COMPACT SUPPORT AND DEAD CORES FOR STATIONARY DEGENERATE DIFFUSION EQUATIONS

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By

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To my parents

Abstract

For a sign-changing function $a(x) \in C^{\alpha}_{loc}(\mathbb{R}^n)$ with bounded $\Omega^+ = \{x \in \mathbb{R}^n \mid a(x) > 0\}$, we study non-negative entire solutions $u(x) \geq 0$ of the semilinear elliptic equation $-\triangle u = a(x)u^q + b(x)u^p$ in $\mathbb{R}^n$ with $n \geq 3$. $0 < q < 1$, $p > q$, and $\lambda > 0$. We consider two types of coefficient $b(x) \in C^{\alpha}_{loc}(\mathbb{R}^n)$, either $b(x) \leq 0$ in $\mathbb{R}^n$, or $b(x) \equiv 1$. In each case, we give sufficient conditions on $a(x)$ for which all solutions must have compact support. In case Ω^+ has several connected components, we also give conditions under which there exist "dead core" solutions which vanish identically in one or more of these components. In the "logistic" case $b(x) \leq 0$, we prove that there can be only one solution with given dead core components. In the case $b(x) \equiv 1$, the question of existence is more delicate, and we introduce a parametrized family of equations by replacing $a(x)$ by $a_\gamma = \gamma a^+(x) - a^-(x)$. We show that there exists a maximal interval $\gamma \in (0, \Gamma]$ for which there exists a stable (locally minimizing) solution. Under some hypotheses on a^- near infinity, we prove that there are two solutions for each $\gamma \in (0, \Gamma)$. Some care must be taken to ensure the compactness of Palais–Smale sequences, and we present an example which illustrates how the Palais–Smale condition could fail for certain $a(x)$. The analysis is based on a combination of comparison arguments, *a priori* estimates, and variational methods.

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Contents

Chapter 1

Introduction

In my thesis I study the following elliptic problem in $\mathbb{R}^n$, $n \geq 3$:

$$\begin{cases} -\Delta u = a(x)u^q + b(x)u^p & \text{in } \mathbb{R}^n , \quad 0 < q < 1, \ p > q, \\ u \geq 0 & \text{in } \mathbb{R}^n , \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \end{cases} \tag{1.0.1}$$

where $a(x)$ and $b(x)$ are Hölder continuous in $\mathbb{R}^n$. By $\mathcal{D}^{1,2}(\mathbb{R}^n)$ we mean the space of functions with finite Dirichlet "energy", more precisely, it is the completion of $C_0^\infty(\mathbb{R}^n)$ under the Dirichlet semi-norm, $(\int_{\mathbb{R}^n} |\nabla u|^2 dx)^{1/2}$. The important feature of this equation is that it combines a non-Lipschitz nonlinearity u^q with a sign-changing coefficient $a(x)$, and it was originally observed by Schatzman [38] that solutions could vanish on large sets and in fact that, under appropriate hypotheses on $a(x)$, there exist solutions with compact support. The goal of this thesis is to study compactly supported solutions to (1.0.1). In particular, we give conditions on $a(x), b(x)$ which ensure that *all* solutions have compact support. We also study the number and support properties of the solution set under different assumptions about the size and shape of $a(x)$ and the sign of $b(x)$.

Equations of the type (1.0.1) arise as stationary solutions to degenerate reaction-diffusion equations of the form which was proposed by Namba [31] as a mathematical model of population dynamics,

$$w_t - \Delta(w^m) = w(a(x) + b(x)w^s), \qquad s > 0 \text{ and } m > 1.$$

Assuming time-independence and making the change of variable $u = w^m$, we arrive at (1.0.1) with $q = 1/m$ and $p = (s+1)/m > q$. If $w(x,t)$ represents a population density, then $a(x)$ represents a sort of growth rate, and the region where $a(x) > 0$ is favorable to population growth, whereas the region where $a(x) < 0$ is hostile to the species. The initial-boundary value problem for the above reaction-diffusion equation is studied in [6], and the attractivity properties of non-negative solutions are investigated in [23].

The sub-linearity hypothesis, $0 < q < 1$, is essential for phenomena which we study. If instead we consider the same equation (1.0.1) with $q \geq 1$, then a simple application of the Strong Maximum Principle (see Lemma 2.1.1) shows that a nonnegative solution must be strictly positive everywhere in $\mathbb{R}^n$, and so compactly supported solutions, and the rich structure of the solution spaces which we study here, would be impossible.

We denote as usual, $a^+(x) = \max(0, a(x))$ and $a^-(x) = \max(0, -a(x))$. The support properties of the solutions depend principally on the regions where $a(x) \geq 0$ and on the size of the positive part $a^+(x)$. Let $\Omega^+ = \{x \in \mathbb{R}^n \mid a(x) > 0\}$, $\Omega^{0+} = \{x \in \mathbb{R}^n \mid a(x) \geq 0\}$ and $\Omega^- = \{x \in \mathbb{R}^n \mid a(x) < 0\}$. As we will see, the Strong Maximum Principle cannot be applied in the region Ω^-, which means that a nonnegative solution may become identically zero in any subregion of Ω^-. This is consistent with the biological interpretation of Ω^- as a region which is hostile to the species, and it is natural both for mathematical reasons and for applications to assume that the following condition is always met:

Basic Hypothesis: *The domain Ω^+ is bounded and non-empty.*

In other words, the favorable region Ω^+ consists of bounded islands, surrounded by the unfavorable Ω^-. When Ω^+ consists of several connected components we must make the following hypothesis about the nature of the favorable region:

$$\begin{cases} \Omega^+ \text{ has } k < \infty \text{ connected components with } \Omega^+ = \cup_{i=1}^k \Omega_i^+, \\ \text{and each connected component } \Omega_i^+ \text{ satisfies an interior ball condition.} \end{cases} \tag{1.0.2}$$

We will see later (see Lemma 2.1.3) that the Strong Maximum Principle *does* apply in each of the components Ω_i^+ of Ω^+. The conclusion is that for any nonnegative solution u of (1.0.1) and for each individual component Ω_i^+, either $u > 0$ in Ω_i^+ or $u \equiv 0$ in Ω_i^+. In principle, given any sub-collection of the components Ω_i, $i \in I \subset \{1, \ldots, k\}$, we could hope to find a solution u with $u > 0$ for those components, and zero in the others. Following Bandle, Pozio, and Tesei [7], we call such solutions, which vanish identically in some part of the favorable region Ω^+, <u>*dead core*</u> solutions.

To organize the space of solutions of (1.0.1) according to the pattern of the supports, we define the following classes of solutions:

Definition 1.0.1. $M = \{1, 2, ..., k\}$

(1) For any non-empty $I \subset M$, denote by S_I the class of solutions of (1.0.1) which are positive in $\Omega_I^+ = \cup_{i \in I} \Omega_i^+$.

(2) N_I denotes the set $\{u \in S_I \mid u \equiv 0 \text{ in } \Omega^+ - \Omega_I^+\}$.

Thus, a solution in S_M is positive in the entire set Ω^+, whereas the elements of N_I, $I \neq \emptyset$, have some dead cores.

The results for (1.0.1) are different depending on the sign of $b(x)$. So we will focus on two cases: first, if $b(x) \leq 0$ we'll call the nonlinearity of "logistic" type. To emphasize this dependence, we denote the equation by $(1.0.1)_0$; for $b(x) \geq 0$ we specialize to the case $b(x) \equiv 1$, and denote the equation by $(1.0.1)_1$. We call this case the "concave plus convex" nonlinearity, as we expect to prove multiplicity results along the lines of Ambrosetti, Brezis, and Cerami [5] (see also [2].)

1.1 Logistic Nonlinearities

In this case equation (1.0.1) always admits a nontrivial nonnegative solution.

Theorem 1.1.1. *There exists a classical maximal solution* $U \in \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *of* $(1.0.1)_0$. *Moreover* $U \leq \omega$, *where* ω *is the unique positive solution to*

$$-\Delta\omega = a^+\omega^q \quad \text{in } \mathbb{R}^n, \quad \lim_{|x|\to\infty}\omega = 0.$$

The existence and uniqueness of $\omega(x) > 0$ follows from Brezis and Kamin [12], and the Theorem is proven by means of monotone iteration method and sub-super solution method.

If $q \geq 1$, by the strong maximum principle there could not be any compactly supported solution at all, so the sub-linearity of q is crucial for the existence of solution with compact support. In order for solutions to have compact support, we also require more information about $a(x)$. In case Ω^{0+} is unbounded, the strong maximum principle leads us to expect that in general solutions will not have compact support. So to obtain compactly supported solutions we will assume,

$$\Omega^{0+} \text{ is nonempty and bounded.} \tag{1.1.1}$$

However, this hypothesis is not sufficient. and in addition we must impose some conditions on the decay of the negative part $a^-(x)$. We prove:

Theorem 1.1.2. *Assume* (1.1.1). *There exist* $\eta > 0$ *and* $\rho_2 > 0$ *so that if*

$$\liminf_{|x|\to\infty} a^-|x|^{(n-2)(1-q)} > \eta, \tag{1.1.2}$$

every weak solution u *of* $(1.0.1)_0$ *is classical and compactly supported, with supp(* u*) contained in* $B(0,\rho_2)$.

To understand the interdependence of the constants in Theorem 1.1.2, think of $a^+(x)$ as being fixed (here $b(x)$ can vary) and consider how the asymptotic behavior of $a^-(x)$ affects the support of the solution. By the hypothesis (1.1.1) we can choose $\rho_1 > 0$ so that $\Omega^{0+} \subset\subset B(0, \rho_1)$. Then, the constant η will depend on q, p, n, ρ_1, and $||a^+||_{L^\infty(\mathbb{R}^n)}$. If a^- is decaying too rapidly to zero, then we may not have compact supported solutions (see Remark 3.1.12 at the end of Chapter 3 section 2.) Nevertheless, we cannot claim that condition (1.1.2) is sharp, as we will see in Theorem 1.1.8 below. The method we use to prove above theorem is based on a priori estimates and comparison method, inspired by Cortázar, Elgueta and Felmer [18] on the constant-coefficient equation $-\Delta u = u^p - u^q$, which is quite different from the one used by Schatzman [38], who used Puel's existence theorem [34] to construct compactly supported solutions.

In case $a(x)$ is bounded away from zero at infinity the proof is somewhat simpler:

Corollary 1.1.3. *Assume* (1.1.1), *and suppose that there exists* $\alpha > 0$ *and* $\rho_1 > 0$ *such that*

$$a^-(x) \geq \alpha \qquad \textit{for all } |x| \geq \rho_1. \tag{1.1.3}$$

Then, there exists $R > 0$ *so that every weak solution* u *has support supp(u)* $\subset B(0,\ R)$. *Moreover,* R *depends only on* $q, p, n,\ \rho_1,$ *and* $||a^+||_{L^\infty(\mathbb{R}^n)}$.

We cannot expect the result of Theorem 1.1.2 to hold in $\mathbb{R}^1$ or $\mathbb{R}^2$. Nevertheless, under the hypothesis (1.1.3) it is true that any $L^t(\mathbb{R}^n)$ solution (with $t \geq 1$) in $\mathbb{R}^n$, for any $n \geq 1$, must be compactly supported (see Theorem 1.2.2.) However, we cannot prove the uniform control on the support in terms of the coefficients as in Corollary 1.1.3.

We also study the structure of the solution set of $(1.0.1)_0$ in case the favorable domain Ω^+ has several components as stated in hypothesis (1.0.2). Recall that a solution in N_I has *dead cores*, as it vanishes identically in part of the favorable set Ω^+. If dead core solutions do exist, the class S_I can contain many elements: see the following proposition and Theorem 1.1.8. However, in many cases the class N_I can have at most one solution. Following the idea in [7], we present a generalization of the uniqueness result of Spruck [39]:

Theorem 1.1.4. *Assume (1.0.2), if $p \geq 1$, then the number of elements in N_I is at most 1 for any non-empty I. In particular if $k = 1$, then the solution to (1.0.1) is unique and its support is connected.*

The method we adopt is from C. Bandle, M.A. Pozio and A.Tesei [7, 8], which they used to show uniqueness for bounded domain with both Dirichelet and Neumann boundary condition. We may also prove uniqueness of solutions in class N_I with $q < p < 1$ under some additional hypotheses on $b(x)$; see Theorem 3.2.13. Actually Spruck [39] imposed a monotonicity condition, $x \cdot \nabla a(x) < 0$, and considered $(1.0.1)_0$ with $b(x) \equiv 0$. He proved uniqueness of compactly supported solutions by means of a special version of Hopf's boundary lemma. He also proved that, under the same hypotheses, the support of the solution is star-shaped with Lipschitz boundary.

When Theorem 1.1.4 applies, the solution space of $(1.0.1)_0$ is completely characterized by the support properties of the solutions. Consider the following example: let Ω_i^+, $i = 1, \ldots, k$ be any smooth, compact and connected sets in $\mathbb{R}^n$, with $\min_{i \neq j} \text{dist}\,(\Omega_i^+, \Omega_j^+) > 0$. Let $b(x) \equiv -1$ and define $a(x) = a_\lambda(x)$ by

$$a(x) = \begin{cases} \lambda, & \text{if } x \in \Omega_i^+,\ i = 1, \ldots, k, \\ -1, & \text{if } x \notin \cup_{i=1}^k \Omega_i^+, \end{cases} \tag{1.1.4}$$

where $\lambda > 0$ is a fixed constant. (See figure 1.1.) Combining the results of Corollary 1.1.3 and Theorem 1.1.4, we have:

Proposition 1.1.5. *Assume $a(x)$ is defined as in (1.1.4) with fixed $\lambda > 0$, $b(x) \equiv -1$, and $p \geq 1$.*

(1) There exists $\delta^* = \delta^*(\Omega^+, \lambda) > 0$ so that if $\min_{i \neq j}$ dist$(\Omega_i^+, \Omega_j^+) \geq \delta^*$, then N_I contains exactly one *solution for each $I \neq \emptyset$.*

(2) *There exists* $\delta_* = \delta_*(\Omega^+, \lambda) > 0$ *so that if* $\max_{i \neq j}$ *dist*$(\Omega_i^+, \Omega_j^+) \leq \delta_*$, *then* (1.0.1) *admits* exactly one *solution in* $\mathbb{R}^n$. *This solution is positive on the set* Ω^+.

Note that in case (1), for Ω^+ with k connected components, the equation (1.0.1) admits exactly $2^k - 1$ nontrivial solutions in all.

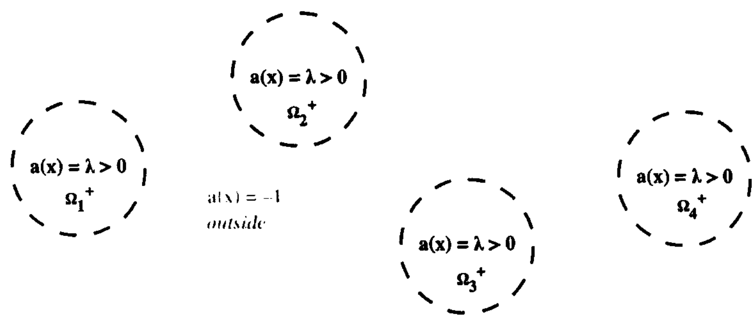

Figure 1.1: $a(x)$ as in the example (1.1.4).

Theorem 1.1.1 established the existence of maximal solution in general, that is, positive in all of the favorable set Ω^+. In addition, under (1.0.2) we can assert the existence of a minimal solution in S_I for each I:

Theorem 1.1.6. *Under the hypothesis (1.0.2),* S_I *has a minimum element* $u_I \in \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *for any non-empty* $I \subset M$.

In a bounded domain, the existence of minimal solution in S_I is proven in Theorem 4 of [33]. There is a connection between the maximum solution of $(1.0.1)_0$ and the minimum energy solution of the functional $E: \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^{q+1}(\mathbb{R}^n) \to \mathbb{R}$, defined by

$$E(v) = \int_{\mathbb{R}^n} |\nabla v|^2 dx - \frac{1}{q+1} \int_{\mathbb{R}^n} a(v^+)^{q+1} dx - \frac{1}{p+1} \int_{\mathbb{R}^n} b(v^+)^{p+1} dx.$$

If we assume $a(x)$ and $b(x)$ are uniformly bounded in $\mathbb{R}^n$, then E is smooth (see [22]). The interesting fact is that $\inf_{v \in \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^{q+1}(\mathbb{R}^n)} E(v)$ is achieved at a non-negative function U, which is a solution of $(1.0.1)_0$. By minimization it is easy to see that $U > 0$ in all connected components of Ω^+, that is $U \in S_M$. By Theorem 1.1.4, when

$p \geq 1$ this minimum energy solution is the (unique) maximal solution in S_M. On the other hand, as pointed out in [2], if there do exist dead core solutions (in some class N_I, $I \neq \emptyset$) then these solutions cannot be energy minimizers even modulo any finite dimensional subspace. As a result, our study of dead core solutions to $(1.0.1)_0$ will rely on comparison arguments, monotone iteration, and *a priori* estimates rather than variational methods.

Another way to influence the support properties of the solutions is by varying the relative strengths of the positive and negative parts of $a(x)$. We introduce a parameter $\lambda > 0$, and set

$$a_\lambda(x) = \lambda a^+(x) - a^-(x).$$

Clearly this does not affect the geometry of the favorable and unfavorable regions, and Ω^+, Ω^{0+} and Ω^- remain the same for the whole family of a_λ. Intuitively, we expect that the size of the support of the solutions of

$$-\Delta u = a_\lambda(x)u^q + bu^p \text{ in } \mathbb{R}^n, \quad u \geq 0 \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \tag{1.1.5}$$

should grow with increasing λ. We will study the asymptotic behavior of solutions for large and small λ. To do so, we require a stronger condition on the geometry of the function $a(x)$ and its sets Ω_i^{0+}:

Definition 1.1.7. *We say Ω^{0+} is admissible if it is also bounded, (1.0.2) holds, Ω^{0+} also has k connected components with $\Omega^{0+} = \cup_{i=1}^k \Omega_i^{0+}$ and $\Omega_i^+ \subset \Omega_i^{0+}$ for $i \in M$. Moreover, $dist(\Omega_i^{0+}, \Omega_j^{0+}) > 0$ for $i \neq j$.*

For admissible Ω^{0+} we have the following Theorem:

Theorem 1.1.8. *Assume Ω^{0+} is admissible, there exists $\lambda_* > 0$ so that for all $\lambda < \lambda_*$, all solutions of (1.1.5) are compactly supported and $N_I^q \neq \emptyset$ for any non-empty collection $I \subset M$.*

Note that for $\lambda < \lambda_*$ the compact support of the solutions follows even in the absence of asymptotic conditions on $a(x)$. showing that condition (1.1.2) cannot be sharp. To emphasize the dependence on λ. equation (1.1.5) is often referred as $(1.1.5)_\lambda$ (the subscript λ is omitted if no confusion arises). For λ big, we have the following Theorem in case $q < p \leq 1$.

Theorem 1.1.9. *Assume (1.0.2), if $q < p \leq 1$, there exists $\lambda_1^* > 0$ so that the equation $(1.1.5)_\lambda$ has a unique solution u_λ, which is positive in $\overline{\Omega^+}$ for all $\lambda > \lambda_1^*$. Moreover,*

$$\lim_{\lambda\to\infty} \frac{u_\lambda}{\lambda^{\frac{1}{1-q}}} = \begin{cases} \omega, & \text{if } p < 1 \\ \omega_b, & \text{if } p = 1, \end{cases}$$

uniformly on $\mathbb{R}^n$, where $\omega, \omega_b > 0$ are as in Theorem 3.1.8 and Lemma 3.3.5.

In particular, combining Theorem 1.1.2 and above theorem we conclude that as $\lambda \to \infty$ the support must grow.

Corollary 1.1.10. *Assume (1.0.2), if $q < p \leq 1$ and $\liminf_{|x|\to\infty} a^-|x|^{(n-2)(1-q)} = \infty$, then there exists $\lambda_2^* > 0$ so that problem (1.1.5) has a unique compactly supported solution u_λ with $u_\lambda > 0$ in Ω^+ for all $\lambda \geq \lambda_2^*$. Moreover, u_λ increases point-wise as λ increases, and so supp (u_λ) expands to $\mathbb{R}^n$ as $\lambda \to \infty$,*

$$\cup_{\lambda>0}\text{supp}(u_\lambda) = \mathbb{R}^n.$$

For the case $p > 1$, the asymptotic behavior is more complicated, and depends strongly on the form of $b(x)$. Some specific results are proven in Theorem 3.3.7 in section 4.

To illustrate our results on the parametrized problem (1.1.5), we return to the previous piecewise constant $a(x)$ from our example (1.1.4). For simplicity, assume $b(x) \equiv 0$. For λ small enough, Theorem 1.1.8 applies and we conclude that (1.1.5) admits a unique solution in N_I for all $I \neq \emptyset$, so (1.1.5) admits exactly $2^k - 1$ solutions in all, and all but one has dead cores. For λ sufficiently large, by Corollary 1.1.10 the equation has exactly one solution which is positive in all of Ω^+.

In [7, 8] C. Bandle, M.A. Pozio and A.Tesei studied dead core solutions for this problem in a bounded domain with both Dirichlet and Neumann boundary condition, and S. Alama [2] used a bifurcation analysis for dead cores in the Neumann problem for similar equations. Most other previous work on equations of the form $(1.0.1)_0$ has been for bounded domains or for constant coefficient equations in the whole space $\mathbb{R}^n$: see [10, 14, 17, 18, 21, 27, 33, 34, 29, 40] and the reference therein.

1.2 Concave Plus Convex Nonlinearity

In this part our equation takes the form,

$$\begin{cases} -\Delta u = a(x)u^q + u^p & \text{in } \mathbb{R}^n,\ n \geq 3, \\ u \geq 0 \quad \text{in } \mathbb{R}^n, & u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \end{cases} \tag{1.2.1}$$

We must further restrict the nonlinear terms and asymptotic behavior of a: we assume that Ω^+ is bounded, non-empty and (1.0.2) holds, moreover we add the following hypothesis:

$$0 < \liminf_{|x|\to\infty} a^-(x) \leq \limsup_{|x|\to\infty} a^-(x) < \infty \qquad \textit{and} \qquad 0 < q < 1 < p < \frac{n+2}{n-2}. \tag{1.2.2}$$

The additional hypotheses on p are important for several reasons, which will be explained later on. The upper bound on p is related to the compactness in the Sobolev embedding theorem, and it is well known that basic *a priori* estimates of solutions of elliptic equations can fail when there is no such assumption on p.

Since $b \equiv 1$ and $p > 1$ the nonlinearity combines convex and concave terms in Ω^+, as has been studied by Ambrosetti, Brezis, and Cerami [5] (see also [2]), so we expect some similar results. In particular, unlike the case discussed in the first part, we no longer expect uniqueness of solutions in each set N_I, and seek a second solution using variational methods.

To illustrate the difficulties which can arise in the concave plus convex case, first we observe that Cortázar, Elgueta, and Felmer [18] have proven that the equation $-\Delta v = v^p - v^q$ in $\mathbb{R}^n$ has a compactly supported solution with connected support, which is unique up to translation. This suggests that (1.2.1) could have a solution whose support lies completely in Ω^-. (Note that we prove this is impossible in the logistic case $b(x) \leq 0$, see Lemma 3.2.10) Indeed, consider the following special example:

Example 1.2.1. *Let $\Omega^+ \subset\subset B(0,r)$ and $a(x) \equiv -1$ in $\mathbb{R}^n \setminus B(0,r)$ for some $r > 0$. Again from [18], we may construct arbitrarily many solutions of (1.2.1) by gluing together the compactly supported solutions of $-\Delta v = v^p - v^q$ in disjoint balls in $\mathbb{R}^n - B(0,r)$. (See figure 1.2.)*

In particular, for such $a(x)$, the variational functional associated to (1.2.1) cannot satisfy the Palais–Smale condition. Furthermore, it leads us to the following difficult question: when can we prove multiplicity of solutions of (1.2.1), so the solutions have connected support? And, do the solutions differ in the set Ω^+?

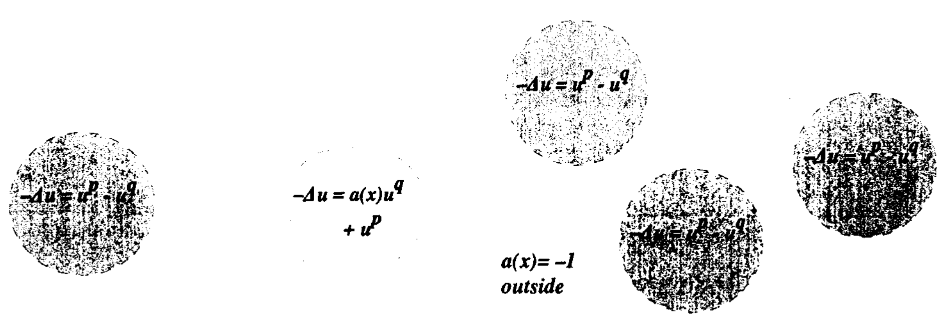

Figure 1.2: $a(x)$ as in Example 1.2.1.

We now state our results in the case $b(x) \equiv 1$. First, all solutions of (1.2.1) must again have compact support:

Theorem 1.2.2. *Every weak solution of (1.2.1) is a compactly supported classical solution.*

The proof of this result also uses the technique from Cortázar, Elgueta, and Felmer [18]. In contrast to Theorem 1.1.2, there is no uniform control on the support of solutions due to lack of uniform a priori decay estimates on the solutions. Again, the failure of such a uniform decay estimate can be seen from Example 1.2.1.

The existence of solutions to (1.2.1) is also more delicate than the logistic case. For a given $a(x)$ satisfying the required conditions there may not be an entire nonnegative solution with support in Ω^+ at all. We are thus led to consider the parametrized family as in the previous part,

$$-\Delta u = a_\gamma u^q + u^p, \quad u \in D^1(\mathbb{R}^n), \quad u \geq 0 \text{ in } \mathbb{R}^n, \tag{1.2.3}$$

where $a_\gamma = \gamma a^+ - a^-$ and $\gamma > 0$. When $\gamma > 0$ is small, we show that there exists a "small" solution, the minimal solution of (1.2.3), but for γ large there is no nonnegative solution at all:

Theorem 1.2.3. *Assume* (1.2.2). *For any non-empty* $I \subset M$, *there exists* $0 < \Gamma_I < \infty$ *such that:*

(1) $S_I \neq \emptyset$ *when* $0 < \gamma \leq \Gamma_I$*;*

(2) $S_I = \emptyset$ *when* $\gamma > \Gamma_I$*;*

(3) S_I *has a minimal element* $u_{I,\gamma}$ *for all* $0 < \gamma \leq \Gamma_I$*;*

(4) $\|u_{I,\gamma}\|_\infty \to 0$ *as* $\gamma \to 0^+$.

In particular S_M is not empty for $0 < \gamma \leq \Gamma_M$. This result is proven in two parts: the existence of the interval $(0, \Gamma_I)$ is proven in section 2 of Chapter 2. The existence of a minimal solution is in section 1 of the same chapter. The crucial step for proof of above theorem is that for small γ, we can construct an explicit super-solution in $\mathbb{R}^n$ depending on γ. We note that the existence of a solution at the endpoint $\gamma = \Gamma_I$ is not trivial, and follows from estimates of the minimal solution (see remark after Theorem 1.2.5.) The condition $p > 1$ is necessary for construction of the super-solution (see the proof of Lemma 4.2.2.)

As in [2, 3] we may view this existence theorem as a bifurcation result in the parameter γ. We expect that family of solutions bifurcates from the trivial solution at $\gamma = 0$, and the extremal value Γ_I is a sort of turning point in a bifurcation curve. The difficulty with making this precise for (1.2.3) is that the linearization is singular at $u = 0$, and so standard continuation methods (see Crandall and Rabinowitz [20]) do not apply.

Just as in Theorem 1.1.8 in the previous section, if Ω^{0+} is admissible, we also can say something about the dead core solutions.

Proposition 1.2.4. *Assume Ω^{0+} is admissible, there exists $\gamma_* > 0$ so that for all $\gamma < \gamma_*$, $N_I \neq \emptyset$ for any non-empty collection $I \subset M$.*

This proposition is proven in section 4.2. Although we have $p > 1$, we can not show that N_M has a unique element. Actually we expect that the element in S_M (or N_M) is not unique!

To study multiplicity of solutions, we adopt a variational framework for our problem. As mentioned in the previous section, variational analysis of solutions with dead cores (in N_I, $I \neq M$) is difficult since these solutions have infinite dimensional negative spaces associated to them. So in the remainder of the results we will only consider the solutions $u \in S_M$, that is $u(x) > 0$ on all of Ω^+. In the following, we denote by $\Gamma = \Gamma_M$ and U_γ the minimal solution in S_M for $0 < \gamma \leq \Gamma$. We also denote by $S_{I,\gamma}$ the class of solutions of $(1.2.3)_\gamma$ from Definition 1.0.1.

Consider the Banach sapce

$$H_q^1 = \{v \in \mathcal{D}^{1,2}(\mathbb{R}^n) \mid \int_{R^n} |v|^{q+1} dx < \infty\}$$

endowed with the norm

$$||v||_{H_q^1} = (\int_{R^n} |\nabla v|^2 dx)^{\frac{1}{2}} + (\int_{R^n} |v|^{q+1} dx)^{\frac{1}{q+1}}.$$

Define the energy functional $I_\gamma : H_q^1 \to R$ associated with (1.2.3) as

$$\begin{aligned} I_\gamma(v) = \frac{1}{2}\int_{R^n} |\nabla v|^2 dx - \frac{\gamma}{q+1}\int_{R^n} a^+(v^+)^{q+1} dx + \frac{1}{q+1}\int_{R^n} a^-(v^+)^{q+1} dx \\ - \frac{1}{p+1}\int_{R^n} (v^+)^{p+1} dx \end{aligned}$$

From [19] we see that I_γ is C^1 from H_q^1 to $\mathbb{R}^1$. Consider the following minimization problem in a convex constraint set

$$\text{Inf } \{I_\gamma(v) \mid v \in Y\} \qquad and \qquad Y = \{v \in H_q^1 \mid 0 \le v \le U_\Gamma \ a.e.\}.$$

From Lemma 4.3.1 the infimum is attained at some function in Y, say v_γ, and $v_\gamma \in S_{M,\gamma}$. Then under the following hypothesis

$$\begin{cases} \Omega^{0+} \text{ has } m < \infty \text{ connected components with } \Omega^{0+} = \cup_{i=1}^m \Omega_i^{0+}, \\ \text{and (1.0.2) holds, } \Omega_i^{0+} \cap \Omega^+ \ne \emptyset \text{ for every i= 1, ..., m,} \end{cases} \tag{1.2.4}$$

we show that these solutions are actually local minimizers of I_γ in the H_q^1 topology:

Theorem 1.2.5. *Assume* (1.2.4). *For* $0 < \gamma < \Gamma$, v_γ *is a local minimizer for* I_γ *in* H_q^1; *that is, there exists* $\delta > 0$ *such that*

$$I_\gamma(v_\gamma) \le I_\gamma(v) \quad for\ all\ v \in H_q^1\ with\ ||v - v_\gamma||_{H_q^1} < \delta.$$

We recall that Brezis and Nirenberg [13] first observed that minimization in the C^1-topology (for example, the sub- and super-solution construction above) yields minima in the weaker H^1-topology for a large class of subcritical elliptic variational problems.

See also [4] for remarks on supercritical problems. Theorem 1.2.5 will be proven in section 4.4.

As we have remarked above, the existence of a solution at the endpoint $\gamma = \Gamma_I$ is not trivial, it is the result of a priori estimates for the family of minimal solutions u_γ as $\gamma \to \Gamma_I^-$. It is an "extremal solution" of the family of stable solutions, and similar results for bounded domains have been obtained by Alama and Tarantello [3]. In addition, Cabré [15] has studied extremal solutions for certain autonomous equations in bounded domains, and has shown that extremal solutions exist for stable solution families even for nonlinearities with super-linear growth, for which usual Palais–Smale type compactness results fail.

Given that we have a local minimizer of I_γ for $\gamma \in (0, \Gamma)$, we expect a second solution by using the celebrated Mountain–Pass Theorem of Ambrosetti and Rabinowitz [35]. As mentioned above (see Example 1.2.1) the main obstacle is the Palais–Smale condition, and we must impose some additional condition on the coefficients in order to apply Concentration–Compactness methods [30] (see also Struwe [41] and [9]). We prove:

Theorem 1.2.6. *Assume* (1.2.4) *and there exists* $a_\infty > 0$ *and* $R > 0$ *so that*

$$\lim_{|x|\to\infty} a^-(x) = a_\infty > 0,$$

$$a^-(x) < a_\infty \quad \text{for all } |x| \geq R.$$

Then, if $0 < \lambda < \Lambda_M$, $S_{M,\gamma}$ *contains at least two elements for all* $\gamma \in (0, \Gamma)$.

Again, the natural question that arises is, do the two solutions in $S_{M,\gamma}$ differ in Ω^+? We conjecture that if we assume strict monotonicity, $\nabla a(x) \cdot x < 0$, then these two solutions should be distinct in Ω^+, but this is still an interesting open question.

Combining the above results, if the hypotheses of the above theorem are met, then (1.2.1) admits at least two nonnegative solutions for $\gamma \in (0, \Gamma)$ and at least one for

$\gamma = \Gamma$, mirroring the results of [2, 8, 5] for boundary-value problems.

Another way to recover compactness in unbounded domains is via radial symmetry, $a(x) = a(|x|)$. With no additional hypothesis, the minimal solution U_γ must be radially symmetric. Let $r_1 = \sup\{r \geq 0 \mid a(r) \geq 0\}$, $0 < r_1 < \infty$ since Ω^+ is bounded and non-empty. We prove the following theorem via the moving planes method of Gidas, Ni and Nerenberg [26].

Theorem 1.2.7. *If $a(r)$ is decreasing in $[0,\ r_1]$ and strictly decreasing in $[r_1,\ \infty)$, any non-zero solution of $(1.0.1)_1$ is radially symmetric and decreases as r increases. In particular all solutions of $(1.0.1)_1$ have connected support.*

Indeed if $a(r)$ is only decreasing, we can show that all radial solutions of $(1.0.1)_1$ must decrease as r increases, therefore they must have connected support.

For radial (but not necessarily monotone) $a(x) = a(|x|)$, after a few adjustment from above method we claim that

Theorem 1.2.8. *For $0 < \lambda \leq \Lambda_M$, if $a(x) = a(|x|)$, then S_M contains at least two elements with radial symmetry.*

From previous theorem and some results from [32] we see that if $a(r)$ is strictly decreasing and smooth, these two radial elements in S_M are different in Ω^+. We could say one is small and the other one is big, in analogy with the results on convex and concave non-linearities by Ambrosetti, Brezis, and Cerami [5]. I conjecture that if Ω^+ is connected, there is a constant $A > 0$ independent of λ so that S_M has at most one element with L^∞-norm less than A. A first step will be to do this when Ω^+ is a ball. The result cannot hold true if Ω^{0+} is admissible and has more one connected component.

Chapter 2

Maximum Principle and Some Applications

In this chapter we present some classical results for the maximum principle and some applications to our equations. Comparison theorems will be very important to our methods, and so we review some well-known theorems which are based on maximum principles.

2.1 The Maximum Principle

Let us consider the following elliptic operator

$$Lv = -\triangle v + c(x)v \quad in\ D \tag{2.1.1}$$

where $c(x)$ is a continuous function and D is a open and bounded domain in $\mathbb{R}^n$.

Lemma 2.1.1 (Hopf's Lemma [24]). *Suppose $v \in C^2(D) \cap C^1(\bar{D})$ and $c(x) \in L^\infty$. Assume*

$$Lv \geq 0 \;\; in\ D \qquad and \qquad v \geq 0 \;\; in\ D.$$

Suppose also v is not identically 0.

(i) *If $x^0 \in \partial D$, $v(x^0) = 0$, and D satisfies the interior ball condition at x^0, then*

$$\frac{\partial v}{\partial n} < 0,$$

where n is the outer unit normal vector of ∂D at x^0.

(ii) *Furthermore,*

$$v > 0 \quad in \quad D$$

Theorem 2.1.2 (Strong Maximum Principle [24]). *Assume $v \in C^2(D) \cap C^1(\bar{D})$ and*

$$c \geq 0 \quad in\ D.$$

Suppose also D is connected.

(i) *If $Lv \leq 0$ in D, and v attains a non-negative maximum over $\bar{D}$ at an interior point, then v is constant within D.*

(ii) *If $Lv \geq 0$ in D, and v attains a non-positive minimum over $\bar{D}$ at an interior point, then v is constant within D.*

We remark that the hypothesis $u \in C^2$ has been weakened by J. Serrin [37] to include C^1 weak solutions.

As an immediate application of the strong maximum principle, we have:

Lemma 2.1.3. *The classical solution u of (1.0.1) is either positive in Ω_i^+ or entirely zero in Ω_i^+ for any $i \in M$.*

Proof. Let us consider the set $S = \{x \in \Omega_i^+ | u(x) = 0\}$. First, we claim that S is open in Ω_i^+. Indeed, if $S \neq \emptyset$, then pick any $x_0 \in S$, we have $u(x_0) = 0$ and $a(x_0) > 0$. Therefore by continuity

$$a(x) + b(x)u^{p-q}(x) > 0 \ in\ B(x_0,\ \epsilon),$$

for small $\epsilon > 0$. Hence we have

$$-\triangle u = a(x)u^q + b(x)u^p = (a(x) + b(x)u^{p-q})u^q \geq 0 \text{ in } B(x_0, \epsilon) \text{ and } u(x) \geq 0 \text{ in } B(x_0, \epsilon).$$

So by maximum principle $u(x) \equiv 0$ in $B(x_0, \epsilon)$, which means S is open in Ω_i^+.

It is also clear by continuity that S is closed in Ω_i^+, therefore $u > 0$ in Ω_i^+ or $u \equiv 0$ in Ω_i^+ due to the connectivity of Ω_i^+. □

2.2 A Comparison Theorem

We will often use the following comparison theorem, which was proven in the case of bounded domains and $b(x) \equiv 0$ by Bandle, Pozio, and Tesei [7] (see also Spruck [39]):

Lemma 2.2.1. *Assume (1.0.2), and let $u_1, u_2 \in C^1(\mathbb{R}^n) \cap W^{2,s}_{loc}(\mathbb{R}^n)$, $s > n$, be two functions such that for some $I \subset M$,*

(1) u_1, u_2 are positive in $\overline{\Omega_I^+}$;

(2) $u_1 = 0$ in $\overline{\Omega^+ - \Omega_I^+}$;

(3) $\lim_{|x|\to\infty} u_1(x) = \lim_{|x|\to\infty} u_2(x) = 0$;

(4) For a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned} -\triangle u_1 &\leq au_1^q + bu_1^p \\ -\triangle u_2 &\geq au_2^q + bu_2^p. \end{aligned}$$

Then we must have $u_1 \leq u_2$ in $\mathbb{R}^n$

It is worth pointing out that if u_2 is positive everywhere in $\mathbb{R}^n$, then the second condition that $u_1 = 0$ in $\overline{\Omega^+ - \Omega_I^+}$ is removable. This Lemma is proven for C^2 solutions in a bounded domain in [7]. Here we provide a proof for completeness; the extension to solutions in $C^1(\mathbb{R}^n) \cap W^{2,p}_{loc}(\mathbb{R}^n)$ is done by applying Serrin's generalization of the maximum principle [37].

Proof. Suppose the contrary, $D := \{x \in \mathbb{R}^n |\ u_1 > u_2\} \neq \emptyset$. Let $U_1 = \frac{1}{1-q}u_1^{1-q}$, $U_2 = \frac{1}{1-q}u_1^{1-q}$, so that

$$U_1 > U_2 \ in\ D.$$

Since $\lim_{|x|\to\infty} u_1(x) = \lim_{|x|\to\infty} u_2(x) = 0$, there exists a point $x_0 \in D$ where the difference $\delta := U_1 - U_2$ attains its maximum. Let us now distinguish two cases.

Case 1: Suppose that $U_2(x_0) > 0$ for some $x_0 \in D$ where δ takes its maximum. Denote by V the maximal connected component of the set $D_1 := \{x \in D :\ U_2(x) > 0\}$ containing x_0, then δ belongs to $W^{2,s}(V)$ and the above calculations show that

$$\Delta\delta = \Delta U_1 - \Delta U_2 \geq (-q)u_1^{q-1}|\nabla U_1|^2 + qu_2^{q-1}|\nabla U_2|^2 - b(u_1^{p-q} - u_2^{p-q}).$$

It is easy to see that from $u_1 > u_2$ *in* D we have

$$u_2^{q-1} > u_1^{q-1} \ in\ D, \qquad u_1^{p-q} > u_2^{p-q} \ in\ D.$$

So

$$\begin{aligned}
& \Delta\delta + qu_2^{q-1}((\nabla U_1 + \nabla U_1), \nabla\delta) \\
\geq\ & (-q)u_1^{q-1}|\nabla U_1|^2 + qu_2^{q-1}|\nabla U_2|^2 + qu_2^{q-1}|\nabla U_1|^2 - qu_2^{q-1}|\nabla U_2|^2 - b(u_1^{p-q} - u_2^{p-q}) \\
\geq\ & (qu_2^{q-1} - qu_1^{q-1})|\nabla U_1|^2 - b(u_1^{p-q} - u_2^{p-q}) \\
\geq\ & 0.
\end{aligned}$$

Since δ assumes its maximum at an interior point of V, the weak maximum principle of Serrin ([37]) ensures that $\delta \equiv$ *constant* in V. It then follows that

$$0 = \nabla\delta = \nabla U_1 - \nabla U_2 \qquad and \qquad \Delta\delta \equiv 0 \ in\ V.$$

But by assumption in V we have

$$0 \equiv \Delta\delta = q|\nabla U_1|^2(u_2^{q-1} - u_1^{q-1}) - b(u_1^{p-q} - u_2^{p-q}) \geq 0,$$

which implies $\nabla U_1 = 0$ in V, in turn we have $\nabla U_2 = 0$ in V, so u_1 and u_2 must be constant in V. But since $\lim_{|x|\to\infty} u_1(x) = \lim_{|x|\to\infty} u_2(x) = 0$, we must have $u_1 \equiv u_2$ in V. That is a contradiction, so case 1 is impossible.

Case 2: Suppose $U_2(x_0) = 0$ for ALL x_0 where δ achieves its maximum. Let

$$C := \{x \in D : \ \delta(x) = \delta(x_0)\}$$

Note by assumption, $U_2 \equiv 0$ in C. Since $\delta = \delta(x_0) > 0$ in C, we have $U_1 > 0$ in C. On the other hand also by assumption $u_1 \equiv 0$ in $\overline{\Omega^+ - \Omega_I^+}$. Hence we have

$$C \cap \overline{(\Omega^+ - \Omega_I^+)} = \emptyset.$$

By hypothesis 1., $U_2 > 0$ in $\overline{\Omega_I^+}$, hence $C \cap \overline{\Omega_I^+} = \emptyset$. So we have

$$C \cap \overline{\Omega^+} = \emptyset,$$

which means that C and $\overline{\Omega^+}$ are at a positive distance to each other. Therefore there exists a neighborhood U of C such that $\overline{U} \cap \overline{\Omega^+} = \emptyset$ and $\delta(x) > 0$ in $\overline{U}$. Then by monotonicity, $\min_{\overline{W}}(u_1 - u_2) > 0$ is attained, where W is a connected component of U.

Thus there exists $b > 0$ such that $\delta(x) \leq b < \delta(x_0)$ for $\forall x \in \partial W$. For $\forall \epsilon > 0$, we define

$$U_{2\epsilon} := \frac{1}{1-q}(u_2 + \epsilon)^{1-q}, \qquad \delta_\epsilon := U_1 - U_{2\epsilon}.$$

Clearly $\delta_\epsilon \leq \delta$ in D. We can pick positive ϵ small enough such that

$$u_1 > u_2 + \epsilon \qquad and \qquad \delta_\epsilon(x_0) > 0.$$

It follows that

$$\delta_\epsilon(x) \leq \delta(x) \leq b < \delta_\epsilon(x_0) \qquad \forall x \in \partial W.$$

Hence δ_ϵ attains its maximum at some interior point in W and is not constant in $\overline{W}$. On the other hand, from assumption (1.0.2) we have $a \leq 0$ in $\overline{W}$, therefore

$$\begin{aligned}
\triangle\delta_\epsilon &\geq (-q)u_1^{q-1}|\nabla U_1|^2 + q(u_2+\epsilon)^{q-1}|\nabla U_{2\epsilon}|^2 + a\frac{u_2^q}{(u_2+\epsilon)^q} - a - b(u_1^{p-q} - \frac{u_2^p}{(u_2+\epsilon)^q}) \\
&\geq (-q)(u_2+\epsilon)^{q-1}(|\nabla U_1|^2 - |\nabla U_{2\epsilon}|^2 + a(\frac{u_2^q}{(u_2+\epsilon)^q} - 1) - b(u_1^{p-q} - \frac{(u_2+\epsilon)^p}{(u_2+\epsilon)^q}) \\
&\geq (-q)(u_2+\epsilon)^{q-1}((\nabla(U_1+U_{2\epsilon})), (\nabla\delta_\epsilon)) \qquad in\ W,
\end{aligned}$$

that is

$$-\triangle\delta_\epsilon + (-q)(u_2+\epsilon)^{q-1}((\nabla(U_1+U_{2\epsilon})), (\nabla\delta_\epsilon) \leq 0 \qquad in\ W.$$

But by the weak maximum principle of Serrin [37], δ_ϵ can not achieve its maximum in W unless it is constant. This is a contradiction, whence the result follows. □

One important use of Theorem 2.2.1 is to prove uniqueness of solutions within the classes N_I (as defined in Definition 1.0.1.)

Another application is to the solutions of:

$$-\triangle\omega = a^+(x)\omega^q, \ \omega(x) \to 0 \ as \ |x| \to \infty,$$

and

$$-\triangle\omega_b = a^+(x)\omega_b^q + b(x)\omega_b, \ \omega(x)_b \to 0 \ as \ |x| \to \infty,$$

with $b(x) \leq 0$. The function ω will be used in Chapter 3 as super-solutions and will appear as a limit of the solutions of the parametrized equations (1.1.5) as $\lambda \to \infty$. The existence of ω follows from Brezis and Kamin [12] (see Theorem 3.1.8), and ω_b exists by the sub and super-solution method, see Lemma 3.3.5. By Lemma 2.2.1 we conclude that the solutions ω, ω_b are unique, and that $\omega_b \leq \omega$ point-wise in $\mathbb{R}^n$.

Chapter 3

Logistic Nonlinearities

In this chapter, we study the semi-linear elliptic problem in $\mathbb{R}^n$, $n \geq 3$:

$$\begin{cases} -\triangle u &= a(x)u^q + b(x)u^p \quad \text{in } \mathbb{R}^n\,, \quad 0 < q < 1, p > q \\ \quad u &\geq 0 \quad\quad\quad\quad\quad\quad\quad\; \text{in } \mathbb{R}^n\,, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \end{cases} \tag{3.0.1}$$

where $a(x)$ and $b(x)$ are locally Hölder continuous, and $b(x) \leq 0$.

3.1 Compact Support

In this section we prove Theorem 1.1.2. The method we use is derived from the approach of Cortázar, Elgueta and Felmer [18] on the constant-coefficient equation $-\triangle u = u^p - u^q$.

First we develop a few useful lemmas. Assume (1.1.1), and pick $\rho_1 > 0$ such that $\Omega^{0+} \subset\subset B(0, \rho_1)$.

Lemma 3.1.1. *Assume (1.1.1). Then, any weak solution u of* (1.0.1) *is a classical solution and* $\lim_{|x|\to\infty} u(x) = 0$.

Proof. The regularity of u follows from standard bootstrap arguments; see Appendix B in Struwe [41] or Theorem 0 in Brezis [11]. Since $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, then $u \in L^{2^*}(\mathbb{R}^n)$. Hence for any $\epsilon > 0$, there exists $R(\epsilon) > \rho_1$ such that

$$||u||_{L^{2^*}(\mathbb{R}^n - B(0,R))} < \epsilon \ for\ all\ R > R(\epsilon).$$

So for any $x \in \mathbb{R}^n - B(0, R(\epsilon)+2)$, we have $B(x,1) \subset\subset \mathbb{R}^n - B(0, R(\epsilon))$ and $-\Delta u(y) \leq 0$ in $B(x,1)$. Therefore by the property of subharmonic function, we have

$$0 \leq u(x) \leq \frac{1}{|B(x,1)|} \int_{B(x,1)} u(y)dy \leq C||u||_{L^{2^*}(B(x,1))} \leq C\epsilon,$$

that is $\lim_{|x|\to\infty} u(x) = 0$. □

Remark 3.1.2. *Note that we only need to assume that Ω^+ is bounded and $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$. On the other hand, obtaining $\lim_{|x|\to\infty} u(x) = 0$ is very crucial to the succeeding arguments. In Section 5, we will discuss solutions whose Dirichlet energy is not assumed to be finite.*

The next Lemma shows that u not only uniformly tends to zero, but also goes to zero with certain speed, as $|x|$ goes to infinity.

Lemma 3.1.3. *Assume (1.1.1), then $u(x) \leq \frac{C}{|x|^{n-2}}$, where $C = ||u||_{L^\infty(\mathbb{R}^n)}\rho_1^{n-2}$ and $x \in \mathbb{R}^n - B(0,\rho_1)$.*

Proof. Let $v = \frac{C}{|x|^{n-2}}$, then by the special choice of C we have:

$$-\Delta v(x) = 0 \text{ in } \mathbb{R}^n - B(0,\rho_1) \text{ and } v(x) \geq u(x) \text{ on } \partial B(0,\rho_1).$$

Now consider $w = u - v$, then w satisfies:

$$-\Delta w(x) \leq 0 \text{ in } \mathbb{R}^n - B(0,\rho_1) \text{ and } w(x) \leq 0 \text{ on } \partial B(0,\rho_1).$$

Moreover we see that $\lim_{|x|\to\infty} w(x) = 0$. We now claim that $w(x) \leq 0$ in $\mathbb{R}^n - B(0,\rho_1)$. Indeed, otherwise there would exist $x_0 \in \mathbb{R}^n - \overline{B(0,\rho_1)}$ such that $u(x_0) > v(x_0) > 0$.

We also notice that $w(x) \le 0$ on $\partial B(0, \rho_1)$ and $\lim_{|x|\to\infty} w(x) = 0$, we may assume w attains maximum at x_0. Therefore we would have

$$0 \le -\Delta w(x_0) = -\Delta u(x_0) < 0,$$

a contraction. Hence, we have $u \le v = \frac{C}{|x|^{n-2}}$ in $\mathbb{R}^n - B(0, \rho_1)$. □

Next, we must estimate the sup-norm of the solution. We prove:

Proposition 3.1.4. *Assume (1.1.1). There exists a constant C so that for any solution u of* (1.0.1) *we have:*

$$||u||_{L^\infty(\mathbb{R}^n)} \le C(q, p, n, ||a^+||_{L^\infty(\Omega^+)}, \Omega^+),$$

where this C tends to zero as $||a^+||_{L^\infty(\Omega^+)}$ tends to zero.

First, the maximum principle yields:

Lemma 3.1.5. *$||u||_{L^\infty(\mathbb{R}^n)}$ can be attained in $\overline{\Omega^+}$, i.e. there exists $x_0 \in \overline{\Omega^+}$ such that $||u||_{L^\infty(\mathbb{R}^n)} = u(x_0)$.*

Proof. Since $\lim_{|x|\to\infty} u(x) = 0$, we may assume $||u||_{L^\infty(\mathbb{R}^n)}$ is attained at x_1, which is not in $\overline{\Omega^+}$. Let Ω be the connected component of $\mathbb{R}^n - \overline{\Omega^+}$, which contains x_1. By the Strong Maximum Principle 2.1.2, $u(x) = u(x_1)$ in $\overline{\Omega}$. Since $\overline{\Omega^+} \cap \overline{\Omega}$ is not empty, we are done.

□

First, we estimate the Dirichlet energy of the solutions. The claim in the following proof will also be useful in our existence proofs in the next section:

Lemma 3.1.6. *Assume (1.1.1), then we have*

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \le \int_{\mathbb{R}^n} a^+ u^{q+1} dx$$

Proof. Since u satisfies the equation $-\triangle u = au^q + bu^p$, multiply both sides of the equation by u and iterate by parts. We have for $R \geq \rho_1$

$$\int_{B(0,R)} |\nabla u|^2 dx - \int_{\partial B(0,R)} \frac{\partial u}{\partial n} u dS = \int_{B(0,R)} a^+ u^{q+1} dx + \int_{B(0,R)} b u^{p+1} dx$$
$$\leq \int_{\mathbb{R}^n} a^+ u^{q+1} dx. \qquad (3.1.1)$$

We now claim that there exists a sequence $\{R_n\}$ and $\lim_{n\to\infty} R_n = \infty$, such that $\int_{\partial B(0,R_n)} \frac{\partial u}{\partial n} u dS \to 0$ as $n \to \infty$. Indeed, we have the following estimate:

$$\begin{aligned}
|\int_{\partial B(0,R)} \frac{\partial u}{\partial n} u dS| &\leq \frac{||u||_{L^\infty(\mathbb{R}^n)} \rho_1^{n-2}}{R^{n-2}} \int_{\partial B(0,R)} |\nabla u| dS \\
&\leq \frac{||u||_{L^\infty(\mathbb{R}^n)} \rho_1^{n-2}}{R^{n-2}} ||1||_{L^2(\partial B(0,R))} ||\nabla u||_{L^2(\partial B(0,R))} \\
&\leq \frac{CR^{\frac{n-1}{2}}}{R^{n-2}} ||\nabla u||_{L^2(\partial B(0,R))} \\
&\leq \frac{C}{R^{\frac{n-3}{2}}} ||\nabla u||_{L^2(\partial B(0,R))}.
\end{aligned}$$

Notice $\infty > ||\nabla u||^2_{L^2(\mathbb{R}^n)} = \int_0^\infty \int_{\partial B(0,R)} |\nabla u|^2 dS dr$, so there should exist a sequence $\{R_n\}$ with $\lim_{n\to\infty} R_n = \infty$, such that $||\nabla u||_{L^2(\partial B(0,R_n))} \to 0$ as $n \to \infty$. Therefore

$$\int_{\partial B(0,R_n)} \frac{\partial u}{\partial n} u dS \to 0 \; as \; n \to \infty,$$

and the claim is proven.

Applying this to (3.1.1), we have

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \leq \int_{\mathbb{R}^n} a^+ u^{q+1} dx.$$

□

The next step is a bootstrap argument. Recall ρ_1 is chosen such that $\Omega^{0+} \subset\subset B(0, \rho_1)$.

Lemma 3.1.7. *For any positive integer $s \geq 2$, there exists a constant*

$$C' = C'(q, p, n, s, ||a^+||_{L^\infty(\Omega^+)}, ||u||_{L^{2^*}(\mathbb{R}^n)}, \Omega^+)$$

so that

$$||u||_{L^{(s+1)\frac{n}{n-2}}(\mathbb{R}^n)} \leq C'.$$

Moreover, C' tends to zero as $||a^+||_{L^\infty(\Omega^+)}$ and $||u||_{L^{2^*}(\mathbb{R}^n)}$ tend to zero.

Proof. We rewrite equation (1.0.1) as

$$-\Delta u + a^- u^q - bu^p = a^+ u^q.$$

Then we just follow the steps in Appendix B in Struwe [41] (or Theorem 0 in Brezis [11].) The term $a^- u^q - bu^p$ are nonnegative and so they may be neglected. Note that the existence of the integrals is assured *a priori* without truncation, since a^+ is continuous and has compact support. □

The uniform estimate on u will be derived by comparison with the solution $\omega \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ of

$$-\Delta\omega = a^+(x)\omega^q, \quad \omega(x) \to 0 \text{ as } |x| \to \infty. \tag{3.1.2}$$

Theorem 3.1.8. *Assuming Ω^+ is bounded and nonempty, there exists a unique nonnegative solution $\omega \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ to (3.1.2).*

Proof. By the Basic Hypothesis, Ω^+ is bounded, so it is easy to see that $-\Delta v = a^+$ in R^n has a solution $v(x) = \int_{R^n} \Phi(x-y)a^+(y)dx$, where Φ is the fundamental solution of Laplace's equation. Furthermore, $\lim_{|x|\to\infty} v(x) = 0$. Thus, the existence of $\overline{\omega}$ follows from Theorem 2' in Brezis-Kamin [12]. □

Note that the conclusions of Lemmas 3.1.5, 3.1.6, and 3.1.7 also hold with ω in place of u.

We are now ready to complete the proof of Proposition 3.1.4.

Proof. As remarked above, ω satisfies the conclusions of the above Lemmas 3.1.5, 3.1.6, and 3.1.7. From Lemma 3.1.6, we know that

$$\int_{\mathbb{R}^n} |\nabla\omega|^2 dx \leq \int_{\mathbb{R}^n} a^+ \omega^{q+1} dx. \tag{3.1.3}$$

By the Sobolev embedding we also have

$$\int_{\mathbb{R}^n} |\nabla\omega|^2 dx \geq C||\omega||^2_{L^{2^*}(\mathbb{R}^n)} \tag{3.1.4}$$

for some constant C independent of ω. Also by Hölder we obtain

$$\int_{\mathbb{R}^n} a^+ \omega^{q+1} dx \leq ||a^+||_{L^t(\mathbb{R}^n)} ||\omega^{q+1}||_{L^{\frac{2^*}{q+1}}(\mathbb{R}^n)} = ||a^+||_{L^t(\mathbb{R}^n)} ||\omega||^{q+1}_{L^{2^*}(\mathbb{R}^n)}, \tag{3.1.5}$$

where t is the conjugate of $\frac{2^*}{q+1}$. Therefore combine (3.1.3), (3.1.4) and (3.1.5), we get

$$C||\omega||^2_{L^{2^*}(\mathbb{R}^n)} \leq ||a^+||_{L^t(\mathbb{R}^n)} ||\omega||^{q+1}_{L^{2^*}(\mathbb{R}^n)},$$

that is

$$||\omega||_{L^{2^*}(\mathbb{R}^n)} \leq C(||a^+||_{L^t(\mathbb{R}^n)})^{\frac{1}{1-q}}.$$

Choosing s so that $(s+1)\frac{n}{n-2} \geq \frac{(n+1)(p+1)}{2}$, from Lemma 3.1.7, we know that $||\omega||_{L^{(s+1)\frac{n}{n-2}}(\mathbb{R}^n)}$ is uniformly bounded. Apply the standard elliptic estimate (see [42]) on the domain $\Omega^+ \subset\subset \hat{O} = \{x \in \mathbb{R}^n \,|dist(x, \Omega^{0+}) < \epsilon\}$, where ϵ is chosen so small that $||a^+||_{L^\infty(\hat{O})} = ||a^+||_{L^\infty(\Omega^+)}$ and volume$(\hat{O}) \leq 2$volume(Ω^+), we have

$$||\omega||_{W^{2,\frac{n+1}{2}}(\Omega^+)} \leq C(||\triangle\omega||_{L^{\frac{n+1}{2}}(\hat{O})} + ||\omega||_{L^{\frac{n+1}{2}}(\hat{O})}).$$

Then by Sobolev embedding theorem, in view of Lemma 3.1.5 we have shown that there exists a constant $C'' = C''(q, p, n, ||a^+||_\infty, \Omega^+)$ so that $||\omega||_\infty \leq C''$.

To conclude, we use Lemma 2.2.1 with $u_1 = u$ and $u_2 = \omega$, notice that ω is positive everywhere. Applying Lemma 2.2.1 we thus obtain $0 \leq u(x) \leq \omega(x)$ holds for all $x \in \mathbb{R}^n$, for any solution u of (3.0.1), and the proposition is proven. □

Corollary 3.1.9. *Assume (1.1.1), then*

$$u(x) \leq \frac{\eta_1 \rho_1^{n-2}}{|x|^{n-2}}$$

for any $x \in \mathbb{R}^n - B(0, \rho_1)$ and η_1 is a number depending on q, p, n and $||a^+||_{L^\infty(\mathbb{R}^n)}$.

Now we are ready to prove Theorem 1.1.2:

Proof. This is a comparison argument using the method from [17]. For positive number M, c, let $w(s)$ be the function defined implicitly by

$$\int_{w(s)}^{M} \frac{dt}{\sqrt{\frac{c}{q+1} t^{q+1}}} = \sqrt{2}s,$$

with constants M and c to be chosen later. Indeed we can write $w(s)$ explicitly in terms of s

$$w(s) = \left[M^{1-\frac{q+1}{2}} - \sqrt{2}s(1 - \frac{q+1}{2}) \sqrt{\frac{c}{q+1}} \right]^{\frac{2}{1-q}}.$$

Notice that since $0 < q < 1$, then $\frac{2}{1-q} > 2$. So $w(s)$ is at least twice continuously differentiable in $[0,\ B]$, where B is defined by

$$M^{1-\frac{q+1}{2}} = \sqrt{2}B(1 - \frac{q+1}{2}) \sqrt{\frac{c}{q+1}}.$$

It is easy to see that $w(s)$ satisfies

$$w''(s) - cw^q(s) = 0 \text{ } in \text{ } (0,\ B).$$

Moreover $w(s)$ is a decreasing function in s, $w(B) = w'(B) = w''(B) = 0$. Therefore, by defining $w(s) \equiv 0$ for $s \in [B,\ \infty)$, we obtain a non-increasing solution of

$$w''(s) - cw^q(s) = 0 \text{ } in \text{ } (0,\ \infty)$$

with $w(0) = M$ and $\text{supp}(w) = [0,\ B]$.

We know from previous proposition that $u(x) \le \frac{\eta_1 \rho_1^{n-2}}{|x|^{n-2}}$ for $|x| > \rho_1$. Let us define $g(x) : \mathbb{R}^n - B(0, \rho_1) \longrightarrow R$ to be the following

$$g(x) = \left[\left(\frac{\eta_1 \rho_1^{n-2}}{(|x|-1)^{n-2}}\right)^{1-\frac{q+1}{2}} - \sqrt{2}\left(1 - \frac{q+1}{2}\right)\sqrt{\frac{\eta}{(q+1)|x|^{(n-2)(1-q)}}} \right],$$

where we pick $\eta > 0$ such that

$$\sqrt{2}\left(1 - \frac{q+1}{2}\right)\sqrt{\frac{\eta}{q+1}} = (\eta_1 \rho_1^{n-2})^{1-\frac{q+1}{2}} + 1.$$

We rewrite $g(x)$ as the following form

$$g(x) = \frac{1}{(|x|-1)^{\frac{(n-2)(1-q)}{2}}} \left[(\eta_1 \rho_1^{n-2})^{1-\frac{q+1}{2}} - \sqrt{2}\left(1 - \frac{q+1}{2}\right)\sqrt{\frac{\eta}{q+1}}\left(\frac{|x|-1}{|x|}\right)^{\frac{(n-2)(1-q)}{2}} \right].$$

By the assumption that $\liminf_{|x|\to\infty} a^- |x|^{(n-2)(1-q)} > \eta$, then there exists $\rho_2 > \rho_1 + 1$, which depends on a, such that

$$g(x) < 0 \ for \ |x| \ge \rho_2 \ and \ a^- \ge \frac{\eta}{|x|^{(n-2)(1-q)}} \ for \ |x| \ge \rho_2 - 1.$$

Now we choose

$$M = \frac{\eta_1 \rho_1^{n-2}}{(\rho_2 - 1)^{n-2}} \quad and \quad c = \frac{\eta}{\rho_2^{(n-2)(1-q)}}.$$

Hence consider the function $f(s) : \ [0, \ 1] \longrightarrow R$ defined by

$$f(s) = M^{1-\frac{q+1}{2}} - \sqrt{2}s\left(1 - \frac{q+1}{2}\right)\sqrt{\frac{c}{q+1}}.$$

We find that $f(0) > 0$ and $f(1) = g(\rho_2) < 0$ from above calculation, then according to the mean-value Theorem

$$0 < \sup_{0 \le t \le 1} \{t \ | f(s) \ge 0 \ for \ s \in [0, \ t]\} < 1.$$

Therefore for the choice of M and c, B is well-defined and $0 < B < 1$.

Let $v(x) = w(|x| - (\rho_2 - 1))$, then we see that v satisfies

$$\begin{aligned} \triangle v - cv^q &\le 0 \ in \ \mathbb{R}^n - B(0, \rho_2 - 1) \\ v &= M \ on \ \partial(\mathbb{R}^n - B(0, \rho_2 - 1)). \end{aligned}$$

Also notice that for $|x| \in [\rho_2 - 1,\ \rho_2)$, $a^- \geq \frac{\eta}{|x|^{(n-2)(1-q)}} > \frac{\eta}{\rho_2^{(n-2)(1-q)}} = c$.

For u we have

$$\begin{aligned} \Delta u + au^q + bu^p &= 0 \; in \; \mathbb{R}^n - B(0, \rho_2 - 1) \\ u &\leq M \; on \; \partial(\mathbb{R}^n - B(0, \rho_2 - 1)). \end{aligned}$$

By subtracting them, we have

$$-\Delta(v - u) \geq -a(x)u^q - cv^q \; for \; x \in \mathbb{R}^n - B(0, \rho_2 - 1).$$

We now claim that $v \geq u \geq 0 \; for \; x \in \mathbb{R}^n - B(0, \rho_2 - 1)$. Otherwise there would exist $x_0 \in (\mathbb{R}^n - \overline{B(0, \rho_2 - 1)})$ such that $u(x_0) > v(x_0) \geq 0$, which implies that $v - u$ attains its global minimum at some point $x_0 \in \mathbb{R}^n - \overline{B(0, \rho_2 - 1)}$. At x_0,

$$\begin{aligned} 0 &\geq -\Delta(v - u)(x_0) \\ &\geq (-a(x_0)u^q(x_0) - cv^q(x_0)) \\ &\geq \begin{cases} -a(x_0)u^q(x_0) & > 0 \;\; if \; v(x_0) = 0 \\ (-a(x_0) - c)v^q(x_0) & > 0 \;\; if \; v(x_0) > 0, \end{cases} \end{aligned}$$

a contradiction, and so the claim is proven.

So we must have $v \geq u \geq 0 \; for \; x \in (\mathbb{R}^n - B(0, \rho_2 - 1))$, which implies u has compact support. Therefore $\text{supp}(u) \subset\subset B(0, \rho_2)$. □

In the end we note that the main ingredient in the above proof is the decay estimate on the solution in the exterior of $B(0,\ \rho_1)$. Any improvement on the required decay (1.1.2) of $a^-(x)$ would require a sharper estimate in Corollary 3.1.9.

Remark 3.1.10. *For solutions of* (1.0.1) *in* $\mathbb{R}^n$, $n = 1, 2$ *we unfortunately do not have decay estimates as in Lemma 3.1.3, principally because the fundamental solution does not decay to zero at infinity in dimension* $n \leq 2$. *Nevertheless, Lemma 3.1.1 still holds for any classical solution in* $L^t(\mathbb{R}^n)$ *for* $t \geq 1$, *so we can prove in very similar way* *that all classical solutions of (3.0.1) in* $L^t(\mathbb{R}^n)$ *have compact supports under strong assumption* $\liminf_{|x|\to\infty} a^-(x) > 0$. *However, we can not uniformly control the size of the support because the Sobolev inequalities are domain-dependent in dimensions* $n = 1, 2$. *The statement we can make in any dimension is the following:*

Theorem 3.1.11. *Assume (1.1.1), if* $\liminf_{|x|\to\infty} a^- > 0$, *all classical solutions in* $L^t(\mathbb{R}^n)$ *for* $t \geq 1$ *must have compact support.*

Proof. The proof is much simpler since we do not need to choose the place where we make the comparison. We may just pick $M = \|u\|_{L^\infty(\mathbb{R}^n)}$ and compare w with u outside $B(0, \rho_1)$. □

Remark 3.1.12. *If* $a^-(x)$ *decays too fast at infinity solutions may not have compact support. Indeed, using the same trick as in [7, 2] we obtain that for any compactly supported solution,*

$$\int_{supp(U)} a(x) + bU^{p-q} dx < 0. \tag{3.1.6}$$

However, if we choose $b \equiv 0$ *and* $a(x) \in L^1$ *satisfying (1.1.1) with* $\int_{\mathbb{R}^n} a(x) > 0$ *then a compactly supported solution could never satisfy* (3.1.6), *and thus no solution can have compact support.*

3.2 Existence and Uniqueness

In this section we present the proof of the basic existence theorems, Theorem 1.1.1 and Theorem 1.1.6, and a more general form of the uniqueness result Theorem 1.1.4. Throughout we assume the dimension $n \geq 3$.

We use the method of sub- and super-solutions (also known as the method of upper and lower solutions). The basic idea is to find a sub-solution (lower solution) $\underline{u}$ and a super-solution (upper solution) $\overline{u}$ which have the following properties: $\underline{u} \leq \overline{u}$ at each point in $\mathbb{R}^n$, and each satisfies the equation (3.0.1) but with inequality replacing equality:

$$-\Delta \underline{u} \leq a(x)\underline{u}^q + b(x)\underline{u}^p, \quad \text{in } \mathbb{R}^n,$$
$$-\Delta \overline{u} \geq a(x)\overline{u}^q + b(x)\overline{u}^p, \quad \text{in } \mathbb{R}^n.$$

The existence of a solution to (3.0.1) will come from iterating the equation starting from either the sub- or super-solution. This process will be monotonic, and is called "monotone iteration" (see e.g. Sattinger [36].) Existence results of this type are well-known in the setting of bounded domains (see [7, 8], for example,) and we adapt the technique here for entire solutions in $\mathbb{R}^n$. It will be essential to be sure that at each step in the iteration process we preserve the uniform boundedness of the solutions and belongingness of the solutions in the function space $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

Recall from Theorem 3.1.8 the definition of $\omega \in \mathcal{D}^{1,2}(\mathbb{R}^n)$. To begin the iteration, we start with the following equation:

$$-\Delta z + a^- z^q - bz^p = a^+ f^q \text{ in } \mathbb{R}^n \text{ , } z \geq 0 \text{ and } z \in \mathcal{D}^{1,2}(\mathbb{R}^n), \tag{3.2.1}$$

where f is Hölder Continuous and $0 \leq f \leq \omega$.

Lemma 3.2.1. *(3.2.1) has a solution $Z \in \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $Z \leq \omega$.*

Proof. We use monotone iteration to prove this result. Let us consider the following equation

$$-\triangle z_n + a^- z_n^q - b z_n^p = a^+ f^q \ in \ B(0,\ n)\ ,\ z \geq 0 \ and \ z \in H_0^1(B(0,\ n)). \qquad (3.2.2)$$

It is easy to see that $\underline{z} = 0$ is a lower solution, $\overline{z} = \int_{\mathbb{R}^n} \Phi(x-y)a^+(y)f^q(y)dy$ is an upper solution, where Φ is the fundamental solution of Laplace's equation. Furthermore we see that $\overline{z} \leq \omega$ since $0 \leq f \leq \omega$. Therefore by lower-upper solution method the above equation has a solution $Z_n \leq \omega$.

Claim 1. *The solution Z_n is unique*

Indeed if there is another solution $\overline{Z_n}$, then we have

$$-\triangle(Z_n - \overline{Z_n})dx + a^-(Z_n^q - \overline{Z_n}^q)dx - b(Z_n^p - \overline{Z_n}^p)dx = 0.$$

Then multiply both sides by $Z_n - \overline{Z_n}$ and integrate by parts, we have

$$\int_{B(0,\ n)} |\nabla(Z_n - \overline{Z_n})|^2 dx + \int_{B(0,\ n)} a^-(Z_n^q - \overline{Z_n}^q)(Z_n - \overline{Z_n})dx - \int_{B(0,\ n)} b(Z_n^p - \overline{Z_n}^p)(Z_n - \overline{Z_n})dx = 0$$

Since they are all nonnegative, we conclude that $Z_n = \overline{Z_n}$.

It is easy to see that Z_{n+1} is an upper solution for (3.2.2), so by the uniqueness $Z_{n+1} \geq Z_n$. Moreover we have the following estimate

$$\int_{B(0,\ n)} |\nabla Z_n|^2 dx + \int_{B(0,\ n)} a^- Z_n^{q+1} dx - \int_{B(0,\ n)} b Z_n^{p+1} dx = \int_{B(0,\ n)} a^+ f^q Z_n dx \leq \int_{\mathbb{R}^n} a^+ \omega^{q+1} dx.$$

Therefore let $Z = \lim_{n\to\infty} Z_n$, then Z is a solution to (3.2.1) and it satisfies

$$\int_{\mathbb{R}^n} |\nabla Z|^2 dx + \int_{\mathbb{R}^n} a^- Z^{q+1} dx - \int_{\mathbb{R}^n} b Z^{p+1} dx \leq \int_{\mathbb{R}^n} a^+ \omega^{q+1} dx. \qquad (3.2.3)$$

and $Z \leq \omega$. ☐

Next, in order to prove the uniqueness of the solution Z, we need to improve Lemma 3.1.3. Let

$$V = \frac{[n(n-2)]^{\frac{n-2}{4}}}{(1+|x|^2)^{\frac{n-2}{2}}}. \qquad (3.2.4)$$

Of course $V(x)$ is (up to scaling and translation) the unique solution of the familiar critical Sobolev exponent equation in $\mathbb{R}^n$, $\triangle V + V^{\frac{n+2}{n-2}} = 0$ (see [16]). Since Ω^+ is bounded, we can pick $\bar{\rho_1} > 0$ so that $\Omega^+ \subset\subset B(0,\ \bar{\rho_1})$. We have the following lemma.

Lemma 3.2.2. *Assume Z in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is a nonnegative smooth solution of $-\triangle Z \leq 0$ in $\mathbb{R}^n - B(0,\ \bar{\rho_1})$. Then, there exists a constant $C > 0$ such that $Z \leq CV$ in $\mathbb{R}^n - B(0,\ \bar{\rho_1})$. Moreover there exists a increasing sequence $\{R_n\}$ with $\lim_{n\to\infty} R_n = \infty$ so that $\int_{\partial B(0,\ R_n)} \frac{\partial Z}{\partial n} Z \to 0$ as $n \to \infty$.*

Proof. From Lemma 3.1.1, we have $\lim_{|x|\to\infty} Z(x) = 0$. Following the proof of Lemma 3.1.3, replace v in Lemma 3.1.3 by CV for some big constant C. Notice that $-\triangle(CV) > 0$, we can show that $Z \leq CV$ in $\mathbb{R}^n - B(0,\ \bar{\rho_1})$. Therefore we have $Z \leq C_1|x|^{-(n-2)}$ for some C_1, then the last part follows from the claim in previous Lemma 3.1.6. □

Lemma 3.2.3. *The solution obtained from previous lemma is unique.*

Proof. Suppose there are two solutions Z, $\overline{Z}$, which satisfy the estimates (3.2.3), then we have

$$-\triangle(Z - \overline{Z})dx + a^-(Z^q - \overline{Z}^q)dx - b(Z^p - \overline{Z}^p)dx = 0.$$

Then multiply both sides by $Z - \overline{Z}$ and integrate by parts over $B(0,\ R)$, we have

$$\int_{B(0,\ R)} |\nabla(Z-\overline{Z})|^2 dx + \int_{B(0,\ R)} a^-(Z^q-\overline{Z}^q)(Z-\overline{Z})dx - \int_{B(0,\ R)} b(Z^p-\overline{Z}^p)(Z-\overline{Z})dx + \int_{\partial B(0,\ R)} \frac{\partial(Z-\overline{Z})}{\partial n}(Z-\overline{Z})dS = 0$$

From Lemma 3.2.2, there exists a sequence $\{R_n\}$ and $\lim_{n\to\infty} R_n = \infty$, such that $\int_{\partial B(0,\ R_n)} \frac{\partial(Z-\overline{Z})}{\partial n}(Z - \overline{Z}) \to 0$ as $n \to \infty$. Let $n \to \infty$, we have

$$\int_{\mathbb{R}^n} |\nabla(Z - \overline{Z})|^2 dx + \int_{\mathbb{R}^n} a^-(Z^q - \overline{Z}^q)(Z - \overline{Z})dx - \int_{\mathbb{R}^n} b(Z^p - \overline{Z}^p)(Z - \overline{Z})dx = 0$$

So we can conclude $Z = \overline{Z}$. □

Next we go to the main iteration process. Let us consider the following iteration equation

$$-\triangle u_{n+1} + a^- u_{n+1}^q - b u_{n+1}^p = a^+ u_n^q \ in\ \mathbb{R}^n \ , \ u_n \geq 0 \ and\ u_n \in \mathcal{D}^{1,2}(\mathbb{R}^n), \qquad (3.2.5)$$

where $u_1 = \underline{u}_\rho$ for small ρ such that $0 \leq \underline{u}_\rho \leq \omega$, and $\underline{u}_\rho$ is constructed in the following manner: take small ball $B \subset\subset \Omega^+$, let $\underline{a} = inf_{x \in B} a(x)$, define

$$\underline{u}_\rho = \begin{cases} \rho\xi_i, & in\ B \\ 0, & elsewhere \end{cases}$$

where $\xi > 0$ is the eigenfunction corresponding to the first eigenvalue of the following problem

$$-\triangle\xi = \lambda\underline{a}\xi \ in\ B \qquad and \qquad \xi = 0 \ on\ \partial B,$$

for details, see [7]. Notice that $\underline{u}_\rho$ is Lipschitz in $\mathbb{R}^n$ and satisfies

$$\int_{\mathbb{R}^n} \nabla\underline{u}_\rho \nabla\phi dx \leq \int_{\mathbb{R}^n} a\underline{u}_\rho^q \phi dx + \int_{\mathbb{R}^n} b\underline{u}_\rho^p \phi dx \ for\ \phi \in C_0^\infty(\mathbb{R}^n),$$

since $\underline{u}_\rho$ is a lower solution.

Lemma 3.2.4. *$\omega \geq u_2 \geq u_1 = \underline{u}_\rho$ and*

$$\int_{\mathbb{R}^n} |\nabla u_2|^2 dx + \int_{\mathbb{R}^n} a^- u_2^{q+1} dx - \int_{\mathbb{R}^n} b u_2^{p+1} dx \leq \int_{\mathbb{R}^n} a^+ \omega^{q+1} dx. \qquad (3.2.6)$$

Proof. The proof is simple, we just go back to proof of Lemma 3.2.1. It is easy to see that $\underline{u}_\rho$ is a lower solution to the equation (3.2.2) and ω is an upper solution, therefore we have desired results. □

Base on the above lemma, we can start our induction process.

Lemma 3.2.5. *$\omega \geq u_{n+1} \geq u_n \geq \underline{u}_\rho$ and*

$$\int_{\mathbb{R}^n} |\nabla u_n|^2 dx + \int_{\mathbb{R}^n} a^- u_n^{q+1} dx - \int_{\mathbb{R}^n} b u_n^{p+1} dx \leq \int_{\mathbb{R}^n} a^+ \omega^{q+1} dx. \qquad (3.2.7)$$

Proof. From above lemma we know the initial step is true, now assume $u_n \geq u_{n-1}$, we have

$$-\triangle u_{n+1} + a^- u_{n+1}^q - bu_{n+1}^p \geq -\triangle u_n + a^- u_n^q - bu_n^p,$$

that is

$$-\triangle(u_n - u_{n+1}) + a^-(u_n^q - u_{n+1}^q) - b(u_n^p - u_{n+1}^p) \leq 0.$$

Then multiply both sides by $(u_n - u_{n+1})^+$. integrate over $B(0,\ R)$, and follow the steps in claim of Lemma 3.2.3, we will have

$$\int_{\mathbb{R}^n} |(u_n - u_{n+1})^+|^2 dx + \int_{\mathbb{R}^n} a^-(u_n^q - u_{n+1}^q)(u_n - u_{n+1})^+ dx - \int_{\mathbb{R}^n} b(u_n^p - u_{n+1}^p)(u_n - u_{n+1})^+ dx \leq 0.$$

From above we conclude $(u_n - u_{n+1})^+ = 0$. which means $u_{n+1} \geq u_n$. □

Now we are in position to prove Theorem 1.1.1:

Proof. We simply take $U = \lim_{n\to\infty} u_n$, then in view of the estimates from the previous we know U is a solution of (3.0.1). So we prove the existence for (3.0.1). As for the maximal solution, we follow the same process. Notice that during the proof for Proposition 3.1.4, we have that $u \leq \omega$ for any solution of (3.0.1) and

$$-\triangle \omega = a^+ \omega^q \geq a\omega^q + b\omega^p \ in\ \mathbb{R}^n.$$

So we pick $u_0 = \omega$ and iterate, only this time $\{u_n\}$ is decreasing sequence and $u_n \leq \omega$. Furthermore we have that $u_n \geq u$ for any solution u of of (3.0.1). Therefore $\bar{U} = \lim_{n\to\infty} u_n$ is the maximal solution. □

Remark 3.2.6. *The boundedness of Ω^+ is not essential for the existence of U; for a more general existence result see [12].*

Corollary 3.2.7. *Under hypothesis (1.1.1), (1.0.1) has a classical compactly supported solution U with its support contained in $B(0, \rho_2)$ if $\lim_{|x|\to\infty} a^-(x)|x|^{(n-2)(1-q)} > \eta$, where η and ρ_2 are from Theorem 1.1.2.*

In terms of the solution class S_I (defined in Definition 1.0.1), we obtain the following existence result:

Corollary 3.2.8. *Assume (1.0.2), then $S_I \neq \emptyset$ for any nonempty $I \in M$.*

Proof. Construct a subsolution $\underline{u}_\rho$ as a superposition of disjointedly supported subsolutions, one for each component of Ω_I^+, as in the proof of Theorem 1.1.1. Monotone iteration then produces a solution which is positive in each component. □

With some minor modifications of these arguments we may now prove the existence of minimal element in S_I as announced in Theorem 1.1.6.

Proof. We first assume that hypothesis (1.0.2) holds. To prove the assertion we may argue as in Theorem 4 of [33] , and define $u_I = \inf_{u \in S_I} u(x)$, which (by [33]) is a nonzero solution □

Finally, we also conclude that, for the parametrized family of problems (1.1.5), the maximal solution $U_\lambda \in S_M$ is monotone.

Corollary 3.2.9. *Assume (1.0.2), the maximum solution $U_\lambda \in S_M$ of (1.1.5) is increasing as λ increases.*

Proof. It is simple, for $0 < \lambda_1 < \lambda_2$, we know that $\omega \geq \max(U_{\lambda_1}, U_{\lambda_2})$. Then we iterate this super-solution $\omega$for $(1.1.5)_{\lambda_2}$, like above it results in a monotone decreasing sequence $\{u_n\}$ and $u_n \geq \max(U_{\lambda_1}, U_{\lambda_2})$. But $U_{\lambda_1} \leq u_n \to U_{\lambda_2}$, we are done! □

We now turn to questions of uniqueness, and the characterization of the solution space of (1.0.1) in terms of the supports of the solutions. First, we note that every solution of (3.0.1) must be positive in at least one connected component of Ω^+.

Lemma 3.2.10. *Assume (1.0.2), if $I = \emptyset$, then $N_I = \emptyset$.*

Proof. If $I = \emptyset$, then for any $u \in N_I$ it is a subharmonic function. Thus, for any $x \in \mathbb{R}^n$, we have

$$0 \le u(x) \le \frac{1}{|\partial B(x,\ R)|} \int_{\partial B(x,\ R)} u(y)dy \to 0 \ as \ R \to \infty$$

since $\lim_{|x| \to \infty} u(x) = 0$, which implies $u(x) = 0$. □

From Lemma 2.1.3 we know that any solution u of (3.0.1) is either positive in Ω_i^+ or entirely zero in Ω_i^+ for $i \in M$, but for uniqueness we require that u should be positive up to the boundary of each component. This will be guaranteed for $p \ge 1$ (see below), but a delicate question for $p < 1$. We define a class of functions P by:

$$P = \{v \in C^1(\mathbb{R}^n) \cap W^{2,s}_{loc}(\mathbb{R}^n)\ |s > n,\ \ v \gneq 0 \text{ in } \mathbb{R}^n,\ v > 0 \text{ in } \overline{\Omega_i^+} \text{ if } v > 0 \text{ in } \Omega_i^+ \text{ for } i \in M\ .\}$$

Lemma 3.2.11. *Assume (1.0.2) and solution u of (3.0.1) is positive in Ω_i^+ for some $i \in M$. Then if $p \ge 1$, $u > 0$ in $\overline{\Omega_i^+}$; if $p < 1$ and $\frac{b(x)}{a(x)}$ is uniformly bounded in $\overline{\Omega^+}$, $u > 0$ in $\overline{\Omega_i^+}$.*

Proof. For $p \ge 1$, hypothesis (1.0.2) ensures that an interior ball condition is satisfied by Ω_i^+, so we may directly apply Hopf's Lemma to the equation

$$-\Delta u - b(x)u^p = a(x)u^q \ge 0 \ in \ \overline{\Omega_i^+} \ and \ u \ge 0 \ in \ \overline{\Omega_i^+}.$$

We conclude that $u > 0$ in $\overline{\Omega^+}$.

For $p < 1$, if $x_0 \in \partial\Omega_i^+$ and $u(x_0) = 0$. since Ω_i^+ satisfies an interior ball condition, we take a small ball $B_\epsilon \subset \Omega_i^+$ with radius ϵ and $x_0 \in \partial B_\epsilon$. For ϵ small we have

$$-\Delta u = a(x)(1 + \frac{b(x)}{a(x)}u^{p-q})u^q \ge 0 \ in \ B_\epsilon.$$

Hopf's Lemma implies that $\nabla u(x_0) \ne 0$. Since u attains minimum at x_0, $\nabla u(x_0) = 0$, a contradiction. □

Definition 3.2.12. *We say that $b(x)$ is compatible with $a(x)$ if any solution u of (1.0.1) lies in the function set P.*

From above lemma we see that any non-positive $b(x)$ is compatible when $p \geq 1$, but $p < 1$ requires some extra assumptions on b near $\partial\Omega_i^+$.

Uniqueness in the classes N_I now follows from the comparison Lemma 2.2.1, since membership in the function set P ensures that the hypotheses are satisfied. The following generalizes the uniqueness result in Spruck [39], Theorem 1.1.4.

Theorem 3.2.13. *Assume (1.0.2), if $b(x)$ is compatible, then the number of elements in N_I is at most 1 for any non-empty I. In particular if $k = 1$, then the solution to (1.0.1) is unique and its support is connected.*

Proof. In order to apply Lemma 2.2.1, we need the elements in N_I decaying to zero at the infinity, this is assured by Lemma 3.1.1. □

We may immediately combine the existence of a maximal solution from Theorem 1.1.6 with the uniqueness result to obtain:

Corollary 3.2.14. *Assume (1.0.2), if b is compatible, the maximal solution U is the unique element in S_M.*

Here we present the proof for Proposition 1.1.5.

Proof. For $b(x) \equiv -1$, define $a_i(x)$ by

$$a_i(x) = \begin{cases} \lambda, & \text{if } x \in \Omega_i^+, \\ -1, & \text{if } x \notin \Omega_i^+, \end{cases}$$

Then we consider the following equation:

$$-\triangle u = a_i(x)u^q - u^p \textit{ in } \mathbb{R}^n, \quad u \geq 0 \textit{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^n). \tag{3.2.8}$$

By regularity results and strong maximum principle, we see that solution to (3.2.8) lies in the set P. Therefore by Lemma 2.2.1 the solution to (3.2.8) is unique, denoted by u_i. By uniqueness when Ω_i^+ moves, so does u_i. Furthermore u_i is also compactly supported by Theorem 1.1.2. So if we choose δ^* big enough so that $supp(u_i) \cap supp(u_j) = \emptyset$ for any $i \neq j$, then by Theorem 3.2.13 N_I contains exactly one solution for each $I \neq$. The first part is done!

For the second part, notice that $\Omega_i^+ \subset\subset supp(u_i)$. For nonempty I, choose any $i \in I$, then the minimum solution u_I in S_I is super-solution for problem (3.2.8). By uniqueness we find that $u_I \geq u_i$ for any $i \in I$. So if we choose δ_* so small that $supp(u_i) \cap \Omega_i^+ \neq \emptyset$ for any $i \neq j$, then (3.0.1) has only one solution, which is positive in Ω^+. □

If we assume further that $a(x)$ and $b(x)$ are radially symmetric, we have the following:

Corollary 3.2.15. *Assume (1.0.2), if a and b are radially symmetric and b is compatible, then the unique element in N_I is radially symmetric. If in addition Ω^+ is a ball centered at the origin and $b(x) = 0$ in Ω^+, then the unique solution of (3.0.1) is radially decreasing.*

Proof. If a and b are radially symmetric, since Laplace's operator is invariant under rotation, we know that the element in N_I is radially symmetric due to Lemma 2.2.1. If Ω^+ is a ball centered at the origin and $b(x) = 0$ in Ω^+, then b is compatible. The result follows from Strong Maximum Principle 2.1.2 and the fact that the unique solution uniformly converges to zero at the infinity. □

Remark 3.2.16. *Note that we do not need to apply the moving planes method in this setting since we have the uniqueness result. Furthermore, the moving plane process would require more stringent hypotheses on a and b.*

3.3 The Parametrized Equation

In this section we consider the effect of the parameter λ on the shape and multiplicity of solutions to the parametrized family,

$$-\triangle u = a_\lambda(x)u^q + bu^p \text{ in } \mathbb{R}^n, \quad u \geq 0 \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \tag{3.3.1}$$

where $a_\lambda(x) = \lambda a^+(x) - a^-(x)$.

First we are going to discuss the asymptotic behavior of problem (3.3.1), actually we already know that from Theorem 1.1.1. it has a maximum solution U_λ in $L^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,2}(\mathbb{R}^n)$ for any λ. The first question we want to ask is what will happen when λ tends to zero? Well assume b is compatible, if Ω^+ is admissible, we see that that the maximum solution U_λ of (3.3.1) breaks into pieces and each piece is also a solution of (3.3.1). Actually We have the following Theorem 1.1.8, whose proof is very similar to Theorem 2.5 in [7], so we briefly repeat it here. Before proving the Theorem, we need two Lemmas.

Lemma 3.3.1. *For any positive constant c, the equation $\triangle \overline{u} = c\overline{u}^q$ in $\mathbb{R}^n$ has a radial solution $\overline{U} = \theta r^{\frac{2}{1-q}}$, where $\theta^{1-q} = \frac{(1-q)^2 c}{2[n-q(n-2)]}$.*

Lemma 3.3.2. *For any ball $B \subset\subset \mathbb{R}^n - \Omega^{0+}$, any $g(x) \geq 0$, the following problem has at most one non-negative classical solution,*

$$-\triangle v = a(x)v^q + bv^p \text{ in } B \qquad v = g \text{ on } \partial B.$$

The proof for the first lemma is by direct calculation, the proof for the second lemma is by direct comparison of two solutions. With the help of the above two lemma, we can show the proof of Theorem 1.1.8:

Proof. By assumption Ω^{0+} is admissible, then $\mathrm{dist}(\Omega_i^{0+}, \Omega_j^{0+}) > 0$ for any $i \neq j$. Let $\delta = \frac{1}{16}\inf_{i\neq j} \, dist(\Omega_i^{0+}, \Omega_j^{0+})$, then $\delta > 0$.

As before, take $\rho_1 > 0$ so that $\Omega^{0+} \subset\subset B(0, \rho_1)$. Let

$$C_i = \{x \in B(0,\ \rho_1 + 48\delta) \mid\ dist(x,\ \Omega_i^{0+}) \leq \delta\}$$

It is easy to see that $C_i \cap C_j = \emptyset$ for any $i \neq j$. Let $C = \cup_{i \in M} C_i$. We define

$$N = \{x \in B(0,\ \rho_1 + 32\delta) \mid\ dist(x,\ \Omega^{0+}) \geq 4\delta\}.$$

For any $x \in N$, $\overline{B(x, \delta)} \cap C_i = \emptyset$ for any $i \in M$. Finally set

$$\underline{a} = \inf_{x \in B(0,\ R+48\delta)-C} a^-(x).$$

By Lemma 3.3.1 we find that the following equation

$$\Delta \overline{u} = \underline{a}\overline{u}^q\ in\ B(x,\ \delta), \qquad \overline{u} = \theta(\delta)^{\frac{2}{1-q}}\ on\ \partial B(x,\ \delta)$$

has a solution $\overline{U} = \theta|y-x|^{\frac{2}{1-q}}$, where $x \in N$.

We now claim that the equation,

$$-\Delta v = a^-(y)v^q + b(y)v^p\ in\ B(x,\ \delta) \qquad v = U\ on\ \partial\ B(x,\ \delta)$$

has a unique solution $v = U_\lambda$ if $||U_\lambda||_{L^\infty(\mathbb{R}^n)} \leq \theta(\delta)^{\frac{2}{1-q}}$, where U_λ is the maximum solution of (3.3.1). Indeed, from Lemma 3.1.4, we have $\lim_{\lambda \to 0} U_\lambda = 0$. Therefore there exists $\lambda_* > 0$ such that $||U_\lambda||_{L^\infty(\mathbb{R}^n)} \leq \theta(\delta)^{\frac{2}{1-q}}$ for $\lambda \leq \lambda_*$. Hence $\overline{U}$ is an upper solution for the above equation and 0 is a lower solution, so the above equation has a solution v. But it is clear the maximum solution U_λ is also a solution, by lemma 3.3.2 we know that the above equation has a unique solution $v = U_\lambda$. Therefore we have $0 \leq U(x) \leq \overline{U}(x) = 0$, which means $U(x) = 0$ for all $x \in N$. Hence we can write

$$U_\lambda = \Sigma_{i \in M} u_i\ and\ supp(u_i) \cap supp(u_j) = \emptyset\ for\ i \neq j,$$

where $\Omega^+ \subset supp(u_i)$, which means that $\Sigma_{i \in I} u_i \in N_I$. □

Remark 3.3.3. *If in addition we assume b is compatible in Theorem 1.1.8, then Lemma 2.2.1 will apply, we have uniqueness for N_I. So for λ small, the support of the maximum solution U_λ will break into k disjoint components, which generate the unique element in N_I.*

We finish the discussion when λ is small, it is natural to ask what will happen when λ gets big. Under the assumption (1.0.2), fix $I \subset M$, by Theorem 1.1.6 the minimum element of S_I exists, denoted by u_λ. Let $v_\lambda = \lambda^{\frac{1}{q-1}} u_\lambda$, then v_λ satisfies the following equation $\mathbb{R}^n$.

$$-\Delta v = a^+ v^q - \frac{a^-}{\lambda} v^q + b\lambda^{\frac{1-p}{q-1}} v^p. \tag{3.3.2}$$

For each fixed λ, let $\overline{S}_I$ be the corresponding set of S_I, associated with the above equation (3.3.2). Let us begin the process of proving Theorem 1.1.9 with a few Lemma. Recall that Theorem 1.1.9 concerns the case $p < 1$.

Lemma 3.3.4. *Assume (1.0.2), then v_λ is the minimum element in $\overline{S}_I$. Moreover, v_λ is increasing as λ increases, and $v_\lambda \leq \omega$, where ω is as in* (3.1.2).

Proof. Suppose $0 < \lambda_1 < \lambda_2$, since $-\frac{1}{\lambda}$ and $b\lambda^{\frac{1-p}{q-1}}$ are increasing, then from Theorem 1.1.1 we see that v_{λ_2} is an upper solution for the equation (3.3.2) at $\lambda = \lambda_1$, by choosing suitable $\underline{u}_\rho$ as lower solution, we get a solution $v \in \overline{S}_I$ of (3.3.2) at $\lambda = \lambda_1$ such that $v_{\lambda_1} \leq v \leq v_{\lambda_2}$. The last part is from Theorem 1.1.6. □

We now identify a limit for v_λ as $\lambda \to \infty$. When $p < 1$, the limit will be ω, as in Lemma 3.1.8. When $p = 1$ the linear term modifies the limit. We have:

Lemma 3.3.5. *For Hölder continuous $b(x) \leq 0$, there exists a unique non-negative solution $\omega_b \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ to the following equation:*

$$-\Delta \omega_b = a^+(x)\omega_b{}^q + b(x)\omega_b, \quad \omega_b(x) \to 0 \text{ as } |x| \to \infty \tag{3.3.3}$$

Proof. Now, since $b(x) \leq 0$, ω is a super-solution for the second equation. As in section 3.2 we may construct a sub-solution supported in a ball compactly contained in Ω^+, and follow the monotone iteration method as developed in that section. Notice that we only needed to assume the Basic Hypothesis (Ω^+ bounded) for these results to hold. The uniqueness follows from Lemma 2.2.1, as we remarked at the end of Chapter 2. ☐

Lemma 3.3.6. *Assume (1.0.2), then*

$$\lim_{\lambda \to \infty} v_\lambda(x) = \begin{cases} \omega(x), & \text{if } q < p < 1; \\ \omega_b(x), & \text{if } p = 1, \end{cases}$$

uniformly in $\mathbb{R}^n$, where ω is from (3.1.2) *and ω_b is from Lemma 3.3.5.*

Proof. From above Lemma we see that v_λ is increasing and uniformly bounded, let $V = \lim_{n\to\infty} v_\lambda$, then it is easy to see that $\lim_{|x|\to\infty} V(x) = 0$, moreover from equation (3.3.2) we have $||v_\lambda||_{C^{1,\alpha}(B(0,R))}$ is uniformly bounded for fixed R, therefore we have v_λ uniformly converges to V. In either case, we may pass to the limit in the distributional formulation of the equation and obtain that the limit solves (3.1.2) if $p < 1$ or (3.3.3) if $p = 1$. Since $\lim_{|x|\to\infty} V(x) = 0$, by uniqueness of the solution (in either case) we obtain our conclusion. ☐

Now we are ready to prove Theorem 1.1.9:

Proof. Since $u_\lambda = \lambda^{\frac{1}{1-q}} v_\lambda$ and v_λ uniformly converges to $V > 0$ in $\mathbb{R}^n$. Since u_λ is minimum element in S_I and the choices for I are finite, all solutions of $(3.3.1)_\lambda$ are positive in $\overline{\Omega^+}$. Therefore Lemma 2.2.1 implies that N_M has only one element! ☐

When we consider the case $p > 1$, it is easy to see that the above process should not work, in part because now $\lim_{\lambda\to\infty} \lambda^{\frac{1-p}{q-1}} = \infty$. To obtain some asymptotic results in this regime we must impose some conditions on $b(x)$ on and around the set Ω^+.

Let u_λ be any solution of (1.1.5). We prove:

Theorem 3.3.7. *Assume (1.0.2) and $p > 1$.*

(1) For any $\sigma > 0$, $\lim_{\lambda\to\infty} ||u_\lambda||^{p-q+\sigma}_{L^\infty(\mathbb{R}^n)}\lambda^{-1} = \infty$.

(2) If $\inf_{x\in\Omega^+}|b(x)| > 0$, then $||u_\lambda||^{p-q}_{L^\infty(\mathbb{R}^n)} \le C\lambda$ for some constant $C > 0$ independent of λ.

(3) If $b(x) = 0$ in a ball $B \subset \Omega_i^+$ for some $i \in M$, then $\liminf_{\lambda\to\infty} ||u_\lambda||_{L^\infty(\mathbb{R}^n)}\lambda^{\frac{1}{q-1}} > 0$, for $I = \{i\}$ and $u_\lambda \in S_I$.

(4) If $\{x \in \mathbb{R}^n \,|b(x) = 0\}\cap\Omega^+$ has a open connected component O such that $\Omega_i^+\cap O \ne \emptyset$ for any $i \in M$, then the problem (1.1.5) has a unique solution, which is positive in Ω^+ for λ large enough.

We first require some Lemmas. Let $v_\lambda = \lambda^{\frac{1}{q-1}}u_\lambda$, so that v_λ satisfies the following modified equation.

$$-\triangle v = a^+v^q - \frac{a^-}{\lambda}v^q + \frac{b}{\lambda^{1-\frac{\epsilon}{1-q}}}u_\lambda^{p-q-\epsilon}v^{q+\epsilon} \text{ in } \mathbb{R}^n, \tag{3.3.4}$$

where we pick $\epsilon > 0$ small so that $\epsilon < 1-q$. Under the assumption (1.0.2) the equation (3.3.4) admits a minimal solution $\overline{v_\lambda}$ in the class $\overline{S_I}$, where $\overline{S_I}$ is the corresponding set of S_I, associated with the equation (3.3.4).

Lemma 3.3.8. *Assume (1.0.2), if for some $\sigma > 0$, $\liminf_{\lambda\to\infty} ||u_\lambda||^{p-q+\sigma}_{L^\infty(\mathbb{R}^n)}\lambda^{-1} < \infty$. Then there exists an increasing sequence $\{\lambda_n\}$ with $\lim_{n\to\infty}\lambda_n = \infty$ so that $||\overline{v_{\lambda_n}}||_{L^\infty(\mathbb{R}^n)}$ has a positive lower bound $C(\Omega_I^+)$ independent of λ for large λ.*

Proof. By assumption we can pick an increasing sequence $\{\lambda_n\}$ with $\lim_{n\to\infty}\lambda_n = \infty$ so that $||u_{\lambda_n}||^{p-q+\sigma}_{L^\infty(\mathbb{R}^n)}\lambda_n^{-1} \le C$ for some $C > 0$. Hence we have $||u_{\lambda_n}||_{L^\infty(\mathbb{R}^n)} \le C\lambda_n^{\frac{1}{p-q+\sigma}}$, so

$$\frac{u_{\lambda_n}^{p-q-\epsilon}}{\lambda_n^{1-\frac{\epsilon}{1-q}}} \le C\lambda_n^{\frac{p-q-\epsilon}{p-q+\sigma}-(1-\frac{\epsilon}{1-q})}.$$

For this fixed $\sigma > 0$, we can choose $\epsilon > 0$ small so that $\frac{p-q-\epsilon}{p-q+\sigma} < 1 - \frac{\epsilon}{1-q}$. Therefore $\frac{b}{\lambda_n^{1-\frac{\epsilon}{1-q}}} u_{\lambda_n}^{p-q-\epsilon} \to 0$ uniformly in $\overline{\Omega^+}$. By the same proof for Lemma 4 in [33], the result follows. □

We are ready to prove the first two assertions in Theorem 3.3.7. For the first, since under hypothesis (1.0.2) the choices for I are finite, we only need to show the Theorem is true for u_λ, which is the minimum element in S_I for each fixed $I \subset M$.

Let us prove by contradiction, assume otherwise for some $\sigma > 0$,

$$\liminf_{\lambda \to \infty} ||u_\lambda||_{L^\infty(\mathbb{R}^n)}^{p-q+\sigma} \lambda^{-1} < \infty,$$

then take ϵ and $\{\lambda_n\}$ from the proof of above Lemma. So there exists $C > 0$, for example $C = C(\Omega_I^+)/2$ from above lemma, so that for n large

$$||v_{\lambda_n}||_{L^\infty(\mathbb{R}^n)} \geq ||\overline{v_{\lambda_n}}||_{L^\infty(\mathbb{R}^n)} \geq C,$$

which implies $||u_{\lambda_n}||_{L^\infty(\mathbb{R}^n)} \geq C\lambda_n^{\frac{1}{1-q}}$. But we assume at the beginning $||u_{\lambda_n}||_{L^\infty(\mathbb{R}^n)} \leq C\lambda_n^{\frac{1}{p-q+\sigma}}$, it is a contradiction! So we must have $\liminf_{\lambda\to\infty} ||u_\lambda||_{L^\infty(\mathbb{R}^n)}^{p-q+\sigma}\lambda^{-1} = \infty$ for any $\sigma > 0$.

For the second part, let us assume $\inf_{x\in\Omega^+} |b(x)| = \bar{b} > 0$, from Lemma 3.1.5 we see that

$$b(x_0)u_\lambda^p(x_0) + a(x_0)u_\lambda^q(x_0) = -\Delta u_\lambda(x_0) \geq 0,$$

where u_λ attains global maximum at $x_0 \in \overline{\Omega^+}$. Hence we have

$$\lambda ||a^+||_{L^\infty(\mathbb{R}^n)} ||u_\lambda||_{L^\infty(\mathbb{R}^n)}^q \geq (-b(x_0)) ||u_\lambda||_{L^\infty(\mathbb{R}^n)}^p \geq \bar{b} ||u_\lambda||_{L^\infty(\mathbb{R}^n)}^p,$$

that is $||u_\lambda||_{L^\infty(\mathbb{R}^n)}^{p-q} \leq \lambda ||a^+||_{L^\infty(\mathbb{R}^n)} \bar{b}^{-1}$. This proves the second assertion of the theorem.

To prove the third assertion, we require the following Lemma from [33]:

Lemma 3.3.9 (Lemma 4 of [33]). *Assume (1.0.2), if $b(x) = 0$ in a ball $B \subset \Omega_i^+$ for some $i \in M$, then $||\overline{v_\lambda}||_{L^\infty(B)}$ has a positive lower bound $C(B)$ independent of λ for large λ.*

This proof of the third statement is then very simple. Take $c = \frac{1}{2}C(B)$, where $C(B)$ is from above lemma. For large λ,

$$||v_\lambda||_{L^\infty(\mathbb{R}^n)} \geq ||\overline{v_\lambda}||_{L^\infty(\mathbb{R}^n)} \geq c,$$

which leads to our conclusion.

Finally, we prove the fourth statement of Theorem 3.3.7 by contradiction. Since Ω^+ has finitely many connected components, we can assume that there exists an increasing sequence $\{\lambda_n\}$ with $\lim_{n\to\infty} \lambda_n = \infty$ so that $u_{\lambda_n} > 0$ in Ω_i^+ for some $i \in M$and $u_{\lambda_n} = 0$ in Ω_j^+ for some $j \neq i$. Take $I = \{i\}$, assume u_{λ_n} is the minimum element in S_I, then like the proof for Theorem 1.1.9, restricting to a subsequence if necessary, we have $v_{\lambda_n} \geq \overline{v_{\lambda_n}} \to \overline{V}$ in O, which solves $-\Delta w = a^+ w^q$ in O, by above Lemma and maximum principle $\overline{V} > 0$ in O, which contradicts our assumption! This concludes the proof of Theorem 3.3.7.

3.4 Solutions not in $\mathcal{D}^{1,2}(\mathbb{R}^n)$

In this section we consider the possibility that u is an entire solution to the P.D.E. $-\Delta u = a(x)u^q + b(x)u^p$ in $\mathbb{R}^n$, without requiring u to lie in the finite energy space $\mathcal{D}^{1,2}(\mathbb{R}^n)$. When $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ we prove the uniform decay estimates in Lemmas 3.1.1 and 3.1.3. Since these estimates are crucial for the proof of compact support (Theorem 1.1.2) in the more general case we must recover them in some way.

In general, we do expect that there are solutions which are not in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. For example, consider the following equation

$$\Delta z = c_1 z^q + b_1(x) z^p \ in \ \mathbb{R}^n - B(0, r_1), \tag{3.4.1}$$

where $c_1 > 0$, $0 < q < p$ and $r_1 > 0$, moreover $b_1(x) = c_2(r - r_1)^{\frac{1+q-2p}{1-q}} r^{-1}$ for $r \geq r_1$ and $c_2 > 0$. We seek the form of solution $z = \theta(r - r_1)^{\frac{2}{1-q}}$ for some $\theta > 0$, plug it into the equation (3.4.1) and calculate, it is easy to see that if the following is satisfied

$$c_1 = \frac{2}{1-q}\left(\frac{2}{1-q} - 1\right)\theta^{1-q} \qquad and \qquad c_2 = \frac{2(n-1)}{1-q}\theta^{1-p},$$

then $z = \theta(r - r_1)^{\frac{2}{1-q}}$ is a solution of the equation (3.4.1), therefore let us look at the equation

$$-\Delta u = a(x)u^q + b(x)u^p \ in \ \mathbb{R}^n$$

where $a(x) = -c_1$ in $\mathbb{R}^n - B(0, R_1)$ for some $R_1 > 0$ and $b(x) \leq 0$.

From Theorem 1.1.2 we find that there exists $\rho_2 > R_1$ such that supp $(u) \subset B(0, \rho_2)$ for any compact support solution u of above equation and ρ_2 is independent of b. Now if we assume $r_1 = \rho_2$ and $b(x) = -b_1(x)$ for $|x| \geq \rho_2$, then the above equation has a solution U of the form

$$U = \begin{cases} \theta(r - \rho_2)^{\frac{2}{1-q}} & for \ r \geq \rho_2 \\ u & for \ r \leq \rho_2 \end{cases}$$

and it is clear that $U \notin \mathcal{D}^{1,2}(\mathbb{R}^n)$!

In the radial case we may prove the following result:

Theorem 3.4.1. *Assume* $a(x) = a(|x|)$, $b(x) = b(|x|)$ *and* $\liminf_{|x|\to\infty} a^-(x) = c > 0$. *Then any smooth radial solution* $u(x) = u(|x|)$ *has the property that*

$$\lim_{|x|\to\infty} u(x) = 0$$

if $u \in L^\infty(\mathbb{R}^n)$ *and* $\lim_{r\to\infty} u_r r^{-1} = 0$.

Proof. First pick $R > 0$ big enough so that $a^-(x) \geq \frac{c}{2}$ for any $|x| \geq R$, then we discuss the behavior of $u(r)$ in the domain $[R, \ \infty)$, we divide into three cases.

Case 1: there exists $r_1 \geq R$ such that $u_r(r_1) = 0$.

First we see that $u(r)$ must attain local minimum at $r = r_1$. Indeed if $u(r_1) = 0$, then we are done! So assume $u(r_1) > 0$, then we have

$$u_{rr}(r_1) = u_{rr}(r_1) + (n-1)u_r(r_1)r^{-1} = \Delta u(r_1) = -a(r_1)u^q(r_1) - b(r_1)u^p(r_1) > 0.$$

Therefore $u(r)$ attains local minimum at $r = r_1$.

We now claim that $u(r) = 0$ for $r \geq r_1$. Otherwise, there would exist $r_2 > r_1$ such that $u(r_2) > 0$. So we should have that $u_r(r) > 0$ for $r > r_2$. If not, there would exist $r_3 > r_2$ such that $u_r(r_3) = 0$, it is easy to see that $u(r)$ attains local minimum at $r = r_3$, therefore $u(r)$ achieves local maximum at some $r_4 \in (r_1,\ r_3)$, but this is impossible because $u_{rr}(r_4) = -a(r_4)u^q(r_4) - b(r_4)u^p(r_4) > 0$. So we must have $u_r(r) > 0$ for $r > r_2$. Since $u \in L^\infty(\mathbb{R}^n)$ we have $u(r) \uparrow M$ as $r \to \infty$ for some positive constant M, this leads to a sequence $\{r_n\}$ and $\lim_{n\to\infty} r_n = \infty$ such that $\lim_{n\to\infty} u_{rr}(r_n) = 0$. But since $\lim_{r\to\infty} u_r r^{-1} = 0$, then we should have

$$\liminf_{n\to\infty} u_{rr}(r_n) = \liminf_{n\to\infty}(-a(r_n)u^q(r_n) - b(r_n)u^p(r_n)) > 0.$$

This is a contradiction! So we have $u(r) = 0$ for $r \geq r_1$.

Case 2: $u_r(r) > 0$ for any $r \in [R,\ \infty)$. From the above proof we see that this case is impossible.

Case 3: $u_r(r) < 0$ for any $r \in [R,\ \infty)$.
Let $m = \lim_{r\to\infty} u(r)$, then we must have $m = 0$. Otherwise $m > 0$, then we can find a sequence $\{r_n\}$ and $\lim_{n\to\infty} r_n = \infty$ such that

$$\lim_{n\to\infty} u_{rr}(r_n) = 0.$$

But

$$\liminf_{n\to\infty} u_{rr}(r_n) = \liminf_{n\to\infty}(-a(r_n)u^q(r_n) - b(r_n)u^p(r_n) - \frac{n-1}{r_n}u_r(r_n)) \geq \liminf_{n\to\infty}(-a(r_n)u^q(r_n)) > 0.$$

This is a contradiction, and hence we must have $\lim_{r\to\infty} u(r) = 0$! □

It should be interesting and possible to prove some results in non-radial settings. In particular, Brezis–Kamin (Lemma A.6 of [12])), if we could show that

$$\frac{1}{|\partial B(0,R)|}\int_{\partial B(0,R)} u \to 0 \qquad \text{as } R \to 0,$$

then any smooth solution $u \in L^\infty(\mathbb{R}^n)$ of (3.0.1) has the property that $\lim_{|x|\to\infty} u(x) = 0$. Many open questions remain.

Chapter 4

Concave Plus Convex Nonlinearities

In this chapter we will deal with the special case when $b(x) = 1$, i.e. we study elliptic problem in $\mathbb{R}^n$, $n \geq 3$:

$$\begin{cases} -\triangle u &= a(x)u^q + u^p \quad \text{in } \mathbb{R}^n, \quad 0 < q < 1 < p < 2^* - 1, \\ u &\geq 0 \quad \text{in } \mathbb{R}^n, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n). \end{cases} \tag{4.0.1}$$

Moreover we always make the following assumption on a,

$$0 < a_\infty = \liminf_{|x| \to \infty} a^- \leq \limsup_{|x| \to \infty} a^- < \infty, \tag{4.0.2}$$

for a positive constant.

Notice that the solutions of problem (4.0.1) could support in the region Ω^- (see example 1.2.1). Therefore in this chapter, if u is a solution of (4.0.1), we always mean that $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ is a weak solution of (4.0.1) and u is positive somewhere in Ω^+. We will see that in this case the property of u is quite different to the one in (3.0.1), partially because that we do not have fixed sign for the right side of the equation in a neighborhood around infinity, hence we can not have results like Lemmas 3.1.3, 3.1.5, 3.1.6 and 3.1.4. Nevertheless, the solutions in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ still have compact supports.

4.1 Compact Support and Minimal Solution

Just as in the logistic case (3.0.1), the solutions u of (4.0.1) still have compact support, although we do not have the control on the size of the support. We first prove a very useful lemma which is inspired by the paper of Cortázar, Elgueta and Felmer [18].

Lemma 4.1.1. *Given a smooth function $v \geq 0$, suppose it satisfies for some $R > 0$*

$$-\Delta v \leq -a^- v^q + v^p \text{ in } \mathbb{R}^n - B(0,\ R) \quad \text{and} \quad \lim_{|x|\to\infty} v(x) = 0,$$

then there exists $\epsilon_0 > 0$ such that the set $\{x \in \mathbb{R}^n - B(0,\ R) \mid 0 < v(x) < \epsilon\}$ is bounded for any $\epsilon \leq \epsilon_0$, i.e. v is compactly supported.

Proof. Define the functions $f(s),\ F(s) : \mathbb{R}^+ \to \mathbb{R}$ to be

$$f(s) = s^p - cs^q, \qquad F(s) = \frac{1}{p+1} s^{p+1} - c\frac{1}{q+1} s^{q+1},$$

where $c = \frac{1}{2}a_\infty$. Let $D > 0$ be the constant such that $D^{p-q} = c\frac{q}{p}$, it is easy to see that $f(s)$ is decreasing in the range $[0,\ D]$. Since $\lim_{|x|\to\infty} v(x) = 0$ and $\liminf_{|x|\to\infty} a^- = a_\infty$, then pick $R_1 > R$ such that

$$a^-(x) \geq c \quad \text{and} \quad v(x) < D \text{ for all } x \in \mathbb{R}^n - B(0,\ R_1).$$

Let $w(r)$ be the function defined implicitly by

$$\int_{w(r)}^{D} \frac{ds}{\sqrt{-F(s)}} = \sqrt{2}r.$$

It is easy to see that $w(r)$ satisfies

$$w''(r) + f(w(r)) = 0 \text{ in } (0,\ A),$$

where A is given by

$$\sqrt{2}A = \int_0^D \frac{ds}{\sqrt{-F(s)}}.$$

Moreover $w(r)$ is a decreasing function in r, and it satisfies

$$w(0) = D, \qquad w(A) = w'(A) = w''(A) = 0.$$

Therefore by defining $w(r) \equiv 0$ for $r \in [A, \infty)$, we obtain a non-increasing solution of

$$w''(r) + f(w(r)) = 0 \; in \; (0, \infty),$$

with $w(0) = D$ and $supp\ w = [0, A]$.
Finally let $V(x) = w(|x| - R_1)$, then we have

$$\begin{aligned} \triangle V - cV^q + V^p &\leq 0 \; in \; \mathbb{R}^n - \overline{B(0,R_1)} \\ V &= D \; on \; \partial(\mathbb{R}^n - B(0,R_1)). \end{aligned}$$

Noticing for v we have

$$\begin{aligned} \triangle v - a^- v^q + v^p &\geq 0 \; in \; \mathbb{R}^n - \overline{B(0,R_1)} \\ v &< D \; on \; \partial(\mathbb{R}^n - B(0,R_1)). \end{aligned}$$

By subtracting them, we have

$$-\triangle(V - v) \geq V^p - cV^q + a^-(x)v^q - v^p \; for \; x \in (\mathbb{R}^n - \overline{B(0,R_1)})$$

Claim: $V \geq v \geq 0 \; for \; x \in \mathbb{R}^n - B(0, R_1)$.

Otherwise there exists $x_0 \in \mathbb{R}^n - \overline{B(0,\ R_1)}$ such that $v(x_0) > V(x_0)$, which implies that $V - v$ attains global minimum at some point in $\mathbb{R}^n - \overline{B(0, R_1)}$. Without loss of generality let us assume $V - v$ achieves minimum at x_0, then we must have

$$\begin{aligned} 0 &\geq -\triangle(V - v)(x_0) \\ &\geq V^p(x_0) - cV^q(x_0) + a^-(x_0)v^q(x_0) - v^p(x_0) \\ &\geq V^p(x_0) - cV^q(x_0) + a^-(x_0)v^q(x_0) - v^p(x_0) + cv^q(x_0) - cv^q(x_0) \\ &\geq (V^p(x_0) - cV^q(x_0)) - (v^p(x_0) - cv^q(x_0)) + (a^-(x_0)v^q(x_0) - cv^q(x_0)) \\ &> 0, \end{aligned}$$

a contradiction, and the claim is done.

Thus we must have $V \geq v \geq 0$ *for* $x \in (\mathbb{R}^n - B(0, R_1))$, which implies v has compact support. Taking $\epsilon_0 = \frac{1}{2}D$, the Lemma then follows.

□

Now we introduce a Lemma which follows easily from an analogous result in [18]. For any ball $B(x,1) \subset\subset B(x,2)$ we have:

Lemma 4.1.2. *There exists a continuous function* h*:* $\mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *, with* $h(0) = 0$*, such that*

$$\| u \|_{L^\infty(B(x,1))} \leq Kh(\| u \|_{H^1(B(x,2))}).$$

The function h *depends on* q*,* p*,* n*, and the constant* K *depends on* q*,* p*,* n *and* $||a||_{L^\infty(B(x,2))}$.

Lemma 4.1.3. *Assume* u *is a solution of (4.0.1), then* $\lim_{|x| \to \infty} u(x) = 0$.

Proof. Since we assume that $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, then for $\epsilon > 0$, there exists $R_1 > 0$, which depends on ϵ, such that

$$||u||_{D^1(\mathbb{R}^n - B(0,R_1))} < \epsilon.$$

Hence for $x \in \mathbb{R}^n - \overline{B(0, R_1 + 2)}$, we have $B(x,1) \subset B(x,2) \subset \mathbb{R}^n - \overline{B(0,R_1)}$. So from Lemma 4.1.2 we get

$$|u(x)| \leq ||u||_{L^\infty(B(x,1))} \leq Kh(||u||_{H^1(B(x,2))}).$$

Notice that $||u||_{H^1(B(x,2))}$ is controlled by $||u||_{\mathcal{D}^{1,2}(\mathbb{R}^n - B(0,R_1))}$, since $h(t)$ is continuous and $h(0) = 0$. Thus the lemma is proved. □

Combining the above two Lemmas, we obtain the Theorem 1.2.2:

Theorem 4.1.4. *Every weak solution of (4.0.1) is classical and has compact support.*

Remark 4.1.5. *By comparing the proofs of Theorems 1.1.2 and 1.2.2 we immediately notice the difference: in the logistic case, the maximum value is attained inside* Ω^{0+} *whereas in for* (4.0.1) *this may not be true, as example 1.2.1 demonstrates.*

We also note that the above theorem relies heavily on the fact that $\liminf_{|x|\to\infty} a^- > 0$.

Now we turn to the issue of minimal element of S_I if it is not empty. We have the following theorem which is part of Theorem 1.2.3

Theorem 4.1.6. *Assume (1.0.2), if* $S_I \neq \emptyset$, *then there exists a minimum element* u_I *in* S_I.

Before we are ready to prove this theorem, we need a few Lemmas.

Lemma 4.1.7. *Assume (1.0.2), the solution* u *of the equation (4.0.1) is either positive in* Ω_i^{0+} *or identically zero in* Ω_i^{0+} *for any* $1 \geq i \leq m$. *If in addition* u *is positive in* Ω_i^{0+}, *then* u *is positive in* $\overline{\Omega_i^{0+}}$.

Proof. The first part is due to Lemma 2.1.3, and for the second statement we know that the following is true

$$-\Delta u = a(x)u^q + u^p \geq 0 \text{ in } \Omega_i^{0+} \quad \text{and} \quad u \geq 0 \text{ in } \Omega_i^{0+},$$

since u is uniformly bounded in Ω_i^{0+}. By the Hopf Lemma 2.1.1 we can conclude our lemma. □

Let $\bar{S}_I$ and $\bar{N}_I$ be the corresponding set of S_I and N_I for the following equation:

$$-\Delta u = au^q, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^n),\ u \geq 0$$

Then by Lemma 2.2.1 we have the following result:

Lemma 4.1.8. *Assume (1.0.2), if $S_I \neq \emptyset$, then we have $u \geq \underline{u}_I$ for any $u \in S_I$, where $\underline{u}_I$ represents the minimum element in $\bar{S}_I$.*

Proof. Since $S_I \neq \emptyset$, pick any $u \in S_I$, then there exists $J \subset M$ such that $I \subset J$ and $u \in N_J$. By the sub and super solution method, we conclude that $\bar{N}_J$ is not empty. Let us denote this unique element in $\bar{N}_J$ by $\underline{u}$, hence we have $\underline{u} \leq u$. Since $I \subset J$, then $\underline{u} \in \bar{S}_I$. So we have $u \geq \underline{u} \geq \underline{u}_I$. This Lemma is done. □

We still need some results about the following equation

$$-\triangle v + a^-(x)v^q = a^+(x)h^q + h^p \ in \ \mathbb{R}^n \quad and \quad v \geq 0 \ in \ \mathbb{R}^n, \tag{4.1.1}$$

where h is nonnegative, smooth and compact supported.

Lemma 4.1.9. *There exists a compactly supported solution of the above equation (4.1.1).*

Proof. For $R > 0$, let us consider the Dirichlet boundary problem

$$-\triangle v + a^-(x)v^q = a^+(x)h^q + h^p \ in \ B(0,\ R) \quad and \quad v = 0 \ on \ \partial B(0,\ R).$$

Since h is nonnegative, then 0 is a lower solution to this equation. We also find out that $\overline{v} = \int_{\mathbb{R}^n} \Phi(x-y)(a^+(y)h^q(y) + h^p)dy$ satisfies

$$-\triangle \overline{v} = a^+h^q + h^p \geq a^+h^q + h^p - a^-\overline{v}^q \ in \ \mathbb{R}^n,$$

where Φ is the fundamental solution of the Laplace's equation. So $\overline{v}$ is an upper solution, then there exist a solution $v \in H_0^1(B(0,\ R))$ of this equation. Since h and a^- are Holder continuous, then this solution v is classical. It is not hard to see that v is also unique.

Next we employ the arguments from Lemma 4.1.1 to show that when R is large enough, actually this solution v is compactly supported in $B(0,\ R)$. Define the functions $f(s),\ F(s) : \mathbb{R}^+ \to \mathbb{R}$ to be

$$f(s) = -cs^q, \qquad F(s) = -c\frac{1}{q+1}s^{q+1},$$

where $c = \frac{1}{2}a_\infty$. Let $D > 0$ be the constant such that $D = ||h||_{L^\infty(\mathbb{R}^n)}$, it is easy to see that $f(s)$ is decreasing in the range $[0,\ D]$. Since $\lim_{|x|\to\infty} \overline{v} = 0$ and $\liminf_{|x|\to\infty} a^- = a_\infty$, then pick $R_1 > 0$ such that

$$a^-(x) \geq c \ for\ all\ x \in \mathbb{R}^n - B(0,\ R_1) \quad and \quad supp\ h \subset\subset B(0,\ R_1).$$

Let $w(r)$ be the function defined implicitly by

$$\int_{w(r)}^{D} \frac{ds}{\sqrt{-F(s)}} = \sqrt{2}r.$$

It is easy to see that $w(r)$ satisfies

$$w''(r) + f(w(r)) = 0 \ in\ (0,\ A),$$

where A is given by

$$\sqrt{2}A = \int_0^D \frac{ds}{\sqrt{-F(s)}}.$$

Moreover $w(r)$ is a decreasing function in r, and it satisfies

$$w(0) = D, \qquad w(A) = w'(A) = w''(A) = 0.$$

Therefore by defining $w(r) \equiv 0$ for $r\ \in\ [A,\ \infty)$, we obtain a non-increasing solution of

$$w''(r) + f(w(r)) = 0 \ in\ (0,\ \infty),$$

with $w(0) = D$ and $supp\ w = [0,\ A]$.

Finally let $V(x) = w(|x| - R_1)$, then we have for $R > R_1 + A + 1$

$$\begin{aligned} \triangle V - cV^q &\leq 0 \ in\ B(0,\ R) - \overline{B(0, R_1)} \\ V &= D \ on\ \partial B(0, R_1)) \ and\ V = 0 \ on\ \partial B(0, R)). \end{aligned}$$

Noticing for v we have

$$\begin{aligned} \triangle v - a^- v^q &= 0 \; in \; B(0,\; R) - \overline{B(0, R_1)} \\ v &< D \; on \; \partial B(0, R_1) \; and \; v = 0 \; on \; \partial B(0, R)). \end{aligned}$$

By subtracting them, we have

$$-\triangle(V - v) \geq -cV^q + a^-(x)v^q \; for x \in (B(0,\; R) - \overline{B(0, R_1)})$$

Claim 2. $V \geq v \geq 0 \; for \; x \; \in B(0,\; R) - B(0, R_1)$.

Otherwise there exists $x_0 \in B(0,\; R) - \overline{B(0,\; R_1)}$ such that $v(x_0) > V(x_0)$, which implies that $V - v$ attains global minimum at some point in $B(0,\; R) - \overline{B(0, R_1)}$. Without loss of generality let us assume $V - v$ achieves minimum at x_0, then we must have

$$\begin{aligned} 0 &\geq -\triangle(V - v)(x_0) \\ &\geq -cV^q(x_0) + a^-(x_0)v^q(x_0) \\ &> 0. \end{aligned}$$

a contradiction, this claim is done.

So we must have $V \geq v \geq 0 \; for \; x \in (B(0,\; R) - B(0, R_1))$, which implies v has compact support in $B(0,\; R)$ for $R > R_1 + A + 1$, then v is clearly also a solution of (4.1.1). So this lemma is proven. □

Lemma 4.1.10. *The compactly supported smooth solution of (4.1.1) is unique.*

Proof. Suppose there are two compactly supported smooth solutions v_1 and v_2, then they satisfies

$$-\triangle v_1 + a^- v_1^q = a^+(x)h^q + h^p \; and \; - \triangle v_2 + a^- v_2^q = a^+(x)h^q + h^p \; in \; \mathbb{R}^n.$$

Subtracting them we have $-\Delta(v_1 - v_2) + (v_1^q - v_2^q) = 0$ in $\mathbb{R}^n$, multiply both sides by $(v_1 - v_2)$ and integrate over $\mathbb{R}^n$, since they are compactly supported, we have

$$\int_{\mathbb{R}^n} |\nabla(v_1 - v_2)|^2 dx + \int_{\mathbb{R}^n} a^-(v_1^q - v_2^q)(v_1 - v_2) dx = 0.$$

So we must have $v_1 = v_2$. This lemma is done. □

Now we start the monotone iteration process, using the minimum element in $\bar{S}_I$ as the starting point. Consider the following iteration equation

$$-\Delta u_{n+1} + a^- u_{n+1}^q = a^+ u_n^q + u_n^p \text{ in } \mathbb{R}^n \text{ and } u_n \geq 0 \text{ in } \mathbb{R}^n. \tag{4.1.2}$$

where u_1 is the minimum element in $\bar{S}_I$.

Lemma 4.1.11. *Assume (1.0.2), then every u_n is well-defined and is compactly supported.*

Proof. From Lemma 4.1.9 and Lemma 4.1.10, we see that every u_n is well-defined and is compactly supported. □

Lemma 4.1.12. *Assume (1.0.2), then $u_2 \geq u_1$.*

Proof. We know that u_1 and u_2 satisfy the following equations

$$-\Delta u_1 + a^- u_1^q = a^+ u_1^q \quad \text{and} \quad -\Delta u_2 + a^- u_2^q = a^+ u_1^q + u_1^p \text{ in } \mathbb{R}^n.$$

Subtract each other, we have $-\Delta(u_1 - u_2) + a^-(u_1^q - u_2^q) = -u_1^p \leq 0$ in $\mathbb{R}^n$, multiply both side by $(u_1 - u_2)^+$ and integrate over $\mathbb{R}^n$, we get

$$\int_{\mathbb{R}^n} |\nabla(u_1 - u_2)^+|^2 dx + \int_{\mathbb{R}^n} a^-(u_1^q - u_2^q)(u_1 - u_2)^+ dx \leq 0.$$

Therefore $(u_1 - u_2)^+ = 0$, which implies $u_2 \geq u_1$ in $\mathbb{R}^n$. □

Lemma 4.1.13. *Assume (1.0.2), then $u_{n+1} \geq u_n$.*

Proof. We show this by induction, from the above lemma, we see that the first step is right. Now we assume that $u_n \geq u_{n-1}$, then for u_n and u_{n+1} we have

$$-\Delta u_n + a^- u_n^q = a^+ u_{n-1}^q + u_{n-1}^p \quad and \quad -\Delta u_{n+1} + a^- u_{n+1}^q = a^+ u_n^q + u_n^p \; in \; \mathbb{R}^n.$$

Subtract each other, we have $-\Delta(u_n - u_{n+1}) + a^-(u_n^q - u_{n+1}^q) \leq 0$ in $\mathbb{R}^n$, multiply both side by $(u_n - u_{n+1})^+$ and integrate over $\mathbb{R}^n$, we get

$$\int_{\mathbb{R}^n} |\nabla(u_n - u_{n+1})^+|^2 dx + \int_{\mathbb{R}^n} a^-(u_n^q - u_{n+1}^q)(u_n - u_{n+1})^+ dx \leq 0.$$

Therefore $(u_n - u_{n+1})^+ = 0$, which implies $u_{n+1} \geq u_n$ in $\mathbb{R}^n$. □

Lemma 4.1.14. *Assume (1.0.2), if $S_I \neq \emptyset$, then $u_n \leq u$ for any $u \in S_I$.*

Proof. We also prove this by induction. Take any $u \in S_I$, then by lemma 4.1.8, we know that $u \geq u_1$, so the first step of the induction is right, now we assume $u \geq u_n$, then we see that u_{n+1} and u satisfy

$$-\Delta u_{n+1} + a^- u_{n+1}^q = a^+ u_n^q + u_n^p \quad and \quad -\Delta u + a^- u^q = a^+ u^q + u^p \; in \; \mathbb{R}^n.$$

Subtract each other, we have $-\Delta(u_{n+1} - u) + a^-(u_{n+1}^q - u^q) \leq 0$ in $\mathbb{R}^n$, multiply both side by $(u_{n+1} - u)^+$ and integrate over $\mathbb{R}^n$. we get

$$\int_{\mathbb{R}^n} |\nabla(u_{n+1} - u)^+|^2 dx + \int_{\mathbb{R}^n} a^-(u_{n+1}^q - u^q)(u_{n+1} - u)^+ dx \leq 0.$$

Therefore $(u_{n+1} - u)^+ = 0$, which implies $u_{n+1} \leq u$ in $\mathbb{R}^n$. □

Finally we are ready to prove the Theorem 4.1.6.

Proof. Take any $u \in S_I$, from above lemma we know that u_n is increasing and $u_n \leq u$, then denote u_I to be the limit function of u_I, it is clear that $u_I \leq u$. So we only need to prove that u_I is a solution of (4.0.1).

Claim 3. *u_I is a solution of (4.0.1).*

Indeed we know that u_n is uniformly bounded by u, which is compact supported, then we know from the equation (4.1.2) that $||u_n||_{C^{1,\alpha}(\mathbb{R}^n)}$ is uniformly bounded, then from the Arzela–Ascoli Theorem we have u_n uniformly converges to u_I, moreover $u_n \rightharpoonup u_I$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Now take any function $\phi \in C_0^\infty(\mathbb{R}^n)$, multiply ϕ to the equation (4.1.2) both side and integrate over $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \nabla u_{n+1} \nabla \phi dx + \int_{\mathbb{R}^n} a^- u_{n+1}^q \phi dx = \int_{\mathbb{R}^n} a^+ u_n^q \phi dx + \int_{\mathbb{R}^n} u_n^p \phi dx.$$

Pass the limit we have

$$\int_{\mathbb{R}^n} \nabla u_I \nabla \phi dx + \int_{\mathbb{R}^n} a^- u_I^q \phi dx = \int_{\mathbb{R}^n} a^+ u_I^q \phi dx + \int_{\mathbb{R}^n} u_I^p \phi dx,$$

which implies u_I is a solution of the equation (4.0.1) in the weak sense, by standard bootstrap arguments, we see that u_I is a classical solution. This theorem is done. □

4.2 Existence for the Parametrized Equation

As mentioned in the introduction, unlike the logistic case where the existence of a solution is true under very weak hypotheses, for equation (4.0.1) the question of existence is more delicate. Our approach is to include a parameter in the equation and vary the strength of the positive part of $a(x)$,

$$-\Delta u = a_\gamma u^q + u^p, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \quad u \geq 0 \text{ in } \mathbb{R}^n, \tag{4.2.1}$$

where $a_\gamma = \gamma a^+ - a^-$ and $\gamma > 0$. We only consider the dependence on γ, so a is fixed. To emphasize the dependence on γ, problem (4.2.1) is often referred to as problem $(4.2.1)_\gamma$. When there is no confusion, the subscript γ will be omitted. For the existence of $(4.2.1)_\gamma$ the idea is not complicated, and has already appeared in the proof of Lemma 4.1.9, namely we find a global super-solution, which is positive in $\mathbb{R}^n$ and uniformly goes to zero at the infinity. This super-solution is also a super-solution of the following problem:

$$-\Delta u = a_\gamma(x) u^q + u^p \text{ in } B(0,\ R) \qquad \text{and} \qquad u \in H_0^1(B(0,\ R)),\ u \geq 0, \tag{4.2.2}$$

for any $R > 0$, then study the solution of above equation for large R.

First define for nonempty $I \in M := \{1, 2, \ldots, k\}$ (where we recall that k denotes the number of connected components of Ω^+):

$$\Gamma_I \equiv \sup\{\gamma > 0 \mid S_I \neq \emptyset \text{ for } (4.2.1)_\gamma\}.$$

Lemma 4.2.1. *Assume (1.0.2), then Γ_I is finite.*

Proof. Otherwise for each Ω_i^+, $i \in I$, we take a small ball B_i such that $B_i \subset\subset \Omega_i^+$. We define φ_i, γ_i to be the first positive eigenfunction and first eigenvalue of the following eigenvalue problem

$$-\Delta \varphi_i = \lambda \varphi_i, \text{ in } B_i, \qquad \varphi_i = 0 \text{ on } \partial B_i.$$

Next multiply $(4.2.1)_\gamma$ with φ_i, then integrate over B_i, we have

$$\begin{aligned}\int_{B_i}(-\Delta u)\varphi_i dx &= \int_{B_i} a_\gamma u^q \varphi_i dx + \int_{B_i} u^p \varphi_i dx \\ &= \int_{B_i} \gamma a_i^+ u^q \varphi_i dx + \int_{B_i} u^p \varphi_i dx.\end{aligned}$$

But

$$\int_{B_i}(\Delta\varphi_i u - \Delta u \varphi_i)dx = \int_{\partial B_i}(\frac{\partial \varphi_i}{\partial n}u - \frac{\partial u}{\partial n}\varphi_i)ds = \int_{\partial B_i}\frac{\partial \varphi_i}{\partial n}u < 0,$$

that is

$$\gamma_i \int_{B_i} u\varphi_i dx = \int_{B_i} -\Delta\varphi_i u dx > \int_{B_i} -\Delta u \varphi_i dx.$$

Therefore

$$\gamma_i \int_{B_i} u\varphi_i dx \geq \int_{B_i} \gamma a_i^+ u^q \varphi_i dx + \int_{B_i} u^p \varphi_i dx$$

i.e.

$$\int_{B_i}(\gamma_i u - \gamma a_i^+ u^q - u^p)\varphi_i dx \geq 0.$$

Let $\underline{a} = \inf_{x \in \cup_{i\in I} B_i} a(x)$, then we have

$$\int_{B_i}(\gamma_i u - \gamma \underline{a} u^q - u^p)\varphi_i dx \geq 0 \ for \ i \in I \ and \ \gamma > 0.$$

We know by assumption u is positive in Ω_I^+, but $\gamma t - \gamma \underline{a} t^q - t^p < 0$ for all $t > 0$ when γ is sufficiently large, so this is a contradiction. Hence we must have $\Gamma < \infty$. □

From the above lemma and remark, we can tell that the equation (3.0.1) and (4.0.1) are quite different, we will see more later on. Next we are going to prove a few lemmas to show that $\Gamma_I > 0$. First recall that in [16] the nonnegative smooth solutions of the following equation

$$\Delta v + v^{\frac{n+2}{n-2}} = 0 \ for \ x \in \mathbb{R}^n \ when \ n \geq 3.$$

are of the form

$$v(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^0|^2)^{\frac{n-2}{2}}},$$

for some $\lambda > 0$ and $x^0 \in \mathbb{R}^n$. Note that

$$v(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^0|^2)^{\frac{n-2}{2}}} \leq \frac{[n(n-2)]^{\frac{n-2}{4}}}{\lambda^{\frac{n-2}{2}}} \equiv c(\lambda),$$

so we pick $\lambda > 0$ such that $c(\lambda) = 1$ and fix some $x^0 \in \Omega^+$. Denoted this special one by V.

Lemma 4.2.2. *There exists $\gamma_* > 0$ and $M > 0$ such that*

$$-\Delta(MV) \geq a_\gamma(x)(MV)^q + (MV)^p,$$

where M depends on a and $\gamma \leq \gamma_*$.

Proof. Set $a^\infty = \sup\{a(x) \mid x \in \mathbb{R}^n\}$, let B^+ be a fixed ball including Ω^+ with center x^0 such that

$$\inf\{a^-(x) \mid x \in \mathbb{R}^n - B^+\} > \frac{1}{2}a_\infty.$$

This could be done because $\liminf_{|x|\to\infty} a^- = a_\infty$. Let $K = \inf\{V(x) \mid x \in B^+\}$, I should mention that when the radius of the ball B^+ tends to infinity, K then goes to zero.

Next we want to show $-\Delta(MV) \geq a_\gamma(x)(MV)^q + (MV)^p$ for some suitable positive constant M and small γ, or equivalently

$$\begin{aligned} MV^{\frac{n+2}{n-2}} &\geq a_\gamma(x)(MV)^q + (MV)^p \text{ in } B^+ \\ MV^{\frac{n+2}{n-2}} &\geq a_\gamma(x)(MV)^q + (MV)^p \text{ in } \mathbb{R}^n - B^+. \end{aligned}$$

First we discuss the part in $\mathbb{R}^n - B^+$, where we need to get

$$M^{1-q}V^{\frac{n+2}{n-2}-q} \geq a_\gamma + (MV)^{p-q}.$$

But we have in $\mathbb{R}^n - B^+$

$$-\frac{1}{2}a_\infty + M^{p-q} \geq a_\gamma + (MV)^{p-q}.$$

So choose M so that $0 < M < (\frac{1}{2}a_\infty)^{\frac{1}{p-q}}$, then we obtain

$$M^{1-q}V^{\frac{n+2}{n-2}-q} > 0 \geq -a_\infty + M^{p-q} \geq a_\gamma + (MV)^{p-q} \; in \; \mathbb{R}^n - B^+.$$

that is for $0 < M < (\frac{1}{2}a_\infty)^{\frac{1}{p-q}}$, we have

$$MV^{\frac{n+2}{n-2}} \geq a_\gamma(x)(MV)^q + (MV)^p \; in \; \mathbb{R}^n - B^+.$$

If $p = \frac{n+2}{n-2}$, then $0 < M < 1$ is enough.

Now we discuss the part in B^+, where we need to get

$$MV^{\frac{n+2}{n-2}} \geq a_\gamma(x)(MV)^q + (MV)^p.$$

But in B^+, we know

$$\begin{aligned} MV^{\frac{n+2}{n-2}} &\geq MK^{\frac{n+2}{n-2}}, \\ \gamma a^\infty M^q + M^p &\geq a_\gamma(x)(MV)^q + (MV)^p. \end{aligned}$$

Therefore we only need to show

$$MK^{\frac{n+2}{n-2}} \geq \gamma a^\infty M^q + M^p \; in \; B^+.$$

Set $A = a^\infty K^{-\frac{n+2}{n-2}}$, $B = K^{-\frac{n+2}{n-2}}$, it is equivalent to show

$$M^{1-q} \geq \gamma A + BM^{p-q} \; in \; B^+.$$

that is

$$M^{1-q} - BM^{p-q} \geq \gamma A \; in \; B^+.$$

After some calculations we have the following results

$$\max\{t^{1-q} - Bt^{p-q} - \gamma A\} > 0 \Longleftrightarrow (\gamma A)^{p-1}B^{1-q} < \frac{(p-1)^{p-1}(1-q)^{1-q}}{(p-q)^{p-q}},$$

the maximum achieves at $t_B = [\frac{(1-q)}{B(p-q)}]^{\frac{1}{p-1}}$. As I mentioned at the beginning of the proof, we can enlarge the radius of the Ball B^+, make K small, in turn B is big and t_B is small. Therefore we can choose B^+ large such that

$$0 < t_B < (\frac{1}{2}a_\infty)^{\frac{1}{p-q}}.$$

Take γ_* such that

$$(\gamma_* A)^{p-1} B^{1-q} = \frac{(p-1)^{p-1}(1-q)^{1-q}}{2(p-q)^{p-q}},$$

and choose $M = t_B$. So for $\gamma \leq \gamma_*$, we have suitable M such that

$$-\triangle(MV) \geq a_\gamma (MV)^q + (MV)^p \ in \ \mathbb{R}^n.$$

This lemma is proved. □

Remark 4.2.3. *Notice that we can choose M so that $M \to 0$ as $\gamma \to 0$.*

Lemma 4.2.4. $\Gamma > 0$

Proof. Now we can see that MV is an upper solution for the equation (4.2.2), that is

$$-\triangle u = a_\gamma(x) u^q + u^p \ in \ B(0,\ R) \qquad and \qquad u \in H_0^1(B(0,\ R)),\ u \geq 0,$$

where $\gamma \leq \gamma_*$. Then we can adapt the proof of Lemma 4.1.1 just like we did for Lemma 4.1.9, as we can always find a suitable lower solution. □

Combining above lemmas together, we reach the following theorem:

Theorem 4.2.5. *Assume (1.0.2), then $0 < \Gamma_I < \infty$.*

For later on we denote the minimal element of S_I at γ as $u_{I,\gamma}$.

Corollary 4.2.6. *For $0 < \gamma_1 < \gamma_2 < \Gamma$, $u_{I,\gamma_1} \leq u_{I,\gamma_2}$. Moreover $\lim_{\gamma \to 0^+} ||u_{I,\gamma}||_{L^\infty(\mathbb{R}^n)} = 0$*

Proof. It is easy to see that u_{I,γ_2} acts naturally as an upper solution for problem $(4.2.2)_{\gamma_1}$, where we choose R_2 such that supp $u_{I,\gamma_2} \subset\subset B(0,\ R)$. With proper lower solution we see that $(4.2.2)_{\gamma_1}$ has a non-negative solution $u \leq u_{I,\gamma_2}$. Since supp $u_{I,\gamma_2} \subset\subset B(0,\ R)$, then u is also a solution of $(4.2.1)_{\gamma_1}$ and $u \in S_{I,\gamma_1}$, so we have $u_{I,\gamma_1} \leq u \leq u_{I,\gamma_2}$. Next result is from remark 4.2.3 since $||u_{I,\gamma}||_{L^\infty(\mathbb{R}^n)} \leq M(\gamma)$. □

Putting Theorem 4.1.6, Theorem 4.2.5 and Corollary 4.2.6 all together we obtain Theorem 1.2.3 from the introduction. The proof of Proposition 1.2.4 is similar to the analogous result Theorem 1.1.8 for the logistic case, but finding an appropriate super-solution is more difficult. In order to prove this proposition, we need two lemmas and a few notations.

Fix $c > 0$, it will be chosen later, let $F(s) = \int_0^s (t^p - \frac{c}{n+1} t^q)dt$ and $\sigma = (\frac{c}{n+1}\frac{q}{p})^{\frac{1}{p-q}}$, for $0 < e \leq \sigma$, it also will be decided later, denote $\delta = \frac{1}{\sqrt{2}} \int_0^e \frac{ds}{\sqrt{-F(s)}}$.

Lemma 4.2.7. *Let $B = \{x \in \mathbb{R}^n \mid |x| < \delta\}$, then the equation*

$$-\triangle v = v^p - cv^q \text{ in } B \quad \text{and} \quad v = e \text{ on } \partial B$$

has a solution $\overline{u}$ such that $\overline{u}(0) = 0$ and $0 \leq \overline{u}(x) \leq e$ in B.

Proof. We are going to use the sub-super solution method to prove this lemma. First construct the super-solution, let $w(r)$ be the function defined implicitly by

$$\int_{w(r)}^{e} \frac{ds}{\sqrt{-F(s)}} = \sqrt{2}r.$$

It is easy to see that $w(r)$ satisfies

$$w''(r) + w^p(r) - \frac{c}{n+1} w^q(r) = 0 \text{ in } (0,\ \delta)$$

where δ is given by above, $w(r)$ is a decreasing function in r, $w(0) = e$ and $w''(\delta) = w'(\delta) = w(\delta) = 0$.

Now let $V(r) = w(\delta - r)$, then $V(0) = V'(0) = V''(0) = 0$, $V(\delta) = e$ and $V(r)$ is increasing in $[0,\ \delta]$, furthermore

$$V''(r) + V^p(r) - \frac{c}{n+1} V^q(r) = 0 \text{ in } (0,\ \delta).$$

So we have

$$V'(r) = \int_0^r V''(s)ds = \int_0^r \frac{c}{n+1} V^q(s) - V^p(s)ds \leq (\frac{c}{n+1} V^q(r) - V^p(r))r.$$

Therefore for $r \neq 0$ we have

$$\triangle V(r) = V''(r) + \frac{n-1}{r}V'(r) \leq \frac{c}{n+1}V^q(r) - V^p + (n-1)(\frac{c}{n+1}V^q(r) - V^p(r))$$
$$= \frac{n}{n+1}cV^q(r) - nV^p(r) \leq cV^q(r) - V^p(r).$$

For $r = 0$, take the limit we see that $\triangle V = 0 = cV^q(0) - V^p(0)$, therefore V satisfies

$$-\triangle V \geq V^p - cV^q \text{ in } B(0,\ \delta) \quad \text{and} \quad V(\delta) = e \text{ on } \partial B(0,\ \delta),$$

which implies $V(r)$ is a super-solution for the equation in the lemma. It is easy to see that 0 is a sub-solution, so we have a solution $\overline{u}$ such that $0 \leq \overline{u} \leq V \leq e$ and $\overline{u}(0) = V(0) = 0$. □

Next pick a point $x_0 \in \mathbb{R}^n$ and a positive number $R_0 > 0$, and choose a continuous function $b(x) \in C(B(x_0,\ R_0),\ R)$ such that $\inf_{x \in B(x_0,\ R_0)} = \underline{b} > 0$, denote $\overline{\sigma} = (\frac{\underline{b}q}{p})^{\frac{1}{p-q}}$, we also choose a continuous function $g(x) \in C(\partial B(x_0,\ R_0),\ R)$ such that $0 \leq g(x) \leq \overline{\sigma}$ for any $x \in \partial B(x_0,\ R_0)$. Then we have the following lemma.

Lemma 4.2.8. *The equation*

$$-\triangle v = v^p - b(x)v^q \text{ in } B(x_0,\ R_0) \quad \text{and} \quad v = g(x) \text{ on } \partial B(x_0,\ R_0)$$

has at most one smooth solution v such that $0 \leq v \leq \overline{\sigma}$.

Proof. We show this result by contradiction. Let us assume there are two different solutions v_1 and v_2, which satisfy above equation. Since they coincide on the boundary, we may assume that there is point x^* such that $v_2(x^*) > v_1(x^*)$. Therefore without loss of generality we may assume $(v_1 - v_2)$ attains minimum at x^*, then we see

$$0 \geq -\triangle(v_1 - v_2)(x^*) = (v_1^p(x^*) - b(x^*)v_1(x^*)) - (v_2^p(x^*) - b(x^*)v_2(x^*)) > 0,$$

a contradiction, this lemma is done. □

Now we are going to use these two lemmas to prove Proposition 1.2.4.

Proof. By assumption Ω^{0+} is admissible, then $\mathrm{dist}(\Omega_i^{0+},\ \Omega_j^{0+}) > 0$ for any $i \neq j$. Let $\overline{\delta} = \inf_{i\neq j}\ dist(\Omega_i^{0+},\ \Omega_j^{0+})$, then $\overline{\delta} > 0$.

Pick $R > 0$ such that $\Omega^{0+} \subset\subset B(0,\ R)$, denote $C_i = \{x \in B(0,\ R+3\overline{\delta}) \mid dist(x,\ \Omega_i^{0+}) \leq \frac{\overline{\delta}}{16}\}$, it is easy to see that $C_i \cap C_j = \emptyset$ for any $i \neq j$. Let $C = \cup_{i\in M} C_i$.

We define

$$N = \{x \in B(0,\ R + 2\overline{\delta}) \mid \ dist(x,\ \Omega^{0+}) \geq \frac{\overline{\delta}}{4}\},$$

then for any $x \in N$, $\overline{B(x,\ \frac{\overline{\delta}}{16})} \cap C_i = \emptyset$ for any $i \in M$. Finally let $\underline{a} = \inf_{x\in B(0,\ R+3\overline{\delta})-C}\, a^-(x)$, then $\underline{a} > 0$. Now we choose the constants c and e mentioned before Lemma 4.2.7. Let $c = \underline{a}$, then $\sigma = (\frac{c}{n+1}\frac{q}{p})^{\frac{1}{p-q}}$. Denote $\delta_1 = \frac{1}{\sqrt{2}}\int_0^\sigma \frac{ds}{\sqrt{-F(s)}}$, we make the following choice for e:

$$\text{If } \delta_1 > \frac{\overline{\delta}}{16},\ \text{choose suitable } e \text{ such that } \delta = \frac{\overline{\delta}}{16},$$

$$\text{If } \delta_1 \leq \frac{\overline{\delta}}{16},\ \text{choose } e \text{ to be } \ \sigma.$$

Since the sup-norm of the minimal element $u_{M,\gamma}$ tends to zero as γ goes to zero, we pick $\gamma_* > 0$ so that

$$||u_{M,\gamma}||_{L^\infty(\mathbb{R}^n)} < e\ for\ \gamma \leq \gamma_*.$$

Claim: If $\gamma \leq \gamma_*$, then $u_{M,\gamma} = 0$ for any $x \in N$. Actually take $x \in N$, then consider the following equation

$$-\Delta v(y) = v^p(y) - a^-(y)v^q(y)\ in\ B(x,\ \delta) \quad and \quad v = u_{M,\gamma}\ on\ \partial B(x,\ \delta), \qquad (4.2.3)$$

we see that from Lemma 4.2.8 this equation has a unique solution $u_{M,\gamma}$. But from Lemma 4.2.7 and Lemma 4.2.8 we know the unique solution $\overline{u}$ of the equation

$$-\Delta v(y) = v^p(y) - cv^q(y)\ in\ B(x,\ \delta) \quad and \quad v = e\ on\ \partial B(x,\ \delta),$$

is an super-solution for equation (4.2.3) and 0 is a sub-solution, therefore we have $0 \leq u_{M,\gamma} \leq \overline{u}$. Since $\overline{u} \leq e$ and $\overline{u}(x) = 0$, we have $u_{M,\gamma} = 0$ for $x \in N$. The claim is done.

Since $u_{M,\gamma}$ is the minimal element in S_M at γ, then $u_{M,\gamma}$ vanishes outside of $B(0,\ R + 2\overline{\delta})$. Thus, it is easy to that the support of $u_{M,\gamma}$ consists of k disjoint components, and thus its restriction to each component gives k compactly supported solutions of (4.0.1). By taking an appropriate union over these pieces we construct an element of N_I for any choice of $I \subset M$. This concludes the proof of Theorem 1.2.4. □

4.3 Existence at Γ

So far we have established an interval of existence of solutions $\gamma \in (0, \Gamma_I)$ in the class S_I, where $I \subset M$ indicates the components of Ω^+ in which these solutions must be positive. Now we assert that a solution of class S_I must exist at the endpoint of the maximal interval of existence, $\gamma = \Gamma_I$. This is the "extremal solution" for this family (see [15]).

First we introduce the Banach space

$$H_q^1 = \{v \in \mathcal{D}^{1,2}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |v|^{q+1} dx < \infty\}$$

endowed with the norm

$$||v||_{H_q^1} = (\int_{\mathbb{R}^n} |\nabla v|^2 dx)^{\frac{1}{2}} + (\int_{\mathbb{R}^n} |v|^{q+1} dx)^{\frac{1}{q+1}}.$$

Define the energy functional $I_\gamma : \ H_q^1 \to \mathbb{R}$ associated with (4.2.1) as

$$I_\gamma(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx - \frac{\gamma}{q+1}\int_{\mathbb{R}^n} a^+(v^+)^{q+1} dx + \frac{1}{q+1}\int_{\mathbb{R}^n} a^-(v^+)^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n} (v^+)^{p+1} dx$$

It is a standard fact that I_γ is a C^1 functional on H_q^1 (see [19].) Denote S_I and N_I at γ by $S_{I,\gamma}$ and $N_{I,\gamma}$, and also denote S_{I,Γ_I} and N_{I,Γ_I} by S_{Γ_I} and N_{Γ_I}.

Lemma 4.3.1. *Suppose $\overline{u} \in N_{I,\overline{\gamma}}$ for some $\overline{\gamma}$, then $N_{I,\gamma}$ admits an element u_γ for every $0 < \gamma \leq \overline{\gamma}$. Moreover $u_\gamma \leq \overline{u}$ and $I_\gamma(u_\gamma) < 0$.*

Proof. For $0 < \gamma \leq \overline{\gamma}$, $\overline{u}$ is an upper solution for the equation $(4.2.1)_\gamma$ and 0 is a lower solution. So we consider the following minimization problem in a convex constraint set

$$\text{Inf } \{I_\gamma(v) \mid v \in X\} \qquad \textit{and} \qquad X = \{v \in H_q^1 \mid 0 \leq v \leq \overline{u} \ a.e.\}$$

By some easy modifications to Theorem 1.2.4 of Struwe [41], the infimum is achieved at some $u_\gamma \in X$ and $(\phi,\ I'_\gamma(u_\gamma)) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, and by routine regularity arguments u_γ is a solution to $(4.2.1)_\gamma$ since $u_\gamma \leq \overline{u}$. Since $u_\gamma \in X$, it vanishes on the components $\Omega^+ - \Omega_I^+$. It remains to show that u_γ does not vanish in Ω_I^+.

Claim: u_γ does not vanish in Ω_I^+.

Indeed suppose for some $i \in I$, $u_\gamma \not> 0$ in Ω_i^+, then by Lemma 4.1.7 we have $u_\gamma \equiv 0$ over $\overline{\Omega_i^+}$. Choose a ball $B \subset\subset \Omega_i^+$ and ϕ with $0 \leq \phi \in C_0^\infty(B)$. Hence for small positive t, $(u_\gamma + t\phi) \in X$ and

$$I_\gamma(u_\gamma + t\phi) = I_\gamma(u_\gamma) + I_\gamma(t\phi) < I_\gamma(u_\gamma),$$

since

$$I_\gamma(t\phi) = \frac{1}{2}t^2 \int_B |\nabla\phi|^2 dx - \frac{1}{q+1}t^{q+1}\gamma \int_B a^+\phi^{q+1}dx - \frac{t^{p+1}}{p+1}\int_B \phi^{p+1}dx < 0$$

for t sufficiently small. This contradicts the choice of u_γ as the infimum of I_γ over X. So we must have $u_\gamma \in N_{I,\overline{\gamma}}$. Also notice that $I_\gamma(t\phi) < 0$ for t small enough, we have $I_\gamma(u_\gamma) < 0$. This lemma is proved. □

Remark 4.3.2. *Given the variational formulation of the problem as an infimum, it is natural to ask whether the solutions obtained by above lemma are local minima of I_γ in any sense. Notice this can not be the case when $I \neq M$. Indeed following the arguments of the last part of the proof, we can decrease the value of I_γ near such solution by small perturbations in each Ω_j^+, where $j \notin I$. So the existence of a second solution in the classes N_I, $I \neq \emptyset$ (with dead cores) remains an open question.*

Corollary 4.3.3. *For $0 < \gamma < \Gamma$, $I_\gamma(u_{I,\gamma}) < 0$, where $u_{I,\gamma}$ is the minimum element in $S_{I,\gamma}$.*

Proof. It is simple, we just need to apply above lemma with $\overline{u} = u_{I,\gamma}, \overline{\gamma} = \gamma$ and some $J \subset M$ such that $I \subset J$ and $u_{I,\gamma} \in N_{J,\gamma}$. Hence by above lemma we get a solution $u_\gamma \in S_{I,\gamma}$ such that

$$I_\gamma(u_\gamma) < 0 \qquad \textit{and} \qquad 0 \leq u_\gamma \leq u_{I,\gamma}.$$

Since $u_{I,\gamma}$ is the minimum element in $S_{I,\gamma}$, then we must have $u_\gamma = u_{I,\gamma}$. This lemma is done. □

In order to show the existence at Γ_I, we need to do some estimates.

Lemma 4.3.4. $||u_{I,\gamma}||_{H_q^1} + ||u_{I,\gamma}||_{L^{p+1}(\mathbb{R}^n)}$ *is uniformly bounded.*

Proof. First we know that u_{I_γ} satisfies the equation

$$-\triangle u_{I,\gamma} = a_\gamma u_{I,\gamma}^q + u_{I,\gamma}^p,$$

then multiply this equation by $u_{I,\gamma}$ and iterate over $\mathbb{R}^n$, notice that it has compact support, we obtain

$$\int_{\mathbb{R}^n} |\nabla u_{I,\gamma}|^2 dx = \gamma \int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1} dx - \int_{\mathbb{R}^n} a^- u_{I,\gamma}^{q+1} dx + \int_{\mathbb{R}^n} u_{I,\gamma}^{p+1} dx. \tag{4.3.1}$$

By above lemma, we also have $I_\gamma(u_{I,\gamma}) < 0$, that is

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_{I,\gamma}|^2 dx + \frac{1}{q+1}\int_{\mathbb{R}^n} a^- u_{I,\gamma}^{q+1} dx < \frac{\gamma}{q+1}\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1} dx + \frac{1}{p+1}\int_{\mathbb{R}^n} u_{I,\gamma}^{p+1} dx. \tag{4.3.2}$$

Put (4.3.1) into (4.3.2), we get

$$\begin{aligned}\frac{\gamma}{2}\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1} dx - \frac{1}{2}\int_{\mathbb{R}^n} a^- u_{I,\gamma}^{q+1} dx + \frac{1}{2}\int_{\mathbb{R}^n} u_{I,\gamma}^{p+1} dx + \frac{1}{q+1}\int_{\mathbb{R}^n} a^- u_{I,\gamma}^{q+1} dx \\ < \frac{\gamma}{q+1}\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1} dx + \frac{1}{p+1}\int_{\mathbb{R}^n} u_{I,\gamma}^{p+1} dx,\end{aligned}$$

that is

$$(\frac{1}{q+1} - \frac{1}{2})\int_{\mathbb{R}^n} a^- u_{I,\gamma}^{q+1} dx + (\frac{1}{2} - \frac{1}{p+1})\int_{\mathbb{R}^n} u_{I,\gamma}^{p+1} dx < \gamma(\frac{1}{q+1} - \frac{1}{2})\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1} dx.$$

Since $\frac{1}{q+1} > \frac{1}{2} > \frac{1}{p+1}$, we have from above inequality

$$(\frac{1}{2} - \frac{1}{p+1})\int_{\mathbb{R}^n} u_{I,\gamma}^{p+1} dx < \gamma(\frac{1}{q+1} - \frac{1}{2})\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1} dx. \tag{4.3.3}$$

From basic hypothesis a^+ is compactly supported, hence we get

$$\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1} dx \leq ||a^+||_{L^\infty(\mathbb{R}^n)} \int_{supp\ a^+} u_{I,\gamma}^{q+1} dx \leq C(a^+)||a^+||_{L^\infty(\mathbb{R}^n)}||u_{I,\gamma}||_{L^{p+1}(\mathbb{R}^n)}^{q+1}, \tag{4.3.4}$$

where $C(a^+)$ is some constant depending on a^+ and Ω^+. Put this back to (4.3.3), we get

$$(\frac{1}{2} - \frac{1}{p+1})||u_{I,\gamma}||_{L^{p+1}(\mathbb{R}^n)}^{p+1} \leq C(a^+)\gamma(\frac{1}{q+1} - \frac{1}{2})||a^+||_{L^\infty(\mathbb{R}^n)}||u_{I,\gamma}||_{L^{p+1}(\mathbb{R}^n)}^{q+1},$$

therefore we have

$$||u_{I,\gamma}||_{L^{p+1}(\mathbb{R}^n)}^{p-q} \leq C(a^+)\gamma(\frac{1}{q+1} - \frac{1}{2})||a^+||_{L^\infty(\mathbb{R}^n)}(\frac{1}{2} - \frac{1}{p+1})^{-1}.$$

which implies that $||u_{I,\gamma}||_{L^{p+1}(\mathbb{R}^n)}$ is uniformly bounded, then plug this and (4.3.4) back to (4.3.2). By basic hypothesis we conclude that $||\nabla u_{I,\gamma}||_{L^2(\mathbb{R}^n)}$ and $||u_{I,\gamma}||_{L^{q+1}(\mathbb{R}^n)}$ are uniformly bounded. This lemma is proved. □

Theorem 4.3.5. *S_{Γ_I} is not empty.*

Proof. Pick an increasing sequence $\{\gamma_n\}$ with limit Γ_I, by above lemma we see that $||u_{I,\gamma}||_{H_q^1} + ||u_{I,\gamma}||_{L^{p+1}(\mathbb{R}^n)}$ is uniformly bounded, then there exists $u_{\Gamma_I} \in H_q^1$ such that,

$$u_{\gamma_n} \rightharpoonup u_{\Gamma_I} \textit{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^n),\ L^{p+1}(\mathbb{R}^n) \textit{ and } L^{q+1}(\mathbb{R}^n).$$

Moreover $u_{\gamma_n} \rightarrow u_{\Gamma_I}$ a.e. in $\mathbb{R}^n$. From Corollary 4.2.6 we know that $u_{I,\gamma}$ is increasing, so by Monotone Convergence Theorem we see that

$$u_{\gamma_n} \rightarrow u_{\Gamma_I} \textit{ strongly in } L^{p+1}(\mathbb{R}^n) \textit{ and } L^{q+1}(\mathbb{R}^n). \tag{4.3.5}$$

Now for u_{γ_n}, it satisfies the equation

$$-\Delta u_{\gamma_n} = a_{\gamma_n} u_{\gamma_n}^q + u_{\gamma_n}^p.$$

Take any $\varphi \in C_0^\infty(\mathbb{R}^n)$, multiply both sides of above equation and integrate over $\mathbb{R}^n$, notice that u_{γ_n} has compact support, we get

$$\int_{\mathbb{R}^n} \nabla u_{\gamma_n} \nabla \varphi dx = \int_{\mathbb{R}^n} a_{\gamma_n} u_{\gamma_n}^q \varphi + \int_{\mathbb{R}^n} u_{\gamma_n}^p \varphi,$$

notice (4.3.5), pass the limit on n, we have

$$\int_{\mathbb{R}^n} \nabla u_{\Gamma_I} \nabla \varphi dx = \int_{\mathbb{R}^n} a_{\Gamma_I} u_{\Gamma_I}^q \varphi + \int_{\mathbb{R}^n} u_{\Gamma_I}^p \varphi,$$

therefore u_{Γ_I} is a weak solution of $(4.2.1)_{\Gamma_I}$. By routine regularity arguments, we know that u_{Γ_I} is a classical solution. □

Corollary 4.3.6. *u_{Γ_I} constructed above is the minimum element in S_{I_Γ}, i.e. $u_{\Gamma_I} = u_{I,\Gamma_I}$.*

Proof. From above theorem we see that S_{Γ_I} is not empty, then pick any $U \in S_{\Gamma_I}$, we just need to apply lemma 4.3.1 to the equation $(4.2.1)_\gamma$ with $\overline{u} = U$, $\overline{\gamma} = \Gamma_I$ and some $J \subset M$ such that $I \subset J$ and $U \in N_{\Gamma_J}$, then we get a solution u_γ for the equation $(4.2.1)_\gamma$ such that $u_\gamma \in S_{\Gamma_I}$, hence we have $U \geq u_\gamma \geq u_{I,\gamma}$, since $\lim_{\gamma \to \Gamma^-} u_{I,\gamma} = u_{\Gamma_I}$, we have $U \geq u_{\Gamma_I}$. This corollary is proved. □

For later on we denote u_{M,Γ_M} by U_Γ and denote $u_{M,\gamma}$ by U_γ. We conclude this section with a simple result:

Corollary 4.3.7. *Assume $a(x) = a(|x|)$. then $U_\gamma(x) = U_\gamma(|x|)$ for $0 < \gamma \leq \Gamma$.*

4.4 Existence of Local Minimizers

Brezis and Nirenberg [13] first observed that minimization in the C^1-topology (for example, the sub- and super-solution construction above) yields minima in the weaker H^1-topology for a large class of subcritical elliptic variational problems. Now we employ similar idea to prove Theorem 1.2.5. Recall that U_γ represents the minimum element in $S_{M,\gamma}$ for $0 < \gamma \leq \Gamma$, here $\Gamma = \Gamma_M$. Now consider the following minimization problem in a convex constraint set

$$\text{Inf}\ \{I_\gamma(v) \mid v \in Y\} \qquad and \qquad Y = \{v \in H_q^1 \mid 0 \leq v \leq U_\Gamma \ a.e.\}. \tag{4.4.1}$$

From the result in Struwe's book [41], the infimum is attained at some function in Y, say v_γ, and $v_\gamma \in S_{M,\gamma}$.

Lemma 4.4.1. *Each connected component of the set $\{x \in \mathbb{R}^n \mid U_\gamma > 0\}$ at least contains one connected component of Ω^{0+}.*

Proof. This lemma is true due to the fact that U_γ is the minimum element in $S_{M,\gamma}$ and $a(x)$ satisfies hypothesis (1.2.4). ☐

Lemma 4.4.2. *For $0 \leq \gamma < \bar{\gamma}$, let u be a solution to $(4.2.1)_\gamma$ such that $0 \leq u \leq U_{\bar{\gamma}}$ in $\mathbb{R}^n$, then $u(x) < U_{\bar{\gamma}}(x)$ for all $x \in A = \{x \in \mathbb{R}^n \mid U_{\bar{\gamma}}(x) > 0\}$.*

Proof. Let $v = U_{\bar{\gamma}} - u \geq 0$ in $\mathbb{R}^n$, since u and $U_{\bar{\gamma}}$ satisfy the following equations

$$-\Delta U_{\bar{\gamma}} = (\bar{\gamma}a^+ - a^-)U_{\bar{\gamma}}^q + U_{\bar{\gamma}}^p \qquad and \qquad -\Delta u = (\gamma a^+ - a^-)u^q + u^p,$$

then we have

$$-\Delta U_{\bar{\gamma}} + a^- U_{\bar{\gamma}}^q = \bar{\gamma}a^+ U_{\bar{\gamma}}^q + U_{\bar{\gamma}}^p \geq \gamma a^+ u^q + u^p = -\Delta u + a^- u^q.$$

So we obtain that in $\mathbb{R}^n$, $-\Delta(U_{\bar{\gamma}} - u) + a^-(U_{\bar{\gamma}}^q - u^q) \geq 0$. We rewrite this as

$$-\Delta v + a^-\left(\frac{U_{\bar{\gamma}}^q - u^q}{U_{\bar{\gamma}} - u}\right)v \geq 0.$$

Assume that for some $x_0 \in A_i$, we have $v(x_0) = 0$, where A_i is one connected component of A. Let $S_i = \{x \in A_i \mid v(x) = 0\}$, then S_i is not empty since $x_0 \in S_i$. Since $U_{\bar{\gamma}}(x_0) > 0$, then take a small ball $B = B(x_0, r)$ such that $\overline{B} \subset\subset A_i$, then over B we have

$$0 \le \frac{U_{\bar{\gamma}}^q - u^q}{U_{\bar{\gamma}} - u} \le \frac{U_{\bar{\gamma}}^q}{U_{\bar{\gamma}}},$$

which implies that $\frac{U_{\bar{\gamma}}^q - u^q}{U_{\bar{\gamma}} - u}$ uniformly bounded in B. Since $v \ge 0$ in $\mathbb{R}^n$, then by maximum principle we have $v \equiv 0$ in B, which means S_i is open in A_i. By continuity S_i is also close in A_i, since S_i is not empty, then $S_i = A_i$. From the above lemma we see that A_i contains some connected component of Ω^+ by (1.2.4). This leads to a contradiction when comparing the equations satisfied by u and U_γ. □

Next we adjust the proof of proposition 5.2 in [2] to our setting and prove the following theorem in detail, which is Theorem 1.2.5.

Theorem 4.4.3. *Assume (1.2.4), for* $0 \le \gamma < \Gamma$, v_γ *is a local minimizer for* I_γ *in* H_q^1*; that is, there exists* $\delta > 0$ *such that*

$$I_\gamma(v_\gamma) \le I_\gamma(v) \quad \text{for all } v \in H_q^1 \text{ with } ||v - v_\gamma||_{H_q^1} < \delta.$$

Proof. From above we see that $Y = \{v \in H_q^1 \mid 0 \le v \le U_\Gamma \ a.e.\}$, U_Γ is an upper solution for $(4.2.1)_\gamma$ and v_γ is a solution to $(4.2.1)_\gamma$ with $v_\gamma \le U_\Gamma$.

Suppose there exists a sequence $\{u_n\} \subset H_q^1$ such that $u_n \to v_\gamma$ strongly in H_q^1 and $I_\gamma(u_n) < I_\gamma(v_\gamma)$. It is easy to see that $u_n \to v_\gamma$ in $H^1(\mathbb{R}^n)$. Let

$$v_n = \max\{0, \min\{u_n, U_\Gamma\}\}, \quad u_n^- = \max\{-u_n, 0\} \quad and \quad w_n = (u_n - U_\Gamma)^+.$$

So $u_n = v_n - u_n^- + w_n$, $v_n \in Y$, $w_n \in H_q^1$, and u_n^- and w_n have disjoint supports. Define the measurable sets

$$S_n = supp(w_n), \qquad T_n = supp(u_n^-) \qquad and \qquad R_n = \{x \in \mathbb{R}^n \mid 0 \le u_n \le U_\Gamma\}.$$

Recall that here $A = \{x \in \mathbb{R}^n \mid U_\Gamma(x) > 0\}$, let $B = \mathbb{R}^n - A$, from the lemma above we see that $U_\Gamma > v_\gamma$ in A, and Lemma 4.1.7 implies that $\overline{\Omega^{0+}} \subset\subset A$, hence $B \in \Omega^-$. Notice that A is bounded in $\mathbb{R}^n$.

We next claim that $|A \cap S_n| \to 0$ as $n \to \infty$. Indeed, let $\epsilon > 0$ be given. For $\delta > 0$, set

$$E_n = \{x \in A \mid \ u_n > U_\Gamma > v_\gamma + \delta\} \quad and \quad F_n = \{x \in A \mid \ u_n > U_\Gamma \ and \ U_\Gamma \le v_\gamma + \delta\}.$$

It is clear that $A \cap S_n \subset E_n \cup F_n$. From lemma 4.4.2, we see that

$$0 = |\{x \in A \mid \ U_\Gamma \le v_\gamma\}| = | \cap_{j=1}^{\infty} \{x \in A \mid \ U_\Gamma \le v_\gamma + \frac{1}{j}\}|$$
$$= \lim_{j \to \infty} \{x \in A \mid \ U_\Gamma \le v_\gamma + \frac{1}{j}\}|.$$

Hence there exists $\delta_1 > 0$ so that $|F_n| \le \{x \in A \mid \ U_\Gamma \le v_\gamma + \delta_1\}| < \frac{1}{2}\epsilon$ for all n. But on the other hand, since $u_n \to v_\gamma$ strongly in H_q^1, then there exists $n_1 > 0$ such that for all $n \ge n_1$

$$\frac{1}{2}\delta_1^2 \epsilon \ge \int_{\mathbb{R}^n} (u_n - v_\gamma)^2 dx \ge \int_{E_n} \delta_1^2 dx = \delta_1^2 |E_n|,$$

so we have $|E_n| < \frac{1}{2}\epsilon$, which implies that

$$|A \cap S_n| \le |E_n| + |F_n| < \epsilon.$$

For convenience set

$$H_\gamma(x, \ v) = \frac{1}{q+1} a_\gamma (v^+)^{q+1} + \frac{1}{p+1} (v^+)^{p+1}.$$

Since $u_n = U_\Gamma + w_n$ and $v_n = U_\Gamma$ in S_n, we have:

$$I_\gamma = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_n|^2 dx - \int_{\mathbb{R}^n} H_\gamma(x, \ u_n) dx$$
$$= \int_{R_n} (\frac{1}{2}|\nabla v_n|^2 - H_\gamma(x, \ v_n)) dx + \int_{S_n} (\frac{1}{2}|\nabla u_n|^2 - H_\gamma(x, \ u_n)) dx + \int_{T_n} (\frac{1}{2}|\nabla u_n^-|^2) dx$$
$$= \int_{R_n} (\frac{1}{2}|\nabla v_n|^2 - H_\gamma(x, \ v_n)) dx + \int_{S_n} (\frac{1}{2}|\nabla v_n|^2 - H_\gamma(x, \ v_n)) dx + \int_{T_n} (\frac{1}{2}|\nabla u_n^-|^2) dx$$

$$+\int_{S_n}\frac{1}{2}(|\nabla(U_\Gamma+w_n)|^2-|\nabla v_n|^2)-(H_\gamma(x,\ U_\Gamma)-H_\gamma(x,\ v_n))dx$$

$$=\int_{R_n}(\frac{1}{2}|\nabla v_n|^2-H_\gamma(x,\ v_n))dx+\int_{S_n}(\frac{1}{2}|\nabla v_n|^2-H_\gamma(x,\ v_n))dx+\int_{T_n}(\frac{1}{2}|\nabla u_n^-|^2)dx$$

$$+\int_{S_n}\frac{1}{2}(|\nabla(U_\Gamma+w_n)|^2-|\nabla U_\Gamma|^2)-(H_\gamma(x,\ U_\Gamma)-H_\gamma(x,\ U_\Gamma))dx$$

$$=\int_{S_n}\frac{1}{2}|\nabla w_n|^2+\nabla U_\gamma\nabla w_n-(H_\gamma(x,\ U_\Gamma)-H_\gamma(x,\ U_\Gamma))dx$$

$$+I_\gamma(v_n)+\frac{1}{2}\int_{T_n}(|\nabla u_n^-|^2)dx.$$

Since U_Γ is an upper solution with respect to $(4.2.1)_\gamma$, then we have

$$-\triangle U_\gamma=a_\Gamma U_\Gamma^q+U_\Gamma^p\geq a_\gamma U_\Gamma^q+U_\Gamma^p,$$

multiply the above by w_n and integrate over $\mathbb{R}^n$, notice that U_Γ has compact support, we get

$$\int_{\mathbb{R}^n}\nabla U_\gamma\nabla w_n\geq\int_{\mathbb{R}^n}a_\gamma U_\Gamma^q w_n+U_\Gamma^p w_n dx=\int_{\mathbb{R}^n}H_{\gamma v}(x,\ U_\Gamma)w_n dx.$$

Therefore we have

$$I_\gamma(u_n)\geq I_\gamma(v_n)+\frac{1}{2}\int_{\mathbb{R}^n}(|\nabla u_n^-|^2+|\nabla w_n|^2)dx$$
$$-\int_{S_n}(H_\gamma(x,\ U_\Gamma)-H_\gamma(x,\ U_\Gamma)-H_{\gamma v}(x,\ U_\Gamma)w_n)dx. \quad (4.4.2)$$

Now we estimate each term in $H_\gamma(x,\ U_\Gamma)$ on the set $A\cap S_n$, by the fact that this set is small for n large,

$$\int_{A\cap S_n}w_n^2dx\leq|A\cap S_n|^{\frac{2}{n}}(\int_{\mathbb{R}^n}w_n^{2^*}dx)^{\frac{n-2}{n}}\leq o(1)(\int_{\mathbb{R}^n}|\nabla w_n|^2dx), \quad (4.4.3)$$

in the same way we also have

$$\int_{A\cap S_n}w_n^{p+1}dx\leq o(1)(\int_{\mathbb{R}^n}|\nabla w_n|^2dx). \quad (4.4.4)$$

Since $U_\Gamma>0$ in Ω^{0+}, then there exists $l>0$ such that $U_\Gamma\geq l$ for all $x\in\overline{\Omega^{0+}}$, hence we have

$$
\begin{aligned}
0 &\le \frac{1}{q+1}(U_\Gamma + w_n)^{q+1} - \frac{1}{q+1}U_\Gamma^{q+1} - U_\Gamma^q w_n = \int_{U_\Gamma}^{U_\Gamma + w_n} s^q ds - \int_{U_\Gamma}^{U_\Gamma + w_n} U_\Gamma^q ds \\
&= \int_{U_\Gamma}^{U_\Gamma + w_n} (s^q - U_\Gamma^q) ds \le \int_{U_\Gamma}^{U_\Gamma + w_n} q l^{q-1}(s - U_\Gamma) ds \\
&\le \frac{1}{2} q l^{q-1} w_n^2 \quad for\ all\ x \in \Omega^{0+},
\end{aligned}
$$

so we have from (4.4.3)

$$
\begin{aligned}
0 \le \int_{A\cap S_n \cap \Omega^{0+}} & a_\gamma(x)(\frac{1}{q+1}(U_\Gamma + w_n)^{q+1} - \frac{1}{q+1}U_\Gamma^{q+1} - U_\Gamma^q w_n) dx \\
&\le \Gamma ||a^+||_{L^\infty(\mathbb{R}^n)} \int_{A\cap S_n \cap \Omega^{0+}} \frac{1}{2} q l^{q-1} w_n^2 dx \le o(1)(\int_{\mathbb{R}^n} |\nabla w_n|^2 dx). \quad (4.4.5)
\end{aligned}
$$

We notice that, On $A \cap \Omega^-$,

$$
a_\gamma(x) < 0 \quad and \quad \frac{1}{q+1}(U_\Gamma + w_n)^{q+1} - \frac{1}{q+1}U_\Gamma^{q+1} - U_\Gamma^q w_n \ge 0,
$$

hence

$$
\int_{A\cap\Omega^-} a_\gamma(\frac{1}{q+1}(U_\Gamma + w_n)^{q+1} - \frac{1}{q+1}U_\Gamma^{q+1} - U_\Gamma^q w_n) dx \le 0. \qquad (4.4.6)
$$

To estimate the other term, we notice that there exists $\theta = \theta(x) \in (0,\ 1)$

$$
0 \le \frac{1}{p+1}(U_\Gamma + w_n)^{p+1} - \frac{1}{p+1}U_\Gamma^{p+1} - U_\Gamma^p w_n = p(U_\Gamma + \theta w_n)^{p-1}\frac{1}{2}w_n^2 \le C(1 + w_n^{p-1})w_n^2,
$$

as a consequence, from (4.4.3) and (4.4.4)

$$
\begin{aligned}
\int_{A\cap S_n} \frac{1}{p+1}(U_\Gamma + w_n)^{p+1} - \frac{1}{p+1}U_\Gamma^{p+1} - U_\Gamma^p w_n dx &\le C \int_{A\cap S_n} w_n^2 + w_n^{p+1} dx \\
&\le o(1)(\int_{\mathbb{R}^n} |\nabla w_n|^2 dx). \quad (4.4.7)
\end{aligned}
$$

Overall from (4.4.6) and (4.4.7), we have

$$\begin{aligned}
&\int_{S_n\cap A}(H_\gamma(x,\ U_\Gamma)-H_\gamma(x,\ U_\Gamma)-H_{\gamma v}(x,\ U_\Gamma)w_n)dx\\
&=\int_{S_n\cap A}(\frac{1}{q+1}a_\gamma(U_\Gamma+w_n)^{q+1}+\frac{1}{p+1}(U_\Gamma+w_n)^{p+1})dx\\
&\quad-\int_{S_n\cap A}(\frac{1}{q+1}a_\gamma(U_\Gamma)^{q+1}+\frac{1}{p+1}(U_\Gamma)^{p+1})dx-\int_{S_n\cap A}(a_\gamma U_\Gamma^q w_n+U_\Gamma^q w_n)dx\\
&=\int_{S_n\cap A}a_\gamma(\frac{1}{q+1}(U_\Gamma+w_n)^{q+1}-\frac{1}{q+1}U_\Gamma^{q+1}-U_\Gamma^q w_n)\\
&\qquad+(\frac{1}{p+1}(U_\Gamma+w_n)^{p+1}-\frac{1}{p+1}U_\Gamma^{p+1}-U_\Gamma^p w_n)dx\\
&=\int_{S_n\cap A\cap\Omega^{0+}}a_\gamma(\frac{1}{q+1}(U_\Gamma+w_n)^{q+1}-\frac{1}{q+1}U_\Gamma^{q+1}-U_\Gamma^q w_n)dx\\
&\quad+\int_{S_n\cap A\cap\Omega^-}a_\gamma(\frac{1}{q+1}(U_\Gamma+w_n)^{q+1}-\frac{1}{q+1}U_\Gamma^{q+1}-U_\Gamma^q w_n)dx\\
&\quad+\int_{S_n\cap A}(\frac{1}{p+1}(U_\Gamma+w_n)^{p+1}-\frac{1}{p+1}U_\Gamma^{p+1}-U_\Gamma^p w_n)dx\\
&\le o(1)(\int_{\mathbb{R}^n}|\nabla w_n|^2dx)+0+o(1)(\int_{\mathbb{R}^n}|\nabla w_n|^2dx)=o(1)(\int_{\mathbb{R}^n}|\nabla w_n|^2dx). \qquad (4.4.8)
\end{aligned}$$

To estimate the terms on the set B, we must notice the fact that $B\subset\Omega^-$. Since $U_\Gamma=0$ on B, then we have that

$$\begin{aligned}
0\ge&\int_B a_\gamma(x)(\frac{1}{q+1}(U_\Gamma+w_n)^{q+1}-\frac{1}{q+1}U_\Gamma^{q+1}-U_\Gamma^q w_n)dx\\
&=\frac{1}{q+1}\int_B a_\gamma(x)w_n^{q+1}dx=-\int_B\frac{a^-}{q+1}w_n^{q+1}dx \qquad (4.4.9)
\end{aligned}$$

It is easy to see that for any $x\in S_n$, $u_n(x)\ge U_\Gamma(x)\ge v_\gamma(x)$, so for any $x\in\mathbb{R}^n$

$$0\le w_n(x)=(u_n-U_\Gamma)^+(x)\le(u_n-v_\gamma)^+(x)\le|u_n(x)-v_\gamma(x)|,$$

which implies that $w_n\to 0$ in $L^{q+1}(\mathbb{R}^n)$.

Claim: $|P_n|=|\{x\in\Omega^-\mid\ w_n\ge(a^-(x))^{\frac{1}{p-q}}\}|\to 0$ as $n\to\infty$.

Indeed, let $\epsilon>0$ be given. For $\delta_2>0$ (to be chosen later), set

$$E_n=\{x\in\Omega^-\mid\ \delta_2\ge w_n(x)\ge(a^-(x))^{\frac{1}{p-q}}\}$$

and

$$F_n = \{x \in \Omega^- \mid (a^-(x))^{\frac{1}{p-q}} \leq w_n(x) \textit{ and } w_n > \delta_2\}$$

It is easy to see that $P_n \subset E_n \cup F_n$ and

$$0 = |\{x \in \Omega^- \mid a^-(x) = 0\}| = |\cap_{j=1}^{\infty} \{x \in \Omega^- \mid a^-(x) \leq \frac{1}{j}\}|$$
$$= \lim_{j\to\infty} |\{x \in \Omega^- \mid a^-(x) \leq \frac{1}{j}\}|,$$

which implies there exists $\delta_2 > 0$, depending on ϵ, such that

$$|E_n| \leq |\{x \in \Omega^- \mid a^-(x) \leq (\delta_2)^{p-q}\}| < \frac{1}{2}\epsilon.$$

On the other hand, since $w_n \to 0$ in $L^{q+1}(\mathbb{R}^n)$, there exists $n_2 > 0$ so that for all $n \geq n_2$

$$\frac{1}{2}\epsilon(\delta_2)^{1+q} \geq \int_{\mathbb{R}^n} w_n^{q+1}dx \geq \int_{F_n} (\delta_2)^{1+q}dx \geq (\delta_2)^{1+q}|F_n|,$$

hence $|F_n| \leq \frac{1}{2}\epsilon$, and $|P_n| \leq |E_n| + |F_n| < \epsilon$. This completes the proof of this claim.

Therefore

$$\begin{aligned}
\int_{B\cap S_n} \frac{1}{p+1} w_n^{p+1}dx &= \int_{B\cap S_n - P_n} \frac{1}{p+1} w_n^{p+1}dx + \int_{B\cap S_n \cap P_n} \frac{1}{p+1} w_n^{p+1}dx \\
&= \frac{1}{p+1}\left(\int_{B\cap S_n - P_n} w_n^{p+1}dx + \int_{B\cap S_n\cap P_n} w_n^{p+1}dx\right) \\
&= \frac{1}{p+1}\left(\int_{B\cap S_n - P_n} w_n^{p-q}w_n^{q+1}dx + \int_{B\cap S_n\cap P_n} w_n^{p+1}dx\right) \\
&\leq \frac{1}{p+1}\int_{B\cap S_n} a^- w_n^{q+1}dx + C(|P_n|)||w_n||^{p+1}_{L^{2^*}(\mathbb{R}^n)} \\
&\leq \frac{1}{p+1}\int_{B\cap S_n} a^- w_n^{q+1}dx + o(1)\left(\int_{\mathbb{R}^n} |\nabla w_n|^2 dx\right).
\end{aligned}$$

So from (4.4.9) and above we have

$$\begin{aligned}
&\int_{S_n\cap B} H_\gamma(x, U_\Gamma + w_n) - H_\gamma(x, U_\Gamma) - H_{\gamma v}(x, U_\Gamma)w_n dx = \int_{S_n\cap B} a_\gamma\left(\frac{1}{q+1}w_n^{q+1}\right)dx \\
&+ \int_{S_n\cap B} \frac{1}{p+1}w_n^{p+1}dx \leq -\frac{1}{q+1}\int_{S_n\cap B} a^- w_n^{q+1}dx + \frac{1}{p+1}\int_{S_n\cap B} a^- w_n^{q+1}dx + o(1)||w_n||^2_{H^1_q} \\
&\leq \left(\frac{1}{p+1} - \frac{1}{q+1}\right)\int_{S_n\cap B} a^- w_n^{q+1}dx + o(1)\left(\int_{\mathbb{R}^n} |\nabla w_n|^2 dx\right). \quad (4.4.10)
\end{aligned}$$

Finally from (4.4.2), (4.4.8) and (4.4.10) we get

$$
\begin{aligned}
&0 > I_\gamma(u_n) - I_\gamma(v_\gamma) \\
&\geq \frac{1}{2}\int_{\mathbb{R}^n}(|\nabla u_n^-|^2 + |\nabla w_n|^2)dx - \int_{S_n}(H_\gamma(x,\ U_\Gamma) - H_\gamma(x,\ U_\Gamma) - H_{\gamma v}(x,\ U_\Gamma)w_n)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^n}(|\nabla u_n^-|^2 + |\nabla w_n|^2)dx - \int_{S_n\cap A}(H_\gamma(x,\ U_\Gamma) - H_\gamma(x,\ U_\Gamma) - H_{\gamma v}(x,\ U_\Gamma)w_n)dx \\
&\qquad - \int_{S_n\cap B}(H_\gamma(x,\ U_\Gamma) - H_\gamma(x,\ U_\Gamma) - H_{\gamma v}(x,\ U_\Gamma)w_n)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^n}(|\nabla u_n^-|^2 + |\nabla w_n|^2)dx - o(1)(\int_{\mathbb{R}^n}|\nabla w_n|^2dx) + (\frac{1}{q+1} - \frac{1}{1+p})\int_{S_n\cap B}a^- w_n^{q+1}dx \\
&\qquad \geq \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u_n^-|^2dx + (\frac{1}{2} - o(1))(\int_{\mathbb{R}^n}|\nabla w_n|^2dx),
\end{aligned}
$$

which implies that $u_n^- = 0$ and $w_n = 0$ in $\mathbb{R}^n$ for n large, hence for n large $u_n = v_n$, from this we derive that $I_\gamma(u_n) = I_\gamma(v_n) \geq I_\gamma(v_\gamma)$. This is a contradiction. This theorem is done. □

4.5 Second Solution in the General Case

From the previous section we know that there exists a family of local minimizers v_γ, $\gamma \in (0,\Gamma)$, for the energy functional I_γ. Here we seek a second solution in the form $u = v_\gamma + v$ with $v \geq 0$ by means of Mountain-Pass Theorem. We define

$$J_\gamma(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx + \frac{1}{q+1} v^{q+1} - \frac{1}{q+1}(v^+)^{q+1} - H(x,v)dx \quad v \in H_q^1,$$

where $H(x,v) = \int_0^v h(x,s)ds$ with

$$h(x,v) = a_\gamma[(v_\gamma + v^+)^q - v_\gamma] + [(v_\gamma + v^+)^p - v_\gamma].$$

Therefore we see that

$$\begin{aligned}
J_\gamma(v) &= \frac{1}{q+1}||v^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)} + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 - H(x,v)dx \\
&= \frac{1}{q+1}||v^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)} + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx \\
&\quad - \int_{\mathbb{R}^n} \frac{1}{q+1} a_\gamma[(v_\gamma + v^+)^{q+1} - v_\gamma^{q+1}] - a_\gamma v_\gamma^q v^+ \\
&\qquad + \frac{1}{p+1}[(v_\gamma + v^+)^{p+1} - v_\gamma^{p+1}] - v_\gamma^p v^+ dx \\
&= \frac{1}{q+1}||v^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)} + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx - \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma (v_\gamma + v^+)^{q+1} dx \\
&\quad + \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma v_\gamma^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n} (v_\gamma + v^+)^{p+1} dx + \frac{1}{p+1}\int_{\mathbb{R}^n} v_\gamma^{p+1} dx \\
&\quad + \int_{\mathbb{R}^n} a_\gamma v_\gamma^q v^+ + v_\gamma^p v^+ dx \\
&= \frac{1}{q+1}||v^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)} + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla (v_\gamma + v^+)|^2 dx \\
&\quad - \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma (v_\gamma + v^+)^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n} (v_\gamma + v^+)^{p+1} dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v_\gamma|^2 dx + \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma v_\gamma^{q+1} dx + \frac{1}{p+1}\int_{\mathbb{R}^n} v_\gamma^{p+1} dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v_\gamma|^2 dx - \frac{1}{2}\int_{\mathbb{R}^n} |\nabla (v_\gamma + v^+)|^2 dx + \int_{\mathbb{R}^n} a_\gamma v_\gamma^q v^+ + v_\gamma^p v^+ dx
\end{aligned}$$

$$= \frac{1}{q+1}||v^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)} + I_\gamma(v_\gamma + v^+) - I_\gamma(v_\gamma) + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^-|^2 dx$$
$$- \int_{\mathbb{R}^n} \nabla v_\gamma \nabla v^+ dx + \int_{\mathbb{R}^n} a_\gamma v_\gamma^q v^+ dx + \int_{\mathbb{R}^n} v_\gamma^p v^+ dx$$
$$= I_\gamma(v_\gamma + v^+) - I_\gamma(v_\gamma) + \frac{1}{2}||\nabla v^-||^2_{L^2(\mathbb{R}^n)} + \frac{1}{q+1}||v^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)}. \tag{4.5.1}$$

Since v_γ is a local minimum (Theorem 4.4.3), from the above formula we immediately conclude that $v = 0$ is a local minimum of J_γ:

Lemma 4.5.1. *Assume (1.2.4), for any fixed* $0 < \gamma < \Gamma$, *there exists* $\delta_1 > 0$ *such that* $J_\gamma(v) \geq J_\gamma(0) = 0$ *with* $||v||_{H^1_q} < \delta_1$.

We also have mountain–pass structure, that is, the functional J_γ takes values strictly less than $0 = J_\gamma(0)$:

Lemma 4.5.2. *Assume* $\gamma \geq 0$, *and let* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *with* $\varphi \gneqq 0$ *and supp*(φ) *disjoints from supp*(v_γ). *Then, there exists a constant* $T > 0$ *such that* $J_\gamma(T\varphi) < 0$.

Proof. By (4.5.1), and since the supports of v_γ, φ are disjoint, we have

$$J_\gamma(T\varphi) = I_\gamma(T\varphi) = T^2 \int \frac{1}{2}|\nabla\varphi|^2 - T^{q+1}\int \frac{a_\gamma(x)}{q+1}|\varphi|^{q+1} - T^{p+1}\int \frac{1}{p+1}|\varphi|^{p+1} < 0$$

for T sufficiently large, since $q < 1 < p$. □

Lemma 4.5.3. *Suppose* $\gamma \geq 0$, $\{v_n\}$ *is a sequence in* H^1_q *such that* $J_\gamma(v_n) \to c_\gamma$ *and* $J'_\gamma(v_n) \to 0$, *then* $\{v_\gamma + v_n^+\}$ *is uniformly bounded in* H^1_q.

Proof. First notice that $J'_\gamma(v_n)v_n^- = -(||\nabla v_n^-||^2_{L^2(\mathbb{R}^n)} + ||v_n^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)})$, then we have

$$\begin{aligned}
||\nabla v_n^-||^2_{L^2(\mathbb{R}^n)} + ||v_n^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)} &\leq ||J'_\gamma(v_n)||(||\nabla v_n^-||_{L^2(\mathbb{R}^n)} + ||v_n^-||_{L^{q+1}(\mathbb{R}^n)}) \\
&\leq ||J'_\gamma(v_n)||(||\nabla v_n^-||^2_{L^2(\mathbb{R}^n)} + ||v_n^-|^{q+1}|_{L^{q+1}(\mathbb{R}^n)} + O(1)) \\
&\leq o(1)(||\nabla v_n^-||^2_{L^2(\mathbb{R}^n)} + ||v_n^-|^{q+1}|_{L^{q+1}(\mathbb{R}^n)}) + o(1),
\end{aligned}$$

hence we derive that

$$(1-o(1))(||\nabla v_n^-||^2_{L^2(\mathbb{R}^n)} + ||v_n^-|^{q+1}|_{L^{q+1}(\mathbb{R}^n)}) \leq o(1),$$

that is $v_n \to 0$ in H^1_q.

Therefore we may take $u_n = v_\gamma + v_n^+$, then we reach that

$$I_\gamma(u_n) \to I_\gamma(v_\gamma) + c \qquad and \qquad I'_\gamma(u_n) \to 0.$$

Since $I_\gamma(v_\gamma) < 0$, we have

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_n|^2 dx + \frac{1}{q+1}\int_{\mathbb{R}^n} a^- u_n^{q+1} dx - \frac{\gamma}{q+1}\int_{\mathbb{R}^n} a^+ u_n^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u_n^{p+1} dx < c. \tag{4.5.2}$$

We also have

$$\begin{aligned} I'_\gamma(u_n)u_n &= \int_{\mathbb{R}^n} |\nabla u_n|^2 dx + \int_{\mathbb{R}^n} a^- u_n^{q+1} dx - \gamma\int_{\mathbb{R}^n} a^+ u_n^{q+1} dx - \int_{\mathbb{R}^n} u_n^{p+1} dx \\ &= o(1)||u_n||_{H_r}. \end{aligned}$$

Pick θ such that $2 < \theta < p+1$, then $\frac{1}{p+1} < \frac{1}{\theta} < \frac{1}{2} < \frac{1}{q+1}$, hence from above we get

$$\frac{1}{\theta}\int_{\mathbb{R}^n} |\nabla u_n|^2 dx + \frac{1}{\theta}\int_{\mathbb{R}^n} a^- u_n^{q+1} dx - \frac{\gamma}{\theta}\int_{\mathbb{R}^n} a^+ u_n^{q+1} dx - \frac{1}{\theta}\int_{\mathbb{R}^n} u_n^{p+1} dx = o(1)||u_n||_{H_r}. \tag{4.5.3}$$

Subtract (4.5.2) from (4.5.3) we get

$$\begin{aligned} (\frac{1}{2}-\frac{1}{\theta})\int_{\mathbb{R}^n} |\nabla u_n|^2 dx + (\frac{1}{q+1}-\frac{1}{\theta})\int_{\mathbb{R}^n} a^- u_n^{q+1} dx - (\frac{\gamma}{q+1}-\frac{\gamma}{\theta})\int_{\mathbb{R}^n} a^+ u_n^{q+1} dx \\ - (\frac{1}{p+1}-\frac{1}{\theta})\int_{\mathbb{R}^n} u_n^{p+1} dx < c + o(1)||u_n||_{H_r}, \end{aligned}$$

that is

$$\begin{aligned} (\frac{1}{2}-\frac{1}{\theta})(\int_{\mathbb{R}^n} |\nabla u_n|^2 dx + \int_{\mathbb{R}^n} a^- u_n^{q+1} dx) + (\frac{1}{\theta}-\frac{1}{p+1})\int_{\mathbb{R}^n} u_n^{p+1} dx \\ \leq \gamma(\frac{1}{q+1}-\frac{1}{\theta})\int_{\mathbb{R}^n} a^+ u_n^{q+1} dx + c + o(1)||u_n||_{H_r}. \end{aligned}$$

From Young inequality we have

$$a^+u_n^{q+1} = (\frac{1}{\epsilon}a^+)(\epsilon u_n^{q+1}) < (\epsilon u_n^{q+1})^{\frac{p+1}{q+1}} + (\frac{1}{\epsilon}a^+)^{(\frac{p+1}{q+1})^*},$$

where we drop the two coefficient when applying Young inequality, $(\frac{p+1}{q+1})^*$ represents the conjugate of $\frac{p+1}{q+1}$ and ϵ is small such that

$$\Gamma\epsilon^{\frac{p+1}{q+1}}(\frac{1}{q+1} - \frac{1}{\theta}) \leq \frac{1}{2}(\frac{1}{\theta} - \frac{1}{p+1}).$$

Pick C such that

$$\Gamma(\frac{1}{q+1} - \frac{1}{\theta})\int_{\mathbb{R}^n}(\frac{1}{\epsilon}a^+)^{(\frac{p+1}{q+1})^*} \leq C.$$

Overall we reach

$$(\frac{1}{2}-\frac{1}{\theta})(\int_{\mathbb{R}^n}|\nabla u_n|^2dx+\int_{\mathbb{R}^n}a^-u_n^{q+1}dx)+\frac{1}{2}(\frac{1}{\theta}-\frac{1}{p+1})\int_{\mathbb{R}^n}u_n^{p+1}dx \leq c+C+o(1)||u_n||_{H_r}. \tag{4.5.4}$$

Claim: There exists small positive $\eta < \min\{\frac{1}{2}a_\infty,\ \frac{1}{2}\}$ and a constant $C_1 > 0$such that

$$\int_{\mathbb{R}^n}|\nabla u_n|^2dx + \int_{\mathbb{R}^n}a^-u_n^{q+1}dx + C_1 \geq \eta(||\nabla u_n||^2_{L^2(\mathbb{R}^n)} + ||u_n||^{q+1}_{L^{q+1}(\mathbb{R}^n)}).$$

Indeed, since $\liminf_{|x|\to 0} a^- = a_\infty$, then there exists $r_1 > 0$ such that $a^-(x) \geq \frac{1}{2}a_\infty$ for $x \in R^n - B(0, r_1)$. Now we see that

$$(1-\eta)||\nabla u_n||^2_{L^2(\mathbb{R}^n)} \geq \frac{1}{2}||\nabla u_n||^2_{L^2(\mathbb{R}^n)} \geq \frac{1}{2}C(n)||u_n||^2_{L^{2^*}(R^n)} \geq \frac{1}{2}C(n)||u_n||^2_{L^{2^*}(B(0,\ r_1))}$$

and

$$\eta||u_n||^{q+1}_{L^{q+1}(\mathbb{R}^n)} - \int_{\mathbb{R}^n}a^-u_n^{q+1}dx \leq \eta||u_n||^{q+1}_{L^{q+1}(B(0,\ r))} \leq \eta C(r_1)||u_n||^{q+1}_{L^{2^*}(B(0,\ r_1))}.$$

So when η is small enough and C_1 is large enough, we have

$$(1-\eta)||\nabla u_n||^2_{L^2(\mathbb{R}^n)} + C_1 \geq \frac{1}{2}C(n)||u_n||^2_{L^{2^*}(B(0,\ r_1))} + C_1$$
$$\geq \frac{1}{2}C(n)||u_n||^{q+1}_{L^{2^*}(B(0,\ r_1))} \geq \eta C(r_1)||u_n||^{q+1}_{L^{2^*}(B(0,\ r_1))} \geq \eta||u_n||^{q+1}_{L^{q+1}(\mathbb{R}^n)} - \int_{\mathbb{R}^n}a^-u_n^{q+1}dx$$

The claim is done.

Now from (4.5.4) and above claim, enlarging the constant C, we get

$$\begin{aligned}(\frac{1}{2}-\frac{1}{\theta})\eta(||\nabla u_n||^2_{L^2(\mathbb{R}^n)}+||u_n||^{q+1}_{L^{q+1}(\mathbb{R}^n)})\\ \leq c+C+o(1)(||\nabla u_n||_{L^2(\mathbb{R}^n)}+||u_n||_{L^{q+1}(\mathbb{R}^n)})\\ \leq c+C+o(1)(||\nabla u_n||^2_{L^2(\mathbb{R}^n)}+||u_n||^{q+1}_{L^{q+1}(\mathbb{R}^n)}+O(1))\\ \leq c+C+o(1)(||\nabla u_n||^2_{L^2(\mathbb{R}^n)}+||u_n||^{q+1}_{L^{q+1}(\mathbb{R}^n)})+o(1),\end{aligned}$$

which implies that $||\nabla u_n||_{L^2(\mathbb{R}^n)}$ and $||u_n||_{L^{q+1}(\mathbb{R}^n)}$ are uniformly bound, going back to (4.5.4), we have $||u_n||_{L^{p+1}(\mathbb{R}^n)}$ also uniformly bounded.

□

Unfortunately the Sobolev embedding from H^1_q to $L^{p+1}(\mathbb{R}^n)$ is not compact, so we must use Concentration Compactness arguments [30] to derive an alternative to the classical Palais–Smale condition. To the end of the section we recall the hypothesis on $a(x)$ from the statement of Theorem 1.2.6 :

$$\lim_{|x|\to\infty} a^-(x)=a_\infty>0 \quad \textit{and} \quad a^-(x)<a_\infty \quad \text{for } x\in\mathbb{R}^n\setminus B(0,R). \tag{4.5.5}$$

First we mention the following result from [18].

Theorem 4.5.4. *The following probelem*

$$-\Delta w=w^p-w^q \textit{ in } \mathbb{R}^n \qquad \textit{and} \qquad w\geq 0 \textit{ in } \mathbb{R}^n$$

has a unique compactly supported radially symmetric solution w_0. Moreover the energy at w_0 is positive, that is

$$I(w_0)=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w_0|^2dx+\frac{1}{q+1}\int_{\mathbb{R}^n}(w_0^+)^{q+1}dx-\frac{1}{p+1}\int_{\mathbb{R}^n}(w_0^+)^{p+1}dx>0.$$

Proof. See [18]. □

Now we define the following energy functional I_∞, which is the energy functional I_γ at infinity,

$$I_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx + \frac{a_\infty}{q+1}\int_{\mathbb{R}^n} (v^+)^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n} (v^+)^{p+1} dx \qquad v \in H_q^1.$$

Corollary 4.5.5. *The following problem*

$$-\Delta w = w^p - a_\infty w^q \text{ in } \mathbb{R}^n \qquad \text{and} \qquad w \geq 0 \text{ in } \mathbb{R}^n \tag{4.5.6}$$

has a unique compact supported radial symmetric solution W. Moreover $I_\infty(W) > 0$.

Proof. Let $c_1 > 0$ so that $c_1^{p-q} a_\infty = 1$, then consider $w_1 = c_1 w$, by calculation, we see that w_1 satisfies

$$-\Delta w_1 = c_1^{1-p}(w^p - w^q) \text{ in } \mathbb{R}^n.$$

Let $c_2 > 0$ so that $c_2^2 c_1^{1-p} = 1$, then consider $w_2(y) = w_1(c_2 y) = w_1(x)$, it is easy to see that $w_2(y)$ satisfies the following equation

$$-\Delta_y w_2(y) = c_1^{1-p} c_2^2 (w_2^p(y) - w_2^q(y)) = w_2^p(y) - w_2^q(y) \text{ in } \mathbb{R}^n.$$

Therefore from the uniqueness of above lemma, we derive the uniqueness of compact supported radial symmetric solution W to (4.5.6). From Pohozaev identity we conclude that $I_\infty(W) > 0$. This corollary is done. □

From now on we denote the unique compact supported radial symmetric solution to (4.5.6) by W, pick some x_0 in R^n such that $\text{supp}(W(x+x_0)) \cap \text{supp}(v_\gamma(x)) = \emptyset$, and $\text{supp}(\ W(x+x_0)) \subset \mathbb{R}^n \setminus B(0, R)$. By the hypothesis (4.5.5), it follows that $a^-(x) < a_\infty$ in the support of $W(x+x_0)$. It is also easy to see that $W(x+x_0)$ is still a solution to (4.5.6) and $\text{supp}(W(x+x_0)) \subset\subset \Omega^-$, we denote $W(x+x_0)$ by W_0. Now define

$$S_\gamma = \{\sigma \in C([0,\ 1],\ H_q^1) \mid \sigma(0) = 0 \text{ and } \sigma(1) = TW_0\}.$$

where $J_\gamma(TW_0) < 0$ and $T > 1$, this is possible because of Lemma 4.5.2.

Let $c_\gamma = \inf_{\sigma\in S_\gamma}\max_{s\in[0,\ 1]} J_\gamma(\sigma(s))$, then from Lemma 4.5.1, we see that $J_\gamma(v)\geq 0$ with $||v||_{H_r} < \delta_1$. Therefore $c_\gamma \geq 0$.

Lemma 4.5.6. $c_\gamma < I_\infty(W_0)$.

Proof. Consider $\sigma(s) = sW_0$, $s\in[0,\ T]$. We have from (4.5.5)

$$J_\gamma(\sigma(s)) = I_\gamma(sW_0+v_\gamma) - I_\gamma(v_\gamma) = I_\gamma(sW_0) + I_\gamma(v_\gamma) - I_\gamma(v_\gamma) = I_\gamma(sW_0) < I_\infty(sW_0).$$

Claim: $I_\infty(sW_0)$ achieves its maximum value at $s=1$.

Indeed, we have

$$I_\infty(sW_0) = \frac{s^2}{2}\int_{\mathbb{R}^n}|\nabla W_0|^2dx + \frac{a_\infty s^{q+1}}{q+1}\int_{\mathbb{R}^n}W_0^{q+1}dx - \frac{s^{p+1}}{p+1}\int_{\mathbb{R}^n}W_0^{p+1}dx,$$

since $-\triangle W_0 = W_0^p - a_\infty W_0^q$ in $\mathbb{R}^n$, we should have

$$\int_{\mathbb{R}^n}|\nabla W_0|^2dx = \int_{\mathbb{R}^n}W_0^{p+1}dx - a_\infty\int_{\mathbb{R}^n}W_0^{q+1}dx,$$

hence we get

$$I_\infty(sW_0) = (\frac{s^2}{2} - \frac{s^{p+1}}{p+1})\int_{\mathbb{R}^n}W_0^{p+1}dx + a_\infty(\frac{s^{q+1}}{q+1} - \frac{s^2}{2})\int_{\mathbb{R}^n}W_0^{q+1}dx$$
$$= d_1(\frac{s^2}{2} - \frac{s^{p+1}}{p+1}) - d_2(\frac{s^2}{2} - \frac{s^{q+1}}{q+1}),$$

where $d_1 = \int_{\mathbb{R}^n}W_0^{p+1}dx$ and $d_2 = a_\infty\int_{\mathbb{R}^n}W_0^{q+1}dx$. For s huge, $I_\infty(sW_0)$ is negative, then we can assume $I_\infty(sW_0)$ attains its maximum value at t, so $I'_\infty(tW_0) = d_1(t - t^p) - (t-t^q)d_2 = 0$, that is $d_1(t-t^p) = d_2(t-t^q)$, since $p>1>q>0$ and $I_\infty(W_0)>0$, we must have $t=1$. This lemma is done.

□

Corollary 4.5.7. *If there exist $u\in S_{M,\gamma}$ so that $c_\gamma \geq I_\gamma(u) - I_\gamma(v_\gamma) + I_\infty(W_0)$, then $u\notin Y$, in particular $u\neq v_\gamma$.*

Proof. By assumption $c_\gamma \geq I_\gamma(u) - I_\gamma(v_\gamma) + I_\infty(W_0)$, from above lemma, we see $c_\gamma < I_\infty(W_0)$, hence we have $c_\gamma > I_\gamma(u) - I_\gamma(v_\gamma) + c_\gamma$, that is $I_\gamma(u) < I_\gamma(v_\gamma)$. But if $u \in Y$, then $I_\gamma(u) \geq I_\gamma(v_\gamma)$, it is a contradiction. So $u \notin Y$. □

Since our ultimate goal is find a second solution to $(4.2.1)_\gamma$, so from now on we assume that for any $u \in S_{M,\gamma}$

$$c_\gamma < I_\gamma(u) - I_\gamma(v_\gamma) + I_\infty(W_0). \tag{4.5.7}$$

Actually under the assumption (4.5.7), we will show the compactness of P-S sequence.

Lemma 4.5.8. *Assume that u_n is uniformly bounded in $L^{q+1}(\mathbb{R}^n)$, ∇u_n is also uniformly bounded in $L^2(\mathbb{R}^n)$ and*

$$\sup_{y\in R^n} \int_{B(y,\ 1)} |u_n|^{q+1} dx \to 0 \quad as\ n \to \infty,$$

then $u_n \to 0$ in$L^\alpha(\mathbb{R}^n)$ for α between $q+1$ and 2^* as $n \to \infty$.

Proof. See Lions [30]. □

Lemma 4.5.9. *Assume $\{u_n\} \subset H^1_q$ and $u_n \rightharpoonup u_0$ weakly in H^1_q for some $u_0 \in H^1_q$ with compact support, then there exists at least a subsequence, still denoted by $\{u_n\}$, such that*

$$|u_n|^{q+1} - |u_0|^{q+1} - |u_n - u_0|^{q+1} \to 0 \quad in\ L^1(\mathbb{R}^n)$$
$$|u_n|^p - |u_0|^p - |u_n - u_0|^p \to 0 \quad in\ L^t(\mathbb{R}^n),$$

where $1 \leq t \leq \frac{2n}{n+2}$

Proof. By the weak convergence, restricting to a subsequence if necessary, we see that $u_n \to u_0$ a.e. in $\mathbb{R}^n$. Pick $r > 0$ big enough so that $\mathrm{supp}(u_0) \subset\subset B(0,\ r)$, then we only need to prove the above claim in $L^1(B(0,\ r))$ and $L^t(B(0,\ r))$, restrict to a subsequence if necessary, we assume $u_n \to u_0$ strongly in L^s with $1 \le s < 2^*$. Notice that

$$||u_n|^{q+1} - |u_0|^{q+1} - |u_n - u_0|^{q+1}| \le C_1(|u_n|^{q+1} + |u_0|^{q+1})$$
$$||u_n|^p - |u_0|^p - |u_n - u_0|^p|^t \le C_2(|u_n|^{pt} + |u_0|^{pt}),$$

then Lebesgue Convergence Theorem applies and the lemma is proved. □

Lemma 4.5.10. *Suppose $\gamma \ge 0$, $\{v_n\}$ is a sequence in H_q^1 such that $J_\gamma(v_n) \to c_\gamma$ and $J'_\gamma(v_n) \to 0$, then $\{v_n\}$ contains a strongly convergent subsequence in H_q^1. Moreover if $v_n \to v_0 \ge 0$, then $u_0 = v_\gamma + v_0$ is a solution to* (4.2.1).

Proof. In wiew of Lemma 4.5.3, we could take $u_n = v_n^+ + v_\gamma$, then we should have

$$I_\gamma(u_n) \to I_\gamma(v_\gamma) + c_\gamma \qquad \textit{and} \qquad I'_\gamma(u_n) \to 0.$$

Since there is no confusion, we put away some of the subscript γ, we denote I_γ as I and γa^+ as a^+, still keep I_∞ and c_γ. We also use L^{q+1}, L^2 and L^{p+1} instead of $L^{q+1}(\mathbb{R}^n)$, $L^2(\mathbb{R}^n)$ and $L^{p+1}(\mathbb{R}^n)$. Therefore we have

$$I(u_n) \to I(v_\gamma) + c_\gamma \qquad \textit{and} \qquad I'(u_n) \to 0.$$

Again from Lemma 4.5.3, we have that $||\nabla u_n||_{L^2} + ||u_n||_{L^{q+1}} + ||u_n||_{L^{p+1}}$ is uniformly bounded, then restricting to a subsequence if necessary, we assume that there exists $u_0 \in H_q^1$ so that

$$u_n \rightharpoonup u_0 \ \ \textit{weakly in } H_q^1.$$

By a diagonal process, restricting to a subsequence if necessary, we may assume $u_n \to u_0$ strongly in $L^{q+1}(B(0,r))$ for any $r > 0$ and $u_n \to u_0$ a.e. in $\mathbb{R}^n$, then by interpolation we have $u_n \to u_0$ strongly in $L^t(B(0,r))$ for any $r > 0$ and $q+1 \le t < 2^*$. By local compactness we see that $u_0 \ge v_\gamma$ in $\mathbb{R}^n$. It is also easy to see that u_0 is a solution to $(4.2.1)_\gamma$, so u_0 has compact support.

Now let us consider $u_n^1 = u_n - u_0$, since u_0 is compactly supported, $(u_n - u_0)^-$ has bounded support.

Claim: $(u_n - u_0)^- \to 0$ in H_q^1.

Indeed, since $(u_n - u_0)^-$ has bounded support, then by local compactness $(u_n - u_0)^- \to 0$ in $L^{p+1} \cap L^{q+1}$. We know that u_0 is a solution to $(4.2.1)_\gamma$, then $I'(u_0) = 0$, hence $(I'(u_n) - I'(u_0))(u_n - u_0)^- \to 0$, that is

$$\begin{aligned} & (I'(u_n) - I'(u_0)(u_n - u_0)^- \\ = & -||\nabla(u_n - u_0)^-||_{L^2}^2 - \int_{\mathbb{R}^n} a(u_n^q - u_0^q)(u_n - u_0)^- dx - \int_{\mathbb{R}^n} (u_n^p - u_0^p)(u_n - u_0)^- dx \\ \to & -||\nabla(u_n - u_0)^-||_{L^2}^2 + o(1), \end{aligned}$$

which implies $||\nabla(u_n - u_0)^-||_{L^2} \to 0$, so the claim is proven.

We may assume $u_n^1 = u_n - u_0 \geq 0$ in $\mathbb{R}^n$, then we find that

$$\begin{aligned} ||\nabla u_n||_{L^2}^2 - ||\nabla u_0||_{L^2}^2 - ||\nabla u_n^1||_{L^2}^2 &= ||\nabla u_n||_{L^2}^2 - ||\nabla u_0||_{L^2}^2 - ||\nabla(u_n - u_0)||_{L^2}^2 \\ &= 2\left(\int_{\mathbb{R}^n} \nabla u_n \nabla u_0 dx - ||\nabla u_0||_{L^2}^2\right) \to 0 \quad (4.5.8) \end{aligned}$$

and

$$||u_n||_{L^{q+1}}^{q+1} - ||u_0||_{L^{q+1}}^{q+1} - ||u_n - u_0||_{L^{q+1}}^{q+1} = \int_{supp(u_0)} |u_n|^{q+1} - |u_0|^{q+1} - |u_n - u_0|^{q+1} dx \to 0$$

The second one is from Lemma 4.5.9. Therefore we have

$$||\nabla u_n||_{L^2}^2 + ||u_n||_{L^{q+1}}^{q+1} = ||\nabla u_0||_{L^2}^2 + ||u_0||_{L^{q+1}}^{q+1} + ||\nabla u_n^1||_{L^2}^2 + ||u_n^1||_{L^{q+1}}^{q+1} + o(1) \quad (4.5.9)$$

Claim: $I(u_n) = I(u_0) + I_\infty(u_n^1) + o(1)$

Since $\lim_{|x| \to \infty} a^- = a_\infty$, u_0 has compact support and $u_n \to u_0$ strongly in $L^{q+1}(B(0,r))$ for any $r > 0$, we get

$$\frac{1}{q+1} \int_{\Omega^-} a^- u_n^{q+1} dx = \frac{1}{q+1} \int_{\Omega^-} a^- u_0^{q+1} dx + \frac{1}{q+1} \int_{\Omega^-} a_\infty (u_n - u_0)^{q+1} dx + o(1),$$

since $\int_{\Omega^{0+}}(u_n-u_0)^{q+1}dx \to 0$ and $\int_{\Omega^{0+}} a^+u_n^{q+1}dx \to \int_{\Omega^{0+}} a^+u_0^{q+1}dx$, we reach

$$\frac{1}{q+1}\int_{\mathbb{R}^n} au_n^{q+1}dx = \frac{1}{q+1}\int_{\mathbb{R}^n} au_0^{q+1}dx - \frac{1}{q+1}\int_{\mathbb{R}^n} a_\infty(u_n-u_0)^{q+1}dx + o(1). \quad (4.5.10)$$

From Lemma 4.5.9 it is also clear that

$$\int_{\mathbb{R}^n} u_n^{p+1}dx = \int_{\mathbb{R}^n} u_n^{p+1}dx + \int_{\mathbb{R}^n}(u_n-u_0)^{p+1}dx + o(1). \quad (4.5.11)$$

Since for $I(u_n)$, $I(u_0)$ and $I_\infty(u_n^1)$, we have

$$\begin{aligned}
I(u_n) &= \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u_n|^2dx - \frac{1}{q+1}\int_{\mathbb{R}^n} au_n^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u_n^{p+1}dx\\
I(u_0) &= \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u_0|^2dx - \frac{1}{q+1}\int_{\mathbb{R}^n} au_0^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u_0^{p+1}dx\\
I(u_n^1) &= \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u_n^1|^2dx + \frac{1}{q+1}\int_{\mathbb{R}^n} a_\infty(u_n^1)^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n}(u_n^1)^{p+1}dx,
\end{aligned}$$

then from (4.5.8), (4.5.10) and (4.5.11) this claim is true.

Claim: $I'_\infty(u_n^1) \to 0$.

First we have for $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}
I'_\infty(u_n^1)\varphi &= \int_{\mathbb{R}^n}\nabla(u_n-u_0)\nabla\varphi dx + a_\infty\int_{\mathbb{R}^n}(u_n-u_0)^q\varphi dx - \int_{\mathbb{R}^n}(u_n-u_0)^p\varphi\\
I'(u_0)\varphi &= \int_{\mathbb{R}^n}\nabla u_0\nabla\varphi dx + \int_{\mathbb{R}^n} a^-u_0^q\varphi dx - \int_{\mathbb{R}^n} a^+u_0^q\varphi dx - \int_{\mathbb{R}^n} u_0^p\varphi dx\\
&= 0\\
I'(u_n)\varphi &= \int_{\mathbb{R}^n}\nabla u_n\nabla\varphi dx + \int_{\mathbb{R}^n} a^-u_n^q\varphi dx - \int_{\mathbb{R}^n} a^+u_n^q\varphi dx - \int_{\mathbb{R}^n} u_n^p\varphi dx\\
&= o(1)||\varphi||_{H_q^1}.
\end{aligned}$$

Since u_n weakly converges to u_0 and u_0 has compact support, we can pick $r>0$ such that supp(u_0) $\subset\subset B(0,\ r)$. Hence

$$\begin{aligned}
I'(u_n)\varphi - I'(u_0)\varphi = \int_{\mathbb{R}^n}\nabla(u_n-u_0)\nabla\varphi dx + \int_{\mathbb{R}^n} a^-(u_n^q-u_0^q)\varphi dx\\
-\int_{\mathbb{R}^n} a^+(u_n^q-u_0^q)\varphi dx - \int_{\mathbb{R}^n}(u_n^p-u_0^p)\varphi dx,
\end{aligned}$$

then

$$
\begin{aligned}
&I'_\infty(u_n^1)\varphi - (I'(u_n)\varphi - I'(u_0)\varphi) \\
&\quad = \int_{\mathbb{R}^n} a_\infty (u_n - u_0)^q \varphi dx - \int_{\mathbb{R}^n} a^-(u_n^q - u_0^q)\varphi dx + \int_{\mathbb{R}^n} a^+(u_n^q - u_0^q)\varphi dx \\
&\quad - \int_{\mathbb{R}^n} [(u_n - u_0)^p + u_0^p - u_n^p]\varphi dx \\
&= \int_{B(0,\ r)} [a_\infty (u_n - u_0)^q - a^-(u_n^q - u_0^q) + a^+(u_n^q - u_0^q)]\varphi dx \\
&\quad + \int_{R^n - B(0,\ r)} (a_\infty - a^-) u_n^q \varphi dx - \int_{B(0,\ r)} [(u_n - u_0)^p + u_0^p - u_n^p]\varphi dx.
\end{aligned}
$$

Therefore for $\epsilon > 0$ pick r enough so that

$$
\left|\int_{\mathbb{R}^n - B(0,\ r)} (a_\infty - a^-) u_n^q \varphi dx\right| \leq \frac{1}{4}\epsilon ||\varphi||_{L^{q+1}},
$$

then by compactness over $B(0,\ r)$ and Lemma 4.5.9, there exists $n_0 > 0$ so that for $n \geq n_0$

$$
\begin{aligned}
&\left|\int_{B(0,\ r)} [a_\infty (u_n - u_0)^q - a^-(u_n^q - u_0^q) + a^+(u_n^q - u_0^q)]\varphi dx\right| \leq \frac{1}{4}\epsilon ||\varphi||_{L^{q+1}} \\
&\left|\int_{B(0,\ r)} [(u_n - u_0)^p + u_0^p - u_n^p]\varphi dx\right| \leq \frac{1}{4}\epsilon ||\nabla\varphi||_{L^2}.
\end{aligned}
$$

Since $I'(u_n) \to 0$, then enlarge n_0 if necessary, assume for $n \geq n_0$, $|I'(u_n)\varphi| < \frac{1}{4}\epsilon ||\varphi||_{H_q^1}$, then we should have for $n \geq n_0$

$$
|I'_\infty(u_n^1)\varphi| < \epsilon ||\varphi||_{H_q^1},
$$

which implies $I'_\infty(u_n^1) \to 0$ as $n \to \infty$.

Now we have

$$
I_\infty(u_n^1) = I(u_n) - I(u_0) + o(1) \qquad \textit{and} \qquad I'_\infty(u_n^1) \to 0.
$$

Let us define

$$
\delta = \overline{\lim_{n \to \infty}} \sup_{y \in \mathbb{R}^n} \int_{B(y,\ 1)} |u_n^1|^{q+1} dx.
$$

If $\delta = 0$, then Lemma 4.5.8 implies that $u_n^1 \to 0$ in L^{p+1}. Since $I'_\infty(u_n^1) \to 0$, then we have

$$I'_\infty(u_n^1)u_n^1 = \int_{\mathbb{R}^n} |\nabla u_n^1|^2 dx + \int_{\mathbb{R}^n} |u_n^1|^{q+1} dx - \int_{\mathbb{R}^n} |u_n^1|^{p+1} dx \to 0,$$

which implies that $\int_{\mathbb{R}^n} |\nabla u_n^1|^2 dx + \int_{\mathbb{R}^n} |u_n^1|^{q+1} dx \to 0$. So u_n strongly converges to u_0 in H_q^1 and we are done.

If $\delta > 0$, we may assume the existence of $\{y_n^1\} \subset \mathbb{R}^n$ such that

$$\int_{B(y_n^1,\ 1)} |u_n^1|^{q+1} dx > \frac{1}{2}\delta.$$

Let us define $v_n^1(x) = u_n^1(x + y_n^1) = u_n^1(. + y_n^1)$. For $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$|I'_\infty(v_n^1)\varphi| = |I'_\infty(u_n^1)\varphi(. - y_n^1)| \leq ||I'_\infty(u_n^1)||||\varphi||_{H_q^1}.$$

Hence we have

$$I_\infty(v_n^1) = I_\infty(u_n^1) = I(u_n) - I(u_0) + o(1) \qquad and \qquad I'_\infty(v_n^1) \to 0.$$

We may assume $v_n^1 \rightharpoonup v^1$ weakly in H_q^1 for some $v^1 \in H_q^1$ and $v_n^1 \to v^1$ a.e. in $\mathbb{R}^n$. Since $\int_{B(0,\ 1)} |v_n^1|^{q+1} dx > \frac{1}{2}\delta$, it follows that

$$\int_{B(0,\ 1)} |v^1|^{q+1} dx > \frac{1}{2}\delta,$$

which implies that $v^1 \neq 0$. But u_n^1 weakly converges to zero in H_q^1, so $\{y_n^1\}$ is unbounded. We may assume that $\lim_{n\to\infty} |y_n^1| = \infty$. It is easy to see that v^1 is a solution to (4.5.6), so v^1 has compact support, more important $I_\infty(v^1) \geq I_\infty(W_0)$.

Claim: $(v_n^1 - v^1)^- \to 0$ in H_q^1.

Indeed, since $(v_n^1 - v^1)^-$ has bounded support, then by local compactness we may assume $(v_n^1 - v^1)^- \to 0$ in $L^{p+1} \cap L^{q+1}$. We know that v^1 is a solution to (4.5.6), then

$I'_\infty(v^1) = 0$, hence $(I'_\infty(v_n^1) - I'_\infty(v^1))(v_n^1 - v^1)^- \to 0$, that is

$$\begin{aligned} & (I'_\infty(v_n^1) - I'_\infty(v^1)(v_n^1 - v^1)^- \\ = & -||\nabla(v_n^1 - v^1)^-||_{L^2}^2 + \int_{\mathbb{R}^n} a_\infty((v_n^1)^q - (v^1)^q)(v_n^1 - v^1)^- dx \\ & - \int_{\mathbb{R}^n} ((v_n^1)^p - (v^1)^p)(v_n^1 - v^1)^- \\ \to & -||\nabla(v_n^1 - v^1)^-||_{L^2}^2 + o(1), \end{aligned}$$

which implies $||\nabla(v_n^1 - v^1)^-||_{L^2} \to 0$, so the claim is proven.

We may assume $u_n^2 = u_n^1 - v^1(. - y_n^1) \geq 0$, then repeat above process again.

Claim:

$$||\nabla u_n^1||_{L^2}^2 = ||\nabla u_n^2||_{L^2}^2 + ||\nabla v^1||_{L^2}^2 + o(1) \quad \textit{and} \quad ||u_n^1||_{L^{q+1}}^{q+1} = ||u_n^2||_{L^{q+1}}^{q+1} + ||v^1||_{L^{q+1}}^{q+1} + o(1)$$

Actually we have

$$\begin{aligned} ||\nabla u_n^1||_{L^2}^2 - ||\nabla u_n^2||_{L^2}^2 - ||\nabla v^1||_{L^2}^2 &= 2\int_{\mathbb{R}^n} \nabla u_n^1 \nabla v^1(. - y_n^1) - |\nabla v^1|^2 dx \\ &= 2\int_{\mathbb{R}^n} \nabla v_n^1 \nabla v^1 - |\nabla v^1|^2 dx \to 0 \end{aligned}$$

and

$$||u_n^1||_{L^{q+1}}^{q+1} - ||u_n^2||_{L^{q+1}}^{q+1} - ||v^1||_{L^{q+1}}^{q+1} = \int_{\mathbb{R}^n} |v_n^1|^{q+1} - |v^1|^{q+1} - |v_n^1 - v^1|^{q+1} dx \to 0.$$

So this claim is done.

Claim: $I_\infty(u_n^1) = I_\infty(u_n^2) + I_\infty(v^1) + o(1)$.

From Lemma 4.5.9 we see that

$$\int_{\mathbb{R}^n} (v_n^1)^{q+1} - (v_n^1 - v^1)^{q+1} - (v^1)^{q+1} dx = o(1)$$

and

$$\int_{\mathbb{R}^n} (v_n^1)^{p+1} - (v_n^1 - v^1)^{p+1} - (v^1)^{p+1} dx = o(1).$$

For $I_\infty(u_n^1)$, $I_\infty(u_n^2)$ and $I_\infty(v^1)$ we have

$$\begin{aligned}
I_\infty(u_n^1) &= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla u_n^1|^2 + \frac{a_\infty}{q+1}|u_n^1|^{q+1} - \frac{1}{p+1}(u_n^1)^{p+1}dx \\
&= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla v_n^1|^2 + \frac{a_\infty}{q+1}|v_n^1|^{q+1} - \frac{1}{p+1}(v_n^1)^{p+1}dx \\
I_\infty(u_n^2) &= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla(u_n^1 - v^1(.-y_n^1))|^2 + \frac{a_\infty}{q+1}|u_n^1 - v^1(.-y_n^1)|^{q+1} \\
&\qquad -\frac{1}{p+1}(u_n^1 - v^1(.-y_n^1))^{p+1}dx \\
&= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla(v_n^1 - v^1)|^2 + \frac{a_\infty}{q+1}|v_n^1 - v^1|^{q+1} - \frac{1}{p+1}(v_n^1 - v^1)^{p+1}dx \\
I_\infty(v^1) &= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla v^1|^2 + \frac{a_\infty}{q+1}|v^1|^{q+1} - \frac{1}{p+1}(v^1)^{p+1}dx,
\end{aligned}$$

then from above claim we see that this claim is true.

Claim: $I'_\infty(u_n^2) \to 0$.

In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}
|I'_\infty(u_n^2)\varphi| &= |\int_{\mathbb{R}^n} \nabla(u_n^1 - v^1(.-y_n^1))\nabla\varphi + a_\infty|(u_n^1 - v^1(.-y_n^1))|^q\varphi - |(u_n^1 - v^1(.-y_n^1))|^p\varphi dx| \\
&= |I'_\infty(v_n^1 - v^1)\varphi(.+y_n^1)| \\
&\le ||I'_\infty(v_n^1 - v^1)||\,||\varphi(.+y_n^1)||_{H_q^1} \le ||I'_\infty(v_n^1 - v^1)||\,||\varphi||_{H_q^1},
\end{aligned}$$

hence for the claim to be true, we only need to show $I'_\infty(v_n^1 - v^1) \to 0$. Indeed for $I'_\infty(v_n^1 - v^1)$, $I'_\infty(v_n^1)$ and $I'_\infty(v^1)$ we have

$$\begin{aligned}
I'_\infty(v_n^1 - v^1)\varphi &= \int_{\mathbb{R}^n} \nabla(v_n^1 - v^1)\nabla\varphi + a_\infty(v_n^1 - v^1)^q\varphi - (v_n^1 - v^1)^p\varphi dx \\
I'_\infty(v_n^1)\varphi &= \int_{\mathbb{R}^n} \nabla v_n^1\nabla\varphi + a_\infty(v_n^1)^q\varphi - (v_n^1)^p\varphi dx \to 0 \\
I'_\infty(v^1)\varphi &= \int_{\mathbb{R}^n} \nabla v^1\nabla\varphi + a_\infty(v^1)^q\varphi - (v^1)^p\varphi dx = 0.
\end{aligned}$$

Hence we obtain

$$(I'_\infty(v_n^1) - I'_\infty(v^1))\varphi = \int_{\mathbb{R}^n} \nabla(v_n^1 - v^1)\nabla\varphi + a_\infty((v_n^1)^q - (v^1)^q)\varphi - ((v_n^1)^p - (v^1)^p)\varphi,$$

then

$$(I'_\infty(v_n^1 - v^1) - (I'_\infty(v_n^1) - I'_\infty(v^1)))\varphi = a_\infty \int_{\mathbb{R}^n} [(v_n^1 - v^1)^q - ((v_n^1)^q - (v^1)^q)]\varphi dx$$
$$- \int_{\mathbb{R}^n} [(v_n^1 - v^1)^p - ((v_n^1)^p - (v^1)^p)]\varphi dx.$$

Notice that v^1 has compact support, from Lemma 4.5.9 we have

$$|(I'_\infty(v_n^1 - v^1) - (I'_\infty(v_n^1) - I'_\infty(v^1)))\varphi| \le o(1)||\varphi||_{L^{q+1}} + o(1)||\nabla\varphi||_{L^2} \le o(1)||\varphi||_{H_q^1},$$

which implies that $I_\infty(v_n^1 - v^1) \to 0$, so this claim is done.

Now we have

$$I_\infty(u_n^2) = I_\infty(u_n^1) - I_\infty(v^1) + o(1) \quad and \quad I'_\infty(u_n^2) \to 0.$$

Again define

$$\delta_1 = \overline{\lim_{n\to\infty}} \sup_{y\in\mathbb{R}^n} \int_{B(y,\ 1)} |u_n^2|^{q+1} dx.$$

We have two cases.

If $\delta_1 = 0$, then from Lemma 4.5.8 we conclude $u_n^2 \to 0$ in L^{p+1}, notice that

$$I'_\infty(u_n^2)u_n^2 = ||\nabla u_n^2||_{L^2}^2 + a_\infty||u_n^2||_{L^{q+1}}^{q+1} - ||u_n^2||_{L^{p+1}}^{p+1} = o(1)||u_n^2||_{H_q^1},$$

so we derive that $||u_n^2||_{H_q^1} \to 0$, in turn we have $I_\infty(u_n^2) \to 0$, which means that

$$I_\infty(u_n^1) = I_\infty(v^1) + o(1) \ge I_\infty(W_0) + o(1),$$

since $I(u_n) = I(v_\gamma) + c_\gamma + o(1)$ and $I(u_n) = I(u_0) + I_\infty(u_n^1)$, we have

$$c_\gamma + I(v_\gamma) \ge I(u_0) + I_\infty(W_0),$$

which contradicts the assumption (4.5.5).

If $\delta_1 > 0$, then there exists $\{y_n^2\} \subset \mathbb{R}^n$ such that

$$\int_{B(y_n^1,\ 1)} |u_n^2|^{q+1} dx \ge \frac{1}{2}\delta_1.$$

Since $I'_\infty(u_n^2) \to 0$, then we get

$$I'_\infty(u_n^2)u_n^2 = ||\nabla u_n^2||_{L^2}^2 + a_\infty ||u_n^2||_{L^{q+1}}^{q+1} - ||u_n^2||_{L^{p+1}}^{p+1} = o(1),$$

in turn we obtain

$$\begin{aligned} I_\infty(u_n^2) &= \frac{1}{2}||\nabla u_n^2||_{L^2}^2 + \frac{a_\infty}{q+1}||u_n^2||_{L^{q+1}}^{q+1} - \frac{1}{p+1}||u_n^2||_{L^{p+1}}^{p+1} \\ &= (\frac{1}{2} - \frac{1}{p+1})||u_n^2||_{L^{p+1}}^{p+1} + a_\infty(\frac{1}{q+1} - \frac{1}{2})||u_n^2||_{L^{q+1}}^{q+1} + o(1) \\ &\geq a_\infty(\frac{1}{q+1} - \frac{1}{2})||u_n^2||_{L^{q+1}}^{q+1} + o(1) \geq \epsilon, \end{aligned}$$

uniformly for all n with some positive ϵ depending on δ_1. So we have

$$\epsilon \leq I_\infty(u_n^2) = I_\infty(u_n^1) - I_\infty(v^1) + o(1) = I(u_n) - I(u_0) - I_\infty(v^1) + o(1),$$

since $I(u_n) = I(v_\gamma) + c_\gamma + o(1)$, we have

$$c_\gamma \geq I(u_0) - I(v_\gamma) + I_\infty(v^1) + \epsilon \geq I(u_0) - I(v_\gamma) + I_\infty(W_0) + \epsilon,$$

which again contradicts the assumption (4.5.5).

So we must have $\delta = 0$, which means $u_n \to u_0$ strongly in H_q^1. This lemma is proved.

□

From above proof we see that the compactness of the support of u_0 and v^1 dramatically reduce the complexity of the proof.

Corollary 4.5.11. *Without the assumption (4.5.5), in the above lemma if* $u_n \rightharpoonup u_0$ *weakly in* H_q^1*, but not strongly, then* $I_\gamma(u_0) < I_\gamma(v_\gamma)$*. In particular* $u_0 \neq v_\gamma$ *and* $u_0 \notin Y$*.*

Proof. Otherwise we have $I_\gamma(u_0) \geq I_\gamma(v_\gamma)$, since $c_\gamma < I_\infty(W_0)$, then $c_\gamma < I_\infty(W_0) \leq I_\gamma(u_0) - I_\gamma(v_\gamma) + I_\infty(W_0)$, so we have assumption (4.5.5) for free. Follow above proof, we conclude the corollary. □

Theorem 4.5.12. *Suppose there exists $\eta_\gamma > 0$ such that for any $\rho \in [0, \ \eta_\gamma]$*

$$\inf\{J_\gamma(v) \mid \ ||v||_{H_q^1} = \rho\} = 0,$$

and $c_\gamma = 0$, then for each $\rho \in (0, \ \eta_\gamma)$. the problem $(4.2.1)_\gamma$ has a solution with $||u - v_\gamma||_{H_q^1} = \rho$.

Proof. For any fixed $\rho \in (0, \ \eta_\gamma)$, the set $F = \partial B(0, \ \rho)$ in H_q^1 satisfies the hypothesis of theorem (1) in Ghoussoub and Preiss [25]. their Theorem (1.bis) asserts the existence of a solution for each $\rho \in (0, \ \eta_\gamma)$ with the help of above Lemma 4.5.10, i.e. the compactness of the P-S sequence. □

Corollary 4.5.13. *Under assumption (4.5.5), the equation* $(4.2.1)_\gamma$ *has two element in $S_{M,\gamma}$ for $0 < \gamma < \Gamma$.*

Proof. If $c_\gamma > 0$, the result is from the Mountain-Pass theorem. If $c_\gamma = 0$, it is from Theorem 4.5.12. □

Corollary 4.5.14. *Under assumption (4.5.5), the equation* $(4.2.1)_0$ *has a compactly supported nonzero solution.*

Proof. We just need to change Y in (4.4.1) to $Y_\gamma = \{v \in H_q^1 \mid \ 0 \leq v \leq U_\gamma \ a.e.\}$, it is easy to see that when γ is small enough, I_0 achieves its infimum at $v = 0$, after that everything is the same like the proof above theorem. □

Finally, we ask the following two questions. First, are these two solutions we obtain in this way distinct in Ω^+? And secondly, if we assume that $a(tx^s)$ is strictly decreasing as $t \geq 0$ increases for any direction $x^s \in S^n$, can we show that the the solutions of $(4.2.1)_\gamma$ have connected support? We believe that this is the case, but it remains an open question in this generality. In the next section we specialize to the radial setting, where can give a positive answer to these questions.

4.6 Radial Symmetry

In this section we are going to use the method of moving plane by Gidas, Ni and Nerenberg [26] to prove the radial symmetry of the solutions of (4.0.1). First assume $a(x) = a(|x|)$, then let us denote $r_1 = \sup\{r \geq 0 \mid a(r) \geq 0\}$, We should point out that we do not assume here that Ω^{0+} is not empty, so r_1 could be zero, but we still keep the assumption (4.0.2). We also make the further assumption on a that

$$\text{a(r) is decreasing in } [0,\ r_1] \text{ and is strictly decreasing in } [r_1,\ \infty) \tag{4.6.1}$$

We have the following main theorem concerning the solutions of (4.0.1).

Theorem 4.6.1. *Under the assumption (4.6.1), any solution u of (4.0.1) is radially symmetric and decreases as r increases.*

We will present a long list of lemmas to prove this theorem. Take any nonzero solution of (4.0.1) in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, say u, by Theorem 1.2.2, u is compact supported, so we can define $r_2 = \sup\{|x| \geq 0 \mid u(x) > 0\}$. It is clear $0 < r_2 < \infty$ and $u(\partial B(0,\ r_2)) = 0$.

Let us consider now an arbitrary direction τ, which for simplicity we can assume to be $\tau = e_1$. For $\lambda \geq 0$ we define

$$\Sigma_\lambda = \{(x_1, x_2, ..., x_n) \mid x_1 > \lambda\} \quad \textit{and} \quad T_\lambda = \partial\Sigma_\lambda.$$

For $x \in R_n$, let $x^\lambda = 2(\lambda - x_1)e_1 + x$ be the reflection of x with respect to T_λ. We define the functions $u_\lambda, a_\lambda, w_\lambda : R_n \to R^1$ as

$$u_\lambda(x) = u(x^\lambda), \qquad a_\lambda(x) = a(x^\lambda), \qquad w_\lambda = u_\lambda(x) - u(x).$$

We set

$$\lambda_0 = \{\lambda \geq 0 \mid w_\lambda(x) \geq 0 \ \forall\ x\ \in \Sigma_\lambda\}.$$

Clearly since u has compact support, $\lambda_0 < \infty$. In view ofLemma 4.1.7, since u is nonzero, we may assume that $u(r_3 e_1) > 0$ for some $r_3 \in (r_1,\ r_2)$.

Lemma 4.6.2. $\Sigma_{\lambda_0} \cap B(0,\ r_2) \neq \emptyset$, *i.e.* $\lambda_0 < r_2$

Proof. From Lemma 4.1.7 we see that $a(r_2) < 0$. Since we also have $u(\partial B(0,\ r_2)) = 0$, then there exists $\delta_1 > 0$ and $\sigma_1 > 0$ such that

$$a^-(r) \geq \sigma_1 \ for\ r \in [r_2 - \delta_1,\ r_2]$$

and

$$u(x) \leq A_1 \ for\ x \in \Sigma_{r_2-\delta_1} \cap B(0,\ r_2),$$

where $A_1^{p-q} = \frac{\sigma_1 q}{p}$.
Now if there exists $x_1 \in \Sigma_{r_2-\delta_1} \cap B(0,\ r_2)$ such that

$$w_{r_2-\delta_1}(x_1) < 0.$$

Since $w_{r_2-\delta_1}(x) \geq 0$ on $\partial[\Sigma_{r_2-\delta_1} \cap B(0,\ r_2)]$, then we may assume $w_{r_2-\delta_1}$ attains minimum at x_1. Therefore we have

$$\begin{aligned} 0 &\geq -\Delta w_{r_2-\delta_1}(x_1) = -\Delta u_{r_2-\delta_1}(x_1) - (-\Delta u(x_1)) \\ &= [a_{r_2-\delta_1}(x_1)u^q_{r_2-\delta_1}(x_1) + u^p_{r_2-\delta_1}(x_1)] - [a(x_1)u^q(x_1) + u^p(x_1)] \\ &\geq [a(x_1)u^q_{r_2-\delta_1}(x_1) + u^p_{r_2-\delta_1}(x_1)] - [a(x_1)u^q(x_1) + u^p(x_1)] \qquad since\ a_{r_2-\delta_1}(x_1) > a(x_1) \\ &> 0 \qquad since\ A_1 > u(x_1) > u_{r_2-\delta_1}(x_1) \geq 0\ and\ a^-(x_1) \geq \sigma_1. \end{aligned}$$

This is a contradiction, this lemma is proven. □

Now if $\lambda_0 = 0$, then Theorem 4.6.1 is proved. But if $\lambda_0 > 0$, then by definition of λ_0, there exists a sequence $\{\lambda_k\}$ such that

$$\lambda_k < \lambda_0 \qquad and \qquad \lambda_k \uparrow \lambda_0$$

and the function w_{λ_k} possesses a negative minimum x_k in $\Sigma_{\lambda_k} \cap B(0,\ r_2)$. It follows that

$$w_{\lambda_k}(x_k) < 0, \qquad \nabla w_{\lambda_k}(x_k) = 0 \qquad and \qquad x_k \in \Sigma_{\lambda_k} \cap B(0,\ r_2).$$

Consequently the sequence $\{x_k\}$ is bounded, restricting a subsequence if necessary, we can assume it converges to $\bar{x} \in \overline{\Sigma_{\lambda_0} \cap B(0, r_2)}$. Moreover

$$w_{\lambda_0}(\bar{x}) \leq 0 \qquad and \qquad \nabla w_{\lambda_0}(\bar{x}) = 0.$$

Lemma 4.6.3. *If we assume that $a^-(\bar{x}) > 0$, then $u(\bar{x}) > 0$.*

Proof. Let us assume otherwise $u(\bar{x}) = 0$, since $a^-(\bar{x}) > 0$, we may assume that there exists $\sigma_2 > 0$ such that

$$\inf_{k \in N} \{a^-(x_k)\} \geq \sigma_2 \qquad since\ x_k \to \bar{x}$$

and

$$\sup_{k \in N} \{u(x_k)\} \leq A_2 \qquad since\ u(x_k) \to 0.$$

Now fix a particular k, then $w_{\lambda_k}(x) \geq 0$ on $\partial[\Sigma_{\lambda_k} \cap B(0, r_2)]$. But

$$\begin{aligned} 0 &\geq -\Delta w_{\lambda_k}(x_k) = -\Delta u_{\lambda_k}(x_k) - (-\Delta u(x_k)) \\ &= [a_{\lambda_k}(x_k) u^q_{\lambda_k}(x_k) + u^p_{\lambda_k}(x_k)] - [a(x_k) u^q(x_k) + u^p(x_k)] \\ &\geq [a(x_k) u^q_{\lambda_k}(x_k) + u^p_{\lambda_k}(x_k)] - [a(x_k) u^q(x_k) + u^p(x_k)] \qquad (since\ a_{\lambda_k}(x_k) \geq a(x_k)) \\ &> 0 \qquad (since\ A_2 \geq u(x_k) > u_{\lambda_k}(x_1) \geq 0\ and\ a^-(x_k) \geq \sigma_2.) \end{aligned}$$

This is a contradiction, this lemma is proved. □

Lemma 4.6.4. *If we assume that $a^-(\bar{x}) > 0$ and $\bar{x} \in \Sigma_{\lambda_0} \cap B(0, r_2)$, then $w_{\lambda_0}(x) \equiv 0$ for all $x \in B(\bar{x}, \epsilon)$, where ϵ is a small positive number.*

Proof. From above lemma we see that $u(\bar{x}) > 0$, then we can choose a small positive ϵ such that

$$B(\bar{x}, \epsilon) \subset\subset \Sigma_{\lambda_0} \cap B(0, r_2) \qquad and \qquad \inf_{x \in B(\bar{x}, \epsilon)} u(x) > 0.$$

Since $w_{\lambda_0}(x) \geq 0$ in $\Sigma_{\lambda_0} \cap B(0,\ r_2)$, then we see that $\inf_{x \in B(\bar{x},\ \epsilon)} u_{\lambda_0}(x) > 0$.
In $B(\bar{x},\ \epsilon)$, the function w_{λ_0} satisfies the following equation

$$\begin{aligned} &-\Delta w_{\lambda_0} + (a(x)u^q + u^p) - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) \\ &\qquad\qquad = -\Delta u_{\lambda_0} - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) + \Delta u + (a(x)u^q + u^p) = 0, \end{aligned}$$

but from $a_{\lambda_0} \geq a$, we have

$$\begin{aligned} &-\Delta w_{\lambda_0} + (a_{\lambda_0}(x)u^q + u^p) - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) \\ &\qquad\qquad \geq -\Delta w_{\lambda_0} + (a(x)u^q + u^p) - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) = 0. \end{aligned}$$

So we obtain

$$-\Delta w_{\lambda_0} + [a_{\lambda_0}(x)\frac{u^q - u^q_{\lambda_0}}{u_{\lambda_0} - u} + \frac{u^p - u^p_{\lambda_0}}{u_{\lambda_0} - u}]w_{\lambda_0} \geq 0 \ in\ B(\bar{x},\ \epsilon),$$

also notice that $c(x) = a_{\lambda_0}(x)\frac{u^q - u^q_{\lambda_0}}{u_{\lambda_0} - u} + \frac{u^p - u^p_{\lambda_0}}{u_{\lambda_0} - u}$ is uniformly bounded in $B(\bar{x},\ \epsilon)$ and $w_{\lambda_0} \geq 0$ in $B(\bar{x},\ \epsilon)$, therefore by the strong maximum principle $w_{\lambda_0}(x) \equiv 0$ for all $x \in B(\bar{x},\ \epsilon)$ since $w_{\lambda_0}(\bar{x}) = 0$. This lemma is proved. □

Lemma 4.6.5. *If we assume that $a^-(\bar{x}) > 0$ and $\bar{x} \in T_{\lambda_0} \cap B(0, r_2)$, then $w_{\lambda_0}(x) \equiv 0$ for all $x \in B(\bar{x},\ \epsilon)$, where ϵ is a small positive number.*

Proof. Since $a^-(\bar{x}) > 0$, it is clear that $u(\bar{x}) > 0$, then we can choose a small positive ϵ such that

$$B(\bar{x},\ \epsilon) \cap \Sigma_{\lambda_0} \subset \Sigma_{\lambda_0} \cap B(0,\ r_2) \qquad and \qquad \inf_{x \in B(\bar{x},\ \epsilon)} u(x) > 0.$$

Since we assume $\bar{x} \in T_{\lambda_0}$, then we see that $\inf_{x \in B(\bar{x},\ \epsilon)} u_{\lambda_0}(x) > 0$.
In $B(\bar{x},\ \epsilon) \cap \Sigma_{\lambda_0}$, the function w_{λ_0} satisfies the following equation

$$\begin{aligned} &-\Delta w_{\lambda_0} + (a(x)u^q + u^p) - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) \\ &\qquad\qquad = -\Delta u_{\lambda_0} - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) + \Delta u + (a(x)u^q + u^p) = 0, \end{aligned}$$

but from $a_{\lambda_0} \geq a$, we have

$$- \triangle w_{\lambda_0} + (a_{\lambda_0}(x)u^q + u^p) - (a_{\lambda_0}(x)u_{\lambda_0}^q + u_{\lambda_0}^p)$$
$$\geq -\triangle w_{\lambda_0} + (a(x)u^q + u^p) - (a_{\lambda_0}(x)u_{\lambda_0}^q + u_{\lambda_0}^p) = 0.$$

So we obtain

$$-\triangle w_{\lambda_0} + [a_{\lambda_0}(x)\frac{u^q - u_{\lambda_0}^q}{u_{\lambda_0} - u} + \frac{u^p - u_{\lambda_0}^p}{u_{\lambda_0} - u}]w_{\lambda_0} \geq 0 \; in \; B(\bar{x}, \; \epsilon),$$

also notice that $c(x) = a_{\lambda_0}(x)\frac{u^q - u_{\lambda_0}^q}{u_{\lambda_0} - u} + \frac{u^p - u_{\lambda_0}^p}{u_{\lambda_0} - u}$ is uniformly bounded in $B(\bar{x}, \epsilon) \cap \Sigma_{\lambda_0}$, $w_{\lambda_0} \geq 0$ in $B(\bar{x}, \epsilon) \cap \Sigma_{\lambda_0}$ and $B(\bar{x}, \epsilon) \cap \Sigma_{\lambda_0}$ satisfies the interior ball condition at $\bar{x}$, therefore if w_{λ_0} is not identically zero in $B(\bar{x}, \epsilon) \cap \Sigma_{\lambda_0}$, then by a refinement of Hopf's Lemma, $\frac{\partial w_{\lambda_0}(\bar{x})}{\partial n} < 0$ since $w_{\lambda_0}(\bar{x}) = 0$, but we already have $\nabla w_{\lambda_0}(\bar{x}) = 0$. So we must have $w_{\lambda_0}(x) \equiv 0$ for all $x \in B(\bar{x}, \epsilon)$. This lemma is proved. □

Lemma 4.6.6. *If $r_1 > 0$, then $\lambda_0 \leq r_1$.*

Proof. Assume otherwise $\lambda_0 > r_1$, then by the definition of r_1 we find out that $a^-(\bar{x}) > 0$, so from Lemma 4.6.3 we see that $u(\bar{x}) > 0$, which means that $\bar{x} \notin \partial B(0, r_2)$. From Lemma 4.6.4 and Lemma 4.6.5 we see either $\bar{x} \in \Sigma_{\lambda_0} \cap B(0, r_2)$ or $\bar{x} \in T_{\lambda_0}$ implies that $w_{\lambda_0}(x) \equiv 0$ in $B(\bar{x}, \epsilon)$ for some small positive ϵ. Since x_k converges to $\bar{x}$, then choose a particular large k such that $x_k \in B(\bar{x}, \epsilon)$ and $a(x_k) < 0$, then we have

$$a(x_k)u^q(x_k) + u^p(x_k) = -\triangle u(x_k) = -\triangle u_{\lambda_0}(x_k) = a_{\lambda_0}(x_k)u_{\lambda_0}^q(x_k) + u_{\lambda_0}^p(x_k).$$

Since $w_{\lambda_k}(x_k) = u_{\lambda_k}(x_k) - u(x_k) < 0$, then $u(x_k) > 0$. So from above we have $a(x_k) = a_{\lambda_0}(x_k)$, but we also have $a(x_k) < a_{\lambda_0}(x_k)$. It is a contradiction. So we should have $\lambda_0 \leq r_1$. The lemma is done. □

From above lemma, we have the following interesting theorem:

Theorem 4.6.7. *Under the assumption (4.6.1), if $r_1 > 0$, then any nonzero solution of (4.0.1) is positive in $\overline{B(0, r_1)}$.*

Proof. From the above lemma, we see that $\lambda_0 \leq r_1$, since $u(r_3e_1) > 0$ and $r_3 > r_1$, we obtain $u(r_1e_1) > 0$. Therefore by continuity and maximum principle, we have $u > 0$ in $\overline{B(0,\ r_1)}$. □

Lemma 4.6.8. *If* $r_1 = 0$, *then* $\lambda_0 = 0$.

Proof. Assume otherwise $\lambda_0 > 0$, then by the definition of r_1 we find out that $a^-(\bar{x}) > 0$, so from Lemma 4.6.3 we see that $u(\bar{x}) > 0$, which means that $\bar{x} \notin \partial B(0,\ r_2)$. From Lemma 4.6.4 and Lemma 4.6.5 we see either $\bar{x} \in \Sigma_{\lambda_0} \cap B(0,\ r_2)$ or $\bar{x} \in T_{\lambda_0}$ implies that $w_{\lambda_0}(x) \equiv 0$ in $B(\bar{x},\ \epsilon)$ for some small positive ϵ. Since x_k converges to $\bar{x}$, then choose a particular large k such that $x_k \in B(\bar{x},\ \epsilon)$, then we have

$$a(x_k)u^q(x_k) + u^p(x_k) = -\triangle u(x_k) = -\triangle u_{\lambda_0}(x_k) = a_{\lambda_0}(x_k)u^q_{\lambda_0}(x_k) + u^p_{\lambda_0}(x_k).$$

Since $w_{\lambda_k}(x_k) = u_{\lambda_k}(x_k) - u(x_k) < 0$, then $u(x_k) > 0$. So from above we have $a(x_k) = a_{\lambda_0}(x_k)$, but now $a(r)$ is strictly decreasing. It is a contradiction. So we should have $\lambda_0 = 0$. The lemma is done. □

Now we are in position to prove Theorem 4.6.1.

Proof. The above lemma already takes care of the case $r_1 = 0$, so we assume $r_1 > 0$, then from above Theorem 4.6.7 there exists $\delta_3 > 0$ such that $u(x) > 0$ for all $x \in \overline{B(0,\ r_1 + \delta_3)}$. From lemma 4.6.3 we see that $u(\bar{x}) > 0$ if $\bar{x} \in \partial B(0,\ r_2)$, but $u(\partial B(0,\ r_2)) = 0$, which means that $\bar{x} \notin \partial B(0,\ r_2)$

Now if $a(\bar{x}) < 0$, we can derive a contradiction like we did in Lemma 4.6.6.

If $a(\bar{x}) \geq 0$, then $u(\bar{x}) > 0$ and $\bar{x} \in B(0,\ r_1 + \delta_3) \cap \Sigma_{\lambda_0}$ or ∂T_{λ_0}. Therefore we can apply the same proof of Lemma 4.6.4 and Lemma 4.6.5 in $B(0,\ r_1 + \delta_3) \cap \Sigma_{\lambda_0}$, then conclude that

$$u(x) \equiv 0 \ for\ all\ x \in B(0,\ r_1 + \delta_3) \cap \Sigma_{\lambda_0}.$$

Hence we can pick a point $x^* \in B(0,\ r_1+\delta_3) \cap \Sigma_{\lambda_0}$ in the e_1 axis such that

$$a(x^*) < 0 \qquad and \qquad u(x^*) = u_{\lambda_0}(x^*) > 0,$$

then we have $a_{\lambda_0}(x^*) > a(x^*)$, but we also have

$$a(x^*)u^q(x^*) + u^p(x^*) = -\Delta u(x^*) = -\Delta u_{\lambda_0}(x^*) = a_{\lambda_0}(x^*)u^q_{\lambda_0}(x^*) + u^p_{\lambda_0}(x^*),$$

which implies $a_{\lambda_0}(x^*) = a(x^*)$. So it is a contraction. We have $\lambda_0 = 0$, this theorem is proved.

□

Here we also have a supplement for Theorem 4.6.1:

Theorem 4.6.9. *Assume $a(r)$ is decreasing in r, then any nonzero radial solution u of (4.0.1) decreases as r increases.*

We still apply the moving plane method and the same notations from above, since u is already radial, we only need to move in e_1 direction. We complete the proof in a few lemmas.

Lemma 4.6.10. *$\lambda_0 < r_2$ and $u(x) > 0$ in $\Sigma_{\lambda_0} \cap B(0,\ r_2)$.*

Proof. $\lambda_0 < r_2$ is simply from Lemma 4.6.2, then $u(r)$ is decreasing in $[\lambda_0,\ r_2]$, by the definition of r_2 we get

$$u(r) > 0 \ in \ [\lambda_0,\ r_2),$$

which implies that $u(x) > 0$ in $\Sigma_{\lambda_0} \cap B(0,\ r_2)$. □

Lemma 4.6.11. *w_{λ_0} is not identically zero on $(\partial B(0,\ r_2)) \cap \Sigma_{\lambda_0}$.*

Proof. From above lemma $u(r) > 0$ in $[\lambda_0,\ r_2)$, since $u(r_2) = 0$, then w_{λ_0} is not identically zero on $(\partial B(0,\ r_2)) \cap \Sigma_{\lambda_0}$. □

Lemma 4.6.12. *If there exists* $\bar{y} \in \Sigma_{\lambda_0} \cap B(0,\ r_2)$ *such that* $w_{\lambda_0}(\bar{y}) = 0$, *then* $w_{\lambda_0} \equiv 0$ *in* $\Sigma_{\lambda_0} \cap B(0,\ r_2)$.

Proof. Let us consider the set $S = \{x \in \Sigma_{\lambda_0} \cap B(0,\ r_2) \mid\ w_{\lambda_0}(x) = 0\}$, by assumption S is not empty.

Claim: S is open in $\Sigma_{\lambda_0} \cap B(0,\ r_2)$.

Now take any $y \in S$, then from Lemma 4.6.10 $u(y) > 0$. So we can choose a small positive ϵ such that

$$B(y,\ \epsilon) \subset\subset \Sigma_{\lambda_0} \cap B(0,\ r_2) \qquad and \qquad \inf_{x \in B(\bar{x},\ \epsilon)} u(x) > 0.$$

Since $w_{\lambda_0}(x) \geq 0$ in $\Sigma_{\lambda_0} \cap B(0,\ r_2)$, then we see that $\inf_{x \in B(y,\ \epsilon)} u_{\lambda_0}(x) > 0$.
In $B(y,\ \epsilon)$, the function w_{λ_0} satisfies the following equation

$$-\triangle w_{\lambda_0} + (a(x)u^q + u^p) - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) = -\triangle u_{\lambda_0} - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) + \triangle u + (a(x)u^q + u^p) = 0,$$

but from $a_{\lambda_0} \geq a$, we have

$$-\triangle w_{\lambda_0} + (a_{\lambda_0}(x)u^q + u^p) - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) \geq -\triangle w_{\lambda_0} + (a(x)u^q + u^p) - (a_{\lambda_0}(x)u^q_{\lambda_0} + u^p_{\lambda_0}) = 0.$$

So we obtain

$$-\triangle w_{\lambda_0} + [a_{\lambda_0}(x)\frac{u^q - u^q_{\lambda_0}}{u_{\lambda_0} - u} + \frac{u^p - u^p_{\lambda_0}}{u_{\lambda_0} - u}]w_{\lambda_0} \geq 0 \ in\ B(\bar{x},\ \epsilon),$$

also notice that $c(x) = a_{\lambda_0}(x)\frac{u^q - u^q_{\lambda_0}}{u_{\lambda_0} - u} + \frac{u^p - u^p_{\lambda_0}}{u_{\lambda_0} - u}$ is uniformly bounded in $B(y,\ \epsilon)$ and $w_{\lambda_0} \geq 0$ in $B(y,\ \epsilon)$, therefore by the strong maximum principle $w_{\lambda_0}(x) \equiv 0$ for all $x \in B(y,\ \epsilon)$ since $w_{\lambda_0}(y) = 0$. So S is open. By continuity S is also close in $\Sigma_{\lambda_0} \cap B(0,\ r_2)$, so $w_{\lambda_0} \equiv 0$ in $\Sigma_{\lambda_0} \cap B(0,\ r_2)$. This lemma is proved. □

Now if $\lambda_0 = 0$, then theorem is proved. But if $\lambda_0 > 0$, just exactly like above we have the same x_k, λ_k and $\bar{x}$.

Lemma 4.6.13. *If we assume that* $\bar{x} \in T_{\lambda_0} \cap B(0,\ r_2)$*, then* $w_{\lambda_0} \equiv 0$ *in* $\Sigma_{\lambda_0} \cap B(0,\ r_2)$*.*

Proof. From Lemma 4.6.10 $u(\bar{x}) > 0$, according to Lemma 4.6.5 there exists a small positive ϵ such that

$$w_{\lambda_0}(x) \equiv 0 \ for\ x \in B(\bar{x},\ \epsilon),$$

then appealing to Lemm 4.6.12 we have $w_{\lambda_0} \equiv 0$ in $\Sigma_{\lambda_0} \cap B(0,\ r_2)$. □

We are ready to give the proof of this Theorem 4.6.9.

Proof. If $\bar{x} \in \partial B(0,\ r_2)$, then $a(\bar{x}) < 0$. by Lemma 4.6.3 we have $u(\bar{x}) > 0$, contradicting the fact that $u(r_2) = 0$, so we must have either $\bar{x} \in \Sigma_{\lambda_0} \cap B(0,\ r_2)$ or $\bar{x} \in T_{\lambda_0} \cap B(0,\ r_2)$, but from Lemma 4.6.12 and lemma 4.6.13 we have

$$w_{\lambda_0} \equiv 0 \ in\ \Sigma_{\lambda_0} \cap B(0,\ r_2),$$

which contradicts Lemma 4.6.11 by continuity. So $\lambda_0 = 0$, the theorem is proved. □

4.7 Second Solution in the Radial Case

In this section we give an independent proof for the existence of a second solution in S_M in the radial case. The symmetry allows us to simplify many steps in the procedure, and we no longer require the assumption (4.5.5) on $a(r)$, because of the uniform decay of radial H^1 functions, as observed by Strauss [40]. Recall that U_γ represents the minimum element in $S_{M,\gamma}$ for $0 < \gamma \leq \Gamma$ with $\Gamma = \Gamma_M$. Consider the following minimization problem in a convex constraint set

$$\operatorname{Inf}\{I_\gamma(v) \mid v \in Y\} \qquad and \qquad Y = \{v \in H_q^1 \mid 0 \leq v \leq U_\Gamma \ a.e.\}. \tag{4.7.1}$$

From Struwe [41] the infimum is attained at some function in Y, say v_γ, and $v_\gamma \in S_{M,\gamma}$. Notice if we replace U_Γ with $U_{\bar{\gamma}}$ for some $\bar{\gamma} > \gamma$, although we still can guarantee the existence of v_γ, this v_γ may be different.

Lemma 4.7.1. *For $\gamma \in (0, \Gamma]$, assume $a(x) = a(|x|)$ is radially decreasing, then v_γ could be chosen so that $v_\gamma(x) = v_\gamma(|x|)$ and it is also radially decreasing.*

Proof. Since $a(|x|)$ is radially decreasing and U_Γ is radial, then by Theorem 4.6.9 U_Γ is also radially decreasing. Hence take the Schwartz Symmetrization of v_γ, denoted by v_γ^*, it is easy to see that $v_\gamma^* \in Y$.

Claim: $\int_{\mathbb{R}^n} a_\gamma (v_\gamma^*)^{q+1} \geq \int_{\mathbb{R}^n} a_\gamma (v_\gamma)^{q+1}$.

Indeed, pick a huge constant C such that $C + a_\gamma > 0$ in R^n, thus we see that $(C + a_\gamma)^* = C + a_\gamma$. So we have, notice that v_γ has compact support,

$$\int_{\mathbb{R}^n} (C + a_\gamma)^* (v_\gamma^*)^{q+1} \geq \int_{\mathbb{R}^n} (C + a_\gamma) v_\gamma^{q+1},$$

that is $\int_{\mathbb{R}^n} a_\gamma (v_\gamma^*)^{q+1} \geq \int_{\mathbb{R}^n} a_\gamma (v_\gamma)^{q+1}$ since $\int_{\mathbb{R}^n} (v_\gamma^*)^{q+1} = \int_{\mathbb{R}^n} v_\gamma^{q+1}$.
But we also know that

$$\int_{\mathbb{R}^n} |\nabla v_\gamma^*|^2 dx \leq \int_{\mathbb{R}^n} |\nabla v_\gamma|^2 dx \qquad and \qquad \int_{\mathbb{R}^n} (v_\gamma^*)^{p+1} dx = \int_{\mathbb{R}^n} v_\gamma^{p+1} dx,$$

therefore we have

$$\begin{aligned} I_\gamma(v_\gamma^*) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v_\gamma^*|^2 dx - \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma (v_\gamma^*)^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n} (v_\gamma^*)^{p+1} dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v_\gamma|^2 dx - \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma (v_\gamma)^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n} (v_\gamma)^{p+1} dx = I_\gamma(v_\gamma). \end{aligned}$$

So we can choose v_γ^* as v_γ. The radially decreasing property of v_γ is from Theorem 4.6.9.

□

Remark 4.7.2. *Here above we only assume that $a(x)$ is radially non-increasing, not strictly decreasing, otherwise the lemma would be trivial due to theorem 4.6.1.*

Now we introduce some notation. Choose a ball B centered at the origin such that $\text{supp}(U_\Gamma) \subset\subset B$, since $\Omega^{0+} \subset\subset \text{supp}(U_\Gamma)$, then we have $\Omega^{0+} \subset\subset B$. Denote H_r as the subspace of H_q^1, which consists of radial symmetric functions with the same norm.

Lemma 4.7.3. *For $\gamma \in (0, \Gamma)$, then v_γ is a local minimizer for I_γ in $H^1(B)$; that is, there exists $\delta > 0$ such that*

$$I_\gamma(v_\gamma) \le I_\gamma(v) \text{ for all } v \in H^1(B) \text{ with } ||v - v_\gamma||_{H^1(B)} < \delta.$$

Proof. From Lemma 4.3.1 we have

$$I_\gamma(v_\gamma) = \inf\{I_\gamma(v) \mid v \in Y\}.$$

Since $\text{supp}(U_\Gamma) \subset\subset B$, we also get

$$I_\gamma(v_\gamma) = \inf\{I_\gamma(v) \mid v \in H^1(B) \text{ and } 0 \le v \le U_\Gamma\}.$$

Then the result follows from Proposition 5.2 in [2]. □

Lemma 4.7.4. *For $\gamma \in (0, \Gamma)$, assume v_γ is radial, then v_γ is also a local minimizer for I_γ in H_r.*

Proof. From above lemma there exists $\delta >$ such that

$$I_\gamma(v_\gamma) \leq I_\gamma(v) \ for\ all\ v \in H^1(B)\ with\ ||v - v_\gamma||_{H^1(B)} < \delta.$$

Since $H_r \hookrightarrow H^1(B)$, there exists $\delta_1 > 0$ such that

$$||v - v_\gamma||_{H^1(B)} < \delta\ if\ ||v - v_\gamma||_{H_r} < \delta_1.$$

Now take any function $v \in C_0^\infty(\mathbb{R}^n)$ such that $v \in H_r$ and $||v - v_\gamma||_{H_r} < \delta_1$, then we have

$$\begin{aligned}
&I_\gamma(v)\\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx - \frac{\gamma}{q+1}\int_{\mathbb{R}^n} a^+(v^+)^{q+1}dx + \frac{1}{q+1}\int_{\mathbb{R}^n} a^-(v^+)^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n}(v^+)^{p+1}dx\\
&= \frac{1}{2}\int_B |\nabla v|^2 dx - \frac{\gamma}{q+1}\int_B a^+(v^+)^{q+1}dx + \frac{1}{q+1}\int_B a^-(v^+)^{q+1}dx - \frac{1}{p+1}\int_B (v^+)^{p+1}dx\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n - B}|\nabla v|^2 dx + \frac{1}{q+1}\int_{\mathbb{R}^n - B} a^-(v^+)^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n - B}(v^+)^{p+1}dx\\
&\geq I_\gamma(v_\gamma) + \frac{1}{2}\int_{\mathbb{R}^n - B}|\nabla v|^2 dx + \frac{1}{q+1}\int_{\mathbb{R}^n - B} a^-(v^+)^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n - B}(v^+)^{p+1}dx.
\end{aligned}$$

Since $\Omega^{0+} \subset\subset B$, then denote $\inf_{x \in R^n - B} a^-$ by c, hence

$$I_\gamma(v) \geq I_\gamma(v_\gamma) + \frac{1}{2}\int_{\mathbb{R}^n - B}|\nabla v|^2 dx + \frac{c}{q+1}\int_{\mathbb{R}^n - B}(v^+)^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n - B}(v^+)^{p+1}dx.$$

Let

$$V = \begin{cases} v(\partial B) & x \in B \\ v & x \in \mathbb{R}^n - B, \end{cases}$$

then $V \in H_r$, so we have

$$I_\gamma(v) \geq I_\gamma(v_\gamma) + \frac{1}{2}\int_{\mathbb{R}^n}|\nabla V|^2 dx + \frac{c}{q+1}\int_{\mathbb{R}^n - B}(V^+)^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n - B}(V^+)^{p+1}dx.$$

that is

$$I_\gamma(v) - I_\gamma(v_\gamma) \geq \frac{1}{2}\int_{\mathbb{R}^n}|\nabla V|^2 dx + \frac{c}{q+1}\int_{\mathbb{R}^n - B}(V^+)^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^n - B}(V^+)^{p+1}dx.$$

We claim that:

$$E(V) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla V|^2 dx + \frac{c}{q+1}\int_{\mathbb{R}^n - B} (V^+)^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n - B} (V^+)^{p+1} dx \geq 0$$

when δ_1 is small enough.

Indeed, By using Hölder inequality, denote $d = \frac{n+2-p(n-2)}{n+2-q(n-2)}$, we have

$$\int_{\mathbb{R}^n - B} |V^+|^{p+1} dx \leq ||V^+||^{d(q+1)}_{L^{q+1}(\mathbb{R}^n - B)} ||V^+||^{2^*(1-d)}_{L^{2^*}(\mathbb{R}^n - B)}. \tag{4.7.2}$$

Since $d + (1-d)\frac{n}{n-2} > 1$, there exists $\alpha > 1$ and $\beta > 1$ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad \bar{\alpha} = d\alpha(q+1) > q+1 \quad and \quad \bar{\beta} = \beta(1-d)2^* > 2.$$

Hence from (4.7.2) and the Young inequality, we get

$$\int_{\mathbb{R}^n - B} |V^+|^{p+1} dx \leq \frac{1}{\alpha} ||V^+||^{\bar{\alpha}}_{L^{q+1}(\mathbb{R}^n - B)} + \frac{1}{\beta} ||V^+||^{\bar{\beta}}_{L^{2^*}(\mathbb{R}^n - B)}.$$

By above and Sobolev inequality, we find

$$\begin{aligned} E(v) &\geq \frac{C(n)}{2} ||V^+||^2_{L^{2^*}(\mathbb{R}^n)} + \frac{c}{q+1} ||V^+||^{q+1}_{L^{q+1}(\mathbb{R}^n - B)} - \frac{1}{\alpha(p+1)} ||V^+||^{\bar{\alpha}}_{L^{q+1}(\mathbb{R}^n - B)} \\ &\quad - \frac{1}{\beta(p+1)} ||V^+||_{L^{2^*}(\mathbb{R}^n - B)} \\ &\geq \frac{C(n)}{2} ||V^+||^2_{L^{2^*}(\mathbb{R}^n - B)} + \frac{c}{q+1} ||V^+||^{q+1}_{L^{q+1}(\mathbb{R}^n - B)} - \frac{1}{\alpha(p+1)} ||V^+||^{\bar{\alpha}}_{L^{q+1}(\mathbb{R}^n - B)} \\ &\quad - \frac{1}{\beta(p+1)} ||V^+||_{L^{2^*}(\mathbb{R}^n - B)}. \end{aligned}$$

Since $\bar{\alpha} > q+1$ and $\bar{\beta} > 2$, then for δ_1 small enough we get

$$\begin{aligned} E(v) &\geq \frac{C(n)}{2} ||V^+||^2_{L^{2^*}(\mathbb{R}^n - B)} + \frac{c}{q+1} ||V^+||^{q+1}_{L^{q+1}(\mathbb{R}^n - B)} - \frac{1}{\alpha(p+1)} ||V^+||^{\bar{\alpha}}_{L^{q+1}(\mathbb{R}^n - B)} \\ &\quad - \frac{1}{\beta(p+1)} ||V^+||_{L^{2^*}(\mathbb{R}^n - B)} \geq 0. \end{aligned}$$

There for we have $I_\gamma(v) - I_\gamma(v_\gamma) \geq 0$ for δ_1 small enough, that is, v_γ is a local minimizer in H_r. This lemma is done. □

Now if v_γ is radial, then from above lemma we see that v_γ is a local minimum for the energy functional

$$I_\gamma = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx - \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma (v^+)^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^n} (v^+)^{p+1} dx \quad v \in H_r,$$

which is associated with the equation $(4.2.1)_\gamma$

$$-\Delta u = (\gamma a^+ - a^-)u^q + u^p.$$

It is easy to see that $I_\gamma(t\varphi) \to -\infty$ as $t \to \infty$ for some positive radially symmetric $\varphi \in C_0^\infty(R^n)$. So we have a Mountain-Pass structure. We expect to find a second solution in the form $u = v_\gamma + v$ with $v \geq 0$.

If v_γ solves the equation, then v should solve

$$-\Delta v = a_\gamma[(v_\gamma + v)^q - v_\gamma^q] + [(v_\gamma + v)^p - v_\gamma^p].$$

Set

$$\begin{aligned} h(x,v) &= a_\gamma[(v_\gamma + v^+)^q - v_\gamma^q] + [(v_\gamma + v^+)^p - v_\gamma^p], \\ H(x,v) &= \int_0^v h(x,s)ds = \int_0^v [(v_\gamma + s^+)^q - v_\gamma^q] + [(v_\gamma + s^+)^p - v_\gamma^p] ds \\ &= \frac{1}{q+1} a_\gamma [(v_\gamma + v^+)^{q+1} - v_\gamma^{q+1}] - a_\gamma v_\gamma^q v^+ + \frac{1}{p+1}[(v_\gamma + v^+)^{p+1} - v_\gamma^{p+1}] - v_\gamma^p v^+. \end{aligned}$$

For $v \in H_r$, define the functional

$$J_\gamma(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx + \frac{1}{q+1} v^{q+1} - \frac{1}{q+1}(v^+)^{q+1} - H(x,v)dx.$$

By the same calculation we reach

$$J_\gamma(v) = I_\gamma(v_\gamma + v^+) - I_\gamma(v_\gamma) + \frac{1}{2}||\nabla v^-||_{L^2(\mathbb{R}^n)}^2 + \frac{1}{q+1}||v^-||_{L^{q+1}(\mathbb{R}^n)}^{q+1}.$$

Lemma 4.7.5. *Assume v_γ is radial, then there exists $\delta_1 > 0$ such that $J_\gamma(v) \geq J_\gamma(0) = 0$ with $||v||_{H_r} < \delta_1$.*

Proof. From above calculation we have

$$J_\gamma(v) = I_\gamma(v_\gamma + v^+) - I_\gamma(v_\gamma) + \frac{1}{2}||\nabla v^-||^2_{L^2(\mathbb{R}^n)} + \frac{1}{q+1}||v^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)},$$

in view of Lemma 4.7.4 we see this lemma is true. □

Lemma 4.7.6. *Assume $\gamma \geq 0$, for any radially symmetric $\varphi \in C_0^\infty(R^n)$ with $\varphi \gneqq 0$, there exists a constant $T > 0$ such that $J_\gamma(T\varphi) < 0$.*

Proof. By direct calculation. □

The most important fact about the radial case is the following lemma, due to Strauss [40] (see also Berestycki and Lions [10]): Next two lemmas deal with the compactness of $P - S$ sequence.

Lemma 4.7.7 (Strauss [40]). *H_r compactly embeds in $L^{p+1}(\mathbb{R}^n)$ for $1 < p < 2^* - 1$.*

With this lemma, we can now prove the Palais–Smale condition holds in the radial case.

Lemma 4.7.8. *Suppose $\gamma \geq 0$, $\{v_n\}$ is a sequence in H_r such that $J_\gamma(v_n) \to c$ and $J'_\gamma(v_n) \to 0$, then $\{v_n\}$ contains a strongly convergent subsequence in H_r. Moreover if $v_n \to v_0 \geq 0$, then $u_0 = v_\gamma + v_0$ is a solution to* (4.2.1).

Proof. In view of Lemma 4.5.3, we could take $u_n = v_n^+ + v_\gamma$, then we should have

$$I_\gamma(u_n) \to I_\gamma(v_\gamma) + c_\gamma \qquad \textit{and} \qquad I'_\gamma(u_n) \to 0.$$

Again from Lemma 4.5.3, we have that $||\nabla u_n||_{L^2} + ||u_n||_{L^{q+1}} + ||u_n||_{L^{p+1}}$ is uniformly bounded, then restricting to a subsequence if necessary, there exist $u_0 \in H_r$ such that

$$u_n \rightharpoonup u_0 \textit{ weakly in } H_r.$$

By Lemma 4.7.7 we also have

$$u_n \to u_0 \; strongly \; in \; L^{p+1}(\mathbb{R}^n).$$

By the weak convergence we can easily see that u_0 is a solution to the equation $(4.2.1)_\gamma$, then u_0 has compact support and $I'_\gamma(u_0) = 0$. So we get

$$(I'_\gamma(u_n) - I'_\gamma(u_0))(u_n - u_0) = \int_{\mathbb{R}^n} |\nabla(u_n - u_0)|^2 dx + \int_{\mathbb{R}^n} a^-(u_n^q - u_0^q)(u_n - u_0)dx$$
$$- \gamma \int_{\mathbb{R}^n} a^+(u_n^q - u_0^q)(u_n - u_0)dx - \int_{\mathbb{R}^n} (u_n^p - u_0^p)(u_n - u_0)dx \to 0 \quad (4.7.3)$$

Since $u_n \to u_0$ strongly in $L^{p+1}(\mathbb{R}^n)$, then we have

$$\int_{\mathbb{R}^n} (u_n^p - u_0^p)(u_n - u_0)dx \to 0 \quad and \quad \int_{\mathbb{R}^n} a^+(u_n^q - u_0^q)(u_n - u_0)dx \to 0.$$

Therefore (4.7.3) reduces to

$$\int_{\mathbb{R}^n} |\nabla(u_n - u_0)|^2 dx + \int_{\mathbb{R}^n} a^-(u_n^q - u_0^q)(u_n - u_0)dx \to 0,$$

which implies $u_n \to u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. The final task is to show $u_n \to u_0$ in $L^{q+1}(\mathbb{R}^n)$.

Claim: $u_n \to u_0$ strongly in $L^{q+1}(\mathbb{R}^n)$.

Indeed, we know that u_0 is a solution to the equation $(4.2.1)_\gamma$ and has compact support, then take a ball B centerred at the origin such that $\Omega^{0+} \subset\subset supp(u_0) \subset\subset B$, since $\int_{\mathbb{R}^n} a^-(u_n^q - u_0^q)(u_n - u_0)dx \to 0$, we have

$$\int_{\mathbb{R}^n - B} a^-(u_n^q - u_0^q)(u_n - u_0)dx \to 0,$$

that is

$$\int_{\mathbb{R}^n - B} a^- u_n^{q+1} dx \to 0.$$

By assumption (4.0.2) we get $\int_{\mathbb{R}^n - B} u_n^{q+1} dx \to 0$, which implies

$$u_n \to u_0 \; strongly \; in \; L^{q+1}(\mathbb{R}^n),$$

since $u_n \to u_0$ strongly in $L^{p+1}(\mathbb{R}^n)$. Therefore

$$u_n \to u_0 \textit{ strongly in } H_r.$$

This lemma is done. □

Remark 4.7.9. *It is worth pointing out that this lemma includes the case $\gamma = 0$.*

Theorem 4.7.10. *Suppose there exists $\eta_\gamma > 0$ such that for any $\rho \in [0, \eta_\gamma]$*

$$\inf\{J_\gamma(v) \mid \ ||v||_{H_r} = \rho\} = 0,$$

and $c_\gamma = 0$, which is defined in the next theorem, then for each $\rho \in (0, \eta_\gamma)$, the problem $(4.2.1)_\gamma$ has a solution with $||u - v_\gamma||_{H_r} = \rho$.

Proof. For any fixed $\rho \in (0, \eta_\gamma)$, the set $F = \partial B(0, \rho)$ in H_r satisfies the hypothesis of theorem (1) in Ghoussoub and Preiss [25], their Theorem (1.bis) asserts the existence of a solution for each $\rho \in (0, \eta_\gamma)$ with the help of above lemma. □

Remark 4.7.11. *In their concave plus convex example, Ambrosetti, Brezis, and Cerami [5] prove uniqueness of a "small" solution for their problem. In order to prove the same kind of result for $(4.2.1)_\gamma$, the possibility of the type of degeneracy in Theorem 4.7.10 must be eliminated.*

Theorem 4.7.12. *Assume $a(x) = a(|x|)$. then for any $\gamma \in (0, \Gamma)$, $S_{M,\gamma}$ has at least two elements.*

Proof. Now for fixed γ, we have U_γ and v_γ in $S_{M,\gamma}$, moreover U_γ is radial. If they are different, then this theorem is done; if they are the same, then v_γ is radial, so consider the following set

$$S_\gamma = \{\sigma \in C([0, 1], H_r) \mid \sigma(0) = 0 \textit{ and } \sigma(1) = T\varphi\}.$$

where $\varphi \in C_0^\infty \cap H_r$, $\varphi \gneq 0$ and $J_\gamma(T\varphi) < 0$, this is possible because of Lemma 4.7.6. Let $c_\gamma = \inf_{\sigma \in S_\gamma} \max_{s \in [0,\ 1]} J_\gamma(\sigma(s))$, then from Lemma 4.7.5, we see that $J_\gamma(v) \geq 0$ with $||v||_{H_r} < \delta_1$. Therefore $c_\gamma \geq 0$.

If there exists some $\rho < \delta_1$ such that $\inf\{J_\gamma(v) \mid ||v||_{H_r} = \rho\} > 0$, then we have $c_\gamma > 0$. By the Mountain-Pass Theorem of Ambrosetti and Rabinowitz, there exists a solution V_γ of $(4.2.1)_\gamma$ with $J_\gamma(V_\gamma) > 0$, i.e. $I_\gamma(V_\gamma) > I_\gamma(v_\gamma)$, which implies that V_γ is different from v_γ.

If not, but $c_\gamma > 0$, we still have the same result like above.

If not and $c_\gamma = 0$, then for all $\rho \in [0,\ \delta_1)$, we have $\inf\{J_\gamma(v) \mid ||v||_{H_r} = \rho\} = 0$, then from Theorem 4.7.10 we see that there are infinite many solutions of $(4.2.1_\gamma$. This theorem is proved.

□

With more assumptions on $a(|x|)$ we can distinguish the two solutions.

Corollary 4.7.13. *Assume $a(x)$ is radial and decreasing, then $S_{M,\gamma}$ has at least two radially decreasing solutions.*

Proof. Since $a(x)$ is radial and decreasing, then from Lemma 4.7.1 we can choose v_γ to be radial and decreasing, so we can find a second solution V_γ, which is radial. From Theorem 4.6.9, V_γ is also decreasing. □

Corollary 4.7.14. *Assume $a(x)$ is radial and decreasing, in addition it is smooth, then $S_{M,\gamma}$ has at least two radially decreasing elements, which are different in Ω^{0+}.*

Proof. From above corollary we have two different radially decreasing elements, v_γ and V_γ, so at least one of them, say V_γ, does not coinside with U_γ.

Claim: $V_\gamma(0) > U_\gamma(0)$.

Indeed, since U_γ is minimum element, then $V_\gamma(0) \geq U_\gamma(0)$. Now if $U_\gamma(0) = V_\gamma(0) = \alpha > 0$, then for $\epsilon < \alpha$, the following initial value problem has at most one solution

$$(r^{n-1}u_r(r))_r + r^{n-1}(a_\gamma(r)u^q(r) + u^p(r)) = 0 \quad u(0) = \alpha, \quad u'(0) = 0 \quad \textit{and} \quad u \geq \epsilon.$$

For proof, see proposition 2.35 in [32]. Also we notice that for ϵ sufficiently small, the set $S_\epsilon = \{x \in \mathbb{R}^n \mid 0 < V_\gamma(x) < \epsilon\} \subset\subset \Omega^-$ by Lemma 4.1.7. Moreover if ϵ shrinks, the set S_ϵ shrinks since V_γ is radially decreasing. Therefore just like the proof of Lemma 4.2.8, we see that the following equation has at most one solution

$$-\Delta v = a^- v^q + v^p \ \textit{in} \ S_\epsilon \qquad v = V_\gamma \ \textit{on} \ \partial S_\epsilon.$$

So we should have $U_\gamma \equiv V_\gamma$ in $\mathbb{R}^n$. It is a contradiction. Hence this theorem is done. □

In the end of this section we show the existence of a solution for $(4.2.1)_\gamma$ with $\gamma = 0$.

Theorem 4.7.15. *Assume $a(x) = a(|x|)$. then $(4.2.1)_\gamma$ with $\gamma = 0$ has a radial solution.*

The proof is very similar to the case $\gamma > 0$, so we briefly present the proof in a few lemma.

Consider the following minimization problem in a convex constraint set

$$\text{Inf}\ \{I_0(v) \mid v \in Y_\gamma\} \qquad \textit{and} \qquad Y_\gamma = \{v \in H^1_q \mid 0 \leq v \leq U_\gamma \ \textit{a.e.}\}. \tag{4.7.4}$$

Lemma 4.7.16. *For γ sufficiently small. (4.7.4) attains infimum at $v = 0$.*

Proof. It is easy to see that for any $v \in Y_\gamma$, we have

$$I_0 \geq C_1||v||^2_{L^{2^*}(\mathbb{R}^n)} - C_2||v||^{p+1}_{L^{2^*}(\mathbb{R}^n)},$$

since U_γ has compact support. But $\lim_{\gamma\to 0^+} ||U_\gamma||_{L^\infty(\mathbb{R}^n)} = 0$, then $I_0(v) \geq 0$ for any $v \in Y_\gamma$ and γ small. In particular the infimum is achieved at $v = 0$ for γ sufficiently small. □

Now fix a small $\gamma > 0$ such that above infimum reaches at $v = 0$, choose a ball B_γ centered at the origin such that $\text{supp}(U_\gamma) \subset\subset B_\gamma$, since $\Omega^{0+} \subset\subset \text{supp}(U_\gamma)$, then we have $\Omega^{0+} \subset\subset B_\gamma$. We have two lemma similar to Lemma 4.7.3 and Lemma 4.7.4.

Lemma 4.7.17. *$v = 0$ is a local minimizer for I_0 in $H^1(B_\gamma)$; that is, there exists $\delta > 0$ such that*

$$I_0(v) \geq 0 \textit{ for all } v \in H^1(B_\gamma) \textit{ with } ||v - ||_{H^1(B_\gamma)} < \delta.$$

Proof. Since $\text{supp}(U_\gamma) \subset\subset B_\gamma$, we also get

$$I_0(0) = \inf\{I_0(v) \mid v \in H^1(B_\gamma) \textit{ and } 0 \leq v < U_\gamma\}.$$

Then the result follows from Proposition 5.2 in [2]. □

Lemma 4.7.18. *$v = 0$ is also a local minimizer for I_0 in H_r.*

The proof is exactly the same as in Lemma 4.7.4.

Next we see that

$$J_0(v) = I_0(v^+) + \frac{1}{2}||\nabla v^-||^2_{L^2(\mathbb{R}^n)} + \frac{1}{q+1}||v^-||^{q+1}_{L^{q+1}(\mathbb{R}^n)}.$$

The following steps is the same as $\gamma > 0$, we just simply replace v_γ with 0, then above lemma and Lemma 4.7.6 assure the Mountain-pass structure, Lemma 4.7.8 gives the compactness of P-S sequence and Lemma 4.7.10 takes care of special case.

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