth almost nothing. From the put-call parity, the put price will be close to the discounted value of the strike price when S_0 is close to zero. For very large S_0, the hazard rate will be very small and the equity volatility will be almost a constant. Consequently, the bond price will be close to that of a default-free bond and the call price will be close to Black-Scholes price with parameters of interest rate r and volatility c. In order to obtain general explicit expressions for these formulae, we need to study the distribution of the pre-default stock price S_T under the hat measure.

4.5 The Dual-Jacobi Process

Since the distribution of the pre-default stock price S_T under the hat measure and the risk-neutral measure are in the same distribution family, we will consider the format in equation (4.17) for the sake of simplicity. Let $Y_t := S_t^p$, where S_t is specified as in equation (4.17): then the SDE for Y_t can be written as follows using Ito's lemma

$$\frac{dY_t}{Y_t} = [(rp + \frac{1}{2}p(p-1)c^2) + (ap + \frac{1}{2}p(p-1)bc^2)Y_t^{-1}]dt + pc\sqrt{1 + bY_t^{-1}}dW_t. \quad (4.27)$$

This SDE takes the form of equation (4.17). Therefore, in order to obtain the distribution of S_T for a general p, it is sufficient to study the case when $p = 1$.

Considering the case when $p = 1$, equation (4.17) is simplified into the following SDE (we use notation X_t instead of S_t here for this particular SDE)

$$dX_t = (a + rX_t)dt + c\sqrt{X_t^2 + bX_t}dW_t, \quad (4.28)$$

with initial value $X_0 := x_0 > 0$ at time $t = 0$. Note that this process has the form resembling a Jacobi process, but in fact it is not. For a Jacobi process, its volatility term typically looks like $c\sqrt{bX_t - X_t^2}$, which restrict the process itself to lie in the finite interval $(0, b)$.[4] In our model, however, the volatility term is $c\sqrt{bX_t + X_t^2}$ and the process lies in the half positive domain $(0, +\infty)$. We will refer to this SDE as the *Dual-Jacobi* process.

The strong unique solution of the Dual-Jacobi process is guaranteed by the Yamada & Watanabe (1971) theorem.[5] This solution is a nonnegative diffusion which, provided $a > \frac{1}{2}c^2b$ has unattainable boundaries on both zero and infinity. When X_t is large, the volatility of the Dual-Jacobi process is approximately cX_t. Therefore, for large X_t, the Dual-Jacobi can be approximated as

$$dX_t = (a + rX_t)dt + cX_tdW_t.$$

This process has a well-known explicit solution.[6] This result allows Linetsky (2006) to write a spectral representation of the transition density of their stock dynamics. The transition density has both discrete and continuous spectrums which are related

[4]It is known as Wright-Fisher diffusion in probability literature, see Feng & Wang (2007)

[5]See Karatzas & Shreve (1991), page 291.

[6]See Karatzas & Shreve (1991), page 360.

to Whittaker functions and Laguerre polynomials. For sufficient large X_t when the effect of a can be neglected, we can also think of GBM as a proxy of the Dual-Jacobi process. When X_t is small, the volatility of the Dual-Jacobi is approximately $c\sqrt{bX_t}$. Consequently the Dual-Jacobi can be approximated by a CIR process which follows non-central chi-square distribution. However, to the author's best knowledge, no explicit formula for the transition density has been found for the Dual-Jacobi process.

One may resort to PDE approach and try to solve numerically. Let $f(t, x)$ denote the probability density function (pdf) of X_t at time t. This pdf satisfies the Kolmogorov Forward Equation, or Fokker-Planck Equation given by[7]

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}[(a + rx)f] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[c^2(x^2 + bx)f], \tag{4.29}$$

for $x > 0$ with initial condition and boundary conditions given by

$$\begin{aligned} f(0, x) &= \delta(x - x_0), \\ f(t, 0) &= f(t, +\infty) = 0, \quad \forall t \geq 0. \end{aligned}$$

Since $f(t, x)$ is a pdf, it is nonnegative and it also satisfies the integrability condition

$$\int_0^\infty f(t, x)dx = 1, \quad \forall t \geq 0.$$

One nice property about Dual-Jacobi SDE is that all its moments can be explicitly calculated. The following proposition gives a recursive equation for the moments of the Dual-Jacobi process.

Proposition 4.5.1. *Let X_t be a Dual-Jacobi process given by (4.28) with initial value $X_0 = x_0$. Let $\alpha_m(t) := \mathbb{E}[X_t^m]$, $m = 0, 1, 2, \ldots$, with $\alpha_0(t) = 1$. Then $\alpha_m(t)$ for*

[7]This PDE is closely related to *singular* Sturm-Liouville problem with *semi-infinite* domain.

$m = 1, 2, ...,$ *satisfies the following iterative equation*

$$\begin{aligned}
\alpha_m(t) &= x_0^m e^{r_m t} + a_m \int_0^t e^{r_m(t-s)} \alpha_{m-1}(s) ds, && (4.30)\\
r_m &:= m[r + \frac{1}{2}c^2(m-1)],\\
a_m &:= m[a + \frac{1}{2}bc^2(m-1)].
\end{aligned}$$

Explicit formulae for all positive moments can therefore be calculated by direct integration or applying Laplace transforms to equation (4.30). Particularly, the first four moments are given by

$$\begin{aligned}
\alpha_1(t) &= (x_0 + \frac{a}{r})e^{rt} - \frac{a}{r},\\
\alpha_2(t) &= (x_0^2 + A_2 - B_2)e^{r_2 t} - A_2 e^{rt} + B_2,\\
\alpha_3(t) &= (x_0^3 + A_3 - B_3 + C_3)e^{r_3 t} - A_3 e^{r_2 t} + B_3 e^{rt} - C_3,\\
\alpha_4(t) &= (x_0^4 + A_4 - B_4 + C_4 - D_4)e^{r_4 t} - A_4 e^{r_3 t} + B_4 e^{r_2 t} - C_4 e^{rt} + D_4.
\end{aligned}$$

where

$$\begin{aligned}
A_2 &= \frac{a_2(x_0 + \frac{a}{r})}{r_2 - r}, \; B_2 = \frac{aa_2}{rr_2},\\
A_3 &= \frac{a_3(x_0^2 + A_2 - B_2)}{r_3 - r_2}, \; B_3 = \frac{a_3 A_2}{r_3 - r}, \; C_3 = \frac{a_3 B_2}{r_3},\\
A_4 &= \frac{a_4(x_0^3 + A_3 - B_3 + C_3)}{r_4 - r_3}, \; B_4 = \frac{a_4 A_3}{r_4 - r_2}, \; C_4 = \frac{a_4 B_3}{r_4 - r}, \; D_4 = \frac{a_4 C_3}{r_4}.
\end{aligned}$$

Proposition 4.5.1 motivates us to approximate the pdf of a Dual-Jacobi process by using these explicit moments.

4.6 The Gram-Charlier Approximation

In this section, we first transform our original stock price dynamics to a Dual-Jacobi process. Gram-Charlier Expansions are then used to obtain explicit asymptotic expansions of both zero coupon bond prices and European call prices.

Proposition 4.6.1. *Assume that the stock price S_t has risk-neutral dynamics specified in equation (4.17). Let $Y_t := S_t^p$ with $Y_0 := S_0^p$. Then under the hat measure, Y_t satisfies the following Dual-Jacobi SDE*

$$\begin{aligned} dY_t &= (\hat{a} + \hat{r}Y_t)dt + \hat{c}\sqrt{Y_t^2 + bY_t}d\hat{W}_t, && (4.31)\\ \hat{a} &= p[a + \frac{1}{2}bc^2(p+1)], && (4.32)\\ \hat{r} &= p[r + \frac{1}{2}c^2(p+1)], && (4.33)\\ \hat{c} &= pc. && (4.34) \end{aligned}$$

This proposition can be shown by applying the hat measure change to SDE (4.27). From Proposition 4.5.1, all the moments of Y_T under the hat measure are finite and have explicit expressions. This fact motivates us to use these finite explicit moments to approximate bond and call prices. The zero coupon bond with zero recovery and the European call price can thus be expressed as

$$\begin{aligned} B_0(S_0, T) &= S_0\hat{\mathbb{E}}[Y_T^{-1/p}], && (4.35)\\ C_K(S_0, T) &= S_0\hat{\mathbb{E}}[(1 - KY_T^{-1/p})^+]. && (4.36) \end{aligned}$$

Jarrow & Rudd (1982) were the first to introduce the Gram-Charlier expansion to the quantitative finance literature.[8] Ever since, the Gram-Charlier expansion has been widely used in option pricing, for example Madan & Milne (1994) and Jondeau & Rockinger (2001) etc. These expansion theories have a very old history in statistics literature and they arise from approximating the distribution of a sum of independent random variables, see Cramer (1946).

Let $f(y)$ denote our unknown target pdf with unknown cdf F. Let $g(y)$ denote our known base pdf with known cdf G. Let α_i^F and α_i^G denote the i-th moments of distribution F and G respectively, for $i = 1, 2, 3, \ldots$. Assume all these moments

[8] Jarrow & Rudd (1982) named Edgeworth series in their paper, but in fact, the series expansions they used are Gram-Charlier.

are finite and have explicit expressions. Let κ_i^F and κ_i^G denote the i-th cumulants of distribution F and G respectively. All κ_i^F and κ_i^G will be known explicitly, since all moments are known. The following gives the algebraic relationships between cumulants and moments up to the fourth order

$$
\begin{aligned}
\kappa_1 &= \alpha_1, \\
\kappa_2 &= \alpha_2 - \alpha_1^2, \\
\kappa_3 &= \alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1^3, \\
\kappa_4 &= \alpha_4 - 4\alpha_3\alpha_1 - 3\alpha_2^2 + 12\alpha_2\alpha_1^2 - 6\alpha_1^4. \\
& \quad \ldots\ldots\ldots\ldots
\end{aligned} \tag{4.37}
$$

Let $\epsilon_i := \kappa_i^F - \kappa_i^G$. The Gram-Charlier expansion considers the following expression as a candidate to approximate the true pdf $f(y)$.

$$
f(y) = g(y) - \frac{\eta_1 g'(y)}{1!} + \frac{\eta_2 g''(y)}{2!} - \frac{\eta_3 g'''(y)}{3!} + \ldots, \tag{4.38}
$$

where $\eta_i, i = 1, 2, \ldots$ are given by

$$
\begin{aligned}
\eta_1 &= \epsilon_1, \\
\eta_2 &= \epsilon_2 + \epsilon_1^2, \\
\eta_3 &= \epsilon_3 + 3\epsilon_2\epsilon_1 + \epsilon_1^3, \\
\eta_4 &= \epsilon_4 + 4\epsilon_3\epsilon_1 + 3\epsilon_2^2 + 6\epsilon_2\epsilon_1^2 + \epsilon_1^4, \\
& \quad \ldots\ldots\ldots
\end{aligned} \tag{4.39}
$$

Jarrow & Rudd (1982) and Cramer (1946) provided a detailed derivation of the above algebraic relationships.

In our case, the target pdf is the pdf of Y_T under the hat measure. For a given function $\psi(y)$, the expectation $\hat{\mathbb{E}}[\psi(Y_T)]$ can be expanded as

$$\begin{aligned}\hat{\mathbb{E}}[\psi(Y_T)] &:= \int_0^{+\infty} \psi(y)f(y)dy, \\ &= \int_0^{+\infty} \psi(y)g(y)dy - \frac{\eta_1}{1!}\int_0^{+\infty} \psi(y)g^{'}(y)dy \\ &+\frac{\eta_2}{2!}\int_0^{+\infty} \psi(y)g^{''}(y)dy - \frac{\eta_3}{3!}\int_0^{+\infty} \psi(y)g^{'''}(y)dy + \ldots\end{aligned} \tag{4.40}$$

The zero coupon bond and the European call can be priced using these expansions. Different choices of the base pdf will yield different expansions. The base pdf is usually chosen to be a normal or log-normal distribution. Considering that Y_T is nonnegative, it is more appropriate to choose log-normal as the base pdf.

Proposition 4.6.2. *Assume the base pdf $g(y)$ is log-normal with parameters $\hat{\mu}$ and $\hat{\sigma}^2$. Then the prices of a zero coupon bond with zero recovery and the European call can be expanded as*

$$\begin{aligned}B_0(S_0,T) &= z_0 - \frac{\eta_1 z_1}{p} + \frac{\eta_2(1+p)z_2}{2!p^2} - \frac{\eta_3(1+p)(1+2p)z_3}{3!p^3} \\ &+\frac{\eta_4(1+p)(1+2p)(1+3p)z_4}{4!p^4} + \ldots \\ C_K(S_0,T) &= [S_0\Phi(d_1) - K\theta_0] + \frac{\eta_1 K\theta_1}{p} - \frac{\eta_2}{2!}\left[\frac{S_0}{pK^p}g(K^p) + \frac{K(1+p)\theta_2}{p^2}\right] \\ &+\frac{\eta_3}{3!}\left[\frac{S_0}{pK^p}g^{'}(K^p) + \frac{S_0(1+p)}{(pK^p)^2}g(K^p) + \frac{K(1+p)(1+2p)\theta_3}{p^3}\right] \\ &-\frac{\eta_4}{4!}\left[\frac{S_0}{pK^p}g^{''}(K^p) + \frac{S_0(1+p)}{(pK^p)^2}g^{'}(K^p) + \frac{S_0(1+p)(1+2p)}{(pK^p)^3}g(K^p)\right. \\ &\left.+\frac{K(1+p)(1+2p)(1+3p)\theta_4}{p^4}\right] + \ldots\end{aligned} \tag{4.41}$$

where

$$\begin{aligned}d_1 &= \frac{-p\log K + \hat{u}}{\hat{\sigma}}, \\ z_n &= S_0 e^{-\frac{(1+np)}{p}\hat{\mu} + \frac{(1+np)^2}{2p^2}\hat{\sigma}^2}, \\ \theta_n &= z_n\Phi(d_1 - \frac{1+np}{p}\hat{\sigma}).\end{aligned}$$

For log-normal base pdf, the m-th moments are all finite and given by

$$\alpha_m^G = e^{m\hat{\mu}+\frac{1}{2}m^2\hat{\sigma}^2}.$$

With explicit expressions of α_m^F and α_m^G, the coefficients η_m in the above proposition can be computed by equations (4.37) and (4.39). These coefficents depend on the model parameters a, r, c, b, p as well as the initial stock price S_0. The following lemma will be needed in order to prove the above proposition.

Lemma 4.6.3. *Assume that the random variable X is log normal with parameters μ and σ^2. Let $g(x)$ denote its pdf. Then, for any real number α and $K > 0$, the following equation holds*

$$\int_K^{+\infty} x^\alpha g(x)dx = e^{\alpha\mu+\frac{1}{2}\alpha^2\sigma^2}\Phi(\frac{-\log K+\mu+\alpha\sigma^2}{\sigma}).$$

Log-normal distributions are determined by two parameters. There are many possible ways to choose these two parameters, such as matching the first two moments of the target distribution. We can also match instantaneous drift and volatility. The following are some examples.

Example 4.6.1. *Asymptotic match.*

The base pdf is chosen to be a GBM with drift $\hat{r}$ and volatility $\hat{c}$. The log-normal parameters for the base pdf are given by

$$\begin{aligned}\hat{\mu} &= p[\log S_0 + (r + \frac{1}{2}c^2)T],\\ \hat{\sigma}^2 &= p^2c^2T.\end{aligned}$$

The first term for $B_0(S_0, T)$ becomes the default-free bond price. The first term for $C_K(S_0, T)$ becomes the Black-Scholes price with no default parameters. The rest of the terms are corrections accounting for default possibilties and local volatility effects.

Example 4.6.2. *Matching the first two moments.*

Parameters $\hat{\mu}$ and $\hat{\sigma}^2$ are chosen to match the first two moments of the target distribution. More specifically, these parameters are given by

$$\begin{aligned}\hat{\mu} &= 2\log[\alpha_1^F] - \frac{1}{2}\log[\alpha_2^F],\\ \hat{\sigma}^2 &= \log[\alpha_2^F] - 2\log[\alpha_1^F].\end{aligned}$$

This specification allows standard Gram-Charlier expansion as follows

$$f(y) = g(y) - \frac{\epsilon_3}{3!}\frac{d^3g(y)}{dy^3} + \frac{\epsilon_4}{4!}\frac{d^4g(y)}{dy^4} - \frac{\epsilon_5}{5!}\frac{d^5g(y)}{dy^5} + ...$$

Example 4.6.3. *Matching instantaneous drift and volatility.*

The base pdf is chosen to be a GBM with drift $\hat{r} + \hat{a}S_0^{-p}$ and volatility $\hat{c}\sqrt{1 + bS_0^{-p}}$. The log-normal parameters for the base pdf are then given by

$$\begin{aligned}\hat{\mu} &= p[\log S_0 + (r + \frac{1}{2}c^2)T + (a + \frac{1}{2}bc^2)TS_0^{-p}],\\ \hat{\sigma}^2 &= p^2c^2T(1 + bS_0^{-p}).\end{aligned}$$

Remark 4.6.1. *It is known that the Gram-Charlier expansion is not guaranteed to converge. Cramer (1946) has studied the Gram-Charlier expansion when the base pdf is standard normal. He showed that if the target pdf $f(y)$ is of bounded vairation in $(-\infty, +\infty)$ and $\int_{-\infty}^{+\infty} e^{y^2/4}f(y)dy$ is convergent, then the Gram-Charlier expansion will converge to $f(x)$ in every continuity point of $f(y)$. However, in reality, there is only a small class of distributions that will validate the Gram-Charlier expansion. In practice, most of these expansions are of the asymptotic type that they will converge for a small number of terms and then diverge, see Barndorff-Nielsen & Cox (1989). But, this does not mean these expansions will not be useful. Because if a small number of terms (usually not more than two or three) suffice to give a good approximation, it does not concern us much whether the infinite series is convergent or divergent. On the other hand, a convergent series is of little practical value if a large number of terms are required to be calculated in order to have a reasonable approximation, see Cramer (1946) for a discussion.*

4.7 Numerical Analysis

As mentioned in the preceding remark, our expansions are not necessarily convergent. However, we will numerically show that a small number of terms suffice to provide very good approximations. Numerical examples are used to study the residual errors and the price sensitivity to our model parameters. The Gram-Charlier approximation is much faster than the finite difference method for pricing securites. We also use numerical examples to illustrate implied volatility surface/skew and credit spreads.

4.7.1 Residual Error and Sensitivity

In this numerical analysis, we study two Gram-Charlier approximations which are decribed in Examples 4.6.2 and 4.6.3. We use three terms for the first approximation and four terms for the second approximation. The parameters in Table 4.1 is the base case parameters used for comparison study. Since no explicit pricing formulae are available, Monte-Carlo prices are used as the benchmark. Prices calculated using a numerical PDE approach (implicit finite difference method) are also provided.

a	r	c	b	p	S_0	T	K	R
3.6421	0.0518	0.2923	23.5930	1.8751	7.55	0.5	7.55	0.3228

Table 4.1: Base Case Parameters for Numerical Analysis.

Table 4.2 illustrates the bond prices for varying parameters. Around 500 points per path and 10000 paths are used for one Monte-Carlo price simulation with variance reduction techniques. For the finite difference method, the step sizes for the stock price and the maturity are \$0.5 and 5/2400-year respectively. The first row of prices are calculated with the base case parameters given in Table 4.1. The rest of the rows are obtained by changing one parameter, while holding the other parameters constant as in the base case. Comparing with Monte-Carlo prices, our approximations are very good as seen in "E1" and "E2" columns: there are only a few bps difference across all our considerations. The Mean Relative Error (MRE), calculated based on the 15 samples, are around 0.314% and 0.24% for the two approximations respectively.

Note that our model prices are consistantly smaller than the default-free bond prices. This is largely due to none-zero default parameters a and p. When a increases or p decreases, the credit quality deteriorates and hence the bond price drops. Our model prices will go down when the interest rate r or maturity T increase. When the volatility parameter c increases, the bond price will go down to reflect more risk. For a larger initial stock price S_0, the bond price will be higher since the company is less likely to default. We also find that the bond price is relatively insensitive to the parameter b.

Table 4.3 shows the European call prices for various scenarios. The Black-Scholes price in Table 4.3 is calculated by imposing $a = b = 0$. Our approximations are reasonably good compared with the Monte-Carlo price. The MREs for the two approximation approaches are around 0.3885% and 1.2290% respectively. The call prices are found to be under valued by Black-Scholes' formula, compared with our model prices. The call prices will go up as interest rate r rises, or volatility c increases, or maturity T increases, or the initial stock price S_0 becomes higher, or the strike price K becomes smaller. This is consistent with the Black-Scholes formula. When a increases or p decreases, or b increases, the call prices will go up to compensate for more risks.

	DF	MC	FD	A1	A2	E1	E2
base case	0.9744	0.9468	0.9472	0.9440	0.9454	-0.2957	-0.1479
$a = 4.6421$	0.9744	0.9404	0.9404	0.9374	0.9348	-0.3190	-0.5955
$a = 2.6421$	0.9744	0.9543	0.9543	0.9508	0.9562	-0.3668	0.1991
$r = 0.0618$	0.9696	0.9425	0.9426	0.9394	0.9407	-0.3289	-0.1910
$r = 0.0418$	0.9793	0.9513	0.9518	0.9486	0.9502	-0.2838	-0.1156
$c = 0.3923$	0.9744	0.9446	0.9451	0.9366	0.9437	-0.8469	-0.0953
$c = 0.1923$	0.9744	0.9485	0.9485	0.9478	0.9496	-0.0738	0.1160
$b = 28.593$	0.9744	0.9468	0.9470	0.9429	0.9437	-0.4119	-0.3274
$b = 18.593$	0.9744	0.9477	0.9474	0.9451	0.9457	-0.2743	-0.2110
$p = 2.0751$	0.9744	0.9558	0.9559	0.9553	0.9550	-0.0523	-0.0837
$p = 1.6751$	0.9744	0.9344	0.9346	0.9307	0.9313	-0.3960	-0.3318
$T = 1.00$	0.9495	0.8968	0.8968	0.8951	0.8938	-0.1896	-0.3345
$T = 0.25$	0.9871	0.9732	0.9734	0.9722	0.9723	-0.1028	-0.0925
$S_0 = 8.55$	0.9744	0.9526	0.9527	0.9502	0.9514	-0.2519	-0.1260
$S_0 = 6.55$	0.9744	0.9394	0.9393	0.9351	0.9367	-0.4577	-0.2874
$R = 0.4228$	0.9744	0.9513	0.9512	0.9485	0.9568	-0.2943	0.5782
$R = 0.2228$	0.9744	0.9432	0.9432	0.9395	0.9407	-0.3923	-0.2651
					MRE	0.3140	0.2411

Table 4.2: Zero coupon bond prices. "DF" denotes the default-free price. "MC" denotes the Monte-Carlo price. "FD" denotes the finite difference price. "A1" denotes the first approximation price. "A2" denotes the second approximation price. The relative error (in %) "E1" is defined as E1:=(A1-MC)/MC. The signed relative error (in %) "E2" is defined as E2:=(A2-MC)/MC. "MRE" denotes the mean relative error (in %).

	BS	MC	FD	A1	A2	E1	E2
base case	0.7148	0.9881	0.9884	0.9856	0.9836	-0.2530	-0.4554
$a = 4.6421$	0.7148	1.0287	1.0249	1.0232	1.0253	-0.5347	-0.3305
$a = 2.6421$	0.7148	0.9542	0.9522	0.9484	0.9388	-0.6078	-1.6139
$r = 0.0618$	0.7336	1.0100	1.0075	1.0048	1.0008	-0.5149	-0.9109
$r = 0.0418$	0.6962	0.9673	0.9694	0.9665	0.9625	-0.0827	-0.4962
$c = 0.3923$	0.9223	1.2351	1.2351	1.2274	1.1998	-0.6234	-2.8581
$c = 0.1923$	0.5078	0.7530	0.7479	0.7540	0.7624	0.1328	1.2483
$b = 28.593$	0.7148	1.0143	1.0143	1.0084	0.9987	-0.5817	-1.5380
$b = 18.593$	0.7148	0.9670	0.9615	0.9620	0.9637	-0.5171	-0.3413
$p = 2.0751$	0.7148	0.9025	0.9001	0.9012	0.8997	-0.1440	-0.3102
$p = 1.6751$	0.7148	1.1167	1.1152	1.1079	1.0992	-0.7880	-1.5671
$T = 1.00$	1.0590	1.4985	1.4979	1.4964	1.4858	-0.1401	-0.8475
$T = 0.25$	0.4873	0.6591	0.6567	0.6605	0.6576	-0.2124	-0.2276
$S0 = 8.55$	1.4160	1.6794	1.6781	1.6794	1.6487	-0.0000	-1.8280
$S0 = 6.55$	0.2565	0.4874	0.4845	0.4833	0.4972	-0.8412	2.0107
$K = 8.55$	0.3354	0.5456	0.5488	0.5485	0.5571	-0.5315	2.1078
$K = 6.55$	1.3356	1.6221	1.6210	1.6237	1.5864	-0.0986	-2.2009
					MRE	0.3885	1.2290

Table 4.3: European call prices. "BS" denotes the Black-Scholes price. "MC" denotes the Monte-Carlo price. "FD" denotes the finite difference price. "A1" denotes the first approximation price. "A2" denotes the second approximation price. The relative error (in %) "E1" is defined as E1:=(A1-MC)/MC. The signed relative error (in %) "E2" is defined as E2:=A2-MC. "MRE" denotes the mean relative error (in %).

4.7.2 Accuracy versus Speed

For the finite difference method in Tables 4.2 and 4.3, the average computational time is 0.1442 second per call price and 0.1353 second per bond price. For the first approximation scheme in Tables 4.2 and 4.3, the average computational time is 0.002 second per call price and 0.0012 second per bond price. The Gram-Charlier approximation scheme is more than 70 times faster than the implicit finite difference method on pricing calls, and more than 100 times faster on pricing bonds.

Table 4.4 illustrate a comparison of the computational time for using finite difference method versus the Gram-Charlier approximation to price bonds. As maturity T increases, while holding other parameters constant, the Gram-Charlier approximation becomes less accurate than the finite difference method. However, the finite difference method becomes much slower than the Gram-Charlier approximation. The computational time for the second approximation is constant 0.0012 second, which does not depend on maturity T. For the finite difference method, the computational time increases almost 10 times, when T increases from 1-year to 10-year. There is a trade off between the accuracy and the computation speed. The accuracies of the approximations are not universally the same. The expansions may diverge for certain parameters if inappropriate terms are used. Therefore, we recommend using Monte-Carlo or PDE approach to check the numerical residual errors before using the asymptotic expansions.

4.7.3 The Implied Volatility Surface and Credit Spreads

In this section, we apply our model to some real observed data and examine its goodness of fit. Figure 4.3(a) plots the observed implied volatility surface against moneyness (strike over spot) and maturity of the option. The real data is taken from Ford Motor Corp. on March 16, 2007 from Bloomberg. Figure 4.3(b) shows the theoretical implied volatility surface computed by using the calibrated parameters. We first calculate the model call price using the first approximation for a given moneyness and maturity and quote this value as a Black-Scholes implied volatility. The stock price of Ford on

	MC	FD	A1	TFD	TA1
T = 0.5	0.9806	0.9805	0.9806	0.1353	0.0012
T = 1	0.9602	0.9613	0.9613	0.2858	0.0012
T = 2	0.9207	0.9237	0.9238	0.5164	0.0012
T = 3	0.8880	0.8872	0.8880	0.8239	0.0012
T = 4	0.8478	0.8520	0.8540	1.0708	0.0012
T = 5	0.8165	0.8178	0.8218	1.3135	0.0012
T = 6	0.7811	0.7848	0.7914	1.5884	0.0012
T = 7	0.7487	0.7530	0.7627	1.8490	0.0012
T = 8	0.7121	0.7223	0.7355	2.1448	0.0012
T = 9	0.6874	0.6928	0.7098	2.3755	0.0012
T = 10	0.6564	0.6644	0.6855	2.6148	0.0012

Table 4.4: Acuracy versus Speed. "MC" denotes the Monte-Carlo price. "FD" denotes the finite difference price. "A1" denotes the second approximation price. "TFD" denotes the computaional time for FD method (in seconds). "TA1" denotes the computational time for A1 method (in seconds).

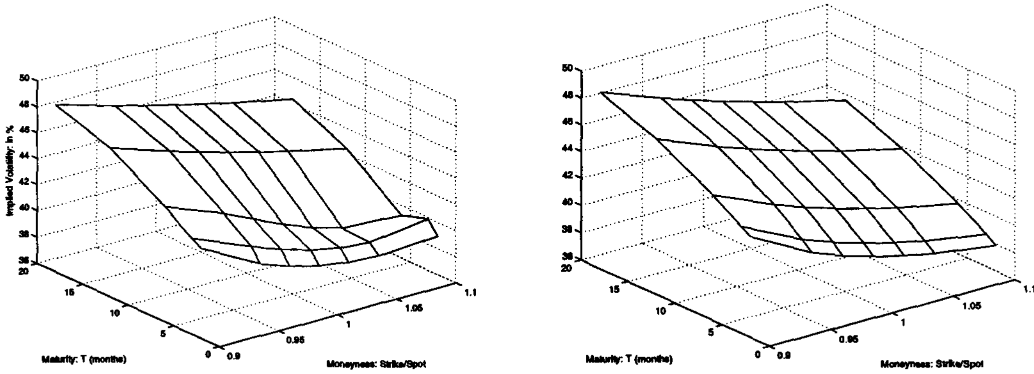
March 16, 2007 is \$7.55. The short rate r is taken to be 5.18%, which is the 1-month U.S Treasury yield on March 16, 2007. The parameters a, c, b, and p are calibrated by minimizing the differences of the observed and theoretical implied volatilities in the sense of the Root of Mean Square Error (RMSE).

As seen in Figures 4.3(a) and 4.3(b), the theoretical surface is able to capture the main two properties of real implied volatility surface: a) negative skew for fixed maturity; b) as maturity increases, the skew becomes less pronounced. Table 4.5 provides numerical details of the observed and model implied surfaces. The RMSE across all the data on the volatility surface is found to be 0.5472%, which is very small.

Figure 4.4 demonstrates implied volatility skews from the model. Both the level and the slope (in absolute value) of the implied volatility skew increase as the creditworthiness deteriorates (i.e. the default parameter a increases or p decreases). For large volatility parameters c and b, the level of the implied volatility will be higher but the slope of the skew is not affected very much. For higher moneyness, the implied volatility asymptotically goes down to the volatility parameter c.

The term structure of CDS on March 16, 2007 from Ford is also taken from Bloomberg. We then estimate the recovery rate R, by minimizing the differences between the market CDS spreads and model CDS spreads, while holding the parameters a, c, b and p constant as calibrated from the volatility surface. The recovery rate is estimated at 0.3228. However, the CDS term structure, generated by the parameters calibrated from the volatility surface, is unable to match the real observed CDS term structure. As seen in Figure 4.5, the market CDS curve is upward sloping, but the model CDS curve implied from the volatility surface is downward sloped. This discrepancy may indicate either the model is wrong for Ford, or there exist arbitrage opportunities.

As an inverse exercise, we can first calibrate the parameters by fitting the CDS curve and then compute the implied volatility surface using these calibrated parameters. For this exercise, we calibrate R, a, c, b, and p using the CDS data. As seen in Figure 4.6, the calibrated CDS curve fits the market curve very well. The RMSE of the calibrated CDS curve is found to be 9.7122 bps, which is very small considering the large credit spreads of Ford. However, the implied volatility surface, calculated using the parameters calibrated from the CDS data, does not provide a good fit to the observed volatility surface. As seen in Table 4.6, this model implied volatility surface is too flat and too high compared with the observed surface.



(a) Observed Implied Volatility Surface. (b) Model Implied Volatility Surface.

Figure 4.3: Implied Volatility Surfaces: Observed vs. Model Implied. The observed surface is from Ford Corp. on March 16, 2007. The calibrated parameters are: $a = 3.6421$, $c = 0.2923$, $b = 23.5930$, and $p = 1.8751$.

Observed	90	95	97.5	100	102.5	105	110
T=2m	42.9223	40.5271	39.7240	39.1576	38.8114	38.6691	38.9445
T=3m	43.3745	41.5105	40.6955	40.0286	39.5712	39.3848	39.9058
T=6m	44.7892	43.0559	42.0286	41.0245	40.1506	39.5141	39.2246
T=12m	47.3736	45.8992	45.2300	44.5992	44.0032	43.4409	42.4144
T=18m	48.8218	47.5070	46.9075	46.3397	45.7997	45.2870	44.3412
Model	90	95	97.5	100	102.5	105	110
T=2m	43.4910	41.0786	40.2266	39.5444	38.9939	38.5458	37.8710
T=3m	43.9048	41.7269	40.9028	40.2144	39.6368	39.1495	38.3823
T=6m	45.2695	43.3181	42.5120	41.8007	41.1722	40.6158	39.6820
T=12m	47.5087	45.7109	44.9216	44.1976	43.5332	42.9232	41.8471
T=18m	49.0884	47.4113	46.6557	45.9506	45.2926	44.6784	43.5690

Table 4.5: Implied Volatility Surfaces: Observed vs. Model Implied (in %). The row with 90,...,110, denotes moneyness (in %). The root mean square error is 0.5472%.

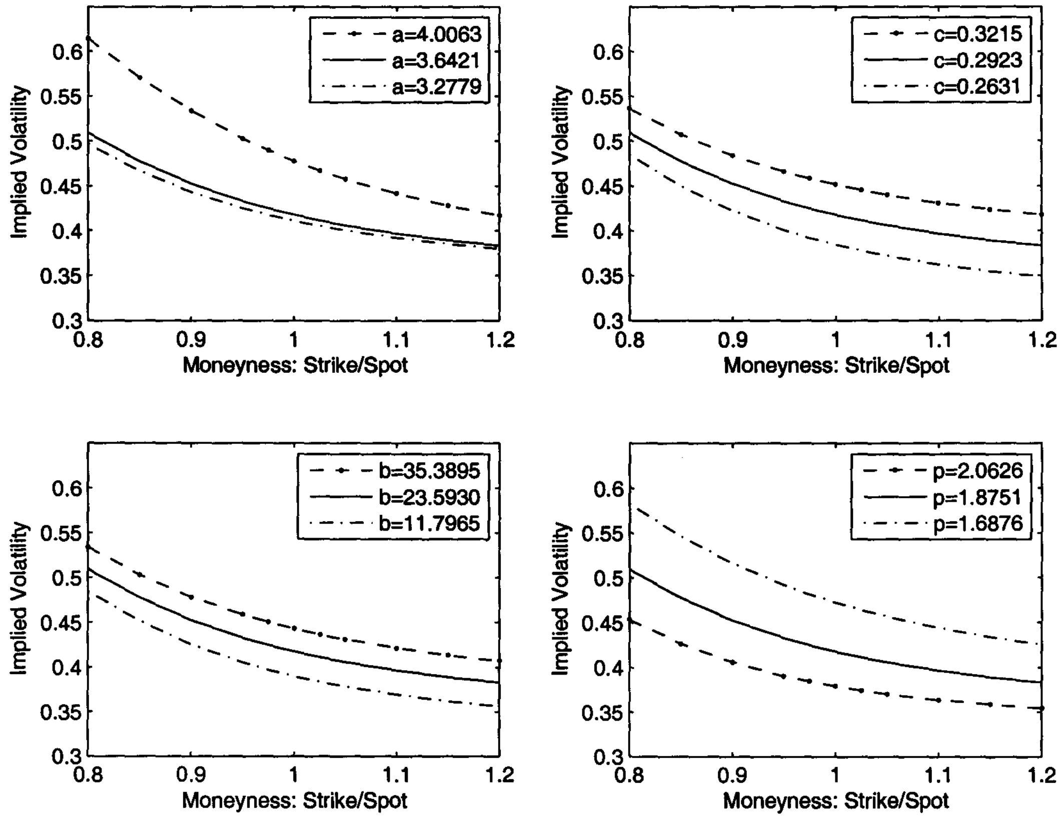


Figure 4.4: Model Implied Volatility Skew. Base parameters are chosen to be the parameters in Table 4.1.

Table 4.7 illustrates the two sets of parameters calibrated using separate datasets of implied volatility and CDS data. The two sets of parameters are very different, except for p. One may want to calibrate the model using implied volatility and CDS data simultaneously. A more advanced calibration scheme is needed to do this, since the naive minimizing RMSE is not feasible here. First, these two datasets have different denominations. The implied volatility data is denominated in percentage, while the CDS data is usually expressed in bps. Even though we can express CDS in percentage, it is not clear how to compare 50% implied volatility to 5% CDS. This is similar to

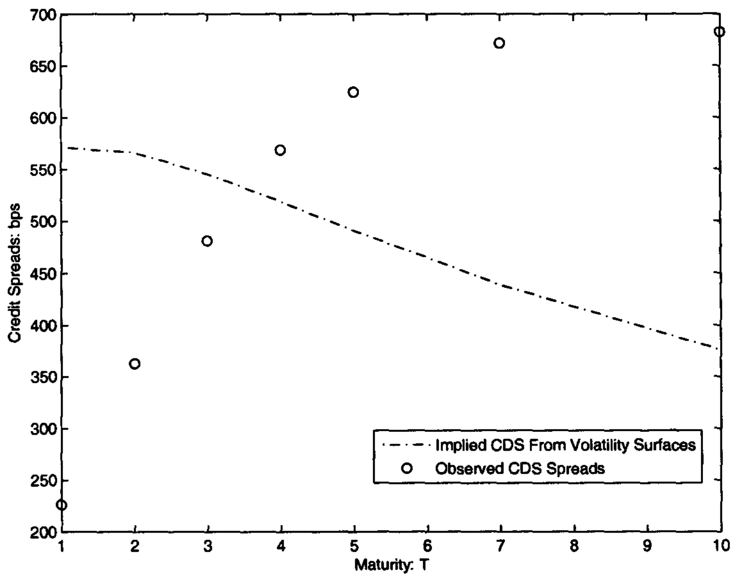


Figure 4.5: Term Structure of CDS Spreads. The dashed curve is calculated using the parameters calibrated from the implied volatility surface on March 16, 2007. The circles show the observed CDS term structure of Ford on March 16, 2007. The recovery rate is estimated at 0.3228.

Model	90	95	97.5	100	102.5	105	110
T=2m	64.0690	63.8675	63.7885	63.7205	63.6614	63.6097	63.5241
T=3m	64.1805	63.9865	63.9071	63.8371	63.7749	63.7195	63.6250
T=6m	64.4496	64.2677	64.1893	64.1179	64.0527	63.9929	63.8871
T=12m	64.6959	64.5432	64.4750	64.4115	64.3523	64.2970	64.1965
T=18m	64.6928	64.5694	64.5135	64.4610	64.4116	64.3650	64.2793

Table 4.6: Implied Volatility Surfaces (in %): using model parameters calibated from CDS spreads. The row with 90,...,110 denotes moneyness (in %). The root mean square error is 9.7122.

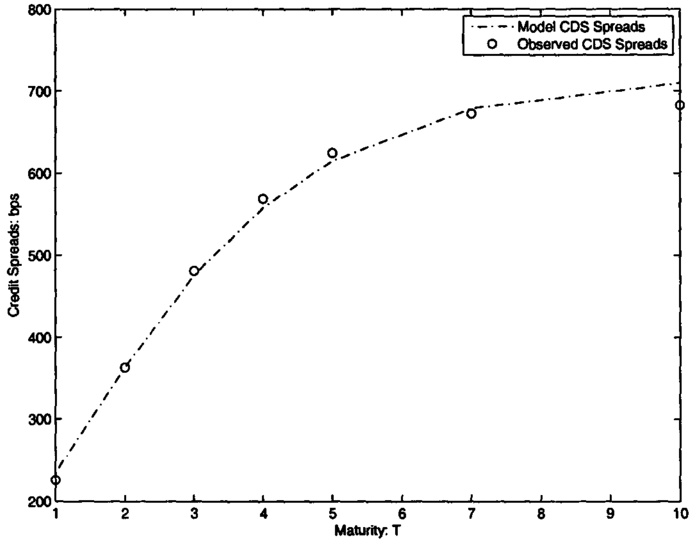


Figure 4.6: Fitting the Term Structure of CDS Spreads. The circles show the observed CDS term structure of Ford on March 16, 2007. The dashed-curve is model predicted CDS curve. Calibrated parameters are: $a = 0.6417$, $c = 0.6252$, $b = 1.0071$, $p = 1.8865$ and $R = 0.0094$.

asking if an orange is more beautiful than an apple. Second, there is no consensus on how to assign relative weights to different datasets. The calibrated parameters will be different for arbitrary choice of weights. The weights we use here for this exercise are 0 for one dataset and 1 for the other.

	a	c	b	p	R
VolPara	3.6421	0.2923	23.5930	1.8751	0.3228
CDSPara	0.6417	0.6252	1.0071	1.8865	0.0094

Table 4.7: Comparison of Calibrated Parameters. "VolPara" denotes the parameters calibrated from the implied volatility data of Ford Motor Corp. on March 16, 2007. "CDSPara" denotes the parameters calibrated from the CDS data of the same company on the same day.

Table 4.8 shows calibrated parameters from daily observed implied volatility surface data in March 2007. The calibrated parameters have large variations. The mean of the calibrated parameters are $a = 1.1105$, $c = 0.1937$, $b = 47.6545$ and $p = 1.2973$. We observe that parameter b is significantly larger than zero. This indicates that the component bS_t^{-p} in the volatility function $\sigma(S_t)$ has large contributions to the observed implied volatility.

In this calibration exercise, we find that our model is able to fit very well the market implied volatility surface and the market CDS curve separately on Ford's data. However, it is unsuccessful in fitting these two datasets jointly. Similar results have been obtained by Hull, Nelken & White (2004), who used implied volatility data to calibrate default parameters in Merton's model. They find that the model CDS spreads, calculated using the parameters calibrated from the implied volatility surface, is about 95 bps higher than the observed CDS spreads on average. Carr & Wu (2006) considered a two factor model with stochastic volatilities and stochastic hazard rates. Carr & Wu (2006) applied the EKF and calibrated their model using combined datasets of implied volatilities and CDS spreads. However, their calibration does not provide satisfactory results either, especially for fitting CDS spreads. About 50 percent of the variation in the CDS spreads on General Motors and only 30 percent on Altria Group can be explained by their model.

4.8 Extension of the Model

In our model, the local equity volatility is chosen to be of the form implied from the Leland-Toft structural model. A natural extension of the model is to choose the hazard rate function to be of the form implied from other structural models with jumps.

One shortcoming of the Leland-Toft model is that the short spreads are zero. A natural approach to raise the short spreads above zero is to add jumps. Hilberink &

Rogers (2002) extended the Leland-Toft model by generalizing the firm value process to be an exponential Levy process with only downward jumps. Chen & Kou (2006) introduced double exponential jumps to Leland-Toft and studied their implication for credit spreads and implied volatility. We are interested in deriving the implied hazard rate in Hilberink & Rogers (2002) and Chen & Kou (2006) as a function of the equity.

Both of the papers obtain an implicit expression of the hazard rate, h, as follows

$$h = \lambda \left(\frac{V_B}{V}\right)^{\eta}, \tag{4.42}$$

where λ denotes the Poisson rate of exponentially distributed downward jumps with parameter η. As before, V_B denotes the default barrier determined endogenously and V denotes the asset value. However, neither of the studies give an explicit formula for h as a function of the equity E.

When V is close to V_B, following Hilberink & Rogers (2002), we can approximate the equity as follows

$$E \approx V - V_B(1 + \log(V/V_B)). \tag{4.43}$$

Then applying Taylor expansion, to the first order approximation, we obtain

$$\frac{V}{V_B} \approx \sqrt{\frac{2E}{V_B}} + 1. \tag{4.44}$$

Therefore, equations (4.42) and (4.44) imply that the hazard rate, $h(E)$, as a function of the equity can be approximated by

$$h(E) \approx \lambda \left(\sqrt{\frac{2E}{V_B}} + 1\right)^{-\eta}. \tag{4.45}$$

Several observations about the above implied hazard rate function are worth noting. First, the implied hazard rate is a decreasing function of E/V_B. This captures a negative relationship of equity price and the hazard rate. Second, the implied hazard rate vanishes to zero as the equity E approaches infinity. This is reasonable since a company with infinite equity will not default. Third, the implied hazard rate is bounded above by the Poisson rate of downward jumps λ. The third observation differs significantly from the specification of hazard rate in Carr & Linetsky (2006) in that their hazard rate is unbounded, when the stock price goes to zero. An upper bounded hazard rate captures the fact that the firm value not only can jump-to-default, but also can diffuse down to the bankruptcy level. Figure 4.7 plots different curves of $h(E)$, for varying η.

Using the implied hazard rate function from Hilberink-Rogers, we are interested in the following model with the pre-default stock price S_t specified as

$$dS_t = [r + \lambda(a\sqrt{S_t} + 1)^{-\eta}]S_t dt + cS_t\sqrt{1 + bS_t^{-p}}dW_t.$$

Gram-Charlier approximation is not feasible for this model, since we do not have explicit expressions for its moments. Instead a numerical PDE approach can be applied to study its properties and implications.

4.9 Summary

Motivated by Linetsky (2006) and Carr & Linetsky (2006), we have analyzed the implied equity volatility as a function of the stock price, from the structural credit risk literature. We have derived that the implied equity volatility in Leland-Toft was a power decay function of the equity, bounded below by a positive constant, which can be thought of as the asset volatility. We then proposed a new jump-to-default model with the local volatility implied from Leland & Toft (1996). All the moments can be calculated explicitly for the transformed pre-default stock process. Gram-Charlier expansions were then used to approximate bond and call prices.

Numerical examples showed that the asymptotic approximation is very accurate and more than 70 times faster than the implicit finite difference method. The model was calibrated separately using one day implied volatility data and CDS data on the

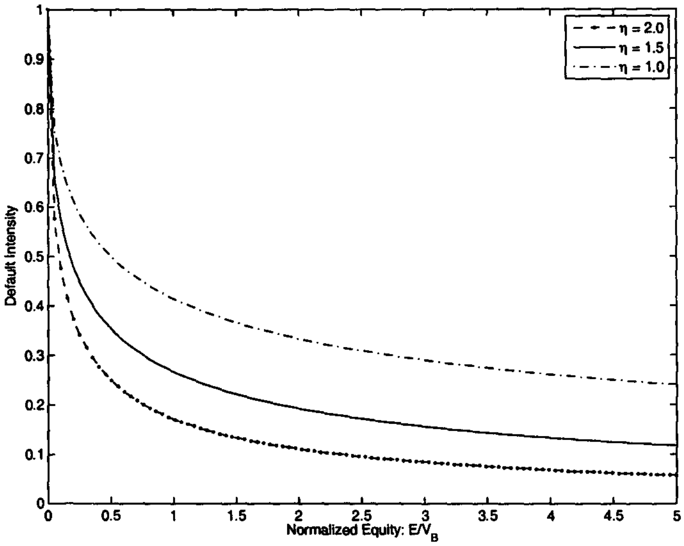


Figure 4.7: Hazard Rate implied from Hilberink & Rogers (2002) as a function of equity. The x-axis denotes the normalized equity, E/V_B. The y-axis denotes the hazard rate with maximum $\lambda = 1$.

same day for Ford Corp. We found that our model could fit the separate datasets very well respectively. However, it is unable to fit both the implied volatility surface and the CDS curve simultaneously. This indicates that either this model is not feasible for Ford or there exist arbitrage opportunities.

Our model is different from Carr & Linetsky (2006) in that our local volatility specification is bounded below by a positive constant, while the volatility of the CEV model goes to zero when the stock price approaches infinity. Our model is more realistic for firms that have large equity but low debt. We also discussed an extension of the model, whose hazard rate function is implied from Hilberink & Rogers (2002) and Chen & Kou (2006).

4.10 Appendix III

- **Proof of Proposition 4.4.1:**

  From equation (4.5) we have

$$S_T = S_0 e^{\int_0^T r+h(S_t)dt+\int_0^T \sigma(S_t)dW_t-\frac{1}{2}\int_0^T \sigma^2(S_t)dt},$$

  and therefore

$$\mathbb{E}[e^{-\int_0^T r+h(S_t)dt}\psi(S_T)] = S_0\mathbb{E}[e^{\int_0^T \sigma(S_t)dW_t-\frac{1}{2}\int_0^T \sigma^2(S_t)dt}S_T^{-1}\psi(S_T)],$$

  The proof is completed by invoking Girsanov's theorem.

- **Proof of Proposition 4.5.1:**

  Ito's lemma implies that

$$d[X^m] = [l_a(m)X^{m-1} + l_r(m)X^m]dt + cmX^{m-1}\sqrt{X^2+bX}dW.$$

  Integrating and then taking expectations will complete the proof.

- **Proof of Proposition 4.6.2:**

  We prove the result for $B_0(S_0, T)$ here; the proof for $C_K(S_0, T)$ is analogous. The Gram-Charlier expansion implies that

$$\begin{aligned} B_0(S_0,T) &= S_0\int_0^{+\infty} y^{-1/p}g(y)dy - \frac{\eta_1 S_0}{1!}\int_0^{+\infty} y^{-1/p}g'(y)dy \\ &\quad + \frac{\eta_2 S_0}{2!}\int_0^{+\infty} y^{-1/p}g''(y)dy - \frac{\eta_3 S_0}{3!}\int_0^{+\infty} y^{-1/p}g'''(y)dy + ... \end{aligned}$$

  We then calculate term by term.

$$\begin{aligned} S_0\int_0^{+\infty} y^{-1/p}g(y)dy &= S_0 e^{-\frac{1}{p}\hat{\mu}+\frac{1}{2p^2}\hat{\sigma}^2}. \\ S_0\int_0^{+\infty} y^{-1/p}g'(y)dy &= S_0\left[y^{-1/p}g(y)|_0^{+\infty} + \frac{1}{p}\int_0^{+\infty} y^{-\frac{1}{p}-1}dy\right] \\ &= \frac{S_0}{p}e^{-\frac{(1+p)}{p}\hat{\mu}+\frac{(1+p)^2}{2p^2}\hat{\sigma}^2}. \end{aligned}$$

  We have used lemma 4.6.3 for the above calculations. The other terms can be calculated similarly.

mm/dd/yy	a	c	b	p
03/02/07	0.1290	0.2253	53.8955	1.4321
03/05/07	0.4328	0.1970	89.9888	1.5359
03/06/07	0.9227	0.2019	53.7872	1.3986
03/07/07	1.2158	0.2597	64.0233	1.8103
03/08/07	1.0758	0.2002	86.7238	1.5823
03/09/07	0.9820	0.1648	73.4251	1.3491
03/12/07	0.2719	0.2007	80.0007	1.4426
03/13/07	0.3554	0.1445	29.8695	0.7924
03/14/07	1.3785	0.2562	30.0186	1.5021
03/15/07	1.2250	0.2501	30.0157	1.4726
03/16/07	3.6421	0.2923	23.5930	1.8751
03/19/07	1.0784	0.1675	24.9472	1.1088
03/20/07	0.2720	0.0913	29.9566	0.5109
03/21/07	1.4006	0.2397	15.0315	1.3369
03/22/07	1.8488	0.1903	51.6305	1.5006
03/23/07	2.0773	0.2529	57.2209	1.7950
03/26/07	1.5601	0.2069	44.2211	1.4781
03/27/07	1.5739	0.2003	44.2211	1.4389
03/28/07	1.5828	0.1913	44.2210	1.4144
03/29/07	0.1413	0.0578	44.1666	0.1973
03/30/07	0.1536	0.0771	29.7862	0.2684
Mean	1.1105	0.1937	47.6545	1.2973
	(0.8372)	(0.0611)	(21.5153)	(0.4688)

Table 4.8: Calibrated parameters for Ford Motor Corp. from daily implied volatility surface data in March, 2007. The numbers in the parentheses are standard deviations.

Chapter 5

Conclusions

The three working papers that comprise the body of this thesis provide new ideas and insights on different perspectives of credit risk modeling.

- **Imperfect Information**

  In reality, the information we receive about the credit quality of a company is always contaminated. The worse the contaminated information, the higher the short spreads. Therefore, it is important to consider the impact of incomplete information when we do quantitative modeling on credit risk. The new idea in the RS model is to take account of imperfect information by randomizing the initial condition of the risk factor. The RM-II model provides an example that admits no default intensity, yet still generates finite positive short spreads. The RBC-II model generates positive short spreads through its positive default intensity. The RS model was also applied to a mean-reverting solvency ratio where again it can be seen to raise short spreads.

- **Rich Correlation Structure versus Nonnegativity**

  For any company, both the interest rate and the short spread should be nonnegative. In a multi-firm set up, both negative and positive default correlations should be possible among different firms. The extant intensity-based factor models are not easily able to generate rich correlation structures among different firms and preserve the nonnegativity constraints. The iCIR framework we present provides a new class of multi-factor semi-affine models that are able to generate rich correlation structures while preserving nonnegativity.

- **Integrated Market-Credit Risk**

  The equity market contains information that is useful for credit risk analysis. Data in the credit market is usually scarce and contaminated, while data from the equity market is plentiful and more reliable. Therefore, it is not wise to ignore the equity market data while analyzing the credit market. Our new hybrid model introduces a way of pricing both equity and credit derivatives in a unified framework. This framework has broad financial implications and inherits the best features of many existing models on credit risk, from Leland-Toft to Carr-Linetsky.

- **Model Usability**

  The essence of credit models is to be applied in real life financial risk management. A good model should fit the facts with as few parameters as possible and take the shortest computational time. The RS models have few parameters and lead to explicit expressions for PD, LGD and CS. The iCIR model has only two CIR factors for each company and admits explicit expressions for both default and default-free bonds. The Gram-Charlier expansion used in the new hybrid model significantly speeds up the computation of bond and call prices.

It would be interesting to empirically test if a firm's short spread will fall after its annual report. In the iCIR model, the calibration has proven to be difficult. Searching for a better calibration scheme is another direction for future research. In the hybrid model, we are not able to fit the equity and credit market data at the same time. Further investigation is needed to either modify this model to fit both market data or use this model to search for arbitrage opportunities if any.

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