LARGE SCALE DIMENSION THEORY OF METRIC SPACES

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By

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To Mom

Abstract

This thesis studies the large scale dimension theory of metric spaces. Background on dimension theory is provided, including topological and asymptotic dimension, and notions of nonpositive curvature in metric spaces are reviewed. The hyperbolic dimension of Buyalo and Schroeder is surveyed. Miscellaneous new results on hyperbolic dimension are proved, including a union theorem, an estimate for central group extensions, and the vanishing of hyperbolic dimension for countable abelian groups. A new quasi-isometry invariant called weak hyperbolic dimension (abbreviated wdim) is introduced and developed. Weak hyperbolic dimension is computed for a variety of metric spaces, including the fundamental computation $\operatorname{wdim} \mathbb{H}^n = n - 1$. An estimate is proved for (not necessarily central) group extensions. Weak dimension is used to obtain the quasi-isometric nonembedding result $\mathbb{H}^4 \not\to \mathrm{Sol} \times \mathrm{Sol}$ and possible directions for further nonembedding applications are explored.

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Contents

Introduction

This thesis introduces a new quasi-isometry invariant of metric spaces called the *weak hyperbolic dimension*. The invariant arose in an attempt to better understand the *hyperbolic dimension* introduced by Buyalo and Schroeder in [BS07a]. The weak hyperbolic dimension shares certain properties with the hyperbolic dimension: in particular it distinguishes between hyperbolic n-space $\mathbb{H}^n$ for which the weak dimension is $n-1$, and n-dimensional Euclidean space $\mathbb{R}^n$, for which the weak dimension is 0. This is in contrast with the asymptotic dimension which assigns the dimension n to both $\mathbb{H}^n$ and $\mathbb{R}^n$ and thus does not distinguish between hyperbolic and Euclidean space.

The hyperbolic dimension is difficult to compute. For example, the dimension of the product $\mathbb{H}^2 \times \mathbb{H}^2$ is unknown[1]. This research project began, therefore, with an effort to build up tools for the computation of hyperbolic dimension. The results are in Chapter 5.

A frequent roadblock comes up when one tries to prove things about the hyperbolic dimension: the *finite union condition* gets in the way. Weak hyperbolic dimension, studied in Chapter 6, is hyperbolic dimension with the finite union condition removed. This makes it easier to produce tools to compute the dimension, and indeed we compute the weak dimension for a broad group of spaces including products with both tree and hyperbolic factors and for finitely generated groups such as a discrete model of

[1] It is either 3 or 4.

solvgeometry, the solvable Baumslag-Solitar groups, a wreath product, and a free product of an infinite cyclic group with a free abelian group.

Originally conceived as a tool to study hyperbolic dimension, the weak dimension turns out, due to its advantage in computability, to be interesting in its own right. Moreover there is evidence that weak dimension may have applications to non-embedding problems. This thesis closes with a discussion of such applications as a direction for future research.

The large scale point of view

In this work we adopt the "large scale" point of view for the geometry of metric spaces. A major motivation for this stance comes from the field of geometric group theory, which studies infinite but finitely generated groups.

Definition 0.0.1. The group Γ is *finitely generated* if there exists a finite set $S \subset \Gamma$ such that each $\gamma \in \Gamma$ can be written as a finite product $\gamma = s_1 \dots s_n$ (where n depends on γ) of elements $s_i \in S \cup S^{-1}$.

The starting point of geometric group theory is that the *word metric* allows one to view a finitely generated group as a geometric object, namely, a metric space.

Definition 0.0.2. Let Γ be generated by the finite set S. The *word metric* on Γ is the function $d\colon \Gamma \times \Gamma \to \mathbb{N}$ defined by the formula

$$d(g,h) = \min \{n \in \mathbb{N} \mid g^{-1}h = s_1 \dots s_n \text{ for } s_i \in S \cup S^{-1}\}.$$

The word metric is indeed a metric turning the group Γ into a metric space (see Definition 1.0.5). The apparent problem is that the word metric produced depends on the choice of finite generating set. If S and T are distinct finite generating sets for Γ

then in general the word metrics d_S and d_T for S and T respectively will not be equal. There is, however, the following relation between the metrics.

Proposition 0.0.3. *Let S and T be finite generating sets for the group Γ and let $\lambda = \max\{\lambda_S, \lambda_T\}$ where*

$$\lambda_S = \max\{d_S(t,e) \mid t \in T\},$$
$$\lambda_T = \max\{d_T(s,e) \mid s \in S\}$$

and where $e \in \Gamma$ is the identity. Then for all $g, h \in \Gamma$

$$\frac{1}{\lambda} d_S(g,h) \leq d_T(g,h) \leq \lambda \cdot d_S(g,h).$$

This relation between the word metrics motivates the following definition, which is fundamental to the large scale point of view.

Definition 0.0.4. A map $f\colon X \to Y$ between metric spaces X and Y is *quasi-isometric* if there exist constants $a \geq 1$, $b \geq 0$ such that

$$\frac{d_X(x_1,x_2)}{a} - b \leq d_Y(f(x_1), f(x_2)) \leq a \cdot d_X(x_1,x_2) + b$$

for all $x_1, x_2 \in X$. If in addition the image $f(X) \subset Y$ is an r-net in Y for some $r > 0$, that is, for all $y \in Y$ there is $x \in X$ such that $d(f(x), y) < r$, then f is a *quasi-isometry.* The spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f\colon X \to Y$.

Quasi-isometry of metric spaces is an equivalence relation: quasi-isometric spaces have the same structure from the large scale point of view. Large scale geometry can thus be described as the study of quasi-isometry classes of metric spaces. Moreover the large scale structure of a finitely generated group is independent of the choice of generating set and is therefore an invariant of the group itself.

The best argument for the usefulness of this point of view is probably Gromov's landmark theorem on groups of polynomial growth, which says that a finitely generated group is virtually nilpotent if and only if it has polynomial growth. In this theorem a purely algebraic, group theoretic property (that of nilpotence) is equated to a purely geometric property (that of growth).

Outline of the thesis

Here is a brief outline of the contents of this thesis. Chapters 1 and 2 serve as a rapid introduction to dimension theory. Chapter 1 lays the technical foundation of the simplicial geometry of covers and nerves, cast into the P-dimension formalism of Buyalo and Schroeder. Readers wishing to skip the technical details and be told the point are directed to the summary in 1.3.3. Chapter 2 introduces the two most fundamental dimensions for metric spaces, the small-scale topological dimension and its large-scale counterpart the asymptotic dimension. Chapter 3 tours the various negative and non-positive curvature conditions most relevant to metric geometry. Then Chapter 4 introduces the hyperbolic dimension and, since the literature on this dimension is not extensive, also functions as a survey. New results on hyperbolic dimension and weak hyperbolic dimension are in Chapters 5 and 6 respectively, and have been summarized at the beginning of this introduction.

Definitions central to the development of large scale geometry are set apart and numbered. Occasionally, a set of related concepts are introduced in quick succession, usually as part of a larger construction; such definitions are given within the normal flow of the text, indicated by italics. Definitions that are either brief or familiar are given within the normal flow of the text and indicated with italics.

Chapter 1

Dimension theory

The basic objects of study in this thesis are *metric spaces*.

Definition 1.0.5. A *metric* or *distance function* on a set X is a function $d\colon X \times X \to \mathbb{R}$ satisfying

- *positivity:* $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
- *symmetry:* $d(x, y) = d(y, x)$ for all $x, y \in X$, and
- *the triangle inequality:* $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A *metric space* is a pair (X, d) where d is a metric on the set X. A map $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is *isometric* if $d_Y(f(x), f(x')) = d_X(x, x')$ for all $x, x' \in X$. If f is also bijective it is called an *isometry* and X and Y are said to be *isometric*.

Geodesic metric spaces form an important class of metric spaces in which any pair of points is joined by a "straight line".

Definition 1.0.6. An isometric map $c\colon I \to X$ is a *geodesic segment* if $I = [a, b]$ for some closed interval $[a, b] \subset \mathbb{R}$, a *geodesic ray* if $I = [0, \infty)$, and a *geodesic line* if

$I = (-\infty, \infty)$. The metric space X is *geodesic* if for all $x, y \in X$ there exists a geodesic segment $c\colon [a, b] \to X$ joining x and y, i.e., such that $c(a) = x$ and $c(b) = y$.

Dimension theory assigns to each metric space X an integer invariant, the *dimension* of X. This will be made precise as we introduce the specific metric space dimensions that play a role in this work: the topological and the asymptotic dimensions (Chapter 2), the hyperbolic dimension (Chapters 4 and 5), and the weak hyperbolic dimension (Chapter 6). To efficiently study these different dimensions and to avoid repeating arguments separately for each one, we adopt the *P-dimension formalism* of Buyalo and Schroeder featured in [BS07b]. The object of this chapter is to explain the P-dimension formalism and in the process offer an introduction to the tools and constructions underlying the dimension theory of metric spaces.

1.1 Simplicial constructions

Dimension theory is, in the details, about whether a metric space admits covers with certain properties. This section describes how covers are turned into geometric objects.

1.1.1 Abstract simplicial complexes and uniform simplicial polyhedra

Definition 1.1.1. An *abstract simplicial complex* K is an ordered pair (V, S) where V is the set of *vertices* and S is a set of finite non-empty subsets of V, called *simplices*, satisfying

- $\{v\} \in S$ for every $v \in V$,
- if $s \in S$ and $\emptyset \neq s_0 \subset s$, then $s_0 \in S$.

The term *abstract simplicial complex* will often be abbreviated to *simplicial complex* and *complex*.

Given a complex K we write $\mathrm{Ver}(K)$ to denote the set of vertices of K and $\mathrm{Sim}(K)$ to denote the set of simplices of K. If $s \in \mathrm{Sim}(K)$ is a simplex of K with $|s| = n+1$ then s is called an *n-simplex* of K. The *combinatorial dimension* of a complex K, denoted $\dim K$, is the maximum $n \in \mathbb{N}$ such that K has an n-simplex[1]. For K a complex and $n \geq 0$ we denote by $\mathrm{Sim}_n(K) \subset \mathrm{Sim}(K)$ the simplices of K having dimension n. Thus $\mathrm{Sim}(K) = \cup_{n=0}^{\infty} \mathrm{Sim}_n(K)$ decomposes as a disjoint union.

We now describe a construction that associates a metric space, denoted $|K|$, to an abstract simplicial complex K. The resulting object is an example of a *uniform simplicial polyhedron* and is called the *geometric realization* of the abstract simplicial complex K.

Let J be an index set (not necessarily finite). By $\mathbb{R}^J$ we denote the space of finitely-supported functions $J \to \mathbb{R}$. For $x \in \mathbb{R}^J$ and $j \in J$ write $x_j = x(j)$ to denote the *coordinates* of x. The distance $d(x,y)$ between points $x, y \in \mathbb{R}^J$ is defined by

$$d(x,y) = \sqrt{\sum_{j \in J} (x_j - y_j)^2}$$

which is a finite sum because $x, y \colon J \to \mathbb{R}$ are finitely-supported. In fact $\mathbb{R}^J$ is an inner product space with inner product

$$x \cdot y = \sum_{j \in J} x_j y_j.$$

There is then the natural identification of J with the orthonormal basis $\{e_j\}_{j \in J}$ where $e_j \in \mathbb{R}^J$ is defined by $e_j(i) = \delta_{ji}$ for each $i \in J$.

When J is a finite set, i.e., $J = \{1, ..., m\}$ for $m \geq 1$, then $\mathbb{R}^J$ is m-dimensional Euclidean space. Just as each finite-dimensional Euclidean space contains a *standard simplex*, so too does $\mathbb{R}^J$.

[1] If K is the empty complex we set $\dim K = -1$.

Definition 1.1.2. For J an index set the *standard simplex* in $\mathbb{R}^J$, denoted Δ^J, is defined to be all those points $x \in \mathbb{R}^J$ such that $x_j \geq 0$ for all $j \in J$ and $\sum_{j \in J} x_j = 1$.

We view the standard simplex as a subspace of $\mathbb{R}^J$. Observe that since Δ^J is a convex subset of $\mathbb{R}^J$, Δ^J is a geodesic metric space when endowed with the restriction of the Euclidean metric from $\mathbb{R}^J$.

Definition 1.1.3. Let K be an abstract simplicial complex and let $J = \mathrm{Ver}(K)$ be the vertex set of K. For a simplex $\sigma = \{j_0, \ldots, j_n\} \in \mathrm{Sim}(K)$ we define the *geometric realization of* σ, denoted $|\sigma|$, to be the subspace of $\mathbb{R}^J$ consisting of all convex combinations $\mathrm{Conv}(\{e_{j_0}, \ldots, e_{j_n}\})$ of the basis vectors $e_{j_0}, \ldots, e_{j_n}$. The *geometric realization of* K, denoted $|K|$, is the union in $\mathbb{R}^J$ of the geometric realizations of each simplex $\sigma \in \mathrm{Sim}(K)$

$$|K| = \cup\{|\sigma| \colon \sigma \in \mathrm{Sim}(K)\} \subset \mathbb{R}^J$$

viewed as a subspace with metric restricted from $\mathbb{R}^J$. We call this metric the *uniform*[2] *metric on* $|K|$, denoted d_u.

Observe that when J is the vertex set of an abstract complex K, the geometric realization $|K|$ is a subspace of the standard simplex Δ^J.

The geometric realization $|K|$ of an abstract complex K is an example of a *uniform simplicial polyhedron*[3], that is, a metric space that is isometric to a subcomplex of the standard simplex Δ^J for some J. We have not said precisely what is meant by *subcomplex* because for our purposes it is enough to know that every geometric realization is a uniform polyhedron and that every uniform polyhedron can be obtained as the geometric realization of some abstract complex.

In general $|K|$ is not convex and not a geodesic metric space. Denote by d_l the length metric for the space $(|K|, d_u)$. The space $(|K|, d_u)$ is bounded in diameter, but

[2]The metric is also called the *affine* metric, denoted d_a.

[3]Abbreviated *uniform polyhedron*.

in general $(|K|, d_l)$ is not bounded. However when K is *locally finite*, which means that each vertex of K belongs to only finitely many simplices, there is the following fact.

Proposition 1.1.4. *Let K be a locally finite path-connected simplicial complex. Then the uniform metric d_u and the length metric d_l define the same topology on $|K|$. Moreover $(|K|, d_l)$ is a complete, proper, geodesic metric space.*

See [Roe03] for a discussion of the proof.

Observe that $|K| \subset \Delta^J$. Since $J = \mathrm{Ver}(K)$ there is the identification of the vertices of $|K|$ with the orthonormal basis $\{e_j\}_{j\in J}$. Thus we have the conventions $\mathrm{Ver}(|K|) = \{e_j\}_{j\in J}$ whereas $\mathrm{Ver}(K) = J$.

Definition 1.1.5. Let $e_j \in \mathrm{Ver}(|K|)$ be a vertex in the uniform polyhedron $|K|$. The *open star of the vertex e_j*, denoted st_{e_j}, is the subset of $|K|$ given by the intersection

$$\{(1-t)e_j + tx \mid 0 \leq t < 1, x \in |K|\} \cap |K|.$$

1.1.2 Barycentric subdivision

Let J be a non-empty vertex set and consider the standard simplex Δ^J. Let $\mathcal{J} = \mathcal{P}_0(J)$ denote the *finite power set* of J, that is, the set of all finite non-empty subsets of J. Consider the standard simplex $\Delta^{\mathcal{J}}$. We will now describe the construction of a uniform polyhedron $\mathrm{ba}\,\Delta^J \subset \Delta^{\mathcal{J}}$ called the *barycentric subdivision* of Δ^J.

Construct an abstract simplex $\mathrm{ba}\,K^J$ by setting $\mathrm{Ver}(\mathrm{ba}\,K^J) = \mathcal{J}$ and defining $\mathrm{Sim}(\mathrm{ba}\,K^J) \subset \mathcal{P}_0(\mathcal{J})$ according to the rule that for $\alpha \in \mathcal{P}_0(\mathcal{J})$ we have $\alpha \in \mathrm{Sim}(\mathrm{ba}\,K^J)$ if and only if for every $s, s' \in \alpha$ either $s \subset s'$ or $s' \subset s$.

Definition 1.1.6. The *barycentric subdivision of the standard simplex* Δ^J, denoted $\mathrm{ba}\,\Delta^J$, is the geometric realization $|\mathrm{ba}\,K^J|$ of the abstract simplex $\mathrm{ba}\,K^J$ constructed above.

Observe that $\mathrm{ba}\,\Delta^J$ is a subset of the standard simplex $\Delta^{\mathcal{J}}$. There is a canonical bijection $\phi\colon\ \Delta^J \to \mathrm{ba}\,\Delta^J$ that identifies Δ^J and $\mathrm{ba}\,\Delta^J$. The bijection ϕ is most conveniently defined by specifying ϕ^{-1}. If $v \in \mathrm{Ver}(\mathrm{ba}\,\Delta^J)$ then $v = e_{\{j_0,\ldots,j_n\}}$ is a basis vector in $\mathbb{R}^{\mathcal{J}}$, where $\{j_0,\ldots,j_n\} \subset J$. Then $\phi^{-1}(v)$ is defined by

$$\phi^{-1}(v) = \frac{1}{n+1}\left(e_{j_0} + \ldots + e_{j_n}\right).$$

Observe that $\phi^{-1}(v) \in \Delta^J$. Then ϕ^{-1} is extended to all of $\mathrm{ba}\,\Delta^J$ by linearity.

Definition 1.1.7. The map $\phi\colon\ \Delta^J \to \mathrm{ba}\,\Delta^J$ is called the *barycentric subdivision map*. It is also called the *canonical bijection* between Δ^J and $\mathrm{ba}\,\Delta^J$.

We use the barycentric subdivision map to construct the barycentric subdivision of an arbitrary uniform simplicial polyhedron.

Definition 1.1.8. Let $|K| \subset \Delta^J$ be a uniform simplicial polyhedron with vertex set J. The *barycentric subdivision of* $|K|$, denoted $\mathrm{ba}|K|$, is the image $\phi(|K|)$ of $|K|$ under the barycentric subdivision map ϕ, viewed as a subcomplex of $\Delta^{\mathcal{J}}$.

Observe that the image $\phi(|K|)$ is contained in $\mathrm{ba}\,\Delta^J$. Thus for $|K| \subset \Delta^J$ we have $\mathrm{ba}|K| \subset \mathrm{ba}\,\Delta^J$.

1.1.3 Simplicial products

Let J_1 and J_2 be non-empty vertex sets. Consider the metric product $\Delta^{J_1} \times \Delta^{J_2}$ of the standard simplices Δ^{J_1} and Δ^{J_2}. At this stage the product $\Delta^{J_1} \times \Delta^{J_2}$ does not have the structure of a uniform simplicial polyhedron. Here we describe a procedure to give the product the structure of a uniform simplicial polyhedron in a canonical way. The resulting polyhedron is called the *simplicial product of the simplices* Δ^{J_1} *and* Δ^{J_2} which we denote here by $\Delta^{J_1} \times_\Delta \Delta^{J_2}$.

Let $J = J_1 \cup J_2$ and let $\mathcal{J} = \mathcal{P}_0(J)$, $\mathcal{J}_1 = \mathcal{P}_0(J_1)$, and $\mathcal{J}_2 = \mathcal{P}_0(J_2)$. Let K be the abstract simplicial complex with vertex set $\mathcal{J}_1 \times \mathcal{J}_2$ and simplices given by the following rule: identify $\mathcal{J}_1 \times \mathcal{J}_2$ with a subset of $\mathcal{J}$ using the map $(s_1, s_2) \mapsto s_1 \cup s_2$. Then a finite collection α of members of $\mathcal{J}_1 \times \mathcal{J}_2$ is a simplex of K if and only if $s, s' \in \alpha$ implies either $s \subset s'$ or $s' \subset s$. Observe that the geometric realization $|K|$ is a uniform simplicial polyhedron that is a subcomplex of $\Delta^{\mathcal{J}_1 \times \mathcal{J}_2}$.

Definition 1.1.9. The *simplicial product of the standard simplices* Δ^{J_1} *and* Δ^{J_2} is the uniform simplicial polyhedron $|K|$ constructed above and is denoted by $\Delta^{J_1} \times_\Delta \Delta^{J_2}$.

There is a canonical bijection, denoted φ_Δ, that identifies the metric spaces $\Delta^{J_1} \times \Delta^{J_2}$ and $\Delta^{J_1} \times_\Delta \Delta^{J_2}$. This map is most conveniently defined by specifying φ_Δ^{-1}. By construction a vertex $v \in \mathrm{Ver}(\Delta^{J_1} \times_\Delta \Delta^{J_2})$ is a basis vector $e_{(s_1,s_2)}$ with $s_1 \in \mathcal{J}_1$ and $s_2 \in \mathcal{J}_2$. Writing $s_1 = \{j_0, \ldots, j_n\}$ and $s_2 = \{k_0, \ldots, k_m\}$ define

$$\varphi_\Delta^{-1}(v) = \left(\frac{1}{n+1}(e_{j_0} + \ldots + e_{j_n}), \frac{1}{m+1}(e_{k_0} + \ldots + e_{k_m}) \right).$$

Observe that $\varphi_\Delta^{-1}(v) \in \Delta^{J_1} \times \Delta^{J_2}$. Then φ_Δ^{-1} is extended to all of $\Delta^{J_1} \times_\Delta \Delta^{J_2}$ by linearity.

Definition 1.1.10. The map

$$\varphi_\Delta \colon \Delta^{J_1} \times \Delta^{J_2} \to \Delta^{J_1} \times_\Delta \Delta^{J_2}$$

is called the *simplicial product map.*

We use the simplicial product map to construct the simplicial product of an arbitrary pair of simplicial polyhedra.

Definition 1.1.11. Let $|K_1| \subset \Delta^{J_1}$ and $|K_2| \subset \Delta^{J_2}$ be uniform simplicial polyhedra with vertex sets J_1 and J_2 respectively. Viewing $|K_1| \times |K_2|$ as a subset of $\Delta^{J_1} \times \Delta^{J_2}$, the image $\varphi_\Delta(|K_1| \times |K_2|)$ is a subcomplex of $\Delta^{J_1} \times_\Delta \Delta^{J_2}$ and is called the *simplicial product of the polyhedra* $|K_1|$ *and* $|K_2|$, denoted $|K_1| \times_\Delta |K_2|$.

1.1.4 Barycentric subdivision of a simplicial product

The identification of $\mathcal{J}_1 \times \mathcal{J}_2$ with a subset of $\mathcal{J}$ is furnished by the map $(s_1, s_2) \mapsto s_1 \cup s_2$. This identification induces an isometric injection

$$i\colon \Delta^{\mathcal{J}_1 \times \mathcal{J}_2} \to \Delta^{\mathcal{J}}\,.$$

Recall that the simplicial product $\Delta^{J_1} \times_\Delta \Delta^{J_2}$ of the standard simplices Δ^{J_1} and Δ^{J_2} is a subcomplex of the standard simplex $\Delta^{\mathcal{J}_1 \times \mathcal{J}_2}$. Thus we may use the isometric injection i to view the simplicial product $\Delta^{J_1} \times_\Delta \Delta^{J_2}$ as the subcomplex $i(\Delta^{J_1} \times_\Delta \Delta^{J_2})$ of the standard simplex $\Delta^{\mathcal{J}}$.

Then the barycentric subdivision map $\phi\colon \Delta^{\mathcal{J}} \to \operatorname{ba}\Delta^{\mathcal{J}}$ allows us to take the image

$$(\phi \circ i)(\Delta^{J_1} \times_\Delta \Delta^{J_2})$$

to obtain a subcomplex of $\operatorname{ba}\Delta^{\mathcal{J}}$.

Definition 1.1.12. The *barycentric subdivision of the simplicial product of the standard simplices* Δ^{J_1} *and* Δ^{J_2}, denoted $\Delta^{J_1} \times_{\operatorname{ba}\Delta} \Delta^{J_2}$, is the uniform simplicial polyhedron $(\phi \circ i)(\Delta^{J_1} \times_\Delta \Delta^{J_2})$. The map $\phi \circ i$ is called the *canonical bijection between* $\Delta^{J_1} \times_\Delta \Delta^{J_2}$ *and* $\Delta^{J_1} \times_{\operatorname{ba}\Delta} \Delta^{J_2}$.

Recall the simplicial product map

$$\varphi_\Delta\colon \Delta^{J_1} \times \Delta^{J_2} \to \Delta^{J_1} \times_\Delta \Delta^{J_2}\,.$$

The canonical bijection $\phi \circ i$ and the simplicial product map compose to produce the canonical bijection

$$\varphi_{\operatorname{ba}\Delta} := \phi \circ i \circ \varphi_\Delta\colon \Delta^{J_1} \times \Delta^{J_2} \to \Delta^{J_1} \times_{\operatorname{ba}\Delta} \Delta^{J_2}\,.$$

1.2 The nerve of a cover and barycentric maps

1.2.1 Covers of a metric space

The concept of a cover, or covering, of a metric space is fundamental in dimension theory.

Definition 1.2.1. The collection $\mathcal{A} = \{A_i\}_{i\in I}$ of subsets $A_i \subset X$ of the metric space X is called a *cover* of X if $\cup_{i\in I} A_i = X$. If each $A \in \mathcal{A}$ is an open set, $\mathcal{A}$ is called an *open* cover. The *multiplicity of* $\mathcal{A}$, denoted $\mu(\mathcal{A})$, is the maximal integer $i \geq 1$ such that there exist elements $A_1, \ldots, A_i \in \mathcal{A}$ with non-empty intersection $A_1 \cap \ldots \cap A_i \neq \emptyset$.

In large scale dimension theory the Lebesgue number of a cover is also of fundamental importance.

Definition 1.2.2. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of the metric space X. The *Lebesgue number of* $\mathcal{U}$ *at the point* $x \in X$, denoted $L(\mathcal{U}, x)$, is defined to be

$$L(\mathcal{U}, x) := \sup\{\text{dist}(x, X - U) \mid U \in \mathcal{U}\}.$$

Since $\mathcal{U}$ is an open cover of X, for each $x \in X$ there is some $U \in \mathcal{U}$ and some $r > 0$ such that $B_r(x) \subset U$, so that $\text{dist}(x, X - U) \geq r$. In particular, $L(\mathcal{U}, x) > 0$. Moreover if $d < L(\mathcal{U}, x)$ then there is some element $U \in \mathcal{U}$ such that $B_d(x) \subset U$.

Definition 1.2.3. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of the metric space X. The *Lebesgue number of* $\mathcal{U}$, denoted $L(\mathcal{U})$, is defined to be

$$L(\mathcal{U}) := \inf_{x\in X} L(\mathcal{U}, x).$$

An open cover $\mathcal{U}$ may indeed have $L(\mathcal{U}) = 0$. However if $L(\mathcal{U}) > d > 0$ then for each $x \in X$ there is an element $U \in \mathcal{U}$ such that $B_d(x) \subset U$.

Definition 1.2.4. For the cover $\mathcal{U} = \{U_j\}_{j \in J}$ of the metric space X let $N = N(\mathcal{U})$ be the abstract simplicial complex with one vertex for every element of $\mathcal{U}$ (so that $\mathrm{Ver}(N)$ is identified with the index set J) and with $\emptyset \neq J_0 \subset J$ a simplex of N if and only if $\cap_{j \in J_0} U_j \neq \emptyset$. The *nerve of the cover* $\mathcal{U}$, denoted $\mathcal{N}(\mathcal{U})$, is the uniform simplicial polyhedron $|N| \subset \Delta^J$, that is, the geometric realization of the abstract complex N.

In some contexts it is convenient to also refer to the abstract complex $N(\mathcal{U})$ as the nerve of the cover $\mathcal{U}$.

1.2.2 Barycentric maps

Definition 1.2.5. A *cut function* for the cover $\mathcal{U}$ of X is a strictly positive real-valued function $d \colon X \to \mathbb{R}$ such that $d(x) \leq L(\mathcal{U}, x)$ for every $x \in X$.

Given a locally finite open cover $\mathcal{U}$ and a cut function $d \colon X \to \mathbb{R}$ for $\mathcal{U}$ we construct a *barycentric map* as follows. For each $j \in J$ define $q_j \colon X \to \mathbb{R}$ by

$$q_j(x) = \min\{d(x), \mathrm{dist}(x, X - U_j)\}.$$

For each $x \in X$

$$\sum_{j \in J} q_j(x) < \infty$$

since $\mathcal{U}$ is a locally finite cover. Moreover, by construction

$$0 < d(x) \leq \sum_{j \in J} q_j(x).$$

Now for each $j \in J$ define $p_j \colon X \to \mathbb{R}$ by

$$p_j(x) = \frac{q_j(x)}{\sum_{i \in J} q_i(x)}.$$

Finally define $p \colon X \to \mathbb{R}^J$ by letting $p(x) \in \mathbb{R}^J$ be the point in $\mathbb{R}^J$ with coordinates $p_j(x)$, $j \in J$. Observe that the image $p(x)$ lands in Δ^J. In fact, the following is true.

Proposition 1.2.6. *Let $\mathcal{U}$ be a locally finite open cover of the metric space X and let $p\colon X \to \Delta^J$ be constructed as above. Then the image $p(X)$ is contained in $\mathcal{N}(\mathcal{U})$.*

Definition 1.2.7. The *barycentric map* associated with the locally finite open cover $\mathcal{U}$ of X and the cut function $d\colon X \to \mathbb{R}$ for $\mathcal{U}$ is the function $p\colon X \to \mathcal{N}(\mathcal{U})$ constructed above.

Remark 1.2.8. When $\mathcal{U}$ is an open locally finite cover of the metric space X and $p\colon X \to \mathcal{N}$ is any barycentric map for $\mathcal{U}$ we have $U_j = p^{-1}(\mathrm{st}_{e_j})$ for each $j \in J$. Thus the cover $\mathcal{U}$ can be precisely recovered from its nerve and any barycentric map; that is,

$$\mathcal{U} = \{p^{-1}(\mathrm{st}_{e_j}) \mid j \in J\}.$$

1.2.3 Covers obtained from simplicial constructions

Remark 1.2.8 shows that a barycentric map $p\colon X \to \mathcal{N}$ can be used to reconstruct the original cover from which the nerve was built. This basic insight points the way to constructing new covers on spaces, using barycentric maps to nerves together with the simplicial constructions we have seen.

Definition 1.2.9. Let $|K|$ be a uniform simplicial polyhedron. The *cover of $|K|$ by its open stars*, denoted $\mathcal{ST}(|K|)$, is the cover

$$\{\mathrm{st}_v \mid v \in \mathrm{Ver}(|K|)\}$$

of $|K|$.

Definition 1.2.10. Let $|K|$ be a uniform simplicial polyhedron and let $\phi\colon |K| \to \mathrm{ba}|K|$ be the barycentric subdivision map. We denote by $\mathcal{ST}_{\mathrm{ba}}(|K|)$ the cover

$$\phi^{-1}(\mathcal{ST}(\mathrm{ba}|K|)) = \{\phi^{-1}(U) \mid U \in \mathcal{ST}(\mathrm{ba}|K|)\}$$

of $|K|$.

Definition 1.2.11. Let $\mathcal{U}$ be a locally finite open cover of the metric space X and let $p\colon X \to \mathcal{N}$ be a barycentric map. The *barycentric subdivision of the cover* $\mathcal{U}$, denoted $\mathrm{ba}\mathcal{U}$, is the cover

$$\mathrm{ba}\mathcal{U} = \{p^{-1}(W) \mid W \in \mathcal{ST}_{\mathrm{ba}}(\mathcal{N})\}$$

of X.

Let $\mathcal{U}$ be a cover of the space X and $\mathcal{V}$ be a cover of the space Y. A natural construction is the *product cover* of $X \times Y$, denoted $\mathcal{U} \times \mathcal{V}$ and defined by

$$\mathcal{U} \times \mathcal{V} := \{U \times V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

The drawback of this construction is that if $\mu(\mathcal{U}) = m + 1$ and $\mu(\mathcal{V}) = n + 1$ then $\mu(\mathcal{U} \times \mathcal{V}) = (m + 1)(n + 1)$. The product cover construction therefore *multiplies* multiplicities, giving estimates for dimension that are too high. The following product construction is an essential improvement for the purpose of dimension theory.

Definition 1.2.12. Let $|K_i|$, $i = 1, 2$, be uniform simplicial polyhedra and let $\varphi_{\mathrm{ba}\,\Delta}\colon |K_1| \times |K_2| \to |K_1| \times_{\mathrm{ba}\,\Delta} |K_2|$ be the canonical bijection. We denote by $\mathcal{ST}_{\mathrm{ba}}(|K_1| \times |K_2|)$ the cover

$$\{\varphi^{-1}_{\mathrm{ba}\,\Delta}(\mathrm{st}_v) \mid v \in \mathrm{Ver}(|K_1| \times_{\mathrm{ba}\,\Delta} |K_2|)\}$$

of $|K_1| \times |K_2|$. *(Warning: the notation* $\mathcal{ST}_{\mathrm{ba}}$ *used here is identical to that of Definition 1.2.10; the correct operation is indicated by whether the argument is a single polyhedron, as in Definition 1.2.10, or the product of two polyhedra, as here.)*

Definition 1.2.13. Let $\mathcal{U}$ be a locally finite open cover of the metric space X with barycentric map $p\colon X \to \mathcal{N}(\mathcal{U})$, and let $\mathcal{V}$ be a locally finite open cover of the metric space Y with barycentric map $q\colon Y \to \mathcal{N}(\mathcal{V})$. The *subdivided simplicial product cover*, denoted $\mathcal{U} \times_{\mathrm{ba}\,\Delta} \mathcal{V}$, is the cover

$$\{(p \times q)^{-1}(W) \mid W \in \mathcal{ST}_{\mathrm{ba}}(\mathcal{N}(\mathcal{U}) \times \mathcal{N}(\mathcal{V}))\}$$

of $X \times Y$.

Now suppose that $\mathcal{U}$ covers X with multiplicity $\mu(\mathcal{U}) = m+1$ and $\mathcal{V}$ covers Y with multiplicity $\mu(\mathcal{V}) = n+1$. Then $\mathcal{N}(\mathcal{U})$ has combinatorial dimension m and $\mathcal{N}(\mathcal{V})$ has combinatorial dimension n, so that the polyhedron $\mathcal{N}(\mathcal{U}) \times_{\mathrm{ba}\,\Delta} \mathcal{N}(\mathcal{V})$ has dimension $m+n$. The cover $\mathcal{U} \times_{\mathrm{ba}\,\Delta} \mathcal{V}$ thus has multiplicity $\mu(\mathcal{U} \times_{\mathrm{ba}\,\Delta} \mathcal{V}) = m+n+1$. Heuristically, this corresponds to the situation $\dim X = m$, $\dim Y = n$, $\dim X \times Y = m + n$.

1.2.4 Key lemmas

We record a number of technical lemmas which quantify the basic properties of the constructions defined above.

Lemma 1.2.14. *Let $\mathcal{U}$ be an open and locally finite cover of the metric space X with finite multiplicity $\mu(\mathcal{U}) = m < \infty$ and strictly positive Lebesgue number $L(\mathcal{U}) \geq d > 0$. Then there exists a Lipschitz barycentric map $p \colon X \to \mathcal{N}$ such that*

$$\mathrm{Lip}(p) \leq \frac{(m+1)^2}{d}.$$

Lemma 1.2.15. *Let $|K| \subset \Delta^J$ be a uniform simplicial polyhedron of finite combinatorial dimension $\dim K < \infty$. Then there exists $C \geq 1$ depending only on $\dim K$ such that the barycentric subdivision map $\phi \colon |K| \to \mathrm{ba}|K|$ is bilipschitz with bilipschitz constant C.*

Remark 1.2.16. There exists $0 < l_m < 1$ such that for any uniform polyhedron $|K|$ with $\dim|K| + 1 \leq m$ the cover $\mathcal{ST}_{\mathrm{ba}}(|K|)$ has Lebesgue number $L(\mathcal{ST}_{\mathrm{ba}}(|K|)) \geq l_m$.

Lemma 1.2.17. *Let $|K_1| \subset \Delta^{J_1}$ and $|K_2| \subset \Delta^{J_2}$ be uniform simplicial polyhedra of finite combinatorial dimension. Then there exists $C \geq 1$ depending only on $\dim K_1$ and $\dim K_2$ such that the canonical bijection $\varphi_{\mathrm{ba}\,\Delta} \colon |K_1| \times |K_2| \to |K_1| \times_{\mathrm{ba}\,\Delta} |K_2|$ is bilipschitz with bilipschitz constant C.*

Remark 1.2.18. There exists $0 < c_m < 1$ be such that given any uniform polyhedra $|K_1|, |K_2|$ with $\dim|K_i| + 1 \leq m$ the Lebesgue number of the cover $\mathcal{ST}_{\mathrm{ba}}(|K_1| \times |K_2|)$ of $|K_1| \times |K_2|$ is $\geq c_m$.

The next lemma shows that for uniform polyhedra $|K_i|$, $i = 1, 2$, the cover $\mathcal{ST}_{\mathrm{ba}}(|K_1| \times |K_2|)$ of $|K_1| \times |K_2|$ refines the product cover $\mathcal{ST}(|K_1|) \times \mathcal{ST}(|K_2|)$.

Lemma 1.2.19. *Let $|K_1|$ and $|K_2|$ be uniform simplicial polyhedra. Then for every vertex $v \in \mathrm{Ver}(|K_1| \times_{\mathrm{ba}\,\Delta} |K_2|)$ there are vertices $v_1 \in \mathrm{Ver}(|K_1|)$ and $v_2 \in \mathrm{Ver}(|K_2|)$ such that*

$$\varphi_{\mathrm{ba}\,\Delta}^{-1}(\mathrm{st}_v) \subset \mathrm{st}_{v_1} \times \mathrm{st}_{v_2}$$

where $\varphi_{\mathrm{ba}\,\Delta} \colon |K_1| \times |K_2| \to |K_1| \times_{\mathrm{ba}\,\Delta} |K_2|$ is the canonical bijection.

1.3 The P-dimension formalism

1.3.1 The P-dimension axioms

We begin by fixing some notation and terminology. Let Met denote the category of metric spaces. By a *metric pair* we mean an ordered pair (X, A) where X is a metric space and $A \subset X$ is a subspace of X. For X a metric space let $\mathfrak{G}(X)$ denote the set of open covers of X. For $m \geq 1$ let $\mathfrak{G}_m(X) \subset \mathfrak{G}(X)$ denote the set of open covers of X having multiplicity $\mu \leq m$.

Definition 1.3.1. A *filter* on a set P is a collection $\mathcal{F} \subset \mathcal{P}(P)$ of subsets of P such that

- $\emptyset \notin \mathcal{F}$,
- $A \in \mathcal{F}$ and $A \subset B$ imply $B \in \mathcal{F}$,

- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

Definition 1.3.2. A *property set* is a set P together with a map having domain $P \times \text{Met}$ sending

$$(p, X) \mapsto p_X \colon \mathfrak{G}(X) \to \{0, 1\}.$$

An open cover $\mathcal{U}$ of the metric space X is said to *have the property* $p \in P$ if $p_X(\mathcal{U}) = 1$. A Lipschitz map $f \colon X \to |K|$ into a uniform polyhedron has property p if the cover $f^{-1}(\mathcal{ST}(|K|))$ has property p.

Definition 1.3.3. A *property space* is a pair $(P, \mathcal{F})$ where P is a property set and $\mathcal{F}$ is a filter on P such that Axioms 1.3.4, 1.3.5, and 1.3.6 below are satisfied.

Axiom 1.3.4. *For each* $m \geq 1$ *there is a map*

$$\mathbf{ba}_m \colon P \to P$$

such that

- $\mathbf{ba}_m(\mathcal{F}) \subset \mathcal{F}$,
- *for all* $p \in P$ *and for every metric space* X *and every* $\mathcal{U} \in \mathfrak{G}_m(X)$ *having property* p, *there exists a barycentric map* $q \colon X \to \mathcal{N}(\mathcal{U})$ *such that* $\text{ba}\mathcal{U} \in \mathfrak{G}(X)$ *has property* $\mathbf{ba}_m(p)$.

Axiom 1.3.5. *For every metric pair* (X, A) *and every* $\mathcal{U} \in \mathfrak{G}(X)$ *having property* $p \in P$, *the restriction* $\mathcal{U}|_A \in \mathfrak{G}(A)$ *also has property* p.

Axiom 1.3.6. *For each* $m \geq 1$ *there is a map*

$$\mathbf{prod}_m \colon P \times P \to P$$

such that

- $\mathbf{prod}_m(\mathcal{F} \times \mathcal{F}) \subset \mathcal{F}$,
- *for every* $p_1, p_2 \in P$*, every pair of metric spaces* X *and* Y*, and every* $\mathcal{U} \in \mathfrak{G}_m(X)$ *having property* p_1 *and every* $\mathcal{V} \in \mathfrak{G}_m(Y)$ *having property* p_2 *there exist barycentric maps* $q_1 \colon X \to \mathcal{N}(\mathcal{U})$ *and* $q_2 \colon Y \to \mathcal{N}(\mathcal{V})$ *such that* $\mathcal{U} \times_{\mathrm{ba}\,\Delta} \mathcal{V} \in \mathfrak{G}(X \times Y)$ *has property* $\mathbf{prod}_m(p_1, p_2) \in P$*.*

(Axiom 1.3.6 implicitly refers to a method for placing a metric on each product $X \times Y$*; specifically, the test for whether* $\mathcal{U} \times_{\mathrm{ba}\,\Delta} \mathcal{V}$ *has property* $\mathbf{prod}_m(p_1, p_2)$ *will depend on the metric of* $X \times Y$*.)*

A subset of a metric space $A \subset X$ has diameter $\mathrm{diam}(A)$ given by the quantity $\sup\{d(a, b) \mid a, b \in A\}$. The *mesh* of a collection $\mathcal{A} := \{A_i\}_{i \in I}$ of subsets $A_i \subset X$ of a metric space X is defined by

$$\mathrm{mesh}(\mathcal{A}) := \sup\{\mathrm{diam}(A) \mid A \in \mathcal{A}\}.$$

.

Example 1.3.7 (Property space for topological dimension)**.** Let $P = (0, \infty)$. For $p \in P$ and a cover $\mathcal{U} \in \mathfrak{G}(X)$ define

$$p_X(\mathcal{U}) = \begin{cases} 0 & \mathrm{mesh}(\mathcal{U}) > p \\ 1 & \mathrm{mesh}(\mathcal{U}) \leq p. \end{cases}$$

The filter $\mathcal{F}$ is generated by the set $\{(0, t) \mid t > 0\}$.

Example 1.3.8 (Property space for asymptotic dimension)**.** Let $P = (0, \infty)$. For $p \in P$ and a cover $\mathcal{U} \in \mathfrak{G}(X)$ define

$$p_X(\mathcal{U}) = \begin{cases} 1 & \mathrm{mesh}(\mathcal{U}) < \infty \text{ and } L(\mathcal{U}) \geq p \\ 0 & \text{otherwise.} \end{cases}$$

The filter $\mathcal{F}$ is generated by the set $\{(t, \infty) \mid t > 0\}$.

For a metric space X and a property space $(P, \mathcal{F})$ we make the following definitions.

Definition 1.3.9. Let $\operatorname{cov}(X, n) \subset P$ denote the set of those $p \in P$ for which there exists a cover $\mathcal{U} \in \mathfrak{G}_{n+1}(X)$ having property p. The *covering P-dimension* of a metric space X is the minimal integer $n \in \mathbb{N}$ such that $\operatorname{cov}(X, n) \in \mathcal{F}$. We write $P \dim_{\text{cov}} X = n$.

Definition 1.3.10. Let $\operatorname{col}(X, n) \subset P$ denote the set of those $p \in P$ for which there exists an $(n+1)$-colored cover $\mathcal{U} \in \mathfrak{G}(X)$ having property p. The *colored P-dimension* of a metric space X is the minimal integer $n \in \mathbb{N}$ such that $\operatorname{col}(X, n) \in \mathcal{F}$. We write $P \dim_{\text{col}} X = n$.

Definition 1.3.11. Let $\operatorname{pol}(X, n) \subset P$ denote the set of those $p \in P$ for which there exists a map $f \colon X \to K$ into an n-dimensional uniform simplicial complex with f Lipschitz and having property p. The *polyhedral P-dimension* of a metric space X is the minimal integer $n \in \mathbb{N}$ such that $\operatorname{pol}(X, n) \in \mathcal{F}$. We write $P \dim_{\text{pol}} X = n$.

Remark 1.2.16 and Axiom 1.3.4 produce the following theorem; see [BS07b] for the proof.

Theorem 1.3.12. *Let $(P, \mathcal{F})$ be a property space satisfying Axioms 1.3.4, 1.3.5, and 1.3.6. Then*

$$P \dim_{\text{cov}} X = P \dim_{\text{col}} X = P \dim_{\text{pol}} X$$

for every metric space X.

In light of this theorem we may refer to *the* P-dimension of a metric space X, denoted $P \dim X$.

1.3.2 The product and monotonicity theorems

Axiom 1.3.5 results in the monotonicity property for a P-dimension.

Theorem 1.3.13 (Monotonicity theorem)**.** *Let* $(P, \mathcal{F})$ *be a property space satisfying Axioms 1.3.4, 1.3.5, and 1.3.6. Then for any metric pair* (X, A)*,*

$$P \dim A \leq P \dim X.$$

See [BS07b] for the proof. Remark 1.2.18 and Axiom 1.3.6 produce the next theorem; the proof again is in [BS07b].

Theorem 1.3.14 (Product theorem)**.** *Let* $(P, \mathcal{F})$ *be a property space satisfying Axioms 1.3.4, 1.3.5, and 1.3.6. Then for any two metric spaces* X *and* Y

$$P \dim X \times Y \leq P \dim X + P \dim Y.$$

1.3.3 Summary of the P dimension formalism

A P-dimension assigns to each metric space X a non-negative integer invariant, denoted $P \dim X$, such that

- $P \dim A \leq P \dim X$ for all metric pairs (X, A) (monotonicity theorem),
- $P \dim X \times Y \leq P \dim X + P \dim Y$ for all spaces X and Y (product theorem).

The usefulness of the P dimension formalism is that we do not need to prove a product theorem and a monotonicity theorem separately for each new dimension that we introduce.

Chapter 2

The topological and asymptotic dimensions

2.1 Topological dimension

Topological dimension is the foremost example of a small scale dimension. It is relevant to large scale dimension theory because of the pattern of theorems relating a large scale dimension of a space to a small scale dimension of a boundary of the space. See for example Theorems 4.4.1 and 6.2.3.

2.1.1 Definition and basic properties of topological dimension

Definition 2.1.1. The *topological dimension* of a metric space X is the minimal integer $n \in \mathbb{N}$ such that for every $\varepsilon > 0$ there exists an open cover $\mathcal{U}$ of X with

- $\mu(\mathcal{U}) \leq n + 1$,
- $\mathrm{mesh}(\mathcal{U}) \leq \varepsilon$.

We write $\dim X = n$.

Proposition 2.1.2. *Topological dimension is a P-dimension; that is, the property space $(P, \mathcal{F})$ for topological dimension (see Example 1.3.7) satisfies Axioms 1.3.4, 1.3.5, and 1.3.6.*

The following examples are offered to give an intuition for the topological dimension.

Example 2.1.3. For $n \geq 1$, $\dim \mathbb{R}^n = n$. Any open subset $U \subset \mathbb{R}^n$ (with the subspace metric) has $\dim U = n$. Any subset $A \subset \mathbb{R}^n$ containing an open set $U \subset A$ has $\dim A = n$.

Example 2.1.4. Any n-dimensional manifold M^n (with metric compatible with the topology of M) has $\dim M = n$. Any open subset or any subset containing an open subset has $\dim = n$.

Example 2.1.5. Any discrete metric space, i.e., a space X for which there exists $\varepsilon > 0$ such that $x \neq y \in X$ implies $d(x, y) > \varepsilon$, has $\dim X = 0$.

Example 2.1.6. The Cantor set $C \subset [0, 1]$ has $\dim C = 0$.

2.1.2 The Hurewicz theorem

A basic tool in classical dimension theory is the Hurewicz theorem:

Theorem 2.1.7 (Hurewicz theorem)**.** *Let $f \colon X \to Y$ be a continuous map between compact metric spaces X and Y, and suppose that for some $n \in \mathbb{N}$*

$$\dim f^{-1}(y) \leq n$$

for every $y \in Y$. Then

$$\dim X \leq \dim Y + n.$$

Our main interest in the Hurewicz theorem here is that a version is true for asymptotic dimension, which we discuss in the next section.

2.2 Asymptotic dimension

The asymptotic dimension, introduced by Gromov [Gro93], is the foremost example of a large scale dimension. An excellent survey is [BD08]. Much more is known about asymptotic dimension than about hyperbolic dimension. In this section we introduce the fundamental properties of asymptotic dimension and afterwards focus on those properties most relevant to our study of the hyperbolic and the weak hyperbolic dimensions in Chapters 4, 5, and 6.

2.2.1 Definition and basic properties of asymptotic dimension

Definition 2.2.1. The collection $\mathcal{A} := \{A_i\}_{i \in I}$ of subsets $A_i \subset X$ of a metric space X is *uniformly bounded* if there exists $M > 0$ such that

$$\operatorname{diam}(A) \leq M$$

for all $A \in \mathcal{A}$.

A collection $\mathcal{A}$ is uniformly bounded if and only if $\operatorname{mesh}(\mathcal{A}) < \infty$.

Definition 2.2.2. The *asymptotic dimension* of a metric space X is the minimal integer $n \in \mathbb{N}$ such that for every $D > 0$ there exists an open cover $\mathcal{U}$ of X such that

- $\mathcal{U}$ is uniformly bounded,
- $\mu(\mathcal{U}) \leq n + 1$,
- $L(\mathcal{U}) \geq D$.

We write asdim $X = n$.

Example 2.2.3. Let X be a bounded metric space. Then the cover $\mathcal{U} = \{X\}$ shows asdim $X = 0$.

Proposition 2.2.4. *Asymptotic dimension is a P-dimension; that is, the property space $(P, \mathcal{F})$ for asymptotic dimension (see Example 1.3.8) satisfies Axioms 1.3.4, 1.3.5, and 1.3.6.*

The proof relies on the technical lemmas of §1.2.4. The following examples are offered to give an intuition for the asymptotic dimension.

Example 2.2.5. For $n \geq 1$, asdim $\mathbb{R}^n = n$. Moreover the discrete space $\mathbb{Z}^n \subset \mathbb{R}^n$ also has asdim $\mathbb{Z}^n = n$.

Example 2.2.6. For $n \geq 2$, asdim $\mathbb{H}^n = n$.

2.2.2 Coarse invariance of asymptotic dimension

The asymptotic dimension is a quasi-isometry invariant: if the spaces X and Y are quasi-isometric then asdim X = asdim Y. Actually asymptotic dimension is an invariant even for a weaker notion of large scale equivalence of spaces, defined below.

Definition 2.2.7. A map $f\colon X \to Y$ between metric spaces X and Y is a *coarse embedding* if there exist functions $\rho_1, \rho_2\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\rho_1(t), \rho_2(t) \to \infty$ as $t \to \infty$ and such that

$$\rho_1(d_X(x_1, x_2)) \leq d_Y(f(x_1), f(x_2)) \leq \rho_2(d_X(x_1, x_2))$$

for all $x_1, x_2 \in X$. If in addition the image $f(X) \subset Y$ is an r-net in Y for some $r > 0$, that is, for all $y \in Y$ there is $x \in X$ such that $d(f(x), y) < r$, then f is a *coarse equivalence*. The spaces X and Y are *coarse equivalent* if there exists a coarse equivalence $f\colon X \to Y$.

Every quasi-isometric map is a coarse embedding. Thus two quasi-isometric spaces are also coarse equivalent.

Proposition 2.2.8. *If X and Y are coarsely equivalent metric spaces then* $\operatorname{asdim} X = \operatorname{asdim} Y$.

2.2.3 Asymptotic dimension and discrete groups

Let Γ be a finitely-generated group and let d_S, d_T be word metrics for two generating sets $S, T \subset \Gamma$. The fact that (Γ, d_S) and (Γ, d_T) are quasi-isometric allows the following definition to be made.

Definition 2.2.9. The *asymptotic dimension of the finitely-generated group* Γ, denoted $\operatorname{asdim} \Gamma$, is the asymptotic dimension of the metric space (Γ, d) where d is a word metric for a finite generating set.

A word metric on a finitely-generated group Γ has two important properties: it is *left-invariant* and *proper*. These nice properties can be had by metrics other than word metrics, so we make the following definition.

Definition 2.2.10. A metric d on a group Γ is *left-invariant* if

$$d(\tau\gamma_1, \tau\gamma_2) = d(\gamma_1, \gamma_2)$$

for all $\gamma_1, \gamma_2, \tau \in G$. A left-invariant metric d on a discrete group G is *proper* if the ball $B_r(e)$ of radius r centered at the identity $e \in \Gamma$ is finite for all $r \geq 0$.

The asymptotic dimension of a countable group Γ is independent of the choice of proper, left-invariant metric due to the next result; see [Smi06] or [Sha04] .

Proposition 2.2.11. *Let Γ be a countably generated group and let d_1, d_2 be proper, left-invariant metrics on Γ. Then (Γ, d_1) and (Γ, d_2) are coarsely equivalent.*

Asymptotic dimension therefore assigns a non-negative integer invariant to every countably generated group.

Definition 2.2.12. The *asymptotic dimension of the countable group* Γ, denoted $\operatorname{asdim}\Gamma$, is the asymptotic dimension of the metric space (Γ, d) where d is any left-invariant, proper metric on Γ. *(The value* $\operatorname{asdim}\Gamma$ *is well-defined by Proposition 2.2.11.)*

The next results are useful tools for computing the asymptotic dimension of discrete groups. Moreover it seems worth investigating whether there are analogous results for the hyperbolic and weak hyperbolic dimensions. Dranishnikov and Smith [DS06] prove the following.

Theorem 2.2.13. *For a countable group* Γ *we have*

$$\operatorname{asdim}\Gamma = \sup\{\operatorname{asdim} F \mid F < \Gamma \text{ is finitely generated}\}.$$

In [BDK04] the following formula is proved for the dimension of a free product.

Theorem 2.2.14. *For* A *and* B *finitely generated groups with* $\operatorname{asdim} A = n$ *and* $\operatorname{asdim} B \leq n$*, there is the formula* $\operatorname{asdim} A * B = \max\{n, 1\}$.

For a free product with amalgamation there is the estimate from [BD04].

Theorem 2.2.15. *For* A *and* B *finitely generated groups with* $\operatorname{asdim} A \leq n$ *and* $\operatorname{asdim} B \leq n$*, there is the estimate* $\operatorname{asdim} A *_C B \leq n + 1$.

2.2.4 Asymptotic Hurewicz theorem

Bell and Dranishnikov show in [BD06] that the asymptotic dimension admits an analogue of the Hurewicz theorem. A family $\{X_i\}_i$ of metric spaces has *asymptotic dimension* $\leq n$ *uniformly* if for every $D > 0$ there exists $R > 0$ such that each space X_i

admits a uniformly R-bounded cover $\mathcal{U}_i$ with multiplicity $\mu(\mathcal{U}_i) \leq n+1$ and Lebesgue number $L(\mathcal{U}_i) \geq D$.

Theorem 2.2.16 (Asymptotic Hurewicz theorem)**.** *Let $f\colon X \to Y$ be a Lipschitz map between metric spaces, where X is geodesic. Suppose that for some $n \in \mathbb{N}$ the family*

$$\{f^{-1}(B_R(y))\}_{y\in Y}$$

has asdim $\leq n$ *uniformly for every $R > 0$. Then*

$$\operatorname{asdim} X \leq \operatorname{asdim} Y + n.$$

The asymptotic Hurewicz theorem leads directly to an upper bound for the asymptotic dimension of a finitely-generated group extension.

Theorem 2.2.17 (Extension theorem for asdim)**.** *Let $\phi\colon G \to H$ be a surjective group homomorphism, where G is finitely generated. Let $K = \ker\phi$. Then*

$$\operatorname{asdim} G \leq \operatorname{asdim} H + \operatorname{asdim} K.$$

Theorem 2.2.17 is extremely useful for computing $\operatorname{asdim} G$ for finitely generated G. It is the motivation for Theorem 5.4.2 for hyperbolic dimension in Chapter 5 and Theorem 6.3.3 for weak hyperbolic dimension in Chapter 6. Those estimates however give upper bounds that are not as tight as that of Theorem 2.2.17.

Chapter 3

Negative curvature in metric spaces

For Riemannian manifolds notions of curvature arise out of the differential machinery of the Levi-Civita connection and the Riemannian curvature tensor. It turns out that many of these natural curvature conditions can be characterized purely in terms of the distance function induced by the Riemannian metric. This makes possible a number of curvature conditions that can be placed on metric spaces, without requiring any additional differential geometric structure. In this chapter we tour two of the most common of those conditions: hyperbolicity conditions in Section 3.2 and CAT(κ) conditions in Section 3.6. Hyperbolic spaces admit the construction of a boundary that encodes the metric structure of the space. The various approaches to this construction are described in Sections 3.3, 3.4, and 3.5. We begin, however, with a class of metric spaces loosely called *trees*.

3.1 Trees

Trees are a good source of clarifying examples and counter examples in metric geometry. In particular, they help to clarify connections between dimension, growth, and negative curvature.

Definition 3.1.1. Let X be a geodesic metric space. A *triangle with vertices* $a, b, c \in X$ is a union of geodesic segments ab, bc, and ca joining a to b, b to c, and c to a respectively.

Above we write *a* triangle and not *the* triangle, because in general many geodesic segments may join the same two points.

Definition 3.1.2. Let X be a geodesic metric space. A *tripod with vertices* $a, b, c \in X$ is a triangle with vertices $a, b, c \in X$ such that there exists a unique point $p \in ab \cap bc \cap ca$ such that

$$ab \cup bc \cup ca = ap \cup bp \cup cp.$$

That is, a tripod is a triangle that can be decomposed into three geodesic segments emanating from the same point.

Definition 3.1.3. The geodesic metric space T is a *real tree* if T is uniquely geodesic and every triangle in T is a tripod.

Real trees are also known as $\mathbb{R}$-*trees*. An alternative characterization is the following.

Definition 3.1.4. The geodesic metric space T is a *real tree* if T is uniquely geodesic and if every union of two geodesic segments meeting only at a common endpoint is again a geodesic segment.

Example 3.1.5. Let $T = \mathbb{R}^n$ and let d_E be the standard Euclidean metric on $\mathbb{R}^n$. For $P, Q \in \mathbb{R}^n$ define $d(P,Q) = d_E(P,Q)$ if P and Q lie on a common line through the origin O, and define $d(P,Q) = d(P,O) + d(O,Q)$ otherwise. Then T is an $\mathbb{R}$-tree.

Example 3.1.6. Let $T = \mathbb{R}^2$ and define $d((a,b),(x,y)) = |y-b|$ if $a = x$ and $d((a,b),(x,y)) = |x-a| + |b| + |y|$ otherwise. Then T is an $\mathbb{R}$-tree.

An interesting feature of the above example is that every point along the x-axis is a "branch" point. A *metric tree*, defined below, has the property that the set of branch points is a discrete set. By the term *graph* we mean the usual notion of an ordered pair (V, E) of vertices and edges, where the set of edges E consists of 2-element subsets of V. The term *graph* will also refer to the *geometric realization* of a graph (V, E), which is obtained as the quotient $X/\sim$ of the disjoint union $X = \cup\{[0,1]_e \mid e \in E\}$ where $\sim$ identifies endpoints of the intervals $[0,1]_e$ corresponding to the same vertices.

Definition 3.1.7. A *metric tree* is a simply-connected graph.

Every metric tree is also a real tree.

Example 3.1.8. Let T be a simply-connected 1-dimensional simplicial polyhedron. Let d_l denote the length metric built out of the uniform metric d_u and let $d_t = d_l/\sqrt{2}$. Then (T, d_t) is a metric tree.

Example 3.1.9. Let $F_n = \langle a_1, \ldots, a_n \rangle$ denote the free group on n generators. Set $A = \{a_1, \ldots, a_n\}$. Then the Cayley graph $C(F_n, A)$ of F_n with respect to the generating set A is a metric tree.

Every real tree has asymptotic dimension 0 or 1:

Proposition 3.1.10. *Let T be a real tree. Then* $\operatorname{asdim} T \leq 1$.

We now offer a simple example of how trees help clarify and separate properties of metric spaces.

Remark 3.1.11. We have $\operatorname{asdim} T \leq 1$ for any real tree T and $\operatorname{asdim} \mathbb{R}^n \geq 2$ for $n \geq 2$. Therefore there exists no quasi-isometric embedding $\mathbb{R}^n \to T$ for any $n \geq 2$ and any real tree T. Asymptotic dimension obstructs such an embedding.

If T is a real tree with superpolynomial growth (for example the Cayley graph of F_n from Example 3.1.9 above) there exists no quasi-isometric embedding $T \to \mathbb{R}^n$ for any $n \geq 1$. Growth obstructs such an embedding.

Now we define the tool that allows one to characterize tree-like behaviour in a metric space that is not geodesic.

Definition 3.1.12. The *Gromov product* of the points $x, y \in X$ with respect to the basepoint $o \in X$, denoted $(x|y)_o$, is given by the expression

$$(x|y)_o = \frac{1}{2}\left(d(o,x) + d(o,y) - d(x,y)\right).$$

Remark 3.1.13. If the space X is an $\mathbb{R}$-tree and $p \in X$ is any base point there is the following inequality:

$$(x|y)_p \geq \min\{(x|z)_p, (z|y)_p\} \tag{3.1}$$

for all $x, y, z \in X$.

The discrete space $\mathbb{Z}$ with metric $d(i,j) = |i - j|$ is not geodesic, but satisfies Inequality 3.1. Another example is the free group $F_n = \langle a_1, \dots, a_n \rangle$ on n generators with word metric induced by the generating set $\{a_1, \dots, a_n\}$.

3.2 Hyperbolicity of metric spaces

Triangles in the hyperbolic disk have an interesting property: they are thin. That property is singled out and made precise in the definition of *hyperbolicity*.

In terms of the technical details, there are multiple ways to approach hyperbolicity. The treatment here is directly influenced by that found in [BS07b].

3.2.1 Hyperbolic geodesic metric spaces

Let X be a geodesic metric space. For $x, y, z \in X$ we denote a triangle with vertices x, y, z by xyz.

Definition 3.2.1. The geodesic metric space X is Rips hyperbolic if there exists $\delta \geq 0$ such that: for every triangle $xyz \subset X$ we have

$$xy \subset N_\delta(yz \cup zx), \quad yz \subset N_\delta(zx \cup xy), \quad zx \subset N_\delta(xy \cup yz);$$

that is, any side of any triangle is contained in the δ-neighborhood of the other two sides.

A triangle having the property that any side is contained in the δ-neighborhood ($\delta \geq 0$) of the other two sides is called *δ-thin*. A Rips hyperbolic space is therefore a space for which there exists $\delta \geq 0$ such that all triangles are δ-thin. The motivating example for this definition is the hyperbolic disk.

Example 3.2.2. In 2-dimensional hyperbolic space $\mathbb{H}^2$, all triangles are δ-thin for $\delta = \frac{1}{2}\log 3$. That is, $\mathbb{H}^2$ is δ-hyperbolic for $\delta = \frac{1}{2}\log 3$. Any triangle in n-dimensional hyperbolic space lies inside a totally geodesic submanifold isometric to $\mathbb{H}^2$. Thus $\mathbb{H}^n$ is also δ-hyperbolic for $\delta = \frac{1}{2}\log 3$.

We include a second (equivalent) characterization of hyperbolicity for geodesic spaces. The characterization is not as intuitive as that of Rips but is effective for technical purposes.

Definition 3.2.3. Let $\delta \geq 0$. The geodesic metric space X is *δ-hyperbolic* if for every triangle $xyz \subset X$ we have: if $y' \in xy$ and $z' \in xz$ are points satisfying

$$d(x, y') = d(x, z') \leq (y|z)_x$$

then

$$d(y', z') \leq \delta.$$

The geodesic metric space X is *hyperbolic* if there exists $\delta \geq 0$ such that X is δ-hyperbolic.

Propositions 3.2.4 and 3.2.5 below demonstrate that, for a geodesic space X, the conditions "X is hyperbolic" and "X is Rips hyperbolic" are equivalent (though the constants for which each condition can be verified will in general be different).

Proposition 3.2.4. *If the geodesic metric space X is δ-hyperbolic for $\delta \geq 0$, then X is Rips hyperbolic with hyperbolicity constant δ.*

Proposition 3.2.5. *If the geodesic metric space X is Rips hyperbolic with hyperbolicity constant $\delta \geq 0$, then X is 4δ-hyperbolic.*

Proposition 3.2.6 below suggests the tool for characterizing hyperbolicity in metric spaces that are not geodesic.

Proposition 3.2.6. *Let the geodesic metric space X be δ-hyperbolic for $\delta \geq 0$. Then for any base point $o \in X$ the inequality*

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta \tag{3.2}$$

is satisfied for all $x, y, z \in X$.

Inequality 3.2 is the δ-*inequality.* In Proposition 3.2.6 the δ-inequality follows from the condition that X is δ-hyperbolic. For non-geodesic metric spaces, the δ-inequality is elevated to the hyperbolicity condition itself.

3.2.2 Gromov hyperbolic metric spaces

Let X be a metric space, not necessarily geodesic.

Definition 3.2.7. The metric space X is *Gromov hyperbolic* if there exists $\delta \geq 0$ such that: for every base point $o \in X$ the inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

is satisfied for all $x, y, z \in X$.

Such a space is also called *Gromov δ-hyperbolic.* By Proposition 3.2.6 a δ-hyperbolic (or a Rips hyperbolic) geodesic space X is Gromov hyperbolic.

The next result simplifies the task of checking that a given space is Gromov hyperbolic.

Proposition 3.2.8. *Suppose there exists a base point $o \in X$ and $\delta \geq 0$ such that*

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

for all $x, y, z \in X$. Then X is Gromov hyperbolic with hyperbolicity constant 2δ.

Thus to check that a metric space X is Gromov hyperbolic it is enough to verify that there is one base point $o \in X$ and some $\delta \geq 0$ for which the Gromov inequality is satisfied.

We have noted that a δ-hyperbolic (or Rips hyperbolic) geodesic space is Gromov hyperbolic. Proposition 3.2.9 provides the other direction: a Gromov hyperbolic space, if it is also geodesic, is δ-hyperbolic (for some $\delta \geq 0$) and Rips hyperbolic.

Proposition 3.2.9. *Let the geodesic metric space X be Gromov hyperbolic in the sense of Definition 3.2.7 with hyperbolicity constant $\delta \geq 0$. Then X is hyperbolic in the sense of Definition 3.2.3 with hyperbolicity constant 4δ.*

3.2.3 Summary of hyperbolicity

We have seen three ways to characterize hyperbolicity in geodesic spaces: δ-hyperbolicity, Rips hyperbolicity, and Gromov hyperbolicity. Any one of the characterizations implies the other two, although the constants involved will differ. Of these three characterizations, Gromov hyperbolicity does not depend on the geodesic property of a space, and therefore can be used to characterize hyperbolicity in not-necessarily-geodesic spaces.

Hyperbolicity is a quasi-isometry invariant for geodesic metric spaces. That is, if X and Y are quasi-isometric geodesic metric spaces and X is hyperbolic then Y is hyperbolic. This fact depends on the stability of quasi-geodesics in geodesic hyperbolic spaces.

3.3 The boundary at infinity

Hyperbolic metric spaces admit a natural construction of a *boundary at infinity.* We describe the various approaches to this construction in this section.

3.3.1 Constructing the boundary

Let X be a Gromov hyperbolic metric space and let $o \in X$ be a base point.

Definition 3.3.1. The sequence $\{x_i\}_{i=1}^{\infty} \subset X$ *converges to infinity* if

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty.$$

Proposition 3.3.2. *Suppose the sequence $\{x_i\}_{i=1}^{\infty} \subset X$ converges to infinity with respect to the base point $o \in X$. Then $\{x_i\}_{i=1}^{\infty}$ converges to infinity with respect to any other base point $o' \in X$.*

Definition 3.3.3. Let $\{x_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty} \subset X$ be two sequences converging to infinity in the Gromov hyperbolic space X. Then $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ are *equivalent*, denoted $\{x_i\}_{i=1}^{\infty} \sim \{y_i\}_{i=1}^{\infty}$, if

$$\lim_{i\to\infty}(x_i|y_i)_o = \infty.$$

The requirement that X be Gromov hyperbolic ensures that Definition 3.3.3 defines an equivalence relation.

Definition 3.3.4. The *boundary at infinity*, denoted $\partial_\infty X$, of the Gromov hyperbolic metric space X, is defined to be the set of equivalence classes under the relation $\sim$ of sequences in X converging to infinity.

The boundary at infinity is also referred to as the *sequential boundary* and as the *Gromov boundary.*

Definition 3.3.5. For $\xi, \eta \in \partial_\infty X$ the *Gromov product of ξ and η with respect to the base point $o \in X$*, denoted $(\xi|\eta)_o$, is defined by

$$(\xi|\eta)_o = \inf \liminf_{i\to\infty}(x_i|y_i)_o$$

where the infimum is taken over all sequences $\{x_i\}_{i=1}^{\infty} \subset X$ representing ξ and all sequences $\{y_i\}_{i=1}^{\infty} \subset X$ representing η.

One can check that $(\xi|\eta)_o = \infty$ if and only if $\xi = \eta$. Thus for $\xi \neq \eta \in \partial_\infty X$ we have $(\xi|\eta)_o \in [0, \infty)$.

3.3.2 Constructing a metric on the boundary

We construct a metric on the boundary at infinity $\partial_\infty X$ of the Gromov hyperbolic metric space X according to the following steps.

Step 1. Choose a basepoint $o \in X$ and a parameter $1 < a < 2^{1/\delta}$ (if $\delta = 0$ then choose $a \in (1, \infty)$).

Step 2. Define a function $\rho = \rho_{o,a} \colon \partial_\infty X \times \partial_\infty X \to \mathbb{R}$ by

$$\rho(\xi, \eta) = a^{-(\xi|\eta)_o}$$

for $\xi, \eta \in \partial_\infty X$. Then ρ is an a^δ-quasi-metric.

Step 3. Apply the *chain construction* to the a^δ-quasi-metric ρ to obtain a metric $d = d(o, a)$ on $\partial_\infty X$.

A *quasi-metric* for a metric space Z is a function $q \colon Z \times Z \to \mathbb{R}$ satisfying

(i) $q(x, y) \geq 0$ for all $x, y \in Z$ and $q(x, y) = 0$ if and only if $x = y$,

(ii) $q(x, y) = q(y, x)$ for all $x, y \in Z$, and

(iii) there exists a constant $k \geq 1$ such that for all $x, y, z \in Z$ there is the inequality

$$q(x, y) \leq k \max\{q(x, z), q(z, y)\}.$$

A quasi-metric with constant $k \geq 1$ is called a *k-quasi-metric*. A 1-quasi-metric is called an *ultrametric*. Note that any metric provides an example of a 2-quasi-metric.

When $q \colon Z \times Z \to \mathbb{R}$ is a quasi-metric, the *chain construction* defines a function $d \colon Z \times Z \to \mathbb{R}$ by

$$d(x, y) = \inf\{\sum_{i=1}^{n} q(x_{i-1}, x_i) \mid x = x_0, x_1, \ldots, x_{n-1}, x_n = y \in Z\}$$

where the infimum is taken over all finite sequences $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ in Z. If the chain construction is carried out on a 2-quasi-metric the resulting function d is always a metric. See [Sch06] for a nice presentation of this fact based on material in the 1937 paper [Fri37].

On the boundary at infinity $\partial_\infty X$ of a Gromov hyperbolic space X it is the *visual metrics*, defined below, that are compatible with the large scale geometry of X.

Definition 3.3.6. A metric d on the boundary at infinity $\partial_\infty X$ of a Gromov hyperbolic metric space X is called *visual with respect to the base point $o \in X$ and the parameter $a > 1$* if there exist constants $c_1, c_2 > 0$ such that

$$c_1 a^{-(\xi|\eta)_o} \leq d(\xi, \eta) \leq c_2 a^{-(\xi|\eta)_o}$$

for all $\xi, \eta \in \partial_\infty X$.

The good news is that metrics constructed according to steps 1, 2, and 3 are visual.

Proposition 3.3.7. *Let X be a Gromov hyperbolic space with hyperbolicity constant $\delta \geq 0$, and let $d = d(o, a)$ be a metric on $\partial_\infty X$ constructed according to steps 1, 2, 3 above. Then d is a visual metric with respect to o and a.*

Propositions 3.3.8 and 3.3.9 quantify how visual metrics behave under change of base point and under change of parameter, respectively.

Proposition 3.3.8. *Let d and d' be metrics on $\partial_\infty X$ that are visual with respect to the same parameter $a > 1$ but different base points o, o'. Then d and d' are bi-Lipschitz equivalent, that is, there is $c \geq 1$ such that*

$$\frac{d(\xi, \eta)}{c} \leq d'(\xi, \eta) \leq c \cdot d(\xi, \eta)$$

for all $\xi, \eta \in \partial_\infty X$.

Proposition 3.3.9. *Let d and d' be metrics on $\partial_\infty X$ that are visual with respect to the same base point $o \in X$ but different parameters $a, a' > 1$. Then d and d' are Hölder equivalent, that is, there is $c \geq 1$ such that*

$$\frac{(d(\xi, \eta))^\alpha}{c} \leq d'(\xi, \eta) \leq c \cdot (d(\xi, \eta))^\alpha$$

for all $\xi, \eta \in \partial_\infty X$, where $\alpha = (\ln a')/(\ln a)$.

The bi-Lipschitz and Hölder equivalence results for visual metrics make possible the following definition.

Definition 3.3.10. For X a Gromov hyperbolic metric space the topology on the boundary $\partial_\infty X$ is the topology induced by a metric on $\partial_\infty X$ visual with respect to some base point $o \in X$ and some parameter $a > 1$. *Due to Propositions 3.3.8 and 3.3.9, the topology produced in this way is independent of the choice of base point $o \in X$ and parameter $a > 1$.*

Propositions 3.3.11 and 3.3.12 below record the basic facts about the boundary at infinity.

Proposition 3.3.11. *If X is a Gromov hyperbolic space and d is a visual metric on $\partial_\infty X$, then the metric space $(\partial_\infty X, d)$ is bounded and complete.*

Proposition 3.3.12. *If X is a proper geodesic Gromov hyperbolic metric space, then $\partial_\infty X$ is compact.*

3.3.3 Constructing a topology on the union $X \cup \partial_\infty X$

For the purpose of topologizing the union $X \cup \partial_\infty X$ it is useful to use an alternative characterization, directly in terms of the Gromov product, for the topology on the boundary $\partial_\infty X$ of a Gromov hyperbolic space. This approach follows the efficient discussion found in [KB02].

For each $\xi \in \partial_\infty X$ and each $r > 0$ define the set

$$U(\xi, r) := \{\eta \in \partial_\infty X \mid [(x_n)_{n\geq 1}] = \xi, [(y_n)]_{n\geq 1} = \eta, \liminf_{i,j\to\infty} (x_i|y_j)_o \geq r\}.$$

The topology on $\partial_\infty X$ is the topology generated by the basis

$$\{U(\xi, r) \mid \xi \in \partial_\infty X, r > 0\}.$$

The topology generated by this family is the same as the topology induced by any visual metric on the boundary.

Next for each $\xi \in \partial_\infty X$ and each $r > 0$ define the set

$$U'(\xi, r) := U(\xi, r) \cup \{y \in X \mid [(x_n)_{n\geq 1}] = \xi, \liminf_{i\to\infty}(x_i|y)_o \geq r\}.$$

Definition 3.3.13. Let X be a Gromov hyperbolic metric space. The topology on the union $X \cup \partial_\infty X$ is the topology generated by the basis

$$\{U'(\xi, r) \mid \xi \in \partial_\infty X, r > 0\} \cup \{B_r(x) \mid x \in X, r > 0\}.$$

3.4 The geodesic boundary

Points in the geodesic boundary are equivalence classes of geodesic rays under the following equivalence relation.

Definition 3.4.1. Let $\gamma_1\colon [0,\infty) \to X$ and $\gamma_2\colon [0,\infty) \to X$ be geodesic rays in the metric space X. Then γ_1 and γ_2 are *equivalent*, denoted $\gamma_1 \sim \gamma_2$, if there exists a constant $C > 0$ such that for all $t \in [0,\infty)$ the inequality

$$d(\gamma_1(t), \gamma_2(t)) \leq C$$

is satisfied.

The equivalence relation makes sense in any metric space. In this section, however, we will always take X to be a hyperbolic geodesic space.

Definition 3.4.2. The *geodesic boundary of X relative to the base point $x \in X$*, denoted $\partial_x^g X$, is the set of equivalences classes, under the relation $\sim$ of Definition 3.4.1, restricted to geodesic rays $\gamma\colon [0,\infty) \to X$ in X such that $\gamma(0) = x$.

We say that a geodesic ray $\gamma\colon [0,\infty) \to X$ with $\gamma(0) = x$ is *based at x*.

Definition 3.4.3. Let X be a hyperbolic geodesic metric space and let $x \in X$ be a basepoint. For $[\gamma] \in \partial_x^g X$ and $r \geq 0$ define a subset $V([\gamma], r)$ of $\partial_x^g X$ by

$$V([\gamma], r) := \{[\xi] \in \partial_x^g X \mid \liminf_{t \to \infty} (\gamma'(t)|\xi(t))_x \geq r, \gamma' \in [\gamma]\}.$$

The topology on the relative geodesic boundary $\partial_x^g X$ is generated by the basis

$$\{V([\gamma], r) \mid [\gamma] \in \partial_x^g X, r \geq 0\}.$$

Definition 3.4.4. The *geodesic boundary of* X, denoted $\partial^g X$, is the set of equivalences classes of geodesic rays in X under the relation $\sim$ of Definition 3.4.1.

For $x \in X$ we denote by $i_x \colon \partial_x^g X \to \partial^g X$ the canonical inclusion of $\partial_x^g X$ into $\partial^g X$.

Proposition 3.4.5. *For any $x \in X$ the natural map $i_x \colon \partial_x^g X \to \partial^g X$ is a bijection.*

Proposition 3.4.6. *Let the hyperbolic geodesic metric space X be proper. For any $x, y \in X$ the map*

$$(i_y)^{-1} \circ i_x \colon \partial_x^g X \to \partial_y^g X$$

is a homeomorphism.

As a consequence of Proposition 3.4.6, when X is proper we may choose any base point $x \in X$ together with its associated bijection i_x to induce a topology on $\partial^g X$.

Definition 3.4.7. Let the hyperbolic geodesic space X be proper. The topology on the geodesic boundary $\partial^g X$ is that induced by the bijection $i_x \colon \partial_x^g X \to \partial^g X$, for any $x \in X$. *Due to Proposition 3.4.6, the topology produced in this way is independent of the choice of $x \in X$.*

Example 3.4.8. Let X be a Hadamard manifold, that is, a complete simply-connected Riemannian manifold with sectional curvatures $K \leq 0$. Then the exponential map

$\exp_o T_oX \to X$ is a diffeomorphism, for any base point $o \in X$. The map $\phi\colon S^{n-1} \to \partial^g_o X$ defined by

$$\phi(v) = t \mapsto \exp_o(tv), t \in [0, \infty)$$

for $v \in S^{n-1}$, is a homeomorphism. Hence the geodesic boundary $\partial^g_o X$ can be identified with S^{n-1}.

3.5 The identification of the geodesic boundary and the boundary at infinity

Denote by $i\colon \partial^g X \to \partial_\infty X$ the map defined by

$$[\gamma] \mapsto [(\gamma(n))_{n\geq 1}]$$

for $[\gamma] \in \partial^g X$. One can check that i is well-defined, and Proposition 3.5.1 below shows that the map i provides the canonical identification between the geodesic and the sequential boundaries.

Proposition 3.5.1. *The map $i\colon \partial^g X \to \partial_\infty X$ is a bijection. The map $i \circ i_x\colon \partial^g_x X \to \partial_\infty X$ is a homeomorphism for every $x \in X$.*

Having identified $\partial^g X$ and $\partial_\infty X$, we can use the topology on $X \cup \partial_\infty X$ constructed in Section 3.3.3 to topologize $X \cup \partial^g X$. For geodesic hyperbolic spaces that are *proper*, i.e., closed balls are compact, the union $X \cup \partial^g X$ with this topology is a compactification.

Proposition 3.5.2. *Let the metric space X be geodesic, hyperbolic, and proper. Then $\partial^g X$ and $X \cup \partial^g X$ are compact.*

3.6 CAT(κ) metric spaces

A useful condition of non-positive curvature is found in the concept of a CAT(0)-space. The acronym CAT, due to Gromov [Gro87], acknowledges the foundational work of Cartan, Aleksandrov, and Toponogov. The standard reference for the study of such spaces is [BH99].

For $\kappa \leq 0$ denote by M_κ the unique simply-connected 2-dimensional Riemannian manifold of constant sectional curvature κ. Let X be a geodesic metric space and let $A, B, C \in X$ be distinct vertices of a triangle ABC. Denote by a, b, and c the lengths of the geodesic sides BC, CA, and AB respectively.

Definition 3.6.1. A *comparison triangle for ABC in the space M_κ* is a triangle with vertices $\overline{A}, \overline{B}, \overline{C} \in M_\kappa$ such that $d(\overline{B}, \overline{C}) = a$, $d(\overline{C}, \overline{A}) = b$, and $d(\overline{A}, \overline{B}) = c$.

Definition 3.6.2. The geodesic space X is CAT(κ) *($\kappa \leq 0$)* if for every triangle $ABC \in X$ and every pair of distances $0 \leq s \leq c$ and $0 \leq t \leq b$ we have

$$d_X(P, Q) \leq d_{M_\kappa}(\overline{P}, \overline{Q})$$

where P and Q are the points on AB and CA with $d_X(A, P) = s$ and $d_X(A, Q) = t$ respectively, and $\overline{P}$ and $\overline{Q}$ are the points on $\overline{AB}$ and $\overline{CA}$ with $d_{M_\kappa}(\overline{A}, \overline{P}) = s$ and $d_{M_\kappa}(\overline{A}, \overline{Q}) = t$ respectively.

Comparison spaces M_κ can be taken with constant curvature $\kappa > 0$, resulting in the notion of a CAT(κ)-space for $\kappa > 0$. The fact that such spaces M_κ have a finite upper bound on diameter places a restriction on the diameter of triangles to be considered in the definition. Such spaces do not appear in this work.

A space is called *non-positively curved* if it is *locally* CAT(0), that is, if every point is contained in a (geodesically convex) neighborhood that is CAT(0).

Since M_κ has δ-thin triangles for some $\delta > 0$ when $\kappa < 0$, a CAT(κ)-space with $\kappa < 0$ is hyperbolic. Two fundamental facts about CAT(0)-spaces are recorded in the propositions below.

Proposition 3.6.3. *Let X be a* CAT(0)*-space. Then for all $x, y \in X$, the geodesic segment joining x and y is unique.*

Proposition 3.6.4. *Let X be* CAT(0) *with $o \in X$ any basepoint. Let $H\colon X \times [0,1] \to X$ be the map defined by taking $H(x,t)$ to be the point on the geodesic ox at distance $(1-t)d(o,x)$ from o. Then H is continuous, so that X is contractible.*

Chapter 4

Hyperbolic dimension

Buyalo and Schroeder introduced hyperbolic dimension in 2007 in [BS07a]. Another complete treatment, from a slightly different point of view, is in [BS07b].

4.1 Large scale doubling

The notion of *large scale doubling* is fundamental for hyperbolic dimension. A useful heuristic is the following: asymptotic dimension regards the *bounded* sets of a metric space as the "small" sets; hyperbolic dimension regards the *doubling* sets of a metric space as the "small" sets.

Let X be a metric space.

Definition 4.1.1. The subset $U \subset X$ is (N, R)-*large-scale-doubling* (for $N \geq 1$ and $R \geq 0$) if for every center $x \in X$ and for every radius $r \geq R$ the intersection $U \cap B_{2r}(x)$ can be covered by $\leq N$ radius-r balls $B_r(x_1), \ldots, B_r(x_N)$ with centers $x_i \in X$.

Example 4.1.2. Let $U \subset X$ be bounded. Then for any $r > \operatorname{diam}(U)$ any ball $B_r(u)$ centered at $u \in U$ covers all of U. Thus U is $(1, R)$-large-scale-doubling for any

$R > \mathrm{diam}(U)$.

Example 4.1.3. For $n \geq 1$, n-dimensional Euclidean space $\mathbb{R}^n$ is (N, R)-large-scale-doubling for every $R \geq 0$ and for $N = N(n)$.

4.1.1 Quasi-isometry invariance of large scale doubling

Suppose $f \colon X \to Y$ is an (a, b)-quasi-isometric map between metric spaces X and Y. Let $B_t(y_0)$ be a ball of radius $t > 0$ centered at $y_0 \in Y$. Suppose we try to contain $f^{-1}(B_t(y_0))$ inside a ball in X. Now y_0 may not be in the image of f. Thus for a center we must choose $f(x_0) \in B_t(y_0)$ (if there is no such $x_0 \in X$ then $f^{-1}(B_t(y_0))$ is empty). Now suppose $x \in f^{-1}(B_t(y_0))$. Then because $d(f(x_0), f(x)) \leq d(f(x_0), y_0) + d(y_0, f(x)) < 2t$ we have

$$d_X(x_0, x) \leq a(d_Y(f(x_0), f(x)) + b) \leq a(2t + b).$$

We summarize this in the remark below.

Remark 4.1.4. Let $f \colon X \to Y$ be an (a, b)-quasi-isometric map between metric spaces X and Y and let $B_t \subset Y$ be a ball of radius t in Y. Then the preimage $f^{-1}(B_t) \subset X$ is contained in a ball of radius $a(2t + b)$.

The other direction is analogous but simpler, because if $B_t(x_0) \subset X$ is a ball of radius t in X then $f(x_0)$ is a natural choice for a center in Y.

Remark 4.1.5. Let $f \colon X \to Y$ be an (a, b)-quasi-isometric map between metric spaces X and Y and let $B_t \subset X$ be a ball of radius t in X. Then the image $f(B_t) \subset Y$ is contained in a ball of radius $at + b$.

Remarks 4.1.4 and 4.1.5 show that balls with radii in the ratio $2 : 1$ will in general correspond to balls with radii in a different ratio after taking a quasi-isometric image or preimage[1]. The following proposition shows how to use doubling to count

[1]The ratio approaches $2 : 1$ as the radius $r \to \infty$.

coverings where the balls involved have radii in ratios other than $2:1$. In addition, it demonstrates the polynomial growth behaviour of large scale doubling sets, where, in Proposition 4.1.6 below, ρ would be regarded as a fixed scale for measurement.

Proposition 4.1.6. *Let $U \subset X$ be (N,R)-large-scale-doubling. Let $r, \rho > 0$ be radii satisfying $r \geq \rho \geq 2R$. Then for every center $x \in X$, the intersection $B_r(x) \cap U$ can be covered by $\leq N^k$ balls B_ρ of radius ρ where $k = \log_2(2r/\rho)$.*

Proof. The intersection $B_r(x) \cap U$ can be covered by N balls of radius $r/2$, or by N^2 balls of radius $r/4$, and so on. After the nth step $B_r(x) \cap U$ is covered by N^i balls of radius $r/2^i$. This can be accomplished so long as $r/2^i \geq R$. Now there is an integer $n \geq 1$ such that

$$\frac{r}{2^n} < \rho \leq \frac{r}{2^{n-1}} \tag{4.1}$$

and since $\rho \geq 2R$ we have that $r/2^n \geq R$. Thus $B_r(x) \cap U$ can be covered by N^n balls $B_{r/2^n}$ of radius $r/2^n$ and therefore also by N^n balls B_ρ of the larger radius ρ. By (4.1) we have $n \leq \log_2(2r/\rho)$. The result follows. □

Here is a useful restatement of Proposition 4.1.6:

Remark 4.1.7. Let $U \subset X$ be (N,R)-large-scale-doubling and let $r \geq \rho \geq 2R$. Then every intersection $B_r \cap U$ contains at most N^k mutually 2ρ-separated points, where $k = \log_2(2r/\rho)$.

With Proposition 4.1.6 we can prove the following result.

Proposition 4.1.8. *Let $f \colon X \to Y$ be an (a,b)-quasi-isometric map ($a \geq 1$, $b \geq 0$), and suppose that $V \subset Y$ is (N,R)-large-scale-doubling. Then $f^{-1}(V) \subset X$ is (N^l, S)-large-scale-doubling where $l = \log_2(16a^2+4)$ and $S = \max\{2ab, a(4R+b)\}$.*

Proof. For $r \geq S$ consider the intersection $B_{2r}(x) \cap f^{-1}(V)$. The image under f of $B_{2r}(x) \cap f^{-1}(V)$ is contained in $f(B_{2r}(x)) \cap V$ which is contained in some intersection

$B_t \cap V$ where $t = 2ar + b$. Let $\rho = \frac{1}{2}\left(\frac{r}{a} - b\right)$. Now $r \geq a(4R + b)$ ensures $\rho \geq 2R$ so that $B_t \cap V$ can be covered by $\leq N^k$ balls B_ρ where

$$\begin{aligned} k &= \log_2 \frac{2t}{\rho} \\ &= \log_2 \frac{2(2ar + b)}{\frac{1}{2}\left(\frac{r}{a} - b\right)} \\ &= \log_2 \frac{4a(2ar + b)}{r - ab} \\ &\leq \log_2(16a^2 + 4) \end{aligned}$$

because $r \geq 2ab$ implies $\frac{4a(2ar+b)}{r-ab} \leq 16a^2 + 4$. Now each preimage $f^{-1}(B_\rho)$ is contained in a ball of radius r. Therefore $B_{2r}(x) \cap f^{-1}(V)$ can be covered by $\leq N^l$ balls B_r of radius r. □

4.2 Definition and basic properties of hyperbolic dimension

Definition 4.2.1. The collection $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets $A_i \subset X$ is *uniformly large-scale-doubling* if there exists $N \geq 1$ and $R > 0$ such that

- each $A \in \mathcal{A}$ is (N, R)-large-scale-doubling,
- every finite union $A_1 \cup \ldots \cup A_j$ of elements of $\mathcal{A}$ is (N, R')-large-scale-doubling (where possibly $R' > R$).

Definition 4.2.2. The *hyperbolic dimension* of the metric space X is the minimal integer $n \in \mathbb{N}$ such that for every $D > 0$ there exists an open cover $\mathcal{U}$ of X such that

- $\mathcal{U}$ is uniformly large-scale-doubling,
- $\mu(\mathcal{U}) \leq n + 1$,

- $L(\mathcal{U}) \geq D$.

We write $\operatorname{hypdim} X = n$.

Example 4.2.3. Let X be a large scale doubling metric space. Then $\operatorname{hypdim} X = 0$.

Remark 4.2.4. If $U \subset X$ is bounded then U is $(1, R)$-large-scale-doubling for any $R > \operatorname{diam}(U)$. If $\mathcal{U}$ is a uniformly bounded cover of X then $\mathcal{U}$ is uniformly $(1, R)$-large-scale-doubling for any $R > \operatorname{mesh}(\mathcal{U})$. There is therefore the basic inequality

$$\operatorname{hypdim} X \leq \operatorname{asdim} X$$

for all metric spaces X.

Example 4.2.5. Let T be a tree. Then $\operatorname{hypdim} T \leq \operatorname{asdim} T \leq 1$.

The property set for hyperbolic dimension is $P = (0, \infty)$. The filter $\mathcal{F}$ on P is the filter generated by the collection $\{(d, \infty) \mid d \in \mathbb{R}, d > 0\}$. A cover $\mathcal{U}$ of a metric space X has the property $d \in (0, \infty)$ if $\mathcal{U}$ is open and uniformly large scale doubling and has Lebesgue number $L(\mathcal{U}) \geq d$.

Proposition 4.2.6. *Hyperbolic dimension is a P-dimension; that is, the property space $(P, \mathcal{F})$ for hyperbolic dimension satisfies Axioms 1.3.4, 1.3.5, and 1.3.6.*

Proof. Axiom 1.3.4: Let $m \geq 1$ be given. By Remark 1.2.16 we may choose $0 < l_m < 1$ such that for any uniform polyhedron $|K|$ with $\dim|K| + 1 \leq m$ the cover $\mathcal{ST}_{\mathrm{ba}}(|K|)$ of $|K|$ has Lebesgue number $L(\mathcal{ST}_{\mathrm{ba}}(|K|)) \geq l_m$. Let $\lambda_m = l_m/(m+1)^2$ and define the map $\mathbf{ba}_m \colon P \to P$ by $p \mapsto \lambda_m p$. Now if $\mathcal{U}$ is a cover of a metric space X with property p and multiplicity $\mu(\mathcal{U}) \leq m$, then the cover $\mathrm{ba}\mathcal{U}$ is open, is uniformly large scale doubling since it is inscribed in $\mathcal{U}$, and has Lebesgue number

$$L(\mathrm{ba}\mathcal{U}) \geq \frac{l_m}{\mathrm{Lip}(q)} \geq \lambda_m p$$

where $q\colon X \to \mathcal{N}$ is a barycentric map for $\mathcal{U}$ with $\mathrm{Lip}(q) \leq (m+1)^2/p$. Thus $\mathrm{ba}\mathcal{U}$ has the property $\mathbf{ba}_m(p)$.

Axiom 1.3.5: If $\mathcal{U}$ is a uniformly large-scale-doubling cover of a metric space X then $\mathcal{U}|_A$ is a uniformly large-scale-doubling cover of A. Moreover $L(\mathcal{U}|_A) \geq L(\mathcal{U})$, so that if $\mathcal{U}$ has property p then $\mathcal{U}|_A$ has property p also.

Axiom 1.3.6: Let $m \geq 1$ be given. By Remark 1.2.18 we may choose $0 < c_m < 1$ such that given any uniform polyhedra $|K_1|, |K_2|$ with $\dim|K_i| + 1 \leq m$ the Lebesgue number of the cover $\mathcal{ST}_{\mathrm{ba}}(|K_1| \times |K_2|)$ of $|K_1| \times |K_2|$ is $\geq c_m$. Let $\mu_m = c_m/(m+1)^2$ and define the map $\mathbf{prod}_m\colon P \times P \to P$ by $(p_1, p_2) \mapsto \mu_m \min\{p_1, p_2\}$. Then $\mathbf{prod}_m(\mathcal{F} \times \mathcal{F}) \subset \mathcal{F}$. Let $\mathcal{U}_1$ be a cover of X with property $p_1 \in P$ and multiplicity $\mu(\mathcal{U}) \leq m$ and let $\mathcal{U}_2$ be a cover of Y with property $p_2 \in P$ and multiplicity $\mu(\mathcal{U}) \leq m$. By Lemma 1.2.14 there exist barycentric maps $q_1\colon X \to \mathcal{N}(\mathcal{U})$ and $q_2\colon Y \to \mathcal{N}(\mathcal{V})$ with $\mathrm{Lip}(f_i) \leq (m+1)^2/p_i$ for $i = 1, 2$. Then the open cover $\mathcal{U}_1 \times_{\mathrm{ba}\,\Delta} \mathcal{U}_2$ has Lebesgue number

$$L(\mathcal{U}_1 \times_{\mathrm{ba}\,\Delta} \mathcal{U}_2) \geq \frac{c_m}{\mathrm{Lip}(q_1 \times q_2)} \geq \mu_m \min\{p_1, p_2\}$$

since $\mathrm{Lip}(q_1 \times q_2) \leq (m+1)^2 \max\{1/t_1, 1/t_2\}$. Also $\mathcal{U}_1 \times_{\mathrm{ba}\,\Delta} \mathcal{U}_2$ refines the product cover $\mathcal{U}_1 \times \mathcal{U}_2$ and so is uniformly large scale doubling. Thus $\mathcal{U}_1 \times_{\mathrm{ba}\,\Delta} \mathcal{U}_2$ has the property $\mathbf{prod}_m(p_1, p_2)$. □

4.3 Quasi-isometry invariance of hyperbolic dimension

We show that hyperbolic dimension is a large scale dimension; in particular, it is a quasi-isometry invariant. Let $f\colon X \to Y$ be a quasi-isometric map between metric spaces X and Y. The following is immediate from Proposition 4.1.8.

Proposition 4.3.1. *Let the collection $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets $A_i \subset Y$ be uniformly large scale doubling. Then the collection*

$$f^{-1}(\mathcal{A}) = \{f^{-1}(A_i)\}_{i \in I}$$

of subsets $f^{-1}(A_i) \subset X$ is uniformly large scale doubling.

Lemma 4.3.2. *Let $\mathcal{V}$ be a cover of the metric space Y with Lebesgue number $L(\mathcal{V}) > D > 0$. Let $f \colon X \to Y$ be an (a, b)-quasi-isometric map (with $a \geq 1$ and $b \geq 0$). If $D > 2(a^2 + 1)b$ then the cover*

$$f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$$

of X has Lebesgue number $\geq a\left(\frac{D}{2a^2} + b\right)$.

Proof. Let $x \in X$. Then there exists $V \in \mathcal{V}$ such that $B_D(f(x)) \subset V$. Let $\lambda = 2a^2$ be a parameter. Then $f^{-1}(B_{D/\lambda}(f(x))) \subset f^{-1}(V)$. Moreover $f^{-1}(B_{D/\lambda}(f(x))) \subset B_{a\left(\frac{D}{\lambda}+b\right)}(x)$. Then

$$f(B_{a\left(\frac{D}{\lambda}+b\right)}(x)) \subset B_{a^2\left(\frac{D}{\lambda}+b\right)+b} f((x))$$

which ball is itself contained in $B_D(f(x))$ so long as

$$a^2\left(\frac{D}{\lambda} + b\right) + b < D,$$

that is, $D > 2(a^2 + 1)b$. Thus $f(B_{a\left(\frac{D}{2a^2}+b\right)}(x)) \subset V$ so that $B_{a\left(\frac{D}{2a^2}+b\right)}(x) \subset f^{-1}(V)$. The result follows. □

Proposition 4.3.3. *Suppose the metric spaces X and Y are quasi-isometric. Then* $\operatorname{hypdim} X = \operatorname{hypdim} Y$.

Proof. Let $n = \operatorname{hypdim} Y$. Let $f \colon X \to Y$ be an (a, b)-quasi-isometric map (with $a \geq 1$, $b \geq 0$). Given $d > 0$. Choose $D > 0$ large enough so that $a\left(\frac{D}{2a^2} + b\right) \geq d$.

Since $\operatorname{hypdim} Y \leq n$ there exists an open uniformly (N, R)-large-scale-doubling cover $\mathcal{V}$ of Y with Lebesgue number $L(\mathcal{V}) > D$ and with multiplicity $\mu(\mathcal{V}) \leq n+1$. By Lemma 4.3.2 the open cover $\mathcal{U} := f^{-1}(\mathcal{V})$ has Lebesgue number $L(\mathcal{U}) \geq d$. By Proposition 4.3.1 $\mathcal{U}$ is uniformly large scale doubling. Since $\mu(\mathcal{U}) \leq \mu(\mathcal{V}) \leq n+1$ we have $\operatorname{hypdim} X \leq n = \operatorname{hypdim} Y$. By symmetry $\operatorname{hypdim} Y \leq \operatorname{hypdim} X$. Thus $\operatorname{hypdim} X = \operatorname{hypdim} Y$. □

4.4 Hyperbolic dimension of δ-hyperbolic metric spaces

Buyalo and Schroeder in [BS07a] prove the following fundamental result, obtaining lower bounds on hyperbolic dimension for a large class of metric spaces.

Theorem 4.4.1. *Let the metric space X be geodesic and δ-hyperbolic with bounded growth at some scale, and suppose $\partial^g X$ is infinite. Then*

$$\operatorname{hypdim} X \geq \dim \partial^g X + 1.$$

The proof of Theorem 4.4.1 has two main ingredients. The first is the following estimate from below for certain subspaces of $\mathbb{H}^n$, proved in [BS07a].

Proposition 4.4.2. *Let $Z \subset \partial^g \mathbb{H}^n$ $(n \geq 2)$ be compact and infinite and let $X \subset \mathbb{H}^n$ be the convex hull of Z. Then*

$$\operatorname{hypdim} X \geq \dim Z + 1.$$

The second ingredient in the proof of Theorem 4.4.1 is a major theorem of Bonk and Schramm [BS00].

Theorem 4.4.3 (Bonk-Schramm Embedding Theorem)**.** *Let the metric space X be geodesic and δ-hyperbolic with bounded growth at some scale. Then for some $n \geq 2$*

there exists $Z \subset \partial^g \mathbb{H}^n$ *homeomorphic to* $\partial^g X$ *such that the convex hull* $\mathrm{Conv}(Z) \subset \mathbb{H}^n$ *is roughly similar to* X.

The metric spaces X, Y are *roughly similar* if there exists a map $f\colon X \to Y$ together with constants $k, \lambda > 0$ such that

$$\lambda d_X(x, x') - k \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + k$$

for all $x, x' \in X$ and the image $f(X) \subset Y$ is an r-net in Y for some $r > 0$. In particular roughly similar spaces are quasi-isometric. The metric space X has *bounded growth at some scale* if there exist fixed scales $0 < r < R$ and an integer $N \geq 1$ such that the open ball $B_R(x)$ of radius R can be covered by $\leq N$ open balls B_r of radius r for all centers $x \in X$.

The proof of Theorem 4.4.1 is straightforward with Proposition 4.4.2 and the Bonk-Schramm Embedding Theorem.

Proof of Theorem 4.4.1. The spaces X and $\mathrm{Conv}(Z)$ are roughly similar and therefore quasi-isometric. Thus

$$\begin{aligned} \operatorname{hypdim} X &= \operatorname{hypdim} \mathrm{Conv}(Z) \\ &\geq \dim Z + 1 \\ &= \dim \partial^g X + 1. \end{aligned}$$

□

4.5 Non-embedding results

In [BS05] Buyalo and Schroeder construct, for each $n \geq 2$, a quasi-isometric embedding

$$\mathbb{H}^n \to T^n = T \times \ldots \times T$$

of n-dimensional hyperbolic space into the product of n-copies of a certain simplicial tree T. They also show, directly, that there exists no quasi-isometric embedding $\mathbb{H}^2 \to T \times \mathbb{R}^m$ for any $m \geq 0$. Hyperbolic dimension, introduced in [BS07a] shortly after, provides the tool to prove this non-embedding fact for all $n \geq 3$: since $\operatorname{hypdim} \mathbb{H}^n = n$ whereas $\operatorname{hypdim} T^{n-1} \times \mathbb{R}^m \leq n-1$, there exists no quasi-isometric embedding $\mathbb{H}^n \to T^{n-1} \times \mathbb{R}^m$ for any $m \geq 0$. Thus the embeddings constructed in [BS05] are optimal in this sense.

In [BDS07] the authors prove, for each finitely generated word hyperbolic group Γ, the existence of quasi-isometric embeddings

$$\Gamma \to T_b^{n+1} = T_b \times \ldots \times T_b$$

where T_b is the binary tree and $n = \dim \partial_\infty \Gamma$ is the topological dimension of the boundary at infinity. By Theorem 4.4.1, when $\partial_\infty \Gamma$ is infinite we have $\operatorname{hypdim} \Gamma \geq n+1$. Thus hyperbolic dimension again provides the tool to show that the embeddings proved to exist in [BDS07] are optimal. See [Buy05] for a third example in which hyperbolic dimension demonstrates that certain constructed embeddings are optimal.

The embeddings studied in [BS07a], [BDS07], [Buy05] motivate the definition of the following quasi-isometry invariant called the ‘t’ rank.

Definition 4.5.1. For X a metric space the *‘t’ rank of* X, denoted $\text{t-rank}\, X$, is the minimal integer $k \geq 0$ such that there exists a quasi-isometric embedding

$$X \to T_1 \times \ldots \times T_k \times \mathbb{R}^m$$

for some $m \geq 0$ and metric trees $T_1, \ldots, T_k$.

There is for every metric space X the inequality

$$\operatorname{hypdim} X \leq \text{t-rank}\, X.$$

The non-embedding information contained in the hyperbolic dimension of a metric space is summarized by the next theorem, which appears in [BS07b].

Theorem 4.5.2. *Let the metric space X have* $\operatorname{hypdim} X \geq p$ *and let $T_1, \dots, T_k$ be any metric trees. Then for $p > k$ there exists no quasi-isometric embedding*

$$X \to T_1 \times \ldots \times T_k \times \mathbb{R}^m$$

for any $m \geq 0$.

We refer to a product of the form $T_1 \times \ldots \times T_k \times \mathbb{R}^m$ as *a product of trees stabilized by a Euclidean factor*. Note that Theorem 4.5.2 still holds when we stabilize by an arbitrary large scale doubling factor.

4.6 Open problems

Buyalo and Schroeder [BS07b] construct embeddings of a hyperbolic space into products of lower dimensional hyperbolic spaces.

Proposition 4.6.1. *Let $n_1, \dots, n_k \geq 2$ be given, and set $n - 1 = n_1 + \ldots + n_k - k$. Then there exists a quasi-isometric embedding*

$$\mathbb{H}^n \to \mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_k}.$$

There is, for example, a quasi-isometric embedding $\mathbb{H}^3 \to \mathbb{H}^2 \times \mathbb{H}^2$. By this embedding and by the product theorem we have $3 \leq \operatorname{hypdim} \mathbb{H}^2 \times \mathbb{H}^2 \leq 4$.

Question 4.6.2. *Is* $\operatorname{hypdim} \mathbb{H}^2 \times \mathbb{H}^2$ *equal to* 3 *or* 4*?*

Question 4.6.3. *For $n_1, \dots, n_k \geq 2$, what is the hyperbolic dimension of the product $\mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_k}$?*

Lebedeva, in [Leb07], has found examples of strict inequality in the product theorem. For each prime p Dranishnikov shows [Dra99], [Dra97] there is a hyperbolic Coxeter group Γ_p with boundary at infinity a Pontryagin surface Π_p. Lebedeva proves, for $p \neq q$, that $\operatorname{hypdim} \Gamma_p \times \Gamma_q < \operatorname{hypdim} \Gamma_p + \operatorname{hypdim} \Gamma_q$.

Chapter 5

Miscellaneous results on hyperbolic dimension

In this section we collect some new results on hyperbolic dimension, including a union theorem (Section 5.2), a proof that hypdim vanishes for countable abelian groups (Section 5.3), and a version of an extension theorem (Section 5.4). It will help to have a characterization of hyperbolic dimension in terms of d-disjoint families.

5.1 Hyperbolic dimension via disjoint families

Large scale dimensions characterized with reference to the Lebesgue number of a cover can alternatively be characterized in terms of d-disjoint families.

Definition 5.1.1. The subsets $A, B \subset X$ of a metric space X are *d-disjoint (for $d > 0$)* if $d(a, b) > d$ for all $a \in A$, $b \in B$. A collection $\{A_i\}_i$ of subsets $A_i \subset X$ of X is *d-disjoint* if each pair A_i, A_j, $i \neq j$, is d-disjoint.

This point of view is standard for large scale dimensions, but has not been spelled out for hyperbolic dimension, so we do so here. For this purpose we use hypdim_1 to refer to the hyperbolic dimension introduced in Definition 4.2.2, and we introduce the following modified versions.

Definition 5.1.2. $\operatorname{hypdim}_2 X \leq n$ if for every $D > 0$ there exists an open cover $\mathcal{U}$ of X such that

- $\mathcal{U} = \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_n$ with each family D-disjoint,
- $\mathcal{U}$ is uniformly large-scale-doubling.

Definition 5.1.3. $\operatorname{hypdim}_3 X \leq n$ if for every $D > 0$ there exists an open cover $\mathcal{U}$ of X such that

- $\mathcal{U} = \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_n$ with each family disjoint,
- $L(\mathcal{U}) \geq D$,
- $\mathcal{U}$ is uniformly large-scale-doubling.

A cover meeting the conditions of Definition 5.1.3 has multiplicity $\leq n+1$.

The following result is contained in the proof of Theorem 3.16 in [Gra06]. There Grave works in the more general setting of coarse spaces; we record the result as it applies to metric spaces.

Proposition 5.1.4. *Let X be a metric space and let $D > 0$. Suppose $\mathcal{U}$ is an open cover with multiplicity $\mu(\mathcal{U}) \leq n+1$ and Lebesgue number $L(\mathcal{U}) > (n+1)D$. For each $1 \leq i \leq n+1$ define*

$$\mathcal{U}_i = \{U_1 \cap \ldots \cap U_i \colon U_1, \ldots, U_i \in \mathcal{U} \text{ pairwise distinct}\},$$

$$\mathcal{S}_i = \cup_{U \in \mathcal{U}_i} N_{-(n+2-i)D}(U),$$

$$\mathcal{V}_i = \{N_{-(n+2-i)D}(U) - \mathcal{S}_{i+1} \colon U \in \mathcal{U}_i\},$$

and define $\mathcal{V} = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_{n+1}$. Then

- *$\mathcal{V}$ is an open cover of X and refines $\mathcal{U}$,*
- *each family $\mathcal{V}_1, \ldots, \mathcal{V}_{n+1}$ is D-disjoint.*

Below we check that a neighborhood of a doubling subset is still doubling.

Lemma 5.1.5. *Let $A \subset X$ be (N, R)-large-scale-doubling, and let $D > 0$. Then $N_D(A) \subset X$ is (N^2, S)-large-scale-doubling for $S = \max\{2R, 2D\}$.*

Proof. For $r \geq 2R$ any intersection $B_{2r} \cap A$ can be covered by N^2 balls of radius $r/2$ so that any intersection $B_{2r} \cap N_D(A)$ can be covered by N^2 balls of radius $r/2 + D$ and therefore by N^2 balls of radius r so long as $r/2 + D \leq r$, i.e., $r \geq 2D$. □

Corollary 5.1.6. *Let the collection $\mathcal{A}$ of subsets of X be uniformly (resp. weakly uniformly) (N, R)-large-scale-doubling, and let $D > 0$. Then the collection $N_D(\mathcal{A})$ is uniformly (resp. weakly uniformly) (N^2, S)-large-scale-doubling for $S = \max\{2R, 2D\}$.*

We can now show the equivalence of Definitions 4.2.2, 5.1.2, and 5.1.3.

Theorem 5.1.7. *Let X be a metric space. Then*

$$\operatorname{hypdim}_1 X = \operatorname{hypdim}_2 X = \operatorname{hypdim}_3 X.$$

Proof. We will show successively that

$$\operatorname{hypdim}_1 X \geq \operatorname{hypdim}_2 X \geq \operatorname{hypdim}_3 X \geq \operatorname{hypdim}_1 X.$$

Assume $\operatorname{hypdim}_1 X = n$. Let $D > 0$ be given. Choose an open, uniformly (N, R)-large-scale-doubling cover $\mathcal{U}$ of X with multiplicity $\mu(\mathcal{U}) = n+1$ and Lebesgue number $L(\mathcal{U}) > (n+1)D$. By Proposition 5.1.4 there is an open cover $\mathcal{V}$ refining $\mathcal{U}$ such that $\mathcal{V}$ is the union of families $\mathcal{V}_0, \ldots, \mathcal{V}_n$ each D-disjoint. Since $\mathcal{V}$ refines $\mathcal{U}$, $\mathcal{V}$ is uniformly (N, R)-large-scale-doubling, so that $\mathcal{V}$ meets the conditions of Definition 5.1.2. Thus $\operatorname{hypdim}_2 X \leq n$.

Assume $\operatorname{hypdim}_2 X = n$. Let $D > 0$ be given. Choose an open, uniformly (N, R)-large-scale-doubling cover $\mathcal{U} = \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_n$ with each family $2D$-disjoint. The open cover $N_D(\mathcal{U}) = \{N_D(U) \colon U \in \mathcal{U}\}$ then has multiplicity $\mu(N_D(\mathcal{U})) \leq n+1$ and Lebesgue number $L(N_D(\mathcal{U})) > D$, and by Corollary 5.1.6 $N_D(\mathcal{U})$ is uniformly (N^2, S)-large-scale-doubling, so that $\mathcal{U}$ meets the conditions of Definition 5.1.3. Thus $\operatorname{hypdim}_3 X \leq n$.

Assume $\operatorname{hypdim}_3 X = n$. Any cover meeting the conditions of Definition 5.1.3 also meets the conditions of Definition 4.2.2. Thus $\operatorname{hypdim}_1 X \leq n$. □

5.2 A union theorem

In [BS07b] a finite union theorem is proved for various P dimensions: asymptotic dimension, Assouad-Nagata dimension, l-dimension, and asymptotic l-dimension. Here we adapt those techniques to prove a union theorem for hyperbolic dimension. Let X be a metric space.

Definition 5.2.1. Let U be a subset of X and let $\mathcal{V}$ be a collection of subsets of X. The *saturation of* U *by* $\mathcal{V}$, denoted $U * \mathcal{V}$, is the union

$$\cup\{V \in \mathcal{V} \mid U \cap V \neq \emptyset\} \cup U.$$

Definition 5.2.2. Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of X. The *saturation of* $\mathcal{U}$ *by* $\mathcal{V}$, denoted $\mathcal{U} * \mathcal{V}$, is the collection

$$\{U * \mathcal{V} \mid U \in \mathcal{U}\}$$

of subsets of X.

The definition of Lebesgue number can be adapted to apply to an open cover of a subset $A \subset X$.

Definition 5.2.3. Let $\mathcal{U}$ be a collection of open subsets of X, covering $A \subset X$. Define

$$L(\mathcal{U}, x) = \sup \{\text{dist}(x, X - U) \mid U \in \mathcal{U}\}$$

and

$$L(\mathcal{U}) = \inf \{L(\mathcal{U}, a) \mid a \in A\}.$$

Lemma 5.2.4. *Suppose $A, B \subset X$. Let A be covered by the m-colored collection*

$$\mathcal{U} = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_m$$

of open sets, where

- *$\mathcal{U}$ is uniformly (N, R)-large-scale-doubing,*
- *$L(\mathcal{U}) \geq d_1 > 0$.*

Let B be covered by the m-colored collection

$$\mathcal{V} = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_m$$

of open sets, where

- $\text{mesh}(\mathcal{V}) \leq L(\mathcal{U})/2$,
- $L(\mathcal{V}) \geq d_2 > 0$.

Let $r = L(\mathcal{U})/2$. Define for $1 \leq i \leq m$ the family

$$\mathcal{W}_i = N_{-r}(\mathcal{U}_i) * \mathcal{V}_i \cup \{V \in \mathcal{V}_i \mid N_{-r}(U) \cap V = \emptyset \text{ for all } U \in \mathcal{U}_i\}$$

and let

$$\mathcal{W} = \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_m.$$

Then:

(0) $\mathcal{W}$ *covers* $A \cup B$*,*

(1) $\mathcal{W}$ *is* m*-colored,*

(2) $L(\mathcal{W}) \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\}$*,*

(3) $\mathcal{W}$ *is uniformly* $(N+1, \max\{R, \text{mesh}(\mathcal{V})\})$*-large-scale-doubling.*

Proof. (0) $N_{-r}(\mathcal{U})$ still covers A so $\mathcal{W}$ covers $A \cup B$.

(1) Suppose $x, y \in V$ for $V \in \mathcal{V}_i$, and that $x \in N_{-r}(U)$, $y \in N_{-r}(U')$, where $U, U' \in \mathcal{U}_i$. Then $B_r(x) \subset U$ and $B_r(y) \subset U'$. If U and U' are distinct this implies $d(x,y) \geq r$, contradicting $\text{mesh}(\mathcal{V}_i) \leq r$ (remember $\mathcal{V}_i$ consists of *open* sets). The point is, no element $V \in \mathcal{V}_i$ can intersect more than one $N_{-r}(U)$ for $U \in \mathcal{U}$. From here it follows that $\mathcal{U}_i$ disjoint, $\mathcal{V}_i$ disjoint imply $\mathcal{W}_i$ disjoint. Therefore $\mathcal{W}$ is m-colored.

(2) Observe that $L(N_{-r}(\mathcal{U})) \geq L(\mathcal{U})/2$. So

$$L(\mathcal{W}) \geq \min\{L(N_{-r}(\mathcal{U})), L(\mathcal{V})\} \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\}.$$

(3) When $N_{-r}(U) \cap V \neq \emptyset$ there is $x \in N_{-r}(U) \cap V$. Now $V \subset B_r(x)$ because V is open with $\text{diam}(V) \leq r$. Also $B_r(x) \subset U$. Thus $V \subset U$. Therefore each $N_{-r}(U) * \mathcal{V}_i \subset U$, so that each $N_{-r}(U) * \mathcal{V}_i$ is (N, R)-large-scale-doubling (inherited from U). Moreover any finite union

$$N_{-r}(U_1) * \mathcal{V}_{i(1)} \cup \ldots \cup N_{-r}(U_k) * \mathcal{V}_{i(k)}$$

is contained in the (N, R')-large-scale-doubling set $U_1 \cup \ldots \cup U_k$. Now any finite union whatsoever of elements from $\mathcal{W}$ consists of elements

$$N_{-r}(U_1) * \mathcal{V}_{i(1)} \cup \ldots \cup N_{-r}(U_k) * \mathcal{V}_{i(k)} \cup V_1 \cup \ldots \cup V_j.$$

This is $P \cup Q$, where

$$P = N_{-r}(U_1) * \mathcal{V}_{i(1)} \cup \ldots \cup N_{-r}(U_k) * \mathcal{V}_{i(k)}$$

and

$$Q = V_1 \cup \ldots \cup V_j.$$

P is (N, R')-large-scale-doubling, as argued. Q is bounded, so is $(1, R'')$-large-scale-doubling. The union $P \cup Q$ is therefore $(N+1, \max\{R', R''\})$-large-scale-doubling.

The collection $\mathcal{W}$ is therefore uniformly $(N+1, \max\{R, \text{mesh}(\mathcal{V})\})$-large-scale-doubling. $\square$

Theorem 5.2.5. *Let the metric space X be the union of $A, B \subset X$, $X = A \cup B$. Then*

$$\text{hypdim}\, X \leq \max\{\text{hypdim}\, A, \text{asdim}\, B\}.$$

Proof. Let $m = \max\{\text{hypdim}\, A, \text{asdim}\, B\}$. We may assume $m < \infty$. Given $d > 0$, there is an open $(m+1)$-colored cover $\mathcal{V}$ of B with $\text{mesh}(\mathcal{V}) < \infty$ and $L(\mathcal{V}) \geq d$. For $d' \geq 2\max\{d, \text{mesh}(\mathcal{V})\}$, there is an open $(m+1)$-colored cover $\mathcal{U}$ of A with $L(\mathcal{U}) \geq d'$ and with $\mathcal{U}$ uniformly (N, R)-large-scale-doubling.

Now $\text{mesh}(\mathcal{V}) \leq L(\mathcal{U})/2$. By Lemma 5.2.4, there is an open $(m+1)$-colored cover $\mathcal{W}$ of X with $L(\mathcal{W}) \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\} \geq d$ and with $\mathcal{W}$ uniformly large-scale-doubling. Therefore $\text{hypdim}\, X \leq m$. $\square$

For $X = A \cup B$ we therefore have the two estimates

$$\text{hypdim}\, X \leq \max\{\text{hypdim}\, A, \text{asdim}\, B\}$$
$$\text{hypdim}\, X \leq \max\{\text{asdim}\, A, \text{hypdim}\, B\}.$$

By the finite union theorem for asymptotic dimension we have

$$\text{hypdim}\, X \leq \text{asdim}\, X = \max\{\text{asdim}\, A, \text{asdim}\, B\}. \tag{5.1}$$

Thus when, for example,

$$\text{asdim}\, B \leq \text{hypdim}\, A < \text{asdim}\, A$$

Theorem 5.2.5 does strictly better. The estimate (5.1) gives only $\operatorname{hypdim} X \le \operatorname{asdim} A$ whereas Theorem 5.2.5 gives $\operatorname{hypdim} X \le \operatorname{hypdim} A$.

Question 5.2.6. *If the metric space X is the union of $A, B \subset X$, $X = A \cup B$, does the inequality*

$$\operatorname{hypdim} X \le \max\{\operatorname{hypdim} A, \operatorname{hypdim} B\}$$

hold?

5.3 Hyperbolic dimension of countable abelian groups

In this section we show that if G is a countable abelian group then $\operatorname{hypdim}(G, d) = 0$ where d is any weighted word metric on G.

5.3.1 Large scale doubling: subsets versus subspaces

We distinguish between two closely related notions of large scale doubling.

Definition 5.3.1. The metric space Y is (N, R)-large-scale-doubling if for every $y \in Y$ and every $r \ge R$, the ball $B_{2r}(y)$ can be covered by N balls of radius r.

Propositions 5.3.2 and 5.3.3 below show the relationship between the two properties

- $U \subset X$ is (N, R)-large-scale-doubling
- the space $(U, d|_U)$ is (N, R)-large-scale-doubling.

Proposition 5.3.2. *Let $U \subset X$. Suppose the metric space $(U, d|_U)$ is (N, R)-large-scale-doubling (Definition 5.3.1). Then $U \subset X$ is (N^2, R)-large-scale-doubling (Definition 4.1.1).*

Proof. For $x \in X$ and $r \geq R$, suppose $u \in B_{2r}(x) \cap U$. Now

$$B_{2r}(x) \cap U \subset B_{4r}(u) \cap U.$$

Since U is (N, R)-large-scale-doubling and $u \in U$, $B_{4r}(u) \cap U$ can be covered by N^2 balls of radius r. Thus the ball $B_{2r}(x) \cap U$ is covered by N^2 balls of radius r. □

Proposition 5.3.3. *Suppose $U \subset X$ is (N, R)-large-scale-doubling (Definition 4.1.1). Then the metric space $(U, d|_U)$ is $(N^2, 2R)$-large-scale-doubling (Definition 5.3.1).*

Proof. For $u \in U$ and $r \geq 2R$, the intersection $B_{2r}(u) \cap U$ can be covered by N^2 balls

$$B_{r/2}(x_1) \cap U, \ldots, B_{r/2}(x_{N^2}) \cap U$$

with centers in X. For each $1 \leq i \leq N^2$ choose $u_i \in B_{r/2}(x_i) \cap U$ (if the intersection is empty, discard the ball from the cover). Then $B_{r/2}(x_i) \subset B_r(u_i)$ so that the N^2 balls

$$B_r(u_1) \cap U, \ldots, B_r(u_{N^2}) \cap U$$

cover $B_{2r}(u) \cap U$. □

5.3.2 Large scale doubling with cushion and excess

We add two slight adjustments to the concept of large scale doubling for technical purposes in proving Theorem 5.3.14.

Proposition 5.3.4. *Let the metric space X be (N, R)-large-scale-doubling. Then for every $r \geq R$ and every $x \in X$, the ball $B_{4r}(x)$ can be covered by $M = N^2$ balls*

$$B_{2r}(x_1), \ldots, B_{2r}(x_M)$$

with the following property: for every $y \in B_{4r}(x)$ there exists $1 \leq i \leq M$ such that

$$y \in B_r(x_i) \subset B_{2r}(x_i).$$

Proof. For $x \in X$ and $r \geq R$ cover $B_{4r}(x)$ by N^2 balls $B_r(x_i)$, $1 \leq i \leq N^2$. Then the balls $B_{2r}(x_i)$, $1 \leq i \leq N^2$, have the desired property. □

Definition 5.3.5. The metric space X is (M, S)-*cushioned-large-scale-doubling* if for every $s \geq S$ and every $x \in X$ the ball $B_{2s}(x)$ can be covered by M balls of radius s

$$B_s(x_1), \dots, B_s(x_M)$$

with the property that for every $y \in B_{2s}(x)$ there is an index $1 \leq i \leq M$ such that

$$y \in B_{s/2}(x_i) \subset B_s(x_i).$$

Proposition 5.3.4 can be stated as follows: for a metric space (N, R)-large-scale-doubling implies $(N^2, 2R)$-cushioned-large-scale-doubling.

Definition 5.3.6. The metric space X is (N, R)-*large-scale-doubling with excess* $\lambda \in (0, 1)$ if for every $t \geq 2R$ and every $x \in X$ the ball $B_{t+\lambda t}(x)$ can be covered by N balls

$$B_{t/2}(x_1), \dots, B_{t/2}(x_N)$$

of radius $t/2$.

Proposition 5.3.7. *Let the metric space X be (N, R)-large-scale-doubling. Then for any $\lambda \in (0, 1)$, X is (N^2, S)-large-scale-doubling with excess λ, for $S = \frac{4R}{1+\lambda}$.*

Proof. For $x \in X$ and $t > 0$ the ball $B_{t+\lambda t}(x)$ can be covered by N balls of radius $\frac{t+\lambda t}{2}$, and therefore by N^2 balls of radius $\frac{t+\lambda t}{4}$, if

$$\frac{t + \lambda t}{2} \geq 2R$$

or $t \geq \frac{4R}{1+\lambda}$. Then $\lambda < 1$ implies $\frac{t+\lambda t}{4} \leq \frac{t}{2}$. This shows X is (N^2, S)-large-scale-doubling with excess λ. □

Proposition 5.3.8. *Let the metric space X be (M, S)-large-scale-doubling with excess $\lambda \in (0, 1)$. Then X is $(M^2, 2S)$-cushioned-large-scale-doubling with excess λ.*

Proof. For $x_0 \in X$ consider $B_{t+\lambda t}(x_0)$. Choose M balls of radius $t/2$ covering $B_{t+\lambda t}(x_0)$, after which we may choose M^2 balls

$$B_{t/4}(x_1), \ldots, B_{t/4}(x_{M^2})$$

covering $B_{t+\lambda t}(x_0)$ so long as $t \geq 4S$. Then the M^2 balls

$$B_{t/2}(x_1), \ldots, B_{t/2}(x_{M^2})$$

have the property that for any $y \in B_{t+\lambda t}(x_0)$ there is an index $1 \leq i \leq M^2$ such that

$$y \in B_{t/4}(x_i) \subset B_{t/2}(x_i).$$

□

5.3.3 Large scale doubling and parallel cosets

Let G be a finitely-generated (infinite) group with word metric d.

Definition 5.3.9. The coset gK of the subgroup $K \triangleleft G$ is *parallel to* K if there is a representative $\tau \in gK$ such that

$$\sup\{d(k, \tau k) \mid k \in K\} < M$$

for some $M = M(\tau) > 0$. The subgroup $K \triangleleft G$ has *parallel cosets* if each coset gK is parallel to K.

Let $gK, hK \in G/K$ be parallel to K with representatives τ_1, τ_2 as in Definition 5.3.9. Then translation by $\tau_2\tau_1^{-1}$ maps $gK \to hK$ and

$$\begin{aligned} d(gk', (\tau_2\tau_1^{-1})gk') &= d(\tau_1 k, \tau_2\tau_1^{-1}(\tau_1 k)) \\ &= d(\tau_1 k, \tau_2 k) \\ &\leq d(\tau_1 k, k) + d(k, \tau_2 k) \\ &< M(\tau_1) + M(\tau_2). \end{aligned}$$

That is, gK and hK are parallel to each other.

Proposition 5.3.10. *Let $K \lhd G$ and let $\tau \in G$ represent a parallel coset. Let $V \subset K$ and suppose the subspace $(V, d|_V)$ is (N, R)-cushioned-large-scale-doubling with excess $\lambda \in (0, 1/4)$. Then the disjoint union $V \sqcup \tau V$ viewed as a subspace with metric restricted from G is (N, S)-large-scale-doubling for $S = \max\{R, M/2\lambda\}$, where $M = M(\tau)$.*

Proof. For $h \in V$ and $r \geq S$ consider the intersection $B_{2r}(h) \cap (V \sqcup \tau V)$. Suppose $g \in B_{2r}(h) \cap \tau V$. Then

$$\begin{aligned} d(h, \tau^{-1}g) &\leq d(h, g) + d(g, \tau^{-1}g) \\ &< 2r + M \end{aligned}$$

so that $\tau^{-1}g \in B_{2r+M}(h) \cap V$. If $r \geq M/2\lambda$ then $2r + M \leq 2r + 2\lambda r$ so that $B_{2r+M}(h) \cap V$ can be covered by N balls $B_r(h_1), \ldots, B_r(h_N)$ with the property that for each $k \in B_{2r+M}(h) \cap V$ there is $1 \leq i \leq N$ such that $k \in B_{r/2}(h_i) \subset B_r(h_i)$. Now

$$\begin{aligned} d(g, h_i) &\leq d(g, \tau^{-1}g) + d(\tau^{-1}g, h_i) \\ &< M + \frac{r}{2} \\ &\leq 2\lambda r + \frac{r}{2} \\ &< r \end{aligned}$$

since $\lambda < 1/4$. Thus $g \in B_r(h_i)$ so that $B_{2r}(h) \cap (V \sqcup \tau V)$ is covered by the N balls $B_r(h_1), \ldots, B_r(h_N)$, which at first we only knew to cover $B_{2r}(h) \cap V$.

When the center h is taken in τV a similar argument shows that $B_{2r}(h) \cap (V \sqcup \tau V)$ can be covered by N balls of radius r which are at first only known to cover $B_{2r}(h) \cap \tau V$. Therefore $V \sqcup \tau V$ viewed as a subspace with metric restricted from G is (N, S)-large-scale-doubling for $S = \max\{R, M/2\lambda\}$. □

The technique of Proposition 5.3.10 carries over to any finite union of translations of a doubling set:

Proposition 5.3.11. *Let $K \triangleleft G$ and let $\tau_1, \ldots, \tau_L$ represent parallel cosets. Let $V \subset K$ and suppose the subspace $(V, d|_V)$ is (N, R)-cushioned-large-scale-doubling with excess $\lambda \in (0, 1/4)$. Then the disjoint union*

$$\tau_1 V \sqcup \ldots \sqcup \tau_L V$$

viewed as a subspace with metric restricted from G is (N, S)-large-scale-doubling for $S \geq \max\{R, C\}$ where

$$C = \max\{(M(\tau_i) + M(\tau_j))/2\lambda \mid 1 \leq i, j \leq L\}.$$

5.3.4 Hyperbolic dimension of countable abelian groups

Dranishnikov and Smith [DS06] extend the definition of asymptotic dimension to apply to all discrete groups. We adapt the techniques of that paper to show in Theorem 5.3.14 that the hyperbolic dimension of a countable abelian group is zero.

Let G be a countable group with $w\colon G \to [0, \infty)$ a weight function. Let $\|\cdot\| = \|\cdot\|_w$ and $d = d_w$ be the corresponding norm and metric on G. A subgroup $F < G$ becomes a metric space with the induced metric.

Lemma 5.3.12. *Let $F < G$ with* $\operatorname{hypdim} F \leq m$*. Suppose F has parallel cosets and that the cosets of F are mutually d-disjoint for some $d > 0$. Then there exists an open, uniformly large-scale-doubling cover $\mathcal{V}$ of G with multiplicity $\mu(\mathcal{V}) \leq m + 1$ and Lebesgue number $L(\mathcal{V}) \geq d$.*

Proof. Since $\operatorname{hypdim} F \leq m$ there exists an open, uniformly large-scale-doubling cover $\mathcal{U}$ of F with multiplicity $\mu(\mathcal{U}) \leq m + 1$ and Lebesgue number $L(\mathcal{U}) \geq d$. Since F is d-disjoint from all other cosets, $\mathcal{U}$ has Lebesgue number $L(\mathcal{U}) \geq d$ when viewed as a cover in G of the subset $F \subset G$.

Let $Z \subset G$ be a system of representatives for the cosets of F. Define

$$\mathcal{V} = \{zU \mid z \in Z, U \in \mathcal{U}\}.$$

$\mathcal{V}$ has multiplicity $\mu(\mathcal{V}) \leq m+1$ because left translation $g \mapsto zg$ is a bijection between F and zF, and all cosets are disjoint.

By Propositions 5.3.2 and 5.3.3 the collection $\mathcal{U}$ is uniformly $(N^4, 2R)$-large-scale-doubling in G. For $z \in Z$ the map

$$\begin{aligned} z\colon F &\to zF \\ k &\mapsto zk \end{aligned}$$

is an isometry. The collection $z\mathcal{U}$ covers zF and is uniformly $(N^4, 2R)$-large-scale-doubling in G, with $L(z\mathcal{U}) \geq d$.

An arbitrary finite collection of elements of $\mathcal{V}$ has the form

$$z_1U_1 \cup \ldots \cup z_jU_j$$

for $z_1, \ldots, z_j \in Z$. Since $\mathcal{U}$ is uniformly large-scale-doubling the finite union $V = U_1 \cup \ldots \cup U_j$ is (N, R')-large-scale-doubling as a subset of F. Thus V is $(N^2, 2R')$-large-scale-doubling as a metric space.

By Propositions 5.3.7 and 5.3.8 a metric space that is (M, S)-large-scale-doubling is $\left(M^4, \frac{8S}{1+\lambda}\right)$-cushioned-large-scale-doubling with excess $\lambda \in (0, 1/4)$. Therefore $V = U_1 \cup \ldots \cup U_j$ with metric induced from G is $\left(N^8, \frac{16R'}{1+\lambda}\right)$-cushioned-large-scale-doubling with excess λ.

By Proposition 5.3.11 $z_1V \cup \ldots \cup z_jV$ is (N^8, S)-large-scale-doubling as a subspace (for some $S > 0$), or (N^{16}, S)-large-scale-doubling as a subset of G. Since

$$z_1U_1 \cup \ldots \cup z_jU_j \subset z_1V \cup \ldots \cup z_jV$$

the collection $\mathcal{V}$ is uniformly $(N^{16}, 2R)$-large-scale-doubling. Finally, a ball of radius d centered at $g \in zF$ is contained in zF because cosets are d-disjoint. Therefore the Lebesgue number for $\mathcal{V}$ still satisfies $L(\mathcal{V}) \geq d$, proving the result. □

Proposition 5.3.13. *Let (G, d) be a countable group with weighted word metric. Suppose all subgroups $F < G$ have parallel cosets. Then*

$$\operatorname{hypdim}(G, d) = \sup\{\operatorname{hypdim}(F, d|_F) \mid F < G \text{ is finitely-generated}\}.$$

Proof. If $\sup\{\operatorname{hypdim} F\} = \infty$ we have $\operatorname{hypdim} G \geq \operatorname{hypdim} F$ for each F by monotonicity, so that $\operatorname{hypdim} G = \infty$, and the equality is satisfied. Thus we assume $\sup\{\operatorname{hypdim} F\} = m < \infty$.

Given $d > 0$. Define $T = \{g \in G \mid w(g) \leq d\}$. Observe that T is finite and let $F = \langle T \rangle$ be the subgroup generated by the finite set T. Then $\operatorname{hypdim} F \leq m$.

Suppose $g^{-1}h \notin F$, that is, $g^{-1}h$ cannot be written as a product of elements of T. Then $d(g, h) = \|g^{-1}h\| > d$. Thus distinct cosets $gF \neq hF$ are d-disjoint.

By Lemma 5.3.12 there is an open, uniformly large-scale-doubling cover $\mathcal{V}$ of G with multiplicity $\mu(\mathcal{V}) \leq m + 1$ and Lebesgue number $L(\mathcal{V}) \geq d$. Since $d > 0$ was arbitrary this shows $\operatorname{hypdim} G \leq m$. Since $F \subset G$ we have $\operatorname{hypdim} G = m$ by monotonicity. □

Theorem 5.3.14. *Let G be countable and abelian with weighted word metric d. Then* $\operatorname{hypdim} G = 0$.

Proof. By the structure theorem for finitely-generated abelian groups, any finitely-generated $F < G$ is quasi-isometric to $\mathbb{Z}^n$ for some n and therefore has $\operatorname{hypdim} F = 0$. Thus $\operatorname{hypdim} G = 0$. □

5.4 Hyperbolic dimension of group extensions

For finitely-generated group extensions

$$1 \to K \to G \to G/K \to 1$$

with parallel cosets there is an estimate for the hyperbolic dimension of the extension G in terms of the hyperbolic dimension of the kernel K and the *asymptotic*[1] dimension of the quotient G/K. We will need the following fact.

Lemma 5.4.1. *Let $N \triangleleft G$ be a normal subgroup of G with G finitely generated. Let d_G be the word metric induced by the finite generating set $\langle S \rangle = G$ and let $d_{G/N}$ be the word metric induced by the generating set $\pi(S)$ where $\pi\colon G \to G/N$ is the quotient projection. Denote by $B_R(gN)$ the ball of radius $R > 0$ with center $gN \in G/N$ and by $N_R(gN)$ the R-neighborhood around the subset $gN \subset G$. Then*

$$N_R(gN) = \pi^{-1}(B_R(gN)).$$

Proof. The quotient projection π is 1-Lipschitz. If $h' \in N_R(gN)$ then there exists $h \in gN$ so that $d_G(h, h') < R$. Then $d_{G/N}(\pi(h), \pi(h')) < R$. Since $\pi(h) = gN$ we have $\pi(h') \in B_R(gN)$ or $h' \in \pi^{-1}(B_R(gN))$. Thus

$$N_R(gN) \subset \pi^{-1}(B_R(gN)).$$

On the other hand if $h \in \pi^{-1}(B_R(gN))$ then $d_{G/N}(gN, hN) < R$ and

$$\|(gN)^{-1}hN\|_{G/N} = l$$

for some integer $l < R$. This means that there exist generators $s_1, \dots, s_l$ such that

$$\begin{aligned} g^{-1}hN &= s_1N \dots s_lN \\ &= s_1 \dots s_l N \\ &= Ns_1 \dots s_l \end{aligned}$$

[1]Recall that the asymptotic dimension also appeared in the union theorem for hyperbolic dimension.

since $N \lhd G$ is normal. Hence since $g^{-1}h \in g^{-1}hN = Ns_1 \ldots s_l$ there is $k \in N$ such that

$$g^{-1}h = ks_1 \ldots s_l$$
$$k^{-1}g^{-1}h = s_1 \ldots s_l$$

which means $\|(gk)^{-1}h\|_G \leq l$ so that $d_G(gk, h) < R$. Since $gk \in gN$ this means $h \in N_R(gN)$. Thus

$$\pi^{-1}(B_R(gN)) \subset N_R(gN).$$

The result follows. □

We can now prove the extension theorem for an extension with parallel cosets.

Theorem 5.4.2. *Let G be finitely-generated and let*

$$1 \to K \to G \to G/K \to 1$$

be a short exact sequence where $K \lhd G$ has parallel cosets. Then

$$\operatorname{hypdim} G \leq (\operatorname{hypdim} K + 1)(\operatorname{asdim} G/K + 1) - 1.$$

Proof. Given $d > 0$. Let $\operatorname{asdim} G/K = \kappa$. There are $\kappa + 1$ families

$$\mathcal{U}_0, \ldots, \mathcal{U}_\kappa$$

covering G/K with each family d-disjoint and uniformly bounded in diameter by some $D > 0$. Now the quotient projection $\pi \colon G \to G/K$ is 1-Lipschitz. It follows that for each $0 \leq i \leq \kappa$ the family

$$\pi^{-1}(\mathcal{U}_i) = \{\pi^{-1}(U) \mid U \in \mathcal{U}_i\}$$

of subsets of G is d-disjoint.

For $U \in \mathcal{U}_i$ consider $\pi^{-1}(U)$. Since $\mathcal{U}_i$ is uniformly D-bounded, U is contained in some ball $B_D(gK) \subset G/K$ so that $\pi^{-1}(U)$ is contained in $N_D(gK) \subset G$ by Lemma 5.4.1.

Let $\operatorname{hypdim} K = n$. Then there is a uniformly (N, R)-large-scale-doubling cover

$$\mathcal{V} = \mathcal{V}_0 \cup \ldots \cup \mathcal{V}_n$$

of K with each family $(d+2D)$-disjoint. By Propositions 5.3.2 and 5.3.3 the collection $\mathcal{V}$ is uniformly $(N^4, 2R)$-large-scale-doubling in G. Then for each $0 \leq i \leq n$ the family $N_D(\mathcal{V}_i)$ is d-disjoint, and by Corollary 5.1.6 the collection

$$N_D(\mathcal{V}) = N_D(\mathcal{V}_0) \cup \ldots \cup N_D(\mathcal{V}_n)$$

is uniformly (N^8, S)-large-scale-doubling in G for some $S = S(2R, D)$.

For each coset gK choose a representative $\tau \in gK$ and $M = M(\tau) > 0$ as in Definition 5.3.9. Define $\tau\mathcal{V}_i = \{\tau V \mid V \in \mathcal{V}_i\}$ and $\tau\mathcal{V} = \tau\mathcal{V}_0 \cup \ldots \cup \tau\mathcal{V}_n$. Since left translation $k \mapsto \tau k$ is an isometry mapping $K \to \tau K$, the collection $\tau\mathcal{V}$ is a uniformly (N, R)-large-scale-doubling cover of τK, and each family $\tau\mathcal{V}_i$ is $(d+2D)$-disjoint. Then (just as was argued above for the collection $N_D(\mathcal{V})$) the collection $N_D(\tau\mathcal{V})$ is uniformly (N^8, S)-large-scale-doubling in G.

We have already observed that each $\pi^{-1}(U)$ is contained in the D-neighborhood of some coset τK where $\tau = \tau_U$ depends on U. For each $U \in \mathcal{U}$, contained in $N_D(\tau K)$, define

$$N_D(\tau\mathcal{V})|_{\pi^{-1}(U)} = \{N_D(\tau V) \cap \pi^{-1}(U) \mid V \in \mathcal{V}\}.$$

Furthermore for each $0 \leq i \leq \kappa$ define

$$\mathcal{W}_i = \cup_{U \in \mathcal{U}_i} N_D(\tau_U \mathcal{V})|_{\pi^{-1}(U)}.$$

Observe that each family $\mathcal{W}_i$ consists of families

$$\mathcal{W}_{i,j} = \cup_{U \in \mathcal{U}_i} N_D(\tau_U \mathcal{V}_j)|_{\pi^{-1}(U)}$$

each d-disjoint. Now define the collection $\mathcal{W} = \mathcal{W}_0 \cup \ldots \cup \mathcal{W}_\kappa$.

We will show that $\mathcal{W}$ is uniformly large-scale-doubling. Indeed, an arbitrary element of $\mathcal{W}$ is of the form

$$N_D(\tau_U V) \cap \pi^{-1}(U)$$

for $V \in \mathcal{V}$, $U \in \mathcal{U}$, and $\tau = \tau_U$, and so is (N^8, S)-large-scale-doubling in G.

Now, an arbitrary finite collection of elements of $\mathcal{W}$ has the form

$$N_D(\tau_{U_1} V_1) \cap \pi^{-1}(U_1), \ldots, N_D(\tau_{U_j} V_j) \cap \pi^{-1}(U_j)$$

for $V_i \in \mathcal{V}$, $U_i \in \mathcal{U}$, and $\tau_i = \tau_{U_i}$, for $1 \leq i \leq j$. Let $V = V_1 \cup \ldots \cup V_j$. Since $\mathcal{V}$ is a uniformly (N, R)-large-scale-doubling cover of K the finite union V is (N, R')-large-scale-doubling as a subset of K, and thus V is $(N^2, 2R')$-large-scale-doubling as a metric space. Recall that by Propositions 5.3.7 and 5.3.8, a metric space that is $(\widetilde{N}, \widetilde{R})$-large-scale-doubling is $(\widetilde{N}^4, 8\widetilde{R}/(1+\lambda))$-cushioned-large-scale-doubling with excess λ. Therefore the subspace V with metric induced from G is $(N^8, 16R'/(1+\lambda))$-cushioned-large-scale-doubling with excess λ.

By Proposition 5.3.11 the union $\tau_1 V \cup \ldots \cup \tau_j V$ is, for some $S > 0$, an (N^8, S)-large-scale-doubling subspace with metric restricted from G. Thus the union $\tau_1 V \cup \ldots \cup \tau_j V$ is an (N^{16}, S)-large-scale-doubling subset of G, whereupon the D-neighbourhood $N_D(\tau_1 V \cup \ldots \cup \tau_j V)$ is (N^{32}, T)-large-scale-doubling for some $T > 0$. Moreover, the finite union

$$N_D(\tau_{U_1} V_1) \cap \pi^{-1}(U_1) \cup \ldots \cup N_D(\tau_{U_j} V_j) \cap \pi^{-1}(U_j)$$

of elements from $\mathcal{W}$ is contained in the (N^{32}, T)-large-scale-doubling subset $N_D(\tau_1 V \cup \ldots \cup \tau_j V)$. Therefore $\mathcal{W} = \{\mathcal{W}_{i,j}\}$ is a uniformly N^{32}-large-scale-doubling cover of G, with each family $\mathcal{W}_{i,j}$ d-disjoint for $0 \leq i \leq \kappa$ and $0 \leq j \leq n$. The result follows. □

Corollary 5.4.3. *Let G be finitely-generated and let*

$$1 \to K \to G \to G/K \to 1$$

be a short exact sequence where $K < Z(G)$*. Then*

$$\operatorname{hypdim} G \leq \operatorname{asdim} G/K.$$

Proof. Since $K < Z(G)$, K has parallel cosets. Moreover K is abelian and countable. By Theorem 5.3.14, $\operatorname{hypdim} K = 0$. The estimate of Theorem 5.4.2 then gives the result. □

Chapter 6

Weak hyperbolic dimension

This chapter introduces *weak hyperbolic dimension*, a new quasi-isometry invariant of metric spaces. It is closely related to hyperbolic dimension, satisfying the basic inequaliy $\operatorname{wdim} X \leq \operatorname{hypdim} X$, and is obtained by modifying the definition of hyperbolic definition to remove the finite union condition. This modification makes weak dimension ineffective for the particular nonembedding applications for which Buyalo and Schroeder introduced hyperbolic dimension, but the advantage is that removing the finite union condition makes weak dimension easier to compute.

6.1 Definition and basic properties of weak hyperbolic dimension

Definition 6.1.1. The collection $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets $A_i \subset X$ is *weakly uniformly large scale doubling* if there exists $N \geq 1$ and $R > 0$ such that each $A \in \mathcal{A}$ is (N, R)-large-scale-doubling.

We are now ready to define the weak hyperbolic dimension.

Definition 6.1.2. The *weak hyperbolic dimension* of the metric space X is the minimal integer $n \in \mathbb{N}$ such that for every $D > 0$ there exists an open cover $\mathcal{U}$ of X such that

- $\mathcal{U}$ is a weakly uniform large-scale-doubling cover,
- $\mu(\mathcal{U}) \leq n + 1$,
- $L(\mathcal{U}) \geq D$.

We write $\operatorname{wdim} X = n$.

The property set for the weak dimension is $P = (0, \infty)$. The filter $\mathcal{F}$ on P is the filter generated by the collection $\{(d, \infty) \mid d \in \mathbb{R}, d > 0\}$. A cover $\mathcal{U}$ of a metric space X has the property $d \in (0, \infty)$ if $\mathcal{U}$ is open and weakly uniformly large scale doubling and has Lebesgue number $L(\mathcal{U}) \geq d$.

Proposition 6.1.3. *Weak hyperbolic dimension is a P-dimension; that is, the property space $(P, \mathcal{F})$ for the weak dimension satisfies Axioms 1.3.4, 1.3.5, and 1.3.6.*

Proof. The proof that hyperbolic dimension is a P-dimension carries over line for line with the notion of a weakly uniform large-scale-doubling cover replacing that of a uniform large-scale-doubling cover. □

Similarly, some other results for hyperbolic dimension carry over immediately to the weak hyperbolic dimension. In particular, we can define weak dimension in terms of d-disjoint families.

Definition 6.1.4. For a metric space X $\operatorname{wdim} X \leq n$ if for every $D > 0$ there exists an open cover $\mathcal{U}$ of X such that

- $\mathcal{U} = \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_n$ with each family D-disjoint,
- $\mathcal{U}$ is weakly uniformly large-scale-doubling.

From the d-disjoint family characterization of weak dimension, it is straightforward to prove quasi-isometry invariance, although the alternative method used here for hyperbolic dimension also carries over line for line.

Proposition 6.1.5. *Suppose the metric spaces X and Y are quasi-isometric. Then* $\operatorname{wdim} X = \operatorname{wdim} Y$.

The union theorem carries over as well.

Theorem 6.1.6. *Let the metric space X be the union of $A, B \subset X$, $X = A \cup B$. Then*

$$\operatorname{wdim} X \leq \max \{\operatorname{wdim} A, \operatorname{asdim} B\}.$$

As for hyperbolic dimension, the union theorem for weak dimension is a hybrid theorem in which the asymptotic dimension shows up. This leads to the question:

Question 6.1.7. *Let the metric space X be the union of $A, B \subset X$, $X = A \cup B$. Is there the inequality*

$$\operatorname{wdim} X \leq \max \{\operatorname{wdim} A, \operatorname{wdim} B\}?$$

6.2 Weak hyperbolic dimension of δ-hyperbolic metric spaces

A key computational tool for weak dimension in hyperbolic geodesic spaces is a lower bound based on the covering dimension of the geodesic boundary. The proof uses a fundamental property of large-scale-doubling subsets of CAT(-1)-spaces, proved in [BS07b] and recorded below. The property is likewise used in the proof of Proposition 4.4.2 of Buyalo and Schroeder which motivates Proposition 6.2.2.

Lemma 6.2.1 (Buyalo and Schroeder, [BS07b]). *Let X be* CAT(-1) *with basepoint $o \in X$. Given $N \in \mathbb{N}$ and $R > 1$. Then for every sufficiently large $\delta > 0$ there exists a positive constant C depending only on N and δ such that for all (N, R)-large-scale-doubling subsets $A \subset X$ satisfying* $\operatorname{dist}(o, A) \geq c > \delta$ *we have*

$$\angle_\delta A \leq C \cdot e^{-c/2}. \tag{6.1}$$

For the definition of the *angle δ-measure* in (6.1) see [BS07b, page 162]. The property of angle δ-measure needed for the proof below is that if $V \subset \mathbb{H}^n$ is connected then $\operatorname{diam}_\angle V \leq \angle_\delta V$, where the angle diameter $\operatorname{diam}_\angle V$ is the diameter of the shadow of V in the boundary $S^{n-1} = \partial^g \mathbb{H}^n$.

Proposition 6.2.2. *Let $Z \subset \partial^g \mathbb{H}^n$ ($n \geq 2$, $|Z| \geq 2$) and let $X = \operatorname{Conv}(Z) \subset \mathbb{H}^n$ be the convex hull of Z. Then*

$$\operatorname{wdim} X \geq \dim Z.$$

Proof. Fix a base point $o \in X$. Suppose for a contradiction that

$$\operatorname{wdim} X < \dim Z = k \in \mathbb{N}.$$

Then there exists an open cover $\mathcal{U}$ of X with each $U \in \mathcal{U}$ (N, R)-large-scale-doubling ($N \geq 1, R > 1$) and with $\mu(\mathcal{U}) \leq k$.

For $r > 0$ we use a construction called the *sphere of radius r over Z* given by

$$\operatorname{Sph}_r(Z) := \{\gamma(r) \mid \gamma \in Z \subset \partial^g_o \mathbb{H}^n\}.$$

Now the space $X = \operatorname{Conv}(Z)$ is geodesic and indeed CAT(-1). Hence we may fix $\delta > 0$ sufficiently large so that there exists $C = C(N, \delta)$ such that $U \subset X$ (N, R)-large-scale-doubling and $\operatorname{dist}(o, U) \geq c > \delta$ imply

$$\angle_\delta U \leq C e^{-c/2}.$$

Given $\tau > 0$. We will construct an open cover of $\mathrm{Sph}_1(Z)$ with mesh $\leq \tau$ and multiplicity $\leq k$. Choose $r > \delta$ so that $Ce^{-r/2} \leq \tau$.

For each $U \in \mathcal{U}$ the set $U - B_r(o)$ is (N, R)-large-scale-doubling (since it is contained in U) and satisfies $\mathrm{dist}(o, U - B_r(o)) \geq r > \delta$. Therefore

$$\angle_\delta(U - B_r(o)) \leq Ce^{-r/2} \leq \tau.$$

For $U \in \mathcal{U}$ let V be a connected component in X of $U - B_r(o)$. Then because V is connected

$$\mathrm{diam}_\angle V \leq \angle_\delta V \leq \angle_\delta(U - B_r(o)) \leq \tau.$$

Moreover

$$\mathrm{diam}_\angle(V \cap \mathrm{Sph}_r(Z)) \leq \mathrm{diam}_\angle V \leq \tau.$$

For each $U \in \mathcal{U}$ let $\{V_i\}_{i \in I}$ be the set of connected components in X of $U - B_r(o)$ and let

$$\mathcal{A}(U) := \{V_i \cap \mathrm{Sph}_r(Z)\}_{i \in I}$$

be the set of connected components of $U - B_r(o)$ restricted to $\mathrm{Sph}_r(Z)$. Then

$$\mathcal{V}_r := \cup_{U \in \mathcal{U}} \mathcal{A}(U)$$

is an open cover of $\mathrm{Sph}_r(Z)$ with multiplicity $\mu(\mathcal{V}_r) \leq \mu(\mathcal{U}) \leq k$ and such that for each $V \in \mathcal{V}_r$, $\mathrm{diam}_\angle V \leq \tau$. Radial contraction $\phi_r \colon (\mathrm{Sph}_r(Z), d_\angle) \to (\mathrm{Sph}_1(Z), d_\angle)$ is an isometry so that

$$\phi_r(\mathcal{V}_r) := \{\phi_r(V) \mid V \in \mathcal{V}_r\}$$

is an open cover of $\mathrm{Sph}_1(Z)$ with multiplicity $\mu(\phi_r(\mathcal{V}_r)) = \mu(\mathcal{V}_r) \leq k$ and with $\mathrm{diam}_\angle W \leq \tau$ for each $W \in \phi_r(\mathcal{V}_r)$. Since $\tau > 0$ was arbitrary this shows $\dim \mathrm{Sph}_1(Z) < k$. Now $(\mathrm{Sph}_1(Z), d_\angle)$ and $(Z, d_\angle)$ are isometric, so that $\dim \mathrm{Sph}_1(Z) = \dim Z$. This contradicts that $\dim Z = k$. Therefore $\mathrm{wdim}\, X \geq \dim Z$. □

Theorem 6.2.3. *Let the metric space X be geodesic and δ-hyperbolic with bounded growth at some scale. Then*

$$\operatorname{wdim} X \geq \dim \partial^g X.$$

Proof. If $\partial^g X$ is finite the result is immediate. Otherwise by the proof of the Bonk-Schramm embedding theorem there is an integer $n \geq 2$ such that $\partial^g X$ is homeomorphic to a set $Z \subset \partial^g \mathbb{H}^n$ and X is roughly similar to the convex hull $\operatorname{Conv}(Z)$ of Z. Recall that rough similarity implies quasi-isometry. Then

$$\begin{aligned} \operatorname{wdim} X &= \operatorname{wdim} \operatorname{Conv}(Z) \\ &\geq \dim Z \\ &= \dim \partial^g X. \end{aligned}$$

□

Corollary 6.2.4. *Let X be an n-dimensional Hadamard manifold with sectional curvatures $K \leq -1$ and with bounded growth at some scale. Then*

$$\operatorname{wdim} X \geq n - 1.$$

Proof. X is geodesic and δ-hyperbolic with bounded growth so that $\operatorname{wdim} X \geq \dim \partial^g X$. But $\partial^g X \cong S^{n-1}$. The result follows. □

In particular

$$\operatorname{wdim} \mathbb{H}^n \geq n - 1$$

for $n \geq 2$.

6.3 A Hurewicz-type extension theorem

The finite union condition makes it unclear how to define a notion of uniform hyperbolic dimension for a family of metric spaces. For weak dimension, however, we can make the following definition.

Definition 6.3.1. The family of metric spaces $\{X_i\}_{i\in I}$ has wdim $\leq n$ *uniformly over* I if for every $d > 0$ there exists $N \in \mathbb{N}$ and $R > 0$ such that each X_i can be covered by $(n+1)$ open d-disjoint families

$$\mathcal{V}^i = \mathcal{V}_0^i \cup \mathcal{V}_1^i \cup \ldots \cup \mathcal{V}_n^i$$

such that each $V \in \mathcal{V}^i$ is an (N, R)-large-scale-doubling subset of X_i.

The following proposition is a Hurewicz-type mapping result.

Proposition 6.3.2. *Let* $\pi\colon X \to Y$ *be a Lipschitz map between metric spaces such that for every* $R > 0$ *the family*

$$\{\pi^{-1}(B_R(y))\}_{y\in Y}$$

of subspaces of X *has* wdim $\leq n$ *uniformly over* Y*. Then*

$$\operatorname{wdim} X \leq (n+1)(\operatorname{asdim} Y + 1) - 1.$$

Proof. Given $d > 0$. Let $\pi\colon X \to Y$ be λ-Lipschitz and let $\operatorname{asdim} Y = \kappa$. Choose an open uniformly bounded cover

$$\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_\kappa$$

of Y such that each family $\mathcal{V}_j$ is (λd)-disjoint. Let $\operatorname{mesh} \mathcal{V} < R$. Then each $V \in \mathcal{V}$ is contained in a ball $B_R(y)$ for some $y \in Y$. Therefore for each $0 \leq j \leq \kappa$

$$\pi^{-1}(\mathcal{V}_j) = \{\pi^{-1}(V)\}_{V\in\mathcal{V}_j}$$

is a d-disjoint family of open subspaces of X having wdim $\leq n$ uniformly. Indeed the union

$$\pi^{-1}(\mathcal{V}) = \pi^{-1}(\mathcal{V}_0) \cup \pi^{-1}(\mathcal{V}_1) \cup \ldots \cup \pi^{-1}(\mathcal{V}_\kappa)$$

is a family of open subspaces of X having wdim $\leq n$ uniformly.

Hence there is $N \in \mathbb{N}$ and $R > 0$ such that for each $V \in \mathcal{V}$ the subspace $\pi^{-1}(V)$ can be covered by $(n+1)$ d-disjoint families

$$\mathcal{U}^V = \mathcal{U}_0^V \cup \mathcal{U}_1^V \cup \ldots \cup \mathcal{U}_n^V$$

where each $U \in \mathcal{U}^V$ is an open and (N, R)-large-scale-doubling subset $U \subset \pi^{-1}(V)$. Observe that such a U is $(N^2, 2R)$-large-scale-doubling as a subspace and therefore as a subset $U \subset X$ it is $(N^4, 2R)$-large-scale-doubling. Then for fixed index $0 \leq j \leq \kappa$ the $(n+1)$ families

$$\mathcal{U}_{ij} := \cup\{\mathcal{U}_i^V \mid V \in \mathcal{V}_j\}$$

indexed by $0 \leq i \leq n$ are d-disjoint and consist of open elements each $(N^4, 2R)$-large-scale-doubling in X. Moreover the families $\mathcal{U}_{0j}, \mathcal{U}_{1j}, \ldots, \mathcal{U}_{nj}$ cover $\pi^{-1}(\mathcal{V}_j)$.

In this way $X = \cup_j \pi^{-1}(\mathcal{V}_j)$ can be covered by $(n+1)(\kappa+1)$ open d-disjoint families each consisting of $(N^4, 2R)$-large-scale-doubling members. Therefore

$$\operatorname{wdim} X \leq (n+1)(\operatorname{asdim} Y + 1) - 1.$$

□

Theorem 6.3.3. *Let G be a finitely generated group with word metric d_G induced by the finite generating set $\langle S \rangle = G$. Let*

$$1 \to N \to G \to G/N \to 1$$

be an exact sequence of groups where $N \triangleleft G$ *has the subspace metric and* G/N *has the word metric* $d_{G/N}$ *induced by the generating set* $\pi(S)$ *where* $\pi\colon G \to G/N$ *is the quotient projection. Then*

$$\operatorname{wdim} G \leq (\operatorname{wdim} N + 1)(\operatorname{asdim} G/N + 1) - 1.$$

Proof. Let $R > 0$ and $gN \in G/N$ be given. Now $N_R(gN) = \pi^{-1}(B_R(gN))$. Let $\operatorname{wdim} N = n$. Consider $N_R(N)$. Observe that the inclusion $N \to N_R(N)$ is $(1,0)$-quasi-isometric and the image is an R-net in $N_R(N)$. Therefore N and $N_R(N)$ are quasi-isometric so that $\operatorname{wdim} N_R(N) = n$. Given $d > 0$ there exist $M \in \mathbb{N}$ and $S > 0$ and a cover of $N_R(N)$ by $(n+1)$ d-disjoint families

$$\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n$$

with each $U \in \mathcal{U}$ open and (M,S)-large-scale-doubling as a subset of $N_R(N)$. Left translation by g isometrically maps $N_R(N) \to gN_R(N) = N_R(gN)$. Thus the family

$$g\mathcal{U} = g\mathcal{U}_0 \cup g\mathcal{U}_1 \cup \ldots \cup g\mathcal{U}_n$$

is a cover of $N_R(gN)$ by $(n+1)$ d-disjoint families with each $U \in g\mathcal{U}$ open and (M,S)-large-scale-doubling as a subset of $N_R(gN)$. Therefore the family

$$\{\pi^{-1}(B_R(gN))\}_{gN \in G/N}$$

has $\operatorname{wdim} \leq n$ uniformly over G/N. By Proposition 6.3.2

$$\operatorname{wdim} G \leq (\operatorname{wdim} N + 1)(\operatorname{asdim} G/N + 1) - 1.$$

□

6.4 Computation of weak hyperbolic dimension of various metric spaces

6.4.1 Sol geometry, Baumslag-Solitar groups, and a wreath product

We now discuss examples of group extensions for which the weak hyperbolic dimension can be computed using Theorem 6.3.3.

The Lie group Sol is obtained from the underlying manifold $\mathbb{R}^3$ with group product given by

$$(a, b, c) \cdot (x, y, z) = (a + e^c x, b + e^{-c} y, c + z).$$

The Riemannian metric

$$ds^2 = e^{-2z} dx^2 + e^{2z} dy^2 + dz^2$$

is left invariant. Recall that in the logarithmic model of hyperbolic space we model $\mathbb{H}^2$ by the xz-plane $\mathbb{R}^2$ with Riemannian metric $e^{-2z} dx^2 + dz^2$. Now the projections

$$\pi_1 \colon \text{Sol} \to \mathbb{H}^2, (x, y, z) \mapsto (x, z)$$
$$\pi_2 \colon \text{Sol} \to \mathbb{H}^2, (x, y, z) \mapsto (y, -z)$$

preserve distances in the sense that

$$d_{\text{Sol}}((x, y_1, z_1), (x, y_2, z_2)) = d_{\mathbb{H}^2}((y_1, -z_1), (y_2, -z_2))$$
$$d_{\text{Sol}}((x_1, y, z_1), (x_2, y, z_2)) = d_{\mathbb{H}^2}((x_1, z_1), (x_2, z_2))$$

(see [BSW12]). Thus Sol contains many isometrically embedded, totally geodesic copies of the hyperbolic plane.

Proposition 6.4.1. *We have* $\operatorname{wdim} \text{Sol} = 1$.

Proof. Since there are isometric copies of $\mathbb{H}^2$ in Sol, we have $\operatorname{wdim} \mathrm{Sol} \geq \operatorname{wdim} \mathbb{H}^2 \geq 1$ because $\mathbb{H}^2$ is not doubling.

For an estimate from above, we use the group extension

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \rtimes_\phi \mathbb{Z} \to \mathbb{Z} \to 0$$

where the homomorphism $\phi \colon \mathbb{Z} \to \operatorname{Aut}(\mathbb{Z}^2)$ is given by

$$\phi(n) \colon \begin{pmatrix} i \\ j \end{pmatrix} \mapsto \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{pmatrix} i \\ j \end{pmatrix}.$$

The finitely generated group $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ is a lattice in Sol, so that, up to quasi-isometry, $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ is a discrete model for Sol. By the extension theorem

$$\begin{aligned} \operatorname{wdim} \mathrm{Sol} &= \operatorname{wdim} \mathbb{Z}^2 \rtimes_\phi \mathbb{Z} \\ &\leq (\operatorname{wdim} \mathbb{Z}^2 + 1)(\operatorname{asdim} \mathbb{Z} + 1) - 1 \\ &= 1 \end{aligned}$$

and the result follows. □

Corollary 6.4.2. *We have* $\operatorname{wdim} \mathbb{H}^2 = 1$.

For $n \geq 2$, let $\mathrm{BS}(1, n)$ denote the *solvable Baumslag-Solitar group* given by the presentation

$$\left\langle a, b \mid aba^{-1} = b^n \right\rangle.$$

The space $\mathrm{BS}(1, n)$ with the word metric contains many quasi-isometrically embedded copies of the hyperbolic plane; see [FM98]. Thus $\operatorname{hypdim} \mathrm{BS}(1, n) \geq 2$ and $\operatorname{wdim} \mathrm{BS}(1, n) \geq 1$ ($\mathrm{BS}(1, n)$, having exponential growth, is not doubling, which also gives the estimate $\operatorname{wdim} \mathrm{BS}(1, n) \geq 1$).

Proposition 6.4.3. *We have* $\operatorname{wdim} \mathrm{BS}(1, n) = 1$.

Proof. We use the group extension

$$0 \to \mathbb{Z}\left[\frac{1}{n}\right] \to \mathrm{BS}(1,n) \to \mathbb{Z} \to 0$$

where the kernel $\mathbb{Z}\left[\frac{1}{n}\right]$ is countable abelian, so that $\operatorname{wdim}\mathbb{Z}\left[\frac{1}{n}\right] \leq \operatorname{hypdim}\mathbb{Z}\left[\frac{1}{n}\right] = 0$ by Theorem 5.3.14. By Theorem 6.3.3 we have

$$\operatorname{wdim}\mathrm{BS}(1,n) \leq (\operatorname{wdim}\mathbb{Z}\left[\frac{1}{n}\right] + 1)(\operatorname{asdim}\mathbb{Z} + 1) - 1 = 1.$$

We have already commented that since $\mathrm{BS}(1,n)$ contains quasi-isometric copies of $\mathbb{H}^2$, $\operatorname{wdim}\mathrm{BS}(1,n) \geq 1$. Therefore $\operatorname{wdim}\mathrm{BS}(1,n) = 1$. □

Our last example of a weak dimension computed using the extension theorem is the wreath product of $\mathbb{Z}$ with itself. This group is constructed as follows. Let H be the set of finitely supported functions mapping $\mathbb{Z} \to \mathbb{Z}$. Define $u \colon H \to H$ by

$$u(f)(n) = f(n) + \delta_{n0},$$

i.e., u changes f by increasing its value on 0 by exactly 1; note that u is a permutation of H. Likewise $v \colon H \to H$ defined by

$$v(f)(n) = f(n+1)$$

(which shifts f by one) is a permutation of H. Together u and v generate the (restricted) *wreath product of* $\mathbb{Z}$ *by* $\mathbb{Z}$.

Proposition 6.4.4. *We have* $\operatorname{wdim}\mathbb{Z}\wr\mathbb{Z} = 1$.

Proof. We use the group extension

$$0 \to \oplus_\infty\mathbb{Z} \to \mathbb{Z}\wr\mathbb{Z} \to \mathbb{Z} \to 0$$

and the fact that $\operatorname{wdim}\oplus_\infty\mathbb{Z} = 0$ since $\oplus_\infty\mathbb{Z}$ is countable abelian. Then

$$\operatorname{wdim}\mathbb{Z}\wr\mathbb{Z} \leq (\operatorname{wdim}\oplus_\infty\mathbb{Z} + 1)(\operatorname{asdim}\mathbb{Z} + 1) - 1 = 1.$$

Since $\mathbb{Z}\wr\mathbb{Z}$ is not doubling (it has exponential growth), $\operatorname{wdim}\mathbb{Z}\wr\mathbb{Z} = 1$. □

The wreath product, containing a subgroup isomorphic to $\mathbb{Z}^n$ for each $n \geq 1$, has $\operatorname{asdim} \mathbb{Z} \wr \mathbb{Z} = \infty$ but finite wdim.

6.4.2 Hyperbolic space

Having determined that $\operatorname{wdim} \mathbb{H}^2 = 1$ we are ready to compute $\operatorname{wdim} \mathbb{H}^n$ for $n \geq 3$. Recall Proposition 4.6.1:

Proposition. *Let $n_1, \ldots, n_k \geq 2$ be given, and set $n - 1 = n_1 + \ldots + n_k - k$. Then there exists a quasi-isometric embedding*

$$\mathbb{H}^n \to \mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_k}.$$

This class of embeddings proves very useful in computing the weak dimension. Below, we use the embeddings with each $n_i = 2$ to compute the weak dimension of n-dimensional hyperbolic space for $n \geq 3$.

Theorem 6.4.5. *For $n \geq 2$,* $\operatorname{wdim} \mathbb{H}^n = n - 1$.

Proof. For $n = 2$ this is Corollary 6.4.2. Let $n \geq 3$. By Proposition 4.6.1 there exists a quasi-isometric embedding

$$\mathbb{H}^n \to \times_{i=1}^{k} \mathbb{H}_i^2$$

where k satisfies $n - 1 = 2k - k$, i.e., $k = n - 1$. By the monotonicity and product theorems we have $\operatorname{wdim} \mathbb{H}^n \leq \sum_{i=1}^{n-1} \operatorname{wdim} \mathbb{H}^2 = n - 1$. By Theorem 6.2.3 we have $\operatorname{wdim} \mathbb{H}^n \geq n - 1$. Therefore $\operatorname{wdim} \mathbb{H}^n = n - 1$ for all $n \geq 2$. □

6.4.3 Products with tree and hyperbolic factors

Now we can use the embeddings of Proposition 4.6.1 to compute the weak dimension of any product of hyperbolic spaces.

Theorem 6.4.6. *For integers $n_1, \ldots, n_k \geq 2$,*

$$\operatorname{wdim} \mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_k} = n_1 + \ldots + n_k - k.$$

Proof. Write $Y = \mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_k}$. By the product theorem and the fact that $\operatorname{wdim} \mathbb{H}^i = i - 1$ we have

$$\operatorname{wdim} Y \leq n_1 - 1 + \ldots + n_k - 1 = n_1 + \ldots + n_k - k.$$

On the other hand for $n \geq 2$ such that $n - 1 = n_1 + \ldots + n_k - k$ there is by Proposition 4.6.1 a quasi-isometric embedding $\mathbb{H}^n \to Y$ so that $\operatorname{wdim} Y \geq \operatorname{wdim} \mathbb{H}^n = n - 1$. The result follows. □

In [BDS07] it is proved that for every finitely-generated word hyperbolic group Γ there exists a quasi-isometric embedding into the product of $n + 1$ binary trees, where $n = \dim \partial_\infty \Gamma$. This lets us compute the weak hyperbolic dimension of a product of binary trees.

Theorem 6.4.7. *Let $X = T_1 \times \ldots \times T_k$ be the k-fold product of binary trees T_i ($1 \leq i \leq k$). Then $\operatorname{wdim} X = k$.*

Proof. For $k = 1$ the result is immediate since T not doubling implies $\operatorname{wdim} T > 0$. Thus $\operatorname{wdim} T = 1$.

Assume $k \geq 2$. By [BDS07] there exists a quasi-isometric embedding $\mathbb{H}^k \to X$. Together with the product theorem (using $\operatorname{wdim} T_i = 1$), this gives $k - 1 \leq \operatorname{wdim} X \leq k$.

Suppose for a contradiction that $\operatorname{wdim} X = k - 1$. Now $X \times X$ is again a product of trees with $k' = 2k$, so that by the above argument $2k - 1 \leq \operatorname{wdim} X \times X \leq 2k$. But now if $\operatorname{wdim} X = k - 1$ then by the product theorem

$$2k - 1 \leq \operatorname{wdim} X \times X \leq (k - 1) + (k - 1) = 2k - 2$$

gives a contradiction. Therefore $\operatorname{wdim} X = k$. □

The same technique allows us to compute the weak dimension for a product of trees and hyperbolic spaces.

Theorem 6.4.8. *For integers $n_1, \ldots, n_k \geq 2$ and $l \geq 1$ let*

$$X = \mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_k} \times T_1 \times \ldots \times T_l.$$

Then $\operatorname{wdim} X = n_1 + \ldots + n_k - k + l$.

Proof. Let $Y = \mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_k}$ and let $n-1 = n_1 + \ldots + n_k - k$. Then $\operatorname{wdim} Y = n-1$. Then

$$\begin{aligned} \operatorname{wdim} X &\leq \operatorname{wdim} Y + \operatorname{wdim} T_1 \times \ldots \times T_l \\ &= n - 1 + l. \end{aligned}$$

When $l \geq 2$ we use the fact that $\mathbb{H}^l$ embeds into $T_1 \times \ldots \times T_l$, so that there is an embedding

$$\mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_k} \times \mathbb{H}^l \to X.$$

Thus $n + l - 2 \leq \operatorname{wdim} X \leq n + l - 1$. When $l = 1$ the inequality $n - 1 \leq \operatorname{wdim} X \leq n$ is immediate.

Suppose for a contradiction that $\operatorname{wdim} X = n + l - 2$. Now $X \times X$ is again a product of hyperbolic spaces and trees with $n' - 1 = 2n_1 + \ldots + 2n_k - 2k = 2n - 2$ and $l' = 2l$ so that by the argument above $n' + l' - 2 \leq \operatorname{wdim} X \times X \leq n' + l' - 1$. Then

$$\begin{aligned} 2n + 2l - 3 &= n' + l' - 2 \\ &\leq \operatorname{wdim} X \times X \\ &\leq \operatorname{wdim} X + \operatorname{wdim} X \\ &= (n + l - 2) + (n + l - 2) = 2n + 2l - 4 \end{aligned}$$

gives a contradiction. Therefore $\operatorname{wdim} X = n + l - 1$. □

6.5 Applications to non-embedding problems

6.5.1 A non-embedding result

Proposition 6.5.1. *There exists no quasi-isometric embedding* $\mathbb{H}^4 \to \mathrm{Sol} \times \mathrm{Sol}$.

We point out that neither the asymptotic dimension nor the hyperbolic dimension (hypdim $\mathrm{Sol} \times \mathrm{Sol}$ is unknown) rule out such an embedding; also considerations of volume growth do not rule out such an embedding since both spaces have exponential growth.

Proof of Proposition 6.5.1. Using the quasi-isometric embedding $\mathbb{H}^3 \to \mathbb{H}^2 \times \mathbb{H}^2$ and the existence of an isometric copy $\mathbb{H}^2 \subset \mathrm{Sol}$, we can conclude there is an isometric embedding $\mathbb{H}^3 \to \mathrm{Sol} \times \mathrm{Sol}$. Thus $2 \leq \mathrm{wdim}\, \mathrm{Sol} \times \mathrm{Sol} \leq 1 + 1 = 2$ by the product theorem. Now $\mathrm{wdim}\, \mathbb{H}^4 = 3$, so that there exists no quasi-isometric embedding $\mathbb{H}^4 \not\to \mathrm{Sol} \times \mathrm{Sol}$. □

6.5.2 A problem for future research

Proposition 6.5.1 suggests that the weak dimension might have applications to other non-embedding problems. In particular there is the following problem discussed in [BS07b].

For $n \geq 2$ let M^n be a compact n-dimensional Riemannian manifold-with-boundary with constant sectional curvature $K = -1$. Assume also that the geodesic boundary $\partial^g M^n$ of M^n is non-empty[1]. Let X^n be the universal cover of M^n.

An example of such a manifold M^2 when $n = 2$ is a torus with a small open disk removed. The cover X^2 in this case is the hyperbolic disk $\mathbb{H}^2$ with a countable collection

[1]Ruling out the case, for example, where M^n is a closed ball in hyperbolic n-space $\mathbb{H}^n$.

of horodisks removed. To see this, cover the punctured torus with the hyperbolic plane $\mathbb{H}^2$. A disk around the puncture has the geometry of a cusp. So the hyperbolic plane $\mathbb{H}^2$ is an infinite sheeted cover of the torus with a cusp attached along the boundary of a removed disk. The preimage of the cusp under the covering projection is a countable collection of horodisks.

Another example is when M^2 is a pair of pants, obtained by gluing two ideal simplices of $\mathbb{H}^2$ together along their boundaries and removing the three resulting cusps. In this case the cover X^2 is the disk $\mathbb{H}^2$ with a countable collection of disjoint open half spaces removed. Similarly there are examples X^n for $n \geq 3$ with X^n isometric to the space $\mathbb{H}^n$ with a countable collection of half spaces removed. For such examples it is pointed out in [BS07b] that $\dim \partial_\infty X^n = n-2$ and $\operatorname{hypdim} X^n \leq \operatorname{asdim} X^n = n-1$. These dimensions immediately give the following non-embedding theorem, again from [BS07b].

Theorem 6.5.2. *Let X^n be a space as above, and let $X^n \times \ldots \times X^n$ be the k-fold product, $k \geq 1$. Then there is no quasi-isometric embedding*

$$\mathbb{H}^p \to X^n \times \ldots \times X^n \times \mathbb{R}^m$$

for $p > k(n-1)$ and any $m \geq 0$.

Consider the space $X^n \times X^n$ (i.e., the case $k = 2$). Theorem 6.5.2 says nothing about whether the embedding exists for the edge case $p = 2(n-1)$. When $n = 2$ there are examples of manifolds M^2 for which the space X^2 is quasi-isometric to a binary tree T_b. By [BDS07] there is then an embedding $\mathbb{H}^2 \to T_b \times T_b$.

Question 6.5.3. *For $n \geq 3$ does there exist a quasi-isometric embedding*

$$\mathbb{H}^{2n-2} \to X^n \times X^n?$$

For $k \geq 3$ the question is also open.

Question 6.5.4. *For $n \geq 2$ and $k \geq 3$ does there exist a quasi-isometric embedding*

$$\mathbb{H}^{k(n-1)} \to X^n \times \ldots \times X^n?$$

For $n \geq 3$ the weak dimension of X^n is unknown. We have $\operatorname{wdim} X^n \leq n-1$ and it may be the case that $\operatorname{wdim} X^n < n-1$. After all, $\dim \partial_\infty X^n = n-2$. For the case $n=2$ we have $\dim \partial_\infty X^2 = 0$, whereas X^2 is not doubling, so that $\operatorname{wdim} X^2 = 1$. For this reason the $n=2$ case may be exceptional; for $n \geq 3$, the value $\dim \partial_\infty X^n$ may be a reasonable estimate for the value of $\operatorname{wdim} X^n$.

If indeed $\operatorname{wdim} X^n = n-2$ for $n \geq 3$ then $\operatorname{wdim} \mathbb{H}^{2n-2} = 2n-3$ whereas $\operatorname{wdim} X^n \times X^n \leq 2n-4$, so that Question 6.5.3 would be answered in the negative. Likewise, in the case $k \geq 3$, if $\operatorname{wdim} X^n = n-2$ for $n \geq 3$ then $\operatorname{wdim} \mathbb{H}^{k(n-1)} = k(n-1)-1$ whereas $\operatorname{wdim} X^n \times \ldots \times X^n \leq k(n-2)$, so that Question 6.5.4 would also be answered in the negative.

For certain examples of manifolds M in dimension $n=3$ the cover X is quasi-isometric to a finitely generated group with the structure of a free product with amalgamation. Based on properties of the factors in the free product and on the results of [BJN09], [Fin74] a natural first step to computing $\operatorname{wdim} X^3$ is to compute the weak dimension of the free product $\mathbb{Z} * \mathbb{Z}^m$ for $m \geq 2$ (if $m=1$ we have $\operatorname{wdim} \mathbb{Z} * \mathbb{Z} = \operatorname{wdim} F_2 = 1$).

6.5.3 Computing $\operatorname{wdim} \mathbb{Z} * \mathbb{Z}^m$ for $m \geq 2$

The computation of $\operatorname{wdim} \mathbb{Z} * \mathbb{Z}^m$ is interesting in that, unlike the examples we have seen so far, the dimension is computed by the direct construction of covers. We assume $m \geq 2$; the case $m=1$ is that of the free nonabelian group $F_2 = \mathbb{Z} * \mathbb{Z}$ which has $\operatorname{wdim} F_2 = 1$ since its Cayley graph is a regular 4-valent tree. To begin observe that $\mathbb{Z} * \mathbb{Z}^m = \pi_1(M)$ where $M = S^1 \vee T^m$ is the wedge sum of a circle S^1 and an m-torus

$T^m = \Pi_{i=1}^m S^1$.

Let $p\colon X \to M$ be a covering projection, with X the universal cover of M. Observe that $p^{-1}(T^m)$ decomposes as a countable disjoint union of subspaces $\sqcup_{a\in A} S_a$ with each $S_a \subset X$ homeomorphic to Euclidean space $\mathbb{R}^m$. The complement $\mathring{I} := X - \sqcup_{a\in A} S_a$ decomposes as a countable union of disjoint open intervals.

It will be useful to have an inductive description of X.

Step 1. Begin with a single Euclidean space S_0 which we think of as the *root space* analogous to the root vertex of a tree. At every integer lattice point $(i_1, \ldots, i_m) \in S_0 \cong \mathbb{R}^m$ we glue a copy of the interval $[-1, 1]$ to the space by identifying $0 \in [-1, 1]$ with the lattice point. Next we attach countably many disjoint copies of $\mathbb{R}^m$ at the endpoints of every attached interval $[-1, 1]$ by identifying the endpoint with the origin in the copy of $\mathbb{R}^m$.

Step i, $i \geq 2$. Having completed step $i-1$, for every space $\mathbb{R}^m$ attached at step $i-1$, we attach countably many copies of the interval $[-1, 1]$ by pairing each integer lattice point with a copy of $[-1, 1]$ and gluing the lattice point to 0 in its paired copy of $[-1, 1]$ (except at the origin of each $\mathbb{R}^m$ where only the interval $[0, 1]$ is attached by identifying 0 and the origin). At the endpoints ± 1 of each attached interval $[-1, 1]$ (and at the endpoint 1 of each attached interval $[0, 1]$) we attach a copy of $\mathbb{R}^m$ by identifying the endpoint with the origin in the copy of $\mathbb{R}^m$.

After denumerably many steps we obtain the space X.

Endow each copy $S_a \cong \mathbb{R}^m \subset X$ with the Euclidean metric. Assign a unit circumference to S^1 in the wedge sum $S^1 \vee T^m$, which endows each attached interval with the canonical metric restricted from the real line. The resulting length metric d on X is geodesic and restricts to the Euclidean metric on each $S_a \cong \mathbb{R}^m$ and to the canonical metric on each interval. By the Švarc-Milnor Lemma X is quasi-isometric to the group $\mathbb{Z} * \mathbb{Z}^m$.

It is convenient to refer also to the metric d^∞ on X induced by the l^∞-norm on each space S_a. We define

$$B_r^\infty(x) = \{y \in X \mid d^\infty(x,y) < r\}$$

(and similarly for closed balls $\overline{B}_r^\infty(x)$). Observe that $B_r(x) \subset B_r^\infty(x)$. The advantage of this metric is that for a closed ball $\overline{B}_n^\infty(v)$ with positive integer radius n centered at a branch point v, every intersection $\overline{B}_n^\infty(v) \cap S_a$ is either empty, a single branch point, or a cube with vertices and center at branch points.

Proposition 6.5.5. *For* $m \geq 2$,

$$\operatorname{wdim} \mathbb{Z} * \mathbb{Z}^m = 1. \tag{6.2}$$

For the proof we make the following definitions and comments. The inductive description of X implicitly chooses a distinguished root space and a distinguished basepoint, the origin o of the root space. Observe that such a choice also determines an origin for each space S_a. Let O denote the set of origins of all spaces S_a, $a \in A$. A point $(i_1, \ldots, i_m) \in \mathbb{Z}^m$ of an integer lattice $\mathbb{Z}^m \subset S_a$ is called a *branch point*. Each origin $o' \in O$ has a unique *parent* branch point denoted $p(o')$.

For $r > 0$ and $A \subset X$, the *outward* r*-neighborhood of* A, denoted $N_r^+(A)$, is the set of all $x \in X$ for which a point $a \in A$ lies on the geodesic segment joining o to x satisfying $d(a,x) \leq r$.

For a branch point v we can identify the set of *descendant spaces of* v. These are precisely those attached spaces $S_a \subset X$ such that for all $p \in S_a$, any path joining o to p must pass through v (this definition makes a space S_a a descendant of its origin). Given $r > 0$ and branch point v the *terrace of radius* r *centered at* v, denoted $\operatorname{terr}_r(v)$, is the union of all descendant spaces S_a of v that intersect the closed ball $\overline{B}_r^\infty(v)$.

Remark 6.5.6. For each $r > 0$ only finitely many spaces S_a intersect the ball $\overline{B}_r^\infty(o)$. Therefore $\operatorname{terr}_r(o)$ is large-scale-doubling because it is the union of finitely many large-scale-doubling spaces S_a. Observe also that every terrace $\operatorname{terr}_r(v)$ is isometric to a subset of the terrace $\mathrm{terr}_r(o)$. Therefore for each fixed $r > 0$ the collection $\{\mathrm{terr}_r(v)\}_v$ of r-terraces over all branch points v of X is weakly uniformly large-scale-doubling.

Proof of Proposition 6.5.5. Given $r \geq 1$ an integer. We will construct a weakly uniform large-scale-doubling cover $\mathcal{U} = \mathcal{U}_e \cup \mathcal{U}_o$ where $\mathcal{U}_e$ and $\mathcal{U}_o$ are r-disjoint. The cover is constructed inductively, but the reader is cautioned that the induction does not synchronize with that of the construction of the space X above.

We first construct disjoint families $\mathcal{V}_0, \mathcal{V}_1, \ldots$ satisfying $\mathrm{dist}(A, B) \geq 3r$ for $A \in \mathcal{V}_i$ and $B \in \mathcal{V}_j$ with $|i - j| \geq 2$. At each stage we denote by X_i the union $\cup\{V \mid V \in \mathcal{V}_0 \cup \ldots \cup \mathcal{V}_i\}$.

Step 0. The collection $\mathcal{V}_0$ contains the single element $\mathrm{terr}_{3r}(v)$. Observe that X_0 covers the ball $B_{2r}(o) - \mathring{I}$.

Step i, $i \geq 1$. Let Λ_i index the set of origins $o' \in O$ with the property that $o' \notin X_{i-1}$ but $p(o) \in X_{i-1}$. The collection $\mathcal{V}_i$ is then defined as

$$\mathcal{V}_i := \{\mathrm{terr}_{3r}(o') \mid o' \in \Lambda_i\}.$$

Observe that $\mathcal{V}_i$ is disjoint and that X_i covers the ball $B_{(i+1)3r}(o)\mathring{I}$.

We have defined families $\mathcal{V}_i$, $i \geq 0$, in which each element is of the form $\mathrm{terr}_{3r}(o')$ for some $o' \in O$. By construction $\mathrm{dist}(A, B) \geq 3r$ for all $A \in \mathcal{V}_i$ and $B \in \mathcal{V}_j$ whenever $|i - j| \geq 2$.

To rectify the fact that the families $\mathcal{V}_i$, $i \geq 0$, are merely disjoint and not r-disjoint, we operate inductively on the families $\mathcal{V}_i$ to obtain families $\mathcal{U}_i$. At each stage we denote by Y_i the union $\cup\{U \mid U \in \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_i\}$.

Step 0. Replace $\mathcal{V}_0 = \{\mathrm{terr}_{3r}(o)\}$ by $\mathcal{U}_0 = \{N_r^+(\mathrm{terr}_{3r}(o))\}$. Observe that $\mathcal{U}_0$ covers the ball $B_{3r+r}(o)$.

Step i, $i \geq 1$. Transform each $V \in \mathcal{V}_i$ by first set-subtracting the portion of the space covered by $\mathcal{V}_0, \ldots, \mathcal{V}_{i-1}$, and then taking the outward r-neighborhood of what remains; that is,

$$\mathcal{U}_i = \{N_r^+(V - Y_{i-1}) \mid V \in \mathcal{V}_i\}.$$

Observe that $\mathcal{U}_i$ is $2r$-disjoint and that X_i covers the ball $B_{(i+1)3r+r}(o)$.

The collection $\mathcal{U} = \cup_{i=0}^{\infty}\mathcal{U}_i$ covers X. Every element of $\mathcal{U}$ is contained in a set of the form $N_r(\mathrm{terr}_{3r}(v))$. By Remark 6.5.6 and Lemma 5.1.6 the collection $\mathcal{U}$ is weakly uniformly large-scale-doubling. By construction $\mathrm{dist}(A, B) \geq r$ for all $A \in \mathcal{U}_i$ and $B \in \mathcal{U}_j$ whenever $|i - j| \geq 2$.

The partition

$$\mathcal{U}_e = \cup_{i=0}^{\infty}\mathcal{U}_{2i}, \quad \mathcal{U}_o = \cup_{i=0}^{\infty}\mathcal{U}_{2i+1}$$

of $\mathcal{U}$ into the two r-disjoint families $\mathcal{U}_e$, $\mathcal{U}_o$ shows that $\mathrm{wdim}\, X \leq 1$. Finally X has exponential growth so that $\mathrm{wdim}\, X \neq 0$. The result follows. □

Free products of free abelian groups

For $n \geq 1$ let F_n denote the free nonabelian group on n generators. By the result of [BJN09], the spaces $F_n * \mathbb{Z}^m$ and $\mathbb{Z} * \mathbb{Z}^m$ are quasi-isometric. Proposition 6.5.5 thus also gives

$$\mathrm{wdim}\, F_n * \mathbb{Z}^m = 1. \tag{6.3}$$

In view of Theorems 2.2.14 and 2.2.15 for the asymptotic dimension, we conclude with two natural questions.

Question 6.5.7. *For A and B infinite finitely generated groups with* $\mathrm{wdim}\, B \leq \mathrm{wdim}\, A = n$ *is there the formula*

$$\mathrm{wdim}\, A * B = \max\{n, 1\}$$

for the weak dimension of the free product?

Question 6.5.8. *Is there an estimate from above for the weak hyperbolic dimension of a finitely generated free product with amalgamation?*

Bibliography

[BD04] G. Bell and A. Dranishnikov, *On asymptotic dimension of groups acting on trees*, Geom. Dedicata **103** (2004), 89–101. MR 2034954 (2005b:20078)

[BD06] G. C. Bell and A. N. Dranishnikov, *A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory*, Trans. Amer. Math. Soc. **358** (2006), no. 11, 4749–4764. MR 2231870 (2007e:20092)

[BD08] G. Bell and A. Dranishnikov, *Asymptotic dimension*, Topology Appl. **155** (2008), no. 12, 1265–1296. MR 2423966 (2009d:55001)

[BDK04] G. C. Bell, A. N. Dranishnikov, and J. E. Keesling, *On a formula for the asymptotic dimension of free products*, Fund. Math. **183** (2004), no. 1, 39–45. MR 2098148 (2005g:20064)

[BDS07] Sergei Buyalo, Alexander Dranishnikov, and Viktor Schroeder, *Embedding of hyperbolic groups into products of binary trees*, Invent. Math. **169** (2007), no. 1, 153–192. MR 2308852 (2008b:57001)

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038)

[BJN09] Jason A. Behrstock, Tadeusz Januszkiewicz, and Walter D. Neumann, *Commensurability and QI classification of free products of finitely generated abelian groups*, Proc. Amer. Math. Soc. **137** (2009), no. 3, 811–813. MR 2457418 (2009i:20051)

[BS00] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. **10** (2000), no. 2, 266–306. MR 1771428 (2001g:53077)

[BS05] Sergei Buyalo and Viktor Schroeder, *Embedding of hyperbolic spaces in the product of trees*, Geom. Dedicata **113** (2005), 75–93. MR 2171299 (2006f:53055)

[BS07a] S. Buyalo and V. Schroeder, *The hyperbolic dimension of metric spaces*, Algebra i Analiz **19** (2007), no. 1, 93–108. MR 2319511 (2008i:57023)

[BS07b] Sergei Buyalo and Viktor Schroeder, *Elements of asymptotic geometry*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2007. MR 2327160 (2009a:53068)

[BSW12] Sara Brofferio, Maura Salvatori, and Wolfgang Woess, *Brownian motion and harmonic functions on* $\mathrm{Sol}(p,q)$, Int. Math. Res. Not. IMRN (2012), no. 22, 5182–5218. MR 2997053

[Buy05] S. V. Buyalo, *Capacity dimension and embedding of hyperbolic spaces into the product of trees*, Algebra i Analiz **17** (2005), no. 4, 42–58. MR 2173936 (2006e:31008)

[Dra97] A. N. Dranishnikov, *On the virtual cohomological dimensions of Coxeter groups*, Proc. Amer. Math. Soc. **125** (1997), no. 7, 1885–1891. MR 1422863 (98d:55001)

[Dra99] ———, *Boundaries of Coxeter groups and simplicial complexes with given links*, J. Pure Appl. Algebra **137** (1999), no. 2, 139–151. MR 1684267 (2000d:20069)

[DS06] A. Dranishnikov and J. Smith, *Asymptotic dimension of discrete groups*, Fund. Math. **189** (2006), no. 1, 27–34. MR 2213160 (2007h:20041)

[Fin74] Benjamin Fine, *The structure of* $\mathrm{PSL}_2(R)$*; R, the ring of integers in a Euclidean quadratic imaginary number field*, Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Princeton Univ. Press, Princeton, N.J., 1974, pp. 145–170. Ann. of Math. Studies, No. 79. MR 0352289 (50 #4776)

[FM98] Benson Farb and Lee Mosher, *A rigidity theorem for the solvable Baumslag-Solitar groups*, Invent. Math. **131** (1998), no. 2, 419–451, With an appendix by Daryl Cooper. MR 1608595 (99b:57003)

[Fri37] A. H. Frink, *Distance functions and the metrization problem*, Bull. Amer. Math. Soc. **43** (1937), no. 2, 133–142. MR 1563501

[Gra06] Bernd Grave, *Asymptotic dimension of coarse spaces*, New York J. Math. **12** (2006), 249–256 (electronic). MR 2259239 (2007f:51023)

[Gro87] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR 919829 (89e:20070)

[Gro93] ———, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR 1253544 (95m:20041)

[KB02] Ilya Kapovich and Nadia Benakli, *Boundaries of hyperbolic groups*, Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math., vol. 296, Amer. Math. Soc., Providence, RI, 2002, pp. 39–93. MR 1921706 (2004e:20075)

[Leb07] N. Lebedeva, *Dimensions of products of hyperbolic spaces*, Algebra i Analiz **19** (2007), no. 1, 149–176. MR 2319513 (2008e:55001)

[Roe03] John Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003. MR 2007488 (2004g:53050)

[Sch06] Viktor Schroeder, *Quasi-metric and metric spaces*, Conform. Geom. Dyn. **10** (2006), 355–360 (electronic). MR 2268484 (2007g:54041)

[Sha04] Yehuda Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. **192** (2004), no. 2, 119–185. MR 2096453 (2005m:20095)

[Smi06] J. Smith, *On asymptotic dimension of countable abelian groups*, Topology Appl. **153** (2006), no. 12, 2047–2054. MR 2237596 (2007g:20044)