HOMOTOPY SELF-EQUIVALENCES OF FOUR-MANIFOLDS

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by

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TO MY GRANDMOTHER

Abstract

In this thesis, we study the group of base-point preserving homotopy classes of homotopy self-equivalences of a four-manifold. Based on the approach of Hambleton and Kreck, an explicit description of this group is obtained when the fundamental group of the manifold is either a free group or a two-dimensional Poincaré duality group. As a byproduct, a classification of such four-manifolds up to s-cobordism is obtained by using the modified surgery theory of Kreck.

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Contents

Introduction

Let X be a connected pointed topological space which has the homotopy type of a CW complex. For such a space let $\mathrm{Aut}_\bullet(X)$ denote the group of base-point preserving homotopy classes of homotopy self-equivalences of X with the multiplication induced from the composition of maps. This group is a geometric version of the group of automorphisms of a group, and these concepts coincide when X is an Eilenberg-MacLane space, i.e. $\mathrm{Aut}_\bullet(K(G,n)) = \mathrm{Aut}\, G$. One can think of $\mathrm{Aut}_\bullet(X)$ as the analogue in the homotopy category of the homeomorphism group, $\mathrm{Homeo}_\bullet(X)$ of a topological space, or in the smooth category as the analogue of the diffeomorphism group, $\mathrm{Diffeo}_\bullet(X)$ of a smooth manifold. The group $\mathrm{Aut}_\bullet(X)$ also plays an important role in the homotopy type classification problem, since spaces of the same homotopy type have isomorphic self-homotopy equivalence groups.

Let $\mathcal{T}_\bullet$ denote the category whose objects are topological spaces with base point and whose morphisms are based homotopy classes of based maps. If $h\colon X \to Y$ and $\varphi \in \mathrm{Aut}_\bullet(X)$, then we have a sequence of obstructions in $H^i(Y, X; \pi_i(Y))$ to the existence of a map $\psi\colon Y \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "h"] & X \arrow[r, "h"] & Y \\ Y \arrow[urr, dashed, "\psi"] & & \end{tikzcd}$$

the diagram is commutative. Even if there exists such a map ψ, it does not have to be a homotopy equivalence. Therefore $\mathrm{Aut}_\bullet(X)$ is not functorial.

Let $\mathcal{C}$ be any category and $\mathrm{Eq}(X)$ denote the set of morphisms $f\colon X \to X$ which are equivalences in $\mathcal{C}$, i.e., there is a morphism $g\colon X \to X$ with $f \circ g = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_X$. Now if $T\colon \mathcal{T}_\bullet \to \mathcal{C}$ is a covariant functor, then T induces a homomorphism $T_*\colon \mathrm{Aut}_\bullet(X) \to Eq(TX)$. Many of the results on $\mathrm{Aut}_\bullet(X)$ are based on this observation. For a particular T, one determines properties of T_* and then attempts to determine $\mathrm{Aut}_\bullet(X)$. For example let $\mathcal{G}$ be the category of groups and $T\colon \mathcal{T}_\bullet \to \mathcal{G}$ be defined as $T(X) = \pi_n(X)$ or $H_n(X;G)$. Our main example is $T\colon \mathcal{T}_\bullet \to \mathcal{CW}$ where $\mathcal{CW}$ is the category of CW-complexes and $T(X) = P_n(X)$ (n-th stage Postnikov tower). Postnikov decompositions are well-suited to study this group since a homotopy self-equivalence f does induce homotopy self-equivalences on the *n-th* stages, on the other hand f need not induce homotopy self-equivalences of the n-skeleta $X^{(n)}$.

Let $\mathcal{E}_\bullet(X)$ denote the space of based maps $X \to X$ which are homotopy equivalences, with base point the identity function and with the compact open topology. It contains the group of basepoint preserving homeomorphisms $\mathrm{Homeo}_\bullet(X)$, and, when X is a smooth manifold, the group of basepoint preserving diffeomorphisms $\mathrm{Diffeo}_\bullet(X)$. From knowledge of $\mathcal{E}_\bullet(X)$, one hopes to obtain information about these subspaces. Also note that we have $\mathrm{Aut}_\bullet(X) = \pi_0(\mathcal{E}_\bullet(X))$. See surveys [1], [60], and [61] for review of results on the group of self-homotopy equivalences and related topics.

Remark. In [1], $E(X)$ and $\mathcal{E}(X)$ are used to denote the space of based homotopy self-equivalences of X and the group of based homotopy classes of based homotopy self-equivalences of X respectively. Meanwhile in [60] and in [61], $E^*(X)$ and $\mathcal{E}^*(X)$ are used for the corresponding terms.

We are going to work with closed, connected, oriented, smooth or topological 4-manifolds. Let M denote such a manifold with a fixed base-point $x_0 \in M$. For technical reasons, we will restrict ourselves to homotopy self-equivalences preserving both the given orientation on M and a base-point x_0. Let $\mathrm{Aut}_\bullet(M)$ denote the group of homotopy classes of such homotopy self-equivalences. At this point let us fix some notation. The fundamental group $\pi_1(M, x_0)$ will be denoted by π. The higher homotopy groups $\pi_i(M, x_0)$ will be denoted by π_i (where $\pi_i = \pi_i(\widetilde{M}, \widetilde{x_0})$ for $i > 1$) and they are naturally endowed with a π-module structure. We write Λ for the integral group ring of π. We will mean homology and cohomology with integral coefficients unless otherwise noted.

We can ignore the base-point and consider the collection of homotopy classes of free (unbased) maps $M \to M$ which are homotopy equivalences. This forms a group under composition of homotopy classes which will be denoted by $\mathrm{Aut}\,(M)$. The evaluation map at x_0 gives a fibration

$$\mathcal{E}_\bullet(M) \longrightarrow \mathcal{E}(M) \xrightarrow{\mathrm{ev}} M\ .$$

We then have a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_1(\mathcal{E}(M)) \xrightarrow{\mathrm{ev}_*} \pi \longrightarrow \mathrm{Aut}_\bullet(M) \longrightarrow \mathrm{Aut}\, M \longrightarrow 1\ .$$

Hence if M is simply connected, then clearly $\mathrm{Aut}_\bullet(M) \cong \mathrm{Aut}\,(M)$.

In 1990, Cochran and Habegger [20], computed $\mathrm{Aut}\,(M)$ for simply connected 4-manifolds:

Theorem (Cochran and Habegger). *Let M be a 1-connected 4-manifold and* $\mathrm{Aut}(H_2(X), \cdot)$ *(respectively* $\mathrm{Aut}(H_2(X), \pm\cdot)$ *) denote the group of au-*

tomorphism of $H_2(M)$ preserving the intersection form (respectively up to sign). Suppose that rank $H_2(M)$ is non-zero, then

$$\mathrm{Aut}\,(M) \cong KH_2(M;\mathbb{Z}/2) \rtimes \mathrm{Aut}(H_2(M), \pm\cdot)$$

and if $\mathrm{Aut}^+(M)$ *denotes those homotopy classes of self- homotopy equivalences which preserve the orientation*

$$\mathrm{Aut}^+(M) \cong KH_2(M;\mathbb{Z}/2) \rtimes \mathrm{Aut}(H_2(M), \cdot)\ ,$$

where $KH_2(M;\mathbb{Z}/2) := \ker(w_2\colon H_2(M;\mathbb{Z}/2) \to \mathbb{Z}/2)$.

Remark. If the rank of $H_2(M)$ is zero, then $M \simeq S^4$ and as a self homotopy equivalence of a sphere is homotopic to the identity if and only if it has degree 1, we have $\mathrm{Aut}\,(M) \cong \mathbb{Z}/2$.

It is a well known result of Milnor [56] and Whitehead [74] that a simply connected 4-dimensional manifold M is classified up to homotopy equivalence by its intersection form, but in the non-simply connected case this form does not detect the homotopy type, one missing invariant is the first k-invariant $k_M \in H^3(\pi;\pi_2)$, see [33, Remark 4.5]. Hambleton and Kreck [33] defined the quadratic 2-type of M as the quadruple $[\pi, \pi_2, k_M, s_M]$ where s_M is the Λ-valued intersection form on $\pi_2 \cong H_2(M;\Lambda)$. The group of isometries of the quadratic 2-type of M, denoted by $\mathrm{Isom}[\pi, \pi_2, k_M, s_M]$, consists of all pairs of isomorphisms

$$\chi\colon \pi \to \pi \quad \text{and} \quad \psi\colon \pi_2 \to \pi_2\ ,$$

such that $\psi(gx) = \chi(g)\psi(x)$ for all $g \in \pi$ and $x \in \pi_2$, which preserve the k-invariant, $\psi_*(\chi^{-1})^*k = k$, and the equivariant intersection form,

$s_M(\psi(x), \psi(y)) = \chi_* s_M(x, y)$. In [33] the authors showed that the quadratic 2-type detects the homotopy type of an oriented 4-manifold M if π is a finite group with 4-periodic cohomology (this result was later extended to finite groups with 4-periodic 2-Sylow subgroups by Bauer [5]).

Using Hambleton and Kreck's classification of 4-manifolds and a spectral sequence argument, in 1996 Hayat and Legrand [39] showed that the group $\mathrm{Aut}\,(M)$ fits in an exact sequence. Let us first point out a construction from [20, p. 425] before we state their theorem. When M is simply connected there exists a homomorphism

$$\Phi\colon H^2(M;\pi_3) \cong \mathrm{Hom}(\pi_2, \pi_3) \cong \pi_2 \otimes \pi_3 \longrightarrow \pi_4$$

given by $\Phi(a \otimes b) = [a, b] + w_2(a)(b \circ \Sigma\eta)$ where $[\ ,\ ]$ is the Whitehead product and η is the Hopf map. When M is not simply connected, the homomorphism Φ associated to the universal cover $\widetilde{M}$ of M induces a homomorphism $U\colon \mathrm{Hom}(\pi_2, \pi_3)^\pi \longrightarrow (\pi_4)_\pi$, where $(\pi_4)_\pi$ is the group of co-invariants of π_4.

Theorem (Hayat and Legrand). *Let M be a compact connected orientable 4-dimensional manifold, having finite fundamental group with a periodic cohomology of period 4. The group $\mathrm{Aut}\,(M)$ fits in the following exact sequence*

$$H^1(\pi;\pi_2) \longrightarrow \mathrm{coker}\,(U) \longrightarrow \mathrm{Aut}\,(M) \longrightarrow \mathrm{Isom}[\pi, \pi_2, k_M, s_M] \longrightarrow 1\ .$$

In their 2004 paper [35], Hambleton and Kreck first defined a suitable thickening $\mathrm{Aut}_\bullet(M, w_2)$ of $\mathrm{Aut}_\bullet(M)$: the class $w_2 \in H^2(M;\mathbb{Z}/2)$ gives a fibration $w_2\colon M \to K(\mathbb{Z}/2,2)$ and we can form the pullback

$$\begin{tikzcd} M\langle w_2\rangle \arrow[r, "j"] \arrow[d, "\xi"'] & M \arrow[d, "w_2"] \\ BSO \arrow[r, "w_2(\gamma)"'] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

where γ denotes the stable universal bundle over BSO. Now define $\mathrm{Aut}_\bullet(M,w_2)$ as the group of equivalence classes of maps $\hat{f}\colon M \to M\langle w_2\rangle$ such that (i) $f := j \circ \hat{f}$ is a base-point and orientation preserving homotopy equivalence, and (ii) $\xi \circ \hat{f} = \nu_M$. The connection between $\mathrm{Aut}_\bullet(M,w_2)$ and $\mathrm{Aut}_\bullet(M)$ is given by the following short exact sequence of groups (see [35])

$$0 \to H^1(M;\mathbb{Z}/2) \to \mathrm{Aut}_\bullet(M,w_2) \to \mathrm{Aut}_\bullet(M) \to 1\ .$$

So for example if π is a finite group of odd order, then since $H^1(M;\mathbb{Z}/2)=0$, we actually have $\mathrm{Aut}_\bullet(M,w_2)\cong \mathrm{Aut}_\bullet(M)$.

Let B denote the 2-type of M. We may construct B by adjoining cells of dimension at least 4 to kill the homotopy groups in dimensions ≥ 3. The natural map $c\colon M\to B$ is given by the inclusion of M into B and is a 3-connected map. By a result of Møller [58], there is an exact sequence

$$0 \longrightarrow H^2(\pi;\pi_2) \longrightarrow \mathrm{Aut}_\bullet(B) \xrightarrow{(\pi_1,\pi_2)} \mathrm{Isom}[\pi,\pi_2,k_M] \longrightarrow 1\ .$$

In [35], the authors established a commutative braid of exact sequences, valid for any closed, oriented smooth or topological 4-manifold M, containing $\mathrm{Aut}_\bullet(M,w_2)$ and got an explicit formula when the fundamental group π is finite of odd order. Before stating their result let us point out that if π is a finite group then the intersection form s_M is determined by $c_*[M]\in H_4(B)$(see [33, p. 90]). As a consequence of Møller's result, we have a finite extension of $\mathrm{Isom}[\pi, \pi_2, k_M, s_M]$ (see [36])

$$0 \longrightarrow H^2(\pi;\pi_2) \longrightarrow \mathrm{Isom}[\pi,\pi_2,k_M,c_*[M]] \longrightarrow \mathrm{Isom}[\pi,\pi_2,k_M,s_M] \longrightarrow 0\ ,$$

where $\mathrm{Isom}[\pi,\pi_2,k_M,c_*[M]] := \{\phi \in \mathrm{Aut}_\bullet(B) | (\phi)_*(c_*[M]) = c_*[M]\}$.

Theorem (Hambleton and Kreck). *Let M be a connected, closed, oriented smooth (or topological) manifold of dimension 4. If $\pi_1(M, x_0)$ has odd order, then*

$$\mathrm{Aut}_\bullet(M) \cong KH_2(M;\mathbb{Z}/2) \rtimes \mathrm{Isom}[\pi,\pi_2,k_M,c_*[M]]\ .$$

In the first part of this thesis, we will extend the above result in two cases, namely when π is a free group or a PD_2-group. Since $H^3(\pi;\pi_2) = 0$ for both cases we have $k_M = 0$. For notational ease we will drop it from the notation and write $\mathrm{Isom}[\pi,\pi_2,s_M]$ for the group of isometries of the quadratic 2-type.

In Chapter 2, we will deal with 4-manifolds with free fundamental group. First note that $H^2(\pi;\pi_2) = 0$ if π is a free group, thus we have $\mathrm{Aut}_\bullet(B) \cong \mathrm{Isom}[\pi,\pi_2]$. Hillman [42] proved that π_2 is a free Λ-module and as a consequence he showed that $c_*[M]$ and s_M uniquely determine each other (see [44]). Hence we have $\mathrm{Isom}[\pi,\pi_2,c_*[M]] \cong \mathrm{Isom}[\pi,\pi_2,s_M]$.

We have an analogous bordism group $\mathrm{Aut}_\bullet(B, w_2)$. Note that there is a similar pullback diagram for B

$$\begin{tikzcd} B\langle w_2\rangle \arrow[r, "j"] \arrow[d, "\xi"'] & B \arrow[d, "w_2"] \\ BSO \arrow[r, "w_2(\gamma)"'] & K(\mathbb{Z}/2,2)\ . \end{tikzcd}$$

We define $\mathrm{Aut}_\bullet(B, w_2)$ as the set of equivalence classes of maps $\hat{f}\colon M \to$

$B\langle w_2\rangle$ such that (i) $f := j \circ \hat{f}$ is a base-point preserving 3-equivalence, and (ii) $\xi \circ \hat{f} = \nu_M$.

We will define an extension $\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$ of $\mathrm{Isom}[\pi, \pi_2, s_M]$ as

$$\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M] := \{\hat{f} \in \mathrm{Aut}_\bullet(B, w_2) : \phi_f \in \mathrm{Isom}[\pi, \pi_2, s_M]\}$$

which sits in an exact sequence of groups

$$0 \longrightarrow H^1(M; \mathbb{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M] \xrightarrow{\hat{j}} \mathrm{Isom}[\pi, \pi_2, s_M] \longrightarrow 1 .$$

We then prove the following for 4-manifolds with free fundamental group.

Theorem A. *Let M be a connected, closed, oriented, smooth or topological manifold of dimension 4. If $\pi := \pi_1(M)$ is a free group, then*

$$\mathrm{Aut}_\bullet(M, w_2) \cong (KH_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)) \rtimes \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M] ,$$

where $KH_2(M; \mathbb{Z}/2) := \ker(w_2 \colon H_2(M; \mathbb{Z}/2) \to \mathbb{Z}/2)$.

In Chapter 4, we work with 4-manifolds with PD_2 fundamental group. Unfortunately, we don't know whether $c_*[M]$ and s_M uniquely determine each other or not, so instead of $\mathrm{Isom}[\pi, \pi_2, s_M]$, we work with the group $\mathrm{Isom}[\pi, \pi_2, c_*[M]]$. We defined

$$\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]] := \{\hat{f} \in \mathrm{Aut}_\bullet(B, w_2) : \phi_f \in \mathrm{Isom}[\pi, \pi_2, c_*[M]]\}$$

and showed that it sits in the following short exact sequence

$$0 \longrightarrow H^1(M; \mathbb{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]] \longrightarrow \mathrm{Isom}[\pi, \pi_2, c_*[M]] \longrightarrow 1 .$$

We obtained the following result:

Theorem C. *Let M be a connected, closed, oriented, smooth or topological manifold of dimension 4. If $\pi := \pi_1(M)$ is a PD_2 group, then*

$$\mathrm{Aut}_\bullet(M, w_2) \cong (KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \rtimes \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, c_*[M]] \ .$$

The second part of this thesis deals with the classification of 4-manifolds up to s-cobordism. The geometric classification techniques, surgery and s-cobordism theorem, are not known to hold for the groups under consideration, the most one can hope at present is to obtain classification up to s-cobordism or up to stabilization by connected sum with copies of $S^2 \times S^2$.

The classification of manifolds is one of the central problems in mathematics. Since any finitely-presented group can be realized as the fundamental group of a compact n-manifold if $n \geq 4$, a complete answer is not possible for manifolds of dimension ≥ 4 as a result of the undecidability of the word problem for finitely-presented groups (there is in general no finite procedure for deciding in all cases whether two groups given by finite sets of generators and relations are isomorphic). Thus as a first invariant one has to fix the fundamental group.

For 4-manifolds with prescribed fundamental group classification up to homeomorphism or diffeomorphism is still a hard problem even if the fundamental group is trivial, because there is not enough room to apply the fundamental Whitney trick in this dimension. In 1964 Wall [72], was able to get around this difficulty at the price of stabilizing. We say that two 4-manifolds are stably diffeomorphic (homeomorphic) if they become diffeomorphic (homeomorphic) after connected sum with $S^2 \times S^2$'s.

Theorem (Wall). *If M_1 and M_2 are smooth, simply connected and h-*

cobordant, then there is an integer k such that we have a diffeomorphism

$$M_1 \sharp k(S^2 \times S^2) \cong M_2 \sharp k(S^2 \times S^2) \ .$$

Definition. A cobordism W^{n+1} between manifolds M_1 and M_2 is called an h-cobordism if the inclusions $M_1 \hookrightarrow W$ and $M_2 \hookrightarrow W$ are homotopy equivalences

If two 4-manifolds are homotopy equivalent, then their intersection forms must be isomorphic. The importance of intersection forms for 4-dimensional manifolds comes from the following theorem of Whitehead [74], as sharpened by Milnor [56].

Theorem (Milnor, Whitehead). *Two simply-connected 4-manifolds are homotopy equivalent if and only if their intersection forms are isomorphic.*

In 1964, using handlebody theory Wall [72] also proved:

Theorem (Wall). *If M_1 and M_2 are smooth, simply connected, and have isomorphic intersection forms, then M_1 and M_2 must be h-cobordant.*

It is important to obtain an h-cobordism: if M_1 and M_2 are compact oriented n-manifolds that are h cobordant through the simply connected $(n+1)$-manifold W and $n \geq 5$, then the h-cobordism theorem of Smale [65] says that W is diffeomorphic to the cylinder over M_1. In particular M_1 and M_2 are diffeomorphic. In 1981 M. Freedman [27] was able to prove the h-cobordism theorem in the topological category for $n = 4$.

Theorem (Freedman). *M_1 and M_2 are compact oriented 4-manifolds that are h cobordant through the simply connected 5-manifold W, then we have a homeomorphism $W \cong M_1 \times [0,1]$, and in particular M_1 and M_2 are homeomorphic.*

Remark. The smooth version of the 5-dimensional h-cobordism theorem is false by Donaldson [24].

The h-cobordism theorem was generalized by Barden [4], Mazur [54] and Stallings [68], to the s-cobordism theorem for non-simply connected manifolds, using Whitehead torsion. The Whitehead torsion of a homotopy equivalence $f\colon M_1 \to M_2$ sits in the Whitehead group, $\tau(f) \in Wh(\pi_1(M_1))$. A homotopy equivalence is simple if $\tau(f) = 0$.

Definition. A cobordism W^{n+1} between manifolds M_1 and M_2 is called an s-cobordism if the inclusions $M_1 \hookrightarrow W$ and $M_2 \hookrightarrow W$ are simple homotopy equivalences

The s-cobordism theorem says that an h-cobordism W^{n+1} between M_1 and M_2 with $n \geq 5$ is diffeomorphic to the trivial cobordism $M_1 \times [0,1]$ if and only if the homotopy equivalence $M_1 \hookrightarrow W$ is simple. Freedman was able to obtain remarkable results on the topological classification of 4-manifolds by proving a version of the Whitney trick in the 4-dimensional topological category where the fundamental group lies in a certain class of good groups, i.e., groups for which the Whitney trick is known. This in turn has led to an s-cobordism theorem for 4-manifolds.

Theorem (Freedman). *A compact s-cobordism of dimension* 5 *with good fundamental group has a topological product structure.*

In Chapter 3, we get the following result for 4-manifolds with free fundamental group.

Theorem B. *Let M_1 and M_2 be two closed, connected, oriented, topological* 4*-manifolds with free fundamental group and have the same Kirby-*

Siebenmann invariant. Then they are s-cobordant if and only if they have isometric quadratic 2-types.

For manifolds with PD_2 fundamental group we had to put an extra condition on the manifolds. A manifold M is said to have w_2-type (I) or (II) if M and $\widetilde{M}$ are spin at the same time (see [34] for the actual definition). We get the following result in chapter 5:

Theorem D. *Let M_1 and M_2 be two closed, connected, oriented, topological 4-manifolds with PD_2 fundamental group. Suppose that they have the same Kirby-Siebenmann invariant and M_1 has w_2-type (I) or (II). Then M_1 and M_2 are s-cobordant if and only if they have isometric quadratic 2-types.*

Remark. Similar results were obtained in [17], [42] and [44], but our method is different.

We finish this introduction by pointing out that our methods are based on the approach of Hambleton and Kreck [35] involving bordism techniques and the modified surgery theory of Kreck [49].

Chapter 1

Background

1.1 Universal Poincaré Duality and the Equivariant Intersection Pairing

The groups that we are interested in are infinite groups, so it will be convenient to start this chapter by recalling some basic facts on Poincaré duality and intersection pairings over the group ring of the fundamental group. The exposition in this section is based on [19], [22], [32], [38] and [73]. Let M be a closed, oriented, smooth or topological 4-manifold. We shall assume that M is provided with a CW-structure, up to homotopy. The covering map $p\colon \widetilde{M} \to M$ induces a CW-structure for $\widetilde{M}$, i.e., if $M^{(k)}$ and $\widetilde{M}^{(k)}$ denote the k-skeletons of M and $\widetilde{M}$ respectively, then $\widetilde{M}^{(k)} = p^{-1}(M^{(k)})$. As the fundamental group π acts on the right cellularly on $\widetilde{X}$ by covering translations, it acts on $\widetilde{M}^{(k)}$ by permuting the k-cells of $\widetilde{M}^{(k)}$ which lie over a fixed k-cell of $M^{(k)}$. Therefore there is an induced free right Λ-module structure on the cellular chain complex $C_*(\widetilde{M})$ of $\widetilde{M}$ with $C_k(\widetilde{M}) = H_k(\widetilde{M}^{(k)}, \widetilde{M}^{(k-1)}) =$ free right Λ-module generated by the k-cells of M.

Let A be a left Λ-module. The tensor product $C_*(\widetilde{M}) \otimes_\Lambda A$ is a chain complex of Abelian groups. The homology of M with local coefficients in A is defined as

$$H_*(M; A) = H_*(C_*(\widetilde{M}) \otimes_\Lambda A)\ .$$

Next, we will give the corresponding definition of cohomology. Since the functor $\mathrm{Hom}_\Lambda(-,-)$ is defined on the category of pairs of right Λ-modules or of pairs of left Λ-modules, we need to either change $C_*(\widetilde{M})$ to a left Λ-module or consider coefficients in right Λ-modules. We opt for the former. The standard involution $\lambda \to \overline{\lambda}$ on Λ is induced by the formula

$$\sum n_g g \to \sum n_g g^{-1}$$

for $n_g \in \mathbb{Z}$ and $g \in \pi$. Let $\overline{A}$ be the corresponding right Λ-module with the conjugate structure given by $a.\lambda := \overline{\lambda} a$, for $\lambda \in \Lambda$ and $a \in A$ (we paid our attention only to the orientable case, so the first Stiefel-Whitney class vanishes). The cohomology of M with local coefficients in A is given by

$$H^*(M; A) = H^*(\mathrm{Hom}_\Lambda(C_*(\widetilde{M}), \overline{A}))\ .$$

For $A = \mathbb{Z}$, we obtain the integral homology and cohomology of M. In fact, the cellular chain complex of M satisfies $C_*(M) \otimes_\Lambda \epsilon^*\mathbb{Z}$, where $\mathbb{Z}$ is regarded as a left Λ-module via augmentation map $\epsilon\colon \Lambda \to \mathbb{Z}$. We have

$$H_*(M) = H_*(C_*(\widetilde{M}) \otimes_\Lambda \epsilon^*\mathbb{Z}) \cong H_*(C_*(M)) = H_*(M)$$

and

$$\begin{aligned} H^*(M) = H^*(\mathrm{Hom}_\Lambda(C_*(\widetilde{M}), \mathbb{Z})) &\cong H^*(\mathrm{Hom}(C_*(\widetilde{M}) \otimes_\Lambda \epsilon^*\mathbb{Z}, \mathbb{Z})) \\ &\cong H^*(\mathrm{Hom}(C_*(M), \mathbb{Z})) = H^*(M)\ . \end{aligned}$$

If $A = \Lambda$, then

$$H_*(M;\Lambda) = H_*(C_*(\widetilde{M}) \otimes_\Lambda \Lambda) \cong H_*(C_*(\widetilde{M})) = H_*(\widetilde{M}) \ .$$

The same fact does not hold for cohomology without some modifications. The cochain complex $C^*(M;\Lambda) = \mathrm{Hom}_\Lambda(C_*(\widetilde{M}),\Lambda)$ is not in general isomorphic to $\mathrm{Hom}(C_*(\widetilde{M}),\mathbb{Z})$. It turns out that

$$H^*(M;\Lambda) \cong H^*_{cpt}(\widetilde{M})$$

the *compactly supported cohomology* of M (the cohomology groups defined by integral cochains with compact support, i.e., taking non-zero values on a finite number of cells, with the induced Λ-module structure). So for finite π, $\widetilde{M}$ is compact and we have $H^*(M;\Lambda) \cong H^*(\widetilde{M})$. However, for infinite π and a compact M the cover $\widetilde{M}$ is non-compact and it is the compactly supported cohomology which is relevant.

Proposition 1.1.1 ([59]). *The homology and cohomology groups of M with local coefficients in Λ are related by cap products*

$$\cap \colon H_i(M) \otimes H^j(M;\Lambda) \to H_{i-j}(M;\Lambda) \qquad (x,y) \to x \cap y$$

such that for every $\lambda \in \Lambda$, we have

$$x \cap \lambda y = (x \cap y)\lambda \in H_{i-j}(M;\Lambda) \ .$$

For any homology class $x \in H_i(M)$ the pairing

$$s \colon H^{i-j}(M;\Lambda) \times H^j(M;\Lambda) \to \Lambda \qquad (a,b) \to a(x \cap b)$$

is sesquilinear and such that

$$s(b,a) = (-1)^{j(i-j)}\overline{s(a,b)} \in \Lambda \ .$$

To have a better understanding of the above cap product, we are going to use the universal covering space $\widetilde{M}$ of M. We know that there are defined cap product pairings

$$\cap\colon H_i^{lf}(\widetilde{M}) \otimes H^j(M;\Lambda) \to H_{i-j}(M;\Lambda)$$

with $H_*^{lf}(M)$ the homology groups defined using the locally finite infinite chains, which are formal sums $\sum_\sigma n_\sigma \sigma$ of singular simplices $\sigma\colon \Delta^n \to M$ with $n_\sigma \in \mathbb{Z}$, such that each $m \in M$ has a neighborhood meeting the images of only finitely many σ's with $n_\sigma \neq 0$ (to define the usual homology, we consider chains that are finite formal linear combinations of simplices of M). Thus for noncompact $\widetilde{M}$ of M there are defined infinite transfer maps $trf\colon H_*(M) \to H_*^{lf}(\widetilde{M})$ which assign to a singular simplex sum of its lifts. Then the cap product with an element in $H_i(M)$ can be expressed as the composite

$$\cap\colon H_i(M) \otimes H^j(M;\Lambda) \xrightarrow{trf\otimes 1} H_i^{lf}(\widetilde{M}) \otimes H^j(M;\Lambda) \xrightarrow{\cap} H_{i-j}(M;\Lambda) \ .$$

We can now state the Poincaré duality in the following way:

Theorem 1.1.2. *For any closed, oriented smooth 4-manifold M and universal cover $\widetilde{M}$, there is a fundamental class $[M] \in H_4(M)$ and cycle σ representing $[M]$ such that the cap product with σ induces Λ-equivalence of Λ-chain complexes*

$$\sigma \cap -\colon C^q(M;\Lambda) \to C_{4-q}(M;\Lambda)$$

As a consequence we obtain the following Λ-module isomorphisms

$$[M] \cap -\colon H^q(M;\Lambda) \to H_{4-q}(M;\Lambda) \ .$$

Next, we are going to recall the definition of the equivariant intersection pairing s_M on $\pi_2(M)$, or the homology intersection pairing of M with respect to $\widetilde{M}$,

$$s_M\colon H_2(M;\Lambda) \times H_2(M;\Lambda) \to \Lambda;\ (a,b) \to s_M(a,b) = a^*(b)\ .$$

This is a sesquilinear pairing where $a^* \in H^2(M;\Lambda)$ is the Poincaré dual of a, such that $s_M(a,b) = \overline{s_M(b,a)} \in \Lambda$. We are going to give two interpretations of this definition:

(I) Any element $a \in \pi_2 \cong H_2(M;\Lambda)$ can be represented as an immersion $S_2 \xrightarrow{a} M$ which does not necessarily preserve the base-point, so first we specify a homotopy class of paths in M joining the base point x_0 to $a(1)$. Note that π acts on π_2 by composing the path with a loop on x_0. For any pair $(a,b) \in H_2(M;\Lambda) \times H_2(M;\Lambda)$, we can assume that the immersed spheres a and b are oriented and intersect transversely in a finite set of points x. Then we set

$$s_M(a,b) = \sum_{x \in a \cap b} \lambda_x g_x$$

where (1) $g_x \in \pi$ is the class of the loop at x_0 which starts along the path to the base-point of b, around b (avoiding other intersection points) to x, around a to its base-point, and back along the given path to x_0. (2) to define λ_x for an intersection point, we note that the orientations of a and b induce an orientation of M at the intersection point $x \in a \cap b$. The chosen orientation of M at the base-point x_0 can be transported along the path to a and through a to the intersection point x. If these orientations agree, we set $\lambda_x = 1$, otherwise $\lambda_x = -1$.

(II) For our second interpretation of the equivariant intersection form, we start with the usual $\mathbb{Z}$ valued intersection form,

$$\begin{tikzcd} s\colon H_2(M;\Lambda)\otimes H_2(M;\Lambda)\cong H^2(M;\Lambda)\otimes H^2(M;\Lambda) \arrow[d, "\cup"] & \\ H^4(M;\Lambda) \arrow[r, "\cap[\tilde{M}]"] & H_0(M;\Lambda)\cong\mathbb{Z} \end{tikzcd}$$

where $[\widetilde{M}]$ is the fundamental class of $\widetilde{M}$, with possibly infinite chains. This form is π-equivariant. Now, we define the equivariant intersection form by

$$s_M(a,b)=\sum_{g\in\pi} s(a,g^{-1}b)g\in\mathbb{Z}[\pi]=\Lambda\ .$$

The cohomology intersection pairing

$$s_M\colon H^2(M;\Lambda)\times H^2(M;\Lambda)\to\Lambda;\quad (a,b)\to a([M]\cap b)$$

coincides with the homology intersection pairing via Poincaré duality isomorphism

$$\begin{tikzcd} H^2(M;\Lambda)\times H^2(M;\Lambda) \arrow[r, "s_M"] \arrow[d, "\cong"'] & \Lambda \arrow[d, equal] \\ H_2(M;\Lambda)\times H_2(M;\Lambda) \arrow[r, "s_M"] & \Lambda \end{tikzcd}$$

The intersection form s_M induces the Λ-homomorphism

$$\widehat{s_M}\colon H_2(M;\Lambda)\to \mathrm{Hom}_\Lambda(H_2(M;\Lambda),\Lambda)$$

defined by $\widehat{s_M}(x)(y)=s_M(x,y)$ [73, p. 47]. The relation between s_M and Poincaré duality can be stated as follows

Lemma 1.1.3. *If* $\mathrm{ev}\colon H^2(M;\Lambda)\to Hom_\Lambda(H_2(M;\Lambda),\Lambda)$ *is the evaluation homomorphism, given by* $\mathrm{ev}(c)(x)=c\cap x=c(x)$*, then the diagram*

$$\begin{tikzcd} H^2(M;\Lambda) \arrow[r, "\mathrm{ev}"] \arrow[d, "{[M]\cap -}"'] & Hom_\Lambda(H_2(M;\Lambda),\Lambda) \\ H_2(M;\Lambda) \arrow[r, equal] & H_2(M;\Lambda) \arrow[u, "\widehat{s_M}"'] \end{tikzcd}$$

commutes, i.e., $\widehat{s_M}\circ([M]\cap -)=\mathrm{ev}$.

1.2 Whitehead's Γ Functor

Next we define Whitehead's quadratic functor Γ, which we use in our homology calculations. This functor is defined for any Abelian group G. We define $\Gamma(G)$, by means of symbolic generators and relations . The elements of $\Gamma(G)$ are equivalence classes of words written on G. Let $w(g)$ denote the word g. The relations for $\Gamma(G)$ are

(i) $w(-g) \equiv w(g)$

(ii) $w(g_1+g_2+g_3)w(g_2+g_3)^{-1}w(g_1+g_3)^{-1}w(g_1+g_2)^{-1}w(g_1)w(g_2)w(g_3) \equiv 0$,

for all elements g, g_1, g_2, $g_3 \in G$.

The group $\Gamma(G)$ is determined by the generators G and the relations given by (i) and (ii). It follows from (ii), with $g_1 = g_2 = g_3 = 0$, that $w(0) \equiv 0$. Hence, with $g_2 = 0$ and $g_1+g_3 = g_4$, we have $w(g_4)w(g_3)^{-1}w(g_4)^{-1}w(g_3) \equiv 0$. Therefore $\Gamma(G)$ is Abelian.

Definition 1.2.1. A function $f\colon G \to H$ between Abelian groups is quadratic if $f(g) = f(-g)$ and if the function $G \times G \to H$ which is given by $(g_1, g_2) \to f(g_1 + g_2) - f(g_1) - f(g_2)$ is bilinear in g_1 and g_2.

Since a quadratic map is consistent with the relations defining $\Gamma(G)$, for each Abelian group G there is a universal quadratic map $\gamma\colon G \to \Gamma(G)$ with the property that for all H and quadratic maps $f\colon G \to H$ there is a unique homomorphism $\bar{f}\colon \Gamma(G) \to H$ with $\bar{f}\gamma = f$.

We can construct the group $\Gamma(G)$ as follows: Consider the map $i\colon G \to F(G)$ where $F(G)$ is the free Abelian group generated by the underlying set of G. The map i is the inclusion of generators. We set

$\Gamma(G) = F(G)/R$ where R denotes the relations (i) and (ii) for $\Gamma(G)$. Now γ is the composite $G \to F(G) \to F(G)/R$ of i and of the quotient map.

Now, Γ is a functor, which carries Abelian groups to Abelian groups since a homomorphism $\varphi\colon G \to H$ yields the quadratic map $\gamma\varphi$ which induces a unique homomorphism $\Gamma(\varphi) = \overline{\gamma\varphi}$ such that the diagram

$$\begin{tikzcd} \Gamma(G) \arrow[r, "\Gamma(\varphi)"] & \Gamma(H) \\ G \arrow[u, "\gamma"] \arrow[r, "\varphi"] & H \arrow[u, "\gamma"'] \end{tikzcd}$$

commutes. We consider the function $\Delta\colon G \to G \otimes G$ given by $\Delta(g) = g \otimes g$. Clearly, Δ is quadratic and yields the canonical homomorphism

$$\overline{\Delta}\colon \Gamma(G) \to G \otimes G \quad \text{with} \quad \overline{\Delta}\gamma = \Delta\ .$$

Next we obtain by the quadratic map γ the bilinear pairing

$$[\ ,\ ] = [1,1]\colon G \otimes G \to \Gamma(G) \quad \text{with} \quad [g_1, g_2] = \gamma(g_1 + g_2)\gamma(g_1)^{-1}\gamma(g_2)^{-1}\ .$$

We write $[f, g] = [1,1](f \otimes g)\colon X \otimes Y \to G \otimes G \to \Gamma(G)$ where $f\colon X \to G$, $g\colon Y \to G$ are homomorphisms. We have

$$[g_1, g_2] = [g_2, g_1] \quad \text{and} \quad \overline{\Delta}[g_1, g_2] = g_1 \otimes g_2 + g_2 \otimes g_1\ .$$

Moreover, the homomorphism $\overline{\Delta}$ is injective in case G is free Abelian [7, p. 14]. For a direct sum $G \oplus H$ we have the isomorphism

$$\Gamma(G \oplus H) = \Gamma(G) \oplus \Gamma(H) \oplus G \otimes H\ ,$$

which is given by $\Gamma(i_1)$, $\Gamma(i_2)$ and $[i_1, i_2]$, where i_1 and i_2 are the inclusions of G and H into $G \oplus H$ respectively. A similar result is true for an arbitrary direct sum where I is an ordered set:

$$\Gamma(\bigoplus_I G_i) = \bigoplus_I \Gamma(G_i) \oplus \bigoplus_{i>j} G_i \otimes G_j\ .$$

Moreover, Γ commutes with direct limits of Abelian groups. If $G = \mathbb{Z}$ then $\Gamma(G) = \mathbb{Z}$ generated by $\gamma(1)$. This shows that for a free Abelian group G, $\Gamma(G)$ is also free Abelian.

We will give two expressions for $\Gamma(G)$ when G is a free Abelian group (see [75, pages 62, 63]). Let $\{g_i\}$ be a set of free generators of G, indexed in (1-1) fashion to a set $\{i\}$ with a total ordering.

Proposition 1.2.2 ([75]). *$\Gamma(G)$ is free Abelian and is freely generated by the set of elements $\gamma(g_i)$, $[g_j, g_k]$, for every $i \in \{i\}$ and every pair j, $k \in \{i\}$ such that $j < k$.*

Let $G^* = \{\phi : G \to \mathbb{Z} \mid \exists N, \phi(g_i) = 0 \text{ for } i > N\}$. Then G^* is a free Abelian group, which is freely generated by $\{g_i^*\}$, where $g_i^* g_j = 1$ or 0 according as $j = i$ or $j \neq i$. We describe a homomorphism $f\colon G^* \to G$ as locally finite if and only if for almost all values of i we have $f(g_i{}^*) = 0$ and f is symmetric if and only if $g^* f(h^*) = h^* f(g^*)$ for every pair $g^*, h^* \in G^*$. Let

$$f(g_j{}^*) = \sum_j f_{ij} g_j = \sum_j (g_j{}^* f g_i{}^*) g_j \ ,$$

then being symmetric is equivalent to the condition $f_{ij} = f_{ji}$. Let S be the additive group of all locally finite, symmetric homomorphisms, $f\colon G^* \to G$.

Proposition 1.2.3 ([75]). $S \cong \Gamma(G)$

Finally, with G still free Abelian, if a group π acts from right on G by linear maps (i.e., G is a right $\mathbb{Z}[\pi]$-module) this induces an action of π on $\Gamma(G)$. If we consider $\Gamma(G)$ as a subgroup of $G \otimes G \cong \operatorname{Hom}(G^*, G)$ this π-action on $\Gamma(G)$ is given by the diagonal action on $G \otimes G$. In terms of homomorphisms $p \in \pi$ maps $f \in S$ to $p \circ f \circ p^*$. With this convention $\Gamma(G)$ is a Λ-submodule of $G \otimes G$.

See [75] or for a recent exposition [6] and [7] for further reading on this subject.

1.3 Manifolds with structure

When we deal with non-spin manifolds we are going to use the language of manifolds with structure. Let $\xi_r\colon E_r \to BSO(r)$ be a fibration. If η is an r-dimensional vector bundle over the space X classified by the map $\eta\colon X \to BSO(r)$, then an (E_r, ξ_r) structure on η is a homotopy class of liftings to E_r of the map η; i.e., an equivalence class of maps $\bar{\eta}\colon X \to E_r$ with $\xi_r \circ \bar{\eta} = \eta$

$$\begin{tikzcd} & E_r \arrow[d, "\xi_r"] \\ X \arrow[ur, "\bar{\eta}"] \arrow[r, "\eta"'] & BSO(r) \end{tikzcd}$$

where $\bar{\eta}$ and $\widetilde{\eta}$ are equivalent if they are homotopic by a homotopy $H\colon X \times I \to E_r$ such that $\xi_r \circ H(x,t) = \eta(x)$ for all $(x,t) \in X \times I$.

We will be interested in the stable normal bundles of manifolds. Since the classifying map $\nu_M\colon M \to BSO$ depends on an embedding of M into $\mathbb{R}^{n+r}$ for r large, we will define an equivalence relation on the sequences of (E_r, ξ_r) structures on the normal bundle of M. Let M^n be a compact, oriented manifold (with or without boundary) and let $i\colon M \to \mathbb{R}^{n+r}$ be an embedding. The normal bundle of i is the quotient of the pullback of the tangent bundle of $\mathbb{R}^{n+r}$, $i^*\tau(\mathbb{R}^{n+r})$, by the subbundle $\tau(M)$. Giving $\tau(\mathbb{R}^{n+r}) = \mathbb{R}^{n+r} \times \mathbb{R}^{n+r}$ the Riemannian metric obtained from the usual inner product in Euclidean space, the total space N of the normal bundle may be identified with the orthogonal complement of $\tau(M)$ in $i^*\tau(\mathbb{R}^{n+r})$, or the fiber of N at m may be identified with the subspace of $\mathbb{R}^{n+r} \times \mathbb{R}^{n+r}$ consisting of vectors (m, x) with x orthogonal to $i_*\tau(M)_m$. The normal map of i is given by sending m into $N_m \in \widetilde{G_r}(\mathbb{R}^{n+r})$, covered by the bundle map $n\colon N \to \widetilde{\gamma^r}(\mathbb{R}^{n+r})$; $\quad (m, x) \to (N_m, x)$. Composing with the inclusion into $\widetilde{\gamma^r}$ provides a map $\nu(i)\colon M \to BSO(r)$ which classifies the normal bundle of the embedding i.

Suppose one is given a sequence (E, ξ) of fibrations $\xi_r\colon E_r \to BSO(r)$ and maps $g_r\colon E_r \to E_{r+1}$ such that the diagram

$$\begin{tikzcd} E_r \arrow[r, "g_r"] \arrow[d, "\xi_r"'] & E_{r+1} \arrow[d, "\xi_{r+1}"] \\ BSO(r) \arrow[r, "i_r"] & BSO(r+1) \end{tikzcd}$$

commutes, i_r being the usual inclusion. An (E_r, ξ_r) structure on the normal bundle of M^n in $\mathbb{R}^{n+r}$ defines a unique (E_{r+1}, ξ_{r+1}) structure via inclusion $\mathbb{R}^{n+r} \subset \mathbb{R}^{n+r+1}$. A *normal* (E, ξ) *structure* on M^n is an equivalence class of sequences of (E_r, ξ_r) structures on the normal bundle $\nu(i)$ of M, two such sequences being equivalent if they become equivalent for sufficiently large r. An (E, ξ) manifold is a pair consisting of a manifold M^n and a normal (E, ξ) structure on M^n.

Now the bordism group $\Omega_n(E)$ can be defined as the group of bordism classes of (E, ξ) manifolds i.e., it consists of triangles

$$\begin{tikzcd} & E \arrow[d, "\xi"] \\ M \arrow[ur, "\bar{\nu}"] \arrow[r, "\nu"'] & BSO \end{tikzcd}$$

where M is a closed, oriented, n-manifold and ν classifies the stable normal bundle of M given by some embedding into Euclidean space (for construction of the classifying map $\nu\colon M \to BSTOP$ see for example [62, p.207]).

The map $\bar{\nu}$ is called a normal (E, ξ) structure on M (for more details on manifolds with structures see [69, Chapter 2]).

Definition 1.3.1. Let $\xi\colon E \to BSO$ be a fibration.

(i) A normal (E, ξ) structure $\bar{\nu}\colon M \to E$ of an oriented manifold M in E is a normal k-smoothing, if it is a $(k+1)$-equivalence.

(ii) We say that E is k-universal if the fibre of the map $E \to BSO$ is connected and its homotopy groups vanish in dimension $\geq k+1$.

For each oriented manifold M, up to fibre homotopy equivalence, there is a unique k-universal fibration E over BSO admitting a normal k-smoothing of M. Thus the fibre homotopy type of the fibration E over BSO is an invariant of the manifold M and we call it the *normal k-type* of M (see [49, p. 711]).

1.4 Minimal Models and Generalized Eilenberg-MacLane Spaces

Minimal models are introduced by Jonathan A. Hillman in [43] and [44]. Before proceeding with the definition, let us introduce another notation here. Throughout this thesis, for any map $f\colon X \to Y$, the map induced on the homotopy groups will be denoted by $\pi_n(f)$, i.e., $\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$.

Definition 1.4.1. A PD_4-complex Z is a *model* for a PD_4-complex X if there is a 2-connected degree-1 map $g_X\colon X \to Z$. The surgery kernel $K_2(g_X) := \ker(\pi_2(g_X))$ is a finitely generated projective $\mathbb{Z}[\pi_1(X)]$-module, and is an orthogonal direct summand of $\pi_2(X)$ with respect to the intersection pairing by [73, theorem 5. 2]. The PD_4-complex X is *minimal* if every such map is a homotopy equivalence, i.e., X is minimal with respect to the partial order on PD_4-complexes determined by setting $X \geq Y$ if there is a 2-connected degree 1-map $f\colon X \to Y$. This is so if the intersection pairing is trivial, by [73, Lemma 2.2] and [53, Theorem 5.5]. We shall then say that X is *strongly minimal.*

Remark 1.4.2. If we assume that $\mathrm{cd}(\pi_1(X)) \leq 2$, then we may drop the qualification strongly for the two notions of minimality are equivalent by [44, Theorem 13].

Next, we are going to state the criterion for the existence of a strongly minimal model. But first recall the evaluation exact sequence: the evaluation homomorphism sits in an exact sequence

$$0 \longrightarrow H^2(\pi;\Lambda) \longrightarrow H^2(X;\Lambda) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_\Lambda(\pi_2,\Lambda) \longrightarrow H^3(\pi;\Lambda) \longrightarrow 0\ .$$

The cohomology intersection pairing s_X is defined by $s_X(u,v) = \mathrm{ev}(v)(PD(u))$ for all $u, v \in H^2(X;\Lambda)$ where PD stands for Poincaré dual. Since $s_X(u,v) = 0$ for all $u \in H^2(X;\Lambda)$ and $v \in H^2(\pi;\Lambda)$, the pairing s_X induces a pairing

$$s'_X\colon H^2(X;\Lambda)/H^2(\pi;\Lambda) \times H^2(X;\Lambda)/H^2(\pi;\Lambda) \to \Lambda\ .$$

Theorem 1.4.3. ([44, Theorem 2]) *Let X be a PD_4-complex and K a finitely generated projective summand of $\pi_2(X)$ such that s'_X restricts to a nonsingular pairing on $K \times K$, then there is a 2-connected degree-1 map to a PD_4-complex Z, $g_X\colon X \to Z$ with $K_2(g_X) = K$.*

Corollary 1.4.4. ([44, p. 2416]) *The PD_4-complex X has a strongly minimal model if and only if $H^2(X;\Lambda)/H^2(\pi;\Lambda)$ is a finitely generated projective Λ-module and s'_X is nonsingular.*

The above conditions hold if $\mathrm{cd}(\pi_1(X)) \leq 2$. We are going to use the notion of strongly minimal models when we deal with 4-manifolds with PD_2 fundamental group π (then we have $\mathrm{cd}(\pi) = 2$). If $\mathrm{cd}(\pi) = 2$, a group π is a PD_2-group if $H^2(\pi;\Lambda)$ is infinite cyclic [11]. The strongly minimal PD_4-complexes with fundamental group a PD_2-group are the total spaces of S^2-bundles over aspherical closed surfaces by [41, Theorem 5.10] and [44, Theorem 13].

Next we are going to give the definitions of *algebraic* 2*-type* and *generalized Eilenberg-Mac Lane spaces* : If X is a CW-complex, let $u_X \colon X \to K(\pi_1(X),1)$ denote the classifying map for the universal covering $\widetilde{X}$ of X and let $c_X \colon X \to B(X)$ denote the second stage of the Postnikov tower for X. We have

$$\begin{tikzcd} & B(X) \arrow[d, "u_{B(X)}"] \\ X \arrow[ur, "c_X"] \arrow[r, "u_X"'] & K(\pi_1(X),1) \end{tikzcd}$$

that is, $u_X = u_{B(X)} \circ c_X$. A map $f \colon X \to K(\pi_1(X),1)$ lifts to a map from X to $B(X)$ if and only if $f^*(k_1(X)) = 0$, where $k_1(X) \in H^3(\pi_1(X);\pi_2(X))$ is the first k-invariant of X. In particular, if $k_1(X) = 0$ then $u_{B(X)}$ has a cross-section. The algebraic two type of X is the triple $[\pi_1(X), \pi_2(X), k_1(X)]$.

Two such triples $[\pi_1(X), \pi_2(X), k_1(X)]$ and $[\pi_1(X'), \pi_2(X'), k_1(X')]$ are equivalent if there are isomorphisms

$$\alpha \colon \pi_1(X) \to \pi_1(X') \quad \text{and} \quad \beta \colon \pi_2(X) \to \pi_2(X')$$

such that $\beta(gm) = \alpha(g)\beta(m)$ for all $g \in \pi_1(X)$, $m \in \pi_2(X)$ and $\beta_*(k_1(X)) = \alpha^*(k_1(X')) \in H^3(\pi_1(X); \alpha^*\pi_2(X'))$ (where $\alpha^*\pi_2(X')$ denotes the induced $\pi_1(X)$-module structure on $\pi_2(X')$ by α). Such an equivalence can be realized by a homotopy equivalence of $B(X)$ and $B(X')$, i.e., the algebraic 2-type $[\pi_1(X), \pi_2(X), k_1(X)]$ and the Postnikov 2-stage determine each other, and $k_1(X) = 0$ if and only if $u_{B(X)}$ has a section.

If P is a $\mathbb{Z}[\pi_1(X)]$-module, then let $L := L_{\pi_1(X)}(P, 2)$ be the space with algebraic 2-type $[\pi_1(X), P, 0]$ and universal covering space $\widetilde{L} \simeq K(P, 2)$. Such objects L always exist and they are unique up to homotopy equivalence ([6, p. 214]). Note that if $P = \pi_2(X)$, then $k_1(X) = 0$ if and only if $B(X) \simeq L$. The space L is a generalized Eilenberg- Mac Lane space: if $f\colon X \to K(\pi_1(X), 1)$ is a map, then there is a natural bijection from the set of homotopy classes of maps $g\colon X \to L$ lifting f to $H^2(X; f^*P)$ (see [6], [41] and [52] for more details).

Chapter 2

Free Fundamental Group

2.1 The structures of π_2 and $\Gamma(\pi_2)$

Let M be a closed, oriented, smooth or topological 4-manifold with fundamental group π, a free group of rank r. Let $\Lambda = \mathbb{Z}[\pi]$ denote the integral group ring of π. The standard involution $\lambda \to \overline{\lambda}$ on Λ is induced by the formula

$$\sum n_g g \to \sum n_g g^{-1}$$

for $n_g \in \mathbb{Z}$ and $g \in \pi$. All modules considered in this thesis will be right Λ-modules, unless otherwise noted. However if L is any left Λ-module, let $\overline{L}$ be the corresponding right Λ-module with the conjugate structure given by $l.\lambda = \overline{\lambda}.l$, for $\lambda \in \Lambda$ and $l \in L$. In particular, if R is a right Λ-module, following [42] we will denote the conjugate dual module $\overline{\mathrm{Hom}_\Lambda(R, \Lambda)}$ by $R^\dagger$.

Lemma 2.1.1 ([42]). *If R is a finitely presentable right Λ-module then $R^\dagger$ is a free Λ-module.*

Proof. Let

$$F_1 \xrightarrow{i} F_0 \longrightarrow R \longrightarrow 0$$

be a presentation for R. Dualizing gives an exact sequence

$$0 \longrightarrow R^{\dagger} \longrightarrow F_0^{\dagger} \xrightarrow{i^{\dagger}} F_1^{\dagger}$$

Now consider the factor module $\mathrm{coker}(i^{\dagger}) = F_1^{\dagger}/\mathrm{im}(i^{\dagger})$. Since π is a free group, $\mathrm{vcd}\,\pi = \mathrm{cd}\,\pi = 1$ and the only finite subgroup of π is 1, and $\widehat{H^*}(\{1\}, -) = 0$. So, every Λ-module M is cohomologically trivial which implies $\mathrm{projdim}_{\Lambda} M \leq 2$ (see [13, p. 287]). Hence, $\mathrm{projdim}_{\Lambda}\,\mathrm{coker}(i^{\dagger}) \leq 2$ and

$$0 \longrightarrow R^{\dagger} \longrightarrow F_0^{\dagger} \longrightarrow F_1^{\dagger} \longrightarrow \mathrm{coker}(i^{\dagger}) \longrightarrow 0$$

implies $R^{\dagger}$ is projective ([13, p.184]). But projective modules over free group rings are free ([2]), so $R^{\dagger}$ is free. □

We may assume that M has the homotopy type of a finite complex because Wall's finiteness obstruction vanishes. Let $C_* = C_*(M;\Lambda)$ be the cellular chain complex of $\widetilde{M}$, with respect to the natural π-equivariant cell structure. This is a complex of free right Λ-modules. Let $B_q \subseteq Z_q$ denote the q-dimensional boundaries and q-cycles in C_q respectively, and let $H_q = H_q(C_*) = Z_q/B_q$, for $q \geq 0$. Then $H_q = H_q(M;\Lambda)$ is isomorphic to $H_q(\widetilde{M};\mathbf{Z})$ with the right Λ-module structure given by the action of π by deck transformations on $\widetilde{M}$. In particular $H_2 \cong \pi_2(M) = \pi_2$. We have the following exact sequences :

$$0 \longrightarrow B_0 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

Since π is a free group, the augmentation ideal is a free Λ-module, so $\mathbb{Z}$ has a short free resolution. By Schanuel's Lemma B_0 is stably free hence free ([2]). Note that $Z_1 = B_1$, since $H_1(\widetilde{X}) = 0$. We have a split exact sequence

$$0 \longrightarrow B_1 \longrightarrow C_1 \longrightarrow B_0 \longrightarrow 0\ .$$

So B_1 is also stably free, hence free.

$$0 \longrightarrow Z_2 \longrightarrow C_2 \longrightarrow B_1 \longrightarrow 0$$

This sequence also splits, so Z_2 is free. In particular, B_0, B_1 and Z_2 are finitely generated free Λ-modules. Hence

$$C_3 \longrightarrow Z_2 \longrightarrow \pi_2 \longrightarrow 0$$

is a finite presentation for π_2 (this presentation will later be used in the proof of Theorem 2.1.4).

Since some of our techniques are homotopy theoretic, we will sometimes work in the category of Poincaré duality complexes. So let us briefly outline the definition of Poincaré duality complexes.

Definition 2.1.2. A finite connected CW-complex X is called a Poincaré duality complex of dimension n (PD_n-complex for short) if there exits a class $[X] \in H_n(X)$ such that the cap product with $[X]$ gives isomorphisms $H^i(X; A) \to H_{n-i}(X; A)$ for any left Λ-module A.

Every n-dimensional manifold is homotopy equivalent to a *CW*-complex and hence determines a PD_n-complex ([48]). But there are PD_n-complexes which do not have the homotopy type of an n-dimensional manifold (Gitler and Stasheff [30] constructed an example of a simply connected finite complex which satisfies 5-dimensional Poincaré duality, but which is not the homotopy type of a closed topological manifold).

Lemma 2.1.3. *Let M be a PD_4-complex with fundamental group π. Then there is an exact sequence*

$$0 \longrightarrow H^2(\pi; \Lambda) \longrightarrow H^2(M; \Lambda) \xrightarrow{\text{ev}} \mathrm{Hom}_\Lambda(\pi_2, \Lambda) \longrightarrow H^3(\pi; \Lambda) \longrightarrow 0$$

Proof. Use the Serre spectral sequence of the fibration $\widetilde{M} \to M \to K(\pi,1)$, whose E_2-term is given by $E_2^{p,q} = H^p(\pi; H^q(\widetilde{M};\Lambda)) \Rightarrow H^{p+q}(M;\Lambda)$. The non-zero terms on the $p+q=2$ line are

$$E_2^{2,0} = H^2(\pi;\Lambda) \cong E_\infty^{2,0}$$

and

$$\begin{aligned} E_2^{0,2} &= H^0(\pi; H^2(\widetilde{M};\Lambda)) \cong H^2(\widetilde{M};\Lambda)^\pi \cong \mathrm{Hom}(H_2(\widetilde{M};\mathbb{Z}),\Lambda)^\pi \\ &\cong \mathrm{Hom}_\Lambda(H_2(\widetilde{M};\mathbb{Z}),\Lambda) \cong \mathrm{Hom}_\Lambda(\pi_2,\Lambda) \ . \end{aligned}$$

There is a differential $d_3\colon E_3^{0,2} \cong E_2^{0,2} \to E_3^{3,0} \cong H^3(\pi;\Lambda)$. This d_3 must be onto, since $H^3(M;\Lambda) \cong H_1(\widetilde{M};\mathbb{Z}) = 0$. Now, when we write the filtration for $H^2(M;\Lambda)$, we get the above exact sequence. □

So for π a free group we have $\overline{\pi_2} \cong H^2(M;\Lambda) \cong \mathrm{Hom}_\Lambda(\pi_2,\Lambda)$. Thus, $\pi_2 \cong \overline{\mathrm{Hom}_\Lambda(\pi_2,\Lambda)} = \pi_2^\dagger$. Recall that $C_3 \longrightarrow Z_2 \longrightarrow \pi_2 \longrightarrow 0$ gives a finite presentation for π_2, then by Lemma 2.1.1, $\pi_2 \cong \pi_2^\dagger$ is a free Λ-module. Also we have $\pi_2 \otimes_\Lambda \mathbb{Z} \cong H_2(C_* \otimes_\Lambda \mathbb{Z}) \cong H_2(M;\mathbb{Z}) \cong \mathbb{Z}^{\beta_2(M)}$ and hence $\pi_2 \cong \Lambda^{\beta_2(M)}$. Summarizing we have proved:

Theorem 2.1.4. *Let M be a closed, oriented, smooth or topological 4-manifold with free fundamental group. Then $\pi_2(M)$ is a free Λ-module.*

Next, we are going to show that $\Gamma(\pi_2)$ is a free Λ-module whenever π is a free group. Let us start with putting a simple ordering $\leq$ on π, i.e., elements g_i of π are indexed $(1-1)$ fashion to a set $\{i\}$ with a simple ordering. Now, for a given $g_i \in \pi$, we define a homomorphism

$$\Lambda \to \Gamma(\Lambda) \subseteq \Lambda \otimes \Lambda \quad \text{by} \quad 1 \to 1 \otimes g_i + g_i \otimes 1 \ .$$

Let $D = \pi - 1$. On D we have a free involution mapping $g_i \to g_i^{-1}$. If we fix for each orbit $\{g_i, g_i^{-1}\}$ of $D/\mathbb{Z}_2$ a representative, say g_i, we obtain a Λ-homomorphism

$$\Lambda^{|D/\mathbb{Z}_2|} = \prod_{|D/\mathbb{Z}_2|} \Lambda \to \Gamma(\Lambda) \subseteq \Lambda \otimes \Lambda$$

by mapping the component corresponding to $\{g_i, g_i^{-1}\}$ via the map $1 \to 1 \otimes g_i + g_i \otimes 1$. We have also a Λ-homomorphism

$$\Lambda \to \Gamma(\Lambda) \quad \text{mapping} \quad 1 \to 1 \otimes 1\ .$$

Lemma 2.1.5. *The maps above give a Λ-isomorphism*

$$\Lambda \oplus \Lambda^{|D/\mathbb{Z}_2|} \xrightarrow{\cong} \Gamma(\Lambda)$$

Proof. As a $\mathbb{Z}$ module Λ has basis π. Thus $\Gamma(\Lambda)$ has a $\mathbb{Z}$ basis

$$\{g_i \otimes g_i |\ i \in \mathbb{Z}\} \cup \{g_i \otimes g_j + g_j \otimes g_i |\ i, j \in \mathbb{Z},\ i < j\}$$

by Proposition 1.2.2 and the injectivity of $\overline{\Delta}\colon \Gamma(G) \to G \otimes G$. All these basis elements are contained in the image of the homomorphism above. The intersection of the image of two different components in the direct sum is $\{0\}$. Thus it is enough to check that the maps on the components are injective. It is easy to check that the $\mathbb{Z}$-basis of Λ is mapped to pairwise different basis elements of $\Gamma(\Lambda)$. We have to show that this $\mathbb{Z}$-basis of $\Gamma(\Lambda)$ is linearly independent in Λ. Let

$$\sum_{i\in\mathbb{Z}} \left[\left(\sum_{k_i\in\mathbb{Z}} n_{k_i} g_{k_i}\right) g_i \otimes g_i\right] + \sum_{i<j} \left[\left(\sum_{k_{ij}\in\mathbb{Z}} n_{k_{ij}} g_{k_{ij}}\right) (g_i \otimes g_j + g_j \otimes g_i)\right] = 0$$

$$\sum_{i\in\mathbb{Z}} \sum_{k_i\in\mathbb{Z}} n_{k_i} (g_{k_i+i} \otimes g_{k_i+i}) + \sum_{(i,j)} \sum_{k_{ij}} n_{k_{ij}} \left(g_{k_{ij}+i} \otimes g_{k_{ij}+j} + g_{k_{ij}+j} \otimes g_{k_{ij}+i}\right) = 0\ .$$

But $\{g_i \otimes g_i | i \in \mathbb{Z}\} \cup \{g_i \otimes g_j + g_j \otimes g_i |\ i, j \in \mathbb{Z}, i < j\}$ forms a $\mathbb{Z}$-basis. $\square$

We showed that $\Gamma(\Lambda)$ is free as a Λ-module. Recall that $\pi_2 \cong \Lambda^{\beta_2(M)}$ and $\Gamma(K \oplus L) \cong \Gamma(K) \oplus \Gamma(L) \oplus K \otimes L$. So, to show that $\Gamma(\pi_2)$ is free it is enough to show that $\underbrace{(\Lambda \oplus \Lambda \oplus \ldots \oplus \Lambda)}_{\beta_2(M)-1} \otimes \Lambda$ is a free Λ-module.

We will show that $\{1 \otimes 1 \otimes \ldots \otimes g_i | \ i \in \mathbb{Z}\}$ is a Λ-basis.

$$\sum_{i\in\mathbb{Z}} \sum_{j_i\in\mathbb{Z}} n_{j_i} g_{j_i} (1 \otimes 1 \otimes \ldots \otimes g_i)$$

$$= \sum \sum n_{j_i} (g_{j_i} \otimes g_{j_i} \otimes \ldots \otimes g_{j_i+i}) = 0 \ .$$

But since tensor product of free modules is free, $n_{j_i} = 0$. Therefore, $H_4(\widetilde{B}) \cong \Gamma(\pi_2)$ is a free Λ-module.

Remark 2.1.6. For the last step, it is enough to show that $\Lambda \otimes \Lambda$ is a free Λ-module and there are two alternative ways to see this: first we can use the following theorem

Theorem 2.1.7. ([13, p. 69]) *Let M be a G-module and let M_0 be its underlying Abelian group. Then $\mathbb{Z}G \otimes M$ (with the diagonal G-action) is canonically isomorphic to the induced module $\mathbb{Z}G \otimes M_0$. In particular, $\mathbb{Z}G \otimes M$ is a free $\mathbb{Z}G$-module if M is free as a $\mathbb{Z}$-module.*

Then take $G = \pi$ and $M = \Lambda$ and now the result follows. Alternatively we can first turn $\Lambda = \mathbb{Z}[\pi]$ into a Hopf algebra by defining

(i) $\Delta\colon \Lambda \to \Lambda \otimes \Lambda$ by $\Delta(g) = g \otimes g$,

(ii) $\varepsilon\colon \Lambda \to \mathbb{Z}$ by $\varepsilon(g) = 1$,

(iii) $\eta\colon \Lambda \to \Lambda$ by $\eta(g) = g^{-1}$

for all $g \in \pi$. Then, thinking Λ both as a projective Λ-module and a free $\mathbb{Z}$-module, we can use the following proposition

Proposition 2.1.8 ([9]). *Suppose P is a projective module and M is an R-free module for a Hopf algebra Λ over R. Then $P \otimes_R M$ is projective.*

Take $P = M = \Lambda$ and $R = \mathbb{Z}$ also recall that the projective modules over free group rings are free by [2].

Remark 2.1.9. Actually we have showed that if F is a free Λ-module, then $\Gamma(F)$ is also a free Λ-module.

2.2 Spin Case

The purpose of this section is to state and prove a theorem calculating the group $\mathrm{Aut}_\bullet(M)$, when M is a spin 4-manifold with free fundamental group. Our main result is Theorem 2.2.14. Throughout the section we mean smooth (or topological) bordism and homology with integral coefficients unless otherwise noted.

Hambleton and Kreck [35] constructed a braid

$$
\begin{tikzcd}
\Omega_5^{Spin}(M) \arrow[dr] \arrow[rr, bend left] & & \widetilde{\mathcal{H}}(M) \arrow[dr] \arrow[rr, bend left] & & \mathrm{Aut}_\bullet(B) \arrow[dr, "\beta"] & \\
& \Omega_5^{Spin}(B) \arrow[ur] \arrow[dr] & & \mathrm{Aut}_\bullet(M) \arrow[ur] \arrow[dr, "\alpha"] & & \Omega_4^{Spin}(B) \\
\pi_1(\mathcal{E}_\bullet(B)) \arrow[ur] \arrow[rr, bend right] & & \widehat{\Omega}_5^{Spin}(B, M) \arrow[ur, "\gamma"] \arrow[rr, bend right] & & \widehat{\Omega}_4^{Spin}(M) \arrow[ur] &
\end{tikzcd}
$$

of exact sequences that is sign commutative, the sub-diagrams are all strictly commutative except for the two composites ending in $\mathrm{Aut}_\bullet(M)$, and valid for any closed, oriented, smooth or topological spin 4-manifold M. We will use this braid to obtain an explicit formula for $\mathrm{Aut}_\bullet(M)$. While we calculate the terms on the braid we will give the necessary definitions (for the details we always refer to [35]).

We will fix a lift $\nu_M\colon M \to BSpin$ of the classifying map for the stable normal bundle of M. The Abelian group $\Omega_n^{Spin}(M)$, with disjoint union as the group operation, denotes the singular bordism group of spin manifolds with a reference map to M, i.e. an element (N, f) of $\Omega_n^{Spin}(M)$ is represented by an n manifold N^n endowed with a spin structure and a continuous map $f\colon N \to M$. We consider (N_1, f_1) and (N_2, f_2) as equivalent provided that they make up the boundary of an $(n+1)$-dimensional spin manifold W with a reference map $F\colon W \to M$ such that the spin structures on N_1 and N_2 induced from the one on W and F restricted to the boundaries give f_1 and f_2. By imposing the requirement that the reference maps to M must have degree zero, we will obtain the modified bordism groups $\widehat{\Omega}_4^{Spin}(M)$.

Proposition 2.2.1. *The relevant spin bordism groups of M are given as follows:*

$$\Omega_4^{Spin}(M) \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}\ ,$$
$$\Omega_5^{Spin}(M) \cong H_1(M) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}/2\ .$$

Proof. This follows from the Atiyah - Hirzebruch spectral sequence, whose E^2-term is $H_p(M;\Omega_q^{Spin}(*))$. The first differential $d_2\colon E^2_{p,q} \to E^2_{p-2,q+1}$ is given by the dual of Sq^2 (if $q=1$) or this composed with reduction mod 2 (if $q=0$), see [70, p.751]. We substitute the values

$$\Omega_q^{Spin}(*) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0 \quad \text{for} \quad 0 \le q \le 5\ .$$

Remark 2.2.2. Defining spin structures for 1- and 2-manifolds requires first stabilization. We have S^1 as the generator of $\Omega_1^{Spin}(*)$ and $S^1 \times S^1$ as the generator of $\Omega_2^{Spin}(*) \cong \mathbb{Z}/2$, see [62, pp.521, 523].

The differential for $(p,q) = (4,1)$ is dual to $Sq^2\colon H^2(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$ which is zero, since M is spin (see [57, p.132]). The differential for $(p,q) = (3,1)$ is zero for $Sq^i(x) = 0$ if $i > \deg(x)$. We obtain a filtration

$$\Omega_4^{Spin}(*) \oplus H_2(M;\Omega_2^{Spin}(*)) \underbrace{\subseteq}_{\mathbb{Z}/2} F_{3,1} \underbrace{\subseteq}_{\mathbb{Z}} \Omega_4^{Spin}(M)$$

we have

$$0 \longrightarrow \Omega_4^{Spin}(*) \oplus H_2(M;\mathbb{Z}/2) \longrightarrow F_{3,1} \longrightarrow H_3(M;\Omega_1^{Spin}(*)) \longrightarrow 0$$

and

$$V \times S^1 \xrightarrow{f \circ pr_1} F_{3,1}$$

gives the splitting of the above short exact sequence, where we consider an embedding $f\colon V \to M$ of a closed spin 3-manifold representing a generator of $H_3(M;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^r$, by Thom Realizability Theorem [12, Theorem 11.16], where S^1 is equipped with the non-trivial spin structure. Finally, since $\mathrm{Ext}(\mathbb{Z},-) = 0$, the line $p+q=4$ gives $\Omega_4^{Spin}(M) \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}$.

For the line $p+q=5$ on the E^2-page, we have $H_1(M)$ in the $(1,4)$ position, $H_3(M;\mathbb{Z}/2)$ in the $(3,2)$ position, and $H_4(M;\mathbb{Z}/2)$ in the $(4,1)$ position and all these terms survive to E^∞. We have a filtration

$$H_1(M;\Omega_4^{Spin}(*)) \subseteq F_{3,2} \subseteq F_{4,1} \subseteq \Omega_5^{Spin}(M)$$

and a short exact sequence

$$0 \longrightarrow H_1(M) \longrightarrow F_{3,2} \longrightarrow H_3(M;\Omega_2^{Spin}(*)) \longrightarrow 0$$

again

$$N \times (S^1 \times S^1) \xrightarrow{f \circ \mathrm{pr}_1} F_{3,2}$$

gives the splitting, where N is a closed spin 3-manifold and for

$$0 \longrightarrow F_{3,2} \longrightarrow F_{4,1} \longrightarrow H_4(M; \Omega_1^{Spin}(*)) \longrightarrow 0$$

$$M \times S^1 \xrightarrow{pr_1} M$$

gives the splitting. Therefore, $\Omega_5^{Spin}(M) \cong H_1(M) \oplus H_3(M; \mathbb{Z}/2) \oplus \mathbb{Z}/2$. □

Corollary 2.2.3. *The modified bordism group is given as*

$$\widehat{\Omega}_4^{Spin}(M) \cong \mathbb{Z} \oplus H_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$$

Proof. By the above proposition, we have

$$\Omega_4^{Spin}(M) \cong \Omega_4^{Spin}(*) \oplus H_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2) \oplus H_4(M) .$$

The first summand is generated by $*\colon K3 \to M$ (see e.g. [62, p. 229]) where $*$ is the constant map to the base point $x_0 \in M$. Note that we have

$$H_2(M; \mathbb{Z}/2) \cong H_0(\pi; H_2(\widetilde{M}; \mathbb{Z}/2)) \cong (\pi_2 \otimes \mathbb{Z}/2) \otimes_\Lambda \mathbb{Z} .$$

Hence any element of $H_2(M; \mathbb{Z}/2)$ can be represented by a map $\sigma\colon S^2 \to M$. As a result the second summand in the above direct sum decomposition is generated by $\sigma_i \circ p_1\colon S^2 \times (S^1 \times S^1) \to M$ where $\sigma_i\colon S^2 \to M$ represents a generator of $H_2(M; \mathbb{Z}/2)$. The third summand is generated by $f_i \circ p_1\colon V^3 \times S^1 \to M$ where $f_i\colon V^3 \to M$ represents a generator of $H_3(M; \mathbb{Z}/2)$. The last summand is generated by id: $M \to M$. Since by definition for $\widehat{\Omega}_4^{Spin}(M)$ reference maps to M must have degree zero, we have to drop the last summand. □

The CW-complex $B := B(M)$ is the 2-type of M(second stage of the Postnikov tower for M), i.e., there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "c"] \arrow[d, "u_M"'] & B \arrow[d, "u_B"] \\ B\pi \arrow[r, equal] & B\pi \end{tikzcd}$$

Here u_M is unique up to homotopy and a classifying map for the universal covering $\widetilde{M}$ of M. Let us briefly explain the construction of the above diagram: We can attach cells of dimension ≥ 4 to obtain a CW-complex structure for B with the following properties

(i) The inclusion map $M \to B$ induces isomorphisms $\pi_k(M) \to \pi_k(B)$ for $k \leq 2$, and

(ii) $\pi_k(B) = 0$ for $k \geq 3$.

Note that the universal covering space $\widetilde{B}$ of B is the Eilenberg-MacLane space $K(\pi_2, 2)$, and the inclusion $\widetilde{M} \to \widetilde{B}$ induces isomorphism on π_2.

Consider the universal π-fibration $E\pi \to B\pi$ and its associated fibration $p\colon E\pi \times_\pi \widetilde{B} \to B\pi$ with typical fiber $\widetilde{B}$. Since π acts freely on $\widetilde{B}$, there is also a fibration $q\colon E\pi \times_\pi \widetilde{B} \to B$ with fiber $E\pi$. The same argument also gives a fibration $r\colon E\pi \times_\pi \widetilde{M} \to M$. The fiber $E\pi$ is contractible, so the maps q and r must be homotopy equivalences. The inclusion $\widetilde{M} \subset \widetilde{B}$ induces an inclusion $i\colon E\pi \times_\pi \widetilde{M} \to E\pi \times_\pi \widetilde{B}$. The classifying map of M is defined as $c := q \circ i \circ r^{-1}\colon M \to B$ which induces isomorphisms $\pi_k(c)\colon \pi_k(M) \to \pi_k(B)$ for any $k \leq 2$; in other words, c is a 3-equivalence. Observe that the map c is homotopic to the inclusion of M into B.

In order to calculate the bordism groups of B, we need to find $H_i(B)$. Note that B is a fibration over $K(\pi, 1) = B\pi$, bouquet of circles, with fibre $K(\pi_2, 2)$, universal covering space $\widetilde{B}$. We will use the Serre spectral sequence of the fibration $\widetilde{B} \to B \to K(\pi, 1)$ whose E^2-term is given by $E^2_{p,q} = H_p(K(\pi, 1); H_q(\widetilde{B}))$, the homology of $K(\pi, 1)$ with local coefficients in the homology of $\widetilde{B}$. So we will need homologies of $\widetilde{B}$. First recall a theorem of Whitehead:

Theorem 2.2.4 ([75]). *Let X be a CW complex and Γ denote the Whitehead's quadratic functor, then there is a functorial Whitehead exact sequence*

$$\pi_4(\widetilde{X}) \xrightarrow{h} H_4(\widetilde{X}) \xrightarrow{b} \Gamma(\pi_2(\widetilde{X})) \longrightarrow \pi_3(\widetilde{X}) \xrightarrow{h} H_3(\widetilde{X}) \longrightarrow 0$$

of right Λ-modules, where h is the Hurewicz homomorphism in degree 3 *and* 4 *and b is the secondary boundary homomorphism (see Chapter I of [7] for a recent exposition).*

We have $H_4(\widetilde{B}) \cong \Gamma(\pi_2)$, since $\pi_i(\widetilde{B}) = 0$ for $i > 3$. A finitely generated free group is countable, so π is countable, therefore Λ and hence π_2 is countable. Let

$$X_0 = *, X_1 = K(\mathbb{Z}, 2), \cdots, X_N = K(\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{N}, 2), \cdots$$

Consider the sequence of maps

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \cdots$$

where i_k's are inclusions. The mapping telescope is the union of the mapping cylinders $M_{i_k} = \frac{X_i \times I \coprod X_{i+1}}{(x_i, 1) \sim f(x_i)}$ with the copies of X_k in M_{i_k} and $M_{i_{k+1}}$ identified for all k. Let T_k be the union of the first k mapping cylinders. This deformation retracts onto X_k by deformation retracting each mapping cylinder onto its right end in turn. Since the inclusion maps i_k are cellular, each mapping cylinder is a CW complex and the telescope is the increasing union of the subcomplexes $T_k \simeq X_k$.

Observe that $\widetilde{B}$ is homotopy equivalent to the mapping telescope of the above sequence and we have, see [38, p. 312]

$$H_n(\widetilde{B}) \cong \varinjlim H_n(X_k)$$

$$H^n(\widetilde{B}; \mathbb{Z}/2) \cong \varprojlim H^n(X_k; \mathbb{Z}/2)$$

Proposition 2.2.5. *Let B denote the 2-type of a spin 4-manifold with free fundamental group. The homology groups of B are given by*

$$H_i(B) \cong \begin{cases} H_i(M) & \textit{if } i = 0, 1 \textit{ or } 2 \\ 0 & \textit{if } i = 3 \textit{ or } 5 \\ \mathbb{Z} \otimes_\Lambda \Gamma(\pi_2(M)) & \textit{if } i = 4 \end{cases}$$

Proof. We are going to use the Serre spectral sequence of the fibration $\widetilde{B} \to B \to K(\pi, 1)$ whose E_2-term is given by $E_2^{p,q} = H_p(\pi; H_q(\widetilde{B}))$. First note that since the natural map $c\colon M \to B$ is 3-connected, obviously we have $H_i(M) \cong H_i(B)$ for $i \leq 2$. If we substitute the values

$$H_i(\widetilde{B}) \cong \begin{cases} 0 & \text{if } i = 1, 3, \text{ or } 5 \\ \pi_2(M) & \text{if } i = 2 \\ \Gamma(\pi_2) & \text{if } i = 4 \end{cases}$$

then we get the following isomorphisms:

$H_3(B) \cong H_1(\pi; H_2(\widetilde{B})) \cong H_1(\pi, \pi_2) = 0$, since π_2 is a free Λ-module,

$H_4(B) \cong H_0(\pi; H_4(\widetilde{B})) \cong H_0(\pi, \Gamma(\pi_2)) \cong \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z}$,

$H_5(B) \cong H_1(\pi; H_4(\widetilde{B}) \cong H_1(\pi; \Gamma(\pi_2)) = 0$, since $\Gamma(\pi_2)$ is a free Λ-module.

□

Proposition 2.2.6. *Let $\Omega_*^{Spin}(B)$ denote the singular bordism group of spin manifolds with a reference map to B. We then have the following:*

$$\Omega_4^{Spin}(B) \subset H_4(B) \oplus \mathbb{Z} \quad \text{and} \quad \Omega_5^{Spin}(B) \cong H_1(B)$$

where the first inclusion is given by $[N,f] \to (f_*[N], \sigma(N))$.

Proof. We use the same spectral sequence, whose E^2-term is $H_p(B;\Omega_q^{Spin}(*))$. For the line $p+q=4$, the non-zero terms on the E^2-page are $H_0(B;\Omega_4^{Spin}(*))$ in the $(0,4)$ position, $H_2(B;\Omega_2^{Spin}(*))$ in the $(2,2)$ position and $H_4(B;\Omega_0^{Spin}(*))$ in the $(4,0)$ position. The differential $d_2\colon E^2_{4,1} \to E^2_{2,2}$ is the dual of $Sq^2\colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2)$. By using the Serre spectral sequence of the fibration $\widetilde{B} \to B \to K(\pi,1)$, we can see that

$$H^2(B;\mathbb{Z}/2) \cong H^0(\mathbb{Z}; H^2(\widetilde{B};\mathbb{Z}/2)) \cong H^2(\widetilde{B};\mathbb{Z}/2)^\pi$$

and this isomorphism is given by p^*. Consider the following commutative diagram

$$\begin{tikzcd} H^2(\widetilde{B};\mathbb{Z}/2) \arrow[r, "Sq^2"] & H^4(\widetilde{B};\mathbb{Z}/2) \\ H^2(B;\mathbb{Z}/2) \arrow[u, "p^*"] \arrow[r, "Sq^2"] & H^4(B;\mathbb{Z}/2) \arrow[u, "p^*"] \end{tikzcd}$$

which implies that $Sq^2\colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2)$ is injective, since the homomorphism $Sq^2\colon H^2(\widetilde{B};\mathbb{Z}/2) \to H^4(\widetilde{B};\mathbb{Z}/2)$ is injective (to see this notice that $Sq^2\colon H^2(X_k;\mathbb{Z}/2) \to H^4(X_k;\mathbb{Z}/2)$ is injective and that $\varprojlim$ is left exact [51, p.164]). Hence $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$ is surjective. Therefore, on the line $p+q=4$, the only groups which survive to E^∞ are $\mathbb{Z}$ in the $(0,4)$ position, and a subgroup of $H_4(B)$ in the $(4,0)$ position.

On the line $p+q=5$, the non-zero terms are $H_4(B;\Omega_1^{Spin}(*))$ in the $(4,1)$ position and $H_1(B;\Omega_4^{Spin}(*))$ in the $(1,4)$ position. The differential

$$d_2\colon E^2_{6,0} \cong H_6(B;\mathbb{Z}) \to E^2_{4,1} \cong H_4(B;\mathbb{Z}/2)$$

is reduction mod 2 composed with the dual of Sq^2. Consider the diagram

$$\begin{tikzcd} H^2(\widetilde{B};\mathbb{Z}/2) \arrow[r, "Sq^2"] & H^4(\widetilde{B};\mathbb{Z}/2) \arrow[r, "Sq^2"] & H^6(\widetilde{B};\mathbb{Z}/2) \\ H^2(B;\mathbb{Z}/2) \arrow[r, "Sq^2"] \arrow[u, "p^*"] & H^4(B;\mathbb{Z}/2) \arrow[r, "Sq^2"] \arrow[u, "p^*"] & H^6(B;\mathbb{Z}/2) \arrow[u, "p^*"] \end{tikzcd}$$

A direct calculation shows that the top row is exact. We have

$$H^4(B;\mathbb{Z}/2) \cong H^0(\mathbb{Z};H^4(\widetilde{B};\mathbb{Z}/2)) \cong H^4(\widetilde{B};\mathbb{Z}/2)^\pi$$

and again this isomorphism is given by p^*. Let $\alpha \in H^4(B;\mathbb{Z}/2)$ such that $Sq^2(\alpha) = 0$ and $p^*(\alpha) = \beta$. There exists $\lambda \in H^2(\widetilde{B};\mathbb{Z}/2)$ such that $Sq^2(\lambda) = \beta$, since the above row is exact. But $\beta \in H^4(\widetilde{B};\mathbb{Z}/2)^\pi$ implies $\lambda \in H^2(\widetilde{B};\mathbb{Z}/2)^\pi$. Therefore the sequence

$$H^2(B;\mathbb{Z}/2) \xrightarrow{Sq^2} H^4(B;\mathbb{Z}/2) \xrightarrow{Sq^2} H^6(B;\mathbb{Z}/2)$$

is exact. With the surjectivity of $H_6(B;\mathbb{Z}) \to H_6(B;\mathbb{Z}/2)$, we can conclude that $d_2\colon H_6(B;\mathbb{Z}) \to H_4(B;\mathbb{Z}/2)$ is surjective onto the kernel of the differential $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$. Thus the only group which survive to E_∞ is $H_1(B) = H_1(M)$ in the $(1,4)$ position. □

Next, we are going to give definitions of the maps α and γ and the modified bordism group $\widehat{\Omega}_5^{Spin}(B,M)$. The map $\alpha\colon \mathrm{Aut}_\bullet(M) \to \widehat{\Omega}_4^{Spin}(M)$ is defined by $\alpha(f) := [M,f] - [M,\mathrm{id}]$. This map is not a homomorphism since $\alpha(f \circ g) = \alpha(f) + f_*(\alpha(g))$. An element (W,F) of $\widehat{\Omega}_5^{Spin}(B,M)$ is a 5-dimensional spin manifold with boundary $(W,\partial W)$, equipped with a reference map $F\colon W \to B$ such that $F|_{\partial W}$ factors through the classifying map $c\colon M \to B$ and $F|_{\partial W}\colon \partial W \to M$ has degree zero.

By taking the boundary connected sum with the zero bordant element $(M \times I, p_1)$ along ∂W and $M \times \{1\}$, we may assume that W has two boundary components $\partial_1 W = -M$ and $\partial_2 W = N$. Since (W, F) is a modified bordism element $g := F|_N$ is a degree-1 map from $N \to M$. The obstructions to lifting the map $F\colon W \to B$ to M, relative to g lie in the groups $H^{i+1}(W, N; \pi_i(X)) \cong H_{4-i}(W, M; \pi_i(X))$, where X denotes the fibre of the map $c\colon M \to B$. These obstructions vanish because X is 2-connected. Let $r\colon W \to M$ be a lift of F and $f := r|_{\partial_1 W}\colon M \to M$. Observe that f is a degree-1 map and a 3-equivalence, so it is a homotopy equivalence. Define

$$\gamma(W, F) := [f\colon M \to M] \in \mathrm{Aut}_\bullet(M)\ .$$

The map γ is well defined. The crucial point is, if (W', F') is another representative for the same bordism class and (T, φ) is a bordism between (W, F) and (W', F') over $\widehat{\Omega}_5^{Spin}(B, M)$, then we may assume that $\varphi\colon T \to B$ is a 3-equivalence by surgery on the interior of T.

Corollary 2.2.7. *There is an isomorphism*

$$\widehat{\Omega}_5^{Spin}(B, M) \cong H_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$$

and the group $\widehat{\Omega}_5^{Spin}(B, M)$ *injects into* $\mathrm{Aut}_\bullet(M)$*. The image of* α *is equal to* $H_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$*.*

Proof. Recall that we have $\Omega_5^{Spin}(M) \cong H_1(M) \oplus H_3(M; \mathbb{Z}/2) \oplus H_4(M; \mathbb{Z}/2)$ and $\Omega_5^{Spin}(B) \cong H_1(B)$. Let $H_1(M) \cong \mathbb{Z}^r$ be generated by $\sigma_i\colon S^1 \to M$ for $i = 1, \ldots, r$, then $c \circ \sigma_i\colon S^1 \to B$ generate $H_1(B)$. The generators of the $H_1(M)$ summand in $\Omega_5^{Spin}(M)$ are represented by $\sigma_i \times \{*\}\colon S^1 \times K3 \to M$ where $\{*\}$ denotes the constant map and the generator of $\Omega_4^{Spin}(*)$ is the $K3$ surface. Note that we are using the non-trivial spin structure on S^1. Similarly $(c \circ \sigma_i) \times \{*\}\colon S^1 \times K3 \to B$ generates $\Omega_5^{Spin}(B)$. Therefore, the map $\Omega_5^{Spin}(M) \to \Omega_5^{Spin}(B)$, which is composing with our reference map $c\colon M \to B$, maps the summand $H_1(M)$ isomorphically to $H_1(B)$ and $H_3(M;\mathbb{Z}/2) \oplus H_4(M;\mathbb{Z}/2)$ to zero. Hence, the map $\Omega_5^{Spin}(M) \to \Omega_5^{Spin}(B)$ is onto, so by the exactness of the braid the map $\Omega_5^{Spin}(B) \to \widehat{\Omega}_5^{Spin}(B,M)$ must be zero. Therefore

$$\widehat{\Omega}_5^{Spin}(B,M) \cong \ker(\widehat{\Omega}_4^{Spin}(M) \to \Omega_4^{Spin}(B)) .$$

By Corollary 2.2.3, we have $\widehat{\Omega}_4^{Spin}(M) \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ and by Proposition 2.2.6, $\Omega_4^{Spin}(B) \subset \mathbb{Z} \oplus H_4(B)$. The map between these groups is composing with c, hence the kernel is $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$.

By the exactness of the braid $\Omega_5^{Spin}(B) \to \widehat{\Omega}_5^{Spin}(B,M)$ is the zero map [35, Lemma 2.2]. Also by the commutativity of the braid the map $\pi_1(\mathcal{E}_\bullet(B)) \to \widehat{\Omega}_5^{Spin}(B,M)$ is zero. Now again by the exactness of the braid $\gamma\colon \widehat{\Omega}_5^{Spin}(B,M) \to \mathrm{Aut}_\bullet(M)$ is an injective map [35, Corollary 2.13].

If $f\colon M \to M$ represents an element of $\mathrm{Aut}_\bullet(M)$, then we have $\alpha(f) = [M,f] - [M,\mathrm{id}]$. The natural map $\Omega_4^{Spin}(M) \to H_0(M)$ sends a spin 4-manifold to its signature. It follows that $\alpha(f) \in H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ since the signature is preserved by a homotopy equivalence. Since, by the exactness of the braid the map $\widehat{\Omega}_5^{Spin}(B,M) \to \widehat{\Omega}_4^{Spin}(M)$ and γ is injective $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \subseteq \operatorname{im}\alpha$ by the commutativity of the braid. Therefore $\operatorname{im}\alpha = H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. □

Before proceeding any further, let us look at $\mathrm{Aut}_\bullet(B)$ more closely. First of all, we can consider $\mathrm{Aut}_\bullet(B)$ as the group of based fibre homotopy classes of based fibre homotopy equivalences of B (see [58, Section 2]). Let $\mathrm{Isom}[\pi,\pi_2]$ be the subgroup of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(\pi_2)$ consisting of all those pairs (χ,ψ) for which $\psi(\eta a) = \chi(\eta)\psi(a)$ for all $\eta \in \pi$, $a \in \pi_2$.

For any $\phi \in \mathrm{Aut}_\bullet(B)$, the homomorphism $\pi_2(\phi)$ is $\pi_1(\phi)$-equivariant by [58, Lemma 2.1]. Now consider the following homomorphism

$$(\pi_1, \pi_2)\colon \mathrm{Aut}_\bullet(B) \to \mathrm{Isom}[\pi, \pi_2] \quad \text{given by} \quad (\pi_1, \pi_2)(\phi) = (\pi_1(\phi), \pi_2(\phi)).$$

We are going to find a section s for (π_1, π_2). Note that $B \simeq E\pi \times_\pi \widetilde{B}$ and for any $(\chi, \psi) \in \mathrm{Isom}[\pi, \pi_2]$, let $E\chi\colon E\pi \to E\pi$ and $\widetilde{\psi}\colon \widetilde{B} \to \widetilde{B}$ be the induced maps. We can define

$$s\colon \mathrm{Isom}[\pi, \pi_2] \to \mathrm{Aut}_\bullet(B) \quad \text{by} \quad s(\chi, \psi)([e, u]) = [E\chi(e), \widetilde{\psi}(u)]\ ,$$

where $[e, u]$ denotes the equivalence class of $(e, u) \in E\pi \times \widetilde{B}$. We have $s(\chi, \psi) \in \mathrm{Aut}_\bullet(B)$ (by Whitehead's theorem) and $(\pi_1, \pi_2)(s(\chi, \psi)) = (\chi, \psi)$. The kernel K of (π_1, π_2) consists of $\phi \in \mathrm{Aut}_\bullet(B)$ with $\pi_i(\phi) = \mathrm{id}$ for $i = 1, 2$. For $\phi \in K$, we have the following diagram

$$\begin{tikzcd} & K(\pi_2, 2) \arrow[d] \\ B \arrow[r, "\phi"] \arrow[dr, "p"'] & B \arrow[d, "p"] \\ & B\pi \end{tikzcd}$$

where the lower triangle is commutative. Associate to any such ϕ, a cohomology class $\delta(\phi, \mathrm{id}) \in H^2(B; \pi_2)$, the primary obstruction to the existence of a homotopy between ϕ and id, see [58, p.25] and [12, Theorem 13.11]. By the Serre spectral sequence of the fibration $\widetilde{B} \to B \to K(\pi, 1)$, we have the following diagram with an exact row,

$$\begin{tikzcd} 0 \arrow[r] & H^2(\pi; \pi_2) \arrow[r] & H^2(B; \pi_2) \arrow[r, "\mathrm{ev}"] & \mathrm{Hom}_\pi(\pi_2, \pi_2) \arrow[r] & 0 \\ & & K \arrow[u, "\delta(-,\mathrm{id})"] & & \end{tikzcd}$$

The vertical arrow is an injective map which takes K into $H^2(\pi;\pi_2)$, since $\mathrm{ev}(\delta(\phi,\mathrm{id})) = \pi_2(\phi) - \mathrm{id} = 0$.

Next we are going to construct a self-equivalence ϕ_x of B for each $x \in H^2(\pi;\pi_2)$. Remember that B is a $K(\pi_2,2)$ fibration over $B\pi$ with a section $\sigma\colon B\pi \to B$ (existence of such a section comes from the fact that $k_M = 0$). Since $K(\pi_2,2) = \Omega K(\pi_2,3)$, we may view B as the union of loop spaces joined together in an appropriate way. Let

$$B \times_{B\pi} B = \{(b_1,b_2) \in B \times B : u_B(b_1) = u_B(b_2)\}$$

and

$$\mu\colon B \times_{B\pi} B \to B$$

be the fibrewise loop multiplication, i.e. the restriction of μ to $u_B^{-1}(x)$ is multiplication of loops in $u_B^{-1}(x) \simeq \Omega K(\pi_2,3)$ for all $x \in B\pi$.

Let $[B\pi,B]_{B\pi}$ be the set of homotopy classes of (over $B\pi$ maps) $f\colon B\pi \to B$ such that $u_B \circ f = \mathrm{id}_{B\pi}$, in other words elements of $[B\pi,B]_{B\pi}$ are homotopy classes of sections of u_B. By Lemma 2.1.3 we have an exact sequence

$$0 \longrightarrow H^2(\pi;\pi_2) \longrightarrow H^2(B;\pi_2) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_\Lambda(\pi_2,\pi_2) \longrightarrow 0\ .$$

Let $\iota \in H^2(B;\pi_2)$ be such that $\mathrm{ev}(\iota) = \mathrm{id}_{\pi_2}$. The function

$$\theta\colon [B\pi,B]_{B\pi} \to H^2(\pi;\pi_2)$$

given by $\theta(f) = f^*(\iota)$ is an isomorphism with respect to the addition on $[B\pi,B]_{B\pi}$ determined by μ, i.e., $\theta(\sigma) = 0$, since $f^*(\iota)$ is the primary obstruction to the existence of a homotopy between f and σ (see [8, Theorem 5.2.4] or [45]), and $\theta(\mu(f,f')) = \theta(f) + \theta(f')$.

Let $x \in H^2(\pi;\pi_2)$ and $f_x = \theta^{-1}(x)$. Define a self-equivalence of B

$$\phi_x \colon B \to B \quad \text{as} \quad \phi_x = \mu(f_x u_B, \mathrm{id}_B)$$

i.e. $\phi_x(b) = f_x u_B(b).b$ where multiplication is multiplication of loops in the fiber over $u_B(b)$. Then $u_B \circ \phi_x = u_B$ and so $\phi_x \in [B,B]_{B\pi}$ where $[B,B]_{B\pi}$ denotes the set of homotopy classes of over $B\pi$ maps. We have $\phi_0 = \mu(\sigma u_B, \mathrm{id}_B) = \mathrm{id}_B$ and $\phi_x^*(\iota) = \iota + u_B^*(x) \in H^2(B;\pi_2)$. Note also that by homotopy associativity of μ we have

$$\phi_{x+y} = \mu(\mu(f_x, f_y)u_B, \mathrm{id}_B) =$$

$$\mu(\mu(f_x u_B, f_y u_B), \mathrm{id}_B) = \mu(f_x u_B, \mu(f_y u_B, \mathrm{id}_B)) \ .$$

Hence we see that

$$\phi_{x+y} = \mu(f_x u_B, \phi_y) = \mu(f_x u_B \phi_y, \phi_y) = \phi_x \phi_y \ .$$

Therefore ϕ_x is a homotopy equivalence for all $x \in H^2(\pi;\pi_2)$ and $x \to \phi_x$ defines a homomorphism from $H^2(\pi;\pi_2)$ to $\mathrm{Aut}_\bullet(B)$. The lift of ϕ_x to the universal cover $\widetilde{B}$ is homotopic to the identity, since the lift of u_B is homotopic to the constant map. Hence ϕ_x acts as identity on π_2. Therefore the homomorphism $x \to \phi_x$ is an isomorphism onto the kernel of the map (π_1, π_2). As a result we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^2(\pi;\pi_2) \arrow[r] & \mathrm{Aut}_\bullet(B) \arrow[r, "(\pi_1,\pi_2)"] & \mathrm{Isom}[\pi,\pi_2] \arrow[r] \arrow[l, bend left, "s"] & 1 \end{tikzcd} \ .$$

In particular we have

$$\mathrm{Aut}_\bullet(B) = H^2(\pi;\pi_2) \rtimes \mathrm{Isom}[\pi,\pi_2] \ ,$$

where the action of $\mathrm{Isom}[\pi,\pi_2]$ on $H^2(\pi;\pi_2)$ is given by $(\chi,\psi).x = \psi^*((\chi^{-1})^*(x))$.

Remark 2.2.8. In [71], it is proved that $H^2(\pi;\pi_2)$ is isomorphic to the group of all based homotopy classes of self-homotopy equivalences of B inducing the identity automorphisms of all homotopy groups.

If π is a free group, then $H^2(\pi;\pi_2) = 0$. Hence for this section we have $\mathrm{Aut}_\bullet(B) \cong \mathrm{Isom}[\pi_1, \pi_2]$.

Next, we look for a relation between $c_*[M]$ and the cohomology intersection pairing s_M on M. Recall that we have defined the homomorphism $\overline{\Delta}\colon \Gamma(\pi_2) \to \pi_2 \otimes \pi_2$ via the universal property $\overline{\Delta}\gamma(a) = a \otimes a$ and $\overline{\Delta}(\Gamma(\pi_2))$ is the subgroup of $\pi_2 \otimes \pi_2$ generated by $\{a \otimes a : a \in \pi_2\}$. Also $\Gamma(\pi_2)$ inherits a Λ-module structure given by the diagonal action on $\pi_2 \otimes \pi_2$. Therefore $\Gamma(\pi_2)$ is a Λ-submodule of $\pi_2 \otimes \pi_2$ inducing an inclusion

$$\Gamma(\pi_2) \otimes_\Lambda \mathbb{Z} \xhookrightarrow{\overline{\Delta}\otimes 1} (\pi_2 \otimes \pi_2) \otimes_\Lambda \mathbb{Z} \cong \pi_2 \otimes_\Lambda \overline{\pi_2}$$

where the bar denotes the left Λ-module structure provided by the canonical anti-automorphism on Λ. For π a free group we have

$$H_4(B) \cong H_0(\pi; H_4(\widetilde{B})) \cong H_4(\widetilde{B}) \otimes_\Lambda \mathbb{Z} .$$

The secondary boundary homomorphism $b\colon H_4(\widetilde{B}) \to \Gamma(\pi_2)$ on the Whitehead's exact sequence is an isomorphism, since $\pi_3(\widetilde{B}) = \pi_4(\widetilde{B}) = 0$. Hence $H_4(B) \cong \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z}$ which is inside the diagonal of $\pi_2 \otimes_\Lambda \overline{\pi_2}$. For any $x \in H_4(B)$, let us denote by $\sum_i x_{2i} \otimes x_{2i}$, the image of x in $\pi_2 \otimes_\Lambda \overline{\pi_2}$. On the other hand we can define

$$\mathrm{ev}_{\pi_2}(x_2)(u,v) = u(x_2)\overline{v(x_2)}$$

for all $x_2 \in \pi_2$ and $u, v \in H^2(B;\Lambda)$ (note that $H^2(B;\Lambda) \cong \pi_2^\dagger = \overline{\mathrm{Hom}_\Lambda(\pi_2, \Lambda)}$ by Lemma 2.1.3). It is easy to see that $\mathrm{ev}_{\pi_2}(x_2)(u,v)$ is quadratic in x_2 and

Hermitian in u and v: for any $g, h \in \pi$,

$$\mathrm{ev}_{\pi_2}(x_2)((g^{-1})^*(u), (h^{-1})^*(v)) = u(g_*^{-1}(x_2))\overline{v(h_*^{-1}(x_2))}$$

$$= u(x_2)g^{-1}\overline{v(x_2)h^{-1}} = gu(x_2)\overline{hv(x_2)} = g\,\mathrm{ev}_{\pi_2}(x_2)(u,v)\overline{h} \; .$$

Let $\mathrm{Her}(H^2(B;\Lambda))$ be the group of Hermitian pairings on $H^2(B;\Lambda)$, then ev_{π_2} determines a homomorphism $B_{\pi_2}\colon H_4(B) \to \mathrm{Her}(H^2(B;\Lambda))$ and in [44, Lemma 10], it is shown that

$$B_{\pi_2}(x)(u,v) = v(x \cap u) = (u \cup v)(x) \; .$$

Consider the image of $c_*[M] \in H_4(B)$, $B_{\pi_2}(c_*[M])(u,v) = (u \cup v)(c_*x) = c^*(u \cup v)[M] = (c^*(u) \cup c^*(v))[M] = s_M(c^*(u), c^*(v))$ where s_M is the cohomology intersection pairing on M. Moreover, by [44, Theorem 7] B_{π_2} is an isomorphism whenever π is a free group. Therefore in this section $c_*[M]$ and s_M will uniquely determine each other.

The map $\beta\colon \mathrm{Aut}_\bullet(B) \to \Omega_4^{Spin}(B)$ is defined by

$$\beta(\phi) = [M, \phi \circ c] - [M, c] \; .$$

But this map is not a homomorphism, for $\beta(\phi \circ \varphi) = \beta(\phi) + \phi_*(\beta(\varphi))$.

Lemma 2.2.9.

$$\ker(\beta\colon \mathrm{Aut}_\bullet(B) \to \Omega_4^{Spin}(B)) = \mathrm{Isom}[\pi, \pi_2, s_M] \; .$$

Proof. Although β is not a homomorphism, we can still define $\ker(\beta) = \beta^{-1}(0)$. The natural map $\Omega_4^{Spin}(B) \to H_4(B)$ sends a bordism element to the image of its fundamental class. If $\phi \in \mathrm{Aut}_\bullet(B)$, and $c\colon M \to B$ is its classifying map, then $\beta(\phi) := [M, \phi \circ c] - [M, c]$. The image of this element in $H_4(B)$ is zero when $\phi_*(c_*[M]) = c_*[M]$. But $ev_{\pi_2}(c_*[M]) = s_M$, so $\ker \beta$ is contained in the group of the isometries of the quadratic 2-type.

Since $\mathrm{Aut}_\bullet(B) \cong \mathrm{Isom}[\pi_1, \pi_2]$ an element in $\mathrm{Isom}[\pi, \pi_2, s_M]$ will be $\phi \in \mathrm{Aut}_\bullet(B)$ such that $\phi_*(c_*[M]) = c_*[M]$. Clearly $\beta(\phi) = 0$. □

Lemma 2.2.10. *For each $\phi \in \mathrm{Aut}_\bullet(B)$ such that $\phi_*(c_*[M]) = c_*[M]$, there is an $f \in \mathrm{Aut}_\bullet(M)$ such that $c \circ f \simeq \phi \circ c$.*

Proof. First, let us assume that $H_2(M;\mathbb{Q}) \neq 0$. Since $\phi_*(c_*[M]) = c_*[M]$, there exists an $f \in \mathrm{Aut}_\bullet(M)$, such that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "c"'] & M \arrow[d, "c"] \\ B \arrow[r, "\phi"] & B \end{tikzcd}$$

commutes up to homotopy, by [33, Lemma 1.3].

For the case $H_2(M;\mathbb{Q}) = 0$ our construction of f depends on the proof of [15, Proposition 9]. First of all the assumption $H_2(M;\mathbb{Q}) = 0$ implies that $H_2(M;\Lambda) \cong H_2(\widetilde{M}) \cong \pi_2 = 0$ and hence $B \simeq K(*_r\mathbb{Z}, 1)$. We also have $H_3(M;\Lambda) \cong H_3(\widetilde{M}) \cong \pi_3(\widetilde{M}) \cong \pi_3(M)$ hence $H_3(M;\Lambda) \otimes_\Lambda \mathbb{Z} \cong H_3(M) \cong H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbb{Z}) \cong \oplus_r\mathbb{Z}$. Therefore any element of $H_3(M)$ can be represented by a map $S^3 \to M$.

Choosing an isomorphism of π with $*_r\mathbb{Z}$ yields a basis of $H_1(M)$. Let $E = \{e_1, e_2, \ldots, e_r\}$ be a basis of $H_1(M)$ according to an isomorphism $\pi \cong *_r\mathbb{Z}$ and $U = \{u_1, u_2, \ldots, u_r\}$ be the dual basis of $H^1(M)$. Let $\{f_1, f_2, \ldots, f_r\}$ and $\{v_1, v_2, \ldots, v_r\}$ be the basis of $H_3(M)$ and $H^3(M)$ corresponding to U and E respectively, via the Poincaré duality.

We choose maps

$$\alpha = \vee_r \alpha_i \colon \vee_r S^1 \to M^{(3)} \subset M$$

and

$$\beta = \vee_r \beta_i \colon \vee_r S^3 \to M^{(3)} \subset M ,$$

where α_i and β_i represent generators of $H_1(M)$ and $H_3(M)$. The wedge map $\alpha \vee \beta \colon \bigvee_r (S^1 \vee S^3) \to M^{(3)}$ induces isomorphism on π_i for any $i = 1, 2, 3$ hence it is a homotopy equivalence by the Whitehead theorem. We have $H^1(M) \cong [M, K(\mathbb{Z}, 1)] \cong [M, S^1]$, and $H^3(M) \cong [M, K(\mathbb{Z}, 3)] \cong [M, K(\mathbb{Z}, 3)^{(4)}] \cong [M, S^3]$, since $K(\mathbb{Z}, 3)$ is obtained from S^3 by attaching cells of dimension ≥ 5. Therefore the cartesian product of the elements u_i and v_i defines a map $u \times v = \prod_{i=1}^r (u_i \times v_i) \colon M \to \prod_1^r (S^1 \times S^3)$. We can assume by construction that $u \times v$ restricts to a map $\gamma \colon M^{(3)} \to \bigvee_r (S^1 \vee S^3)$ such that $\gamma \circ (\alpha \vee \beta)$ is homotopic to the identity. The composition $M^{(3)} \xrightarrow{\gamma} \bigvee_r (S^1 \vee S^3) \subset \bigvee_r (S^1 \times S^3)$ extends to a map $\varphi \colon M \to \bigvee_r (S^1 \times S^3)$ [16, Lemma 4.2]. Now we consider the wedge $\bigvee_r (S^1 \times S^3)$ as the connected sum with $(r-1)$ 4-dimensional discs adjoined along the 3-spheres which serve to define the connected sums. In other words, $\sharp_r (S^1 \times S^3)$ embeds into $\bigvee_r (S^1 \times S^3)$, up to homotopy. The map φ can be deformed into a degree 1 map $h \colon M \to \sharp_r (S^1 \times S^3)$ [15, Lemma 13]. The map h induces a surjective homomorphism on π, see [14, Proposition 1.2], but since π is free it must be an isomorphism. Hence h is a homotopy equivalence.

Remark 2.2.11. This method is used in [15], to prove that an oriented 4-dimensional Poincaré space M with $\pi_1(M) \cong *_r \mathbb{Z}$ and $H_2(M; \mathbb{Q}) = 0$ is homotopy equivalent to the connected sum $\sharp_r (S^1 \times S^3)$.

Note that $\pi_1(\phi)$ induces an automorphism of π. Composing $\pi_1(\phi)$ with the previous isomorphism on $\pi \cong *_r \mathbb{Z}$, we get a new basis $E' = \{e'_1, e'_2, \ldots, e'_r\}$ of $H_1(M)$. The same construction gives us another homotopy equivalence $h'\colon M \to \sharp_r(S^1 \times S^3)$. Since h and h' are degree 1 maps, we can construct an orientation preserving homotopy self equivalence of M by

$$f := h \circ h'^{-1}\colon M \to M\ .$$

Now, it is easy to see that by construction $c \circ f \simeq \phi \circ c$. $\square$

Finally, we are going to find the images of $\mathrm{Aut}_\bullet(M)$ and $\widetilde{\mathcal{H}}(M)$ in $\mathrm{Aut}_\bullet(B)$. Let us start by going briefly through the definitions of the bordism groups $\widetilde{\mathcal{H}}(M)$ and $\widetilde{\mathcal{H}}(B)$. The group $\widetilde{\mathcal{H}}(M)$ is defined as the bordism group of objects (W, F) where W is a compact 5-dimensional spin manifold with $\partial_1 W = -M$ and $\partial_2 W = M$, and $F\colon W \to M$ is a continuous map such that $F|_{\partial_1 W} = \mathrm{id}_M$ and $F|_{\partial_2 W} = f$ is a base-point and orientation-preserving homotopy equivalence. We have an analogous bordism group $\widetilde{\mathcal{H}}(B)$. It is defined as the bordism group of objects (W, F) where W is a compact 5-dimensional spin manifold with $\partial_1 W = -M$ and $\partial_2 W = M$, and $F\colon W \to B$ is a continuous map such that $F|_{\partial_1 W} = c$ and $F|_{\partial_2 W} = f$ is a base-point preserving 3-equivalence.

Lemma 2.2.12. ([35, Lemma 2.6]) $\widetilde{\mathcal{H}}(M) \cong \widetilde{\mathcal{H}}(B)$.

Corollary 2.2.13. *The images of* $\mathrm{Aut}_\bullet(M)$ *and* $\widetilde{\mathcal{H}}(M)$ *in* $\mathrm{Aut}_\bullet(B)$ *are precisely equal to* $\mathrm{Isom}[\pi, \pi_2, s_M]$.

Proof. For each $[f] \in \mathrm{Aut}_\bullet(M)$, we have a base-point preserving homotopy equivalence $\phi_f\colon B \to B$ such that $c \circ f = \phi_f \circ c$, since the obstructions to extending f to B lie in the groups $H^{i+1}(B, M; \pi_i(B))$=0 for all i. Note that $\mathrm{Aut}_\bullet(B) \cong \mathrm{Isom}[\pi, \pi_2]$ and by the naturality of ev_{π_2} all we have to show is $(\phi_f)_*(c_*[M]) = c_*[M]$, since $ev_{\pi_2}(c_*[M])$ is just the intersection form. We have

$$(\phi_f)_*(c_*[M]) = (\phi_f \circ c)_*[M] = (c \circ f)_*[M] = c_*[M]$$

since the fundamental class in $H_4(M)$ is preserved by an orientation preserving homotopy equivalence. We see that $\mathrm{im}(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$ is contained in $\mathrm{Isom}[\pi, \pi_2, s_M]$. The other inclusion follows from the lemma above.

The result for the image of $\widetilde{\mathcal{H}}(M)$ follows by the exactness of the braid and the fact that $\ker(\beta) = \mathrm{Isom}[\pi, \pi_2, s_M]$ □

We can now put the pieces together to establish our main result. Here are the relevant terms of our braid diagram:

$$\begin{tikzcd}
\begin{smallmatrix} H_1(M) \oplus H_3(M;\mathbb{Z}/2) \\ \oplus\, \mathbb{Z}/2 \end{smallmatrix} \arrow[dr] \arrow[rr, bend left] & & \widetilde{\mathcal{H}}(M) \arrow[dr] \arrow[rr, bend left] & & \mathrm{Isom}[\pi, \pi_2, s_M] \arrow[dr] & \\
& H_1(M) \arrow[ur] \arrow[dr, "0"] & & \mathrm{Aut}_\bullet(M) \arrow[ur] \arrow[dr, "\alpha"] & & 0 \\
\pi_1(\mathcal{E}_\bullet(B)) \arrow[ur] \arrow[rr, bend right, "0"] & & \begin{smallmatrix} H_2(M;\mathbb{Z}/2) \\ \oplus \\ H_3(M;\mathbb{Z}/2) \end{smallmatrix} \arrow[ur, "\gamma"] \arrow[rr, bend right, "\cong"] & & \begin{smallmatrix} H_2(M;\mathbb{Z}/2) \\ \oplus \\ H_3(M;\mathbb{Z}/2) \end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

Before we state the main result, let us point out that there is an action of $\mathrm{Isom}[\pi, \pi_2, s_M]$ on the normal subgroup

$$K_1 := \ker(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$$

Let $[f] \in K_1$ and $\phi \in \mathrm{Isom}[\pi, \pi_2, s_M]$. We have $c \circ f \simeq c$ and there is a homotopy equivalence $h\colon M \to M$, such that $c \circ h \simeq \phi \circ c$ (see Lemma 2.2.10). Now define

$$\phi.f := h \circ f \circ h^{-1} .$$

Since $c \circ (h \circ f \circ h^{-1}) \simeq \phi \circ c \circ f \circ h^{-1} \simeq \phi \circ c \circ h^{-1} \simeq c \circ h \circ h^{-1} \simeq c$, this action is well defined. Now we can state the main theorem of this section:

Theorem 2.2.14. *Let M be a connected, closed, oriented smooth or topological spin manifold of dimension 4. If the fundamental group $\pi := \pi_1(M)$ is a free group, then*

$$\mathrm{Aut}_\bullet(M) \cong (H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \rtimes \mathrm{Isom}[\pi, \pi_2, s_M] \ .$$

Proof. From the braid diagram, we have

$$\ker(\widetilde{\mathcal{H}}(M) \to \mathrm{Isom}[\pi, \pi_2, s_M]) \cong H_1(M) \ ,$$

so $\mathrm{Isom}[\pi, \pi_2, s_M] \cong \widetilde{\mathcal{H}}(M)/H_1$. This gives the splitting of the short exact sequence

$$0 \to K_1 \to \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}[\pi, \pi_2, s_M] \to 1 \ .$$

It follows that

$$\mathrm{Aut}_\bullet(M) \cong K_1 \rtimes \mathrm{Isom}[\pi, \pi_2, s_M]$$

with the conjugation action of $\mathrm{Isom}[\pi, \pi_2, s_M]$ on the normal subgroup K_1 defining the semi-direct product structure.

We already know that γ is injective, see Corollary 4.2.7. By the commutativity of the braid, to show that it is actually an injective *homomorphism*, it is enough to show that α is a homomorphism on the image of γ. Let $\gamma(W, F) = f$ and $\gamma(W', F') = g$. Recall that

$$\alpha(f \circ g) = \alpha(f) + f_*(\alpha(g)) \ .$$

We have to show that $f_*(\alpha(g)) = \alpha(g)$. By Corollary 4.2.7,

$$\alpha(g) \in H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \ .$$

Any element f, in the image of γ is trivial in $\mathrm{Aut}_\bullet(B)$, i.e., the image $\phi_f = \mathrm{id}_B$ and $c \circ f = c$. Since $H_3(M;\mathbb{Z}/2) \cong H^1(M;\mathbb{Z}/2)$ and c induces isomorphisms on $H_2(M;\mathbb{Z}/2)$ and $H^1(M;\mathbb{Z}/2)$, f acts as the identity on $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. Now a diagram chase shows that γ is a homomorphism. Therefore we have a short exact sequence of groups and homomorphisms

$$0 \to (H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \to \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}[\pi, \pi_2, s_M] \to 1\ .$$

Moreover, $K_1 = \mathrm{im}\,\gamma$ (by the exactness of the braid) and K_1 is mapped isomorphically onto $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ by the map α. Finally, we apply the above formula to obtain the relations

$$0 = \alpha(id_M) = \alpha(g \circ g^{-1}) = \alpha(g) + g_*(\alpha(g^{-1}))$$

for any $[g] \in \mathrm{Aut}_\bullet(M)$, and

$$\alpha(g \circ f \circ g^{-1}) = \alpha(g \circ f) + g_*(\alpha(g^{-1})) = g_*(\alpha(f))$$

for any $[f] \in K$. Therefore the conjugation action of $\mathrm{Isom}[\pi, \pi_2, s_M]$ on K_1 agrees with the induced action on homology under the identification $K_1 \cong H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ via α. It follows that

$$\mathrm{Aut}_\bullet(M) \cong (H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \rtimes \mathrm{Isom}[\pi, \pi_2, s_M]$$

as required, with the action of $\mathrm{Isom}[\pi, \pi_2, s_M]$ on the normal subgroup $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ given by the induced action of homotopy self-equivalences on homology. □

Remark 2.2.15. If we take $M = S^1 \times S^3$, then from the above theorem we get $\mathrm{Aut}_\bullet(M) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Recall that these are orientation preserving homotopy self-equivalences. Let $\varphi\colon S^1 \times S^3 \to S^1 \times S^3$ be defined by $\varphi(x, y) = (-x, y)$. Now if we compose orientation preserving self-equivalences with φ we get also the orientation reversing homotopy self-equivalences. Therefore the based homotopy classes of based self homotopy equivalences of $S^1 \times S^3$ is isomorphic to $(\mathbb{Z}/2)^3$, which is consistent with the results of [10, p. 25] and [60, p. 58].

2.3 Non-spin Case

The purpose of this section is to state and prove a theorem calculating the group $\mathrm{Aut}_\bullet(M, w_2)$. Our main result is Theorem A on page 74. For the non-spin case of our braid we will use the language of manifolds with structure. The class $w_2 \in H^2(B; \mathbb{Z}/2)$ gives a fibration $w_2\colon B \to K(\mathbb{Z}/2, 2)$ and we can form the pullback

$$\begin{tikzcd} B\langle w_2\rangle_n \arrow[r, "j_n"] \arrow[d, "\xi_n"'] & B \arrow[d, "w_2"] \\ BSO(n) \arrow[r, "w_2(\gamma^n)"] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

where γ^n is the universal oriented vector bundle over $BSO(n)$. Now we can form a sequence of fibrations $\xi_r\colon B\langle w_2\rangle_r \to BSO(r)$ and maps such that the diagram

$$\begin{tikzcd} B\langle w_2\rangle_r \arrow[r, "g_r"] \arrow[d, "\xi_r"'] & B\langle w_2\rangle_{r+1} \arrow[r] \arrow[d, "\xi_{r+1}"] & \cdots \\ BSO(r) \arrow[r, "i_r"] & BSO(r+1) \arrow[r] & \cdots \end{tikzcd}$$

commutes, i_r being the usual inclusion. We know that

$$BSO = \operatorname*{colim}_{r \to \infty} BSO(r) = \bigcup_r BSO(r) \ ,$$

and define

$$B\langle w_2\rangle := \operatorname*{colim}_{r\to\infty} B\langle w_2\rangle_r\ .$$

We have the following pullback diagram

$$\begin{tikzcd}
BSpin \arrow[r, "i"] \arrow[d, equal] & B\langle w_2\rangle \arrow[r, "j"] \arrow[d, "\xi"] & B \arrow[d, "w_2"] \\
BSpin \arrow[r] & BSO \arrow[r, "w"] & K(\mathbb{Z}/2,2)
\end{tikzcd}$$

The map $w = w_2(\gamma)$ pulls back the second Stiefel-Whitney class for the universal oriented vector bundle γ over BSO. The fibration $B\langle w_2\rangle$ over BSO is called the normal 2-type of M. The "James" spectral sequence used to compute $\Omega_*(B\langle w_2\rangle) = \pi_*(M\xi)$ has the same E_2-term as the one used above for $w_2 = 0$, but the differentials are twisted by w_2. In particular, d_2 is the dual of Sq^2_w, where $Sq^2_w(x) := Sq^2(x) + x \cup w_2$ (see [70, Section 2]).

There is a corresponding non-spin version of $\Omega_*^{Spin}(M)$, namely the bordism groups $\Omega_*(M\langle w_2\rangle) := \pi_*(M\xi)$ of the Thom space associated to the fibration:

$$\begin{tikzcd}
BSpin \arrow[r, "i"] \arrow[d, equal] & M\langle w_2\rangle \arrow[r, "j"] \arrow[d, "\xi"] & M \arrow[d, "w_2"] \\
BSpin \arrow[r] & BSO \arrow[r, "w"] & K(\mathbb{Z}/2,2)
\end{tikzcd}$$

Again the E_2-term of the James spectral sequence is unchanged from the spin case, but the differentials are twisted by w_2 with the above formula for Sq^2_w. As in the spin case, we choose a particular representative for the map w_2 such that $w_2 = w \circ \nu_M$. Next we will define a suitable "thickening" of $\mathrm{Aut}_\bullet(M)$ for the non-spin case:

Definition 2.3.1. Let $\mathrm{Aut}_\bullet(M, w_2)$ denote the set of equivalence classes of maps $\widehat{f}\colon M \to M\langle w_2\rangle$ such that (i) $f := j \circ \widehat{f}$ is a base-point and orientation preserving homotopy equivalence, and (ii) $\xi \circ \widehat{f} = \nu_M$. Two such maps $\widehat{f}$ and $\widehat{g}$ are *equivalent* if there exists a homotopy $\widehat{h}\colon M \times I \to M\langle w_2\rangle$ such that $h := j \circ \widehat{h}$ is a base-point preserving homotopy between f and g, and $\xi \circ \widehat{h} = \nu_M \circ p_1$, where $p_1\colon M \times I \to M$ denotes projection on the first factor.

Given two maps $\widehat{f}, \widehat{g}\colon M \to M\langle w_2\rangle$ as above, we define

$$\widehat{f} \bullet \widehat{g}\colon M \to M\langle w_2\rangle$$

as the unique map from M into the pull-back $M\langle w_2\rangle$ defined by the pair $f \circ g\colon M \to M$ and $\nu_M\colon M \to BSO$. Since $w_2 \circ f \circ g = w_2 \circ g = w_2$, this pair of maps is compatible with the pull-back. The following lemma is proved in [35, Lemma 3.3] for which we give a more detailed proof.

Lemma 2.3.2. $\mathrm{Aut}_\bullet(M, w_2)$ *is a group under this operation.*

Proof. To check that the operation just defined passes to equivalence classes, suppose that $\widehat{h}$ is a homotopy as above between $\widehat{g}$ and $\widehat{g'}$ representing the same element of $\mathrm{Aut}_\bullet(M, w_2)$. Let $h := j \circ \widehat{h}$, $g := j \circ \widehat{g}$ and $g' := j \circ \widehat{g'}$, then h is a base-point preserving homotopy between g and g' and $\xi \circ \widehat{h} = \nu_M \circ p_1$. Notice that $f \circ h$ is a homotopy between $f \circ g$ and $f \circ g'$. Also we have $w_2 \circ (f \circ h) = w_2 \circ h = w_2 \circ p_1$. Therefore $(f \circ h, \nu_M \circ p_1)\colon M \times I \to M\langle w_2\rangle$ gives a homotopy between $\widehat{f} \bullet \widehat{g}$ and $\widehat{f} \bullet \widehat{g'}$. A similar argument in the case when $\widehat{f}$ is varied by a homotopy to $\widehat{f'}$ shows that $\widehat{f} \bullet \widehat{g} \simeq \widehat{f'} \bullet \widehat{g}$.

Let $\widehat{\mathrm{id}}_M\colon M \to M\langle w_2\rangle$ denote the map defined by the pair of maps $(\mathrm{id}_M\colon M \to M, \nu_M\colon M \to BSO)$. This map will represent the identity element in our group structure.

Given $\widehat{f}$ representing an element of $\mathrm{Aut}_\bullet(M, w_2)$, let f^{-1} denote the homotopy inverse of $f := j \circ \widehat{f}$. By the Dold-Whitney theorem [23], there is an isomorphism $(f^{-1})^*(\nu_M) \cong \nu_M$. We have a base-point preserving homotopy $h\colon M \times I \to BSO$ between $\nu_M \circ f^{-1} \simeq \nu_M$. Define $\widehat{f^{-1}}\colon M \to M\langle w_2\rangle$ lifting $\nu_M \circ f^{-1}$ by $\widehat{f^{-1}}(x) := (f^{-1}(x), \nu_M \circ f^{-1}(x))$ for all $x \in M$, this makes sense because $w_2 = w \circ \nu_M$. We apply the homotopy lifting property to get $\widehat{h}\colon M \times I \to M\langle w_2\rangle$ lifting h, such that $\xi \circ (\widehat{h}|_{M\times\{1\}}) = \nu_M$. Let $g\colon M \to M$ be defined by $g := j \circ (\widehat{h}|_{M\times\{1\}})$. Then $g \simeq f^{-1}$ by the homotopy $h' := j \circ \widehat{h}$, and we have $w_2(g(x)) = w_2(j(\widehat{h}|_{M\times\{1\}}(x))) = w(\xi(\widehat{h}\,|\,_{M\times\{1\}}(x))) = w(\nu_M(x)) = w_2(x)$ for all $x \in M$. Hence $\widehat{g}\colon M \to M\langle w_2\rangle$ defined by $\widehat{g}(x) = (g(x), \nu_M(x))$ is an element of $\mathrm{Aut}_\bullet(M, w_2)$. Let $H := f \circ h'$ and note that H is a homotopy between $f \circ g$ and id ($H(m,0) = (f \circ g)(m)$ and $H(m,1) = m$). We want to be able to say that $\widehat{f} \bullet \widehat{g} \simeq \widehat{\mathrm{id}}$. Consider the following diagram

$$\begin{tikzcd} M \times \{0\} \arrow[r, "\widehat{f}\bullet\widehat{g}"] \arrow[d, hook] & M\langle w_2\rangle \arrow[d, "{(j,\xi)}"] \\ M \times I \arrow[r, "{(H,\nu_M\circ p_1)}"'] \arrow[ur, dashed, "\widehat{H}"] & M \times BSO \end{tikzcd}$$

By the covering homotopy theorem we get $\widehat{H}\colon M \times I \to M\langle w_2\rangle$ lifting H, with the property that $j \circ \widehat{H}|_{M\times\{1\}}(m) = H(m,1) = m$ and $\xi \circ \widehat{H} = \nu_M \circ p_1$. We therefore have a homotopy $\widehat{H}$ between $\widehat{f} \bullet \widehat{g} \simeq \widehat{\mathrm{id}}$. □

Next comes the connection between $\mathrm{Aut}_\bullet(M, w_2)$ and $\mathrm{Aut}_\bullet(M)$.

Lemma 2.3.3. *There is a short exact sequence of groups*

$$0 \longrightarrow H^1(M;\mathbb{Z}/2) \longrightarrow \mathrm{Aut}_\bullet(M, w_2) \longrightarrow \mathrm{Aut}_\bullet(M) \longrightarrow 1$$

Proof. Let $f \in \mathrm{Aut}_\bullet(M)$, then $f^*(\nu_M) \cong \nu_M$ by the Dold-Whitney Theorem [23]. Choose a base-point preserving homotopy $h\colon M \times I \to BSO$ between $\nu_M \circ f \simeq \nu_M$. Define $\widehat{f}\colon M \to M\langle w_2\rangle$ by $\widehat{f}(x) = (f(x), \nu_M(f(x)))$ lifting

$\nu_M \circ f$, this makes sense because $w_2(f(x)) = (w \circ \nu_M)(f(x))$ (recall that we choose $w_2 = w \circ \nu_M$). Apply the homotopy lifting property to get $\widehat{h}\colon M \times I \to M\langle w_2\rangle$ lifting h such that $\xi \circ (\widehat{h}|_{M\times 1}) = \nu_M$. Let $f' := j \circ (h|_{M\times 1})$, then $f' \simeq f$ by the homotopy $j \circ \widehat{h}$, and we have $w_2(f'(x)) = w_2(x)$ for all $x \in M$. As a consequence, $\widehat{f'}(x) = (f'(x), \nu_M(x)) \in \mathrm{Aut}_\bullet(M, w_2)$. Therefore the natural map $\mathrm{Aut}_\bullet(M, w_2) \to \mathrm{Aut}_\bullet(M)$ defined by sending $\widehat{g}$ to $g := j \circ \widehat{g}$ induces a surjective homomorphism.

Next, let $\widehat{f}, \widehat{g} \in \mathrm{Aut}_\bullet(M, w_2)$ and $f := j \circ \widehat{f}$, $g := j \circ \widehat{g}$. Suppose there exists a homotopy h between f and g. If we can lift h to a homotopy $\widehat{h}\colon M \times I \to M\langle w_2\rangle$ such that $j \circ \widehat{h} = h$ and $\xi \circ \widehat{h} = \nu_M \circ p_1$, then $\widehat{f}$ and $\widehat{g}$ are equivalent in $\mathrm{Aut}_\bullet(M, w_2)$. We have the following lifting problem:

$$\begin{tikzcd}
 & K(\mathbb{Z}/2,1) \arrow[d] \\
\partial(M \times I) \arrow[r, "\widehat{f} \sqcup \widehat{g}"] \arrow[d, hook] & M\langle w_2\rangle \arrow[d, "{(j,\xi)}"] \\
M \times I \arrow[r, "{(h, \nu_M \circ p_1)}"'] \arrow[ur, dashed] & M \times BSO
\end{tikzcd}$$

The obstructions to lifting $(h, \nu_M \circ p_1)$ lie in the groups

$$H^{i+1}(M \times I, \partial(M \times I); \pi_i(K(\mathbb{Z}/2,1))) \cong H^i(M; \pi_i(K(\mathbb{Z}/2,1))) \ .$$

So the only non-zero obstructions are in $H^1(M;\mathbb{Z}/2)$. Let $\widehat{f} \in \mathrm{Aut}_\bullet(M, w_2)$, for any $\alpha \in H^1(M;\mathbb{Z}/2)$, we are going to construct a $\widehat{g} \in \mathrm{Aut}_\bullet(M, w_2)$ with the property that $f \simeq g$ and the obstruction to $\widehat{f}$ and $\widehat{g}$ being equivalent is α. Note that different maps $M \times I \to K(\mathbb{Z}/2,2)$ relative to the given maps on the boundary are also classified by $H^1(M;\mathbb{Z}/2)$. So we may think $\alpha\colon M \times I \to K(\mathbb{Z}/2,2)$ such that $\alpha|_{M\times\{0\}}$ and $\alpha|_{M\times\{1\}}$ is the constant map to the base point $\{*\}$ of $K(\mathbb{Z}/2,2)$. Consider the following diagram

$$\begin{tikzcd} & M\langle w_2\rangle \arrow[d, "(j,\xi)"] \\ M\times\{0\} \arrow[r, "(f,\nu_M)"] \arrow[d, hook] & M\times BSO \arrow[d, "\rho"] \\ M\times I \arrow[r, "\alpha"] \arrow[ur, dashed, "\widehat{\alpha}"] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

where the fibration $\rho\colon M\times BSO \to K(\mathbb{Z}/2,2) = \Omega K(\mathbb{Z}/2,3)$ is given by $(x,y)\to w_2(x)-w(y)$, for which the fiber over the base point is by definition $M\langle w_2\rangle$. By the homotopy lifting property we have $\widehat{\alpha}\colon M\times I\to M\times BSO$ making the diagram commutative. Let $\widehat{g} := \widehat{\alpha}|_{M\times\{1\}}$, then since $w_2(p_1\circ\widehat{g}(x)) = w(p_2\circ\widehat{g}(x))$, where p_1 and p_2 are projections to the first and second components respectively, $\widehat{g}$ actually gives us a map $M\to M\langle w_2\rangle$. Observe that $p_1\circ\widehat{\alpha}\colon M\times I\to M$ is a homotopy between f and g. In order to lift this homotopy to $M\langle w_2\rangle$ so that $\widehat{f}$ and $\widehat{g}$ are equivalent we should have $w_2((p_1\circ\widehat{\alpha})(x,t)) = w((p_2\circ\widehat{\alpha})(x,t))$ for all $x\in M$ and $t\in I$, which is possible if and only if α represents the trivial map. Hence α is the obstruction to lifting this homotopy to $M\langle w_2\rangle$. □

Remark 2.3.4. We start with the fibration $M\langle w_2\rangle \hookrightarrow M\times BSO \xrightarrow{\rho} K(\mathbb{Z}/2,2)$. Let F be the homotopy fiber of $M\langle w_2\rangle\hookrightarrow M\times BSO$, so $F\to M\langle w_2\rangle\to M\times BSO$ is a fibration (up to homotopy). Then F is homotopy equivalent to the loop space $\Omega K(\mathbb{Z}/2,2) = K(\mathbb{Z}/2,1)$. That is how we get the first fibration $K(\mathbb{Z}/2,1)\to M\langle w_2\rangle\to M\times BSO$ (see, for example, [22, p. 38]).

To define an analogous group $\mathrm{Aut}_\bullet(B,w_2)$ of self-equivalences, we will first state a lemma from [35, Lemma 3.8].

Lemma 2.3.5. *Given a base-point preserving map $f\colon M \to B$, there is a unique extension (up to base-point preserving homotopy) $\phi_f\colon B \to B$ such that $\phi_f \circ c = f$. If f is a 3-equivalence then ϕ_f is a homotopy equivalence. Moreover, if $w_2 \circ f = w_2$, then $w_2 \circ \phi_f = w_2$.*

Definition 2.3.6. Let $\mathrm{Aut}_\bullet(B, w_2)$ denote the set of equivalence classes of maps $\widehat{f}\colon M \to B\langle w_2\rangle$ such that (i) $f := j \circ \widehat{f}$ is a base-point preserving 3-equivalence, and (ii) $\xi \circ \widehat{f} = \nu_M$.

Two such maps $\widehat{f}$ and $\widehat{g}$ are *equivalent* if there exists a homotopy $\widehat{h}\colon M \times I \to B\langle w_2\rangle$ such that $h := j \circ \widehat{h}$ is a base-point preserving homotopy between f and g, and $\xi \circ \widehat{h} = \nu_M \circ p_1$, where $p_1\colon M \times I \to M$ denotes projection onto the first factor. Given two maps $\widehat{f}, \widehat{g}\colon M \to B\langle w_2\rangle$ as above, $\widehat{f} \bullet \widehat{g}\colon M \to B\langle w_2\rangle$ is defined as the pair $(\phi_f \circ \phi_g \circ c, \nu_M)$.

Lemma 2.3.7. ([35, Lemma 3.10]) $\mathrm{Aut}_\bullet(B, w_2)$ *is a group under this operation.*

When $w_2 := w_2(M) \neq 0$, Hambleton and Kreck modified the bordism groups in the braid in order to carry out the arguments used to establish commutativity. As before we will first give the braid and while we calculate the terms, we will give the necessary definitions.

Theorem 2.3.8. ([35, Theorem 3.15]) *Let M be a closed, oriented smooth or topological 4-manifold. Then there is a sign-commutative diagram of exact sequences.*

$$\begin{tikzcd}
\Omega_5(M\langle w_2\rangle) \arrow[dr] \arrow[rr, bend left] & & \widetilde{\mathcal{H}}(M,w_2) \arrow[dr] \arrow[rr, bend left] & & \mathrm{Aut}_\bullet(B,w_2) \arrow[dr, "\beta"] & \\
& \Omega_5(B\langle w_2\rangle) \arrow[ur] \arrow[dr] & & \mathrm{Aut}_\bullet(M,w_2) \arrow[ur] \arrow[dr, "\alpha"] & & \Omega_4(B\langle w_2\rangle) \\
\pi_1(\mathcal{E}_\bullet(B,w_2)) \arrow[ur] \arrow[rr, bend right] & & \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \arrow[ur, "\gamma"] \arrow[rr, bend right] & & \widehat{\Omega}_4(M\langle w_2\rangle) \arrow[ur] &
\end{tikzcd}$$

such that the two composites ending in $\mathrm{Aut}_\bullet(M, w_2)$ *agree up to inversion, and the other sub-diagrams are strictly commutative.*

Proposition 2.3.9. *Let* $B\langle w_2\rangle$ *denote the normal* 2*-type of a* 4*-manifold* M *with free fundamental group. Then we have*

$$\Omega_4(M\langle w_2\rangle) \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}$$

$$\Omega_5(M\langle w_2\rangle) \cong H_1(M) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}/2$$

$$\Omega_4(B\langle w_2\rangle) \subset \mathbb{Z} \oplus \mathbb{Z}/2 \oplus H_4(B)$$

$$\Omega_5(B\langle w_2\rangle) \cong H_1(M) \ .$$

Proof. As before, we only need to compute the d_2 differentials. Since M is orientable, w_2 is also the second Wu class of M. So we have $Sq^2_w(x) := Sq^2(x) + x \cup w_2 = 0$. Now, everything works exactly the same as in the spin case.

For the bordism groups of $B\langle w_2\rangle$, first consider the following commutative diagram

$$\begin{tikzcd}
H^2(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[r, "Sq^2_w"] & H^4(\widetilde{B};\mathbb{Z}/2)^\pi \\
H^2(B;\mathbb{Z}/2) \arrow[u, "p^*", "\cong"'] \arrow[r, "Sq^2_w"] & H^4(B;\mathbb{Z}/2) \arrow[u, "p^*", "\cong"']
\end{tikzcd}$$

Let $x \in H^2(B;\mathbb{Z}/2)$ such that $Sq^2_w(x) = 0$. By the commutativity of the diagram, we have

$$0 = Sq^2_w(p^*(x)) = Sq^2(p^*(x)) + p^*(x) \cup w_2(\widetilde{B})$$

where $w_2(\widetilde{B}) = p^*(w_2(B))$. But since $Sq^2\colon H^2(\widetilde{B};\mathbb{Z}/2) \to H^4(\widetilde{B};\mathbb{Z}/2)$ is injective, $p^*(x) = p^*(w_2(B))$. Hence $x = w_2(B)$ which we can be further identified with w_2 via our reference map c. Therefore

$$\ker(Sq^2_w\colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2)) \cong \langle w_2 \rangle \cong \mathbb{Z}/2$$

$$\cong \operatorname{coker}(d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)) \ .$$

Since all the other differentials are zero, this gives the $\mathbb{Z}/2$ in the $E^{\infty}_{2,2}$ position.

To see that $H_1(B) \cong H_1(M)$ is the only group on the line $p+q=5$ which survives to E_∞, we use the following commutative diagram

$$\begin{tikzcd}
H^2(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[r, "Sq^2_w"] & H^4(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[r, "Sq^2_w"] & H^6(\widetilde{B};\mathbb{Z}/2)^\pi \\
H^2(B;\mathbb{Z}/2) \arrow[r, "Sq^2_w"] \arrow[u, "p^*", "\cong"'] & H^4(B;\mathbb{Z}/2) \arrow[r, "Sq^2_w"] \arrow[u, "p^*", "\cong"'] & H^6(B;\mathbb{Z}/2) \arrow[u, "p^*", "\cong"']
\end{tikzcd} \ .$$

We are going to show that the top row is exact by first considering the sequence

$$H^2(X_k;\mathbb{Z}/2) \xrightarrow{Sq^2_w} H^4(X_k;\mathbb{Z}/2) \xrightarrow{Sq^2_w} H^6(X_k;\mathbb{Z}/2)$$

where $X_k = K(\underbrace{\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}}_{k}, 2)$ and then taking the inverse limit (remember that $\varprojlim$ is left exact). Let $x \in H^2(X_k;\mathbb{Z}/2)$, then we have

$$Sq^2_w(x^2 + x \cup w_2) = Sq^2(x) \cup x + Sq^1(x) \cup Sq^1(x) + x \cup Sq^2(x) +$$

$$Sq^2(x) \cup w_2 + Sq^1(x) \cup Sq^1(w_2) + x \cup Sq^2(w_2) + x^2 \cup w_2 + x \cup w_2^2$$

$$= Sq^1(x) \cup Sq^1(x) + Sq^1(x) \cup Sq^1(w_2)$$

where Sq^1 is the Bockstein homomorphism. Since $H^3(X_k;\mathbb{Z}/2) = 0$, Sq^1 is the zero map. Therefore the image of $Sq^2_w\colon H^2(X_k;\mathbb{Z}/2) \to H^4(X_k;\mathbb{Z}/2)$ is contained in the kernel of $Sq^2_w\colon H^4(X_k;\mathbb{Z}/2) \to H^6(X_k;\mathbb{Z}/2)$. To see the other inclusion let $y \in H^4(X_k;\mathbb{Z}/2)$ be such that

$$Sq^2_w(y) = Sq^2(y) + y \cup w_2 = 0\ .$$

By the cohomology Künneth formula we have

$$\begin{aligned}
H^2(X_k;\mathbb{Z}/2) &\cong \sum_{i=1}^{k} H^2(\mathbb{C}P^\infty_i;\mathbb{Z}/2)\ ,\\
H^4(X_k;\mathbb{Z}/2) &\cong \sum_{i=1}^{k} H^4(\mathbb{C}P^\infty_i;\mathbb{Z}/2) \oplus \sum_{i\neq j} H^2(\mathbb{C}P^\infty_i;\mathbb{Z}/2) \otimes H^2(\mathbb{C}P^\infty_j;\mathbb{Z}/2)\ ,\\
H^6(X_k;\mathbb{Z}/2) &\cong \sum_{i=1}^{k} H^6(\mathbb{C}P^\infty_i;\mathbb{Z}/2) \oplus \sum_{i\neq j} H^2(\mathbb{C}P^\infty_i;\mathbb{Z}/2) \otimes H^4(\mathbb{C}P^\infty_j;\mathbb{Z}/2)\\
&\quad \oplus \sum_{i\neq j\neq l} H^2(\mathbb{C}P^\infty_i;\mathbb{Z}/2) \otimes H^2(\mathbb{C}P^\infty_j;\mathbb{Z}/2) \otimes H^2(\mathbb{C}P^\infty_l;\mathbb{Z}/2)\ .
\end{aligned}$$

So y must be in one of the following forms:

(i) $y = z^2$ for some $z \in H^2(X_k;\mathbb{Z}/2)$. If this is the case then

$$0 = Sq^2_w(y) = Sq^2_w(z^2) = z^2 \cup w_2 \neq 0\ ,$$

a contradiction.

(ii) $y = \sum_{i,j} z_i \cup z_j$ for some $z_i, z_j \in H^2(X_k;\mathbb{Z}/2)$. If this is the case then $0 = Sq^2_w(y) = Sq^2_w(\sum_{i,j} z_i \cup z_j) = \sum_{i,j} z_i^2 \cup z_j + z_i \cup z_j^2 + z_i \cup z_j \cup w_2 \neq 0$, a contradiction.

(iii) $y = z^2 + \sum_{i,j} z_i \cup z_j$. If this is the case then

$$\begin{aligned} 0 = Sq^2_w(y) &= Sq^2_w(z^2 + \sum_{i,j} z_i \cup z_j) \\ &= Sq^2(z^2) + z^2 \cup w_2 + \sum_{i,j}(z_i^2 \cup z_j + z_i \cup z_j^2 + z_i \cup z_j \cup w_2) \\ &= z^2 \cup w_2 + \sum_{i,j}(z_i^2 \cup z_j + z_i \cup z_j^2 + z_i \cup z_j \cup w_2) \ . \end{aligned}$$

So we should have

$$\begin{aligned} z^2 \cup w_2 &= \sum_{i,j}(z_i^2 \cup z_j + z_i \cup z_j^2 + z_i \cup z_j \cup w_2) \\ &= a^2 \cup b + a \cup b^2 + a \cup b \cup w_2 \ , \end{aligned}$$

where $a = \sum_{i=1}^k x_i$ and $b = \sum_{j=1}^k$. For this to be the case, z is equal to either a and b is equal to w_2 or z is equal to b and a is equal to w_2 or $a = b = z$ (otherwise each of the terms in the summation above will live in different direct summand of $H^6(X_k;\mathbb{Z}/2)$). But in all of these cases $y = Sq^2_w(z)$.

We have $\{H^2(X_k;\mathbb{Z}/2), i_k^*\}$, $\{H^4(X_k;\mathbb{Z}/2), i_k^*\}$ and $\{H^6(X_k;\mathbb{Z}/2), i_k^*\}$ inverse system of modules, where $i_k \colon X_{k-1} \to X_k$ is the inclusion map. Consider the commutative diagram with exact rows

$$\begin{tikzcd} H^2(X_k;\mathbb{Z}/2) \arrow[r, "Sq^2_w"] \arrow[d, "i_k^*"] & H^4(X_k;\mathbb{Z}/2) \arrow[r, "Sq^2_w"] \arrow[d, "i_k^*"] & H^6(X_k;\mathbb{Z}/2) \arrow[d, "i_k^*"] \\ H^2(X_{k-1};\mathbb{Z}/2) \arrow[r, "Sq^2_w"] & H^4(X_{k-1};\mathbb{Z}/2) \arrow[r, "Sq^2_w"] & H^6(X_{k-1};\mathbb{Z}/2) \ . \end{tikzcd}$$

Then the sequence

$$\varprojlim H^2(X_k;\mathbb{Z}/2) \xrightarrow{Sq^2_w} \varprojlim H^4(X_k;\mathbb{Z}/2) \xrightarrow{Sq^2_w} \varprojlim H^6(X_k;\mathbb{Z}/2)$$

is exact. Let $a \in H^2(B;\mathbb{Z}/2)$, then

$$\begin{aligned} Sq^2_w(a^2 + a \cup w_2) = a^2 \cup w_2 + Sq^1(a) \cup Sq^1(w_2) \\ + a \cup w_2^2 + (a^2 + a \cup w_2) \cup w_2 = 0 \end{aligned}$$

since reduction mod 2, $H^3(B) \to H^3(B;\mathbb{Z}/2)$ is onto, $Sq^1 = 0$. Hence the image of $Sq^2_w\colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2)$ is contained in the kernel of $Sq^2_w\colon H^4(B;\mathbb{Z}/2) \to H^6(B;\mathbb{Z}/2)$. Now, let $b \in H^4(B;\mathbb{Z}/2)$ such that $Sq^2_w(b) = Sq^2(b) + b \cup w_2 = 0$ and let $p^*(b) = y$. Then by the commutativity of the above diagram $Sq^2_w(y) = 0$. There exists a $z \in H^2(\widetilde{B};\mathbb{Z}/2)$ such that $Sq^2_w(z) = y$. Then we also have a $c \in H^2(B;\mathbb{Z}/2)$ such that $p^*(c) = z$ and $Sq^2_w(c) = b$. Therefore the sequence

$$H^2(B;\mathbb{Z}/2) \xrightarrow{Sq^2_w} H^4(B;\mathbb{Z}/2) \xrightarrow{Sq^2_w} H^6(B;\mathbb{Z}/2)$$

is exact. Also, since $H_5(B) = 0$, we have $H_6(B) \to H_6(B;\mathbb{Z}/2)$ is surjective. Hence $d_2\colon H_6(B) \to H_4(B;\mathbb{Z}/2)$ is onto the kernel of the differential $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$. □

Let $\widehat{c}\colon M \to B\langle w_2\rangle$ denote the map defined by the following pair $(c\colon M \to B, \nu_M\colon M \to BSO)$. Consider the diagram

$$\begin{tikzcd} M\langle w_2\rangle \arrow[r, "c \circ j"] \arrow[d, "\xi"'] & B \arrow[d, "w_2"] \\ BSO \arrow[r, "w"] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

we have $(w_2 \circ c) \circ j = w_2 \circ j$ and since the pullback satisfies the universal property, there exists a map $\overline{c}\colon M\langle w_2\rangle \to B\langle w_2\rangle$.

Next we are going to give definitions of the maps α and γ (by paying special attention to the definition of γ) and describe the elements of

$\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$. Let $\widehat{\mathrm{id}}\colon M \to M\langle w_2\rangle$ denote the map defined by the pair $(\mathrm{id}_M\colon M \to M, \nu_M\colon M \to BSO)$. Given $[\widehat{f}] \in \mathrm{Aut}_\bullet(M, w_2)$, we define $\alpha\colon \mathrm{Aut}_\bullet(M, w_2) \to \widehat{\Omega}_4(M\langle w_2\rangle)$ by $\alpha(\widehat{f}) := [M, \widehat{f}] - [M, \widehat{\mathrm{id}}_M]$ where the modified bordism groups are defined by letting the degree of a reference map $\widehat{g}\colon N^4 \to M\langle w_2\rangle$ to be the ordinary degree of $g := j \circ \widehat{g}$.

An element $(W, \widehat{F})$ of $\widehat{\Omega_5}(B\langle w_2\rangle, M\langle w_2\rangle)$ is a 5-dimensional manifold with boundary $(W, \partial W)$, equipped with a reference map $\widehat{F}\colon W \to B\langle w_2\rangle$ such that $\widehat{F}|_{\partial W}$ factors through $\bar{c}$. The definition of γ in the non-spin case is similar to the one in the spin case: Let $(W, \widehat{F}) \in \Omega_5(B\langle w_2\rangle, M\langle w_2\rangle)$, by taking the boundary connected sum with $(M \times I, \widehat{\mathrm{id}} \circ p_1)$, we may assume that W has two boundary components $\partial_1 W = -M$ and $\partial_2 W = N$. The obstructions to lifting $\widehat{F}$ to $M\langle w_2\rangle$ relative to $\widehat{F}|N$ vanishes, the lifting arguments take place over the fixed map $\nu_M\colon M \to BSO$. We have $\widehat{r}\colon W \to M\langle w_2\rangle$, lift of $\widehat{F}$ and $\widehat{f} := \widehat{r}|\partial_1 W\colon M \to M\langle w_2\rangle$. Define

$$\gamma(W, \widehat{F}) := \widehat{f}\colon M \to M\langle w_2\rangle\ .$$

To see that the map γ is well-defined, suppose that $(W', \widehat{F'})$ is another representative for the same relative bordism class and that we have already found liftings $\widehat{r}$ and $\widehat{r'}$ of the maps $\widehat{F}$ and $\widehat{F'}$ respectively. Let (T, φ) denote a bordism over $\Omega_5(B\langle w_2\rangle, M\langle w_2\rangle)$ between $(W, \widehat{F})$ and $(W', \widehat{F'})$. Since any element in the kernel of $\pi_2(\varphi)\colon \pi_2(T) \to \pi_2(B\langle w_2\rangle)$ can be represented by an embedded 2-sphere with trivial normal bundle, we may assume that the reference map $\varphi\colon T \to B\langle w_2\rangle$ is a 3-equivalence by surgeries on the interior of T, see [49, Proposition 4].

Corollary 2.3.10. *There is an isomorphism*

$$\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \cong KH_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$$

and the group $\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$ *injects into* $\mathrm{Aut}_\bullet(M, w_2)$. *The image of* α,

$$\operatorname{im}\alpha = KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)\ .$$

Proof. Consider the maps $S^1 \times K3 \xrightarrow{x_i \times \{*\}} M$ where $x_i\colon S^1 \to M$ is a generator of $H_1(M) \cong \mathbb{Z}^r$ for each $i = \{1, 2, \ldots, r\}$ and S^1 is equipped with a non-trivial spin structure. Since $S^1 \times K3$ is a spin manifold the maps

$$(x_i \times \{*\}, \nu_{S^1 \times K3})\colon S^1 \times K3 \to M\langle w_2\rangle$$

are well-defined and $[S^1 \times K3, (x_i \times \{*\}, \nu_{S^1\times K3})]$ generate the $H_1(M)$ summand in $\Omega_5(M\langle w_2\rangle)$. Similarly, the generators of $\Omega_5(B\langle w_2\rangle)$ are of the form $[S^1 \times K3, (c \circ x_i \times \{*\}, \nu_{S^1\times K3})]$. The homomorphism

$$\Omega_5(M\langle w_2\rangle) \to \Omega_5(B\langle w_2\rangle) \cong H_1(M)$$

is defined by composing with the reference map $\bar{c}$. Note that

$$\bar{c}([S^1 \times K3, (x_i \times \{*\}, \nu_{S^1\times K3})]) = [S^1 \times K3, (c \circ x_i \times \{*\}, \nu_{S^1\times K3})]\ .$$

Hence $\Omega_5(M\langle w_2\rangle) \to \Omega_5(B\langle w_2\rangle)$ is onto and by the exactness of the braid the map $\Omega_5(B\langle w_2\rangle) \to \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$ must be zero. Thus

$$\begin{aligned}\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) &\cong \ker(\widehat{\Omega}_4(M\langle w_2\rangle) \to \Omega_4(B\langle w_2\rangle)) \\ &\cong KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)\end{aligned}$$

where $KH_2(M;\mathbb{Z}/2) := \ker(w_2\colon H_2(M;\mathbb{Z}/2) \to \mathbb{Z}/2)$.

The map $\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \to \widehat{\Omega}_4(M\langle w_2\rangle)$ is injective, by the exactness of the braid again, so the map $\pi_1(\mathcal{E}_\bullet(B, w_2)) \to \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$ must be zero, by the commutativity of the braid. Therefore

$$\gamma\colon \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \to \mathrm{Aut}_\bullet(M, w_2)$$

is injective.

If $\widehat{f}\colon M \to M\langle w_2\rangle$ represents an element of $\mathrm{Aut}_\bullet(M, w_2)$, then

$$\alpha(\widehat{f}) = [M, \widehat{f}] - [M, \widehat{\mathrm{id}}] \ .$$

We have

$$\Omega_4(M\langle w_2\rangle) \cong H_0(M) \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \ .$$

The natural map $\Omega_4(M\langle w_2\rangle) \to H_0(M)$ sends a 4-manifold to its signature. Since the class $w_2 \in H^2(M;\mathbb{Z}/2)$ is a characteristic element for the cup product form (mod 2), it is preserved by the induced map of a self-homotopy equivalence of M. Therefore, the image of $\mathrm{Aut}_\bullet(M, w_2)$ in $\Omega_4(M\langle w_2\rangle)$ lies in the subgroup $KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. Since, the map γ is injective we should have $KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \subseteq \operatorname{im}\alpha$ by the commutativity of the braid. Therefore $\operatorname{im}\alpha = KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. □

Next, we are going to define a homomorphism

$$\widehat{\jmath}\colon \mathrm{Aut}_\bullet(B, w_2) \to \mathrm{Aut}_\bullet(B) \ .$$

For any $\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2)$, $f := j \circ \widehat{f}\colon M \to B$ is a 3-equivalence. There is a unique homotopy equivalence (up to base point preserving homotopy) $\phi_f\colon B \to B$ such that $\phi_f \circ c \simeq f$ (see [35, Lemma 3.8]). We define

$$\widehat{\jmath}(\widehat{f}) := \phi_f \ .$$

Let $\widehat{g}$ be another element of $\mathrm{Aut}_\bullet(B, w_2)$, then $\widehat{f} \bullet \widehat{g}$ is defined by the pair $(\phi_f \circ \phi_g \circ c, \nu_M)$. Therefore $\widehat{\jmath}(\widehat{f} \bullet \widehat{g}) = \phi_f \circ \phi_g$. Let

$$\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M] := \{\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2) \mid \phi_f \in \mathrm{Isom}[\pi, \pi_2, s_M]\} \ .$$

Lemma 2.3.11. *There is a short exact sequence of groups*

$$0 \longrightarrow H^1(M;\mathbb{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M] \xrightarrow{\widehat{\jmath}} \mathrm{Isom}[\pi, \pi_2, s_M] \longrightarrow 1$$

Proof. For any $\phi \in \mathrm{Isom}[\pi, \pi_2, s_M]$, we have an $f \in \mathrm{Aut}_\bullet(M)$ such that $c \circ f \simeq \phi \circ c$ (by Lemma 2.2.10). We may assume that the pair (f, ν_M) determines an element of $\mathrm{Aut}_\bullet(M, w_2)$ (by Lemma 2.3.11). Then the pair $(c \circ f, \nu_M)$ determines an element $\widehat{f}$ of $\mathrm{Aut}_\bullet(B, w_2)$ and by definition $\widehat{\jmath}(\widehat{f}) = \phi$. Suppose now that $\widehat{f}$, $\widehat{g} \in \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$ such that $h : \phi_f \simeq \phi_g$. We have the following diagram

$$\begin{tikzcd}
 & K(\mathbb{Z}/2, 1) \arrow[d] \\
\partial(M \times I) \arrow[r, "\widehat{f} \sqcup \widehat{g}"] \arrow[d, hook] & B\langle w_2 \rangle \arrow[d, "(j,\xi)"] \\
M \times I \arrow[r, "(h \circ c \times \mathrm{id}, \nu_M \circ p_1)"'] \arrow[ur, dashed] & B \times BSO
\end{tikzcd}$$

The obstructions to lifting $(h \circ c \times \mathrm{id}, \nu_M \circ p_1)$ lie in the groups

$$H^{i+1}(M \times I, \partial(M \times I); \pi_i(K(\mathbb{Z}/2, 1))) \cong H^i(M; \pi_i(K(\mathbb{Z}/2, 1))) \; .$$

So the only non-zero obstructions are in $H^1(M; \mathbb{Z}/2)$. The rest of the proof follows exactly the same as the proof of Lemma 2.3.3. □

Corollary 2.3.12. *The image of* $\mathrm{Aut}_\bullet(M, w_2)$ *in* $\mathrm{Aut}_\bullet(B, w_2)$ *is precisely equal to* $\mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$.

Proof. Let $\widehat{f} \in \mathrm{Aut}_\bullet(M, w_2)$ and $\phi_{\widehat{f}}$ denote the image of $\widehat{f}$ in $\mathrm{Aut}_\bullet(B, w_2)$. Then $\widehat{\jmath}(\phi_{\widehat{f}}) = \phi_f$ satisfies $\phi_f \circ c = c \circ f$. So ϕ_f preserves $c_*[M]$ and hence $\phi_f \in \mathrm{Isom}[\pi, \pi_2, s_M]$. Now suppose that $\phi \colon B \to B$ is an element of $\mathrm{Aut}_\bullet(B)$ contained in $\mathrm{Isom}[\pi, \pi_2, s_M]$. Then there exists $f \in \mathrm{Aut}_\bullet(M)$ such that

$\phi \circ f \simeq c \circ f$ [33, Lemma 1.3]. There exists $f'\colon M \to M$ such that $f \simeq f'$ with $w \circ \nu_M = w_2 \circ f'$ [35, Lemma 3.1]. As a consequence we have $\widehat{f'} = (f', \nu_M) \in \mathrm{Aut}_\bullet(M, w_2)$. If $\phi_{\widehat{f'}} \in \mathrm{Aut}_\bullet(B, w_2)$ denotes the image of $\widehat{f'}$, then since $\phi \circ c \simeq c \circ f' = j \circ \phi_{\widehat{f'}}$ we have $\widehat{j}(\phi_{\widehat{f'}}) = \phi$. Hence the image of $\mathrm{Aut}_\bullet(M, w_2)$ in $\mathrm{Aut}_\bullet(B, w_2)$ is $\mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$. □

Let us briefly outline the definitions of the bordism groups $\widetilde{\mathcal{H}}(M, w_2)$ and $\widetilde{\mathcal{H}}(B, w_2)$. The elements of the group $\widetilde{\mathcal{H}}(M, w_2)$ are pairs $(W, \widehat{F})$, where W is a compact, oriented 5-manifold with $\partial_1 W = -M$ and $\partial_2 W = M$. The map $\widehat{F}\colon W \to M\langle w_2 \rangle$ restricts to $\widehat{\mathrm{id}}_M$ on $\partial_1 W$, and on $\partial_2 W$ to a map $\widehat{f} \in \mathrm{Aut}_\bullet(M, w_2)$. Similarly $\widetilde{\mathcal{H}}(B, w_2)$ is the bordism group of pairs $(W, \widehat{F})$, where W is a compact, oriented 5-manifold with $\partial_1 W = -M$ and $\partial_2 W = M$. The map $\widehat{F}\colon W \to B\langle w_2 \rangle$ restricts to $\widehat{c}$ on $\partial_1 W$, and on $\partial_2 W$ to a map $\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2)$.

Lemma 2.3.13. ([35, Lemma 3.13]) $\widetilde{\mathcal{H}}(M, w_2) \cong \widetilde{\mathcal{H}}(B, w_2)$.

In the non-spin case, the map $\beta\colon \mathrm{Aut}_\bullet(B, w_2) \to \Omega_4(B\langle w_2 \rangle)$ is defined by $\beta(\widehat{f}) := [M, \widehat{f}\,] - [M, \widehat{c}]$.

Lemma 2.3.14. $\ker(\beta\colon \mathrm{Aut}_\bullet(B, w_2) \to \Omega_4(B\langle w_2 \rangle)) = \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$ *and the image of* $\widetilde{\mathcal{H}}(M, w_2)$ *in* $\mathrm{Aut}_\bullet(B, w_2)$ *is equal to* $\mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$.

Proof. Let $\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2)$ and suppose first that $\widehat{f} \in \ker \beta$. Note that the map $\Omega_4(B\langle w_2 \rangle) \to H_4(B)$ sends a bordism element $[N, \widehat{g}]$ to $g_*[N]$ where $g := j \circ \widehat{g}$. Therefore the image of $\widehat{f}$ in $H_4(B)$ is zero when $(j \circ \widehat{f})_*[M] = c_*[M]$. But since $(j \circ \widehat{f})$ is a 3-equivalence, by [35, Lemma 3.8] there exists $\phi \in \mathrm{Aut}_\bullet(B)$ with $\phi \circ c = j \circ \widehat{f}$. So, $\phi_*(c_*[M]) = c_*[M]$ which means $\widehat{j}(\widehat{f}) = \phi \in \mathrm{Isom}[\pi, \pi_2, s_M]$. Therefore $\ker(\beta) \subseteq \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$.

To see the other inclusion let $\widehat{g} \in \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$ and remember that

$$\Omega_4(B\langle w_2 \rangle) \subset \Omega_4(*) \oplus \mathbb{Z}/2 \oplus H_4(B) \ ,$$

where $\mathbb{Z}/2$ summand is coming from

$$\mathrm{coker}(d_2 \colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)) \cong \langle w_2 \rangle \ .$$

Clearly the image of $\widehat{g}$ in $\Omega_4(*)$ is zero and by the above argument the image of $\widehat{g}$ in $H_4(B)$ is also zero. Since by Corollary 2.3.12 the image of $\mathrm{Aut}_\bullet(M, w_2)$ in $\mathrm{Aut}_\bullet(B, w_2)$ is precisely equal to $\mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$, there is an $\widehat{f} \in \mathrm{Aut}_\bullet(M, w_2)$ whose image in $\mathrm{Aut}_\bullet(B, w_2)$ is $\widehat{g}$. But since the class w_2 is preserved by a self-homotopy equivalence of M, the image of $\widehat{f}$ under the map $\bar{c} \circ \alpha \colon \mathrm{Aut}_\bullet(M, w_2) \to \Omega_4(B\langle w_2 \rangle)$ in $\mathrm{coker}(d_2)$ is zero. By the commutativity of the braid, we have the image of $\beta(\widehat{g})$ in $\mathrm{coker}(d_2) = 0$. Hence $\widehat{g} \in \ker(\beta)$.

The result about the image of $\widetilde{\mathcal{H}}(M, w_2)$ follows from the exactness of the braid ([35, Lemma 2.7]) and the fact that $\ker(\beta) = \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$.

□

The relevant terms of our braid are now:

$$\begin{tikzcd}
\begin{array}{c} H_1(M) \oplus H_3(M;\mathbb{Z}/2) \\ \oplus\, \mathbb{Z}/2 \end{array} \arrow[rr, bend left] \arrow[dr] & & \widetilde{\mathcal{H}}(M, w_2) \arrow[rr, bend left] \arrow[dr] & & \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M] \arrow[dr, "0"] & \\
& H_1(M) \arrow[ur] \arrow[dr, "0"] & & \mathrm{Aut}_\bullet(M, w_2) \arrow[ur] \arrow[dr, "\alpha"] & & 0 \\
\pi_1(\mathcal{E}_\bullet(B)) \arrow[ur] \arrow[rr, bend right, "0"] & & \begin{array}{c} KH_2(M;\mathbb{Z}/2) \\ \oplus \\ H_3(M;\mathbb{Z}/2) \end{array} \arrow[ur, "\gamma"] \arrow[rr, bend right, "\cong"] & & \begin{array}{c} KH_2(M;\mathbb{Z}/2) \\ \oplus \\ H_3(M;\mathbb{Z}/2) \end{array} \arrow[ur] &
\end{tikzcd}$$

There is an action of $\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]$ on the normal subgroup

$$\widehat{K_1} := \ker(\mathrm{Aut}_\bullet(M,w_2)\to \mathrm{Aut}_\bullet(B,w_2))$$

which is defined as follows: Let $\widehat{f}\in\widehat{K_1}$, then $c\circ f\simeq c$. Also let $\widehat{\phi}\in \mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]$ and $\phi=\widehat{\jmath}(\widehat{\phi})$, then $\phi\in\mathrm{Isom}[\pi,\pi_2,s_M]$. There is a homotopy equivalence $h\colon M\to M$ such that $c\circ h=\phi\circ c$. Then since $h\circ f\circ h^{-1}$ preserves w_2, we can define $\widehat{\phi}.\widehat{f}:=(h\circ f\circ h^{-1},\nu_M)$.

We can now state the main theorem of this chapter. It is a generalization of Theorem 2.2.14.

Theorem A. *Let M be a connected, closed, oriented topological manifold of dimension 4. If $\pi:=\pi_1(M)$ is a free group, then*

$$\mathrm{Aut}_\bullet(M,w_2)\cong (KH_2(M;\mathbb{Z}/2)\oplus H_3(M;\mathbb{Z}/2))\rtimes \mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]\ .$$

Proof. We have a split short exact sequence

$$0\longrightarrow \widehat{K_1}\longrightarrow \mathrm{Aut}_\bullet(M,w_2)\longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]\longrightarrow 1\ .$$

Any element $\widehat{f}$ will act as identity on $\mathrm{im}(\alpha)=KH_2(M;\mathbb{Z}/2)\oplus H_3(M;\mathbb{Z}/2)$, so λ is a homomorphism. Also $\widehat{K_1}\cong KH_2(M;\mathbb{Z}/2)\oplus H_3(M;\mathbb{Z}/2)$ and the rest of the proof follows as in the spin case. □

Remark 2.3.15. We have

$$H_2(M;\mathbb{Z}/2)\cong H_0(\pi;H_2(\widetilde{M};\mathbb{Z}/2))\cong \mathbb{Z}\otimes_\Lambda(\pi_2\otimes\mathbb{Z}/2)\ .$$

Therefore any element of $H_2(M;\mathbb{Z}/2)$ can be represented by a map $S^2\to M$. Let $0\neq x\in KH_2(M;\mathbb{Z}/2)$ and $\alpha\colon S^2\to M$ corresponds to x via the above isomorphism. Choose an embedding $D^4\hookrightarrow M$. Shrink ∂D^4 to a point to give a map $M \to M \vee S^4$. Now let $\eta\colon S^3 \to S^2$ be the Hopf map, $S\eta\colon S^4 \to S^3$ its suspension and $\eta^2\colon S^4 \to S^2$ the composition $\eta^2 = \eta \circ S\eta$. Let f be the composite map

$$M \longrightarrow M \vee S^4 \xrightarrow{\mathrm{id} \vee \eta^2} M \vee S^2 \xrightarrow{\mathrm{id} \vee \alpha} M$$

f induces identities on π_1 and on $H_i(\widetilde{M})$, so f is homologous to the id_M (but it is not homotopic to the identity, for γ is injective). Hence it is a homotopy equivalence, by [53, Theorem 5.5].

To realize $H_3(M;\mathbb{Z}/2)$ as homotopy equivalences, first observe that $H_3(M) \cong \mathbb{Z} \otimes_\Lambda H_3(\widetilde{M})$ and reduction mod 2 is onto, so by Hurewicz theorem for any element of $H_3(M;\mathbb{Z}/2)$ there exists a map $\beta\colon S^3 \to M$. Now the following composite map

$$M \longrightarrow M \vee S^4 \xrightarrow{\mathrm{id} \vee S\eta} M \vee S^3 \xrightarrow{\mathrm{id} \vee \beta} M$$

is again a homotopy-equivalence.

Chapter 3

s-cobordism theorem in the free case

3.1 Preliminaries

In this chapter we are going to show that the *quadratic 2-type* with the *Kirby-Siebenmann invariant* determines a classification of topological 4-manifolds with free fundamental group, up to *s-cobordism.*

Before we state our theorem, let us first point out that the signature of a closed, oriented 4-manifold M, $\sigma(M)$ is given by the signature of the usual intersection form

$$s_M^{\mathbb{Z}}\colon H_2(M) \otimes H_2(M) \to \mathbb{Z}\ .$$

Note that, when π is a free group $H_2(M) \cong H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}$. Also we have $\mathrm{Hom}_\Lambda(\pi_2, \Lambda) \otimes_\Lambda \mathbb{Z} \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_2 \otimes_\Lambda \mathbb{Z}, \mathbb{Z})$. Therefore

$$s_M \otimes_\Lambda \mathbb{Z}\colon (H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}) \otimes (H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}) \to \Lambda \otimes_\Lambda \mathbb{Z} \cong Z$$

is the integral intersection form $s_M^{\mathbb{Z}}$, since $H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}$ is the largest quotient of $H_2(M;\Lambda)$ on which π acts trivially. Therefore the signature of M is determined by the formula

$$\sigma(M) = \sigma(s_M^{\mathbb{Z}}) = \sigma(s_M \otimes_\Lambda \mathbb{Z}) \ .$$

The next lemma will tell us that for M with free fundamental group π, there are basically two cases : $w_2(M) \neq 0$ and $w_2(M) = 0$. Following [34], we will call them as type (I) case and type (II) case respectively.

Lemma 3.1.1. *M is spin if and only if $\widetilde{M}$ is spin.*

Proof. Let $u\colon M \to K(\pi,1)$ be a classifying map for the fundamental group π. Consider the homotopy fibration

$$\widetilde{M} \xrightarrow{p} M \xrightarrow{u} K(\pi,1)$$

which induces a short exact sequence

$$0 \longrightarrow H^2(K(\pi,1);\mathbb{Z}/2) \xrightarrow{u^*} H^2(M;\mathbb{Z}/2) \xrightarrow{p^*} H^2(\widetilde{M};\mathbb{Z}/2) \ \cdot$$

When π is a free group, we have $H^2(K(\pi,1);\mathbb{Z}/2) = 0$ moreover since $w_2(\widetilde{M}) = p^*(w_2(M))$, $\widetilde{M}$ is spin if and only if M_1 is spin. □

Finally, note that the Whitehead group $Wh(\pi)$ is trivial for $\pi \cong *_r\mathbb{Z}$ (see [29] or [67]), hence in this case being s-cobordant is equivalent to being h-cobordant. Here is our main result for this chapter:

3.2 Main Result

Theorem B. *Let M_1 and M_2 be two closed, connected, oriented, topological 4-manifolds with free fundamental group and have the same Kirby-Siebenmann invariant. Then they are s-cobordant if and only if they have isometric quadratic 2-types.*

The proof of Theorem B. If M_1 and M_2 are s-cobordant, then the inclusion of M_1 into an s-cobordism between M_1 and M_2 and the homotopy inverse of the inclusion from M_2 is an orientation preserving homotopy equivalence and thus induces an isometry between the intersection forms. So, M_1 and M_2 have isometric *quadratic 2-types.*

Suppose now that M_1 and M_2 have isometric *quadratic 2-types.* For the type (I) case, let us recall that

$$\Omega_4^{STOP}(K(\pi,1)) \cong \Omega_4^{STOP}(*) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

where $\pi \cong *_r\mathbb{Z}$. The isomorphism can be given by associating the pair $(\sigma(M), KS(M))$ to M, where $KS(M)$ is the *Kirby-Siebenmann invariant* of M, and it is a characteristic class that vanishes exactly when the topological tangent bundle of M reduces to a vector bundle bundle, in particular, it vanishes if M admits a smooth structure. The latter invariant $\sigma(M)$ is the signature of the 4-manifold M. Since M_1 and M_2 have isomorphic intersection forms, we have $\sigma(M_1) = \sigma(M_2)$. The hypotheses imply that we have a cobordism W between M_1 and M_2 over $K(\pi,1)$. For the type (II) case, note that

$$\Omega_4^{TOPSPIN}(K(\pi,1)) \cong \Omega_4^{TOPSPIN}(*) \cong \mathbb{Z}\ .$$

The isomorphism can be given by $\frac{\sigma(N)}{8}$. So two spin 4-manifolds with the same signature will be spin cobordant over $K(\pi,1)$ without the assumption on the Kirby-Siebenmann invariant. In both cases we may assume that W is connected.

Choose a handle decomposition of W [28]. Since W is connected, we can cancel all 0- and 5-handles. Further, we may assume by low-dimensional surgery that $M_1 \hookrightarrow W$ is a 2 equivalence. So we can trade all 1-handles for 3-handles, and upside-down, all 4-handles for 2-handles. We end up with a handle decomposition of W that only contains 2- and 3-handles, and view W as

$$W = M_1 \times [0,1] \cup \{2 - \mathit{handles}\} \cup \{3 - \mathit{handles}\} \cup M_2 \times [-1,0]$$

which we split into two halves : on one side, M_1 and the 2-handles, on the other, M_2 and the 3-handles. Looking upside-down at the upper half of W, instead of seeing the 3-handles as glued to the lower half, we can view them as 2-handles glued upwards to $M_2 \times [-1,0]$. Let $3/2$ be the level in W that appears immediately after all 2-handles have been attached, but before any 3-handle is attached. Thus the ascending cobordism $W_{3/2}$ contains just M_1 and all 2-handles. In its 4-dimensional upper boundary $M_{3/2}$, are located both the belt spheres of the 2-handles and the attaching spheres of the 3-handles.

The strategy for the remainder of the proof is the following: We will cut W into two halves, then glue them back after sticking in an h-cobordism of $M_{3/2}$. This cut and reglue procedure will create a new cobordism from M_1 to M_2. If we choose the right h-cobordism, then the 3-handles from the upper half will cancel algebraically the 2-handles from the lower half. This means that the newly created cobordism between M_1 and M_2 will have no homology relative to its boundaries, and so it will indeed be an s-cobordism from M_1 to M_2 (see [49, p. 722]) .

Let us first clarify the shape of $M_{3/2}$: a 5-dimensional 2-handle is a copy of $D^2 \times D^3$, to be attached by gluing $S^1 \times D^3$ to M_1. To attach such a 2-handle to M, we need to specify where the attaching circle $S^1 \times 0$

is being sent, but a null-homotopic circle in a 4-manifold is isotopic to any other embedded circle. We also need to specify how the thickening of the attaching circle is to be glued to M_1. Since $\pi_1 SO(3) = \mathbb{Z}/2$, there are only two ways of doing that, depending on whether the 3-disk D^3 in M_1 twists an even or an odd number of times around the attaching circle. Therefore to fully describe $M_{3/2}$ all we need is to specify how many odd and how many even 2-handles are to be attached. Since the cobordism is over $K(\pi, 1)$ the attaching map $S^1 \times D^3 \to M_1$ is null-homotopic, so attaching an even 2-handle is the same as connect summing with $S^2 \times S^2$ and if the 2-handle is odd then attaching it is the same as connect summing with $S^2 \tilde{\times} S^2$ (see [62, p.157]). In conclusion, we have

$$M_{3/2} \approx M_1 \sharp k(S^2 \times S^2) \sharp k'(S^2 \tilde{\times} S^2) \ .$$

We may assume that no $S^2 \tilde{\times} S^2$-terms are present. We will argue that there are no $S^2 \tilde{\times} S^2$-summands again in two cases;

Type (I) Case: First suppose that M_1 is not spin, so $\widetilde{M_1}$ is not spin. $\widetilde{M_1}$ is simply connected, so every element of $H_2(\widetilde{M_1}; \mathbb{Z}/2)$ is represented by an immersed sphere. Since $w_2(\widetilde{M_1}) \neq 0$, it has nonzero value on some 2-sphere Σ, whose normal bundle is twisted. Since the normal bundle is preserved when we push down into M_1, we may conclude $w_2(M_1)$ has nonzero value on some immersed 2-sphere in M. Note that $M_1 \sharp S^2 \times S^2$ is obtained from M_1 by surgery on a circle C bounding some 2-disk $D \subset M_1$, with framing determined by the unique normal framing of D, and $M_1 \sharp S^2 \tilde{\times} S^2$ is obtained by surgery on C with the other framing. If we take D to be the north polar cap of Σ, then isotoping C over Σ to the south polar cap will interchange the framings, since the normal bundle of Σ is twisted (see [31, Proposition

5.2.4.]). It follows that in this case we have $M_1\sharp S^2 \times S^2 \approx M_1\sharp S^2\tilde{\times} S^2$.

Type (II) Case: If M_1 is spin, two manifolds $M_1\sharp S^2 \times S^2$ and $M_1\sharp S^2\tilde{\times} S^2$ are different, since the latter one has no spin structure. But in this case W can be chosen to be spin, so we can assume that W does not contain any odd handles.

From the lower half of W we have $M_{3/2} \approx M_1\sharp m_1(S^2 \times S^2)$, while from the upper half we have $M_{3/2} \approx M_2\sharp m_2(S^2 \times S^2)$, since $M_{3/2}$ can also be obtained by attaching even 2-handles upwards to M_2. We have $\mathrm{rank}(H_2(M_1)) = \mathrm{rank}(H_2(M_2))$, since $H_2(M_1) \xrightarrow[\cong]{(c_2)_*^{-1}\circ(c_1)_*} H_2(M_2)$, so it follows that $m = m_1 = m_2$. Hence we have a homeomorphism

$$\zeta\colon M_2\sharp m(S^2 \times S^2) \xrightarrow{\approx} M_1\sharp m(S^2 \times S^2) \ .$$

Remark 3.2.1. We can conclude that M_1 and M_2 are stably homeomorphic by using the notion of *normal* 1-*type* of a compact manifold (to remember the definition see Definition 1.3.1). Kreck [49, Corollary 3] by using a new operation, he called *subtraction of tori*, proved that two closed 4-dimensional manifolds with the same Euler characteristic and the same normal 1-type, are stably homeomorphic, by adding the same number of $S^2 \times S^2$, if they are bordant over the normal 1-type. In the spin case the normal 1-type is given by $E := BSpin \times K(\pi,1) \xrightarrow{\xi} BTOP$ and hence

$$\Omega_*(E) = \Omega_4^{TOPSPIN}(K(\pi,1)) \cong \Omega_4^{TOPSPIN} \cong \mathbb{Z} \ .$$

On the other hand, in the non-spin case the normal 1-type is given by $E := BSO \times K(\pi,1) \xrightarrow{\xi} BTOP$ and

$$\Omega_*(E) = \Omega_4^{STOP}(K(\pi,1)) \cong \Omega_4^{STOP}(*) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \ .$$

Therefore M_1 and M_2 are bordant over the normal 1-type and so they must be stably homeomorphic.

Let $B(M_i)$ denote the 2-types of M_i and $c_i\colon M_i \to B(M_i)$ corresponding 3-equivalences for $i = 1, 2$. We are going to construct a homotopy equivalence between $B(M_1)$ and $B(M_2)$. Note that, we have isomorphisms $\pi_2(c_i)\colon \pi_2(M_i) \xrightarrow{\cong} \pi_2(B(M_i))$ for $i = 1, 2$. Since M_1 and M_2 have isometric quadratic 2-types, we also have the following isomorphisms

$$\chi\colon \pi_1(M_1) \to \pi_1(M_2) \quad \text{and} \quad \psi\colon \pi_2(M_1) \to \pi_2(M_2)$$

such that

$$s_{M_2}(\psi(x), \psi(y)) = \chi_*(s_{M_1}(x, y)) \ .$$

Start with the composition

$$\pi_2(c_2) \circ \psi \circ \pi_2(c_1)^{-1}\colon \pi_2(B(M_1)) \xrightarrow{\cong} \pi_2(B(M_2)) \ .$$

We can think of any Abelian group G as a topological group with discrete topology. Then we can define $K(G, 1) = BG$, which is also an Abelian topological group, and $K(G, 2) = BK(G, 1) = B^2G$. This construction is functorial. Hence we have a homotopy equivalence

$$B^2(\pi_2(c_2) \circ \psi \circ \pi_2(c_1)^{-1})\colon K(\pi_2(B(M_2)), 2) \to K(\pi_2(B(M_2)), 2)$$

which is π_1-equivariant, since ψ is π_1-equivariant. We also have another π_1-equivariant homotopy equivalence, namely $E\chi\colon E\pi_1(M_1) \to E\pi_1(M_2)$, where the contractible space $E\pi_1(M_i)$ is the total space of the universal bundle over $B\pi_1(M_i)$ for $i = 1, 2$. Let

$$\theta := E(\chi) \times B^2(\pi_2(c_2) \circ \psi \circ \pi_2(c_1)^{-1})$$

and recall that $B(M_i) \simeq E\pi_1(M_i) \times_{\pi_1(M_i)} K(\pi_2(B(M_i)), 2)$. Then we have

$$\theta\colon B(M_1) \to B(M_2) \ .$$

Also since $B(M_i)$ is a fibration over $B\pi_1(M_i)$ with fiber $K(\pi_2(B(M_i)), 2)$ by five lemma, we can see that θ is a homotopy equivalence. Summarizing we have a homotopy equivalence θ with the following commutative diagram:

$$\begin{tikzcd}
\pi_2(M_1) \arrow[r, "\pi_2(c_1)"] \arrow[d, "\psi"'] & \pi_2(B(M_1)) \arrow[d, "\pi_2(\theta)"] \\
\pi_2(M_2) \arrow[r, "\pi_2(c_2)"'] & \pi_2(B(M_2)) \ .
\end{tikzcd}$$

Also observe that $\theta_\sharp(s_{M_2}) = s_{M_1}$. Now let

$$M := M_1 \sharp m(S^2 \times S^2) \quad \text{and} \quad M' := M_2 \sharp m(S^2 \times S^2)$$

with the following quadratic 2-types,

$$[\pi, \pi_2, s_M] := [\pi_1(M_1), \pi_2(M_1) \oplus \Lambda^{2m}, s_{M_1} \oplus H(\Lambda^m)]$$

and

$$[\pi_1(M_2), \pi_2(M_2) \oplus \Lambda^{2m}, s_{M_2} \oplus H(\Lambda^m)]$$

where $H(\Lambda^m)$ is the hyperbolic form on $\Lambda^m \oplus (\Lambda^m)^*$. Next, observe that

$$(\pi_1(\zeta) \circ \chi, \pi_2(\zeta) \circ (\psi \oplus \mathrm{id})) = (\mathrm{id}, \pi_2(\zeta) \circ (\psi \oplus \mathrm{id}))$$

gives us an element in $\mathrm{Isom}[\pi, \pi_2, s_M]$ since it is the composition of isometries. Let $B := B(M)$ denote the 2-type of M. Remember that we have an exact sequence of the form

$$0 \longrightarrow H^2(\pi; \pi_2) \longrightarrow \mathrm{Aut}_\bullet(B) \xrightarrow{(\pi_1, \pi_2)} \mathrm{Isom}[\pi, \pi_2] \longrightarrow 1 \ .$$

Therefore we can find a $\phi \in \mathrm{Aut}_\bullet(B)$ such that

$$\pi_1(\phi) = \mathrm{id} \quad \text{and} \quad \pi_2(\phi) = \pi_2(\zeta) \circ (\psi \oplus \mathrm{id})$$

Also recall that we have the following short exact sequence by Lemma 2.3.11

$$0 \longrightarrow H^1(M; \mathbb{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M] \xrightarrow{\widehat{j}} \mathrm{Isom}[\pi, \pi_2, s_M] \longrightarrow 1$$

Choose

$$\widehat{f} \in \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M] \quad \text{such that} \quad \widehat{j}(\widehat{f}) = \phi$$

i.e. $\widehat{f}\colon M \to B\langle w_2 \rangle$ and $j \circ \widehat{f} \simeq \phi \circ c$. Recall that the image of $\widetilde{\mathcal{H}}(M, w_2)$ in $\mathrm{Aut}_\bullet(B, w_2)$ is precisely equal to $\mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$. So from the braid diagram for M we know that there exists $(W, \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2)$ which maps to $\widehat{f}$, i.e., $\widehat{F}\colon W \to B\langle w_2 \rangle$ and $F|_{\partial_2 W} = \widehat{f}$. We are using the fact that $\widetilde{\mathcal{H}}(M, w_2) \cong \widetilde{\mathcal{H}}(B, w_2)$. We will use a comparison of C.T.C. Wall's surgery program for studying homotopy equivalences with M. Kreck's modified surgery program for studying stable homeomorphisms. We have a commutative diagram of exact sequences (see [35, Lemma 4.1])

$$\begin{tikzcd}
\tilde{L}_6(\mathbb{Z}[\pi_1]) \arrow[r, equal] \arrow[d] & \tilde{L}_6(\mathbb{Z}[\pi_1]) \arrow[d] & \\
\mathcal{S}(M \times I, \partial) \arrow[r] \arrow[d] & \mathcal{H}(M) \arrow[r] \arrow[d] & \mathrm{Aut}_\bullet(M, w_2) \\
\mathcal{T}(M \times I, \partial) \arrow[r] \arrow[d] & \widetilde{\mathcal{H}}(M, w_2) \arrow[r, two heads] \arrow[d] & \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M] \\
L_5(\mathbb{Z}[\pi_1]) \arrow[r, equal] & L_5(\mathbb{Z}[\pi_1]) &
\end{tikzcd}$$

where the left-hand vertical sequence is from Wall's surgery exact sequence [73, Chap.10]. To obtain the right-hand vertical sequence we use the modified surgery theory of Kreck [49].

The group $\mathcal{H}(M)$ consists of oriented h-cobordisms W^5 from M to M, under the equivalence relation induced by h-cobordism relative to the boundary. The tangential structures $\mathcal{T}(M \times I, \partial)$, is the set of degree 1 normal maps $F\colon (W, \partial W) \to (M \times I, \partial)$, inducing the identity on the boundary [73, Prop.10.2]. The group structure on this set is defined as for $\widetilde{\mathcal{H}}(M, w_2)$. The map

$$\mathcal{T}(M \times I, \partial) \to \widetilde{\mathcal{H}}(M, w_2)$$

takes such an element to $(W, \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2)$, where $\widehat{F} = \widehat{p_1} \circ F$. This map factors through $\Omega_5(M\langle w_2 \rangle)$ by sending such an element to the bordism class of $(W \cup M \times I, \widehat{F} \cup \widehat{p_1})$.

Let $\sigma_5 \in L_5(\mathbb{Z}[\pi_1])$ be the image of $(W, \widehat{F})$. The map

$$\mathcal{T}(M \times I, \partial) \to L_5(\mathbb{Z}[\pi_1])$$

is onto, see for example [17, Theorem 8] or [41, Lemma 6.9]. Let $(W', F') \in \mathcal{T}(M \times I, \partial)$ maps to σ_5 and let $(W', \widehat{F'}) \in \widetilde{\mathcal{H}}(M, w_2)$ be the image of (W', F'). Since $\widehat{F'}|_{\partial_2(W')} = \widehat{\mathrm{id}} \in \mathrm{Aut}_\bullet(M, w_2)$ it will be mapped to $\widehat{c} \in \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$ and remember that $\widehat{j}(\widehat{c}) = \mathrm{id}_B$. Consider the difference of these elements in $\widetilde{\mathcal{H}}(M, w_2)$,

$$(W'', \widehat{F''}) := (W', \widehat{F'}) \bullet (-W, \widehat{f}^{-1} \bullet \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2)\ .$$

At this point let us quickly review how the group operation is defined in $\widetilde{\mathcal{H}}(M, w_2)$: The group structure on $\widetilde{\mathcal{H}}(M, w_2)$ is given by the formula

$$(W, \widehat{F}) \bullet (W', \widehat{F}') := (W \cup_{\partial_2 W = \partial_1 W'} W', \widehat{F} \cup \widehat{f} \bullet \widehat{F}')\ .$$

The inverse of $(W, \widehat{F})$ is represented by $(-W, \widehat{f^{-1}} \bullet \widehat{F})$ where $\widehat{f^{-1}}$ represents the inverse for $\widehat{f} = \widehat{F}|_{\partial_2 W}$ in $\mathrm{Aut}_\bullet(M, w_2)$.

Remark 3.2.2. Let us point out the fact that the map

$$\mathcal{T}(M \times I, \partial) \to \widetilde{\mathcal{H}}(M, w_2)$$

factors through $\Omega_5(M\langle w_2\rangle)$ by sending $(W', F') \in \mathcal{T}(M \times I, \partial)$ to the bordism class of $(W' \cup M \times I, \widehat{F'} \cup \widehat{p_1})$, see [35, Lemma 4.1]. If $(W', \widehat{F'})$ maps exactly to $(W, \widehat{F})$, exactness of the braid tells us that $\widehat{f} \simeq \widehat{c}$ which in turn implies ϕ induces identity homomorphisms on π_1 and π_2. In particular we would have, $\pi_2(\zeta^{-1}) = \psi \oplus \mathrm{id}$, which means in W 3-handles cancel 2-handles. But then W would be an h-cobordism between M_1 and M_2.

The element $(W'', \widehat{F''}) \in \widetilde{\mathcal{H}}(M, w_2)$ maps to $0 \in L_5(\mathbb{Z}[\pi_1])$. By the exactness of the right-hand vertical sequence there exists an h-cobordism T of M which maps to $(W'', \widehat{F''})$. Let f denote the induced homotopy self equivalence of M. By construction we have a homotopy commutative diagram

$$\begin{tikzcd} B \arrow[r, "\phi"] & B \\ M \arrow[u, "c"] \arrow[r, "f"] & M \arrow[u, "c"'] \end{tikzcd}$$

where $c \circ f = j \circ \widehat{f}$.

Remark 3.2.3. Before we found an $f \in \mathrm{Aut}_\bullet(M)$ we made a choice, we chose an $\widehat{f} \in \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$. If we had chosen a different representative, say $\widehat{g}$, then we would end up with a $g \in \mathrm{Aut}_\bullet(M)$ such that $c \circ f \simeq c \circ g$. But then $\pi_i(f) = \pi_i(g)$ for $i = 1, 2$ and this is what is important for us.

Note that $\pi_2(\zeta^{-1} \circ f) = \psi \oplus \mathrm{id}$ and also $\zeta^{-1} \circ f$ gives us a self-equivalence of $M_{3/2}$ which we will denote by Φ. Now, we put the s-cobordism T in between two halves of W and see what happens to 2- and 3-handles.

To translate everything into algebra, we proceed as follows: We view $M_{3/2}$ as $M_{3/2} \approx M_1 \sharp m(S^2 \times S^2)$ and we denote by x_k the class of $S^2 \times 1$ and by $\overline{x_k}$ the class of $1 \times S^2$ in the k^{th} $S^2 \times S^2$-summand. The classes $\overline{x_k}$ are the classes of the belt spheres of the lower 2-handles, and they bound in the lower cobordism. We write

$$H_2(M_{3/2}) \cong H_2(M_1) \oplus \mathbb{Z}\{x_1, \overline{x_1}, \ldots, x_m, \overline{x_m}\}$$

Now look at $M_{3/2}$ from upwards as $M_{3/2} \approx M_2 \sharp m(S^2 \times S^2)$. This decomposition is obtained by adding upside-down 2-handles to M_2 in the upper half of W. Respective summands in the decomposition do not correspond by a homeomorphism unless $M_1 \approx M_2$. Denote by y_k the class of $S^2 \times 0$ and by $\overline{y_k}$ the class of $0 \times S^2$ in the k^{th} $S^2 \times S^2$-summand of this latter splitting. The classes y_k are the classes of the attaching spheres of the upper 3-handles, and they bound in the upper cobordism. We write

$$H_2(M_{3/2}) \cong H_2(M_2) \oplus \mathbb{Z}\{y_1, \overline{y_1}, \ldots, y_m, \overline{y_m}\}$$

A good self-equivalence of $M_{3/2}$ will be the one that sends the class y_k onto x_k, thus guaranteeing that the attaching sphere y_k of each 3-handle has algebraic intersection 1 with the belt sphere $\overline{x_k}$ of the corresponding 2-handle. But since $\pi_2(\zeta^{-1} \circ f) = \psi \oplus \mathrm{id}$, we have

$$\Phi_*(y_k) = x_k \text{ and } \Phi_*(\overline{y_k}) = \overline{x_k} \ .$$

□

Chapter 4

PD_2 Fundamental Group

4.1 The structures of π_2 and $\Gamma(\pi_2)$

We start this section by giving the definition of Poincaré duality groups.

Definition 4.1.1. A group G is said to be of type FP if the augmentation $\mathbb{Z}[G]$-module $\mathbb{Z}$ admits a finite projective resolution over $\mathbb{Z}[G]$. G is said to be an n-dimensional Poincaré duality (PD_n) group if it is FP, $H^i(G,\mathbb{Z}[G]) = 0$ for $i \neq n$ and the *dualizing module* $D = H^n(G;\mathbb{Z}[G]) \cong \mathbb{Z}$, where $n = \mathrm{cd}G$.

There are natural isomorphisms $H^i(G,-) \cong H_{n-i}(G, D \otimes -)$ [13, Theorem10.1, p.220]. If G acts trivially on D, then G is said to be *orientable* (i.e., if D is isomorphic to the augmentation module $\mathbb{Z}$). Note that if G is an orientable PD_n group then we have

$$H^i(G;M) \cong H_{n-i}(G;M)$$

as in Poincaré duality for closed, orientable manifolds.

Remark 4.1.2. If G is a group such that there exists a closed $K(G,1)$-

manifold Y, then G is a Poincaré duality group and is orientable if and only if Y is orientable.

Let us further recall that a Poincaré duality complex of dimension n (or PD_n-complex for short) is a finitely dominated CW-complex exhibiting n-dimensional equivariant Poincaré duality. One has the following relation between Poincaré duality groups and Poincaré duality complexes: the classifying space $K(G,1)$ is a Poincaré duality complex if and only if G is a Poincaré duality group [46]. The only PD_1-group is $\mathbb{Z}$, hence every PD_1-complex is homotopy equivalent to S^1. Eckmann, Linnell and Müller showed that every PD_2-group is the fundamental group of a closed aspherical surface (see [25]), hence every PD_2-complex is homotopy equivalent to a closed surface.

Let M be a closed, connected, oriented, smooth or topological 4-manifold with fundamental group a PD_2 group, $\pi \cong \pi_1(F)$, where F is a closed, oriented aspherical surface, i.e., $F = K(\pi, 1) = B\pi$.

Let $C_* = C_*(M; \Lambda)$ be the cellular chain complex of $\widetilde{M}$, and let $B_i \subset Z_i$ denote the i-dimensional boundaries and i-cycles in C_i respectively. We have $H_i = H_i(C_*) = H_i(\widetilde{M}) = Z_i/B_i$. The complex C_* gives a resolution of the augmentation module

$$0 \longrightarrow B_1 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \ .$$

Since the $\mathrm{cd}\,\pi = 2$, B_1 is a projective Λ-module [13, p.184]. Also since $\widetilde{M}$ is simply connected $H_1 = 0$. In particular, $Z_1 = B_1$ is projective, so $C_2 \cong Z_1 \oplus Z_2$ and hence Z_2 is also a projective Λ-module. Dualizing the following partial projective resolution of π_2

$$C_3 \xrightarrow{\partial} Z_2 \longrightarrow \pi_2 \longrightarrow 0$$

gives us the exact sequence

$$0 \longrightarrow \pi_2^\dagger \longrightarrow Z_2^\dagger \longrightarrow C_3^\dagger \longrightarrow \mathrm{coker}(\partial^\dagger) \longrightarrow 0$$

Now π is a PD_2 group, so $\mathrm{vcd}\,\pi = \mathrm{cd}\,\pi = 2$. The only finite subgroup of π is 1 and $\widehat{H}^*(\{1\}, -) = 0$. So, $\mathrm{coker}(\partial^\dagger)$ is cohomologically trivial which implies $\mathrm{projdim}_\Lambda \mathrm{coker}(\partial^\dagger) \leq 2$ [13, p. 287]. Therefore $\pi_2^\dagger := \overline{\mathrm{Hom}_\Lambda(\pi_2, \Lambda)}$ is a projective Λ-module, by [13, Lemma 2.1, p.184]. We have another short exact sequence

$$0 \longrightarrow \overline{H^2(\pi;\Lambda)} \longrightarrow \overline{H^2(M;\Lambda)} \longrightarrow \pi_2^\dagger \longrightarrow 0$$

arising from the Serre spectral sequence of the fibration $\widetilde{M} \to M \to K(\pi, 1)$ (recall Lemma 2.1.3). Since π is a PD_2 group $\overline{H^2(\pi;\Lambda)} \cong \mathbb{Z}$. Let P denote the projective module $\pi_2^\dagger$, then by Poincaré duality we have

$$\pi_2 \cong \mathbb{Z} \oplus P\,.$$

By [44], there exists a 2-connected degree-1 map $g_M \colon M \to Z$ such that $\ker(g_M) = P$ (see also section 2.4). Then by [73, Lemma 2.3], P is actually stably free. Summarizing, we have proved:

Proposition 4.1.3. *Let M be a closed, connected, oriented, smooth or topological 4-manifold with fundamental group a PD_2 group and let π_2 denote $\pi_2(M)$. Then we have*

$$\pi_2 \cong \mathbb{Z} \oplus P$$

where P is a stably free Λ-module.

We now turn our attention to $\Gamma(\pi_2)$ where Γ is the Whitehead's quadratic functor.

Proposition 4.1.4. *$\Gamma(\pi_2) \cong Q \oplus \mathbb{Z}$, where Q is a projective Λ module.*

Proof. Recall that $\Gamma(G)$ is the Whitehead's quadratic functor defined on Abelian groups. If G is a Λ-module, then $\Gamma(G)$ inherits from G a Λ-module structure. We have $\Gamma(P \oplus \mathbb{Z}) \cong \Gamma(P) \oplus \mathbb{Z} \oplus P$ so we need to show that $\Gamma(P)$ is a projective Λ-module. Let $P \oplus P' \cong F$ where F is a free Λ- module. We have the isomorphism $\Gamma(F) \cong \Gamma(P) \oplus \Gamma(P') \oplus P \otimes P'$. We know that $\Gamma(F)$ is a free Λ-module (see Remark 2.1.9) and by Proposition 2.1.8 $P \otimes P'$ is a projective Λ-module. Hence $Q := \Gamma(P)$ is a projective Λ-module. $\square$

4.2 Spin Case

The purpose of this section is to state and prove a theorem calculating the group $\mathrm{Aut}_\bullet(M)$, when M is a spin 4-manifold with PD_2 fundamental group π. Our main result is Theorem 4.2.11.

Proposition 4.2.1. *The relevant spin bordism groups of M are given as follows:*

$$\Omega_4^{Spin}(M) \cong \Omega_4^{Spin}(*) \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z} \ ,$$
$$\Omega_5^{Spin}(M) \cong H_1(M) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}/2 \ .$$

Proof. For the line $p+q = 4$ in the E_2-term, we have $\Omega_4^{Spin}(*) \cong \mathbb{Z}$ in the $(0,4)$ position, $H_2(M;\mathbb{Z}/2)$ in the $(2,2)$ position, $H_3(M;\mathbb{Z}/2)$ in the $(3,1)$ position, and $H_4(M) \cong \mathbb{Z}$ in the $(4,0)$ position. The differential for $(p,q) = (3,1)$ is zero for $Sq^i(x) = 0$ if $i > \deg(x)$ and since M is spin the differential for $(p,q) = (4,1)$ is also zero. So all these terms survive to E_∞.

For the line $p+q = 5$, we have 3 non-zero terms : $H_1(M)$ in the

$(1,4)$ position, $H_3(M;\mathbb{Z}/2)$ in the $(3,2)$ position and $H_4(M;\mathbb{Z}/2)\cong\mathbb{Z}/2$, in the $(4,1)$ position. Again all these terms survive to E_∞. □

Since a finitely generated group is necessarily countable, π is countable. Then any finitely generated projective Λ-module is countable. Hence we have that π_2 is countable. Let

$$X_0 = *, X_1 = K(\mathbb{Z},2), \cdots, X_N = K(\mathbb{Z}^N,2), \cdots$$

where $\mathbb{Z}^N$ is the N-fold product of $\mathbb{Z}$. Consider the sequence of maps

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \cdots$$

where i_k's are inclusions. We can take $\widetilde{B}$ to be homotopy equivalent to the mapping telescope of the above sequence and we have,

$$H_n(\widetilde{B}) \cong \varinjlim H_n(X_k)$$

$$H^n(\widetilde{B};\mathbb{Z}/2) \cong \varprojlim H^n(X_k;\mathbb{Z}/2)$$

Proposition 4.2.2. *Let B denote the 2-type of a spin 4-manifold M with PD_2 fundamental group. Then*

$$H_i(B) \cong \begin{cases} \mathbb{Z} & \textit{if } i=0 \\ H_2(M) & \textit{if } i=2 \\ H_1(\pi_1) & \textit{if } i=1,\ 3 \textit{ or } 5 \\ \mathbb{Z}\otimes_\Lambda \Gamma(\pi_2)\oplus\mathbb{Z} & \textit{if } i=4 \end{cases}$$

Proof. Using the Serre spectral sequence of the fibration

$$\widetilde{M} \to M \to K(\pi,1)$$

we have the following isomorphisms:

$$H_3(B) \cong H_1(\pi_1, \pi_2) \cong H_1(\pi_1) \ ,$$

$$H_4(B) \cong H_0(\pi_1, \Gamma(\pi_2)) \oplus H_2(\pi_1, \pi_2) \cong \mathbb{Z} \otimes_\Lambda \Gamma(\pi_2) \oplus \mathbb{Z} \ ,$$

$$H_5(B) \cong H_1(\pi_1, \Gamma(\pi_2)) \cong H_1(\pi_1) \ .$$

□

In the calculations of $\Omega_5^{Spin}(B)$, we will have to understand the Steenrod operation $Sq^2\colon H^3(B;\mathbb{Z}/2) \to H^5(B;\mathbb{Z}/2)$. Hence before we start calculating the bordism groups, we will first find $H^3(B)$ and $H^5(B)$. Recall that we have a 2-connected degree-1 map $f\colon M \to Z$ where Z is an S^2 bundle over F (Z is the minimal model of M and in the spin case by [55] Z is trivial, i.e. $Z \simeq F \times S^2$). Also since $T\colon \mathcal{T}_\bullet \to \mathcal{CW}$ defined by $T(X) = B(X)$ (second stage in the Postnikov tower) is a covariant functor, we have a 2-connected map $g\colon B \to B(Z)$, that is

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "c_M"'] & Z \arrow[d, "c_Z"] \\ B \arrow[r, "g"] & B(Z) \end{tikzcd}$$

$g \circ c_M = c_Z \circ f$. The map g is a fibration with fibre $K(P,2)$. We will use the cohomology Serre spectral sequence associated to g.

The non-zero terms on the $p+q=3$ line are $H^1(B(Z);H^2(K(P,2)))$ and $H^3(B(Z))$. We have

$$\begin{aligned} H^1(B(Z);H^2(K(P,2))) &\cong H^1(\pi;\mathrm{Hom}(H_2(K(P,2)),\mathbb{Z})) \\ &\cong H^1(\pi;\mathrm{Hom}(P,\mathbb{Z})) \cong \mathrm{Ext}^1_\pi(P,\mathbb{Z}) = 0 \end{aligned}$$

by [13, Proposition 2.2, p.61] and since P is a projective module. Hence we have,

$$H^3(B) \cong H^3(B(Z)) \cong H^1(F; H^2(\widetilde{B(Z)}))$$
$$\cong H^1(\pi; H^2(K(\mathbb{Z},2)) \cong H^1(\pi)\ .$$

Next, we consider the line $p+q = 5$: non-zero terms are $H^1(B(Z); H^4(K(P,2)))$, $H^3(B(Z); H^2(K(P,2)))$ and $H^5(B(Z))$. By Whitehead exact sequence we have $H_4(K(P,2)) \cong \Gamma(P)$ which is a projective Λ-module since P is projective. As a consequence

$$H^1(B(Z); H^4(K(P,2))) \cong H^1(\pi; H^4(K(P,2)))$$
$$\cong H^1(\pi; \mathrm{Hom}(H_4(K(P,2)))) \cong H^1(\pi; \Gamma(P)) = 0$$

by [13, Proposition 2.2, p.61]. Also

$$H^3(B(Z); H^2(K(P,2))) \cong H^1(\pi; H^2(K(P,2))) = 0\ .$$

Hence we have,

$$H^5(B) \cong H^5(P_2(Z)) \cong H^1(\pi; H^4(K(\mathbb{Z},2))) \cong H^1(\pi)\ .$$

Summarizing we have proved:

Lemma 4.2.3. *The third and fifth cohomology groups of B are given as*

$$H^3(B) \cong H^1(\pi; H^2(K(\mathbb{Z},2))) \quad \textit{and} \quad H^5(B) \cong H^1(\pi; H^4(K(\mathbb{Z},2)))\ .$$

Remark 4.2.4. We can get the above result by simply considering the fibration $\widetilde{B} \to B \to K(\pi,1)$. Actually this is the reason why we didn't check any differentials in our calculations above, because for example, we know that $H^3(B) \cong H^1(\pi; H^2(\widetilde{B})) \cong H^1(\pi; \mathrm{Hom}(P \oplus \mathbb{Z}, \mathbb{Z})) \cong H^1(\pi)$. But to see the cohomology ring structure we used the fibration g.

Next, we are going to calculate the spin bordism groups of B.

Proposition 4.2.5. *Let B denote the 2-type of a spin 4-manifold with PD_2 fundamental group. We have*

$$\Omega_4^{Spin}(B) \subset \mathbb{Z} \oplus \mathbb{Z}/2 \oplus H_4(B) \quad \textit{and} \quad \Omega_5^{Spin}(B) \subset H_1(B) \oplus H_5(B)\ .$$

Proof. For the line $p+q=4$, the non-zero terms on the E^2-page are $H_4(B)$ in the $(4,0)$ position, $H_3(B;\mathbb{Z}/2)$ in the $(3,1)$ position, $H_2(B;\mathbb{Z}/2)$ in the $(2,2)$ position, and $\mathbb{Z}$ in the $(0,4)$ position . The differential $d_2\colon E^2_{4,1} \to E_2^{2,2}$ becomes the homomorphism $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$, which is the dual of $Sq^2\colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2)$. Now consider the following short exact sequence:

$$0 \longrightarrow H^2(F;H^0(\widetilde{B};\mathbb{Z}/2)) \longrightarrow H^2(B;\mathbb{Z}/2) \longrightarrow H^2(\widetilde{B};\mathbb{Z}/2)^{\pi} \longrightarrow 0$$

arising from the filtration for $H^2(B;\mathbb{Z}/2)$, where $H^2(\widetilde{B};\mathbb{Z}/2)^{\pi}$ is the largest submodule of $H^2(\widetilde{B};\mathbb{Z}/2)$ on which π acts trivially. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^2(F;H^0(\widetilde{B};\mathbb{Z}/2)) \arrow[r] \arrow[d, "Sq^2=0"] & H^2(B;\mathbb{Z}/2) \arrow[r, "p^*"] \arrow[d, "Sq^2"] & H^2(\widetilde{B};\mathbb{Z}/2)^{\pi} \arrow[r] \arrow[d, "Sq^2"] & 0 \\
0 \arrow[r] & H^2(F;H^2(\widetilde{B};\mathbb{Z}/2)) \arrow[r] & H^4(B;\mathbb{Z}/2) \arrow[r, "p^*"] & H^4(\widetilde{B};\mathbb{Z}/2)^{\pi} \arrow[r] & 0
\end{tikzcd}$$

The leftmost Sq^2 is given by the $Sq^2\colon H^0(\widetilde{B};\mathbb{Z}/2) \to H^2(\widetilde{B};\mathbb{Z}/2)$ (see [63] and [64]), which is zero. Also note that $Sq^2\colon H^2(\widetilde{B};\mathbb{Z}/2) \to H^4(\widetilde{B};\mathbb{Z}/2)$ is injective. Let $\alpha \in H^2(B;\mathbb{Z}/2)$, if $\alpha \in \ker(Sq^2)$ then

$$0 = Sq^2(\alpha) \Rightarrow p^*(Sq^2(\alpha)) = 0 \Rightarrow Sq^2(p^*(\alpha)) = 0 \Rightarrow p^*(\alpha) = 0$$

Together with the naturality of Sq^2, we have

$$\ker(Sq^2\colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2)) \cong H^2(F;\mathbb{Z}/2) \cong \mathbb{Z}/2\ .$$

Note that $H^2(B;\mathbb{Z}/2) \cong \mathrm{Hom}_{\mathbb{Z}/2}(H_2(B;\mathbb{Z}/2),\mathbb{Z}/2)$, by the universal coefficient theorem, so for $0 \neq \alpha \in \ker(Sq^2) \cong H^2(F;\mathbb{Z}/2) \subset H^2(B;\mathbb{Z}/2)$ we have a unique $x \in H_2(B;\mathbb{Z}/2)$ such that $\alpha(x) = 1$ and $\alpha(y) = 0$ for any $y \neq x$. Consider the following equation $0 = Sq^2(\alpha)(z) = \alpha(d_2(z))$ which implies x can not be equal to $d_2(z)$ for any $z \in H_4(B;\mathbb{Z}/2)$. So x represents a non-zero class in $\mathrm{coker}(d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2))$. Conversely let x represent a non-zero class in $\mathrm{coker}(d_2)$ and α be the dual cohomology class, then $Sq^2(\alpha)(z) = \alpha(d_2(z)) = 0$ for all $z \in H_4(B;\mathbb{Z}/2)$, hence $\alpha \in \ker(Sq^2)$. Also note that $H_2(B;\mathbb{Z}/2) \cong \mathbb{Z} \otimes_\Lambda H_2(\widetilde{B};\mathbb{Z}/2) \oplus H_2(\pi;\mathbb{Z}/2)$ and we have

$$\mathrm{coker}(d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)) \cong H_2(\pi;\mathbb{Z}/2) \cong \mathbb{Z}/2\ .$$

Hence $E^3_{2,2} \cong \mathbb{Z}/2$. Let $u_B\colon B \to F$ be the classifying map. Consider the following lifting problem

$$\begin{tikzcd} & B \arrow[d, "u_B"] \\ F \arrow[ur, dashed] \arrow[r, "\mathrm{id}"] & F \end{tikzcd}$$

The lifting obstructions lie in $H^{i+1}(F;\pi_i(\widetilde{B}))$. However, since $\mathrm{cd}\,\pi = 2$, note that $H^{i+1}(F;\pi_i(\widetilde{B})) = 0$ for all i. Let $s\colon F \to B$ be a lift of id, i.e. $u_B \circ s = \mathrm{id}$. Consider $s \circ pr_1\colon F \times (S^1 \times S^1) \to B$ where we use the non-trivial spin structure on the $(S^1 \times S^1)$ factor. Therefore $E^\infty_{2,2} \cong \mathbb{Z}/2$.

For the $(3,1)$ position, we know that $d_2\colon H_5(B;\mathbb{Z}) \to H_3(B;\mathbb{Z}/2)$ is reduction mod 2 composed with the dual of Sq^2, where by Lemma 4.2.3 we have

$$Sq^2\colon H^1(\pi; H^2(K(\mathbb{Z},2);\mathbb{Z}/2)) \to H^1(\pi; H^4(K(\mathbb{Z},2);\mathbb{Z}/2))$$

which is an isomorphism by [63] and [64], so its dual is also an isomorphism and reduction mod 2, $H_5(B;\mathbb{Z}) \to H_5(B;\mathbb{Z}/2)$, is also surjective. Hence

d_2 is surjective in this case. Therefore, on the line $p+q=4$, the groups which survive to E^∞ are $\mathbb{Z}$ in $(0,4)$ position, $\mathbb{Z}/2$ in $(2,2)$ position, and a subgroup of $H_4(B)$ in the $(4,0)$ position.

The non-zero terms on the line $p+q=5$ are: $H_5(B)$, $H_4(B;\mathbb{Z}/2)$, $H_3(B;\mathbb{Z}/2)$ and $H_1(B)$. We know that $Sq^2\colon H^3(B;\mathbb{Z}/2) \to H^5(B;\mathbb{Z}/2)$ is injective, hence its dual $d_2\colon H_5(B;\mathbb{Z}/2) \to H_3(B;\mathbb{Z}/2)$ is surjective, i.e. $E^\infty_{3,2}=0$. We will show that the differential $d_2\colon H_6(B) \to H_4(B;\mathbb{Z}/2)$ is onto the kernel of $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$. This will follow from the exactness of the sequence

$$H^2(\widetilde{B};\mathbb{Z}/2) \xrightarrow{Sq^2} H^4(\widetilde{B};\mathbb{Z}/2) \xrightarrow{Sq^2} H^6(\widetilde{B};\mathbb{Z}/2)$$

by considering the following commutative diagram associated to the fibration $\widetilde{B} \xrightarrow{p} B \xrightarrow{u_B} F$

$$\begin{tikzcd}
0 & 0 & 0 \\
H^2(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[u] \arrow[r, "Sq^2"] & H^4(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[u] \arrow[r, "Sq^2"] & H^6(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[u] \\
H^2(B;\mathbb{Z}/2) \arrow[u, "p^*"] \arrow[r, "Sq^2"] & H^4(B;\mathbb{Z}/2) \arrow[u, "p^*"] \arrow[r, "Sq^2"] & H^6(B;\mathbb{Z}/2) \arrow[u, "p^*"] \\
H^2(F;H^0(\widetilde{B};\mathbb{Z}/2)) \arrow[u, "u^*"] \arrow[r, "Sq^2=0"] & H^2(F;H^2(\widetilde{B};\mathbb{Z}/2)) \arrow[u, "u^*"] \arrow[r, "Sq^2"] & H^2(F;H^4(\widetilde{B};\mathbb{Z}/2)) \arrow[u, "u^*"] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

We want to show that the middle row is exact:

$$Sq^2(y^2) = y^2 \cup y + Sq^1(y) \cup Sq^1(y) + y \cup y^2 = 0\ ,$$

since $Sq^1 = 0$ ($H^2(B) \to H^2(B;\mathbb{Z}/2)$ is onto), hence im $\subset$ ker. Let $x \in H^4(B;\mathbb{Z}/2)$ such that $Sq^2(x) = 0$ and $p^*(x) = \widetilde{x}$. There exists

$\widetilde{y} \in H^2(\widetilde{B};\mathbb{Z}/2)$ such that $Sq^2(\widetilde{y}) = \widetilde{x}$, since the top row is exact. But $\widetilde{x} \in H^4(\widetilde{B};\mathbb{Z}/2)^\pi$ implies $\widetilde{y} \in H^2(\widetilde{B};\mathbb{Z}/2)^\pi$. From the filtration for $H^2(B;\mathbb{Z}/2)$ we know that $p^*\colon H^2(B;\mathbb{Z}/2) \to H^2(\widetilde{B};\mathbb{Z}/2)^\pi$ is onto. Hence there exists a $y \in H^2(B;\mathbb{Z}/2)$ with $p^*(y) = \widetilde{y}$. If the difference $y^2 - x$ is not 0, then there is a non-zero $z \in H^2(F; H^2(\widetilde{B};\mathbb{Z}/2))$ such that $y^2 - x = u^*(z)$. Then $0 = Sq^2(y^2 - x) = u^*(Sq^2(z))$. But since both Sq^2 at the bottom row and u^* are injective, $u^*(Sq^2(z))$ can't be zero. Hence $y^2 = Sq^2(y) = x$. As a result the following sequence is also exact

$$H_6(B;\mathbb{Z}/2) \longrightarrow H_4(B;\mathbb{Z}/2) \xrightarrow{d_2} H_2(B;\mathbb{Z}/2)$$

where the first map is the dual of $Sq^2\colon H^4(B;\mathbb{Z}/2) \to H^6(B;\mathbb{Z}/2)$. With the surjectivity of reduction mod 2, $H_6(B;\mathbb{Z}) \to H_6(B;\mathbb{Z}/2)$, we can conclude that $d_2\colon H_6(B;\mathbb{Z}) \to H_4(B;\mathbb{Z}/2)$ is surjective onto the kernel of the differential $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$. Finally note that we have $\ker(d_2\colon H_5(B) \to H_3(B;\mathbb{Z}/2)) = \ker(H_5(B) \to H_5(B;\mathbb{Z}/2))$ reduction mod 2. Therefore, on the line $p + q = 5$, the only groups which survive to E^∞ are $H_1(B)$ in the $(1,4)$ position, and a subgroup of $H_5(B)$ in the $(5,0)$ position. □

Let A denote the remaining subgroup of $H_5(B)$ in $\Omega_5^{Spin}(B)$. We are going to show that $A \subset \mathrm{im}(\pi_1(\mathcal{E}_\bullet(B)) \to \Omega_5^{Spin}(B))$. Let us review the basic information that will be used. Recall that the fundamental group, π, of F has a one-relator presentation

$$\pi = \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \ldots [x_g, y_g] = 1 \rangle \,,$$

where $[x_i, y_i]$ denotes $x_i y_i x_i^{-1} y_i^{-1}$. Let $x_i\colon S^1_{x_i} \to B$ and $y_i\colon S^1_{y_i} \to B$ for $i = 1, \ldots, g$, also denote generators of $H_1(B) \cong H_1(F) \cong \mathbb{Z}^{2g}$, here g denotes the genus of F. By Serre spectral sequence $H_5(B) \cong H_1(\pi) \cong H_1(F)$, and then by the Poincaré duality and universal coefficient theorem $H_1(\pi) \cong H^1(\pi) \cong \mathrm{Hom}(H_1(F), \mathbb{Z})$. A basis for $H_1(F)$ determines a dual basis for $\mathrm{Hom}(H_1(F), \mathbb{Z})$, so dual to x_i is the cohomology class x_i^* assigning the value 1 to x_i and 0 to the other basis elements, and similarly we have a cohomology class y_i^* dual to y_i. We are going to construct a map $\varphi_{x_i}\colon B \times S^1 \to B$ cell by cell and skeleton by skeleton. Note that, since we want φ_{x_i} to be an adjoint map for an element in $\pi_1(\mathcal{E}_\bullet(B))$, on $B \vee S^1 = B \times \{s_0\} \cup \{b_0\} \times S^1$, φ_{x_i} should be defined as

$$\varphi_{x_i}(b, s_0) = b \quad \text{and} \quad \varphi_{x_i}(b_0, s) = b_0 \ .$$

The 1-skeleton of B is $B^{(1)} \simeq \bigvee_{k=1}^{g}(S^1_{x_k} \vee S^1_{y_k})$. To construct $B \times S^1$ from $B \vee S^1$, first we have to attach 2-cells of the form $D^1_{x_k} \times D^1$ and $D^1_{y_k} \times D^1$. So, we are going to define φ_{x_i} first on $S^1_{x_k} \times S^1$ and $S^1_{y_k} \times S^1$. Since on $\{b_0\} \times S^1$ the map should be constant, it factors through $S^1_{x_k} \times S^1/\{b_0\} \times S^1 \cong S^1_{x_k} \vee S^2_{x_k}$. Also we know that our map should be identity on $S^1_{x_k} \times \{s_0\}$. So the only freedom we have is on the part $S^2_{x_k}$. Recall that $\pi_2 \cong \mathbb{Z} \oplus P$ and let $\sigma\colon S^2 \to B$ be the generator of $\mathbb{Z} \subset \pi_2$. Define φ_{x_i} on $\{b\} \times S^2_{x_k}$ as

$$\varphi_{x_i}(\{b\} \times S^2_{x_k}) = \begin{cases} \sigma & \text{if } k = i \\ * & \text{if } k \neq i \ . \end{cases}$$

i.e. we have a commutative diagram

$$\begin{tikzcd} S^1_{x_i} \times S^1 \arrow[r, "\varphi_{x_i}"] \arrow[d] & B \\ S^1_{x_i} \vee S^2_{x_i} \arrow[ur, "x_i \vee \sigma"'] & \end{tikzcd}$$

and on $S^1_{y_k} \times S^1$ we define φ_{x_i} as

$$\varphi_{x_i}(S^1_{y_k} \times S^1) = * \ .$$

By Serre spectral sequence, $H_2(B) \cong H_2(\pi) \oplus H_0(\pi, H_2(\widetilde{B})) \cong H_2(\pi) \oplus \mathbb{Z} \oplus H_0(\pi, P)$, where $H_2(\widetilde{B}) \cong \pi_2 \cong \mathbb{Z} \oplus P$. We remark that $\sigma_*[S^2] \in H_2(B)$, which for notational ease we will again denote by σ, generates the $\mathbb{Z}$ summand and we have $(\varphi_{x_i})_*\colon H_2(B\times S^1) \to H_2(B)$ with $(\varphi_{x_i})_*(x_i \otimes z) = \sigma$, where z denotes the generator of $H_1(S^1)$. Now, we have defined our map on $B \vee S^1 \cup B^{(1)} \times S^1$. Next, we are going to attach 3-cells of the form $D^2_t \times D^1$, where D^2_t represents a 2-cell of B with attaching map $\psi_t\colon S^1_t \to B^{(1)}$. Note that $S^2 = \partial(D^2_t \times D^1) = S^1_t \times D^1 \cup D^2_t \times \{0,1\}$. If a 2-cell of B is attached nullhomotopically, then $\varphi_{x_i} \circ \psi_t|_{\partial(D^2_t \times D^1)} = 0 \in \pi_2(B)$. There should be only one 2-cell $D^2_{t_0}$ of B which is not attached nullhomotopically. We need to consider only $D^2_{t_0} \times D^1$ where $D^2_{t_0}$ is attached along the loop given by the product of the commutators of the generators, $[x_1, y_1] \dots [x_g, y_g]$. We are going to show that $[\varphi_{x_i} \circ \psi_{t_0}] = 0 \in \pi_2(B)$. To start with we have $\psi_{t_0}(S^2) = ((S^1_{x_1} \vee S^1_{y_1}) \vee (-S^1_{x_1} \vee -S^1_{y_1}) \vee \cdots \vee (S^1_{x_g} \vee S^1_{y_g}) \vee (-S^1_{x_g} \vee -S^1_{y_g})) \times S^1 \cup \Phi_t(D^2_t)$, where $(-)$ denotes the opposite orientation and $\Phi_t\colon D^2_t \to B$ is the characteristic map. Then $\varphi_{x_i} \circ \psi_{t_0}(S^2) = \varphi_{x_i}((S^1_{x_i} \vee -S^1_{x_i})) \times S^1 \cup \Phi_t(D^2_t) \equiv S^1_{x_i} \vee \sigma - S^1_{x_i} \vee \sigma \equiv \sigma - \sigma = 0 \in \pi_2(B)$. So φ_{x_i} extends over 3-cells. Since $\pi_i(B) = 0$ for $i \geq 3$, there are no more obstructions to extend φ_{x_i} to $B \times S^1$. To use cup products we will pass to cohomology. We will first consider the map

$$\varphi^*_{x_i}\colon H^1(B) \to H^1(B \times S^1) \cong H^1(B) \otimes H^1(S^1)$$

which is given by $\varphi^*_{x_i}(\alpha) = \alpha$ for any $\alpha \in H^1(B)$, i.e., φ_{x_i} is projection to the first component in this dimension. Also we will consider

$$\varphi^*_{x_i}\colon H^2(B) \to H^2(B \times S^1) \cong H^2(B) \oplus H^1(B) \otimes H^1(S^1)\ .$$

By Serre spectral sequence, $H^2(B) \cong H^2(\pi) \oplus H^0(\pi; H^2(\widetilde{B})) \cong H^2(\pi) \oplus$

$H^2(\widetilde{B})^\pi \cong H^2(\pi) \oplus \mathrm{Hom}(H_2(\widetilde{B}), \mathbb{Z})^\pi \cong H^2(\pi) \oplus \mathrm{Hom}_\pi(\mathbb{Z} \oplus P, \mathbb{Z}) \cong H^2(\pi) \oplus \mathbb{Z} \oplus \mathrm{Hom}_\pi(P, \mathbb{Z})$. We have σ^* dual to σ which generates the $\mathbb{Z}$ summand in $H^2(B)$. Also let z^* denote the dual cohomology class in $H^1(S^1)$. Then we have $\varphi^*_{x_i}(\sigma^*) = \sigma^* + x_i^* \otimes z^*$. Now, take $y_i^* \in H^1(B)$ and consider $y_i^* \cup (\sigma^*)^2 \in H^5(B) \cong H^1(\pi; H^4(\widetilde{B}))$. We have $\varphi^*_{x_i}(y_i^* \cup (\sigma^*)^2) = y_i^* \cup (\sigma^* + x_i^* \otimes z^*)^2 = y_i^* \cup ((\sigma^*)^2 + 2x_i^* \cup \sigma^* \otimes z^*) = y_i^* \cup (\sigma^*)^2 + 2x_i^* \cup y_i^* \cup \sigma^* \otimes z^*$. Observe that $y_i^* \cup (\sigma^*)^2 \in H^1(\pi; H^4(\widetilde{B})) \cong H^5(B)$ and $2x_i^* \cup y_i^* \cup \sigma^* \otimes z^* \in H^2(\pi; H^2(\widetilde{B})) \otimes H^1(S^1) \subset H^4(B) \otimes H^1(S^1)$. Note that $x_i^* \cup y_i^*$ generates $H^2(\pi)$ and $c^*(x_i^* \cup y_i^* \cup \sigma^*) = M^*$, where M^* denotes the dual cohomology class of the fundamental class $[M]$ of M. In summary we have

$$\begin{array}{ccccc} H^5(B) & \xrightarrow{p_2 \circ \varphi^*_{x_i}} & H^4(B) \otimes H^1(S^1) & \xrightarrow{c^* \otimes id} & H^4(M) \otimes H^1(S^1) \\ & & & & \\ y_i^* \cup (\sigma^*)^2 & \longrightarrow & (2x_i^* \cup y_i^* \cup \sigma^*) \otimes z^* & \longrightarrow & M^* \otimes z^* \end{array}$$

Now if we dualize back to homology

$$H_4(M) \otimes H_1(S^1) \xrightarrow{c_* \otimes id} H_4(B) \otimes H_1(S^1) \xrightarrow{(\varphi_{x_i})_*} H_5(B)$$

we see that $(\varphi_{x_i})_*(c_*[M] \otimes [S^1]) = 2x_i \otimes (\sigma \otimes \sigma) \equiv 2x_i$.

Remark 4.2.6. To consider $\pi_1(\mathcal{E}_\bullet(B))$, we have to choose a base point $f \in \mathcal{E}_\bullet(B)$. Then $\pi_1(\mathcal{E}_\bullet(B)) = \pi_1(\mathcal{E}_\bullet(B)^f, f)$ where by $\mathcal{E}_\bullet(B)^f$ we mean all homotopy self-equivalences of B homotopic to f. This is because $\mathcal{E}_\bullet(B)$ is not path connected, in fact $\pi_0(\mathcal{E}_\bullet(B)) = \mathrm{Aut}_\bullet(B)$ is the group of homotopy classes of homotopy equivalences of B. We have chosen our base point to be the identity map of B.

By the exactness of the braid A is mapped injectively into $\widehat{\Omega}_5^{Spin}(B, M)$ and we are going to identify A with its image in $\widehat{\Omega}_5^{Spin}(B, M)$. By the commutativity of the braid, $\mathrm{im}(\pi_1(\mathcal{E}_\bullet(B)) \to \widehat{\Omega}_5^{Spin}(B, M)) = A$. Furthermore, again by the exactness of the braid we have $\gamma(A) = \mathrm{id} \in \mathrm{Aut}_\bullet(M)$ and $\ker(\widehat{\Omega}_5^{Spin}(B,M) \to \widehat{\Omega}_4^{Spin}(M)) = A$.

Corollary 4.2.7. *The quotient group $\widehat{\Omega}_5^{Spin}(B,M)/A$ is isomorphic to the direct sum $\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M};\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ and it injects into $\mathrm{Aut}_\bullet(M)$. The image of α is equal to $\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M};\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$.*

Proof. The map $\Omega_5^{Spin}(M) \xrightarrow{c\circ-} \Omega_5^{Spin}(B)$ sends $H_1(M)$ isomorphically and $H_3(M;\mathbb{Z}/2) \oplus H_4(M;\mathbb{Z}/2)$ to zero. Therefore

$$\widehat{\Omega}_5^{Spin}(B,M)/A \cong \ker(\widehat{\Omega}_4^{Spin}(M) \to \Omega_4^{Spin}(B)) .$$

Recall that $\widehat{\Omega}_4^{Spin}(M) \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ and $\Omega_4^{Spin}(B) \subset \mathbb{Z} \oplus \mathbb{Z}/2 \oplus H_4(B)$ where the $\mathbb{Z}/2$ summand in $\Omega_4^{Spin}(B)$ is isomorphic to $H_2(\pi;\mathbb{Z}/2)$. Note that $H_2(M;\mathbb{Z}/2) \cong \mathbb{Z}\otimes_\Lambda H_2(\widetilde{M};\mathbb{Z}/2) \oplus H_2(\pi;\mathbb{Z}/2)$. Hence

$$\ker(\widehat{\Omega}_4^{Spin}(M) \to \widehat{\Omega}_4^{Spin}(B)) \cong \mathbb{Z} \otimes_\Lambda H_2(\widetilde{M};\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) .$$

The map $\pi_1(\mathcal{E}_\bullet(B)) \to \widehat{\Omega}_5^{Spin}(B,M)/A$ is zero by the commutativity and the exactness of the braid, since the map $\widehat{\Omega}_5^{Spin}(B,M)/A \to \widehat{\Omega}_4^{Spin}(M)$ is injective. Therefore $\gamma\colon \widehat{\Omega}_5^{Spin}(B,M)/A \to \mathrm{Aut}_\bullet(M)$ is injective.

For the last part of the corollary, let $f\colon M \to M$ represent an element of $\mathrm{Aut}_\bullet(M)$. The natural map $\Omega_4^{Spin}(M) \to H_0(M)$ sends a spin 4-manifold to its signature. The signature is preserved by a homotopy equivalence, it follows that $\alpha(f) \in H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. Moreover f induces identity on $H_2(\pi;\mathbb{Z}/2) \cong \mathbb{Z}/2$, so we have $\mathrm{im}\,\alpha \subseteq \mathbb{Z} \otimes_\Lambda H_2(\widetilde{M};\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. On the other hand since, the map $\widehat{\Omega}_5^{Spin}(B,M)/A \to \widehat{\Omega}_4^{Spin}(M)$ and γ is injective $\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M};\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \subseteq \mathrm{im}\,\alpha$, by the commutativity of the braid. Therefore $\mathrm{im}\,\alpha = \mathbb{Z} \otimes_\Lambda H_2(\widetilde{M};\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. □

Once again, we look for a relation between $c_*[M]$ and the equivariant cohomology intersection pairing s_M. We first remark that the map B_{π_2}, defined for free fundamental groups, gives us a map $\mathbb{Z} \otimes_\Lambda \Gamma(\pi_2) \to \mathrm{Her}(\pi_2^\dagger)$ which does not have to be injective for PD_2 groups. We are going to use a different map $F\colon H_4(B) \to \mathrm{Her}(H^2(B;\Lambda))$ which is defined in [19], to get s_M. Recall that the equivariant cohomology intersection pairing is defined by

$$s_M\colon H^2(M;\Lambda) \times H^2(M;\Lambda) \to \Lambda; \quad (u,v) \to v([M] \cap u) = (u \cup v)[M] .$$

Let $\mathrm{Her}(H^2(B;\Lambda))$ be the group of Hermitian pairings on $H^2(B;\Lambda)$. The right action of $\pi_1(B)$ on the chains of the universal cover $\widetilde{B}$ induces a left Λ-module structure on $H^2(B;\Lambda)$. We are going to define a natural map $F\colon H_4(B) \to Her(H^2(B;\Lambda))$. To define F we need an equivariant chain approximation to the diagonal. We will use the Alexander-Whitney diagonal chain approximation map (any two diagonal approximations are naturally chain homotopic [12, p.327]), which is a Λ-module chain map

$$\widetilde{\Delta}\colon C_*(\widetilde{B}) \to C_*(\widetilde{B}) \otimes C_*(\widetilde{B})$$

defined as $\widetilde{\Delta}(\sigma) = \sum_{p+q=*} \sigma\rfloor_p \otimes {}_p\lfloor\sigma$ where $\sigma\rfloor_p$ and ${}_p\lfloor\sigma$ denote the front p-face and back q-face of σ respectively, with Λ acting by

$$C_*(\widetilde{B}) \otimes C_*(\widetilde{B}) \times \mathbb{Z}[\pi] \to C_*(\widetilde{B}) \otimes C_*(\widetilde{B})$$
$$\left(\sigma \otimes \sigma', \sum_{g \in \pi} n_g g\right) \to \sum_{g\in\pi} n_g(\sigma g \otimes \sigma' g) .$$

Apply $- \otimes_\Lambda \mathbb{Z}$ to obtain a $\mathbb{Z}$-module chain map

$$\Delta = \widetilde{\Delta} \otimes 1\colon C_*(\widetilde{B}) \otimes_\Lambda \mathbb{Z} = C_*(B;\mathbb{Z}) \to (C_*(\widetilde{B}) \otimes C_*(\widetilde{B})) \otimes_\Lambda \mathbb{Z} = C_*(\widetilde{B}) \otimes_\Lambda \overline{C_*(\widetilde{B})}$$

where $\overline{C_*(\widetilde{B})}$ denotes the left Λ-module cellular chain complex with the same additive structure and

$$\Lambda \times \overline{C_*(\widetilde{B})} \to \overline{C_*(\widetilde{B})} \ ; \quad (a, y) \to y\bar{a} \ .$$

Given a 4-chain $x \in C_4(B;\mathbb{Z})$, we have

$$\Delta(x) = \sum_i x_i' \otimes x_i'' \in (C_*(\widetilde{B}) \otimes_\Lambda \overline{C_*(\widetilde{B})})_4 = \sum_{p+q=4} (C_p(\widetilde{B}) \otimes_\Lambda \overline{C_q(\widetilde{B})}) \ .$$

The cap product of x and a 2-cochain $v \in C^2(\widetilde{B})$ is the 2-chain

$$x \cap v = \sum_i x_i'' \overline{v(x_i')} \in C_2(\widetilde{B})$$

with $v(x_i') = 0 \in \mathbb{Z}[\pi]$ if the dimension of x_i' is not 2. Now, define F by

$$F(x)(u, v) = u(x \cap v) = (u \cup v)(x) \ .$$

Recall that $c\colon M \to B$ is a 3-equivalence. The construction of F applied to M yields s_M, the usual cup product form. Thus the naturality implies that $F(c_*[M]) = s_M$. In other words we have a commutative diagram

$$\begin{tikzcd}
H^2(B;\Lambda) \times H^2(B;\Lambda) \arrow[rr, "F(c_*[M])"] \arrow[d, "c^* \times c^*"', "\cong"] & & \Lambda \\
H^2(M;\Lambda) \times H^2(M;\Lambda) \arrow[r, equal] & H^2(M;\Lambda) \times H^2(M;\Lambda) \arrow[ur, "s_M"'] &
\end{tikzcd}$$

Therefore for any $u, v \in H^2(B;\Lambda)$, we have

$$F(c_*[M])(u, v) = s_M(c^*(u), c^*(v)) = s_M(u, v) = (u \cup v)([M])$$

where we identify $H^2(B;\Lambda)$ with $H^2(M;\Lambda)$ via c^*. Following [7], we use the notation $\mathcal{P}_2(\pi)$ for the class of all oriented 4-dimensional Poincaré complexes X for which there is an isomorphism $\pi \cong \pi_1(X)$.

Theorem 4.2.8 ([19]). *Let $M_i \in \mathcal{P}_2(\pi)$, $i = 1, 2$, and let $c_i\colon M_i \to B$ are 3-equivalences over a 2-stage Postnikov system B. Then M_1 and M_2 have the same intersection pairing over Λ if and only if $F((c_1)_*[M_1]) = F((c_2)_*[M_2])$.*

Therefore any $\phi \in \mathrm{Aut}_\bullet(B)$ which preserves $c_*[M]$, also preserves the intersection form s_M. Let us define

$$\mathrm{Isom}[\pi, \pi_2, c_*[M]] := \{\phi \in \mathrm{Aut}_\bullet(B) \mid \phi_*(c_*[M]) = c_*[M]\} .$$

Lemma 4.2.9.

$$\ker(\beta\colon \mathrm{Aut}_\bullet(B) \to \Omega_4^{Spin}(B)) = \mathrm{Isom}[\pi, \pi_2, c_*[M]] .$$

Proof. If $\phi \in \mathrm{Aut}_\bullet(B)$ and $c\colon M \to B$ is the classifying map, then we have $\beta(\phi) = [M, \phi \circ c] - [M, c]$. The natural map $\Omega_4^{Spin}(B) \to H_4(B)$ sends a bordism element to the image of its fundamental class. The image of this element in $H_4(B)$ is zero when $\phi_*(c_*[M]) = c_*[M]$, so $\ker(\beta) \subseteq \mathrm{Isom}[\pi, \pi_2, c_*[M]]$. To see the other inclusion, first recall that by Proposition 4.2.5 we have

$$\Omega_4^{Spin}(B) \subset \Omega_4^{Spin}(*) \oplus \mathbb{Z}/2 \oplus H_4(B) ,$$

where $\mathbb{Z}/2 \cong \mathrm{coker}(d_2\colon H_4(B; \mathbb{Z}/2) \to H_2(B; \mathbb{Z}/2))$. If $\phi \in \mathrm{Isom}[\pi, \pi_2, c_*[M]]$, it is easy to see that the projection of $\beta(\phi)$ to the first and third components of the direct sum above are zero. Since ϕ is a homotopy equivalence it will induce the identity map on the $\mathbb{Z}/2$ summand, so the projection of $\beta(\phi)$ to the middle component is also zero. Hence we have $\phi \in \ker(\beta)$ and $\mathrm{Isom}[\pi, \pi_2, c_*[M]] \subseteq \ker(\beta)$. □

Corollary 4.2.10. *The images of* $\mathrm{Aut}_\bullet(M)$ *or* $\widetilde{\mathcal{H}}(M)$ *in* $\mathrm{Aut}_\bullet(B)$ *are precisely equal to* $\mathrm{Isom}[\pi, \pi_2, c_*[M]]$.

Proof. For each $[f] \in \mathrm{Aut}_\bullet(M)$, we have a base-point preserving self-equivalence $\phi_f \colon B \to B$ such that $c \circ f = \phi_f \circ c$. Hence ϕ_f will satisfy the following

$$(\phi_f)_*(c_*[M]) = (\phi_f \circ c)_*[M] = (c \circ f)_*[M] = c_*[M] \ ,$$

since the fundamental class in $H_4(M)$ is preserved by an orientation preserving homotopy equivalence. We see that $\mathrm{im}(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$ is contained in $\mathrm{Isom}[\pi, \pi_2, c_*[M]]$.

To see the other inclusion, first observe that $H_2(M;\mathbb{Q}) \neq 0$. Let $\phi \colon B \to B$ be an element of $\mathrm{Aut}_\bullet(B)$ such that $\phi_*(c_*[M]) = c_*[M]$. By [33, 1.3], there exists a lifting $f \colon M \to M$ such that $c \circ f \simeq \phi \circ c$. It follows (as in [33, p. 88]) that f is a homotopy equivalence and the image of f is ϕ.

The result for the image of $\widetilde{\mathcal{H}}(M)$ follows by the exactness of the braid ([35, Lemma 2. 7]) and the fact that $\ker(\beta) = \mathrm{Isom}[\pi, \pi_2, c_*[M]]$. $\square$

Recall that A denotes the remaining subgroup of $H_5(B)$ in $\Omega_5^{Spin}(B)$ and $A \subset \mathrm{im}(\pi_1(\mathcal{E}_\bullet(B)) \to \Omega_5^{Spin}(B))$. Before we put the pieces of braid together, we will divide both $\Omega_5^{Spin}(B)$ and $\widehat{\Omega}_5^{Spin}(B, M)$ by A, which we can do by the commutativity and the exactness of the braid. Here are the relevant terms of our braid diagram:

$$\begin{tikzcd}
\begin{matrix} H_1(M) \oplus \mathbb{Z}/2 \\ \oplus\, \mathbb{Z}/2 \end{matrix} \arrow[rr, bend left] \arrow[dr] & & \widetilde{\mathcal{H}}(M) \arrow[rr, bend left] \arrow[dr] & & \mathrm{Isom}[\pi, \pi_2, c_*[M]] \arrow[dr, "0"] & \\
& H_1(M) \arrow[ur] \arrow[dr, "0"] & & \mathrm{Aut}_\bullet(M) \arrow[ur] \arrow[dr, "\alpha"] & & \mathbb{Z}/2 \\
\pi_1(\mathcal{E}_\bullet(B)) \arrow[ur] \arrow[rr, bend right, "0"] & & \begin{matrix} \mathbb{Z} \otimes_\Lambda H_2(\widetilde{M};\mathbb{Z}/2) \\ \oplus \\ H_3(M;\mathbb{Z}/2) \end{matrix} \arrow[ur, "\gamma"] \arrow[rr, bend right] & & \begin{matrix} H_2(M;\mathbb{Z}/2) \\ \oplus \\ H_3(M;\mathbb{Z}/2) \end{matrix} \arrow[ur] &
\end{tikzcd}$$

There is an action of $\mathrm{Isom}[\pi, \pi_2, c_*[M]]$ on the normal subgroup $K_1 := \ker(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$. Let $[f] \in K_1$, then $c \circ f \simeq c$ also let $\phi \in \mathrm{Isom}[\pi, \pi_2, c_*[M]]$, then by [33, Lemma 1.3] there is $h\colon M \to M$, a homotopy equivalence , such that $c \circ h \simeq \phi \circ c$. Now define $\phi.f := h \circ f \circ h^{-1}$. Since $c \circ (h \circ f \circ h^{-1}) \simeq \phi \circ c \circ f \circ h^{-1} \simeq \phi \circ c \circ h^{-1} \simeq c \circ h \circ h^{-1} \simeq c$, this action is well defined.

Now we can state the main theorem of this section:

Theorem 4.2.11. *Let M be a connected, closed, oriented smooth or topological spin manifold of dimension 4. If $\pi := \pi_1(M)$ is a PD_2 group, then*

$$\mathrm{Aut}_\bullet(M) \cong (\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M}; \mathbb{Z}/2)) \oplus H_3(M; \mathbb{Z}/2)) \rtimes \mathrm{Isom}[\pi, \pi_2, c_*[M]]$$

where $s_M\colon H_2(M; \Lambda) \times H_2(M; \Lambda) \to \Lambda$ is the intersection form on π_2.

Proof. Let $K_2 := \ker(\widetilde{\mathcal{H}}(M) \to \mathrm{Isom}[\pi, \pi_2, c_*[M]])$, then

$$\mathrm{Isom}[\pi, \pi_2, c_*[M]] \cong \widetilde{\mathcal{H}}(M)/K_2 \ .$$

By the exactness and the commutativity of the braid we have

$$K_2 \cong \mathrm{im}(\Omega_5^{Spin}(B) \to \widetilde{\mathcal{H}}(M)) \cong H_1(M)$$

and it maps to $1 \in \mathrm{Aut}_\bullet(M)$ by the commutativity of the braid. This gives the splitting of the short exact sequence

$$0 \to K_1 \to \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}[\pi, \pi_2, c_*[M]] \to 1 \ .$$

It follows that

$$\mathrm{Aut}_\bullet(M) \cong K_1 \rtimes \mathrm{Isom}[\pi, \pi_2, c_*[M]]$$

with the conjugation action of $\mathrm{Isom}[\pi, \pi_2, c_*[M]]$ on the normal subgroup K_1 defining the semi-direct product structure.

The braid diagram shows that the map γ is an injective *homomorphism.* To see this, it is enough to show that α is a homomorphism on the image of γ. Let $\gamma(W, F) = f$ and $\gamma(W', F') = g$. Recall that $\alpha(f \circ g) = \alpha(f) + f_*(\alpha(g))$. We have to show that $f_*(\alpha(g)) = \alpha(g)$. By Corollary 4.2.7, $\alpha(g) \in \mathbb{Z} \otimes_\Lambda H_2(\widetilde{M}; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$. Any element f in the image of γ is trivial in $\mathrm{Aut}_\bullet(B)$, i.e., the image $\phi_f = \mathrm{id}_B$ and $c \circ f = c$. Since $c_*\colon H_2(M; \mathbb{Z}/2) \to H_2(B; \mathbb{Z}/2)$ is an isomorphism, f acts as the identity on $H_2(M; \mathbb{Z}/2)$, so it also acts as identity on $\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M}; \mathbb{Z}/2) \subset H_2(B; \mathbb{Z}/2)$. Moreover since $H_3(M; \mathbb{Z}/2) \cong H^1(M; \mathbb{Z}/2)$ and c induces isomorphism on $H^1(M; \mathbb{Z}/2$, so f also acts as identity on $H_3(M; \mathbb{Z}/2)$. Now a diagram chase shows that γ is a homomorphism.

Therefore we have a short exact sequence of groups and homomorphisms

$$0 \to (\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M}; \mathbb{Z}/2)) \oplus H_3(M; \mathbb{Z}/2) \to \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}[\pi, \pi_2, c_*[M]] \to 1\,.$$

Moreover, $K_1 = \operatorname{im} \gamma$ ([35, Corollary 2.13]) and K_1 is mapped isomorphically onto $\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M}; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$ by the map α.

The conjugation action on K_1 agrees with the induced action on homology under the identification $K_1 \cong \mathbb{Z} \otimes_\Lambda H_2(\widetilde{M}; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$ via α (see the proof of Theorem A). It follows that

$$\mathrm{Aut}_\bullet(M) \cong (\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M}; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)) \rtimes \mathrm{Isom}[\pi, \pi_2, c_*[M]]$$

as required, with the action of $\mathrm{Isom}[\pi, \pi_2, c_*[M]]$ on the normal subgroup $\mathbb{Z} \otimes_\Lambda H_2(\widetilde{M}; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$ is given by the induced action of homotopy self-equivalences on homology. $\square$

Remark 4.2.12. By Serre spectral sequence we have

$$H_2(M) \cong \pi_2 \otimes_\Lambda \mathbb{Z} \oplus H_2(F)$$

hence

$$H_2(M;\mathbb{Z}/2) \cong \pi_2 \otimes_\Lambda \mathbb{Z}/2 \oplus H_2(F;\mathbb{Z}/2)\ .$$

Therefore for any $x \in \pi_2 \otimes_\Lambda \mathbb{Z}/2$ there exists a map $\alpha\colon S^2 \to M$. As in the free fundamental group case, let $\eta\colon S^3 \to S^2$ be the Hopf map, then the following composition is a homotopy self-equivalence of M.

$$M \xrightarrow{\text{pinch}} M \vee S^4 \xrightarrow{\mathrm{id}\vee\eta^2} M \vee S^2 \xrightarrow{\mathrm{id}\vee\alpha} M\ .$$

To realize $H_3(M;\mathbb{Z}/2)$ as homotopy equivalences, first observe that

$$H_3(\widetilde{M}) \cong H_3(M;\Lambda) \cong H^1(M;\Lambda) \cong H^1(\pi;\Lambda) \cong H^1(F;\Lambda) \cong H_1(F;\Lambda) \cong 0$$

since π is a PD_2 group and F is an aspherical surface. So the Hurewicz homomorphism is trivial this time. We are going to construct the homotopy self-equivalences by using the minimal model for M. The strongly minimal PD_4-complexes with fundamental group a PD_2-group are the total spaces of S^2-bundles over aspherical closed surfaces. When M is a spin manifold, the minimal model Z will be trivial, i.e., $Z \simeq F \times S^2$ (see [55, Theorem 2]). We have

$$\begin{aligned} H_3(M;\mathbb{Z}/2) &\cong H^1(M;\mathbb{Z}/2) \cong H^1(F;\mathbb{Z}/2) \\ &\cong [F, \mathbb{R}P^\infty] \cong [F, \mathbb{R}P^3] \cong [F, SO(3)]\ . \end{aligned}$$

Hence we can represent any $\alpha \in H_3(M;\mathbb{Z}/2)$ as $\alpha\colon F \to SO(3)$. Define a self-equivalence φ of Z as $\varphi\colon F\times S^2 \to F\times S^2$ such that $\varphi(x,y) = (x, \alpha(x)y)$. This is clearly a homotopy self-equivalence of Z. Now, by [44, Theorem 11] we have a homotopy self-equivalence f of M such that $g_M \circ f = \varphi \circ g_M$.

4.3 Non-spin Case

The purpose of this section is to state and prove a theorem calculating the group $\mathrm{Aut}_\bullet(M, w_2)$. Our main result is Theorem C on page 118. Through out this section M denotes a 4-manifold with PD_2 fundamental group.

Proposition 4.3.1. *The relevant bordism groups of $M\langle w_2\rangle$ and $B\langle w_2\rangle$ are given as follows:*

$$\Omega_4(M\langle w_2\rangle) \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z} \ ,$$

$$\Omega_5(M\langle w_2\rangle) \cong H_1(M) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}/2 \ ,$$

$$\Omega_4(B\langle w_2\rangle) \subset \mathbb{Z} \oplus \mathbb{Z}/2 \oplus H_4(B) \ ,$$

$$\Omega_5(B\langle w_2\rangle) \subset H_1(B) \oplus H_5(B) \ .$$

Proof. Recall that if M is not a spin manifold, then the differential d_2 is the dual of $Sq^2_w(x) = Sq^2(x) + x \cup w_2$. Once again orientability of M implies that $Sq^2_w = 0$ ($w_2 = v_2$, the second Wu class). Hence, for $M\langle w_2\rangle$ everything works exactly the same as in the spin case.

For the bordism group $\Omega_4(B\langle w_2\rangle)$, non-zero terms on the E^2-page are $H_0(B;\Omega_4^{Spin}(*)) \cong \mathbb{Z}$ in the $(0,4)$ position, $H_2(B;\mathbb{Z}/2)$ in the $(2,2)$ position, $H_3(B;\mathbb{Z}/2)$ in the $(3,1)$ position and $H_4(B)$ in the $(4,0)$ position. To see the $\mathbb{Z}/2$ summand consider the fibration $\widetilde{B} \xrightarrow{p} B \xrightarrow{u_B} F$ and the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^2(F;H^0(\widetilde{B};\mathbb{Z}/2)) \arrow[r, "u_B^*"] \arrow[d, "Sq^2_w"] & H^2(B;\mathbb{Z}/2) \arrow[r, "p^*"] \arrow[d, "Sq^2_w"] & H^2(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[r] \arrow[d, "Sq^2_w"] & 0 \\
0 \arrow[r] & H^2(F;H^2(\widetilde{B};\mathbb{Z}/2)) \arrow[r, "u_B^*"] & H^4(B;\mathbb{Z}/2) \arrow[r, "p^*"] & H^4(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[r] & 0 \ .
\end{tikzcd}$$

Let $\widetilde{w_2} = p^*(w_2)$, where $w_2 = w_2(B)$. We have to consider two cases:

(i) $\widetilde{w_2} = 0$ and

(ii) $\widetilde{w_2} \neq 0$.

If $\widetilde{w_2} = 0$, then the leftmost $Sq_w^2 = 0$ and the rightmost Sq_w^2 is injective. Note that $w_2 \in \ker(Sq_w^2)$: to see this first observe that there exists a $y \in H^2(F; H^0(\widetilde{B}; \mathbb{Z}/2)) \cong H^2(F; \mathbb{Z}/2)$ with $u_B^*(y) = w_2$, since $p^*(w_2) = 0$. By naturality we have $Sq_w^2(w_2) = Sq_w^2(u_B^*(y)) = 0$. Conversely if $x \in H^2(B; \mathbb{Z}/2)$ such that $Sq_w^2(x) = 0$, then $0 = p^*(Sq_w^2(x)) = Sq_w^2(p^*(x))$, hence $p^*(x) = 0$. Observe that if $x \neq 0$, then $x = w_2$, for $\ker(p^*) \cong H^2(F; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Hence in this case,

$$\ker(Sq_w^2 \colon H^2(B; \mathbb{Z}/2) \to H^4(B; \mathbb{Z}/2)) \cong \langle w_2 \rangle \cong \mathbb{Z}/2 \,.$$

If $\widetilde{w_2} \neq 0$, then the leftmost Sq_w^2 is injective, it is just the cup product with $\widetilde{w_2}$, and $\ker(Sq_w^2 \colon H^2(\widetilde{B}; \mathbb{Z}/2)^\pi \to H^4(\widetilde{B}; \mathbb{Z}/2)^\pi) \cong \langle \widetilde{w_2} \rangle$. If $x \in H^2(B; \mathbb{Z}/2)$ such that $Sq_w^2(x) = 0$, then $p^*(x) = 0$ or $p^*(x) = \widetilde{w_2}$. If $p^*(x) = 0$, then there exists a $0 \neq y \in H^2(F; \mathbb{Z}/2)$ with $u_B^*(y) = x$. But $0 = Sq_w^2(x) = Sq_w^2(u_B^*(y)) = u_B^*(Sq_w^2(y)) \neq 0$. Thus $p^*(x) = \widetilde{w_2}$, so x must be of the form $x = w_2 + u_B^*(y)$ for some $y \in H^2(F; \mathbb{Z}/2)$. We have $0 = Sq_w^2(x) = Sq_w^2(w_2) + Sq_w^2(u_B^*(y)) = 0 + u_B^*(Sq_w^2(y))$. But then $y = 0$, since both $Sq_w^2 \colon H^2(F; H^0(\widetilde{B}; \mathbb{Z}/2)) \to H^2(F; H^2(\widetilde{B}; \mathbb{Z}/2))$ and u_B^* are injective $x = w_2$. Hence also in this case we have

$$\ker(Sq_w^2 \colon H^2(B; \mathbb{Z}/2) \to H^4(B; \mathbb{Z}/2)) \cong \langle w_2 \rangle \cong \mathbb{Z}/2 \,.$$

Since Sq_w^2 and d_2 are dual to each other, we have

$$\operatorname{coker}(d_2 \colon H_4(B; \mathbb{Z}/2) \to H_2(B; \mathbb{Z}/2)) \cong \mathbb{Z}/2 \,.$$

For (3, 1) position, we consider $d_2\colon H_5(B) \to H_3(B;\mathbb{Z}/2)$, which is reduction mod 2 composed with the dual of $Sq^2_w\colon H^3(B;\mathbb{Z}/2) \to H^5(B;\mathbb{Z}/2)$. By Serre spectral sequence we have

$$H^3(B;\mathbb{Z}/2) \cong H^1(\pi; H^2(\widetilde{B};\mathbb{Z}/2)) \quad \text{and} \quad H^5(B;\mathbb{Z}/2) \cong H^1(\pi; H^4(\widetilde{B};\mathbb{Z}/2)) .$$

We should understand the Sq^2_w on the coefficients, which is $Sq^2_w(x) = Sq^2(x) + x \cup \widetilde{w_2}$ for $x \in H^2(\widetilde{B};\mathbb{Z}/2)$. By the universal coefficient theorem,

$$H^2(\widetilde{B};\mathbb{Z}/2) \cong \mathrm{Hom}(H_2(\widetilde{B}),\mathbb{Z}/2) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2) \oplus \mathrm{Hom}(P,\mathbb{Z}/2) .$$

We are going to show that $\widetilde{w_2}$ can not come from

$$\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2) \cong \mathrm{Hom}(H^2(\pi;\Lambda);\mathbb{Z}/2) \cong H^2(K(\mathbb{Z},2);\mathbb{Z}/2) .$$

If we can show this, then the above Sq^2_w becomes Sq^2 and since by Lemma 4.2.3, we have

$$H^3(B;\mathbb{Z}/2) \cong H^1(\pi; H^2(K(\mathbb{Z},2);\mathbb{Z}/2))$$

and

$$H^5(B;\mathbb{Z}/2) \cong H^1(\pi; H^4(K(\mathbb{Z},2);\mathbb{Z}/2)$$

we can conclude that Sq^2_w is an isomorphism. Let us start by recalling the evaluation exact sequence (see Lemma 2.1.3). There is an exact sequence

$$0 \longrightarrow H^2(\pi;\Lambda) \xrightarrow{u_M^*} H^2(M;\Lambda) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_\Lambda(\pi_2,\Lambda) \longrightarrow H^3(\pi;\Lambda) \longrightarrow 0$$

where ev is the evaluation homomorphism given by $\mathrm{ev}(v)(x) = v(x)$ and $u_M\colon M \to K(\pi,1)$ is the classifying map. So we have, $\mathrm{ev}(v)(x) = 0$ for any $v \in H^2(\pi;\Lambda)$ and $x \in \pi_2 \cong H_2(\widetilde{B}) \cong H_2(\widetilde{M})$. Now suppose $\widetilde{w_2}$ comes from $\mathrm{Hom}(H^2(\pi;\Lambda);\mathbb{Z}/2)$. This implies $w_2(\widetilde{M})$ also comes from

$\mathrm{Hom}(H^2(\pi;\Lambda);\mathbb{Z}/2)$, since $c\colon M \to B$ is 3-connected. Also remember that $w_2(\widetilde{M}) \in H^2(\widetilde{M};\mathbb{Z}/2) \cong \mathrm{Hom}(H_2(\widetilde{M}),\mathbb{Z}/2)$. Then there exists $0 \neq u \in H^2(\pi;\Lambda)$ and $x \in H_2(\widetilde{M})$such that $u(x) \neq 0$ which contradicts the above exact sequence. Therefore $Sq^2_w = Sq^2$ which is an isomorphism, so it is dual is also an isomorphism and reduction mod 2, $H_5(B) \to H_5(B;\mathbb{Z}/2)$ is surjective. Hence $d_2\colon H_5(B) \to H_3(B;\mathbb{Z}/2)$ is surjective.

For the calculations of $\Omega_5(B\langle w_2\rangle)$, first observe that the following sequence

$$H^2(X_k;\mathbb{Z}/2) \xrightarrow{Sq^2_w} H^4(X_k;\mathbb{Z}/2) \xrightarrow{Sq^2_w} H^6(X_k;\mathbb{Z}/2)$$

is exact, which can be shown by the same technique as in the free fundamental group case, and then by taking the inverse limit we can see that

$$H^2(\widetilde{B};\mathbb{Z}/2) \xrightarrow{Sq^2_w} H^4(\widetilde{B};\mathbb{Z}/2) \xrightarrow{Sq^2_w} H^6(\widetilde{B};\mathbb{Z}/2)$$

is also exact. Consider the following diagram

$$\begin{tikzcd}
0 & 0 & 0 \\
H^2(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[u] \arrow[r, "Sq^2_w"] & H^4(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[u] \arrow[r, "Sq^2_w"] & H^6(\widetilde{B};\mathbb{Z}/2)^\pi \arrow[u] \\
H^2(B;\mathbb{Z}/2) \arrow[u, "p^*"] \arrow[r, "Sq^2_w"] & H^4(B;\mathbb{Z}/2) \arrow[u, "p^*"] \arrow[r, "Sq^2_w"] & H^6(B;\mathbb{Z}/2) \arrow[u, "p^*"] \\
H^2(F;H^0(\widetilde{B};\mathbb{Z}/2)) \arrow[u, "u^*"] \arrow[r, "Sq^2_w"] & H^2(F;H^2(\widetilde{B};\mathbb{Z}/2)) \arrow[u, "u^*"] \arrow[r, "Sq^2_w"] & H^2(F;H^4(\widetilde{B};\mathbb{Z}/2)) \arrow[u, "u^*"] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

We want to show that the middle row is exact. Let $x \in H^2(B;\mathbb{Z}/2)$, then we have

$$Sq^2_w(x^2 + x \cup w_2) = Sq^2(x) \cup x + Sq^1(x) \cup Sq^1(x) + x \cup Sq^2(x) +$$

$$Sq^2(x) \cup w_2 + Sq^1(x) \cup Sq^1(w_2) + x \cup Sq^2(w_2) + x^2 \cup w_2 + x \cup w_2^2$$

$$= Sq^1(x) \cup Sq^1(x) + Sq^1(x) \cup Sq^1(w_2)$$

where Sq^1 is the Bockstein homomorphism. Since $H^2(B) \to H^2(B;\mathbb{Z}/2)$ is onto, Sq^1 is the zero map. We therefore have

$$\mathrm{im}(Sq^2_w\colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2))$$

$$\subseteq \ker(Sq^2_w\colon H^4(B;\mathbb{Z}/2) \to H^6(B;\mathbb{Z}/2))\ .$$

To see the other inclusion let $y \in H^4(B;\mathbb{Z}/2)$ such that $Sq^2_w(y) = Sq^2(y) + y \cup w_2 = 0$. Let $p^*(y) = \widetilde{y}$, then by the commutativity of the above diagram $Sq^2_w(\widetilde{y}) = 0$. By the exactness of the top row, there exists a $\widetilde{z} \in H^2(\widetilde{B};\mathbb{Z}/2)$ such that $Sq^2_w(\widetilde{z}) = \widetilde{y}$. By the exactness of the leftmost sequence, we have an $x \in H^2(B;\mathbb{Z}/2)$ with $p^*(x) = \widetilde{x}$. Suppose that $Sq^2_w(x) = y' \neq y$, then there exists a $0 \neq b \in H^2(F; H^2(\widetilde{B};\mathbb{Z}/2))$ such that $u^*(b) = y - y'$ and $Sq^2_w(b) = 0$. Then there exists an $a \in H^2(F; H^0(\widetilde{B};\mathbb{Z}/2))$ with $Sq^2_w(a) = b$ and let $x' = u^*(a)$. Now consider

$$\begin{aligned} Sq^2_w(x+x') &= Sq^2_w(x) + Sq^2_w(x') = y' + Sq^2_w(u^*(a)) \\ &= y' + u^*(Sq^2_w(b)) = y' + y - y' = y\ . \end{aligned}$$

Hence $y \in \mathrm{im}(Sq^2_w)$ and we have

$$\mathrm{im}(Sq^2_w\colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2))$$

$$= \ker(Sq^2_w\colon H^4(B;\mathbb{Z}/2) \to H^6(B;\mathbb{Z}/2))\ .$$

Since $H_5(B)$ is torsion free, $H_6(B) \to H_6(B;\mathbb{Z}/2)$ is surjective. Therefore $d_2\colon H_6(B) \to H_4(B;\mathbb{Z}/2)$ is onto the kernel of $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$.

The differential $d_2\colon E^2_{5,1} \to E^2_{3,2}$ is the dual of $Sq^2_w\colon H^3(B;\mathbb{Z}/2) \to H^5(B;\mathbb{Z}/2)$ and by the above argument $Sq^2_w = Sq^2$ which is an isomorphism and so its dual d_2 is also an isomorphism in this case. Hence the kernel of the differential $d_2\colon H_5(B) \to H_3(B;\mathbb{Z}/2)$ is equal to the kernel of the reduction mod 2. $\square$

Let A denote the remaining subgroup of $H_5(B)$. Recall the maps $\varphi_{x_i}\colon B \times S^1 \to B$ that we constructed in the spin case. Since $\varphi_{x_i}(-,s)$ is homotopic to the identity for each $s \in S^1$, φ_{x_i} will preserve w_2. We also have $w_2 = w \circ \nu_M$ and hence

$$(\varphi_{x_i} \circ (c \times \mathrm{id}), \nu_M \circ p_1)\colon M \times S^1 \to B\langle w_2\rangle$$

gives an element of $\Omega_5(B\langle w_2\rangle)$. Therefore

$$A \subset \mathrm{im}(\pi_1(\mathcal{E}_\bullet(B, w_2)) \to \Omega_5(B\langle w_2\rangle)) \ .$$

As in the spin case, we are going to identify A with its image in $\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$.

Corollary 4.3.2. *We have the following isomorphism,*

$$\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)/A \cong KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \ ,$$

where $KH_2(M;\mathbb{Z}/2) := \ker(w_2\colon H_2(M;\mathbb{Z}/2) \to \mathbb{Z}/2)$. *Moreover, the group* $\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)/A$ *injects into* $\mathrm{Aut}_\bullet(M, w_2)$ *and the image of* α,

$$\mathrm{im}(\alpha) = KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \ .$$

Proof. Consider the maps $S^1 \times K3 \xrightarrow{x_i \times \{*\}} M$ where $x_i\colon S^1 \to M$ is a generator of $H_1(M) \cong \mathbb{Z}^r$ for each $i = \{1, 2, \dots, r\}$ and S^1 is equipped with a non-trivial spin structure. Since $S^1 \times K3$ is a spin manifold the maps $(x_i \times \{*\}, \nu_{S^1 \times K3})\colon S^1 \times K3 \to M\langle w_2\rangle$ are well-defined and

$$[S^1 \times K3, (x_i \times \{*\}, \nu_{S^1 \times K3})]$$

generate the $H_1(M)$ summand in $\Omega_5(M\langle w_2\rangle)$. Similarly, elements of the form

$$[S^1 \times K3, (c \circ x_i \times \{*\}, \nu_{S^1 \times K3})]$$

generate the $H_1(B)$ summand of $\Omega_5(B\langle w_2\rangle)$. The homomorphism

$$\Omega_5(M\langle w_2\rangle) \to \Omega_5(B\langle w_2\rangle) \subset H_1(M) \oplus H_5(B)$$

is defined by composing with the reference map $\bar{c}$. Note that

$$\bar{c}([S^1 \times K3, (x_i \times \{*\}, \nu_{S^1 \times K3})]) = [S^1 \times K3, (c \circ x_i \times \{*\}, \nu_{S^1 \times K3})] \ .$$

Hence $\Omega_5(M\langle w_2\rangle) \to H_1(B) \subset \Omega_5(B\langle w_2\rangle)$ is onto and by the exactness of the braid we have

$$\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)/A \cong \ker(\widehat{\Omega}_4(M\langle w_2\rangle) \to \Omega_4(B\langle w_2\rangle)) \ .$$

Recall that $\widehat{\Omega}_4(B\langle w_2\rangle) \subset H_0(B) \oplus \mathbb{Z}/2 \oplus H_4(B)$ where $\mathbb{Z}/2 \cong \langle w_2\rangle$ Also recall that we have, $\widehat{\Omega}_4(M\langle w_2\rangle) \cong H_0(M) \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. Hence we get,

$$\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)/A \cong KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \ .$$

By the commutativity of the braid,

$$\mathrm{im}(\pi_1(\mathcal{E}_\bullet(B)) \to \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)) = A \ .$$

Moreover, by the exactness of the braid $\gamma(A) = \widehat{\mathrm{id}} \in \mathrm{Aut}_\bullet(M, w_2)$. Therefore the map $\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)/A \to \widehat{\Omega}_4(M\langle w_2\rangle)$ is injective, by the exactness of the braid again, so the map $\pi_1(\mathcal{E}_\bullet(B, w_2)) \to \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)/A$ must be zero, by the commutativity of the braid. Therefore

$$\gamma\colon \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)/A \to \mathrm{Aut}_\bullet(M, w_2)$$

is injective.

If $\widehat{f}\colon M \to M\langle w_2\rangle$ represents an element of $\mathrm{Aut}_\bullet(M, w_2)$, then $f := j \circ \widehat{f}$ is a homotopy equivalence and $\alpha(\widehat{f}) = [M, \widehat{f}] - [M, \widehat{\mathrm{id}}]$. We have

$$\widehat{\Omega}_4(M\langle w_2\rangle) \cong H_0(M) \oplus H_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)\ .$$

The natural map $\widehat{\Omega}_4(M\langle w_2\rangle) \to H_0(M)$ sends a 4-manifold to its signature which is preserved by a homotopy equivalence. Also since the class $w_2 \in H^2(M; \mathbb{Z}/2)$ is a characteristic element for the cup product form (mod 2), it is preserved by the induced map of a self-homotopy equivalence of M. Therefore, the image of $\mathrm{Aut}_\bullet(M)$ in $\Omega_4(M\langle w_2\rangle)$ lies in the subgroup $KH_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$. The other inclusion follows as in the proof of Corollary 4.2.7. $\square$

Recall that in chapter 3, we defined a homomorphism

$$\widehat{j}\colon \mathrm{Aut}_\bullet(B, w_2) \to \mathrm{Aut}_\bullet(B) \quad \text{by} \quad \widehat{j}(\widehat{f}) = \phi_f$$

where $\phi_f\colon B \to B$ is the unique homotopy equivalence with $\phi_f \circ c \simeq f$. We will use the same map again. Let

$$\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]] := \{\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2) \mid \phi_f \in \mathrm{Isom}[\pi, \pi_2, c_*[M]]\}\ .$$

Lemma 4.3.3. *There is a short exact sequence of groups*

$$0 \longrightarrow H^1(M; \mathbb{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]] \xrightarrow{\widehat{j}} \mathrm{Isom}[\pi, \pi_2, c_*[M]] \longrightarrow 1$$

Proof. This follows exactly the same as the proof of Lemma 2.3.11. $\square$

Lemma 4.3.4. $\ker(\beta\colon \mathrm{Aut}_\bullet(B, w_2) \to \Omega_4(B\langle w_2\rangle)) = \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]]$.

Proof. The proof of Lemma 2.3.14 works also in this case. $\square$

Corollary 4.3.5. *The images of* $\mathrm{Aut}_\bullet(M, w_2)$ *or* $\widetilde{\mathcal{H}}(M, w_2)$ *in* $\mathrm{Aut}_\bullet(B, w_2)$ *are precisely equal to* $\mathrm{Isom}[\pi, \pi_2, c_*[M], w_2]$.

Proof. This also follows from the proof of Corollary 2.3.12. $\square$

Before we put the pieces together on our braid, we will divide out $\Omega_5(B\langle w_2\rangle)$ and $\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$ by A. The relevant terms on our braid are now:

$$\begin{tikzcd}
\begin{matrix} H_1(M) \oplus H_3(M;\mathbb{Z}/2) \\ \oplus\, \mathbb{Z}/2 \end{matrix} \arrow[dr] \arrow[rr, bend left] & & \widetilde{\mathcal{H}}(M, w_2) \arrow[dr] \arrow[rr, bend left] & & \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]] \arrow[dr, "0"] & \\
& H_1(M) \arrow[ur] \arrow[dr, "0"] & & \mathrm{Aut}_\bullet(M, w_2) \arrow[ur] \arrow[dr, "\alpha"] & & 0 \\
\pi_1(\mathcal{E}_\bullet(B)) \arrow[ur] \arrow[rr, "0"] & & \begin{matrix} KH_2(M;\mathbb{Z}/2) \\ \oplus \\ H_3(M;\mathbb{Z}/2) \end{matrix} \arrow[ur, "\gamma"] \arrow[rr, "\cong"] & & \begin{matrix} KH_2(M;\mathbb{Z}/2) \\ \oplus \\ H_3(M;\mathbb{Z}/2) \end{matrix} \arrow[ur] &
\end{tikzcd}$$

There is an action of $\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]]$ on the normal subgroup $\widehat{K_1} := \ker(\mathrm{Aut}_\bullet(M, w_2) \to \mathrm{Aut}_\bullet(B, w_2))$, which is defined as follows: Let $\widehat{f} \in \widehat{K_1}$, then $c \circ f \simeq c$. Also let $\widehat{\phi} \in \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]]$ and $\phi = \widehat{\jmath}(\widehat{\phi})$, then $\phi \in \mathrm{Isom}[\pi, \pi_2, c_*[M]]$ with $\phi_*(c_*[M]) = c_*[M]$. There is a homotopy equivalence $h\colon M \to M$ such that $c \circ h = \phi \circ c$ ([33, Lemma 1.3]). Then since $h \circ f \circ h^{-1}$ preserves w_2, we can define $\widehat{\phi}.\widehat{f} := (h \circ f \circ h^{-1}, \nu_M)$.

We can now state the main theorem of this chapter:

Theorem C. *Let M be a connected, closed, oriented topological manifold of dimension 4. If $\pi := \pi_1(M)$ is a PD_2 group, then*

$$\mathrm{Aut}_\bullet(M, w_2) \cong (KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \rtimes \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]] \ .$$

The proof of Theorem C. We have a split short exact sequence

$$0 \longrightarrow \widehat{K_1} \longrightarrow \mathrm{Aut}_\bullet(M, w_2) \longrightarrow \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, c_*[M]] \longrightarrow 1 \ .$$

Any element $\widehat{f}$ will act as identity on $\mathrm{im}(\alpha) = KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$, so λ is a homomorphism. Also $\widehat{K_1} \cong KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ and the rest of the proof follows as in the spin case. $\square$

Chapter 5

s-cobordism theorem in the PD_2 case

5.1 Preliminaries

Let M be a closed, connected, oriented, topological 4-manifold with PD_2 fundamental group π. As in Chapter 3, we start by pointing out the relation between equivariant s_M and the integral $s_M^{\mathbb{Z}}$ intersection forms. We have $H^2(M;\Lambda) \cong H^2(\pi;\Lambda) \oplus H^2(\widetilde{M};\Lambda)^\pi$ such that $H^2(\pi;\Lambda)$ is totally isotropic under the cup product pairing (see Lemma 2.1.3). Therefore s_M induces a nonsingular pairing on $H^2(\widetilde{M};\Lambda)^\pi \cong P$. On the other hand we have

$$\begin{aligned} H_2(M) &\cong (\mathbb{Z} \oplus P) \otimes_\Lambda \mathbb{Z} \oplus H_2(F) \\ &\cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus (P \otimes_\Lambda \mathbb{Z}) \, . \end{aligned}$$

Now, recall that we have a 2-connected degree-1 map $g_M \colon M \to Z$. This induces the splitting of the integral intersection form as

$$s_M^{\mathbb{Z}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus (s_M \otimes_\Lambda \mathbb{Z})$$

when M is a spin manifold, and

$$s_M^{\mathbb{Z}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \oplus (s_M \otimes_\Lambda \mathbb{Z}) \ ,$$

if M is not spin, see for example [18]. Therefore, also when π is a PD_2 group, the signature of M is determined by the formula

$$\sigma(M) = \sigma(s_M^{\mathbb{Z}}) = \sigma(s_M \otimes_\Lambda \mathbb{Z}) \ .$$

Remember that in the free case, we have a cobordism W over $K(\pi, 1)$, which does not contain any odd handles. This basically follows from Lemma 3.1.1. Let us consider the following homotopy fibration, $\widetilde{M} \xrightarrow{p} M \xrightarrow{u} K(\pi, 1)$ which induces a short exact sequence

$$0 \longrightarrow H^2(K(\pi,1);\mathbb{Z}/2) \xrightarrow{u^*} H^2(M;\mathbb{Z}/2) \xrightarrow{p^*} H^2(\widetilde{M};\mathbb{Z}/2) \ .$$

This short exact sequence tells us that when M is spin $\widetilde{M}$ is also spin, but if M is not spin $\widetilde{M}$ may or may not admit a spin structure. But we want them to be spin at the same time. As a consequence we will impose an extra condition on our manifolds in this case.

Definition 5.1.1 ([34]). We say that a manifold M has w_2-type (I), (II), or (III) if one of the following holds:

(I) $w_2(\widetilde{M}) \neq 0$,

(II) $w_2(M) = 0$, or

(III) $w_2(M) \neq 0$ and $w_2(\widetilde{M}) = 0$.

Finally, note that the Whitehead group $Wh(\pi)$ is trivial whenever $\pi \cong \pi_1(F)$, where F is a closed, oriented aspherical surface [26], hence also in this chapter being s-cobordant is equivalent to being h-cobordant. Now, let us state our main result:

5.2 Main Result

Theorem D. *Let M_1 and M_2 be two closed, connected, oriented, topological 4-manifolds with PD_2 fundamental group. Suppose that they have the same Kirby-Siebenmann invariant and M_1 has w_2-type (I) or (II). Then M_1 and M_2 are s-cobordant if and only if they have isometric quadratic 2-types.*

Proof. If M_1 and M_2 are s-cobordant, then they are homotopy equivalent and hence they have isometric quadratic 2-types.

The strategy for the rest of the proof is the same as in the free fundamental group case. We know that oriented topological bordism group over $K(\pi, 1)$ is

$$\Omega_4^{STOP}(K(\pi,1)) \cong \Omega_4^{STOP}(*) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

via the signature and the ks-invariant. Therefore in the type (I) case, there is an oriented cobordism W between M_1 and M_2. On the other hand, in the type (II) case we have

$$\Omega_4^{TOPSPIN}(K(\pi,1)) \cong \mathbb{Z} \oplus H_2(\pi; \mathbb{Z}/2) \; .$$

The invariants are the signature and an invariant in $H_2(\pi;\mathbb{Z}/2)$. We have M_1 and M_2 with the same signature, so to be bordant over $K(\pi,1)$ they should also have the same invariant in $H_2(\pi;\mathbb{Z}/2)$. Let us investigate this bordism invariant. First, we define a degree-1 normal map

$$\varphi'\colon F\times(S^1\times S^1)\to F\times S^2$$

by collapsing $S^1\vee S^1\subset S^1\times S^1$. This is a degree-1 normal map, since the normal bundle of a sphere is trivial. By low dimensional surgery we replace φ' by a 2-connected map $\varphi\colon N\to F\times S^2$. Then by [21, Theorem 3.10], $[N,\varphi]-[F\times S^2,\mathrm{id}]$ maps to

$$\mathrm{kervaire}^2(\varphi)\cap[F\times S^2]\in E^{\infty}_{2,2}=H_2(F\times S^2;\Omega_2^{Spin}(*))\cong\mathbb{Z}/2$$

where $\mathrm{kervaire}^2(\varphi)\in H^2(F\times S^2;\mathbb{Z}/2)$ is the codimension-2 Kervaire invariant. The bordism class $[N,\mathrm{p}_1\circ\varphi]-[F\times S^2,\mathrm{p}_1]$ maps to

$$\varsigma:=(\mathrm{p}_1)_*(\mathrm{kervaire}^2(\varphi)\cap[F\times S^2])$$

which is the the generator of $E^{\infty}_{2,2}=H_2(F;\Omega_2^{Spin}(*))\cong H_2(\pi;\mathbb{Z}/2)$. Now if M_1 and M_2 do not map to the same element in $H_2(\pi;\mathbb{Z}/2)$, then we would have

$$[M_1]=[N\sharp rE_8]\quad\text{and}\quad[M_2]=[F\times S^2\sharp rE_8]$$

over $K(\pi,1)$ (we take connected sum with r copies of E_8 to make the signatures equal). Next we consider the restriction of the surgery assembly map

$$\kappa_2\colon H_2(\pi;\mathbb{Z}/2)\to L_4(\Lambda)$$

which is injective (see [21, Corollary 1.5]). By the characteristic class formula [21, Proposition 3.6] $\kappa_2(\varsigma)$ is the surgery obstruction of φ, and the kernel form is just the reduced intersection form s'_N of N which is non-trivial. On the other hand the equivariant intersection form of $F \times S^2$ is trivial. Therefore $[N\sharp rE_8]$ and $[F \times S^2\sharp rE_8]$ map to different elements of $L_4(\Lambda)$. This is a contradiction since M_1 and M_2 have isomorphic equivariant intersection forms and hence have isomorphic reduced intersection forms. Therefore they also represent the same class in $H_2(\pi; \mathbb{Z}/2)$. As a result we have a cobordism W between M_1 and M_2 over $K(\pi, 1)$, which is a spin cobordism in the type (II) case. We may view W as

$$W = M_1 \times [0,1] \cup \{2-handles\} \cup \{3-handles\} \cup M_2 \times [-1,0] \ .$$

We will split W into two halves : on one side, M_1 and the 2-handles, on the other, M_2 and the 3-handles. As in the free fundamental group case, let $W_{3/2}$ be the ascending cobordism that contains just M_1 and the 2-handles and $M_{3/2}$ be its 4-dimensional upper boundary. Since M_1 and $\widetilde{M_1}$ are spin at the same time, we can assume that there are no $S^2\tilde{\times}S^2$-terms are present in $M_{3/2}$. From the lower half of W we have $M_{3/2} \approx M_1\sharp m_1(S^2 \times S^2)$, while from the upper half we have $M_{3/2} \approx M_2\sharp m_2(S^2 \times S^2)$, since $M_{3/2}$ can also be obtained by attaching even 2-handles upwards to M_2. Since $\text{rank}(H_2(M_1)) = \text{rank}(H_2(M_2)))$, it follows that $m = m_1 = m_2$. Hence we have a homeomorphism

$$\zeta\colon M_2\sharp m(S^2 \times S^2) \xrightarrow{\approx} M_1\sharp m(S^2 \times S^2) \ .$$

Since M_1 and M_2 have isometric quadratic 2-types, we have

$$\chi\colon \pi \to \pi' \quad \text{and} \quad \psi\colon \pi_2(M_1) \to \pi_2(M_2)$$

a pair of isomorphisms such that $\psi(gx) = \chi(g)\psi(x)$ for all $g \in \pi$, $x \in \pi_2(M_1)$

and preserving the intersection form i.e.,

$$s_{M_2}(\psi(x), \psi(y)) = \chi_*(s_{M_1}(x, y)) \ .$$

As in the free fundamental group case, we can construct a homotopy equivalence $\theta\colon B(M_1) \to B(M_2)$ such that $\theta_\sharp(s_{M_1}) = s_{M_2}$ and we have the following commutative diagram

$$\begin{tikzcd}
\pi_2(M_1) \arrow[r, "\pi_2(c_1)"] \arrow[d, "\psi"'] & \pi_2(B(M_1)) \arrow[d, "\pi_2(\theta)"] \\
\pi_2(M_2) \arrow[r, "\pi_2(c_2)"'] & \pi_2(B(M_2)) \ .
\end{tikzcd}$$

Now let

$$M := M_1 \sharp m(S^2 \times S^2) \quad \text{and} \quad M' := M_2 \sharp m(S^2 \times S^2)$$

with the following quadratic 2-types,

$$[\pi, \pi_2, s_M] := [\pi_1(M_1), \pi_2(M_1) \oplus \Lambda^{2m}, s_{M_1} \oplus H(\Lambda^m)]$$

and

$$[\pi_1(M_2), \pi_2(M_2) \oplus \Lambda^{2m}, s_{M_2} \oplus H(\Lambda^m)] \ ,$$

where $H(\Lambda^m)$ is the hyperbolic form on $\Lambda^m \oplus (\Lambda^m)^*$. Next, observe that as in the free fundamental group case

$$(\pi_1(\zeta) \circ \chi, \pi_2(\zeta) \circ (\psi \oplus \mathrm{id})) = (\mathrm{id}, \pi_2(\zeta) \circ (\psi \oplus \mathrm{id}))$$

gives us an element in $\mathrm{Isom}[\pi, \pi_2, s_M]$. Let $B = B(M)$ denote the 2-type of M. Remember that we have an exact sequence of the form

$$0 \longrightarrow H^2(\pi; \pi_2) \longrightarrow \mathrm{Aut}_\bullet(B) \xrightarrow{(\pi_1, \pi_2)} \mathrm{Isom}[\pi, \pi_2] \longrightarrow 1 \ .$$

Therefore we can find a $\phi \in \mathrm{Aut}_\bullet(B)$ such that

$$\pi_1(\phi) = \mathrm{id} \quad \text{and} \quad \pi_2(\phi) = \pi_2(\zeta) \circ (\psi \oplus \mathrm{id}) \ .$$

Note that the homotopy self-equivalence ϕ preserves the intersection form s_M, so $\phi \in \mathrm{Isom}[\pi, \pi_2, s_M]$. For π a PD_2 group, on the braid we see $\mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, c_*[M]]$, not $\mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$. Therefore to be able to use the braid, we need to construct a self homotopy equivalence of B which preserves $c_*[M]$.

First of all, by a result of Hillman [44, Theorem2], we know that there is a 2-connected degree-1 map $g_M \colon M \to Z$ with $\ker(\pi_2(g_M)) = P$ where $\pi_2 \cong P \oplus \mathbb{Z}$ and $\pi_2(Z) \cong \mathbb{Z}$. We may assume that $\pi_2(g_M)$ corresponds projection to the second factor and $c_Z \circ g_M = g \circ c$ for some 2-connected map $g \colon B \to B(Z)$. The map g is a fibration with fibre $K(P, 2)$, and the inclusion of $\mathbb{Z}$ into $\pi_2(M_2)$ determines a section s for g. After composing ϕ with a self homotopy equivalence of B if necessary we may assume that $g \circ \phi = g$.

Let $L := L_\pi(P, 2)$ be the space with algebraic 2-type $[\pi, P, 0]$ and universal covering space $\widetilde{L} \simeq K(P, 2)$. We may construct L by adjoining 3-cells to M to kill the kernel of the projection from π_2 to P and then adjoining higher dimensional cells to kill the higher homotopy groups. The splitting $\pi_2 \cong P \oplus \mathbb{Z}$ also determines a projection $q \colon B \to L$. Summarizing we have the diagram below with a commutative square

$$\begin{tikzcd}
& M \arrow[r, "g_M"] \arrow[d, "c"'] & Z \arrow[d, "c_Z"] \\
\widetilde{L} \arrow[r] & B \arrow[r, "g"] & B(Z) \arrow[l, bend left, "s"]
\end{tikzcd}$$

To begin with we have the following isomorphisms

$$\begin{aligned}
H_4(B) &\cong \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z} \oplus H_2(\pi;\pi_2) \cong \Gamma(\mathbb{Z} \oplus P) \otimes_\Lambda \mathbb{Z} \oplus H_2(\pi) \\
&\cong (\Gamma(\mathbb{Z}) \oplus \Gamma(P) \oplus \mathbb{Z} \otimes P) \otimes_\Lambda \mathbb{Z} \oplus H_2(\pi) \\
&\cong \Gamma(P) \otimes_\Lambda \mathbb{Z} \oplus \Gamma(\mathbb{Z}) \otimes_\Lambda \mathbb{Z} \oplus H_2(\pi) \oplus (\mathbb{Z} \otimes P) \otimes_\Lambda \mathbb{Z} \\
&\cong H_4(L) \oplus H_4(B(Z)) \oplus P \otimes_\Lambda \mathbb{Z} \,.
\end{aligned}$$

Let us start with projecting the homology classes $\phi_*(c_*[M])$ and $c_*[M]$ to $H_4(L) \cong \Gamma(P) \otimes_\Lambda \mathbb{Z}$. Recall that by Lemma 2.1.3, we have an exact sequence

$$0 \longrightarrow H^2(\pi;\Lambda) \longrightarrow H^2(M;\Lambda) \xrightarrow{\text{ev}} \mathrm{Hom}_\Lambda(\pi_2,\Lambda) \longrightarrow 0 \,.$$

The cohomology intersection pairing is defined by $s_M(u,v) = ev(v)(PD(u))$ for all $u, v \in H^2(M;\Lambda)$ where PD is the Poincaré duality isomorphism. Since $s_M(u,v) = 0$ for all $u \in H^2(M;\Lambda)$ and $v \in H^2(\pi;\Lambda)$, the pairing s_M induces a nonsingular pairing

$$s'_M \colon H^2(M;\Lambda)/H^2(\pi;\Lambda) \times H^2(M;\Lambda)/H^2(\pi;\Lambda) \to \Lambda \,.$$

Note that $\mathrm{Hom}_\Lambda(\pi_2,\Lambda) \cong H^2(M;\Lambda)/H^2(\pi;\Lambda)$. If we further restrict s'_M to $\mathrm{Hom}_\Lambda(P,\Lambda) \cong H^2(L;\Lambda)/H^2(\pi;\Lambda)$, we get a Hermitian pairing s''_M. We want to work with right modules, so we can take $s''_M \in \mathrm{Her}(P^\dagger)$. Therefore, we have the following commutative diagram,

$$\begin{tikzcd}
H_4(B) \arrow[r, "F"] \arrow[d, "q_*"'] & \mathrm{Her}(H^2(B;\Lambda)) \arrow[d, "q_\sharp"] \\
\Gamma(P) \otimes_\Lambda \mathbb{Z} \arrow[r, "\cong", "B_P"'] & \mathrm{Her}(P^\dagger) \,.
\end{tikzcd}$$

Hillman [44], showed that B_P is an isomorphism whenever P is a finitely generated Λ-module. By the commutativity of the above diagram we have $B_P(q_*(c_*[M])) = B_P(q_*(\phi_*(c_*[M]))) = s''_M$, hence $q_*(c_*[M]) = q_*(\phi_*(c_*[M]))$.

Before we continue, let us review a lemma which was proved in [44, Lemma 8]. Let P and Q be right Λ-modules such that P is finitely generated and projective. Let

$$d\colon P \to P^{\dagger\dagger} \quad \text{and} \quad t\colon P^{\dagger} \otimes_{\Lambda} \overline{Q} \to \mathrm{Hom}_{\Lambda}(P, Q)$$

be given by

$$d(p)(\mu) = \overline{\mu(p)} \quad \text{and} \quad t(\mu \otimes q)(p) = \mu(p).q = q\overline{\mu(p)}\ ,$$

for all $p \in P$, $\mu \in P^{\dagger}$ and $q \in Q$. Since P is finitely generated and projective these functions are isomorphisms of right Λ-modules and Abelian groups respectively. Let $B_P\widetilde{(\gamma(p) \otimes 1)}$ be the adjoint of $B_P(\gamma(p) \otimes 1)$, for all $p \in P$.

Lemma 5.2.1 ([44]). *Let P be a finitely generated Λ-module and $\sigma\colon P \to Q$ be a Λ-module homomorphism. Let $\psi_{\sigma}(p, q) = (p, q + \sigma(p))$ for all $(p, q) \in \pi_2 = P \oplus Q$, and let d and t be the isomorphisms defined above. Then ψ_{σ} is an automorphism of π_2 and*

$$\Gamma(\psi_{\sigma})(\gamma(p)) - \gamma(p) \equiv (d \otimes 1)^{-1}[(\widetilde{B_P(\gamma(p))} \otimes 1)(t^{-1}(\sigma))] \ mod\ \Gamma(Q)$$

for all $p \in P$.

Back to our proof. Note that $B(Z)$ is a retract of B. Comparison of the spectral sequences for the classifying maps u_B and $u_{B(Z)}$ shows that $\mathrm{coker}(H_4(s))$ is isomorphic to

$$H_0(\pi; H_4(\widetilde{B}))/H_0(\pi; H_4(\widetilde{B(Z)})) \cong (\Gamma(\pi_2)/\Gamma(\mathbb{Z})) \otimes_{\Lambda} \mathbb{Z}\ .$$

Since B_P is injective and ϕ preserves the intersection form the images of $\phi_*(c_*[M])$ and $c_*[M]$ in $(\Gamma(\pi_2)/\Gamma(\mathbb{Z})) \otimes_{\Lambda} \mathbb{Z}$ differ by an element of the subgroup $(P \otimes \mathbb{Z}) \otimes_{\Lambda} \mathbb{Z}$. Let p represent this difference and let $\eta \in \Gamma(P)$

represent $(b \otimes 1)[q_*(c_*[M])]$, where $b\colon H_4(L;\Lambda) \to \Gamma(P)$ is the secondary boundary homomorphism (see Theorem 2.2.4). Since $B_P(\eta \otimes 1) = s''_M$ is nonsingular $\widetilde{B_P(\eta \otimes 1)}\colon P^\dagger \to P^{\dagger\dagger}$ is surjective. We have

$$\begin{tikzcd} P^\dagger \otimes_\Lambda \mathbb{Z} \arrow[rr, "\widetilde{B_P(\eta\otimes 1)}"] & & P^{\dagger\dagger} \otimes_\Lambda \mathbb{Z} \\ \mathrm{Hom}_\Lambda(P, \mathbb{Z}) \arrow[u, "t^{-1}", "\cong"'] & & P \otimes_\Lambda \mathbb{Z} \cong (P \otimes \mathbb{Z}) \otimes_\Lambda \mathbb{Z} \arrow[u, "\cong", "d\otimes \mathrm{id}"'] \end{tikzcd}$$

so we may choose a homomorphism $\sigma\colon P \to \mathbb{Z}$ such that

$$(\widetilde{B_P(\eta \otimes 1)} \otimes 1)(t^{-1}(\sigma)) = (d \otimes 1)(p)$$

and $\Gamma(\psi_\sigma)(\eta) - \eta \equiv p \bmod \Gamma(\mathbb{Z})$, by the above lemma. Let ϕ_σ be the corresponding self homotopy equivalence of B. Then $g \circ \phi_\sigma = g$ and

$$(\phi_\sigma \circ \phi)_*(c_*[M]) \equiv c_*[M] \bmod \Gamma(\mathbb{Z}) \otimes_\Lambda \mathbb{Z}\ .$$

Note that ϕ_σ preserves the intersection form and $\pi_2(\phi_\sigma) = \psi_\sigma$ is the identity map, when we see it as a map on the free direct summand, i.e. $\Lambda^{2m} \to \Lambda^{2m}$. Since

$$g_*(c_*[M]) = g_*(\phi_*(c_*[M])) = g_*((\phi_\sigma \circ \phi)_*(c_*[M])) \text{ in } H_4(B(Z))\ ,$$

we have

$$(\phi_\sigma \circ \phi)_*(c_*[M]) = c_*[M] \text{ in } H_4(B(Z))\ .$$

The self homotopy equivalence $\phi_\sigma \circ \phi$ of B satisfies the properties we want. By abuse of notation we will denote this homotopy equivalence also by ϕ. Note that $\phi \in \mathrm{Isom}[\pi, \pi_2, c_*[M]]$. Also recall that we have the following short exact sequence by Lemma 4.3.3

$$0 \longrightarrow H^1(M;\mathbb{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]] \xrightarrow{\hat{\jmath}} \mathrm{Isom}[\pi, \pi_2, c_*[M]] \longrightarrow 1\ .$$

Choose $\widehat{f} \in \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, c_*[M]]$ such that $\widehat{j}(\widehat{f}) = \phi$. Comparison of C. T. C. Wall's surgery program with M. Kreck's modified surgery program gives a commutative diagram of exact sequences (see [35], Lemma 4. 1)

$$\begin{tikzcd}
\tilde{L}_6(\mathbb{Z}[\pi_1]) \arrow[r, equal] \arrow[d] & \tilde{L}_6(\mathbb{Z}[\pi_1]) \arrow[d] & \\
\mathcal{S}(M \times I, \partial) \arrow[r] \arrow[d] & \mathcal{H}(M) \arrow[r] \arrow[d] & \mathrm{Aut}_\bullet(M) \\
\mathcal{T}(M \times I, \partial) \arrow[r] \arrow[d] & \widetilde{\mathcal{H}}(M, w_2) \arrow[r, two heads] \arrow[d] & \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, c_*[M]] \\
L_5(\mathbb{Z}[\pi_1]) \arrow[r, equal] & L_5(\mathbb{Z}[\pi_1]) &
\end{tikzcd}$$

Now the rest of the proof follows exactly the same as in the free fundamental group case. $\square$

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