On Closure Operator for Interval Order Structures

ON CLOSURE OPERATOR FOR INTERVAL ORDER STRUCTURES

BY

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Dedications

I dedicate this thesis to my mother Marina and my late grandmother Raisa.

Abstract

Formal studies of models of concurrency are usually focused on two major models: Interleaving abstraction (Bergstra, 2001; Milner, 1990) and partially ordered causality (Diekert and Rozenberg, 1995; Jensen, 1997; Reisig, 1998). Although very mature, these models retain a known limitation: Neither of them can model the "not later than" relationship effectively, which causes problems with specifying priorities, error recovery, time testing, inhibitor nets, etc. See for reference: Best and Koutny (1992); Janicki (2008); Janicki and Koutny (1995); Juhas *et al.* (2006); Kleijn and Koutny (2004). A solution, proposed independently (in this order) in (Lamport, 1986; Gaifman and Pratt, 1987) and (Janicki and Koutny, 1991), suggests to model concurrent behaviours by *ordered structure*, i.e. a triple $(X, \prec, \sqsubset)$, where X is the set of event occurrences, and $\prec$ and $\sqsubset$ are two binary relations on X. The relation $\prec$ is interpreted as "causality", i.e. an abstraction of the "earlier than" relationship, and $\sqsubset$ is interpreted as "weak causality", an abstraction of the "not later than" relationship. For ordered structures' model, the following two kinds of relational structures are of special importance: *stratified order structures (SO-structures)* and *interval order structures (IO-structures)*. The SO-structures can fully model concurrent behaviours when system executions (operational semantics) are described in terms of stratified orders, while the IO-structures can fully model concurrent behaviours when system executions are described in terms of interval orders (Janicki, 2008; Janicki and Koutny, 1997). It was argued in (Janicki and Koutny, 1993), and also implicitly in a 1914 Wiener's paper Wiener

(1914), that any execution that can be observed by a single observer must be an interval order. Thus, IO-structures provide a very definitive model of concurrency. However, the theory of IO-structures remains far less developed than its simpler counterpart - the theory of SO-structures.

One of the most important concepts lying at the core of partial orders and algebraic structures theory is the concept of *transitive closure* of relations. The equivalent of transitive closure for SO-structures, called $\diamondsuit$-*closure*, has been proposed in (Janicki and Koutny, 1995) and consequently used in (Janicki and Koutny, 1995; Juhas *et al.*, 2006; Kleijn and Koutny, 2004) and others. However, a similar concept for IO-structures has not been proposed. In this thesis we define that concept.

We introduce the transitive closure for IO-structures, called the $\blacklozenge$-*closure*. We prove that it has same properties as the standard transitive closure for partial orders and $\diamondsuit$-closure for SO-structures (published in Janicki and Zubkova (2009); Janicki *et al.* (2009)), and provide some comparison of different versions of transitive closure used in various relational structures. Some properties of another recently introduced $\bigstar$-closure (Janicki *et al.*, 2013) are also discussed.

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Contents

Chapter 1

Introduction and Motivations

The concept of *closure* (c.f. Cohn (1981); Rosen (2002)) is one of the basic constructions used in abstract algebra and its applications. In principle, it can be described as follows. There is a set R of some type (R could be a family of interrelated relations) that may, or may not have some property $\mathbf{P}$. If there is a set S of the same type with property $\mathbf{P}$ containing R such that S is a subset of every set (of the same type as R) with property $\mathbf{P}$ containing R, then *S is called the* **closure** *of R with respect to* $\mathbf{P}$.

Closures are most often used for relations (transitive, symmetric closures, etc.), but they are also powerful tools for relational systems (c.f. Cohn (1981); Janicki and Koutny (1995); Janicki *et al.* (2013)) as well. They allow to construct desired object by defining their skeletons and then applying appropriate closure operator.

In Janicki and Koutny (1995) the $\diamondsuit$-closure was introduced and used to derive *stratified order structures* (a model of concurrent behaviours) from quite general initial relational structures. This construction allowed to define a relationship between monoidal comtraces and stratified order structures. The $\diamondsuit$-closure can be interpreted as some specialized generalization of transitive closure.

The $\blacklozenge$-closure, the main contribution of this thesis, is an equivalent of $\diamondsuit$-closure but for more advanced and general model of concurrent behaviours, namely *interval order structures*. Introduction of the $\blacklozenge$-closure and proving its properties in the main contribution of this thesis. Results of this thesis were successfully published in Janicki and Zubkova (2009) and Janicki *et al.* (2009). Moreover, the $\blacklozenge$-closure was consequently used to construct *interval traces*, resulting in the follow-up publication Janicki *et al.* (2012).

The rest of this chapter will be devoted to providing some motivations that lead to the concept of interval order structures, to justify the introduction of the $\blacklozenge$-closure in the first place.

The study of reasoning about concurrent computations is a subject of great complexity and interval order structures is a mathematical tool used to model complex concurrent behaviours. The notion of interval order structures is studied in this thesis and has two different underlying motivations. One was introduced by Lamport in 1986 and concerned with proofs of correctness of solution to a mutual exclusion problem, and the other introduced by Janicki and Koutny in 1991 gives a partial order semantics for concurrency. We will briefly present both of the approaches.

Janicki and Koutny have a general motivation to study and model concurrent systems using various types of partial orders and their generalizations. For that purpose they needed to have a mathematical structure that can uniquely represent a concurrent history when all system runs are modelled by interval orders.

In Lamport's case, the problem of managing shared resources and synchronization for multiple processes in a concurrent system was interesting rom both practical and theoretical perspective. Mutual exclusion problem is an abstraction of the shared resources management problem and lies at the heart of concurrency theory.

Both motivations have resulted in introduction of a single mathematical object, an interval order structure.

We will describe the developments in reasoning about concurrent processes in this chapter, starting from Dijkstra's "cyclic" computations and going into Lamport's more refined view of those computations, separating stages of concurrent process into noncritical, critical and trying sections. At the end we present a case for use of main subject of this thesis, relational structures, as a tool to devise constructive correctness proofs for a version of bakery algorithm.

But first we will recall some basic mathematical definitions that are used in motivation.

1.1 Mathematical basics: Relations, Partial Orders and Transitive Closure

In this section we present relatively well known mathematical concepts and results that will be used frequently in the thesis, cf. Fishburn (1985) and Rosen (2002).

Definition 1. *Let X be a set and $R_1, R_2, R, Q \subseteq X \times X$ be relations on X. We define*

1. *the* composition operator $\circ$ *on these two relations as*

$$R_1 \circ R_2 \overset{df}{=} \big\{(a,b) \mid \exists x \in X.\ (a,x) \in R_1 \wedge (x,b) \in R_2\big\}.$$

2. *the* identity *relation as*

$$id_X \overset{df}{=} \{(a,a) \mid x \in X\}.$$

3. *$R^0 = id_X$ and $R^n = R^{n-1} \circ R$, for all $n \geq 1$.*

4. *a composition operator over Q, $\circ_Q$, as*

$$R_1 \circ_Q R_2 \stackrel{df}{=} \{(a,b) \mid \exists (x,y) \in Q. \big((a,x) \in R_1 \wedge (x,b) \in R_2\big) \bigwedge \big((a,y) \in R_1 \wedge (y,b) \in R_2\big)\}.$$

5. *the* inverse *of R as* $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

Definition 2 (Basic Classification)**.** *A relation $R \subseteq X \times X$ is:*

1. Reflexive *if $id_X \subseteq R$;*

2. Irreflexive *if $id_X \cap R = \varnothing$;*

3. Symmetric *if $R = R^{-1}$;*

4. Asymmetric *if $R \cap R^{-1} = \varnothing$;*

5. Transitive *if $R \circ R \subseteq R$;*

6. *An* equivalence relation *if it is symmetric, transitive and reflexive;*

The most important relation in this thesis is *partial order*, which is defined below.

Definition 3 (Partial Order)**.**

1. *A relation $< \subseteq X \times X$ is called a (*strict*)* partial order*, if it is irreflexive and transitive, i.e. for all $a,b,c \in X$ we have: $a \not< a$ and $a < b < c \implies a < c$.*

2. *For a given partial order $<$, a relation $\frown_<$, defined as*

$$a \frown_< b \stackrel{df}{\iff} \neg(a < b) \wedge \neg(b < a) \wedge a \neq b$$

is interpreted and incompatibility.

3. *For a given partial order* $<$*, a relation* $<^{\frown}$*, defined as*

$$a <^{\frown} b \overset{df}{\iff} a < b \lor a \frown_{<} b$$

is interpreted as 'not less than'.

The theory of partial orders is very rich and complex, however in this thesis we will use only three types defined below.

Definition 4 (Type of Partial Orders)**.** *Let* $<$ *be a partial order on a set* X*. Then* $<$ *is:*

1. Total *if* $\frown_{<} = \emptyset$*. In other words,* $\forall a,b \in X$*:* $a < b \lor b < a \lor a = b$*;*

2. Stratified *if* $\forall a,b,c \in X$*:* $a \frown_{<} b \frown_{<} c \implies a \frown_{<} c \lor a = c$*, i.e., the relation* $\frown_{<} \cup id_X$ *is an equivalence relation on* X*;*

3. Interval *if* $\forall a,b,c,d \in X$*:* $a < c \land b < d \implies a < d \lor b < c$*.*

It follows directly from these definitions that every total order is also stratified and every stratified order is also interval. Figure 2.1 illustrates the above definition. We will reserve the symbol $\lhd$ to denote total orders.

Definition 5 (Extension)**.** *Given a partial order* $< \subseteq X \times X$*, a relation* $<' \subseteq X \times X$ *is an* extension *of* $<$ *if* $< \subseteq <'$*.*

For convenience, we separately define the set

$$\mathsf{Total}(<) \overset{df}{=} \{\lhd \subseteq X \times X \mid \lhd \text{ is a total order and } < \subseteq \lhd\}.$$

In other words, the set $\mathsf{Total}(<)$ consists of all the *total order extensions* of $<$.

Definition 6 (Closures)**.** *Let* $R \subseteq X \times x$ *be a relation. We define:*

1. *The* reflexive closure *of R as $R \cup id_X$;*

2. *The* transitive closure *of R as $R^+ \stackrel{df}{=} \bigcup_{i \geq 1} R^i$;*

3. *The* reflexive and transitive closure *of R as $R^* \stackrel{df}{=} \bigcup_{i \geq 0} R^i = R^+ \cup id_X$; and*

4. *The* irreflexive and transitive closure *of R as $R^{\curlywedge} \stackrel{df}{=} R^+ \setminus id_X = R^* \setminus id_X$;*

5. *The* symmetric closure *of R as $R^{\bowtie} \stackrel{df}{=} R \cup R^{-1}$.*

Definition 7 (Acyclicity)**.** *A relation R is* acyclic *if its transitive closure is asymmetric.*

Note, that we can now define stratified orders using symmetric closure.

Corollary 1. *A relation R is a* stratified order *if it is a partial order, such that $X \times X \setminus R^{\bowtie}$ is an equivalence relation.* ■

1.2 Lamport's Model

1.2.1 Mutual Exclusion Problem

The *mutual exclusion (ME)* problem was first formally stated and solved by Edsger Dijkstra in (Dijkstra, 1965).

The problem: Given N processes involved in a computation, and each process's computation path containing a *critical section (CS)*, i.e. the a region in computation, that at any given moment in time, only one process can access this section, devise a fair algorithm that can guarantee every process exclusive access to its CS.

Definition 8 (Mutual Exclusion Property)**.** *Given N processes in the system, for no two processes i and j the execution of their critical section CS_i and CS_j can be observed concurrently.*

The high-level algorithm for a process that executes the computation is given below. Keep in mind that in a concurrent system there are N such processes and each of them executes the code of the same structure.

initial declaration;
repeat forever
  noncritical section;
  trying;
  critical section;
  exit;
end repeat

Later developments of the subject established standard requirements for such an algorithm. It must possess the basic set of properties defined formally below.

Deadlock freedom: If there exists a nonterminating trying section execution, then there exists an infinite number of critical section executions.

The deadlock freedom (later referred to as DF) guaranties the continual progress of the computation, but it does not include the case, when one of the processes in stuck in the infinite repeat of a trying section without eventual access to CS. This condition is stated as:

Livelock freedom: Every trying operation execution must terminate.

The livelock freedom (later referred to as LF) guarantees that every process that is trying to access CS with eventually be granted that access. The next requirement is the strongest fairness requirement that can be imposed on concurrent processes:

First-Come-First-Served: For any pair of processes i and j and any execution of CS_j, if i requested access to CS earlier than j, then execution of CS_i is granted earlier than execution of CS_j.

We will later refer to that requirement as FCFS, and in the literature, especially queuing theory, it is sometimes called FIFO (First-In-First-Out). We now present a useful result showing relations between those requirements.

Theorem 1 (Lamport (1986)). *If both Deadlock freedom and FCFS are satisfied, then Livelock freedom is also satisfied.* ■

1.2.2 Bakery Algorithm

The bakery algorithm presented here describes the procedure for fair, in the sense of Definitions 1.2.1-1.2.1, access to shared resources and solves the mutual exclusion problem for the most general case of N processes.

We assume processes are communicating through reading and writing into communication variables, or, simply, shared variables. Every shared variable has an owner - the process that can write into this shared variable, and we restrict that no other process can write into that variable. When these variables grow in size, it becomes possible to execute read and write concurrently, but this discussion is outside of the scope of the thesis.

When process wants to enter a critical section, it picks a ticket with number on it and waits until his number is being called. The next process that wants to enter the critical section takes the consecutive ticket with a consecutive number. If two processes come to request access to critical section simultaneously, we break the ties by comparing an issued number (num_i, num_j) with process's own number (i, j), so that the process with lower number accesses the critical section first. Formally, processor i cannot enter critical section until the one of the conditions $num_j > num_i$ or $num_j = num_i \wedge j > i$ is met.The pseudocode is given in Algorithm 1. Lines $2-9$ represent the trying protocol to access the critical section, where line $2-4$ are the bakery's "doorway", and $4-9$ are "the bakery". Line 10 represents the exit protocol.

Algorithm 1 Lamport's Bakery Algorithm

shared variables:

$num[1..N]$ ▷ array of integers initially all 0's (ticket number)

$dw[1..N]$ ▷ array of Booleans, initially all False (doorway)

Process i owns num_i and dw_i

Program for process i

local variables:

$j : 1..N$

repeat

NCS ▷ noncritical section

$dw_i \leftarrow True$ ▷ open and enter the doorway

$num_i \leftarrow 1 + max\{num_j : 1 \leq j \leq N\}$ ▷ take the ticket

$dw_i \leftarrow False$ ▷ close the doorway

for $j \leftarrow 1$ **to** N **do begin** ▷ start trying

  while dw_j **do skip**

  while $num_j \neq 0$ **and** $\langle num_j, j \rangle < \langle num_i, i \rangle$ **do skip**

end

CS ▷ critical section

$num_i \leftarrow 0$ ▷ exit

forever

1.2.3 Formalization of the Bakery Algorithm

Definition 9 (Process execution)**.** *A* process execution *is a tuple* $\Gamma = (X, \prec, \sqsubset)$*, where:*

1. $X = \bigcup_{1 \leq i \leq n} X_i$ *and every* $X_i \in X$ *is a countable set of operations of the process i;*

2. $\prec$ *and* $\sqsubset$ *are binary relations on* X*.*

Two binary relations $\prec$ and $\sqsubset$ represent causal (temporal) ordering of operations. We say $\prec$ is a causality and $a \prec b$ iff a is "earlier than" b, i.e. a ends before b. And $\sqsubset$ is a weak causality and $a \sqsubset b$ iff a is "not later than" b, i.e. a starts "not later than" b or a is earlier than or concurrent with b. In other terms, if a causally affects b we say $a \prec b$, and if a could causally affect b, we say $a \sqsubset b$.

Lamport suggests the following general axiomatization of these relations.

Definition 10 (Lamport's axiomatization)**.** *Given two binary relations* $\prec$ *and* $\sqsubset$ *on* X*, where* X *is a set of all elementary operations of every process, and an execution* Γ*, given by relational structure* $\Gamma = (X, \prec, \sqsubset)$*, for every elementary operation* a, b, c, d*:*

A1: $a \not\prec a$

A2: $a \prec b \implies a \sqsubset b$

A3: $a \prec b \implies b \not\sqsubset a$

A4: $a \sqsubset b \prec c \vee a \prec b \sqsubset c \implies a \sqsubset c$

A5: $a \prec b \sqsubset c \prec d \implies a \prec d$ ∎

Axiom $A1$ states that $\prec$ is an irreflexive partial order. Axiom $A2$ states that if one operation is earlier than the other it is also not later then the other. Axiom $A3$ states that if it is known that one operation precedes the other, it cannot be that the other one is concurrent or in reverse order. Axioms $A4$ and $A5$ give tools for reasoning about interplay of two relations.

Axioms $A6-A7$ are additional axioms to construct proofs for reasoning about sets of operations, rather than elementary ones.

A6: $\{e \mid e \sqsubset a\}$ is finite

A7: $a, b \in X_i \implies a \prec b \vee b \prec a$ ∎

Axiom $A6$ asserts that all events in the system execution have started at initial point in time and this moment is not infinitely back into the past. Axiom $A7$ states that if operations a and b belong to the same process, then one must precede the other. It is fairly straightforward, that operations in one single process must be linearly ordered.

We follow with the rest of Lamport axiomatic characterization for process' communication and presenting axioms describing the behaviour of operations $Read(x)$ and $Write(x)$. Let variable x be a process' shared variable.

A8: $\exists \mathsf{W} \in Write(x).\forall \mathsf{R} \in Read(x) : \mathsf{W} \prec \mathsf{R}$

A9: Let $\mathsf{R} \in Read(x).\forall \mathsf{W} \in Write(x) : \mathsf{W} \prec \mathsf{R} \vee \mathsf{R} \prec \mathsf{W}$. Let W^{max} be the maximum element over relation $\prec$ of the set $X = \{\mathsf{W} \mid \mathsf{W} \in Write(x) \wedge \mathsf{W} \prec \mathsf{R}\}$. Then R returns the value x, written by operation W^{max}. ∎

Axiom $A8$ states that there is always a write operation that sets the initial value of the shared variable and it precedes the first read operation for that value. Axiom $A9$ states that any read operation that is mutually exclusive with write operation for the same variable must return the value written to that variable by the last read operation.

Now lets extend this axiomatization to reason about sets of operations, i.e. we get rid of atomicity restriction. Then the causality $\prec$ and weak causality $\sqsubset$ over arbitrary operations will be redefined as follows.

Definition 11. *Let $A \subseteq X$ and $B \subseteq X$ be arbitrary sets of operations. Then:*

1. $A \prec B \overset{df}{=} \{\forall a \in A, \forall b \in B \mid a \prec b\}$

2. $A \sqsubset B \overset{df}{=} \{\exists a \in A, \exists b \in B \mid a \sqsubset b\}$

Definition 11 provides tools for reasoning about correctness of algorithms at a higher abstraction levels. Given that the system is defined by the set of its executions, the correctness of bakery algorithm can be extended from single execution level to the level of the whole distributed system.

Axioms $A1 - A5$ continue to hold for sets of operations. Axiom $A7$ does not hold, and $A6$ holds in weaker form (see Proposition 1(4)).

Definition 12 (Immediate successor)**.** *Let $(X, \prec)$ be a partial order. Let a and b be two elements in X and let $a \prec b$. They are called* successive *iff there is no such element c that $a \prec c \prec b$.*

Element b is sometimes called an immediate successor of a.

Proposition 1. *Let two operations $\mathsf{W}, \mathsf{W}' \in Write(x)$ s.t. $\mathsf{W} \prec \mathsf{W}'$, s.t. W' is an immediate successor of W (within one process). Then:*

1. *Let $\mathsf{R} \in Read(x)$, s.t. $\mathsf{W} \prec \mathsf{R} \prec \mathsf{W}'$. Then R returns the value written by W.*

2. *Let W be the last $Write(x)$ operation and R is a read operation s.t. $\mathsf{W} \prec \mathsf{R}$, then R returns the value written by W.*

3. *If an execution has finitely many $Write(x)$ operations and infinitely many $Read(x)$ operations, then the set of $Read(x)$ operations that return a value other than that written by the last $Write(x)$ is finite.*

4. *For any operation A (not necessarily terminating) the set $\{B \mid B \prec A\}$ is finite.* ■

Note, that this model assumes that concurrent reads of shared resources do not affect each other and there is only one writer. So the correctness proof essentially checks the case of reads overlapping one or more writes.

1.2.4 Proof of Correctness

We first provide some useful notation:

1. n_i is a line n of bakery algorithm executed by process i

2. $n_i : \mathsf{R}(x = v)$ represents a process i reading variable x in line n of i's algorithm and the value returned by this read is v.

3. $n_i : \mathsf{W}(x \leftarrow v)$ represents a process i writing value v into variable x in line n of i's algorithm.

Also note, that in Algorithm 1, line numbering $1-10$ provides a useful insight into linear ordering behind the operations within the same process.

Lemma 1. *Suppose some process i is in the CS, while some other process k, s.t. $i \neq k$, is in the bakery. Formally, $4_k \sqsubset 9_i \sqsubset 9_k$. For $p \in \{i,k\}$, let v_p be the value that process p writes into num_p in line 3. Then $\langle v_i, i\rangle < \langle v_k, k\rangle$*

Proof. Since i is in CS, we know it has already completed its "trying" cycle at 6_i (since operations within the process are linearly ordered, see $A7$). Thus, it has executed the read $6_i : \mathsf{R}(dw_k = False)$, and so it cannot be that i did it in this order:

$$2_k \prec 6_i : \mathsf{R}(dw_k = False) \prec 4_k. \quad (^*)$$

Because if that would be the case, by Proposition 1(1), i would have found $dw_k = True$ in line 6, i.e. the operation would have been $6_i : \mathsf{R}(dw_k = True)$, not $6_i : \mathsf{R}(dw_k = False)$. The register dw_k's value could be $True$ only while k was in the "doorway" of the bakery (lines 2_k to 4_k), and it must have been not the case at the time of i "trying". Meaning, it has to be that i has entered the CS either before/concurrent with k's "doorway" or after/concurrent with k's "doorway". Hence, by $A3$ and (*), either $6_i \sqsubset 2_k$ or $4_k \sqsubset 6_i$. Then we have two cases to consider:

(1) Case 1. Looking at num_k's behaviour. Suppose $4_k \sqsubset 6_i : \mathsf{R}(dw_k = False)$. Then by $A7$ we can extend it to

$$3_k : W(num_k \leftarrow v_k) \prec 4_k \sqsubset 6_i : \mathsf{R}(dw_k = False) \prec 7_i : \mathsf{R}(num_k = v)$$

for some value v. By applying $A5$ we get

$$3_k : W(num_k \leftarrow v_k) \prec 7_i : \mathsf{R}(num_k = v). \quad (^{**})$$

But we also have $9_i \sqsubset 9_k$ from problem statement, so by applying $A7$ we get

$$7_i : \mathsf{R}(num_k = v) \prec 9_i \sqsubset 9_k \prec 10_k.$$

So, by $A5$, we get $7_i \prec 10_k$ and therefore by $A7$ and $(^{**})$ we have

$$3_k : \mathsf{W}(num_k \leftarrow v_k) \prec 7_i : \mathsf{R}(num_k = v) \prec 10_k : \mathsf{W}(num_k \leftarrow 0).$$

So it must be by Proposition 1(1) that $v = v_k$. That is, i read the value written by k into num_k in line 3_k. Note, that $v_k > 0$, because k always writes a positive number into num_k and 3_k must have executed by now. Since i is in CS, it must have already executed $7_i : \mathsf{R}(num_k = v)$ for some v, and it read either $v = 0$ or $\langle v_i, i\rangle < \langle v, k\rangle$. But we just showed $v = v_k \neq 0$, so it must be that $\langle v_i, i\rangle < \langle v, k\rangle = \langle v_k, k\rangle$, as we wanted.

(2) Case 2. Looking at num_i's behaviour. Suppose $6_i : \mathsf{R}(dw_k = False) \sqsubset 2_k$. By applying $A7$ we have

$$3_i : \mathsf{W}(num_i \leftarrow v_i) \prec 6_i : \mathsf{R}(dw_k = False) \sqsubset 2_k \prec 3_k : \mathsf{R}(num_i = v)$$

for some v. Note, that k could only read num_i in 3_k. And by $A5$ it becomes

$$3_i : \mathsf{W}(num_i \leftarrow v_i) \prec 3_k : \mathsf{R}(num_i = v). \quad (^{***})$$

From the problem statement we have $4_k \sqsubset 9_i$, and by applying $A7$ we get

$$3_k : R(num_i = v) \prec 4_k \sqsubset 9_i \prec 10_i : \mathsf{W}(num_i \leftarrow 0).$$

So by $A5$, we get

$$3_k : \mathsf{R}(num_i = v) \prec 10_i : \mathsf{W}(num_i \leftarrow 0).$$

Combining it with $(^{***})$ we get

$$3_i : \mathsf{W}(num_i \leftarrow v_i) \prec 3_k : \mathsf{R}(num_i = v) \prec 10_i : \mathsf{W}(num_i \leftarrow 0).$$

By Proposition 1(1), $v = v_i$. So k read value v_i in 3_k. Therefore, it must be $v_k > v_i$, since algorithm sets the next picked number to be greater than the maximum preceded number. Since $v_i > 0$ after write at 3_i and before write in 10_i, we have $\langle v_i, i\rangle < \langle v_k, k\rangle$, as desired.

■

Theorem 2. *The bakery algorithm satisfies mutual exclusion (ME), deadlock freedom (DF) and first-come-first-served (FCFS) condition.*

Proof. Let v_p denote the value written by p into num_p in line 3, i.e. $3_p : \mathsf{R}(num_p \leftarrow v_p)$.

1. **ME.** This property follows directly from Lemma 1.2.4, since if two processes i and k are both in critical section, then both $\langle v_i, i\rangle < \langle v_k, k\rangle$ and $\langle v_k, k\rangle < \langle v_i, i\rangle$ would hold, which is a contradiction.

2. **FCFS.** Suppose, by contradiction, k finishes "the doorway" before i starts it, but FCFS doesn't hold and i enters the CS before k does, i.e. formally $4_k \prec 2_i \wedge 9_i \prec 9_k$. Given some value v that i reads in num_k on line 3, we apply $A7$ and get

   $$3_k : \mathsf{W}(num_k \leftarrow v_k) \prec 4_k \prec 2_i \prec 3_i : \mathsf{R}(num_k = v) \prec 3_i : \mathsf{W}(num_i \leftarrow v_i) \prec 9_i. \; (^*)$$

   Furthermore, by applying $A7$ to problem statement,

   $$9_i \prec 9_k \prec 10_k : \mathsf{W}(num_k \leftarrow 0).$$

   By combining this with $(^*)$, and applying $A1$ and $A5$,

   $$3_k : \mathsf{W}(num_k \leftarrow v_k) \prec 3_i : R(num_k = v) \prec 10_k : \mathsf{W}(num_k \leftarrow 0)$$

   and therefore, by Proposition 1(1), $v = v_k$. Since the value v_i that i writes into num_i in line 3 is larger than any value it reads on line 3 (see $(^*)$), it follows that $v < v_i$ and therefore, $v_k < v_i$. By the problem statement, we have $4_k \prec 2_i \prec 9_i \prec 9_k$, and by $A1$ and $A2$ this implies $4_k \sqsubset 9_i \sqsubset 9_k$. By Lemma 1.2.4 this, in turn, implies $\langle v_i, i\rangle < \langle v_k, k\rangle$, which contradicts that $v_k < v_i$, thus FCFS holds.

3. **DF.** Suppose, by contradiction, that there is a deadlock. Thus, eventually there is a set of processes T executing nonterminating trying protocols, while the remaining processes $\overline{T}$ are executing nonterminating noncritical sections (NCS). We claim, that for each process i, there is only a finite number of $Write(num_i)$ and $Write(dw_i)$ operations. To see this, first consider a process $i \in \overline{T}$. Since there are no write operations of i into num_i or dw_i in NCS, we have by $A7$ and $A8$ that $\mathsf{W} \prec$ NCS, where W is any write operation of i into num_i or dw_i and NCS is a nonterminating operation of process i. By Proposition 1(4) the set $\{\mathsf{W} \mid \mathsf{W} \prec NCS\}$ is finite, so none of the processes in $\overline{T}$ cause a deadlock in this algorithm. Regarding the case of process $i \in T$ the claim follows similarly. Lets observe that there are no write operations for num_i or dw_i in lines 6_i and 7_i, which are the only two potentially nonterminating operations inside the trying protocol. Next, we claim that no $i \in T$ can be executing a nonterminating 6_i. For, as was just argued, for any j there are only finitely many operations for dw_j and the last such operation sets $dw_j \leftarrow False$. Therefore, by Proposition 1(3), there can't be infinitely many $Read(dw_j)$ operations that return $True$. And only nonterminating $True$ as the value of dw_j would have made $i \in T$ to executed a nonterminating line 6. Therefore, if there still exists a nonterminating process, it is a $i \in T$ executing a nonterminating 7_i. Now pick $i \in T$ such that $\forall k \in T \setminus \{i\}$, $\langle v_i, i\rangle < \langle v_k, k\rangle (^*)$. As just argued, i must be executing a nonterminating 7_i. Thus, for some j, there are infinitely many $7_i : \mathsf{R}(num_j = v)$ operations where $v \neq 0 (^{**})$ and $\langle v, j\rangle < \langle v_i, i\rangle (^{***})$. As we argued before, there is only a finite number of assignments to num_j.

   (1) Case 1. $j \in \overline{T}$. Then the last assignment to num_j is in line 10_j and sets $num_j \leftarrow 0$. Therefore, by Proposition 1(3), there can't be infinitely many $7_i : \mathsf{R}(num_j = v)$ operations with $v \neq 0$, which contradicts $(^{**})$.

   (2) Case 2. $j \in T$. Then the last assignment to num_j is in line 3_j and sets $num_j \leftarrow v_j$.

Therefore, by Proposition 1(3), we must have $v = v_j$. But by choice of i in $(^*)$ and the fact that $j \in T$, we have $\langle v_i, i\rangle \leq \langle v_j, j\rangle = \langle v, j\rangle$, contradicting $(^{***})$. Therefore, there can't be a deadlock.

∎

Note, that since bakery algorithm satisfies DF and FCFS, by Theorem 1 it also satisfies live lock freeness (LF).

The problem with the bakery algorithm is that shared variables num_i has to be of an unbounded size. Even if we assume that any read must return a value that was previously written or is being written concurrently with that read, in an execution in which the bakery is never empty, the value of num_i chosen by each process i will keep increasing forever. But for the theoretical setting this restriction can be neglected.

There exists a number of mutual exclusion algorithms that work in the model of nonatomic read and write operations that have been developed after presented algorithm. The ultimate goal of those algorithms is to achieve maximum possible degree of fairness (ideally, FCFS) with as few shared bits per process as possible. The bakery algorithm presented here achieves the first goal, but requires shared variables that can hold unboundedly large integers.

We have successfully showcased the proof of correctness that requires use of relational structures to prove mutual exclusion property of an algorithm for N concurrent processes. We believe that this example gives a good motivation for an in-depth investigation of relational structures and understanding how this theory allows to reason about concurrent behaviours.

1.3 Janicki-Koutny Motivation

Consider the following simple program written using Dijkstra's *cobegin*'s and *coend*'s, which is also illustrated in Figure 1.1:

```
Q: cobegin
     a :  begin worka; lock(r) end;
     b :  begin unlock(r); workb end;
     [ ]
     c :  workc
     coend
```

Assume that the subroutines a, b and c are atomic, *worka*, *workb* and *workc* require the resource r, which can be used simultaneously by any finite number of subroutines. The resource r is initially unlocked and available to use.

The program Q illustrates the difficulties of modelling 'simultaneity' and 'not later than' relationships when no restrictions on the shape of the system runs is assumed.

The program's inhibitor Petri net[1] representation N_Q is given in Figure 1.1. For both the program Q and the net N_Q, all possible observations (system runs) that involve all three events a, b, c are represented by the set of partial orders $Obs(Q) = \{<_1, <_2, <_3, <_4\}$.

How can we say which observations are equivalent without knowing the details of particular events? Such sets of equivalent (from concurrency viewpoint) observations are often called *concurrent histories* or *concurrent behaviours*. How can they be defined without going to the details of particular models? The key to solution is the idea of an *observational invariant* (c.f. Janicki and Koutny (1993)).

[1] Inhibitor Petri nets are now a part of popular folklore knowledge. They are simply the nets with inhibitor arcs, where the inhibitor arc forbids the execution of transition c if there is a token in place s_3.

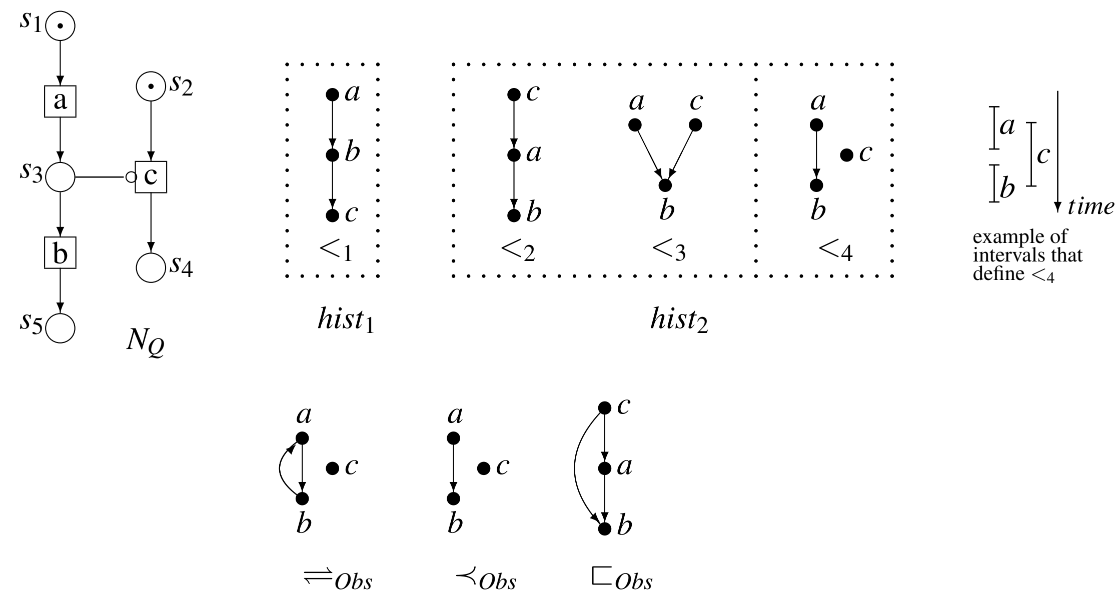


Figure 1.1: Inhibitor net N_Q and all behaviours involving one occurrence of a, b and c. The net N_Q generates $Obs(Q) = \{<_1, <_2, <_3, <_4\}$, and two concurrent histories: $hist_1$ and $hist_2$. Partial orders are represented by Hasse diagrams. Note that $\rightleftharpoons_{Obs} = \prec_{Obs} \cup \prec_{Obs}^{-1}$ and $\prec_{Obs} = \rightleftharpoons_{Obs} \cap \sqsubset_{Obs}$.

For a given set of observations $Obs = \{<_1, \ldots, <_n\}$, where each $<_i$ is an interval order over some set of event occurrences X, an *observational invariant* is a property that can be defined in terms of partial orders and is satisfied by all elements of *Obs*. It was shown in Janicki and Koutny (1993) that we have only two such independent invariants,

- *interleaving*, $\rightleftharpoons_{Obs}$, defined as $x \rightleftharpoons_{Obs} y \iff \forall <_i \in Obs.\ x <_i y \vee y <_i x$, and

- *weak causality*, $\sqsubset_{Obs}$, defined as $x \sqsubset_{Obs} y \iff \forall <_i \in Obs.\ x <_i y \vee x \frown_{<_i} y$.

The most obvious and popular invariant, *causality*, $\prec_{Obs}$, is defined as

$$x \prec_{Obs} y \iff \forall <_i \in Obs.\ x <_i y,$$

can be derived from interleaving and weak causality as

$$x \prec_{Obs} y \iff x \rightleftharpoons_{Obs} y \wedge x \sqsubset_{Obs} y.$$

Let $<$ be any interval order on X. We say that $<$ is an *extension* of

- the invariant $\rightleftharpoons_{Obs}$ iff $x \rightleftharpoons_{Obs} y \iff x < y \vee y < x$,
- the invariant $\sqsubset_{Obs}$ iff $x \sqsubset_{Obs} y \iff x <^{\frown} y$, and
- the invariant $\prec_{Obs}$ iff $x \prec_{Obs} y \iff x < y$.

Clearly $<$ is an extension of $\prec_{Obs}$ iff it is and extension of both $\rightleftharpoons_{Obs}$ and $\sqsubset_{Obs}$. Note also that every $<_i$ from *Obs* is an extension of both $\rightleftharpoons_{Obs}$ and $\sqsubset_{Obs}$.

Let Obs^{inv} be the set of all interval orders over the set X that are extensions of both $\rightleftharpoons_{Obs}$ and $\sqsubset_{Obs}$. The set Obs^{inv} is called a *closure of Obs with respect to the observational invariants*. Clearly $Obs \subseteq Obs^{inv}$.

We say that a set of observations *Obs* is a *concurrent history* (i.e. all elements of *Obs* are equivalent observations) iff $Obs = Obs^{inv}$.

The above equality can be verified by inspection that the set $Obs(Q)$ is split into *two* concurrent histories $hist_1$ and $hist_2$, both shown in Figure 1.1.

Note that for the case of Figure 1.1 we have $\rightleftharpoons_{Obs} = \prec_{Obs} \cup \prec_{Obs}^{-1}$. This is not true in general, but occurs in many models of concurrency.

Another important factor that must be included in any general model of concurrency is a treatment of *simultaneity*. This can be done by using the concept of *concurrency paradigms* proposed in Janicki and Koutny (1993).

A *concurrency paradigm* is a supposition or statement about the structure of a history involving a treatment of simultaneity. Lets assume that *Obs* is a *concurrent history* (with the domain X). The classical causality based approach usually stipulates that if there is a run $< \in Obs$ such that $a \frown_{<} b$, then there must be a run such that a precedes b and a run such that b precedes a, and vice versa.

Formally, *concurrency paradigms* are logical formulas defined for event variables x and y by a simple grammar

$$\omega := true|false|\exists < . \, x < y|\exists < . \, x > y|\exists < . \, x \frown_{<} y|\neg\omega|\omega \vee \omega|\omega \wedge \omega|\omega \Rightarrow \omega,$$

A history Obs satisfies a paradigm ω if for all distinct $a, b \in X$, $\omega(a,b)$ holds. It was shown in Janicki and Koutny (1993) that in the study of concurrent histories, we only need to consider eight non-equivalent paradigms, denoted by $\pi_1, \cdots, \pi_8$. Of those eight, only π_1, π_3 and π_8 are important for modelling concurrency. The most general paradigm, $\pi_1 = true$, admits all concurrent histories. The most restrictive paradigm, π_8, admits only concurrent histories Obs such that

$$(\exists <\in Obs. \, x \frown_{<} y) \iff (\exists <\in \Delta. \, x < y) \wedge (\exists <\in \Delta. \, x > y).$$

The paradigm π_3, which is general enough to deal with most problems that cannot be dealt with under π_8, admits concurrent histories Obs such that

$$(\exists <\in \Delta. \, x < y) \wedge (\exists <\in \Delta. \, x > y) \Rightarrow (\exists <\in \Delta. \, x \frown_{<} y).$$

Clearly, $\pi_8 \Rightarrow \pi_3 \Rightarrow \pi_1$, The paradigms determine the way histories can be represented by their relational invariants, see Janicki and Koutny (1993) for detailed treatment.

When the paradigm π_3 holds then for every concurrent history Obs, we have

$$\rightleftharpoons_{Obs} = \prec_{Obs} \cup \prec_{Obs}^{-1},$$

so we can use more natural invariants, causality $\prec_{Obs}$ (i.e. an abstraction of "earlier than") and weak causality $\sqsubset_{Obs}$ (i.e. and abstraction of “not later than") as a unique representation of *Obs*.

In this thesis, except Chapter 8, we will assume that the paradigm π_3 does hold.

It was shown in Janicki and Koutny (1993) that for concurrent histories that assume interval orders as observations, the relations $\prec_{Obs}$ and $\sqsubset_{Obs}$ satisfy all Lamport' s axioms from Definition 10 and an additional axiom that was added later. In Janicki and Koutny (1997) it was proved that two relations $\prec$ and $\sqsubset$ can be interpreted as concurrency invariants if and only if they satisfy these six axioms.

1.4 Thesis structure

The thesis is structured as follows. Chapter 1 provides two major motivations for our research, introduce standard transitive closure operator and provide necessary notation. In Chapter 2 we give the rest of mathematical basis for partial order theory and relational structures used in this thesis. Chapter 3 recaps stratified order structures and their properties, as well as provides analysis for its closure operator $\diamondsuit$-closure. Chapter 4 introduces main achievement of this thesis, a transitive closure for interval order structures, called $\blacklozenge$-closure. We present our main contributions, i.e. proofs of properties of $\blacklozenge$-closure for interval orders structures. Chapter 5 is devoted to generalized mutex order structures structures, a generalization of stratified order structures, and a $\bigstar$-closure operator on them. Chapter 6 gives some concluding remarks on the thesis.

Chapter 2

Mathematical Basics

2.1 Representation Theorem for Partial Orders

Now we are ready to present two most important representation theorems used in this thesis.

Szpilrajn's Theorem (Szpilrajn, 1930) provides the link between a partial order and the set of its total order extensions. By Szpilrajn's Theorem, we know that every partial order $<$ is uniquely represented by the the set of its total order extensions $\mathsf{Total}(<)$. Szpilrajn's Theorem can be stated as follows:

Theorem 3 (Szpilrajn (1930))**.** *For every partial order* $<$*,*

$$< \;=\; \bigcap_{\lhd \in \mathsf{Total}(<)} \lhd.$$

$\blacksquare$

In the most general case, the proof of Szpilrajn Theorem requires use of Axiom of Choice (c.f. Fishburn (1985)).

Stratified orders can also be defined in an alternative way, namely, a partial order $<$ on X is stratified if and only if there exists a total order $\lhd$ on some Y and a mapping $\phi : X \to Y$ such that $\forall x,y \in X.\ x < y \iff \phi(x) \lhd \phi(y)$. This definition is illustrated in Figure 2.1 with partial orders $<_2$ and $\lhd_2$. Mapping ϕ is defined as $\phi : \{a,b,c,d\} \to \{\{a\},\{b,c\},\{d\}\}$ with $\phi(a) = \{a\}$, $\phi(b) = \phi(c) = \{b,c\}$, $\phi(d) = \{d\}$. Note that for all $x,y \in \{a,b,c,d\}$ we have $x <_2 y \iff \phi(x) \lhd_2 \phi(y)$, where the total order $\lhd_2$ can be concisely represented by a *step sequence* $\{a\}\{b,c\}\{d\}$. As a consequence, stratified orders and step sequences can uniquely represent each other (cf. (Janicki and Koutny, 1995; Janicki and Le, 2008; Le, 2008)).

For the representation of interval orders, the name and intuition follow from Fishburn's Theorem:

Theorem 4 (Fishburn (1970))**.** *A partial order $<$ on X is* interval *iff there exists a total order $\lhd$ on some T and two mappings $B,E : X \to T$ such that for all $x,y \in X$,*

1. *$B(x) \lhd E(x)$, and*
2. *$x < y \iff E(x) \lhd B(y)$.* ∎

Usually $B(x)$ is interpreted as the beginning and $E(x)$ as the end of an *interval* x. Totally ordered set T usually represents time and taken as the real line $\mathbb{R}$. The intuition of Fishburn's theorem is illustrated in Figure 2.1 with partial orders $<_3$ and $\lhd_3$. We define mappings B and E as $B,E : \{a,b,c,d\} \to \{B(a),E(a),B(b),E(b),B(c),E(c),B(d),E(d)\}$. Then for all $x,y \in \{a,b,c,d\}$, we have $B(x) \lhd_3 E(x)$ and $x <_3 y \iff E(x) \lhd_3 B(y)$.

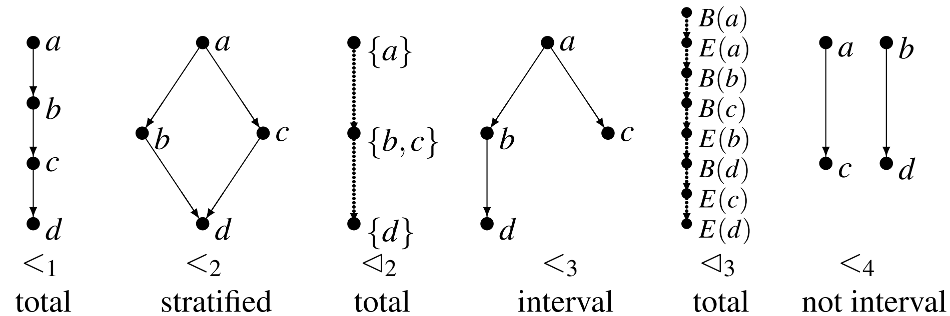


Figure 2.1: Various types of partial orders (represented as Hasse diagrams). The partial order $<_1$ is an extension of $<_2$, $<_2$ is an extension of $<_3$, and $<_3$ is and extension of $<_4$. Note that order $<_1$, being total, is uniquely represented by a sequence *abcd*, the stratified order $<_2$ is uniquely represented by a step sequence $\{a\}\{b,c\}\{d\}$, and the interval order $<_3$ is (*not* uniquely) represented by a sequence that represents $\lhd_3$, i.e. $B(a)E(a)B(b)B(c)E(b)B(d)E(c)E(d)$.

2.2 Properties of Transitive Closure

The following, well known, results regarding transitive closure R^+ and reflexive transitive closure R^* will often be used in our proofs (see Hopcroft *et al.* (2003); Rosen (2002) for details).

Lemma 2. *For every binary relation $R \subseteq X \times X$, we have*

1. $(R^*)^* = R^*$, $R^* \circ R^* = R^*$,
2. $R^+ = R^* \circ R = R \circ R^*$
3. $R^+ = (R^+ \circ R) \cup R$, $R^+ \circ R = R \circ R^+$,
4. $R^* \setminus id_X = R^+ \setminus id_X$,
5. $(R^* \setminus id_X)^* \setminus id_X = R^* \setminus id_X$. ■

The below propositions summarize properties of transitive closure operator $^+$. These properties were extended to the $\Diamond$-closure operator for stratified order structures in (Janicki and Koutny, 1995) and will be extended to the $\blacklozenge$-closure operator for interval order structures in Chapter 3 (which constitute main achievements of this thesis), and $\bigstar$-closure in Chapter 4.

Proposition 2 (Properties of transitive closure +)**.** *Let $R \subseteq X \times X$. Then*

1. *If R is irreflexive then $R \subseteq R^+ \setminus id_X$;*
2. *$(R^+)^+ = R^+$;*
3. *R^+ is a partial order $\iff$ R is acyclic $\iff$ R^+ is irreflexive,*
4. *If R is a partial order then $R^+ = R$;*
5. *If R is a partial order and $\hat{R} \subseteq R$, then $\hat{R}^+$ is a partial order and $\hat{R}^+ \subseteq R$.* ■

Now, we recall definition of relational order structures.

Definition 13 (Relational structures)**.** *A tuple of relations $S = (X, R_1, R_2, \ldots, R_n)$ where each $R_i \subseteq X \times X$ is a binary relation on some X, with $1 \leq i \leq n$, is an n-ary* relational structure*. The set X is called the* domain *of a relation structure S.*

We also generalize the definition of extensions for partial orders (see Definition 5) to relational structures.

Definition 14 (Extension of Relational Structures)**.** *An* extension *of relational structure $S = (X, R_1, R_2, \ldots, R_n)$, where each $R_i \subseteq X \times X$, is any relational structure $S' = (X, R'_1, R'_2, \ldots, R'_n)$, satisfying $R_i \subseteq R'_i$, for every $1 \leq i \leq n$. We write $S \subseteq S'$ to denote extension of S.*

Chapter 3

Closure Operator for Stratified Order Structures

In this chapter we introduce the original concept of stratified order structures and present their basic properties. Next we will discuss the concept and properties of $\Diamond$-closure, introduced in Janicki and Koutny (1995).

3.1 Stratified Order Structures and $\Diamond$-Closure

In this section we will present the basic properties of stratified order structures and give relevant $\Diamond$-closure definitions and properties.

In this thesis we are particularly interested in relational structures of two relations. For reference we give a simplified definition. A relational structure of (order 2) is a triple $S = (X, R_1, R_2)$, where $R_1, R_2 \subseteq X \times X$ are binary relations on X. A relational structure $S' = (X, R_1', R_2')$ is an *extension* of $S = (X, R_1, R_2)$ if $R_1 \subseteq R_1'$ and $R_2 \subseteq R_2'$. If S' is an extension of S, we write $S \subseteq S'$.

Definition 15 (Gaifman and Pratt (1987); Janicki and Koutny (1991))**.** *A* stratified order structure (SO-structure) *is a relational structure* $S = (X, \prec, \sqsubset)$, *such that for all* $a, b, c \in X$, *the following hold:*

S1: $a \not\sqsubset a$

S2: $a \prec b \implies a \sqsubset b$

S3: $a \sqsubset b \sqsubset c \wedge a \neq c \implies a \sqsubset c$

S4: $a \sqsubset b \prec c \vee a \prec b \sqsubset c \implies a \prec c$ ■

SO-structures were independently introduced in (Gaifman and Pratt, 1987) and (Janicki and Koutny, 1991). Their comprehensive theory has been presented in (Janicki and Koutny, 1997) and Janicki (2008). They have been successfully applied to model inhibitor and priority systems, asynchronous races, synthesis problems, etc., see Kleijn and Koutny (2004) and Janicki (2008) for more references.

The relation $\prec$ is called *causality* and represents the "earlier than" relationship, and the relation $\sqsubset$ is called *weak causality* and represents the "not later than" relationship. The axioms $S1$ - $S4$ model the mutual relationship between "earlier than" and "not later than" relations, provided that the system runs are defined as stratified orders.

In principle, event hough different language was used, the motivation of both Gaifman and Pratt (1987) and Janicki and Koutny (1991) was the same, a concise representation of a concurrent history, i.e. a set of equivalent system runs/observations.

Proposition 3 (Janicki and Koutny (1993))**.**

1. $(X, \prec)$ is a poset, $a \prec b \Rightarrow b \not\sqsubset a$, and $a \sqsubset b \Rightarrow b \not\prec a$.

2. If $(X, <)$ is a stratified order then $(X, <, <^\frown)$ is a SO-structure. ■

Proposition 3(2) allows an introduction of a *stratified order extension of stratified order structure*.

Definition 16. *1. A stratified order $<$ on X is a* stratified extension *of a SO-structure $S = (X, \prec, \sqsubset)$ if $\prec \subseteq <$ and $\sqsubset \subseteq <^{\frown}$.*

2. The set of all stratified extensions of S will be denoted by $\mathsf{Strat}(S)$. ■

Clearly if a stratified order $<$ is an extension of $S = (X, \prec, \sqsubset)$, then $(X, <, <^{\frown})$ is a relational structure extension of S, in the sense of Definition 14.

One of the main properties of stratified order structures is the following generalization of Szpilrajn's Theorem.

Theorem 5 (Janicki and Koutny (1997))**.** *For every SO-structure $S = (X, \prec, \sqsubset)$:*

$$S = \left(X, \bigcap_{<\in\mathsf{Strat}(S)} <, \bigcap_{<\in\mathsf{Strat}(S)} <^{\frown}\right)$$

. ■

The above theorem is often interpreted as the proof of the claim that SO-structures uniquely represent sets of equivalent system runs provided that the system operational semantics can be fully described in terms of stratified orders (see Janicki (2008); Janicki and Koutny (1997) for details).

Transitive closure operator for relational structures $\diamondsuit$-closure (called "diamond closure"), was first introduced and studied in Janicki and Koutny (1995). We will now present the concept of $\diamondsuit$-closure that plays a substantial role in most of the applications of SO-structures for modelling concurrent systems (cf. Janicki and Koutny (1997); Juhas *et al.*

(2006); Kleijn and Koutny (2004)).

Definition 17 (Janicki and Koutny (1995)). *For every relational structure $S = (X, R_1, R_2)$ we define $S^\Diamond$, the* $\Diamond$-closure *as*

$$S^\Diamond \overset{df}{=} \left(X, \prec^\Diamond_{R_1,R_2}, \sqsubset^\Diamond_{R_1,R_2}\right) = \left(X, (R_1 \cup R_2)^* \circ R_1 \circ (R_1 \cup R_2)^*, (R_1 \cup R_2)^* \setminus id_X\right).$$

■

Intuitively, the $\Diamond$-closure is a generalization of transitive closure for relations to SO-structures. More intuitive definition is that given a relational structure $S = (X, \prec, \sqsubset)$, the $\Diamond$-closure is defined as $(X, \prec', (\prec \cup \sqsubset)^* \setminus id_X)$, where $\prec'$ is a composition of three relations: $(\prec \cup \sqsubset)^*$, $\prec$ and, then again, $(\prec \cup \sqsubset)^*$. In other words, $a \prec' b$ if $aR_1 \circ \cdots \circ R_k b$, where each R_i is either $\prec$ or $\sqsubset$, and at least one R_i is $\prec$. The $\Diamond$-closure is used to construct SO-structures.

The theorem below shows that the $\Diamond$-closure has all the properties formulated for transitive closure in Proposition 2.

Theorem 6 (Properties of $\Diamond$-closure, Janicki and Koutny (1995)). *Let $R_1, R_2 \subseteq X \times X$ be two relations and $S = (X, R_1, R_2)$ be a relational structure. Then*

1. *If R_2 is irreflexive then $S \subseteq S^\Diamond$.*

2. $\left(S^\Diamond\right)^\Diamond = S^\Diamond$.

3. *$S^\Diamond$ is an SO-structure if and only if $\prec^\Diamond_{R_1,R_2} = (R_1 \cup R_2)^* \circ R_1 \circ (R_1 \cup R_2)^*$ is irreflexive.*

4. *If S is a stratified order structure then $S = S^\Diamond$.*

5. *Let S be a SO-structure and let $\hat{S} \subseteq S$. Then $\hat{S}^{\lozenge} \subseteq S$ and $\hat{S}^{\lozenge}$ is a SO-structure.* ■

Among others, Theorem 6 helps us to show a relationship between the SO-structures and *comtraces*, an extension of Mazurkiewicz traces that allows us to model the "not later than" relationship using quotient monoids (Diekert and Rozenberg, 1995; Janicki and Koutny, 1995) of step sequence monoids.

Chapter 4

Closure Operator for Interval Order Structures

4.1 Interval Order Structures and ⧫-closure

This chapter contains the major contribution of this thesis.

First we describe interval order structures, its axiomatization and basic properties. Next we will introduce the concept and properties of ⧫-closure, a counterpart of $\Diamond$-closure for interval order structures, which is the main contribution of this thesis.

We start with a short presentation of some properties on *interval order structures (IO-structures)*, then we define ⧫-closure, the main concept studied in this thesis, and prove its equivalence to Theorem 6. Because IO-structures are more complex than SO-tructures, the proofs are more involved than that of Theorem 6.

Definition 18 (Lamport (1986); Janicki and Koutny (1991))**.** *An* IO-structure *is a relational structure $S = (X, \prec, \sqsubset)$, such that for all $a, b, c, d \in X$, the following hold:*

I1: $(a \not\sqsubset a)$

I2: $a \prec b \implies a \sqsubset b$

I3: $a \prec b \prec c \implies a \prec c$

I4: $a \prec b \sqsubset c \vee a \sqsubset b \prec c \implies a \sqsubset c$

I5: $a \prec b \sqsubset c \prec d \implies a \prec d$

I6: $a \sqsubset b \prec c \sqsubset d \implies a \sqsubset d \vee a = d$ ■

Here, just like in the case of SO-structures, the *causality* relation $\prec$ also represents the "earlier than" relationship, and the *weak causality* relation $\sqsubset$ represents the "not later than" relationship, but under the assumption that the system runs are defined as interval orders. Axioms I1 – I5 describe relationship between interval "earlier than" and "not later than".

As we have already presented in Chapter 1, IO-structures were independently introduced in (Lamport, 1986) and (Janicki and Koutny, 1991), however the motivations and intuitions were different. Some of their properties has been presented in (Janicki and Koutny, 1997), yet their theory is not as well-developed and, consequently, less often applied for modelling concurrency than the theory of SO-structures (Janicki, 2008). The lack of a construction equivalent to the $\Diamond$-closure prevented us from building a working relationship between IO-structures and linguistic models for concurrency such as Mazurkiewicz traces (Diekert and Rozenberg, 1995) and comtraces (Janicki and Koutny, 1995; Kleijn and Koutny, 2004).

It was noted earlier, that every stratified order is also an interval order, so the similar relationship holds for their corresponding relational structures.

Proposition 4 (Janicki and Koutny (1997))**.** *Every SO-structure is an IO-structure.* ■

Due to above theorem, many properties of SO-structures hold for IO-structures as well.

Proposition 5 (Janicki and Koutny (1993))**.**

1. *If* $(X, \prec)$ *is a partially ordered set, then* $a \prec b \Rightarrow b \not\sqsubset a$*, and* $a \sqsubset b \Rightarrow b \not\prec a$*.*
2. *If* $(X, <)$ *is an interval order, then* $(X, <, <^{\frown})$ *is an IO-structure.* ■

Proposition 5(2) allows an introduction of a *interval order extension of interval order structure*.

Definition 19. 1. *An interval order* $io = (X, <)$ *is an* interval order extension *of an IO-structure* $S = (X, \prec, \sqsubset)$ *if* $\prec \subseteq <$ *and* $\sqsubset \subseteq <^{\frown}$*.*

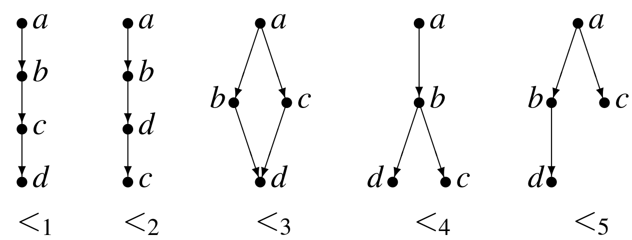
2. *The set of all interval order extensions of S will be denoted by* $\mathsf{Interv}(S)$*.* ■

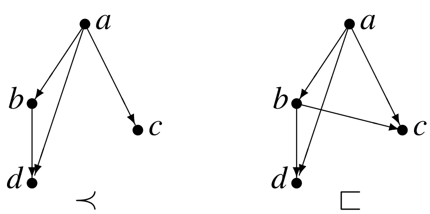
Clearly if an interval order $<$ is an extension of $S = (X, \prec, \sqsubset)$, then $(X, <, <^{\frown})$ is a relational structure extension of S, in the sense of Definition 14.

We also have an analogue of Theorem 5 for representation of interval orders and IO-structures.

Theorem 7 (Janicki and Koutny (1997))**.** *For each IO-structure* $S = (X, \prec, \sqsubset)$*, we have*

$$S = \Big(X, \bigcap_{<\in \mathsf{Interv}(S)} <, \bigcap_{<\in \mathsf{Interv}(S)} <^{\frown}\Big).$$





```
P: begin int x,y,z:
     a:  begin x:=0; y:=0; z:=0 end;
     cobegin
       begin
         b:  x=0 → y:=y+1;
         d:  z:=z+1
       end,
       c:  x:=x+1
     coend
   end P
```

Figure 4.1: An example of a simple interval order structure $S = (X, \prec, \sqsubset)$, with $X = \{a,b,c,d\}$ and its set of all interval extensions $\mathsf{Interv}(S) = \{<_1, <_2, <_3, <_4, <_5\}$. The orders $<_1$ and $<_2$ are total, $<_3$ and $<_4$ are stratified and $<_5$ is interval but not stratified. The elements of $\mathsf{Interv}(S)$ are all equivalent runs (executions) of the program P involving the actions a, b, c and d, so the interval order structure uniquely defines a concurrent behaviour (history) of P (see for details). The elements of $\mathsf{Interv}(S)$ are represented as Hasse diagrams, while $\prec$ and $\sqsubset$ are represented as graphs of their entire relations. In this case $\prec$ equals $<_5$, as there are not so many partial orders over the four elements set, but the interpretations of $<_5$ and $\prec$ are different. The incomparability in $<_5$ is interpreted as *simultaneity* while in $\prec$ as *having no casual relationship*.

■

The above theorem is a generalization of the Szpilrajn Theorem (Theorem 3) to IO-structures. It is interpreted as the proof of the claim that IO-structures uniquely represent sets of equivalent system runs, provided that the system's operational semantics is fully described in terms of interval orders, c.f. (Janicki, 2008; Janicki and Koutny, 1997) for details. An example of simple IO-structure which illustrates the main ideas behind this concept is shown in Figure 4.1.

Before defining the concept of ⧫-closure for IO-structures and proving its properties, we need to introduce some auxiliary notions and prove some useful preliminary results.

Definition 20 (Sequence notation). *For every non-empty* sequence of relations $seq = \langle S_1, \ldots, S_k \rangle$, *we define:*

1. $first(seq) = S_1$*, i.e. the first relation in the sequence,*
2. $last(seq) = S_k$*, i.e. the last relation in the sequence,*
3. $pos(i, seq) = S_i$*, i.e. the relation at the position i in the sequence, and*
4. $length(seq) = k$*, i.e. the length of the sequence.* ■

Definition 21. *For two sequences of relations* $seq_S = \langle S_1, \ldots, S_k \rangle$ *and* $seq_Q = \langle Q_1, \ldots, Q_s \rangle$, *we define:*

1. *Relational* sequence concatenation $seq_S \cdot\ seq_Q$ *as*

$$\langle S_1, \ldots, S_k \rangle \cdot \langle Q_1, \ldots, Q_l \rangle = \langle S_1, \ldots, Q_l \rangle;$$

2. λ *to be an empty sequence, or simply* $\langle \rangle$*;*
3. $seq \cdot \lambda = \lambda \cdot seq = seq,$ *for any sequence* seq*;*
4. Subsequence *of k first consecutive elements of seq:* $seq^k = seq^{k-1} \cdot seq$ *for each sequence* seq *and for all* $k > 0$*, and if* $k = 0$ *then* $seq^0 = \lambda$. ■

Every set of sequences is called a *language*. For more details regarding sequences, languages, and appropriate proof techniques the reader is referred to, for instance, Hopcroft *et al.* (2003).

Definition 22. *Let $R_1, R_2 \in X \times X$ be two relations and let $seq = \langle S_1, \ldots, S_k \rangle$ be a nonempty sequence of relations, such that $S_i \in \{R_1, R_2\},\ i = 1, \ldots, k$. Then seq has $\prec_{R_1,R_2}$*-property *in the following cases:*

1. *If $length(seq) = 1$, then $first(seq) = S_1 = R_1$,*
2. *If $length(seq) > 1$, then $first(seq) = last(seq) = S_1 = S_k = R_1$ and for each $i = 1, \ldots, length(seq) - 1$, if $pos(i, seq) = R_2$, then $pos(i+1, seq) = R_1$.*
3. *A language L has $\prec_{R_1,R_2}$-property iff each $seq \in L$ has $\prec_{R_1,R_2}$-property.*
4. *$\mathbb{S}^{\prec}_{R_1,R_2}$ denotes the set of all sequences that have $\prec_{R_1,R_2}$-property.* ∎

Definition 23. *Let $R_1, R_2 \in X \times X$ be two relations and let $seq = \langle S_1, \ldots, S_k \rangle$ be a nonempty sequence of relations, such that $S_i \in \{R_1, R_2\}, i = 1, \ldots, k$. Then seq has $\sqsubset_{R_1,R_2}$*-property *in the following cases:*

1. *If $length(seq) > 1$, then for each $i = 1, \ldots, length(seq) - 1$, if $pos(i, seq) = R_2$ then $pos(i+1, seq) = R_1$.*
2. *A language L has $\sqsubset_{R_1,R_2}$-property iff each $seq \in L$ has $\sqsubset_{R_1,R_2}$-property.*
3. *$\mathbb{S}^{\sqsubset}_{R_1,R_2}$ denotes the set of all sequences that have $\sqsubset_{R_1,R_2}$-property.* ∎

In other words, a sequence $\langle S_1, \ldots, S_k \rangle$ has $\sqsubset_{R_1,R_2}$-property if it *does not* contain a subsequence $\langle R_2, R_2 \rangle$, and it has $\prec_{R_1,R_2}$-property if it has $\sqsubset_{R_1,R_2}$-property and, additionally, $S_1 = S_k = R_1$. For example, the sequence $\langle R_1, R_2, R_2, R_1, R_2 \rangle$ has neither $\sqsubset_{R_1,R_2}$-property nor $\prec_{R_1,R_2}$-property. The sequence $\langle R_2, R_1, R_1, R_2, R_1, R_2, R_1 \rangle$ has $\sqsubset_{R_1,R_2}$-property, but not

$\prec_{R_1,R_2}$-property, while the sequence $\langle R_1,R_2,R_1,R_1,R_2,R_1,R_2,R_1\rangle$ has both $\sqsubset_{R_1,R_2}$-property and $\prec_{R_1,R_2}$-property. The basic properties of those kind of sequences are given by the following lemma.

Lemma 3. *Let R_1 and R_2 be relations on X. Then:*

1. *If seq has $\prec_{R_1,R_2}$-property, then it has $\sqsubset_{R_1,R_2}$-property.*
2. *If $\langle S_1,\ldots,S_k\rangle$ and $\langle Q_1,\ldots,Q_l\rangle$ have $\prec_{R_1,R_2}$-property, then concatenation $\langle S_1,\ldots,S_k\rangle\cdot\langle Q_1,\ldots,Q_l\rangle$, the sequence $\langle S_1,\ldots,Q_l\rangle$ has $\prec_{R_1,R_2}$-property.*
3. *If seq has $\prec_{R_1,R_2}$-property, then $R_2\cdot seq$, $seq\cdot R_2$ and $R_2\cdot seq\cdot R_2$ have $\sqsubset_{R_1,R_2}$-property.*
4. *If seq_1 has $\prec_{R_1,R_2}$-property, and seq_2 has $\sqsubset_{R_1,R_2}$-property, then $seq_1\cdot seq_2$ and $seq_2\cdot seq_1$ have $\sqsubset_{R_1,R_2}$-property.*
5. *If seq_1 and seq_3 have $\prec_{R_1,R_2}$-property, and seq_2 has $\sqsubset_{R_1,R_2}$-property, then $seq_1\cdot seq_2\cdot seq_3$ has $\prec_{R_1,R_2}$-property.*
6. *If languages L_1 and L_3 have $\prec_{R_1,R_2}$-property, and a language L_2 has $\sqsubset_{R_1,R_2}$-property, then the language $(L_1^+\cdot L_2)^*\cdot L_3^+$ has $\prec_{R_1,R_2}$-property.*
7. *If seq_1 and seq_3 have $\sqsubset_{R_1,R_2}$-property, and seq_2 has $\prec_{R_1,R_2}$-property, then $seq_1\cdot seq_2\cdot seq_3$ has $\sqsubset_{R_1,R_2}$-property.*
8. *If languages L_1, L_3 and L_5 have $\sqsubset_{R_1,R_2}$-property, and languages L_2, L_4 have $\prec_{R_1,R_2}$-property, then the languages $L_1\cdot(L_2^+\cdot L_3)^*\cdot L_4^+$, $(L_2^+\cdot L_3)^*\cdot L_4^+\cdot L_5$, $L_1\cdot(L_2^+\cdot L_3)^*\cdot L_4^+\cdot L_5$ have $\sqsubset_{R_1,R_2}$-property.*

Proof.

(1)-(5) and (7) follow directly from Definitions 22 and 23.

(6) It suffices to show that for all $i,k \geq 1$, $j \geq 0$, the language $(L_1^i \cdot L_2)^j \cdot L_3^k$ has $\prec_{R_1,R_2}$-property.

The property is obvious for $j = 0$, so assume $j > 0$. From (2) it follows L_1^i and L_3^k have $\prec_{R_1,R_2}$-property. Hence by (5) $L_1^i \cdot L_2 \cdot L_3^k$ has $\prec_{R_1,R_2}$-property, which means $(L_1^i \cdot L_2)^j \cdot L_3^k$ has $\prec_{R_1,R_2}$-property for $j = 1$. Assume $j > 1$. Now by (7) $L_1^i \cdot L_2 \cdot L_1^i$ has $\prec_{R_1,R_2}$-property, i.e. $(L_1^i \cdot L_2)^s \cdot L_1^i$ has $\prec_{R_1,R_2}$-property, for $s = 1$. Suppose this holds for s, consider inductive case of $s+1$. We have $(L_1^i \cdot L_2)^{s+1} \cdot L_1^i = (L_1^i \cdot L_2)^s \cdot L_1^i \circ L_2 \cdot L_1^i$. Since by induction assumption $(L_1^i \cdot L_2)^s \cdot L_1^i$ has $\prec_{R_1,R_2}$-property, then by (5), $(L_1^i \cdot L_2)^s \cdot L_1^i \cdot L_2 \cdot L_1^i$ also has $\prec_{R_1,R_2}$-property, which means $(L_1^i \cdot L_2)^s \cdot L_1^i$ has $\prec_{R_1,R_2}$-property for all $s \geq 1$. Hence $(L_1^i \cdot L_2)^j \cdot L_3^k = (L_1^i \cdot L_2)^{j-1} \cdot L_1^i \cdot L_2 \cdot L_3^k$.
Since $(L_1^i \cdot L_2)^{j-1} \cdot L_1^i$ has $\prec_{R_1,R_2}$-property, then by (5), $(L_1^i \cdot L_2)^j \cdot L_3^k$ has it as well.

(8) It suffices to show that for all $i,k \geq 1$, $j \geq 0$, the languages $L_1 \cdot (L_2^i \cdot L_3)^j \cdot L_4^k$, $(L_2^i \cdot L_3)^j \cdot L_4^k \cdot L_5$ and $L_1 \cdot (L_2^i \cdot L_3)^j \cdot L_4^k \cdot L_5$ have $\sqsubset_{R_1,R_2}$-property. From the proof of (6) and (5) it follows that $L_1 \cdot (L_2^i \cdot L_3)^j \cdot L_4^k$ and $(L_2^i \cdot L_3)^j \cdot L_4^k \cdot L_5$ have $\sqsubset_{R_1,R_2}$-property, from the proof of (6) and (7) it follows that $L_1 \cdot (L_2^i \cdot L_3)^j \cdot L_4^k \cdot L_5$ have $\sqsubset_{R_1,R_2}$-property.

■

Lemma 3 provides principal tools for proving most of the remaining results. We may now formally define the relations $\prec^{\blacklozenge}_{R_1,R_2}$ and $\sqsubset^{\blacklozenge}_{R_1,R_2}$, the basic components of $\blacklozenge$-closure definition.

Definition 24. *Let R_1 and R_2 be relations on X. We define:*

1. $\prec^{\blacklozenge}_{R_1,R_2} = \bigcup_{\langle S_1,\dots,S_k\rangle \in \mathbb{S}^{\prec}_{R_1,R_2}} S_1 \circ \dots \circ S_k$;

2. $\sqsubset^{\blacklozenge}_{R_1,R_2} = \left(\bigcup_{\langle S_1,\ldots,S_k\rangle \in \mathbb{S}^{\sqsubset}_{R_1,R_2}} S_1 \circ \cdots \circ S_k\right) \setminus id_X$. ■

In other words, $\prec^{\blacklozenge}_{R_1,R_2}$ is the union of all compositions $S_1 \circ \cdots \circ S_n$, where $n \geq 1$, $S_1 = S_n = R_1$, $S_i = R_2$ implies $S_{i+1} = R_1$, for $i = 1, \ldots, n-1$. And relation $\sqsubset^{\blacklozenge}_{R_1,R_2}$ is the union of all compositions $S_1 \circ \cdots \circ S_n$, where $n \geq 1$, and $S_i = R_2$ implies $S_{i+1} = R_1$, for $i = 1, \ldots, n-1$, and $(a,a) \notin \sqsubset^{\blacklozenge}_{R_1,R_2}$ for all $a \in X$.

To express $\prec^{\blacklozenge}_{R_1,R_2}$ and $\sqsubset^{\blacklozenge}_{R_1,R_2}$ in more operational manner, but also in a compact from, we will use the following definition.

Definition 25 (Operational definition)**.** *Let $R_1, R_2 \subseteq X \times X$ be two relations. Then*

1. $R_1 \oplus R_2 = (R_1^+ \circ R_2)^* \circ R_1^+$,

2. $R_1 \uplus R_2 = (R_2 \cup id_X) \circ (R_1 \cup R_1 \circ R_2)^*$,

3. $R_1 \oplus R_2 \subseteq R_1 \uplus R_2$,

4. $(R_1 \oplus R_2) \oplus (R_1 \uplus R_2) \subseteq (R_1 \oplus R_2)$,

5. $(R_1 \oplus R_2) \uplus (R_1 \uplus R_2) \subseteq (R_1 \uplus R_2)$,

Proof. Follows immediately from the Definition 24. ■

Immediately from the above definitions we have the following useful result.

Corollary 2. *Let R_1 and R_2 be relations on X. Then from above definitions we have:*

1. $\prec^{\blacklozenge}_{R_1,R_2} = R_1 \oplus R_2$;

2. $\sqsubset^{\blacklozenge}_{R_1,R_2} = (R_1 \uplus R_2) \setminus id_X$;

3. $R_1 \oplus R_2 = \bigcup_{\langle S_1,\ldots,S_k\rangle \in \mathbb{S}^{\prec}_{R_1,R_2}} S_1 \circ \cdots \circ S_k$;

4. $R_1 \uplus R_2 = \left(\bigcup_{\langle S_1,\ldots,S_k\rangle \in \mathbb{S}^{\sqsubset}_{R_1,R_2}} S_1 \circ \cdots \circ S_k\right) \setminus id_x$;

5. $(R_1 \subseteq Q_1 \wedge R_2 \subseteq Q_2) \implies (R_1 \oplus R_2 \subseteq Q_1 \oplus Q_2) \wedge (R_1 \uplus R_2 \subseteq Q_1 \uplus Q_2)$. ■

We can now define the main concept of investigation, the concept of ♦-closure.

Definition 26 (♦-closure)**.** *For every relational structure $S = (X, R_1, R_2)$, we define the structure $S^{\blacklozenge}$, a* ♦-closure *of S, as*

$$S^{\blacklozenge} = (X, \prec^{\blacklozenge}_{R_1,R_2}, \sqsubset^{\blacklozenge}_{R_1,R_2}) = (X, R_1 \oplus R_2, (R_1 \uplus R_2) \setminus id_X).$$ ■

The ♦-closure is an extension of ◊-closure of SO-structures and classical transitive closure of relations to IO-structures. We start with proving equivalences of Theorem 6(1) and 6(2).

Proposition 6. *Given two relations $R_1, R_2 \in X \times X$, an IO-structure $S = (X, R_1, R_2)$ and its closure $S^{\blacklozenge}$, we have*

1. *If R_2 is irreflexive then $S \subseteq S^{\blacklozenge}$.*

2. $\left(S^{\blacklozenge}\right)^{\blacklozenge} = S^{\blacklozenge}$.

Proof. (1) By the definition $R_1 \subseteq R_1 \oplus R_2 = \prec^{\blacklozenge}_{R_1,R_2}$ and $R_2 \subseteq R_1 \uplus R_2$. Hence, if R_2 is irreflexive, $R_2 \setminus id_X \subseteq (R_1 \uplus R_2) \setminus id_X = \sqsubset^{\blacklozenge}_{R_1,R_2}$.

(2) ($\supseteq$) Since $\sqsubset^{\blacklozenge}_{R_1,R_2}$ is irreflexive, by (1) we have $S^{\blacklozenge} \subseteq \left(S^{\blacklozenge}\right)^{\blacklozenge}$.

($\subseteq$) We need to show that $\prec^{\blacklozenge}_{\prec^{\blacklozenge}_{R_1,R_2},\sqsubset^{\blacklozenge}_{R_1,R_2}} \subseteq \prec^{\blacklozenge}_{R_1,R_2}$ and $\sqsubset^{\blacklozenge}_{\prec^{\blacklozenge}_{R_1,R_2},\sqsubset^{\blacklozenge}_{R_1,R_2}} \subseteq \sqsubset^{\blacklozenge}_{R_1,R_2}$.

Let $\{a,b\} \in X$ and $a \prec^{\blacklozenge}_{\prec^{\blacklozenge}_{R_1,R_2},\sqsubset^{\blacklozenge}_{R_1,R_2}} b$, i.e. (by Corollary 2) a pair $(a,b) \in (R_1 \oplus R_2)$. This means there must exist a sequence of relations $\langle P_1,\ldots,P_n\rangle$, such that $aP_1 \circ \cdots \circ P_n b$ for $P_i \in \{\prec^{\blacklozenge}_{R_1,R_2}, \sqsubset^{\blacklozenge}_{R_1,R_2}\}$ and $i = 1,\ldots,n$, where $n \geq 1$. From the Definition of $\oplus$ (Definition 25(1)) it follows that $\langle P_1,\ldots,P_n\rangle = \Big(\langle\prec^{\blacklozenge}_{R_1,R_2}\rangle^i \cdot \langle\sqsubset^{\blacklozenge}_{R_1,R_2}\rangle\Big)^j \cdot \langle\prec^{\blacklozenge}_{R_1,R_2}\rangle^k$, where $i,k \geq 1$, and $j \geq 0$. If $x \prec^{\blacklozenge}_{R_1,R_2} y$ then there is a sequence $\langle S_1,\ldots,S_s\rangle$ such that $S_i \in \{R_1,R_2\}$ and $xS_1 \circ \cdots \circ S_s y$. The sequence $\langle S_1,\ldots,S_s\rangle$ obviously has a $\prec^{\blacklozenge}_{R_1,R_2}$-property. Let language $L(\prec^{\blacklozenge}_{R_1,R_2})$ be set of all such sequences that for each x,y we have $x \prec^{\blacklozenge}_{R_1,R_2} y$. Let language $L(\sqsubset^{\blacklozenge}_{R_1,R_2})$ be the language of similar sequences generated by the relation $\sqsubset^{\blacklozenge}_{R_1,R_2}$. Clearly, from Definitions 22(2,3) and 23(2,3), $L(\prec^{\blacklozenge}_{R_1,R_2}) \subseteq \mathbb{S}^{\prec^{\blacklozenge}}_{R_1,R_2}$ and $L(\sqsubset^{\blacklozenge}_{R_1,R_2}) \subseteq \mathbb{S}^{\sqsubset^{\blacklozenge}}_{R_1,R_2}$, and we can conclude
$\langle P_1,\ldots,P_n\rangle \in \Big(L(\prec^{\blacklozenge}_{R_1,R_2})^+ \cdot L(\sqsubset^{\blacklozenge}_{R_1,R_2})\Big)^* \cdot L(\prec^{\blacklozenge}_{R_1,R_2})^+$.

Since $L(\prec^{\blacklozenge}_{R_1,R_2})$ has $\prec^{\blacklozenge}_{R_1,R_2}$-property and $L(\sqsubset^{\blacklozenge}_{R_1,R_2})$ has $\sqsubset^{\blacklozenge}_{R_1,R_2}$-property, by Lemma 3(6), the language $(L(\prec^{\blacklozenge}_{R_1,R_2})^+ \cdot L(\sqsubset^{\blacklozenge}_{R_1,R_2}))^* \cdot L(\prec^{\blacklozenge}_{R_1,R_2})^+$ has $\prec^{\blacklozenge}_{R_1,R_2}$-property, i.e. $\langle P_1,\ldots,P_n\rangle \in \mathbb{S}^{\prec^{\blacklozenge}}_{R_1,R_2}$. Therefore by Definition 24(1), $P_1 \circ \cdots \circ P_n \subseteq \prec^{\blacklozenge}_{R_1,R_2}$, so $a \prec^{\blacklozenge}_{R_1,R_2} b$.

Lets assume $a \sqsubset^{\blacklozenge}_{\prec^{\blacklozenge}_{R_1,R_2},\sqsubset^{\blacklozenge}_{R_1,R_2}} b$. This means that if $a \neq b$, there exists a sequence of relations $\langle P_1,\ldots,P_n\rangle$ such that $aP_1 \circ \cdots \circ P_n b$ where $P_i \in \{\prec^{\blacklozenge}_{R_1,R_2}, \sqsubset^{\blacklozenge}_{R_1,R_2}\}$ for $i = 1,\ldots,n$, and $n \geq 1$.

From the definition of $\sqsubset^{\blacklozenge}_{\prec^{\blacklozenge}_{R_1,R_2},\sqsubset^{\blacklozenge}_{R_1,R_2}}$ and Definition 25(2) of $\uplus$ it follows that $\langle P_1,\ldots,P_n\rangle$ is in one of the following forms: $\langle\sqsubset^{\blacklozenge}_{R_1,R_2} \cup id_X\rangle \cdot \langle\prec^{\blacklozenge}_{R_1,R_2} \cup \prec^{\blacklozenge}_{R_1,R_2} \circ \sqsubset^{\blacklozenge}_{R_1,R_2}\rangle^*$

Reasoning similarly as in the case of $\prec^{\blacklozenge}_{\prec^{\blacklozenge}_{R_1,R_2},\sqsubset^{\blacklozenge}_{R_1,R_2}}$, but using Lemmas 3(8), 3(2) and 3(6) instead of Lemma 3(6) only, one can show that $\langle P_1,\ldots,P_n\rangle \in \mathbb{S}^{\sqsubset^{\blacklozenge}}_{R_1,R_2}$. Therefore by Corollary 2(3) $P_1 \circ \cdots \circ P_n \subseteq R_1 \uplus R_2$, so $(a,b) \in R_1 \uplus R_2$. Since $a \neq b$ then by Corollary 2(2) $a \sqsubset^{\blacklozenge}_{R_1,R_2} b$.

■

Proposition 6(2) shows that $\blacklozenge$-closure is *idempotent*, i.e. multiple application of the operator returns the same result as initial application, and this is what justifies the use of name *closure*, see Rosen (2002).

Unfortunately the exact replica of Theorem 6(3) is false. To show this, consider domain of two elements $X = \{a,b\}$, and two relations on the domain $R_1 = \{(a,b)\}$ and $R_2 = \{(b,a)\}$. In this case $\prec^{\blacklozenge}_{R_1,R_2} = R_1 = \{(a,b)\}$, $\sqsubset^{\blacklozenge}_{R_1,R_2} = R_1 \cup R_2 = \{(a,b),(b,a)\}$, so $\prec^{\blacklozenge}_{R_1,R_2}$ is irreflexive, but $(X, \prec^{\blacklozenge}_{R_1,R_2}, \sqsubset^{\blacklozenge}_{R_1,R_2})$ is not an IO-structure, because $a \prec^{\blacklozenge}_{R_1,R_2} b$ and $b \sqsubset^{\blacklozenge}_{R_1,R_2} a$ contradicts Proposition 5(1).

However we can still prove its slightly weaker version.

Definition 27 (i-directed)**.** *A relational structure $S = (X, R_1, R_2)$ is* i-directed *if the following conditions are satisfied:*

1. *$R_1 \oplus R_2$ is irreflexive,*

2. *$\forall a,b \in X : (a,b) \in R_2 \implies (b,a) \notin R_1 \oplus R_2$.* ■

Proposition 7. *$S^{\blacklozenge}$ is an IO-structure if and only if S is i-directed.*

Proof. ($\Rightarrow$) If $S^{\blacklozenge}$ is an interval order structure then by (I1) and (I2), $\prec^{\blacklozenge}_{R_1,R_2} = R_1 \oplus R_2$ is irreflexive. Suppose $(a,b) \in R_2$ and $(b,a) \in R_1 \oplus R_2$. Since $R_2 \subseteq \sqsubset^{\blacklozenge}_{R_1,R_2}$, we have $a \prec^{\blacklozenge}_{R_1,R_2} b$ and $b \sqsubset^{\blacklozenge}_{R_1,R_2} a$, thus contradicting Proposition 5(1).

($\Leftarrow$) We need to show that the conditions of Definition 18 are satisfied.

(I1) Directly from Definition 24(2)

(I2) From Definition 25(2) we have $\prec^{\blacklozenge}_{R_1,R_2} \subseteq R_1 \uplus R_2$. Since $\prec^{\blacklozenge}_{R_1,R_2}$ is irreflexive, $\prec^{\blacklozenge}_{R_1,R_2} \subseteq (R_1 \uplus R_2) \setminus id_X = \sqsubset^{\blacklozenge}_{R_1,R_2}$.

(I3) Let $a \prec^{\blacklozenge}_{R_1,R_2} b$ and let $b \prec^{\blacklozenge}_{R_1,R_2} c$. This means there are two sequences of relation $\langle S_1,\ldots,S_k\rangle$ and $\langle Q_1,\ldots,Q_l\rangle$, such that $aS_1\ldots S_kb$ and $bQ_1\ldots Q_lc$, where $\langle S_1,\ldots,S_k\rangle$ and $\langle Q_1,\ldots,Q_l\rangle$ both satisfy $\prec^{\blacklozenge}_{R_1,R_2}$-property. By Lemma 3(2), concatenation of two sequences $\langle S_1,\ldots,S_k\rangle\cdot\langle Q_1,\ldots,Q_r\rangle$ satisfies $\prec^{\blacklozenge}_{R_1,R_2}$-property and $aS_1\ldots S_kbQ_1\ldots Q_lc$, i.e. $a \prec^{\blacklozenge}_{R_1,R_2} c$.

(I4) Let $a \prec^{\blacklozenge}_{R_1,R_2} b$ and let $b \sqsubset^{\blacklozenge}_{R_1,R_2} c$. This means $aS_1\ldots S_kb$ and $bQ_1\ldots Q_lc$, where $\langle S_1,\ldots,S_k\rangle$ satisfies $\prec^{\blacklozenge}_{R_1,R_2}$-property, and $\langle Q_1,\ldots,Q_l\rangle$ satisfies $\sqsubset^{\blacklozenge}_{R_1,R_2}$-property. By Lemma 3(4), concatenation $\langle S_1,\ldots,S_k\rangle\cdot\langle Q_1,\ldots,Q_l\rangle$ satisfies $\sqsubset^{\blacklozenge}_{R_1,R_2}$-property and $aS_1\ldots S_kbQ_1\ldots Q_lc$, i.e. $(a,c)\in R_1\uplus R_2$. Suppose $a=c$. Since $a \prec^{\blacklozenge}_{R_1,R_2} b$ and $b \prec^{\blacklozenge}_{R_1,R_2} c$, this means $aS_1\ldots S_kb$, and $bQ_1\ldots Q_la$, where $S_i,Q_i\in\{R_1,R_2\}$. Either Q_1 or Q_l are equal to R_2, otherwise $b \prec^{\blacklozenge}_{R_1,R_2} a$, i.e. $a \prec^{\blacklozenge}_{R_1,R_2} a$, a contradiction since $\prec^{\blacklozenge}_{R_1,R_2}$ is irreflexive. Suppose $Q_1=R_2$. This means $Q_2=R_1$. Now we have bR_2b_1 and $b_1R_1b_2\ldots b_{s-1}Q_{l-1}b_lQ_aR_1a_1S_1\ldots S_ka_kR_1b$, which means $(b_1,b)\in R_1\oplus R_2$, a contradiction to Definition 27(2). Hence $Q_1=R_1$ and $Q_l=R_2$, i.e. by Definition 23, $Q_{l-1}=R_1$. Now we have $b_{l-1}R_2a$ and $aR_1a_1S_1\ldots S_ka_kR_1bQ_1b_1Q_2\ldots Q_{l-1}b_{l-1}R_1b_l$, which means $(a,b_l)\in R_1\oplus R_2$, a contradiction to Definition 27(2). Therefore $a\neq c$, i.e. $(a,c)\in(R_1\uplus R_2)\setminus id_X=$ $\sqsubset^{\blacklozenge}_{R_1,R_2}$.

For $a \sqsubset^{\blacklozenge}_{R_1,R_2} b \prec^{\blacklozenge}_{R_1,R_2} c$ we proceed almost identically.

(I5) Let $a \prec^{\blacklozenge}_{R_1,R_2} b \sqsubset^{\blacklozenge}_{R_1,R_2} c \prec^{\blacklozenge}_{R_1,R_2} d$. This means there are sequences $\langle S_1,\ldots,S_k\rangle$, $\langle P_1,\ldots,P_m\rangle$ and $\langle Q_1,\ldots,Q_l\rangle$, such that $aS_1\ldots S_kb$, $bP_1\ldots P_mc$ and $cQ_1\ldots Q_ld$, where $\langle S_1,\ldots,S_k\rangle$, and $\langle Q_1,\ldots,Q_l\rangle$ have $\prec^{\blacklozenge}_{R_1,R_2}$-property and $\langle P_1,\ldots,P_m\rangle$ has $\sqsubset^{\blacklozenge}_{R_1,R_2}$-property. By Lemma 3(5) the sequence concatenation $\langle S_1,\ldots,S_k\rangle\cdot\langle Q_1,\ldots,Q_l\rangle\cdot\langle P_1,\ldots,P_m\rangle$ has $\prec^{\blacklozenge}_{R_1,R_2}$-property and, clearly, $aS_1\ldots P_md$, so by Definition 24(1), $a \prec^{\blacklozenge}_{R_1,R_2} d$.

(I6) Let $a \sqsubset^{\blacklozenge}_{R_1,R_2} b \prec^{\blacklozenge}_{R_1,R_2} c \sqsubset^{\blacklozenge}_{R_1,R_2} d$. This means there are sequences $\langle S_1,\dots,S_k\rangle$, $\langle P_1,\dots,P_m\rangle$ and $\langle Q_1,\dots,Q_l\rangle$, such that $aS_1\dots S_kb$, $bP_1\dots P_mc$ and $cQ_1\dots Q_ld$, where $\langle S_1,\dots,S_k\rangle$, and $\langle Q_1,\dots,Q_l\rangle$ have $\sqsubset^{\blacklozenge}_{R_1,R_2}$-property and $\langle P_1,\dots,P_m\rangle$ has $\prec^{\blacklozenge}_{R_1,R_2}$-property. By Lemma 3(7) the sequence concatenation $\langle S_1,\dots,S_k\rangle\cdot\langle Q_1,\dots,Q_l\rangle\cdot\langle P_1,\dots,P_m\rangle$ has $\sqsubset^{\blacklozenge}_{R_1,R_2}$-property and, clearly, $aS_1\dots P_sd$, so by Definition 24(2), either $a \sqsubset^{\blacklozenge}_{R_1,R_2} d$ or $a=d$.

■

The fact that the above result is slightly weaker than Theorem 6(3) does not seem to matter much as in virtually all applications of $\lozenge$-closure in Janicki and Koutny (1995) and Kleijn and Koutny (2004), the relations R_1 and R_2 satisfy the equivalence of the conditions of Definition 27 for SO-structures.

We now prove an equivalence of Theorem 6(4), which states that each IO-structure is $\blacklozenge$-closed.

Proposition 8. *If S is an IO-structure, then $S=S^{\blacklozenge}$.*

Proof. Let $S=(X,R_1,R_2)$.

($\subseteq$) If S is an IO-structure then R_2 is irreflexive, so by Proposition 6(1), $S\subseteq S^{\blacklozenge}$.

($\supseteq$) We need to show $R_1\oplus R_2\subseteq R_1$ and $(R_1\uplus R_2)\setminus id_X\subseteq R_2$. To prove that $R_1\oplus R_2=(R_1^+\circ R_2)^*\circ R_1^+\subseteq R_1$ it suffices to show that for each $i\geq 1$, $j\geq 0$, $k\geq 1$, $(R_1^i\circ R_2)^j\circ R_1^+\subseteq R_1$. From (I3) it follows $R_1^i\subseteq R_1$ and $R_1^k\subseteq R_1$, so $(R_1^i\circ R_2)^j\circ R_1^+\subseteq (R_1\circ R_2)^j\circ R_1$. Clearly $(R_1\circ R_2)^j\circ R_1\subseteq R_1$ for $j=0$. Suppose it holds for j and consider the case of $j+1$. We have $(R_1\circ R_2)^{j+1}\circ R_1=(R_1\circ R_2)^j\circ R_1\circ R_2\circ R_1$. By induction assumption $(R_1\circ R_2)^j\circ R_1\subseteq R_1$, so $(R_1\circ R_2)^{j+1}\circ R_1\subseteq R_1\circ R_2\circ R_1\subseteq^{\text{(by (I5))}} R_1$. Hence $R_1\oplus R_2\subseteq R_1$.

Since R_2 is irreflexive than it suffices to show that $(R_1\uplus R_2)\subseteq R_2$. From $R_1\oplus R_2\subseteq$

R_1 it follows that $(R_1 \uplus R_2) \subseteq (R_2 \cup R_1) \cup (R_2 \circ R_1) \cup (R_1 \circ R_2) \cup (R_2 \circ R_1 \circ R_2)$. We have $R_2 \cup R_1 \subseteq^{\text{(by (I2))}} R_2$, $R_2 \circ R_1 \subseteq^{\text{(by (I4))}} R_2$, $R_1 \circ R_2 \subseteq^{\text{(by (I4))}} R_2$, and $R_2 \circ R_1 \circ R_2 \subseteq^{\text{(by (I6))}} R_2$. Hence $R_1 \uplus R_2 \subseteq R_2$.

■

Directly from Proposition 8 we obtain the below result which will be used in the proof of equivalence of Theorem 6(6).

Corollary 3. *Every IO-structure is i-directed.* ■

Proposition 9. *Let $S = (X, R_1, R_2)$ be an IO-structure and let $\hat{S} = (X, \hat{R}_1, \hat{R}_2)$be any relational structure, s.t. $\hat{S} \subseteq S$. Then $\hat{S}^{\blacklozenge} \subseteq S$ and $\hat{S}^{\blacklozenge}$ is an IO-structure.*

Proof. From Corollary 2(3) and Proposition 8 it follows,

$$\hat{S}^{\blacklozenge} \subseteq S^{\blacklozenge} = S.$$

Due to Proposition 7 it suffices to show that $\hat{S}$ is i-directed.
Let $S = (X, R_1, R_2)$, $\hat{S} = (X, \hat{R}_1, \hat{R}_2)$. We have $\hat{R}_1 \oplus \hat{R}_2 \subseteq R_1 \oplus R_2 =^{\text{(Proposition 8)}} R_1$. Since S is interval, R_1 is irreflexive so $\hat{R}_1 \oplus \hat{R}_2$ is irreflexive as well.
Let $(a, b) \in \hat{R}_2$. Since $\hat{R}_2 \subseteq R_2$, we have $(a, b) \in R_2$ which by Corollary 3, implies $(b, a) \notin R_1 \oplus R_1$. Since $\hat{R}_1 \oplus \hat{R}_2 \subseteq R_1 \oplus R_2$, $(b, a) \notin \hat{R}_1 \oplus \hat{R}_2$. Therefore $\hat{S}$ is i-directed. ■

We will now show that $\blacklozenge$-closure is really a generalization of $\lozenge$-closure.

Proposition 10. *If $S = (X, R_1, R_2)$ is an SO-structure, then $S = S^{\Diamond} = S^{\blacklozenge}$.*

Proof. A consequence of Theorem 6(4), Theorem 4 and Proposition 8. ■

Summing up, results proved in this chapter can be presented in the following cumulative theorem:

Theorem 8. *Let $R_1, R_2 \in X \times X$ be two relations of X, and $S = (X, R_1, R_2)$ be an relational structure. Then:*

1. *If R_2 is irreflexive then $S \subseteq S^{\blacklozenge}$.*
2. $\left(S^{\blacklozenge}\right)^{\blacklozenge} = S^{\blacklozenge}$.
3. *$S^{\blacklozenge}$ is an interval order structure if and only if S is i-directed.*
4. *If S is an IO-structure then $S = S^{\blacklozenge}$.*
5. *Let S be an IO-structure and let $\hat{S} \subseteq S$. Then $\hat{S}^{\blacklozenge} \subseteq S$ and $\hat{S}^{\blacklozenge}$ is an IO-structure.*
6. *If S is an SO-structure then $S = S^{\Diamond} = S^{\blacklozenge}$.* ■

4.2 Introduction of Interval Traces

A concept of $\blacklozenge$-closure has been defined for IO-structures. It is an equivalence of $\Diamond$-closure of SO-structures (Janicki and Koutny, 1995) and classical transitive closure of relations. It has also been proved that $\blacklozenge$-closure has in principle the same properties as $\Diamond$-closure, and, in fact, as a standard transitive closure. Because the definition of $\blacklozenge$-closure was more elaborate, the proofs were substantially more complex than their counterparts for

$\Diamond$-closure. Notwithstanding, only one proven property of $\blacklozenge$-closure is slightly weaker than its $\Diamond$-closure counterpart.

Although not discussed in details here, $\blacklozenge$-closure has already proved useful for construction of new type of traces - interval traces (see Janicki *et al.* (2012)).

Following the ideas of partial order representation theorems presented in Section 2.1 of Chapter 2, the counterpart of comtraces for IO-structures has been fully developed based on above results. Fishburn's Theorem (Theorem 4) states that each interval order can be represented by an appropriate total order of the interval beginnings and ends. Representation foundation for interval traces is established by Theorem 9 below. It states that each IO-structure can be represented by an appropriate partial order (not necessarily interval) of beginnings and ends.

Theorem 9 (Abraham *et al.* (1990))**.** *A relational structure $S = (X, \prec, \sqsubset)$ is an IO-structure iff there exists a partial order $\lhd$ on some Y and two mappings $B, E : X \to Y$ such that $B(X) \cap E(X) = \emptyset$ and for each $x, y \in X$:*

1. $B(x) \lhd E(x)$,
2. $x \prec y \iff E(x) \lhd B(y)$,
3. $x \sqsubset y \iff B(x) \lhd E(y)$. ■

Additionally, Szpilrajn's Theorem allows us to represent each partial order by its total extensions and so the combination of these three theorems (including Theorem 4 and Theorem 7) made it possible to construct "interval traces", a version of Mazurkiewicz traces over an appropriate monoid of sequences (called *legal*) of beginnings and ends. Then we use "interval traces" to represent IO-structures via Theorem 9. The topic of trace theory is beyond the scope of this thesis, however, interested reader is directed to the paper by Janicki *et al.* (2012) to see how $\blacklozenge$-closure is used for construction interval traces representation of interval order structures.

Chapter 5

Causal Structures for General Concurrent Behaviours

In this chapter we will present the very recent results (from Janicki *et al.* (2013)), that are in one sense a generalization of those from Chapter 3, and a restriction in another sense. The generalization is that we are no longer conforming to the paradigm π_3 (so $\prec_{Obs}$ and $\sqsubset_{Obs}$ no longer describe a concurrent history *Obs*, we have to use less intuitive and less natural $\rightleftharpoons_{Obs}$ and $\sqsubset_{Obs}$), while the restriction is that we assume all observations to be stratified orders, not more general interval orders as in Chapter 4. The $\bigstar$-Closure discussed in this chapter is a generalization of $\Diamond$-closure, but its relationship to $\blacklozenge$-closure - the main contribution of this thesis, is not fully understood at this point of time.

5.1 Mutex Order Structures and Generalized Mutex Order Structures

While most of concurrent behaviours conforms to the paradigm π_3, there are some that do not. If the paradigm π_3 does not hold, we cannot use invariants $\prec$ and $\sqsubset$ to characterize concurrent histories, we have to use $\rightleftharpoons$ and $\sqsubset$ instead. The relationship between $\rightleftharpoons$ and $\sqsubset$ is less intuitive and less understood than the more natural one between $\prec$ and $\sqsubset$. Finding axioms for $\rightleftharpoons$ and $\sqsubset$ turned out to be problematics and complicated. The first attempt was given in Guo and Janicki (2002), later refined in Janicki (2008) and Janicki and Le (2008), and called *generalized stratified order structures* and *generalized interval order structures*, respectively. However, it was shown in Kleijn and Koutny (2011) that the model of Guo and Janicki (2002) is not as general as originally anticipated. The solution to this problem was recently provided in (Janicki *et al.*, 2013), but only for the case where all observations are represented by *stratified orders*.

The solution explores the relationship $\prec = \rightleftharpoons \cap \sqsubset$ and starts with an alternative set of axioms for the case of paradigm π_3 and observations represented by stratified orders, i.e. the case represented by *stratified order structures*. This new order structure is called *mutex order structure*.

Definition 28 (Mutex Order Structures)**.** *A mutex order structure (MO-structure) is a relational structure $M = (X, \rightleftharpoons, \sqsubset)$, where $\rightleftharpoons$ and $\sqsubset$ are binary relations of X, such that $\forall a, b, c \in X$ we have:*

M1: $a \not\sqsubset a$

M2: $a \rightleftharpoons b \implies b \rightleftharpoons a$

M3: $a \rightleftharpoons b \implies a \sqsubset b \vee b \sqsubset a$

M4: $a \sqsubset b \sqsubset c \wedge a \neq c \implies a \sqsubset c$

M5: $a \sqsubset b \sqsubset c \wedge (a \rightleftharpoons b \vee b \rightleftharpoons c) \implies a \rightleftharpoons c$ ■

The following simple result show the plain relationship between mutex order structures and stratified orders.

Proposition 11 (Janicki *et al.* (2013))**.** *If* $(X, <)$ *is a stratified order then* $(X, < \cup >, <^{\frown})$ *is a MO-structure.* ■

Proposition 11 allows an introduction of a *stratified order extension of mutex order structure*.

Definition 29.

1. *A stratified order* $<$ *on* X *is a* stratified extension *of a MO-structure* $M = (X, \rightleftharpoons, \sqsubset)$ *if* $\rightleftharpoons \subseteq < \cup >$ *and* $\sqsubset \subseteq <^{\frown}$.
2. *The set of all stratified extensions of* M *will be denoted by* $\mathsf{Strat}(M)$. ■

Clearly if a stratified order $<$ is an extension of $M = (X, \rightleftharpoons, \sqsubset)$, then $(X, < \cup >, <^{\frown})$ is a relational structure extension of M, in the sense of Definition 14.

The relationship between mutex order structures and stratified order structures is the following.

Theorem 10 (Janicki *et al.* (2013))**.**

1. *For every mutex order structure* $M = (X, \rightleftharpoons, \sqsubset)$*, the relational structure* $S_M = (X, \rightleftharpoons \cap \sqsubset, \sqsubset)$ *is a stratified order structure and* $\mathsf{Strat}(M) = \mathsf{Strat}(S_M)$.
2. *For every stratified order structure* $S = (X, \prec, \sqsubset)$*, the relational structure* $M_S = (X, \prec^{\bowtie}, \sqsubset)$ *is a mutex order structure and* $\mathsf{Strat}(S) = \mathsf{Strat}(M_S)$. ■

Theorem 10 just states that mutex order structures and stratified order structures can be considered as equivalent, however, as opposed to the stratified order structures, the mutex order structures can be extended in a natural way, so they can model the cases that do not conform to paradigm π_3.

Definition 30 (Generalized Mutex Order Structure, Janicki *et al.* (2013))**.** *A* generalized mutex order structure (GMO-structure) *is a relational structure* $gmos = (X, \rightleftharpoons, \sqsubset)$*, where* $\rightleftharpoons$ *and* $\sqsubset$ *are binary relations of* X *such that, for all* $a,b,c,d \in X$*:*

G1: $a \not\sqsubset a \wedge a \not\rightleftharpoons a$

G2: $a \rightleftharpoons b \implies b \rightleftharpoons a$

G3: $a \sqsubset b \sqsubset c \wedge a \neq c \implies a \sqsubset c$

G4: $a \sqsubset b \sqsubset c \wedge (a \rightleftharpoons b \vee b \rightleftharpoons c) \implies a \rightleftharpoons c$

G5: $a \sqsubset b \sqsubset c \wedge a \rightleftharpoons c \implies b \rightleftharpoons a$

G6: $a \sqsubset b \sqsubset c \wedge a \sqsubset d \sqsubset b \wedge c \rightleftharpoons d \implies a \rightleftharpoons b$ ■

The following results show the relationship between generalized mutex order structures and mutex/stratified order structures.

Proposition 12 (Janicki *et al.* (2013))**.**

1. *Every mutex order structure is a generalized mutex order structure structure.*
2. *If* $(X, \rightleftharpoons, \sqsubset)$ *is a generalized mutex order structure structure, then* $(X, \rightleftharpoons \cap \sqsubset, \sqsubset)$ *is a stratified order structure.*

■

It also can be proved that every generalized stratified order structure of Guo and Janicki (2002) is a generalized mutex order structure structure.

5.2 ★-Closure

For the closures $+$, $\diamondsuit$ and $\blacklozenge$, discussed in the previous sections we have:

- for every relation R, R^+ is a transitive relation,
- for every relational structure $S = (X, R_1, R_2)$, if $(R_1 \cup R_2)^* \circ R_1 \circ (R_1 \cup R_2)^*$ is irreflexive then $S^\diamondsuit$ is a stratified order structure,
- for every relational structure $S = (X, R_1, R_2)$, if S is i-directed (see Definition 27) then $S^\blacklozenge$ is an interval order structure.

While transitive closure always give a desired result, $\diamondsuit$ and $\blacklozenge$-closures are partial, but still can be used to produced desired results. The ★-closure is a counterpart of the three closures presented above for generalized mutex order structures. Since mutex order structures can be simulated by stratified order structures, we do not need any special closure operator for them.

Before defining ★-closure we need to introduce some auxiliary concepts.

Definition 31 (Largest Equivalence Relation)**.** *Let R be relation. We define $R^\circledast$, the largest equivalence relation contained in R^* as:*

$$R^\circledast = R^* \cap (R^*)^{-1}.$$

If $\prec$ is interpreted as an "earlier than", $\sqsubset_{so}$ would be a "not later than", $\rightleftharpoons_{so}$ would be a "not simultaneously" and $\sqsubset_{so}^{\circledast}$ would be "simultaneously".

We can now provide the definition of $\bigstar$-closure.

Definition 32 ($\bigstar$-Closure, Janicki *et al.* (2013))**.**
Let $S = (X, R_1, R_2)$ be a relational structure and let $R_1 \star R_2$ be the relation derived from R_1 and R_2 as follows

$$R_1 \star R_2 = R_2^{\circledast} \circ (R_1 \cup (R_2^{*} \circ_{R_1} R_2^{*})^{\bowtie}) \circ R_2^{\circledast}.$$

Then $\bigstar$-closure of S is given by $S^{\bigstar} = (X, R_1 \star R_2, R_2^{\curlywedge})$.

The definition of relation $R_1 \star R_2$ follows from the requirement that $S^{\bigstar}$ should be a GMO-structure and the axioms of GMO-structures: $(R_2^{*} \circ_{R_1} R_2^{*})^{\bowtie}$ is derived from axioms $G4$ and $G6$, while the $R_2^{\circledast} \circ_{R_1} R_2^{\circledast}$ corresponds to axiom $G5$.

As $\lozenge$ and $\blacklozenge$-closure, the $\bigstar$-closure give expected results only for some special order structures.

Definition 33 (Separable Relational Structure)**.** *A relational structure $S = (X, R_1, R_2)$ is called* separable *if:*

1. *$R_1 \cap R_2^{\circledast} = \varnothing$; and*
2. *R_1 is symmetric and R_2 is irreflexive.*

Proposition 13 (Janicki *et al.* (2013))**.** *Every generalized mutex order structure is separable.* ■

The fundamental properties of $\bigstar$-closure are the following.

Proposition 14 (Janicki *et al.* (2013))**.** *If $S = (X, R_1, R_2)$ is a separable relational structure, then:*

1. *$S^{\bigstar}$ is separable,*

2. *$S \subseteq S^{\bigstar}$,*

3. *$\left(S^{\bigstar}\right)^{\bigstar} = S^{\bigstar}$,*

4. *If S is a GMO-structure, then $S^{\bigstar} = S$,*

5. *$S^{\bigstar}$ is a GMO-structure.* $\blacksquare$

As already mentioned, while $\bigstar$-closure is a clear generalization of $\Diamond$-closure, its relationship to $\blacklozenge$-closure is not obvious. The generalized mutex order structures are generalization of stratified order structures but not interval order structures, as they cannot handle concurrent histories containing interval orders that are not stratified orders. Both $\bigstar$-closure and $\blacklozenge$-closure are extensions of $\Diamond$-closure, but in different directions. The counterpart of generalized mutex order structures that can deal with observations that are not stratified orders does not exist yet.

Chapter 6

Conclusion

In this thesis we have investigated the approach for modelling concurrency using relational order structures. It is an axiomatic approach to system specification, where the behaviour of the system is specified by the set of properties it has to satisfy. Then any implementation that satisfies those properties is considered acceptable. This restrictive approach is opposed to the prescriptive approach in which every possible execution of the implemented system must be presented by a corresponding execution in the specification. One can easily see the benefits and (relative) simplicity of the restrictive approach.

We have provided a motivational use case for proving correctness algorithm that requires the use of IO-structures. Then we presented the mathematical model behind the relational structures with a special focus on closure operator. We presented a spectrum of closure operators for various order structures. A new type of closure operator, $\blacklozenge$-closure, was introduced and analyzed. Its properties were compared with that of a regular transitive closure for partial orders, a $\lozenge$-closure for stratified order structures and their generalization, $\bigstar$-closure for generalized mutex order structures.

On the span of several relational structures we can see how the complexity of closure operator reflects the complexity of the underlying relational structure. The complexity is the result of a growing expressive power of the underlying relational structure, and, thus, its ability to capture a very fine grained concurrent behaviours. The properties of closures vary and can be found in Proposition 2, Theorem 6, Theorem 8 and Proposition 14.

The contributions of the thesis and results can be summarized as follows:

1. We defined the $\blacklozenge$-closure of relational structure $S = (X, R_1, R_2)$ in terms of composition of relations R_1 and R_2, where ${R_1^{\blacklozenge}}_{R_1,R_2}$ is a union of all compositions of relations R_1 and R_2, s.t. R_1 is necessarily the first and the last element, and ${R_2^{\blacklozenge}}_{R_1,R_2}$ is a union of all compositions of relations R_1 and R_2, s.t. R_2 is irreflexive and every R_2 is strictly followed by R_1.

2. We proved that given any relational structure $S = (X, R_1, R_2)$:

   (a) If relation R_2 is irreflexive, then relational structure S is the subset of the $\blacklozenge$-closure of that structure, i.e. $S^{\blacklozenge}$.

   (b) The $\blacklozenge$-closure idempotent, i.e. it is closed under itself.

   (c) The sufficient condition for construction of an IO-structure from any relational structure S by taking $\blacklozenge$-closure of it, i.e. $S^{\blacklozenge}$, is that S must be i-directed (see Definition 27).

   (d) If S is an IO-structure, then its $\blacklozenge$-closure is also an IO-structure.

   (e) If $\hat{S}$ is some subset of an IO-structure S, then $\blacklozenge$-closure of $\hat{S}$ is also an IO-structure. Moreover, $\hat{S}^{\blacklozenge}$ also remains a subset of S.

   (f) If S is an IO-structure, then $\lozenge$-closure and $\blacklozenge$-closure are equivalent.

The $\blacklozenge$-closure in a necessary step for constructing a new variation of traces - interval traces. The usefulness of the contributed $\blacklozenge$-closure was verified in the follow up publication Janicki *et al.* (2012), where $\blacklozenge$-closure was used to show relationship between interval traces and interval order structures.

A detailed look at relationship between ⧫-closure and ★-closure is an opportunity for future research. Also, it will be interesting to compare different axiomatizations of IO-structures and how this variability influences constructive proofs for mutual exclusion problems on the example of Bakery algorithm.

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