Models and Algorithms for Procurement Combinatorial Auctions

MODELS AND ALGORITHMS FOR PROCUREMENT COMBINATORIAL AUCTIONS

BY

BAHAREH MANSOURI

A THESIS

SUBMITTED TO THE SCHOOL OF COMPUTATIONAL SCIENCE & ENGINEERING

OF MCMASTER UNIVERSITY

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Ph.D. Thesis (2014) McMaster University
(Computational Science & Engineering) Hamilton, Ontario, Canada

TITLE: Models and Algorithms for Procurement Combinatorial Auctions

AUTHOR: Bahareh Mansouri
M.Sc., (Applied Mathematics)

SUPERVISOR: Prof. Elkafi Hassini

NUMBER OF PAGES: xvii, 213

Dedications

My lovely son,

What a wonderful coincidence to celebrate the termination of my doctoral studies with your first steps on this beautiful journey.

I would like to dedicate this work, on behalf of myself and daddy who has been a great help to accomplish this work, with the hope to see you rise and shine on top of the highest mountains of success.

Acknowledgements

This thesis is the outcome of exciting years of work and research at the School of Computational Science and Engineering (CSE), McMaster University.

First and foremost, I am indebted to my supervisor, Professor Elkafi Hassini, for his trust, support, thoughtful suggestions, fruitful discussions, and his willingness to share his time, wisdom, and expertise despite all other academic and professional commitments. I am grateful for all that I learned from him. I would like to express my sincere gratitude and appreciation to my supervisory committee members, Professor Tim Davidson, Professor Antoine Deza, and Professor Kai Hunag for their enthusiastic engagement in the discussions and insightful comments that made this work better than it otherwise would have been. My appreciation goes to the professors from whom I have learnt a lot throughout the graduate courses required for the completion of my doctoral degree. I am also grateful to the financial support of the Alexander Graham Bell Canada (CGS)-Doctoral Scholarship, the Ontario Graduate Scholarship (OGS) Program, Professor Elkafi Hassini, and the School of Computational Science and Engineering.

I can not thank my husband, Ali, enough for his love, dedication, support and endurance throughout the years of this program. It was most heart-warming to count on his presence and help during all the ups and downs. Specially, this work would not have been possible without his continuous support ever since we welcomed the most amazing creature of the world, our son, to our life. I would also like to thank my beautiful son, Ashkan, who endured the long hours of daycare starting from a very early age. His smile was indeed most encouraging and inspiring to make my hours at work as efficient as possible.

I am wordless to thanks my parents for giving me the gift of life and their unconditional love. Even though far away, their optimism, encouragement and emotional support made me feel their presence all along. My sincere appreciation goes to my parents-in-law whose support specially during the last year of this program was priceless. Their parent-like help was invaluable to help me concentrate better on my research. Finally, I would like to thank all my family and friends involved in the creation of this thesis.

Last but not least, I would like to take this opportunity to thank all the teachers and staff at McMaster Children Center. Leaving my little son in the good hands of the teachers at this center, from the start to this very moment, gave the comfort to cope with the anxiety to separate from my precious little angle.

Abstract

A key problem in designing marketplaces is how to efficiently allocate a collection of goods amongst multiple people. Auctions have emerged as a powerful tool with the promise to increase market efficiency by allocating goods to those who value them the most. Nevertheless, traditional auctions are unable to handle real-world market complexities. Over the past decade, there has been a trend towards allowing for package bids and other types of multidimensional bidding techniques that enable suppliers to take advantage of their unique abilities and put forth their best offers. In particular the application of iterative combinatorial auctions in procurement saves negotiation costs and time. Conceptually these auctions show a potential for improving the overall market efficiency. However, in practice they host several new challenges and difficulties.

One challenge facing the auctioneer in an iterative combinatorial auction environment is to quickly find an acceptable solution for each round of the auction. Bidders require time to precisely evaluate, price, and communicate different possible combinations based on their current information of item prices. The auctioneer requires time to solve the underlying mathematical problem formulation based on the bids received, report back the feedback information and initiate a new round of the auction.

In Chapter 3, we propose a Lagrangian-based heuristic to solve the auctioneer's winner determination problem. After generating the Lagrange multipliers from the solution of a linear relaxation, the heuristic applies several procedures to fix any potentially infeasible optimal Lagrange solutions. In addition to providing an efficient way of solving the winner determination problem, as compared with the leading commercial solver CPLEX, our approach provides Lagrange multipliers. The latter are used as proxies for prices in the auction feedback mechanism.

In Chapter 4 we develop a model for the bidders pricing problem, an issue that has received much less attention in the literature. Using the auctioneer feedback, that includes the Lagrange multipliers, the pricing model maximizes the bidders' profit while at the same time keeping their bids competitive. We derive several optimality results for the underlying optimization problem. Interestingly, we analytically show that the auction converges to a point where no bidder is able to submit a bid that yields strictly better profit for him and is not less competitive than his previous bids submitted. We experimentally observe that this approach converges in an early stage. We also find that this iterative auction allows the bidders to improve their profit while providing lower and competitive prices to the auctioneer.

In Chapter 5, we introduce a flexible auction model that allows for partial bids. Rather than the regular all-or-nothing indivisible package bids, divisible bids provide flexibility for the auctioneer with the possibility to accept parts of the bids and yet allow the suppliers to capture synergies among the items and provide quantity

discounts. We show numerically that this approach improves the overall efficiency of the auction by increasing the suppliers' profit while decreasing the auctioneer's total price of procurement. In addition, we find that computationally the flexible auction outperforms the regular auction.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

From long ago, auctions have been used as means of trading goods and services with unknown prices. Herodotus reports using auctions in Babylon as early as 500 B.C. [57]. Ever since then, auctions have been used in commercial trades to liquidate properties. Today both the range and value of objects sold by auctions has grown significantly. Through auctions, numerous kinds of commodities ranging from fresh flowers to gold bullion, art objects and antiques are transferred to the hands of the people who value them the most.

As well as individuals and private organizations, the public sector also benefits immensely from auctions in transferring assets to private hands. Examples include sales of industrial enterprises, transportation systems, natural resources such as off-shore oil leases, and electromagnetic spectrum for communication. In today's modern era, wide easy access to internet has intensified the implementation of auctions. By means of various internet auction websites, individuals set up items for sale and apply common auction rules to allocate them to those who offer the best prices.

While *multi-unit auctions* facilitate negotiations on large quantities of a single item, *multi-item auctions* enable bidders to express interest on several heterogeneous ones. As many organizations started adopting some sort of multi-unit multi-item auctions, they realized a fundamental shortcoming of these mechanisms which is their inability to allow for complex bid structures which exploit complementarities and economies of scale. This emerged interest to extend the basic auction types to support bids with a more complex set of preferences.

Combinatorial Auctions (CA) offered one potential solution by permitting bidders to select subsets of items to bid upon and thus fully express their sub- or super-additive valuations over those items. The most important reasons for the increased popularity of combinatorial auctions in commerce include increased sellers' revenue, elevated bidders' preference elicitation opportunities, and eventually improved market efficiency.

As opposed to *single-round* auction designs, *multi-round* or *iterative CAs (ICAs)* have been selected in a number of industrial applications, since they help bidders to express their preferences by providing feedback, such as provisional pricing and allocation information, in each round. ICAs have several advantages over single-round auctions. Whereas single-round combinatorial auctions are usually followed by aftermarket negotiations to overcome the inefficiencies [32], empirical studies suggest that in complex economic environments iterative auctions enhance the ability of the participant to detect competition and learn when and how high to bid in order to produce better results than single-round auctions [89].

In a *reverse auction*, commonly referred to as *procurement auction*, the role of the buyer and seller is changed in that a single buyer offers a contract out for bidding and multiple sellers attempt to offer lower bids than their competitors in order to obtain the business. Industrial procurement is a domain where combinatorial auctions have emerged as a powerful mechanism. In addition to economical advantages of implementing these auctions, industrial procurement benefits from significant cost savings in complex negotiation scenarios as well as improved time efficiency in uploading large data sets and processing them. Combinatorial auctions provide the possibility to impact the market structure and involve small businesses by splitting large contracts into small ones. As reported by Beall et al.[10], more than forty percent of the large firms (spending over 100 million dollars) in North America were using procurement combinatorial auction in 2003.

Alongside their attractive attributes, application of iterative combinatorial auctions (in procurement and otherwise) is not an easy task. The goal in this dissertation is to address some of the challenges faced by the auctioneer and/or the bidders when implementing these auctions.

1.1 Chapter 2

As the number of items increases in an auction, the number of possible combinations for bidders to evaluate grows exponentially. This imposes serious computational challenges when trying to find an allocation of packages to the bidders that provides the auctioneer with the greatest payoff value (greatest revenue in a direct and lowest procurement cost in a reverse auction) [95].

Chapter 2 of this thesis is devoted to reviewing most important research findings relevant to the subject of of our research. Our primary goals in this chapter are to

- build a consistent terminology to be used throughout this thesis,
- review the theoretical principals and foundations,
- review the related literature,
- highlight how the contributions of this thesis fill in some of the gaps in the literature.

1.2 Chapter 3

In an iterative auction environment where the NP-hard problem of the winner determination problem has to be solved repeatedly, time becomes a constraint [51]. The focus in Chapter 3 is to design a Lagrangian-based heuristic for solving this problem. The Lagrangian multipliers provide useful economic interpretation in the context of combinatorial auctions and have the potential to initiate multiple rounds for auctions.

The contribution of this chapter is twofold. First, we describe a novel technique to solve for the Lagrangian optimal solution in a single iteration. As described in Section 2.3.4, traditional solution methodologies applied for solving the Lagrangian relaxation problem suffer from a slow convergence rate as well as sensitivity to the initialization of the parameters involved. Our proposed technique surpasses these difficulties by optimal initialization of the Lagrangian multipliers.

Second, we propose a heuristic to project the Lagrangian optimal solution into the feasible region of the primal problem. On an average of 7,500 problem instances that we generated for problems with 10, 20, 30 items and 100, 200,...,1000 bids, our proposed algorithm is on average 219.86 times faster than CPLEX 12 to find a solution within 10% of the optimality. Nevertheless, we found our approach is best suited for the class of problems for which the maximum quantity of items offered in each package is less than half of the demand (for that item). In this case, for 10, 20, and 30 items, our algorithm provides a near optimal solution respectively 5, 6, and 7 percent off from the optimal in around 1/3, 1/227, and 1/1065 of the CPLEX run-time.

1.3 Chapter 4

Solving for the WDP has received intensive attention in the literature. However, one issue not discussed quite as much is devising mathematical methods for bidders to optimize their bid generation technique. Most auctions assume that bidders know how to determine optimal packages and even derive fundamental theoretical equilibrium properties for such auctions. From the bidders' perspective though, the evaluation and pricing process is a difficult task. They need to explicitly determine how to select and more importantly price the best set of items to bid from an exponential number of possible combinations so as to maximize their profit while increasing their chances of winning the next round of the auction. In this chapter we develop an integrated iterative combinatorial auction that deals with how to integrate the auctioneers' optimization problem with the bidders' optimal bid generation. Specifically, we look at the bidders' pricing problem in an ICA. Pricing in ICAs is subtle. In one line of research, prices are constructed by the auctioneer in a way that in an iterative auction environment they converge to the market equilibrium prices. Bidders respond to these prices with the quantities they are willing to supply at the announced prices. Convergence to market equilibrium prices, incentivizes the bidders to reveal their true valuations in the auction. As reviewed in Section 2.3.5, despite strong theoretical foundations, Scheffel et al. [12, 101] showcase that experimentally these approaches lack efficiency since the presumptions are too strong to implement in realistic settings.

In Chapter 4, we use Lagrangian multipliers to help bidders determine item prices so as to keep communication complexity to a minimum. We use these prices as a *guide* and proceed to formulate the bidders' problem with the objective to maximize their profit subject to generation of that are competitive.

There have been multiple proposals on how to design the pricing scheme in ICAs including approximate linear, non-linear, and personalized non-linear prices as reviewed in Section 2.2.5.2. As of now, there is no general *consensus* on a single *best* design. Each pricing scheme is proven useful for a certain valuation structures. In our research, we focus on ICA designs with linear ask prices in which each item is assigned an individual price, and the price of a package of items is the sum of the single-item prices. Linear prices are easy to understand for bidders, and provide a good guidance for computing the price of any bundle even if no bid was submitted for it.

Every time the auctioneer solves the winner determination problem, the Lagrangian optimal multipliers are revealed to the bidders. Bidders use the announced prices to maximize their profit while generating a more competitive bid. We show that the auction converges to a point where no supplier is able to make a strictly better deal, causing the auction to terminate.

1.4 Chapter 5

With indivisible package bids suppliers either win all that they offer or nothing. Thus, suppliers would have to provide exponentially many bids (with respect to the number of items in the auction) to completely describe their valuation structure. For 10 items, this could lead to over 1,000 bids per supplier. Even if suppliers could determine complete sets of combinatorial bids, they would probably be unwilling to provide this information. Presence of multiple units of each item in a combinatorial auction amplifies this complexity by providing the possibility for bidders to make different choices of quantities for each combination of items chosen in each package.

One approach to get around this complexity is to allow for partial bid acceptance. Rather than pricing every different desirable package of quantities of items, suppliers prepare and submit cost or price functions with per unit prices of the items they include in each package. Despite their wide applicability in real scenarios, auctions with divisible bids are rare in the literature.

The bidding languages currently deployed to address divisible bids suffer from deficiencies. For most cases, the unit price as well as the number of quantities declared remain the same for all assets included in a package. Moreover, some studies ignore how bidders could take advantage of providing quantity discounts for provision of larger quantities of items in order to make more competitive bids.

In Chapter 5 we introduce a bidding language which overcomes these shortcomings and experimentally, illustrate the efficiency of allowing for partial bid acceptance. Analysis of the Lagrangian relaxation properties of the WDP with divisible bid submissions (DWDP) reveals that the properties observed for WDP with indivisible bids holds for DWDP. Looking at the suppliers' optimization problem in a divisible environment, we develop quantity-based (QPMBD) as well as risk-based (RPMBD) profit maximization models for suppliers. QPMBD seeks the optimal price and quantity values that maximize suppliers' profit. QPMBD is modelled as a nonlinear mixed integer programme and then transformed into a mixed integer programme. RPMBD maximizes the profit for suppliers' with different levels of riskiness. We investigate the optimality conditions for the RPMBD problem formulations as well as how the riskiness of the suppliers' affects their overall gain. On average, our imperial work suggests that divisible bids improve the overall auction efficiency by increasing the suppliers' profit while decreasing their price offerings and consequently the auctioneer's total cost of procurement.

1.5 Chapter 6

This Chapter includes our concluding remarks. We also propose research directions to expand on the research questions presented in this thesis.

Chapter 2

Literature Review

Auctions are celebrated as one of the triumphs of game theory in economy. Even so, they have attracted several scientists from operations research and computer science when solving for an efficient solution to the underlying resource allocation problem with self-interested agents.

Ever since the successful application of auctions in the sales of spectrum rights, auctions started to be adopted by many modern market environments in their major trades [71]. As many organizations began to realize the efficacy of auctions, interest has grown from basic auction types to combinatorial auctions which support negotiations on subsets of items. Combinatorial auctions exploit economies of scale in bidders' valuation structure by allowing for complex bid structures. Whereas forward auctions are used for selling, reverse (or procurement) auctions are deployed for procurement of goods or services.

In an auction that aggregates iterative and combinatorial auctions (ICAs), bidders submit bundle bids iteratively and the auctioneer computes allocations and ask prices in each round of the auction. Iterative auctions dynamically collect information about bidders valuations and set the prices of a trade within the auction. Despite achieving desirable economic efficiency, such auction design involves dealing with several computational, communicational complexity which make implementation of these auctions a difficult task.

To review the literature on ICAs and the challenges it brings about, we open this chapter by introducing auction theory and providing an overview of the game theoretic perspective of auctions. Mechanism design is described next to define allocations and payment rules in such a way that rational bidders would follow certain desired strategies. We describe various mechanism designs that define different auction types and briefly go over some primary and secondary auction types we use in our work. Specifically, we will review the foundations of the design of combinatorial and iterative auctions. The rest of this chapter is devoted to describing some of the major challenges faced for the design and implementations of iterative combinatorial auctions. In each case, we address the existing methodologies to handle these difficulties and highlight the contribution of our work whenever applicable.

2.1 Auctions Thoery

Auction theory has been one of the most widely studied fields in economics over the past fifty years. It concerns the design of auctions and the set of rules governing them. Various auction designs (or sets of rules) define different types of auctions.

Only subtle changes in auction rules can cause significant differences in their outcome. Auction theory deals with studying the efficiency of auction designs under the implementation of different sets of rules.

The participants of an auction include the ***auctioneer*** who runs the auction and sets its specific rules and the ***bidders*** who compete to buy his products as he sells them off. The competition among bidders reverts to selling products to the auctioneer when he announces his demand on the items. The bidders' competition is through submission of ***bids***. Bids are expression of the bidders willingness to pay particular monetary amounts for various outcomes. Bidders formulate bids according to their private preferences, bidding strategies as well as auction rules.

From the game theoretic perspective, auctions are defined as ***mathematical games*** in which

- the auctioneer and the bidders constitute the set of ***players***,
- the auctioneer sets the ***rules*** relative to his objective which is mostly revenue maximization or cost minimization,
- the set of ***moves*** (or ***actions***) available to each bidder is his bid function which maps his value (in the case of a buyer) or cost (in the case of a seller) to the bid price he submits,
- the ***payoff*** of each bidder is his expected utility,
- the ***strategy*** bidders follow is to maximize their utility.

In a mathematical game, bidders are known as ***rational*** if they are capable to think through all possible outcomes and choose the one that results in the best possible outcome, and the game is known as ***non-cooperative game*** when rational players are able to make decisions independently.

A strategy is defined as ***dominant*** when it gives as good or better outcome as any other strategy, regardless of how the player's opponents play. It is defined as ***strictly dominant*** if it always gives a better outcome than any other strategy no matter what the components do. Alternatively, it is called ***weakly dominant*** if there is at least one set of opponents' actions for which this strategy yields superior outcome (as compared to the rest of strategies available). A weakly dominant strategy produces similar payoff on all other strategies available to the player for any choice of the opponents' strategies.

A non-cooperative game reaches the ***Nash equilibrium*** state when rational players have chosen a strategy and no player is better off by changing his strategy unilaterally, given that other players keep theirs unchanged. If there exists a strictly dominant strategy for a player, that player will play that strategy in each of the game's Nash equilibria. If all players have a strictly dominant strategy, the game retains a unique Nash equilibrium. Weakly dominated strategies can also constitute Nash equilibria. For instance, assume a non-cooperative game with two players, strategies A and B available to each, and the *payoff matrix* as

Player 1 \ Player 2	A	B
A	(2,2)	(1,1)
B	(1,1)	(1,1)

For both players strategy A *weakly dominates* strategy B. This outcome constitutes a Nash equilibrium, since no player is better off by unilaterally changing his strategy.

2.1.1 Bidders' Valuation Function

Upon entrance in an auction each bidder makes an evaluation of the item(s) being set up for bids. A valuation function is a real valued function that allocates each item i and bidder j to a real number ν_{ij} which is bidder j's personal evaluation of item i. We assume valuation functions to be

1. monotone, that is for sets $S and T, where S \subseteq T$, we have $\nu(S) \leq \nu(T)$,
2. normalized, that is $\nu(\emptyset) = 0$.

Bidders are assumed to have quasi-linear utility (or payoff) function defined as

$$u_{ij} = \nu_{ij} - p_{ij}$$

with p_{ij} as the price bidder j pays on item i.

In a ***private value*** model, each bidder knows the value of the item to himself at the time of bidding and this value does not depend on the private information of other bidders. In many auctions, however, the object's value is unknown to the bidder himself at the time of the auction. He may only have an estimate or some privately known signals correlated with the true value. Other bidders may have additional information that if known, would affect the value that a particular bidder attaches to the object. This structure of item values is in general known as ***interdependent values***. ***Common value*** structure is a special case of this specification in which the value of the item is the same for all bidders, however, the values remains unknown to all of them. The private value model assumption is typical when auctioning pieces of artwork. An example of a common value model is auctioning financial products on the stock exchange. Auctioning wireless spectrum is a commonplace for an interdependent value model where private valuations are driven by the underlying population demographics and technological basis.

Once the bidders participating in an auction evaluate the commodities, they attach a price to them. This price may not necessarily reflect their real valuations and thus result in an intense reduction of the auction revenue. William Vickrey [109], the winner of the 1996 Nobel Prize for Economics, has shown that there exists a particular pricing scheme for a private value model with a single item or multiple homogeneous items in which a winning bidder can never affect the price he pays. He demonstrates that this gives bidders no incentive to misrepresent their values and thus achieves superior performance by making it a dominant strategy for bidders to report their values truthfully.

2.1.2 Mechanism Design

A mechanism design is defined by a set of rules describing the

- ***auction protocol***, including the sequence, syntax and semantics of messages exchanged throughout the auction,

- ***allocation rules***, including constraints ensuring the overall objective of the allocation as well as additional allocation constraints,
- ***payment rules***, determining the payment from or to the winner(s).

The primary goals in the design of an auction concerns the outcome of an auction. One goal is to achieve ***allocative efficiency*** in which the auction mechanism implements a solution that maximizes the total payoff across all agents. Another goal is the ***revenue maximization*** in which the auction achieves a solution that maximizes the payoff to a particular participant, usually the auctioneer.

From the game theoretic perspective, auction design studies a system of self-interested players following different strategies. Auction design rules may restrict bidders to certain strategies which enforce a certain outcome. One important design goal is to encourage strategies that lead to *efficient outcomes*. More specifically, auction designers try to construct ***incentive compatible mechanisms*** in which bidders are self-interested in reporting truthful information about their preferences.

To ensure incentive compatibility, the monetary transfer to each bidder has to be set so that the expected utility of bidding truthfully is always greater than or equal to the utility when the valuation is misrepresented. This is also considered as redistribution of the trade surplus. A dominant strategy is given if the players payoff maximizing strategy is independent from the strategies of the other players. Mechanisms with the dominant strategy equilibrium are called ***strategy-proof***. In a strategy-proof mechanism no assumptions about the information available to the agents about each other are made, and every bidder selects his own optimal strategy without requiring the others to act rational.

Vickery's auction design offers great insight into this for single-item auctions. In his design, the players' payoff maximizing strategy is independent from the strategies of the other players and truthful representation of the valuation is a *weakly dominant strategy*. A mechanism with these characteristics is very desirable from an economics perspective.

2.1.3 Primary Auction Types

The word auction is derived from the Latin word *augere* which means to increase or augment. Yet, not all auctions are based on increasing the price. In fact, they may take up many different types depending on the rules governing their mechanism design. Researchers primarily recognize auctions as either ***oral*** (a.k.a. ***open***) or ***written*** (a.k.a. ***closed sealed-bid***). Oral Auctions are those auctions where all bidders are present, they hear each other's bids and can make offers. In written auctions bidders submit their bids simultaneously without revealing them to the others. Oral and written auction determine the following most widely practiced auction types.

English Auctions (a.k.a. Ascending-Price Auctions)

English auctions are one of the oldest and most frequent auction forms. They are considered oral auctions in which the auctioneer begins by calling out a low price and gradually increases it. The bidders express their interest to buy the product at the announced price usually by raising their hand. These auctions continue until only one bidder is left interested in the object.

Dutch Auctions (a.k.a. Descending-Price Auctions)

Dutch auctions are the counterpart to the English auction wherein the auctioneer begins by a price high enough so that no bidder is interested to buy the object at that price. He gradually decreases it until the first person admits to purchase the object at the announced price. This open auction is made famous by the Amsterdam flower auctions and was designed to rapidly terminate the auction due to the perishable nature of the product on auction.

First-Price Sealed-bid Auctions

First-price sealed-bid auctions are a variation of written auctions in which the bidder who submits the highest bid wins the object. The winner in this auction pays the price he submits.

Second-Price Sealed-bid (a.k.a. Vickrey Auctions)

Second-price sealed-bid auctions are conducted in the same manner as first-price sealed-bid auctions. The only difference is that the winner pays the amount of the second highest bidder. The payment scheme of a second-price sealed-bid auction with a single item or multiple homogeneous units of a single item is the result of Vickrey's study on the auction's equilibrium state from the game theoretic perspective. Thus, this auction design is mostly referred to as Vickrey auction.

Vickrey [109] demonstrates that the bidders pay the amount of the opportunity cost for what they win, rather than the price they bid. Thus, they are only capable of determining whether they win or not. From the bidders' perspective, even though the amount they bid determines the efficient allocation of goods in the auction, it cannot affect the amount they pay. Only by bidding true values can the bidders be sure to win exactly when they are willing to pay the price they bid and so bidding truthfully becomes their (weakly) dominant strategy.

2.1.4 Revenue Equivalence Theorem

The revenue equivalence theorem was shown for the first time by Vickrey in his 1961 seminal paper [109] through an example and was later proved by Myerson [78]. It states that different auction mechanisms that result in the same allocation of goods yield the same revenue to the seller.

Wolfstetter [111] observes that the Dutch open descending auction is strategically equivalent to the first-price sealed-bid auction. In a first-price sealed-bid auction, a bidder maps his private information to a bid. The useful information revealed in Dutch auctions is that some bidder agrees to buy the item at the current price which causes the auction to end. Bidding in a first-price sealed-bid auction is equivalent to offering to buy it at this price in a Dutch auction.

When values are private, the English open ascending auction is equivalent to the second-price sealed-bid auction. With private values the optimal strategy for both is to bid up or stay in until the value. With interdependent values, seeing some bidders drop out in an English auction early on may bring bad news that may cause a bidder to reduce his own estimate of the object's value. Thus, with interdependent values, the two auctions are not necessarily equivalent.

2.2 Secondary Auction Types

Auctions are recognized not only by the rules of the auction, but also by their environment. Important features including the number of sellers and buyers, the number of items being traded, the preferences of the parties, and the form of the private information participants have about preferences determine different auction types. Some environment-dependant secondary auction types relevant to this thesis are described below.

2.2.1 Single-Item Auctions

Putting up a single item for bids classifies the auction as a single-item auction. In the case of having multiple units of the same item, the auction is referred to as a multi-unit single-item auction. Multi-unit auctions facilitate negotiations on large homogeneous quantities of the same item.

As suggested by the equivalence of the primary auction types for a single object, applying the classic English auction can be considered as a sealed bid second price auction. For an auction with multiple homogeneous units of a single item, Ausubel [4] formulated a dynamic ascending price auction format which resembles the outcome of the sealed-bid Vickrey auction with private values, and yet has the advantage of simplicity and privacy reservation.

2.2.2 Multi-Item Auctions

Involving several heterogeneous items in multi-item auctions produces a multi-item auction environment. Similarly, having multiple identical units of each item identifies the auction as multi-unit multi-item auction. In a setting where each bidder is interested in receiving at most one item, sealed-bid multi-item auctions are considered as a generalization of Vickrey auctions [28, 65, 29]. This mechanism requires each bidder to submit a sealed bid listing his valuation of all the items. Like the Vickrey auction, submitting true valuations is a dominant strategy for the bidders.

Another important feature of the multi-item auctions is the possibility to achieve the minimum equilibrium price allocation by *dynamic* or *progressive* auctions rather than a single round. Demange, Gale, and Sotomayor [30] study dynamic auction mechanisms based on price increase for the minimal overdemanded sets of items. They prove that prices converge to the minimum equilibrium price.

Economists have extended the Vickrey auction to encompass more general models. Clarke [20] and Groves [36] considered auctions with multiple heterogeneous items. The new mechanism is usually referred to as the ***Vickrey-Clarke-Grove (VCG) mechanism***. The pay price in a VCG auction is also called ***VCG payment***. VCG assigns goods efficiently and charges bidders the opportunity cost of the items they win. Truthful reporting is a dominant strategy for each bidder in the VCG mechanism. Mishra and Veeramini [75] study a multi-item ascending price auction wherein each supplier is able provide one or more of the items. They prove that their method implements the VCG outcome.

VCG auctions exhibit several appealing theoretical properties. Nevertheless, they suffer from serious short comings. For instance, they are vulnerable to collusion by a coalition of losing bidders, the auctioneer revenue can be very low or zero, and that determination of the VCG payments itself is a computationally hard problem (for more details on this see [5]). Moreover, Ausubel and Milgrom [5] showed that the VCG auction loses its dominant-strategy property when bidders face effective budget constraints.

The shortcomings of the VCG auction are strong enough for it to be hardly used in practice. This has turned some researches towards the design of alternative auction mechanisms that overcome some of the VCG auction drawbacks. For instance, Ausubel and Milgrom [7] proposed an ascending proxy auction and proved that there is no coalition of bidders that can trade among themselves in a way that generates strictly more revenue for the seller and equally or more preferred outcomes for all the bidders of the coalition. In [5] the authors show that the Vickrey auction leads to such outcome only under special conditions. Nevertheless, VCG auctions constitute an important theoretical structure that provides insights into fundamental properties of auction mechanisms in general.

2.2.3 Combinatorial Auction (CA)

One drawback of multi-item auctions is their failure to identify that a bidder's valuation for a combination of items that are for sale is more (or less) than the sum of the individual items' valuations when the items are complementarities (or substitutes). Therefore, bidders need to bid upon the set of items they require individually.

This disadvantages the bidders since *first*, in order to increase their chances of winning all that they require, they are frequently willing to bid above their true valuations leading to the ***winners' curse*** problem: the winner has to overpay for the item he wins. *Second*, bidders are only interested to acquire a whole combination of items. An incomplete package is undesirable and is not worth the money they have to pay for the winning items. This problem is known as the ***exposure problem***.

Combinatorial auctions allow bidders to place bids on any subset of the items known as ***packages*** or ***bundles***. This enables bidders to express complex valuations on the packages of items and thus more precisely report their preferences. Combinatorial auctions often lead to greater auction revenue as well as market economic efficiency in that items would be allocated to those who value them the most.

Combinatorial auctions were first proposed, by Rassenti, Smith, and Bulfin [90] for the allocation of airport landing slots. Their paper introduced principal ideas on the design of mathematical programming formulation of the auctioneers problem, the computational complexity of the winner determination problem, the use of testing techniques from experimental economics, and incentive compatibility of combinatorial auctions.

In procurement, combinatorial auctions have emerged as a powerful tool to automate complex negotiations on multiple items. Over the past few years, they have been employed in a variety of industries saving millions of dollars. Logistics.com, Accesstranspota.com (Canada), Translogistica.com (Uk) are some websites which report the use of combinatorial auctions for long term contracts [104].

2.2.3.1 Valuation and Allocation in CAs

This section generalizes the valuation on combinatorial auctions. According to the private value model assumption, we denote the private valuation of the bidder j for the bundle S by $\nu_j(S)$. The valuations of different bidders are assumed independent and satisfying the free disposal condition, i.e., if $S \subseteq T$ then $\nu_j(S) \leq \nu_j(T)$. The rest of this section is devoted to defining the terms we commonly use.

Definition 2.1 (Value Model)**.**
A value model $\boldsymbol{V} = \{\nu_j(S)\}$ is a set of the private valuations of all bidders for all bundles.

Definition 2.2 (Bid Price)**.**
The price bidder j attaches to bundle S is called bid price and is denoted by $P_{bid,j}(S)$.

In Chapters 3- 5 of this thesis we deal with bid prices and for this reason drop this index.

Definition 2.3 (Pay Price)**.**

Pay price $\boldsymbol{P}_{pay} = (P_{pay,1}(S), \ldots, P_{pay,j}(S), \ldots, P_{pay,n}(S))$ *defines the prices to be paid by each bidder* $j, (j \in \{1, \ldots, n\})$ *for bundle* S.

Definition 2.4 (Bidder Utility or Bidder Payoff)**.**
Bidder j's utility $\pi_j(S, P_{pay,j}(S))$ *expresses his satisfaction of getting bundle* S *at the pay prices* $P_{pay,j}(S)$. *We assume quasi-linear bidder utilities* $\pi_j(S, P_{pay,j}(S)) = \nu_j(S) - P_{pay,j}(S)$ *and* $\pi_j(\emptyset, P_{pay,j}) = 0$.

Definition 2.5 (Allocation)**.**
An allocation X *is a tuple* $(S_1, \ldots, S_n)$ *that assigns a bundle (possibly empty) to every bidder.* S_j *denotes the bundle assigned to bidder* j.

In a single-unit combinatorial auction problem, the allocated bundles do not intersect, i.e., $\forall j, j', \quad S_j \cap S'_j = \emptyset$. With the auctioneer defined as a bidder, some items may remain unallocated.

An allocation X can also be defined by a set of binary variables $x_j(S)$ such that

$$\begin{array}{ll} x_j(S) & \in \{0,1\}, \\ \forall j, S & x_j(S) = 1 \Leftrightarrow S_j = S, \\ \forall j & \sum_{S \subseteq M} x_j(S) \leq 1, \\ \forall j & \sum_{S \subseteq M} x_j(S) = 0 \Rightarrow S_j = \emptyset. \end{array} \tag{2.1}$$

In other words, $x_j(S) = 1$ means that the bidder j receives bundle S. We denote the set of all possible allocations by χ.

Definition 2.6 (Total Bidders Utility or Total Bidders Payoff)**.**
Total bidders utility $\pi_{all}(X, \boldsymbol{P}_{pay})$ *is defined as* $\sum_j \pi_j(S_j, P_{pay,j}(S_j))$.

Definition 2.7 (Auctioneer's Revenue)**.**

For M denoting the set of items and N the set of bidders, the auctioneer's revenue at announced pay prices $\boldsymbol{P}_{pay}$ and allocation X is defined as $\Pi(X, \boldsymbol{P}_{pay}) = \sum_{j \in N} P_{pay,j}(S_j) = \sum_{S \subseteq M, j \in N} x_j(S) P_{pay,j}(S)$.

The auctioneer's revenue is usually considered to be his gain, since his costs are assumed to be 0.

Definition 2.8 (Feasible & Efficient Allocations)**.**

A feasible allocation is an allocation that satisfies properties (2.1)*. An efficient allocation is the optimal feasible allocation.*

Thus, an efficient allocation is an allocation that maximizes the overall gain and is usually denoted as $X^* = (S_1^*, \ldots, S_n^*)$.

2.2.3.2 Combinatorial Allocation Problem (CAP) v.s. the Winner Determination Problem (WDP)

Obtaining an efficient allocation is a typical auction design goal. Given the private bidder valuations for all possible bundles, an efficient allocation can be found by solving the Combinatorial Allocation Problem (CAP)

$$\max_{X=(S_1,\ldots,S_n)} \sum_{j \in N} \nu_j(X). \tag{CAP}$$

CAP has a straightforward integer linear programming formulation. Using the binary decision variables $\{x_j(S)\}$, we can reformulate CAP as

$$\begin{array}{lll} \max & \sum_{S\subseteq M}\sum_{j\in N} x_j(S)\nu_j(S) & \\ s.t. & \sum_{S\subseteq M} x_j(S) \leq 1 & \forall j \in N \\ & \sum_{S\ni i}\sum_{j\in N} x_j(S) \leq 1 & \forall i \in M \\ & x_j(S) \in \{0,1\} & \forall j \in N, S \subseteq N. \end{array} \tag{CAP}$$

The objective function maximizes the overall gain. The first set of constraints guarantees that at most one bundle can be allocated to each bidder. The second set of constraints ensures that each item is not sold more than once.

Usually, the auctioneer does not know the bidders' private valuations needed for solving CAP. Instead, he selects the optimal allocation on the basis of the submitted bids. This problem formulation is referred to as the Winner Determination Problem (WDP).

$$\begin{array}{lll} \max & \sum_{S\subseteq M}\sum_{j\in N} x_j(S)P_{bid,j}(S) & \\ s.t. & \sum_{S\subseteq M} x_j(S) \leq 1 & \forall j \in N \\ & \sum_{S\ni i}\sum_{j\in N} x_j(S) \leq 1 & \forall i \in M \\ & x_j(S) \in \{0,1\} & \forall j \in N, S \subseteq N. \end{array} \tag{WDP}$$

WDP is very similar to the CAP. The only difference is the use of bid prices instead of valuations in the objective function. It is important to be aware of the difference between the the two problems in a real auction. In WDP bidders may or may not truly reflect their true valuations. Nonetheless, implementing a VCG mechanism gives the bidders the incentive not to misrepresent their true valuations. Submitting bids equaling bidders' real valuations makes the *optimal allocation* found by WDP equal to the efficient allocation in CAP. Often researchers are not concerned by the fact that the price offers reported may not reflect the bidders' true valuation of bundles. Rather, they use the announced prices and try to find an appropriate modeling and an efficient algorithm that would find the optimal allocation in a reasonable time. Allocating bundles to the winning bidders and modifying the winners' payment rules by asking them to pay the amount that second best winner has bid, implements a VCG mechanism that gives the bidders the incentive to bid their true valuations from the start. Our solution methodology for handling the CAP problem in Chapter 3 follows with this strategy.

There might exist multiple optimal solutions for the WDP with the same objective function value, in which case multiple efficient allocations exist. ***Tie-breaking rules*** determine which optimal solution to select. For example, allocations determined earlier in time or the ones which posses maximum/minimum number of bidders can be preferred. Alternatively, additional constraints can be added to the model to elucidate a unique winner.

Beyond the standard rules of the WDP, additional allocation rules called ***business constraints*** or more generally ***side constraints*** are of practical importance. These constraints may need to be defined or removed dynamically throughout the auction. Below are examples of common constraints in industrial procurement.

- The number of winning suppliers should be greater than a certain number to avoid depending too heavily on just a few suppliers, but not too large to avoid too much administrative overhead.

- The maximum/minimum amount purchased from each supplier is bounded by a certain limit.

- The auctioneer's or the bidders' budget constraints or contractual obligations is met.
- At least one supplier from a target group is chosen.
- Spend from a set of *preferred suppliers* is maximized. Preferred suppliers are those that meet some predefined standards (e.g., consistent delivery, high quality).
- A maximum α % of suppliers account for at least β % of spend. These constraints are usually referred to as *spend constraints.* A practical configuration is to set α and β respectively to 10 and 90.

2.2.3.3 Bidding Languages

A bidding language defines the format of the communicated messages and the interpretation rules by which bidders are allowed to formulate their bids. Here we introduce a few of the most widely used languages. All prices in this section correspond to bid prices, and are all submitted by the same bidder. For the sake of simplicity, we drop the corresponding indices.

Definition 2.9 (Atomic or Single-Minded Bids)**.**
An atomic bid is a pair $(S, P(S))$ *that a bidder submits with* S *as the subset he is willing to bid on for the price* $P(S)$*. Atomic bids cannot be used to represent the simple additive valuation on two items.*

Definition 2.10 (Additive-OR or OR Bids)**.**
Additive-OR bids are a collection of an arbitrary number of atomic bid pairs $(S_k, P(S_k))$,

where S_k is a subset of the items a bidder bids on, and $P(S_k)$ is the maximum price he is willing to pay for it.

The bidder will be willing to obtain any number of atomic bids for the sum of their prices. A set of OR bids is represented as $\nu = (S_1, P(S_1))OR...OR(S_k, P(S_k))$. Define a valid collection W as $W = \{t| \bigcap S_t = \phi\}$. The value of ν is defined as $Max_W \Sigma_{t \in W} P(S_t)$. It can be concluded from the above definition that it is not possible to express substitutabilities via OR bidding language. OR can only be used to represent those valuations where $\forall S \cap T = \emptyset, \nu(S \cup T) \geq \nu(S) + \nu(T)$ and only them [81]. Thus, it is sufficient if no subadditive valuations exist. Unfortunately, this is often not the case, e.g., in the presence of budget restrictions (if the bidder can not afford every combination of bundles he bid for) or when auctioning substitute goods.

Definition 2.11 (Exclusive-OR or XOR Bids)**.**
Exclusive-OR bids are the submission of an arbitrary number of atomic bids, or pairs $(S_i, P(S_i))$, where each S_i is a subset of items and $P(S_i)$ is the maximum price the bidder is willing to pay for it. Unlike OR bids, bidders would only be willing to obtain at most one of the bids they submit.

For the valuation $\nu = (S_1, P(S_1))XOR...XOR(S_k, P(S_k))$, the value ν would be $Max_{i|S_i \subseteq M} P(S_i)$. An XOR bid is capable of representing substitutabilities as well as complementarities among items. However, it suffers from the communicative complexity caused by the exponential number of bundles to be evaluated and monitored.

Definition 2.12 (OR of XORs & XOR of ORs)**.**
OR of XORs is the language that represents OR of a set of XOR bids. Similarly, XOR of ORs represents XOR of a set of OR bids.

The combination of the two languages in the form of OR of XORs and XOR of ORs deploys the power of both languages to make more expressive, yet concise bids.

Definition 2.13 (OR* Bids)**.**

OR represents XOR bids as a variant of OR bids by introducing dummy items.*

This language is first introduced by Fujishima, Layton-Brown, and Shoham [35] and later explored extensively by Nisan [80]. As an example, $(S_1, P(S_1))XOR(S_2, P(S_2))$ can be represented as $(S_1 \cup \{d\}, P(S_1))OR(S_2 \cup \{d\}, P(S_2))$ where d is a dummy item.

2.2.4 Procurement Auctions

The process of procurement via competitive bidding is an auction in which bidders compete for the right to sell their products. In this case, it is the person bidding the lowest who wins the contract. Defining $P_{pay,j(S)}$ as the price that the auctioneer pays to bidder j for providing bundle S, Definition 2.7 is adjusted to represent the auctioneer's ***total cost of procurement(TCP)***.

Chen et al. [19] discuss sealed-bid auctions in procurement settings. Their auctions are incentive compatible and incorporate transportation costs and other variables involved in production.

Demange et al. [30] designed an iterative ascending price auction for the single-seller model which translates to a descending price reverse auction for the single-buyer model. The authors showed that such a descending price auction converges to the maximum competitive equilibrium price in the single-buyer model and thus each supplier gets his VCG payoff.

Ausubel and Milgrom [7], de Vries et al. [27] and Parkes et al. [88] have proposed ascending price auctions implementing the VCG outcome for a single-seller model when buyers demand multiple items. A proper transpose of these auctions will give descending price reverse auctions.

One major common application of reverse auctions is for E-procurement (electronic procurement). The ***electronic auction*** (e-Auction) is an auction between the auctioneer and the bidders which takes place in an electronic marketplace. In this electronic commerce, the auctioneer offers his goods, commodities or services in an auction and interested parties can submit their bids for the product being auctioned.

Using the internet in the so-called ***online auctions*** has propelled the outreach of auctions. The application of the internet surpasses the complications of having (a large number of) participants' physical attendance at a certain place and for a certain period of time. The influx in reachability of (online) procurement auctions makes them appealing for various application domains. Some first applications of these auctions are in *industrial procurements*. Interesting examples include the procurement auctions for school meals in Chile [33], and the one for packing materials (raw and otherwise) for different manufacturing locations at Mars, Incorporated (chocolate manufacturer) [40] developed with IBM T.J. Watson Research Lab.

Another major application of procurement auctions is in *truckload transportation (TL)*. In this auction, the auctioneer and the bidders are respectively the shipper and the carriers. The shipper needs to outsource a number of transportation services to some external carriers. His request is a transportation contract which specifies the pick-up and delivery location pair (also known as a lane), a volume to be shipped on this lane, and some other information on shipping conditions, specific equipments, etc. Several carriers are invited to participate in the auction. They compete by submitting bids on the shippers' requests. Carriers are permitted to submit bids on packages of lanes to express their preferences for any combination of lanes they want to acquire. For instance, to reduce empty repositioning costs, a carrier may prefer to move shipments to a destination and back rather than serving each lane separately.

Ledyard [61] reports the experience of iterative combinatorial auction at Sears Logistics Services for procurement of TL services. The authors report a 13% saving on Sears service costs.

Elmagherabi and Keskinocak [32] document the experience of Home Depot in using single-round combinatorial auctions for outsourcing TL when moving freight between Home Depot stores and design centres. The authors report that not only did the auction provide Home Depot with better rates, many bidders also expressed increased satisfaction with the results. Yet, the single-round auction caused inefficiency which led to aftermarket negotiations.

Sheffi [102] reports the use of procurement combinatorial auctions by many leading companies beyond Sears and Home Depot with the goal to lower transportation costs. These companies include the Colgate-Palmolive Company, Compaq Computers Inc. Wal-mart Stores Inc., Nestle S.A., and Ford Motor Company.

Caplice [18], Caplice and Sheffi [16, 17], Song [103], Song and Regan [105, 106, 105, 91], Crainic and Gendreau [23, 22], Crainic et al. [24], Rekik et al. [92], Remli and Rekik [93], and [62] are among the researchers who have studied combinatorial auctions in transportation procurement.

2.2.5 Iterative Combinatorial Auctions (ICAs)

Single-round combinatorial auctions often lead to aftermarket to remove possible inefficiencies [59]. ***Iterative*** (or ***Progressive***) ***Combinatorial Auctions (ICAs)*** are those auctions that unlike ***one-shot (or single-round) auctions*** proceed in several ***rounds*** (or ***iterations***) providing the bidders with informative feedback. The classical English auction is one example of progressive auctions.

ICAs have many advantages over their one-shot counterpart, especially in procurement:

- Bidders do not have to evaluate, price, and communicate all possible bundles in one shot. The information in ICAs is decentralized and only required information is exchanged on a need-to-know basis. In many auction scenarios, the bid preparation can be very costly and complex when carried in a single round.
- Bidders can revise and modify their bids based on the information revealed in the auction. The bidders witness the progress of the auction and get feedback

through the information from the auctioneer at the end of each bidding round. They have the chance to reassess their bidding strategy several times before the auction closes at the final round.

- Only prospective winners need to work on bid preparation. If a bidder finds himself *non-competitive*, he can save the costly bid preparation process. For a bidder, who finds himself to be a prospective winner, the feedback from the auction will help prepare the bids for the subsequent bidding rounds.

After the auctioneer receives the bids, he solves the winner determination problem to identify the *winners* at the current round known as the ***provisional winners***. He then provides some kind of feedback to support the bidders in improving their bids in the next round. Based on the kind of information he reveals to coordinate the bidding process, ICAs are further divided into ***prices-based*** or ***non-price-based***.

Price-based iterative auctions are *centralized auctions* in which the auctioneer provides the bidders' current winning bids and ***ask prices*** as feedback information. Ask prices specify either a minimum bid price allowable for a bundle, or a minimum percentage improvement over the highest current bid on a bundle. We denote the ask price for bidder j and bundle S as $P_{ask,j}(S)$.

Alternatively, non-price-based iterative protocols are *decentralized auctions* which ask for bidders' cooperation in finding a better allocation in each round. Two well known members of this family are the Adaptive User Selection Mechanism (AUSM) [9] and the Progressive Adaptive User Selection Environment (PAUSE) [55]. Though these auctions avoid the exposure problem, they require full information revelation and introduce high complexity at the bidders' side.

Having bidders cooperate on bid allocation makes the auction vulnerable to the so-called ***threshold problem*** which raises when two small bidders bidding on separate packages (implicitly) collaborate to overcome a third bidder bidding on a package that contains both of their packages. The small bidders are interested in determining what price each of them should pay to ascertain that the sum of both bids exceeds the third bidder's package price.

Despite the benefits of decentralized auctions, centralization is currently considered more promising in the literature. Vagstad [108] claims that decentralization leads to biased decisions (a discriminatory auction). Maurer and Barroso [70] discuss the higher efficiency of centralized auctions for fostering competition in the market. For this reason the concentration of this thesis is on centralized auction designs.

2.2.5.1 State of an ICA

The state of an ICA clarifies the specifications of the auction dynamics. This section describes some main ingredients of the state of an auction.

Timing Issues

In consideration of the bidders' bid submission timing, ICAs are regarded as either ***continuous auctions*** or ***multi-round auctions***(a.k.a. ***round-based auctions*** or ***discrete auctions***). In continuous auctions, bids are evaluated on arrival of every new bid with continual updates to the current provisional winners and prices. Continuous auctions contribute to a more dynamic environment, since the feedback information is kept up to date at every point in time throughout the auction. However, continuous combinatorial auctions are usually considered impractical, since they lead to high computational costs for the auctioneer (the winner determination must be done whenever a new bid is submitted) and to high monitoring and participation costs for bidders. Alternatively, in multi-round auctions bids are collected over a period of a round before the bid evaluation is performed. All iterative auction designs discussed in this thesis are round-based.

Information Feedback

The key challenge in the design of ICAs is providing information feedback to the bidders after each auction iteration to guide bidding towards an efficient solution. Information feedback about the state of the auction can contain pricing information, the provisional allocation (if any), the list of bids submitted by other bidders, etc. Information hiding can also be used to limit the possibilities of signaling between bidders. While the purpose of providing this feedback is to help make the bidding process more effective, it should be noted that the information released should not facilitate bidders manipulation through signalling or formation of coalition.

Bidding Rules

Bidding rules define, what bids can be submitted or revoked in the current auction state and how the auction state evolves throughout the auction. Ask prices are a common form of bid improvement rules.

Activity Rules (a.k.a. Eligibility Rules)

Activity rules enforce active bidding throughout the auction as opposed to the *wait-until-auction-end-and-snipe* strategy often used by online auctions. Activity rules were introduced in the early Federal Communications Commission (FCC) wireless spectrum auctions and proved important. Decisions about appropriate activity rules are often guided by a tradeoff between allowing for straightforward bidding strategies and encouraging early bidding [87].

Allocation Rules

Allocation rules regulate the selection of the winning bids from the set of submitted ones. Specifically, they determine the formulation of the winner determination problem where the auctioneer's procurement cost is usually minimized subject to the bidding language rules and the inability to sell the same item more than once. In spite of their practical importance, there is a gap in the theory of ICAs in regard to business constraints. Kalagnanam et al. [44], Sandholm and Suri [99], and Collins et al. [21] analyzed the impact of business constraints on the solution and complexity of the WDP.

Termination Condition (a.k.a. Closing Condition)

For a round-based auction design two termination conditions are of importance: conditions to terminate each round known as ***round closing rules***, and conditions to terminate the whole auction known as ***auction closing rules***. With round closing we denote the point in time at which a specific auction round is declared closed, and no more bids are accepted until the start of the next round. We call the time period between the round start and the round closing ***round duration***. After the round is closed, the bid evaluation process, called ***round clearing***, starts.

Setting the round closing rules to control the duration of a round is not an easy task. While giving time is necessary for the bidders to reveal preference elicitation in every round, too much time harms the auctions by reducing the bidders' interest and concentration on the bid preparation process. This problem magnifies towards the final rounds of the auction when preference elicitation is expected to take less time. The round duration is usually set to a ***fixed-time round period***. A ***Ready-in-round*** strategy mitigates the problem by letting bidders communicate their state to the auctioneer when they are ready. The round is then closed as soon as all participating bidders have indicated their readiness.

After a round is closed and cleared, the auction either moves to the next round or terminates according to the auction closing rules. In the latter case, the auction is first ***closed*** (the bidders are informed that no more bids can be submitted) and the final bid evaluation, called ***auction clearing***, starts. The time period between the auction start and the auction closing is called ***auction duration***. Some common auction termination rules include closure

- at a fixed deadline,
- in a limited time duration,
- after a maximal number of rounds,
- when no competitive bid is submitted,
- when the allocation does not change for a certain number of rounds.

The first three conditions are examples of fixed deadlines and the last two items demonstrate a ***rolling closure*** in which the auction remains open for as long as competitive bids are submitted. Roth and Ockenfels [94] have studied the use of deadlines versus rolling closures, as respectively practiced on eBay and Amazon Internet auctions. Bidders on Amazon (with rolling closer) bid earlier than on eBay (with fixed deadline). In fact, many bidders on eBay wait until the last seconds of the auction to bid while Amazon auctions encourage earlier bidding.

Thus, fixed deadlines are useful in settings where bidders are impatient and unwilling to wait a long time for an auction to terminate. However, they require stronger activity rules to prevent the auction from reducing to a sealed-bid auction with all bids delayed until the final round. In comparison, auctions with a rolling closure encourage earlier bidding since they remain open for just as long as competitive bids are submitted. In this thesis we use a rolling closure with the auction open for as long as competitive bids are submitted.

Proxy Agents

With proxy agents bidders can provide direct value information to an automated bidding agent that bids on their behalf. Usually bidders enter the maximum amount they are willing to pay for a package to a proxy machine. This value is kept confidential. The proxy agent places bids on behalf of the bidder using a specified automatic bid increment amount. The proxy agent bids only as much as necessary to make sure that the bidder remains the winner up to his maximum amount. If another bidder places the same maximum bid or higher, the proxy sends out notification to the bidder to either raise his price or that he will lose. In order to realize the elicitation and price discovery benefits of an iterative auction, the bidder-to-proxy language should allow bidders to express partial and incomplete information to guarantee refining during the auction.

Proxy auctions facilitate faster convergence with rapid automated proxy rounds, restricting the strategy space available to bidders. In particular, proxy auctions usually have better control to prevent ***shill bidding***. Shill bidding happens when a person publicly helps another person or organization without disclosing that they have a close relationship. Shill bidders seek to provoke the *bidding war* among other participants by submitting fake bids. eBay runs proxy auctions as a type of an English second-price auction with the difference that the current highest bid is not sealed and is always displayed. eBay forbids shilling; its rules do not allow friends or employees of a person selling an item to bid on the item. Ausubel and Milgrom [7] study ascending price proxy auctions with package bids. They show that compared to the Vickrey auction, the proxy auctions generate higher equilibrium revenues and are less vulnerable to shill bidding and collusion.

Forthcoming concerns with the design of proxy auctions include when to allow proxy information to be revised, what increment to use when increasing the prices, and how to ensure trust and transparency since the bidding activity is transferred to automated agents. Studying the effects of proxy bidding is out of the scope of this thesis. For more information on the topic see Parkes and Ungar [85] and Ausubel and Milgrom [7].

2.2.5.2 Pricing Equilibria

As discussed in Section 2.2.5 a crucial ingredient of an iterative auction design is the definition of *bidding rules*. The literature on ICAs mostly focuses on designing these rules to lead the auction to an *efficient*, rather than *optimal*, outcome. To coordinate the bidding process toward an efficient outcome, price-based ICA implement various *price-update* methods which characterize the rules by which prices are computed in each round. Some major price update rules include:

- ***Greedy update.*** The price is increased on some arbitrary set (perhaps all) of the *over-demanded* items or bundles.
- ***Minimal update.*** The price is increased on a minimal set of *over-demanded* items, or based on the bids from a set of *minimally under-supplied* bidders.
- ***LP-based update.*** Given current bids, a linear program is formulated to find prices that are good approximations for the equilibrium prices.

Based on the characterization of the ask prices, the hierarchical structure of the ***pricing schemes*** is:

1. linear anonymous ask prices,
2. non-linear anonymous ask prices,
3. non-linear non-anonymous ask prices.

Below we define the linearity and anonymity of prices.

Definition 2.14 (Linear (Additive) Ask Prices)**.**

A set of ask prices are called linear if the price of a bundle is always equal to the sum of the prices of its items, i.e., for a set of ask prices $P_{ask,j}(S)$*,*

$$P_{ask,j}(S) = \sum_{k \in S} P_{ask,j}(k) \qquad \forall j, S.$$

Non-linear ask prices are also called ***bundle ask prices***.

Definition 2.15 (Anonymous Ask Prices)**.**

A set of ask prices are called anonymous if the prices of the same bundle are equal for every bidder, i.e., for bidders j, j' *and bundle* S*,*

$$P_{ask,j}(S) = P_{ask,j'}(S).$$

Non-anonymous ask prices are referred to as ***discriminatory*** or ***personalized*** ask prices.

The *linear anonymous* pricing scheme is the simplest, easily understandable, and usually considered fair by the bidders. The communication costs are also minimized, since the amount of information to be transferred is linear in the number of items. *Non-linearity* is essential to allow bidders to express super- or subadditivity in their valuations. Yet, it is often perceived as too complex by the bidders. Also, it increases the communication costs, i.e., in the worst case, an exponential number of prices need to be exchanged. Sometimes, non-linear anonymous prices are not sufficient to lead the auction to competitive equilibrium. In this case *non-linear non-anonymous* prices are introduced.

Definition 2.16 (Competitive Equilibrium (CE) Prices).

The prices $\boldsymbol{P}_{pay}$ *and allocation* $X^* = (S_1^*, \ldots, S_n^*)$ *are in a competitive equilibrium (CE) if:*

$$\begin{aligned} \pi_j(S_j^*, \boldsymbol{P}_{pay}) &= \max_{S \subseteq M} [\pi_j(S, \boldsymbol{P}_{pay}), 0] \qquad \forall j \in N \\ \Pi(X^*, \boldsymbol{P}_{pay}) &= \max_{X \in \chi} \Pi(X, \boldsymbol{P}_{pay}) \end{aligned}$$

where bidder $j \in N = \{1, \ldots, n\}$*, set* S *is a subset of the set of items* $M = \{1, \ldots, m\}$*, and* S_j^* *denotes the bundle allocated to* j*.*

Theorem 2.1 (Bikhchandani and Ostroy [13]). *Allocation* X^* *is in the competitive equilibrium* if and only if S^* *is an efficient allocation.*

The idea of competitive equilibrium prices is to determine prices that characterize an efficient allocation. In CE the utility of every bidder and the auctioneer revenue are maximized at the given prices and the auction will effectively end since the bidders will not be willing to change the allocation by submitting any further bids.

Non-linear discriminatory competitive equilibrium prices always do exist for CAP [87], however, these prices can result in additional non-linearity complexity and are often considered unfair by bidders [14]. This motivates the auctioneer to construction ICAs that update ask prices in the direction of possibly linear, or non-linear but anonymous CE prices. Ideally, such prices converge not just to CE prices, but minimal CE prices defined as follows.

Definition 2.17 (Minimal CE Prices).

Minimal CE prices minimize the auctioneer's revenue $\Pi(X^*, \boldsymbol{P}_{pay})$ *on an efficient allocation* X^* *across all CE prices.*

Minimal CE prices construct an upper bound on the VCG payments. On a restricted class of valuations, minimal CE prices are equivalent to VCG payments which is a desirable property, since it imposes the incentive compatibility of the auction design [87]. But, as suggested experimentally, implementation of the required bidders' valuation condition that implies equivalence of minimal CE prices and VCG prices often fails in realistic ICA settings, making the minimal CE prices incapable of supporting the VCG payments.

Parkes and Ungar [84] and Mishra and Parkes [73] designed a price-based ICAs by characterizing universal CE (UCE) prices. UCEs imply stronger restriction on CE and reveal enough information to determine VCG payments from these prices.

Definition 2.18 (Universal CE (UCE) Prices)**.**
Prices $\boldsymbol{P}_{pay}$ *are universal competitive prices if:*
a) Prices $\boldsymbol{P}_{pay}$ *are CE prices.*
b) Prices $P_{-j,pay}$ *are CE prices for* $CAP(N\backslash j)$*, meaning that they support some efficient allocation in* $CAP(N\backslash j)$*.*

In words, UCE prices are CE prices in the main economy and in every marginal economy. Note that UCE prices must support *some* efficient, not necessarily the *same*, allocation in every marginal economy.

Theorem 2.2 (Parkes and Ungar [84]; Lahaie and Parkes [60]))**.**
A combinatorial auction realizes the VCG outcome if and only if the auction also realizes a set of UCE prices and an allocation in the price equilibrium of the main as well as all marginal economies.

In general, UCE prices are greater than the minimal CE prices because they must consider competition in the marginal economies in addition to the main economy. Mishra [73] note that minimal CE prices are universal for a restricted set of valuations.

2.3 Major Challenges in the Implementation of Iterative Combinatorial Auctions

Alongside their advantages, ICAs bring in a host of new challenges and questions. This section is devoted to exploiting some major challenges faced by the bidders and the auctioneer.

On the bidders' side, the complexity is more of an issue in practice. Bidders must elicit their valuation for an exponential number of items. Next, they need to communicate their preferences with the auctioneer in an efficient and yet concise way. Above all, bidders should adopt an appropriate bidding strategy to pick and price new bundles that increase their chances of winning.

Determining the optimal bid prices in various auction designs has been a concern in classic game-theoretic auction research, but it is even more challenging in ICAs. For example, it is possible that a losing bid becomes winning in a subsequent round without changing the bid. The bidders face the problem of choosing appropriate bundles and bid prices.

Once the bidders evaluate and report their desired bundles in CAP, the auctioneer's job starts to find out the *efficient allocation* of bundles to the bidders, including the possibility that he retains some items that maximize his revenue. As discussed in Section 2.2.3, this reduces to solving WDP to decide what bundles to allocate to which bidders based on their reported bid prices.

Intuitively, solving WDP seems hard because one would need to check for each subset of the bids whether the subset is feasible (no bids within the subset share items) and how much revenue that subset of bids provides. A feasible subset that yields the highest revenue is an optimal solution. With K being the number of bids all bidders submit, there are 2^K subsets of bids, so enumerating them is infeasible. This problem is even more critical in an iterative auction setting when WDP has to be solved several times during the auction. This highlights the need for the auctioneer to adopt appropriate solution methodology that takes reasonable time to investigate the solution space. Decreasing the execution time required at each round of an ICA, reduces the total auction duration and thus improves upon the overall auction efficiency.

In order to computationally evaluate various solution algorithms for WDP, it is necessary for the auctioneer to be aware of how the performance of algorithms compare in a laboratory environment. Accessing records of the bidders' previous bidding behavior is inapplicable *first* due to the limited access to the actual records of bidders' previous behavior, and *second* due to the scaling problem that arises when acquiring this data for various problem sizes. This brings up the need for the auctioneer to simulate bidders' bidding behavior, as realistically as possible, to better evaluate and compare various solution methodologies.

The auctioneer's next bottleneck is determining the prices of items and/or bundles in ICAs for accounting and costing purposes. Since in combinatorial auctions not all items may show up individually in singleton bundles, the auctioneer is interested in determining the price of individual items after the winner determination problem is solved.

2.3.1 The Preference Elicitation Problem (PEP)

In an auction environment, bidders are usually unwilling to reveal their valuations on all bundles. *First*, due to their privacy settings, agents may prefer not to reveal their valuation information [96]. *Second*, solving the ***bidders' valuation problem*** requires selection and valuation of the bundles to bid for from an exponentially large set of possible bundles. Determining the value for a single bundle can be computationally demanding for the bidders in many environments. For instance, in the airport landing slot scenario, airlines would need to solve local scheduling, marketing, and revenue management problems to determine their values for different combination of slots [8]. Requiring this computation for exponentially many bundles is in many cases impractical.

Iterative auctions are a promising platform to mitigate this hurdle by providing the bidders with enough time to put together their valuations on the most interesting bundles.

2.3.2 Strategic Bidding

Once the valuations are determined, bidders need to strategically formulate their new packages. This problem is also termed as the ***Bid Generation Problem (BGP)***. BGP has attracted little attention in the auction literature. If bidders do not know which bundles are most beneficial and how much they should bid for them, the resulting allocation cannot be efficient. Hence, the BGP has to be addressed before a combinatorial auction can be claimed to achieve efficiency.

Wang [110] stresses the need for research on the bid generation problem. He investigates the problem that bidders face for providing services in a transportation combinatorial auction with OR bidding language and developed necessary conditions for bid generation, but did not address the pricing problem. Lee et al. [62] consider the carrier's optimal bid generation problem in combinatorial auctions for transportation procurement and design a carrier optimization model that integrates the generation and selection of routes. Their model has not been validated in a multi-round setting. Lorentziadis [69] considers the bidder pricing problem in the presence of a fixed cost as a function of the unit costs, auction fixed cost, order quantities and the minimum profit margin. Hsieh [41] proposes a model for finding both bundle prices and quantities for the bidder. The prices are the Lagrange multipliers derived from a Lagrangian relaxation of the winner determination problem. He formulates a profit maximization model for the bidders based on these prices. This model is studied in more detail in Chapter 4.

2.3.3 Communication Complexity

Having prepared all the bids to be submitted, bidders need to communicate (in the worst case an exponential number of) their bids to the auctioneer. This adds a comminution cost on the bidders to report their bids to the auctioneer. Nisan [80] addresses this problem through the design of ***bidding languages***. Careful design of bidding languages allows for compact representation of the bidders preferences to ease the communication process. More details on the bidding languages is provided in Section 2.2.3.3.

2.3.4 Solving the Winner Determination Problem (WDP)

Considering bids as subsets of a set of items and their weights as the prices attached to them reduces WDP to an integer linear programming model of a *weighted set packing problem* which seeks the largest total weight corresponding to pairwise disjoint weighted subsets of a set of items. NP-hardness of WDP can be deduced from that of the weighted set packing problem [95]. There has been various methodologies proposed for solving the winner determination problem in combinatorial auctions. We classify and describe some major approaches below.

2.3.4.1 Implementation of Economical Limitations

One approach to solve WDP optimally and provably fast is restricting bundles upon which bids can be submitted [95, 107, 76, 63, 14]. For instance, Bikhchandani and Ostroy [14] represent the problem of assigning a set of items to some bidders with the condition that each bidder receives at most one subset as a linear programming formulation with integer optimal solution. The significance of their model is the existence of dual variables that reflect items' prices and the bidders' marginal product. Necessary and sufficient condition for this link to exist is that the *bidders are substitutes* meaning that the marginal product of a group of bidders is more than the sum of the contributions of individual members of the group. The decentralized primal dual procedure for solving the LP approaches to the pricing equilibrium corresponding to social opportunity cost and yields an incentive compatible auction.

Despite the computational attractiveness due to the bidders' limited preference elicitation opportunity, this approach suffers from economic inefficiencies, the same as non-combinatorial auctions.

2.3.4.2 Application of Exact Solution Algorithms

Because WDP is NP-hard, an optimal algorithm for the problem will be slow on some problem instances (unless P = NP). Exact algorithms solve the problem to the optimal solution. Much of the research on solving WDP to optimality has been carried out by applying artificial intelligence (AI) techniques such as intelligent search. Direct application of commercial integer linear programming solvers also belongs to this class of approaches.

Leyton-Brown et al. [67] work on multi-unit combinatorial auctions and introduce CAMUS (Combinatorial Auction Multi-Unit Search), an algorithm for determining the optimal set of winning bids in multi-unit combinatorial auctions. The method they propose is based on a branch and bound technique. Sandholm [100] proposes a Depth-First Branch and Bound algorithm (DFBnB) that branches on bids. Their algorithm, also called CABOB (Combinatorial Auction Branch On Bids), beats CPLEX 8.0 execution time on some test sets, but there are several cases where it is drastically slower.

2.3.4.3 Application of Approximation Algorithms

An approximation algorithm strives to solve an NP-hard problem in polynomial time to an almost optimal solution with provable performance guarantee. Sandholm [98] and Lehman at al. [64] derive several approximation impossibility results for the winner determination problem. In fact, application of approximation algorithms on WDP can sometimes produce solutions which are quite far from the optimal [63, Chapter 12] and can lead to reducing the total revenue in a direct auction or increasing the total cost in a reverse auction.

2.3.4.4 Application of Heuristic Algorithms

Heuristic algorithms trade off the expected cost of the additional computation (cost of the computational resources and the cost associated with delaying the result) for finding an exact optimal solution against finding a reasonably good near-optimal one. Although heuristics do not generally provide a guarantee on the solution quality or runtime, experimentally they prove useful.

One stream of research in this category concentrates on applying the Lagrangian relaxation (LR) followed by heuristic solution algorithms that maps the optimal Lagrangian solution on to the feasible region of WDP. Guo et al. [39] models the combinatorial auction problem as a set packing and applies the Lagrangian relaxation method. Song et al. [104] show application of Lagrangian relaxation on business-to-business (B2B) procurement combinatorial auctions. Kameshwaran et al. [47] demonstrate the design of progressive auctions using Lagrangian relaxation. They consider procurement of multiple units of a single item with linear and piecewise linear supply curves.

Hsieh [41] adopts a subgradient method to solve the Lagrangian relaxation problem in a multi-unit multi-item reverse WDP. Based on the revelation of Lagrange multipliers, the author presents a scheme to help bidders generate potential winning bids in a multi-round auction format and proceeds to propose a heuristic algorithm to fix possible infeasibilities resulting from the Lagrangian optimal solution. In Chapter 4 we comment on several inconsistencies within the author's proposed problem formulation, heuristic algorithm, numerical example, and empirical experiments.

A major advantage of implementing Lagrangian relaxation on ICAs is that the Lagrange multipliers act as approximates of the item prices. Once revealed to the bidders as feedback in an iterative auction, they can help bidders modify their bids accordingly. However, a considerable drawback of applying this methodology is that classical methodologies such as the Lagrangian decomposition method and subgradient method for solving the Lagrangian relaxation problem, usually consume too much execution time. This time is further added to the time required by a heuristic algorithm which runs to fix infeasibilities from the Lagrangian solution, in case they exist. In addition to the slow convergence rate, the convergence of these algorithms is highly sensitive to the initialization of the parameters involved, i.e., an improper initialization can cause the algorithm to diverge.

As much as the optimal Lagrangian values are valuable to us, solving the direct relaxation problem is unappealing. Our proposed solution algorithm presented in Chapter 3 provides a technique for solving the Lagrangian dual problem associated with the WDP in a single iteration. Consequently, this method quickly provides the desired information about the optimal Lagrangian objective value, multipliers, and solution. This is followed by introducing an aggregate heuristic algorithm for adjusting the solution to a near optimal one. Our extensive numerical experiments illustrate the class of problems on which application of this technique provides near optimal solutions in much less time as compared to the CPLEX solver.

2.3.4.5 Auction Simulation

It is necessary to generate artificial data that is representative of the sort of scenarios the auctioneer is likely to encounter. Sandholm [97, 98, 100], Fujishima et al [35], Boutilier et al. [15], deVries and Vohra [27] have suggested bid generation techniques based on the number of bids and goods. These methods have drawbacks according to [67].

- Which goods to request?

  Most data generation techniques assume equal likelihood for items to appear in bundles of the same size.

- How many goods to request in a bundle?

  These methods determine how many goods to include in a bundle independent of which goods have already been selected for the bundle.

- What price to offer for each generated bundle?

  These procedures suggest drawing prices randomly from

  1. interval [0,1] [98],
  2. interval [0,g] for g being the number of goods requested [98],
  3. normal distribution with mean 16 and standard deviation 3 [15],
  4. interval [g(1-d),g(1+d)] with d=0.5 [35],
  5. a quadratic function of the prices of items included in the bundle [27].

  Techniques 1 and 3 suffer from the fact that the price is independent of the number of goods. In 2, mean and range are parameterized by the same variable. Methods 1 to 4 do not consider which goods have been considered in the bundle when pricing them. In technique 5, bundle prices are expressed as a function of the length of the bundle which is hard to control as the number of the items in the bundle increases.

Leyton-Brown et al. [67] provide a Combinatorial Auctions Test Suite (CATS) which attempts to model realistic bidding behavior via a set of distributions. This facilitates the study of algorithmic performances and their comparison against the previously published results. The suite includes distributions based on real-world applications. In most of the distributions, bids are generated from a graph that depicts the economical relationships between items. In each bid, ceratin goods are more likely to appear together when there exists complementarities between them. In addition, the number of goods to be included in each bundle relates to which goods it contains. Once the bids are generated, price offers are related to goods included in the bundle. CATS has the flexibility to constitute subadditive, additive, or superadditive price offer values in the number of goods requested.

The empirical study of CATS reveals that several CATS instances are quite easy for ILOG CPLEX. Since CATS tries to simulate realistic bidding behavior, this implies that practical problems are usually easy to solve. Also, the hardness level can vary significantly from instance to instance despite fixing the problem size and the distribution [66].

We use CATS throughout the experimental study of our proposed algorithm to generate combinations of items with various cardinalities of the bids submitted, and a real-valued number associated to each bid to describe the price asked for it. We are not concerned with the hardness level of the data generated by CATS since our algorithm is compared against CPLEX on the same data set. We consider a subadditive environment to simulate the objects' complementarities in a reverse combinatorial auction environment.

2.3.5 The Pricing Problem

After an auction terminates, it is valuable to determine the value of the good and/or bundles for clarification of auction results. These prices can also help as guidelines for future auctions. Based on the preliminaries discussed in Section 2.2.5.2, we categorize the literature on pricing ICAs into two major categories: bundle pricing and item pricing approaches.

2.3.5.1 Bundle-Pricing Approaches

With bundle bids, setting ask prices for individual items is not obvious and often even impossible [14]. Additionally, ask prices may need to be personalized, i.e., different bidders get different prices for the same items or bundles, as opposed to the traditional anonymous prices. Bundle pricing, as its name suggests, is used to compute a final price for each bundle. Due to the overlap of some bundles, bundle-pricing approaches require additional assumptions that:

- every bidder must bid on every bundle,
- each bidder gets, at most, one bundle in the resulting allocation.

In order to avoid evaluating all bundles, a bidder can report only valuations for interesting bundles and have a computerized agent fill in valuations for the remaining bundles according to some rule. Condition two implements XOR condition to ensure that the bidders are able to receive multiple items only when they have paid for the complementarities among them. The ideal goals for determining bundle prices for combinatorial auctions are to reach

- *market clearing prices* at which the price of a winning package is not less than the sum of all prices of the goods it includes, and the price of a losing package is less than this sum.
- *incentive compatibility prices* that allows straightforward bidding with truthful valuation revelation.

Leonard [65] investigates incentive compatible prices of the well-studied assignment problem in operations research. The assignment problem is an integer programming problem that is totally unimodular and hence can be solved as a linear programming problem. Leonard's paper identifies a set of shadow prices which maximize the sum of the dual variables. The author shows that not only do these prices clear the market, but also provide incentive compatibility.

Wurman and Wellman [112] discuss the bundle pricing problem for WDP with XOR bidding Language. They implement an LP-based method to update non-linear but anonymous price approximations. Prices converge to market clearing prices, but they are not incentive compatible. After the procedure, every winning or losing bundle is assigned a price. This method prices winning as well as losing bundles. The prices for winning bundles are the result of the dual of an assignment problem. For the losing bundles, they can be infinitely large so that no bidder is interested to buy them. As a result, prices for the losing bundles are considered less informative of the true value of the bundles than those for the winning ones. Without the information about which bundles are won and which ones are not, these prices have little value for helping the potential bidders valuate different bundles.

Bikhchandani and Ostroy [14] discuss the package assignment problem with the XOR bidding Language. They reserve *packages*, rather than bundle, for a specific bidder instead of everyone. The authors add auxiliary variables to the original integer programming (IP) problem. Using the linear programming (LP) duality they provide the sufficient and necessary condition on the buyers' valuation for the packages to have a market-clearing and incentive compatible property. This valuations requirement is known as *buyers are substitutes* and means that the marginal product of any buyer subset is greater than the sum of its individual buyer's.

The authors prove that under the condition that buyers are substitutes, the LP relaxation gives integer solutions and the optimal value of the dual variables give the Vickrey discount for each bidder. Another paper by Bikhchandani et al. [13] proposes a model that also has an integral optimal solution. Under the condition that *buyers are substitutes*, the dual variable are exactly the Vickrey discount of the bidder. The prices for the winning bundles are derived by subtracting the Vickrey discount from the winners bid. Despite the economical attractiveness, since both models introduce a variable for every feasible integer solution, the number of variables needed for the WDP is exponential in the number of bids.

Parkes [86] proposes a bundle pricing scheme as part of an iterative ascending auction model which approximates the VCG outcome. Parkes argues that while the procedure in Wurman and Wellman provides the VCG outcome that incentivizes truthful bidding, it needs complete information of bidder evaluation on every bundle. Bidders may be unable and/or unwilling to reveal their full valuation.

In this approach, Parkes imposes the same set of assumptions as Wurman and Wellman to propose a two-stage procedure with iterative ascending combinatorial auctions. An iterative combinatorial auction runs in the first stage which terminates with an outcome close to the optimal allocation. In the second stage Vickrey discount is computed for the winning bidders whose absence from the auction causes some other winning bidders to lose. While the process terminates with an approximation of the VCG prices, bidders do not need to reveal complete information of their valuation at the beginning of the auction. VCG approximation is only perfect when the bid increment goes to zero in which case the iterative bidding process gets too long.

Mishra and Parkes [74] design an ascending price auction which seeks for non-anonymous universal competitive prices (UCE). Their auction design calculates payments of buyers from the final UCE price and allocation and implements the VCG outcome under any valuation profile of buyers. This work assumes XOR bidding language with no assumption on bidders' valuations. The proposed approach is particularly useful when no CE price supporting VCG payments is observed, i.e., substitutes condition on valuations does not hold. The authors relate this work to the work of de Vries et al. [31] and show that when buyers are substitutes, their auction converges to a UCE price.

deVries, Schummer and Vohra [27] construct non-anonymous prices for an ascending auction for heterogeneous objects. The auction assigns prices to bundles and asks bidders to report their preferred bundles in each round. Bidders' prices follow a minimal update pattern on a set of *minimally undersupplied* bidders. All bidders in a minimally undersupplied set face higher prices on the bundles for which they submitted a bid. The auction realizes VCG outcome for a limited class of problems. The authors also show in [26] that a stronger condition on the submodularity of the coalition values needs to be satisfied to achieve the VCG outcomes with an ascending price auction.

Bichler et al. [12] experimentally compare three selected linear price ICA formats based on allocative efficiency and revenue distribution using different bidding strategies and bidder valuations. The authors discuss the efficiency of computational sciences and methods as strong alternatives to complement the theory and experiment in combinatorial auctions, and understand phenomena that have been shown to be difficult to analyze. Specifically, using computational methods, they show that ICA designs with linear prices perform very well for different valuation models even in cases of high synergies among the valuations. They observe, however, significant differences in efficiency and the revenue distributions of the three ICA formats they use.

Scheffel et al. [12, 101] propose an interesting discussion on the comparison of the auction designs imposing linear versus non-linear package prices. They analyze aggregate metrics such as efficiency and auctioneer revenue for small- and medium-sized value models. As already seen, based on strong theoretical foundations, auction formats such as ascending proxy auction and iBundle result in Vickrey payoffs when the coalitional value function satisfies buyer submodularity conditions and bidders bid their best responses. These auction formats are based on nonlinear and personalized ask prices. The authors show that experimentally these approaches lack efficiency as compared to some other linear price auctions. In a lab environment, iBundle requires a large number of auction rounds and fails to meet the buyer submodularity conditions in most realistic settings. The bidders' also fail to strictly follow best-response strategies in difficult decision situations. In fact, the bidder find it difficult to choose one or more bundles from a set that is exponential in the number of items and attach a bid price to each bundle). To remove this extra complexity, we also follow a linear price structure in this thesis.

2.3.5.2 Item-Pricing Approaches

The bundle pricing approach gives prices only for bundles. So, if no winning bundle contains only a single component of interest, bundle pricing is unable to provide a way to determine an individual component price. Several researchers have worked on pricing individual items in a non-combinatorial environment.

Kelso and Crawford [56] study the labor markets with heterogeneous firms and workers and perfect information. They show that equilibrium in such markets exists and is stable provided that all workers are gross substitutes from each firm's standpoint. They use a greedy update method to provide non-anonymous prices for individual items.

Demange, Gale and Sotomayor [30] define a minimal price update, increasing the prices on a minimal overdemanded set of items for the assignment model. Minimal price updates are adopted to drive individual item prices towards minimal CE prices.

Gul and Stacchetti [38] study the problem of efficient production and allocation of indivisible objects among a set of consumers. They propose conditions which are equivalent to the gross substitute (GS) condition of Kelso and Crawford with the assumption of quasilinearity. Under the GS condition, the auction converges to the smallest Walrasian prices which in turn corresponds to the VCG payments.

Milgrom [72] reviews the uses of economic theory in the design and improvement of the *Simultaneous Ascending Auction (SAA)* which was developed initially for the sale of radio spectrum licenses in the United States. At each round bidders simultaneously make sealed bids for any items in which they are interested.

Ausubel [5] proposes a dynamic design for auctioning multiple heterogeneous commodities. An auctioneer wishes to allocate one or more units of each of K heterogeneous commodities to n bidders. The auctioneer announces a vector of current prices, bidders report back quantities demanded at these prices, and the auctioneer adjusts the prices. Nevertheless, with pure private values, the proposed auction yields Walrasian equilibrium prices.

Ausubel et al. [6] propose a two stage combinatorial auction called *Clock-Proxy Auction*. At the clock stage only single-item bids are allowed and at the proxy stage a round of sealed-bid second-price auction takes place. This auction also reveals price information on individual items.

Cramton [25] designs Simultaneous Multi-Round Design (SMR) which allows only single-item bids, is iterative, and has an eligibility-based stopping rule (i.e., *a use-it-or-lose-it* feature) driven by a minimum increment requirement for new bids. This auction runs multiple single-item auctions simultaneously and was extensively used by the FCC to run early bandwidth auctions. SMR auctions can take a very long time to complete.

Determining the individual prices in a combinatorial environment is not a trivial task. Kwasnica et al. [58] adopts an LP-based price update method and adjusts prices to find good approximations to CE prices given current bids and the current provisional allocation in a multi-object auctions. The authors merge the better features of the Simultaneous Multiple Round (SMR) [25] and the Adaptive User Selection Mechanism (AUSM) of Banks et al. [9] and add one additional feature. The new design is called the Resource Allocation Design (RAD) auction process. For substitutes valuations, this auction reduces to a simultaneous ascending price auction. Like SMR design, RAD is iterative, has an eligibility-based stopping rule, forces a minimum bid increment, and computes prices for each item for sale and like AUSM design, it allows package bidding. To approximate the item prices, this approach attempts to compute a set of prices such that for any winning bundle the sum of the prices of its comprising individual items is equal to the bid and for losing bundles the difference is minimized. The authors show that RAD performs better than both SMR and AUSM achieving higher efficiencies, lower bidder losses, higher net revenues, and faster times for completion without increasing the complexity of a bidders problem. However, formal convergence properties have not been proved for RAD.

O'Neill et al. [83] discusses pricing a more general resource allocation problem with a non-convex objective function. The main idea is to associate a cost with each positively valued integer variable. They show that the optimal solution to a linear program that solves the mixed integer program has dual variables that have the traditional economic interpretation as prices and clear the market in the presence of non-convexities. After solving the resource allocation problem, a new equality constraint is added for each positively valued, optimal integer variable that sets the variable to its optimal value. Next, they solve the dual for the new LP which has the same optimal solution as the original IP. The price is the sum of all composing items plus the additional dual variable corresponding to the integer variable.

Xia et al. [113] show that ONeill [83] prices are not unique, and the prices corresponding to the unallocated items are always zero. The authors also prove that under special circumstances the ONeill [83] and DeMartini prices in [58] are in fact equal. This includes when all the goods are sold in the optimal allocation as well as when some goods remain unallocated with zero DeMartini prices.

Jones and Andrews [43] use the maximum likelihood to estimate the distribution of item prices based on final winning bundle prices. Aparicio et al. [3] present an algorithm for solving an iterative multi-unit combinatorial auction. At each round of the auction, the auctioneer computes a linear anonymous price for each item using a DEA model and pushes bidders to express bids according to them.

Most auction designs providing item-price feedbacks concentrate on combinatorial auctions with a single unit of multiple heterogeneous items. Iftekhar et al. [42] address this gap by evaluating several feedback schemes or algorithms in the context of multi-unit auctions. They numerically evaluate the algorithms corresponding to different scenarios that vary in bidder package selection strategies and in the degree of competition.

Despite the research on item/bundel pricing methodologies as described in Section 2.3.5 and the Section 2.3.2 suppliers' bidding strategy techniques, there is a lack of studies that look at the integration of the two methodologies: develop an appropriate pricing scheme for the auctioneer based on which the bidders package and submit new more competitive bids. In Chapter 4, we simultaneously look at the auctioneer's and the bidders' utility maximization optimization problems and propose a provably convergent iterative auction.

2.4 Divisible-Bid Auctions

When auctioning off multiple items, it is crucial to understand the intrinsic physical nature of the item(s) that are up for bids. When a seller offers some *amount* of a good for sale, the auction is called ***divisible*** or ***continuous*** since the feasible volumes of offers are continuous. Auctions of divisible goods are commonplace in markets for financial securities, energy products, and environmental permits. In such auctions, the bids specify quantities of the divisible goods: The shares of stock, the megawatt-hours of electricity, or the tons of emissions.

When all the goods are divisible, the WDP is a linear programming problem. The value of LP dual variables also gives individual prices, and the bundle prices from the dual problem are asymptotically incentive compatible. When goods are ***indivisible*** or ***discrete***, the duality gap of integer programming (IP) assures that equilibrium prices exist only in special cases [79].

Abrache et al. [1] pointed out that the languages previously considered in the literature (as described in Section 2.2.3.3) were formalized for combinatorial auctions with indivisible items. In this work, the authors propose a two-level bidding language appropriate for intrinsically *divisible items* (e.g., electricity power, telecommunication capacity, assets in financial markets). The authors claim that the new language raises theoretical and practical challenging issues; for instance, the solutions times for large problems is huge which is inefficient. Kaleta [45] introduce three families of bidding languages for *divisible goods* based on the concepts derived from combinatorial auctions.

It is noteworthy to mention that ***item divisibility*** should be clearly distinguished from ***bid divisibility*** where the latter refers to partial acceptance of the bids submitted. In fact, for most of the literature on combinatorial auctions, researchers have concentrated on the design of bidding languages for indivisible assets and only very few have considered divisible goods.

While auctions for divisible goods allow for partial acceptance of a continuous asset, auctions with divisible bids permit partial acceptance of a package of items. Partial acceptance of bids mitigates the burden on suppliers to provide exponentially many indivisible bids (with respect to the number of items in the auction) to completely describe their cost structure.

For multiple units of the same item, suppliers are able to express a volume discount property that is *buy more and pay less* in a so-called ***Discount Volume Auctions***. Several discount volume auction studies have assumed the suppliers use piecewise linear cost functions to express their bids [47, 54, 53, 52]. In [47] Kameshwaran and Narahari propose a Lagrangian-based heuristic algorithm to solve the winner determination in a procurement discount volume auction. Kameshwaran et al. [54] design multi-attribute procurement auction allowing the bidders to bid on multiple attributes. In [53, 52] Kameshwaran and Narahari consider several solution algorithms for the underlying NP-hard winner determination problem. Proposed procedures in [47, 54, 53, 52] do not allow inclusion of more than one item in the auction.

As opposed to discount volume auctions, ***Discount Auctions*** are proposed to facilitate submission of bids which consists of individual costs of heterogeneous items and a discount function specifying the discount over the number of items. Some studies focused on the case of discount auctions for procurement of a single unit of multiple heterogeneous items [48, 49, 50, 46]. The discount function is solely dependent on the number of items included in a package, i.e., the auction facilitates paying less for buying more. This bidding language disregard the complementarities of items. Suppliers are no longer able to convey their desire to receive a package of items simultaneously, and so this is a non-combinatorial auction. Moreover, the discount rate is equivalent among all items. The suppliers' offer a discount rate for procurement of a ceratin number of items, irrespective of which items.

Extending partial bid acceptance on auctions with multiple units of multiple items, Bichler et al. [11] introduce an expressive bidding language which allows for quantity discount. Based on the introduced bidding language, they design and analyse a supplier quantity selection problem (SQS) which allows the buying managers to decide which suppliers should provide how many units of what items. Despite consideration of multiple units of heterogeneous items, this work does not address package bidding. Suppliers are unable to communicate their interest in receiving packages of items and thus suffers from the exposure problem inherent to non-combinatorial auctions.

An important marketplace that considers divisible bids on packages of multiple units of multiple items is in truckload transportation (TL) auctions where the carriers express preferences for serving transportation lanes. Remli and Rekik [93] consider the winner determination problem in the context of TL. They assume carrier t submits a set of bids B_t. With a flexible type of bidding language, a bid $b \in B_t$ is defined by a tuple $(\ell_{tb}, [LB_{tb}, UB_{tb}], c_{tb})$ where ℓ_{tb} is the set of lanes that carrier t offers to serve in bid b, LB_{tb} is the minimum volume that the shipper guarantees to the carrier if bid b wins, UB_{tb} is the maximum volume that the carrier can ship if bid b wins, and c_{tb} is the price asked by carrier t in bid b for transporting one unit volume on each lane $l \in \ell_{tb}$. As an example, the first bid submitted by carrier t_1 may look like $b_1 = (\{4, 5\}, [2, 4], 10)$ which indicates that t_1 offers to ship a volume varying between 2 and 4 units on each of lanes 4 and 5 with a price of 10 for a unit volume of shipment. With the proposed bidding language,

- the carrier is assigned to serve equivalent volumes on all lanes he acquires in the bid he submits. In the above example for instance, the volume he will have to serve is either 2, 3, or 4 for both lanes 4, and 5. It is not possible for him to express greater/lower capacity to serve some of the lanes in the package.

- the carrier has to serve all the lanes which are complimentary to him at an equal price. For instance, he will have to serve lane 4 and 5 both at the fixed cost of $10 per unit. It is possible that serving a lane costs the bidder more/less.
- the bidding language does not allow for a precise declaration of quantity discounts. For instance, a bidder is not able to express a bid that lets him cover lane 4, with a quantity interval of [6,8] and lane 5 with [6,10], charging $8 per unit of shipment. In other words, the quantity discount comes with serving lanes (that are complimentary) with the exact same lower and upper bounds. It does not allow for expressing non-identical lower and upper shipment bounds when providing quantity discounts on lanes in the same package.

Lim et al. [68] design a shipper's transportation procurement model. The auctioneer announces the requirements for freight services for a planning duration. The carriers who are assumed to serve different groups of lanes respond to the auctioneer with quotes of freight rates that best suit their bidding strategy. This work preserves the complementarity of serving groups of lanes, however, it does not account for the explicit representation of quantity discounts.

In the study of Chilean auction for school meals, Olivares et al. [82] state that package bidding should be flexible enough so that firms can express their cost synergies due to economies of scale, and take advantage of this flexibility by discounting package bids for strategic reasons. The authors perform analytical studies on the submitted bids to better understand the bidding behaviour of the bidders. However, the study is performed on single-unit first bid auction and is valid only on this type of auction.

The Mars-IBM team [40] created a procurement auction Web site that enables buyers to incorporate complex bid structures (such as bundled all-or-nothing bids and quantity-discounted bids) and business constraints into strategic-sourcing auctions. The Mars procurement auction does consider divisible bids with quantity discount. However, the designed model is geared for multi-units single-item auctions. Even though suppliers benefit from the opportunity to give a discount for the provision of larger units of items, they are deprived from taking advantage of complementarities among different products.

Caplice and Sheffi [17] classify the bidding languages for TL auctions as either *static* or *flexible*. Static bids reflect the indivisibility of bids, i.e., it is the carriers who determine the specific volume level awarded on each lane, not the shipper. With flexible package bids, by contrast, the shipper determines the volume level awarded to each bidder taking into account the carriers minimum/maximum shipment volume as well as per load rates.

In Chapter 5 we extend this language to allow suppliers to express quantity discounts for provision of large quantities of items. We consider suppliers' bids as piecewise linear price functions that specifies, for each item in the package, the per-unit price for the proposed range of supply. Based on this bid submission rule, this chapter compares the computational efficiency of WDP with divisible as opposed to indivisible bid submission. We observe that a much greater CPU time is required by CPLEX 12 to solve the indivisible formulation. For two suppliers, each submitting two sets of bids for two items, the time ratio is 1.1. This ratio increases to 856.05 for five suppliers, each submitting five sets of bids on five items. Next, we generate multiple profit maximization formulations for the suppliers based on their risk-taking attitude and proceed to compare the suppliers' profit maximization problem in a divisible versus an indivisible auction environment. Empirically, we observed that a divisible-bid auction provides suppliers with a higher average profit value and lower bundle prices.

Chapter 3

A Lagrangian Heuristic for the Winner Determination in Procurement Combinatorial Auctions

Reverse combinatorial auctions have been widely used in various real world applications. In companies which run combinatorial auctions in procurement of goods or services, the procurement manager serves as the auctioneer. He usually preserves a *reservation cost* which is the maximum amount he is willing to pay for one unit of an item. The organizations that compete to provide the items needed by the auctioneer are the suppliers. Through the bid submission, suppliers clearly specify which bundles of items they are willing to provide, how many units of each item they would include, and how much they charge in order to provide what they offer. The price each bidder attaches to his package is known as the *price offer*.

We start this chapter by introducing the winner determination problem in procurement. As discussed in Chapter 2, the problem of finding the winners in a combinatorial auction is computationally NP-hard and inapproximable in polynomial time. We highlight the virtue of implementing Lagrangian relaxation on WDP and compare the optimal solution it provides against that of linear relaxation.

In order to solve the Lagrangian relaxation problem, we propose an efficient initialization of the Lagrangian multipliers. Unlike traditional methodologies for solving the Lagrangian relaxation problem that involve several iterations to attain the optimal Lagrangian multipliers, our techniques solves the Lagrangian relaxation problem in a single iteration. The optimal Lagrangian solution attained is further deployed as the start point of a heuristic algorithm framework. The heuristic algorithm systematically adjusts this solution until feasibility of all primal constraints is attained. The results of this approach are compared against CPLEX 12.

3.1 Problem Formulation

Assume $M = \{1, 2, .., m\}$ is the set of items and $N = \{1, ..., n\}$ is the set of bidders who compete to purchase them. Each bidder $j \in N$ submits a set of package bids $S \subseteq M$ each of which contains a subset of items selected by bidder j and a corresponding price value P_{jS}, also known as *price offer* or simply *price*. The bidders' prices on bundles are *non-linear*. Thus, for an arbitrary bidder j and sets $S, S_1, S_2 (S, S_1, S_2 \subseteq M)$, with $S = S_1 \cup S_2$, we have $p_{jS} \neq p_{jS_1} + p_{jS_2}$, where p_{jS} is the price for bundle S offered by bidder j. The prices are *non-anonymous* meaning that they allow *discriminatory pricing* so that $p_{jS} \neq p_{j'S}$ for bidder $j \neq j'$. Let q_{ijS} be a non-negative integer that represents the quantity of item i that bidder j offers in bundle S and d_i be the auctioneer's demands of item i with $0 \leq q_{ijS} \leq d_i$.

The auctioneer's problem of deciding which bidders should supply how many units of what items and at what price with the objective to minimize the total cost of procurement while satisfying the auctioneer's demand is known as the reverse (or procurement) winner determination problem (WDP). When bidders are allowed to win any number of the packages they bid on, the winner determination problem is referred to as WDP_{OR}. In a procurement setting WDP_{OR} is formulated as the following integer programme.

$$\begin{aligned} \min \quad & \Sigma_{j\in N}\Sigma_{S\subseteq M}\ p_{jS}x_{jS} & \\ s.t. \quad & \Sigma_{j\in N}\Sigma_{S\ni i}\ q_{ijS}x_{jS} \geq d_i & \forall i \in M \\ & x_{jS} \in \{0,1\} & \forall j \in N, \forall S \subseteq M. \end{aligned} \tag{3.1}$$

The set of constraints (known as demand constraints) state that at least d_i units of each item has to be provided at the optimal solution. We note that we are assuming free disposal, i.e., an optimal allocation may over satisfy demand.

In some applications of combinatorial auctions it is preferred to implement an XOR bidding language to allow each bidder to win at most one bundle. To take into account this requirement, we need to add one more constraint in (3.1) to formulate WDP_{XOR}

as follows.

$$\begin{array}{lllll} \min & \Sigma_{j\in N}\Sigma_{S\subseteq M}\ p_{jS}x_{jS} & & & \\ s.t. & \Sigma_{j\in N}\Sigma_{S\ni i}\ q_{ijS}x_{jS} \geq d_i & \forall i \in M & (1) & \\ & \Sigma_{S\subseteq M}\ x_{jS} \leq 1 & \forall j \in N & (2) & \\ & x_{jS} \in \{0,1\} & \forall j \in N, \forall S \subseteq M. & & \end{array} \tag{3.2}$$

Constraints (1) and (2) in (3.2) are respectively referred to as the demand and supply constraints. The XOR bidding language in a procurement WDP setting has the potential to increase the bidders' precision since at most one of their submitted bids gets accepted. Moreover, it does not allow bidders to obtain a set of items as singleton bids without having paid for the complementarities.

3.2 Application of Lagrangian Relaxation on Procurement WDP

Lagrangian relaxation is a technique which approximates a difficult problem by relaxing it to a simpler one. The method removes some of the problem constraints and penalizes their violations by adding them to the objective with weight parameters known as the *Lagrangian multipliers.* The Lagrangian multipliers impose a cost on the relaxed constraints' violations. Thus, each time a solution does not satisfy a removed constraint, a penalty is added to the objective. The choice of the constraints to relax is made such that the relaxed problem is simpler to solve. In other words, the Lagrangian relaxation aims to relax some hard constraints, so that the optimization over the remaining set of constraints is easy.

Applying the Lagrangian relaxation on reverse combinatorial auctions provides us with an approximate solution to the problem and a lower bound on the total cost of procurement. The Lagrangian multipliers corresponding to the first set of constraints determine how much it costs to have one more unit of an item and thus can be interpreted as the item prices. We make use of this information in an iterative auction framework, to give feedback to the bidders about the item prices so that they will have a better sense about each other's valuations. This information is economically valuable since it can initiate an iterative auction wherein the bidders are provided with the opportunity to modify their bids several times throughout the auction before finalizing them.

As mentioned, Lagrangian relaxation finds an approximation to the problem but does not guarantee providing an optimal one. In fact, it may not even produce a feasible solution. In case the solution provided by the Lagrangian relaxation is not feasible, we apply a heuristic algorithm to fix the infeasibilities and find a feasible (near) optimal solution.

3.2.1 Formulating the Lagrangian Relaxation Problem

Consider the winner determination problem as modeled in (3.1). Let $\mathbf{X} = [x_{jS}]_{j\in N, S\subseteq M}$ be the vector of assignments and $\boldsymbol{\lambda}$ be the non-negative Lagrange multipliers assigned to the first set of constraints. This results in the formulation of the *Lagrangian relaxation function* as

$$L(\mathbf{X}, \boldsymbol{\lambda}) = \sum_{j\in N}\sum_{S\subseteq M} p_{jS}x_{jS} + \sum_{i\in N}\lambda_i(d_i - \sum_{j\in N}\sum_{S\ni i}\; q_{ijS}x_{jS}),$$

and the *Lagrange dual function* or simply the *dual function* as

$$
\begin{aligned}
g(\boldsymbol{\lambda}) = \quad & \min \quad \Sigma_{j\in N}\Sigma_{S\subseteq M}\ p_{jS}x_{jS} + \Sigma_{i\in N}\lambda_i(d_i - \Sigma_{j\in N}\Sigma_{S\ni i}\ q_{ijS}x_{jS}) \\
& s.t. \quad x_{jS} \in \{0,1\} \qquad \forall j \in N, \forall S \subseteq M.
\end{aligned}
\tag{3.3}
$$

The Lagrangian dual function is also referred to as the subproblem. For each value of the Lagrange multiplier, we obtain a lower bound on the problem's optimal value. The optimization problem that seeks to find this value is the *Lagrangian dual problem.* For the Lagrange function (3.3) the dual problem is formulated as

$$
\begin{aligned}
& max \quad g(\boldsymbol{\lambda}) \\
& s.t. \quad\ \ \boldsymbol{\lambda} \geq 0.
\end{aligned}
\tag{3.4}
$$

Relaxation of the demand constraint in an XOR bid setting can be done in a similar way with the exception that the subproblem (3.3) is solved in the existence of the additional supply constraints.

3.2.2 Solving the Lagrangian Relaxation Problem

One approach for solving the Lagrange dual problem is applying the *Lagrangian Decomposition Method* [37]. This method starts with arbitrary values of Lagrange multipliers $\boldsymbol{\lambda}$ to achieve an optimal solution to subproblem (3.3) which can be reformulated as

$$
\begin{aligned}
g(\boldsymbol{\lambda}) = \quad & \min \quad \Sigma_{j\in N}\Sigma_{S\subseteq M}(p_{jS} - \Sigma_{i\in N}\lambda_i q_{ijS})x_{jS} + \Sigma_{i\in N}d_i\lambda_i \\
& s.t. \quad x_{jS} \in \{0,1\} \qquad \forall j \in N, \forall S \subseteq M.
\end{aligned}
\tag{3.5}
$$

Define θ as

$$\theta = min_{\mathbf{X}}(\Sigma_{j\in N}\Sigma_{S\subseteq M}(p_{jS} - \Sigma_{i\in M}q_{ijS}\lambda_i)x_{jS}).$$

Once the subproblem is solved, Lagrangian decomposition proceeds to reformulate the Lagrangian dual problem as the following linear programming

$$\begin{aligned} \max \quad & \theta + \Sigma_{i\in N}d_i\lambda_i \\ s.t. \quad & \theta + \Sigma_{i\in M}\lambda_i\Sigma_{j\in N}\Sigma_{S\ni i}q_{ijS}x_{jS} \leq \Sigma_{j\in N}\Sigma_{S\subseteq M}\ p_{jS}x_{jS} \\ & \lambda_i \geq 0 \qquad i \in N. \end{aligned} \tag{3.6}$$

Let $\mathbf{X}^*$ be the optimal solution from the subproblem (3.2). This solution is inserted into the dual problem (3.2) to find out the multipliers that would maximize the objective value at that point. The optimal multipliers derived are inserted back in the subproblem to find the optimal solution. The procedure continues iterating between solving (3.2) and (3.6), changing the objective value of the first and adding one more constraint to the second, until the objective values of the two problems converge. The bound achieved is the Lagrangian dual bound.

3.2.3 Analysis of the Lagrangian Relaxation Bound

This section compares the Lagrangian and linear relaxations of a multi-unit multi-item reverse WDP with and without free disposal in an OR and an XOR bidding environment. Let us define problem (3.2) as

$$\begin{aligned} \min \quad & \mathbf{PX} \\ s.t. \quad & Q\mathbf{X} \geq \mathbf{d} \\ & R\mathbf{X} \leq \mathbf{1} \\ & \mathbf{X} binary, \end{aligned}$$

where, the quantity matrix Q and the supply matrix R are defined as

$$Q = [\tilde{Q}_1, \ldots, \tilde{Q}_n] \quad \tilde{Q}_j = [q_{ijS}]_{i \in M, S \subseteq M} \quad \forall j = \{1, \ldots, n\},$$

$$R = \begin{bmatrix} \tilde{R}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{R}_n \end{bmatrix} \quad \tilde{R}_j = [r_{jS}]_{S \subseteq M} \quad \forall j = \{1, \ldots, n\}.$$

With K being the total number of subsets, matrix Q is $m \times Kn$ and R is $n \times Kn$. Each quantity value q_{ijS} identifies the quantity of item i that bidder j supplies in bundle S. Also, $r_{jS} = 1$ if supplier j submits a bid on bundle S and is 0 otherwise.

Also, let $\mathbf{q}_{jS}$ and $\mathbf{r}_{jS}$ respectively define the columns of Q and R corresponding to bidder j's bids on subsets $S, S \subseteq M$, i.e.,

$$\mathbf{q}_{jS} = \begin{bmatrix} q_{1jS} \\ \vdots \\ q_{mjS} \end{bmatrix}_{1\times m}, \mathbf{r}_{jS} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_{jS} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{1\times n}.$$

The following propositions provide some properties of relaxing demand constrains in an OR, demand only and both supply and demand constraints in an XOR setting. Note that the Lagrangian multipliers associated with the demand constraints are interpreted as item prices for an auction environment. Thus, we are not interested in solely relaxing the supply constraints in XOR.

Proposition 3.1. *Let $\boldsymbol{\lambda}^*$ and $\boldsymbol{\gamma}^*$ be the optimal values of the Lagrangian multipliers respectively associated with demand and supply constraints. We can find closed form optimal solution for the Lagrangian relaxation problem of*

1. *WDP_{OR} with relaxed demand constraints as:*

$$x^*_{jS} = \begin{cases} 1 & p_{jS} - \boldsymbol{\lambda}^{*^t} \boldsymbol{q}_{jS} < 0 \\ 0 & \textit{otherwise,} \end{cases}$$

2. WDP_{XOR} with relaxed demand and supply constraints as:

$$x^*_{jS} = \begin{cases} 1 & p_{jS} - \boldsymbol{\lambda}^{*^t}\boldsymbol{q}_{jS} + \boldsymbol{\gamma}^{*^t}\boldsymbol{r}_{jS} < 0 \\ 0 & \textit{otherwise,} \end{cases}$$

3. WDP_{XOR} with relaxed demand constraints as

$$x^*_{jS} = \begin{cases} 1 & p_{jS} - \boldsymbol{\lambda}^{*^t}\boldsymbol{q}_{jS} = \min_S\{p_{jS} - \boldsymbol{\lambda}^{*^t}\boldsymbol{q}_{jS}\} < 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First, let us assume a free disposal state where the Lagrangian multipliers corresponding to demand constraints take on non-negative values. To prove part 1 we consider the Lagrangian optimization problem

$$\begin{aligned} g(\boldsymbol{\lambda}) &= \inf_{\mathbf{X}} L(\mathbf{X}, \boldsymbol{\lambda}) \\ &= \inf_{\mathbf{X}}\{\mathbf{P}\mathbf{X} + \boldsymbol{\lambda}^t(\mathbf{d} - Q\mathbf{X})\} \\ &= \mathbf{d}^t\boldsymbol{\lambda} + \inf_{\mathbf{X}}(\mathbf{P} - \boldsymbol{\lambda}^t Q)X \\ &= \mathbf{d}^t\boldsymbol{\lambda} + \begin{cases} \sum_{j,S}(p_{jS} - \boldsymbol{\lambda}^t\mathbf{q}_{jS}) & p_{jS} - \boldsymbol{\lambda}^t\mathbf{q}_{jS} < 0 \\ 0 & o.w. \end{cases} \end{aligned} \tag{3.7}$$

$\max_{\boldsymbol{\lambda}} g(\boldsymbol{\lambda}) = g(\boldsymbol{\lambda}^*)$ provides the optimal solution for (3.7). Therefore, $g(\boldsymbol{\lambda})$ is maximized when x^*_{jS} is set to one for negative values of $p_{jS} - \mathbf{q}_{jS}\boldsymbol{\lambda}^*$ and zero otherwise.

To prove part 2, consider the WDP_{XOR} as formulated in (3.2.3). Assigning non-negative Lagrangian multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ respectively to the demand and supply constraints in WDP_{XOR} leads to the Lagrangian subproblem

$$\begin{aligned} g(\boldsymbol{\lambda},\boldsymbol{\gamma}) &= \inf_{\mathbf{X}} L(\mathbf{X},\boldsymbol{\lambda},\boldsymbol{\gamma}) \\ &= \inf_{\mathbf{X}}\{\mathbf{PX} + \boldsymbol{\lambda}^t(\mathbf{d} - Q\mathbf{X}) + \boldsymbol{\gamma}^t(R\mathbf{X} - \mathbf{1})\} \\ &= \mathbf{d}^t\boldsymbol{\lambda} - \mathbf{1}^t\boldsymbol{\gamma} + \begin{cases} \sum_{j,S}(p_{jS} - \boldsymbol{\lambda}^t\mathbf{q}_{jS} + \boldsymbol{\gamma}^t\mathbf{r}_{jS}) & p_{jS} - \boldsymbol{\lambda}^t\mathbf{q}_{jS} + \boldsymbol{\gamma}^t\mathbf{r}_{jS} < 0 \\ 0 & o.w. \end{cases} \end{aligned} \tag{3.8}$$

$g(\boldsymbol{\lambda}^*,\boldsymbol{\gamma}^*)$ optimally solves the Lagrangian relaxation problem.

Finally, for part 3 the Lagrangian subproblem is

$$\begin{aligned} g(\boldsymbol{\lambda}) &= \inf_{\mathbf{X}}\{\mathbf{PX} + \boldsymbol{\lambda}^t(\mathbf{d} - Q\mathbf{X}) | R\mathbf{X} \leq \mathbf{1}, \mathbf{X} \quad binary\} \\ &= \mathbf{d}^t\boldsymbol{\lambda} + \inf_{\mathbf{X}}\{(\mathbf{P} - \boldsymbol{\lambda}^t Q)\mathbf{X} | R\mathbf{X} \leq \mathbf{1}, \mathbf{X} \quad binary\}. \end{aligned} \tag{3.9}$$

In order to maximize $g(\boldsymbol{\lambda})$ it suffices to consider the minimum value of $p_{jS} - \boldsymbol{\lambda}^{*^t}\mathbf{q}_{jS}$ for each bidder $j \in N$. If this values is negative, we set the corresponding variable to 1 and otherwise to 0. This associates at most one bundle to each bidder j and thus $g(\boldsymbol{\lambda}^*)$ provides us with the optimal solution. □

Proposition 3.2. *The Lagrangian and linear relaxations of reverse WDP yield equivalent bounds.*

Proof. According to Theorem 16.10 in [2], the linear and Lagrangian relaxation bounds equal if the Lagrangian subproblem satisfies the integrality property, i.e., LP relaxation finds integral solution for the Lagrangian subproblem for any choice of objective function coefficients.

The linear relaxation of subproblems (3.7) and (3.8) minimize the objective function subject to the constraint that each variable is less than or equal to 1. This forms a totally unimodular matrix of coefficients and so the subproblems satisfy the integrality property. Also, from the definition of matrix R, each of its columns have at most one +1 and they are 0 elsewhere. Thus the integrality of subproblem (3.9) can be deduced from the totally unimodularity of matrix R. □

Theorem 3.1. *For the reverse WDP, the dual variables associated with the demand (or supply) constraints of the linear relaxation problem correspond to the Lagrangian multipliers associated with the demand (or supply) constraints of the Lagrangian relaxation problem.*

Proof. We can reformulate the Lagrangian subproblem (3.8) corresponding to the relaxation of all constraints in an XOR bid setting as

$$g(\boldsymbol{\lambda}, \boldsymbol{\gamma}) = \boldsymbol{\lambda}^t \mathbf{d} - \boldsymbol{\gamma}^t \mathbf{1} + \Sigma_{j,S} \min\{0, p_{jS} - \boldsymbol{\lambda}^t \mathbf{q}_{jS} + \boldsymbol{\gamma}^t \mathbf{r}_{jS}\}.$$

Based on definition of matrix R, $\boldsymbol{\gamma}^t \mathbf{r}_{jS} = \gamma_j$. The corresponding Lagrangian dual problem maximizes $g(\boldsymbol{\lambda}, \boldsymbol{\gamma})$ for nonnegative values of $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ as

$$\sup_{\boldsymbol{\lambda}, \boldsymbol{\gamma}} g(\boldsymbol{\lambda}, \boldsymbol{\gamma}) = \max_{\boldsymbol{\lambda}, \boldsymbol{\gamma}} \{d^t \boldsymbol{\lambda} - 1^t \boldsymbol{\gamma} + \Sigma_{j,S} \min\{0, p_{jS} - \boldsymbol{\lambda}^t \mathbf{q}_{jS} + \boldsymbol{\gamma}_j\}, \boldsymbol{\lambda} \geq 0, \boldsymbol{\gamma} \geq 0\},$$

or equivalently

$$\begin{aligned} \max \quad & d^t \boldsymbol{\lambda} - 1^t \boldsymbol{\gamma} + \Sigma_{j,S} \min\{0, p_{jS} - \boldsymbol{\lambda}^t \mathbf{q}_{jS} + \gamma_j\} \\ s.t. \quad & \boldsymbol{\lambda} \geq 0, \boldsymbol{\gamma} \geq 0. \end{aligned}$$

Linearizing $\min\{0, p_{jS} - \boldsymbol{\lambda}^t \mathbf{q}_{jS} + \gamma_j\}$ with variable l_{jS}, we obtain

$$\begin{aligned} \max \quad & d^t\boldsymbol{\lambda} - 1^t\boldsymbol{\gamma} + \Sigma_{jS} l_{jS} & \\ s.t. \quad & l_{jS} + \boldsymbol{\lambda}^t \mathbf{q}_{jS} - \gamma_j \leq p_{jS} & \forall j, S \\ & l_{jS} \leq 0 & \forall j, S \\ & \boldsymbol{\lambda} \geq 0, \boldsymbol{\gamma} \geq 0. & \end{aligned}$$

Let $l_{jS} = -t_{jS}$. The corresponding dual problem is formulated as

$$\begin{aligned} \min \quad & \Sigma_{j,S} p_{jS} x_{jS} & \\ s.t. \quad & \Sigma_{j,S} \mathbf{q}_{jS} x_{jS} \geq \mathbf{d} & \\ & \Sigma_S x_{jS} \leq \mathbf{1} & \forall j \\ & 0 \leq x_{jS} \leq 1 & \forall j, S. \end{aligned}$$

Multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ which were initially defined as Lagrangian multipliers associated with demand and supply constraints also serve as dual variables to the corresponding constraints in the linear relaxation problem. The equivalence in the OR setting can be concluded similarly. □

Corollary 3.1. *Initializing Lagrangian multipliers at optimal dual values obtained from the linear relaxation solves the Lagrangian function to optimality.*

We note that based on Theorem 3.1 initializing the Lagrangian multipliers at the optimal dual values is in fact equivalent to initializing them at their optimal values. As stated in Proposition 3.1 this initialization solves the Lagrangian subproblem, as formulated in (3.5) to optimality without recourse to the dual problem (3.6).

3.3 Solution Algorithms

This section consists of our proposed solution algorithms for solving the procurement WDP to (near-) optimality. Based on the analysis proposed in Section 3.2.3, we generate an algorithm to efficiently solve the Lagrangian relaxation problem to optimality. Based on the solution derived we propose a heuristic method in section 3.3.2 for solving the underlying primal problem.

3.3.1 Lagrangian Relaxation Solution Algorithm

In Theorem 3.1 we provide optimal values of Lagrange multipliers. As stated in Corollary 3.1, by initializing the Lagrange multipliers at their optimal values, we can solve the Lagrangian relaxation problem by solving its subproblem and using Proposition 3.1. Algorithm 1 summarizes our methodology for efficiently solving the Lagrangian relaxation problem.

Algorithm 1 Lagrangian relaxation solution algorithm

Step1. Solve the corresponding LP relaxation.
Step2. Initialize Lagrange multipliers at the LP's optimal dual values.
Step3. Solve Lagrangian subproblem using the closed form solution in Proposition 3.1.

Table 1 illustrates the amount of execution time saved when solving the Lagrangian relaxation problem using this methodology as compared to the implementation of a traditional decomposition method. Each pair of (item, bid) is averaged on 13 runs of different problem instances. The average ratio of 88.9 on a total of 520 problems indicates that our proposed methodology solves the Lagrangian relaxation problem to optimality in about %1 of the time required by the Lagrangian decomposition method.

Table 1: Execution time comparison of Algorithm 1 and the Lagrangian decomposition method

(#items, #bids)	Average ratio of execution time of Lagrangian decomposition to Algorithm 1
(4,20)	31.02
(4,30)	61.88
(5,20)	51.46
(5,30)	65.81
(5,40)	80.35
(5,50)	68.02
(5,100)	59.35
(6,20)	33.87
(6,30)	61.42
(6,40)	83.34
(6,50)	46.17
(6,60)	74.30
(6,70)	85.35
(6,100)	68.56
(6,150)	72.72
(6,200)	81.17
(6,250)	98.92
(7,100)	69.07
(7,150)	99.51
(7,200)	93.15
(7,250)	104.09
(7,300)	123.88
(7,350)	135.70
(8,100)	88.80
(8,200)	59.53
(8,300)	159.96
(8,400)	87.28
(8,500)	98.67
(9,50)	122.55
(9,70)	112.96
(9,100)	93.83
(9,150)	86.98
(9,200)	90.24
(9,250)	108.43
(9,300)	106.14
(9,350)	134.22
(9,400)	112.55
(9,450)	165.41
Average	**88.86**

3.3.2 Aggregate Heuristic Solution Algorithm

The Lagrangian relaxation provides an integer solution in step 3 of Algorithm 1 which may not be feasible in the primal problem. In order to fix it, we developed a heuristic algorithm, named as the Aggregate heuristic, which consists of several subheuristics and improvement procedures. Starting from the Lagrangian optimal, we observe satisfiability of demand constraints at the current solution. Violation of a demand constraint at the current solution indicates insufficient assignment of quantities of the corresponding item. Each subheuristic initially fixes its current solution $\overline{X}$ at the Lagrangian optimal solution ${X^*}^{LR}$ and then systematically selects a constraint and a variable to set to 1. Setting additional variables to 1 continues until the feasibility of all constraints is achieved. Algorithm 2 describes the generic structure of how each subheuristic procedure functions for a multi-unit multi-item auction with XOR bidding language and the free disposal condition as formulated in model (3.2).

More specifically, in this Algorithm parameter Sh_i identifies possible shortages corresponding to each item (constraint). Set I is defined as the set of all unsatisfied constraints and set J as the set of all bidders who are not currently winning any of their packages. With the existence of an unsatisfied demand, Aggregate heuristic proceeds to run sub-heuristics each consisting of a *constraint selection rule* and a *variable selection rule* (as described below) to set a new variable to 1. The shortage is calculated again only for those constraints which were previously recognized as having a positive shortage.

Note that firstly the algorithm continues the search for as long as there exists bidders who are eligible to win packages. If all bidders are assigned an item and the shortage is not yet fully satisfied, then the algorithm exits the loop and returns infeasibility. At this point either the auctioneer decides to supply the remaining shortage from spot market or he induces another auction inviting more participants.

Secondly, with a non empty set of suppliers we do not necessarily rule out the previously chosen constraint i from the set I since the shortage may not be fully resolved. After setting a variable to 1, we may satisfy none, some, or all of the unsatisfied constraints. For this reason, we update the set I every time a new variable is selected. Set J is also updated to remove the new winner. This guarantees the feasibility of supply constraints in the optimal solution. As soon as a bidder receives a package he is prevented from winning anymore.

Algorithm 2 Subheuristics' Procedure Structure

$\overline{X} \leftarrow X^{*^{LR}}$
$Sh_i \leftarrow d_i - \Sigma_{j \in N} \Sigma_{S \ni i} \, q_{ijS} \overline{x}_{jS} \quad$ for $\quad \forall i \in M$
$I \leftarrow \{i \in M | Sh_i \geq 0\}$
$J \leftarrow \{j \in N | \overline{x}_{jS} = 0 \quad \forall S\}$
while $I \neq \emptyset$ **do**
    if $J = \emptyset$ **then**
        exit loop 'Infeasible problem Instance'
    else
        Select constraint $i \in I$ based on the Constraint Selection Rules
        Select variables $j \in J, S \subseteq M$ based on the Variable Selection Rules
        $\overline{x}_{jS} \leftarrow 1$
        update Sh_i for $\forall i \in I$
        $I = I \backslash \{i | Sh_i < 0\}$
        $J = J \backslash \{j | j$ selected in line 7$\}$
    end if
end while

3.3.2.1 Constraint Selection Rules

The constraint selection rules include selecting constraint $i \in I$ with

Rule 1. the largest shortage

$i = argmax_{i \in I}\{Sh_i | Sh_i = d_i - \Sigma_{j \in N} \Sigma_{S \ni i}\ q_{ijS} \overline{X}_{jS}\}$,

Rule 2. the minimum slackness value

$i = argmin_{i \in I}\{\frac{\Sigma_{j \in N} \Sigma_{S \ni i}\ q_{ijS}}{d_i}\}$,

Rule 3. the maximum slackness value

$i = argmax_{i \in I}\{\frac{\Sigma_{j \in N} \Sigma_{S \ni i}\ q_{ijS}}{d_i}\}$,

Rule 4. the costliest shortage

$i = argmax_{i \in I}\{\lambda_i . Sh_i | Sh_i = d_i - \Sigma_{j \in N} \Sigma_{S \ni i}\ q_{ijS} \overline{X}_{jS}\}$.

Rule 1 looks for the item which has the largest shortage. Rules 2 and 3 search for the items with maximum or minimum slackness values where slackness values refer to the ratio of total quantity offers of an item to its demand. In other words, we are interested to identify the items which receive the largest or the smallest total quantity offers with respect to their demand. Finally, rule 4 searches to supply the items whose shortage is the costliest for the auctioneer with respect to the Lagrangian multipliers. Note that these rules are applied only on the set of violated constraints.

3.3.2.2 Variable Selection Rules

For each constraint selection rule we perform the following 4 variable selection rules to form a sub-heuristic procedure. Note that each constraint corresponds to an item and each variable of this constraint indicates a package containing this item. Once an item is selected, from all the unallocated packages containing this item we choose the one that provides the largest value according to one of the following variable selection rules.

Rule 1. $$\frac{\sum_{i \in I} min(q_{ijS}, Sh_i)}{p_{jS}}$$

Rule 2. $$\frac{\sum_{i \in I} \lambda_i^*.min(q_{ijS}, Sh_i)}{p_{jS}}$$

Rule 3. $$\frac{\sum_{i \in I} \frac{min(q_{ijS}, Sh_i)}{Sh_i}}{p_{jS}}$$

Rule 4. $$\frac{\sum_{i \in I} \frac{\lambda_i^*.min(q_{ijS}, Sh_i)}{Sh_i}}{p_{jS}}.$$

The above ratios are used to provide us with a proxy for the value of an option based on the shortage and pricing information. For each rule we calculate its value for all items with unsatisfied demand and pick the variable that corresponds to the largest value. We use $min(q_{ijS}, Sh_i)$ so that we do not value bundles with quantities exceeding the shortage.

3.3.2.3 Improvement Procedure

Each subheuristic is followed by an improvement procedure to systematically swap a selected variable with a non-selected one. Since the combination of constraint selection rule 1 and variable selection rule 1 provides the best results in our experiments we use this combination to switch variables.

To eliminate a variable already selected, the improvement procedure selects a constraint based on the constraint selection rule 1, and a variable with the *lowest* value of variable selection rule 1. To set a new variable to 1, constraint selection rule 1 and variable selection rule 1 are used once again.

3.3.2.4 Aggregate Heuristic Algorithm

Let c, v be alternatively the indices on the constraint and variable selection rules. We define each sub-heuristic as $H_{c,v}$ and the improvement procedure applied on it as $IH_{c,v}$. Considering all combinations of our constraint and variable selection rules each followed by an improvement procedure, provides 32 computationally efficient subroutines. The Aggregate heuristic algorithm executes all subroutines and extracts the solution corresponding to the minimum objective value as the solution from the heuristic.

3.4 Computational Experiments of Multi-unit Combinatorial Auction

In this section we describe the computational steps taken to simulate a multi-unit auction environment. All our computational experiments are performed on an Intel

Xeon E5440 @2.83 Ghz 2.83 Ghz (2 processors) machine with memory (RAM) 12.0 Gb running on a 64-bit operating system on a standard windows 2008 server.

3.4.1 Using CATS for Data Generation

To imitate actual bidding behaviour in combinatorial auctions we used CATS (Combinatorial Auction Test Suite) [67]. Given the number of goods, bids and required distribution type, CATS determines how many items to include in a bid, what items to include, and what price to attach to the whole package. Once all the items are enumerated, CATS starts counting *dummy bids.* Dummy bids are used to determine what packages are received from which bidders, i.e., packages sharing similar dummy goods are received from a single bidder.

3.4.2 Algorithm Coding

We used General Algebraic Modeling System (GAMS) platform for coding the multi-unit reverse combinatorial auction problem. All required information regarding the bidders submitting bids, the items included in each package, and the price associated with it are extracted from CATS output file within our GAMS code, after several necessary preprocessing of input data as explained in the next section.

We use CPLEX 12 for solving our problem instances directly. The execution time and optimal solution is compared against application of linear relaxation, Lagrangian decomposition method, and our proposed solution methodology for solving the Lagrangian relaxation problem and eventually our heuristic algorithm for fixing the infeasibility of the Lagrangian optimal solution, if needed.

3.4.3 Automating Transformation of CATS Produced Data to GAMS-Compatible Input Files

Despite the benefits of using CATS for generating realistic data, there is no convenient interface between CATS and GAMS. To incorporate this test suite in our methodology, changes need to be applied on the text file generated by CATS before introducing it as an input file for GAMS. In a large scale set of data, it is impractical to apply these changes manually. For this reason, we generated VBA codes in Excel and produced several macros to automate implementing all the required modifications on the text file produced by CATS.

3.4.4 Adjusting CATS Single-Unit Data to Represent Multi-Unit Environments

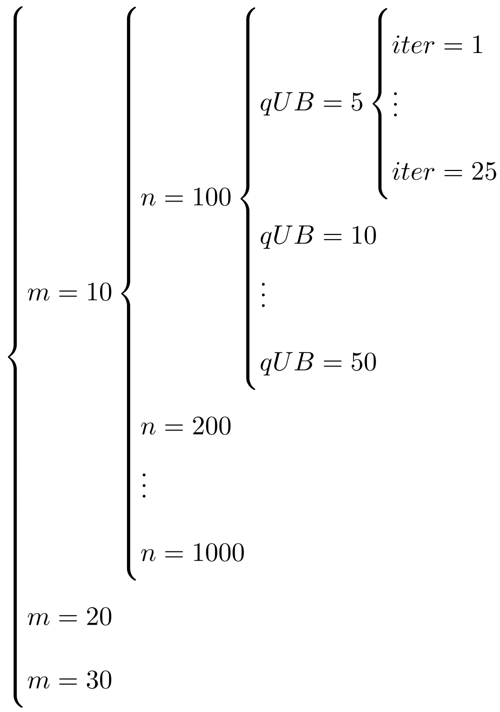
Data generated by CATS represents a single-unit bidding environment wherein the auctioneer requires a single unit of each item. Consequently, suppliers submit bids on single units of items. In other words, suppliers only choose the items they are willing to include in bundles and assign prices to them. Due to the auctioneer's multi-unit demand requirement in a multi-unit combinatorial auction, suppliers are concerned about the additional task of deciding how many units of each item to include in each package they submit. Although the creators of CATS provide a source file to represent a multi-unit bidding behaviour in their first release of CATS (CATS 1.0), it was not carried in their newer version (CATS 2.0). CATS 1.0 does not consistently generate feasible data for multi-unit auctions [98]. Thus, based on the single-unit packages created by CATS we use a random data generator to define the suppliers' number of units and the auctioneers' demand with a normal distribution. We define the new price corresponding to each package P as

$$P_{\text{New price}} = \frac{P_{\text{CATS price}} \times \text{Total number of quantities included in P}}{\text{Total number of items included in P}}.$$

3.4.5 Size of Problem Instances

We explore generation of $10, 20,$ and 30 items each with $100, 200, \ldots, 1000$ number of bids. We fixed the upper bound for demand generation at 50, and varied the upper bound for quantities of items included in the packages at $5, 10, 15, \ldots, 50$ to represent a supply capacity of $10\%, 20\%, \ldots, 100\%$ of demand. In order to get more reliable results, for each combination of number of items, bids, and the specified value of the quantities' upper-bound, we averaged results over 25 problem instances. This adds up to a generation of 7500 problem instances. For more clarification, let m represent the number of items, n the number of bids, qUB the upper-bound considered for the random quantity generation and $iter$ the number of iterations a new instance is generated. Figure 1 illustrates the size of our data generated.

Figure 1: Data Size Diagram



3.4.6 Adjusting Lagrangian Constraint Satisfaction Ratio

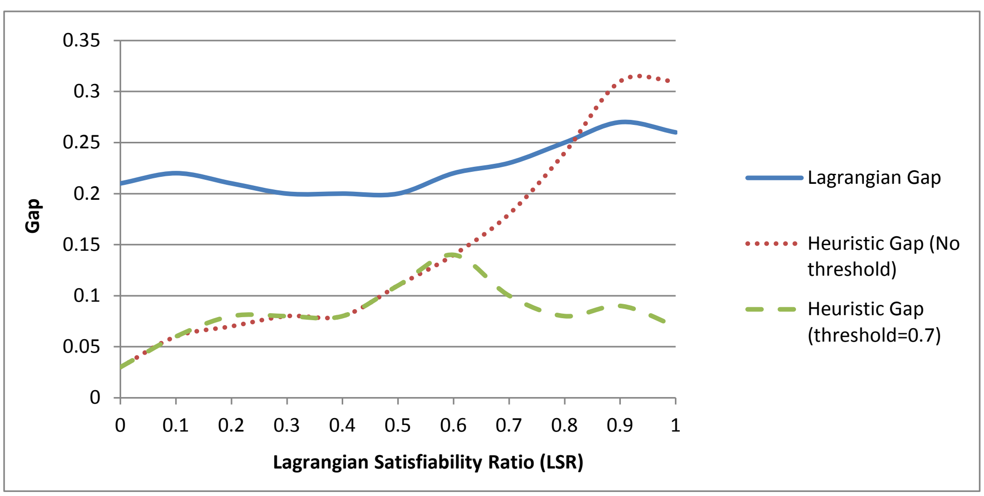
To have a better understanding of the quality of the objective value that the heuristic produces, we record the heuristic optimality gap as $(Z_H - Z_{IP})/Z_{IP}$ and the Lagrangian duality Gap as $(Z_{IP} - Z_{LR})/Z_{LR}$. Table 2 displays these gaps as LR Gap and H Gap respectively.

In our next set of experiments, we investigate how the quality of the Aggregate heuristic optimality gap relates to the *Lagrangian Satisfiability Ratio (LSR)*. We define LSR as the ratio of the primal constraints that are satisfied at the Lagrangian optimal solution to the total number of constraints.

We implemented our experiments on 2,500 problem instances generated for 10 items and bids ranging from 100 to 1000 in increments of 100. For each combination of item and bids, we generated random supply quantities covering a maximum of 10%,20%,...,100% of demand. Finally, for each problem combination we generated 25 different data instances so as to have more significant statistics.

We observed that an increase in the LSR value increases the optimality gap obtained from the Lagrangian relaxation as well as the Aggregate heuristic. In other words, as illustrated in Figure 2, the more constraints the Lagrangian optimal solution satisfies, the larger the optimal Lagrangian and heuristic gaps become. A possible explanation of this observation is that when LSR is close to 1 the heuristic spends less effort in improving the Lagrangian solution and we end up with a relatively large gap.

Figure 2: Lagrangian and Aggregate heuristic optimality gaps when reducing satisfiability ratios below the threshold



For this reason, in our next set of experiments whenever the Lagrangian optimal solution satisfies $\alpha\%$ or more of the primal constraints, we systematically select and zero out positive variables (from the Lagrangian optimal solution) until the LSR value drops down below our threshold value. Algorithm 3 describes this procedure for the Lagrangian optimal solution X^*.

Algorithm 3 Satisfiability Violation

1: $\overline{X} \leftarrow X^*$
2: $Sh_i \leftarrow d_i - \Sigma_{j\in N}\Sigma_{S\ni i}\, q_{ijS}\overline{x}_{jS} \quad$ for $\quad \forall i \in M$
3: $I \leftarrow \{i \in M | Sh_i \geq 0\}$
4: $J \leftarrow \{j \in N | \overline{x}_{jS} = 0 \quad \forall S\}$
5: $Ratio \leftarrow \frac{card(M)-card(I)}{card(I)}$
6: **while** $Ratio \geq threshold$ **do**
7:     Select constraint $i \in M - I$ such that $i = argmax_i\{Sh_i\}$
8:     Select $j \in J, S \subseteq M$ such that $jS = argmax_{jS}\{\frac{\sum_{i\in I} min(q_{ijS}, Sh_i)}{P_{jS}}\}$
9:     $\overline{x}_{jS} \leftarrow 0$
10:     update Sh_i
11:     $I = I \backslash \{\widetilde{i} | Sh_{\widetilde{i}} < 0\}$
12:     $J = J \backslash \{\widetilde{j} | \widetilde{j}$ selected in line 8$\}$
13:     update $Ratio$
14: **end while**

As illustrated in Figure 2, we observe an improvement in the average heuristic optimality gap for $LSR \geq \alpha = 0.7$. The overall average optimality gap drops down from 11% with no violation to 7% with the implementation of this violation.

Tightening α to 0.6 decreases the optimality gap to 6% while increasing the heuristic execution time. In fact, the average execution time ratio of CPLEX to the Aggregate heuristic decreases from 19.45, for $\alpha = 0.7$, to 6.32 when $\alpha = 0.6$. Thus, while decreasing α to 0.6 slightly improves the optimality gap, it yields much larger execution time. For this reason we keep the threshold value α at 0.7 for the rest of our experiments.

3.4.7 Efficiency of the Proposed Algorithm

Define $Z_{IP}, Z_{LR},$ and Z_H respectively as the optimal value of the CPLEX solver, the optimal Lagrangian relaxation, and the solution from the Lagrangian heuristic and T_{IP}, T_{LR} and T_H as their execution times. Let w be the ratio of the number of positive variables (corresponding to distinct winners) in the CPLEX optimal solution to that of Aggregate heuristic. In this section, we compare the efficiency of the Aggregate heuristic algorithm against CLPEX 12 on the data set explained in Section 3.4.5. Table 2 summarizes our results.

For more clarification on this Table, we illustrate the diagram of the 250 problem instances generated for its first row in Figure 3. This row demonstrates average results for instances with 10 items, total number of bids ranging from 100 to 1000 in increments of 100 and a supply of maximum of 10% of demand for each item in the bundles generated. For each class of items, we include the total average, the standard deviation, and the coefficient of variation values in order to provide better understanding of our original data. These values are respectively denoted as Ave, STDev, and CV in Table 2. The following subsections are devoted to explaining our computational results in more details.

Figure 3: Data size diagram of the first row of Table 2

$$m = 10, \quad qUB = 10\% \quad \text{of demand} \begin{cases} n = 100 \begin{cases} iter = 1 \\ \vdots \\ iter = 25 \end{cases} \\ n = 200 \\ \vdots \\ n = 1000 \end{cases}$$

3.4.7.1 Optimality Gap

Our experiments show that the quality of the heuristic optimality gap improves with the decrease in the maximum percentage of the demand supplied in the submitted bids. To show this pattern more clearly we designed Table 2 to classify problem instances based on the maximum percentage of the demand satisfied by submitted bids. Figure 4(a) shows the growth of average optimality gap (along side the average standard deviation) with the increase in the bids' supplied quantity for $10, 20,$ and 30 items. The plot also compares the average optimality gap with its overall expected value when fixing the number of items. For $10, 20,$ and 30 items the average optimality gap derived is respectively $8, 10,$ and 12 percent.

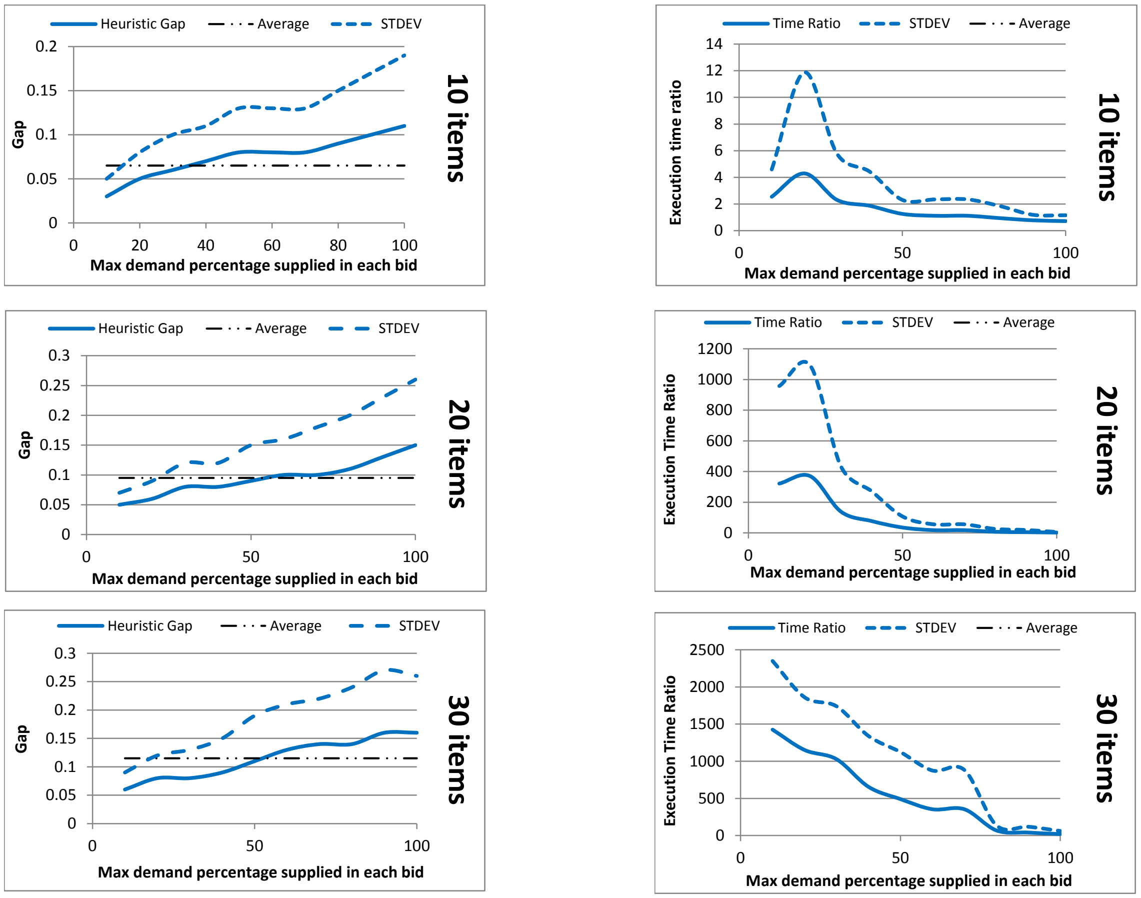
As observed in Figure 4, the heuristic provides lower gaps on the class of problems in which each bidder offers a low percentage of demand. Specifically, when the bidder offer to supply at most 50% of demand, as shown in Table 3, the average heuristic gap for $10, 20, 30$ items and submission of 100 to 1000 bids (in increments of 100)

Table 2: Efficiency comparison of Aggregate heuristic versus CPLEX 12

Demand %	LR Gap	H Gap	T_{IP} (sec)	T_H (sec)	T_{IP}/T_H	w
			10 item			
10	0.02	0.03	0.89	0.4	2.54	0.93
20	0.05	0.05	2.03	0.44	4.29	0.9
30	0.1	0.06	1.03	0.44	2.3	0.87
40	0.15	0.07	0.86	0.43	1.86	0.87
50	0.19	0.08	0.54	0.43	1.25	0.85
60	0.24	0.08	0.5	0.44	1.11	0.88
70	0.24	0.08	0.5	0.44	1.11	0.88
80	0.34	0.09	0.36	0.42	0.93	0.89
90	0.41	0.1	0.29	0.42	0.77	0.88
100	0.44	0.11	0.25	0.41	0.71	0.86
Ave	**0.22**	**0.08**	**0.73**	**0.43**	**1.69**	**0.88**
STDev	**0.07**	**0.05**	**1.3**	**0.22**	**2.26**	**0.18**
CV	**0.32**	**0.62**	**1.8**	**0.51**	**1.34**	**0.21**
			20 items			
10	0.02	0.05	137.02	0.56	321.59	0.9
20	0.07	0.06	189.79	0.57	370.28	0.89
30	0.12	0.08	76.75	0.55	138.72	0.86
40	0.17	0.08	37.79	0.56	77.18	0.87
50	0.22	0.09	18.7	0.53	35.25	0.87
60	0.27	0.1	8.29	0.52	18.03	0.85
70	0.27	0.1	8.29	0.52	18.03	0.85
80	0.38	0.11	3.5	0.52	7.29	0.85
90	0.43	0.13	2.27	0.5	4.84	0.83
100	0.49	0.15	1.04	0.5	2.37	0.84
Ave	**0.24**	**0.1**	**48.34**	**0.53**	**99.36**	**0.86**
STDev	**0.053**	**0.06**	**90.15**	**0.28**	**203.1**	**0.15**
CV	**0.22**	**0.66**	**1.86**	**0.52**	**2.04**	**0.18**
			30 items			
10	0.03	0.06	783.43	0.66	1425.32	0.9
20	0.07	0.08	758.21	0.73	1151.58	0.87
30	0.12	0.08	672.66	0.72	1027.39	0.86
40	0.17	0.09	421.74	0.68	655.54	0.86
50	0.22	0.11	307.56	0.64	490.61	0.85
60	0.27	0.13	223.21	0.65	353.49	0.86
70	0.27	0.14	223.21	0.65	353.49	0.86
80	0.41	0.14	39.12	0.56	68.65	0.86
90	0.44	0.16	24.01	0.6	39.89	0.89
100	0.51	0.16	11.44	0.59	19.38	0.87
Ave	**0.25**	**0.12**	**346.46**	**0.65**	**558.53**	**0.87**
STDev	**0.08**	**0.07**	**264.64**	**0.34**	**490.21**	**0.22**
CV	**0.31**	**0.63**	**0.76**	**0.52**	**0.88**	**0.26**
Overall Ave	**0.24**	**0.1**	**131.84**	**0.54**	**219.86**	**0.87**

Figure 4: Comparison of Lagrangian and Aggregate heuristic when reducing satisfiability ratios beyond the threshold

(a) Optimality gap comparison (b) Execution time comparison



drops down to respectively 6, 7, and 8 percent.

3.4.7.2 Execution Time

In order to observe how fast Aggregate heuristic solves a problem instance to a solution, we plot the execution time ratio of CPLEX to Aggregate heuristic in Figure 4(b). This figure illustrates that when bidders supply low percentages of demand, CPLEX

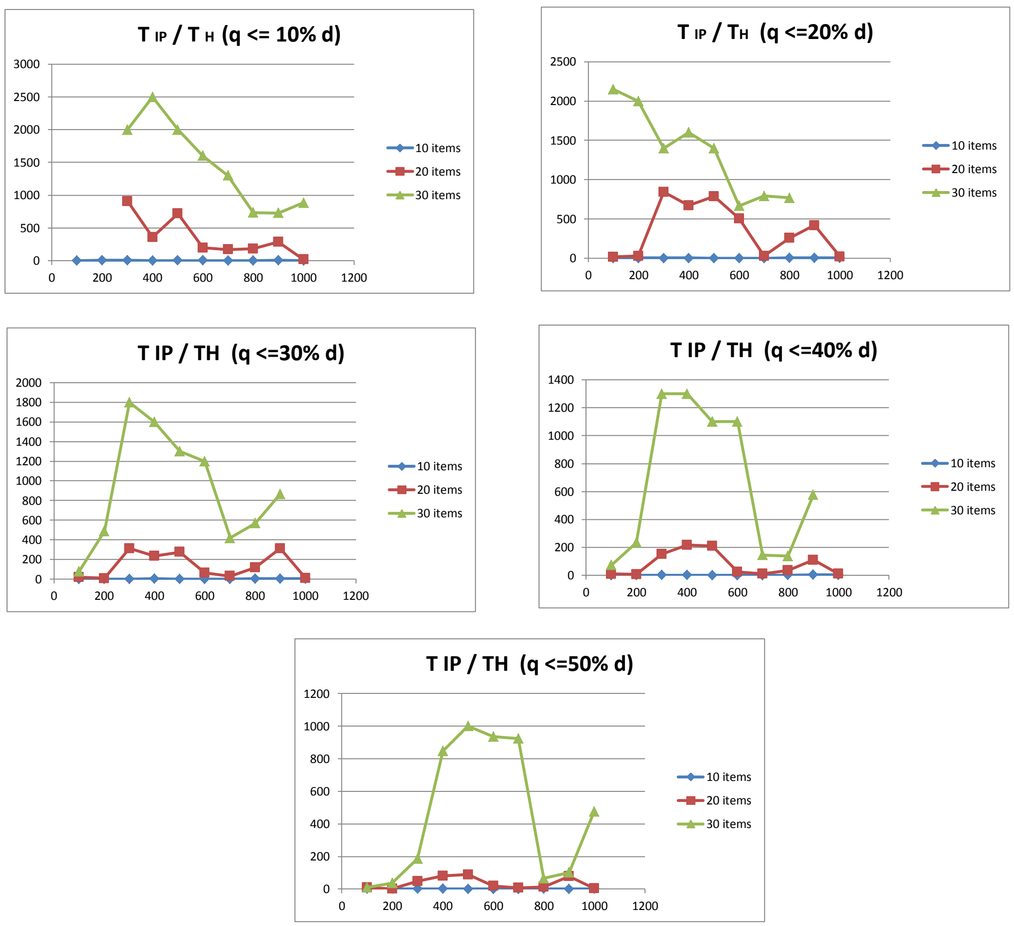
Table 3: Comparison of Aggregate heuristic and CPLEX 12 with supplied quantity of less than half of demand

Demand %	LR Gap	H Gap	T_{IP} (sec)	T_H (sec)	T_{IP}/T_H	w
			10 item			
10	0.02	0.03	0.89	0.4	2.54	0.93
20	0.05	0.05	2.03	0.44	4.29	0.9
30	0.1	0.06	1.03	0.44	2.3	0.87
40	0.15	0.07	0.86	0.43	1.86	0.87
50	0.19	0.08	0.54	0.43	1.25	0.85
Ave	**0.1**	**0.06**	**1.07**	**0.43**	**2.45**	**0.88**
CV	**0.7**	**0.32**	**0.53**	**0.04**	**0.47**	**0.04**
			20 items			
10	0.02	0.05	137.02	0.56	321.59	0.9
20	0.07	0.06	189.79	0.57	370.28	0.89
30	0.12	0.08	76.75	0.55	138.72	0.86
40	0.17	0.08	37.79	0.56	77.18	0.87
50	0.22	0.09	18.7	0.53	35.25	0.87
Ave	**0.1**	**0.07**	**92.01**	**0.55**	**188.6**	**0.88**
CV	**0.79**	**0.23**	**0.77**	**0.03**	**0.79**	**0.02**
			30 items			
10	0.03	0.06	783.43	0.66	1425.32	0.9
20	0.07	0.08	758.21	0.73	1151.58	0.87
30	0.12	0.08	672.66	0.72	1027.39	0.86
40	0.17	0.09	421.74	0.68	655.54	0.86
50	0.22	0.11	307.56	0.64	490.61	0.85
Ave	**0.12**	**0.08**	**588.72**	**0.69**	**950.09**	**0.87**
CV	**0.63**	**0.23**	**0.36**	**0.06**	**0.4**	**0.02**
Overall Ave	**0.11**	**0.07**	**227.27**	**0.56**	**380.38**	**0.88**

12 takes much more time as compared to Aggregate heuristic. In Figure 5, we have a closer look at this ratio for the class of problems in which each bidder offers to supply at most 50% of the demand. In this figure, the x axis shows the number of bids submitted and the y axis shows the time ratio).

It can be seen that this ratio exceeds 2500 for 30 items and 400 bids and when no bid contains more than 10% of the demand (note that we use the term *exceed*, since each problem instance is characterized by a specific combination of the number of items, the number of bids and the percentage of the demand offered in the package, is averaged over 25 generations of random problem instances). As observed in Table 3, averaging over all number of bid submissions, this ratio exceeds 950 for 30 items.

Figure 5: CPU Time Ratio of CPLEX to the Aggregate heuristic for quantity offers less than half of the demand



3.4.7.3 Robustness

To show the extent of variability in relation to the mean of our data, we use the coefficient of variation (CV) defined as the ratio of the standard deviation σ to the mean, i.e., $CV = \sigma/\mu$.

CV combines information about the mean and standard deviation of the system. A system associated with large values of CV is considered weakly robust, due to the large variation of standard deviation as compared to the mean. Conversely, low values of CV indicate strong robustness of the system. Comparison of the CV values for the CPLEX and the Aggregate heuristic execution times shows that the time-wise performance of the Aggregate heuristic is more stable and thus robust compared to CPLEX.

3.4.7.4 Aggregate Heuristic Solution

Since the heuristic algorithms for solving procurement WDP provide a feasible solution which is not necessarily optimal, the heuristic value is greater than (or equal to) the objective value of CPLEX. Thus, it is likely that we get more positive solution variables in the heuristic solution. In other words, the heuristic algorithms introduce more winners in the auction than CPLEX. Thus it is important for auctioneers to know how many more winners they should expect, and whether this additional number of winners is dependent on the problem size or the percentage of demand supplied in each bid.

Figure 6 and Figure 7 illustrate this correspondence respectively with the increase in the maximum percentage of demand offered in the packages and the increase in the number of bids received in the auction.

As observed from the figures, ratio w lacks any particular increasing or decreasing pattern. Thus, our first conclusion is that w may increase or decrease irrespective of the increase in the problem size, the number of bids, and the percentage of the

Figure 6: Comparison of the number of winners in Aggregate heuristic versus CPLEX 12 with respect to the increase in maximum demand percentage supplied in the bids

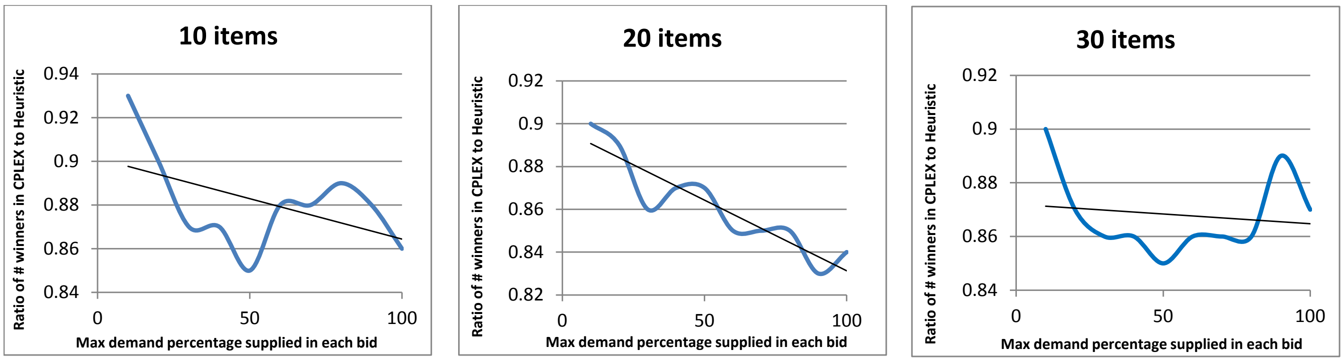
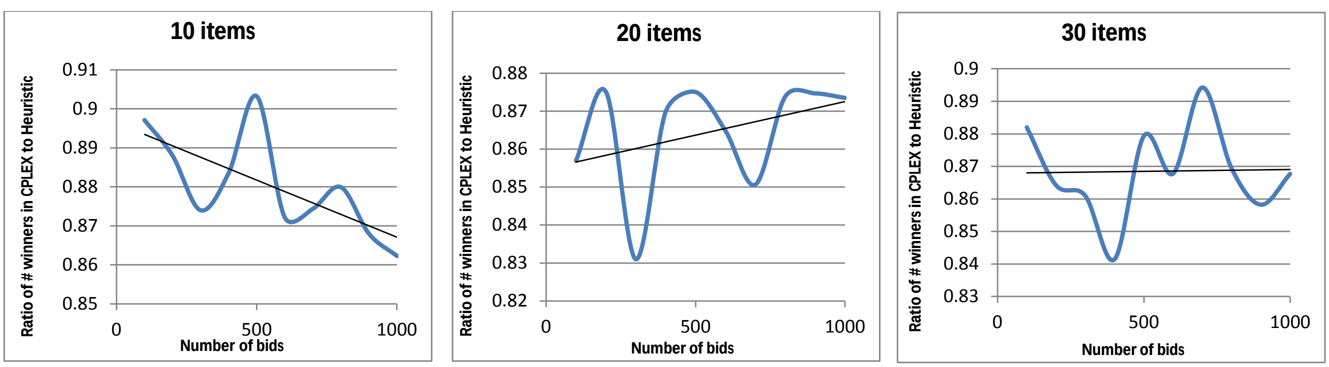


Figure 7: Comparison of the number of winners in Aggregate heuristic versus CPLEX 12 with respect to the increase the number of bids received



demand supplied in each bid.

Secondly, w is always below 1, indicating a larger number of positive variables in the solution obtained from the Aggregate heuristic. For the Aggregate heuristic this average ratio remains above 83%. Thus, the auctioneers knows he could expect obtaining a maximum of 17% extra suppliers in the auction. This helps the auctioneers involve more suppliers in the auction. For auctioneers who do not want a large number of suppliers, it is possible to imply further restricting constraints in advance in order to limit the total number of winning suppliers in the auction.

3.4.7.5 Efficiency of sub-heuristics

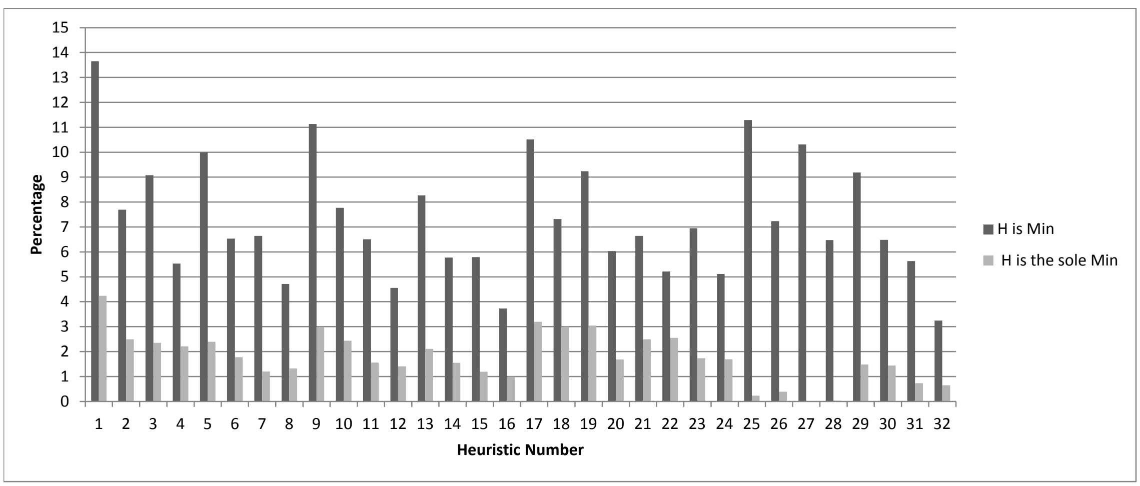
In order to determine which combination of constraints and variable selection rule performs best, we record the total number of times a sub-heuristic provides the best solution. Note that multiple sub-heuristic can provide equal minimum values at the same time. Therefore, we also recorded the number of times a sub-heuristic is the sole minimum, meaning that it is performing strictly better than all other procedures. Table 4 summarizes our results.

Table 4 shows that variable selection rule 1 provides the highest percentages with all different constraint selection rules. However, while H_{42} provides the best solution in over 10% of times, it never produces a strict minimum value. Therefore, it is possible to remove this procedure and reduce execution time. Figure 8 visualizes the efficiency comparison of sub-heuristic procedures.

Table 4: Efficiency comparison of sub-heuristics

H Number	H Name	% H is min	% H is the sole min
1	H_{11}	13.65	4.24
2	IH_{11}	7.69	2.49
3	H_{12}	9.08	2.35
4	IH_{12}	5.53	2.21
5	H_{13}	10	2.39
6	IH_{13}	6.53	1.77
7	H_{14}	6.64	1.2
8	IH_{14}	4.71	1.32
9	H_{21}	11.13	2.99
10	IH_{21}	7.77	2.43
11	H_{22}	6.51	1.56
12	IH_{22}	4.55	1.41
13	H_{23}	8.27	2.11
14	IH_{23}	5.77	1.55
15	H_{24}	5.79	1.19
16	IH_{24}	3.73	0.99
17	H_{31}	10.51	3.19
18	IH_{31}	7.32	3
19	H_{32}	9.24	3.04
20	IH_{32}	6.03	1.68
21	H_{33}	6.64	2.49
22	IH_{33}	5.21	2.55
23	H_{34}	6.95	1.73
24	IH_{34}	5.11	1.69
25	H_{41}	11.29	0.23
26	IH_{41}	7.23	0.39
27	H_{42}	10.31	0
28	IH_{42}	6.47	0.03
29	H_{43}	9.19	1.48
30	IH_{43}	6.48	1.44
31	H_{44}	5.63	0.73
32	IH_{44}	3.24	0.65

Figure 8: Efficiency comparison of subheuristics



Chapter 4

Models for Bidders Pricing Problem

The literature on combinatorial auction optimization has mostly focused on the problem of maximizing the revenue for the auctioneer (see for instance [27]) which in a procurement setting translates to minimizing the total price of procurement. Yet, given the suppliers' wide range of bundling and pricing options, there is a lack of focus on the problem facing them: how to determine and price the optimal quantities to offer. Particularly, in an iterative auction framework, suppliers need to take into account how to make use of the feedback information disclosed by the auctioneer in order to understand the bidding behaviour of their competitors. This helps them develop an insight on their competitiveness level as compared to the rest of the bidders and thus make more tangible decisions in each round of the auction.

To understand this problem, we start by defining relevant terms and notations and introduce a generic profit maximization model (GPMB) for the suppliers. For an iterative auction framework, we use the Lagrangian multipliers to help the bidders find the optimal price and lot sizes. In this chapter our focus is on *i*ndivisible bundles where the auctioneer selects from predetermined fixed bidders bundles. The case of *d*ivisible bundles, where the auctioneer can choose from a defined continuum of bundles and prices, will be discussed in Chapter 5. We describe the work of Hsieh [41] in this area. Even though this work is amongst the first to highlight the importance of this field of research, it suffers from several inconsistencies within the problem formulation, solution algorithm and the numerical implementation. The deficiencies of this model motivated us to proceed by generating more practical problem formulations which consider suppliers with fixed or variable per-unit costs. Using the integrality property we show that we can find closed form solutions for some of our models.

Comparing prices corresponding to two consecutive rounds of auctions, we provide the suppliers with the knowledge on whether to withdraw from the auction or bid more aggressively. Investigation of suppliers' optimal pricing problem and the auctioneer's optimal allocation problem leads us to the design of an iterative auction which determines the rules of how the auctioneer and suppliers interact as well as the level of information that they communicate. We show that our auction is convergent.

Our next phase of work consists of conducting numerical experiments first to empirically observe our proposed analytical results, and second to study the dynamics of the suppliers' and auctioneer's profits.

4.1 Notations

We use the following notation:

i	Index on suppliers, $i \in I = \{1, \ldots, m\}$
j	Index on the suppliers' bids $j \in J = \{1, \ldots, m_i\}$
j'	Index on the suppliers' new bid in Hsieh's model
j^*	Index on the suppliers' best previous bid
k	Index on items, $k \in \{1, \ldots, K\}$
d_k	Number of units demanded for item k
λ_k^*	Optimal Lagrange multipliers for product k
$\hat{\lambda}_k$	Auctioneer's reservation price on product k
P_{ij}	The price that bidder i requests to provide bundle j
PI	Profitability index
c_{ijk}	The unit cost that bidder i affords to provide item k in bundle j
Π_i	The net profit that bidder i expects to get from the auction
q_{ijk}	The quantity of product k offered in the jth bid of supplier i
L_{ik}	Supplier i's minimum capacity to produce item k
U_{ik}	Supplier i's upper bound on production of item k
γ_k	The difference between the Lagrangian price and cost of product k
δ_k	Binary variable that is 1 if product k is selected and 0 otherwise
w	Index on the suppliers' cost scenario, $w = 1, \ldots, W$
(n)	Iteration n of the auction
$R1, R2$	Round 1 and 2 of an iterative auction
RS	Stabilization Round

4.2 Generic Profit Maximization Model

With the information about the Lagrangian multipliers from the previous round of the auction, the suppliers seek to attain appropriate bundle prices that will meet their internal production constraints, maximize sales profits, as well as increase chances of winning in the auction.

$$
\begin{array}{lll}
\textit{max} & \textit{Profit} & \\
\textit{s.t.} & \textit{satisfaction of minimum profitability condition} & (1) \\
 & \textit{satisfaction of competitiveness condition} & (2) \\
 & \textit{satisfaction of pricing consistency condition} & (3) \\
 & \textit{satisfaction of production upper bound constraint} & (4) \\
 & \textit{price variables} \geq 0 & \\
 & \textit{integer quantity variables} \in \mathbb{N} \cup \{0\}. &
\end{array}
\qquad \text{(GPMB)}
$$

In this model, constraint (1) guarantees that a minimum profitability is attained with the generation of a new bundle. The most commonly used profitability indexes (denoted as PI) deployed in industries include

1. Net Profit = Revenue − Cost,
2. Net Profit Margin = $\frac{\text{Net Profit}}{\text{Revenue}}$,
3. Profit Percentage = $\frac{\text{Net Profit}}{\text{Cost}}$.

These measures are financial metrics that are used to assess the business ability to generate earnings despite the expenses and other relevant costs incurred during a specific period of time. For most of these ratios, having a higher ratio than a competitor's is indicative that the company is doing better.

Net profit, also referred to as the *bottom line*, *net income*, or *net earnings* is the revenues less the costs. Profit margin is an indicator of a company's pricing strategies and how well it controls costs. It is mostly used for internal comparisons. A low profit margin indicates a low margin of safety and a higher risk that a decline in sales erases profits and results in a net loss, or a negative margin. Profit percentage defines profit as the percentage of cost and ensures that a company receives the proper amount of gross profit when spending a certain cost. In a survey of nearly 200 senior marketing managers, 91% reported on the efficaciousness of the Net Profit metric [34], and for this reason we take up this index for the rest of our study.

Constraint (2) in GPMB absorbs information about submitted bundles and prices by all other suppliers into each suppliers' profit maximization model via deploying the Lagrangian multipliers. This constraint aims to device a more competitive bid than the best previous bid based on the announced price proxies. In constraint (3), we require the model to produce reasonable consistent prices. Specifically, with this constraint we prevent the model from adopting positive prices for a 0 quantity offer. Constraint (4) takes into account the suppliers' production capacity limit. We adopt the generic problem formulation (GPMB) and customize it to formulate the suppliers' profit maximization problems in Chapters 4 and 5.

4.3 Comments on Hsieh's [41] Proposed Solution Methodology

When competing to supply multiple units of items required by the auctioneer via competing bids, suppliers are faced with several challenges to determine appropriate optimal quantities and prices for each round of the auction that keeps them competitive in the auction while at the same time guaranteeing the maximum possible profit should they win the auction.

Perhaps because of the tilt of the power towards the auctioneer, the bidders' problem has received little attention in the literature. Hsieh [41] proposes a heuristic algorithm for solving the auctioneer's winner determination problem as well as a mathematical programming to maximize the suppliers' profit. This study has motivated our models in this Chapter. However, before we outline our results we would like to highlight some issues that we find with the model and results reported in Hsieh [41].

4.3.1 Algorithmic Issues

Hsieh implements a heuristic algorithm to fix possible infeasibilities of the Lagrangian optimal solution. In Algorithm 4 we summarize the steps of the proposed procedure. This heuristic algorithm starts with the optimal Lagrange solution X^* and defines the set of all constraints violated at this point as K^o. The algorithm proceeds to select *first* the violated constraint corresponding to the item with the lowest shortage and *second* the variable corresponding to the bundle that contains this item and is offered at the lowest price. The value of the corresponding variable is set to one and

Algorithm 4 Hsieh's Heuristic Algorithm for solving the WDP

Initialize X^* at $\overline{X}$.
Define the set of violated demand constraints as K^o.
Define the set of losing bidders as I^o.
while $K^o \neq \emptyset$ **do**
    Select a violated constraint $k \in K$ with the minimum value of shortage.
    Choose bundle x_{ij} which contains item k, is submitted by a currently losing bidder, and attains the lowest bundle price.
    Set $\overline{x}_{ij} = 1$.
    Remove bidder i from I^o.
end while

the winner of this bundle is removed from the set of losing bidders.

Remark 1: Choosing the constraint based on the minimum violation can substantially prolong the execution time, as the problem size increases. For instance, assume a demand shortage of {2, 3, 4, 5, 7, 50}. The Hsieh's algorithm starts with the satisfaction of the least critical shortages (2) and so can take long to satisfy all the constraints.

Remark 2: The algorithm is silent on how to update K^0. Thus, there is the lack of clarification on how to update K^0. An obvious way to update this set is to remove the selected constraint. In Chapter 3 we reevaluate satisfaction of all previously violated constraints every time a new variable is set to 1 to account for cases when setting one variable to 1 satisfies more than one constraint.

Remark 3: The author calculates the optimal duality gap as $\frac{f(\overline{x})-L(\lambda^*)}{f(\overline{x})}$ where $f(\overline{x})$ is the objective value at the feasible solution obtained from the heuristic and $L(\lambda^*)$ is the optimal lower bound that the Lagrangian relaxation provides. This way of calculating the gap leads to an underestimated gap and is uncommon in the optimization literature. The standard for calculating a gap is $\frac{f(\overline{x})-L(\lambda^*)}{L(\lambda^*)}$.

4.3.2 Experimental Issues

We reproduce Example 2 from [41] below to pinpoint some deficiencies in the numerical analysis provided in that paper.

Example 4.1 ([41]). *Let* $I = 3; J = 2; K = 4; d_1 = 2; d_2 = 1; d_3 = 2; d_4 = 1$. *The six bids submitted by the three bidders, two each, are as follows:*

$$q_{111} = 1; q_{112} = 0; q_{113} = 1; q_{114} = 0; P_{11} = 70;$$

$$q_{121} = 1; q_{122} = 1; q_{123} = 0; q_{124} = 0; P_{12} = 75;$$

$$q_{211} = 0; q_{212} = 0; q_{213} = 1; q_{214} = 0; P_{21} = 40;$$

$$q_{221} = 0; q_{222} = 1; q_{223} = 0; q_{224} = 1; P_{22} = 80;$$

$$q_{311} = 0; q_{312} = 0; q_{313} = 1; q_{314} = 0; P_{31} = 45;$$

$$q_{321} = 0; q_{322} = 0; q_{323} = 0; q_{324} = 1; P_{32} = 50;$$

The optimal Lagrangian solution is derived as $x^*_{12} = 1, x^*_{22} = 1, x^*_{32} = 1$, *and the optimal solution from the heuristic as* $\bar{x}_{11} = 1, \bar{x}_{21} = 1, \bar{x}_{22} = 1, \bar{x}_{32} = 1$.

Remark 1: For this example the WDP is infeasible since product 1's demand can not be satisfied in an XOR formulation. This is true since $d_1 = 2$ and the only seller that provides that product is seller 1. Giving the author the benefit of the doubt, we have looked at the possibility that there might have been a typo in the data or solution. However, any 'fixing' of the data of the solution would be significant and involves more than one change.

Remark 2: Comparing the Lagrange optimal solution and the feasible solution from the heuristic, we realize that variable x_{12} that is set to one in the Lagrange optimal solution is missing in the heuristic feasible solution. This contradicts the fact that the heuristic does not have a dropping procedure, i.e., once a variable is set to 1, it should stay in the optimal solution.

Remark 3: The heuristic feasible solution proposed by the author is in fact infeasible since $\bar{x}_{21} + \bar{x}_{22} = 2 > 1$.

4.3.3 Modelling Issues

Hsieh [41] formulates supplier i's profit maximization problem for the generation of a new bid indexed as j' as follows.

$$\begin{array}{lllr} \max & P_{ij'} - \sum_{k=1}^{K} c_{ij'k} q_{ij'k} & & \\ \text{s.t.} & P_{ij'} - \sum_{k=1}^{K} c_{ij'k} q_{ij'k} \geq \Pi_i & & (1) \\ & P_{ij'} - \sum_{k=1}^{K} \lambda_k^* q_{ij'k} \leq P_{ij} - \sum_{k=1}^{K} \lambda_k^* q_{ijk} & \forall j & (2) \\ & P_{ij'} \geq 0, \qquad q_{ij'k} \in \mathbb{N}^+ \cup \{0\} & \forall k \in K. & \end{array} \qquad \text{(PMB)}$$

where λ_k^* stands for the optimal Lagrangian multiplier associated with product k in the previous round. We drop the index j' by assuming the problem at round (n) and define

$$\begin{aligned} j^* &= \text{argmin}_j \{P_{ij}^{(n-1)} - \textstyle\sum_{k=1}^{K} \lambda_k^{*(n-1)} q_{ijk}^{(n-1)}\}, \\ P_i^{*(n-1)} &= P_{ij^*}^{(n-1)}, \\ q_{ik}^{*(n-1)} &= q_{ij^*k}^{(n-1)}, \\ g_i^* &= P_i^{*(n-1)} - \textstyle\sum_{k=1}^{K} \lambda_k^{*(n-1)} q_{ik}^{*(n-1)}. \end{aligned}$$

Since (PMB) is formulated for each supplier i, we also drop the index i for simplicity (we adopt this index as needed). In addition, consideration of g^* for representing the optimal values from the previous rounds leaves all variables and parameters involved in the formulation of (PMB) at the current round (n). This allows us to drop index (n) as well (we will adopt this index whenever clarification on auction round is needed). Hence, (PMB) simplifies as

$$
\begin{array}{lll}
\max & P - \sum_{k=1}^{K} c_k q_k & \\
\text{s.t.} & P - \sum_{k=1}^{K} c_k q_k \geq \Pi & (1) \\
 & P - \sum_{k=1}^{K} \lambda_k^* q_k \leq g^* & (2) \\
 & P \geq 0,\ \ q_k \in \mathbb{Z} \cup \{0\}\ \ \forall k \in K. &
\end{array}
\tag{PMB2}
$$

Defining $\gamma_k = \lambda_k^* - c_k$, in Proposition 4.1 we identify the closed-form optimal solution for (PMB2).

Proposition 4.1. *The optimal solution to* (PMB2) *is as follows:*

1. *If* $\gamma_k \leq 0\ \forall k \in K$ *then,*
   - *(a) If* $g^* < \Pi$*,* (PMB2) *is infeasible.*
   - *(b) If* $g^* \geq \Pi$*,* (PMB2) *yields trivial solution* $q_k^* = 0\ \forall k \in K,$ *and* $P^* = g^*$*.*
2. *If* $\exists k \in K$ *s.t.* $\gamma_k > 0$*, then* (PMB2) *is unbounded.*

Proof. Note that constraints (1) and (2) in (PMB2) can be rewritten as

$$
\Pi + \sum_{k=1}^{K} c_k q_k \ \ \leq P \ \ \leq g^* + \sum_{k=1}^{K} \lambda_k q_k, \tag{4.1}
$$

or equivalently,

$$
\begin{aligned}
\Pi \ \ \leq P - \sum_{k=1}^{K} c_k q_k \ \ &\leq g^* + \sum_{k=1}^{K} \lambda_k^* q_k \\
&\leq g^* + \sum_{k=1}^{K} \gamma_k q_k.
\end{aligned}
\tag{4.2}
$$

Assume that for all k, $\gamma_k \leq 0$. When $g^* < \Pi$ it is easy to see that no values of q can satisfy condition (4.2) and thus (PMB2) is infeasible. With $g^* \geq \Pi$ the optimal value is obtained when all quantity values are set to zero, since assigning positive values to any of the quantity values would decrease the objective function. Thus, $P^* = g^*$.

If there exists $k \in K$ that satisfies $\gamma_k > 0$ then it is possible to increase the associated quantity q and price P infinitely large. This satisfies the constraints and maximizes the objective function. In this case (PMB2) is unbounded with the optimal solution

$$P^* = +\infty$$
$$q_k^* = \begin{cases} +\infty & \gamma_k > 0 \\ 0 & \gamma_k \leq 0. \end{cases}$$

□

Proposition 4.1 implies that for Model (PMB2) we have either an infeasible, an unbounded, or a trivial solution which is not practical for suppliers. To make this formulation more constructive, we consider the following refinements in the generation of the profit maximization model.

1. For the trivial solution $q_k^* = 0$ for $\forall k$, (PMB) yields a positive price value, which translates to asking for a positive price for the supply of nothing. We fix this by making prices consistent with quantity offers.

2. (PMB2) considers neither the suppliers' production capacities nor the auctioneer's demand. Realistically, suppliers production capacity for different products

is bounded. We deploy the term upper bound U_k as the minimum of the actual production capacity and the auctioneer's demand.

3. In addition to the production capacity, due to the costs associated with starting production lines, suppliers usually require an order to supply at least a minimum amount which we will consider in our model as L_k.

4. We facilitate either the production of item k on the suppliers' capacity range, or not producing this item at all. This gives the suppliers the opportunity to withdraw some items in the new package they submit if their production is not profitable.

5. As often is the case, the suppliers' cost function is not a linear function of quantity. In other words, the unit production cost can vary with respect to the quantity produced. Suppliers' cost function is usually considered piecewise linear with lower costs corresponding to larger production units. We will consider both scenarios in Sections 4.4 and 4.5.

4.4 Fixed-cost Profit Maximization Model (FPMB)

With fixed per-quantity cost, c_k, corresponding to the production of item k, we formulate the bidder's pricing problem as follows:

$$
\begin{aligned}
\max \quad & P - \textstyle\sum_{k=1}^{K} c_k q_k \\
\text{s.t.} \quad & P - \textstyle\sum_{k=1}^{K} c_k q_k \geq \Pi && (1) \\
& P - \textstyle\sum_{k=1}^{K} \lambda_k^* q_k \leq g^* && (2) \\
& \delta_k L_k \leq q_k \leq \delta_k U_k \quad \forall k \in K && (3) \\
& P \leq M \textstyle\sum_k q_k && (4) \\
& P \geq 0, \;\; q_k \in \mathbb{N} \cup \{0\}, \;\; \delta_k \in \{0,1\} \quad \forall k \in K.
\end{aligned} \tag{FPMB}
$$

The binary variable δ_k ensures that either product k is supplied with an optimal quantity on the range $[L_k, U_K]$ or the supplier will not supply this product at all. δ_k^* attains 1 in the former and 0 in the latter. For a sufficiently large parameter M, constraint (4) ensures the consistency of price and quantity offers. For zero quantity values, the constraint enforces a zero optimal price while it becomes redundant for positive ones. Next we study some properties of (FPMB).

4.4.1 Optimality

(FPMB) defines a mixed integer programming problem (MIP). In proposition 4.2 we identify the closed-form optimal solution to this problem.

Proposition 4.2. *The optimal solution to the suppliers' capacitated profit maximization problem* (FPMB) *with fixed unit production cost c_k is as follows:*

1. *If $\gamma_k \leq 0 \ \forall k \in K$ then,*

   (a) If $g^* < \Pi$, (FPMB) *is infeasible.*

   (b) If $g^* \geq \Pi$, (FPMB) *yields the trivial solutions*

$$P^* = 0, \quad \delta_k^* = q_k^* = 0 \ \forall k \in K$$

2. If $\exists k \in K \;\; s.t. \;\; \gamma_k > 0$*, then* (FPMB) *is feasible and the optimal solution is*

$$\begin{cases} \delta_k^* = 1, & q_k^* = U_k, \quad P^* = g^* + \sum_k \lambda_k^* U_k \quad \gamma_k > 0 \\ \delta_k^* = 0, & q_k^* = 0, \quad P^* = 0 \qquad\qquad\qquad \gamma_k \leq 0 \end{cases}$$

Proof. 1.a and 1.b can be shown similarly to Proposition 4.1, except constraint (4) enforces zero optimal price values in 1.b. In part 2, the optimal solution q^* is obtained by increasing all quantity variables associated with positive $\gamma_k (= \lambda_k^* - c_k)$ values to the upper bound U_k and setting the rest of the variables to zero. With the increase of the right-hand-side of equation

$$P - \sum_{k=1}^{K} c_k q_k \leq g^* + \sum_{k=1}^{K} \gamma_k q_k$$

P gets large enough to satisfy the inequality and yet maximize the objective function.

□

Proposition 4.2 offers several interpretations:

1. Suppliers' bidding withdrawal condition: When the announced Lagrangian prices on each item is at most as large as its production cost, then (FPMB) is either infeasible or yields a trivial solution. This makes submission of a new bid unprofitable for the supplier.

2. Suppliers' bidding condition: If the cost of production for at least one item is strictly less than its Lagrangian price, then the supplier is able to submit a package by supplying this product. It is important to note here that a supplier is able to supply all its capacity because the auctioneer is willing to accept extra

items. This situation is not uncommon in practice as suppliers often impose minimum shipping quantities due to costly production setup costs. A minimum quantity can also be viewed as part of a quantity discount contract where the price of the undiscounted quantity is infinity [77].

3. Lower bound independency of the optimal solution: Sellers will either choose not to offer a certain product or they will offer it at capacity. This conclusion reduces (FPMB) to

$$
\begin{aligned}
\max \quad & P - \textstyle\sum_{k=1}^{K} c_k q_k \\
\text{s.t.} \quad & P - \textstyle\sum_{k=1}^{K} c_k q_k \geq \Pi \\
& P - \textstyle\sum_{k=1}^{K} \lambda_k^* q_k \leq g^* \\
& q_k \in \{0, U_k\} \qquad \forall k \in K \\
& P \leq M \textstyle\sum_k q_k \\
& P \geq 0.
\end{aligned}
\tag{BFPMB}
$$

Proposition 4.3. *The linear relaxation of* (BFPMB) *yields an integral optimal solution equivalent to the mixed integer programming* (FPMB)*.*

Proof. Considering the linear relaxation of (BFPMB) formulated as

$$
\begin{aligned}
\max \quad & P - \textstyle\sum_{k=1}^{K} c_k q_k \\
\text{s.t.} \quad & P - \textstyle\sum_{k=1}^{K} c_k q_k \geq \Pi & (1)\\
& P - \textstyle\sum_{k=1}^{K} \lambda_k^* q_k \leq g^* & (2)\\
& q_k \leq U_k \qquad \forall k \in K & (3)\\
& P \leq M \textstyle\sum_k q_k & (4)\\
& P \geq 0, \;\; q_k \geq 0 \qquad \forall k \in K.
\end{aligned}
\tag{FPMB2}
$$

the result is obvious from Proposition 4.2. Having optimal values of q_k^* as either 0 or the upper bound U_k, is equivalent to having the binary variable δ_k either set to 0 or 1. □

Therefore, even though consideration of the production lower bound is crucial for companies, from Proposition 4.2 we see that in this auction setting it is not necessary for the suppliers to enforce minimum quantities.

4. Profitability of allowing bid submission on strict subsets of products: Imposing $\delta_k = 1$ for $\forall k \in K$ in (FPMB) allows the supplier to bid on quantities in the range $[L_k, U_k]$. Mathematically this is defined as

$$\begin{aligned}
\max \quad & P - \textstyle\sum_{k=1}^{K} c_k q_k & & \\
\text{s.t.} \quad & P - \textstyle\sum_{k=1}^{K} c_k q_k \geq \Pi & & (1) \\
& P - \textstyle\sum_{k=1}^{K} \lambda_k^* q_k \leq g^* & & (2) \\
& L_k \leq q_k \leq U_k & \forall k \in K \quad & (3) \\
& P \leq M \textstyle\sum_k q_k & & (4) \\
& P \geq 0, \;\; q_k \in \mathbb{N} \cup \{0\} & \forall k \in K. &
\end{aligned} \tag{FPMB3}$$

Intuitively, the feasible solution to (FPMB3) is a subset of that of (FPMB). We formally state this result in Proposition 4.4 and provide an exact form for the optimal solution. The proof for this proposition is omitted as we have already outlined the arguments above and in the proof of Proposition 4.2.

Proposition 4.4. *The optimal value of* (FPMB3) *is at most as good as* (FPMB). *Furthermore the optimal solution is*

(a) If $\gamma_k \leq 0 \ \forall k \in K$ *then,*

i. If $g^* + \sum_k \gamma_k L_k < \Pi$*,* (FPMB3) *is infeasible.*

ii. If $g^* + \sum_k \gamma_k L_k \geq \Pi$*,* $q_k^* = L_k$ *for* $\forall k$*.*

(b) If $\exists k \in K \;\; s.t. \;\; \gamma_k > 0$*,*

$$q_k^* = \begin{cases} U_k & \gamma_k > 0 \\ L_k & \gamma_k \leq 0. \end{cases}$$

In fact, (FPMB3) enforces a minimum production of the items for which the cost of production is more than the announced Lagrangian prices which shrinks the suppliers' total optimal profit. This situation may arise from requirement from the auctioneer to have a minimum shipment from each supplier to justify the necessary order processing and unloading costs.

4.4.2 Comparison of the Bidder's Prices in Successive Auction Rounds

A question that both the auctioneer and the bidders will be interested in looking at is how the price of the new bundle would compare with the optimal price of the prior round. To answer this question, in Proposition 4.5 we study the optimal price obtained from the suppliers' capacitated profit maximization model (FPMB).

As defined in Section 4.3, for an arbitrary supplier $P^{*^{(n-1)}}$, $q_k^{*^{(n-1)}}$ represent the price and quantity values corresponding to his most competitive previous bid. Thus, in case the supplier submits a single bid in the previous round $P^{*^{(n-1)}}$, $q_k^{*^{(n-1)}}$ are the actual values of the bid he submits. For the suppliers with multiple previous bid submissions, $P^{*^{(n-1)}}$, $q_k^{*^{(n-1)}}$ correspond to the price and quantity values associated with bid j^* where

$$j^* = \text{argmin}_j \{P_j^{(n-1)} - \sum_{k=1}^{K} \lambda_k^{*^{(n-1)}} q_{jk}^{(n-1)}\}.$$

With $\gamma_k^{(n)} = \lambda_k^{*^{(n-1)}} - c_k$, let $K^{1^{(n)}}$ and $K^{2^{(n)}}$ be the sets for which $\gamma_k^{(n)}$ is respectively positive and non-positive. So, $K^{1^{(n)}} = \{k|\gamma_k^{(n)} > 0\}$ and $K^{2^{(n)}} = \{k|\gamma_k^{(n)} \leq 0\}$ and $K^{(n)} = K^{1^{(n)}} \cup K^{2^{(n)}}$.

Proposition 4.5. *Under the* (FPMB) *model, the suppliers' optimal price obtained at round (n) compares with the optimal price obtained at round (n-1) as follows:*

$$(a) \quad P^{*^{(n)}} = P^{*^{(n-1)}} \quad \textit{if}$$

$$\begin{cases} \forall k \quad \gamma_k^{(n)} > 0 \quad \& \quad q_k^{*(n-1)} = U_k, \\ \textit{or} \\ \sum_{k \in K^{1^{(n)}}} \lambda_k^{*^{(n-1)}} (U_k - q_k^{*^{(n-1)}}) - \sum_{k \in K^{2^{(n)}}} \lambda_k^{*^{(n-1)}} q_k^{*^{(n-1)}} = 0 \& (K^{1^{(n)}}, K^{2^{(n)}} \neq \emptyset). \end{cases}$$

$$(b) \quad P^{*^{(n)}} > P^{*^{(n-1)}} \quad \textit{if}$$

$$\begin{cases} \forall k \quad \gamma_k^{(n)} > 0 \quad \& \quad \exists k \in K^{1^{(n)}} \quad s.t. \quad q_k^{*^{(n-1)}} < U_k, \\ \textit{or} \\ \sum_{k \in K^{1^{(n)}}} \lambda_k^{*^{(n-1)}} (U_k - q_k^{*^{(n-1)}}) - \sum_{k \in K^{2^{(n)}}} \lambda^{*^{(n-1)}} q_k^{*^{(n-1)}} > 0 \& (K^{1^{(n)}}, K^{2^{(n)}} \neq \emptyset). \end{cases}$$

$$(c) \quad P^{*^{(n)}} < P^{*^{(n-1)}} \quad \textit{if}$$

$$\sum_{k \in K^{1^{(n)}}} \lambda^{*^{(n-1)}} (U_k - q_k^{*^{(n-1)}}) - \sum_{k \in K^{2^{(n)}}} \lambda_k^{*^{(n-1)}} q^{*^{(n-1)}} < 0 \& (K^{1^{(n)}}, K^{2^{(n)}} \neq \emptyset).$$

Proof. In Proposition 4.2 we describe optimal price values for the capacitated profit maximization problem at round (n) as $P^{*^{(n)}} = g^{*^{(n)}} + \sum_k \lambda_k^{*^{(n-1)}} q_k^{*^{(n)}}$ where $g^{*^{(n)}} = \min_j\{P_j^{(n-1)} - \sum_{k=1}^{K} \lambda_k^{*^{(n-1)}} q_{jk}^{(n-1)}\}$. Thus,

$$\begin{aligned}
P^{*^{(n)}} &= P^{*^{(n-1)}} - \textstyle\sum_k \lambda_k^{*^{(n-1)}} q_k^{*^{(n-1)}} + \sum_k \lambda_k^{*^{(n-1)}} q_k^{*^{(n)}} \\
&= P^{*^{(n-1)}} - \textstyle\sum_k \lambda_k^{*^{(n-1)}} q_k^{*^{(n-1)}} + \begin{cases} \sum_k \lambda_k^{*^{(n-1)}} U_k & \gamma_k^{(n)} > 0 \\ 0 & \gamma_k^{(n)} \leq 0 \end{cases} \\
&= P^{*^{(n-1)}} + \begin{cases} \sum_k \lambda_k^{*^{(n-1)}} (U_k - q_k^{*^{(n-1)}}) & \gamma_k^{(n)} > 0 \\ -\sum_k \lambda_k^{*^{(n-1)}} q_k^{*^{(n-1)}} & \gamma_k^{(n)} \leq 0. \end{cases} \\
&= P^{*^{(n-1)}} + \textstyle\sum_{k \in K^{1^{(n)}}} \lambda_k^{*^{(n-1)}} (U_k - q_k^{*^{(n-1)}}) - \sum_{k \in K^{2^{(n)}}} \lambda_k^{*^{(n-1)}} q_k^{*^{(n)}}.
\end{aligned} \tag{4.3}$$

To compare $P^{*^{(n)}}$ and $P^{*^{(n-1)}}$, we consider the following cases.

1. $\gamma_k^{(n)} \leq 0 \quad \forall k \in K^{(n)}, \quad (K^{1^{(n)}} = \emptyset).$

   When $\gamma_k^{(n)} \leq 0$ for all $k \in K^{(n)}$, then (FPMB) is infeasible if $g^{*^{(n)}} < \Pi$, or it yields the trivial solution $q_k^{*^{(n)}} = 0$ if otherwise. Thus, $P^{*^{(n)}} = 0$.

2. $\gamma_k^{(n)} > 0 \quad \forall k \in K^{(n)}, \quad (K^{2^{(n)}} = \emptyset).$

   If the quantities submitted in the best previous package are all at the production capacity level, then $q_k^{*^{(n-1)}} = U_k$ and so $P^{*^{(n)}} = P^{*^{(n-1)}}$. However, if for at least one of the items, the previously submitted quantity is strictly less than the capacity U_k then $P^{*^{(n)}} > P^{*^{(n-1)}}$.

3. $\exists k_1 \in K^{1^{(n)}}, k_2 \in K^{2^{(n)}} \quad s.t. \quad \gamma_{k_1}^{(n)} > 0 \;\text{ and }\; \gamma_{k_2}^{(n)} \leq 0, \quad (K^{1^{(n)}}, K^{2^{(n)}} \neq \emptyset).$

Based on the equality (4.3), when

$$\sum_{k\in K^{1(n)}} \lambda_k^{*(n-1)}(U_k - q_k^{*(n-1)}) > \sum_{k\in K^{2(n)}} \lambda_k^{*(n-1)} q_k^{*(n-1)},$$

the total value of extra production of items whose quantity offers are below production capacity and production cost is lower than the announced item prices, is greater than the total value of the quantities of items whose costs are above the announced item prices. Thus, the optimal price $P^{*(n)}$ should be strictly greater than $P^{*(n-1)}$.

If $\sum_{k\in K^{1(n)}} \lambda_k^{*(n-1)}(U_k - q_k^{*(n-1)}) < \sum_{k\in K^{2(n)}} \lambda_k^{*(n-1)} q_k^{*(n-1)}$, the optimal price $P^{*(n)}$ is strictly less than the previous best price $P^{*(n-1)}$. This can be seen considering that the model suggests zero optimal quantities for items whose production costs are greater than the announced prices.

□

Proposition 4.5 is useful for the auctioneer and bidders as it shows the value of the information being shared between the auction rounds. This proposition shows dependence of bundle prices on several factors such as: how other bidders value different units of products (λ_k), the suppliers' costs (c_k), minimum profit expectations (Π), price and quantities corresponding to the most competitive previous bid (g^*), internal production capacity as well as the auctioneer's demand (considering that the minimum of the two is U_k).

4.5 Variable-cost Profit Maximization Model (VPMB)

For suppliers with variable production cost, the expenses for the supply of products depends on the quantity delivered to the auctioneer. To meet suppliers' variable per-quantity cost, let us define c_{kw} as the per-quantity cost for provision of q_{kw} units of product k for the supplier's wth cost scenario $w = 1, \ldots, W$. For each cost scenario, the supplier considers production of q_{kw} units of items where $a_{kw} \leq q_{kw} \leq b_{kw}$. We can formulate the suppliers' profit maximization model as

$$
\begin{aligned}
\max \quad & P - \textstyle\sum_{k,w} c_{kw} q_{kw} & & \\
\text{s.t.} \quad & P - \textstyle\sum_{k} c_{kw} q_{kw} \geq \Pi & \forall w \quad & (1) \\
& P - \textstyle\sum_{k} \lambda_k^* q_{kw} \leq g^* & \forall w \quad & (2) \\
& a_{kw}\delta_w \leq q_{kw} \leq b_{kw}\delta_w & \forall k, w \quad & (3) \\
& P \leq M \textstyle\sum_{k} q_k & & (4) \\
& \textstyle\sum_{w} \delta_w = 1 & & (5) \\
& P \geq 0 & & \\
& q_{kw} \in \mathbb{N} \cup \{0\}, \delta_w \in \{0, 1\} & \forall k, w &
\end{aligned}
\qquad \text{(VPMB)}
$$

Note that for simplicity we drop (n) for all values, since they either correspond to the current round or constant throughput the auction. VPMB is formulated for future comparison against proposed models in Chapter 5.

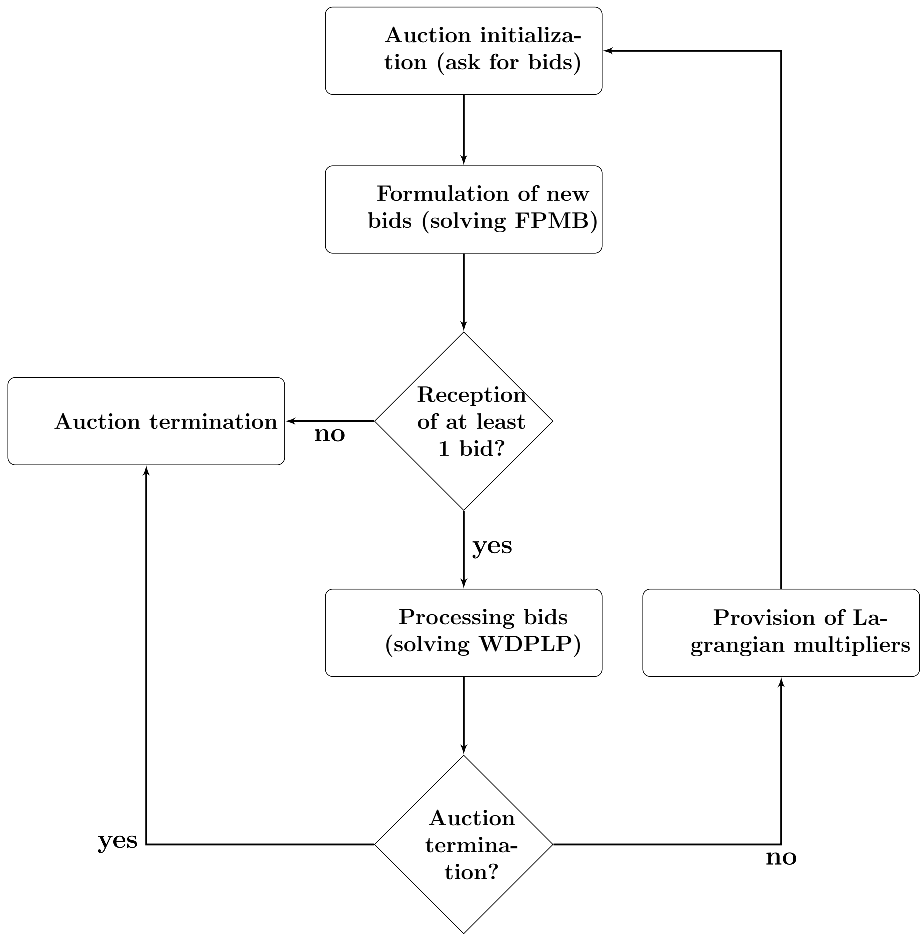
4.6 Lagrangian-based Iterative Auction Design (LIAD)

Suppliers' pricing corrections based on the auctioneer's feedback on the Lagrangian multipliers motivates the design of a discrete (as opposed to continuous) auction. With the assumption that the first round prices and bids are initiated using CATS, the winner determination problem (WDP) and its Lagrangian relaxation (WDPLR) are solved to provide the auctioneer with the optimal allocation and Lagrangian multipliers. As discussed in Chapter 3, rather than directly solving the Lagrangian relaxation problem, the auctioneer is able to solve the linear relaxation problem (WD-PLP) to access the Lagrangian optimal multipliers, objective value, and solution.

The Lagrangian multipliers provide proxies for the products' prices at the current round and hence providing them to the suppliers helps them in revaluating their bid prices in the subsequent auction round. Moreover, solving the WDP problem to (near) optimality will further help the auctioneer keep track of his profit dynamics throughout the auction.

Based on the announced products' price proxies, at the start of all following rounds, suppliers solve their profit maximization model to generate their optimal bids. As described before, the competitiveness of the new bid is seen in the design of the suppliers' PMB model. Once all suppliers pass their new bids to the auctioneer, he solves the winner determination problem WDP and the linear relaxation WDPLP to access the local winners and the Lagrangian multipliers. The bidding language considered is XOR. In Figure 9 we summarize the main steps of the auction procedure.

Figure 9: Iterative auction LIAD flowchart



In Proposition 4.8 we show that the auction iterates as long as at least one supplier submits a new bid incrementing his profit. Thus, at termination

$$\sum_{k \in K^1} \lambda_k^{*(n-1)} (U_k - q_k^{*(n-1)}) = \sum_{k \in K^2} \lambda_k^{*(n-1)} q_k^{*(n-1)}. \tag{4.4}$$

and so $P^{*(n)} = P^{*(n-1)}$.

This proposition provides the auctioneer with a tool to check whether all suppliers have reached their maximum profitability. Equality (4.4) holds when for each supplier and product k, if $\gamma_k^* > 0$, the product is supplied at its upper bound in the previous round, and if $\gamma_k^* < 0$, it is not supplied at all. Consequently, the Lagrangian multipliers of the current and previous rounds yield the same sign for γ_k.

This gives rise to the question of whether the auction converges. In other words, is it affirmative that the auction reaches a point where all suppliers become unable to submit a more competitive bid. To answer this question consider the dual variables (λ_k, δ_i) respectively assigned to the first and second set of constraints in WDPLP. The dual problem of WDPLP at iteration (n) of the auction is

$$\begin{array}{lll} \max & \sum_k \lambda_k^{(n)} d_k + \sum_i \delta_i^{(n)} & \\ s.t. & P_i^{(n)} - \sum_k \lambda_k^{(n)} q_{ik}^{(n)} - \delta_i^{(n)} \geq 0 \quad \forall i & (1) \\ & \lambda_k^{(n)} \geq 0, \delta_i^{(n)} \leq 0 \qquad\qquad \forall i, k. & \end{array} \tag{4.5}$$

Define $\Gamma^{(n)}$ as a sequence whose nth term specifies the optimal objective of the dual problem at round (n).

$$\Gamma^{(n)} = \{\sum_k \lambda_k^{*(n)} d_k + \sum_i \delta_i^{*(n)}, \quad \text{for} \quad n = 1, 2, 3, \ldots\}. \tag{4.6}$$

Proposition 4.6. *For an auctioneer with predetermined reservation prices, sequence* $\Gamma^{(n)}$ *as defined in* (4.6) *is convergent.*

Proof. In order to see the convergence of $\Gamma^{(n)}$, we show that this sequence is bounded and monotonically increasing.

Let $\hat{\lambda}_k$ determine the reservation price on product $k, (k = 1, 2, \ldots, K)$, i.e., $\hat{\lambda}_k$ is the highest price that the auctioneer is willing to pay for product k. It is easy to see that sequence $\Gamma^{(n)}$ is bounded from above by $\sum_k \hat{\lambda}_k d_k$. To see the monotonicity of the sequence, recall that supplier i's competitiveness condition at iteration n of the auction ensures that

$$P_i^{(n)} - \sum_k \lambda_k^{*^{(n-1)}} q_{ik}^{(n)} \leq P_i^{(n-1)} - \sum_k \lambda_k^{*^{(n-1)}} q_{ik}^{(n-1)}.$$

Let LHS and RHS respectively denote the left- and right-hand-side values of this condition. As discussed in the results of Proposition 4.2, at iteration (n) of the auction, PMB provides the suppliers with $(P_i^{*^{(n)}}, q_{ik}^{*^{(n)}})$ in such a way that the LHS of the competitiveness equation grows as large as the RHS value. Hence,

$$P_i^{*^{(n)}} - \sum_k \lambda_k^{*^{(n-1)}} q_{ik}^{*^{(n)}} = P_i^{*^{(n-1)}} - \sum_k \lambda_k^{*^{(n-1)}} q_{ik}^{*^{(n-1)}}. \tag{4.7}$$

Let $(\lambda^{*^{(n-1)}}, \delta^{*(n-1)})$ correspond to the optimal solutions of the dual problem at the previous round. At the previous optimal values, constraint(1) in (4.5) equals

$$P_i^{(n)} - \sum_k \lambda_k^{*^{(n-1)}} q_{ik}^{(n)} - \delta_i^{*^{(n-1)}} \geq 0,$$

which based on the equality (4.7) reformulates as

$$P_i^{(n-1)} - \sum_k \lambda_k^{*^{(n-1)}} q_{ik}^{(n-1)} - \delta_i^{*^{(n-1)}} \geq 0.$$

The left-hand-side of the above inequality is equivalent to the reduced cost associated to supplier i in the primal LP relaxation problem at round (n-1). Since the primal problem is a minimization problem, all the reduced costs at optimality are non-negative. Therefore, the optimal solution of the dual LP relaxation problem at round (n-1) satisfies the constraints of this problem at round (n) and thus belongs to its feasible region.

Consequently at round (n), (4.5) is only able to improve upon the previous optimal value with an optimal objective value either greater than or equivalent to that of the previous round. In other words, the optimal value of the WDPLP at iteration (n) is at least as good as that of iteration (n-1). This implies that sequence $\Gamma^{(n)}$ is monotonically increasing.

□

Note that in Proposition 4.6 we discuss the non-decreasing change pattern of the linear (or equivalently Lagrangian) relaxation lower bound as the auction proceeds in rounds. This result is inconclusive of the change pattern of the optimal MIP objective value, referred to as ZIP*. In fact, as we discuss in Section 4.7, we experimentally observe a non-monotone change pattern for ZIP*. In the next two propositions we characterise the suppliers profits.

Proposition 4.7. *The suppliers' profit is a nondecreasing function of the auction rounds.*

Proof. Considering the equality (4.7), the set of optimal solution ($P_i^{*(n-1)}$, $q_{ik}^{*(n-1)}$) obtained at round (n-1) is a trivial feasible solution at round (n). Therefore, the optimal objective value at round (n) is at least as good as round ($n-1$). Consequently, at every round of the auction suppliers either maintain or improve upon the profit they made at the previous round. □

Proposition 4.8. *For an auction terminating at round (n), $\delta_i^{*(n)} = 0$ for each supplier i.*

Proof. For an auction problem terminating at round (n) assume on the contrary that there exists supplier $\hat{i}$ for whom $\delta_{\hat{i}}^{*(n)} \neq 0$, thus $\delta_{\hat{i}}^{*(n)} < 0$. The complementary slackness condition $\delta_{\hat{i}}^{*(n)}(x_{\hat{i}}^* - 1) = 0$ necessitates $x_{\hat{i}}^* = 1$. Thus, supplier $\hat{i}$ is among the winners of the primal LP relaxation problem with a zero value of the associated reduced cost $P_{\hat{i}}^{*(n)} - \sum_k \lambda_k^{*(n)} q_{\hat{i}k}^{*(n)} - \delta_{\hat{i}}^{*(n)}$. Hence,

$$\delta_{\hat{i}}^{*(n)} = P_{\hat{i}}^{*(n)} - \sum_k \lambda_k^{*(n)} q_{\hat{i}k}^{*(n)} < 0,$$

and

$$g^{*(n)} = P_{\hat{i}}^{*(n)} - \sum_k \lambda_k^{*(n)} q_{\hat{i}k}^{*(n)} < 0.$$

Moreover, since $P^{(n)} - \sum_k c_k q_k^{*(n)} > 0$,

$$\begin{array}{ll} P_{\hat{i}}^{(n)} - \sum_k c_{\hat{i}k} q_{\hat{i}k}^{*(n)} & > P_{\hat{i}}^{(n)} - \sum_k \lambda_k^{*(n)} q_{\hat{i}k}^{*(n)} \\ \sum_k \lambda_k^{*(n)} q_{\hat{i}k}^{*(n)} & > \sum_k c_{\hat{i}k} q_{\hat{i}k}^{*(n)} \\ \sum_k (\lambda_k^{*(n)} - c_{\hat{i}k}) q_{\hat{i}k}^{*(n)} & > 0, \end{array}$$

which implies that

$$\exists \hat{k} \qquad s.t. \qquad (\lambda_k^{*(n)} - c_{\hat{i}k}) > 0. \tag{4.8}$$

Considering the competitiveness condition as

$$\Pi \quad \leq P_{\hat{i}}^{(n)} - \sum_{k=1}^{K} c_{\hat{i}k} q_{\hat{i}k}^{*(n)} \quad \leq g^{*(n)} + \sum_{k=1}^{K} (\lambda_k^{*(n)} - c_{\hat{i}k}) q_{\hat{i}k}^{*(n)},$$

supplier $\hat{i}$ can improve his profit by increasing its offering $q_{\hat{i}k}^{*(n)}$ (and the corresponding price) which contradicts the fact that the auction terminated at round (n).

□

In conclusion, the auction terminates if no supplier is able to formulate a new bid that will strictly improve his profit. Proposition 4.8 aids the auctioneer in verifying if all suppliers have achieved maximum profitability at the current round. Specifically, the auctioneer is able to detect whether all suppliers have reached their maximum profitability at the current round by checking on their corresponding dual variables δ_i^*. The fact that the auction runs so long as the suppliers are able to improve upon their optimal profit incentivizes the suppliers' participation in the auction. At each iteration, suppliers take the marginal prices obtained from the Lagrange multipliers to formulate their new package offer.

4.7 Numerical Experiments

To gain more insights from the models developed in the previous sections we conducted numerical experiments. Based on the bundles generated via CATS, we formulate and solve the corresponding LP relaxation problem to access the optimal dual variables associated with the demand constraints. With $\Pi = 0$, we conduct our experiments for an auction of 3 products and 15 bids received from the suppliers. We use CATS to simulate the items included in each bid and the corresponding bundle prices. CATS bids are illustrated in Table 5. As explained in Section 3.4, any quantity value above the maximum number of items (here fixed at 3) corresponds to a dummy bid. Bids with identical dummies are received from identical bidders.

Table 5: CATS bids generated for 3 items and 15 bids

Bids	Prices	Quantities of items in the Package		
1	329.632	1	2	3
2	86.3849	1		
3	218.796	1	2	
4	196.905	1	3	
5	236.71	2	3	
6	135.677	2		
7	115.175	3		
8	145.904	1	3	364
9	242.745	2	3	364
10	168.918	1	2	364
11	245.51	2	3	393
12	118.07	1	3	393
13	241.602	2	3	515
14	179.878	1	2	515
15	168.828	1	2	514

Since CATS generates a single unit of each item in the package, we use a uniform distribution to generate multi-unit packages. The quantity of each item offered by suppliers and the auctioneer's item demand are respectively generated uniformly from [1,15] and [20,40]. CATS prices for the bundles with single unit items are adjusted as explained in Section 3.4 to reflect the prices on the bundles containing multiple units.

Note that in the generation of the bids' prices we rely on CATS to generate the price for the whole bundle and later scale it. The initial prices reflect bidders valuation of the packages without information on their competitors valuations. As the auction progresses they would adjust their prices to take that information into account as it becomes available to them. Furthermore, we do not explicitly include the items costs information in the initial prices. The item costs are uniformly generated from [0.5,1.5] and then scaled by their corresponding Lagrange multipliers. To remain consistent with the quantities included in each bid, for each supplier i we generate the upper and lower bounds for item k as

$$L_k \sim \lfloor U[1, \min(5, q_k^*)] \rfloor$$
$$U_k \sim \lfloor U[\max\{6, q_k^*\}, 15] \rfloor$$

where $U[a, b]$ denotes a uniform distribution on the interval $[a.b]$. Thus,

$$q_k^* \leq 5 \Rightarrow \begin{cases} L_k \sim \lfloor U[1, q_k^*] \rfloor \\ U_k \sim \lfloor U[6, 15] \rfloor \end{cases}, q_k^* > 5 \Rightarrow \begin{cases} L_k \sim \lfloor U[1, 5] \rfloor \\ U_k \sim \lfloor U[q_k^*, 15] \rfloor. \end{cases}$$

Table 6 records the optimal values attained from solving different suppliers' profit maximization models as well as the auctioneer's winner determination problem at the first and second rounds of the auction.

At the first round of the auction wherein suppliers' bids are generated using CATS data generator, suppliers' optimal price and quantity $(P^{*^{(1)}}, q_k^{*^{(1)}})$ are associated with the corresponding values derived for the suppliers' most competitive bid. At the second round, $(P^{*^{(2)}}, q_k^{*^{(2)}})$ are the optimal values obtained from the PMB model. We record the suppliers' profit from their most competitive bid at round (1) or from solving the profit maximization problem in round (2) respectively as $PP^{*^{(1)}}$, $PP^{*^{(2)}}$.

At the end of each round, the winner determination problem is solved based on the bids received. The profit suppliers make when winning in the auction is referred to as $WP^{*^{(1)}}$, $WP^{*^{(2)}}$. We denote the auctioneer's total cost of procurement at each round as $TCP^{*^{(1)}}$, $TCP^{*^{(2)}}$. When a supplier is a winner in either of the auction rounds, he will actually make the profit as calculated by the profit maximization models. Otherwise, the actual profit gained from the winner determination problem is 0. For each supplier, we calculate $\gamma_k^{(n)}$ for product k as $\lambda_k^{*^{(n-1)}} - c_k$. Finally, we define

$$\theta^{(n)} = sign(\sum_{k \in K^1} \lambda_k^{*^{(n-1)}}(U_k - q_k^{*^{(n-1)}}) - \sum_{k \in K^2} \lambda_k^{*^{(n-1)}} q_k^{*^{(n-1)}})$$

in order to compare the price from the supplier's best previous bid and the optimal price derived from PMB models.

4.7.1 Further Insights on Proposition 4.1-Proposition 4.5

From the results of Table 6, we can make the following observations on the proposed models:

1. Even thought the model (PMB2) introduced by Hsieh [41] (PBM2 rows in Table 6) does provide a new package at the price $P^{*^{(2)}}$ much higher than the previous best price $P^{*^{(1)}}$, the model suggests impractical quantities that are as many as almost 10 times the supplier's capacity for products with positive $\gamma_k^{(2)}$. The optimal quantity of items with negative $\gamma_k^{(2)}$ is 0.

2. Model (FPMB) provides a lower price (compared to (PMB2)). Quantities with positive $\gamma_k^{(2)}$ are offered at the capacity level. Quantities with non positive $\gamma_k^{(2)}$

Table 6: Comparison of bidders' pricing models for indivisible auctions

i		U_k			$q_k^{*(1)}$									$\Upsilon_k^{(2)}$		Feasibility		$q_k^{*(2)}$					
	Model	k=1	k=2	k=3	k=1	k=2	k=3	$P^{*(1)}$	$PP^{(1)}$	$WP^{(1)}$	$TTP^{(1)}$	$\Theta^{(2)}$	k=1	k=2	k=3	Code	k=1	k=2	k=3	$P^{*(2)}$	$PP^{(2)}$	$WP^{(2)}$	$TTP^{(2)}$
1	PMB2	13	13	6	13	13	1	296.67				-	0.51	-4.33	-1	1	100	0	0	389.96			
	FPMB	13	13	6	13	13	1	296.67	36.35	0						1	13	0	0	126.35	93.62	0	
	FPMB2	13	13	6	13	13	1	296.67								1	13	0	0	126.35			
	FPMB3	13	13	6	13	13	1	296.67								1	13	4	4	221.77			
2	PMB2	12	6	7	12	0	0	103.66				+	1.05	-5.53	5.4	1	100	0	100	1535.36			
	FPMB	12	6	7	12	0	0	103.66	79.9	0						1	12	0	7	185.22	117.7	0	
	FPMB2	12	6	7	12	0	0	103.66								1	12	0	7	185.22			
	FPMB3	12	6	7	12	0	0	103.66								1	12	3	7	221.83			
3	PMB2	14	9	8	14	4	0	196.92				+	-0.96	-0.23	3.52	1	0	0	100	1270.73			
	FPMB	14	9	8	14	4	0	196.92	91.4	0						1	0	0	8	198.88	133.87	0	
	FPMB2	14	9	8	14	4	0	196.92								1	0	0	8	198.88			
	FPMB3	14	9	8	14	4	0	196.92								1	1	4	8	250.73			
4	PMB2	12	10	11	12	0	11	226.44				+	0.5	3.93	0.82	1	100	100	100	2750.52			
	FPMB	12	10	11	12	0	11	226.44	77.01	0						1	12	10	11	348.49	116.33	0	
	FPMB2	12	10	11	12	0	11	226.44								1	12	10	11	348.49			
	FPMB3	12	10	11	12	0	11	226.44								1	12	10	11	348.49			
5	PMB2	9	7	7	0	2	4	71.01				-	-1.42	-3.19	-2.04	1	0	0	0	0			
	FPMB	9	7	7	0	2	4	71.01	-14.55	0						1	0	0	0	0	0	0	
	FPMB2	9	7	7	0	2	4	71.01								1	0	0	0	0			
	FPMB3	9	7	7	0	2	4	71.01								10	0	0	0	0			
6	PMB2	9	7	9	0	5	0	67.84				+	-0.63	1.67	1.79	1	0	100	100	2392.41			
	FPMB	9	7	9	0	5	0	67.84	15.16	0						1	0	7	9	197.1	34.58	34.58	
	FPMB2	9	7	9	0	5	0	67.84								1	0	7	9	197.1			
	FPMB3	9	7	9	0	5	0	67.84								1	1	7	9	200.13			
7	PMB2	6	6	9	0	0	9	103.66				-	1.02	1.34	-3.68	1	100	100	0	1522.34			
	FPMB	6	6	9	0	0	9	103.66	-34.36	-34.36						1	6	6	0	90.21	13.02	13.02	
	FPMB2	6	6	9	0	0	9	103.66								1	6	6	0	90.21			
	FPMB3	6	6	9	0	0	9	103.66								1	6	6	3	125.17			
8	PMB2	9	10	8	2	10	0	101.35				-	1.09	-1.96	4.55	1	100	0	100	1441.29			
	FPMB	9	10	8	2	10	0	101.35	-44.14	-44.14						1	9	0	8	93.71	19.46	0	
	FPMB2	9	10	8	2	10	0	101.35								1	9	0	8	93.71			
	FPMB3	9	10	8	2	10	0	101.35								1	9	3	8	130.33			
9	PMB2	12	8	7	12	0	6	106.26				-	-0.8	-2.36	3.03	1	0	0	100	1165.06			
	FPMB	12	8	7	12	0	6	106.26	8.55	8.55						1	0	0	7	81.55	21.21	0	
	FPMB2	12	8	7	12	0	6	106.26								1	0	0	7	81.55			
	FPMB3	12	8	7	12	0	6	106.26								1	5	5	7	157.73			
10	PMB2	9	8	7	1	5	0	53.96				+	-0.18	2.54	4.43	1	0	100	100	2375.5			
	FPMB	9	8	7	1	5	0	53.96	2.44	2.44						1	0	8	7	169.1	41.28	41.28	
	FPMB2	9	8	7	1	5	0	53.96								1	0	8	7	169.1			
	FPMB3	9	8	7	1	5	0	53.96								1	5	8	7	184.25			
11	PMB2	12	10	8	12	10	0	185.71				-	0.01	-5.06	-3.06	1	100	0	0	330.29			
	FPMB	12	10	8	12	10	0	185.71	-23.25	0						1	12	0	0	63.66	27.36	27.36	
	FPMB2	12	10	8	12	10	0	185.71								1	12	0	0	63.66			
	FPMB3	12	10	8	12	10	0	185.71								1	12	2	1	99.72			
											365.2												520.1

are not included in the new package.

3. In model (FPMB3), inclusion of all quantities in the new package enforces a minimum quantity for the items with negative $\gamma_k^{(2)}$ values which decreases the optimal profit for the suppliers. Moreover, while all other models result in feasible prices for all supplies, (FPMB3) is infeasible for supplier 5. An infeasible supplier pricing problem suggests that the supplier is no longer able to offer better prices and would drop from the auction.

4. In comparison of new and old prices, we observe that for supplier 4 whose $\gamma_k^{(2)} > 0$ for all k, $P^{*^{(1)}} < P^{*^{(2)}}$. For supplier 5 whose $\gamma_k^{(2)} < 0$ for all k, $P^{*^{(1)}} > P^{*^{(2)}}$. For suppliers 2, 3, 6, 10 with $\theta(2) > 0$, $P^{*^{(1)}} < P^{*^{(2)}}$ and eventually for suppliers 1, 7, 8, 9, 11 with $\theta^{(2)} < 0$, $P^{*^{(1)}} > P^{*^{(2)}}$.

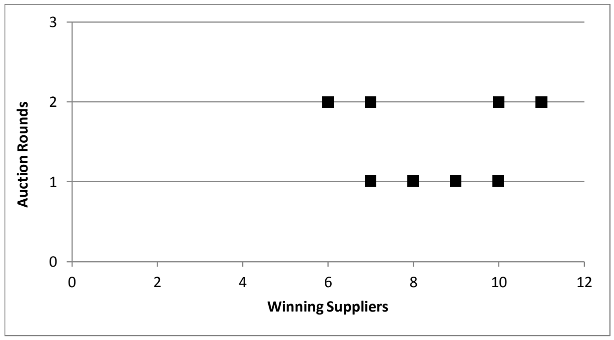
4.7.2 The Auctioneer's and Suppliers' Profit Dynamics at the Second Round of the Auction

This section studies the auctioneer's and the suppliers' profit changes as the auction proceeds to the second round. Define GG as the gross growth and GP as the growth percentage. Based on the results from Table 6, in Figure 10(a) we plot the winning suppliers in the first and second rounds. Figure 10(b) and 10(c) illustrate the gross growth of suppliers' WDP and FPMB profits.

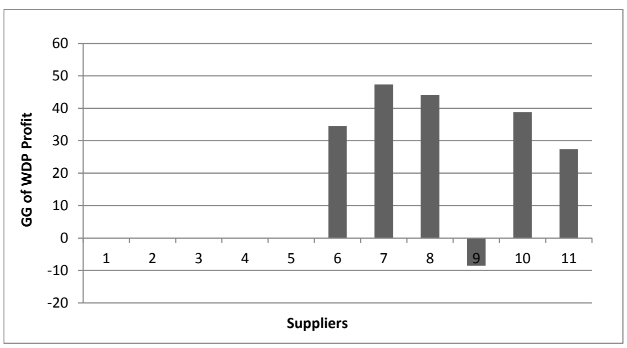
As can be seen in Figure 10(c), the suppliers' FPMB profit increases as they go from the first to the second round of the auction. From Figure 10(a) we see that suppliers 1, 2, 3, 4, and 5 are winning in neither of the auction rounds. Suppliers 6 and 11 who are

Figure 10: Two-round auction

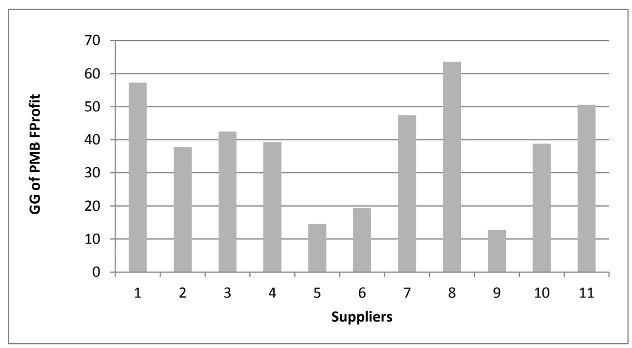
(a) Winning Suppliers



(b) The gross growth in WDP Profit



(c) The gross growth in FPMB Profit



losing in the first round are able to win in the second round despite the gross growth of respectively 47.3 and 50.6 in their FPMB profit. Suppliers 7 and 10 win both rounds of the auction. They both increase their FPMB gross profit. Changing his profit from 2.4 to 41.2, supplier 10 gains 1591.8 % WDP profit increase. Suppliers 8 and 9 who win the first round, lose in the second round of the auction. However, since supplier 8 gains a negative profit in the first round, not winning the auction in the second round implies a positive WDP profit growth in Figure 10(b). Supplier 9 who wins in round 1, fails to win in round 2 and gains negative WDP profit in the second round.

As Table 6 suggests, half of the winning suppliers (suppliers 7 and 8) in round 1 gain negative profit with the supply of the proposed qualities of items. As explained before, this is due to the fact that suppliers do not price the bundles based on the production costs, rather costs are initially generated via CATS as multipliers of average product prices submitted by all suppliers. Nonetheless, suppliers are able to adjust their bids according to the feedback from the auctioneer and maintain positive profit in the future rounds of the auction. In practice, this happens when suppliers are not completely confident how to price products and can therefore face negative profit on the first round. Based on the information revealed by the auction they are able to decide whether to continue bidding while maintaining a minimum desired profit or to drop out of the auction.

It is not unusual then to expect an increase in the auctioneer's total price of procurement on the second round as the suppliers with negative profit correct their pricing to guarantee their minimum expected profit. Terminating the auction at the first stage can result in assigning packages to suppliers with negative profit and increase the suppliers' delivery failure risk.

4.7.3 The Auctioneer's and Suppliers' Profit Dynamics at the Stabilization Round (RS) of the Auction

Results in the previous sections suggest that the auctioneer's total cost of procurement increases at the second round of the auction when suppliers get the opportunity to correct their bid submissions. This raises the question of whether the auctioneer's total cost of procurement keeps increasing until the end of the auction?

To answer this question, we run the auction for a maximum of 10 rounds. Table 7 illustrates the results of the first 5 rounds of the auction and Figures 11(a), 11(b), and 11(c) respectively show the winning suppliers, the auctioneer's total cost of procurement, and the suppliers' cumulative WDP profit (denoted as SCWP).

Results from Table 7 imply that the auctioneer's total cost of procurement does not necessarily increase at every round of the auction. Indeed, it can decrease from round to round (Table 7 demonstrates a decrease in TCP from 476.31 in round 2 to 439.62 in round 3) and eventually remain constant from a relatively early round (round 4 in Table 7). At the start of round 4 all suppliers derive $\theta = 0$. Thus, as discussed in Proposition 4.5 no supplier is able to submit a more competitive bid and the auction terminates. We will refer to this round as the *stabilization round*, and denote it as RS. Note that RS is the round preceded by the round in which we first observe the identical results. In other words, RS is the first round at which repeated results appear.

To go beyond this example, we repeat the iterative auction on 20 different problem instances. The results as shown in Table 8, confirm the convergence of the auctioneer's total cost of procurement, the Lagrangian multipliers as well as the suppliers' profit at a relatively early stage.

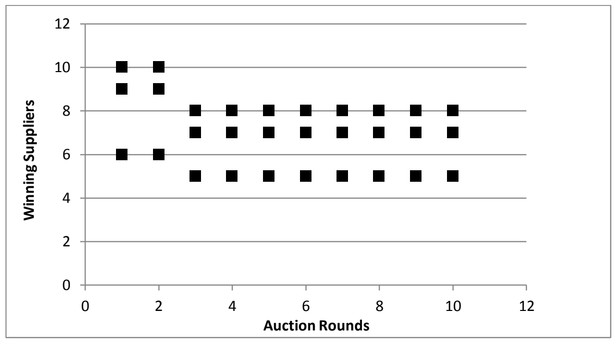
The first column of this table shows the problem instance and the second column

Table 7: Iterative auction

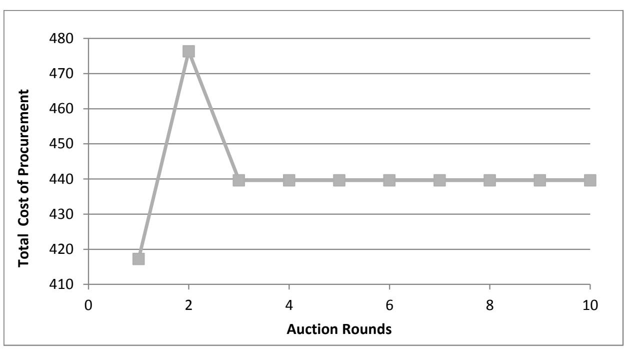
round	i	θ	Υ_k			q_k^*			U_k			P*	PP	WP	TPP	λ_k		
			k=1	k=2	k=3	k=1	k=2	k=3	k=1	k=2	k=3					k=1	k=2	k=3
1	1					14	4	5				252.72	42.02		417.19	8.36	9.13	6.63
	2					6	0	0				51.83	7.85					
	3					5	12	0				185.98	25.17					
	4					8	0	12				196.91	45.45					
	5					0	6	7				153.86	90.75					
	6					0	7	0				94.97	61.97	90.75				
	7					0	0	1				11.52	3.83					
	8					8	1	0				76.01	19.95					
	9					4	0	12				94.46	-21.98	19.95				
	10					2	9	0				98.93	13.14	-21.98				
	11					3	8	0				92.86	-35.55					
SCWP														88.72				
2	1	-	-0.8	-4.4	0.96	0	0	10	14	7	10	132.29	75.59		476.31	8.36	8.86	6.91
	2	+	1.03	-1.46	0.66	10	0	6	10	10	6	125.05	15.95					
	3	-	-2.71	0.35	-2.86	0	12	0	7	12	7	144.18	38.74					
	4	-	-2.8	-4.53	1.45	0	0	12	9	8	12	130.03	67.87					
	5	+	-2.76	4.51	1.58	0	8	8	8	8	8	178.76	101.35					
	6	+	0.97	4.42	3.07	8	7	6	8	7	6	201.63	88.16	88.16				
	7	+	3.39	4.26	-1.05	7	9	0	7	9	9	145.62	66.96					
	8	-	2.63	-1.07	-0.19	8	0	0	8	10	7	66.88	21.02					
	9	-	0.84	-1.29	-0.57	8	0	0	8	7	12	66.88	6.7	6.7				
	10	+	2.38	0.93	2.25	6	10	10	6	10	10	207.81	46.07	46.07				
	11	-	-0.5	-3.59	0.77	6	10	10	10	10	6	207.81	-31.26					
SCWP														140.93				
3	1	0	-0.8	-4.68	1.24	0	0	10	14	7	10	132.29	75.59		439.62	8.36	9.68	6.91
	2	0	1.03	-1.74	0.94	10	0	6	10	10	6	125.05	15.95					
	3	0	-2.71	0.07	-2.58	0	12	0	7	12	7	144.18	38.74					
	4	0	-2.8	-4.81	1.73	0	0	12	9	8	12	130.03	67.87					
	5	0	-2.76	4.23	1.86	0	8	8	8	8	8	178.76	101.35	101.35				
	6	0	0.97	4.14	3.35	8	7	6	8	7	6	201.63	88.16					
	7	0	3.39	3.98	-0.78	7	9	0	7	9	9	145.62	66.96	66.96				
	8	+	2.63	-1.35	0.09	8	0	7	8	10	7	115.24	21.65	21.65				
	9	0	0.84	-1.57	-0.29	8	0	0	8	7	12	66.88	6.7					
	10	0	2.38	0.65	2.53	6	10	10	6	10	10	207.81	46.07					
	11	-	-0.5	-3.87	1.05	0	0	6	10	10	6	41.45	6.27					
SCWP														189.96				
4	1	0	-0.8	-3.86	1.24	0	0	10	14	7	10	132.29	75.59		439.62	8.36	9.68	6.91
	2	0	1.03	-0.91	0.94	10	0	6	10	10	6	125.05	15.95					
	3	0	-2.71	0.89	-2.58	0	12	0	7	12	7	144.18	38.74					
	4	0	-2.8	-3.98	1.73	0	0	12	9	8	12	130.03	67.87					
	5	0	-2.76	5.05	1.86	0	8	8	8	8	8	178.76	101.35	101.35				
	6	0	0.97	4.96	3.35	8	7	6	8	7	6	201.63	88.16					
	7	0	3.39	4.8	-0.78	7	9	0	7	9	9	145.62	66.96	66.96				
	8	0	2.63	-0.52	0.09	8	0	7	8	10	7	115.24	21.65	21.65				
	9	0	0.84	-0.75	-0.29	8	0	0	8	7	12	66.88	6.7					
	10	0	2.38	1.47	2.53	6	10	10	6	10	10	207.81	46.07					
	11	0	-0.5	-3.05	1.05	0	0	6	10	10	6	41.45	6.27					
SCWP														189.96				
5	1	0	-0.8	-3.86	1.24	0	0	10	14	7	10	132.29	75.59		439.62	8.36	9.68	6.91
	2	0	1.03	-0.91	0.94	10	0	6	10	10	6	125.05	15.95					
	3	0	-2.71	0.89	-2.58	0	12	0	7	12	7	144.18	38.74					
	4	0	-2.8	-3.98	1.73	0	0	12	9	8	12	130.03	67.87					
	5	0	-2.76	5.05	1.86	0	8	8	8	8	8	178.76	101.35	101.35				
	6	0	0.97	4.96	3.35	8	7	6	8	7	6	201.63	88.16					
	7	0	3.39	4.8	-0.78	7	9	0	7	9	9	145.62	66.96	66.96				
	8	0	2.63	-0.52	0.09	8	0	7	8	10	7	115.24	21.65	21.65				
	9	0	0.84	-0.75	-0.29	8	0	0	8	7	12	66.88	6.7					
	10	0	2.38	1.47	2.53	6	10	10	6	10	10	207.81	46.07					
	11	0	-0.5	-3.05	1.05	0	0	6	10	10	6	41.45	6.27					
SCWP														189.96				

Figure 11: The auctioneer's and suppliers' payoff in an iterative auction

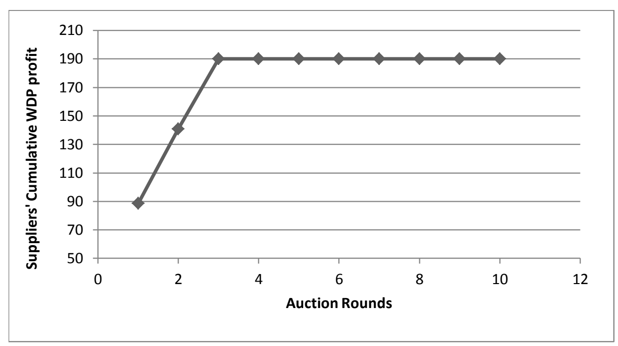
(a) Winning Suppliers



(b) Total Cost of Procurement



(c) Suppliers' cumulative WDP Profit



determines the stabilization round RS. We capture the GP of the auctioneer's TCP, and Lagrangian multipliers, as well as the GG of suppliers' cumulative profit at round RS with respect to round 1 and round 2 (denoted as R1 and R2). The growth percentage GP or the gross growth GG for an index X (which could be either the

Table 8: 20 Instances of iterative auction implementation-comparison results

Instance	RS	GP_TCP (RS,R1)	GP_TCP (RS,R2)	GP_λ1 (RS,R1)	GP_λ1 (RS,R2)	GP_λ2 (RS,R1)	GP_λ2 (RS,R2)	GP_λ3 (RS,R1)	GP_λ3 (RS,R2)	GG_SCWP (RS,R1)	GG_SCWP (RS,R2)
1	4	5.4	-7.7	0	0	6	9.3	4.2	0	136.8	49
2	3	9.3	0	8.5	0	9.4	0	-13.2	0	146.4	0
3	6	20.3	12.6	19	19	5.5	1.1	-4.7	-1	169.9	25.6
4	4	3.2	-2.1	-8.1	0	6.2	0	-2.8	0	172.6	-12.2
5	3	35.1	0	1.7	0	47.4	0	-27.8	0	283.1	0
6	4	34	20.5	0	0	0	0	5	0	187	1.6
7	8	19.2	8.5	-8.6	22.9	34	16.5	0	-0.9	146.8	26.2
8	4	5.6	5	-1	0	-13.1	0	0.5	0	61.3	56.6
9	4	14.4	-2.1	0	0	4.2	0	0.2	0	148.1	29
10	3	-0.8	0	-8.1	0	-18.8	0	19.7	0	195.1	0
11	3	2.6	5.9	8.4	0	-4.4	0	0	0	61.2	0
12	3	9.4	-4.5	28.2	0	3.9	0	0	0	111.1	27.5
13	6	15.9	5.4	0	8.3	14	-0.1	-4.7	12.4	49	0.2
14	4	32.7	1.3	0	10.5	-15.7	4.3	8.8	-7.5	93.7	26.1
15	3	-2.3	-8.3	7.2	0	-0.9	0	12.5	0	205.3	75.2
16	4	33.9	6.4	-25.3	0	11.8	0	6.1	0	56.8	-46.8
17	4	33.9	6.4	12.9	0	-6.4	0	19.4	0	269.4	35.5
18	5	14.9	-2.3	1.6	-9.4	4.3	1.7	9.7	2.8	293	-3.6
19	4	15.3	-3.5	-9.1	0	7.8	0	5.4	0	305.2	-34.4
20	4	5.6	0	23	0	4.3	0	-16.7	0	278.6	0
Average	4	15.4	2.1	2.5	2.6	5	1.6	1.1	0.3	168.5	12.8

$TCP, \lambda_1, \lambda_2, \lambda_3$, or $SCWP$) at round Rn as compared to round Rm is defined as

$$GG\_X(Rm, Rn) = X_{Rn} - X_{Rm}, \qquad GP\_X(Rm, Rn) = \frac{X_{Rn} - X_{Rm}}{X_{Rm}}.$$

Compared to the first round, the auctioneer's TCP growth percentage is positive among most problem instances (except for instances 10 and 15) with an average of 15.4%. This is excepted due to some suppliers' inaccurate valuations at the initial round of the auction resulting in their negative profit. However, comparing the final round to the second round in which suppliers are able to maintain a reasonable minimum profit shows a much less growth percentage with an average of 2.1% which suggests that even though the growth percentage of TCP is positive, the auctioneer can appreciate the little growth of TCP at the final round (around 2%) as compared to the second round.

Table 9: 20 Instances of iterative auction implementation-comparison results

Items	RS vs. R1							RS vs. R2						
	GP^+			GP^-			GP^0	GP^+			GP^-			GP^0
	%	Ave	stdev	%	Ave	stdev	%	%	Ave	stdev	%	Ave	stdev	%
1	45	12.3	9.3	30	-10	8.1	25	20	15.2	6.9	5	-9.4	-	75
2	65	12.2	13.3	30	-9.9	7	5	25	3.3	3.7	5	-0.1	-	60
3	55	8.3	6.6	30	-11.7	9.6	15	10	7.6	6.8	15	-3.1	3.8	75
Average	55	10.9	9.7	30	-10.5	8.2	15	18.3	8.7	5.8	8.3	-4.2	3.8	70

In Table 8 we observed the growth percentage of items' prices over 20 problem instances. We extend this analysis in Table 9 to understand what percentage of this population possesses a positive, negative, or zero price growth (respectively represented as GP^+, GP^-, GP^0). We also look for the amount of average price growth percentage in each class. Define

$\bar{\lambda}$	Average price over $\lambda_1, \lambda_2, \lambda_3$
$PGP^+_{\bar{\lambda}}, PGP^-_{\bar{\lambda}}, PGP^0_{\bar{\lambda}}$	The percentage of instances with respectively positive negative, and zero GP of $\bar{\lambda}$
$GP^+_{\bar{\lambda}}, GP^-_{\bar{\lambda}}$	Positive and negative GP of $\bar{\lambda}$.

Studying the results we observe that,

- Going from R1 to RS we realize a positive average price growth percentage for the majority of problem instances (55%). There is a negative price growth on nearly half of this population (30%). This analogy holds as we compare RS to R2 (18.3 % of the instances have positive growth and 8.3% have negative), i.e.,

$$PGP^{+}_{\bar{\lambda}}(RS, R1) \sim 2PGP^{-}_{\bar{\lambda}}(RS, R1)$$
$$PGP^{+}_{\bar{\lambda}}(RS, R2) \sim 2PGP^{-}_{\bar{\lambda}}(RS, R2).$$

Thus, at rounds R1 and R2 it is more likely that the prices increase than to decrease as compared to RS.

- The percentage of the instances with positive price growth is much larger at R1 as compared to R2. Comparing R1 to RS, 55% of the instances have a positive price growth. This percentage shrinks down to 18.3 when comparing the prices at R2 against RS. Similar analogy holds for instances with a negative price growth, i.e.,

$$PGP^{+}_{\bar{\lambda}}(RS, R1) \sim 3PGP^{+}_{\bar{\lambda}}(RS, R2)$$
$$PGP^{-}_{\bar{\lambda}}(RS, R1) \sim 4PGP^{-}_{\bar{\lambda}}(RS, R2).$$

  Thus, the likelihood that the prices change (increase or decrease) is less at R2 as compared to R1.

- The prices on the majority of problem instances in R2 (around 70%) remain unchanged compared to RS, i.e.,

$$PGP^{0}_{\bar{\lambda}}(RS, R2) >> PGP^{+}_{\bar{\lambda}}(RS, R2) > PGP^{-}_{\bar{\lambda}}(RS, R2).$$

  Thus, at the second round of the auction when bidders prepare their bids based on the feedback from the auction, it is likely that the item prices do not change too much till the end of the auction.

- The growth rate of the average prices for the population with either positive or negative price growth is below 10%. Thus, the final prices have a growth percentage ranging from -10% to +10%. Also, the absolute growth percentage is larger when comparing RS to R1 than to R2, i.e.,

$$\begin{aligned} 0 &\leq GP^{+}_{\bar{\lambda}}(RS, R1) & \leq 10\% \\ -10\% &\leq GP^{-}_{\bar{\lambda}}(RS, R1) & 0\%, \end{aligned}$$

and

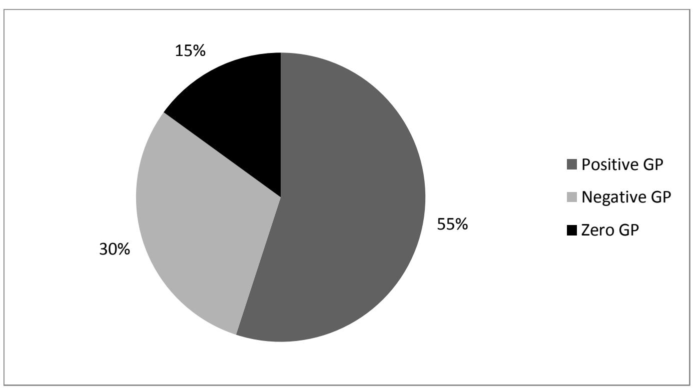
$$\begin{aligned} &GP^{+}_{\bar{\lambda}}(RS, R2) \leq GP^{+}_{\bar{\lambda}}(RS, R1) \\ &|GP^{-}_{\bar{\lambda}}(RS, R2)| \leq |GP^{-}_{\bar{\lambda}}(RS, R1)|. \end{aligned}$$

For more clarification, Figure 12 illustrates $PGP^{+}_{\bar{\lambda}}, PGP^{-}_{\bar{\lambda}}, PGP^{0}_{\bar{\lambda}}$ when comparing RS to R1 and R2.

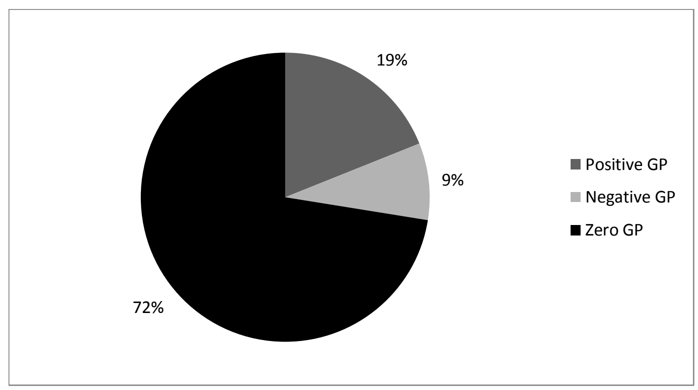
In comparison of the suppliers cumulative profit from the auction, referred to as SCWP profit in Table 8, we detect a large growth of RS compared to R1. This is expected considering the fact that in round 1 suppliers can make small or even negative profit. Compared to R2, suppliers' cumulative WDP profit makes an average gross growth of around 13% implying the limited increase in suppliers' WDP profit as the auction proceeds from R2.

Figure 12: Percentage of problem instances with positive, negative, or 0 growth percentage

(a) Comparison of RS to R1



(b) Comparison of RS to R2



Chapter 5

Divisible-Bid Auctions

When the demand on each item on auction is relatively large, suppliers can find it challenging to carefully combine and price different units of items in a package considering their internal capacity and production costs. In such auctions it would help to provide the bidders with more flexibility in constructing their bundles. In this chapter we consider the case when the bidder can reveal their price functions to the auctioneer allowing the auctioneer a continuum of order options. We refer to this type of bidding as *divisible bidding*.

Suppliers' consent for revealing price functions eases out their bundle evaluation and bid submission processes by enabling them to more efficiently communicate innumerous variations of pricing multiple units of items via concise bids. Once bids are reported, the auctioneer formulates his winner determination problem, represented as a mixed integer programming (MIP) formulation, to decide whom to assign how many units of what items.

In this chapter we differentiate the combinatorial auctions with divisible bids from the ones with indivisible bids. The former is referred to as *divisible-bid auctions* and the latter as *indivisible-bid auctions* hereafter. We study the auctioneers' winner determination problem for divisible bid auctions, namely WDPD, and how it helps both the suppliers and the auctioneer to mitigate combinatorial auctions' computational bottlenecks. We investigate the application of Lagrangian relaxation and analyse the optimal Lagrangian bound, solution, and multipliers derived.

The rest of this chapter is devoted to formulating appropriate profit maximization problems for suppliers to help them identify how many units of what products and at what price to offer in the next round of the auction in order to remain competitive with the rest of the suppliers. In the formulation of profit maximization problems we take into account various levels of suppliers risk-taking attitudes.

As part of our empirical experiments, we simulate a divisible auction environment and computationally implement and compare our presented models. The results are further compared against the optimal prices and quantities obtained from an auction with indivisible bids.

5.1 Notations and Definitions

The definition of i, k, d_k, $\lambda^*(k)$ is consistent with Chapter 4. Other notations frequently used throughout this chapter are as follows.

j	Index on the suppliers' price functions $j \in J = \{1, \ldots, n_i\}$
p_{ijk}	Supplier i's price for item k in price function j
p'_{ijk}	p_{ijk} from the previous round
c_{ijk}	Supplier i's unit cost of item k in price function j
q_{ijk}	Supplier i's quantity of item k in price function j
q'_{ijk}	q_{ijk} from the previous round
f_{ij}	The jth price function submitted by supplier i
b_i	The bid submitted by supplier i
a_{ijk}	The minimum amount of item k supplier i offers in price function j, ($a_{ijk} \in \mathbb{N}^+ \cup \{0\}$)
b_{ijk}	The maximum amount of item k supplier i offers in price function j, ($b_{ijk} \in \mathbb{N}^+ \cup \{0\}$)
L_{ik}	Supplier i's minimum capacity to produce item k, ($L_{ik} \in \mathbb{Z}^+$)
U_{ijk}	Supplier i's maximum capacity to produce item k, ($U_{ik} \in \mathbb{Z}^+$)
Π_i	The minimum net profit that supplier i expects to take from the auction
α_i	The minimum net profit margin that supplier i expects to take from the auction
β_i	The minimum profit percentage that supplier i expects to take from the auction
δ_{ij}	The binary variable which equals 1 if the jth price function submitted by supplier i is selected and 0 otherwise

5.2 The Winner Determination Problem for Divisible-Bid Auctions (WDPD)

Assume each supplier submits bids containing several linear price functions which explicitly define distinct ranges of quantities and their corresponding per unit prices.

Rather than deciding whether or not to select a bid as in conventional WDP problems, in a WDPD, the auctioneer solves for the optimal quantities from suppliers. In this section we introduce the bids as well as the problem formulation in a divisible-bid auction. We discuss how this auction is beneficial for both parties in an auction supply chain framework. Finally, we look at the problem's optimal Lagrangian relaxation values.

5.2.1 Bid Formulation

Let b_i define the bid submitted by supplier i as

$$\begin{aligned} b_i = \{f_{ij} | f_{ij} = \textstyle\sum_k p_{ijk} q_{ijk} \quad \text{for} \quad & q_{ijk} \in [a_{ijk}, b_{ijk}], \\ & a_{ijk} \geq L_{ik}, b_{ijk} \leq U_{ik}, \\ & [a_{ijk}, b_{ijk}] \cap [a_{ij'k}, b_{ij'k}] = \emptyset \quad \forall i, j \neq j', k\}. \end{aligned} \tag{5.1}$$

In bid representation b_i, we assume that suppliers determine prices with respect to an all-units cost function. Meaning that the cost per unit drops when the order size is greater than or equal to the discount break points. For supplier i and product k the cost function is defined as

$$c = T + \begin{cases} c_1 q_1 & a_1 \leq q_1 \leq b_1 \\ c_2 q_2 & a_2 \leq q_2 \leq b_2 \\ \dots & \\ c_{n_i} q_{n_i} & a_{n_i} \leq q_{n_i} \leq b_{n_i} \end{cases}$$

where K denotes the fixed production cost. Note that for simplicity we drop the indices i, k to formulate the cost with respect to the supplier i's jth cost function.

Selection of all-units cost function for suppliers in a divisible-bid auction allows suppliers to provide appropriate production costs on an arbitrary range of quantities without supplying the product in quantities lower than the lower bound of that range.

Proposition 5.1. *Bid formulation* (5.1) *preserves*

- *synergies among products,*
- *discounts on provision of larger quantity units,*
- *the XOR bidding Language.*

Proof. Each price function included in bid b_i is in fact representative of a bundle with the additional flexibility on the number of units included in the package from each item and the associated price. Suppliers are able to indicate complementarity among items by grouping them in a function and assigning low per-quantity prices. Discount on provision of large quantities is represented by assigning low per-unit price coefficients.

The suppliers' all-units cost function allows for the supplier to have only one of the cost functions selected. Once the price function j is selected, the supplier is required to supply certain quantities of the items included in this function. Needless to say, the quantities assigned need to satisfy the ranges defined in the function. This is in line with the XOR bidding environment wherein at most 1 bid is accepted from each bidder. □

5.2.2 WDPD Problem Formulation

With the bid representation (5.1), the winner determination problem for a divisible-bid auctions is formulated as

$$\begin{aligned}
\min \quad & \textstyle\sum_i \sum_j \sum_k p_{ijk} q_{ijk} & & \\
s.t. \quad & \textstyle\sum_{i,j} q_{ijk} \geq d_k & \forall k \quad & (1) \\
& a_{ijk}\delta_{ij} \leq q_{ijk} \leq b_{ijk}\delta_{ij} & \forall i,j,k \quad & (2) \\
& \textstyle\sum_j \delta_{ij} \leq 1 & \forall i \quad & (3) \\
& \delta_{ij} \in \{0,1\} & \forall i,j & \\
& q_{ijk} \in \mathbb{N}^+ \cup \{0\} & \forall i,j,k. &
\end{aligned} \tag{WDPD}$$

The objective function minimizes the total price of procurement, constraint (1) ensures the demand on each product is satisfied. Constraint (2) determines whether bidder i's function j is selected or not. With the selection of this function positive quantities of the items contained in this function are selected on the ranges introduced. Otherwise, all quantities from this function remain at level 0. Constraint (3) makes sure that at most one function is selected for each supplier (XOR condition).

5.2.3 The Virtue of Implementing WDPD

As discussed in Section 2.2.3 there are several complexities inherent in the application of combinatorial auctions. With the condition that suppliers reveal their price functions, implementing divisible-bid auctions helps reduce these complexities. Below, we discuss this in more details.

5.2.3.1 Suppliers' Complexity on Bundle Evaluation

In a combinatorial auction environment determining the value of each bundle necessitates solving pricing, marketing, and revenue management problems. Requiring this computation to be done for exponentially many combinations of multiple units of multiple products is almost impractical. In a problem formulation with K items and demand d_k, each supplier will have $\sum_{i=1}^{K} \binom{K}{i}.d_k^i$ different options. With 5 items and 10 units of demand for each item this yields 161,050 combinations.

Specifically, in an iterative auction framework, due to the short bid-submission time, the bundle evaluation process becomes challenging. Suppliers will only have a limited time to carefully evaluate and price the combinations they are willing to compete on. Divisible-bid auctions help reduce this complexity by providing the opportunity for suppliers to determine prices for ranges of quantities rather than explicit quantity values. As stated in Proposition 5.1, they will still be able to express synergies among different products as well as quantity discounts when providing more units of the same product.

5.2.3.2 Suppliers' Complexity on Bundles' Communication

Once the valuations are determined, bidders need to communicate exponentially many bids to the auctioneer. Assuming the pricing stage is done, the suppliers' next bottleneck is to communicate an exponentially large number of bundles to the auctioneer (161,050 bids for a relatively small auction with 5 items and 10 units of each). Submission of price functions based on the intervals of quantities provides a more concise bid representation format which reduces suppliers' communication complexity.

5.2.3.3 Auctioneers' Complexity on Solving WDPD

In order to investigate the computational efficiency of divisible-bid auction formulation, we conduct numerical experiments to: generate divisible-bid auction problem, convert it to the equivalent indivisible-bid counterpart, and record the CPU time that CPLEX 12 solver consumes to solve each formulation.

To simulate a divisible-bid auction, we use a uniform distribution to generate cost values on the interval [50,100]. To represent the interval of quantities, we generate a random number $m_{ijk} \in [5, 15]$ and derive the lower and upper bounds a_{ijk} and b_{ijk} of the quantity q_{ijk} as

$$a_{ijk} \in [0, m_{ijk}], b_{ijk} \in [m_{ijk} + 1, 20].$$

Each generated instance is next converted to distinct packages in an indivisible-bid auction. For instance, bid b_1 submitted by supplier 1, consisting of two functions and 3 products is represented as

$$\begin{aligned} b_1 = \{ \quad & f_{11} = \textstyle\sum_k^3 p_{11k} q_{11k} \quad q_{11k} \in [a_{11k}, b_{11k}] \quad \text{for} \quad k = 1, 2, 3, \\ & f_{12} = \textstyle\sum_k^3 p_{12k} q_{12k} \quad q_{12k} \in [a_{12k}, b_{12k}] \quad \text{for} \quad k = 1, 2, 3\}. \end{aligned}$$

b_1 is equivalent to the submission of the following indivisible bid:

$$\begin{aligned} &\{\{a_{111}, a_{112}, a_{113}\}, && p = a_{111}q_{111} + a_{112}q_{112} + a_{113}q_{113}\} \\ &\{\{a_{111}, a_{112}, a_{113} + 1\}, && p = a_{111}q_{111} + a_{112}q_{112} + (a_{113} + 1)q_{113}\} \\ &\vdots \\ &\{\{b_{121}, b_{122}, b_{123}\}, && p = b_{121}q_{121} + b_{122}q_{122} + b_{123}q_{123}\}. \end{aligned}$$

Table 10 summarizes our results for i suppliers, each submitting j price functions on k products. Column 2 illustrates the number of bundles equivalent to each instance of the divisible-bid auction problem generated. Columns 3 and 4 record the CPLEX CPU time for solving the two equivalent problem instances in seconds, and column 5 is the time ratio of the divisible-bid to the indivisible-bid problems CPU time.

Table 10: Comparison of CPU time for problems with divisible and indivisible bids

(i,j,k)	number of packages	Divisible CPU time	Indivisible CPU time	Ratio
(2,2,3)	632	0.102	0.113	1.11
(2,3,3)	1248	0.1	0.107	1.07
(3,2,3)	1072	0.099	0.237	2.39
(3,3,3)	1184	0.101	0.126	1.25
(2,2,5)	5052	0.101	0.452	4.48
(3,2,5)	6498	0.105	0.703	6.7
(4,2,5)	39358	0.101	17.889	177.12
(5,2,5)	28954	0.153	20.857	136.32
(2,3,5)	6320	0.105	0.496	4.72
(2,4,5)	27272	0.107	4.19	39.16
(2,5,5)	37892	0.119	7.651	64.29
(3,3,5)	26379	0.104	6.024	57.92
(3,4,5)	33333	0.118	12.604	106.81
(3,5,5)	40773	0.122	15.262	125.1
(4,4,5)	61408	0.213	28.825	135.33
(4,5,5)	76912	0.208	105.402	506.74
(5,5,5)	81876	0.207	177.202	856.05

As observed, CPLEX takes much more time to solve the indivisible-bid problem formulations. In addition, the CPU time for the indivisible-bid problem instances grows quickly with the increase in the size of the problem. Comparison of the smallest and largest problem instances shows that while the CPU time in the largest divisible-bid problem instance takes almost twice as much as the smallest problem instance, this ratio increases to as large as 1737.3 on the equivalent indivisible-bid problem.

5.2.4 Analysis of the Lagrangian Relaxation Bound

Assigning the nonnegative Lagrangian multiplier vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K)$ to the demand constraint and relaxing it produces the following dual function and optimization problem.

$$
\begin{aligned}
L(\boldsymbol{\lambda}) = \quad \min \quad & \textstyle\sum_i \sum_j \sum_k p_{ijk} q_{ijk} + \sum_k \lambda_k (d_k - \sum_i \sum_j q_{ijk}) & \\
s.t \quad & a_{ijk}\delta_{ij} \leq q_{ijk} \leq b_{ijk}\delta_{ij} & \forall i, j, k \\
& \textstyle\sum_j \delta_{ij} \leq 1 & \forall i \\
& \delta_{ij} \in \{0, 1\} & \forall i, j \\
& q_{ijk} \in \mathbb{N} \cup \{0\} & \forall i, j, k.
\end{aligned}
\tag{5.2}
$$

The Lagrangian subproblem can be rewritten as $L(\boldsymbol{\lambda}) = \sum_i L_i(\boldsymbol{\lambda}) + \sum_k \lambda_k d_k$ where

$$
\begin{aligned}
L_i(\boldsymbol{\lambda}) = \quad \min \quad & \textstyle\sum_j \sum_k q_{ijk}(p_{ijk} - \lambda_k) & \\
s.t \quad & a_{ijk}\delta_{ij} \leq q_{ijk} \leq b_{ijk}\delta_{ij} & \forall j, k \\
& \textstyle\sum_j \delta_{ij} \leq 1 & \\
& \delta_{ij} \in \{0, 1\} & \forall j \\
& q_{ijk} \in \mathbb{N} \cup \{0\} & \forall j, k.
\end{aligned}
\tag{5.3}
$$

The dual problem seeks optimal values of $\boldsymbol{\lambda} \geq 0$ that maximize $L(\boldsymbol{\lambda})$. As observed in Proposition 3.1, the Lagrangian subproblem for an indivisible-bid WDP satisfies the integrality property, i.e. given any choice of coefficients in the objective function, it has an integer optimal solution even if the integrality constraints are relaxed. We study this property for WDPD in Proposition 5.2, Corollary 5.1, and Theorem 5.1.

Proposition 5.2. *The Lagrangian subproblem corresponding to the relaxation of the demand constraints for a divisible-bid auction environment as formulated in* (5.2) *satisfies the integrality property.*

Proof. To see the integrality property of (5.2), it suffices to find the closed form optimal integral solution. The objective function of (5.2) is minimized if for each supplier i at most one function corresponding to the most negative value of $\sum_k (p_{ijk} - \lambda_k) q_{ijk}$ is chosen. The least value of $\sum_k (p_{ijk} - \lambda_k) q_{ijk}$ is attained by fixing quantities q_{ijk} at respectively their upper and lower bounds for negative and positive values of $p_{ijk} - \lambda_k$. Let us define these bounds by parameter τ_{ijk} as

$$\tau_{ijk} = \begin{cases} b_{ijk} & \text{if } p_{ijk} - \lambda_k^* \leq 0, \\ a_{ijk} & \text{if } p_{ijk} - \lambda_k^* > 0. \end{cases}$$

Let $g_i = \min_j \{\sum_k (p_{ijk} - \lambda_k) \tau_{ijk}\}$ obtain the lowest value derived from the functions, and $j^* = \text{argmin}_j \{\sum_k (p_{ijk} - \lambda_k) \tau_{ijk}\}$. Then for each supplier i,

$$\delta_{ij}^* = \begin{cases} 1 & \text{if } g_i \leq 0, j = j^*, \\ 0 & o.w. \end{cases}$$

The optimal quantity values are equivalent to τ_{ijk} for the chosen function, and they are defined zero elsewhere.

$$q_{ijk}^* = \begin{cases} \tau_{ijk} & \text{if } \delta_{ij}^* = 1, \\ 0 & o.w. \end{cases}$$

□

Corollary 5.1. *The Lagrangian and linear relaxations WDPD yield equivalent bounds.*

Proof. According to Theorem 16.10 in [2], the linear and Lagrangian relaxation bounds equal if the Lagrangian subproblem satisfies the integrality property which is the case according to Proposition 5.2. □

Theorem 5.1. *For WDPD, the dual variables associated with the demand constraints of the linear relaxation problem correspond to the Lagrangian multipliers associated with relaxation of demand constraints of the Lagrangian relaxation problem.*

Proof. The Lagrangian dual function $L(\lambda)$ as formulated in (5.2) is equivalent to

$$g(\lambda) = \sum_k \lambda_k \mathbf{d}_k + \sum_i \min\{0, \min_j\{\sum_k (\min(p_{ijk} - \lambda_k, 0) b_{ijk} + \max(p_{ijk} - \lambda_k, 0) a_{ijk})\}\}.$$

It can be seen that function g produces equivalent optimal solution as (5.2). Let,

$$g_1(j) = \min(p_{ijk} - \lambda_k, 0) b_{ijk} + \max(p_{ijk} - \lambda_k, 0) a_{ijk}.$$

$g_1(j)$ sets products corresponding to negative values of $p_{ijk} - \lambda_k$ to their upper bounds. More clearly,

$$p_{ijk} - \lambda_k < 0 \quad \Rightarrow \begin{cases} \min(p_{ijk} - \lambda_k, 0) = p_{ijk} - \lambda_k \\ \max(p_{ijk} - \lambda_k, 0) = 0. \end{cases}$$
$$\Rightarrow g_1(j) = (p_{ijk} - \lambda_k) b_{ijk}.$$

Alternatively,

$$p_{ijk} - \lambda_k > 0 \quad \Rightarrow g_1(j) = (p_{ijk} - \lambda_k) a_{ijk}.$$

If for supplier i, $\min_j g_1(j)$ is nonnegative, the optimal quantity values are set to zero. This defines the optimal solution for $g_(\lambda)$ that is equivalent to the optimal solution for $L(\lambda)$ as stated in Proposition 5.2. Now, let

$$v_{ijk} = \min(p_{ijk} - \lambda_k, 0), w_{ijk} = \max(p_{ijk} - \lambda_k, 0).$$

Obviously $v_{ijk} \leq p_{ijk} - \lambda_k, v_{ijk} \leq 0$, and $w_{ijk} \geq p_{ijk} - \lambda_k, w_{ijk} \geq 0$. This transforms the Lagrangian dual function as

$$g(\lambda) = \sum_k \lambda_k d_k + \sum_i \min\{0, \min_j\{\sum_k (v_{ijk} b_{ijk} + w_{ijk} a_{ijk})\}\}.$$

With the definition of $s_i = \min\{0, \min_j\{\sum_k (v_{ijk} b_{ijk} + w_{ijk} a_{ijk})\}\}$, the Lagrangian dual problem can be formulated as

$$\begin{array}{lll} \max & \sum_k \lambda_k d_k + \sum_i s_i & \\ s.t. & s_i - \sum_k (v_{ijk} b_{ijk} + w_{ijk} a_{ijk}) \quad \leq 0 & \forall i, j \\ & \lambda_k + v_{ijk} + w_{ijk} \quad = p_{ijk} & \forall i, j, k \\ & v_{ijk} \leq 0, w_{ijk} \geq 0, \lambda_k \geq 0, s_i \leq 0 & \forall i, j, k. \end{array}$$

Assigning positive dual variables δ_{ij} and q_{ijk} to the first and second constraints of the above linear programme leads to the dual problem

$$
\begin{array}{lllll}
\min & \sum_{i,j,k} p_{ijk} q_{ijk} & & & \\
s.t. & \sum_{ij} q_{ijk} & \geq d_k & \forall k & \\
 & q_{ijk} & \leq b_{ijk}\delta_{ij} & \forall i,j,k & \\
 & q_{ijk} & \geq a_{ijk}\delta_{ij} & \forall i,j,k & \\
 & \sum_j \delta_{ij} & \leq 1 & \forall i & \\
 & q_{ijk} \quad \text{free variable}, \delta_{ij} \geq 0 & & \forall i,j,k. &
\end{array}
\tag{5.4}
$$

Note that, $\sum_j \delta_{ij} \leq 1$ implies that $\delta_{ij} \leq 1$ and $a_{ijk}\delta_{ij} \leq q_{ijk} \leq b_{ijk}\delta_{ij}$, implies that the free variable $q_{ijk} \geq 0$. This makes formulation (5.4) equivalent to the linear relaxation (WDPD) and the dual variables of the linear relaxation problem equivalent to the Lagrangian multipliers λ_k. □

5.3 Suppliers' Profit Maximization Model (PMBD)

In order to package a new bundle in a divisible-bid auction setting we assume that each supplier fixes the bounds of the quantities he offers at the beginning of the first round of the auction based on his production capacity and costs. Throughout different versions of the PMBD model, each supplier seeks an optimal pricing scheme on the corresponding intervals considering internal conditions. The new bundle is formulated in such a way that supply of the new bid is profitable for the supplier and yet remains competitive in the auction.

Since the PMBD models are designed for each supplier i, we drop this index from all variables and constants we use in this section. Also, index j previously assigned to suppliers' bids now associates with the functions included in each bid. This slightly changes our notations. p'_{jk}, q'_{jk} are the price and quantity from the previous round of the auction and we define

$$\begin{aligned} \mu_{jk} &= (a_{jk} + b_{jk})/2 \\ r_{jk}, t_{jk} &\in T = \{a_{jk}, \mu_{jk}, b_{jk}\} \\ m_{jk} &= \min(\lambda^*_k, p'_{jk}) \\ M_{jk} &= \max(\lambda^*_k, p'_{jk}). \end{aligned}$$

We study quantity- and risk- based profit maximization formulations for the suppliers pricing problems.

5.3.1 Quantity-based Profit Maximization Model

We keep our first problem formulation (QPMBD) consistent with the formulation (VPMB) defined in Section 4.5 for the indivisible-bid problems with variable price functions as

$$\max \sum_j \sum_k q_{jk}(p_{jk} - c_{jk}) \qquad \text{(QPMBD)}$$

$$s.t. \sum_k q_{jk}(p_{jk} - c_{jk}) \geq \Pi \qquad \forall j \qquad (5.5a)$$

$$\sum_k q_{jk}(p_{jk} - \lambda^*_k) \leq \sum_k a_{jk}(p'_{jk} - \lambda^*_k) \qquad \forall j \qquad (5.5b)$$

$$a_{jk}\delta_{jk} \leq q_{jk} \leq b_{jk}\delta_{jk} \qquad \forall j, k \qquad (5.5c)$$

$$m_{jk}\delta_{jk} \leq p_{jk} \leq M_{jk}\delta_{jk} \qquad \forall j, k \qquad (5.5d)$$

$$p_{jk} \geq 0, q_{jk} \in \mathbb{N}^+ \cup \{0\}, \delta_{jk} \in \{0, 1\} \qquad \forall j, k.$$

Constraint (5.5a) defines the *minimum profitability condition* as described in Equation GPMB. It ensures that with the selection of each of the suppliers' cost functions, he will earn the minimum net profit of Π. Constraints (5.5b) enforces the *competitiveness condition*, and constraint (5.5c) ensures the *production capacity* for each item at the corresponding prices. The use of binary variable δ_{jk} ensures the possibility for the supplier to withdraw some products in his new bid. Therefore, he will choose either not to supply a quantity, or supply it on a certain range with predetermined cost values.

Constraint (5.5d) guarantees *pricing consistency condition.* For positive quantity values with $\delta_{jk} = 1$ we enforce each new price to be greater than the minimum of the previous price and the Lagrangian multiplier and lower than the maximum of the two. Thus, if the previous prices are already competitive with values lower than the Lagrangian multipliers, the supplier is able to slightly increase prices as long as they do not exceed the Lagrangian multipliers. On the contrary, if previous prices are not competitive in the previous round of the auction, meaning that they are greater than the announced Lagrangian prices, the supplier needs to cut down on his price submission as long as they are not less than the Lagrangian multipliers. Without this condition, we experimentally observe that prices can get very large on a few products and 0 on the rest. $\delta_{jk} = 0$ ensures that the new item price is zero if the bid is not including the corresponding item. Note that an optimal quantity of an item q^*_{jk} can be zero despite positive optimal value of δ_{jk} when the corresponding lower bound a_{jk} is set to zero.

(QPMBD) is a mixed nonlinear integer programming problem. In the following we discuss solution techniques that can be used to solve the problem.

5.3.1.1 Technique 1: Linearization by the Change of variables

One solution approach is to linearize (QPMBD) by the change of variables as $s_{jk} = q_{jk}p_{jk}$. This problem can then be formulated as follows:

$$
\begin{aligned}
\max \sum_j \sum_k & s_{jk} - q_{jk}c_{jk} \\
s.t. \sum_k & s_{jk} - q_{jk}c_{jk} \geq \Pi && \forall j && (5.6a) \\
\sum_k & s_{jk} - q_{jk}\lambda^*_k \leq \sum_k a_{jk}(p'_{jk} - \lambda^*_k) && \forall j && (5.6b) \\
& a_{jk}\delta_{jk} \leq q_{jk} \leq b_{jk}\delta_{jk} && \forall j,k && (5.6c) \\
& m_{jk}q_{jk} \leq s_{jk} \leq M_{jk}q_{jk} && \forall j,k && (5.6d) \\
& s_{jk} \geq 0, q_{jk} \in \mathbb{N}^+ \cup \{0\}, \delta_{jk} \in \{0,1\} && \forall j,k.
\end{aligned}
$$

Once the model is solved, optimal prices are derived as

$$
p^*_{jk} = \begin{cases} s^*_{jk}/q^*_{jk} & \text{if } q^*_{jk} > 0, \\ 0 & \text{if } q^*_{jk} = 0. \end{cases}
$$

Note that s_{jk} replaces $q_{jk}p_{jk}$ in the objective function and constraints (5.6a) and (5.6b). To formulate (5.6d) we multiply all sides of (5.5d) by q_{jk}. The new constraint is linear and equivalent to (5.5d):

1. For $\delta_{jk} = 0$ in (5.6c) $\Rightarrow q_{jk} = 0 \Rightarrow s_{jk} = 0$ and $p_{jk} = 0$.
2. For $\delta_{jk} = 1$ in (5.6c) $\Rightarrow q_{jk} > 0 \Rightarrow m_{jk}q_{jk} \leq s_{jk} \leq M_{jk}q_{jk}$ and $m_{jk} \leq p_{jk} \leq M_{jk}$.

This technique transforms (QPMBD) into a linear mixed integer programming problem with binary, nonnegative integer and nonnegative real variables: $\delta_{jk}, q_{jk},$ and s_{jk}.

5.3.1.2 Technique 2: Defining closed-form solution

Let a rational bidder submit bids which satisfy $p_{jk} \geq c_{jk}$. Proposition 5.3 provides the solution for (QPMBD).

Proposition 5.3. *For all j, k, the profit maximization problem* (QPMBD) *for a rational bidder*

1. *yields integer optimal solution:*
$$\delta^*_{jk} = 1,\ q^*_{jk} = \begin{cases} b_{jk} & \text{if } \gamma_k \geq 0, \\ a_{jk} & \text{if } \gamma_k < 0. \end{cases},\ p^*_{jk} = (1 - \tfrac{a_{jk}}{q^*_{jk}})\lambda^*_k + (\tfrac{a_{jk}}{q^*_{jk}})p'_{jk}.$$
2. *is feasible for* $\Pi \leq \min_j\{\sum_k a_{jk}(m_{jk} - c_{jk})\}$ *and infeasible if there exists j for which* $\Pi > \sum_k a_{jk}(p'_{jk} - \lambda^*_k) + \sum_k b_{jk}(\lambda^*_k - c_{jk})$.

Proof. Combining constraints (5.5a), (5.5b), we have:

$$\Pi \quad \leq \sum_k q_{jk}(p_{jk} - c_{jk}) \quad \leq \sum_k a_{jk}(p'_{jk} - \lambda^*_k) + \sum_k q_{jk}(\lambda^*_k - c_{jk}) \quad \forall j. \tag{5.7}$$

For an arbitrary k define

$$OB_{jk}(q_{jk}, p_{jk}) = q_{jk}(p_{jk} - c_{jk})$$
$$UB_{jk}(q_{jk}) = a_{jk}(p'_{jk} - \lambda^*_k) + q_{jk}(\lambda^*_k - c_{jk}).$$

This defines the objective function as $\sum_j \sum_k OB_{jk}(q_{jk}, p_{jk})$. Also, from constraint (5.7) $\sum_k OB_{jk}(q_{jk}, p_{jk}) \leq \sum_k UB_{jk}(q_{jk})$ for all j. To see part (1) of the proposition consider the following cases:

(i) $\lambda^*_k \geq c_{jk}$.

With this condition the upper bound $UB_{jk}(q_{jk})$ attains its maximum when $q_{jk} = b_{jk}$. Also, defining $0 \leq \eta_{jk} \leq 1$ as $\eta_{jk} = \frac{a_{jk}}{b_{jk}}$, $p_{jk} = (1 - \eta_{jk})\lambda^*_k + \eta_{jk} p'_{jk}$ provides a convex combination of $\{p'_{j,k}, \lambda^*_k\}$. Clearly, $\lambda^*_k \leq p_{jk} \leq p'_{jk}$ if $\lambda^*_k \leq p'_{jk}$, and $p'_{jk} \leq p_{jk} \leq \lambda^*_k$ if otherwise. In either case,

$$\begin{aligned}
OB_{jk}(b_{jk}, p_{jk}) &= b_{jk}((1 - \tfrac{a_{jk}}{b_{jk}})\lambda^*_k + (\tfrac{a_{jk}}{b_{jk}})p'_{jk} - c_{jk}) \\
&= (b_{jk} - a_{jk})\lambda^*_k + a_{jk}p'_{jk} - c_{jk}b_{jk} \\
&= a_{jk}(p'_{jk} - \lambda^*_k) + b_{jk}(\lambda^*_k - c_{jk}) \\
&= UB_{jk}(b_{jk}).
\end{aligned}$$

Thus, for the defined p_{jk} and q_{jk}, $OB_{jk}(b_{jk}, p_{jk})$ realizes the upper bound $UB_{jk}(b_{jk})$. Clearly, the objective function can not improve any further beyond the upper bound without violating the feasibility of the constraints.

(ii) $p'_{jk} \leq p_{jk} \leq \lambda^*_k$ and $\lambda^*_k < c_{jk}$.

In this case $p'_{jk} \leq p_{jk} \leq \lambda^*_k < c_{jk}$, and thus $p_{jk} < c_{jk}$. This contradicts the rationality of the bidders. With the assumption of non-rationality, the optimal price, quantity and thus the objective function is 0.

(iii) $\lambda_k^* \leq p_{jk} \leq p'_{jk}$ and $\lambda_k^* < c_{jk}$.

Condition (iii) implies that $\lambda_k^* < c_{jk} \leq p_{jk} \leq p'_{jk}$. While setting q_{jk} to 0 makes $OB_{jk} = 0$, $q_{jk} = a_{jk}$ improves OB_{jk} to a strictly positive value:

$$\begin{aligned} UB_{jk}(a_{jk}) &= a_{jk}(p'_{jk} - \lambda_k^*) + a_{jk}(\lambda_k^* - c_{jk}) \\ &= a_{jk}(p'_{jk} - c_{jk}). \end{aligned}$$

With $p_{jk} = p'_{jk}$, OB_{jk} attains the upper bound $UB_{jk}(a_{jk})$, i.e., $OB_{jk}(a_{jk}, p'_{jk}) = UB_{jk}(a_{jk})$. Due to the negativity of $\lambda_k^* - c_{jk}$, increasing q_{jk} beyond this point deteriorates $UB_{jk}(q_{jk})$. To see this define $a_{jk} < \zeta_{jk} \leq b_{jk}$.

$$\begin{aligned} UB_{jk}(\zeta_{jk}) &= a_{jk}(p'_{jk} - \lambda_k^*) + \zeta_{jk}(\lambda_k^* - c_{jk}) \\ &< a_{jk}(p'_{jk} - \lambda_k^*) + a_{jk}(\lambda_k^* - c_{jk}) \\ &< UB_{jk}(a_{jk}). \end{aligned}$$

For $\eta_{jk} = \frac{a_{jk}}{\zeta_{jk}}$ and $p_{jk} = (1 - \eta_{jk})\lambda_k^* + \eta_{jk}p'_{jk}$, $OB_{jk}(\zeta_{jk}, p_{jk})$ realizes the upper bound $UB_{jk}(\zeta_{jk})$:

$$\begin{aligned} OB_{jk}(\zeta_{jk}, p_{jk}) &= \zeta_{jk}((1 - \tfrac{a_{jk}}{\zeta_{jk}})\lambda_k^* + (\tfrac{a_{jk}}{\zeta_{jk}})p'_{jk} - c_{jk}) \\ &= (\zeta_{jk} - a_{jk})\lambda_k^* + a_{jk}p'_{jk} - c_{jk}\zeta_{jk} \\ &= a_{jk}(p'_{jk} - \lambda_k^*) + \zeta_{jk}(\lambda_k^* - c_{jk}) \\ &= UB_{jk}(\zeta_{jk}). \end{aligned}$$

Since $UB_{jk}(\zeta_{jk}) < UB_{jk}(a_{jk})$, $q_{jk}^* = a_{jk}$ and $p_{jk}^* = p'_{jk}$.

To see part (2) note that $q_{jk} = a_{jk}$ and $p_{jk} = p'_{jk}$ always satisfy Constraint 5.5b. Thus, for the model to be feasible it suffices that parametrization of Π is appropriate. Based on part (1) of this proposition for rational bidders $p_{jk} > 0$ for all j, k. So, the minimum value that it attains is $m_{j,k}$. Thus,

$$\begin{aligned} \Pi & \leq \textstyle\sum_k a_{jk}(p_{jk} - c_{jk}) \qquad \forall j \\ & \leq \min_j \{\textstyle\sum_k a_{jk}(p_{jk} - c_{jk})\}. \end{aligned}$$

Since for all j, k, $p_{jk} \in [m_{jk}, M_{jk}]$, $\Pi = \sum_k a_{jk}(m_{jk} - c_{jk})$ guarantees the feasibility of the problem. Any lower value of Π maintains the feasibility of constraint 5.5a. Moreover, based on constraint (5.7) if there exists j for which the minimum required profitability exceeds the maximum that the supplier can get, then the problem becomes infeasible. Mathematically this happens when $\Pi > \sum_k a_{jk}(p'_{jk} - \lambda^*_k) + \sum_k b_{jk}(\lambda^*_k - c_{jk})$. □

Note that part (2) in Proposition 5.3 defines the interval that guarantees feasibility. While (QPMBD) can be feasible for values greater than the upper end of this interval, having a maximum value that ensures feasibility is helpful for the suppliers to know for what values of the minimum expected profit the model promises feasible solutions.

Solving for a closed form solution is relatively easy and is preferred over the linearization derived by the change of variables. Moreover, the closed-form solution guarantees that the supplier will price all different supply scenarios with/without quantity discount. If the supplier is unwilling to include a product, he is able to do so by adjusting the production bounds (for instance, setting the lower bound of an item to zero, considers the possibility of excluding the item from the package).

5.3.2 Risk-based Profit Maximization Model (RPMBD)

In our next set of formulations, we determine optimal pricing scheme irrespective of the optimal quantity orders $q_{jk} \in [a_{jk}, b_{jk}]$. Instead, items are priced based on a minimum, average, or maximum quantity offerings for suppliers with different risk-taking level. To do so, we customize the objective function of (GPMB) to maximization of profit with respect to reception of either minimum, average, or maximum order. The *profitability condition* is adjusted to guarantee minimum net profit, minimum net profit margin, and minimum profit percentage. The *competitiveness condition* is reflected for the suppliers when they receive minimum, average or maximum order of quantities in constraint (5.8b) and constraint (5.8c) maintains *pricing consistency*, i.e., the item is either supplied in the corresponding interval or not at all when the price is zero. The profit maximization problem is formulated as

$$\max \sum_j \sum_k r_{jk}(p_{jk} - c_{jk}) \qquad \text{(RPMBD)}$$

$$s.t. \sum_k a_{jk}(p_{jk} - c_{jk}) \geq PI \qquad \forall j \qquad (5.8a)$$

$$\sum_k t_{jk}(p_{jk} - \lambda_k^*) \leq \sum_k a_{jk}(p'_{jk} - \lambda_k^*) \qquad \forall j \qquad (5.8b)$$

$$m_{jk}\delta_{jk} \leq p_{jk} \leq M_{jk}\delta_{jk} \qquad \forall j, k \qquad (5.8c)$$

$$p_{jk} \geq 0, \delta_{jk} \in \{0, 1\} \qquad \forall j, k.$$

Depending on the values that r_{jk} and t_{jk} take up from the set T, the profit maximization model can be customized to reflect suppliers' levels of risk-taking attitudes.

Figure 13: Risk-taking levels in the profit maximization model

$$
\begin{cases}
\text{Risk Averse}, r_{jk} = a_{jk} \begin{cases} \text{level } 1, R_{11} : t_{jk} = a_{jk} \\ \text{level } 2, R_{12} : t_{jk} = \mu_{jk} \\ \text{level } 3, R_{13} : t_{jk} = b_{jk} \end{cases} \\
\text{Risk Neutral}, r_{jk} = \mu_{jk} \begin{cases} \text{level } 1, R_{21} : t_{jk} = a_{jk} \\ \text{level } 2, R_{22} : t_{jk} = \mu_{jk} \\ \text{level } 3, R_{23} : t_{jk} = b_{jk} \end{cases} \\
\text{Risk Seeking}, r_{jk} = b_{jk} \begin{cases} \text{level } 1, R_{31} : t_{jk} = a_{jk} \\ \text{level } 2, R_{32} : t_{jk} = \mu_{jk} \\ \text{level } 3, R_{33} : t_{jk} = b_{jk}. \end{cases}
\end{cases}
$$

As illustrated in Figure 13, we define the profit maximization models for risk averse, risk neutral, and risk seeking suppliers based on the quantity order they adopt in the maximization of their profit. In the risk averse scenario, the model seeks the optimal pricing scheme to maximize the minimum profit. In the risk neutral and risk seeking scenarios the model adjusts to respectively maximize average or maximum profit. Each risk-based model is defined as R_{rt} where $r, t \in \{1, 2, 3\}$. The values 1, 2, and 3 respectively indicates initialization of r_{jk} or t_{jk} at a_{jk}, μ_{jk}, and b_{jk} (for instance R_{23} defines the model wherein $r_{jk} = \mu_{jk}$ and $t_{jk} = b_{jk}$).

5.3.2.1 Optimality of RPMBD

To analyse (RPMBD) at optimality, we first investigate how suppliers risk-taking attitude affects their expected profit. First we observe that increase of the suppliers' risk-taking level produces more profit for the problems when t_{jk} is fixed. This means that defining the objective function in (RPMBD) as $Z(r_{jk}, t_{jk})$, for an arbitrary t_{jk}, $Z^*(r_{jk} = a_{jk}, t_{jk}) \leq Z^*(r_{jk} = \mu_{jk}, t_{jk}) \leq Z^*(r_{jk} = b_{jk}, t_{jk})$. However, on the same risk-taking level, increase of t_{jk} does not constitute an increasing or decreasing pattern for suppliers' profit. The optimal profit from each subclassification depends on the suppliers' previous price values and the Lagrangian prices from the auction. In Proposition 5.4, we show the scenarios where a monotonic change of function Z is detected.

Proposition 5.4. *With the increase of t_{jk} for an arbitrary risk-taking attitude r_{jk}, the optimal profit gained follows*

1. *a decreasing pattern if supplier's initial prices are no less than the Lagrangian multipliers, i.e.,*
   $\lambda_k^* \leq p'_{jk} \quad \forall j, k \Rightarrow Z^*(r_{jk}, t_{jk} = a_{jk}) \geq Z^*(r_{jk}, t_{jk} = \mu_{jk}) \geq Z^*(r_{jk}, t_{jk} = b_{jk})$.

2. *an increasing pattern if supplier's initial prices are no greater than the Lagrangian multipliers, i.e.,*
   $p'_{jk} \leq \lambda_k^* \quad \forall j, k \Rightarrow Z^*(r_{jk}, t_{jk} = a_{jk}) \leq Z^*(r_{jk}, t_{jk} = \mu_{jk}) \leq Z^*(r_{jk}, t_{jk} = b_{jk})$.

Proof. Let $\gamma_{jk} = p'_{jk} - \lambda_k^*$. We examine the following cases:

1. $\gamma_k > 0$ for all j, k.
   $p'_{jk} > \lambda_k^*$ produces a positive value on the right hand side of constraint (5.8b).

This condition implies that new prices are selected to satisfy $\lambda_k^* \leq p_{jk} \leq p'_{jk}$ and thus the left hand side of this constraint is also positive. Increase of t_{jk} values tightens this constraint and thus restricts the optimal price values and consequently the objective function.

2. $\gamma_k < 0$ for all j, k.

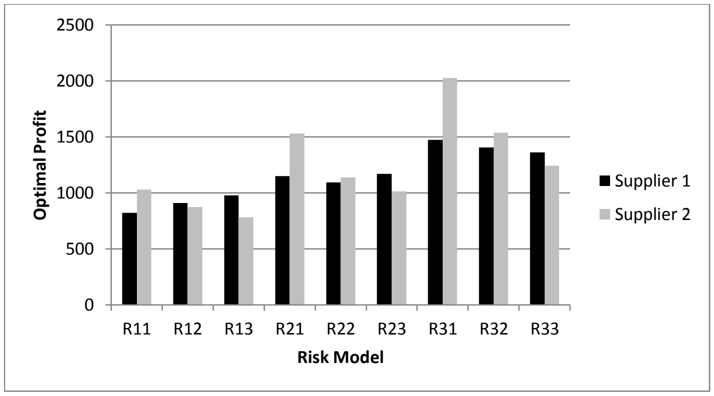
   Similarly, $p'_{jk} < \lambda_k^*$ produces a negative value on the right hand side of constraint (5.8b). This condition implies that new prices are selected to satisfy $\lambda_k^* \geq p_{jk} \geq p'_{jk}$ and so the left hand side of this constraint is also negative. Increase of t_{jk} values loosens this constraint and thus increases the optimal price values and consequently the objective function.

□

Therefore, if all the previous prices a supplier submits are greater than the Lagrange multipliers, maximizing the profit for the minimum quantity orders maximizes the supplier's profit. Nonetheless, if all the previous prices are lower than the Lagrange multipliers, maximizing the profit for the maximum quantity orders maximizes the supplier's profit.

Note that, other than the cases studied in Proposition 5.4, the increase or decrease in the objective function among the risk subclassification models depends on how much the increase or decrease of products improves the total profit, and does not necessarily follow a monotonic pattern. Figure 14 illustrates a counter example where the optimal profit for supplier 1 is non-monotonic.

Figure 14: Suppliers' non-monotonicity example on the risk-taking sub-levels



Once the suppliers determine the risk taking level that best suits them, they have the additional option to have their minimum profit at least as much as their desired net profit Π, net profit margin α, and profit percentage β as defined in Section 4.2. Defining

$$\alpha^* = \alpha \textstyle\sum_k a_{jk} p_{jk} \quad \forall j$$
$$\beta^* = \beta \textstyle\sum_k a_{jk} c_{jk} \quad \forall j,$$

adjusts constraint (5.8a) in (RPMBD) as

$$\begin{aligned} &\textstyle\sum_k a_{jk}(p_{jk} - c_{jk}) \geq \Pi \quad \forall j \\ &\textstyle\sum_k a_{jk}(p_{jk} - c_{jk}) \geq \alpha^* \quad \forall j \\ &\textstyle\sum_k a_{jk}(p_{jk} - c_{jk}) \geq \beta^* \quad \forall j. \end{aligned} \tag{5.9}$$

Proposition 5.5. *For $\forall j, k$, the profit maximization problem* (RPMBD)*,*

1. *yields integer optimal solution $\delta^*_{jk} = 1$.*

2. *is feasible for*

(a) $\Pi = \sum_k a_{jk}(m_{jk} - c_{jk})$

(b) $\alpha = \frac{\sum_k a_{jk}(m_{jk}-c_{jk})}{\sum_k M_{jk}}$

(c) $\beta = \frac{\sum_k a_{jk}(m_{jk}-c_{jk})}{\sum_k c_{jk}}$.

Proof. To see part (1) recall that constraint (5.8a) defines a lower bound on the problem. In order to show that for all j, k, $\delta^*_{j,k} = 1$, it suffices to show that constraint (5.8b) always holds when initializing price variables at their minimum positive values. To see this, consider the following cases for arbitrary values of j, k.

(i) $\lambda^*_k \leq p_{jk} \leq p'_{jk}$.
Setting $p_{jk} = \lambda^*_k$ transforms constraint (5.8b) to $t_{jk}(\lambda^*_k - \lambda^*_k) \leq a_{jk}(p'_{jk} - \lambda^*_k)$. By assumption $p'_{jk} - \lambda^*_k \geq 0$ and so the equation holds.

(ii) $p'_{jk} \leq p_{jk} \leq \lambda^*_k$.
Setting $p_{jk} = p'_{jk}$, we obtain $t_{jk}(p'_{jk} - \lambda^*_k) \leq a_{jk}(p'_{jk} - \lambda^*_k)$. By assumption $p'_{jk} - \lambda^*_k \leq 0$ and so the equation holds.

Therefore, one feasible solution is attained when setting the price variables to their minimum values. Clearly, further increase of the price variables, so long as it does not violate feasibility of constraint (5.8b), improves the optimal objective value.

In part (2), we use the results from part (1). Since $\delta^*_{jk} = 1$ for all j, k, optimal price values are nonzero. From constraint (5.8a), we have

$$\begin{aligned} \Pi &\leq \textstyle\sum_k a_{jk}(p_{jk} - c_{jk}) && \forall j \\ &\leq \min_j\{\textstyle\sum_k a_{jk}(p_{jk} - c_{jk})\}. \end{aligned}$$

Since, $P_{jk} \in [m_{jk}, M_{jk}]$, $\pi = \sum_k a_{jk}(m_{jk} - c_{jk})$ is the minimum value that ensures feasibility of the (RPMBD) for $PI = \Pi$. Similarly, with $PI = \alpha^*$ in constraint (5.8a), we have

$$\begin{aligned} \alpha &\leq \tfrac{\sum_k a_{jk}(p_{jk} - c_{jk})}{\sum_k a_{jk} p_{jk}} && \forall j \\ &\leq \min_j\{\tfrac{\sum_k a_{jk}(p_{jk} - c_{jk})}{\sum_k a_{jk} p_{jk}}\}. \end{aligned}$$

Considering the minimum value of p_{jk}, m_{jk}, in the nominator and the maximum value of p_{jk}, M_{jk}, in the denominator produces the minimum value of α that guarantees feasibility of the problem. Similar analogy proves (c).

□

5.4 Empirical Experiments

In order to simulate a divisible-bid auction environment we randomly generate data, using uniform distributions for an auction with 2 suppliers, each submitting 2 price functions to supply 5 products. We generate cost values for suppliers' cost function on the interval [50,100]. To demonstrate price discount on provision of larger quantities in the second function, we fix constant *dr* (standing for discount rate) at 0.1 and multiply the costs of the second function by (1-*dr*). To generate corresponding prices, we set constant *mp* (standing for marginal profit) to 0.2. Price values are defined as the product of (1+*mp*) by corresponding costs.

To generate the lower and upper bounds of quantity offers for each supplier and each item, we produce intervals of data on [0,20]. To do so, a middle point υ_{jk} is first randomly chosen on [5,15]. The interval is next created with a lower bound generated on [0,υ_{jk}] and an upper bound on [$\upsilon_{jk}+1$,20]. The linear relaxation of the (WDPD) is solved for a demand of 20 units for each product. The dual variables are extracted as equivalents of the Lagrangian multipliers.

With the initialization explained above we solved the profit maximization problems for each supplier. Below we describe our computational results.

5.4.1 Comparison of QPMBD models

Our first set of experiments is for the numerical observation derived for the quantity-based PMBD models. Define Q_0 as (QPMBD), Q_1 as the problem (5.6), and Q_2 as the problem solved by the direct solution. The results are summarized in Table 11.

As observed all 3 models yield equivalent results. The optimal value for q^* is either the lower bound a (when $\gamma^* < 0$), or the upper bound b (when $\gamma^* > 0$). Also, although the optimal prices often take up their lower/upper bound values, there exists cases when they are set to a middle value (for instance supplier 1, function 2, product 1).

5.4.2 Comparison of RPMBD models

With initial values of Lagrangian multipliers and suppliers' prices as presented in Table 12, we illustrate (RPMBD) results in Table 13.

Table 11: Comparison of optimal prices and profit from Q_0, Q_1, Q_2

i	model	j	k	m	p*	M	a	q*	b	Υ
1	Q0	1	1	0	72.6	72.6	6	6	10	-66
			2	80.3	96.03	96.03	8	9	9	23.03
			3	71.28	71.28	79.2	1	1	5	-0.72
			4	75.9	75.9	170.94	4	5	5	101.94
			5	55.44	85.62	90.2	5	5	7	-26.56
		2	1	0	62.46	65.34	11	11	13	-59.4
			2	72.27	96.03	96.03	10	13	13	30.33
			3	71.28	71.28	71.28	6	16	16	6.48
			4	68.31	68.31	170.94	6	6	6	108.84
			5	55.44	55.44	81.18	8	8	18	-18.36
	Q1	1	1	0	72.6	72.6	6	6	10	-66
			2	80.3	96.03	96.03	8	9	9	23.03
			3	71.28	71.28	79.2	1	1	5	-0.72
			4	75.9	75.9	170.94	4	5	5	101.94
			5	55.44	85.62	90.2	5	5	7	-26.56
		2	1	0	62.46	65.34	11	11	13	-59.4
			2	72.27	96.03	96.03	10	13	13	30.33
			3	71.28	71.28	71.28	6	16	16	6.48
			4	68.31	68.31	170.94	6	6	6	108.84
			5	55.44	55.44	81.18	8	8	18	-18.36
	Q2	1	1	0	72.6	72.6	6	6	10	-66
			2	80.3	96.03	96.03	8	9	9	23.03
			3	71.28	71.28	79.2	1	1	5	-0.72
			4	75.9	75.9	170.94	4	5	5	101.94
			5	55.44	85.62	90.2	5	5	7	-26.56
		2	1	0	62.46	65.34	11	11	13	-59.4
			2	72.27	96.03	96.03	10	13	13	30.33
			3	71.28	71.28	71.28	6	16	16	6.48
			4	68.31	68.31	170.94	6	6	6	108.84
			5	55.44	55.44	81.18	8	8	18	-18.36
2	Q0	1	1	0	0	62.7	0	0	10	-57
			2	96.03	96.03	106.7	5	5	6	-0.97
			3	71.28	71.28	108.9	3	3	6	-27.72
			4	69.3	149.36	170.94	3	5	5	107.94
			5	55.44	55.44	61.6	5	5	9	-0.56
		2	1	0	56.43	56.43	11	11	20	-51.3
			2	96.03	96.03	96.03	7	20	20	8.73
			3	71.28	71.28	98.01	7	7	18	-17.82
			4	62.37	139.99	170.94	6	15	15	114.24
			5	55.44	55.44	55.44	10	17	17	5.04
	Q1	1	1	0	0	62.7	0	0	10	-57
			2	96.03	96.03	106.7	5	5	6	-0.97
			3	71.28	71.28	108.9	3	3	6	-27.72
			4	69.3	149.36	170.94	3	5	5	107.94
			5	55.44	55.44	61.6	5	5	9	-0.56
		2	1	0	56.43	56.43	11	11	20	-51.3
			2	96.03	96.03	96.03	7	20	20	8.73
			3	71.28	71.28	98.01	7	7	18	-17.82
			4	62.37	139.99	170.94	6	15	15	114.24
			5	55.44	55.44	55.44	10	17	17	5.04
	Q2	1	1	0	0	62.7	0	0	10	-57
			2	96.03	96.03	106.7	5	5	6	-0.97
			3	71.28	71.28	108.9	3	3	6	-27.72
			4	69.3	149.36	170.94	3	5	5	107.94
			5	55.44	55.44	61.6	5	5	9	-0.56
		2	1	0	56.43	56.43	11	11	20	-51.3
			2	96.03	96.03	96.03	7	20	20	8.73
			3	71.28	71.28	98.01	7	7	18	-17.82
			4	62.37	139.99	170.94	6	15	15	114.24
			5	55.44	55.44	55.44	10	17	17	5.04

Table 12: RPMBD Initializations

		j=1					j=2				
	Supplier	k=1	k=2	k=3	k=4	k=5	k=1	k=2	k=3	k=4	k=5
λ		118.12	79.8	83.22	59.28	74.1	118.12	79.8	83.22	59.28	74.1
P'	1	111.6	80.4	87.6	80.4	79.2	106.02	76.38	83.22	76.38	75.24
	2	87.6	84	87.6	62.4	78	83.22	79.8	83.22	59.28	74.1

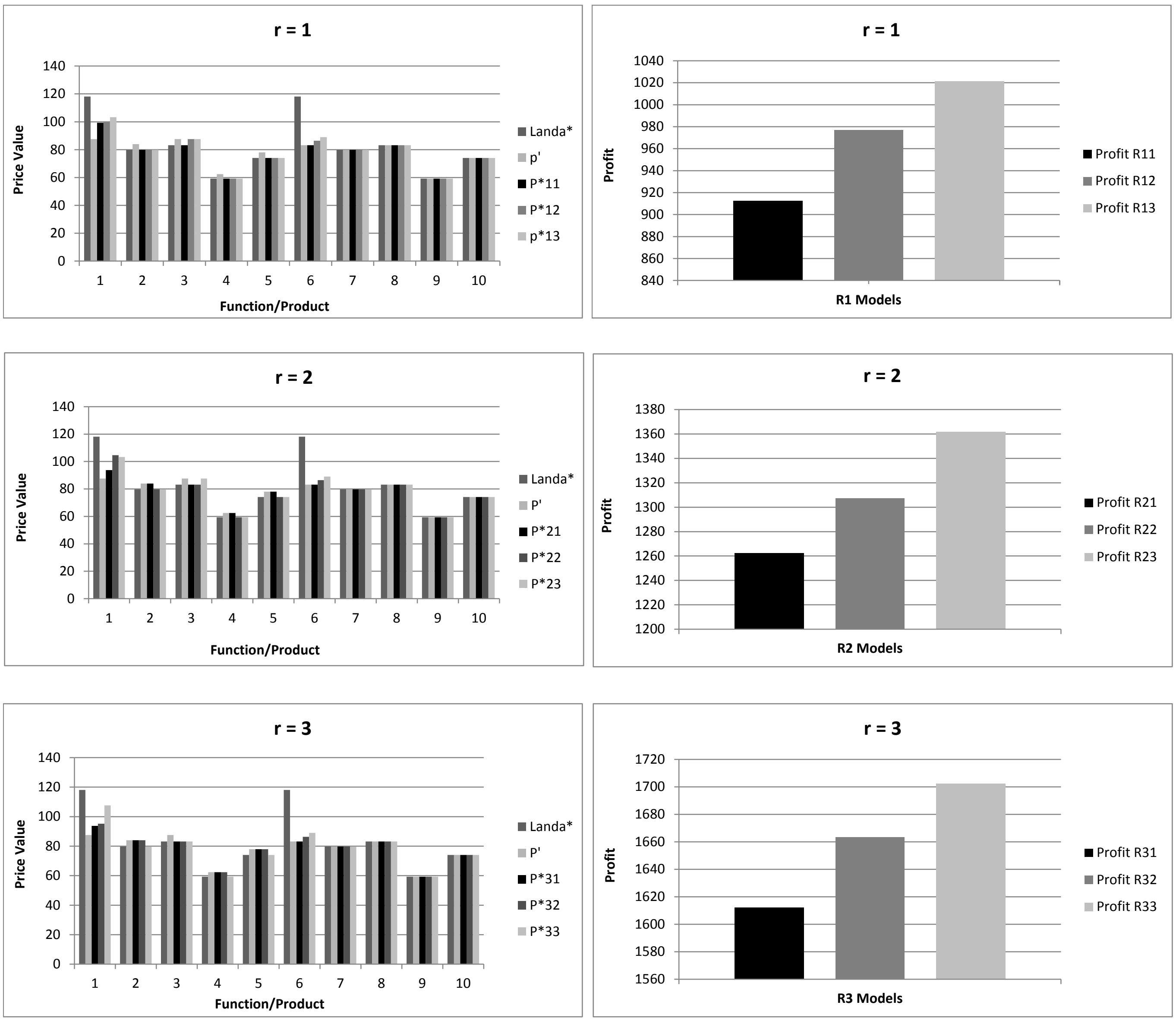
Table 13: Comparison of RPMBD Models

Supplier i	Model	Profit	Pi11	Pi12	Pi13	Pi14	Pi15	Pi21	Pi22	Pi23	Pi24	Pi25
1	R11	912.65	99.28	79.8	83.22	59.28	74.1	83.22	79.8	83.22	59.28	74.1
	R12	976.93	99.65	79.8	87.6	59.28	74.1	86.39	79.8	83.22	59.28	74.1
	R13	1021.46	103.27	79.8	87.6	59.28	74.1	89.04	79.8	83.22	59.28	74.1
	R21	1262.41	93.73	84	83.22	62.4	78	83.22	79.8	83.22	59.28	74.1
	R22	1307.51	104.66	79.8	83.22	59.28	74.1	86.39	79.8	83.22	59.28	74.1
	R23	1361.92	103.27	79.8	87.6	59.28	74.1	89.04	79.8	83.22	59.28	74.1
	R31	1612.18	93.73	84	83.22	62.4	78	83.22	79.8	83.22	59.28	74.1
	R32	1663.37	95.19	84	83.22	62.4	78	86.39	79.8	83.22	59.28	74.1
	R33	1702.38	107.65	79.8	83.22	59.28	74.1	89.04	79.8	83.22	59.28	74.1
2	R11	900.04	118.12	80.4	87.6	71.27	74.1	118.12	79.8	83.22	65.24	74.1
	R12	861.88	118.12	80.4	87.6	66.94	74.1	118.12	76.38	83.22	64.98	75.24
	R13	841.49	118.12	80.4	87.6	64.78	74.1	118.12	76.38	83.22	64.11	75.24
	R21	1155.11	111.6	80.4	87.6	80.4	79.2	106.02	79.8	83.22	75.14	74.1
	R22	1051.1	111.6	79.8	83.22	77.62	74.1	106.02	76.38	83.22	74.46	74.1
	R23	1021.83	118.12	80.4	87.6	64.78	74.1	118.12	76.38	83.22	64.11	75.24
	R31	1410.17	111.6	80.4	87.6	80.4	79.2	106.02	79.8	83.22	75.14	74.1
	R32	1282.02	111.6	79.8	83.22	75.92	79.2	106.02	79.8	83.22	72.52	74.1
	R33	1202.17	111.6	79.8	83.22	73.36	74.1	106.02	76.38	83.22	72.99	74.1

Based on the results in Table 13, in Figure 15 and Figure 16, we compare the optimal prices and profit obtained for each (RPMBD) models. To represent the prices on the 5 products from the 2 functions on the x axis, we index the products as $5(j-1)+k$. Thus, unit 6 on the x axis refers to the first item of the second function. Definition of the R_{rt} models (with $r, t = 1, 2, 3$) is compatible with the description in Section 5.3.2. For each supplier i, $P^{*}rt$ identifies the optimal prices corresponding to a model with r, t. Specifically, each figure fixes the value of r and attains the results for models with $t = 1, 2, 3$.

Figure 15: Price/Profit Comparison-Supplier 1

(a) Optimal Price Comparison (b) Optimal Profit Comparison

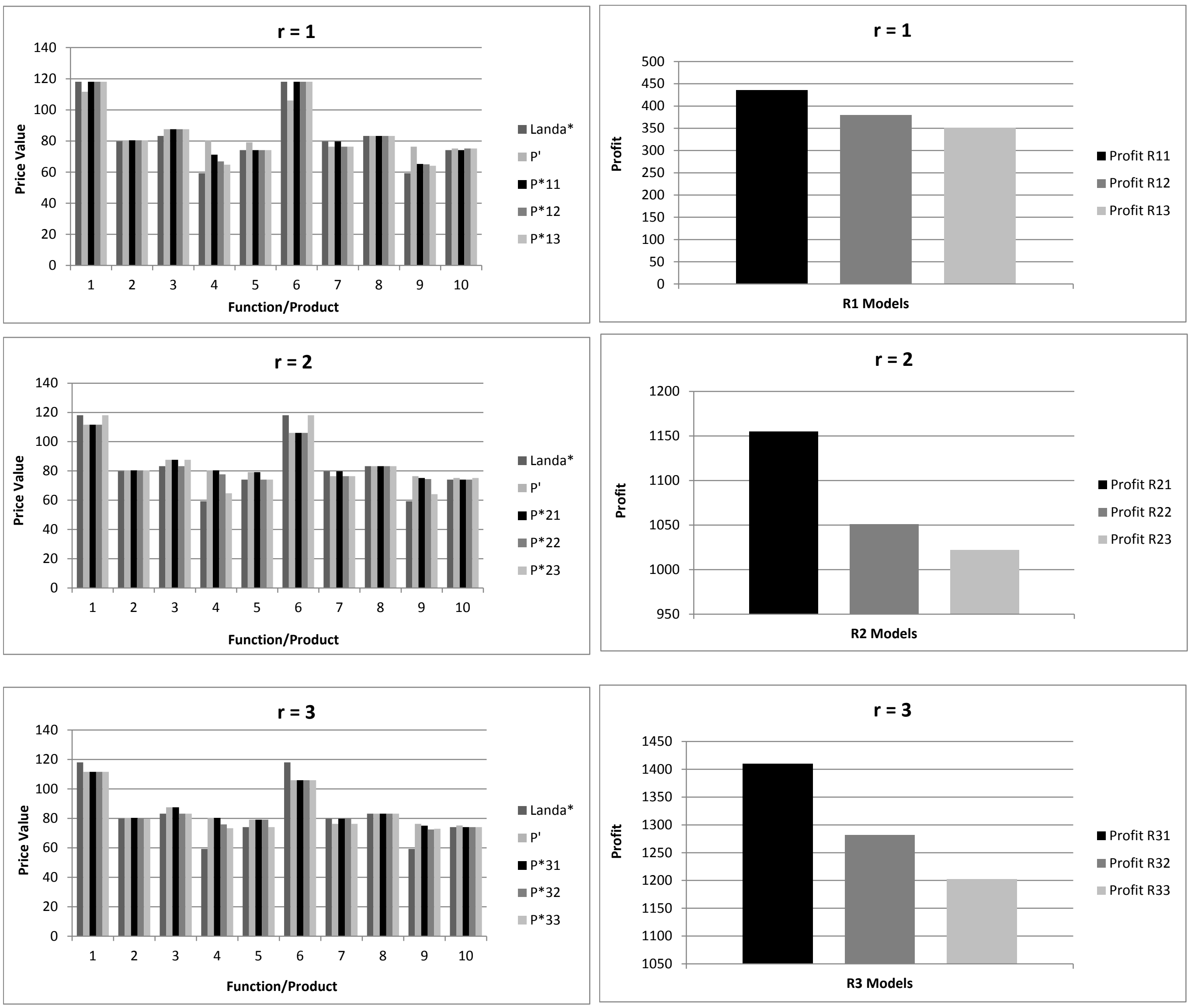


Also, we recorded the right hand side values of constraint (5.8b) in (RPMBD) as

	j=1	j=2
supplier 1	-94	-349
supplier 2	88	65

Figure 16: Price/Profit Comparison-Supplier 2

(a) Optimal Price Comparison (b) Optimal Profit Comparison



As observed from the figures, supplier 1's optimal profit maintains an increasing pattern as we increase the r value. The supplier's initial prices on product 1 both in his first and second function are mostly lower than the Lagrangian multipliers. This produces a negative term on the right hand side of constraint (5.8b) in (RPMBD).

Thus, increasing the t_{jk} values loosens this constraints letting the optimal prices and profit increase.

Supplier 2's initial prices are not as low as supplier 1's. In fact, in many cases they are larger than the Lagrangian multipliers producing a positive term on the right hand side of constraint (5.8b) in (RPMBD). Thus, constraint (5.8b) tightens with the increase of t_{jk}, producing lower optimal prices and profit.

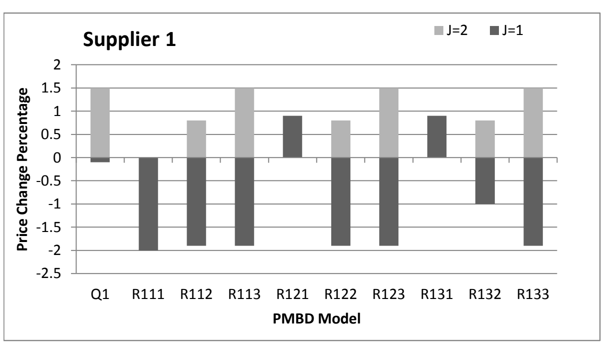
Figure 17 shows the percentage by which each supplier needs to adjust his prices in order to submit a competitive bid. We defined the change percentage as

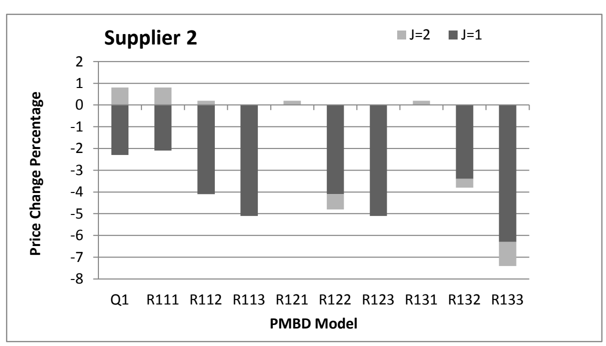
$$\frac{\text{Initial price} - \text{New price}}{\text{Initial price}} * 100$$

.

$J = 1$ and $J = 2$ define the price functions and $Rirt$ defines the problem formulation corresponding to supplier i, and parameters r, t. As observed, supplier 1 mostly decreased prices to enter the second round as for supplier 2, prices are often increased.

Figure 17: Price change percentage for each supplier





5.4.3 Comparison of RPMBD and VPMB Models

Implementation of divisible-bid auctions, if possible, is advantageous for both the suppliers and the auctioneer. In studying the suppliers' pricing scheme, one plausible question that raises is how prices compare in a divisible-bid versus an indivisible-bid auction.

To answer this question we generated divisible-bid problem instances and converted them to equivalent indivisible-bid problems. For the sake of consistency, we compared the quantity-based profit maximization problem (QPMBD) from divisible-bid auctions against the variable-cost profit maximization problem for indivisible-bid auctions problem (VPMB). Table 14 summarizes our results on 25 feasible problem instances.

WDPD_ Z* and WDPD_ T represent the optimal objective value and the corresponding CPU time for solving an instance of the (WDPD). QPMBD_Z*, and QPMBD_T show the optimal objective and CPU time for (QPMBD). We solve this problem using the second linearization technique based on initialization of q_{jk}, δ_{jk} and p_{jk} at their optimal closed-form values.

Similarly, WDPID_Z* and WDPID_T represent the optimal objective value and the corresponding CPU time for solving an instance of problem (3.2), and VPMB_Z*, VPMB_P*, and VPMB_T show the optimal objective, price and CPU time for (VPMB).

Based on the prices derived from the (QPMBD), we find the equivalent value of the bundle that VPMB formulates for submission in the next round of the auction, and denote it by QPMBD_P*. Table 14 confirms the equivalence of WDPD_Z* and WDPID_Z*, and the shorter amount of execution time required by WDPD_Z* to find the optimal solution.

Moreover, the results indicate that on average, divisible-bid auctions provide more profit for the suppliers and yet produce lower bundle prices. While the higher profit expected from divisible-bid auctions make them appealing for suppliers, production of lower bundle prices reduces the total price of procurement and therefore attracts auctioneers' interest to implement this type of auctions.

It can be seen that solving QPMBD is slightly more expensive on the suppliers than VPMBD. However, this difference, around 0.06 seconds on an average of 25 problem instances, is a good trade-off for suppliers' with the less complexity they face for evaluating and communicating an exponential number of bundles with the auctioneer.

Table 14: Comparison of QPMBD and VPMB Models

Instance	Supplier i	WDPD_Z*	WDPD_T	QPMBD_Z*	QPMBD_P*	QPMBD_T	WDPID_Z*	WDPIS_T	VPMB_Z*	VPMB_P*	VPMB_T
1	1	1024.8	0.19	1160.61	5290.38	0.22	1024.8	31.65	1024.8	6277.38	0.23
	2	1024.8	0.19	1424.82	2830.08	0.22	1024.8	31.65	1024.8	5202.49	0.23
2	1	1155.6	0.19	1190.32	2979.6	0.24	1155.6	31.84	1155.6	5662.6	0.22
	2	1155.6	0.19	1362.89	1491	0.44	1155.6	31.84	1155.6	5604.95	0.27
3	1	980.4	0.2	1294.58	6002.34	0.49	980.4	32.72	980.4	7634.28	0.23
	2	980.4	0.2	1118.31	6815.64	0.22	980.4	32.72	980.4	7007.68	0.23
4	1	1548	0.19	1590.34	10644.36	0.26	1548	33.53	1548	10460.64	0.23
	2	1548	0.19	1197.77	5485.14	0.23	1548	33.53	1548	5856.84	0.23
5	1	949.2	0.2	1551.61	9231.24	0.48	949.2	32.53	949.2	9262.39	0.23
	2	949.2	0.2	1325.88	6618.78	0.23	949.2	32.53	949.2	7032.43	0.23
6	1	1942.8	0.19	1412.14	5511.24	0.5	1942.8	32.64	1942.8	6505.81	0.22
	2	1942.8	0.19	1316.65	6836.94	0.22	1942.8	32.64	1942.8	7441.32	0.23
7	1	777.6	0.36	1289.98	6723.66	0.51	777.6	32.45	777.6	7532.29	0.23
	2	777.6	0.36	1412.97	6192.9	0.22	777.6	32.45	777.6	7389.41	0.23
8	1	1303.2	0.2	1027.65	4742.22	0.22	1303.2	32.81	1303.2	5584.76	0.24
	2	1303.2	0.2	1232.66	4404.36	0.22	1303.2	32.81	1303.2	5025.02	0.23
8	1	542.4	0.2	896.02	6881.16	0.31	542.4	33.37	542.4	6872.51	0.25
	2	542.4	0.2	1322.91	6907.44	0.25	542.4	33.37	542.4	7396.8	0.23
10	1	1191.6	0.19	1028.59	4119.9	0.24	1191.6	32.7	1191.6	4478.22	0.22
	2	1191.6	0.19	1297.65	8668.62	0.22	1191.6	32.7	1191.6	8521.5	0.31
11	1	898.8	0.2	1192.88	3424.38	0.24	898.8	33.24	898.8	4914.76	0.23
	2	898.8	0.2	1124.76	5733.36	0.23	898.8	33.24	898.8	6690.19	0.23
12	1	1441.2	0.2	1347.6	4590.18	0.23	1441.2	31.83	1441.2	6776.16	0.23
	2	1441.2	0.2	1029.02	4081.44	0.23	1441.2	31.83	1441.2	6505.92	0.23
13	1	1082.4	0.36	1238.6	5116.98	0.23	1082.4	32.73	1082.4	5780.91	0.23
	2	1082.4	0.36	1262.78	8673.12	0.24	1082.4	32.73	1082.4	8498.24	0.22
14	1	1143.6	0.2	1319.78	2304.72	0.23	1143.6	31.55	1143.6	5009.17	0.23
	2	1143.6	0.2	1036.74	2360.88	0.43	1143.6	31.55	1143.6	4951.19	0.23
15	1	968.4	0.2	1116.38	3503.88	0.32	968.4	31.83	968.4	4604.45	0.23
	2	968.4	0.2	1078.2	6301.32	0.22	968.4	31.83	968.4	6290.73	0.22
16	1	1786.8	0.19	1413.76	6700.5	0.23	1786.8	32.76	1786.8	7950	0.22
	2	1786.8	0.19	1679.5	7383.36	0.48	1786.8	32.76	1786.8	9186.34	0.23
17	1	1484.4	0.12	856.67	5248.32	0.36	1484.4	31.82	1484.4	5399.16	0.23
	2	1484.4	0.12	1275.78	8677.08	0.23	1484.4	31.82	1484.4	8506.68	0.23
18	1	1402.8	0.19	1327.79	8442.54	0.23	1402.8	33.26	1402.8	8367.72	0.23
	2	1402.8	0.19	1431.51	9468.6	0.22	1402.8	33.26	1402.8	9310.16	0.23
19	1	618	0.2	510.44	4542.12	0.24	618	33.2	618	5491.93	0.23
	2	618	0.2	1147.51	5921.94	0.25	618	33.2	618	7265.2	0.23
20	1	1212	0.2	1356.09	9010.26	0.45	1212	32.73	1212	8864.64	0.22
	2	1212	0.2	849.4	6931.26	0.23	1212	32.73	1212	6538.56	0.26
21	1	1155.6	0.2	1390.56	9273.96	0.23	1155.6	32.95	1155.6	9118.86	0.23
	2	1155.6	0.2	1374	8041.32	0.22	1155.6	32.95	1155.6	8097.9	0.23
22	1	1234.8	0.2	1369.01	6231.78	0.22	1234.8	32.73	1234.8	7183.27	0.23
	2	1234.8	0.2	1422.91	5493.12	0.23	1234.8	32.73	1234.8	6635.85	0.23
23	1	1117.2	0.19	908.61	6411.72	0.23	1117.2	31.56	1117.2	6323.58	0.23
	2	1117.2	0.19	1010.48	6909.24	0.47	1117.2	31.56	1117.2	6768.18	0.23
24	1	2016	0.19	1513.28	7045.86	0.56	2016	33.21	2016	8175.51	0.23
	2	2016	0.19	1707.3	9576.78	0.22	2016	33.21	2016	10149.06	0.28
25	1	777.6	0.22	1289.98	6723.66	0.47	777.6	32.04	777.6	7532.29	0.23
	2	777.6	0.22	1412.97	6192.9	0.22	777.6	32.04	777.6	7389.41	0.23
Average		1190.21	0.21	1248.97	6173.87	0.29	1190.21	32.55	1190.21	7020.51	0.23

Chapter 6

Conclusions and Future Extensions

6.1 Conclusion

This section summarizes our concluding remark classified by the chapters.

6.1.1 Chapter 3

A combinatorial auction allows several bidders to submit bids on different selections of items based on their personal preferences. From a computational point of view this problem is difficult to solve due to the exponential growth of the number of combinations.

An interesting design for determining the winners and the item prices involves application of the Lagrangian relaxation on the winner determination problem. Studies based on the Lagrangian relaxation initially solve the Lagrangian relaxation and then focus on development of a heuristic that improves the solution to (ideally) an optimal one. Revelation of the Lagrangian multipliers guides sellers to adjust their prices on the bundles they require and eventually improves auction results.

In this chapter we analytically established the equivalence of Lagrangian and linear relaxations for a multi-item multi-unit winner determination problem with OR and XOR bidding languages, with and without the free disposal condition. The results also indicate equivalence of the Lagrangian multipliers and the dual variables of the LP relaxation. Therefore, solving the linear relaxation of WDP provides fast access to the Lagrangian multipliers as approximates for item prices and a lower bound on the total price of procurement.

Based on this equivalence, we propose a solution method which determines the Lagrangian solution by solving a single subproblem. This method saves significant amount of time in finding the optimal solution as compared to traditional Lagrangian relaxation solution methods. In order to adjust infeasibility of the Lagrangian optimal solution, we design an Aggregate heuristic consisting of 32 computationally efficient heuristic procedures. The best solution obtained from the heuristics is extracted as the optimal value obtained from the Aggregate heuristic.

Our extensive numerical experiments indicate that on the class of problems whose maximum quantity of items included in each package is less than or equal to half of demand, the Aggregate heuristic provides a near optimal solution for respectively 10, 20, and 30 items which is on average 6, 7, and 8 percent off from the optimal in around 1/3, 1/189, and 1/950 of CPLEX time.

This solution scheme is most efficient for iterative combinatorial auctions in large marketplaces with capacitated suppliers. While application of iterative combinatorial auctions increases the auctioneers' overall payoff, it raises the urge for them to quickly attain a (near) optimal solution in each round of the auction. CPLEX 12 behaves poorly in terms of the amount of time it requires to provide the optimal solution with suppliers who are able to supply the manufacturers demand partially (specifically around 50% and below). Application of the Aggregate heuristic scheme can provide a good quality of the solution in relatively much less time.

6.1.2 Chapter 4

In this chapter we stressed the importance of investigating the problem of pricing and bundling for a bidder in a combinatorial auction setting. We use the Lagrangian multipliers as a means to inform the bidders about their valuations and those of the competition. In case suppliers find themselves competitive, the formulated profit maximization problem (PMB) helps them understand how to optimally bundle and price their new package. The bidders goal is to maximize their profit while maintaining their competitive advantage and respecting their capacity and/or the auctioneer minimum order requirements.

The closed-form solution results enable us to compare prices of the two consecutive rounds, discussing the conditions of when to expect prices to grow or shrink or stay the same. Based on the interaction between the suppliers and the auctioneer, we propose an iterative auction which we analytically show that is convergent.

We performed several empirical experiments to study the dynamics of the prices and profits in the auction. On the suppliers' side, we show that the suppliers profit increases as the auction proceeds. For the auctioneer, we observe an increase in cost as the auction goes from the first to the second round. This is not unusual due to suppliers' imprecise first round pricing scheme. The auctioneer cost ultimately reaches a steady state after a practical number of auction rounds.

This auction mechanism brings about a win-win environment in which the auctioneer benefits from the intense competition among the suppliers who try to meet his required demand. The suppliers also take advantage of supplying products that not only preserve their desired minimum profit, but also maximize it throughout their bidding process.

6.1.3 Chapter 5

In a traditional auction problem, suppliers submit bids which are either accepted as a whole package or rejected by the auctioneer. Given the wide range of variations suppliers face in pricing units of the items, in this chapter we investigate whether accepting partial bids increases the auctioneer's and/or the suppliers' efficiency in maximizing their payoff from the auction.

To this end, we define a new auction environment wherein suppliers submit bids as price functions defined on disjoint intervals of quantities. Formulating the winner determination problem, we explore the Lagrangian relaxation properties of the defined mixed integer programming at optimality. The computational efficiency of the derived mathematical programming is numerically compared against the equivalent binary integer programming with indivisible bids. With equivalent optimal objective values, we found that CPLEX 12 solves the divisible-bid problem more efficiently.

We also studied the correspondence of the suppliers' risk-taking attitude with the profit they gain from the auction. Our numerical work on random problem instances showed higher average profit value for the suppliers from the divisible-bid auction while producing lower bundle prices. While the suppliers are able to expect higher profit from the divisible-bid auctions, the lower prices along with lower execution times makes them an appealing alternative for auctioneers as well.

It is worth mentioning that the application of divisible-bid auctions requires the consent from the suppliers for revealing their price functions on intervals of quantities. Moreover, on a low-demand per-product basis where suppliers are willing to submit a limited number of each product, restricting the submitted bids on intervals of quantities is not as practical.

6.2 Extension Opportunities

This section proposes natural extensions of this thesis classified by the chapters. In the final section, we propose interesting research directions beyond the ideas discussed in this thesis.

6.2.1 Chapter 3

- Studying the solutions to the linear and Lagrangian relaxations, we noticed that in most cases both relaxations share similar solution elements with the exact solution. Recognizing some of these elements and elevating their values to 1 can produce the exact solution. This property motivates us to study the solution space of the LP relaxation more carefully to recognize variables which are more likely to appear in the IP solution. In a branch and bound-based bid ordering heuristic scheme, this helps us find a (good quality) feasible solution quite early on, and further improve it via more branching. In other words, the objective is to use the linear relaxation to extract valuable information about the IP model which within a branch and bound tree that would help us make smart choices of branching variables.

- A Lagrangian-based heuristic algorithm for solving the winner determination problem starts with solving the Lagrangian relaxation problem. As seen earlier, this solution is integer, however, not necessarily primal feasible. One possible algorithm design is to remove all variables with positive optimal Lagrangian values and solve the relaxation for the reduced problem. One research line is to test the execution time and the optimality gap of this algorithm.

- In Chapter 3 we established the equivalence of the Lagrangian and linear relaxation optimal bounds. One interesting extension is to investigate the correlation of the linear and Lagrangian optimal solutions when the linear relaxation provides integer optimal solution.

6.2.2 Chapter 4

Possible extensions for this chapter include the following:

- It is often not easy to measure the complexity of a mechanism or the costs of the length of time to complete the auction term. Analytical results on the convergence rate of the auction will help us further establish the practicality of our proposed pricing scheme.

- The mechanism proposed uses Lagrangian multipliers to compel more competitiveness in future auction rounds. Factors that diminish or enhance their predictive value could be investigated.

- The inability of individual pricing models to capture complementarities is another restriction. Non-convex programming methods for working with duality gaps might prove useful for providing estimates and/or bounds on complementarities.

- Even though the incentive behind the auction mechanism design as proposed in Chapter 4 is to provide an interaction between the auctioneer and suppliers while both maximize their utilities, thinking of the economical properties at convergence is valuable.

- Studying the effect of parameters involved in suppliers' optimization problem, i.e., the suppliers' minimum profitability or internal costs, on the final prices and equilibrium is important. Conceptually, different settings can affect the convergence rate as well as the final procurement price.

6.2.3 Chapter 5

In extension of Chapter 5 of this thesis one might consider investigating the following issues:

- Elaborating on devising more expressive bidding Languages for auctions with divisible bids. The truckload transportation literature has some good initiatives in this area, however, as described in section 2.4 suppliers are not yet fully able to communicate their complete preferences through these languages.

- Observing the optimal solution to the winner determination problem for divisible bids (WDPD) reveals that in many cases CPLEX 12 fixes the quantity variables at either of their lower and upper bounds and there are very few cases when an interior point (between the minimum and maximum) is detected for the optimal quantity value. This feature can be investigated further to develop more efficient solution procedures.

- In this chapter we observed that the equivalence of Lagrangian and linear relaxation bounds previously proved for indivisible bids holds for the divisible bids as well. It is worthwhile investigating whether this result holds for a wider class of problems.

6.2.4 General Extension Venues

Considering the concerns of industries and firms these days to reduce costs, one interesting research plan is to look at jointly optimizing sourcing and making inventory decisions using combinatorial auctions. Accepting bids in the phase of ordering materials or components allows suppliers to specify clearly during which periods they are willing to deliver, how many units they are willing to supply, and how much they would charge for providing them.

In most real world environments, the holding cost of an item depends on the price of the item and can vary from period to period. One option to determine the price of an item in a certain period is using the associated Lagrangian multiplier. Multiplying this variable by the surplus develops nonlinearity in the objective function of the problem which adds complexity.

The focus of most lot sizing problems is to assist decision makers with short-term production plans. In medium- to long-term decision plans, it is interesting to take into account the cost of setting up auctions and including suppliers to determine the optimal number of auctions to hold in a long time horizon and the number of suppliers to invite in each.

The manufacturers's poor estimates of demand values, and the suppliers uncertain capacity of supply quantities can result in considerable losses in a deterministic model. This can lead to thinking of a stochastic counterpart model with uncertainty in supply and/or demand.

Finally, another research ground is to consider the possibility of delivery delays and quantity defects in a multi-attribute procurement auction context.

Bibliography

[1] Abrache, J., Crainic, T. G., and Gendreau, M. Models for bundle trading in financial markets. *European Journal of Operational Research 160*, 1 (2005), 88–105.

[2] Ahuja, R. K., Magnanti, T. L., and Orlin, J. B. *Network Flows: Theory, Algorithms, and Applications.* Prentice Hall, 1993.

[3] Aparicio, J., Landete, M., Monge, J. F., and Sirvent, I. A new pricing scheme based on DEA for iterative multi-unit combinatorial auctions. *Top 16*, 2 (2008), 319–344.

[4] Ausubel, L. An efficient ascending-bid auction for multiple objects. *The American Economic Review 94*, 5 (2004), 1452–1475.

[5] Ausubel, L. An efficient dynamic auction for heterogeneous commodities. *The American Economic Review 96*, 3 (2006), 602–629.

[6] Ausubel, L., Cramton, P., and Milgrom, P. The clock-proxy auction: A practical combinatorial auction design. In *Combinatorial Auctions*, P. Cramton, Y. Shoham, and R. Steinberg, Eds. The MIT Press, 2006, pp. 115–138.

[7] Ausubel, L., and Milgrom, P. Ascending auctions with package bidding. *The BE Journal of Theoretical Economics 1*, 1 (2002), 1–42.

[8] Ball, M. O., Donohue, G. L., and Hoffman, K. Auctions for the safe, efficient, and equitable allocation of airspace system resources. In *Combinatorial Auctions*, P. C. Cramton, Y. Shoham, and R. Steinberg, Eds. The MIT Press, 2006, pp. 507–538.

[9] Banks, J. S., Ledyard, J. O., and Porter, D. P. Allocating uncertain and unresponsive resources: An experimental approach. *The Rand Journal of Economics 20*, 1 (1989), 1–25.

[10] Beall, S. *The role of reverse auctions in strategic sourcing.* PhD thesis, 2003.

[11] Bichler, M., Schneider, S., Guler, K., and Sayal, M. Compact bidding languages and supplier selection for markets with economies of scale and scope. *European Journal of Operational Research 214*, 1 (2011), 67–77.

[12] Bichler, M., Shabalin, P., and Pikovsky, A. A computational analysis of linear price iterative combinatorial auction formats. *Information Systems Research 20*, 1 (2009), 33–59.

[13] Bikhchandani, S., de Vries, S., Schummer, J., and Vohra, R. Linear programming and Vickrey auctions. *Mathematics of the Internet: E-auction and Markets 127* (2002), 75–116.

[14] Bikhchandani, S., and Ostroy, J. M. The package assignment model. *Journal of Economic Theory, 107*, 2 (2002), 377–406.

[15] Boutilier, C., Goldszmidt, M., and Sabata, B. Sequential auctions for the allocation of resources with complementarities. In *International Joint Conference on Artificial Intelligence* (1999), vol. 16, pp. 527–534.

[16] Caplice, C., and Sheffi, Y. Optimization based procurement for transportation services. *Journal of Business Logistics 24*, 2 (2003), 109–128.

[17] Caplice, C., and Sheffi, Y. Compbinatorial auctions for truckload transportation. In *Combinatorial Auctions*, P. C. Cramton, Y. Shoham, and R. Steinberg, Eds. The MIT Press, 2006, pp. 539–571.

[18] Caplice, C. G. *An optimization based bidding process: A new framework for shipper-carrier relationships.* PhD thesis, Massachusetts Institute of Technology, 1996.

[19] Chen, R. R., Roundy, R. O., Zhang, R. Q., and Janakiraman, G. Efficient auction mechanisms for supply chain procurement. *Management Science 51*, 3 (2005), 467–482.

[20] Clarke, E. Multipart pricing of public goods. *Public Choice 11*, 1 (1971), 17–33.

[21] Collins, J., Demir, G., and Gini, M. Bidtree ordering in IDA* combinatorial auction winner-determination with side constraints. In *Agent-Mediated Electronic Commerce IV. Designing Mechanisms and Systems.* Springer, 2002, pp. 17–33.

[22] Crainic, T., and Gendreau, M. Advanced fleet management systems and advisors: Converging decision technologies for its and e-business. In *10th World Congress on Intelligent Transport Systems* (2003).

[23] Crainic, T., and Gendreau, M. Freight exchanges and carrier operations: Issues, models, and tools. Tech. rep., École Polytechnique, Montréal, 2007.

[24] Crainic, T., Gendreau, M., Rekik, M., and Robert, J. *Cost Allocations in Combinatorial Auctions for Bilateral Procurement Markets*. CIRRELT, 2009.

[25] Cramton, P. Ascending auctions. *European Economic Review 5* (1998), 745–756.

[26] de Vries, S., Schummer, J., and Vohra, R. V. On ascending vickrey auctions for heterogeneous objects. *Journal of Economic Theory 132*, 1 (2007), 95–118.

[27] de Vries, S., and Vohra, R. V. Combinatorial Auctions: A Survey. *INFORMS Journal on Computing 15*, 3 (2003), 284–309.

[28] Demange, G. Strategyproofness in the assignment market game. Tech. rep., École Polytechnique, Montréal, 1982.

[29] Demange, G., and Gale, D. The strategy structure of two-sided matching markets. *Econometrica: Journal of the Econometric Society* (1985), 873–888.

[30] Demange, G., Gale, D., and Sotomayor, M. Multi-item auctions. *The Journal of Political Economy 94*, 4 (1986), 863–872.

[31] Devries, S., Schummer, J., and Vohra, R. On ascending Vickrey auctions for heterogeneous objects. *Journal of Economic Theory 132*, 1 (2007), 95–118.

[32] Elmaghraby, W., and Keskinocak, P. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Science 49*, 10 (2003), 1287–1309.

[33] Epstein, R., Henriquez, L., Catalán, J., Weintraub, G. Y., Martinez, C., and Espejo, F. A combinatorial auction improves school meals in chile: A case of or in developing countries. *International Transactions in Operational Research 11*, 6 (2004), 593–612.

[34] Farris, P. W., Bendle, N. T., Pfeifer, P. E., and Reibstein, D. J. *Marketing Metrics: The Definitive Guide to Measuring Marketing Performance.* Pearson Education, 2010.

[35] Fujishima, Y., Leyton-Brown, K., and Shoham, Y. Taming the computational complexity of combinatorial auctions: Optimal and approximate approaches. In *International Joint Conference on Artificial Intelligence* (1999), vol. 16, pp. 548–553.

[36] Groves, T. Incentives in teams. *Econometrica: Journal of the Econometric Society 41*, 4 (1973), 617–631.

[37] Guignard, M., and Kim, S. Lagrangean decomposition: A model yielding stronger lagrangean bounds. *Mathematical Programming 39*, 2 (1987), 215–228.

[38] Gul, F. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory 87*, 1 (1999), 95–124.

[39] Guo, Y., Lim, A., Rodrigues, B., and Tang, J. Using a Lagrangian heuristic for a combinatorial auction problem. In *17th IEEE International Conference on Tools with Artificial Intelligence (ICTAI'05)* (2005), pp. 5–103.

[40] Hohner, G., Rich, J., Ng, E., and Reid, G. Combinatorial and quantity-discount procurement auctions benefit Mars, Incorporated and its suppliers. *Interfaces 33*, 1 (2003), 23–35.

[41] Hsieh, F.-S. Combinatorial reverse auction based on revelation of Lagrangian multipliers. *Decision Support Systems 48*, 2 (2010), 323–330.

[42] Iftekhar, M., Hailu, A., and Lindner, R. Item price information feedback in multiple unit combinatorial auctions: Design issues. *IMA Journal of Management Mathematics 22*, 3 (2011), 271–289.

[43] Jones, J. L., and Andrews, R. W. Price discovery in combinatorial auctions using Gibbs sampling. *Decision Support Systems 42*, 2 (2006), 958–974.

[44] Kalagnanam, J. R., Davenport, A. J., and Lee, H. S. Computational aspects of clearing continuous call double auctions with assignment constraints and indivisible demand. *Electronic Commerce Research 1*, 3 (2001), 221–238.

[45] Kaleta, M. Bidding languages for continuous auctions. In *New Trends in Databases and Information Systems*. Springer, 2013, pp. 211–220.

[46] Kameshwaran, S., and Benyoucef, L. Branch on price: A fast winner determination algorithm for discount auctions. In *Algorithmic Aspects in Information and Management*. 2006, pp. 375–386.

[47] Kameshwaran, S., Benyoucef, L., and Xie, X. Design of progressive auctions for procurement based on Lagrangian relaxation. In *Seventh IEEE International Conference on E-Commerce Technology (CEC'05)* (2005), pp. 9–16.

[48] Kameshwaran, S., Benyoucef, L., and Xie, X. Discount auctions for procuring heterogeneous items. In *Proceedings of the 7th International Conference on Electronic Commerce* (2005), ACM, pp. 244–249.

[49] Kameshwaran, S., Benyoucef, L., and Xie, X. Winner determination in discount auctions. In *Internet and Network Economics*. Springer, 2005, pp. 868–877.

[50] Kameshwaran, S., Benyoucef, L., and Xie, X. Branch-and-cut algorithms for winner determination in discount auctions. *Algorithmic Operations Research 2*, 2 (2007), 112.

[51] Kameshwaran, S., and Narahari, Y. A Lagrangian heuristic for bid evaluation in e-procurement auctions. In *IEEE International Conference on Automation Science and Engineering (CASE 2005)* (2005), IEEE, pp. 220–225.

[52] Kameshwaran, S., and Narahari, Y. Benders-based winner determination algorithm for volume discount procurement auctions. *International Journal of Logistics Systems and Management 5*, 1 (2009), 21–40.

[53] Kameshwaran, S., and Narahari, Y. Nonconvex piecewise linear knapsack problems. *European Journal of Operational Research 192*, 1 (2009), 56–68.

[54] Kameshwaran, S., Narahari, Y., Rosa, C., Kulkarni, D., and Tew, J. Multiattribute electronic procurement using goal programming. *European Journal of Operational Research 179*, 2 (2007), 518–536.

[55] Kelly, F., and Steinberg, R. A combinatorial auction with multiple winners for universal service. *Management Science 46*, 4 (2000), 586–596.

[56] Kelso, A. S., and Crawford, V. P. Job matching, coalition formation, and gross substitutes. *Econometrica: Journal of the Econometric Society 50*, 6 (1982), 1483.

[57] Krishna, V. *Auction Theory*. Academic Press, 2002.

[58] Kwasnica, A. M., Ledyard, J. O., Porter, D., and DeMartini, C. A new and improved design for multiobject iterative auctions. *Management Science 51*, 3 (2005), 419–434.

[59] Kwon, R., Anandalingam, G., and Ungar, L. Iterative combinatorial auctions with bidder-determined combinations. *Management Science 51*, 3 (2005), 407–418.

[60] Lahaie, S. M., and Parkes, D. C. Applying learning algorithms to preference elicitation. In *Proceedings of the 5th ACM Conference on Electronic Commerce* (2004), ACM, pp. 180–188.

[61] Ledyard, J. O., Olson, M., Porter, D., Swanson, J. A., and Torma, D. P. The first use of a combined-value auction for transportation services. *Interfaces 32*, 5 (2002), 4–12.

[62] Lee, C.-G., Kwon, R. H., and Ma, Z. A carriers optimal bid generation problem in combinatorial auctions for transportation procurement. *Transportation Research Part E: Logistics and Transportation Review 43*, 2 (2007), 173–191.

[63] Lehmann, D., Müller, R., and Sandholm, T. The winner determination problem. In *Combinatorial Auctions*, Y. S. Peter Cramton and R. Steinberg, Eds. The MIT Press, 2006, pp. 297–317.

[64] Lehmann, D., O'Callaghan, L. I., and Shoham, Y. Truth revelation in approximately efficient combinatorial auctions. *Journal of the ACM 49*, 5 (2002), 577.

[65] Leonard, H. B. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy 91*, 3 (1983), 461.

[66] Leyton-Brown, K., Nudelman, E., and Shoham, Y. Learning the empirical hardness of optimization problems: The case of combinatorial auctions. In *Principles and Practice of Constraint Programming-CP 2002* (2002), Springer, pp. 91–100.

[67] Leyton-Brown, K., Pearson, M., and Shoham, Y. Towards a universal test suite for combinatorial auction algorithms. In *Proceedings of the 2nd ACM conference on Electronic Commerce* (2000), ACM, pp. 66–76.

[68] Lim, A., Rodrigues, B., and Xu, Z. Transportation procurement with seasonally varying shipper demand and volume guarantees. *Operations Research 56*, 3 (2008), 758–771.

[69] LORENTZIADIS, P. L. Pricing in multiple-item procurement auctions with a common to all items fixed cost. *European Journal of Operational Research 190*, 3 (2008), 790–797.

[70] MAURER, L., AND BARROSO, L. A. *Electricity auctions: An overview of efficient practices*. World Bank Publications, 2011.

[71] MCMILLAN, J. Selling spectrum rights. *The Journal of Economic Perspectives 8*, 3 (1994), 145–162.

[72] MILGROM, P. Putting auction theory to work: The simultaneous ascending auction. *Journal of Political Economy 108*, 2 (2000), 245–272.

[73] MISHRA, D. Simple primal-dual auctions are not possible. *Proceedings of the 5th ACM conference on Electronic Commerce* (2004), 250–251.

[74] MISHRA, D., AND PARKES, D. Ascending price Vickrey auctions for general valuations. *Journal of Economic Theory 132*, 1 (2007), 335–366.

[75] MISHRA, D., AND VEERAMANI, D. Vickrey-Dutch procurement auction for multiple items. *European Journal of Operational Research 180*, 2 (2007), 617–629.

[76] MÜLLER, R. Tractable cases of the winner determination problem. In *Combinatorial Auctions*, P. Cramton, Y. Shoham, and R. Steinberg, Eds. The MIT Press, 2006, pp. 319–336.

[77] MUNSON, C. L., AND ROSENBLATT, M. J. Theories and realities of quantity discounts: An exploratory study. *Production and Operations Management 7*, 4 (1998), 352–369.

[78] Myerson, R. B. Optimal auction design. *Mathematics of Operations Research 6*, 1 (1981), 58–73.

[79] Nemhauser, G. L., and Wolsey, L. A. *Integer and Combinatorial Optimization*. Wiley New York, 1988.

[80] Nisan, N. Bidding and allocation in combinatorial auctions. In *2nd ACM Conference on Electronic Commerce* (2000), ACM, pp. 1–12.

[81] Nisan, N., and Segal, I. The communication requirements of efficient allocations and supporting prices. *Journal of Economic Theory 129*, 1 (2006), 192–224.

[82] Olivares, M., Weintraub, G. Y., Epstein, R., and Yung, D. Combinatorial auctions for procurement: An empirical study of the chilean school meals auction. *Management Science 58*, 8 (2012), 1458–1481.

[83] O'Neill, R. P., Sotkiewicz, P. M., Hobbs, B. F., Rothkopf, M. H., and Stewart Jr, W. R. Efficient market-clearing prices in markets with nonconvexities. *European Journal of Operational Research 164*, 1 (2005), 269–285.

[84] Parkes, D. Price-based information certificates for minimal-revelation combinatorial auctions. In *Agent-Mediated Electronic Commerce IV. Designing Mechanisms and Systems*. Springer, 2002, pp. 103–122.

[85] Parkes, D., and Ungar, L. Iterative combinatorial auctions: Theory and practice. In *Proceedings of the National Conference on Artificial Intelligence* (2000), The MIT Press, pp. 74–81.

[86] Parkes, D. C. An iterative generalized vickrey auction: Strategy-proofness without complete revelation. In *AAAI Spring Symposium on Game Theoretic and Decision Theoretic Agents* (2001), pp. 78–87.

[87] Parkes, D. C. Iterative combinatorial auctions. In *Combinatorial Auctions*, P. Cramton, Y. Shoham, and R. Steinberg, Eds. The MIT Press, Cambridge, MA, USA, 2006.

[88] Parkes, D. C., Mishra, D., and Ungar, L. H. A simple ascending generalized vickrey auction, 2004. Working paper.

[89] Porter, D., Rassenti, S., Roopnarine, A., and Smith, V. Combinatorial auction design. *Proceedings of the National Academy of Sciences of the United States of America 100*, 19 (2003), 11153–11157.

[90] Rassenti, S. J., Smith, V. L., and Bulfin, R. L. A combinatorial auction mechanism for airport time slot allocation. *The Bell Journal of Economics 13*, 2 (1982), 402.

[91] Regan, A., and Song, J. An auction based collaborative carrier network. Tech. rep., Institute of Transportation Studies, UC Irvine, 2003.

[92] Rekik, M., and Mellouli, S. Reputation-based winner determination problem for combinatorial transportation procurement auctions. *Journal of the Operational Research Society 63*, 10 (2012), 1400–1409.

[93] Remli, N., and Rekik, M. A robust winner determination problem for combinatorial transportation auctions under uncertain shipment volumes. *Transportation Research Part C: Emerging Technologies 35* (2013), 204–217.

[94] Roth, A. E., and Ockenfels, A. Last minute bidding and the rules for ending second-price auctions: Theory and evidence from a natural experiment on the internet. Tech. rep., National Bureau of Economic Research, 2000.

[95] Rothkopf, M. H., Pekec, A., and Harstad, R. M. Computationally manageable combinational auctions. *Management Science 44*, 8 (1998), 1131–1147.

[96] Rothkopf, M. H., Teisberg, T. J., and Kahn, E. P. Why are vickrey auctions rare? *Journal of Political Economy 98*, 1 (1990), 94–109.

[97] Sandholm, T. An implementation of the contract net protocol based on marginal cost calculation. *Eleventh National Conference on Artificial Intelligence* (1993), 256–262.

[98] Sandholm, T. Algorithm for optimal winner determination in combinatorial auctions. *Artificial Intelligence 135*, 1-2 (2002), 1–54.

[99] Sandholm, T., and Suri, S. Side constraints and non-price attributes in markets. *Games and Economic Behavior 55*, 2 (2006), 321–330.

[100] Sandholm, T., Suri, S., Gilpin, A., and Levine, D. CABOB: A fast optimal algorithm for winner determination in combinatorial auctions. *Management Science 51*, 3 (2005), 374–390.

[101] Scheffel, T., Pikovsky, A., Bichler, M., and Guler, K. An experimental comparison of linear and nonlinear price combinatorial auctions. *Information Systems Research 22*, 2 (2011), 346–368.

[102] Sheffi, Y. Combinatorial auctions in the procurement of transportation services. *Interfaces 34*, 4 (Aug. 2004), 245–252.

[103] Song, J. *Combinatorial auctions: Applications in freight transportatiuon contract procurement*. PhD thesis, UC Irvine, 2003.

[104] Song, J., Nandiraju, S., and Regan, A. C. Optimization models for auctions for transportation service contract procurement. Tech. rep., Institute of Transportation Studies, UC Irvine, 2004.

[105] Song, J., and Regan, A. Combinatorial auctions for transportation service procurement: The carrier perspective. *Transportation Research Record: Journal of the Transportation Research Board 1833*, 1 (2003), 40–46.

[106] Song, J., and Regan, A. Approximation algorithms for the bid construction problem in combinatorial auctions for the procurement of freight transportation contracts. *Transportation Research Part B: Methodological 39*, 10 (2005), 914–933.

[107] Tennenholtz, M. Some tractable combinatorial auctions. In *Association for the Advancement of Artificial Intelligence/Innovative Applications of Artificial Intelligence (AAAI/IAAI)* (2000), pp. 98–103.

[108] Vagstad, S. Centralized vs. decentralized procurement: Does dispersed information call for decentralized decision-making? *International Journal of Industrial Organization 18*, 6 (2000), 949–963.

[109] Vickrey, W. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance 16*, 1 (1961), 8–37.

[110] Wang, X., and Xia, M. Combinatorial bid generation problem for transportation service procurement. *Transportation Research Record: Journal of the Transportation Research Board 1923*, 1 (2005), 189–198.

[111] Wolfstetter, E. Auctions: An introduction. *Journal of Economic Surveys 10*, 4 (1996), 367–420.

[112] Wurman, P. R., and Wellman, M. P. Equilibrium prices in bundle auctions. In *Association for the Advancement of Artificial Intelligence(AAAI), Workshop on Artificial Intelligence for Electronic Commerce* (1999), pp. 56–61.

[113] Xia, M., Koehler, G. J., and Whinston, A. B. Pricing combinatorial auctions. *European Journal of Operational Research 154*, 1 (2004), 251–270.