VELOCITY-DEPENDENT NUCLEON-NUCLEON POTENTIALS

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SCOPE AND CONTENTS:

An investigation is carried out to see whether a nonsingular two-nucleon velocity-dependent potential can give similar results as those of a model hard-core potential in the s-state in nuclear-matter. Reasons for failure of some earlier velocity-dependent model potentials to give saturation in nuclear-matter at reasonable densities are examined and a new velocity-dependent potential is proposed which gives results similar to the model hard-core potential.

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INTRODUCTION

Neutrons and protons, which may be called nucleons, constitute a nucleus. The forces between these particles are of a very short range character, as is known from Rutherford's classic experiments on scattering of α-particles by the atomic nucleus. These forces should be quite strong too, when compared to electromagnetic forces, in order to keep the nucleus bound at the observed density inspite of coulomb repulsion between the protons and the repulsion coming about from surface effects in finite nuclei. The nuclear forces, then, are of a fundamentally different character from electromagnetic or gravitational forces. This force can be made up of two-body forces and many-body forces. In this work, we shall neglect many-body forces. The point of view is taken that all the observed phenomena should be attempted to be explained by two-body forces alone, since these are so much simpler. If we are convinced that these are not sufficient, then three-body and other forces can be introduced, which will make the situation much more complex.

The nucleons obey the Fermi-statistics. This means that there is a built-in many-body force which forbids two protons or two neutrons to be in the same quantum state. This also means that p-p or n-n can interact only in singlet-even or triplet odd states while n-p can interact in all states. We shall be assuming that the nuclear force is charge-independent. This means that if the coulomb force is

disregarded, p-p, n-p, and n-n forces are the same. The fact that only n-p bound-state is found is because n-p can interact in the triplet even states, whereas this is not possible for n-n or p-p.

We have noted, then, that nuclear forces are of short-range ($\doteq 2 \times 10^{-13}$ cm $\doteq$ 2 fm), strong, and charge-independent. In order to study this force in detail, one can proceed in two ways. One is the fundamental approach, where one tries to get the nuclear force due to exchange of some particle or particles between the nucleons. In order to give the right range, these exchange particles should be massive, and they should have strong nuclear interactions. The π-meson has these properties. Yukawa (Y35) was the first to take this approach. The tail of the nuclear potential arises due to the single exchange of a pion. This is the One-Pion-Exchange-Potential, or the O.P.E.P. The internucleon forces in the range less than 1.5 fm can arise due to multiple exchange of pions or due to some heavier particles. In the present state of nuclear theory, one is only sure of the O.P.E.P. part of the two-body potential beyond about 1.5 fm.

In the absence of any complete fundamental theory of nuclear forces, one falls back on the phenomenological approach. One is attempting to describe the force between two nucleons in the nonrelativistic range by a potential (local or nonlocal) which can be plugged into the Schrödinger equation.

Any two-body potential must obey some invariance conditions. Okubo and Marshak (OM58) have given the most general form of the

potential from these considerations. The considerations are:-

(1) Translational invariance in configuration space.

(2) Galilean invariance - which is the nonrelativistic limit of Lorentz invariance.

(3) Symmetry between particle 1 and particle 2.

(4) Rotational invariance.

(5) Space-reflection invariance - which is valid for strong interactions.

(6) Time-reversal invariance

and

(7) Hermiticity of the potential.

Due to the properties of the Pauli spin matrices, one can write V as the sum of a spin independent term, a term linear in σ a term bilinear in σ. The spin independent term can be expressed as $V_o\,(r^2,\ p^2,\ L^2)$, where $\vec{r} = (\vec{r}_1 - \vec{r}_2)$, $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ and $\vec{L} = (\vec{r} \times \vec{p})$. All the potentials that we shall be considering will obey the above invariance restrictions.

Let us now consider how the repulsive core was introduced in the two-nucleon potential (P62).

It was observed in high-energy p-p scattering experiments that the p-p scattering differential cross-section remains amazingly constant at about 4mb/sterad from 170 Mev - 500 Mev.

This type of behaviour can not be explained by an ordinary attractive central force. Consider the even orbital-states first (since singlet even forces are stronger than triplet odd forces). The contribution from the s-state to $\sigma(\theta)$ is proportional to $\sin^2\delta_o$, where δ_o is the s-wave phase-shift. The d-wave gives $\sin^2\delta_2 \left[P_2(\cos\theta)\right]^2$, where $P_2(\cos\theta)$ is Legendre polynomial of

order two. This term alone will give a forward peak, with a zero at about $\theta = 55^{\circ}$. Beyond this $P_2(\cos\theta)$ goes negative. In order to compensate in the backward direction, the interference term $\sin\delta_0 \sin\delta_2 . P_2(\cos\theta)$ must play a role by giving a positive contribution. This is only possible in the backward angles if δ_0 and δ_2 have opposite signs, since $P_2(\cos\theta)$ is negative in this region. Now it is known that the s-wave can 'feel' all the details of the potential, including the short range part, while the higher partial waves mainly 'feel' the tail of the potential, seeing more and more of the close range part, as the energy increases. Since one knows that the O.P.E.P. tail is attractive, it follows that the d-wave phase-shift δ_2 must be positive. Hence the s-wave phase-shift δ_0 must be negative at these energies. This would mean that the potential must be strongly repulsive at short ranges.

The s-wave feels more and more of it as its wave-function gets packed in more and more with increasing energy, while the d - and higher waves, at the energies concerned, do not experience repulsion. Actually, to fill the 'dip' around 55° in $\sigma(\theta)$, one has to consider the triplet-odd forces, and introduce a tensor force.

Jastrow (J51) noted that this behaviour of the singlet even potential could be described by a hard-core of very small radius (0.4 fm) surrounded by an attractive tail. Thus, in this model, the p-p scattering data could be explained by a central potential in singlet-even states, which is a function of $|r|$ only, but has a hard-core; and a noncentral potential in triplet-odd states. It was noted only much later by Peierls (P60) that the phase-shift

behaviour of s- and d - waves could be described by velocity-dependent potentials, which also obey the invariance restrictions, and which are non-singular.

Since the time Jastrow first suggested the hard-core, more and more high energy scattering data have become available. Accurate sets of phase-shifts have been obtained by Stapp et. al. (SYM57) and more recently by Breit and coworkers (B60,61), which fit these experimental data. Gammel and Thaler (GT57) constructed a hard-core potential which more or less fitted Stapp's solution I, which is the most acceptable of the eight sets found, for p-p scattering data at 310 Mev. In the singlet even state, this potential was central, with a hard-core of radius 0.4 fm, followed by a Yukawa type attraction. In the triplet-odd states, however, a pure central plus a tensor force could not reproduce the phase-shift data at high energies. This is because the experiments show that for ranges within 0.7 fm, the $^3p_2$ wave feels much less repulsion than the $^3p_0$ or $^3p_1$ wave. This sort of behaviour is not reproducible by a tensor force, but a potential of $(\vec{L}.\vec{S})$ type does give such a behaviour. Consequently, they introduced a very short-range, but strong $(\vec{L}.\vec{S})$ potential in the triplet-odd state, repulsive in the $^3p_0$ state. Of course the tensor force is also there, but no central attraction was needed in the triplet-odd states. This made the singlet-even state potential much stronger than the triplet odd state. This feature of the even-ℓ state potentials being stronger than those of odd-ℓ state is quite general.

To fit the n-p scattering data, singlet-odd and triplet-even potentials were also constructed by them. These data were relatively few, they retained (L.S) force in triplet-even states also; the hard-core of radius 0.4 fm was retained in all states. These combinations

gave good fit of n-p scattering data at 90 Mev and at lower energies, but it was seen that the parameters had to be changed in order to give good fits at 156 Mev and 310 Mev. Thus the parameters of the G.T. potential were not strictly energy independent for the n-p case. Also, in the singlet even case, the $^1d_2$ - phase shift at 310 Mev is much too large when compared to Stapp's I solution. Both these defects of the Gammel-Thaler potential have been remedied recently by Hamada and Johnston (HJ62), who introduced an additional quadratic (L.S) potential in the interaction. Their model consists of four terms instead of three of Gammel and Thaler:

$$V = V_c + V_T S_{12} + V_{LS} (\vec{L}.\vec{S}) + V_{LL} L_{12}$$

where $L_{12} = \delta_{LJ} + (\sigma_1 . \sigma_2)\ L^2 - (L.S)^2$, and all the V's are functions of r only. The phases produced by the Hamada-Johnston potential are in fair agreement with YLAM (T = 1) and YLAN3M (T = 0) solutions of Breit's group. It is undoubtedly the best two-nucleon potential, so far as the scattering-data is concerned. However, the hard-core radius chosen is about 0.5 fm for all states, and this results in giving too low a binding energy of nuclear matter at too low an equilibrium density (Ra63).

Before going into velocity-dependent potentials, it would be desirable to discuss the motivation of trying to replace the hard-core by velocity-dependence in the potential. Afterall, the hard-core potential reproduces the phases very well. Why then replace it? The objection about it is the infinity it introduces in the potential. It is not very satisfying, physically, to think of a potential which changes abruptly from infinitely repulsive to infinitely attractive. It is too nonsmooth. And this imposes many practical difficulties

when one tries to calculate any properties of the nucleus using such a model. Ordinary Rayleigh Schrödinger perturbation theory fails since individual matrix-elements diverge. One has to use modified perturbation techniques, and sum up a whole class of matrix-elements, in order to get any finite results. These calculations are very complicated, by the very nature of the two-body potential chosen. Brueckner et al. (B58) have done such calculations on nuclear matter and on some finite nuclei using the reaction-matrix method. Gomes, Walecka and Weisskopf (G58) have discussed the nature of the two-body wave function inside nuclear matter when such hard-core potentials are taken in the two-body potential. They took some highly simplified models for simplicity in calculations. Moszkowski and Scott (MS60) developed a very elegant formalism to calculate the properties of nuclear matter when they noted that at the energies concerned, the repulsive core really cancels off a lot of the attraction in the potential, leaving a relatively weak tail which is attractive. We shall consider this method of calculating properites of nuclear matter in detail in Chapter IV when we apply this to velocity-dependent potentials also. Recently Bethe et al. have devised a 'reference spectrum' method of calculating the B.E. of nuclear matter (BBP63).

Having realized that calculations with a non-singular two-body potential would be much simpler, one tries to replace the hard-core by a velocity-dependent potential. The hard-core, as we noted, was introduced to turn the ^1s_o phase-shift negative. This could also be achieved by putting a short-range repulsive velocity-dependent potential, followed by an attractive tail. The

repulsive part will increase in magnitude with increasing energy because of its velocity dependence, ultimately turning the ^1s_o phase-shift negative. It would not, however, affect the $^1d_2$ phase, because of its short-range.

Green (G62) of Birmingham used a potential of the type $V(r) + (p^2w(r) + w(r)p^2)$ for the central part, and added $(\vec{L}.\vec{S})$ and tensor parts in triplet states. $w(r)$ was short-range and repulsive, whereas $V(r)$ of longer range and attractive. We shall consider this potential in detail in Chapter I. Levinger's group at Louisiana State University concentrated mainly on constructing potentials in the singlet-even state. Razavy (Ra61) considered in addition potentials of type $V_o(r) + \vec{p}.w(r)\vec{p}$. If both $V_o(r)$ and $w(r)$ are taken as square-wells, the Schrödinger equation can be solved analytically. Levinger and Simmons (LS61) solved the neutron gas problem, taking a potential of the above type ($V_o(r)$ had an O.P.E.P. tail) and using perturbation method directly. They got substantially the same result as of Brueckner, Gammel and Kubis (BGK60) who used the Gammel Thaler potential and reaction-matrix method. Herendon et al. (HST63) used a velocity-dependent potential to calculate the binding energy of the α-particle by the variational method. They got again results similar to those of hard-core, indicating that in a light nucleus where the average inter-nucleon distance is somewhat larger than the range of the two-body potential used, the binding energy is insensitive to the detailed nature of the potential used. Baker (Ba62) showed, by a unitary transformation,

that outside the range of velocity-dependence, the two-body wave-function is the same as that produced by an angular-momentum dependent potential outside a hard-core. This equivalence of velocity-dependent potential and hard-core holds only outside the range of the velocity-dependent potential, but as we shall see, it is the inside region that is important for giving saturation in nuclear matter.

In order to make a precise comparison with a hard-core potential, Green (G62) used a very simple model. He constructed a velocity dependent potential of type $V(r) + (p^2w(r) + w(r)p^2)$, which fitted the s-wave phase-shift and low energy parameters of the standard-hard-core-potential (S.H.C.P.) of Moszkowski and Scott (MS60). It was assumed that this potential acts only in the s-state and is spin-independent. Green calculated the binding energy of nuclear matter using such a potential. He compared his results with that of Moszkowski and Scott, who had used similar assumptions. Green's results were completely different from the hard-core case, in as much as he did not get any saturation at all. The major part of this thesis will be concerned with this question, and in trying to get a velocity-dependent potential which will give similar results as those of the hard-core potential.

Peischl and Werner (PW63) have recently used a simple velocity-dependent potential very similar in form to one used by Levinger, and have obtained saturation in nuclear-matter. Both the static and the velocity-dependent parts had equal range in their potential, and were simple square-wells. We shall investigate,

in some detail, in Chapters III and IV why their potential gives saturation while Green's s-state potential fails.

In Chapter I, we shall consider Green's full potential (in all states) in detail. We shall briefly review the application of this potential to nuclear matter, to the neutron gas and finally to the optical model. In Chapter II, we shall take Green's simple s-state v.d. potential, and give the results he obtained with it on nuclear matter. We shall consider whether one should expect different results from the hard-core case on the basis of Beg's (B61) work. In Chapter III, we shall see that the off-shell matrix elements of the scattering amplitude can be quite different for the two potentials (hard core and velocity dependent), even though phase-shifts are the same. In order to get similar results in nuclear-matter, the off-shell matrix-elements of the scattering amplitude should also be comparable for the two potentials concerned, in addition to the phase-shift fit. In Chapter IV, we shall apply the Moszkowski-Scott separation method for calculating the binding energy of nuclear-matter to Green's potential considered in Chapter II. This would check the convergence of the perturbation method used by Green, and show where exactly the differences come in. After seeing this, in Chapter V, we shall construct a velocity-dependent potential, which will give more distortion of the two-body wave function in the near region. This potential, which acts only in the s-state, and fits the same phase-shift data as the S.H.C.P. of Moszkowski and Scott, gives saturation in nuclear matter, just as the S.H.C.P. does.

CHAPTER I

VELOCITY-DEPENDENT NUCLEON-NUCLEON POTENTIAL OF GREEN

A. M. Green (G62) has considered two-nucleon potentials of the type

$$U(r,p) = V(r) + \frac{p^2}{m} w(r) + w(r) \frac{p^2}{m}$$

where V(r) consists of central, tensor and spin-orbit parts. p, of course, is the relative momentum, and m the nucleon mass. His objective was to see whether such a potential could reproduce phase-shifts comparable to the Gammel-Thaler potential, and then to apply it to some problems of physical interest.

(a) The Two-body Schrödinger Equation: The potential in the triplet state is

$$U = V(r) + \frac{p^2}{m} w(r) + w(r) \frac{p^2}{m} + V_T(r) S_{12} + V_{LS}(r)\, L.S \qquad (1)$$

where S_{12} is the tensor operator.

We have, $(T + U)\,\Psi = E\Psi$ where Ψ is the wave-function. In the centre-of-mass system, writing (for the triplet case, which is the more complicated)

$$\Psi = \frac{1}{r} \sum_L u_{JL}(r)\, \mathcal{Y}^M_{JL1}, \quad \text{we get}$$

$$\frac{\hbar^2}{m}\left(-\frac{1}{r}\frac{d^2}{dr^2} r + \frac{L^2}{r^2}\right)\sum_L \frac{u_{JL}(r)}{r} \mathcal{Y}^M_{JL1} + V(r)\frac{1}{r}\sum_L u_{JL}(r)\mathcal{Y}^M_{JL1}$$

$$+\frac{\hbar^2}{m}\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{L^2}{r^2}\right)\omega(r)\sum_L \frac{1}{r}u_{JL}(r)\,y^M_{JL1}$$

$$+\,\omega(r)\frac{\hbar^2}{m}\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{L^2}{r^2}\right)\sum_L \frac{1}{r}u_{JL}(r)\,y^M_{JL1}$$

$$+\,V_T(r)\sum_L S_{12}\,\frac{u_{JL}}{r}\,y^M_{JL1}$$

$$+\,V_{LS}\sum_L(\vec{L}.\vec{S})\,\frac{u_{JL}(r)}{r}\,y^M_{JL1}=E\sum_L\frac{u_{JL}(r)}{r}\,y^M_{JL1}$$

Premultiplying by y^{*M}_{JL1} and integrating over solid-angles $d\Omega$, we get

$$-\frac{\hbar^2}{m}\frac{1}{r}\frac{d^2}{dr^2}u_{JL}(r)+\frac{\hbar^2}{m}\frac{L(L+1)}{r^2}\frac{u_{JL}}{r}+V(r)\frac{1}{r}u_{JL}(r)$$

$$-\frac{\hbar^2}{m}\frac{1}{r}\frac{d^2}{dr^2}\left[\omega(r)u_{JL}(r)\right]+\frac{\hbar^2}{m}\frac{L(L+1)}{r^2}\omega(r)\frac{1}{r}u_{JL}(r)$$

$$-\omega(r)\frac{\hbar^2}{m}\frac{1}{r}\frac{d^2}{dr^2}u_{JL}(r)+\omega(r)\frac{\hbar^2}{m}\frac{L(L+1)}{r^2}\frac{u_{JL}(r)}{r}$$

$$+\,V_T(r)\sum_{L'=J-1}^{J+1}\int y^{*M}_{JL1}\,S_{12}\,y^M_{JL'1}\,d\Omega\,\frac{u_{JL'}}{r}$$

$$+\,V_{LS}(r)\sum_{L'}\int y^{*M}_{JL1}\,(\vec{L}.\vec{S})\,y^M_{JL'1}\,d\Omega\,\frac{u_{JL'}}{r}=E\,\frac{u_{JL}(r)}{r}$$

Note that,

$$\left\langle y^{M}_{J,J-1,1} \middle| S_{12} \middle| y^{M}_{J,J-1,1} \right\rangle = -\frac{2(J-1)}{(2J+1)}$$

$$\left\langle y^{M}_{J,J+1,1} \middle| S_{12} \middle| y^{M}_{J,J-1,1} \right\rangle = \frac{6\sqrt{J(J+1)}}{(2J+1)}$$

$$\left\langle y^{M}_{J,J+1,1} \middle| S_{12} \middle| y^{M}_{J,J+1,1} \right\rangle = -\frac{2(J+2)}{(2J+1)}$$

and $$\left\langle y^{M}_{J,J,1} \middle| S_{12} \middle| y^{M}_{J,J,1} \right\rangle = 2$$

Let L = (J-1), then the Schrödinger equation reduces to

$$-\frac{\hbar^2}{m}\left\{\frac{1}{r}\frac{d^2}{dr^2}u_{J,J-1} + \frac{1}{r}\frac{d^2}{dr^2}\left[\omega(r)\,u_{J,J-1}\right] + \frac{\omega(r)}{r}\frac{d^2}{dr^2}u_{J,J-1}\right\}$$

$$+\frac{\hbar^2}{m}\frac{L(L+1)}{r^2}\frac{u_{J,J-1}}{r}(1+2\omega) + V(r)\frac{u_{J,J-1}}{r} - \frac{2(J-1)}{2J+1}V_T(r)\frac{u_{J,J-1}}{r}$$

$$+\frac{6\left[J(J+1)\right]^{1/2}}{(2J+1)}V_T(r)\frac{u_{J,J+1}}{r} + V_{LS}(r)\frac{1}{2}\left[J(J+1)-(J-1)J-2\right]\frac{u_{J,J-1}}{r}$$

$$= E\,\frac{u_{J,J-1}}{r}$$

simplifying this, and using units of $\frac{\hbar^2}{m} = 1$, we get

$$\frac{d^2}{dr^2}u_{J,J-1} + \frac{\omega'' + k^2 - V(r) + \left[\frac{2(J-1)}{2J+1}\right]V_T - (J-1)V_{LS}(r)}{(1+2\omega)}u_{J,J-1}(r)$$

$$+\frac{2\omega'}{(1+2\omega)}\frac{du_{J,J-1}}{dr} - \frac{(J-1)J}{r^2}u_{J,J-1}(r)$$

$$-\frac{6\left[J(J+1)\right]^{1/2}}{(2J+1)}\frac{V_T(r)}{(1+2\omega)}u_{J,J+1}(r) = 0$$

We have put $E = k^2$ above,

For $L = (J + 1)$, the equation is

$$\frac{d^2 u_{J,J+1}}{dr^2} + \frac{\omega'' + k^2 - V(r) + \frac{2(J+2)}{2J+1} V_T(r) + (J+2) V_{LS}(r)}{(1+2\omega)} u_{J,J+1}$$

$$+ \frac{2\omega'}{(1+2\omega)} \frac{du_{J,J+1}}{dr} - \frac{(J+1)(J+2)}{r^2} u_{J,J+1}(r)$$

$$- \frac{6\left[J(J+1)\right]^{1/2}}{(2J+1)} \frac{V_T(r)}{(1+2\omega)} u_{J,J-1}(r) = 0$$

For J = L, however, the equation remains uncoupled,

$$\frac{d^2 u_{J,J}}{dr^2} + \frac{\omega'' + k^2 - V(r) - 2V_T(r) + V_{LS}(r)}{(1+2\omega)} u_{J,J}$$

$$+ \frac{2\omega'}{(1+2\omega)} \frac{du_{J,J}}{dr}$$

$$- \frac{J(J+1)}{r^2} u_{J,J}(r) = 0$$

These equations can be put in a more simplified form by making the substitutions $\eta^2 = (1+2\omega)$

$$\begin{aligned} y_{J,J-1} &= \eta u_{J,J-1} \\ y_{J,J} &= \eta u_{J,J} \\ y_{J,J+1} &= \eta u_{I,J+1} \end{aligned} \qquad (2)$$

Then, after some manipulations, the equations can be reduced to the form

$$\frac{d^2 y_{J,J-1}}{dr^2} = g_1(r)\, y_{J,J-1} + n(r)\, y_{J,J+1} \tag{3}$$

$$\frac{d^2 y_{J,J+1}}{dr^2} = g_2(r)\, y_{J,J+1} + n(r)\, y_{J,J-1} \tag{4}$$

and $$\frac{d^2 y_{J,J}}{dr^2} = g(r)\, y_{J,J} \tag{5}$$

where

$$g_1(r) = \left[\frac{J(J-1)}{r^2} - \frac{\omega'^2}{(1+2\omega)^2} - \frac{k^2 - V(r) + \frac{2(J-1)}{2J+1} V_T - (J-1) V_{LS}}{(1+2\omega)} \right]$$

$$g_2(r) = \left[\frac{(J+1)(J+2)}{r^2} - \frac{\omega'^2}{(1+2\omega)^2} - \frac{k^2 - V(r) + \frac{2(J+2)}{2J+1} V_T + (J+2) V_{LS}}{(1+2\omega)} \right]$$

$$g(r) = \left[\frac{J(J+1)}{r^2} - \frac{\omega'^2}{(1+2\omega)^2} - \frac{k^2 - V(r) - 2V_T(r) + V_{LS}(r)}{(1+2\omega)} \right]$$

and $$n(r) = \frac{6\left[J(J+1)\right]^{1/2}}{(2J+1)} \frac{V_T(r)}{(1+2\omega)} \tag{6}$$

For the singlet case, the equation is identical to (5), except that one puts $V_{LS} = V_T = 0$ in g(r).

(b) Phase-shifts: We shall discuss the more complicated triplet case briefly first. In the asymptotic region, the coupled equations (3) and (4) become uncoupled, as $V_T(r) \to 0$, and the solutions take

the following form:

$$y_{J-1,J} \xrightarrow{r\to\infty} A_1 e^{-i\left(kr-\frac{1}{2}(J-1)\pi\right)} - B_1 e^{i\left(kr-\frac{1}{2}(J-1)\pi\right)} \tag{7}$$

$$y_{J+1,J} \xrightarrow{r\to\infty} A_2 e^{-i\left(kr-\frac{1}{2}(J+1)\pi\right)} - B_2 e^{i\left(kr-\frac{1}{2}(J+1)\pi\right)}$$

here A's, B's are constants, and in general complex. One defines the S-matrix by the relation

$$B = SA, \tag{8}$$

where

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$, and S is a 2 x 2 matrix

using the unitarity and symmetry of the S-matrix, one can parametrize it using three real parameters (HS.57). These are the two coupled phases and the coupling parameter. In the Blatt-Biedenharn (BB52) form, the S-matrix is written as

$$S = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} \exp(2i\delta_-) & 0 \\ 0 & \exp(2i\delta_+) \end{pmatrix} \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}$$

These are the Blatt-Biedenharn phase-parameters, which can be easily transformed to the 'bar phases', using the formulae given by Stapp et. al.(SYM57). Although we have written four complex constants in (7), only two of these are independent. It can be shown that $\mathrm{Re}A_1 = -\mathrm{Re}B_1$; $\mathrm{Im}\,A_1 = \mathrm{Im}\,B_1$. Hence, there are only two complex constants, or four real constants, in (7).

Using (7) and (8), it can be shown that

$$\tan \delta_- = \frac{\mathrm{Re}\ A_1 + \mathrm{Re}\ A_2 \tan\epsilon}{\mathrm{Im}\ A_1 + \mathrm{Im}\ A_2 \tan\epsilon}$$

and

$$\tan \delta_+ = \frac{\mathrm{Re}\ A_2 - \mathrm{Re}\ A_1 \tan\epsilon}{\mathrm{Im}\ A_2 - \mathrm{Im}\ A_1 \tan\epsilon} \qquad (9)$$

It is more convenient to write the asymptotic solutions in the form

$$\begin{aligned} \psi_{J-1,J} &\to r \left[A\, j_{J-1}(kr) + B\, \eta_{J-1}(kr) \right] \\ \psi_{J+1,J} &\to r \left[A'\, j_{J+1}(kr) + B'\, \eta_{J+1}(kr) \right] \end{aligned} \qquad (10)$$

Then (9) can be written as

$$\begin{aligned} \tan \delta_- &= -\frac{(B + B' \tan\epsilon)}{(A + A' \tan\epsilon)} \\ \tan \delta_+ &= \frac{(-B' + B \tan\epsilon)}{(A' - A \tan\epsilon)} \end{aligned} \qquad (11)$$

In (10), all the constants are real.

Equations (3) and (4) can be solved with two different sets of initial values, giving two sets of independent solutions. By solving the coupled differential equations numerically and going to the asymptotic region, one can find the values of δ_+, δ_- and $\tan\epsilon$. These, in turn, can be transformed to bar-phase-shifts. These phases for both triplet and singlet states were found first by Green (G61), and we have extended the calculations to a large set of phase-shifts (PAB62). Our results are shown in appendix 1 where comparison has been made with Gammel-Thaler, Hamada-Johnston and Breit's phases at

90 Mev, 156 Mev and 310 Mev. The singlet and triplet-even phases are as good as for the Gammel-Thaler case, but triplet-odd phases needs improving. The numerical parameters and the detailed form of the potential is also given in appendix 1.

(c) Application of Nuclear-matter: Green applied the above potential for calculating the binding energy of nuclear matter. Ordinary Rayleigh-Schrödinger perturbation theory was used. The results were not at all satisfactory, since it was found that the convergence is poor. Applying just first-order theory, saturation was obtained around the right value of equilibrium density, ($k_F \sim 1.5$ fm^{-1}), but there was too little binding (~ -2 Mev/particle). However, these first order values do not mean much, since second order contributions are large. The second order direct term from the central forces was not too large, (singlet even potential contributed about -4 Mev/particle at $k_F = 1.5$ fm^{-1}). The exchange term from the central forces was not calculated, but a rough estimate indicated it to be small. The main trouble arose from the tensor force, which starts contributing in the second order. The triplet even tensor potential, for example, contributes about -11 Mev/particle at $k_F = 1.5$ fm^{-1}. In order to reach any conclusion about the convergence of the series at all, it would be necessary to calculate higher order terms. It is clear, however, that simple Rayleigh-Schrödinger type perturbation theory is not very suitable for binding energy calculations even for a non-singular potential like Green's, specially for the tensor part of the potential (DaP63).

(d) Neutron Gas: This problem is a neat one, because only the singlet-even forces are important here, the triplet-odd forces being very weak.

Simple perturbation theory is more reliable in this case if one neglects the noncentral forces in the triplet-odd states, and also because the kinetic energy of the neutron gas, $T \gg \langle v \rangle$, the average potential energy. This is because triplet-even interaction is absent, for this case. In nuclear matter, the situation is much less favourable, since there T and $\langle v \rangle$ are comparable, and their difference is a small quantity, which can be drastically modified by higher order terms. Green calculated the average energy per particle of the neutron gas in the range $k_F = 0.5$ to 1.5 fm^{-1}, and his results were in agreement with those calculated by Levinger and Simmons (LS61). The results of the neutron gas binding energy calculations with velocity-dependent potentials and with the hard-core potential of Gammel and Thaler agree well at small densities ($0.5 < k_F < 1.0$ fm^{-1}), but wider divergences appear at higher densities. This, as Levinger and Simmons have pointed out, may be expected on the basis of Beg's (B61) work. We shall discuss this point in some detail in chapter II. The hard-core calculations were done by Brueckner, Gammel and Kubis (BGK60). There is no experimental data on the neutron gas, but all calculations indicate that it is unbound.

(e) <u>The Optical-model potential</u>: Kerman et. al (KMT59) had previously calculated, among other quantities, the optical model potential at high energies, using Gammel-Thaler phases. They made two important approximations:

(i) The multiple-scattering approximation, according to which the total T-matrix could be written as a sum of two-body t-matrices, and

(ii) The impulse approximation, by which one replaces the actual two-body t-matrix in nuclear matter by the two-body t-matrix in free space. These approximations are good at high energies ($E_{lab} > 90$ Mev).

With the above two approximations, the optical model potential can be expressed in terms of phase-shifts alone. The higher partial waves upto J = 5 are important in this calculation at 310 Mev, although Kerman et. al. cut off earlier.

We (PAB62) have applied the above formalism to calculate the integrated optical potential using Green's two-body velocity-dependent potential. The results are given in appendix 1. It will be seen that the agreement with experimental values is as good as in the Gammel-Thaler case. Of course, since the optical potential is expressible in terms of phase-shifts alone, two potentials which give the same phases would also give identical optical potential.

CHAPTER II

CALCULATIONS ON NUCLEAR MATTER IN s-STATE: DIFFERENCES IN RESULTS

The nuclear matter calculation with the complete velocity-dependent potential was considerably complicated by non-central forces, as we saw in chapter I. No definite comparison with the hard-core potential could be made because the convergence of the perturbation series was so poor, mainly due to the tensor forces. The question of interest is, would a velocity-dependent potential, which has the same two-body scattering data as a hard-core potential, give similar results as those of the hard-core potential in nuclear-matter calculations? To decide this question, Green (G62 I, II) chose a very simple model. In this model, the two-nucleon potential was assumed to be spin-independent and acting only in the s-state. The low-energy parameters and the phase-shifts of such a potential upto 300 Mev were chosen to be identical to the standard-hard-core potential (SHCP) of Moszkowski and Scott (MS61). The effective range is 2.5 fm and scattering length is infinity, so that there is no two-body bound-state. The phase-shift characteristic is very similar to that of the singlet ^{1}s-state, and is shown in fig. (II-1) together with the S.H.C.P. phase-shifts. Specifically, the form of potential chosen by Green was again given by

$$V(r) = -A \exp(-a^2r^2) + \frac{\hbar^2}{m}\left\{p^2 \exp(-b^2r^2) + \exp(-b^2r^2)\, p^2\right\} \qquad (1)$$

where

$A = 1.062\ fm^{-2}$, $a = 0.627\ fm^{-1}$, $\beta^2 = 1.131$, $b = 1.44\ fm^{-1}$ in units of $\frac{\hbar^2}{m} = 1$. This particular potential, which Green calls No. 3 potential, fits the S.H.C.P. data best.

Green applied this simple velocity-dependent potential to calculate the binding energy of nuclear-matter. In his first paper (G62 I), he applied simple Rayleigh-Schrödinger type perturbation theory, modifying the propagator by a one-body potential derived from the first-order calculation. The convergence was not too good, the second-order contribution being as large as -11 Mev/particle at $k_f = 1.5\ fm^{-1}$. In the next paper (G62 II) he used a modified perturbation theory, using a transformed wave function and an effective potential. The convergence was much better now, the second-order contribution being about -3 Mev/particle at $k_f = 1.5\ fm^{-1}$. The binding energy curves are shown in fig. (IV 4) together with the curves of Scott and Moszkowski (SM62) who calculated the same with the standard hard-core potential. The results of the two potentials are very different. Whereas the hard-core potential gives saturation around $k_f = 1.40\ fm^{-1}$, with a binding energy of about -10.6 Mev/particle, the velocity-dependent potential shows no sign of saturation even at $k_f = 1.8\ fm^{-1}$. Green's curve will presumably turn up at some higher value of k_f, and the binding will be much too large. This result of Green we have checked by using a completely different method of calculating the binding energy of nuclear matter - the separation method of Moszkowski and Scott. The hard-core

calculations were done by Moszkowski and Scott using this method, and by doing a similar calculation with the velocity-dependent potential, we could appreciate where and why the differences come in. These details will be given in chapter IV, where the separation method will also be discussed in some detail.

Green comments (GI62) that these results are in disagreement with Beg's (B61) conclusions about equivalent potentials. Our point of view is just opposite: we think that different results may be expected for the potentials concerned on the basis of Beg's work. In order to see this, let us go over the salient points in Beg's work. From the previous work of Gel'fand and Levitan (GL51), and Jost and Kohn (JK52) one knows that a local potential in an angular momentum state is uniquely determined if

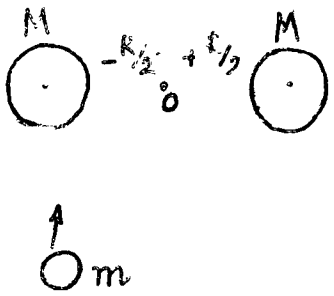
(a) The phase-shifts in that angular momentum state are known for all positive energies;

(b) The energies of the two-body bound-states are known; and

(c) The residua of the S-matrix at the poles corresponding to the bound-states are known.

Since these residua can not be fixed by two-body scattering experiments, one can construct a set of equivalent potentials which give the same two-body scattering data. The situation is less complicated if there is no two-body bound-state - but in any case one needs to know the phase-shifts for a prescribed ℓ for all positive energies to determine the potential uniquely. The objective of Beg's paper is to investigate whether one can distinguish between

equivalent potentials by looking at the three-body amplitude. To this purpose, Beg uses the following simplified model. The scatterer consists of two bound particles, of mass M each. The incident particle, of mass m (m<<(M)



interacts identically with these two centres of scattering. The centre of mass of the scatterers are supposed to be fixed at O, although their distance, R, can vary according to the density of the system concerned. The collision operator for the system can be written as

$$T = (V_1 + V_2)\left[1 - G\,(V_1 + V_2)\right]^{-1}, \quad \text{where}$$

V_1, V_2 are the interactions of the incident particle with the targets. G, for the outgoing wave is

$G = (E + i0 - H_o)^{-1}$, where E is the total energy of the system (including the interaction energy, and $H_o = H_{target} + H_{incident}$, without any interaction. One can then define the composite scattering amplitude by

$$f(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\,\langle \vec{k}',0|\ T\ |\vec{k},0\rangle$$

where 0 denotes the ground-state of the target system and $\vec{k}$, $\vec{k}'$ the initial and final states of the incident particle. It is thus assumed that the target remains unexcited.

One can write

$$\mathcal{F}(\vec{k}',\vec{k}) = -2 \int_0^\infty R^2\, dR\, \rho(R)\ \mathcal{T}(\vec{k}', \vec{k}, R)$$

where $\rho(R) = |\phi_o(R)|^2$, ϕ_o being the ground-state target wavefunction, and $\mathcal{T}$ can be split up into two parts, using a multiple scattering expansion:-

$$\mathcal{T}(\vec{k}',\vec{k}; R) = j_o(qR)\ \langle\vec{k}'|\ t\ |\vec{k}\rangle + \int \frac{d^3p}{(2\pi)^3}\ \frac{\langle\vec{k}'|\ t|\vec{p}\rangle\ \langle\vec{p}|\ t|\vec{k}\rangle j_o(|\vec{K}-\vec{p}|\ R)}{(k^2 + i0 - p^2)}$$

here

$\vec{K} = \vec{q} + \vec{k}$; $\vec{q} = \frac{1}{2}(\vec{k} - \vec{k}')$; j_o is spherical Bessel function of zero order, and t is the one-centre collision operator located at the origin. It will be noticed that $\mathcal{T}$ has been split up into a single-centre scattering part and a double-centre scattering part. Specializing to the case of s-wave scattering alone (where expressions are simpler), Beg writes

$$\mathcal{T}_o(k', k; R) = j_o(qR)\ \langle k|\ t_o|\ k\rangle + \frac{j_o(kR)}{2\pi^2}\ I_o(k, R)$$

where $I_o = -\frac{i}{2R} \int_{-\infty}^{+\infty} pdp\ \frac{\exp(ipR)\ \langle p|to|k\rangle}{(k^2 + i0 - p^2)}^2$.

We notice that $\mathcal{T}_o$ is expressible in two terms - one of which is the on-the-energy shell one-centre amplitude part, whereas the other contains off-shell elements. By specializing further to the case of interactions of finite range (i.e., the interaction between the incident particle and a target centre vanishes exactly beyond a range 'a'), Beg shows that I_o <u>also</u> is expressible in terms of on-shell one-centre amplitude $\langle k|\ t_o\ |\ k\rangle$, <u>provided R>2a</u>. This means that if there is no 'overlap' of the target particles, the

off-shell contributions to the composite scattering amplitude can be simply expressed in terms of two-body physical (on-shell) amplitudes. So, for R>2a, no additional information can be gained from 3-body amplitude. One can write

$\mathcal{F}(\vec{k}', \vec{k}) = \mathcal{F}_1(\vec{k}', \vec{k}) + \mathcal{F}_2(\vec{k}', \vec{k})$, where

$\mathcal{F}_1(\vec{k}', \vec{k}) = -2 \int_0^{2a} R^2 \, dr \, \rho(R) \, \tau(\vec{k}', \vec{k}, R)$ is the part containing off-shell two-body amplitudes, and hence containing additional information about the potential. Beg then goes on to construct two potentials, one local and the other nonlocal separable (i.e., of the form $\langle p | V | k \rangle = v(p)v(k)$), which have the same scattering matrix, but the potential sensitive parts $\mathcal{F}_1(\vec{k}', \vec{k})$ for the two are quite different.

For interactions with infinite tails, the composite amplitude always contains off-shell amplitudes. Thus it is seen that one can hope to distinguish between potentials which give the same phase-shifts by observing quantities which involve the composite amplitude - specially at high densities where there will be 'overlap' between target particles. At equilibrium densities of nuclei, the average distance between nucleons is about 1.7 fm, whereas the range of nuclear force is ~ 2 fm, and overlap can be expected. Thus there is no reason why one should not expect different results in the binding-energy curves of nuclear matter at higher densities, when calculations are done with velocity-dependent and hard-core potentials. This will

become more evident when we calculate, in the next chapter, off-shell two-body scattering amplitudes for the two types of potential which have the same on-shell amplitudes. It will be seen that the off-shell parts are quite different for the types of velocity-dependent potentials considered, and the hard-core potential, because of the two-body wavefunction at short distances being different. We shall see (chapter V), that modifications can be made in the velocity-dependent potential, such that off-shell amplitudes will also be approximately the same as the hard-core potential. Then one can expect similar results in many-body calculations from the two potentials.

CHAPTER III

ON-AND OFF-THE-ENERGY-SHELL MATRIX-ELEMENTS

We saw in chapter II that two potentials (with no two-body bound-state) having the same phase-shift characteristics upto about 300 MeV gave different results when applied to the many-body-problem. On the other hand, Peischl and Werner (PW63), using a simple Levinger type velocity-dependent potential, get saturation in nuclear-matter. In this chapter, we shall see that in order to get similar results in nuclear-matter, not only the phase-shifts but also the off-shell elements of the scattering amplitude should be comparable. This is so since these off-shell amplitudes do come into nuclear-matter calculations. When we do nuclear-matter calculations in the next chapter, this will become clear.

If two potentials have the same phase-shifts, the asymptotic parts of the two-body wavefunction arising from the two potentials must be identical. Any differences, then, arise from different behaviour of the wavefunctions at short ranges. The only restriction to the short-range part of the wavefunction in elastic scattering comes from the effective-range integral at zero energy: $r_o = 2 \int_0^\infty (v_o^2 - u_o^2)\, dr$, where v_o is the asymptotic and u_o the actual wavefunction at zero energy. But otherwise, the wavefunctions within the potential range can be quite different,

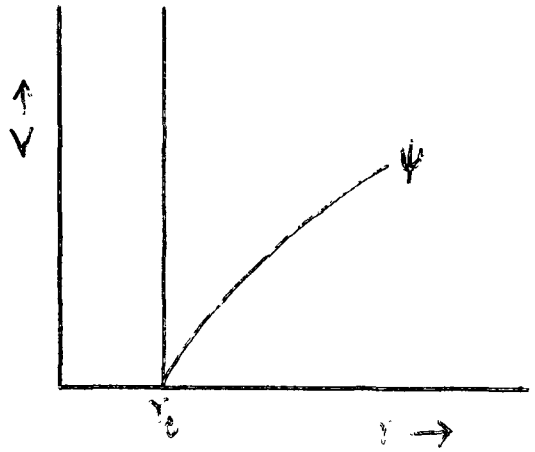
although giving the same phase-shifts.

It is known that on the energy shell, the scattering amplitude, $f(\vec{k},\vec{k})$ for two-body scattering can be expressed in terms of phase-shifts alone. It is the off-shell part that depends sensitively on the nature of the wavefunction within the potential range. A condition for a velocity-dependent potential to give similar results in nuclear-matter to those of a hard-core potential is that it should have comparable off-shell elements. We shall see that a simple Levinger type velocity-dependent potential, with square-wells of equal range for both the static and velocity-dependent part, gives off-shell matrix-elements of scattering amplitude, comparable to those of a hard-core Gomes-Walacka-Weisskopf type of potential. This is exactly the type considered by Peischl and Werner and they get saturation. On the other hand, in the next chapter, we shall see that the off-shell scattering amplitudes for Green's s-wave potential are very much smaller than the standard-hard-core potential of Scott and Moszkowski. That is why saturation was not obtained with Green's potential. In the last chapter, we shall modify Green's potential such that off-shell elements will be comparable, and saturation in nuclear-matter will be obtained.

As a first orientation, we shall take a pure hard-core and a velocity-dependent potential of the type considered by Levinger et. al. The phases of these will be adjusted to be identical upto about 200 MeV. We shall then compute the off-shell elements to see how different these are for the two potentials. Next, a more realistic case will

be treated.

Let us consider the scattering problem with a pure hard-core, in the s-state.



The Schrödinger equation for

$r < r_c$ is (with $\frac{\hbar^2}{m} = 1$)

$$\frac{d^2u^I}{dr^2} + k^2u^I = V u^I \quad - (1)$$

and for $r > r_c$, it is

$$\frac{d^2u^{II}}{dr^2} + k^2u^{II} = 0 \qquad (2).$$ Here $u = r\psi(r)$.

Consider the first equation, with V not infinite, but very large, such that $V >> k^2$. Then we can write (1) as

$$\frac{d^2u^I}{dr^2} \approx Vu^I$$

or

$$\left.\frac{du^I}{dr}\right|_r = \int_0^r Vu^I \, dr \qquad (3)$$

But the solution outside the core is given by

$$u^{II} = \frac{e^{i\delta_o}}{k} \sin(kr + \delta_o), \text{ with } \delta_o \to -kr_c \text{ as } V \to \infty,$$

so $$\frac{du^{II}}{dr} = e^{i\delta_o} \cos(kr + \delta_o) \qquad (4)$$

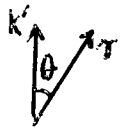
Equating (3) and (4) at $r = r_c$, and remembering that $\cos(kr_c + \delta_o) \to 1$ and $V \to \infty$, we get

$$\int_0^{r_c} V u^I \, dr = e^{i\delta_o} \quad \text{as } V \to \infty.$$

One solution of this is obviously

$$V u^I = e^{i\delta_o} \, \delta(r - r_c) \quad \text{as } V \to \infty. \tag{5}$$

Now, to compute the off-shell scattering amplitude,



$$f(\vec{k}', \vec{k}) = -(4\pi)^{-1} \int e^{-i\vec{k}'\cdot\vec{r}} \, V\psi_k(r) \, \vec{dr} \tag{6}$$

$$= -(4\pi)^{-1} \int e^{-i\vec{k}'\cdot\vec{r}} \, V \frac{u_k(r)}{r} \, r^2 \, d\Omega \, dr$$

$$= -(4\pi)^{-1} \int e^{-i\vec{k}'\cdot\vec{r}} \, e^{i\delta_o(k)} \, \delta(r - r_c) \, r \, d\Omega \, dr, \text{ from (5).}$$

$$= -(4\pi)^{-1} \int e^{-ik'r\cos\theta} \, e^{i\delta_o(k)} \, \delta(r - r_c) \, r \, d(\cos\theta) \, d\varphi \, dr.$$

$$= -\int \frac{\sin k'r}{k'r} \, e^{i\delta_o(k)} \, \delta(r - r_c) \, r \, dr$$

$$= -\frac{\sin k'r_c}{k'} \, e^{i\delta_o(k)}$$

But since $(-k'r_c)$ is the phase-shift for a wave with wave-number k', this can be written in the alternative form

$$f(\vec{k}', \vec{k}) = \frac{\sin \delta_o(k')}{k'} \, e^{i\delta_o(k)} \tag{7}$$

This result is remarkable in that the off-shell scattering amplitude for a pure hard-core is expressible in terms of phase-shifts alone. This will not be true when the hard-core is surrounded by a static finite potential. Also, we note that for a hard-core,

$$\text{the scattering length} \quad a_o = r_c$$

$$\text{effective range} \quad r_e = \frac{2}{3} r_c \tag{8}$$

Now we shall try to get the same scattering characteristics as that of a hard-core by taking a velocity-dependent potential of the Levinger type:

$$V = V_o J_1(r) + \frac{\lambda}{m} \vec{p} . J_2(r) \vec{p} \tag{9}$$

For simplicity, we take

$J_1(r) = J_2(r) = \left[1 - \theta (r-b)\right]$ where $\theta (r-b)$ is the unit step function. That is, both the static and the velocity-dependent parts are square-wells of the same range. Such simple potentials have been considered by Razavy (Ra61).

The Schrödinger equation with such a potential takes the form, for $r<b$, $(\frac{\hbar}{m}^2 = 1)$,

$$u_l'' - k_1^2 u_l = 0, \text{ where } k_1^2 = \frac{1}{(1+\lambda)} (V_o - k^2) \tag{10}$$

In order to get expressions for scattering-length and effective-range with such a potential, we are interested in the region $k \to o$, so that k_1^2 can be assumed positive.

So the wavefunction is of the form

$$u_I = A \sinh k_1 r \quad \text{for} \quad r<b$$
$$u_{II} = B \sin (kr + \delta_o) \quad \text{for} \quad r>b \tag{11}$$

It can be shown (Ra61) that

$$\tan (kb + \delta_o) = \frac{k \sinh k_1 b}{(1+\lambda) k_1 \cosh k b - \frac{\lambda}{b} \sinh k_1 b}$$

From this, we can get a closed expression for the scattering length which is of a different form from Rozavy's. This can be done by calculating $k \cot \delta_o$, as $k \to 0$. We get the scattering-length

$$a_o = b - \frac{b}{(1+\lambda) b K' \coth bK' - \lambda} \tag{12}$$

where

$$K' = \sqrt{\frac{V_o}{1+\lambda}}$$

Also, by making use of the expression

$r_o = 2 \int_0^\infty (v_o^2 - u_o^2)\, dr$, we get,

$$r_o = 2 \left(b + \frac{b^3}{3a_o^2} - \frac{b^2}{a_o}\right) - \left(1 - \frac{b}{a_o}\right)^2 \left[\frac{1}{K'} \coth K'b - \frac{b}{\sinh^2 K'b}\right] \tag{13}$$

To get the same low-energy parameters as those of a hard-core, we use equations (8), (12) and (13) to fix two of the three adjustable

parameters of the velocity-dependent potential (9). The other parameter can be varied to get the best fit in phase-shift upto 200 MeV.

To simulate a hard-core of radius $r_c = 0.4$ fm, the parameters in the velocity-dependent potential (9) were found to be:

$$b = 1 \text{ fm}, \quad V_o = 2.82 \text{ fm}^{-2}, \quad \lambda = -0.5 \qquad (14)$$

The s-wave phase-shift is plotted against wave number k for the two cases in fig.(III-1). Using (7), we can at once find $f(\vec{k}', \vec{k})$ for the hard-core. Our next aim is to get an expression for $f(\vec{k}', \vec{k})$ for the velocity-dependent case. We can then make numerical estimates for the two cases.

In order to do this, we shall follow a derivation due to Fulton and Schwed (FS59) who express $f(\vec{k}', \vec{k})$ in terms of $f(\vec{k}, \vec{k})$ and an integral of the distortion of the wavefunction from its unperturbed form over the potential range. The Schrödinger equation is

$$(\nabla^2 + k^2)\, \psi_k = V\psi_k \,, \quad \text{where again } \frac{\hbar^2}{m} = 1$$

But

$$f(\vec{k}', \vec{k}) = -(4\pi)^{-1} \int e^{-i\vec{k}'\cdot\vec{r}}\, V(r)\psi_k(r)\, d^3r$$

$$= -(4\pi)^{-1} \int e^{-i\vec{k}'\cdot\vec{r}}\, (k^2 + \nabla^2)\, \psi_k(r)\, d^3r$$

For <u>s-wave</u> scattering,

$$\psi_k(r) \xrightarrow[r \to \infty]{} \frac{v_k(r)}{r} = e^{i\delta(k)}\, \frac{\sin[kr + \delta(k)]}{kr} = \eta_k(r), \text{ say.}$$

Since $(k^2 + \nabla^2)\, \eta_k(r) = 0$, one can write

$$f(\vec{k}',\vec{k}) = -(4\pi)^{-1} \int e^{-i\vec{k}'\cdot\vec{r}}\, (k^2 + \nabla^2)\, (\Psi_k(r) - \eta_k(r))\, d^3r$$

Integrating by parts, after some manipulation, one can write

$$f(\vec{k}',\vec{k}) = (k'^2 - k^2) \int_0^\infty \frac{\sin k'r}{k'} \left[u_k(r) - v_k(r) \right] dr + f(\vec{k},\vec{k}) \qquad (15)$$

Here $u_k(r)$ is the actual radial wavefunction and $v_k(r)$ its asymptotic form. The integrand vanishes outside the potential range. Note that for a pure hard-core, $u_k = 0$ for $r \leqslant r_c$, and equation (15) can be reduced to the simple form of (7).

In order to evaluate $f(\vec{k}',\vec{k})$ for the velocity-dependent potential that we are considering in this chapter, we note, from (11), that

$$u_k(r) = A \sinh k_1 r, \quad \text{where} \quad k_1^2 = \frac{1}{(1+\lambda)} (V_o - k^2)$$

and $v_k(r) = B \sin(kr + \delta(k))$.

If we choose $B = \frac{e^{i\delta(k)}}{k}$, then, since the wavefunction is continuous at $r = b$,

$$A = \frac{e^{i\delta(k)}}{k} \frac{\sin(kb + \delta_o)}{\sinh(k_1 b)}$$

Making these substitutions and carrying on the integrations, which are of standard forms, we finally get

$$f(\vec{k}',\vec{k}) = \frac{(k'^2 - k^2)}{(k'^2 + k^2)} \frac{e^{i\delta_o(k)}}{kk'} \sin(kb + \delta_o)$$

$$\left\{k_1 \sin(k'b) \coth(k_1 b) - k' \cos(k'b)\right\}$$

$$- \frac{e^{i\delta_o(k)}}{kk'} \left\{k \sin k'b \cos(kb + \delta_o)\right.$$

$$\left. -k' \sin(kb + \delta_o) \cos(k'b) + k' \sin\delta_o \right\} + f(\vec{k},\vec{k}). \quad (15b)$$

Note that the first two terms in the right-hand side vanish for $k = k'$.

Now let us see how much different $f(\vec{k}',\vec{k})$ is, for a velocity-dependent potential, given by (15b), and a pure hard-core, given by (7). The $f(\vec{k},\vec{k})$'s for the two are of course the same upto at least $k = 1.5$ fm^{-1}.

For a hard-core:

$$f(\vec{k}',\vec{k}) - f(\vec{k},\vec{k}) = e^{i\delta_o(k)} \left\{\frac{\sin\delta_o(k')}{k'} - \frac{\sin\delta_o(k)}{k}\right\}$$

Taking $r_c = 0.4$ fm, $k' = 0.5$ fm^{-1}, $k = 1.0$ fm^{-1}, a simple calculation yields,

$$f(\vec{k}',\vec{k}) - f(\vec{k},\vec{k}) \Big|_{\substack{k' = .5 \text{ fm}^{-1} \\ k = 1.0 \text{ fm}^{-1}}} = -e^{-i(0.4)} \ (.0080) \text{ fm. (hard-core)}$$

For the velocity-dependent potential with parameters given by (14), the same quantity, when evaluated according to (15b), is

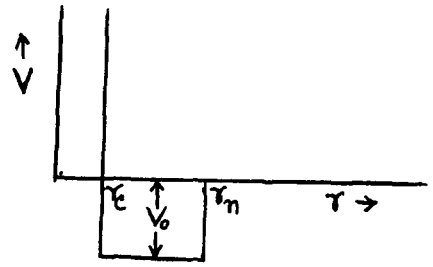
$$f(\vec{k}',\vec{k}) - f(\vec{k},\vec{k})\Big|_{\substack{k' = .5\ \text{fm}^{-1} \\ k = 1.0\ \text{fm}^{-1}}} = -e^{-i(0.4)}\,(.0179)\ \text{fm} \qquad \begin{matrix}\text{(velocity-}\\ \text{dependent}\\ \text{potential)}\end{matrix}$$

which is about double the hard-core value. Numerical calculations will be done in some detail for the next case, where a hard-core is surrounded by an attractive square-well.

Hard-core potential of Gomes-Walecka-Weisskopf type:

Again, we put $\frac{\hbar^2}{m} = 1$.

Define $k_1^2 = (k^2 + V_o)$, then the radial wave-function in s-state is



$$u_I = 0 \text{ for } r < r_c$$

$$u_{II} = A \sin k_1(r-r_c),\ r_c < r < r_n, \text{ where } A = \frac{e^{i\delta_o}}{k}\ \frac{\sin(kr_n + \delta_o)}{\sin k_1(r_n - r_c)}$$

$$u_{III} = \frac{e^{i\delta_o}}{k}\ \sin(kr + \delta_o),\quad r > r_n \qquad (16)$$

One can easily derive

$$\tan\delta_o(k) = \frac{k \tan k_1(r_n - r_c) - k_1 \tan kr_n}{k_1 + k \tan k_1(r_n - r_c) \tan kr_n} \qquad (17)$$

Using the expansion of $k \cot\delta_o$, it can be seen that the scattering length is

$$a = r_n - \frac{1}{K}\ \tan K(r_n - r_c) \qquad (18)$$

where $K^2 = V_o$ (since $\frac{\hbar^2}{m} = 1$)

Making use of the effective-range integral, the effective range

$$r_o = r_n - \frac{r_n^3}{3a^2} - \frac{(a-r_c)}{a^2K^2} + \frac{r_c\,(r_n - a)^2}{a^2} \qquad (19)$$

Gomes, Walecka and Weisskopf take the scattering length to be infinity, same as the standard-hard-core of Moszkowski and Scott. From (18), then

$$K\,(r_n - r_c) = \frac{\pi}{2}$$

from (19), if $a \to \infty$,

$$r_o = (r_n + r_c)$$

The values used by Gomes et. al. are,

$r_n = 2.3$ fm, $r_c = 0.4$ fm, $K^2 = 0.682$ fm^{-2}.

This gives $a \to \infty$ and $r_o = 2.7$ fm. We shall now calculate the off-shell scattering amplitude $f(\vec{k}',\vec{k})$ for such a potential.

We have,

$$f\,(k',k) = -(4\pi)^{-1} \int_{r=0}^{\infty} e^{-i\vec{k}'.\vec{r}}\; V\, u_k(r)\, r\, d\Omega\; dr$$

This can be split-up into two parts,

$$f\,(\vec{k}',\vec{k}) = f_c\,(\vec{k}',\vec{k}) + f_n\,(\vec{k}',\vec{k}) \qquad (20)$$

where f_c corresponds to the core part, and f_n for the surrounding attractive region.

$$f_c = -(4\pi)^{-1} \int_{r=0}^{r_c} e^{-i\vec{k}'.\vec{r}}\; V_c\, u_k\,(r)\, r\, d\Omega\; dr \quad \text{where } V_c \to \infty$$

and $f_n = + (4\pi)^{-1} \int_{r_c}^{r_n} e^{-i\vec{k}'\cdot\vec{r}} \; V_o \; u_k(r) \; r d\Omega \; dr$

Making use of the fact that for $r \langle r_c$,

$$V_c \; u_k \underset{V_c \to \infty}{=} \frac{e^{i\delta_o(k)}}{k} \; \frac{\sin (kr_n + \delta_o)}{\sin k_1 (r_n - r_c)} \; k_1 \; \delta(r-r_c),$$

where $k_1 = \sqrt{k^2 + V_o} = \sqrt{k^2 + K^2}$, we get

$$f_c (\vec{k}', \vec{k}) = - \frac{k_1}{kk'} \sin (k'r_c) \; e^{i\delta_o(k)} \; \frac{\sin (kr_n + \delta_o)}{\sin k_1 (r_n - r_c)} \qquad (21)$$

To calculate f_n, we use the wave-function (16) for the region $r_c \langle r \langle r_n$, and get

$$f_n(\vec{k}'\vec{k}) = \frac{V_o A}{k'} \int_{r_c}^{r_n} \sin (k'r) \sin k_1 (r - r_c) \; dr$$

where $A = \frac{e^{i\delta_o(k)}}{k} \; \frac{\sin (kr_n + \delta_o)}{\sin k_1 (r_n - r_c)}$

Solving the integral, one gets,

$$f_n (\vec{k}', \vec{k}) = \frac{K^2}{k' (k'^2 - k_1^2)} \; \frac{e^{i\delta_o(k)}}{k} \; \frac{\sin (kr_n + \delta_o)}{\sin k_1 (r_n - r_c)}$$

$$\left[k_1 \sin k'r_n \cos k_1 (r_n - r_c) - k' \sin k_1 (r_n - r_c) \cos k' r_n - k_1 \sin k'r_c \right]$$

The total expression for $f (\vec{k}', \vec{k})$ in <u>s-state</u>, for a repulsive core + square-well potential is,

$$f(\vec{k}',\vec{k}) = \frac{e^{i\delta_o(k)}}{k'k} \quad \frac{\sin(kr_n + \delta_o)}{\sin k_1 (r_n - r_c)} \left[- k_1 \sin(k'r_c) + \right.$$

$$+ \frac{K^2}{(k'^2 - k_1^2)} \left\{ k_1 \sin k'r_n \cos k_1 (r_n - r_c) - k' \sin k_1 (r_n - r_c) \cdot \right.$$

$$\left. \left. \cdot \cos k'r_n - k_1 \sin k'r_c \right\} \right] \qquad (22)$$

where, of course, $K^2 = V_o$

$$k_1 = \sqrt{k^2 + K^2}$$

Our next task is to find a velocity-dependent potential which will reproduce the scattering-length, effective-range and phases upto a certain energy, and then compute $f(\vec{k}',\vec{k})$ for such a potential. If we get similar values of $f(\vec{k}',\vec{k})$ also, then such a potential can be expected to give saturation. The form of velocity-dependent potential that we consider now has been used by Levinger and Razavy and later by Peischl and Werner.

$$V(r,p) = -V_o J_1(r) + \frac{\lambda}{m} \vec{p} \cdot J_2(r) \vec{p}, \qquad (23)$$

where $J_1(r) = J_2(r)$ is the unit step-function of range b.
We put $K^2 = V_o$, $K'^2 = \frac{1}{(1+\lambda)} V_o$, and obtain, as before
(see equation (12); coth bK' $\rightarrow$ cot bK' since V_o is attractive)

$$a_o = b - \frac{b}{(1+\lambda) K'b \cot bK' - \lambda}$$

We want $a_o \rightarrow \infty$, which yields

$$\frac{\tan K'b}{K'b} = \frac{1+\lambda}{\lambda} \tag{24}$$

The effective range r_o, as $a_o \rightarrow \infty$, is given by

$$r_o = b - \frac{\lambda}{(1+\lambda)^2 bK'^2} \tag{25}$$

To fit the low-energy characteristics of the hard-core potential of Gomes et. al., we should therefore have

$$2.7 = b\left[1 - \frac{\lambda}{(1+\lambda)^2 b^2 K'^2}\right] \tag{26}$$

Equations (24) and (26) fix two of the three parameters in the potential (23); the other parameter can be used to give the best possible phase-shift fit. The parameters chosen by us were

$$\lambda = 0.1, \quad b = 2.8 \text{ fm}, \quad K' = \sqrt{\frac{V_o}{1+\lambda}} = 0.539 \text{ fm}^{-1} \tag{27}$$

This potential gave a fairly good phase-shift fit upto about $E_{lab} = 100$ MeV, and agreement of low energy parameters with the potential of Gomes et. al. The phase-shift characteristic of this potential, along with the hard-core potential of Gomes et. al., are shown in figure (III-2). We shall now calculate $f(\vec{k}', \vec{k})$ for such a potential in the range in which the phases fit and compare with the result of the hard-core potential of Gomes et. al.

Calculation of f(k',k) for velocity-dependent potential (23):

We shall again use formula (15)

$$f(\vec{k}',\vec{k}) = (k'^2-k^2) \int_0^\infty \frac{\sin k'r}{k'} \left[u_k(r) - v_k(r) \right] dr + f(\vec{k},\vec{k}) \qquad (15)$$

where $f(\vec{k},\vec{k}) = \dfrac{e^{i\delta_o(k)}}{k} \sin \delta_o(k)$

and $v_k(r) = \dfrac{e^{i\delta_o}}{k} \sin (kr + \delta_o)$

For the velocity-dependent potential (23), the radial wave-function is given by

$$u_k^I = A \sin (k_1 r) \quad \text{for} \quad r<b,$$

where $A = \dfrac{e^{i\delta_o(k)}}{k} \dfrac{\sin (kb+\delta_o)}{\sin k_1 b}$; $\quad k_1^2 = \dfrac{k^2 + V_o}{(1+\lambda)}$

and $u_k^{II} = \dfrac{e^{i\delta_o(k)}}{k} \sin (kr + \delta_o)$, for $r>b$

Put $B = \dfrac{e^{i\delta_o(k)}}{k}$

Then a straight-forward calculation using these wavefunctions yields,

$$f(\vec{k}',\vec{k}) = \frac{(k'^2-k^2)}{2kk'} e^{i\delta_o(k)} \left[\frac{\sin (kb+\delta_o)}{\sin (k_1 b)} \left\{ \frac{\sin (k'-k_1)b}{(k'-k_1)} - \frac{\sin(k'+k_1)b}{(k'+k_1)} \right\} \right.$$

$$- \cos \delta_o \left\{ \frac{\sin (k'-k)b}{(k'-k)} - \frac{\sin (k'+k)b}{(k'+k)} \right\}$$

$$\left. + \sin \delta_o \left\{ \frac{\cos (k'-k)b}{(k'-k)} + \frac{\cos (k'+k)b}{(k'+k)} \right\} \right] \qquad (28)$$

Note that this reduces to the usual expression for $f(\vec{k},\vec{k})$ when we put k'=k on the right-hand side.

We have evaluated (28) and the corresponding expression (22) for the hard-core potential for k = 0.7 fm^{-1} and k = 1.0 fm^{-1} and various values of k'. There is no point in comparing beyond values of k' = 1.2 fm^{-1} or so, since the phases do not fit for larger values. The values are shown in table (1) and the plots in fig. (III-3). It will be seen that the values are very close. Since $f(\vec{k}',\vec{k})$ is the quantity that enters into nuclear-matter calculations (as we shall see in the next chapter), we should expect such a velocity-dependent potential to give results comparable to those of a hard-core. In fact Peischl and Werner have got saturation in nuclear-matter with a potential of the same form. Their singlet-even potential had the following parameters -

$$\lambda = 0.15,\ b = 2.3\ fm.,\ K' = .6554\ fm^{-1}$$

as compared to (27). We can not compare the off-shell elements of this potential with those of the Gomes-Walecka-Weisskopf one, since the low-energy characteristics as well as phases of the two are different. (The potential of Gomes et. al. is a hypothetical 'test' potential, with scattering length of infinity).

All this calculation shows is that in addition to fitting the phase-shift data for scattering, one should also pay attention to the two-body scattering wave-function inside the potential range, in order to get reasonable results in nuclear matter. Establishing

the equivalence of two potentials outside the range of velocity-dependence (Ba62), and saying that the equivalent core-radius of the velocity-dependent potential is too small (GII62) is not a satisfactory explanation for different results in the many-body case.

COMPARISON OF OFF-THE-ENERGY-SHELL MATRIX-ELEMENTS $f(\vec{k}',\vec{k})$ FOR s-STATE. HARD-CORE POTENTIAL IS THAT OF GOMES ET. AL. VELOCITY-DEPENDENT POTENTIAL IS GIVEN BY III-(23), III-(27).

TABLE I

k (fm^{-1})	k' (fm^{-1})	$f(k',k)/e^{i\delta(k)}$ in fm velocity-dependent potn.	$f(k',k)/e^{i\delta(k)}$ in fm hard-core potn.
0.7	0.4	1.0748	1.1294
	0.5	0.9960	1.0537
	0.6	0.9056	0.9647
	0.8	0.7024	0.7542
	0.9	0.5962	0.6370
	1.0	0.4912	0.5151
	1.1	0.3904	0.3908
	1.5	0.0755	-0.0850
1.0	0.4	0.7497	0.6622
	0.5	0.6841	0.6158
	0.6	0.6096	0.5616
	0.7	0.5290	0.4991
	0.8	0.4454	0.4313
	0.9	0.3618	0.3588
	1.1	0.2060	0.2057
	1.5	-0.0038	-0.0949

CHAPTER IV

SEPARATION-METHOD CALCULATION WITH GREEN'S POTENTIAL

In this chapter, we shall apply the separation method of Moszkowski and Scott to calculate the binding energy of the ground-state of nuclear-matter. Green's velocity-dependent s-state potential is used, which simulated the two-body scattering data of the standard hard-core potential of Moszkowski and Scott. It was noted in chapter II that this velocity-dependent potential failed to give saturation even at $k_f = 1.8$ fm^{-1}. By recalculating this binding-energy curve by the separation method, we shall be

(i) checking the result of Green who used perturbation theory upto the second order, and

(ii) seeing exactly how the short-range part of the two-body wavefunction enters into the calculations giving different results from the hard-core potential.

Also, it will be easier to see how the velocity-dependent potential should be modified in order to give saturation at a reasonable density. First, we shall discuss the separation method in some detail, deriving the formulae that we use.

The separation method of calculating the binding-energy of nuclear-matter: This method was developed by Moszkowski and Scott (MS61) and modified by Köhler (K61). They noted that

although the nuclear force is very strong, only a relatively weak part of it is responsible for giving the binding in nuclear matter. This is because, at the energies concerned, a large part of the attraction is spent up in 'cancelling' the short-range repulsion, leaving the weak tail to give the binding.

If such a separation could be made, then ordinary perturbation theory could be applied directly to the long-range part of the potential. The short-range part would contribute only in the second-order. One should be more explicit about what exactly is meant by 'cancellation' of the repulsive and attractive parts of the potential. One knows that for free particles, the diagonal element of the reaction-matrix is proportional to the phase-shift. And the diagonal elements of the reaction matrix determine the pair-interaction energy. If the phase-shift is negative, the potential must be repulsive, giving positive interaction energy. A positive phase-shift implies negative interaction energy and net attraction. If, then, the division of the potential is made at the point where the local phase of the (free) two-body scattering wavefunction is zero, the potential within the separation distance would give zero interaction energy for free two-body scattering. By this kind of separation, then, the cancellation is exact for free two-body scattering and the short-range part does not contribute at all to the interaction energy. If we remember that positive phase-shift means pulling the wave function in, and negative phase-shift means pushing the wavefunction out, with respect to the

unperturbed wavefunction, then it is clear that at the separation distance, the properly normalized two-body wavefunction for free scattering will be tangential to the unperturbed wavefunction. Regardless of normalization, the logarithmic derivatives of the unperturbed and actual wavefunctions will be equal at the separation distance. This separation distance, which we denote by 'd', will depend both on the angular momentum and the relative momentum of the state concerned. As the energy of the particles increases, more and more of the attraction would be needed to cancel the repulsion, and 'd' would increase. This kind of separation is not possible when the total phase-shift goes negative.

In nuclear-matter, of course, conditions are different from free two-particle scattering. Even if three-body clusters are neglected (the importance of these have been recently emphasized by Rajaraman at Cornell, Raj.63), two main differences come in. The Pauli principle drastically cuts down any scattering by the long-range part (which is weak) and so there is actually no phase-shift inside nuclear-matter. Also, the energy-momentum relation inside nuclear-matter is modified by the average one-body potential that a nucleon feels due to the presence of all the other nucleons. These two effects give rise to two correction terms in the second order. Since the short-range part of the potential is strong, it distorts the wavefunction in this region drastically, and primarily mixes high momenta

components into it. Since these are mostly above the fermi-sea, the Pauli exclusion principle has only a small effect on the short-range part of the potential. For this reason, one expects the Pauli correction term to be small. There is another correction term which arises out of the coupling of the short-range and the long-range parts of the potential. For the kind of separation made, the two parts are essentially decoupled for free two-body scattering, but in nuclear-matter, interference terms appear in the second-order. We shall now derive an expression for the nuclear reaction-matrix t^N by making such a separation.

Let the two-body wavefunction for free nucleon-nucleon scattering for the given two-body interaction be $\Psi^F = \frac{R^F}{r}$. The unperturbed wavefunction is $\Phi = \frac{R_o}{r}$. At the separation distance d,

$$\frac{1}{R^F} \left.\frac{dR^F}{dr}\right|_{r=d} = \frac{1}{R_o} \left.\frac{dR_o}{dr}\right|_{r=d} \qquad (1)$$

Define the nuclear reaction-matrix in momentum-space by

$$t^N_{kk} = v_{kk} + \int v_{kk'} \frac{Q(P,k')}{e(k,k',P)} t^N_{k'k} \frac{\vec{dk'}}{(2\pi)^3} \qquad (2)$$

where $Q(P,k')$ is the Pauli operator: it is unity if the intermediate states of both the nucleons are outside the Fermi-sea, and zero otherwise. $\vec{k},\vec{k'}$ are the relative momenta

$$\vec{k} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2),$$

$\vec{k}' = \frac{1}{2}(\vec{k}_1' - \vec{k}_2')$, and $\vec{P}$ is the total momentum,

$$\vec{P} = (\vec{k}_1 + \vec{k}_2). \qquad (3)$$

$e(k,k',P)$ is the two-particle propagator where the many-body effects have been taken care of by incorporating in it the one-body potential arising out of all the other particles -

$$e(k,k',P) = \frac{\hbar^2}{m}(k^2-k'^2) + U(k_1) + U(k_2) - U(k_1') - U(k_2') \qquad (4a)$$

The one-body potential is defined by

$$U(k_1) = V^{-1} \sum_{k_2<k_F} t^N_{kk} \qquad (4b)$$

$\vec{k}$ being given by (3) and V is the nuclear volume.

Equation (2) can be written in operator form as

$$t^N = v + v \frac{Q}{e} t^N \qquad (5)$$

Now making the separation

$$v = v_\ell + v_s \qquad (6)$$

the above equation becomes

$$t^N = v_\ell + v_s + v_\ell \frac{Q}{e} t^N + v_s \frac{Q}{e} t^N$$

Now define two reaction-matrices due to the short-range part of the potential v_s:

$$t^D_s = v_s + v_s \frac{1}{e} t^D_s \qquad (7)$$

$$t^F_s = v_s + v_s \frac{1}{e_o} t^F_s \tag{8}$$

where $e_o = \frac{\hbar^2}{m} (k^2 - k'^2)$. Note that the Pauli principle is neglected in both these operators; but whereas t^D_s has the actual propagator e, t^F_s describes free two-body scattering due to v_s with a propagator e_o which ignores the average field of other particles. In the following derivation, we shall assume v_s to be Hermitian, which is strictly true only if d be independent of k. Since this is actually not so, additional terms arise in third-order, which we ignore. Although d does vary with k, its variation is very slow in the region $k = 0.6$ fm^{-1} to 1.0 fm^{-1}, which is the most important region for nuclear-matter calculation.

From (7), then, $t^D_s = v_s + t^D_s \frac{1}{e} v_s$

inverting, $v_s = (1 + t^D_s \frac{1}{e})^{-1} t^D_s$ (9)

Substituting (9) in (5), and pre-multiplying throughout by $(1 + t^D_s \frac{1}{e})$, we get,

$$t^N = v_\ell + t^D_s + t^D_s \frac{1}{e} v_\ell - t^D_s \frac{1}{e} t^N + t^D_s \frac{Q}{e} t^N + v_\ell \frac{Q}{e} t^N$$

Now, make the approximation on the right-hand side

$$t^N \approx v_\ell + t^D_s$$

Then the above equation becomes

$$t^N = v_\ell + t^D_s + t^D_s \left(\frac{Q-1}{e}\right) t^D_s + v_\ell \frac{Q}{e} v_\ell + v_\ell \frac{Q}{e} t^D_s + t^D_s \frac{Q}{e} v_\ell \tag{10}$$

To simplify (10), we shall express t_s^D in terms of t_s^F.

Define the 'wave-matrices' Ω's by

$$t_s^D = v_s \Omega_s^D$$
$$t_s^F = v_s \Omega_s^F \tag{11}$$

Note that

$$\frac{1}{e_o} t_s^F = (\Omega_s^F - 1)$$
$$\frac{1}{e} t_s^D = (\Omega_s^D - 1) \tag{12}$$

From (7), (8), one can write

$$t_s^D = (1 + t_s^F \frac{1}{e_o})^{-1} t_s^F + (1 + t_s^F \frac{1}{e_o})^{-1} t_s^F \frac{1}{e} t_s^D$$

or $$t_s^D = t_s^F + t_s^F (\frac{1}{e} - \frac{1}{e_o}) t_s^D$$

Using (12),

$$t_s^D = t_s^F + (\Omega_s^F - 1)(e_o - e)(\Omega_s^D - 1) \tag{13}$$

$$\approx t_s^F + (\Omega_s^F - 1)(e_o - e)(\Omega_s^F - 1) \tag{14}$$

Substituting this in the expression (10) for t^N, we get,

$$t^N = v_\ell + t_s^F + v_\ell \frac{Q}{e} v_\ell + v_\ell \frac{Q}{e_o} t_s^F + t_s^F \frac{Q}{e_o} v_\ell$$
$$+ (\Omega_s^F - 1)(e_o - e)(\Omega_s^F - 1) + (\Omega_s^F - 1) e (Q-1) (\Omega_s^F - 1) \tag{15}$$

where we have used $(\Omega_s^D - 1) \approx (\Omega_s^F - 1)$.

Expression (15) is identical with the one derived by Köhler (K61) in a slightly different way. We have retained, in (15) only upto second-order terms in v_ℓ and t_s^F.

If one arranges the normalization such that

$$\int_V (\Psi^N)^* \underline{\Phi} \; dr = \int_V \underline{\Phi}^* \underline{\Phi} \; dr = 1 \qquad (16)$$

where Ψ^N is the actual two-particle wavefunction in nuclear-matter and $\underline{\Phi}$ the unperturbed wavefunction, then the total energy of the system can be written as

$$E = \sum_i T_i + \sum_{i<j} \sum \Delta E_{ij} \qquad (17)$$

where

$$\Delta E_{ij} = V^{-1} (t^N_{kk}), \qquad (18)$$

T_i is the kinetic energy of the i^{th} particle, V is the nuclear volume, and k the relative-momentum of the interacting particles i, j. Thus, in order to find the energy of the system, we need to know the diagonal matrix-elements of t^N, which in turn is given approximately by (15). In (15), the first term on the right is just the first-order Born approximation of the long-range part of the potential. With the kind of separation made, $(t_s^F)_{kk} = 0$, since the phase-shift due to this part of the potential is zero. The third term, $v_\ell \frac{Q}{e} v_\ell$, is just the second-order Born-term, with the

actual propagator e, and where Q reduces the magnitude of this term drastically. The next two terms in (15) are the interference terms, arising because the short-range and the long-range parts of the potential are not completely uncoupled in nuclear-matter. For free-space, Q = 1, and these terms vanish. The term $(\Omega_s^F - 1)\ (e_o - e)\ (\Omega_s^F - 1)$ is called the 'dispersion' term. It has appeared because the propagator in nuclear-matter is e, and not e_o. We shall see that this term is extremely important for saturation of the binding energy curve. The last term in (15), $(\Omega_s^F - 1)e(Q-1)(\Omega_s^F - 1)$ is the Pauli-correction term mentioned earlier. For free-space, Q = 1, and this term also vanishes. We thus see that for free nucleon-nucleon scattering, t^N is given by only the first three terms of (15), upto second-order.

In order to evaluate ΔE_{ij} from equation (18), we shall express t^N_{kk} in the form

$$t^N_{kk} = \sum_{\ell} (2\ell + 1)\ c_\ell\ t^\ell_{kk} \qquad (19)$$

where ℓ is the angular momentum of the state concerned, c_ℓ is a constant taking care of the statistics involved, and t^ℓ_{kk} is the contribution to t^N_{kk} from the ℓ^{th} partial-wave. In our calculations, we are only interested in a model potential which acts only in the ℓ = o state. Further, the potential is spin-independent. To determine c_ℓ, consider the total number of

spin and iso-spin states. There are the following combinations:

$$\left.\begin{matrix} T=0 \\ S=0 \end{matrix}\right\} \text{ 1 state} \qquad \left.\begin{matrix} T=0 \\ S=1 \end{matrix}\right\} 3 \text{ states} \qquad \left.\begin{matrix} T=1 \\ S=0 \end{matrix}\right\} 3 \text{ states} \qquad \left.\begin{matrix} T=1 \\ S=1 \end{matrix}\right\} 9 \text{ states}$$

In the $\ell = o$ state, since the total wavefunction must be antisymmetric, interaction can take place only in the second and third conbinations above. This means that the weighting factor is $\frac{6}{16}$. But in a spin-independent potential, the contribution of the exchange term is just the same as that of the direct term, so $c_\ell = \frac{6 \times 2}{16} = \frac{3}{4}$. Hence, for our case of interest, (19) reduces to

$$t^N_{kk} = \frac{3}{4}\, t^o_{kk} \tag{20}$$

Using (15), we can further write (we omit the superscript o hence-forth)

$$t^o_{kk} = (v_\ell)_{kk} + (t^F_s)_{kk} + \Delta t_{kk}(D) + \Delta t_{kk}(I) + \Delta t_{kk}(P) + \Delta t_{kk}(V) \tag{21}$$

where (D), (I), (P), (V) denote the dispersion, interference, Pauli and second-order Born terms respectively. With the separation made, $(t^F_s)_{kk}$ is identically zero.

Now we shall express the terms in (21) in a manner more convenient for numerical calculation. This will also show up the differences in the velocity-dependent and hard-core potentials.

Treatment of the Propagator:- The propagator e has been defined in equation (4) as

$$e(k,k',P) = \frac{\hbar^2}{m}(k^2 - k'^2) + U(k_1) + U(k_2) - U(k'_1) - U(k'_2)$$

We shall follow Moszkowski and Scott in simplifying this to a form suitable for computation. Making transformations (3),

$$e(k,k',P) = \frac{\hbar^2}{m}(k^2 - k'^2) + U\left(\left|\frac{\vec{P}}{2} + \vec{k}\right|\right) + U\left(\left|\frac{\vec{P}}{2} - \vec{k}\right|\right) - U\left(\left|\frac{\vec{P}}{2} + \vec{k}'\right|\right) - U\left(\left|\frac{\vec{P}}{2} - \vec{k}'\right|\right) \qquad (22)$$

We have defined the one-body potential in (4b),

$$U(k_1) = V^{-1} \sum_{k_2 < k_f} \langle \vec{k}_1 \vec{k}_2 | \, t^N \, | \vec{k}_1 \vec{k}_2 \rangle$$

$$= V^{-1} \sum_{k_2 < k_f} t^N \left(\frac{\vec{k}_1 - \vec{k}_2}{2}, \quad \frac{\vec{k}_1 - \vec{k}_2}{2} \right)$$

We replace the sum by the reaction-matrix evaluated at some average momentum. This average was taken at $\frac{k_f}{\sqrt{2}}$. In the first approximation, $t^N \approx t_s^F + v_\ell$, and the diagonal elements of t_s^F vanish. Doing the angular integrations,

$$U(k_1) = \frac{2k_f^3}{3\pi^2} \, v_\ell \left(\sqrt{\frac{k_f^2}{8} + \frac{k_1^2}{4}}, \sqrt{\frac{k_f^2}{8} + \frac{k_1^2}{4}} \right) \qquad (23)$$

Following Moszkowski, if one further assumes that

$$\left|\left(\frac{\vec{P}}{2} + \vec{k}\right)\right| = \left|\left(\frac{\vec{P}}{2} - \vec{k}\right)\right| , \quad \left|\left(\frac{\vec{P}}{2} + \vec{k}'\right)\right| = \left|\left(\frac{\vec{P}}{2} - \vec{k}'\right)\right| ,$$

and $|\vec{P}| = k_f$, then (22) simplifies to

$$e(k,k',P) = \frac{\hbar^2}{m}(k^2 - k'^2) + \frac{4k_f^3}{3\pi^2} \left[v_\ell \left(\sqrt{\frac{3k_f^2}{16} + \frac{k^2}{4}}, \sqrt{\frac{3k_f^2}{16} + \frac{k^2}{4}} \right) - v_\ell \left(\sqrt{\frac{3k_f^2}{16} + \frac{k'^2}{4}}, \sqrt{\frac{3k_f^2}{16} + \frac{k'^2}{4}} \right) \right] \qquad (24)$$

It should be noted here that this expression is not sensitive to the value of $|P|$ chosen above. If one takes $P = 0$, then we get $\frac{k_f^2}{8}$ instead of $\frac{3k_f^2}{16}$ under the square-root sign, and this changes the numerical values only slightly in $e(k,k')$. It was seen that $v_\ell(k,k)$ could be approximated quite well by an expression of the form

$$v_\ell(k,k) = -\frac{\hbar^2}{m} \frac{a + bk^2}{1 + dk^2} \tag{25}$$

Then, after some manipulation, one can write

$$U(k) = -\frac{\hbar^2}{m} \frac{2k_f^3}{3\pi^2} \frac{A + Bk^2}{D + k^2} \tag{26}$$

where $A = \frac{4}{d}\left(a + \frac{3}{16} k_f^2 b\right)$, $B = \frac{b}{d}$, $D = \frac{4}{d} + \frac{3}{4} k_f^2$

$$\text{and } F = \frac{4(ad-b)}{d^2} \tag{27}$$

substituting this form in (24), we finally get

$$e(k,k',P) = \frac{\hbar^2}{m}(k^2 - k'^2) - \frac{\hbar^2}{m} \frac{4k_f^3}{3\pi^2} F \left(\frac{1}{D+k^2} - \frac{1}{D+k'^2}\right) \tag{28}$$

This expression for the propagator will be used to compute the second-order terms.

<u>The Dispersion Term:</u> We have seen that

$$\Delta t_{kk}(D) = \langle k | (\Omega_s^F - 1)(e_o - e)(\Omega_s^F - 1) | k \rangle$$
$$= \langle \Psi_s - \Phi_k | (e_o - e) | \Psi_s - \Phi_k \rangle$$

where Ψ_s is the two-body wavefunction for free nucleon-nucleon

scattering due to the short-range part of the potential alone (for wave number k) and Φ_k the unperturbed wavefunction.

Now,

$$\Delta t_{kk}(D) = \sum_{k'} \langle \Psi_s - \Phi_k | k' \rangle \langle k' | (e_o - e) | k' \rangle \langle k' | \Psi_s - \Phi_k \rangle$$

$$\text{or } \Delta t_{kk}(D) = \sum_{k'} \left| \langle \Psi_s - \Phi_k | k' \rangle \right|^2 \langle k' | (e_o - e) | k' \rangle$$

$$= \sum_{k'} \left| \sum_{r} \langle \Psi_s - \Phi_k | r \rangle \langle r | k' \rangle \right|^2 \langle k' | (e_o - e) | k' \rangle$$

$$\text{Put } \sum_{r} \langle \Psi_s - \Phi_k | r \rangle \langle r | k' \rangle = \Delta \Psi_{k'k} \qquad (29)$$

There, replacing the sum over k' by an integration,

$$\Delta t_{kk}(D) = \frac{4\pi}{(2\pi)^3} \frac{\hbar^2}{m} \int_0^\infty |\Delta \Psi_{k'k}|^2 \frac{4k_f^3}{3\pi^2} F \left(\frac{1}{D+k^2} - \frac{1}{D+k'^2} \right) k'^2 \, dk'$$

$$= \frac{1}{2\pi^2} \frac{4k_f^3}{3\pi^2} F \int_0^\infty |\Delta \Psi_{k'k}|^2 \left(\frac{1}{D+k^2} - \frac{1}{D+k'^2} \right) k'^2 \, dk' \qquad (30)$$

in units of $\frac{\hbar^2}{m} = 1$.

Note that $(\Delta \Psi_{k'k})$ for s-wave scattering is

$$\Delta \Psi_{k'k} = 4\pi \int_0^{d(k)} \frac{\sin k'r}{k'r} \left[\frac{\chi_k^s(r)}{Nr} - \frac{\sin kr}{kr} \right] r^2 \, dr$$

where the normalization N is such that

$$\frac{\chi_k^s(d)}{N} = \frac{\sin dk}{k} \quad \text{at the separation distance d.}$$

We have put $\Psi_s(r) = \dfrac{\chi_k^s(r)}{r}$ above.

So $$\Delta\Psi_{k'k} = 4\pi \int_0^{d(k)} \frac{\sin k'r}{k'} \left[\frac{\chi_k^s(r)}{N} - \frac{\sin kr}{k} \right] dr \qquad (31)$$

From (III-15), it will be at once seen that

$$\frac{1}{4\pi} (k'^2-k^2) \Delta\Psi_{k'k} = f_s(\vec{k}',\vec{k}) \qquad (32)$$

where $f_s(\vec{k'k})$ is the off-shell element of the scattering amplitude due to the short-range part of the potential alone. $f_s(\vec{k},\vec{k})$ vanishes because the local phase of v_s is zero. Beyond a certain value of k, when separation can no longer be made,

$$\frac{1}{4\pi} (k'^2-k^2) \Delta\Psi_{k'k} = f(\vec{k}',\vec{k}) \qquad (33)$$

We see here clearly how this quantity enters into many-body calculations, and how its value depends sensitively on the nature of the wave-function at short distances. From (30), it will be seen that the contribution of the dispersion term to the binding energy is positive and that it increases very fast with increasing k_f. It is the term primarily responsible for pushing up the binding energy curve for larger values of k_f and thus giving saturation. We have plotted the quantity $\left[(\Delta\Psi_{k'k})\, k'\right]^2$, (fig. V-2) which appears in the dispersion term, for the

potential (II-1) of Green which failed to give saturation and the standard-hard-core potential of Moszkowski which gave saturation. We choose the value $k_f = 1.6$ fm^{-1}, $k = 0.8$ fm^{-1}. It will be seen that the 'distortion' for Green's potential is very much smaller - which in turn gives a much smaller dispersion term. In the same figure, we also show $\left[(\Delta\Psi_{k'k})\, k'\right]^2$ for the new potential that we shall propose in chapter V, and which is seen to be quite comparable to the standard-hard-core potential distortion. We shall come back to this point in greater detail in chapter V.

The Pauli term: This is given by

$$\Delta t_{kk}(P) = \langle k \mid \Omega_s^F - 1)\ (Q-1)\ e\ (\Omega_s^F - 1) \mid k\rangle$$

$$= \langle \Psi_s - \Phi_k \mid\ (Q-1)\ e\ \mid \Psi_s - \Phi_k\rangle$$

$$= \sum_{k'} \left|\Delta\Psi_{k'k}\right|^2 \left[Q(P,k')-1\right]\ e\,(k,k')$$

$$\text{or } \Delta t_{kk}(P) = \frac{1}{2\pi^2}\int_0^\infty k'^2\, dk' \left[Q(P,k')-1\right]\ (\Delta\Psi_{k'k})^2$$

$$\times \left\{(k^2-k'^2) - \frac{4k_f^3}{3\pi^2}\, F\left(\frac{1}{D+k^2} - \frac{1}{D+k'^2}\right)\right\} \qquad (33b)$$

in units of $\frac{\hbar^2}{m} = 1$.

In this chapter, we use, for the operator Q,

$$Q\,(P,k') = 0, \quad k'^2 + \frac{P^2}{4} < k_f^2$$

$$Q(P,k') = 1 \qquad k' - \frac{P}{2} > k_f \tag{34}$$

$$= \frac{k'^2 + P^2/4 - k_f^2}{k'P} \qquad \text{otherwise}$$

P is replaced by its average value over the Fermi-sea, for two particles having a relative momentum k'.

This is equal to

$$\frac{P^2}{4} = \frac{3}{5} k_f (k_f - k') \left[1 + \frac{k'^2}{3k_f (2k_f + k')} \right], \qquad k' < k_f$$

$$= 0 \qquad \text{for } k' > k_f \tag{35}$$

It will be seen from (33b) that the contribution to the Pauli term comes primarily from inside the Fermi-sea. There is partial cancellation in the integrand, and moreover, k'^2 dampens it appreciably. All this goes to make the Pauli correction very small. For a potential which is strong at short-ranges, this is quite understandable physically.

<u>The Interference term</u>: For this,

$$\Delta t_{kk}(I) = \langle k | v_\ell \frac{Q}{e_o} t_s^F | k \rangle + \langle k | t_s^F \frac{Q}{e_o} v_\ell | k \rangle$$

using (12),

$$\Delta t_{kk}(I) = \langle k | v_\ell Q (\Omega_s^F - 1) | k \rangle + \langle k | (\Omega_s^F - 1) Q v_\ell | k \rangle$$

This can again be put, after some manipulation, as

$$\Delta t_{kk}(I) = \frac{1}{2\pi^2} \int_0^\infty (v^{\ell}_{k'k} + v^{\ell}_{kk'}) \left[Q(k',P)-1 \right] (\Delta\Psi_{k'k})\, k'^2\, dk' \qquad (36)$$

in units of $\frac{\hbar^2}{m} = 1$;

where

$$v^{\ell}_{k'k} = \langle k' | v_{\ell} | k \rangle$$

$$v^{\ell}_{kk'} = \langle k | v_{\ell} | k' \rangle \quad ,$$

the superscript ℓ denoting the long-range part of the potential, and the separation has been done for the wave number k. In (36), we have used the fact that $\langle k | v_{\ell} \frac{1}{e_o} t^F_s | k \rangle$ is identically zero, and we have subtracted this from the original integral. This makes computation easier, since the contribution comes then from inside the Fermi-sea.

It will be seen from (36) that the contribution of the interference term to the binding energy is negative. It also increases with increasing k_f. It partly cancels out the effect of the dispersion term. Its magnitude depends on how big the 'distortion' $\Delta\Psi_{k'k}$ is, and also how slowly the off-diagonal terms of the potential fall off. If this term is too big, saturation cannot be obtained. For Green's potential that we are considering, it will turn out that this term is actually as big as the dispersion term, cancelling its effect on the binding energy.

$(v^{\ell}_{k'k} + v^{\ell}_{kk'})$ for velocity-dependent potential:-

Writing the full nucleon-nucleon potential as

$v(r,p) = v^{(1)}(r) + (\frac{p^2}{m}\omega(r) + \omega(r)\frac{p^2}{m})$, where $v^{(1)}(r)$ denotes the static part, we shall find expressions for the off-diagonal matrix elements of the potential. We shall work in units of $\frac{\hbar^2}{m} = 1$. For the s-state, put $\chi(k,r) = \frac{\sin kr}{k}$. Then, for the velocity-dependent part, the integral $\langle k' | \frac{p^2}{m}\omega(r) | k \rangle$ is

$$-\int_{d(k)}^{\infty} \chi(k',r)\frac{d^2}{dr^2}\left(\omega(r)\chi(k,r)\right) 4\pi\, dr$$

$$= -4\pi\left[\chi(k',r)\frac{d}{dr}\left(\omega(r)\chi(k,r)\right)\Big|_{d(k)}^{\infty} - \int_{d(k)}^{\infty}\chi'(k',r)\frac{d}{dr}\left(\omega(r)\chi(k,r)\right)\right]$$

$$= -4\pi\Big[-\omega'(d)\chi(k',d)\chi(k,d) + \omega(d)\left\{\chi(k,d)\chi'(k',d) - \chi'(k,d)\chi(k',d)\right\} - k'^2\int_{d(k)}^{\infty}\chi(k',r)\,\omega(r)\,\chi(k,r)\,dr\Big]$$

$$= 4\pi\,\omega'(d)\chi(k,d)\chi(k',d) + 4\pi\,\omega(d)\left\{\chi'(k,d)\chi(k',d) - \chi(k,d)\chi'(k',d)\right\} + 4\pi k'^2\int_{d(k)}^{\infty}\chi(k',r)\,\omega(r)\,\chi(k,r)\,dr$$

The other integral $\langle k' \mid \omega(r) \frac{p^2}{m} \mid k \rangle$ is simply

$$= 4\pi k^2 \int_{d(k)}^{\infty} \chi(k',r)\, \omega(r)\, \chi(k,r)\, dr$$

The static part is simple -

$$\langle k' \mid v^{(1)}(r) \mid k \rangle = 4\pi \int_{d(k)}^{\infty} \chi(k',r)\, v^{(1)}(r)\, \chi(k,r)\, dr$$

The complete expression for $\langle k' \mid v^{\ell}(r,p) \mid k \rangle$ is,

$$\langle k' \mid v^{\ell}(r,p) \mid k \rangle = 4\pi \int_{d}^{\infty} \chi(k',r)\, v^{(1)}(r)\, \chi(k,r)\, dr$$

$$+ 4\pi \omega'(d)\, \chi(k,d)\, \chi(k',d) + 4\pi \omega(d) \Big\{ \chi'(k,d)\, \chi(k',d) -$$

$$- \chi(k,d)\, \chi'(k',d) \Big\} + 4\pi (k^2 + k'^2) \int_{d}^{\infty} \chi(k',r)\, \omega(r)\, \chi(k,r)\, dr \qquad (37)$$

It will be noted that the d in the above expression is always for the wave-number k. From (37), we can write down the expression for

$$(v^{\ell}_{k'k} + v^{\ell}_{kk'}) = 8\pi \int_{d(k)}^{\infty} \chi(k',r)\, v^{(1)}(r)\, \chi(k,r)\, dr$$

$$+ 8\pi \omega'(d)\, \chi(k,d)\, \chi(k',d)$$

$$+ 8\pi (k^2 + k'^2) \int_{d(k)}^{\infty} \chi(k',r)\, \omega(r)\, \chi(k,r)\, dr \qquad (38)$$

Green's potential had

$$v^{(1)}(r) = -Ae^{-a^2 r^2}$$

$$\omega(r) = \beta^2 e^{-b^2 r^2} \qquad (39)$$

where A = 1.062 fm^{-2}, a = 0.627 fm^{-1}, β^2 = 1.131, b = 1.44 fm^{-1}, in units of $\frac{\hbar^2}{m}$ = 1. It is quite easy to evaluate (38) for such a potential. Taking the diagonal elements of (38), we also get the first-order Born term v^{ℓ}_{kk} in (15).

Second-Order Born term: This term is given by

$\Delta t_{kk}(V) = \langle k | v_\ell \frac{Q}{e} v_\ell | k \rangle$ where now the subscript denotes the long-range part of the potential. Or,

$$\Delta t_{kk}(V) = \int_0^\infty \frac{(v^{\ell}_{k'k})\,(v^{\ell}_{kk'})\,Q(P,k')}{e(k,k')} \frac{4\pi k'^2 dk'}{(2\pi)^3} \qquad (40)$$

where e(k,k') is given as before by (28). The contribution to this term comes from outside the Fermi-sea. It is quite small since the off-diagonal matrix-elements fall of rapidly with increasing k', and the k'^2 in the denominator takes care of the k'^2 in the numerator.

Binding Energy:

In order to calculate the binding energy per particle, we multiply t^N_{kk} by the probability of finding a pair of particles with a relative momentum k, and integrate over the Fermi-sphere:

$$\frac{\text{Potential Energy}}{\text{Particle}} = \frac{8}{\pi^2} \int_0^{k_f} t^N_{kk} \left(1 - \frac{3}{2}\frac{k}{k_f} + \frac{1}{2}\frac{k^3}{k_f^3}\right) k^2\, dk, \qquad (41)$$

It will be seen from the above expression that the relative weights for k = 0 and k = k_f are zero. Actually the most important contribution to binding energy at a given k_f comes from around k = $0.5k_f$. The kinetic energy, in the Gas-model, is given by

$$\frac{\text{kinetic energy}}{\text{particle}} = \frac{3}{10} \frac{\hbar^2}{m} k_f^2 \qquad (42)$$

Numerical Results: In figure (IV-1), we have plotted $(v^{\ell})_{kk}$, the first-order Born term in (15), for Green's potential (39), along with the $(v^{\ell})_{kk}$ for the S.H.C.P. It will be seen that they are almost identical, giving the same first-order contribution to the binding energy. This is so because the phase-shifts have been adjusted to be the same, and it is the long-range part of the potential that is giving the phase-shift. All the differences, then, arise from the second-order terms in (15), and primarily from the dispersion term. The same fact has been noted by Moszkowski (Mo-63) in a recent paper. We also show, in figure (IV-2), the variation of the separation distance d(k) with k. It is very similar to the hard-core case. The separation distance d(k) can be given the interpretation of the 'healing distance', in the sense of Gomes et. al. (GWW58). In Table (2) we show the

contribution of the first-order and the various second order terms to the binding energy of nuclear-matter. It will be seen that the total second-order contribution is very small compared to the first-order term, showing good convergence of the method. Why the dispersion term is so much smaller with this potential can be seen by looking at figure (V-2), where we have plotted $(k' \frac{\Delta\Psi_{k'k}}{4\pi})^2$ against k' for k = 0.8fm^{-1}.

Our task in chapter V will be to get a velocity-dependent potential which gives a distortion comparable to the hard-core case. In figure(IV-3), we have plotted the various second-order contributions to the binding energy. Finally, in figure (IV-4) we compare our final binding energy curve with that of Green's, who calculated it by using perturbation theory (G62II). We also give the curve due to Scott and Moszkowski (SM62) for their S.H.C.P. obtained by the same method we have used. It will be seen that our result agrees well with Green's result, although it would seem that Green overestimated the binding energy per particle by about 1 MeV at k_f = 1.6 fm^{-1}. This is not surprising, since Green was using perturbation theory upto the second-order. In this chapter, then, we have found

out why the results of Green are so different from the standard-hard-core case, in addition to checking the earlier results of Green.

Before concluding this chapter, we note that the separation method in this form can not be applied when attraction and repulsion in the two-nucleon potential are not localized. In the Peischl-Werner potential (PW63), the static and the velocity-dependent parts of the potential are of the same range and this method can not be applied. Also, the separation method is much less reliable in the presence of non-central forces (BM62, SM61). This is because it is much harder to get self-consistency in the one-body potential due to larger second-order corrections.

BINDING ENERGY PER PARTICLE WITH GREEN'S VELOCITY-DEPENDENT POTENTIAL

(All unmarked figures in MeV.)

TABLE II

$k_f(fm^{-1})$	K.E./A	First Order P.E./A	First Order B.E./A	Dispersion Term	Pauli Term	Second Order Born	Interference Term	Total Second Order	Total Binding Energy/A
1.0	12.44	-18.46	-6.02	0.11	0.005	-1.30	-0.29	-1.48	-7.50
1.2	17.91	-27.81	-9.90	0.31	0.02	-1.01	-0.65	-1.33	-11.23
1.4	24.38	-38.00	-13.62	0.72	0.09	-0.75	-1.19	-1.13	-14.75
1.6	31.85	-48.33	-16.48	1.49	0.27	-0.62	-1.88	-0.74	-17.22

CHAPTER V

A MODIFIED VELOCITY-DEPENDENT POTENTIAL

In the last chapter, we noted that the 'distortion' $\Delta\Psi_{k'k}$, defined in equation (IV-31):

$$\Delta\Psi_{k'k} = 4\pi \int_0^{d(k)} \frac{\sin k'r}{k'} \left[\frac{\chi_k^s(r)}{N} - \frac{\sin kr}{k} \right] dr$$

was too small for Green's s-state potential. This resulted in too small a dispersion term, and hence no saturation in the binding-energy curve. It is also true that all the binding came from the long-range part of the potential. The short-range part should be responsible for pushing the binding-energy curve up for increasing k_f and giving saturation, if the distortion is large. We also notice, from equation (IV-38), that the velocity-dependent potential contributes an attractive term to v_{kk}^{ℓ} which is directly proportional to the strength β^2. This contribution can be quite large if the velocity-dependent potential is strong.

Our objective in this chapter is to get a velocity-dependent potential which gives a large distortion, in the wavefunction. This distortion depends, for given k and k', on two factors: on the magnitude of the 'wound' $\left[\frac{\chi_k^s(r)}{N} - \frac{\sin kr}{k} \right]$ in the wave-function, and on the location of this in r-space.

If the wave-function $\frac{\chi_k^s(r)}{N}$ differs from the unperturbed wave $\frac{\sin kr}{k}$ only mostly near the origin (for very small values of r), then the factor $\frac{\sin k'r}{k'}$ will damp the integrand to a small value, giving a small distortion $\Delta\Psi_{k'k}$. It is clear that with a velocity-dependent potential which is finite everywhere, the actual magnitude of $\left[\frac{\chi_k^s(r)}{N} - \frac{\sin kr}{k}\right]$ can never be as large as for the hard-core potential. We notice, however, that for the hard-core potential, this quantity is maximum near the origin, upto the radius of the hard-core, and then dies off fast. One way of getting comparable values of $(\Delta\Psi_{k'k})$ for a velocity-dependent potential, then, is to shift the 'wound' $\left[\frac{\chi_k^s(r)}{N} - \frac{\sin kr}{k}\right]$ outward. Then $\frac{\sin k'r}{k'}$ would not damp out the integrand as much, and the distortion may be as large as for the hard-core case. This can be done by having the peak of the velocity-dependent potential $\omega(r)$ at some suitable value of r, and not at the origin. This peak can not be pushed out too much, because of the following reasons:-

(i) The separation distance 'd' should remain around the value of 1 fm for $k = 0.7$ fm^{-1}. This is so because 'd' can be given the interpretation of the healing distance, and this should be much smaller than the average spacing between the nucleons inside the nucleus, which is about 1.7 fm.

(ii) We have to fit the s-wave phase-shift data with such a potential, and this is not possible if we push $\omega(r)$ too far out.

(iii) Also, we must arrange matters such that the interference term (IV-36) remains small. This would mean that $(\Delta\Psi_{k'k})$ should be small within the fermi-sphere for $k' < k_f$ and then get as big as possible beyond k_f, to give a large dispersion term.

It was seen that $(\Delta\Psi_{k'k})$ gets too big for $k'<k_f$ if one pushes the peak of $\omega(r)$ too far out. Of course, $\Delta\Psi_{k'k}$ would become very small for large values of k', when more than half of a cycle of sin k'r can come within d.

These considerations made us take a form for $\omega(r)$ of the following type: $\omega(r) = \beta^2 e^{-b^2 (r-\alpha)^2}$, with its peak at $r = \alpha$. The actual value of α taken was 0.36 fm. In order to make $\Delta\Psi_{k'k}$ large, we should also try to make $\left[\frac{\chi_k^s(r)}{N} - \frac{\sin kr}{k}\right]$ as large as possible. This can be done by increasing the strength β^2 of the velocity-dependent potential. Doing this, however, means introducing a lot of attraction in the long-range part too, as we have already noted. The static attractive potential at the short ranges only counteracts the effects of the repulsive part of the potential. With the form of $\omega(r)$ that we have chosen now, there is first attraction from it for very small values of r, followed by strong repulsion, and then there is attraction again. We realized that we could dispense with the static attractive part completely at short-ranges, keeping only the velocity-dependent part at this range. Since it is known from field theory and phase-shift fits that there is a Yukawa-type tail for the long-range part, we keep a static Yukawa potential of inverse range $\mu = .7082$ fm^{-1} beyond 1.4 fm. The s-state potential that we choose, then, is of the following form -

$$V(r,p) = V_1(r) + \frac{1}{m}\left(p^2\omega(r) + \omega(r)\, p^2\right)$$

where $V_1(r) = 0$ for $r < 1.4$ fm

$$V_1(r) = -0.35 \frac{e^{-.7082r}}{.7082\, r} \text{ for } r > 1.4 \text{ fm} \qquad (1)$$

$$\omega(r) = 2.95 \exp.(-5.2\,(r-.36)^2) \text{ in units of } \frac{\hbar^2}{m} = 1.$$

We kept the distance r = 1.4 fm beyond which $V_1(r)$ is nonzero fixed, and had actually four variable parameters when choosing the above values. These were the strength, range, and the position of the peak of $\omega(r)$, and the strength of the static Yukawa part. This potential gave the same two-body low energy scattering data as the S.H.C.P. of Moszkowski and Scott-scattering length of infinity and effective range = 2.5 fm, and approximately the same phase-shifts in the range 80 MeV to 230 MeV, as can be seen in fig. (V-1). Actually, one should not vary the strength of the Yukawa tail, but take its value from field-theory while construcing an actual potential in any-state. The S.H.C.P., however, is a model spin-independent potential with hypothetical low-energy parameters. We are of the opinion that the actual singlet-even scattering data can be fitted approximately by varying only the three parameters of $\omega(r)$.

With this potential (1), we repeated the nuclear-matter calculation of chapter IV. We have plotted, in figure (V-2), $(k' \frac{\Delta\Psi_{k'k}}{4\pi})^2$ against k' for this new potential and we have given the corresponding graphs for Green's s-state potential and the S.H.C.P. It will be seen that now this quantity is quite comparable to the hard-core case, although it falls off faster.

Thus, we have succeeded in making the distortion larger by pushing out the peak of $\omega(r)$ and increasing its strength at

the same time.

We want to compare our results with those of Scott and Moskowski who used the same equation (IV-15) for evaluating the nuclear reaction matrix t^N. This is what they call the new separation method. They use the approximation P = 0 in the Pauli operator and in the energy denominators. This makes the Pauli operator Q(k,P) a simple step-function in k-space:

$$Q(k,P) = 0 \text{ for } k<k_f$$
$$= 1 \text{ for } k>k_f \qquad (2)$$

instead of equations (IV-34). The form of the propagator can again be written as in equation (IV-28), with F given by (IV-27), and $D = \frac{4}{d} + .5k_f^2$. The net effect of making this approximation is to bring about saturation at a slightly lower value of k_f. This is because Q(k,P) is now a step-function, and this cuts off the interference term slightly.

In table III, we show the first-order and various second-order contributions to the binding-energy of nuclear matter, upto $k_f = 1.60$ fm^{-1}. It will be seen that the total second-order contribution to the potential energy at $k_f = 1.6$ fm^{-1} is about 3.5 MeV, compared to the first order contribution of about -45 MeV at the same density. The convergence of the method, on this basis, seems to be as good as the hard-core case. In figure (V-3), we show the variation of the separation distance against k; it has the reasonable

value of $\sim$ 1.2 fm over most of the range. In table IV, we compare the various second order contributions Δt_{kk} to the diagonal elements of t^N_{kk} with those of S.H.C.P. of Scott and Moszkowski. These figures are given for k = 0.2, 0.8 and k_f = 1.0 and 1.4 fm^{-1}. All unmarked figures are in MeV-fm^3. The main differences are the following:-

(1) The dispersion term is still smaller, by about 20%, roughly. This is because the distortion falls off faster with increasing k' than the hard-core case.

(2) The second-order Born term is appreciably bigger. The contribution to the binding energy per particle, for k_f = 1.0, due to this term for the hard-core potential is -1.3 MeV/A, while for the velocity-dependent potential is -2.4 MeV/A. This fact, that a potential without a hard-core gave larger second-order Born term, was also noted by Moszkowski and Scott in their first paper (MS61). The reason for this is that the off-diagonal matrix-elements of the long-range potential, $v^{\ell}_{k'k}$ die off less fast with increasing k' when there is no hard-core. We can see this in figure (V-4), where we have plotted $(v^{\ell}_{k'k} + v^{\ell}_{kk'})$ against k' for k = 0.8 fm^{-1} for the two potentials. This is also the reason why $\Delta t_{kl}(I)$ is bigger for the velocity-dependent potential. The net result of this is that whereas Scott and Moszkowski get saturation at k_f = 1.40 fm^{-1}, we get this for a larger value of k_f, at about k_f = 1.47 fm^{-1}. Their equilibrium

binding energy is -10.4 MeV/particle, while ours is slightly less, -9.9 MeV/particle. We think that if one does not make the approximation P = 0, the equilibrium k_f will be pushed out to a larger value, possibly around 1.60 fm^{-1}.

We conclude from the results of this chapter that a velocity-dependent nucleon-nucleon potential can give substantially the same results in nuclear matter as a hard-core potential, provided one pays enough attention to the behaviour of the two-body wave-function at short-ranges while constructing the potential. Green's s-state potential failed because only the phase-shifts and the low-energy data were fitted with it, but the distortion was so much smaller than the hard-core potential. Two potentials should give substantially the same results in nuclear matter, if they have the same on-shell matrix-elements of the scattering amplitude, $f(\vec{k},\vec{k})$ upto about $k = 1.5\ fm^{-1}$, as well as approximately the same off-shell elements $f(\vec{k}',\vec{k})$ (i.e., the same distortion), upto about $k' = 4\ fm^{-1}$. However, in order to make any reliable calculation in the binding energy, one has still to calculate the reaction-matrix, or at least to use a modified perturbation theory upto the second-order.

BINDING ENERGY/A FOR (POTENTIAL V-1)

ALL ENERGIES IN MEV. k_f in fm^{-1}

TABLE III

k_f	First Order P.E./A	K.E./A	Pauli	Dispersion	Interference	Second Order Born	Total B.E./A
1.0	-16.40	12.44	+.02	.70	-.51	-2.40	-6.15
1.2	-25.14	17.91	+.07	1.91	-1.27	-2.09	-8.61
1.4	-34.76	24.38	+.21	4.36	-2.60	-1.40	-9.81
1.5	-39.76	27.99	+.34	6.26	-3.51	-1.19	-9.87
1.6	-44.86	31.85	+.57	8.69	-4.59	-1.13	-9.47

COMPARISON OF VARIOUS SECOND-ORDER TERMS OF THE NUCLEAR REACTION-MATRIX FOR THE NEW VELOCITY-DEPENDENT POTENTIAL (V-1) AND S.H.C.P. ALL UNMARKED UNITS IN MEV-FM^3. CENTRE-OF-MASS MOMENTUM P = 0. THE FIGURES FOR S.H.C.P. ARE TAKEN FROM (SM 62) Nuclear Physics 29, (1962) 665.

TABLE IV

k (fm^{-1})	k_f (fm^{-1})	$t_{kk}(V)$		$t_{kk}(P)$		$t_{kk}(D)$		$t_{kk}(I)$		Total second order	
		Potential V-1	SHCP	Potn. V-1	SHCP	Potn. V-1	SHCP	Potn. V-1	SHCP	Potn. V-1	SHCP
0.2	1.0	-98.8	-71.3	1.17	0.4	28.9	40.7	-22.2	-20.1	-90.9	-50.3
	1.4	-28.3	-16.6	5.54	3.0	68.4	91.3	-46.3	-38.2	-.66	39.5
0.8	1.0	-93.0	-74.7	0.34	-0.1	26.0	32.8	-17.2	-14.3	-83.8	-56.3
	1.4	-19.2	-13.9	2.86	1.3	62.3	75.0	-36.9	-28.5	9.06	33.9

FIGURE CAPTIONS

Fig. II-1 Phase-shift characteristics for Green's potential II-1 and S.H.C.P. for s-wave.

Fig. III-1 Phase-shift characteristics for a pure hard-core of radius 0.4 fm and the velocity-dependent potential (III-14).

Fig. III-2 Phase-shift characteristics of the Gomes-Walecka-Weisskopf type hard-core potential and the velocity-dependent potential (III-27).

Fig. III-3 Off-shell elements of the scattering amplitude, $f(k',k)$ against k' for Gomes-Walecka-Weisskopf type potential and velocity-dependent potential III-27.

Fig. IV-1 Plot of $(v^{\ell})_{kk}$ the first order Born term against k, for Green's potential II-1, (or IV-39) and S.H.C.P.

Fig. IV-2 Variation of separation distance d with k for Green's potential II-1.

Fig. IV-3 Various second-order contributions to the binding energy/A in separation-method calculation for Green's potential II-1.

Fig. IV-4 Total binding energy/A for Green's potential (II-1) by separation method, modified perturbation method and the corresponding curve for S.H.C.P.

Fig. V-1 Phase-shift characteristics of the new potential (V-1) and S.H.C.P.

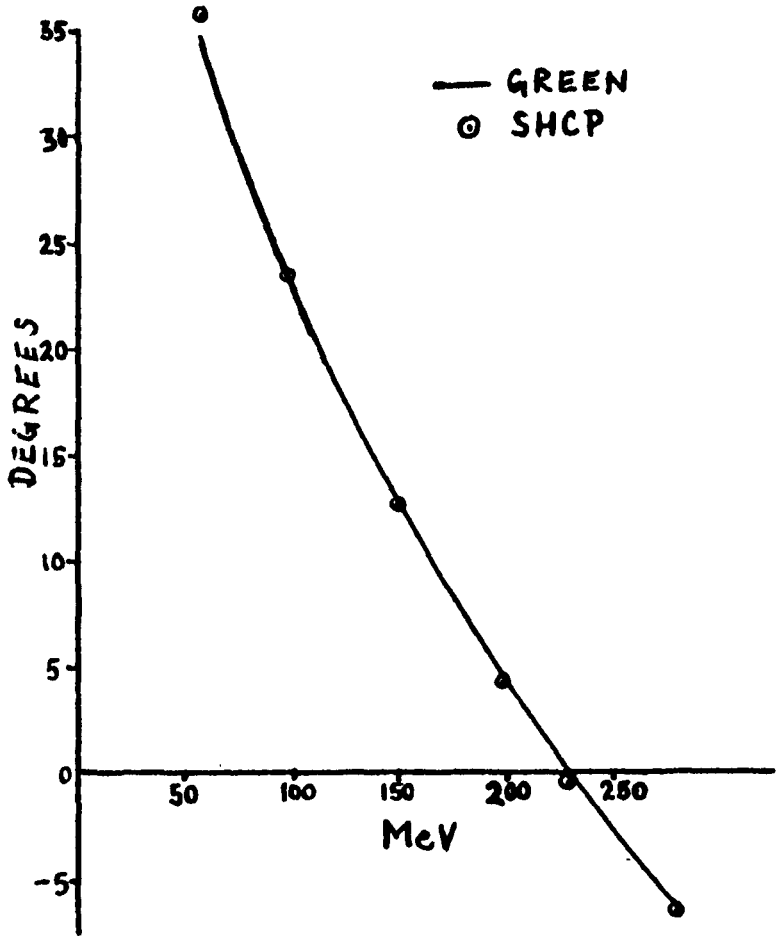
Fig. V-2 Plot of $(k' \frac{\Delta\Psi_{k'k}}{4\pi})^2$ against k' for Green's potential (II-1), S.H.C.P. and the new potential (V-1).

Fig. V-3 Variation of separation distance d against k for new potential (V-1).

Fig. V-4 Plot of $(v^{\ell}_{k'k} + v^{\ell}_{kk'})$ against k' for $k = 0.8$ fm^{-1} for new potential (V-1) and S.H.C.P.

Fig. V-5 Total binding energy/A against k_f for new potential (V-1), and S.H.C.P., calculated by new separation method assuming $P = 0$.

FIG II-1



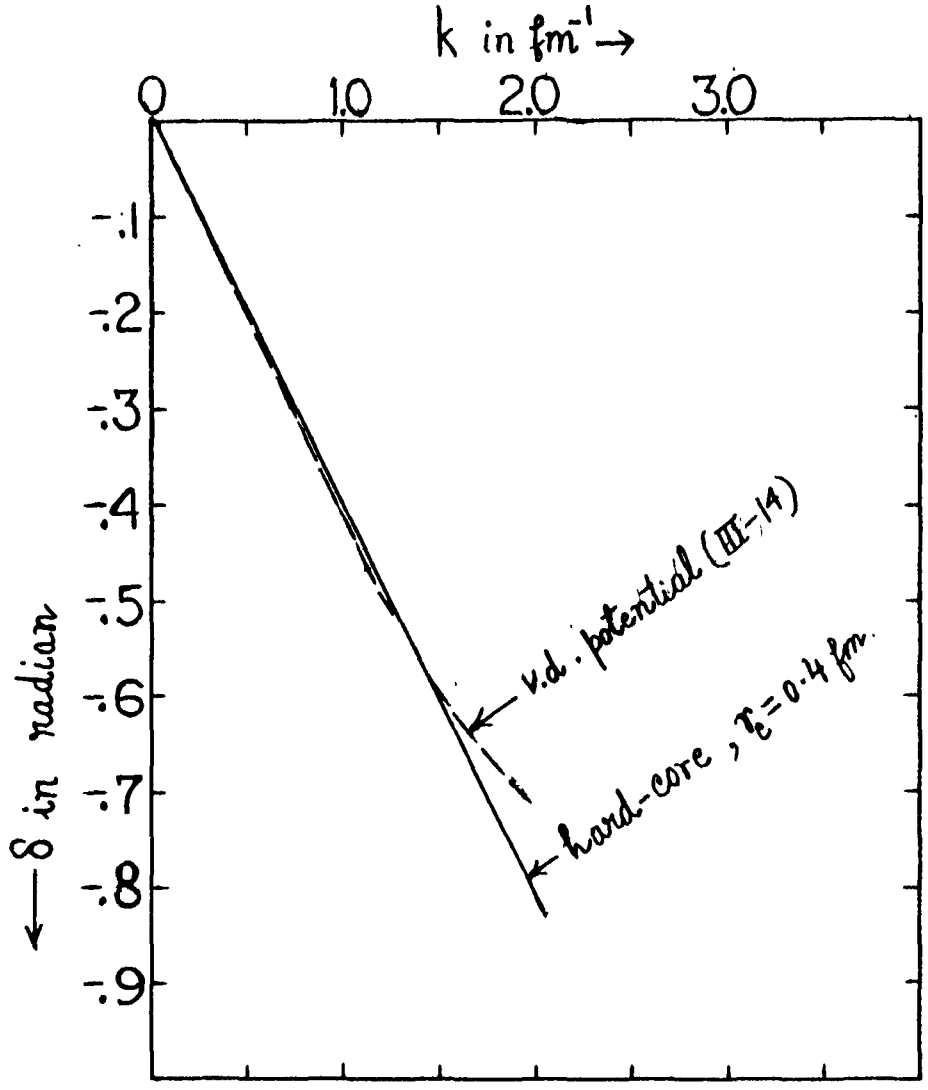

k in fm⁻¹ →
0
1.0
2.0
3.0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
← δ in radian
v.d. potential (III-14)
hard-core, r_c = 0.4 fm.

FIG. III-1

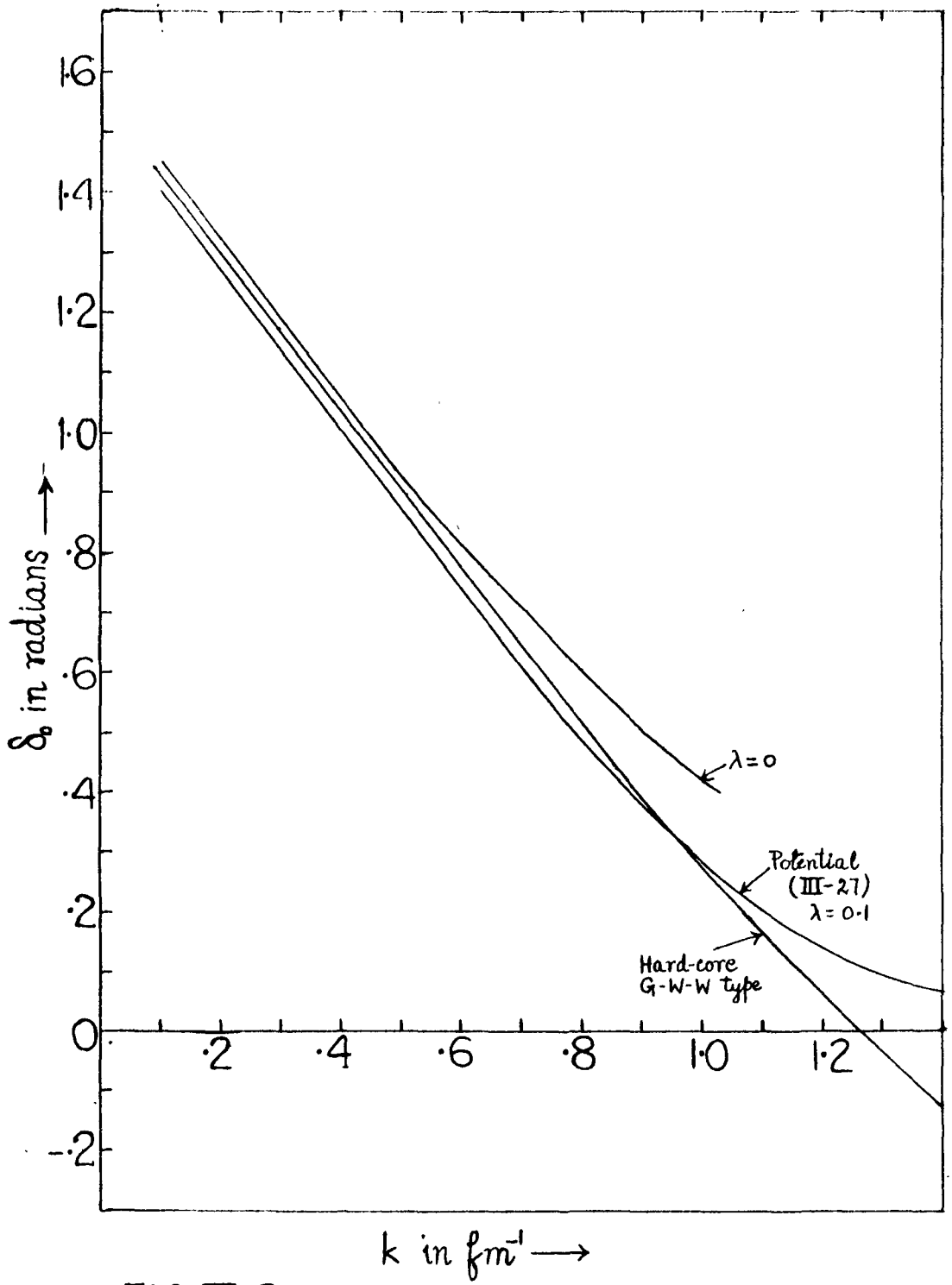

1·6
1·4
1·2
1·0
·8
·6
·4
·2
0
-·2
·2
·4
·6
·8
1·0
1·2
δ_0 in radians
k in fm^{-1}
λ=0
Potential
(III-27)
λ=0·1
Hard-core
G-W-W type

FIG. III-2

FIG III-3

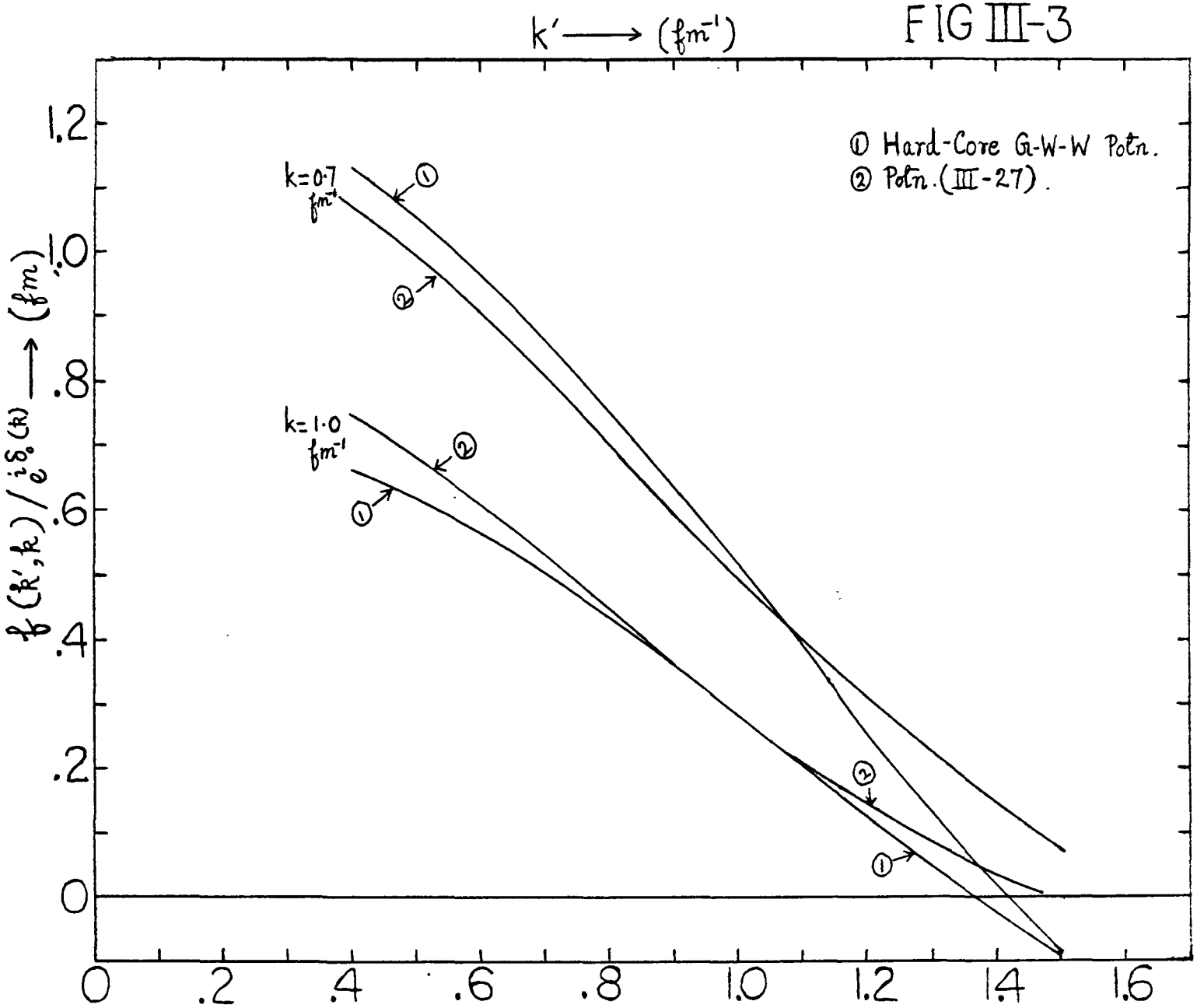

k' ⟶ (fm⁻¹)
f_0(k',k) / e^{iδ_0(k)} ⟶ (fm)
1.2
1.0
.8
.6
.4
.2
0
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
① Hard-Core G-W-W Potn.
② Potn. (III-27).
k=0·7 fm⁻¹
k=1·0 fm⁻¹
①
②

FIG. IV-2

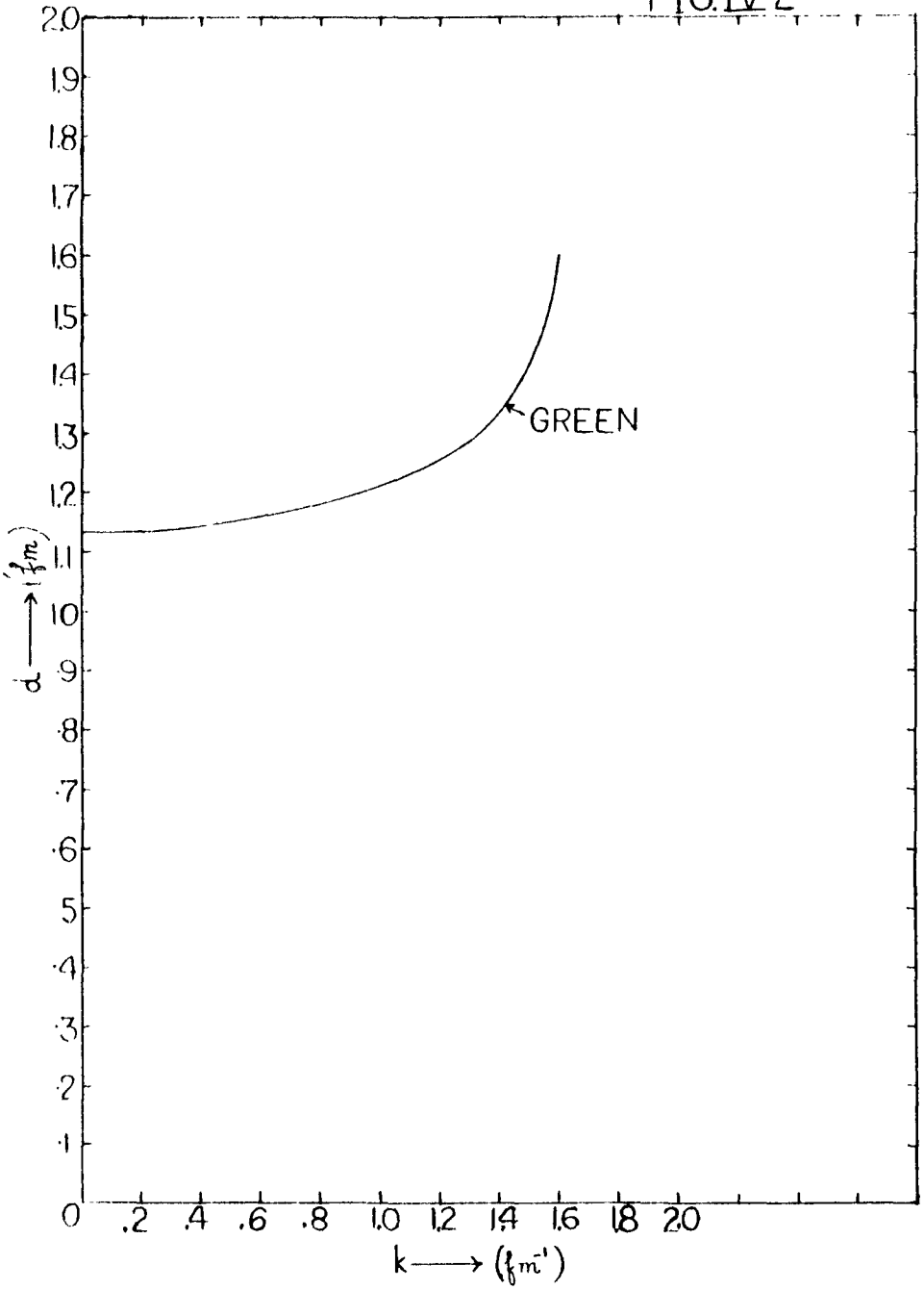

2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
GREEN
d ⟶ (fm)
k ⟶ (fm⁻¹)

FIG. IV-1

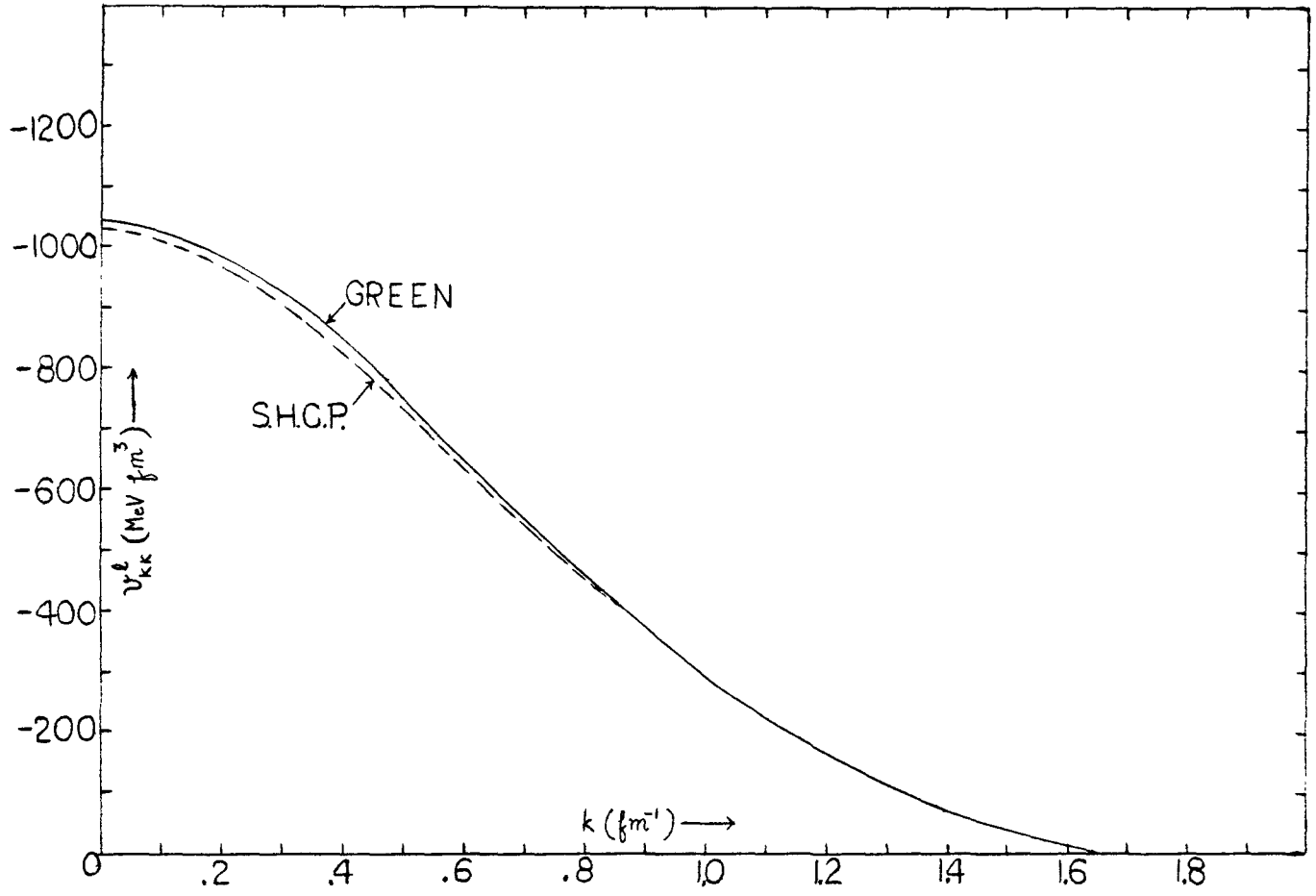

-1200
-1000
-800
-600
-400
-200
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
GREEN
S.H.C.P.
v^{ℓ}_{kk} (MeV fm^3) ⟶
k (fm^{-1}) ⟶

FIG. IV-3

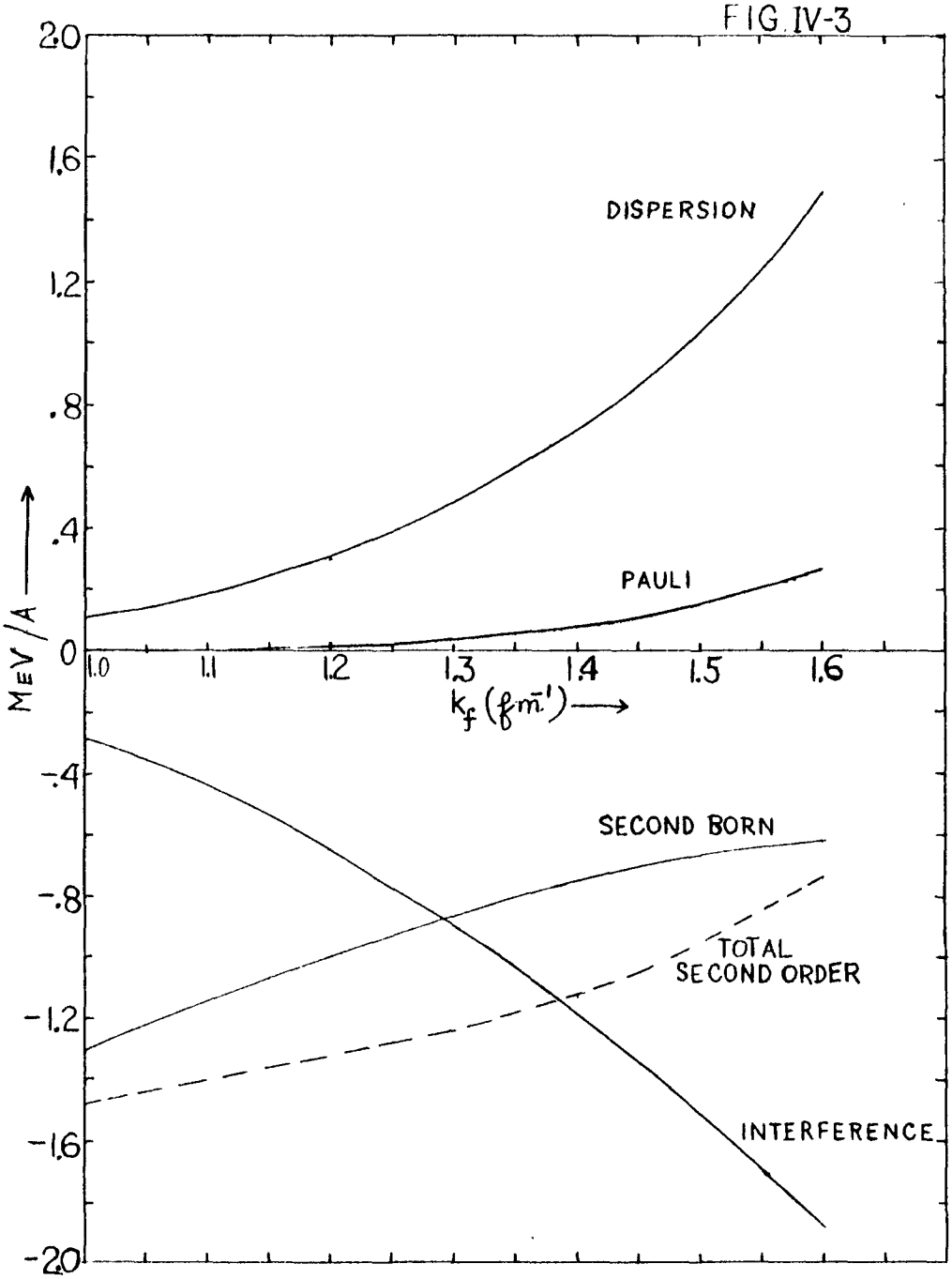

2.0
1.6
1.2
.8
.4
0
-.4
-.8
-1.2
-1.6
-2.0
MEV/A
DISPERSION
PAULI
1.0
1.1
1.2
1.3
1.4
1.5
1.6
k_f (fm^{-1})
SECOND BORN
TOTAL
SECOND ORDER
INTERFERENCE

FIG IV-4

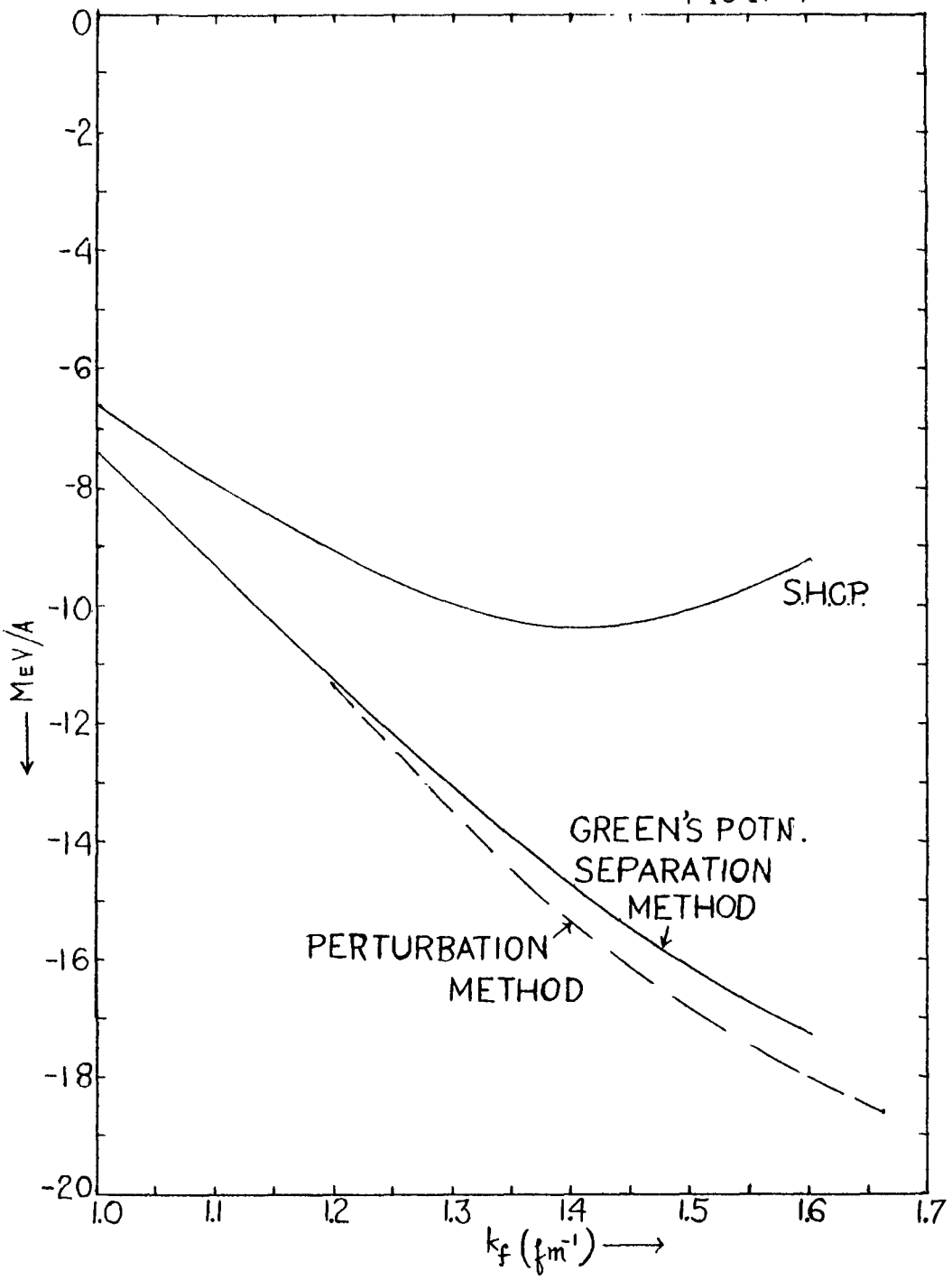

0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
← MEV/A
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
k_f (fm^{-1}) →
S.H.C.P.
GREEN'S POTN. SEPARATION METHOD
PERTURBATION METHOD

FIG. V-1

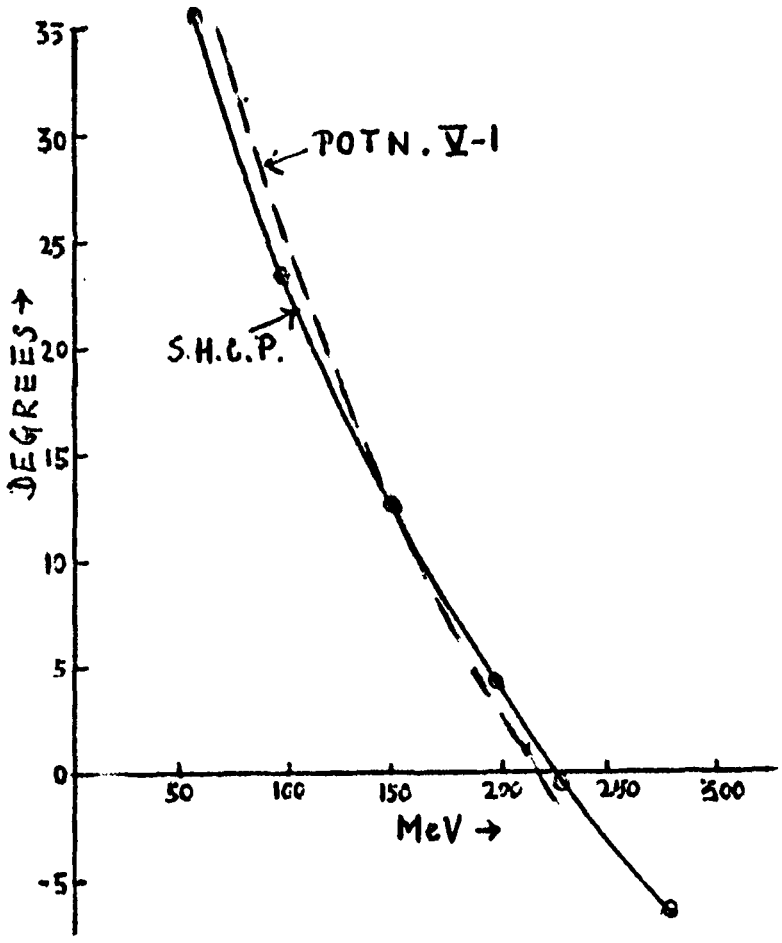

35
30
25
20
15
10
5
0
-5
DEGREES →
POTN. V-1
S.H.C.P.
50
100
150
200
250
300
MeV →

FIG. V-2

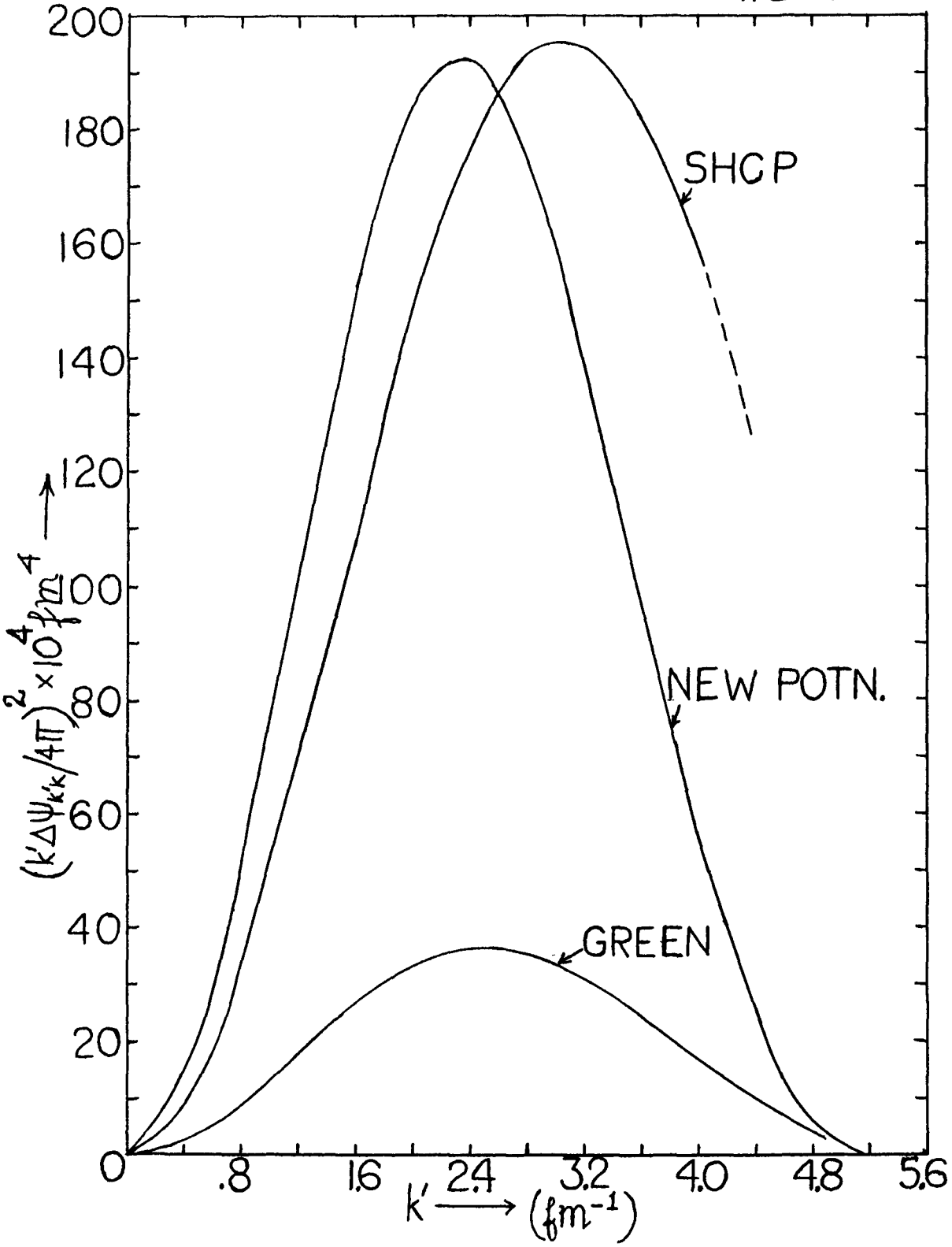
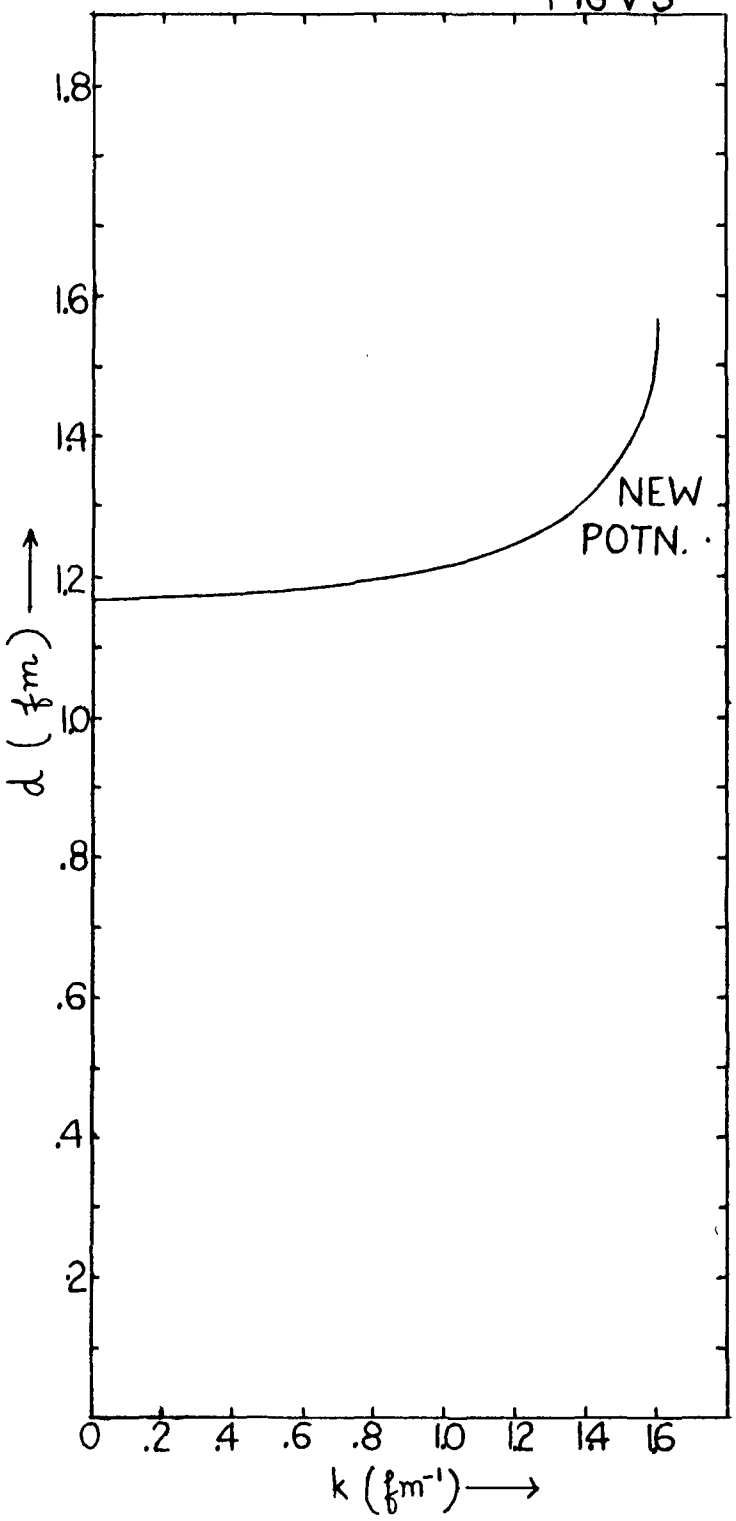

200
180
160
140
120
100
80
60
40
20
0
.8
1.6
2.4
3.2
4.0
4.8
5.6
SHCP
NEW POTN.
GREEN
$(k'\Delta\psi_{k'k}/4\pi)^2 \times 10^4 fm^4$
$k' \longrightarrow (fm^{-1})$

FIG V-3



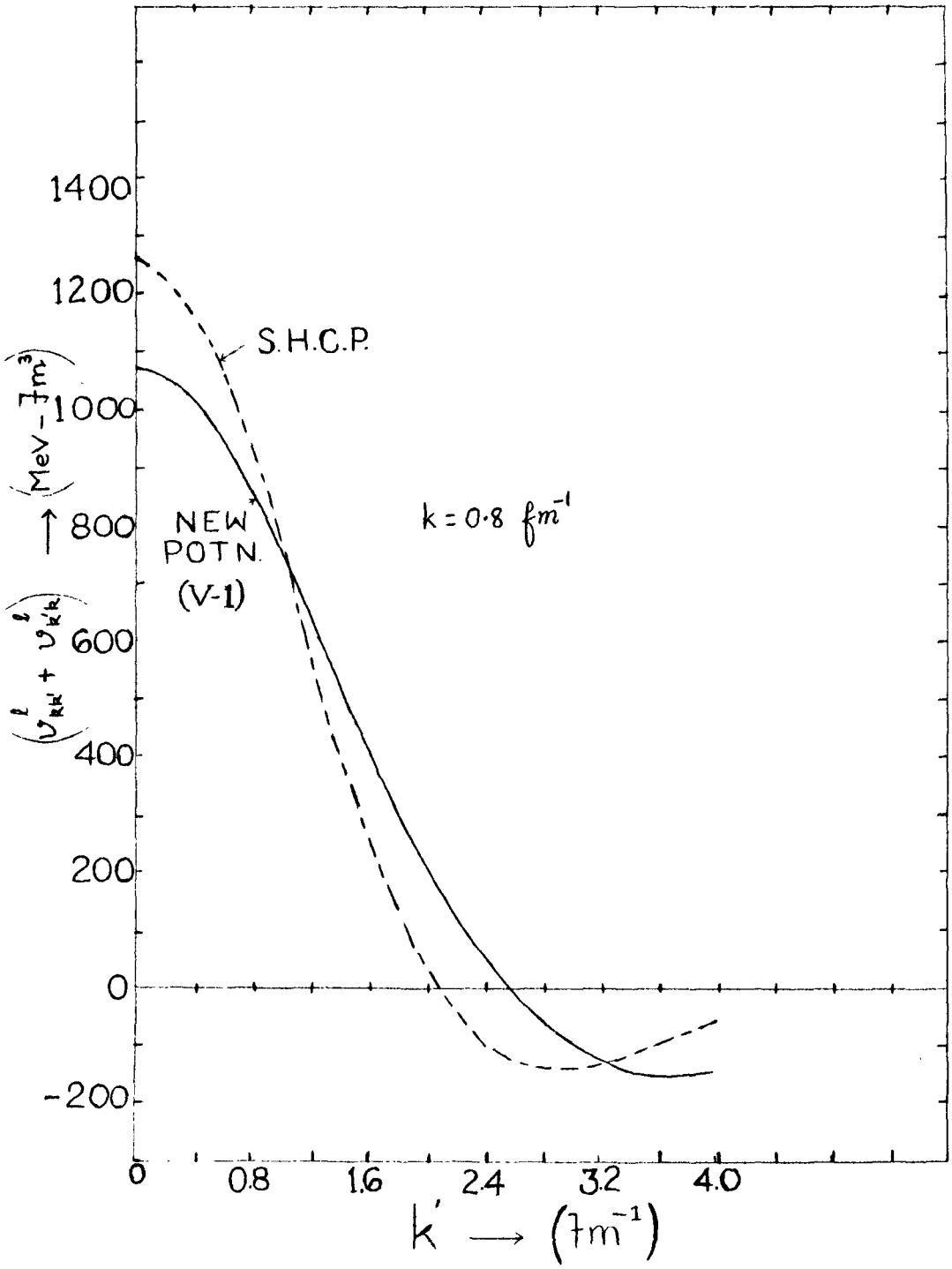
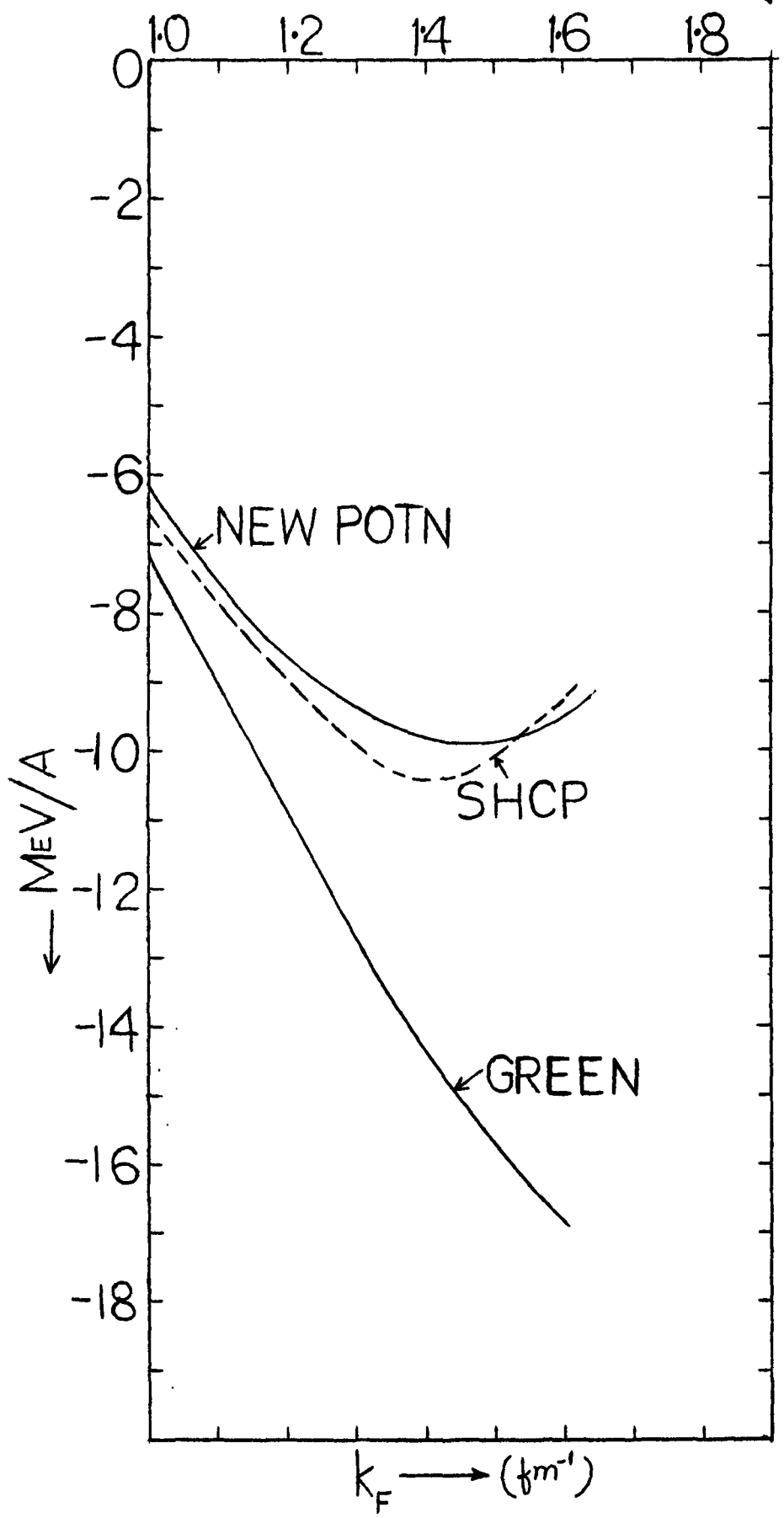

1400
1200
1000
800
600
400
200
0
-200
S.H.C.P.
NEW POTN. (V-1)
k = 0·8 fm⁻¹
(MeV - fm³)
$(v^{l}_{kk'} + v^{l}_{k'k})$
0
0.8
1.6
2.4
3.2
4.0
k' ⟶ (fm⁻¹)

FIG. V-4

Fig. V-5


1·0
1·2
1·4
1·6
1·8
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
NEW POTN
SHCP
GREEN
← MEV/A
k_F ⟶ (fm^{-1})

me 2, number 4 PHYSICS LETTERS 15 September 1962

ON THE VELOCITY-DEPENDENT NUCLEON-NUCLEON POTENTIAL OF GREEN

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Received 27 August 1962

'he object of the present note is to investigate, ɔme detail, the following nucleon-nucleon ve-:y-dependent potential, suggested by Green [1]. potential is of the form

$$V(r) + \frac{p^2}{m_0}\,\omega(r) + \omega(r)\,\frac{p^2}{m_0}\ ,$$

re $V(r)$ consists of the central, tensor and spin-t potentials, p is the relative momentum, r the tive coordinate, and m_0 the nucleon mass. ɜn suggested the following form for the radial ndence of $V(r)$:

$$= -A\exp\left[-(0.6772\,a\mu r)^2\right] - \frac{B\exp(-\mu r)}{\mu r}\left[1-\exp(-\alpha\mu r)\right]\ ,$$

·e $\mu = 0.7082\ \mathrm{fm}^{-1}$. The subtraction in the ɪwa part is done to eliminate the singularity = 0, which makes computation simpler. For 'elocity-dependent part

$$\omega(r) = C\exp\left[-(0.6772\,c\mu r)^2\right]$$

ɲ adjusted the above parameters to fit the low-gy data, i.e., scattering length, effective range ɥuadrupole moment, and also to fit the lower al wave phases with the Breit [2] phase-para-rs at 300 MeV laboratory energy. Green cal-ed the singlet nuclear phase-shifts fairly com-ly, i.e., for $l = 0, 1, 2, 4$; but for the triplet , he gives all the nuclear-bar phase-parameters for $J = 1$; and some (though not all) phase-neters up to $J = 3$. The potential of Green that ıdertook for more complete investigation is ɪllowing:

ŋglet-even: $A = 1.185\ \mathrm{fm}^{-2}$, $a = 1.645$, $B =$ fm^{-2}, $\alpha = 6$, $C = 1.14$, $c = 3$, the scattering ı $a_S = -23.9$ fm, and effective range $r_S =$ 'm with this potential.

ŋglet-odd: $A = 0$, $B = -0.798\ \mathrm{fm}^{-2}$, $\alpha = 0.3$, .3, $c = 2$.

'iplet-even: $B = 0$ throughout for central and r. No $(\boldsymbol{L}\cdot\boldsymbol{S})$ force. Central: $A = 2.6\ \mathrm{fm}^{-2}$, 3. Tensor: $A = 0.985\ \mathrm{fm}^{-2}$, $a = 1.15$. Velocity-dependent: $C = 0.70$, $c = 3$. The scattering length: $a_T = 5.6$ fm. Effective range: $r_T = 1.74$ fm. Quadrupole moment: $Q = 0.287$ ef^2.

Triplet-odd: Central: $A = 0$, $B = -0.089\ \mathrm{fm}^{-2}$, $\alpha = 6$, $C = 0$. Tensor: $A = 0$, $B = -0.35\ \mathrm{fm}^{-2}$, $\alpha = 6$. Spin-orbit: $A = 2.0\ \mathrm{fm}^{-2}$, $a = 2$, $B = 0$.

We have calculated all the nuclear-bar phase-parameters up to $J = 5$, solving the coupled Schrödinger equations numerically, at laboratory energies of 90 MeV, 156 MeV and 310 MeV. Also, all the singlet phase-shifts up to $l = 5$ have been calculated at the same energies. Finally, as an application and a test of this potential to a nuclear problem where all the phase-parameters (including the higher partial waves) are needed, we have calculated the integrated optical potential (the central as well as the spin-orbit part) at the above energies, using the formalism of Kerman et al. [3].

In the graphs, the phase-parameters of Green; Gammel and Thaler [4] and Hamada and Johnston [5] are compared with Breit's values (YLAM or YLAN 3). Breit's values are normalised to one in the graphs. It will be seen that the singlet phase-shifts and the triplet-even phase-parameters of Green are in reasonable agreement with Breit's values. But the triplet-odd potential of Green is definitely worse than the hard-core potentials. Since Green suggested the above potentials only by fitting the low-energy data and the phases at 300 MeV, but not by making a systematic search of all the parameters for the best fit, it is certain that one can improve upon this potential.

If one writes the optical potential in the following form:

$$V(r) = V_C(r) + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\,V_{SO}(r)\,\boldsymbol{l}\cdot\boldsymbol{s}\ ,$$

then the integrated optical potentials are defined by Kerman et al. [3] as

$$U = -\frac{1}{N}\int V_C(r)\,\mathrm{d}r\ , \qquad W_{SO} = +\frac{1}{N}\int V_{SO}(r)\,\mathrm{d}r\ ,$$

where the integration is over the volume of the nucleus, and N is the number of nucleons in the nucleus. These integrated potentials can be expressed

in terms of the forward scattering amplitudes, using impulse and multiple-scattering [3] approximations. These, in turn, are related to the phases obtained. This calculation, then, deals with matrix-elements on the energy-shell and will not differentiate between potentials whose off-shell matrix-elements are different. However, the calculation of integrated optical potentials involves all the phases obtained from he potential, including the higher partial waves. One can, therefore, find out how sensitive the integrated optical potential is to these phase-parameters.

Table 1
Values of U and W_{so}
($N \to \infty$; neutron number = proton number).

E	310 MeV	156 MeV	90 MeV
U (MeV fm^3)	60.6+i211 (76.3+i212 59+i203 (Stapp)	194+i222 (221+i187)	264+i246 (289+i217)
W (MeV fm^5)	47-i7 25-i6 (Stapp)	63-i16.5 50-i13	78-i28 58-i20

The figures calculated from G-T potentials by Kerman et al. are shown in brackets. At 310 MeV, they actually used phase-shifts of Stapp's solution (1), and used fewer phases than we do, cutting off at $J = 4$.

The "experimental" values, quoted by Kerman et al. (an average over a series of nuclei) for U are given in table 2.

Table 2

E	310 MeV	156 MeV	90 MeV
U (MeV fm^3)	24+i165	183+i118	300+i119

We also calculated U and W_{so} for C^{12}, and compared these values with some recent experimental values given by Batty [8] for C^{12} (see table 3). Batty's figures are given in brackets. The values for Im W_{so} by Batty are rather uncertain, but the sign agrees with ours. The agreement with experimental values is not bad, when one considers all the approximations involved [6]. The calculated spin-orbit part of the integrated potential is seen to be higher than experimental values.

Kerman et al., while calculating the scattering amplitudes from the phase-shifts using the formulae of Stapp et al. [7], have cut off at $J = 4$ (including $J = 5$ partially) even at 310 MeV. In our calculations of the scattering amplitude, the contributions to the real part of the scattering amplitude at 310 MeV from $J = 5$ to $J = 7$ and from $l = 6$, were as large as 20%. (These phases were calculated using Born approximation.) Leaving them out will alter the results appreciably. In the triplet-even case, the tensor force in Green's potential has a much longer range than the central or velocity-dependent part, and is the only one which contributes appreciably to the phase-shifts of the higher partial waves. The same situation exists for the triplet even case of the G-T potential (the inverse range $\mu_T = 1.045$ fm^{-1}, whereas $\mu_c = 2.09$ fm^{-1}). One would, therefore, expect that the tensor force would again give an appreciable contribution to the higher partial wave phase-shifts, just as in Green's potential. One should not, therefore, cut off at $J = 3$ at 310 MeV for triplet-even, but should at least go up to $J = 5$. Hence the figures for U and W_{so} at 310 MeV given by Kerman et al. are not reliable and should not be compared with experiments.

Table 3

E	310 MeV	156 MeV	90 MeV
U (MeV fm^3)	55+i193 (64±9 +i183±2)	178+i203 (107±7 +i132.5±2.5)	242+i225 (225±9 +i150±4)
W_{so} (MeV fm^5)	48-i7	64-i16.6	77-i28
Re W_{so}	$(33^{+2.5}_{-5})$	(61±5.5)	(54±4)

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