

ADAPTIVE NUCLEAR REACTOR CONTROL
BASED ON OPTIMAL LOW-ORDER LINEAR MODELS

ADAPTIVE NUCLEAR REACTOR CONTROL
BASED ON OPTIMAL LOW-ORDER LINEAR MODELS

By

GEORGE T. BEREZNAI, B.E., M.Eng.

A Thesis

Submitted to the Faculty of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Doctor of Philosophy

McMaster University

December 1971

DOCTOR OF PHILOSOPHY (1971)
(Electrical Engineering)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Adaptive Nuclear Reactor Control Based
on Optimal Low-order Linear Models

AUTHOR: George Thomas Bereznai,
B.E. (University of Adelaide)
M.Eng. (McMaster University)

SUPERVISOR: Professor N.K. Sinha,
B.Sc.Eng. (Banaras)
Ph.D. (University of Manchester)

NUMBER OF PAGES: xi, 186

ABSTRACT:

The problem of adaptively controlling the power level changes of a nuclear reactor, by the use of a digital computer, is considered. It is established, that for the application of modern control theory a low-order linear model of the reactor is needed, but that the existing models are not sufficiently accurate for the desired purpose. A new technique is therefore developed for finding low-order linear models of a given high-order system. Such models are shown to be suitable for the suboptimal control of the original system, subject to cost functions normally encountered in practice. The proposed methods of modelling and suboptimal control are applied to the adaptive control of a nuclear reactor. In order to emphasize practical realization, a model of an operating nuclear power plant is considered, with emphasis on the physical limitations imposed by the controller mechanism. It is shown, that despite wide variations in the model parameters as a function of the operating power level and of the temperature coefficient, the model can

be updated on-line to a sufficient accuracy to produce negligible deviations between optimal model and suboptimal system performance. Apart from the realization of the adaptive controller, it is indicated that the proposed technique is also suitable for the fully computerized design of optimal and suboptimal feedback controllers for a wide variety of cost functions.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. N.K. Sinha, for his guidance, encouragement and assistance throughout the course of this work and in the preparation of this thesis. His extensive knowledge of mathematics and control systems facilitated much of the research effort.

I am also very thankful to the members of my supervisory committee for their helpful advice, in particular, Dr. R. Kitai on the digital and practical sides of the work and Dr. G.L. Keech regarding the nuclear aspects.

Thanks are due to Dr. J.W. Bandler for making the author aware of computer-aided design techniques, and for useful discussions on related subjects. Dr. A.A. Harms has been instrumental in improving my knowledge of nuclear power systems, for which I would like to express my sincere appreciation.

The leadership provided by Dr. S.S. Haykim as Chairman of the department during the course of my tenure has greatly contributed to maintaining the scholarly atmosphere necessary for research.

The financial assistance provided by the National Research Council of Canada and the Department of Electrical Engineering is gratefully acknowledged.

The competence and reliability of Mrs. K. Paulin, who typed this thesis, has been very much appreciated.

Finally, I would like to thank my wife for her love, patience and understanding throughout this endeavour.

TABLE OF CONTENTS

	Page
CHAPTER 1 - INTRODUCTION	1
CHAPTER 2 - THE OPTIMAL CONTROL PROBLEM	7
2.1 - The Linear Regulator	8
2.2 - Optimal Feedback Control of a Second-order System	11
2.3 - The Linear Servomechanism	13
CHAPTER 3 - OPTIMAL CONTROL OF NUCLEAR REACTORS	17
3.1 - The Six Delayed Neutron Group Model	18
3.2 - Classical Low-order Reactor Models	21
3.2.1 - One Delayed Neutron Group	22
3.2.2 - Prompt-jump Approximation	23
3.2.3 - Infinite Delay-time Approximation	24
3.2.4 - Weighted Average Neutron Generation Time	24
3.2.5 - Comparison of Responses for a Step-change in Reactivity	25
3.3 - A Realistic Nuclear Reactor Model	28
3.4 - The Cost Function	37
3.5 - Review of Recent Publications	39
3.6 - The Proposed Method of Solution	42
CHAPTER 4 - OPTIMAL LOW-ORDER MODELS	45
4.1 - Pattern Search	46
4.2 - The New Fletcher Algorithm	48
4.3 - Statement of the Problem	49
4.4 - Error Criteria	50
4.5 - The Shortest Distance Minimax Criterion	52
4.6 - Starting Parameters and Essential Features of the Computer Programme	58
4.7 - Optimal Models for a Linear Seventh-order System	63
4.8 - Application to System Identification	74
CHAPTER 5 - SUBOPTIMAL CONTROL OF HIGH-ORDER LINEAR SYSTEMS	89
5.1 - Problem Formulation	89
5.2 - Computational Aspects	94
5.3 - Modelling Criteria for Suboptimal Control	97
5.4 - Optimal Control for Least pth and Minimax Cost Function	104
5.4.1 - Frequency-domain Derivation	106
5.4.2 - Time-domain Derivation	109

CHAPTER 5 - continued	Page
5.5 - Optimal Control with Zero Steady-State Error	111
5.6 - Results for Least-pth and Minimax Cost Functions	115
CHAPTER 6 - SUBOPTIMAL CONTROL OF NUCLEAR REACTOR - RESULTS	123
6.1 - Optimal Second-order Linear Models	124
6.1.1 - Effect of Nonlinear Reactor Kinetics	126
6.1.2 - Effect of Parameter Changes	132
6.2 - Responses for Integral Quadratic Cost Functions	134
6.3 - Responses for Least pth and Minimax Cost Functions	140
6.3.1 - Monotonic Reactivity Insertion	140
6.3.2 - Ramp Change of Demanded Power	142
CHAPTER 7 - ADAPTIVE CONTROL OF NUCLEAR REACTOR	146
7.1 - The Adaptive Loop	147
7.1.1 - The Identification Interval	149
7.1.2 - The Adaptation Interval	150
7.2 - Results for Integral Quadratic Cost Functions	152
7.3 - Results for a Desired Step Response	162
7.4 - Results for Least pth and Minimax Cost Functions	162
CHAPTER 8 - CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK	169
REFERENCES	174
BIBLIOGRAPHY	180
PUBLICATIONS	184
APPENDIX	186

LIST OF FIGURES

Figure		Page
2-1	Block diagram representation for establishing the equivalence of the regulator and servomechanism problems for a step change in input.	15
3-1	Response of six delayed neutron group reactor kinetic model, for step changes in reactivity.	20
3-2	Comparison of responses of the six delayed neutron group model and various lower-order kinetic models, for a 0.3mk step change in reactivity.	26
3-3	Comparison of responses of the six delayed neutron group model and various linearized kinetic models, for a 0.3mk step change in reactivity.	27
3-4	Block diagram of the reactor and parts of the power level control system at Douglas Point.	29
3-5	Simplified block diagram of reactor and control system.	32
3-6	Closed-loop response of nuclear reactor to step-change in demanded power, using various nonlinear models.	35
3-7	Closed-loop response of nuclear reactor to step-change in demanded power, for various linear models and the reference system.	36
4-1	Piece-wise linear reference response with specified error bounds, and responses of approximating second-order systems.	54
4-2	Possible responses of a stable system to a unit step input.	59
4-3	Responses of seventh-order system and second-order $M e_p $ model.	75
4-4	Responses of seventh-order system and second-order $\sum e^2$ model.	76
4-5	Responses of seventh-order system and fourth-order $\sum e^2$ model.	77

Figure		Page
4-6	Unit step responses of fourth-order system and second-order model.	80
4-7	Parameter changes during identification of the fourth-order system.	82
4-8	Changes in the objective function during identification of the fourth-order system.	83
4-9	Unit step response of second-order system.	85
4-10	Identification of second-order system.	86
5-1	Block-diagram representation of optimal model controller and suboptimal system controller.	91
5-2	Suboptimal responses of seventh-order system and optimal responses of second-order model.	93
5-3	Comparison of open- and closed-loop responses of second-order model and seventh-order system.	103
5-4	Block-diagram representation of optimum systems.	108
5-5	Block-diagram representation of closed-loop system for realization of zero steady-state error.	113
5-6	Responses for example 5.2.	118
5-7	Responses for example 5.3, cases (a) and (b).	120
5-8	Responses for example 5.3, cases (c) and (d).	121
6-1	Comparison of responses of one delayed neutron group and optimal second-order model to that of the reference system.	125
6-2	Changes in model parameters as a function of the initial power level, for 20% FP increases in demanded power.	128
6-3	Changes in model parameters as a function of the initial power level, for 20% FP decreases in demanded power.	129
6-4	Effect of magnitude of demanded power increase on model parameters. Initial power level 50% FP.	130

Figure		Page
6-5	Effect of magnitude of demanded power decrease on model parameters. Initial power level 100% FP.	131
6-6	Effect of temperature coefficient on optimal model parameters, for an increase in demanded power from 80% to 100% FP.	133
6-7	Block diagram representation of suboptimal system and optimal model controllers.	135
6-8	Suboptimal system, optimal and updated model responses for 90% - 100% FP demanded change.	136
6-9	Suboptimal system, optimal and updated model responses for 50% - 100% FP demanded change.	138
6-10	Responses of reoptimized model and corresponding suboptimal system.	141
6-11	Suboptimal reactor responses for the cost function in equations (6-3) and (6-4).	143
6-12	Suboptimal reactor responses for ramp changes in demanded power.	145
7-1	Adaptive nuclear reactor responses for unlimited identification interval.	157
7-2	Adaptive nuclear reactor responses, when identification interval is limited to 0.2 second.	158
7-3	Adaptive nuclear reactor responses, starting with reoptimized model parameters.	159
7-4	Adaptive nuclear reactor response to coincide with desired critically damped response.	163
7-5	Adaptive and suboptimal reactor responses for monotonic reactivity insertion.	165
7-6	Adaptive reactor responses for ramp changes in demanded power.	166
7-7	Response of Douglas Point nuclear reactor to a 90% - 100% FP change in demanded power.	168

LIST OF TABLES

Table		Page
4-1	Comparison of second-order models for seventh-order system, obtained by previously published methods.	67
4-2	Errors associated with the models in Table 4-1.	69
4-3	Comparison of optimum second-order models for seventh-order system, obtained by the pattern search programme.	71
4-4	Errors associated with the models in Table 4-3.	72
4-5	Optimal $\sum e^2$ models of increasing order for seventh-order system.	73
4-6	Effect of reducing number of samples over a given time interval, on the number of function evaluations.	87
4-7	Effect of using decreasing number of initial sample points, on the accuracy of identification.	88
5-1	Suboptimal system and optimal model performance.	92
5-2	Modelling accuracy in suboptimal control, as a function of the number of samples used over a given time interval.	96
5-3	Performance of optimal second-order models in the suboptimal control of seventh-order system.	99
5-4	Performance of optimal second-order models with fixed gains ($b_0=Ka_0$), in the suboptimal control of seventh-order system.	101
5-5	Comparison of errors for open- and closed-loop responses of system and model.	102
5-6	Results for example 5.1.	116
5-7	Comparison of results obtained for the four cases of example 5.3.	119

Table		Page
6-1	Optimal model parameters for given reactor responses.	126
6-2	Reoptimization of model parameters for suboptimal reactor responses, and corresponding changes in model and system costs.	139
7-1	Changes in k_1 for a 90% - 100% FP change in power level.	154
7-2	Changes in k_1 for a 90% - 100% FP change in power level.	155
7-3	Optimal model costs and suboptimal system costs using adaptive control.	156

CHAPTER 1

INTRODUCTION

In March 1971, the first reactor at the Pickering Nuclear Power Plant went critical. A historical first had occurred: a large, 500 megawatt installation started operation, entirely under the control of digital computers. The uniqueness of this event is particularly surprising in view of the parallel development of nuclear reactors and digital computers.

When the first Canadian nuclear power station began commercial operation in 1967 at Douglas Point, Ontario, a control computer was installed, but it has been used only for the regulation of nonessential parameters. The experience gained in that application was, however, sufficiently favourable to entrust the complete automatic control of the next nuclear power station, the one at Pickering, to digital computers.

The step, of going from a custom-built analogue controller to a digital computer, was as great as the designers of the control system could take, in view of the stringent safety requirements of the nuclear industry. The design philosophy of the controller remained the same, the system that has been found reliable at Douglas Point, was realized on digital machines, except for the modifications inherent in going from a smaller, essentially point controlled system, to a large, multi-region one.

Once the digital computer has proven itself in the field, the next step of the designer should be to take advantage of the unique

abilities of the computer, and to attempt to improve the performance of the system under control.

The classical methods of controller design, typically in the frequency-domain [1],[2], are inadequate for the task of optimizing the performance of a large system, such as the nuclear reactor. Recent advances in optimal control theory and state-variable analysis in the time-domain [3],[4], which necessitate the presence of an on-line computer for their realization, have been applied up to date mainly in the aerospace industry. It is the purpose of this thesis, to investigate the on-line digital computer control of nuclear reactors, with a view to applying optimal control theory. Since it is known that the nuclear reactor is a nonlinear system having time-varying parameters, it is expected that the optimal controller will not be constant, but will have to be adapted to prevailing operating conditions in the plant [5],[6].

The inherent difficulty of field testing a new control system for a nuclear power reactor necessitates the development of a valid model for the reactor. Since an accurate digital computer simulation program has already been written by the author for the Douglas Point plant [5] and because for this smaller reactor, spatial effects may be neglected, it has been selected as the basis for the present work.

The earlier investigation by the author (reference [5]) of an adaptive nuclear reactor controller had been restricted by the adherence to the demanded power level change programs of the Douglas Point plant. The control scheme was developed using semi-empirical simulation methods, and is limited to the minimization of the integral of the absolute value of the error between actual and demanded power output.

The aim in the present work is to realize near-optimum performance of the reactor and controller mechanism, for a variety of inputs and cost functions, in an adaptive configuration. The problem is solved by applying some of the well-known results of modern control theory as well as more recent numerical optimization techniques. At the heart of the proposed method is the computation of optimal low-order linear models, which represent, in a piece-wise manner, the nonlinear nuclear reactor and controller mechanism. The model parameters are updated on-line, on the basis of the observed reactor performance.

Important differences, from a practical point of view, between the present research effort and those described by other investigators, are not only in the form of the assumed low-order model and in the method of updating its parameters, but also that the resultant optimal control law is tested on the actual, appreciably higher-order system dynamics. It is established that the difference between optimal model performance and the corresponding suboptimal system cost are within the usual engineering accuracies.

Since an extensive literature survey prior to 1969 has already been presented by the author, and another work published at about the same time also gives a thorough exposé of earlier research efforts^[7], only the most recent papers on nuclear reactor control are reviewed in this thesis. With the nonspecialist reader in mind, description and criticism of relevant papers will be undertaken in the appropriate chapters, after the necessary background has been established.

In writing this thesis, an elementary appreciation of the basic

processes that take place in a nuclear reactor is assumed. A simple explanation of essential aspects of the type of natural uranium fueled and heavy-water moderated reactor that we are considering has been given by the author elsewhere^[8]. Several excellent texts also exist on the subject, to which the interested reader may turn^{[6],[9]}. The reactor kinetic equations are therefore presented without detailed developments, and only those aspects of the Douglas Point reactor simulation program are described which are essential to the present work.

In Chapter 2, the basic optimal control problem is stated, and its solution for integral quadratic cost functions is indicated. Relationships between the regulator and servomechanism problems are established, and formulae for the optimal feedback coefficients for a second-order system are derived.

The problems associated with applying the results of optimal control theory to the nuclear reactor control problem are discussed in Chapter 3. The various reactor kinetic models which have been described in the literature are considered, and their behaviour for step-changes in reactivity presented. The effect of temperature changes on reactivity and practical limitations on the controller mechanism are next observed, leading up to the model of the Douglas Point plant. It is established that none of the classical low-order reactor kinetic models are adequate for representing the behaviour of a realistic system. The most important parameters of a nuclear power plant are considered, to form a meaningful cost function that reflects the performance of the reactor and associated control system. A critical

review of recent research proposals for optimal reactor control concludes the chapter.

Since the on-line realization of the optimal feedback controller derived in Chapter 2 necessitates the availability of a low-order linear model of the system to be controlled, and as the classical low-order reactor models have been found inadequate in Chapter 3, the problem of deriving better models is considered in Chapter 4. In particular, a method based on search routines is presented, which is capable of giving low-order linear models, such that the deviation between system and model responses, to the same input, is minimized in any desired sense. It is shown that the technique is applicable to the on-line identification of the model parameters, making it suitable for adaptive control applications.

The use of a low-order model to derive the optimal controller for a high-order system will result in suboptimal performance of the latter. It is shown in Chapter 5, that the difference between optimal model cost and suboptimal system cost is sufficiently small to make the method useful in practice. Another significant contribution that is presented in this chapter, is the derivation of the optimal controller for least pth and minimax cost functions.

In Chapter 6 the methods developed in the last two chapters are applied to the suboptimal control of the nuclear reactor. The types of responses obtained from various cost functions are presented, and the effect of nonlinear system characteristics and plant parameter changes are considered.

The solution of our original problem, the adaptive control of

the nuclear reactor is presented in Chapter 7. The on-line identification of the model parameters is considered as well as the corresponding updating of the feedback controller. Results for a wide variety of operating conditions and cost functions are presented to illustrate the versatility of the proposed method.

In the concluding chapter, the main results and contributions of the thesis are summarized. By considering the various assumptions made in deriving our results, several recommendations for future research are made.

Apart from the references cited in the thesis, selected bibliographies of the nuclear reactor and the control literature are presented. A list of papers by the author related to this thesis and accepted for publication is also given for the benefit of the interested reader.

CHAPTER 2

THE OPTIMAL CONTROL PROBLEM

The majority of the applications of optimal control theory are based on either Dynamic Programming^[10] or on Pontryagin's Maximum Principle^[11]. The major difficulty associated with the former is the very large memory requirements for systems of appreciable order (four or greater), while the latter technique necessitates the solution of a two-point boundary value problem. While no general solution is known to the optimal control problem, the special case of the linear regulator, with integral quadratic cost function, is readily treated by both methods^[12]. In this thesis the maximum principle will be used, as it appears to have the greater potential for the type of application we are considering.

Efforts in the past, to apply modern control theory to such practical problems as nuclear reactor control have centered on finding solutions, usually numerical, to the two-point boundary value problem^[7],^[13],^[14]. The essence of the approach to be pursued in this thesis is to somehow reduce or transform the system equations to a simpler form, such that the solution of the two-point boundary value problem is avoided.

The reactor control problem to be considered is essentially a servo-mechanism problem, since the final value of the output is always greater than zero. However, since the mathematical formalism of the regulator problem (final value of the state vector zero) is more convenient, this will be first considered. The equivalence of the two

problems, for a step input, will be established subsequently.

Since the full derivation of the optimal linear regulator is well-known^[12], only a brief outline is presented in the next section, for the sake of completeness.

2.1 The Linear Regulator

It is desired to find the feedback control law for an n th order single input single output linear dynamic system

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \quad (2-1)$$

In phase-variable notation^[3] x_1 is the output and x_2, \dots, x_n its first and subsequent derivatives, A is the $n \times n$ system matrix, \underline{b} the n -dimensional control vector, and u is the input which is to provide optimum performance by minimizing the cost function

$$J = \int_0^{t_f} \frac{1}{2} (\underline{x}^T Q \underline{x} + p u^2) dt \quad (2-2)$$

while the system is driven from a given initial state to a desired final state in the time interval $[0, t_f]$; Q is an $n \times n$ positive semi-definite symmetric matrix and p a positive number.

To solve the above problem via the maximum principle, the Hamiltonian is first formed

$$H(\underline{x}, u, \underline{\lambda}, t) = \frac{1}{2} \underline{x}^T Q \underline{x} + \frac{1}{2} p u^2 + \underline{\lambda}^T A \underline{x} + \underline{\lambda}^T \underline{b} u \quad (2-3)$$

where $\underline{\lambda}$ is the n -dimensional costate vector.

The maximum principle states that for the optimal control

$$\frac{\partial H}{\partial u} = p u + \underline{b}^T \underline{\lambda} = 0 \quad (2-4)$$

and

$$\frac{\partial H}{\partial \underline{x}} = Q \underline{x} + A^T \underline{\lambda} = -\dot{\underline{\lambda}} \quad (2-5)$$

with the boundary condition on $\underline{\lambda}$ given at the final time

$$\underline{\lambda}(t_f) = 0 \quad (2-6)$$

The optimum control is given by equation (2-4)

$$\underline{u} = -p^{-1} \underline{b}^T \underline{\lambda} \quad (2-7)$$

and since a feedback control law is desired, let

$$\underline{\lambda} = R \underline{x} \quad (2-8)$$

to give

$$\underline{u} = -p^{-1} \underline{b}^T R \underline{x} \quad (2-9)$$

It only remains now to determine the elements of the matrix R . Note, that at this stage all the quantities are functions of time.

Differentiating equation (2-8) leads to

$$\dot{\underline{\lambda}} = \dot{R} \underline{x} + R \dot{\underline{x}} \quad (2-10)$$

and using the expression for $\dot{\underline{\lambda}}$ as given by equation (2-5) we obtain

$$\dot{R} \underline{x} + R \dot{\underline{x}} = -Q \underline{x} - A^T R \underline{x} \quad (2-11)$$

The substitution of equation (2-9) into equation (2-1) leads to

$$\dot{\underline{x}} = (A - \underline{b} p^{-1} \underline{b}^T R) \underline{x} \quad (2-12)$$

and it is now possible to eliminate $\dot{\underline{x}}$ from equation (2-11):

$$(\dot{R} + RA - R \underline{b} p^{-1} \underline{b}^T R + Q + A^T R) \underline{x} = 0 \quad (2-13)$$

Since this equation must hold for arbitrary \underline{x} , the term in brackets must be zero. The result is a matrix Riccati differential equation, which must be solved for the elements of R :

$$\dot{R} + RA - R \underline{b} p^{-1} \underline{b}^T R + Q + A^T R = 0 \quad (2-14)$$

The boundary condition is obtained from equations (2-6) and (2-8):

$$R(t_f) = 0 \quad (2-15)$$

Hence the matrix Riccati equation has to be solved backward in time from the given terminal time, t_f . The resultant time-varying matrix R has to be stored, and then used in equation (2-9) to give the desired closed-loop control law.

In evaluating R , it is useful to note that, provided Q is a symmetric matrix, so is R . Another important requirement on R is that it be positive definite. This can be most readily seen by an alternative derivation of equation (2-14). The maximum principle is again used, but the Hamilton-Jacobi approach is followed^[4]. The only step of interest to us is the requirement

$$\underline{x}^T R \underline{x} = \int_0^{t_f} (\underline{x}^T Q \underline{x} + p u^2) dt \quad (2-16)$$

which is proved in the Hamilton-Jacobi method. Since the integrand of the performance index is a positive definite function, R must also be positive definite. Equation (2-16) also indicates a simple method for evaluating the cost function.

The main problem associated with the practical usefulness of equation (2-14) in providing the optimal feedback controller is, that for high-order systems having time-varying parameters, the repeated evaluation of the matrix Riccati equation cannot be accomplished in sufficiently short time for on-line control applications. A useful computational simplification arises if the terminal time t_f is equated to infinity, and A , b , Q and p in equations (2-1) and (2-2) are restricted to be time invariant. In that case, the R matrix becomes a constant, as can be seen from equation (2-16): for a given \underline{x} the performance index does not change for a finite time translation. In practice, for the control of transients and changes in input, the

terminal time may be regarded as infinite as soon as steady state is established.

For a constant R , the matrix Riccati equation (2-14) reduces to a matrix algebraic equation

$$RA - R\underline{b} p^{-1} \underline{b}^T R + Q + A^T R = 0 \quad (2-17)$$

The solution of this equation, in general, is not readily accomplished, because it is nonlinear. There is a special case, however, for which analytical expressions for the elements of the R matrix can be found. These are derived in the next section.

2.2 Optimal Feedback Control of a Second-Order System

Consider the second-order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} u \quad (2-18)$$

and a quadratic cost function in the form of equation (2-2) with

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

Performing the matrix operations indicated by equation (2-17), and remembering the symmetric nature of R , the following three equations are obtained:

$$\frac{b_0^2}{p} r_{12}^2 + 2a_0 r_{12} - q_1 = 0$$

$$\frac{b_0^2}{p} r_{22} + 2a_1 r_{22} - 2r_{12} - q_2 = 0$$

$$a_1 r_{12} + a_0 r_{22} + \frac{b_0^2}{p} r_{12} r_{22} - r_{11} = 0$$

Solving these equations in the above order, and invoking the positive

definite requirement on R when selecting the roots of the quadratics which arise, we obtain the desired analytical expressions for the elements of the R matrix.

$$r_{12} = -\frac{a_0 p}{b_0^2} + \frac{p}{b_0^2} \left[a_0^2 + \frac{b_0^2}{p} q_1 \right]^{\frac{1}{2}} \quad (2-19)$$

$$r_{22} = -\frac{a_1 p}{b_0^2} + \frac{p}{b_0^2} \left[a_1^2 + \frac{b_0^2}{p} (q_2 + 2r_{12}) \right]^{\frac{1}{2}} \quad (2-20)$$

$$r_{11} = a_1 r_{12} + a_0 r_{22} + \frac{b_0^2}{p} r_{12} r_{22} \quad (2-21)$$

In order to obtain the expressions for the feedback coefficients, rewrite equation (2-9) in the form

$$\underline{u} = -\underline{k}^T \underline{x} \quad (2-22)$$

where

$$\underline{k}^T = p^{-1} \underline{b}^T R \quad (2-23)$$

or

$$\begin{bmatrix} k_0 \\ k_1 \end{bmatrix} = \frac{b_0}{p} \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix}$$

Hence

$$k_0 = -\frac{a_0}{b_0} + \frac{1}{b_0} \left[a_0^2 + \frac{b_0^2}{p} q_1 \right]^{\frac{1}{2}} \quad (2-24)$$

$$k_1 = -\frac{a_1}{b_0} + \frac{1}{b_0} \left[a_1^2 + \frac{b_0^2}{p} q_2 + 2b_0 k_0 \right]^{\frac{1}{2}} \quad (2-25)$$

The value of r_{11} is useful in evaluating the cost function, as indicated by equation (2-16). Given the initial state

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix}$$

the cost incurred in driving the system to the origin of state space in

an optimum manner is

$$\begin{aligned}
 J &= \int_0^{\infty} \frac{1}{2} (\underline{x}^T Q \underline{x} + p u^2) dt \\
 &= \frac{1}{2} \underline{x}_0^T R \underline{x}_0 = \frac{1}{2} A^2 r_{11}
 \end{aligned} \tag{2-26}$$

The use of equation (2-26) is considerably simpler than numerically integrating the cost function.

2.3 The Linear Servomechanism

The regulator problem considered in the previous sections is convenient from an analysis point of view, but is not the problem commonly encountered in actual practice. The more typical control problem is to drive the system from a given initial state at $t = t_0$ to a desired finite terminal state at $t = t_f$, such that the cost function

$$J = \int_{t_0}^{t_f} [(\underline{x} - \underline{x}^*)^T Q (\underline{x} - \underline{x}^*) + p(u - u^*)^2] dt \tag{2-27}$$

is minimized.

For a step change in input, the solution of the servomechanism problem is obtained directly from the case of the linear regulator by the addition of a feedforward block, having a gain that ensures zero steady-state error, i.e., $\underline{x}^* = \underline{x}(t_f)$ and $u^* = u(t_f)$. Clearly, all that is necessary, is a linear translation from having $\underline{x}(t_f)=0$ and $u(0^+)=0$ to both of these quantities being finite. Furthermore, to prevent the cost function from becoming infinite, \underline{x}^* and u^* must be introduced to ensure that the integrand of equation (2-27) tends to zero as t_f approaches infinity.

The block diagram representation of the regulator and servo-mechanism problems are shown in Figure 2-1. Note that the feedback blocks are identical in both cases.

The use of the transfer function notation leads to a simple evaluation of the feedforward gain ℓ_0 from the requirement that $x(t_f) = A$, where x is the output and A is the value of the input at $t = 0^+$.

In going from the time domain to the frequency domain, the following notation has been used:

$$\underline{x}(t) = [1 \ 0 \ 0 \ \dots \ 0] \underline{x}(t)$$

$$X(s) = L[\underline{x}(t)]$$

$$U(s) = L[u(t)]$$

where L is the Laplace transform operator and s the complex frequency variable.

To find the value of ℓ_0 , assume that steady state has been reached after the application of the step input. Referring to Figure 2-1b, the following relations hold:

$$X(s) = \frac{b_0}{a_0} U(s)$$

$$U(s) = \frac{\ell_0 A}{s} - k_0 X(s)$$

Hence

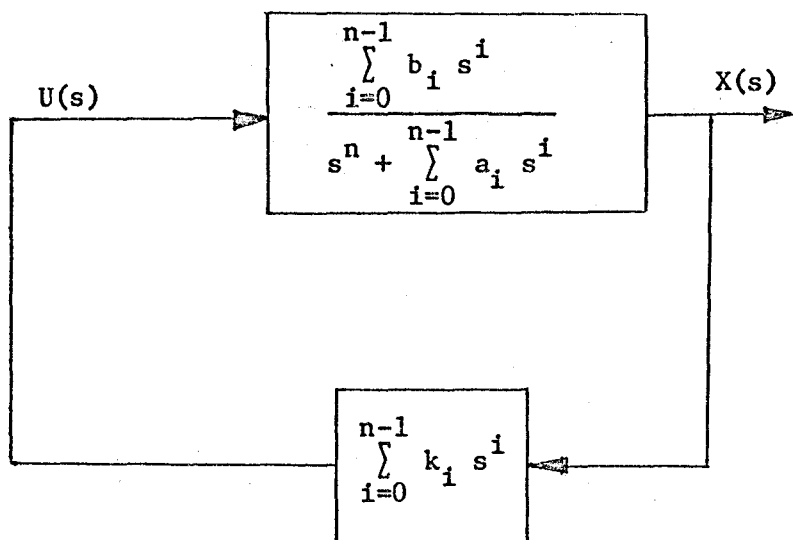
$$X(s) = \frac{b_0}{a_0} \frac{\ell_0 A}{s} - \frac{b_0 k_0}{a_0} X(s)$$

i.e.,

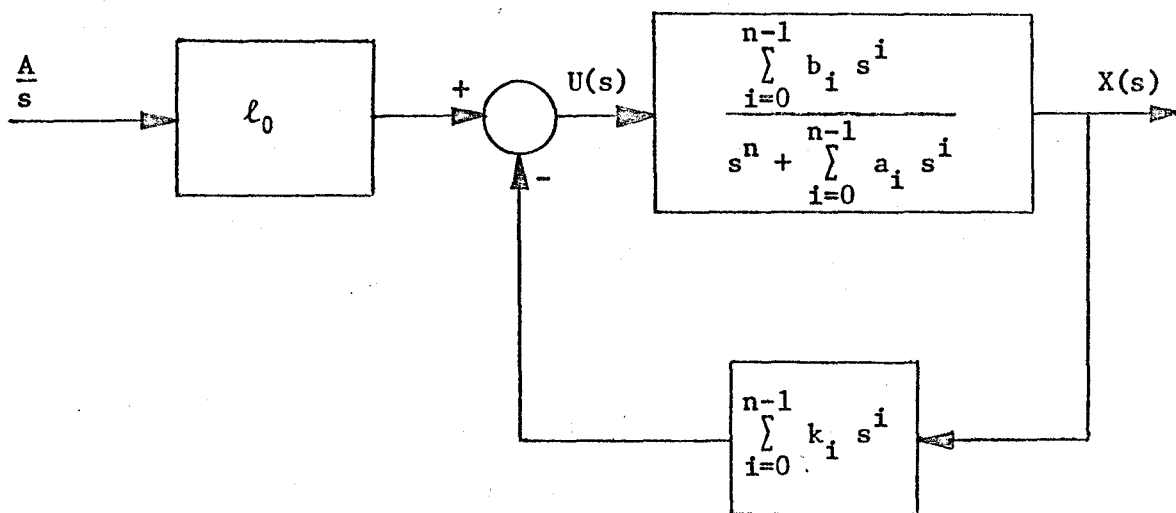
$$X(s) \left[1 + \frac{b_0 k_0}{a_0} \right] = \frac{b_0 \ell_0}{a_0} \frac{A}{s}$$

and since it is required that $X(s) = \frac{A}{s}$ in the steady state, we must have

$$\ell_0 = \frac{a_0}{b_0} + k_0 \quad (2-28)$$



a. The linear regulator



b. The linear servomechanism

Figure 2-1. Block-diagram representation for establishing the equivalence of the regulator and servomechanism problems for a step change in input.

The value of $u^*(t)$ is also readily found using the requirement that in the steady state

$$X(s) = \frac{A}{s}$$

It follows from Figure 2-1b that

$$U^*(s) = \frac{a_0}{b_0} \frac{A}{s}$$

In the case where $u(t)$ is not a step but some other time function, the above result is still applicable as long as the value of $u(t)$ approaches a constant value after a finite time interval has elapsed. This will always be the case for the control problems considered in this thesis. For the nuclear reactor in particular, the task is always to transfer the power level from one given constant value to another.

Before the theories developed in this chapter can be applied to the reactor control problem, we must consider the mathematical description of the system. Since the plant equations will be found to be nonlinear, it will be necessary to find a linear model of the plant, in order to derive the desired optimal feedback controller.

CHAPTER 3

OPTIMAL CONTROL OF NUCLEAR REACTORS

The optimal control problem, as discussed in the previous chapter, may be viewed as consisting of two parts: the system equations and the cost function. In the general problem formulation, no restrictions are placed on either, but nor is an analytical solution known. For the special case of a linear system and integral quadratic cost function the optimal feedback controller has been developed in the previous chapter. Solutions have also been found for certain types of nonlinear systems and/or particular nonquadratic cost functions [15],[16],[17]. Numerical optimization techniques have also been found useful in solving various optimal control problems, for which analytic solutions are not readily obtained [18],[19].

For the particular case of nuclear reactor control, both the system equations and the cost function aspects of the optimal control problem are considered in this chapter. In the present work, the mathematical model used to describe the system dynamics has a dual role: it must be sufficiently accurate to be a valid representation of the physical plant, but at the same time be simple enough to make the computation of the feedback coefficients realizable in practice. These requirements are conflicting, and research efforts in the past invariably used oversimplified models to represent the reactor, and no apparent attempts were made to evaluate the performance of the proposed controller on a more complete mathematical model. As will be shown in this chapter, these reduced order models are far from adequate

in describing the behaviour of the nuclear reactor.

3.1 The Six Delayed Neutron Group Model

The point kinetic equations, that adequately describe the dynamic behaviour of a nuclear reactor for the purposes of control system design, lead to the so-called six delayed neutron group model^[20]:

$$\frac{dn(t)}{dt} = \frac{\delta k(t) - \beta}{\ell} n(t) + \sum_i \lambda_i c_i(t) \quad (3-1)$$

$$\frac{dc_i(t)}{dt} = \frac{\beta_i}{\ell} n(t) - \lambda_i c_i(t) \quad i=1, \dots, 6 \quad (3-2)$$

where

$n(t)$ = neutron power as a fraction of full power

t = time

$\delta k(t)$ = reactivity

β = total delayed neutron fraction

ℓ = mean effective lifetime of a prompt neutron

λ_i = decay constant of the i th neutron precursor

$c_i(t)$ = concentration of i th precursor

β_i = fraction of delayed neutrons due to i th precursor

Numerical values for these constants, appropriate to the Douglas Point reactor, are given in Appendix I.

It is important to note that, in equation (3-1) the input (δk) and the output (n) appear as a product, making the input dependent on the operating power level. Hence, even though for a given reactivity step the kinetic equations are linear, the output varies in a nonlinear manner as different values of reactivity are applied. These nonlinear

characteristics are clearly illustrated in Figure 3-1, where the results of solving equations (3-1) and (3-2) for various constant values of reactivity input have been plotted.

Further nonlinearities arise when closed loop control is attempted, since in that case the reactivity is an explicit function of the instantaneous neutron population. In the so-called power range (20% - 100% of full power), reactivity also becomes a function of the operating temperature. Both of these nonlinearities will be discussed in this chapter.

Since the optimal controller equations derived in Chapter 2 are applicable only to linear systems, it is desirable to have a linearized form of equations (3-1) and (3-2). Linearization may be achieved in the neighbourhood of a given operating point, by considering small perturbations in neutron level ($n_0 + \delta n$) and precursor concentrations ($c_{i0} + \delta c_i$). Since at equilibrium the reactivity is zero, the change in reactivity is still denoted by δk . Substitution of these small deviations in equations (3-1) and (3-2) leads to the following:

$$\frac{d(\delta n)}{dt} = \frac{\delta k - \beta}{\ell} (n_0 + \delta n) + \sum_i \lambda_i (c_{i0} + \delta c_i) \quad (3-3)$$

$$\frac{d(\delta c_i)}{dt} = \frac{\beta_i}{\ell} (n_0 + \delta n) - \lambda_i (c_{i0} + \delta c_i) \quad (3-4)$$

At $t = 0$ equation (3-4) gives

$$0 = \frac{\beta_i}{\ell} n_0 - \lambda_i c_{i0}$$

$$\text{i.e.,} \quad \frac{\beta_i}{\ell} n_0 = \lambda_i c_{i0} \quad (3-5)$$

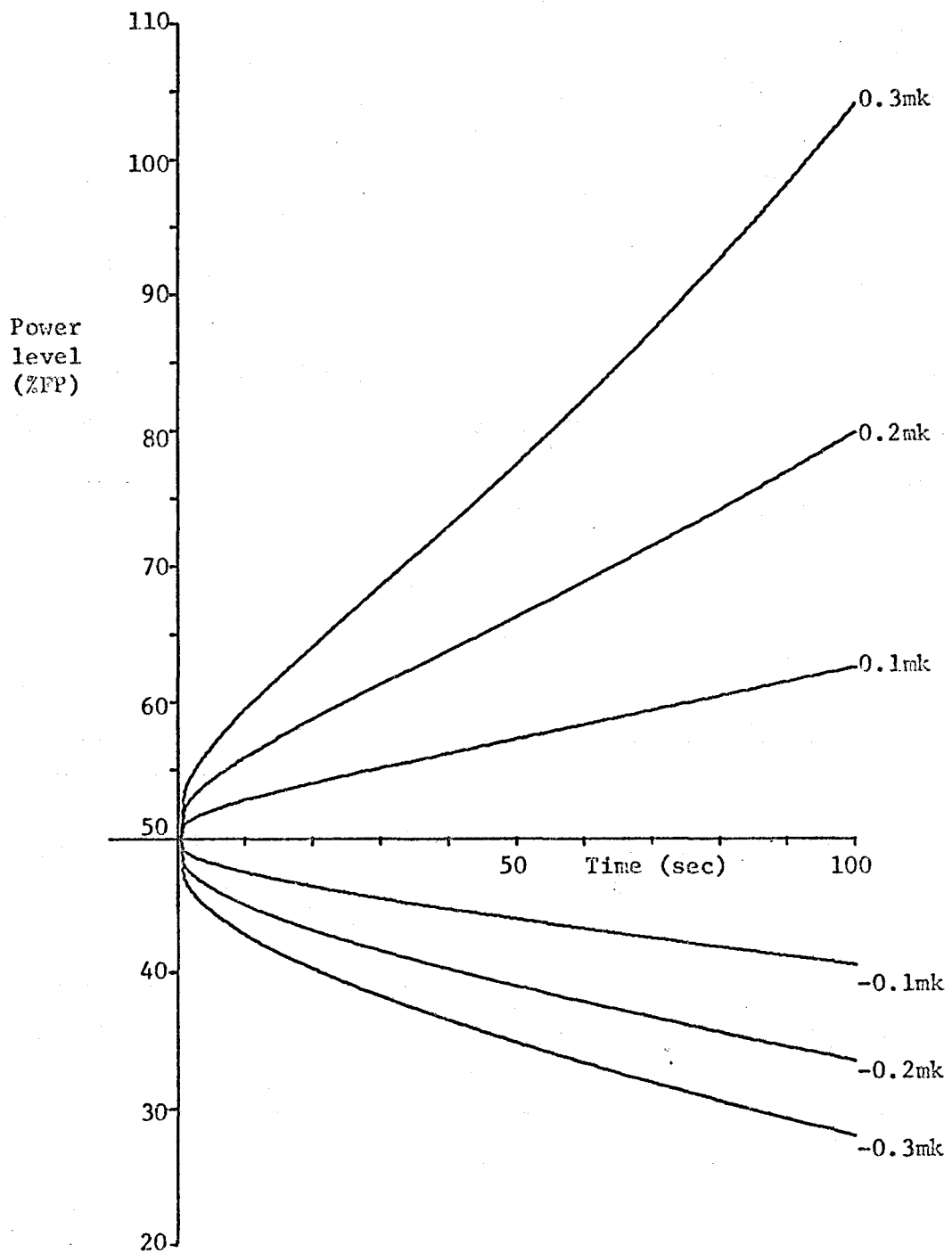


Figure 3-1. Response of six delayed neutron group reactor kinetic model, for step changes in reactivity. (FP = Full Power)

and

$$\frac{\beta}{\ell} n_0 = \sum_i \lambda_i c_{i0} \quad (3-6)$$

since

$$\beta = \sum_i \beta_i$$

Expanding equations (3-3) and (3-4) yields

$$\frac{d(\delta n)}{dt} = \frac{\delta k}{\ell} n_0 - \frac{\beta}{\ell} n_0 + \frac{\delta k \delta n}{\ell} - \frac{\beta \delta n}{\ell} + \sum_i \lambda_i c_{i0} + \sum_i \lambda_i \delta c_i$$

$$\frac{d(\delta c_i)}{dt} = \frac{\beta_i}{\ell} n_0 + \frac{\beta_i}{\ell} \delta n - \lambda_i c_{i0} - \lambda_i \delta c_i$$

Using the relationships of (3-5) and (3-6) and neglecting terms of second order, the linearized kinetic equations are obtained.

$$\frac{d(\delta n)}{dt} = \frac{\delta k}{\ell} n_0 - \frac{\beta}{\ell} \delta n + \sum_i \lambda_i \delta c_i \quad (3-7)$$

$$\frac{d(\delta c_i)}{dt} = \frac{\beta_i}{\ell} \delta n - \lambda_i \delta c_i \quad (3-8)$$

Comparison of the responses produced by the linearized, and the original, essentially nonlinear kinetic equations, will be presented in the next section, after several other kinetics models have been derived.

3.2 Classical Low-Order Reactor Models

Based on the physical understanding of the nuclear process, the order of the point kinetic equations may be reduced considerably^[6]. The resultant models, typically of order one or two, may be referred to as the "classical" reactor models. Their main use is in deriving approximate solutions to reactor kinetic problems, and they have been

used extensively in optimal control studies. However, as will be illustrated in the subsequent sections, the deviation between the responses of these simplified models and the six delayed neutron group representation is excessive for practical control system design.

3.2.1 One Delayed Neutron Group

The basic problem in trying to reduce the order of the six group model is how to account for the delayed neutrons. A very useful simplification results from assuming all the neutrons to belong to a single group, resulting in the one delayed neutron group model

$$\frac{dn(t)}{dt} = \frac{\delta k(t) - \beta}{\ell} n(t) + \lambda c(t) \quad (3-9)$$

$$\frac{dc}{dt} = \frac{\beta}{\ell} n(t) - \lambda c(t) \quad (3-10)$$

where the average decay constant λ is computed from

$$\frac{1}{\lambda} = \frac{1}{\beta} \sum_{i=1}^6 \frac{\beta_i}{\lambda_i}$$

The linearization of the one delayed neutron group model is made along similar lines to the previous case, except that the subscripts and summation sign are omitted. By analogy, the result can be written down by inspection of equations (3-7) and (3-8):

$$\frac{d(\delta n)}{dt} = \frac{\delta k}{\ell} n_0 - \frac{\beta}{\ell} \delta n + \lambda \delta c \quad (3-11)$$

$$\frac{d(\delta c)}{dt} = \frac{\beta}{\ell} \delta n - \lambda \delta c \quad (3-12)$$

Further simplifications of the one delayed neutron group model are possible by setting the derivative in either equation (3-9) or in

(3-10) to zero. The following is the physical reasoning behind obtaining these first order models.

3.2.2 Prompt-Jump Approximation

Reference to Figure 3-1 indicates that an almost instantaneous change in neutron population occurs at $t=0$. At this stage, the prompt neutrons are the major influencing factor, with a time-constant in the order of a millisecond. For most control applications such short time-constants may be neglected, and the assumption that there is a step change in neutron level at $t=0$ does not introduce significant errors. The result is the so-called prompt-jump model.

Setting the derivative equal to zero in equation (3-9) gives

$$\frac{dn}{dt} = 0 = \frac{\delta k - \beta}{\ell} n + \lambda c$$

or

$$c = \frac{\beta - \delta k}{\ell \lambda} n \quad (3-13)$$

and

$$\frac{dc}{dt} = \frac{\beta - \delta k}{\ell \lambda} \frac{dn}{dt} \quad (3-14)$$

Substituting the last two relationships into equation (3-10) leads to a first-order differential equation for the neutron level.

$$\frac{dn}{dt} = \frac{\lambda \delta k}{\beta - \delta k} n \quad (3-15)$$

The linear model is again found by considering a perturbation about the operating neutron level $n_0 + \delta n$.

$$\frac{d(\delta n)}{dt} = \frac{\lambda \delta k}{\beta - \delta k} (n_0 + \delta k)$$

Neglecting the term of second order, we obtain the linearized prompt

jump model

$$\frac{d(\delta n)}{dt} = \frac{\lambda n_0 \delta k}{\beta - \delta k} \quad (3-16)$$

3.2.3 Infinite Delay-Time Approximation

Returning to the one delayed neutron group model, consider the effect of setting to zero the derivative in equation (3-10). In physical terms, the assumption is that the delayed neutrons take an infinite time to appear. In other words, the precursor concentration remains constant at its original level at $t=0^-$.

Using equation (3-10)

$$\frac{dc}{dt} = 0 = \frac{\beta}{\ell} n_0 - \lambda c_0$$

or

$$\frac{\beta}{\ell} n_0 = \lambda c_0$$

Substituting for λc in equation (3-9) gives the infinite delay time model

$$\frac{dn}{dt} = \frac{\delta k - \beta}{\ell} n + \frac{\beta}{\ell} n_0 \quad (3-17)$$

Writing $n = n_0 + \delta n$ and again neglecting the second-order term, the linear model is obtained

$$\frac{d(\delta n)}{dt} = \frac{\delta k}{\ell} n_0 - \frac{\beta}{\ell} \delta n \quad (3-18)$$

3.2.4 Weighted Average Neutron Generation Time

The simplest reactor model is obtained if the delayed neutrons are completely neglected.

$$\frac{dn(t)}{dt} = \frac{\delta k(t)}{\ell} n(t) \quad (3-19)$$

This model gives a reasonably good description of system behaviour if the reactor is critical on prompt neutrons alone. For control purposes this is never the case, and therefore for reactivity changes below prompt critical ($\delta k < \beta$) it is desirable to replace ℓ by the weighted average neutron generation time, ℓ^* [20].

$$\ell^* = \ell(100 - \beta) + \sum_{i=1}^6 \frac{\beta_i}{\lambda_i} \quad (3-20)$$

The weighted average neutron generation time model is therefore written in differential equation form as

$$\frac{dn(t)}{dt} = \frac{\delta k(t)}{\ell^*} n(t) \quad (3-21)$$

3.2.5 Comparison of Responses for a Step-Change in Reactivity

A comparison of the responses of the various simplified reactor kinetic models to the six delayed neutron group representation is given in this section. The initial power level is assumed to be at 50% FP, and a step increase of 0.3 mk is applied. The responses of the six group and the various low-order models are shown in Figure 3-2. The prompt-jump and the one delayed neutron group model have responses very close together, but at an appreciable distance from the correct output. The performance of the infinite delay-time model is very poor, except during the prompt response, as expected from the derivation.

The response of each of the linearized models is shown in Figure 3-3. The deviations from the correct response are even greater than previously. It is clear, that none of these models give an accurate representation of the reactor kinetic process, at least for step changes in reactivity.

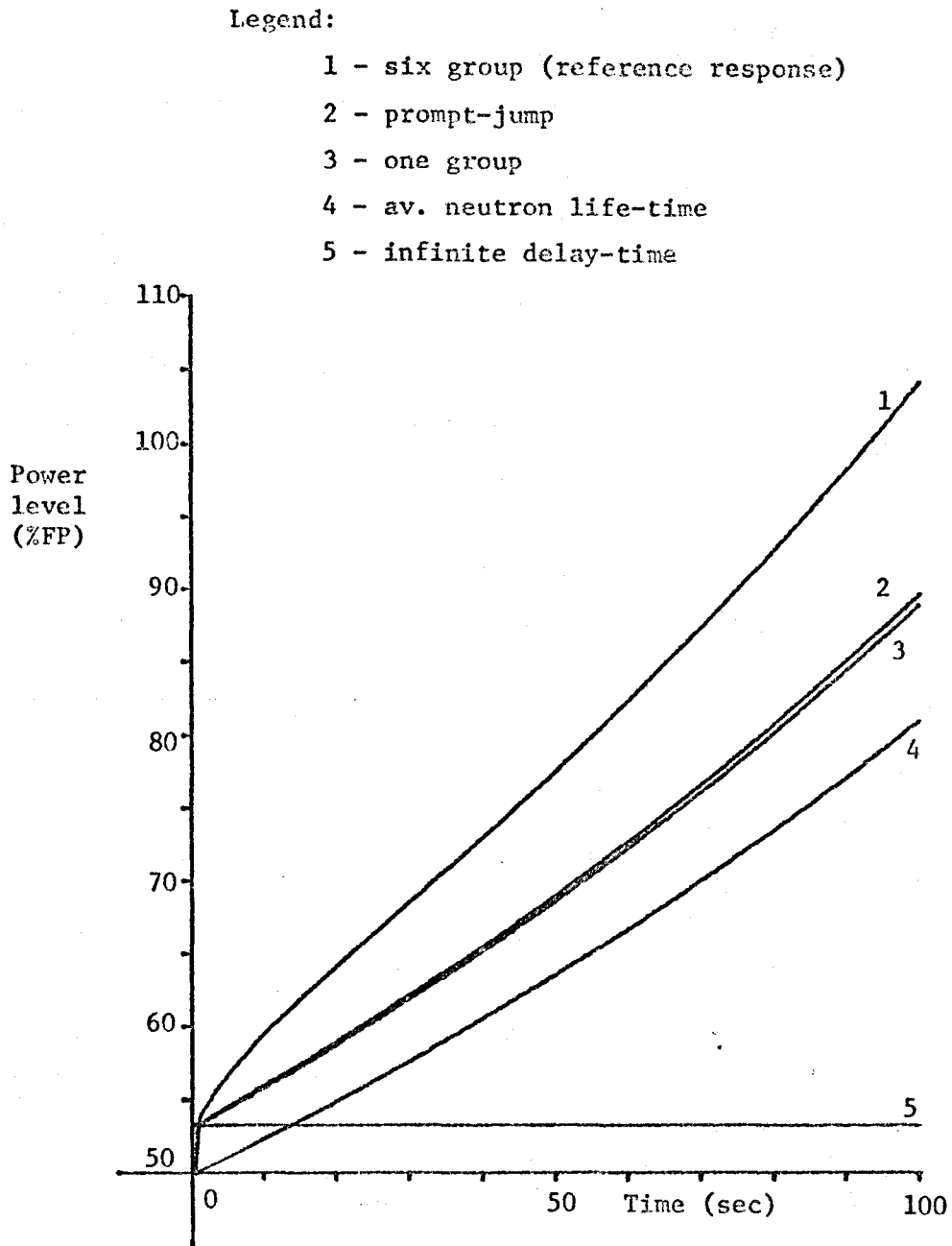


Figure 3-2. Comparison of responses of the six delayed neutron group model and various lower-order kinetic models, for a 0.3mk step change in reactivity.

Legend:

- 1 - six group (nonlinear, reference response)
- 2 - linearized six group
- 3 - linearized one group
- 4 - linearized prompt-jump
- 5 - linearized infinite delay-time

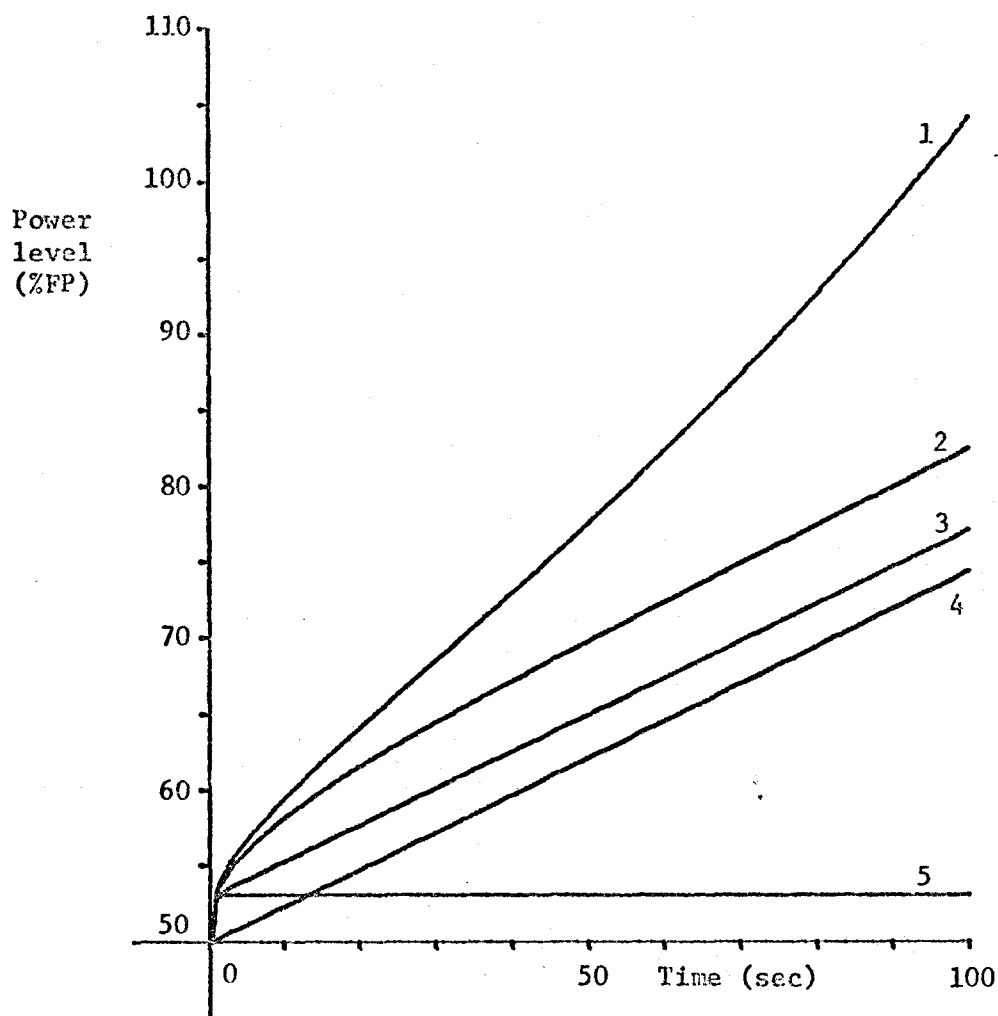


Figure 3-3. Comparison of responses of the six delayed neutron group model and various linearized kinetic models, for a 0.3mk step change in reactivity.

3.3 A Realistic Nuclear Reactor Model

The mathematical models discussed so far in this chapter have pertained only to the neutron kinetics. Furthermore, the effect of temperature on reactivity has been neglected, and only open loop operation has been considered. While the addition of these factors does give a reasonably accurate description of the neutron dynamics, elements of the controller mechanism cannot be neglected if a realistic reactor model is desired. In particular, attention must be given to the power level sensing elements on the one hand, and to the reactivity actuating devices on the other.

To appreciate the task of the reactor control system, and of the designer who hopes to improve on it, one should look at the operation of the existing feedback controller, in our case the one used at Douglas Point. The block diagram representation of the reactor and parts of the present control system are shown in Figure 3-4. Since the operation of this plant has already been described by the author in considerable detail, only those aspects that are most relevant to the present work will be considered here.

The reactivity change (δk_T) brought about by the change in temperature of the reactor core as the neutron level varies, is represented by a feedback loop around the point kinetics model. In differential equation form, the temperature effect may be approximated by:

$$\frac{d(\delta k_T)}{dt} = -\frac{\delta k_T}{\tau_T} + \frac{T_c}{\tau_T} n \quad (3-22)$$

where T_c is the temperature coefficient of reactivity, and τ_T is the time-constant associated with this effect. For the Douglas Point reactor

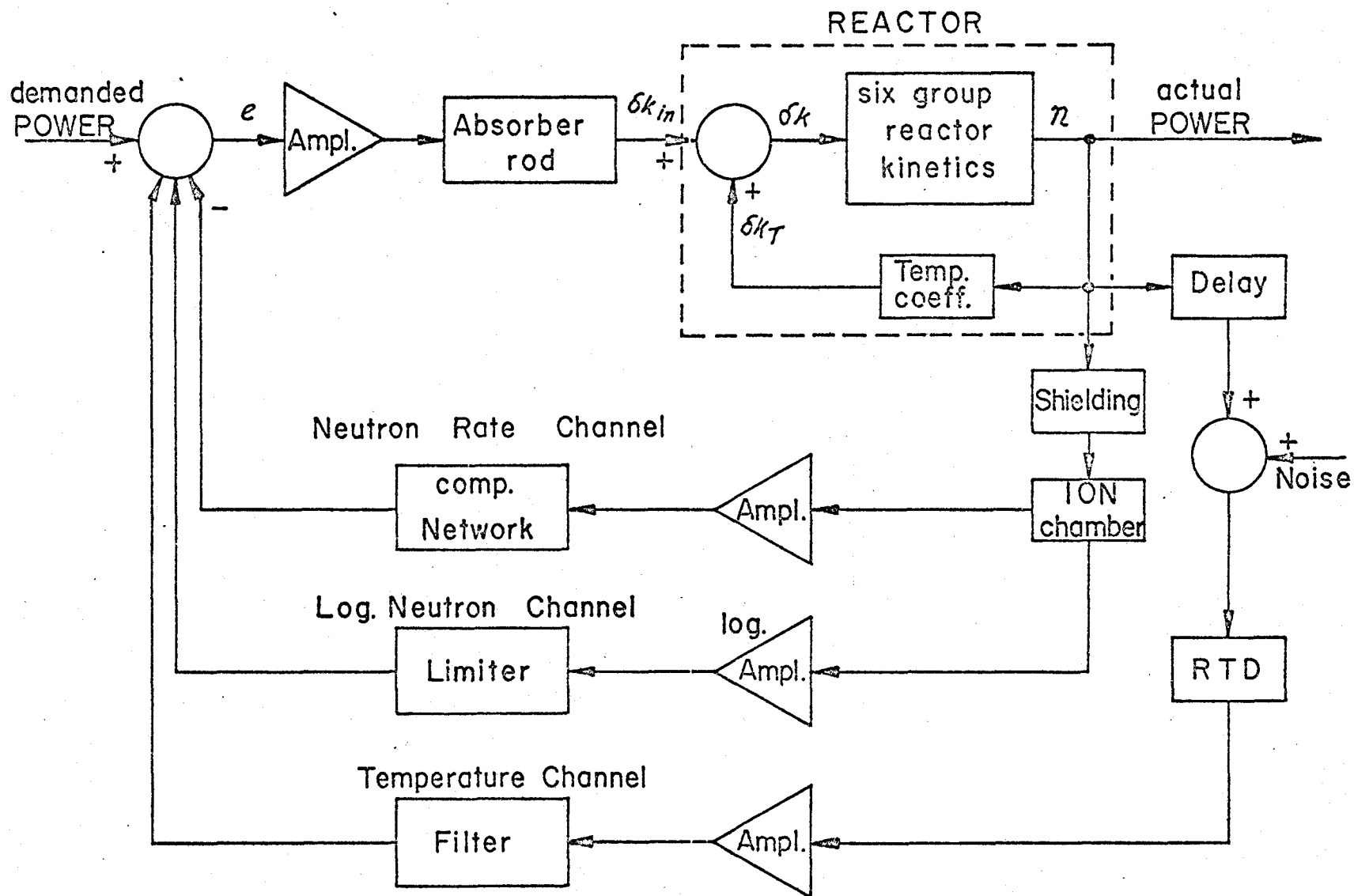


Figure 3-4. Block diagram of the reactor and parts of the power level control system at Douglas Point.

design values for these constants are

$$T_c = -0.0454 \frac{mk}{\%FP}$$

$$\tau_T = 12.5 \text{ sec}$$

The operating power level is measured by two independent transducers. Ion-chambers detect the neutron flux directly, while the change in temperature of the coolant as it flows through the reactor gives an indication of the actual power level. While the neutron levels are synonymous, neither of the above measures are accurate. A neutron reading taken by the ion-chamber may be less than the true neutron level, by a constant, slowly varying amount, due to the "shielding"*. The temperature channel signal indicates the power level of the reactor at an earlier instant, because of the finite transport time between the reactor core and the temperature transducer (RTD). Furthermore, this signal has a considerable noise component due to the turbulent coolant flow. While neither of these indicators give a precise reading of the reactor power level, they may be processed to achieve this aim^[5]. For the purpose of the present work therefore, we can assume that an instantaneous noise-free measure of the reactor level is available.

The error signal (e), that indicates the deviation between the demanded and actual power levels, is amplified, and the output is applied to the absorber rod drive motor, which is represented by the first order differential equation

$$\frac{dv}{dt} = \frac{-v}{\tau_m} + Ce \quad (3-23)$$

*Shielding is due to the depression of the neutron flux by the poison dissolved in the moderator.

where G is the gain of the amplifier, τ_m the motor time-constant and v the effective voltage that determines the rate of reactivity insertion. The latter introduces a further nonlinearity into the system, as the motor reaches its full speed at 15 volts, corresponding to a power error of 10% FP.

The nonlinear absorber rod insertion rate characteristic is represented by the function

$$\begin{aligned} f(v) &= 0.02 & |v| \leq 15 \\ &= \frac{0.3}{v} & |v| > 15 \end{aligned} \quad (3-24)$$

The units of $f(v)$ are mk/sec/volt, hence the effect of a power level error is to cause a certain rate of reactivity insertion (or removal) by the movement of the absorber rod.

By deleting the details of the present feedback controller from Figure 3-4 and including the temperature effect in a single REACTOR block, a simplified block diagram is obtained, as shown in Figure 3-5. The block indicating the feedback controller represents the optimal control problem: its coefficients are to be determined such that a meaningful cost function is minimized.

Combining equations (3-1), (3-2), (3-22), (3-23) and (3-24) results in a ninth-order dynamic model for the nuclear reactor and the reactivity actuating mechanisms. The model is open-loop and contains multiple nonlinearities. In state variable form it may be written as follows:

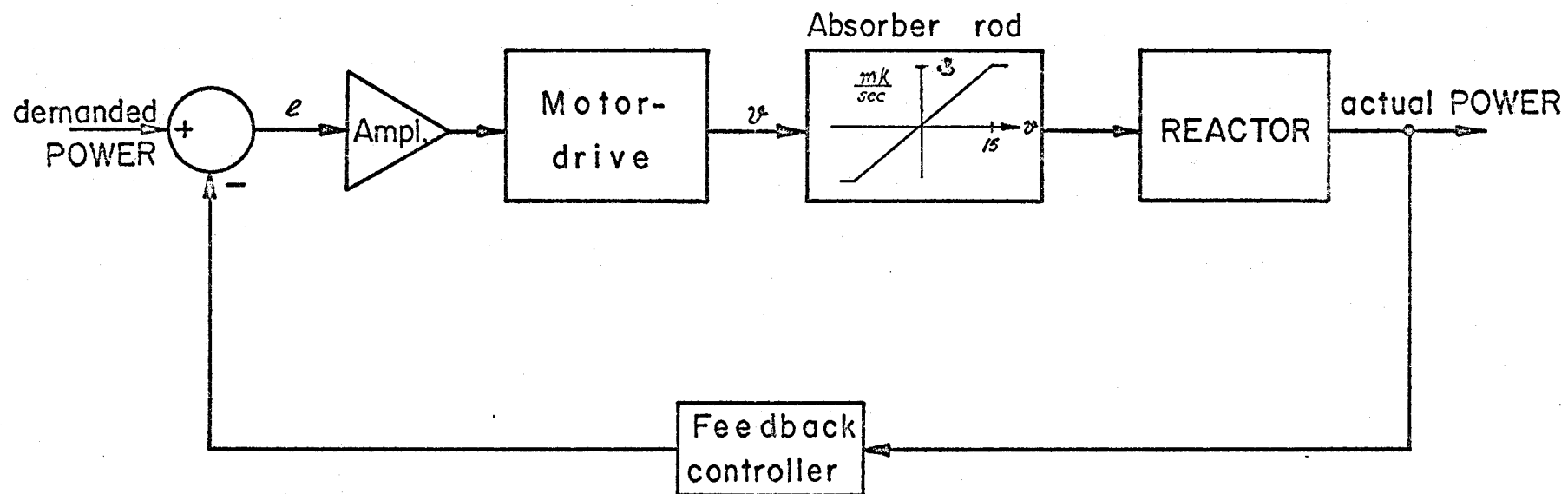


Figure 3-5. Simplified block diagram of Reactor and control system.

$$\begin{bmatrix} \frac{dn}{dt} \\ \frac{dc_1}{dt} \\ \frac{dc_2}{dt} \\ \vdots \\ \frac{dc_6}{dt} \\ \frac{d\delta k_T}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-\beta}{\ell} & \lambda_1 & \lambda_2 & \dots & \lambda_6 & \frac{n}{\ell} & \frac{ntf(v)}{\ell} \\ \frac{\beta_1}{\ell} & -\lambda_1 & 0 & \dots & 0 & 0 & 0 \\ \frac{\beta_2}{\ell} & 0 & -\lambda_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \frac{\beta_6}{\ell} & 0 & 0 & \dots & -\lambda_6 & 0 & 0 \\ \frac{T_c}{\tau_T} & 0 & 0 & \dots & 0 & \frac{-1}{\tau_T} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{\tau_m} \end{bmatrix} \begin{bmatrix} n \\ c_1 \\ c_2 \\ \vdots \\ c_6 \\ \delta k_T \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ G \end{bmatrix} \quad (3-25)$$

or using matrix notation

$$\underline{\dot{x}} = A_r \underline{x} + \underline{b}_r e \quad (3-26)$$

and the system output is given by

$$y = \underline{d}^T \underline{x} \quad (3-27)$$

where

$$\underline{d}^T = [1 \ 0 \ \dots \ 0]$$

The behaviour of the nuclear reactor model, as represented by equation (3-25) has been studied extensively by the author in reference [5]. The responses obtained by digital computer simulation studies have been verified by the designers of the Douglas Point control system as adequately representative of the true behaviour of the reactor.

Our main interest in the present chapter is to evaluate the performance of the various low-order classical models in comparison to the six delayed neutron group model. For this purpose, consider unity

feedback, and let it be desired to increase the power level from 50% to 100% of full power. The step responses of the closed-loop system, assuming the various kinetic models are used to represent the neutron kinetics, are shown in Figure 3-6. The error between the reference response and the one group model is not as great as in the open-loop case, but the maximum error is still greater than 10% FP. It is interesting to note, that the infinite delay-time model gives the second best approximation, the prompt-jump and average neutron generation time models being totally inadequate.

The linearized models show even greater deviations from the correct response (Figure 3-7). None of the low-order models give an acceptable approximation to the transient part of the response, and the maximum error of even the six group linear model is over 15% FP.

The usefulness of these low-order linear reactor kinetic models for the purpose of optimal control computation is further reduced when we realize, that the addition of the effect of temperature on reactivity and the absorber rod drive motor, still results in a fourth-order system.

What is desired is a second-order model of the form given by equation (2-18), which accurately depicts the input-output relationship of the overall system. Clearly, the classical approach of reducing the order of the reactor model based on physical insight cannot give such a representation, and some other method for reducing the order of a system must be used. The development of a suitable technique is presented in Chapter 4.

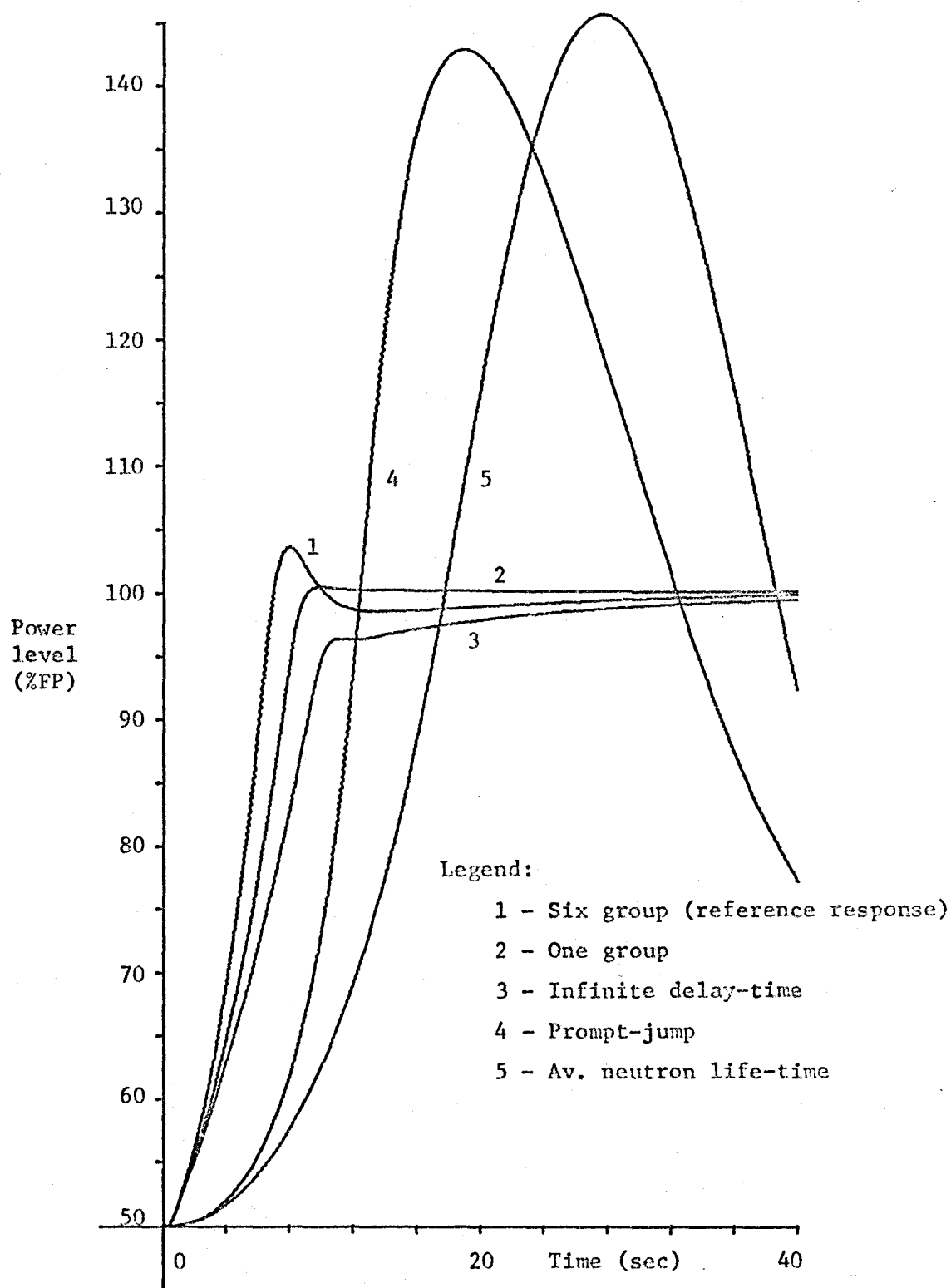


Figure 3-6. Closed-loop response of nuclear reactor to step-change in demanded power, using various nonlinear models.

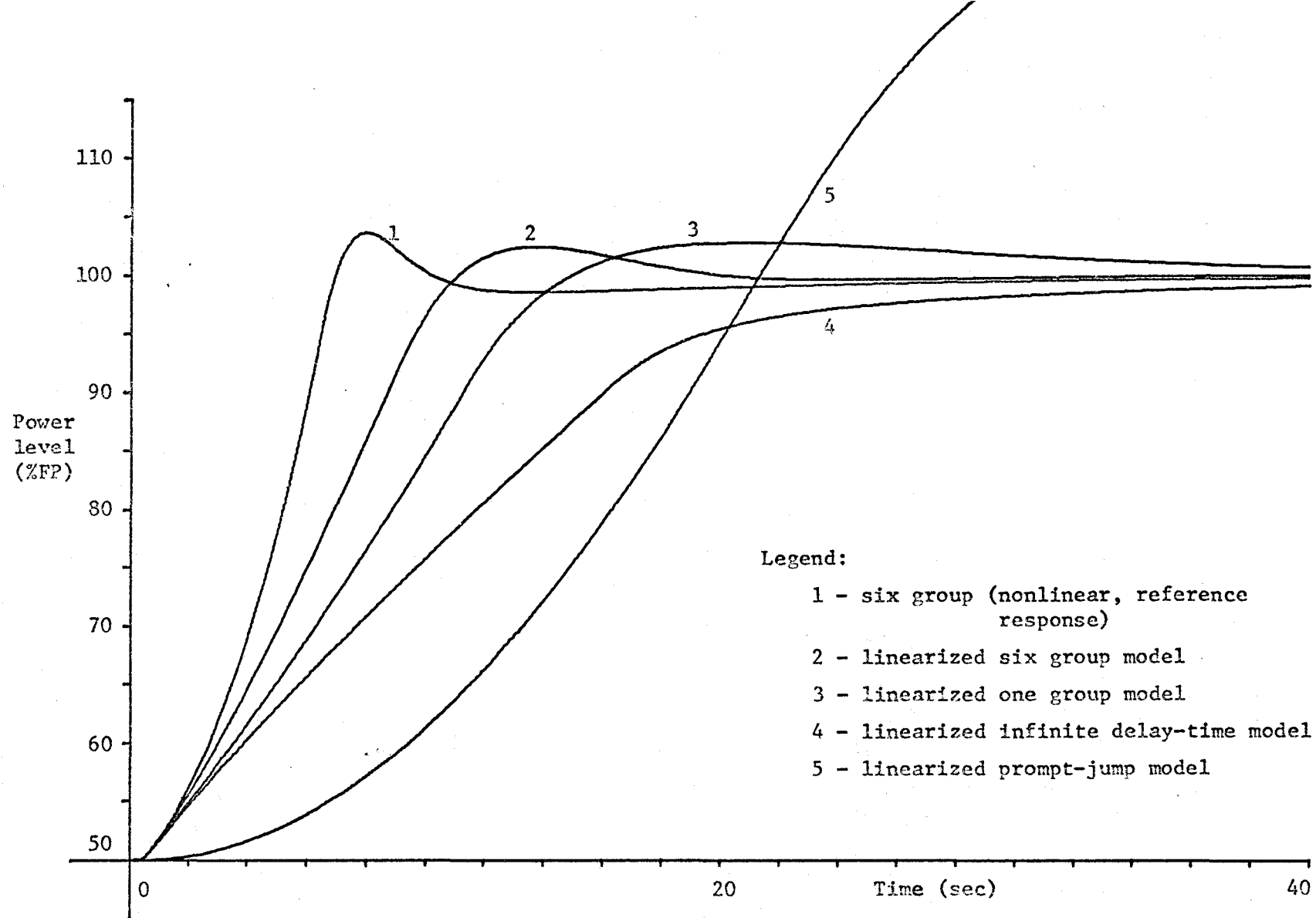


Figure 3-7. Closed-loop response of nuclear reactor to step-change in demanded power, for various linear models and the reference system.

3.4 The Cost Function

Having considered the dynamic representation of the nuclear reactor, we turn our attention to the other important aspect of the control problem: choosing an appropriate cost function. This is one of the most difficult aspects of an optimization problem, requiring considerable experience and insight. Clearly, the final optimal design is only as good as the assigned measure of performance. Ideally, every system variable should be included in the cost function, appropriately weighted to reflect its relative importance. This could be readily achieved, for example, if the actual dollar value, say on a per unit basis, were known for all the variables. Such a case is rarely encountered in practice.

For the nuclear model that has been described in this chapter, it is not practical to include all the model parameters in the cost function. The criterion of choosing the elements of the cost function is two-fold: every term must reflect an important aspect of system performance and each must be a readily measurable quantity.

For a point kinetics model, terms which meet the above criteria must be associated with either the input or the output. They relate to the instantaneous value and rate of change of the neutron population, and the extent of the control effort, with particular emphasis on the mechanical movements of the absorber rod. A cost function that includes all the terms found significant in practice has the following form:

$$\begin{aligned}
 J = f\{ & \alpha |n^*(t) - n(t)|, \gamma \left| \frac{\partial n^*(t)}{\partial t} - \frac{\partial n(t)}{\partial t} \right|, \\
 & \eta |e^*(t) - e(t)|, \theta \left| \frac{\partial e^*(t)}{\partial t} - \frac{\partial e(t)}{\partial t} \right|, \\
 & \nu |\operatorname{sgn}[e(t_i)] - \operatorname{sgn}[e(t_{i-1})]| \} \quad (3-27)
 \end{aligned}$$

The weighting factors α , γ , η , θ and μ may be constant for all instants of time, or they may emphasise a given portion of the time response. Their values reflect the relative importance of the terms they are associated with. The significance of each variable in the cost function is described below. The superscript * is used to denote the desired value of the appropriate quantity.

$n(t)$ is the neutron or power level. It is usually desired to follow the demanded value, and is the quantity often of prime importance. By assigning relatively large weights to α near the desired terminal time, the steady-state error can be made to approach zero.

$\frac{\partial n(t)}{\partial t}$ gives the instantaneous rate of change of the neutron population. It is related to the period of the reactor, and hence to stability, and via the rate of change of temperature, to the thermal stress during a power level change.

$e(t)$ can be regarded as the input to the reactivity control mechanism. Its value determines the extent of the control effort, and relates to the burn up of the absorber rod. Since the quantity of actual interest is the reactivity, the nonlinear dependence of δk on e must be appreciated.

$\frac{\partial e(t)}{\partial t}$ is the rate of change of the error signals. The acceleration of the absorber rod is a function of this quantity, and therefore is a contributing factor to the mechanical wear of the rod drive mechanism.

$\text{sgn}[e(t_i)] - \text{sgn}[e(t_{i-1})]$ is also included with the purpose of reducing mechanical movements of the absorber rod. It tends to eliminate reversals in the direction of the travel of the absorber rod during a change of power level. The sgn function has the value plus or minus one, depending on the sign of the argument; $e(t_i)$ is the value of the error signal at the i th sampling instant. Considering a step change of demanded power, this last term of the cost function will have a zero value only for a monotonic decrease of the power error.

The functional relationship represented by equation (3-27) usually implies the summation of the terms at a given sampling instant. The cost function is evaluated either by summing the functional values, each raised to the power p , over the time-interval of interest, or the maximum value of the function amongst all the sample points is taken. Accordingly, the objective of the optimization problem is either to minimize a sum or the maximum value. While integral cost functions are useful in theory, they are seldom used in practice, because of the problem of numerical integration on the digital computer. The exact form of the cost function to be used will be discussed as each optimization problem is encountered.

3.5 Review of Recent Publications

The problem of optimally controlling the power level changes in a nuclear reactor has been formulated in previous sections of this thesis. A realistic reactor model has been chosen and a practical cost

function has been proposed. It is appropriate at this stage to briefly review the results of recent research efforts, and to evaluate their usefulness in solving the problem we have formulated.

The paper by Murray, Bingham and Martin^[21] is based on the assumption that the precise form of the power response is known, and it is desired to find the reactivity variation that will bring about this response. The method is restricted to power functions whose Laplace transforms exist. The six delayed neutron group model is used, and the effect of temperature on reactivity is included in the treatment.

While the basic concept of starting with a known or desired output and computing the corresponding input is a sound one, the method as presented has great disadvantages from the control point of view, in that only a certain class of power functions are admissible, and that the resultant control is open-loop in nature. Using essentially the same concept, both of these limitations will be overcome by a new procedure presented in this thesis.

Stacey^[22] has considered the use of variational synthesis techniques for the optimal control of a point reactor model. A cost function of the form of equations (3-27) is considered, and the sum of the terms raised to the second power is used. It is assumed that the input can be expanded into a sum of orthogonal functions over the time interval of interest. The coefficients in the expansion are chosen to minimize the cost function while satisfying the point kinetic equations. The method is illustrated for the one delayed neutron group model and using Legendre polynomials as the approximating functions.

Apart from the greatly increased computational complexity that would result if the more complete reactor model was used, this method also results in open-loop control.

In a paper published by Enns^[23], linear programming is used to minimize the maximum deviation of the state variables from their preassigned values. The reactor equations are linearized, thereby permitting the computation of open-loop optimal trajectories. These are applied in a feedforward sense, while plant parameter changes are to be compensated by a fixed, nonoptimal feedback controller. The author illustrates the improvements afforded by the addition of the feedforward signal to the previous feedback control system, but no attempts are made to evaluate the effect of plant nonlinearities, parameter changes and optimizing the feedback controller.

Weaver and Schultz^[24] have proposed an interesting approach to nuclear reactor control. It is based on state variable feedback design, and necessitates the reduction of the high-order nonlinear system to a low-order linear model. This reduction is made in an apparently arbitrary manner, approximately cancelling poles and zeros and neglecting poles far from the $j\omega$ axis. Because of several gross approximations, the authors rely on limiting the input signal, but even then the responses of the linear model and of the nonlinear system differ considerably. The basic shortcomings of the method are the technique used to derive the low-order model and that optimization of general cost functions is not possible.

Lipinski and Vacroux^[25] describe the application of linear optimal control theory and nonlinear estimation to derive feedback

control for a quadratic performance index. A second-order reactor model is used by considering only a single group of delayed neutrons and the prompt jump approximation. Since a sampling interval of one second is assumed, the authors can select the delayed neutron decay constant so that it gives the best approximation to the six group response over a one second interval. The reactor equations are linearized in order to derive the optimal feedback controller, and the result is applied to the nonlinear model, but not to the six group one. Furthermore, neither the effect of temperature on reactivity, nor the absorber rod characteristics are considered.

3.6 The Proposed Method of Solution

It is apparent that none of the techniques reviewed in the previous section are capable of solving the reactor control problem as it has been formulated in this thesis. However, each of these earlier papers contributes one or more significant concepts towards finding a practically realizable solution to the optimal reactor control problem. Some of these concepts are: assuming suitable functions to approximate the desired response, proceeding backwards to compute the corresponding input, the use of a low-order linear model to compute the feedback controller, optimizing the parameters of the model so that it provides the best fit to the actual system.

In addition to the above concepts, the following observations must be made prior to a detailed solution of the control problem. It is clearly impractical to feed back all the state variables, as required by optimal control theory. The system performance obtained

by using a lower order controller can be regarded as optimal only for the specified constraint of having a limited number of state variables to feed back. In the control literature such a controller is usually described as being suboptimal.

When deriving a low-order linear model for the nonlinear reactor, it is advantageous to include in the high-order system the effect of temperature on reactivity and the characteristic of the absorber rod, instead of attempting to simplify the kinetic equations alone.

Since the behaviour of the reactor is a function of the instantaneous power level, and since the plant parameters vary widely over the operating life of the reactor, the parameters of the model must be adjusted to reflect the changes in the plant characteristics. Furthermore, this identification should be performed without introducing extraneous disturbances. The form of the model should be such that it readily leads to the evaluation of the optimal feedback controller.

To find the optimal control for nonquadratic performance indices, it is proposed to approximate the desired response by a function that has the same form as the low-order model, which is used to replace the system for the purpose of analysis. The parameters of this approximating function are varied to minimize the appropriate cost function.

In the next two chapters methods will be developed to derive the parameters of a low-order model on the basis of the response of the high-order system such that the deviations between their respective outputs are minimized. The same basic techniques will also be used to

obtain the parameters of the optimal feedback controller for quite general cost functions.

CHAPTER 4

OPTIMAL LOW-ORDER MODELS

It has been shown in the previous chapter, that the low-order reactor kinetic models which have been derived on the basis of the physical nature of the nuclear process, fall far short of the accuracy required for control purposes. Linearization of these simplified models results in the further degradation of performance, as compared to the six delayed neutron group kinetic equations. It is necessary, therefore, to investigate alternative methods for deriving low-order models for a given high-order system. Because of the time-varying nature of the parameters of the nuclear reactor, the technique must be efficient enough for on-line applications, and only require the availability of signals which are already provided by the monitoring facilities, viz. the input and the output. It is also highly desirable that the model be optimal, hence the approximating parameters should be derived such that a suitable cost function is minimized.

The problem of approximating high-order systems by low-order models has received considerable attention in the recent literature [26], [27]. The proposed methods fall into two broad classes:

- a. The given system equations are manipulated such that only the dominant poles and zeros are retained, or a reduced pole-zero pattern is found. In all of these cases the approximation to the original system is qualitative, i.e., no measure of their goodness of fit can be ascertained.
- b. The remaining methods minimize the mean square error

between the responses of the original and reduced sets of equations. While a cost function is minimized by these techniques, they are limited to the one specific objective function.

The use of the least squares error criterion arises because of mathematical convenience. Although in many cases it reflects the energy of the system, and is therefore a desirable objective to be minimized, it is not necessarily the most appropriate cost function for modelling purposes.

The present chapter is aimed at developing a method that will provide models for any conceivable error criterion, as long as it can be evaluated by a digital computer. Since analytical solutions are known only for the mean square error case, and appear feasible for a limited number of error criteria, a solution based on search techniques is proposed^{[28],[29]}. A direct as well as a gradient search method is to be investigated.

4.1 Pattern Search

The pattern search strategy was devised by Hooke and Jeeves^[30], and is one of the most popular of the multidimensional direct search methods. (Search techniques which do not require information regarding the gradient of the objective function are considered direct.) Its main feature is that it attempts to establish the "pattern" of successful search points. It is, therefore, particularly adept in following a valley, once its direction has been established. Since several well documented descriptions of pattern search are available in the

literature, only its basic characteristics are reviewed here.

Two sequences make up the pattern search algorithm: exploratory moves and pattern moves. During the former, starting from a base point \underline{b} , each variable is incremented in turn, by specified amounts, first in one, and if a decrease in the function value is not found, in the opposite direction. Exploration with the next variable begins with the best point established by the previous ones. Having incremented every variable, exploration is completed, and a new base point, \underline{b}^* has been found. A pattern move is now made in the direction of the improvement from \underline{b} to \underline{b}^* to the point given by $2\underline{b}^* - \underline{b}$. The new base point is not established until exploration has been completed about the end of the last pattern move. In this manner the size of the pattern move changes after each successful exploration. If the latter fails, the last base point is used to start a new exploration with reduced step sizes. The pattern is, however, destroyed. The search is terminated when the step sizes have been reduced below the specified levels.

The computational inefficiency of the pattern search method arises from the necessity of having to evaluate the objective function after every exploratory step. Hence, for a problem having n variables, each base point requires at least n and at most $2n$ function evaluations. In a typical optimization problem, there will be many base points, particularly since convergence near the minimum is rather slow. On the other hand, since the computation of derivatives is avoided, pattern search is very suitable for feasibility studies, and the initial evaluation of new concepts, where the exact value of the optimum is not

significant. It may also be found useful for certain off-line optimization problems, where computer time is not at premium.

4.2 The New Fletcher Method

In a recent paper^[31], Fletcher proposed a new gradient search algorithm, which dispenses with the linear search used in the highly regarded Fletcher-Powell method^[32], with a corresponding reduction in the number of function evaluations. The method is applicable to the minimization of a function of n variables $J(\underline{\phi})$, when the vector of first derivatives $\underline{g}(\underline{\phi}) = \nabla_{\underline{\phi}} J(\underline{\phi})$ is available explicitly, but the matrix of second partial derivatives G (the hessian) is not. ($\underline{\phi}$ is the n -dimensional parameter vector.) The method is based on approximating G^{-1} by a matrix H , in an iterative manner, such that the eigenvalues of H tend monotonically to those of G^{-1} .

The H matrix is used to give the direction of parameter changes that will result in the continuous decrease of J . Using the superscript $*$ to denote values appropriate to the next iteration, the following set of formulae are relevant:

$$\underline{\delta} = -\lambda H \underline{g}$$

$$\underline{\phi}^* = \underline{\phi} + \underline{\delta}$$

$$\underline{\gamma} = \underline{g}^* - \underline{g}$$

The scaling factor λ is based on a simple test, which ensures that the decrease in J is sufficiently large to guarantee ultimate convergence, but rarely requires more than one evaluation of J and \underline{g} .

The updating of H is based on one of the following two formulae, the choice depending on another simple test:

$$H^* = H + \frac{\underline{\delta} \underline{\delta}^T}{\underline{\delta}^T \underline{\gamma}} - \frac{H \underline{\gamma} \underline{\gamma}^T H}{\underline{\gamma}^T H \underline{\gamma}}$$

$$H^* = H - \frac{\underline{\delta} \underline{\delta}^T H}{\underline{\delta}^T \underline{\delta}} - \frac{H \underline{\gamma} \underline{\delta}^T}{\underline{\delta}^T \underline{\gamma}} + \left(1 + \frac{\underline{\gamma}^T H \underline{\gamma}}{\underline{\delta}^T \underline{\gamma}}\right) \frac{\underline{\delta} \underline{\delta}^T}{\underline{\delta}^T \underline{\gamma}}$$

At the start of the first iteration H is usually not known, and is assumed to be the unit matrix.

4.3 Statement of the Problem

The problem of approximating a high-order system by a low-order model such that the deviations between their outputs to the same input is minimized, may be formulated in the following manner.

Let the given high-order single-input single-output system be described by the usual state-variable notation:

$$\dot{\underline{x}} = A \underline{x} + \underline{b} u \quad (4-1)$$

and

$$y = \underline{d}^T \underline{x} \quad (4-2)$$

Consider a discrete set of values of y taken over a suitable interval of time:

$$Y = \{y_0, y_1, y_2, \dots, y_i, \dots, y_I\} \quad (4-3)$$

where $y_i = y(t_i)$, i.e., the output at the i th sampling instant. This set represents samples of the response of the system described by equations (4-1) and (4-2) to a specified input $u(t)$. It may have been obtained by solving the system equations on a digital computer, or it may have been observed by a measuring instrument connected to the actual system. In either case, the continuous output $y(t)$ is sampled

at sufficiently close intervals of time so that no significant information is lost.

The objective is to find another output set Y^* , associated with a model of order m (where $1 \leq m < n$) described by the equations

$$\dot{\underline{x}} = A^* \underline{x}^* + \underline{b}^* u \quad (4-4)$$

$$y^* = \underline{d}^{*T} \underline{x}^* \quad (4-5)$$

such that, for the same input, one of the following two objectives is satisfied,

- (i) for a given m , the error function J is minimized, where

$$J = f[w_i(y_i - y_i^*)] \quad (4-6)$$

which is some suitable function of the errors $y_i - y_i^*$ with a vector weight w_i attached at each sampling instant, or

- (ii) for a given value of J , the lowest order model (m) is determined such that the resultant error J^* satisfies the inequality

$$J - J^* \geq 0 \quad (4-7)$$

4.4 Error Criteria

The functional form that the error criterion expressed by equation (4-6) takes, has a vital bearing on the parameters of the approximating model. Since the purpose of the objective function is to measure the extent to which the model deviates from the actual system, the main problem is how to express numerically this deviation.

The error criterion expressed by equation (4-6) states that the error is a function of the difference between the outputs of the original and reduced systems, and a weighting sequence. It is usual practice to take the absolute value of the output error, and to raise it to some

power p . In addition, the weighting sequence normally appears as a multiplying factor. A more specific form of equation (4-6), but one that includes all the relevant error criteria is

$$J = f(w_i |y_i - y_i^*|^p) \quad (4-8)$$

We can now turn our attention to the functional relationship involved. Again, we can distinguish two alternatives in current practice. One involves a summation, over all i , for the time-interval of interest; the other retains the value at one particular sample only, where the deviation is a maximum. Accordingly, the objective is either to minimize a sum of the errors, or to minimize some maximum deviation.

The two alternative forms of equation (4-8) may therefore be written as

$$J = \sum_{i=0}^I w_i |y_i - y_i^*|^p \quad (4-9)$$

or

$$J = \max_{i=0, I} \{w_i |y_i - y_i^*|\} \quad (4-10)$$

Considering a weighting sequence of unity, we see that the criterion of equation (4-9) gives a measure of the area between the curves when $p=1$, and the mean square error if $p=2$ (except for appropriate scaling factors). As the value of p increases the result of (4-9) (after taking the p th root of the summation) tends to that of (4-10). This is, in fact, a convenient method for minimizing a criterion of the form of equation (4-10), and it will be used, with slight modifications, in later parts of this work.

Turning our attention now to the weighting sequence, it should be noted that all error criteria weigh some aspects of a response

against others. For example, a least squares objective tends to emphasize the transient portion at the expense of the steady-state value.

The tendency of every error criterion to give a bias to the resultant model may of course be further emphasized or, on the other hand, compensated for by assigning appropriate weights by which each sample error is multiplied before it is included in the summation, or compared with the previous maximum.

The desire to obtain a criterion that is free of an apparent bias has led to the development of a new objective function, based on measuring the shortest or perpendicular distance between the reference and the approximating responses.

4.5 The Shortest Distance Minimax Criterion

Since the aim of our approximation problem is to find a model whose output closely resembles that of the original system, the error criterion used in optimizing this closeness must give an accurate measure of the proximity of the two responses. The most frequent choice of minimizing the mean square error arises out of mathematical convenience, rather than of regard to any particular physical aspect of the problem.

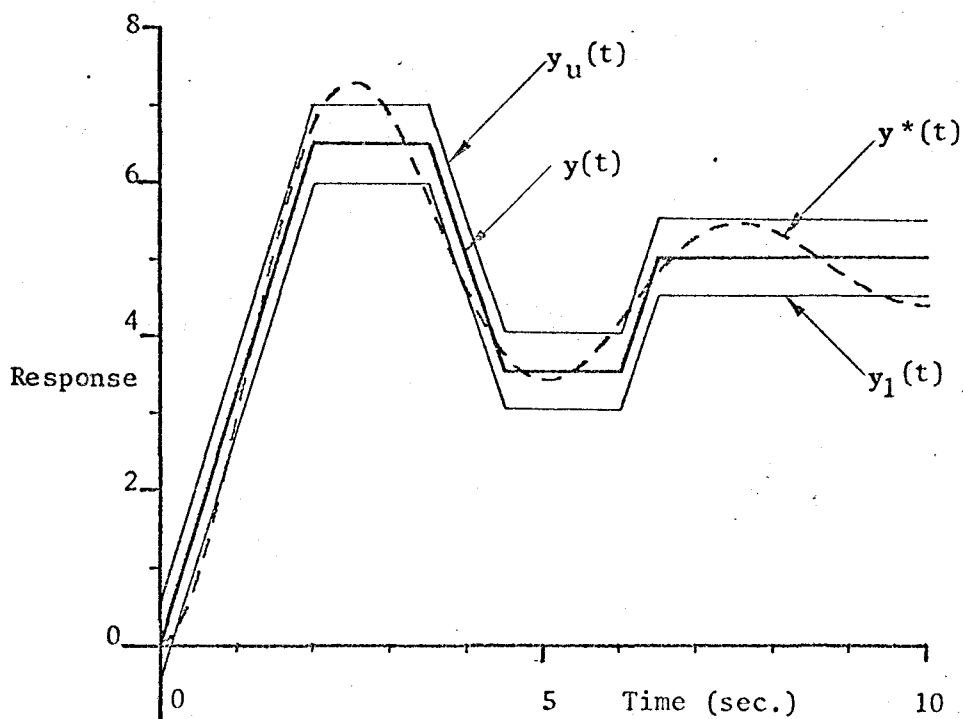
Perhaps the most common physical measure of the proximity of two functions is the area between them. This criterion is suitable for comparing various models, or finding one of a specified order that produces the closest response. However, for the more typical engineering specifications, that express the deviations at various points between the

actual and desired responses, and set the limit on these as a tolerance of $\pm x\%$, criteria that involve summation are not suitable, and we must turn to minimax type objectives.

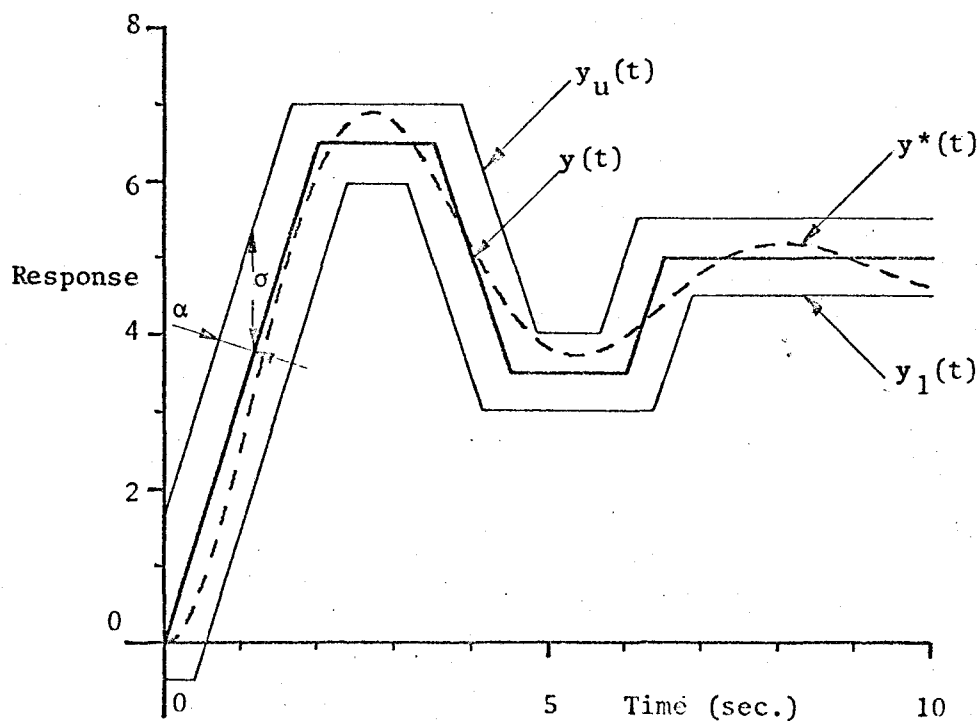
The typical minimax objective used in approximation problems measures the difference between two curves at the same value of the independent variable, which may be, for example, distance, time or frequency. While this formulation is appropriate in the frequency-domain, it may not always be so in the time-domain. Particularly during rapid transients, the difference between two responses is considerably exaggerated by taking the values at the same sampling instants. This practice, like the use of the mean square error criterion, arises more from convenience than from physical regards for the problem.

The most appropriate measure of the maximum deviation of two time-functions appears to be the shortest distance between them, i.e., the perpendicular line drawn from the reference curve to a point on the approximating response.

The basic difference between taking the sample error or the perpendicular error as a measure of the deviation of two curves, is illustrated in Figure 4-1. The desired response is a piece-wise linear function of time, and it is required that the response of the approximating model be everywhere within α of the given curve. For the example, α has been chosen to be 10% of the steady state value of the desired response. In part (a) of the diagram, α has been interpreted in the conventional, sample error sense, and the corresponding bounds on the error have been drawn. Note the apparent closeness of



a. Errors measured at same instant of time.



b. Errors measured everywhere perpendicular to reference curve.

Figure 4-1. Piece-wise linear reference response with specified error bounds, and responses of approximating second-order systems.

the curves along the lines of finite slope. In part (b), the error bounds are drawn at a perpendicular distance of $\pm\alpha$ from the reference function, and the width of the region, into which the approximating curve is to fit, appears to be uniform.

To illustrate the type of responses that result from the alternative problem formulations we have presented, consider approximating the reference function in Figure 4-1 by the output of a second-order system of the form:

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^2 + a_1s + a_0} \quad (4-11)$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the output and input functions respectively. An input step of magnitude five is applied to the approximating system, and the response computed over an interval of ten seconds, using 501 sample points. The initial conditions are assumed to be zero. Using a pattern search algorithm, the following parameter values have been obtained:

1. Minimizing the sample error:

$$b_0 = 1.5464 \quad a_1 = 0.5020 \quad a_0 = 1.6315$$

and the maximum sample error is $J_s = 0.764$.

2. Minimizing the perpendicular error:

$$b_0 = 1.3862 \quad a_1 = 0.5797 \quad a_0 = 1.4664$$

and the maximum perpendicular error is $J_p = 0.408$.

The corresponding responses are also plotted in Figure 4-1. Evidently, using the sample error interpretation of the tolerance ($\alpha = \pm 0.5$), the model as given by equation (4-11) cannot satisfy the specifications. If, on the other hand, the errors are measured in the perpendicular

sense, the second order system obtained by the minimization procedure does fulfill the design requirements.

The basic problem associated with the use of the perpendicular measure of the error is its computation by an automated scheme. Since in most applications the sampling intervals are uniform and the same for both responses, no direct measure of the perpendicular distance between them is available.

Consider first the problem of establishing the error bounds on a desired response for a given tolerance α . In the case of piece-wise linear responses, such as the one considered in Figure 4-1b, the perpendicular error bounds are readily formed by line-segments parallel to each linear portion of the reference response, and at a perpendicular distance of α from it. Referring to Figure 4-1b, we can establish a relationship between the tolerance α and the corresponding limits on the sample error (σ). Taking an arbitrary point t_i , let the slope of $y(t)$ at that point be

$$\left. \frac{dy}{dt} \right|_{t=t_i} = \tan \theta_i \quad (4-12)$$

Then the bound on the sample error, σ_i , that corresponds to a specified tolerance α at $t = t_i$ is given by:

$$\sigma_i = \frac{\alpha}{\cos \theta_i} \quad (4-13)$$

The following points should be noted with reference to equation (4-13):

- a. The slope of the specified function must be finite in the region of interest, i.e.,

$$0 \leq \theta_i < \frac{\pi}{2}$$

Step-changes may, however, be accommodated in an automated scheme by appropriate programming.

- b. Equation (4-13) is exact only if θ_i is further restricted to be constant over the interval $[t_i - \delta_i, t_i + \delta_i]$, where

$$\delta_i = \alpha \sin \theta_i$$

If θ_i varies slowly over the interval in question, the approximation given by equation (4-13) is good. The accuracy is further enhanced by making α small, which will be the case whenever a close-fitting approximation is required.

- c. At places of zero slope, such as the peak overshoot and the steady state, $\sigma = \alpha$, and the above restrictions on θ do not apply.

If equation (4-13) is written in the form

$$\alpha = \sigma_i \cos \theta_i \quad (4-14)$$

σ_i can be associated with the sample error at t_i , to give

$$J = \max_{i=1,N} \{ \cos \theta_i |y(t_i) - y^*(t_i)| \} \quad (4-15)$$

which is the proposed minimax objective.

Comparing equation (4-15) to the formulation of the general minimax objective function (equation (4-10)), the $\cos \theta_i$ term may be identified as a weighting sequence, that de-emphasizes the sample error along a transient as compared to the deviations where the reference response has a steady value. The overall effect is to produce a response that approximates the desired one in such a manner

that the maximum deviation between the two, measured in a direction perpendicular to the reference curve, is minimized. Comments a, b and c above apply to equation (4-15) also, but it should be noted that the inaccuracy, which results from changes in the slope of the reference function, diminishes as the optimum is approached.

While the advantage of the shortest distance error criterion is most apparent in the case of minimax objectives, the concept could equally well be used for least pth criteria.

4.6 Starting Parameters and Essential Features of the Computer Programme

An important problem associated with every search technique is the selection of the starting parameters, since these have considerable influence on the convergence of the process, and on the probability of locating the global optimum. It is proposed that the starting parameters be determined on the basis of a simple first-order model if the step-response of the system has no overshoot, and from a simple second-order model with a pair of complex conjugate poles if it has an overshoot. For the two responses to a unit step input shown in Figure 4-2, the appropriate models are given below.

For (a)

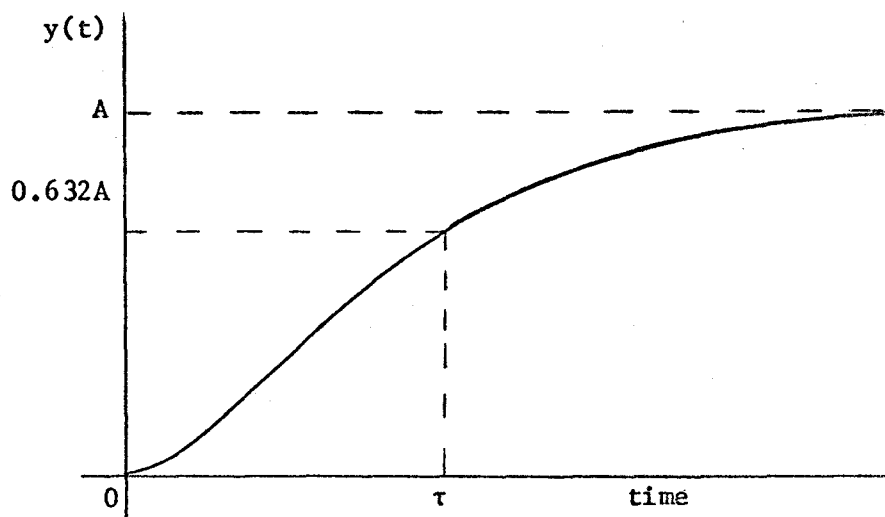
$$G(s) = \frac{b_0}{s + a_0} = \frac{Y(s)}{U(s)} \quad (4-16)$$

where

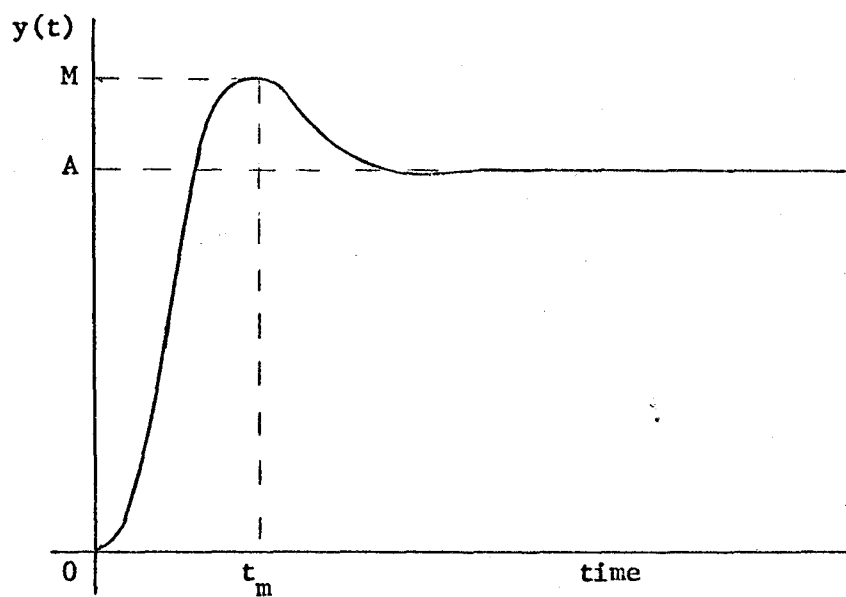
$$a_0 = \frac{1}{\tau}$$

and

$$b_0 = A a_0 .$$



a. Response with no overshoot



b. Response with overshoot

Figure 4-2. Possible responses of a stable system to a unit step input.

For (b)

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0} \quad (4-17)$$

where

$$a_0 = \alpha^2 + \beta^2$$

$$a_1 = 2\alpha$$

$$b_0 = A a_0$$

and

$$a = \frac{1}{t_m} \ln \left(\frac{A}{M-A} \right)$$

$$\beta = \frac{\tau}{t_m}$$

In general, models of higher order are required. Having found the optimum set of parameters by the pattern search programme for a given model, the order is then increased by one and a new pattern search initiated. This process is then continued until the error criterion is satisfied, or the desired order is reached.

To increase the order of the model by one, an additional term, with s raised to the appropriate power is introduced. For example, the first order model obtained from equation (4-16) is replaced by

$$G(s) = \frac{b_0}{s^2 + s + a_0}$$

This result is a consequence of having chosen a phase-space representation for the model in the computer programme. In this form, the general transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad (4-18)$$

is written as

$$\dot{\underline{x}} = A \underline{x} + \underline{d} u \quad (4-19)$$

and

$$\underline{y} = \underline{b}^T \underline{x} \quad (4-20)$$

where

$$\left. \begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -a_0 & -a_1 & -a_2 & -a_3 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \\ \underline{d} &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned} \right\} \quad (4-21)$$

The advantages of the phase-variable representation are twofold. It can be written down by inspection given the transfer function or vice versa. In addition, for an n th order system, at most $2n$ parameters need be varied instead of n^2+n parameters in the more general state-space representation of a single-input single-output system. Moreover, standard techniques for transformation to the phase-variable form from the general case are well known^[33]. The number of variables for the search program is further reduced in those cases where the steady-state error to step, ramp or higher order inputs is constrained to

equal that of the system.

A computer programme has been written that uses a pattern search subroutine to find optimum low-order models for high-order systems. The programme has the following features:

- (a) The original system may be specified either in the phase variable form, or only its output response to a given input function may be available at discrete uniform intervals of time.
- (b) The parameters of the model to be used for starting values may be given by the user, otherwise the programme will compute these as outlined at the beginning of the section.
- (c) The weighting sequence may also be specified in a variety of ways. It may be given as an input matrix of two columns (time and corresponding weights) and as many rows as there are weights to be assigned. Alternatively, a weighting sequence may be requested from the programme, one that gives a measure of the shortest distance between the responses of the model and the actual system. If neither of the above is specified, the programme assumes a uniform weighting sequence.
- (d) The objective function to be minimized may be chosen as the sum of the absolute values of the sample error raised to the p th power ($1 \leq p \leq 10$); or as the maximum value of the sample error. Unfortunately, the minimax formulation results in discontinuous partial derivatives of the error criterion with respect to the parameters of the model. In such cases, the pattern search technique will often fail to find the optimum^[34].

To overcome this difficulty, when a minimax objective is to be satisfied, a least pth formulation, with $p = 10$ is first undertaken. At each function evaluation, however, the maximum deviation is also noted, and the set of parameters that give a minimum of this maximum deviation is stored^[29]. This set is then used at the end of the least 10th minimization as the starting point for a new pattern search to further reduce the minimax objective.

- (e) The order of the time input function $u(t) = t^k$, to which the steady-state response of the model is to coincide with that of the original system, can also be specified. Care must, of course, be taken that the order k is not greater than the type of the system.

4.7 Optimal Models for a Linear Seventh-Order System

Before an attempt is made at developing optimum low-order models for the nuclear reactor, it is desirable to compare the proposed search technique with already existing system order reduction methods. A comparison of some of the earlier techniques has already been attempted, but it was done on an entirely qualitative basis: the responses of the various models are plotted, and the reader is invited to judge for himself that the response of one model is closer to the original than all the others. The measure of closeness, however, has never been specified. In this section, therefore, after a brief review of the existing methods and description of the test system, the quantitative comparison of the models is presented. Having thus

established a frame of reference, it is shown that, irrespective of the error criterion, an optimal model can be derived by the use of search methods, having a lower cost than any of the models obtained by previous techniques.

As it has been pointed out at the beginning of this chapter, presently known methods of system order reduction fall into two broad categories: those that provide only a qualitative fit, and those that give a least squares approximation. Of the six published techniques, four belong to the first group.

1. Davison's method [26],[35] is based on retaining only the dominant eigenvalues of the system, since in most cases poles far removed from the $j\omega$ axis have only negligible effects on the transient response.
2. Mitra's approach [36], often referred to as "the optimal projection method", also retains the dominant eigenvalues, but it utilizes a weighting matrix, such that the projection error, that arises from the linear transformation from the space of the system equations to the subspace that forms a basis for the model, is minimized.
3. Chen and Shieh [37] describe a method based on the continued fraction expansion of the original system's transfer function, that has been arranged in ascending powers of s . Using the final value theorem, it can be shown that the quotients in the expansion have a decreasing effect on the steady-state response. Instead of retaining the dominant poles, this method produces a new pole-zero pattern representative of the original one.

4. Sinha and Wismath^[27] have proposed a technique based on certain properties of the unit step response of a system, such as initial slope, maximum overshoot, steady-state value, etc. Of the four possible pole zero configurations for a second-order system, only the case of a pair of complex poles and no finite zero can be solved analytically. This happens to be the case for the model of the seventh-order system chosen by them, and the model parameters are obtained in a very straightforward manner. Finding the model in the other three cases entails recourse to a search program on the digital computer.
5. Anderson's method^[38] provides a least squares fit between the responses of system and model. It is based on the orthogonal projection theorem in the theory of linear vector spaces, and uses only the output response at discrete points of time.
6. Sinha and Pille^[39] also describe a least squares approach to system order reduction, but their method is based on the use of the matrix pseudoinverse. Theirs is an iterative process, suitable for on-line identification, and the technique is unique in this aspect.

A linear test system, suitable for comparing the models produced by various reduction techniques has been proposed by Sinha and Wismath. The system is of order seven, and has an even distribution of poles in the left half s -plane. It represents the flight control system of a supersonic transport plane^[40].

The parameters of the system transfer function, in ascending powers of s are

numerator:	31237.5
	375000.0
denominator:	281250.0
	3310875.0
	2814271.0
	853703.0
	70341.9
	4097.4
	83.635

The six models which have already been published for the above seventh-order system are listed in Table 4-1, along with the names of their authors. The pole locations, as well as the steady-state value of the response to a unit step are displayed. For the original system, the steady-state value is 0.11111. The method of Sinha and Wismath, and the one due to Chen and Shieh are seen to produce accurate steady-state responses. Looking at the pole locations, the models of Davison and Mitra have poles closely located to one another. This is not surprising, since they are similar methods. Comparing the models produced by the two least-squares approaches, Anderson's and the one of Sinha and Pille, a considerable difference is apparent in the pole locations. Clearly, at least one of the optima is not the global one.

For a quantitative comparison of the various models an infinite variety of cost functions could be used, depending on the purpose for which the model has been developed. If the objective is to match the output of the system by the response of the model to the identical input, the following four cost functions appear to be most appropriate:

Method	Transfer function	Pole locations	Steady-state value
Davison	$\frac{-0.0505s + 0.5558}{s^2 + 4.1126s + 5.0297}$	$-2.053 \pm j0.895$	0.1105
Mitra	$\frac{-0.0159s + 0.5648}{s^2 + 4.0488s + 5.0277}$	$-2.024 \pm j0.965$	0.1123
Chen & Shieh	$\frac{0.1299s + 0.01105}{s^2 + 1.1464s + 0.0994}$	$-1.048, -0.098$	0.1112
Sinha & Wismath	$\frac{0.2098}{s^2 + 1.6904s + 1.8879}$	$-0.845 \pm j1.083$	0.1111
Anderson	$\frac{0.3096}{s^2 + 1.9026s + 2.6879}$	$-1.344 \pm j0.316$	0.1152
Sinha & Pille	$\frac{0.3302}{s^2 + 2.0954s + 2.8886}$	$-1.048 \pm j1.338$	0.1142

Table 4-1. Comparison of second-order models for seventh-order system, obtained by previously published methods.

- a. the sum of the absolute values of the error ($\sum |e|$),
- b. the sum of the square of the errors ($\sum e^2$),
- c. the maximum sample error ($M|e_s|$),
- d. the maximum perpendicular error ($M|e_p|$).

These four measures of the proximity of the two responses have been computed for a unit step input over an interval of 20 seconds taking 501 sample points. The results obtained for the six models already discussed are shown in Table 4-2. In addition, an "average" measure of the error, designed to give a convenient, quantitative comparison of the various techniques, is also displaced. To find the average error represented by the various column entries, they must all be of the same dimension. The mean error is readily obtained for the $\sum |e|$ criterion by dividing with the number of sample points. The $\sum e^2$ term similarly yields

$$e_s = \sqrt{\frac{\sum e^2}{501}}$$

where 501 is the number of samples taken. These two errors are shown in Table 4-2 in brackets under the appropriate entries. To find an overall measure of the error, we could sum all four measures for a given model. However, since two of these are maximum values, while the others are average ones, the result would be weighted too much in favour of the former two. For this reason, the maximum sample error has been omitted, and an average of the remaining three taken.

A comparison of the various methods may now be made on the basis of any one particular measure of the error. For example, Chen and Shieh's model produces the smallest $\sum |e|$ error, while the method of Sinha and Pille results in the smallest $\sum e^2$ error. On the basis of

Method	$\sum e $	$\sum e^2$	$M e_s $	$M e_p $	Average error
Davison	2.8325 (0.00567)	0.0193 (0.00621)	0.0107	0.0105	0.00746
Mitra	1.9675 (0.00393)	0.0106 (0.00460)	0.00898	0.00831	0.00561
Chen & Shieh	0.9500 (0.00190)	0.0122 (0.00494)	0.0278	0.0135	0.00678
Sinha & Wismath	2.4950 (0.00499)	0.0191 (0.00618)	0.0202	0.00827	0.00648
Anderson	1.0625 (0.00212)	0.00405 (0.00284)	0.00852	0.00707	0.00401
Sinha & Pille	1.0250 (0.00205)	0.00362 (0.00269)	0.00590	0.00588	0.00354

Table 4-2. Errors associated with the
models in Table 4-1.

the average error, the technique of Sinha and Pille is the best, and that of Davison the worst. It is also interesting to note that the three methods that attempt to produce an optimum model in some sense, do in fact result in the smallest average errors.

Having established various measures for the proximity of two responses, and observed the ability of previously published techniques to produce accurate models of a given system, let us turn our attention to finding models that are optimum with respect to specific criteria. The pattern search program has been used to find models that minimize one of the following three error criteria:

1. maximum perpendicular error,
2. sum of the absolute values of the errors,
3. sum of the squares of the errors.

For each objective, two models are of interest: one that is constrained to have the same steady-state value to a step input as the original system, and one that is free of this requirement. The six resultant models are shown in Table 4-3. The last three are seen to have the correct steady-state value, as required.

The pole locations for the models that minimize the $\sum |e|$ and the $\sum e^2$ criteria respectively, can be observed to be quite close to one another. The fourth model (minimax perpendicular error with steady-state constraint) is noticeable in having poles much closer to the origin than any of the other eleven models.

The errors produced by the pattern search models are shown in Table 4-4. As expected, each model has the smallest error for the criterion it has been minimized with respect to, and the error is

Objective function	Transfer function	Pole locations	Steady-state value
$\sum e $	$\frac{0.1536s + 0.01329}{s^2 + 1.3456s + 0.1196}$	-0.0957, -1.250	0.1112
$\sum e^2$	$\frac{0.3960}{s^2 + 2.6569s + 3.4191}$	-1.328 \pm j1.286	0.1158
$M e_p $	$\frac{0.0254s + 0.2967}{s^2 + 2.4257s + 2.5581}$	-1.213 \pm j1.043	0.1160
$\sum e $ with s.s. constraint	$\frac{0.1536s + 0.01329}{s^2 + 1.3432s + 0.1196}$	-0.0959, -1.247	0.1111
$\sum e^2$ with s.s. constraint	$\frac{0.1019s + 0.05359}{s^2 + 1.0718s + 0.4823}$	-0.536 \pm j0.442	0.1111
$M e_p $ with s.s. constraint	$\frac{0.0960s + 0.04545}{s^2 + 1.0432s + 0.4091}$	-0.522 \pm j0.370	0.1111

Table 4-3. Comparison of optimum second-order models for seventh-order system, obtained by the pattern search programme.

Objective function	$\sum e $	$\sum e^2$	$ e _s$	$ e _p$	Average error
$\sum e $	0.769 (0.00154)	0.001268 (0.00159)	0.02954	0.00479	0.00264
$\sum e^2$	0.836 (0.00167)	0.001915 (0.00196)	0.00445	0.00373	0.00245
$M e_p $	1.086 (0.00217)	0.003898 (0.00279)	0.00934	0.00293	0.00263
$\sum e $ with s.s. constraint	0.783 (0.00157)	0.01265 (0.00503)	0.02955	0.01079	0.00580
$\sum e^2$ with s.s. constraint	1.963 (0.00393)	0.01075 (0.00464)	0.01879	0.00735	0.00531
$M e_p $ with s.s. constraint	1.942 (0.00392)	0.01174 (0.00485)	0.0172	0.00691	0.00523

Table 4-4. Errors associated with
the models in Table 4-3.

smaller if the steady-state constraint is not imposed. In comparison to the models of Table 4-1, for each of the error criteria considered, there is an optimum model with smaller error, produced by the pattern search program, than by any of the previous approaches.

Comparing the average errors of Tables 4-2 and 4-4, the unconstrained optimum models given by the pattern search program all have considerably smaller errors than the best of the previously available methods. The constrained models are also better than the corresponding ones in Table 4-2.

While a second-order model of the seventh-order system should be adequate for most purposes, in certain special applications a model of higher order may be required. To indicate the type of improvement that may be expected, third- and fourth-order models have also been obtained for the $\sum e^2$ criterion. These are given in Table 4-5, including the second-order model for comparison (no steady-state error constraint):

Model Order	Transfer function	$\sum e^2$
2	$\frac{0.3960}{s^2 + 2.6569s + 3.4191}$	1.915×10^{-3}
3	$\frac{-0.1142s^2 + 0.8546s + 0.4}{s^3 + 6.6677s^2 + 9.6505s + 3.4836}$	6.407×10^{-4}
4	$\frac{-0.0372s^3 + 0.2924s^2 + 2.8028s + 0.4}{s^4 + 8.0944s^3 + 24.1124s^2 + 25.7607s + 3.5412}$	3.078×10^{-5}

Table 4-5. Optimal $\sum e^2$ models of increasing order for seventh-order system.

As a final illustration of the nature and accuracy of the approximations obtained by the use of the pattern search method, the responses of some of the optimal low-order models have been plotted, and compared to the output of the seventh-order system. In every case, a unit step input has been applied. Figure 4-3 shows the response of the second-order $|e|_p$ model. The equal ripple variation of the perpendicular error is evident from the diagram. The use of the $\int e^2$ criterion to obtain a second-order model results in an approximation that is a particularly good fit along the rapid initial rise, but is achieved at the expense of a larger error once the peak of the response is passed, as seen in Figure 4-4. The considerable accuracy that may be achieved by the proposed method is well illustrated by Figure 4-5, where the response of the fourth-order $\int e^2$ model has been compared to the original seventh-order system output.

4.8 Application to System Identification

An important feature of the use of search methods to find optimal low-order models for a given system is, that the system equations need not be known, only the response to a given excitation is necessary to derive the model. The technique can therefore be used to identify the parameters of the given system. Basically two problems may be distinguished: in one of these, the order of the system and the approximate values of the parameters are known, and the search programme is used to identify the parameters precisely. The second, and more difficult example is, when the above information is not available. The proposed solution in this case is to begin with a

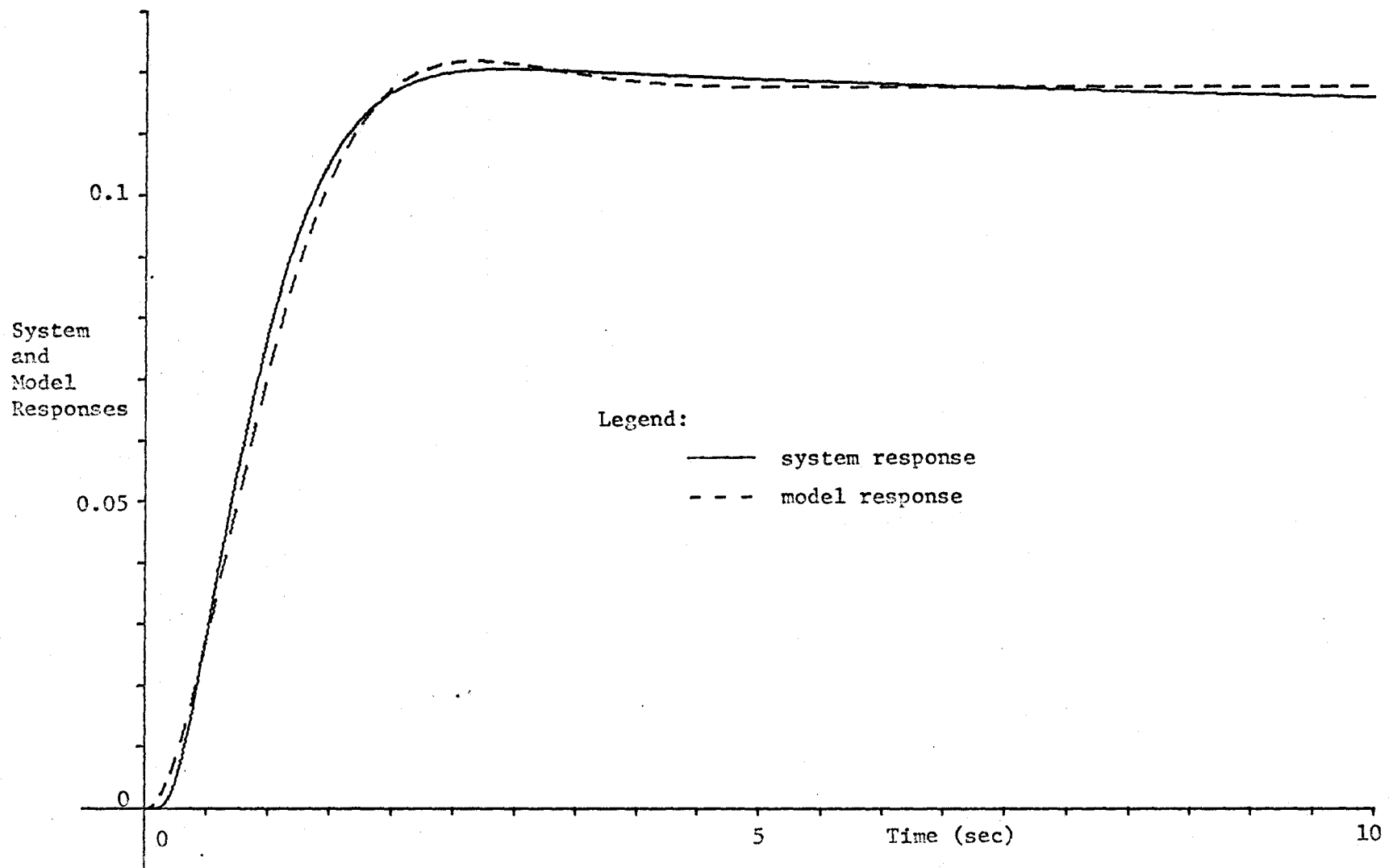


Figure 4-3. Responses of seventh-order system and second-order $M|e_p|$ model.

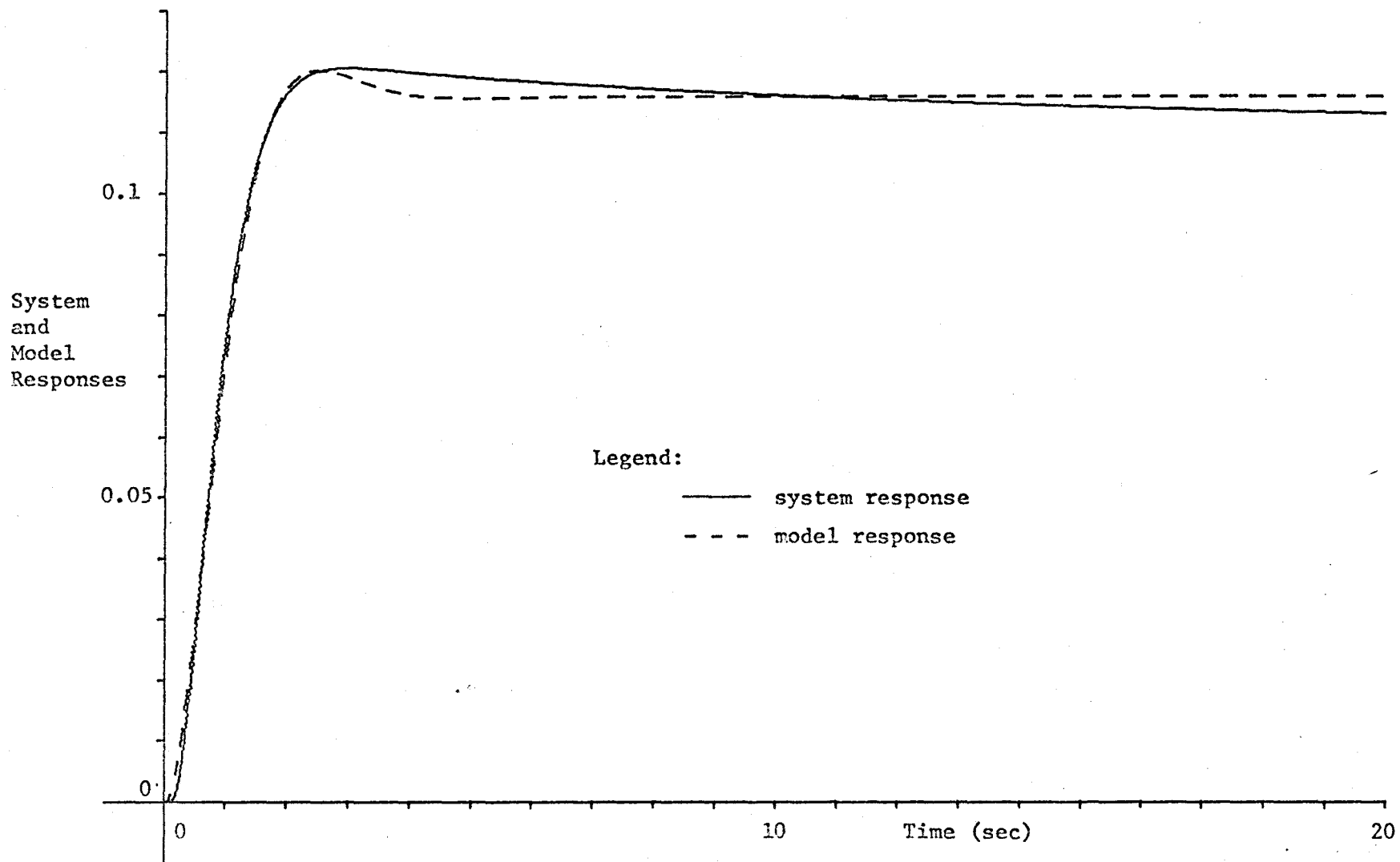


Figure 4-4. Responses of seventh-order system and second-order $\sum e^2$ model.

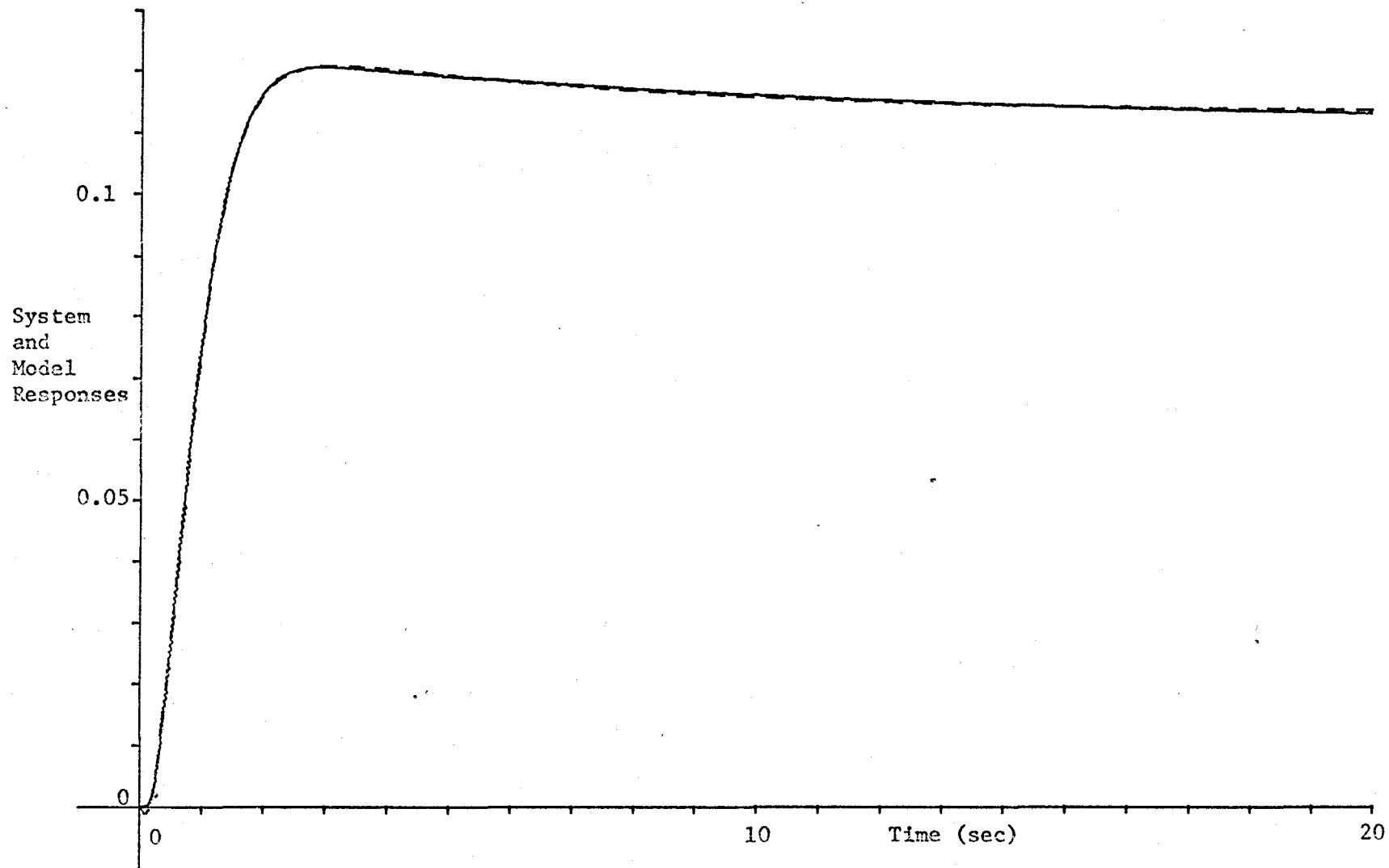


Figure 4-5. Responses of seventh-order system and fourth-order $\sum e^2$ model.

low-order model, optimize its parameters, and if the deviation between the given and the approximating responses is too great, increase the order of the model and reoptimize the parameters, repeating this process until the necessary accuracy is attained. This second problem is the one considered in this section.

In mathematical terms the parameter identification problem appears as follows. Let the input to the plant be denoted by $u(t)$ and the output at the k th sampling instant as $y(kT)$, where T is the uniform sampling interval. Let the model of unknown parameters be in the phase-variable form:

$$\dot{\underline{x}}(t) = A \underline{x}(t) + \underline{d} u(t)$$

$$\hat{y}(t) = \underline{b}^T \underline{x}(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_n \end{bmatrix}$$

$$\underline{d}^T = [0 \quad 0 \quad 0 \quad \dots \quad 1]$$

$$\underline{b}^T = [b_0 \quad b_1 \quad b_2 \quad \dots \quad b_n]$$

$\underline{x}(t)$ is the state vector and $\hat{y}(t)$ the output of the model. The parameters $a_i, b_i, i=0, n$ are to be identified, such that an error criterion of either of the following forms is minimized:

$$J_p = \sum_{k=1}^N |y(kT) - \hat{y}(kT)|^p$$

$$J_m = \max_{k=1, N} \{|y(kT) - \hat{y}(kT)|\}$$

To find the parameter values, it is assumed that the output of the system to be identified has been observed at regular sampling intervals, for a unit step excitation. Depending on whether the response has an overshoot or not, the appropriate formula [equation (4-16) or (4-17)] is used to obtain a set of starting parameters. Alternatively, particularly for inputs other than a step, the initial parameter values are set equal to zero.

Two examples have been considered, a fourth-order system displaying an overshoot in its step response, and a second-order overdamped system. In the case of the latter, both the pattern search and the Fletcher method have been used.

The transfer function of the fourth-order system is given by

$$\frac{Y(s)}{U(s)} = \frac{2s + 1}{s^4 + 3s^3 + 4s^2 + 3s + 1}$$

and its response to a unit step is shown in Figure 4-6. [The one to one relationship of the phase-variable and transfer function notations has been established in equations (4-18) to (4-21).] The response of the initial second-order model in the form of equation (4-17) has also been displayed in Figure 4-6.

The identification process begins with the parameters of the second-order model, and these are varied by the pattern-search program to minimize the error function. For the examples presented in this section, a least squares objective has been used. Once the optimum second-order model has been found, higher order ones are obtained by introducing an s^n term with $n = 3$ and after optimization, $n = 4$.

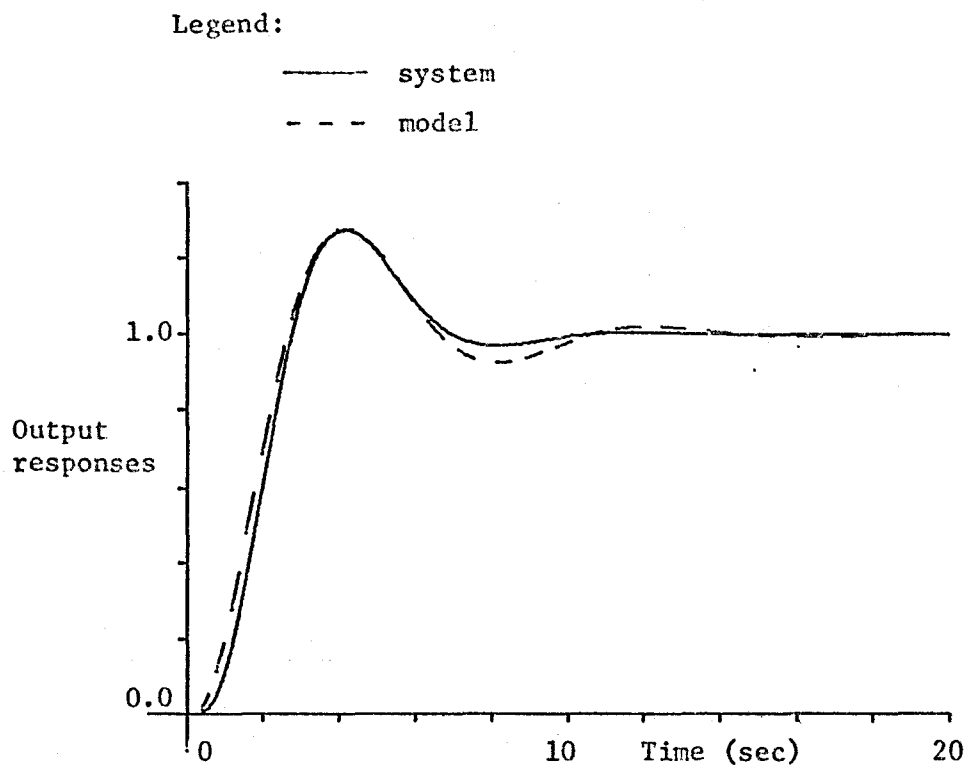


Figure 4-6. Unit step responses of fourth-order system and second-order model.

The manner in which the pattern-search progresses is shown in Figure 4-7. The stages where the order of the model increases to third and then to fourth, are indicated by the $0 \rightarrow 1$ step-changes in parameters a_2 and a_3 . The same changes show up even more sharply in Figure 4-8, where the objective function has been plotted against the number of function evaluations. While the initial second-order model is near the optimum, the subsequent higher-order ones, obtained in each case from the one-lower optimum model, display a rather high initial error. Final convergence is obtained after 1270 function evaluations for a minimum step-size of 0.0001, the values of the model parameters at that stage being:

$$\frac{\hat{Y}(s)}{U(s)} = \frac{2.0010s + 0.9986}{s^4 + 3.0011s^3 + 3.9988s^2 + 2.9995s + 0.9986}$$

The average parameter error is 0.07% and the sum of the squares of the response error is 2.63×10^{-8} . Better accuracy could be obtained by further reducing the minimum step-size.

While the problem of parameter identification has received considerable attention in the past^{[41],[42]}, the author is aware of only one previous example of an identification scheme that is also applicable to system order reduction problems, the one proposed by Sinha and Pille^[43]. Their method is useful for the on-line identification of discrete systems, and is based on an iteratively evaluated form of the matrix pseudoinverse. To identify a continuous system, it is first discretized, and the parameters of the corresponding model, in the z-domain, are estimated from the response to a unit step. The method provides a discrete model with a least-squares fit, the final

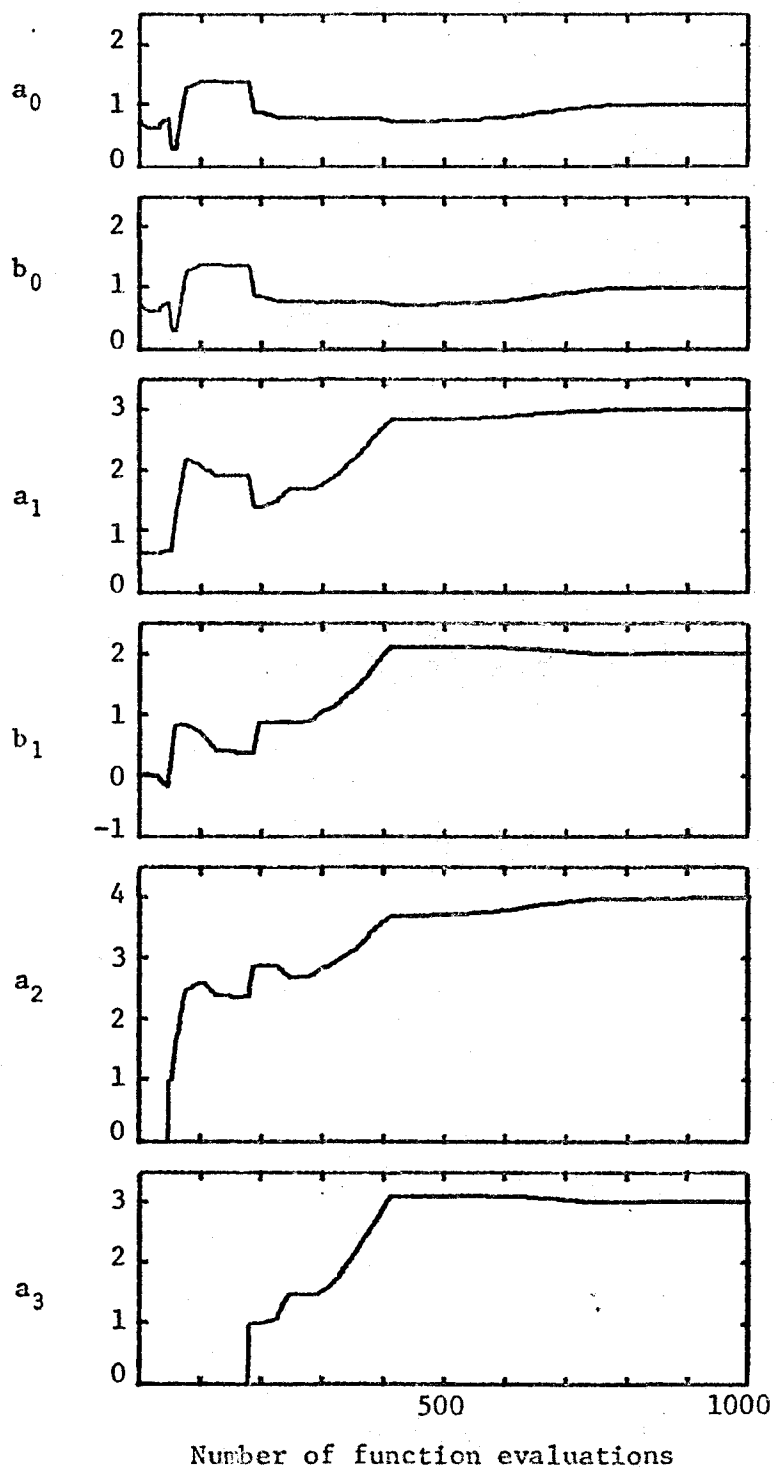


Figure 4-7. Parameter changes during identification of the fourth-order system.

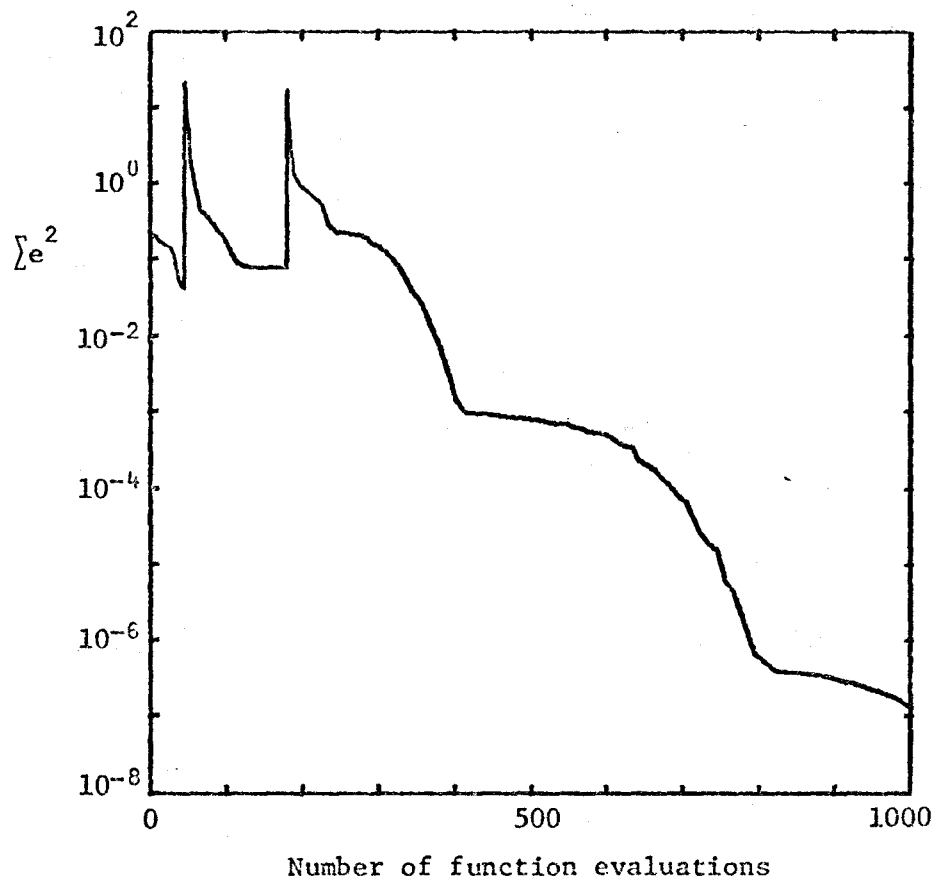


Figure 4-8. Changes in the objective function during identification of the fourth-order system.

error being dependent on the number of iterations before steady-state is reached. For a given system, this is directly proportional to the sampling interval. A continuous equivalent of the discrete model is subsequently obtained.

The second-order system considered by Sinha and Pille has the following transfer function:

$$\frac{\hat{Y}(s)}{U(s)} = \frac{1}{s^2 + 3s + 2}$$

Its response to a unit step input is shown in Figure 4-9. Starting with all parameters assumed to be zero, and a sampling interval of 0.04 second, after 100 iterations the model parameters have been identified as

$$\frac{\hat{Y}(s)}{U(s)} = \frac{0.939}{s^2 + 2.786s + 1.902}$$

The average parameter error is quite large at 5.87%. Using their technique a considerable reduction of the sampling interval would be necessary to obtain more accurate identification.

The above problem has also been attempted using the pattern search program, for the same sampling interval of 0.04 second. The parameters of the model agreed with those of the system to four significant figures after 73 function evaluations. The manner in which the parameters change as well as the reduction of the sum of the squares of the errors are shown in Figure 4-10.

Up to this stage, no emphasis has been placed on the computation time required by the program, which is an important measure of the suitability of the method for on-line applications. The obvious

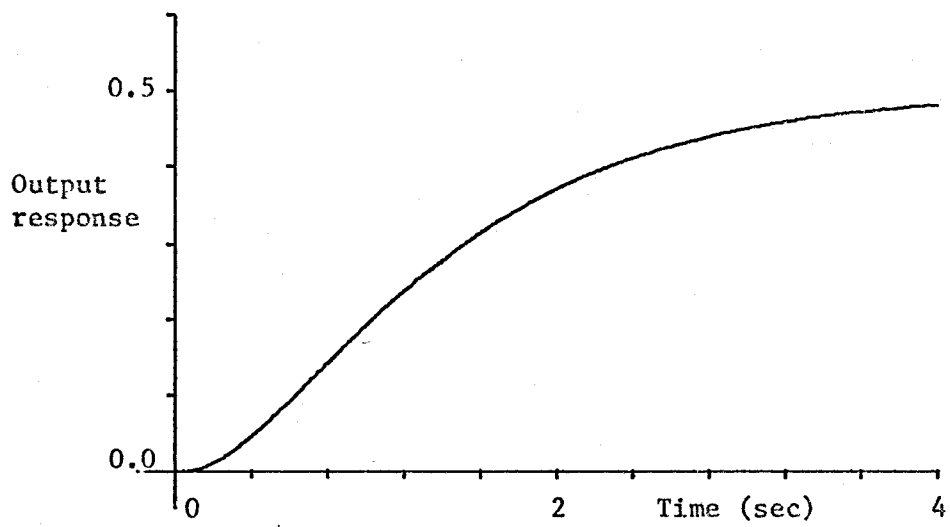


Figure 4-9. Unit step response of a second-order system.

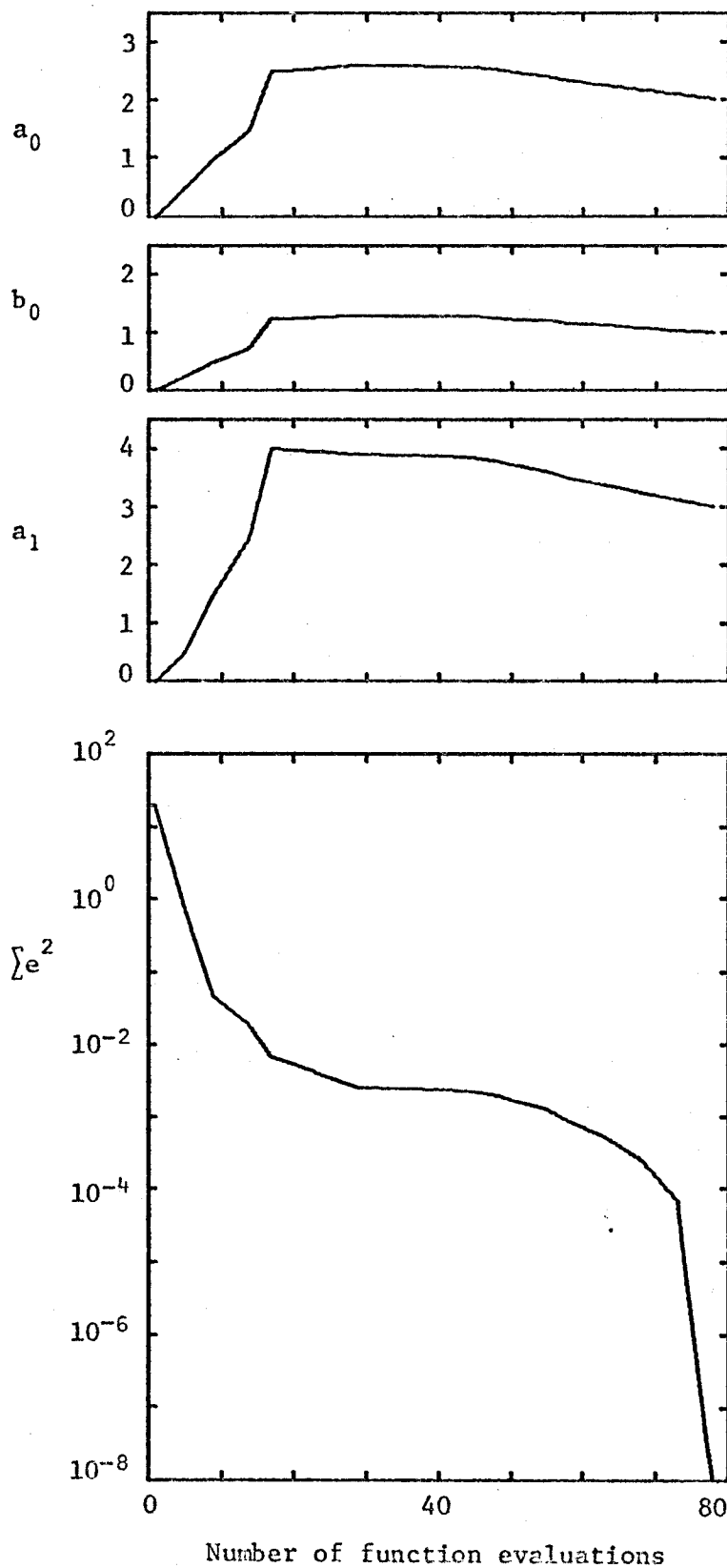


Figure 4-10. Identification of second-order system.

way to improve the efficiency of an identification scheme is to reduce the number of sample points at which the given response and the output of the model are compared. Two cases need to be considered: when the sample points are distributed over the entire response, and when only the initial portion of the response has been observed. The latter is the situation usually encountered in on-line applications.

In many practical cases the steady-state response of the system to a step-input is known. If this is the case, the model to be identified may be assumed to have the form:

$$\frac{\hat{Y}(s)}{U(s)} = \frac{A a_0}{s^2 + a_1 s + a_0}$$

where A is the steady-state gain of the system.

Making the above assumption and applying the Fletcher routine, successively decreasing number of uniformly distributed samples of the response of the second-order system were taken over the 4 second interval. Table 4-6 shows the number of function evaluations required to identify the model parameters to four significant figures.

Number of samples	Number of function evaluations
101	27
51	26
21	38
11	34
6	31
3	37

Table 4-6. Effect of reducing number of samples over a given time interval, on the number of function evaluations.

Note that in the last case in fact only two samples are used, at $t=2$ seconds and $t=4$ seconds respectively, since at $t=0$ the responses are fixed.

The effect of using consecutively smaller initial segment of the response to identify the parameters, is shown in Table 4-7, where the number of sample points used (N), the number of function evaluations (I) and the final model parameters are given.

N	I	a_0	a_1
101	28	3.0000	2.0000
51	21	3.0001	2.0001
21	20	3.0000	2.0000
11	22	3.0005	2.0000
6	26	2.9979	1.9998
3	32	3.0111	2.0036

Table 4-7. Effect of using decreasing number of initial sample points, on the accuracy of identification.

Clearly, after only a few samples of the system have been taken, a reasonably accurate estimate of the system parameters is obtained. These are then improved as more samples are accumulated.

An important extension of the identification scheme presented in this section is the consideration of noise in the observations. Since the proposed method essentially involves optimal smoothing, no undue difficulties are expected. Since in the particular reactor control problem we are considering, a noise-free measure of the output is available, this aspect of the identification problem is not considered further.

CHAPTER 5

SUBOPTIMAL CONTROL OF HIGH-ORDER LINEAR SYSTEMS

In the previous chapter a new method was developed for deriving optimal low-order models of high-order systems. Our interest was focused on approximating the open-loop response of a given system with that of a lower order model. We now turn our attention to using the model to compute the suboptimal feedback control law for the system. The reason why such an approach produces suboptimal performance should be evident from Chapter 2: using the controller computed for the model, not all the state variables of the system are fed back, as would be required for true optimal control.

The advantage of considering the suboptimal control of a linear system prior to tackling the nuclear reactor control problem is, that basic relationships between system and model performance are more readily established. Furthermore, the seventh-order system we have used in the last chapter has been considered by other authors for modelling as well as suboptimal control investigations [44], [45].

5.1 Problem Formulation

Using the notation established in Chapter 2, consider an n th order single input single output linear system

$$\dot{\underline{x}}_n = \underline{A}_n \underline{x}_n + \underline{b}_n u_n \quad (5-1)$$

and an m th order model of this system ($m < n$), derived on the basis of open-loop responses

$$\dot{\underline{x}}_m = A_m \underline{x}_m + b_m u_m \quad (5-2)$$

where the states of the model correspond to the first m states of the system.

For an integral quadratic cost function

$$J_m = \int_0^{\infty} \left(\sum_{i=1}^m q_i x_{mi}^2 + p u_m^2 \right) dt \quad (5-3)$$

the feedback parameters \underline{k}_m that will result in optimal feedback control can be computed, as indicated in Chapter 2. If these same feedback coefficients are used for the control of the system, the performance of the latter will be suboptimal. We are interested in measuring the extent of this suboptimality by evaluating the cost function

$$J_n = \int_0^{\infty} \left(\sum_{i=1}^m q_i x_{ni}^2 + p u_n^2 \right) dt \quad (5-4)$$

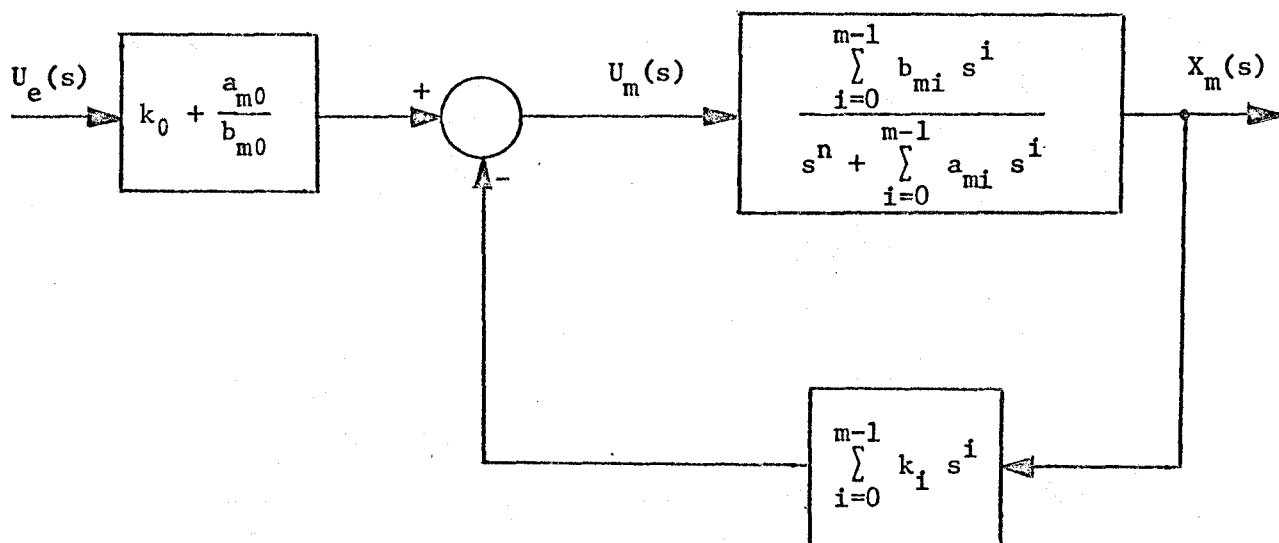
Considering the more general servomechanism problem, the block diagram representation of the optimal model controller and the corresponding suboptimal system controller appears as in Figure 5-1. $U_e(s)$ is the Laplace transform of the externally applied input.

To illustrate the type of responses to be expected using the above scheme, consider the seventh-order system and the second-order $\sum e^2$ model given in Table 4-5:

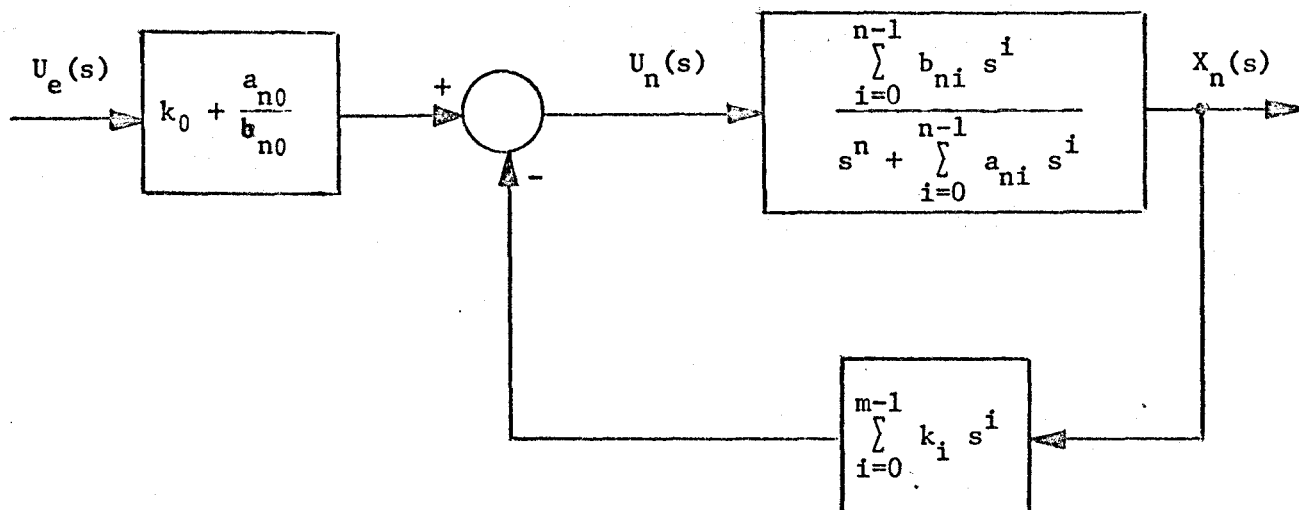
$$\frac{X(s)}{U_e(s)} = \frac{0.3960}{s^2 + 2.6569s + 3.4191}$$

The responses will be evaluated for the following cost functions:

$$J_1 = \int_0^{100} [(x_1 - 1)^2 + 10 x_2^2 + 0.1(u - u^*)^2] dt \quad (5-5)$$



a. Optimal model control



b. Suboptimal system control

Figure 5-1. Block-diagram representation of optimal model controller and suboptimal system controller.

$$J_2 = \int_0^{100} [10(x_1-1)^2 + x_2^2 + 0.1(u-u^*)^2] dt \quad (5-6)$$

(u^* is computed from equation (2-29).)

The weights attached to the terms in the cost function reflect the type of transient response one desires^[4]. A relatively large value associated with the input (u) tends to produce the unforced response of the system. Since this is rarely desired in practice, the corresponding weight is usually the smallest. This being the case, equations (5-5) and (5-6) represent the only other alternatives for a second-order model: attaching a greater weight to either the output or to its first derivative.

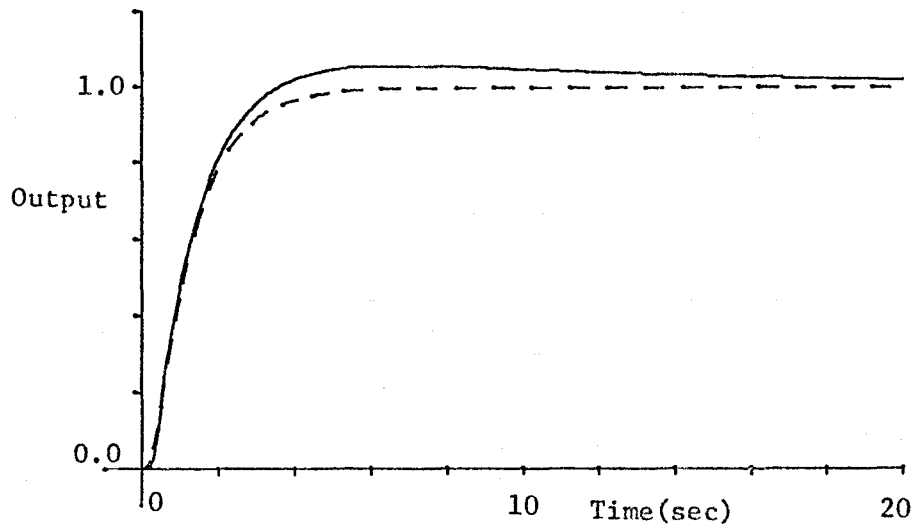
Using equations (2-24) and (2-25), the optimal feedback parameters are readily computed for the model. Their values, as well as the optimal model cost and suboptimal system costs are given in Table 5-1.

Cost function	k_0	k_1	Model cost	System cost
J_1	0.561	5.450	5.388	6.247
J_2	4.578	2.130	5.885	6.362

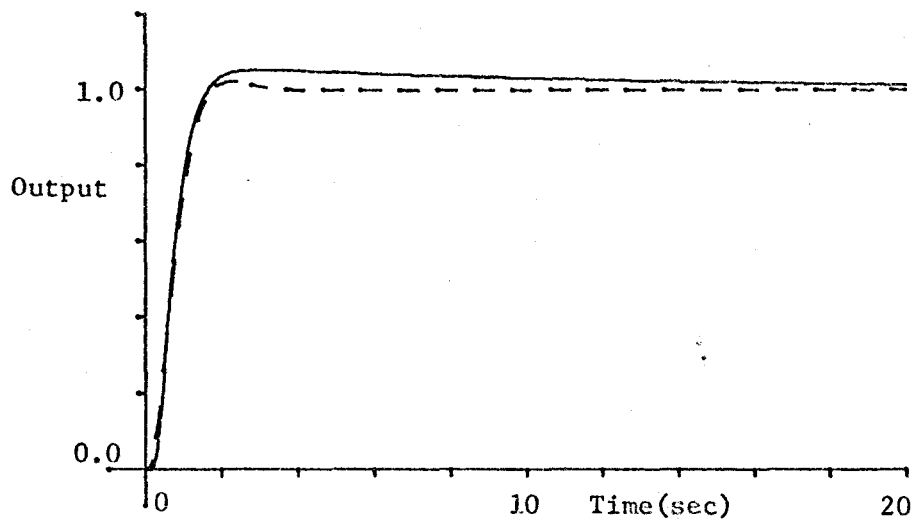
Table 5-1. Suboptimal system and optimal model performances.

The plot of the above optimal and suboptimal responses is shown in Figure 5-2, over a time-interval of 20 seconds. Clearly, the settling time of the system is greater than 20 seconds, hence the computation of the integral to 100 seconds.

Legend: — system response
 ---- model response



$$a. J_1 = \int_0^{100} [(x_1 - 1)^2 + 10x_2^2 + 0.1(u - u^*)^2] dt$$



$$b. J_2 = \int_0^{100} [10(x_1 - 1)^2 + x_2^2 + 0.1(u - u^*)^2] dt$$

Figure 5-2. Suboptimal responses of seventh-order system and optimal responses of second-order model.

It is interesting to note, that the true optimal cost for the seventh-order system has been computed in reference [45] in the case of J_1 , and is given as 6.175. The error in suboptimal cost, as a percentage of the correct value, is 1.17%. This is quite acceptable for most engineering applications, and is a small price to pay for the convenience of needing only two state variables instead of seven.

5.2 Computational Aspects

Since the main objective in the present work is to use low-order models and suboptimal control in an on-line adaptive configuration, it is highly desirable to reduce the computing effort as much as possible. The model should therefore be derived on the basis of the least number of sample points, it should have as few variable parameters as possible, and be of a form that allows the computation of the controller parameters analytically.

The model that satisfies the last of the above criteria has been described in Chapter 2, and has the form

$$\frac{X(s)}{U(s)} = \frac{b_0}{s^2 + a_1 s + a_0} \quad (5-7)$$

Furthermore, by letting $b_0 = Ka_0$, where K is the steady state gain of the system, the number of variable parameters is reduced to two.

Since in the servomechanism problem the steady state error is usually desired to be zero, the above simplification is a particularly valid one. In fact, it will be shown that models derived on the basis of fixing the steady state gain yield lower suboptimal costs than if this assumption is not made.

The models presented in the previous chapter were derived by considering 501 points. Since much of the computing effort is taken up by evaluating the response of the model at each sample point, it is highly desirable to keep these to a minimum. The effects of decreasing the number of samples, on the model parameters as well as on the feedback parameters and on the closed loop performance, are indicated in Table 5-2. The seventh-order system has been used, and a model of the form

$$\frac{X(s)}{U(s)} = \frac{K a_0}{s^2 + a_1 s + a_0} \quad (5-8)$$

has been computed, by minimizing the sum of the squares of the response errors for a unit step input, over an interval of 20 seconds. Two cost functions in the form of equation (5-4) have been evaluated to $t_f = 100$ seconds, the values of the weighting factors being, respectively, $q_1 = 1$, $q_2 = 10$, $p = 0.1$, and $q_1 = 10$, $q_2 = 1$, $p = 0.1$. In both cases the system is transferred from the state $[1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$ to the origin of state space.

The results indicate that there is no change in system or model cost as the number of samples are reduced from 501 to 101. Furthermore, in reducing the points to 21, while the maximum parameter change is 5.36% and the corresponding error in feedback coefficient is 2.51%, the model cost has a 1.81% deviation, but the system cost has changed by less than 0.02%. Hence, as far as comparing model and system costs, 100 sample points suffice, but if the suboptimal system cost is the subject of our interest, 20 or less sample points will give a good approximation. Since a time-interval of 20 seconds has been considered,

			$\int_0^{100} (x_1^2 + 10x_2^2 + 0.1u^2)dt$				$\int_0^{100} (10x_1^2 + x_2^2 + 0.1u^2)dt$			
N	a_0	a_1	k_0	k_1	Model cost	System cost	k_0	k_1	Model cost	System cost
501	3.2031	2.2857	0.539	5.589	5.678	6.245	4.454	2.311	5.970	6.357
201	3.2031	2.2858	0.539	5.589	5.678	6.245	4.454	2.311	5.970	6.357
101	3.2028	2.2855	0.539	5.590	5.678	6.245	4.454	2.311	5.970	6.357
51	3.1966	2.2818	0.539	5.589	5.678	6.245	4.454	2.314	5.974	6.357
21	3.3748	2.3868	0.539	5.610	5.694	6.246	4.454	2.253	5.866	6.356

Table 5-2. Modelling accuracy in suboptimal control, as a function of the number of samples used over a given time interval.

100 points imply a sampling interval of 0.2 second, while 20 samples give 1.0 second intervals.

5.3 Modelling Criteria for Suboptimal Control

In the previous chapter, attention was given to deriving optimal low-order models for a system on the basis of matching the output response only. For optimal control, however, the derivatives of the output must also be considered. When deriving a low-order model for the purpose of designing an optimal controller for a particular cost function, the relative values attached to the terms in the cost function should be reflected in the modelling criterion.

For a cost function that includes the output and its first derivative

$$J = \int_0^{\infty} (q_1 \dot{x}_1^2 + q_2 \dot{x}_2^2 + p u^2) dt \quad (5-9)$$

the following two modelling criteria appear most appropriate:

$$\Delta J_1 = \int_0^{\infty} [q_1 (\dot{x}_{n1} - \dot{x}_{m1})^2 + q_2 (\dot{x}_{n2} - \dot{x}_{m2})^2] dt \quad (5-10)$$

$$\Delta J_2 = \int_0^{\infty} [q_1 |\dot{x}_{n1}^2 - \dot{x}_{m1}^2| + q_2 |\dot{x}_{n2}^2 - \dot{x}_{m2}^2|] dt \quad (5-11)$$

Note that for open-loop control the term involving the input is not present, since $u_n = u_m$.

To investigate the effect of the modelling criterion on the corresponding suboptimal control, eight different objectives have been used to find optimal models for the seventh-order system. A 10 second interval has been used, with samples taken at every 0.2 second. The second-order model used had no finite zeros, and the effect of assuming

zero steady-state error is also to be investigated.

The following are the eight modelling criteria, along with the short-hand notations used to identify them:

1. sum of the absolute values of the errors $[\sum |e|]$
2. sum of the squares of the errors $[\sum e^2]$
3. the maximum sample error $[M|e_s|]$
4. the maximum perpendicular error $[M|e_p|]$
5. the criterion given by equation (5-10) with $q_1 = 1$ and $q_2 = 10$
 $[\Delta J_1(1,10)]$
6. as in 5., but with $q_1 = 10$ and $q_2 = 1$ $[\Delta J_1(10,1)]$
7. the criterion given by equation (5-11) with $q_1 = 1$ and $q_2 = 10$
 $[\Delta J_2(1,10)]$
8. as in 7., but with $q_1 = 10$ and $q_2 = 1$ $[\Delta J_2(10,1)]$

For each of the models obtained, the optimal model cost and suboptimal system costs have been computed, for transferring the states from the origin to $[1 \ 0 \ \dots \ 0]$. Cost functions as given by equations (5-5) and (5-6) have been evaluated, up to 10 seconds. The notation used to indicate integral quadratic cost functions is: $J = [q_1, q_2, p]_0^{t_f}$.

The results shown in Table 5-3 have been obtained with no steady-state constraint on the model parameters, i.e., a_0 , a_1 and b_0 are all independent variables. For each of the eight criteria, the optimal model parameters are shown, as well as the model and system costs for the cost functions indicated. The results show, that while there is appreciable change in the model parameters and optimal model costs as a function of the modelling criterion, the corresponding changes in suboptimal costs are very slight indeed.

Objective function				$J = [1, 10, 0.1]_0^{10}$		$J = [10, 1, 0.1]_0^{10}$	
	b_0	a_1	a_0	Model cost	System cost	Model cost	System cost
$\sum e $	0.3950	2.7132	3.3541	5.267	6.235	5.946	6.310
$\sum e^2$	0.4162	2.8543	3.5316	5.263	6.235	5.858	6.313
$M e_s $	0.3730	2.5552	3.1691	5.282	6.235	6.043	6.307
$M e_p $	0.3702	2.5071	3.1932	5.369	6.233	6.015	6.305
$\Delta J_1(1,10)$	0.4229	2.7994	3.5872	5.343	6.234	6.597	6.311
$\Delta J_1(10,1)$	0.4180	2.8563	3.5468	5.271	6.235	5.847	6.313
$\Delta J_2(1,10)$	0.4869	2.9577	4.1375	5.537	6.236	5.491	6.313
$\Delta J_2(10,1)$	0.3551	2.5023	3.0187	5.224	6.236	6.160	6.307

Table 5-3. Performance of optimal second-order models in the suboptimal control of seventh-order system.

To test the effect of assuming $b_0 = Ka_0$ where K is the steady-state gain of the system, the same eight criteria were used to obtain new models for the seventh-order system. The model and system costs are shown in Table 5-4. Once again, there is little difference between the various suboptimal costs. Comparison with the entries in Table 5-3 indicates that making the steady-state assumption ($b_0 = Aa_0$) results in usually lower suboptimal cost, or at least no worse than if b_0 is an independent variable. Keeping in mind that if b_0 is linearly dependent on a_0 , one less variable needs to be considered, this assumption is highly desirable for deriving low-order models for the suboptimal control of high-order systems. The model that will be used for suboptimal, and subsequently adaptive control, has the form

$$\frac{X(s)}{U(s)} = \frac{Ka_0}{s^2 + a_1s + a_0} \quad (5-12)$$

The advantage of using a model which emphasises the matching of the system response at and near steady state for the purpose of suboptimal control is not surprising, since the feedback parameters are computed on the assumption that the cost function is evaluated to $t = \infty$, i.e., until steady-state is reached.

It should be noted that the response of the seventh-order system approaches its final value to within five significant figures only in the neighbourhood of 100 seconds. On the other hand, the closed-loop model responses settle down within 10 seconds. Several of the cost functions given in Tables 5-3 and 5-4 were evaluated to 100 seconds, but no relative changes in the costs were observed.

Objective function				$J = [1, 10, 0.1]_0^{10}$		$J = [10, 1, 0.1]_0^{10}$	
	b_0	a_1	a_0	Model cost	System cost	Model cost	System cost
$\sum e $	0.3599	2.2804	3.2391	5.711	6.232	5.941	6.300
$\sum e^2$	0.3559	2.2856	3.2028	5.678	6.232	5.970	6.300
$M e_s $	0.3560	2.6172	3.2042	5.340	6.235	6.080	6.302
$M e_p $	0.3730	2.7792	3.3568	5.300	6.237	6.015	6.305
$\Delta J_1(1,10)$	0.4012	2.6449	3.6111	5.600	6.231	5.769	6.301
$\Delta J_1(10,1)$	0.3683	2.3712	3.3143	5.667	6.232	5.906	6.300
$\Delta J_2(1,10)$	0.4528	2.6786	4.0754	5.857	6.236	5.495	6.301
$\Delta J_2(10,1)$	0.3758	2.3975	3.3826	5.689	6.232	5.863	6.300

Table 5-4. Performance of optimal second-order models with fixed d-c gain ($b_0 = Ka_0$), in the suboptimal control of seventh-order system.

Since the $\int e^2$ objective has computational advantages over the other criteria for deriving low-order models, as well as being the one most susceptible to analytical techniques, it will be the one to be considered in the remainder of this thesis. For the seventh-order system, this model is

$$\frac{X(s)}{U(s)} = \frac{0.3559}{s^2 + 2.2856s + 3.2028} \quad (5.13)$$

as given by the first row of Table 5-4. The optimal feedback parameters for this model, and for the two cost functions we have been considering, are given below:

	k_0	k_1
$J = [1, 10, 0.1]_0^\infty$	0.539	5.590
$J = [10, 1, 0.1]_0^\infty$	4.454	2.311

The responses of the system and the model in both the open-loop and the closed-loop configuration are shown in Figure 5-3. It is interesting to note that the closed-loop responses are closer than the open-loop ones. The extent of the reduction of the error between the responses may be seen from Table 5-5.

Mode of Control	$\int e^2$ (0-10 sec)
Open-loop	0.5748
Closed-loop $J = [1, 10, 0.1]_0^\infty$	0.3898
Closed-loop $J = [10, 1, 0.1]_0^\infty$	0.2693

Table 5-5. Comparison of errors for open- and closed-loop responses of system and model.

Legend: — system response

--- model response

103.

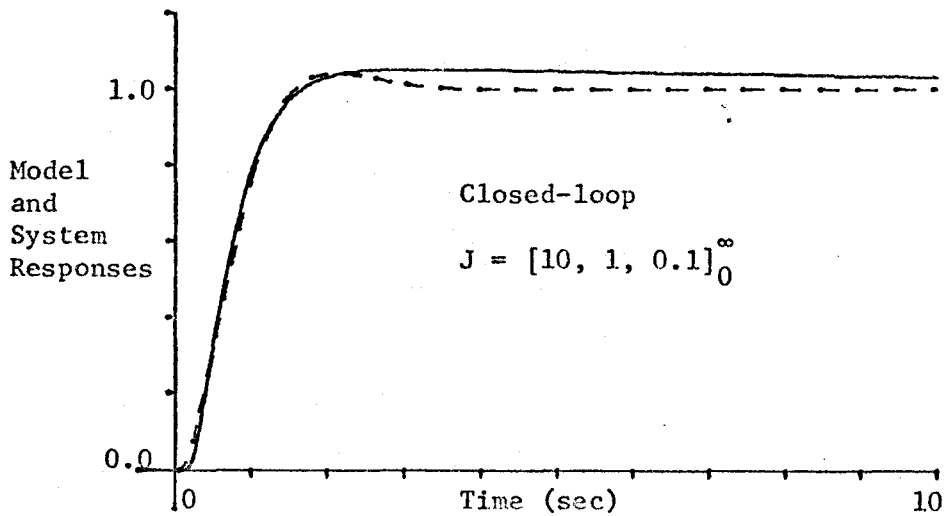
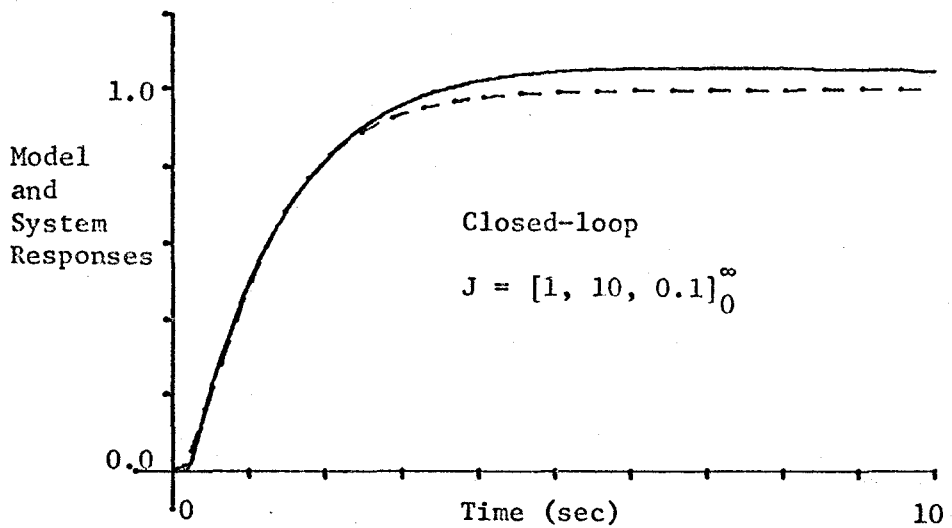
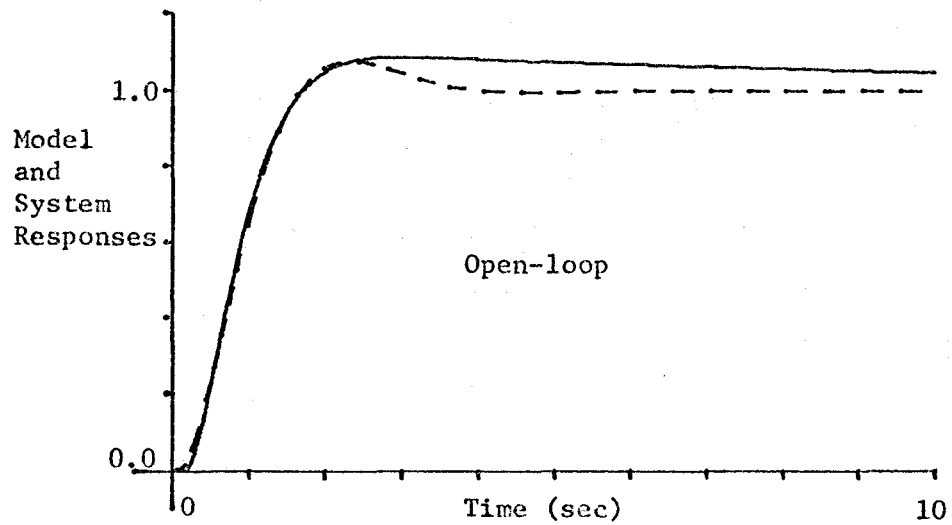


Figure 5-3. Comparison of open- and closed-loop responses of second-order model and seventh-order system.

This reduction of the modelling error in going from the open-loop to the closed-loop responses has been observed for every model that was derived on the basis of the steady-state constraint to a step input. It indicates that if a model of sufficient accuracy can be found to match the open-loop response of the system, the error between the corresponding optimal model output and suboptimal system response will be no greater than in the open-loop case.

5.4 Optimal Control for Least-pth and Minimax Cost Functions

Up to the present stage in this thesis, optimal control has been considered only for the case of integral quadratic cost functions. In Chapter 4, optimal models were derived for not only quadratic objectives, but for the more general least pth and minimax error criteria also. In this section we consider the possibility of synthesizing optimal feedback controllers for the often desirable least pth and minimax objectives.

The attractive feature of the quadratic cost function is, that when the differentiation indicated by equations (2-4) and (2-5) is performed, a linear differential equation results. Analytical solution in the general case is not known to exist, although for certain specific nonquadratic cost functions the optimal control law has been found^{[17], [46]}.

In this section it is shown that the application of search methods will yield the optimal feedback parameters for least pth and minimax cost functions. The only restriction is that the system to be controlled cannot have finite zeros.

The cost functions, for which the optimal controller is to be

found, may take one of the following forms:

$$J_I = \int_0^T \left[\sum_{i=1}^n q_i |x_i - x_i^*|^p + \sum_{k=1}^m r_k |u_k - u_k^*|^p \right] dt \quad (5-14)$$

$$J_S = \sum_{j=1}^N \left[\sum_{i=1}^n q_i |x_{ij} - x_{ij}^*|^p + \sum_{k=1}^m r_k |u_{kj} - u_{kj}^*|^p \right] \quad (5-15)$$

$$J_M = \max_{j=1, N} \left\{ \sum_{i=1}^n q_i |x_{ij} - x_{ij}^*| + \sum_{k=1}^m r_k |u_{kj} - u_{kj}^*| \right\} \quad (5-16)$$

where q_i and r_k are non-negative weights, p a positive number, usually an integer, T the time interval of interest that contains N sample points, x_i is the i th component of the state vector and u_k the $(k-1)$ th derivative of the input; the superscript $*$ denotes desired functions of the appropriate variables, and the subscript j refers to the j th sample of the variable in question.

Note that the previously considered integral quadratic cost function is a special case of equation (5-14), with $p=2$. An important practical feature of the above cost functions is the inclusion of the derivatives of the input. In many applications, the velocity and acceleration of the input must be limited for physical reasons. These are readily dealt with using the above formulation.

The philosophy of the proposed method evolved from the modelling work presented in Chapter 4. There we considered approximating the open-loop unforced response of the system by using rational functions. The question arises: can the optimal response be also approximated in this fashion, and is the result realizable in a feedback fashion? The affirmative answer is presented below.

5.4.1 Frequency-Domain Derivation

Let the system to be controlled have the transfer function

$$\frac{X(s)}{U(s)} = \frac{b_0}{s^n + \sum_{i=0}^{n-1} a_i s^i} \quad (5-17)$$

where b_0 and a_i are constants, and n is the order of the model.

Assume that the desired optimum output trajectory may be approximated by the following rational function:

$$\hat{X}(s) = \frac{g_0 U_e(s)}{s^n + \sum_{i=0}^{n-1} h_i s^i} \quad (5-18)$$

where $U_e(s)$ is the externally applied driving function, and g_0 and h_i are the constant parameters that are to be determined by a search routine such that the appropriate cost function is minimized. A suitable starting point is obtained by a direct substitution of $g_0=b_0$ and $h_i=a_i$, $i=0,1,\dots,n-1$ from equation (5-17).

The corresponding input to the model found by combining equations (5-17) and (5-18):

$$\hat{U}(s) = \frac{g_0 [s^n + \sum_{i=0}^{n-1} a_i s^i] U_e(s)}{b_0 [s^n + \sum_{i=0}^{n-1} h_i s^i]} \quad (5-19)$$

Since the time functions that correspond to equations (5-18) and (5-19) are differentiable with respect to time, all the desired terms in a cost function of the form of equations (5-14) to (5-16) are known. It can therefore be minimized by a search routine that systematically varies the parameters g_0 and h_i .

The solution of the problem is open-loop at this stage, as illustrated in Figure 5-4a. What has evidently been done is to cancel the poles of the model, and to introduce new ones, such that the resultant system behaviour is optimum. The disadvantage of this arrangement is well-known: it is rarely possible to exactly cancel the poles of the model, and changes in the model parameters directly affect the overall performance. The solution which is usually desired is a closed-loop one, as shown in Figure 5-4b, and our task is to express the coefficients ℓ_0 and k_i ($i=0,1,\dots,n-1$), in terms of the constants found by the search routine and the parameters of the system. This is readily accomplished, as shown below.

Referring to Figure 5-4b, we can write

$$\hat{U}(s) = \ell_0 U_e(s) - \sum_{i=0}^{n-1} k_i s^i \hat{X}(s) \quad (5-20)$$

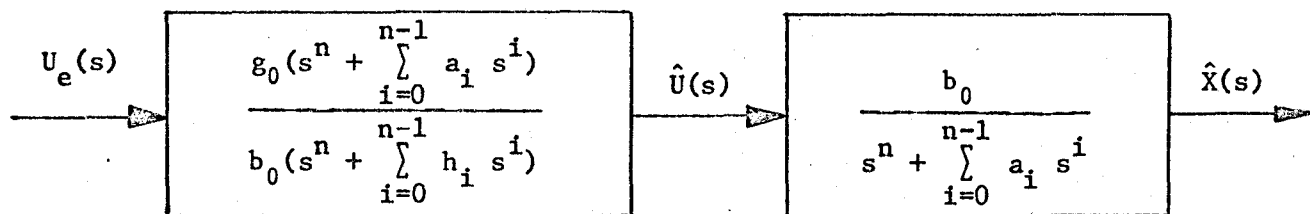
Substituting for $\hat{X}(s)$ from equation (5-18):

$$\hat{U}(s) = \frac{[\ell_0(s^n + \sum_{i=0}^{n-1} h_i s^i) - g_0 \sum_{i=0}^{n-1} k_i s^i] U_e(s)}{s^n + \sum_{i=0}^{n-1} h_i s^i} \quad (5-21)$$

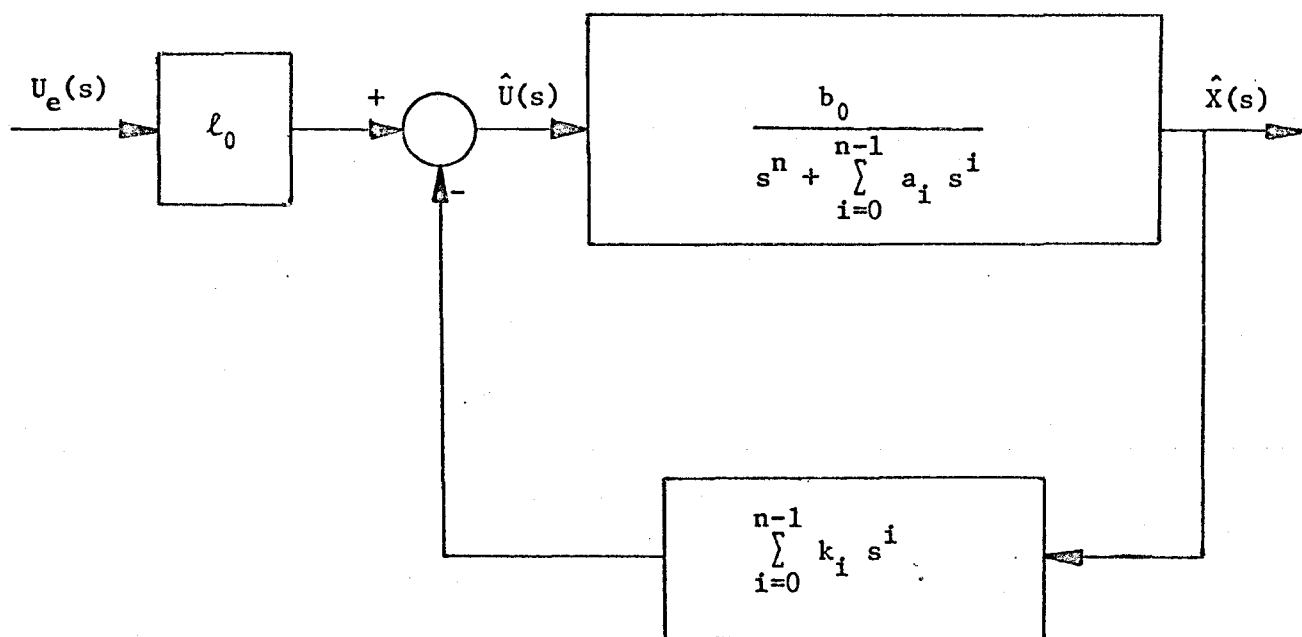
Equating now coefficients in the two expressions for $\hat{U}(s)$, equations (5-19) and (5-21), we obtain the desired formulae for the feedback coefficients:

$$k_i = \frac{h_i - a_i}{b_0} \quad i = 0, 1, \dots, n-1 \quad (5-22)$$

$$\ell_0 = \frac{g_0}{b_0}$$



a. Open-loop realization



b. Closed-loop realization

Figure 5-4. Block-diagram representation of optimum systems.

5.4.2 Time-Domain Derivation

It is instructive, from the point of view of appreciating the proposed method, to repeat the previous derivation in the time-domain. Accordingly, let the system be represented by

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}u \quad (5-23)$$

and it is desired to find the feedback control law

$$u = -\underline{k}^T \underline{x} \quad (5-24)$$

where, for the sake of simplicity, the regulator problem is considered.

Substituting for u in equation (5-23) from equation (5-24) we obtain

$$\dot{\underline{x}} = (\underline{A} - \underline{b} \underline{k}^T) \underline{x} \quad (5-25)$$

In order to find the value of \underline{k} such that the appropriate cost function in the form of equations (5-14) to (5-16) is minimized, the use of a search routine is proposed.

We assume an approximation to equation (5-25) of the following form:

$$\dot{\underline{x}} = \underline{H} \hat{\underline{x}}(t) \quad (5-26)$$

and initially put $\underline{H}=\underline{A}$, which corresponds to the unforced case. By varying the parameters of \underline{H} using a suitable search routine, the desired cost function is minimized.

The least number of parameters will need to be determined if both \underline{H} and \underline{A} are in phase-variable form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \quad H = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -h_0 & -h_1 & -h_2 & \dots & -h_{n-1} \end{bmatrix} \quad (5-27)$$

Since it is also assumed that the system has no finite zeros, the control vector may be written as $\underline{b} = [0 \ 0 \ 0 \ \dots \ b_0]^T$.

At each iteration, it is necessary to solve equation (5-26). Because of the phase-variable notation a closed form solution is readily obtained, thus avoiding the necessity of numerically solving n simultaneous differential equations. The input and its derivatives must also be known at each iteration. Making use of the originally specified model equations, we can substitute into equation (5-23) the approximating relationship of equation (5-26) to give:

$$H \hat{\underline{x}}(t) = A \hat{\underline{x}}(t) + \underline{b} \hat{u}(t)$$

Rearranging, we obtain

$$\underline{b} \hat{u}(t) = -(A - H) \hat{\underline{x}}(t)$$

which on evaluation reduces to

$$\hat{u}(t) = -\underline{k}^T \hat{\underline{x}} \quad (5-27)$$

where

$$k_i = \frac{h_i - a_i}{b_0} \quad i = 0, n-1 \quad (5-28)$$

The values obtained for k_i are, as expected, the same in equations (5-22) and (5-28). Since they are constants, the realization of the controller is readily achieved. It should also be noted that the above results are consistent with modern control theory: optimum

performance is accomplished by feeding back all the phase variables.

5.5 Optimal Control with Zero Steady-State Error

It was shown in section 2.3 how the regulator problem may be converted into the servomechanism problem, by requiring that the steady-state error between a step input and the response of the system be zero. For a system having the transfer function

$$\frac{X(s)}{U(s)} = \frac{b_0}{s^n + \sum_{i=0}^{n-1} a_i s^i} \quad (5-29)$$

the desired result was achieved by the addition of a feedforward block of gain ℓ_0 . The value of ℓ_0 was found to be

$$\ell_0 = \frac{a_0}{b_0} + k_0 \quad (5-30)$$

In section 5.4.1, the same requirements led to a value of

$$\ell_0 = \frac{g_0}{b_0} \quad (5.31)$$

It is desirable to establish the equivalence of equations (5-30) and (5-31), and to seek the conditions under which zero steady-state errors to ramp and higher-order inputs may also be realized.

Using equation (5-22),

$$k_0 = \frac{h_0 - a_0}{b_0}$$

and substituting this value of k_0 in equation (5-30) gives

$$\ell_0 = \frac{h_0}{b_0}$$

Since the requirement is that the d-c gain of the closed-loop system is unity, $h_0 = g_0$, as desired.

The technique used in section 2.3 to convert the regulator problem into the servomechanism by requiring that the steady-state value of the output equals that of the input when the latter is a step, can be extended to inputs which are higher order functions of time. For example, to have zero steady-state error to a ramp input, the feedforward block becomes $\ell_0 + \ell_1 s$. In general, for an input of

$$u(t) = t^m \quad (5-32)$$

the feedforward block takes the form

$$H(s) = \sum_{i=0}^m \ell_i s^i \quad (5-33)$$

The block diagram representation of an n th order system with all phase-variables fed back and a feedforward block of the form of equation (5-33), is shown in Figure 5-5. The coefficients ℓ_i , $i=0, m-1$ are to be determined such that the steady-state error, between the output of the closed-loop system and inputs in the form of equation (5-32) is equal to zero.

Referring to Figure 5-5,

$$X(s) = \frac{b_0}{s^n + \sum_{i=0}^{n-1} a_i s^i} E(s) \quad (5-34)$$

$$E(s) = \sum_{i=0}^m \ell_i s^i U(s) - \sum_{i=0}^{n-1} k_i s^i X(s) \quad (5-35)$$

Substituting in equation (5-34) for $E(s)$ and separating the variables, leads to

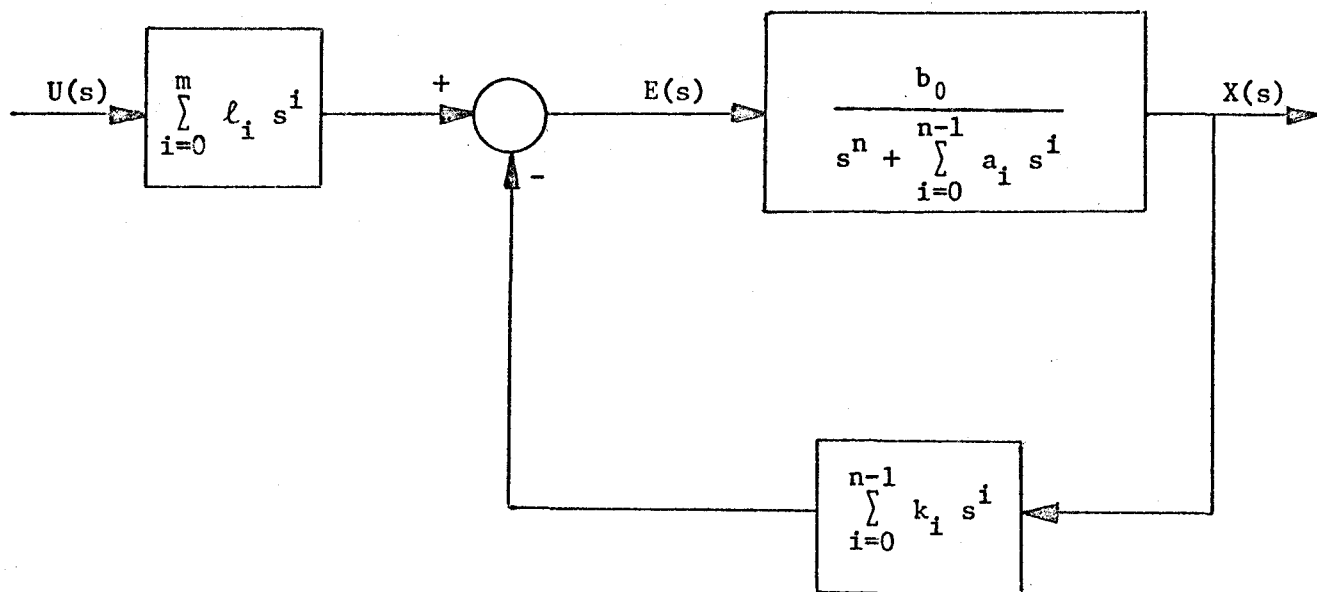


Figure 5-5. Block-diagram representation of closed-loop system for realization of zero steady-state error.

$$X(s) = \frac{b_0 \sum_{i=0}^m \ell_i s^i}{s^n + \sum_{i=0}^{n-1} (a_i + b_0 k_i) s^i} U(s) \quad (5-36)$$

Equation (5-36) is the open-loop representation of the closed-loop system in Figure 5-5.

For an input as given by equation (5-32)

$$U(s) = \frac{m!}{s^{m+1}}$$

Using the final value theorem, we require

$$\lim_{s \rightarrow 0} \{s[X(s) - U(s)]\} = 0$$

i.e.,

$$\lim_{s \rightarrow 0} \left\{ \frac{m!}{s^m} \left[\frac{-s^n - \sum_{i=m+1}^{n-1} (a_i + b_0 k_i) s^i + \sum_{i=0}^m (b_0 \ell_i - a_i - b_0 k_i) s^i}{s^n + \sum_{i=0}^{n-1} (a_i + b_0 k_i) s^i} \right] \right\} = 0$$

This will be satisfied if

$$b_0 \ell_i - a_i - b_0 k_i = 0$$

i.e.,

$$\ell_i = \frac{a_i}{b_0} + k_i \quad \begin{matrix} i = 0, m-1 \\ m \leq n-1 \end{matrix} \quad (5-37)$$

Checking our result, for $i=0$

$$\ell_0 = \frac{a_0}{b_0} + k_0$$

as already established by equation (2-28)

Basically, equation (5-37) can be applied in two configurations.

Either the response is optimized on the basis of feedback alone, and

the appropriate feedforward coefficients added when the loop is closed, or they are included in the open-loop optimization stage. The former approach tends to reduce the steady-state error, while the latter will produce a smaller value of cost.

One other alternative is possible, which is to vary the ℓ_i coefficients also at the stage of the open-loop optimization. It can be readily shown by the procedure used in this section, that the feedforward coefficients remain invariant under the transformation from open- to closed-loop configuration provided $g_0 = b_0$.

5.6 Results for Least-pth and Minimax Cost Functions

To illustrate the method proposed in the last two sections, three examples are now considered, using each type of cost function specified in equations (5-14) to (5-16). The task is to compute the suboptimal controller for a high-order system, based on the optimal controller computed for a low-order model of the system. The seventh-order system and the second-order $\int e^2$ model with steady-state constraint, given in equation (5-13) are considered, and pattern-search used to find the optimum.

Example 5.1: Transfer the system from the state

$$[1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

to the origin of state space so that the cost function

$$J_I = \int_0^{10} (x_1^2 + x_2^2 + 0.1u^2) dt$$

is minimized. For this special case of an integral quadratic cost

function, the optimal feedback control may be obtained analytically for the model, as discussed in Chapter 2. This example, therefore, provides a useful check on the proposed method. The feedback coefficients obtained by the two techniques and the costs incurred by using these controller parameters for both the model and the system are shown in Table 5-6.

	k_0	k_1	J_m	J_s
Analytical solution	0.53944	0.94505	1.24788	1.34008
Proposed solution	0.53936	0.94498	1.24788	1.34008

Table 5-6. Results for example 5.1.

Clearly, the results obtained by the proposed method are accurate for all practical purposes. To obtain the desired optimal response, a minimum step-size of 0.00001 was used, and convergence occurred after 220 function evaluations.

Example 5.2: Transfer the system from the origin of state space to

$$[1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

so that the cost function

$$J_s = \sum_{i=0}^{100} (|x_{1i} - 1| + 2|x_{2i}| + 0.1|u_i - u_i^*|)$$

is minimized over the time interval $[0,10]$ seconds. $[u^*$ is given by equation (2-29).] The results after 61 function evaluations and a minimum step-size of 0.0001, are as follows:

$$k_0 = 5.845$$

$$k_1 = 4.320$$

$$\ell_0 = 14.844$$

$$\text{model cost} = 30.25$$

$$\text{system cost} = 35.60$$

Both the system and the model responses have been plotted in Figure 5-6. The agreement between optimal model response and suboptimal system performance should be acceptable for most practical applications.

Example 5.3: Transfer the system from the origin of state space to

$$[1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

so that the cost function

$$J_M = \max_{i=0,200} \{5|x_{1i} - x_{1i}^*| + |x_{2i} - x_{2i}^*| + 0.1|u_i - u_i^*|\}$$

is minimized over the time interval $[0,10]$ seconds, where

$$\begin{aligned} x_1^* &= 0.2t & x_2^* &= 0.2 & \text{for } 0 \leq t < 5 \text{ sec} \\ x_1^* &= 1.0 & x_2^* &= 0.0 & \text{for } 5 \leq t \leq 10 \text{ sec} \end{aligned}$$

The final value of the input, u_1^* is once again computed from equation (2-29), such that

$$u^*(t) = \frac{a_0}{b_0} x_1^*(t)$$

The externally applied input is given by

$$\underline{u}_e(t) = \underline{x}^*(t)$$

This problem has been solved by considering four different possibilities:

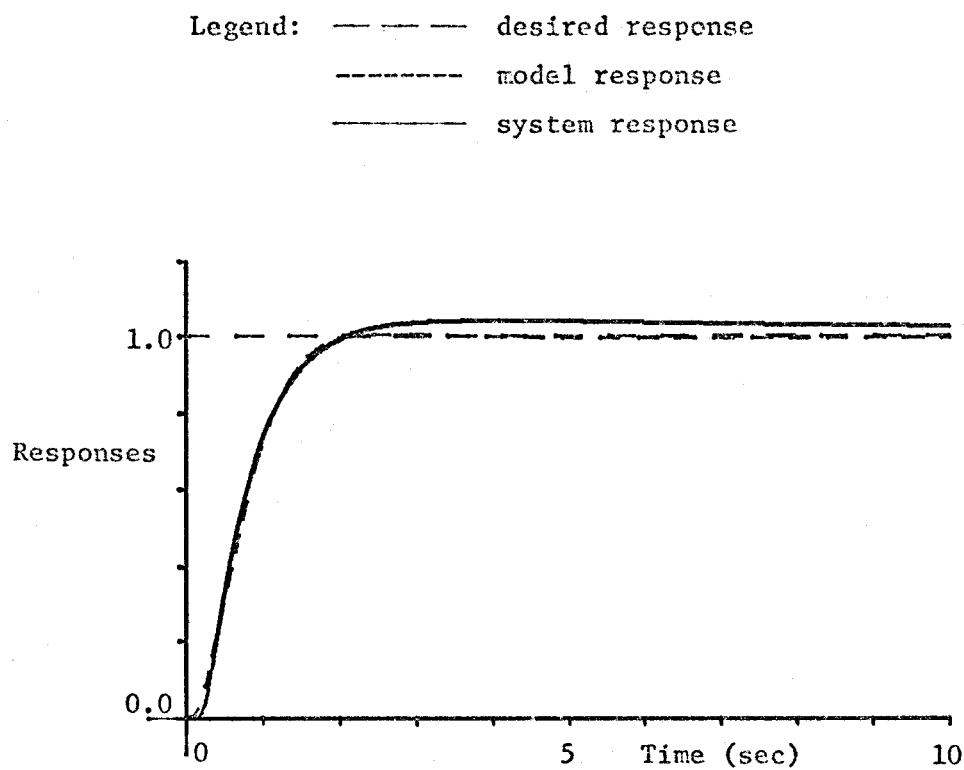


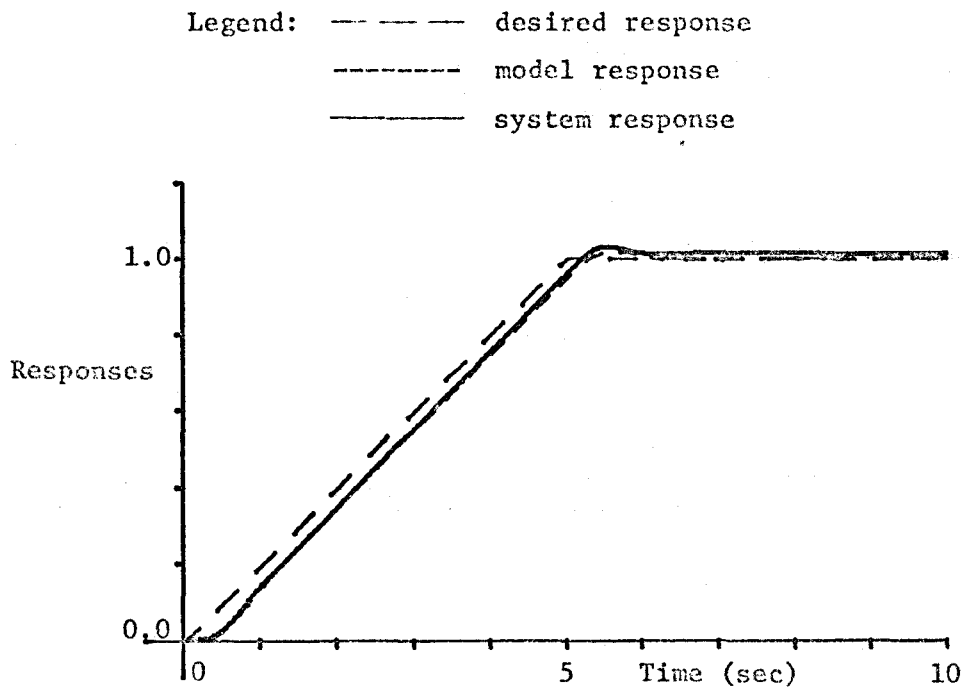
Figure 5-6. Responses for example 5.2.

- a. The open-loop parameters h_0 and h_1 are varied to minimize the cost function, assuming that $g_0 = h_0$. The closed-loop parameters are then computed from equation (5-22).
- b. The same procedure as above is used, but ℓ_1 , as given by equation (5-37), is also included in the closed-loop realization.
- c. The feedforward parameters ℓ_0 and ℓ_1 are evaluated at each stage of the search process [equation (5-37) is applicable]. h_0 and h_1 are again the only variables, and $g_0 = b_0$.
- d. In addition to h_0 and h_1 , ℓ_1 is also varied by the search routine. Once again, $g_0 = b_0$, in order to make the feedforward parameters invariant in going from the open- to the closed-loop configuration.

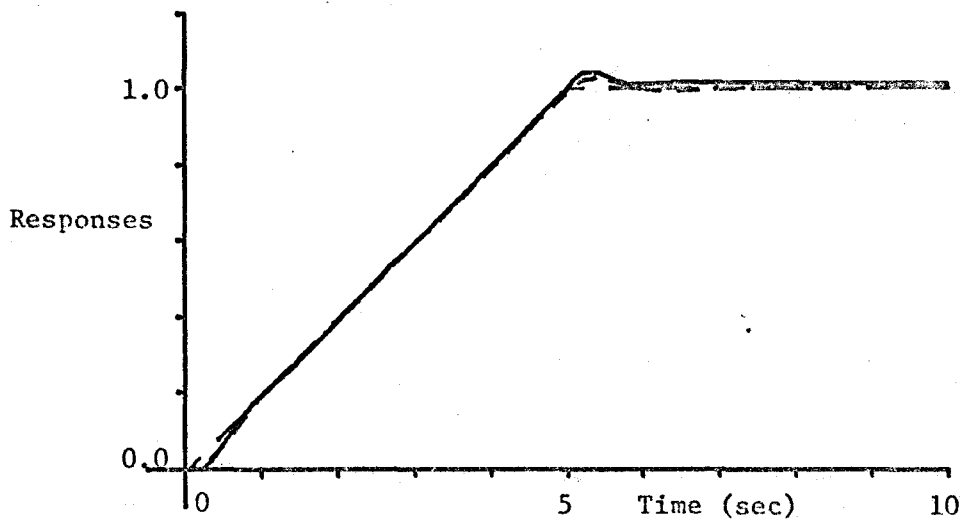
The results obtained for the above four cases are displaced in Table 5-7. The corresponding responses are plotted in Figures 5-7 and 5-8.

Case	k_0	k_1	ℓ_0	ℓ_1	Model cost	System cost
a	32.10	2.963	41.10	0.0	0.524	0.569
b	32.10	2.963	41.10	9.385	0.428	0.538
c	0.0	3.067	9.000	9.489	0.393	0.526
d	0.0	3.633	9.000	9.157	0.388	0.456

Table 5-7. Comparison of results obtained for the four cases of example 5.3.

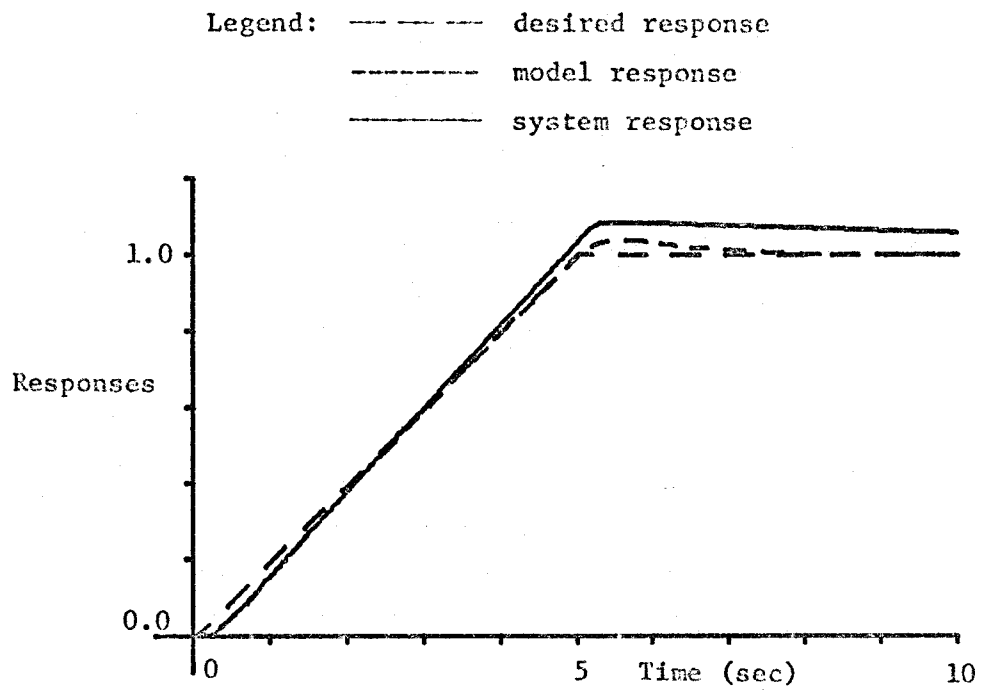


Case (a).

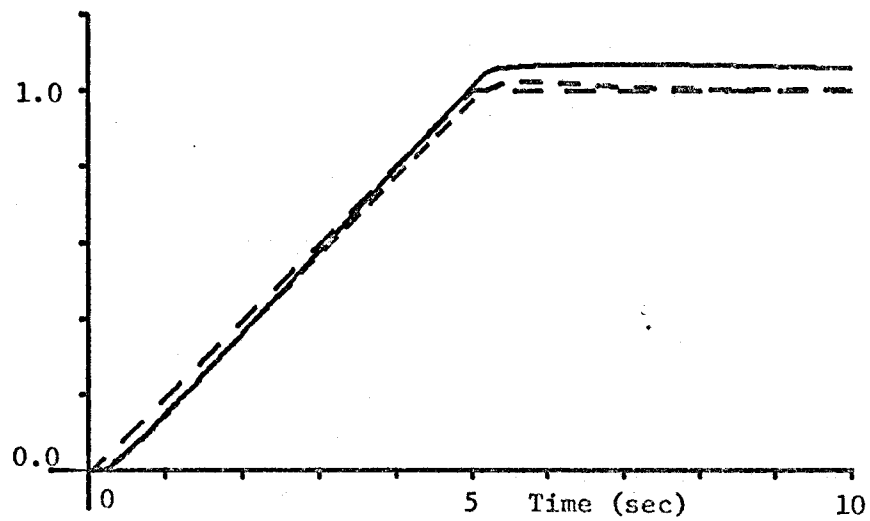


Case (b).

Figure 5-7. Responses for example 5.3,
cases (a) and (b).



Case (c).



Case (d).

Figure 5-8. Responses for example 5.3,
cases (c) and (d).

Both the model and the system costs decrease as one progresses from case (a) to case (d), as expected. Inspection of the diagrams, on the other hand, suggests, that case (b) is the most desirable one. This example very well illustrates the care one must take in selecting the cost function and the weighting coefficients to realize a desired transient response.

CHAPTER 6

SUBOPTIMAL CONTROL OF NUCLEAR REACTOR - RESULTS

Having developed a method for finding optimal low-order models for high-order systems, and having demonstrated the use of such models in the suboptimal control of a linear system, we turn our attention to the nuclear reactor control problem. Since it is impractical to feed back all the state variables of this system, true optimal control will not be attempted. Knowing that the actual plant is successfully operating by feeding back only the output and its derivative, we consider the suboptimal control of the nuclear reactor on the basis of these two signals. To find the feedback parameters, an optimal second-order model for the system is derived, using the method proposed in Chapter 4. If the performance index is quadratic, the optimal controller for the model is computed from equations (2-24) and (2-25). For other cost functions, the technique presented in sections 5.4 and 5.5, is used.

Since the system is nonlinear and has time-varying parameters, it is expected that a new model will have to be found every time there is a change in the operating power level, depending on the magnitude and direction of the demanded power excursion, and as the system parameters vary. The extent of the necessary changes in model parameters are investigated in this chapter, in order to establish the need for the on-line updating of the model.

The nonlinear nature of the system is also expected to affect the agreement between optimal model and suboptimal system performance. Since initially the model is derived on the basis of unity feedback, once the suboptimal controller is applied, the model may no longer reflect accurately the behaviour of the system, and its parameters will have to be modified.

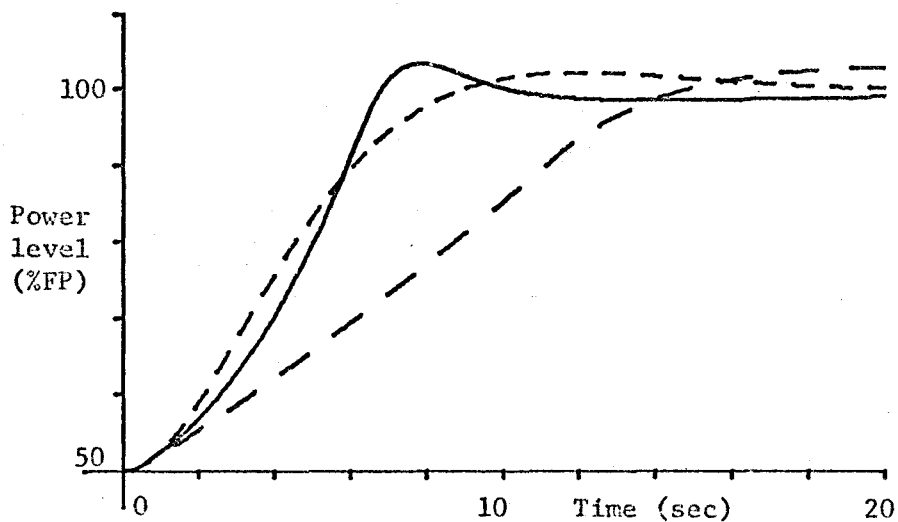
6.1 Optimal Second-order Linear Models

To find the desired optimal models for the nuclear reactor and associated reactivity controller mechanism, the system considered in section 3.3 and depicted in block diagram form in Figure 3-5 is used. It is assumed, that the response of the system to a step change in demanded power has been observed over a suitable time-interval. For the models to be derived in this chapter, the d-c gain is assumed to be the same as that of the system, viz. unity, and the Fletcher method is used to minimize the sum of the squares of the deviations between system and model responses, over 101 sample points.

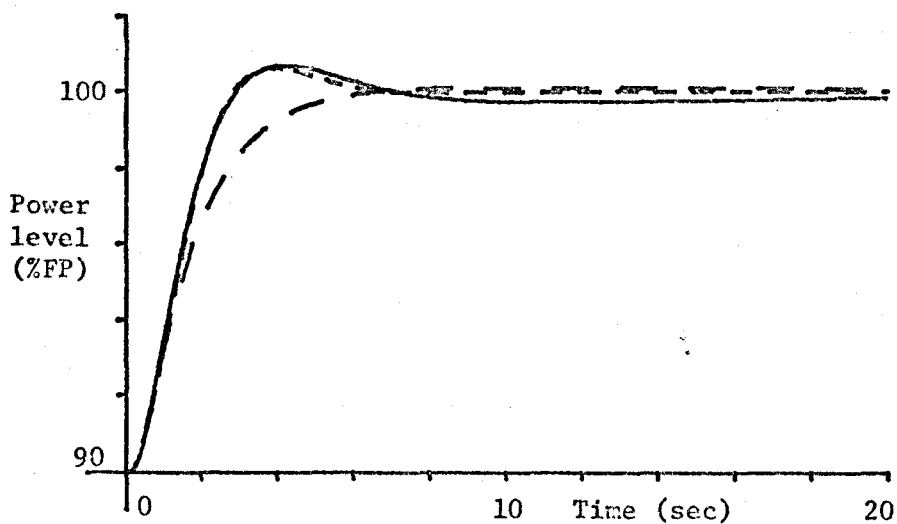
In order to relate the accuracy of the proposed technique of system modelling to the previously discussed classical reactor models, reference must be made to the responses presented in section 3.3. Since a low-order linear model is desired, the linearized one delayed neutron group model has been chosen for comparison. For the same power level change as in Figure 3-7 (50%-100% FP), the responses of the optimal second-order model and of the one delayed neutron group model (which becomes fourth-order with the addition of temperature feedback and absorber rod drive motor), are shown in Figure 6-1a.

Legend:

- reference system
- - - optimal second-order model
- - - one delayed neutron group model



a. Response for 50% - 100% FP.



b. Response for 90% - 100% FP.

Figure 6-1. Comparison of responses of one delayed neutron group and optimal second-order model to that of the reference system.

The considerable improvement in accuracy using the optimal model, despite the lower order, is evident. That even better approximation of the system is possible for smaller power level changes, responses of the system and of the two models for a power level change from 90% to 100% FP are shown in Figure 6-1b. The parameters of the optimal model [as defined in equation (5-8)], as well as the sum of the squares of the errors, are given in Table 6-1.

Response	a_1	a_0	$\sum e^2$
50%-100%	0.5239	0.1392	2.193×10^{-1}
90%-100%	1.4471	1.2098	8.291×10^{-4}

Table 6-1. Optimal model parameters for
given reactor responses.

As expected, the model parameters change considerably from one response to another. It is necessary, therefore, to investigate the variations in model parameters as a function of power level changes and as the reactor parameters vary, before the models are used for the suboptimal control of the nuclear reactor.

6.1.1 Effect of Nonlinear Reactor Kinetics

The nonlinear nature of the reactor kinetic equations has already been discussed in Chapter 3. Our present interest is to observe the form and extent of the changes in the optimal model parameters as a function of the operating power level of the reactor, and of the magnitude and direction of the demanded change in power. In particular, we seek to find a single model, or at least a well

defined set of model parameters, in terms of the observable characteristics of the plant, in order to make the design of a fixed or at least a preprogrammed suboptimal controller possible.

Figure 6-2 illustrates the effect of the initial power level on the model parameters. From various operating points, a 20% increase was demanded. The wide range of variations of a_1 and a_0 are evident from the diagram. A decrease of 20% in power from the same initial point calls for a different set of model parameters, as shown in Figure 6-3. Although the curve for a_1 has not changed appreciably, the one for a_0 is down by about 0.2 from the corresponding one in Figure 6-2. Such a change, however, could be accounted for in a preprogrammed controller algorithm.

Turning our attention to the effect of different magnitudes of power changes from a given initial level, rather different forms of model parameter variations are observed. For increases of various magnitudes from an initial power level of 50% FP, Figure 6-4 shows the changes in model parameters. For decreases from 100%, two results have been plotted in Figure 6-5. To establish a functional relationship between these sets of curves and the ones shown in Figures 6-2 and 6-3, appears to be very difficult, particularly when one takes into account the several remaining combinations of initial power levels, as well as magnitudes and directions of demanded changes. The problem is further complicated by changes in plant parameters, as discussed in the next section.

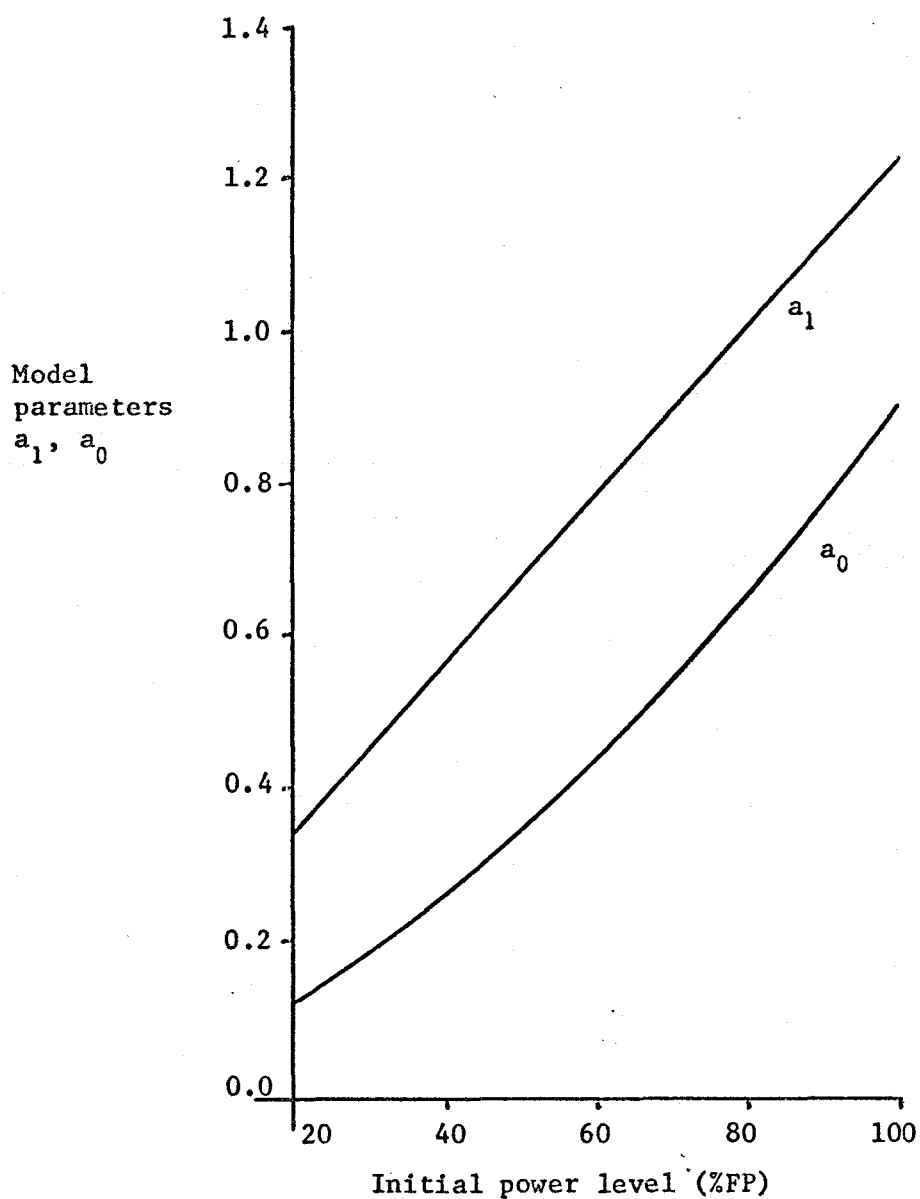


Figure 6-2. Changes in model parameters as a function of the initial power level, for 20% FP increases in demanded power.

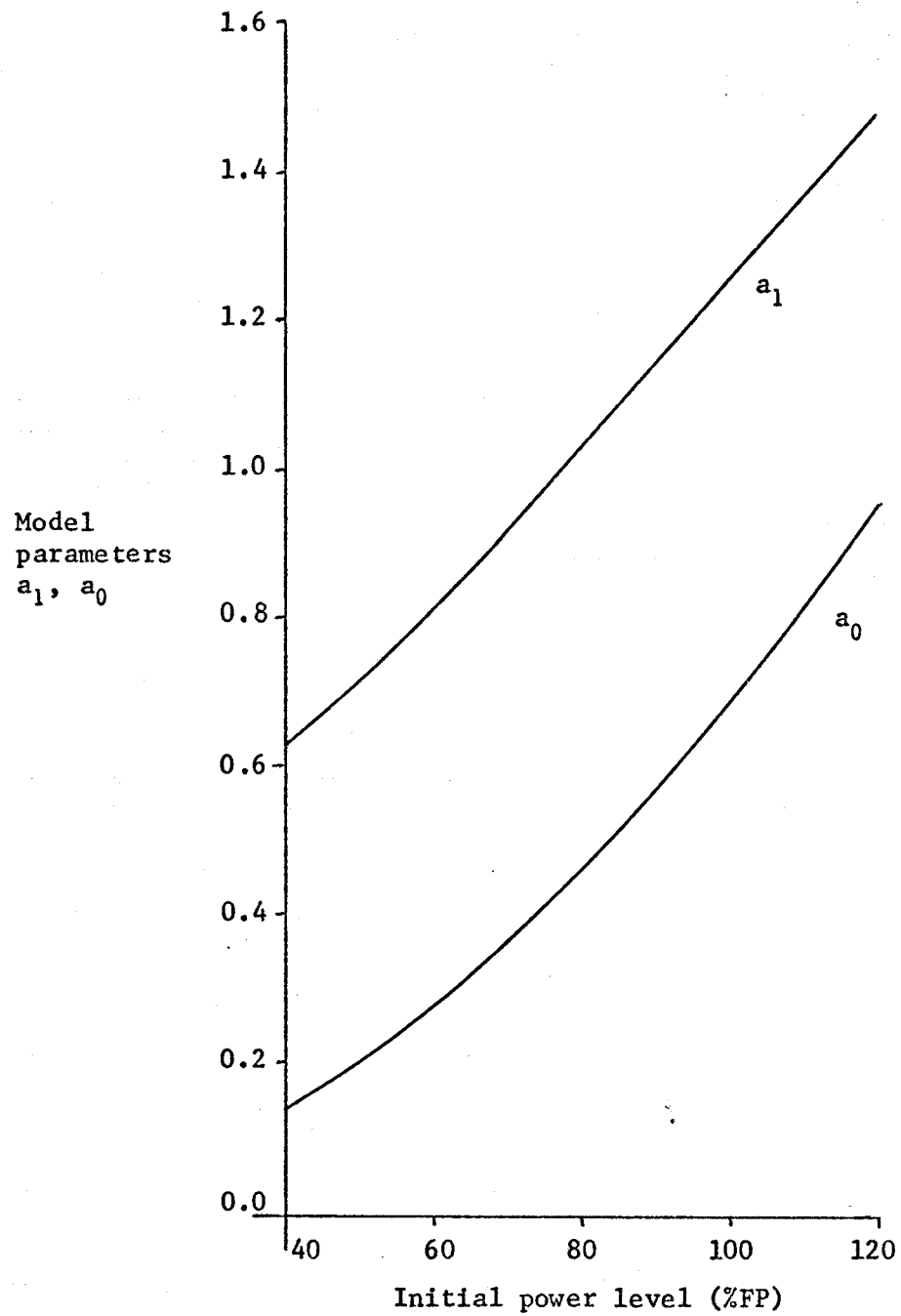


Figure 6-3. Changes in model parameters as a function of the initial power level, for 20% FP decrease in demanded power.

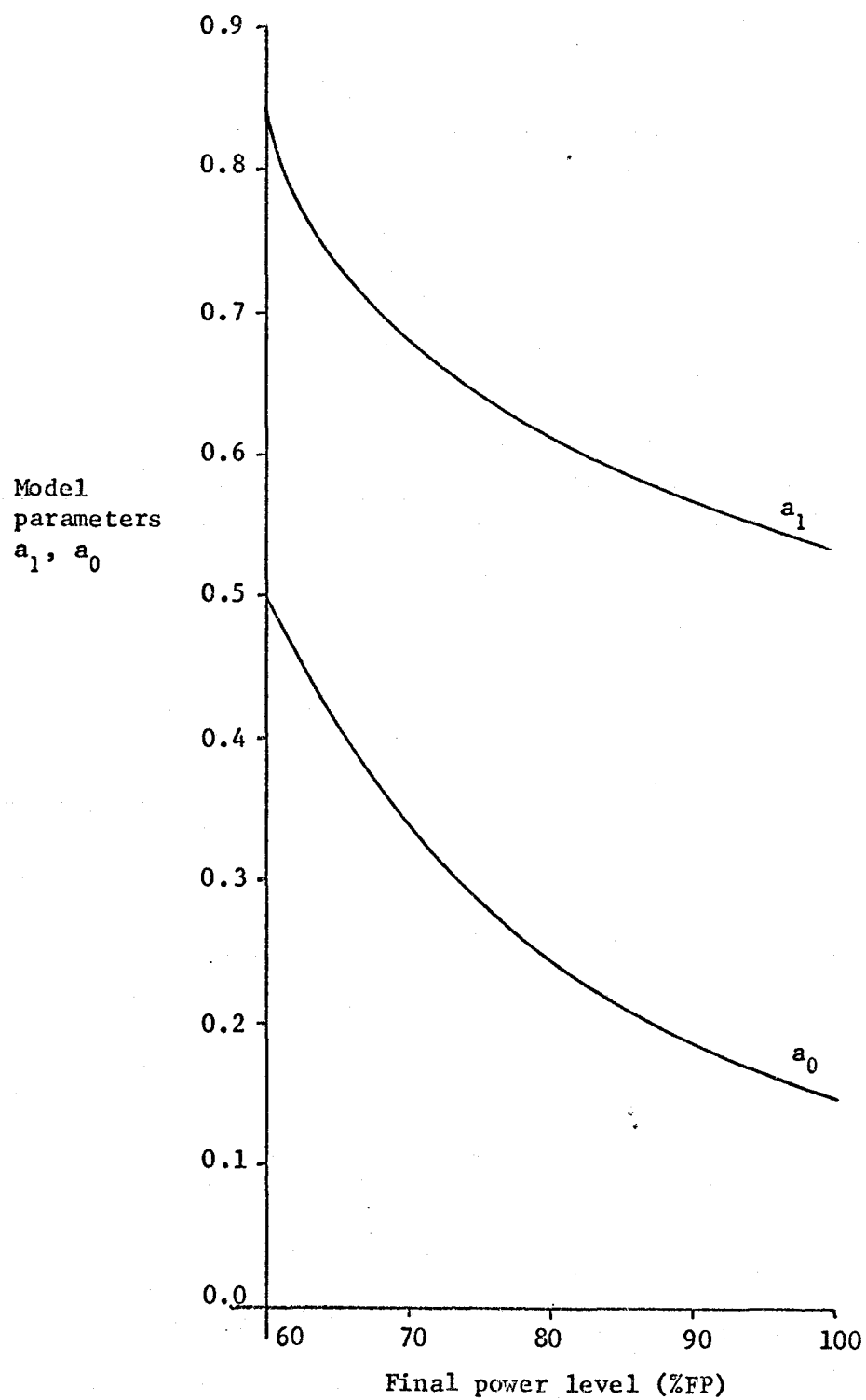


Figure 6-4. Effect of magnitude of demanded power increase on model parameters. Initial power level 50% FP.

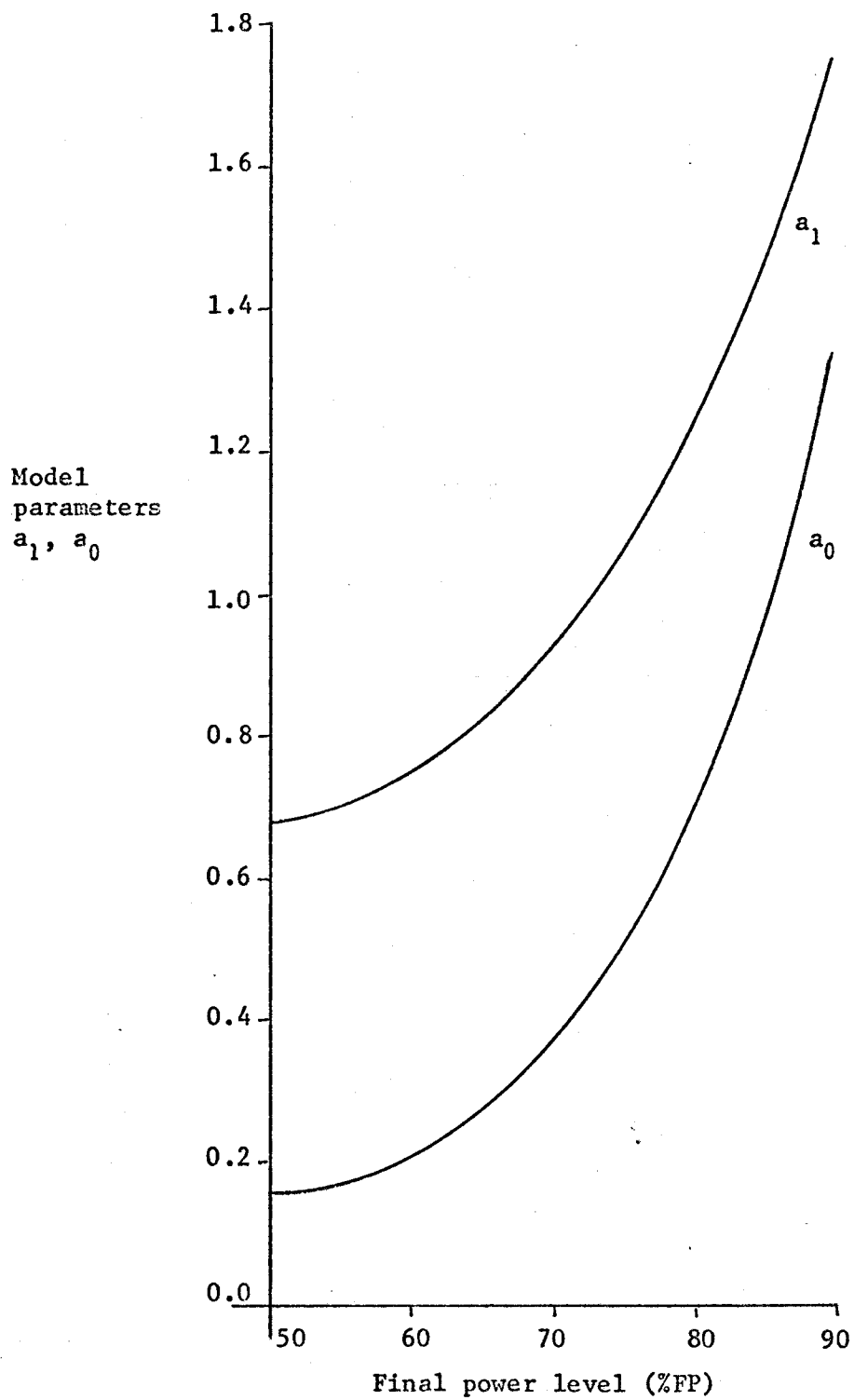


Figure 6-5. Effect of magnitude of demanded power decrease on model parameters. Initial power level 100% FP.

6.1.2 Effect of Parameter Changes

The parameters which are time-varying in our reactor model are the temperature coefficient (T_c) and the concentrations of the various delayed neutron groups (β_i). The range of variations in the temperature coefficient is from -10mk to +5mk, while the maximum changes in delayed neutron concentrations are $\pm 20\%$ of the design values given in Appendix I. It was found by the author earlier^[5], that changes in T_c have a much greater effect on the system response than variations in the β_i , and that it is not feasible to distinguish which of these parameters has changed, from the effect on the system response. Since our interest is only in the change of the overall plant performance, and its effect on the model parameters, the exact nature of the internal change is not important.

The variations in the optimal parameters for the range of T_c are shown in Figure 6-6. The change in a_1 is particularly large. Similar responses could be obtained for various other initial power levels and demanded changes, but it should be apparent that it is not practical to attempt to find model parameters for all possible stages of operation. It is more desirable to continually update the model parameters on-line, in order to take care of both the nonlinear nature of the plant and changes in the system parameters. Before considering such an adaptive scheme, it is useful to investigate the suboptimal control of the reactor, in order to establish the type of performance one can expect from the proposed technique.

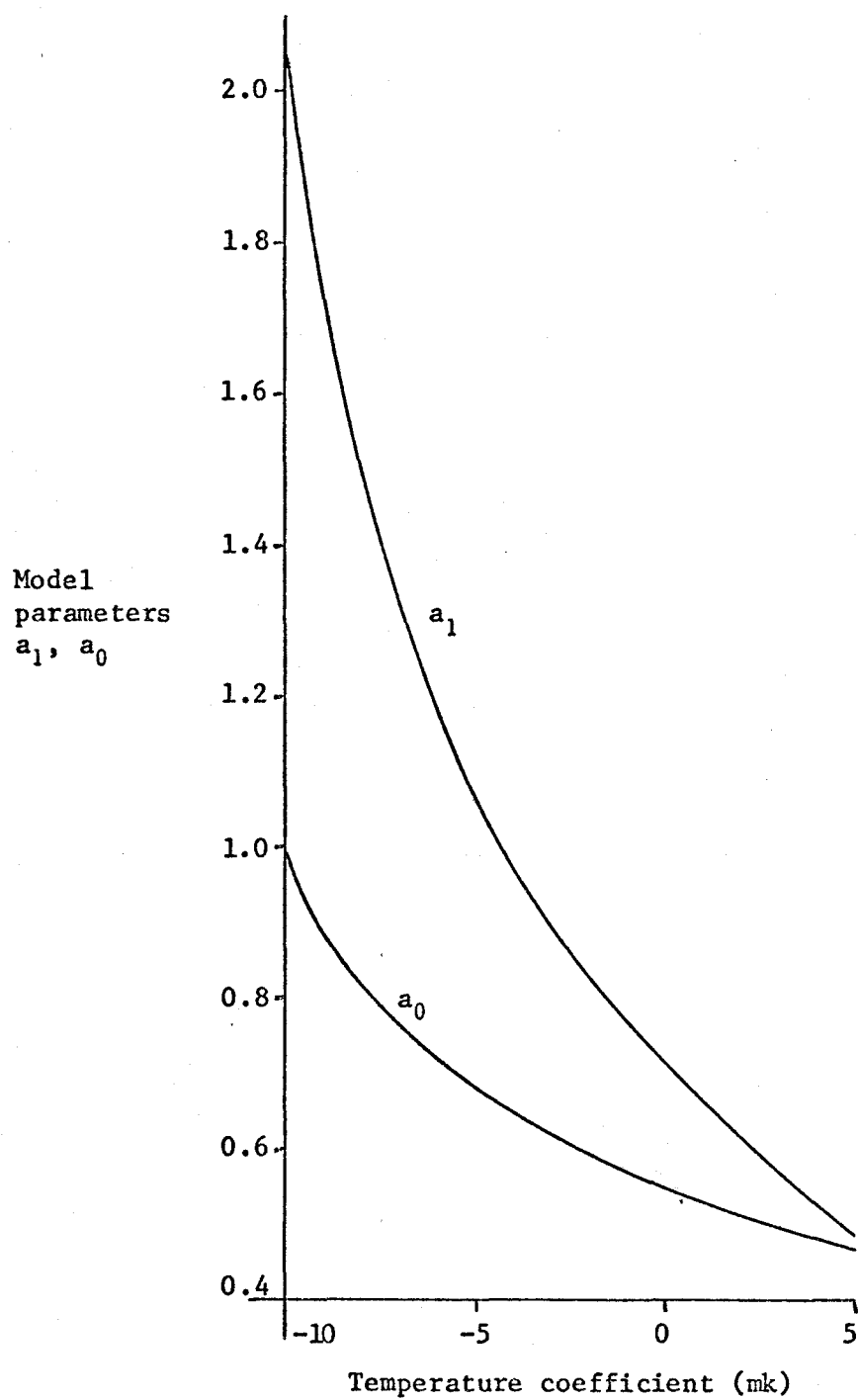


Figure 6-6. Effect of temperature coefficient on optimal model parameters, for an increase in demanded power from 80% to 100% FP.

6.2 Responses for Integral Quadratic Cost Functions

The suboptimal control of the nuclear reactor follows the same pattern as has been described in the previous chapter for a high-order linear system. The main difference is, that instead of finding the optimal model on the basis of the open-loop response, the reactor system equations are initially evaluated for a unity feedback configuration. This is advantageous, since the open-loop response of the reactor for a positive power demand change would be unbounded, except for physical limitations.

Using the examples considered in section 6.1, and the model parameters given in Table 6-1, optimal feedback controllers for the models have been computed for the following cost functions:

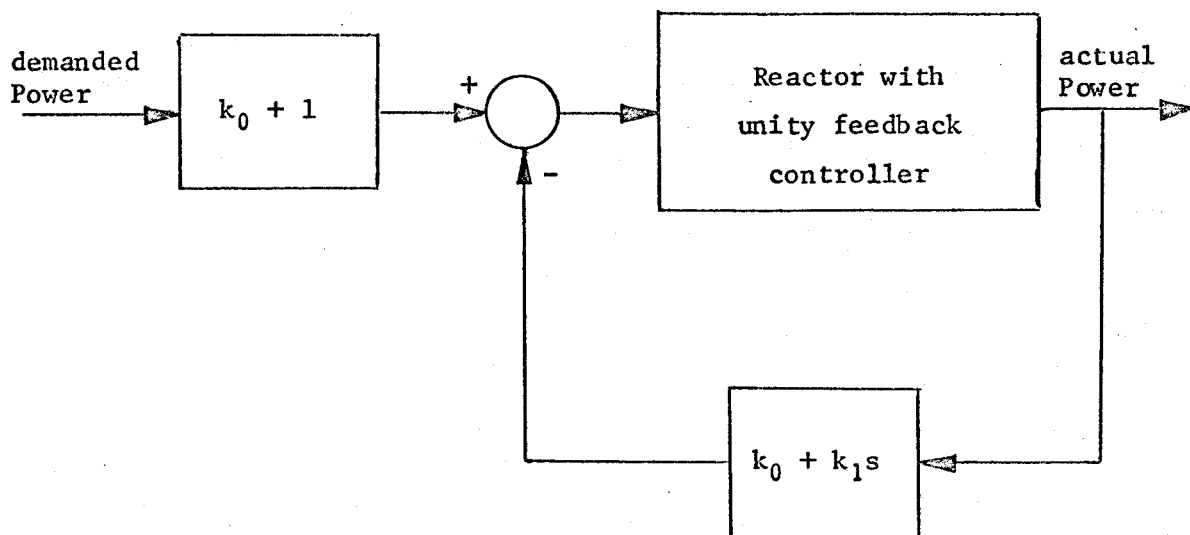
$$J_1 = \int_0^{\infty} [10(x_1 - x_1^*)^2 + x_2^2 + 0.1(u - u^*)^2] dt \quad (6-1)$$

$$J_2 = \int_0^{\infty} [(x_1 - x_1^*)^2 + 10x_2^2 + 0.1(u - u^*)^2] dt \quad (6-2)$$

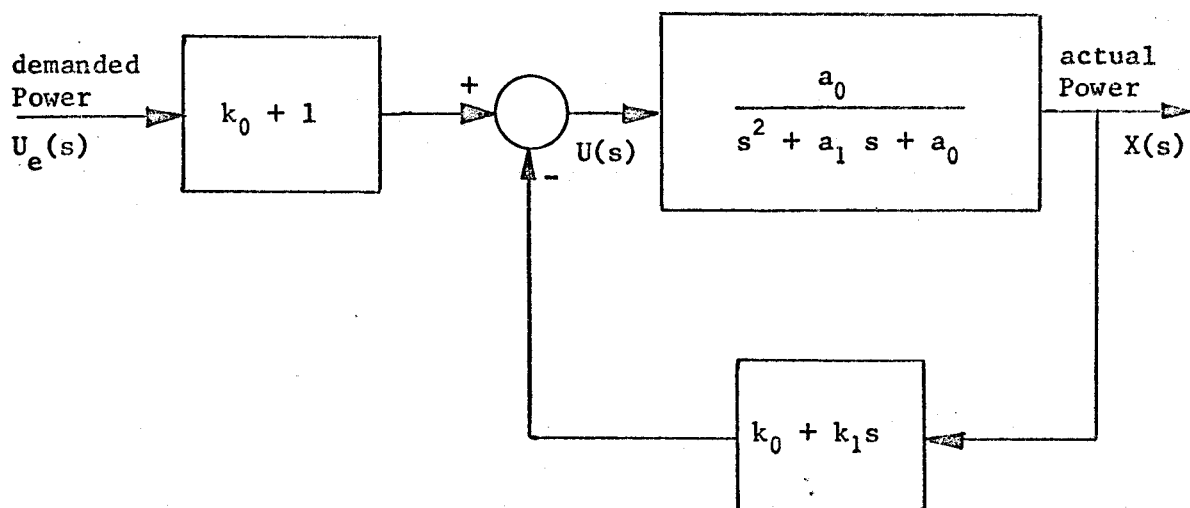
where both x_1^* and u^* are equal to the new value of the demanded power level.

The block diagram realization of the suboptimal reactor controller and the corresponding optimal model controller are shown in Figure 6-7. Note that the feedforward block is k_0+1 , since both the reactor and the model have unity gain.

The responses for a demanded change in power from 90%-100% FP and for the cost functions given by equations (6-1) and (6-2) are shown in Figure 6-8. The effect of the relative weights attached to the response and to its derivative are well illustrated. The most important



a. Reactor and suboptimal controller

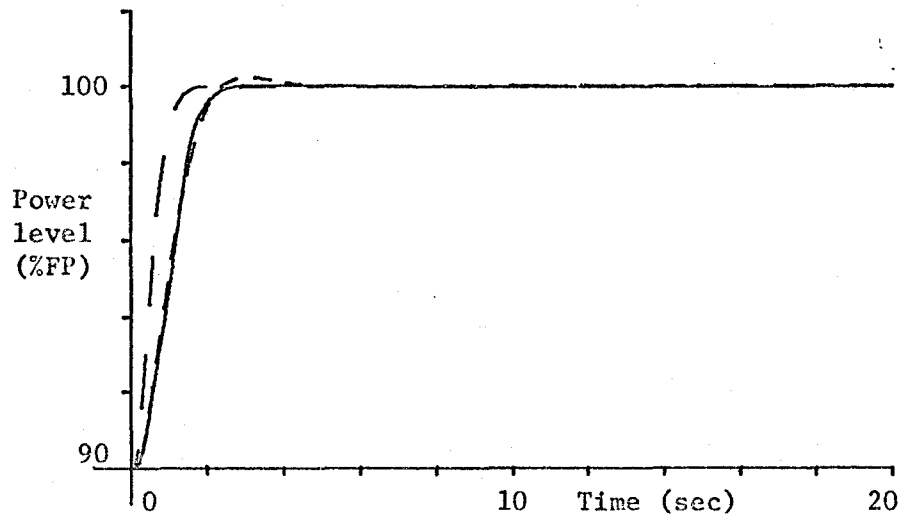


b. Model and optimal controller

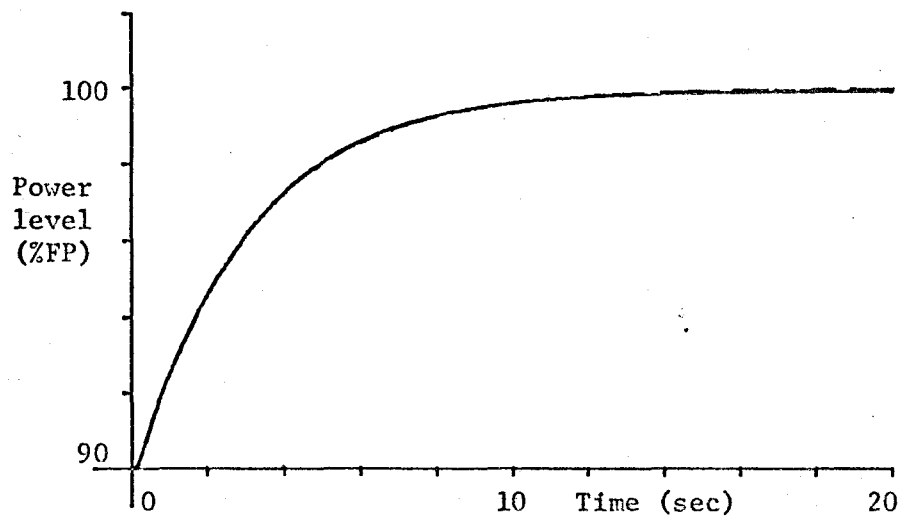
Figure 6-7. Block diagram representation of suboptimal system and optimal model controllers.

Legend:

- suboptimal system response
 — — optimal model response
 - - - updated model response



$$a. \quad J_1 = \int_0^{\infty} [10(x_1 - x_1^*)^2 + x_2^2 + 0.1(u - u^*)^2] dt$$



$$b. \quad J_2 = \int_0^{\infty} [(x_1 - x_1^*)^2 + 10x_2^2 + 0.1(u - u^*)^2] dt$$

Figure 6-8. Suboptimal system, optimal and updated model responses for 90%-100% FP demanded change.

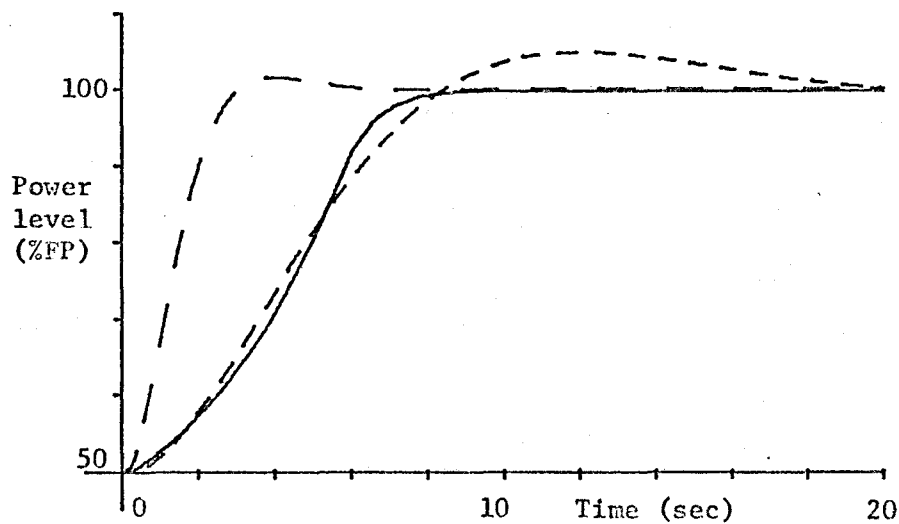
thing to note, is that while for J_2 the model and system responses are identical for practical purposes, there is a considerable deviation in the case of J_1 . This discrepancy is due to the nonlinear nature of the plant, and will be even more apparent for larger changes in power level. It is desirable, therefore, to reoptimize the model parameters, on the basis of the suboptimal response. The response of the updated model is also shown in Figure 6-8. The corresponding change in suboptimal control as well as the effect of repeated re-optimizations will be considered after the responses for a 50%-100% FP power change have been discussed.

Using again the model parameters from Table 6-1 and the cost functions of equations (6-1) and (6-2), the responses of the reactor and the model for a 50%-100% FP demanded power change are shown in Figure 6-9. The difference between optimal model and suboptimal system response is quite large in both cases, being worse, once again, for J_1 . The responses of the updated models are also shown.

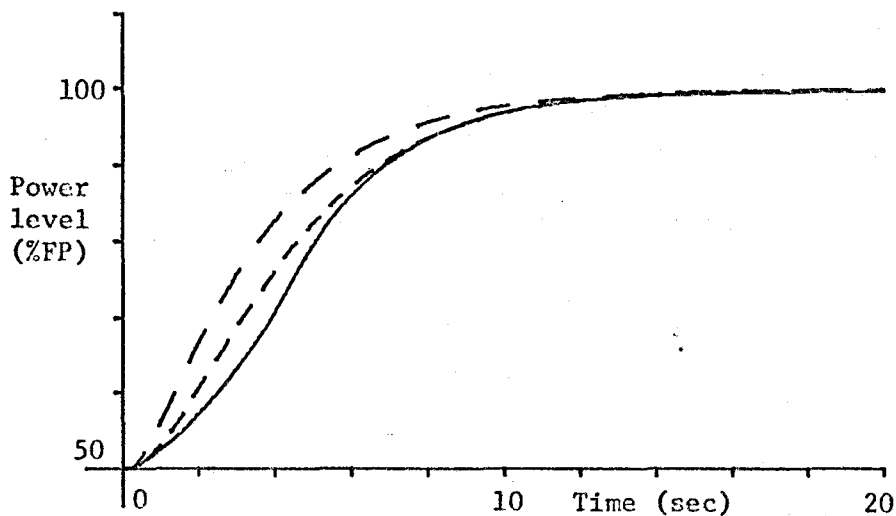
The convergence of the model parameters and hence the optimal and suboptimal costs, following subsequent cycles of updating the model and finding the new reactor response, are illustrated by the results in Table 6-2. Since in the case of Figure 6-8b, no change in model parameters took place, only the remaining three cases are considered. In the upper portion of the table, the model parameters are displaced, starting with the ones obtained from the unity feedback configuration, and already given in Table 6-1. The corresponding optimal model costs and suboptimal system costs are shown in the first row of the lower part of Table 6-2. The changes in model parameters

Legend:

- suboptimal system response
 - - - optimal model response
 - - - updated model response



$$a. J_1 = \int_0^{\infty} [10(x_1 - x_1^*)^2 + x_2^2 + 0.1(u - u^*)^2] dt$$



$$b. J_2 = \int_0^{\infty} [(x_1 - x_1^*)^2 + 10x_2^2 + 0.1(u - u^*)^2] dt$$

Figure 6-9. Suboptimal system, optimal and updated model responses for 50% - 100% FP demanded change.

Number of reoptimi- zations	90% - 100%		50% - 100%			
	$J_1 = [10, 1, 0.1]_0^{20}$		$J_1 = [10, 1, 0.1]_0^{20}$		$J_2 = [1, 10, 0.1]_0^{20}$	
	a_1	a_0	a_1	a_0	a_1	a_0
0	1.4471	1.2098	0.5239	0.1392	0.5239	0.1392
1	1.4157	0.2561	0.2960	0.0107	0.3816	0.0591
2	1.3136	0.2633	0.2318	0.0111	0.2294	0.0342
3	1.2972	0.2643	0.2231	0.0112	0.2172	0.0353
4	1.2972	0.2643	0.2220	0.0112	0.2158	0.0354

	Model cost	System cost	Model cost	System cost	Model cost	System cost
0	0.0602	0.1158	3.26	12.91	0.932	1.254
1	0.1103	0.1114	12.65	11.43	1.088	1.254
2	0.1066	0.1108	11.20	11.24	1.248	1.228
3	0.1062	0.1107	11.03	11.22	1.226	1.226
4	0.1062	0.1107	11.01	11.22	1.224	1.226

Table 6-2. Reoptimization of model parameters for suboptimal reactor responses, and corresponding changes in model and system costs.

as well as in the cost functions at each reoptimization are seen to converge after four cycles. The final response for the 50%-100% FP change are shown in Figure 6-10, indicating good final correspondence between system and model responses.

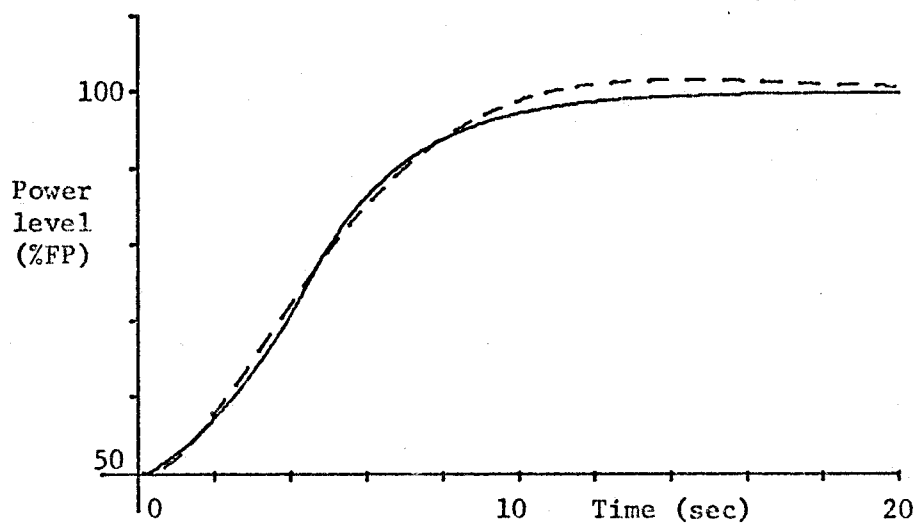
While the above discussed use of repeated reoptimizations is not practical in an on-line situation, the examples serve to illustrate the potential of the proposed approach. In an on-line application, the model parameters begin to be updated on the basis of the closed-loop suboptimal response as soon as sufficient samples are available, and as the updating progresses, the reoptimization of the model parameters is performed automatically. Since this process is the essence of the adaptive controller to be used, it will be discussed in detail in the next chapter.

6.3 Responses for Least pth and Minimax Cost Functions

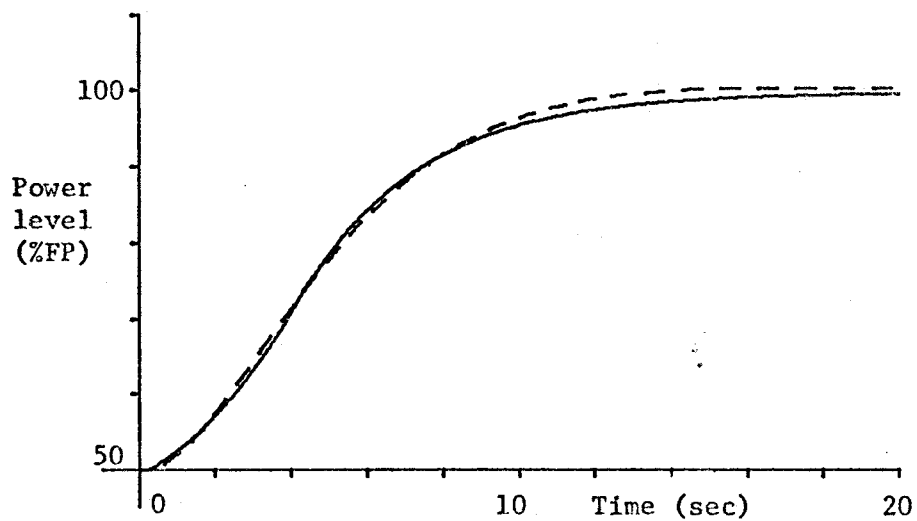
6.3.1 Monotonic Reactivity Insertion

It was pointed out in Chapter 3, that in the reactor control problem it is important to include in the cost function terms relating not only to the output but also to the input. In particular, the number of sign changes of the input signal influences the mechanical wear of the absorber rod mechanism, and should therefore be minimized. The addition of such a term to the cost function renders the Riccati matrix method ineffective, and the technique proposed in sections 5.4 and 5.5 has to be used.

The problem may be illustrated by considering a 90%-100% FP change in power level, and the cost function to be minimized is:



$$a. \quad J_1 = \int_0^{\infty} [10(x_1 - x_1^*)^2 + x_2^2 + 0.1(u - u^*)^2] dt$$



$$b. \quad J_2 = \int_0^{\infty} [(x_1 - x_1^*)^2 + 10x_2^2 + 0.1(u - u^*)^2] dt$$

Figure 6-10. Responses of reoptimized model and corresponding suboptimal system.

$$J_1 = \sum_{i=0}^N 10(x_{1i} - x_{1i}^*)^2 + x_{2i}^2 + 0.1(u_i - u_i^*)^2 \quad (6-3)$$

The optimum response is shown in Figure 6-11a, and the necessary reactivity variation to bring about such a power level change is indicated in part b of the diagram. It is apparent, that following a sudden reactivity insertion, two reversals in the direction of travel of the absorber rod are required to realize the desired response.

To realize a monotonic reactivity insertion, while still minimizing the cost function of equation (6-3), a term is added to reflect the number of sign changes of the input:

$$J_2 = J_1 + \sum_{i=1}^N |\operatorname{sgn}(u_i - u_i^*) - \operatorname{sgn}(u_{i-1} - u_{i-1}^*)| \quad (6-4)$$

The resultant power level change and corresponding reactivity variation are shown in Figure 6-11. The degradation in performance, as far as the power response is concerned, appears to be acceptable in view of realizing a monotonic reactivity input. The numerical change in system cost is from $J_1 = 1.029$ to $J_2 = 1.368$, subsequent to one cycle of reoptimization in each case.

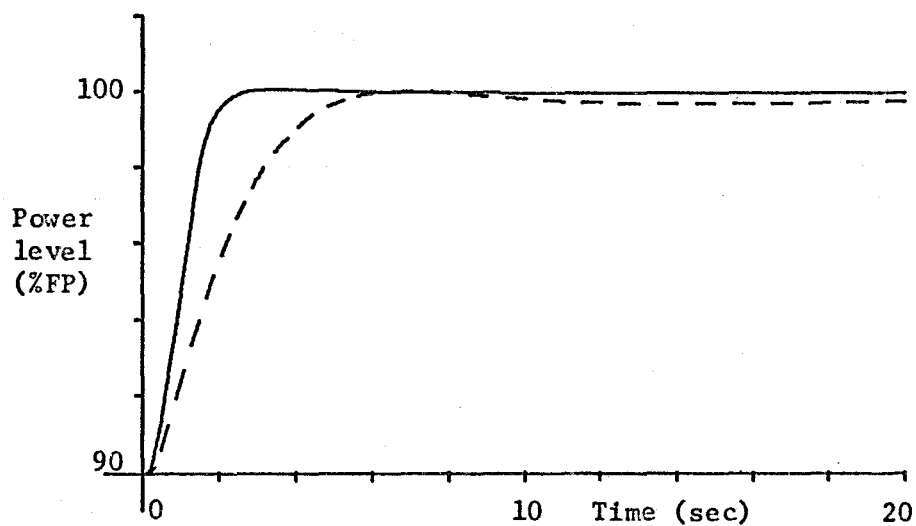
6.3.2 Ramp Change of Demanded Power

It has been assumed, up to this stage in this thesis, that the demanded power change appears as a step. While this assumption may be valid for future reactors, at the Douglas Point plant the demanded power varies as a ramp between the initial and the desired levels.

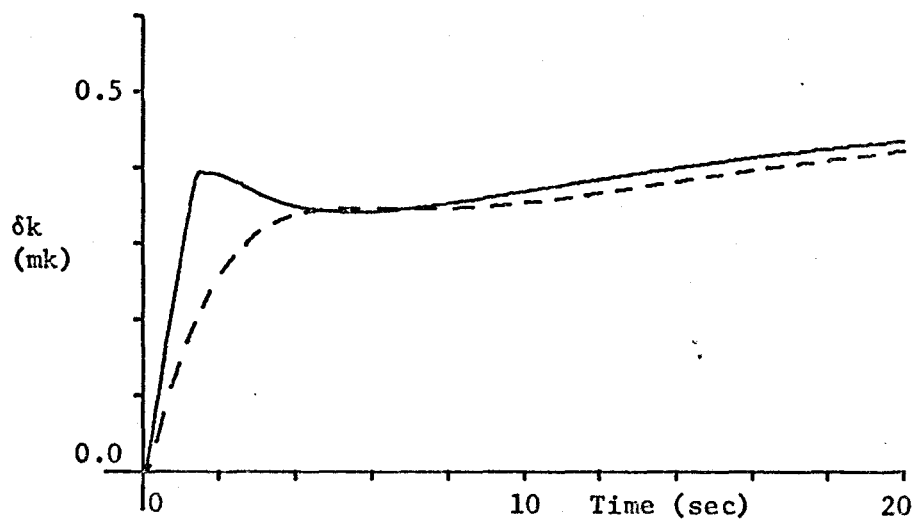
The Riccati matrix approach is once again not suitable, and the proposed

Legend:

— arbitrary reactivity insertion
- - - monotonic reactivity insertion



a. Optimum power level changes



b. Reactivity input

Figure 6-11. Suboptimal reactor responses for the cost functions in equations (6-3) and (6-4).

optimization method is to be used to find the suboptimal controller for the case of a specified rate of power level change. In order to further emphasize the versatility of the new technique, a least pth cost function with $p=1$ as well as a minimax objective are considered for the ramp change in power level.

The rate of change of the demanded power has been assumed to be 2% FP per second. The cost functions to be minimized have been selected as:

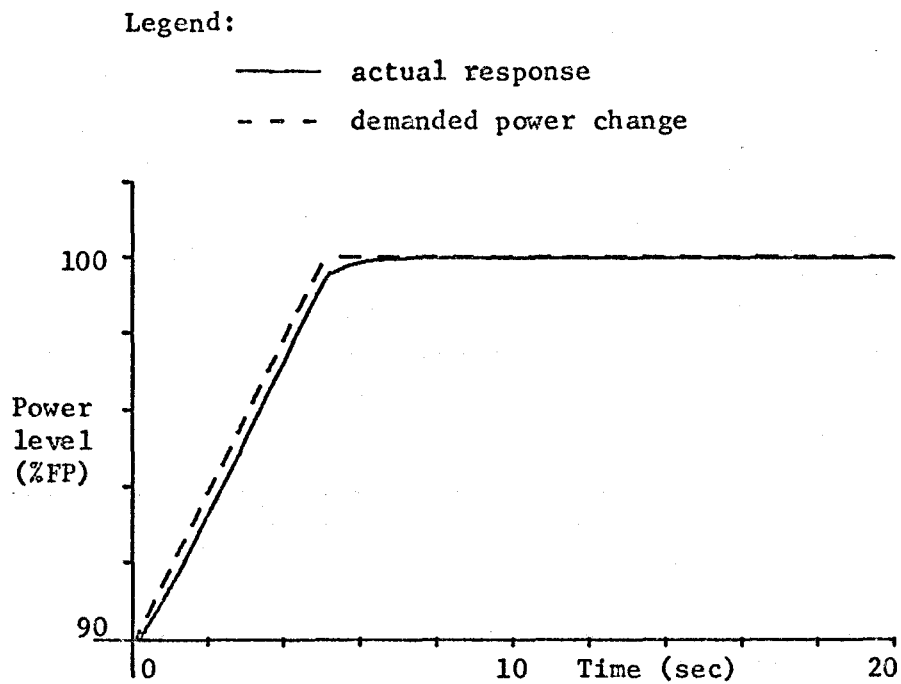
$$J_1 = \sum_{i=0}^{101} (|x_{1i} - x_{1i}^*| + |x_{2i} - x_{2i}^*| + 0.1|u_i - u_i^*|) \quad (6-5)$$

$$J_2 = \max_{i=0,101} \{|x_{1i} - x_{1i}^*| + |x_{2i} - x_{2i}^*| + 0.1|u_i - u_i^*|\} \quad (6-6)$$

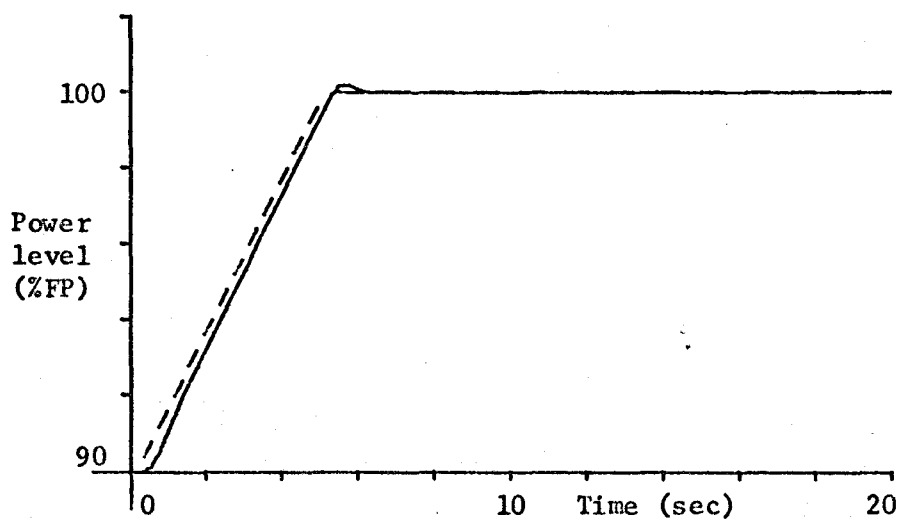
For a change in power level from 90% to 100%, the controller parameters and the suboptimal system costs are shown in Table 6-3, and the responses are plotted in Figure 6-12. While there is a considerable change in the controller parameters, particularly in k_1 , the difference between the corresponding responses is not very significant.

	k_0	k_1	ℓ_0	ℓ_1	Cost
J_1	11.43	8.270	12.43	6.190	0.420
J_2	9.264	0.965	10.26	0.624	0.0281

Table 6-3. Results for ramp change
in demanded power.



a. Response for least pth cost function, $p=1$



b. Response for minimax cost function

Figure 6-12. Suboptimal reactor responses for ramp changes in demanded power.

CHAPTER 7

ADAPTIVE CONTROL OF NUCLEAR REACTOR

The need for continually updating the parameters of the optimal low-order model of the reactor on-line, as successive samples of the given response are obtained, has been indicated several times in the previous chapter. Since following each updating, the feedback controller parameters are recomputed, the resulting control system is adaptive. Because of the nonlinear nature of the plant, the model parameters change not only with the operating characteristics of the system, but are also a function of the feedback parameters used to obtain the response on the basis of which the new model is found. This interaction between model and controller parameters has been found to give converging values of each of the quantities involved in the case of suboptimal operation. The same should also hold for the adaptive mode.

Before considering in detail the performance of the adaptive controller, it is interesting to note that the proposed method, based on a second-order model of the system to be controlled, provides a link between modern optimal control theory and classical techniques of controller design. A fundamental difference between the two approaches is in the nature of specifying the desired performance. In optimal control, a cost function is used for this purpose, which includes not only the output, but also its derivatives, as well as their effect on the input. The classical techniques, on the other hand, aim basically at realizing a given transient output response. This can usually be specified in terms of the natural frequency and damping constant of

the response of a second-order system, to a unit step input.

Using the approach presented in Chapter 5, both of the above objectives may be realized. In fact, if the desired response to a step input is already given, and the optimal model of the system has been derived, the controller parameters are given immediately by equations (5-22) and (5-30). The approximation of the system response to the desired one will once again depend on the accuracy of the model, which may need to be updated to improve the closed-loop response. An example to illustrate the above application is considered in this chapter, along with adaptive control for integral quadratic cost functions, for least pth sums and for minimax objectives.

As in Chapter 6, the second-order model for the reactor and the controller is assumed to have the form

$$\frac{X(s)}{U(s)} = \frac{a_0}{s^2 + a_1s + a_0}$$

and the Fletcher routine is used to identify the parameters a_1 and a_0 , such that the sum of the squares of the sample errors between system and model responses is minimized. This aspect of the identification problem has already been discussed in Chapter 4, and will not be further considered.

7.1 The Adaptive Loop

The implementation of the concepts introduced in previous parts of this thesis for the adaptive controller may be divided into four stages. Initially, the reactor is assumed to be operating at a steady power level; at time $t=0$ a change to a new operating level is initiated.

Using the model parameters appropriate for the demanded change and the desired cost function (this information being stored in the computer) the controller parameters are computed, and the system begins the transition to the new power level. Stage one continues usually for several sampling intervals, the value of the response being stored at each sample point. This may be called the observation interval, since neither updating of the model nor recomputation of the controller coefficients takes place. The goodness of fit, between model and system responses during this stage, depends entirely on the accuracy of the initial model for the operating conditions within the system.

The observation interval is completed after a predetermined number of samples have been collected, and the second stage, the updating of the model, commences. This is essentially the identification interval, since the model parameters are "identified" or the model is reoptimized on the basis of the observed system response.

Having found the new low-order model, the next step is to recompute the controller coefficients. Since this is the stage that provides the adaptation of the control system, it is referred to as the adaptive interval. While theoretically it is desirable to complete both the identification and the adaptation during one sampling interval, so that the next observation stage may begin with the new controller parameters, it is possible in practice to realize a variety of combinations of the relative duration of each of these three intervals. In particular, the length of the adaptation interval is greatly dependent on the choice of the cost function. If a search routine needs to be used to find the new controller parameters, the time will be much longer

than if simple analytical expressions give the desired quantities.

Since the observation interval is not in fact part of the adaptive controller, and needs little explanation, it will not be considered further. Before presenting some examples, however, it is useful to summarize the formulae used in the identification and the adaptation stages.

7.1.1 The Identification Interval

It has been assumed throughout this thesis, that the system to be controlled is replaced by a low-order open-loop model for the purpose of computing the controller coefficients. However, when realizing the suboptimal system and optimal model responses, both appear in the closed-loop form, as indicated in Figure 6-7. Since the same configuration is used in the adaptive scheme, it is necessary to be able to identify the equivalent open-loop parameters of a closed-loop model, given the controller coefficients, and vice versa.

Referring to Figure 6-7 and using the same procedure as in section 5.5, it can be readily shown that for a model in the form

$$\frac{X(s)}{U(s)} = \frac{a_0}{s^2 + a_1s + a_0} \quad (7-1)$$

having a feedback loop $k_0 + k_1s$ and a feedforward block of $k_0 + 1$ (Figure 6-7b), the equivalent open-loop model is

$$\frac{X(s)}{U(s)} = \frac{b_0}{s^2 + b_1s + b_0} \quad (7-2)$$

where

$$b_0 = a_0 + a_0k_0 \quad (7-3)$$

$$b_1 = a_1 + a_0k_1 \quad (7-4)$$

Conversely, if a model in the form of equation (7-2) has been used to represent a closed-loop system having feedback parameters k_0^* and k_1^* , the equivalent open-loop model, on the basis of which the new controller coefficients are to be determined, are given by

$$a_0 = \frac{b_0}{1 + k_0^*} \quad (7-5)$$

$$a_1 = b_1 - a_0 k_1^* \quad (7-6)$$

For the case of ramp or higher-order inputs as well as for systems of order greater than two, equations (7-3) to (7-6) can be readily extended by using equations (5-34) and (5-36).

7.1.2 The Adaptation Interval

It was pointed out during the description of the adaptive loop, that the nature of the cost function has a vital bearing on the amount of computer time that is necessary to determine the parameters of the control system.

For an integral quadratic cost function in the form of

$$J_1 = \int_0^{\infty} [q_1(x_1 - x_1^*)^2 + q_2 x_2^2 + p(u - u^*)^2] dt \quad (7-7)$$

and the model given by equation (7-2), the optimal controller coefficients to transfer the model from a given initial state to the state $[x_1^* \ 0]^T$, are computed from equations (2-24), (2-25) and (2-28), giving

$$k_0 = -1 + \left[1 + \frac{q_1}{p}\right]^{1/2} \quad (7-8)$$

$$k_1 = -\frac{a_1}{a_0} + \left[\left(\frac{a_1}{a_0}\right)^2 + \frac{q_2}{p} + \frac{2k_0}{a_0}\right]^{1/2} \quad (7-9)$$

$$\ell_0 = 1 + k_0 \quad (7-10)$$

It is important to note, that k_0 and hence ℓ_0 are functions of the performance index only, and hence remain constant during adaptation.

In the case of the system performance being specified in terms of the step response of a second-order system, the controller coefficients are again obtained by simple formulae. Let the desired response be

$$\hat{X}(s) = \frac{h_0}{s^2 + h_1 s + h_0} U_e(s) \quad (7-11)$$

where $U_e(s) = \frac{1}{s}$. Equations (5-22) and (5-30) are now applicable giving

$$k_0 = \frac{h_0 - a_0}{a_0} \quad (7-12)$$

$$k_1 = \frac{h_1 - a_1}{a_0} \quad (7-13)$$

$$\ell_0 = 1 + k_0 \quad (7-14)$$

Clearly, no appreciable amount of computer time will be used in evaluating the last three equations.

The time required for computation becomes a problem for on-line application when the coefficients of equation (7-11) are not specified, but need to be determined by a search routine such that a general least pth or minimax cost function is optimized. The duration of the adaptation interval will be a function of the accuracy to which the optimum is approached and of the efficiency of the search routine.

It is interesting to note in this third case, that during the identification interval the feedback coefficients are held constant, and the model parameters are updated, while in the adaptation interval that follows, the model is invariant, but the controller coefficients are optimized. At the beginning of each identification and adaptation

interval the initial conditions for the model response are set equal to the values of the system response at that particular sampling instant.

7.2 Results for Integral Quadratic Cost Functions

Because of the importance of the integral quadratic cost function in optimal control theory, the problem of adaptively controlling the nuclear reactor, such that a cost function of this form is minimized, has been chosen to demonstrate the basic capabilities of the proposed scheme. In particular, the effect of the initial model parameters, the duration of the identification interval and a change in temperature coefficient are to be considered. Two sets of responses will be used:

1. Step change in demanded power from 90%-100% FP; the cost function being

$$J_1 = \int_0^{20} [10(x_1 - x_1^*)^2 + x_2^2 + 0.1(u - u^*)^2] dt$$

2. A 50%-100% FP change, subject to minimizing

$$J_2 = \int_0^{20} [(x_1 - x_1^*)^2 + 10x_2^2 + 0.1(u - u^*)^2] dt$$

The following six cases are considered:

- I. The open-loop model parameters, as given in Table 6-2 are used for computing the initial controller coefficients. Samples are taken every 0.2 second, and each observation interval is of 2 second duration. No limit is placed on the identification interval, but the updating is assumed to be completed in 0.2 second.

- II. As in I., but the number of function evaluations is limited to 10, ensuring that the identification interval does not exceed 0.2 second.
- III. As in II., but the starting model parameters are the reoptimized ones, as given in Table 6-2. These same parameters are used initially for the remaining three cases.
- IV. The sampling interval is reduced to 0.1 second, the observation interval to 1.0 second and the number of function evaluations during identification is limited to 5.
- V. The sampling and observation intervals are increased to 0.5 and 5 seconds, respectively, the limit on the number of function evaluations being 25.
- VI. Using the time and starting parameter specifications as in case III., a 20% increase in the temperature coefficient is assumed, prior to the demanded change in power level.

The results for these six conditions and the two power level changes are displaced in Tables 7-1 to 7-3 and Figures 7-1 to 7-3. Tables 7-1 and 7-2 show the adaptation of k_1 with time (0-20 seconds) for each of the six cases. The total costs for the 20 second interval are given in Table 7-3.

Considering case I, the adaptation of k_1 is seen to be completed after one observation interval for the 90%-100% power change, while it takes three intervals and a considerable variation in k_1 for the 50%-100% response. Figure 7-1 shows the actual responses. Three curves

Time (sec)	I	II	III	IV	V	VI
1	3.941	3.941	5.220	5.220	5.220	5.220
2				5.003		
3	5.900	1.815	5.951	5.709		5.965
4				5.709		
5	5.900	2.400	5.951	5.709	5.220	5.965
6				5.709		
7	5.900	2.400	5.951	5.709		5.965
8				5.709		
9	5.900	2.400	5.951	5.709		5.965
10				5.709		
20	5.900	2.400	5.951	5.709	5.220	5.965

Table 7-1. Changes in k_1 for a 90%-100% FP change in power level.

Time (sec)	I	II	III	IV	V	VI
1	8.379	8.379	10.276	10.276	10.276	10.276
2				11.156		
3	3.806	3.806	3.940	10.680		4.004
4				10.680		
5	9.104	5.028	7.882	12.283	12.895	8.613
6				7.880		
7	9.574	14.915	9.469	14.141		9.653
8				8.006		
9	9.574	15.415	9.469	13.295		9.674
10				9.797		
20	9.574	9.817	9.469	9.797	9.039	9.674

Table 7-2. Changes in k_1 for 50%-100% FP change in power level.

	I	II	III	IV	V	VI
90%-100% FP $J_1 = [10, 1, 0.1]_0^{20}$						
Model cost	0.124	0.102	0.119	0.110	0.148	0.118
System cost	0.126	0.126	0.121	0.111	0.151	0.121
50%-100% FP $J_2 = [1, 10, 0.1]_0^{20}$						
Model cost	1.330	1.361	1.301	1.249	1.330	1.299
System cost	1.311	1.341	1.291	1.223	1.314	1.284

Table 7-3. Optimal model costs and suboptimal system costs using adaptive control.

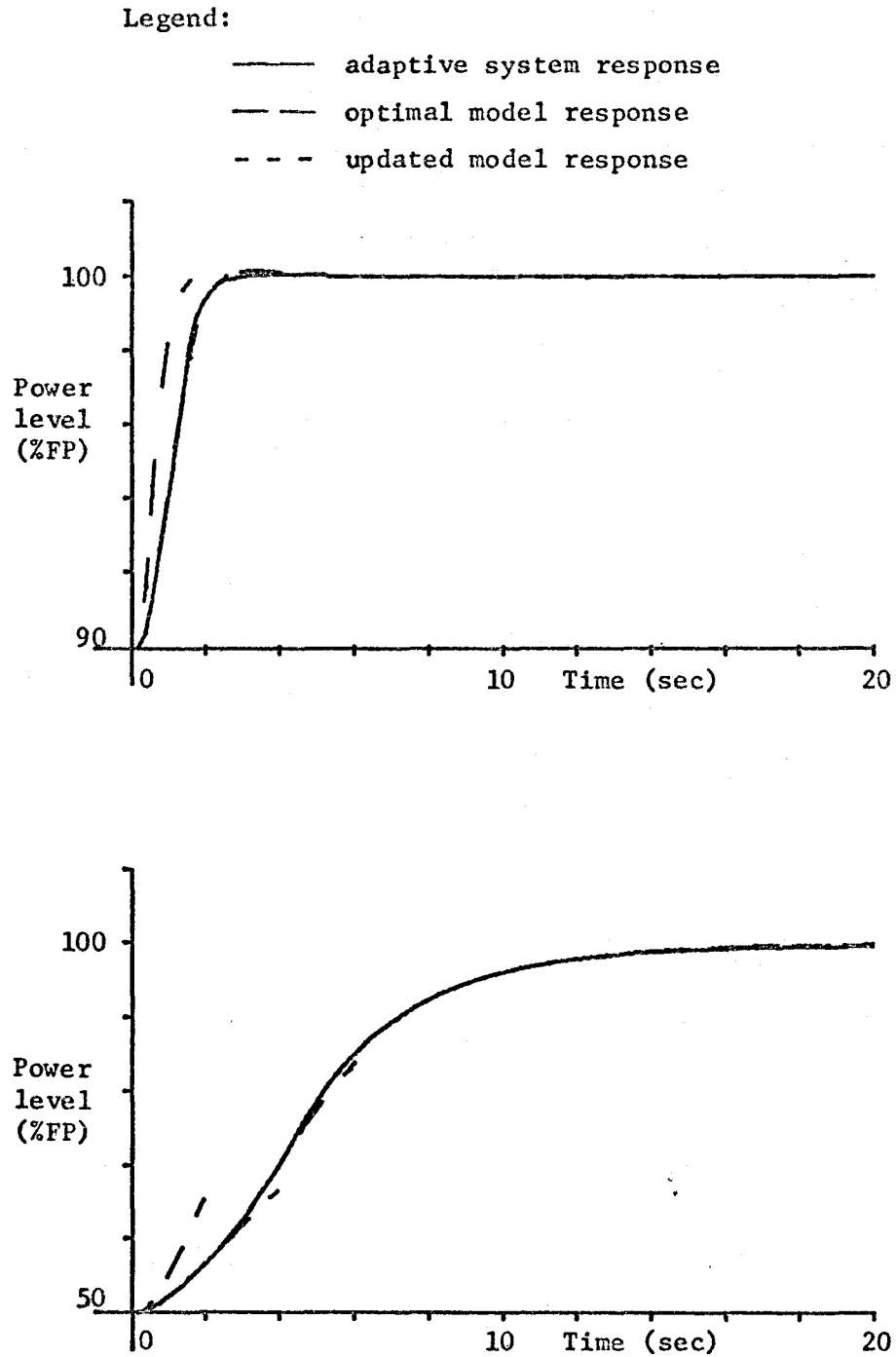


Figure 7-1. Adaptive nuclear reactor responses for unlimited identification interval.

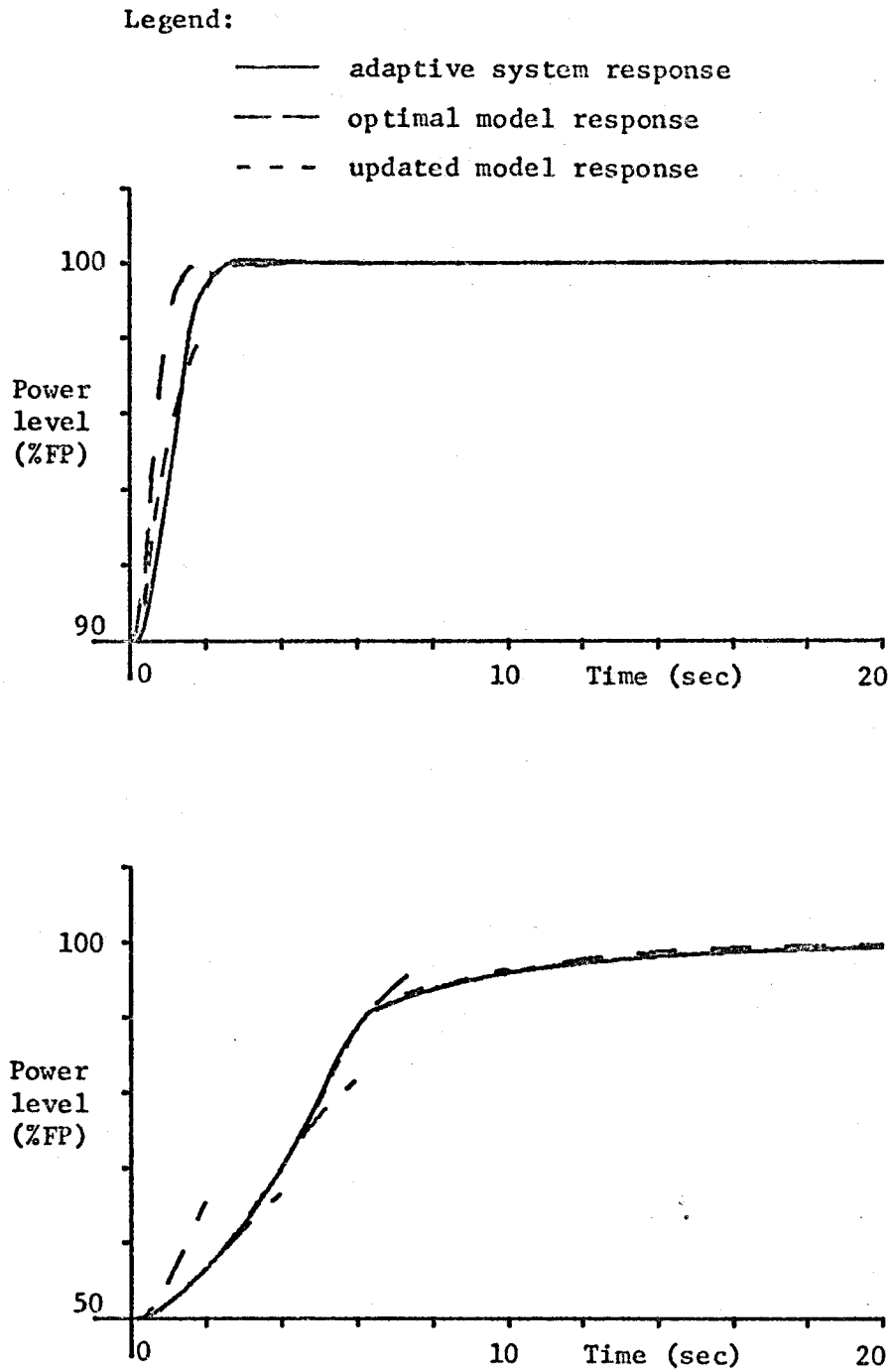


Figure 7-2. Adaptive nuclear reactor responses, when identification interval is limited to 0.2 second.

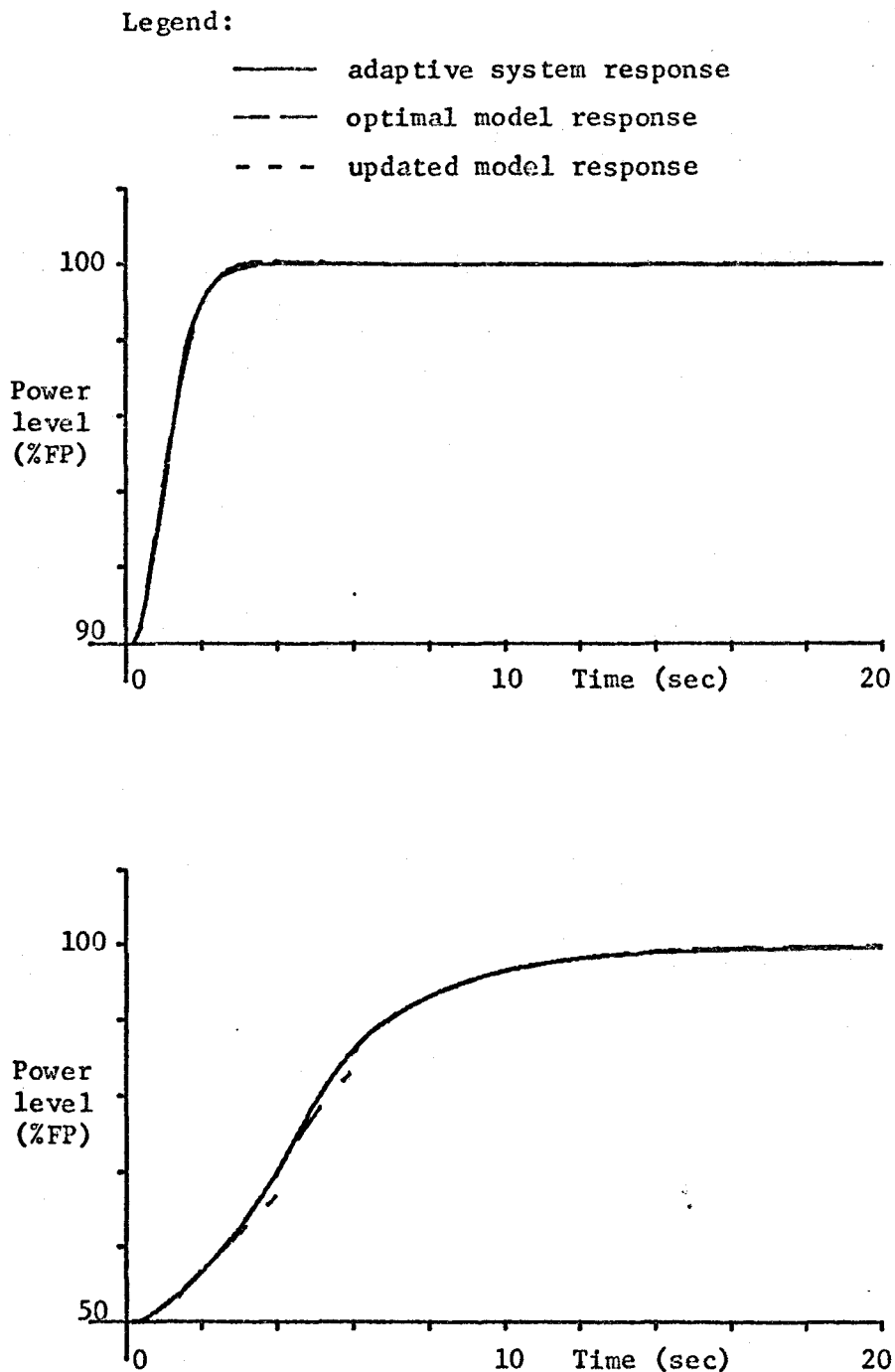


Figure 7-3. Adaptive nuclear reactor responses, starting with reoptimized model parameters.

are indicated: the actual reactor output, the closed-loop model response, and the approximation provided by the updated model, the latter being computed during the identification interval. Note the large initial discrepancy between system and closed-loop model responses. Following one stage of identification, this is greatly reduced, and it becomes difficult to distinguish the three responses.

The good identification in the above example could be achieved because the duration of the identification interval was not limited. If it is desired to update the model parameters within one sampling interval, the number of function evaluations in the search program had to be restricted to 10. The resulting changes in k_1 are seen in Tables 7-1 and 7-2 column II, and the corresponding responses in Figure 7-2. The considerable deviation between model and system responses is evident, as is the sudden change of slope of the reactor output at the end of the third identification interval, where k_1 changes from 5.028 to 14.915. It is important to note though, that no appreciable change in system costs has taken place in going from case I to case II.

The large initial discrepancy between system and model responses may be reduced considerably by using as initial model parameters the values found from repeated reoptimizations of suboptimal responses. Using these values from Table 6-2, the resulting performance is indicated in Figure 7-3, showing good agreement between system and model responses.

The effects of reducing and increasing the observation interval from the above 2 second value, are indicated, respectively, by columns

IV and V of the tables. For the 90%-100% only small changes in k_1 are observed, and for the 5 second case, no updating takes place at all. This is not surprising, since the response has virtually reached the desired value after 5 seconds. In the case of the 50%-100% response, considerable variations in k_1 are observed, as the program attempts to find the best low-order model for each segment of the actual response. For both responses, using one-second observation intervals results in the lowest system cost.

The last case to be considered is the effect of a 20% increase in the temperature coefficient. While it is possible to determine long-range changes in the temperature coefficient^[5], and adjust the model parameters accordingly, this has not been done in this case, in order to illustrate the ability of the adaptive controller to compensate for undetected parameter changes. In comparing the results of column VI to those in III, both the values of k_1 and those of system costs are very close together, indicating the successful compensation of the parameter change.

The above examples illustrate well the performance of the adaptive controller. The advantage of using an integral quadratic cost function is that once the model parameters are known, the controller coefficients are readily computed from simple formulae.

In the examples to follow, the same detailed study of the adaptive controller is not undertaken; they serve merely to illustrate the versatility of the basic approach.

7.3 Results for a Desired Step Response

While it is possible to choose the relative weights in an integral cost function to realize a desired transient response, no direct relationship between the two exists, and particularly for a nonlinear system, the response for a given cost function is dependent on the operating characteristics of the plant. If the objective of the control system designer is to realize a particular transient response, and if this can be specified in terms of a rational transfer function with no finite zeros, the formulae given by equations (5-22) and (5-30) can be used directly to obtain the desired controller coefficients. For the particular case of a second-order system with unity d-c gain, these relationships are given by equations (7-12) to (7-14).

To illustrate the realization of a desired transient response by an adaptive configuration, the critically damped case has been chosen, making $h_0 = 1$ and $h_1 = 2$ in equation (7-11). For the 90%-100% FP power level change and an observation interval of 2 seconds (samples being taken at every 0.2 second), the response of the reactor is shown in Figure 7-4. The good agreement between the actual and desired responses is evident from the diagram.

7.4 Results for Least pth and Minimax Cost Functions

In the general case of least pth and minimax cost functions, analytical results are not available, as for the particular objectives used in the last two sections. In order to apply the formulae of equations (7-12) to (7-14), the optimal response must first be

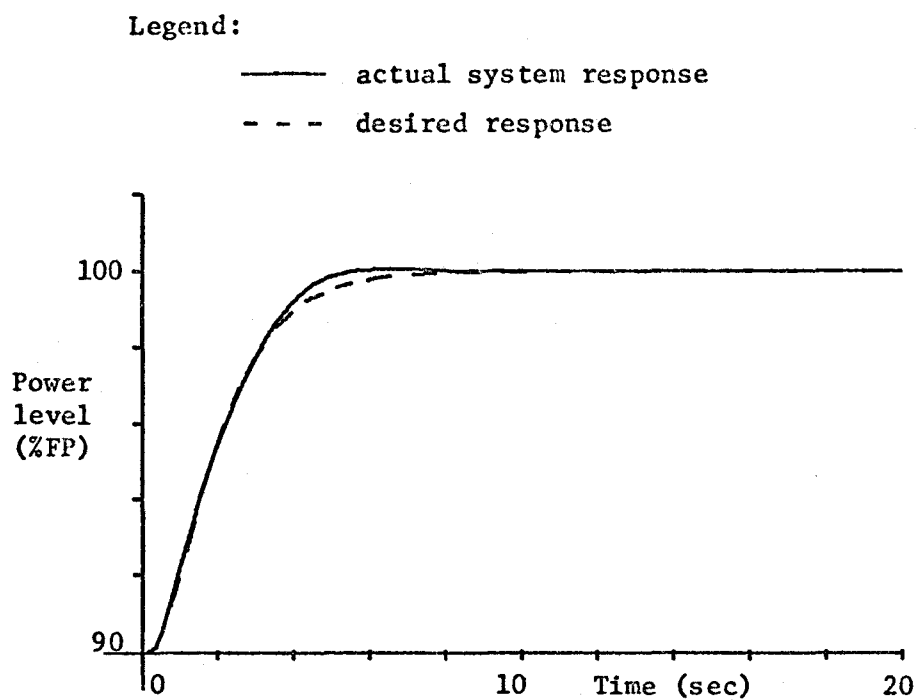


Figure 7-4. Adaptive nuclear reactor response to coincide with desired critically damped response.

approximated by a function in the form of equation (7-11). This has been achieved by the use of search routines, but at the present, these may take a considerable amount of computer time, and create problems with on-line realization. While this time can be reduced by using fewer sample points and initial values in the proximity of the optimum, considerable work is yet to be done in developing fast search methods which can approximate the optimum to a reasonable accuracy in a few steps. For the results in this section, pattern search has been used, which is suitable to illustrate the principle of the technique, although inefficient for on-line applications. Adaptive responses are shown for the three suboptimal examples considered in the last chapter: (i) a monotonic reactivity insertion, such that the cost function in equation (6-4) is minimized, (ii) ramp increases in reactivity, subject to the objective of equation (6-5) and (iii) of equation (6-6). The power level changes from 90% to 100% FP in each case.

The reactor response as well as the corresponding reactivity insertion are shown in Figure 7-5. The suboptimal response has also been shown for comparison, and no significant deviations are observed. The actual cost has somewhat deteriorated from the suboptimal case i.e., 1.368 to 1.412. Considering that only 10 function evaluations were permitted (requiring approximately 1.0 second), the closeness of the responses is quite good.

For the ramp change in power level and the cost function containing the sum of the absolute values of the errors, the result is shown in Figure 7-6a, while the response for the minimax objective

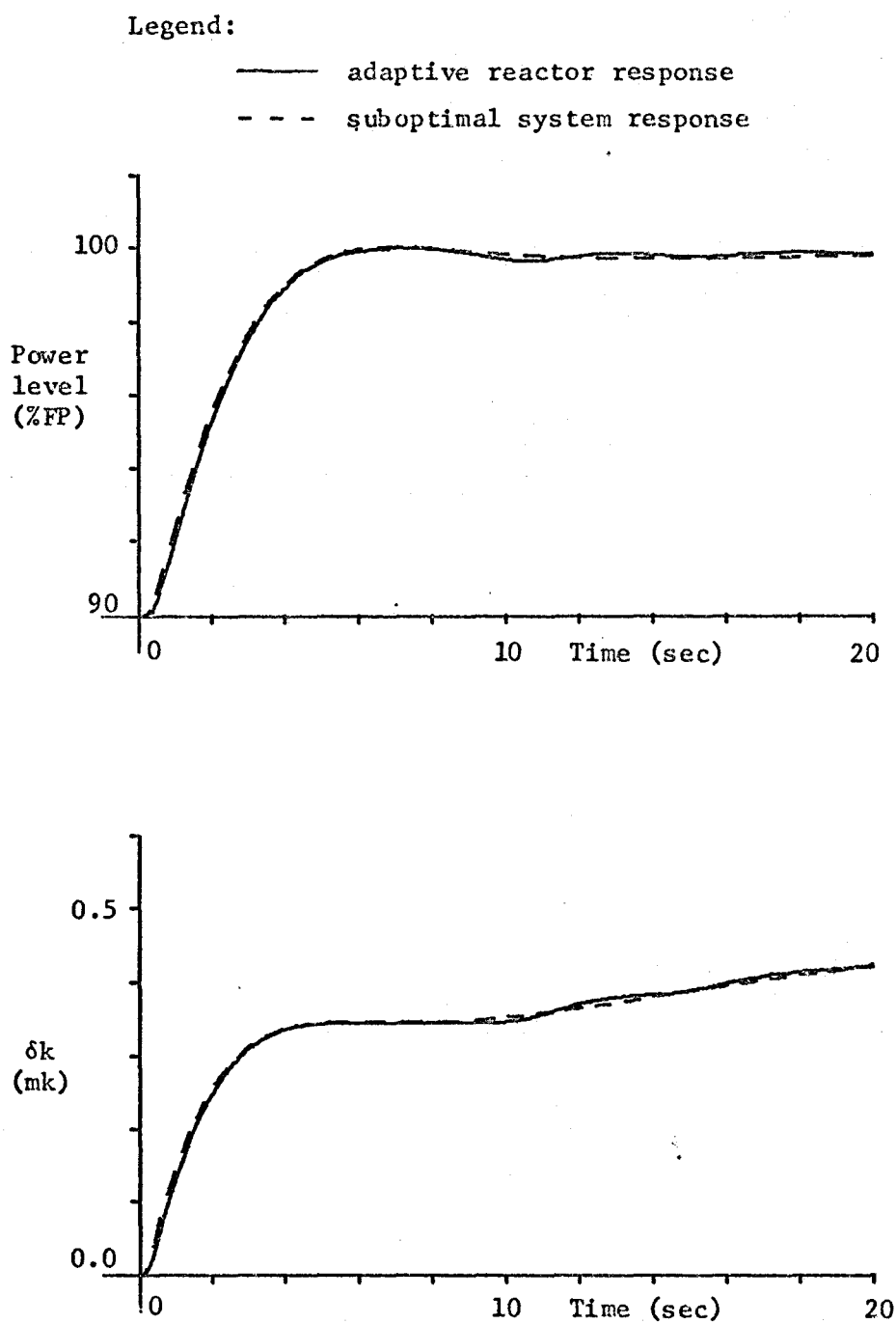
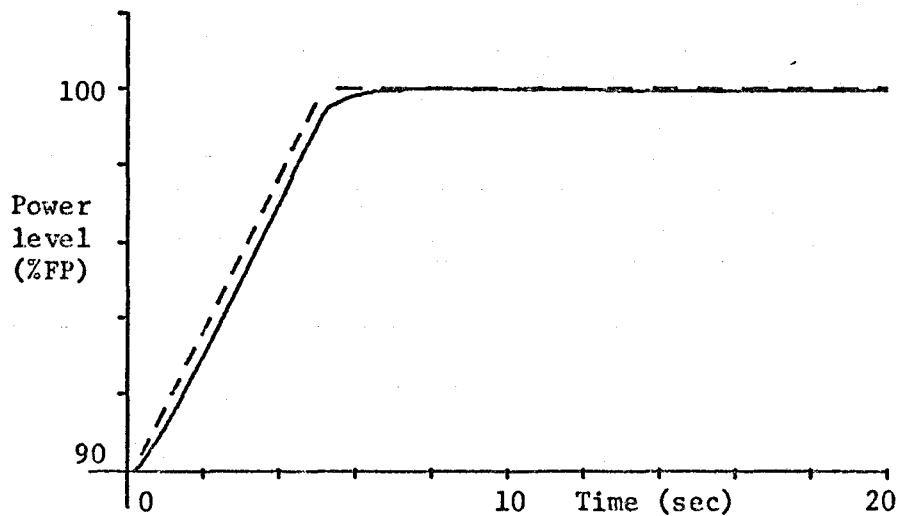


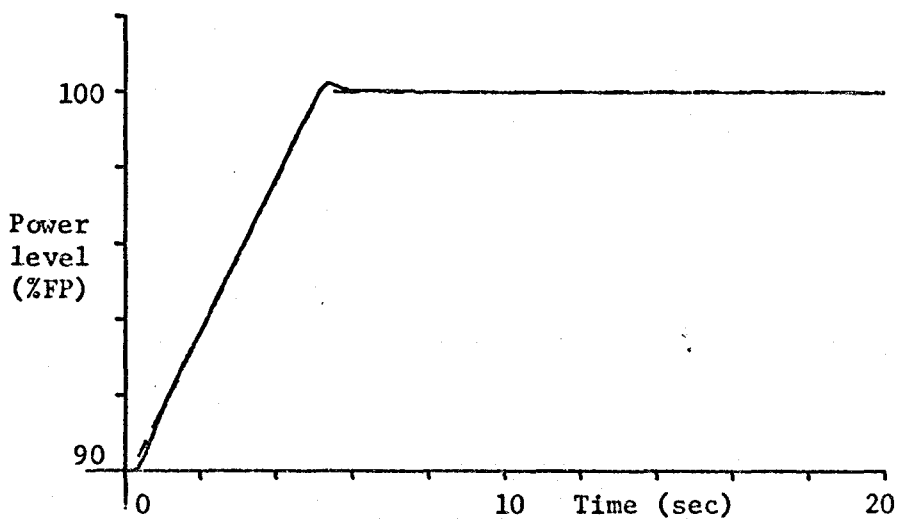
Figure 7-5. Adaptive and suboptimal reactor responses for monotonic reactivity insertion.

Legend:

— adaptive reactor response
- - - demanded power change



a. Response for least pth cost function, $p=1$



b. Response for minimax cost function

Figure 7-6. Adaptive reactor responses for ramp changes in demanded power.

is indicated by Figure 7-6b. The latter one is seen to give particularly good following of the desired response. The changes in system costs, in comparison to the suboptimal ones, are:

	Least pth	Minimax
Suboptimal	0.420	0.0281
Adaptive	0.388	0.0318

Good agreement between the costs is evident, a small decrease for the least pth case, and a slight increase in minimax cost.

While it is difficult to choose a basis of comparison between the proposed adaptive controller and the existing control system at the Douglas Point reactor, since the design objective for the latter is not known, the response of the actual plant is shown in Figure 7-7, as obtained in reference [5]. Clearly, at least as far as the ability of the system to follow the demanded power change is concerned, the proposed scheme is far superior to the existing one.

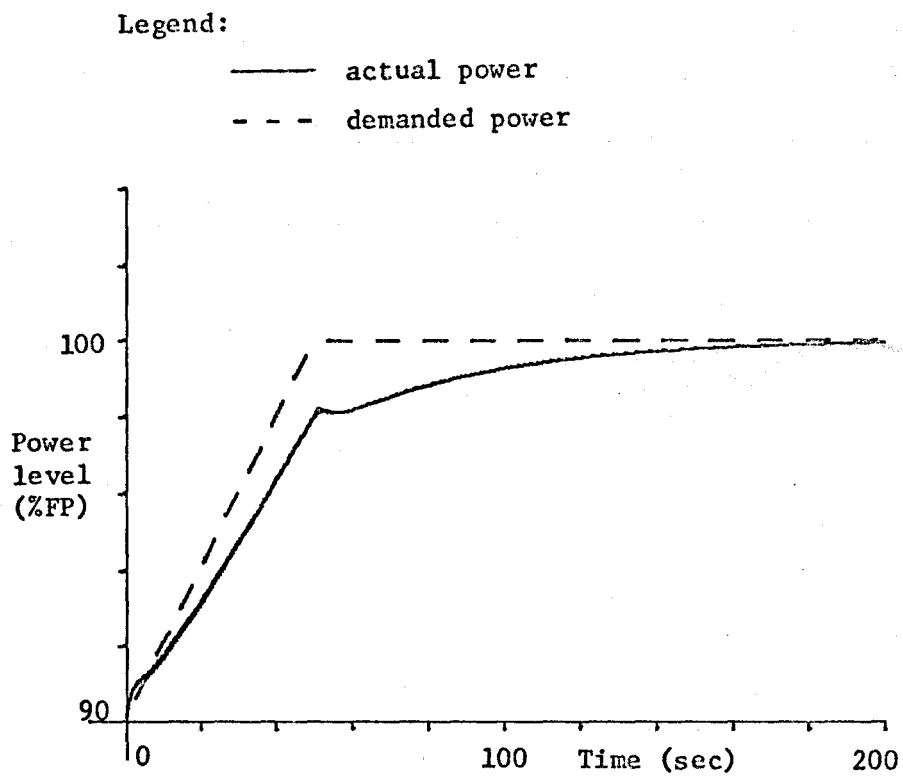


Figure 7-7. Response of Douglas Point nuclear reactor to a 90%-100% FP change in demanded power.

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

While at the outset of this thesis considerable emphasis was placed on the particular control system at the Douglas Point power station, a great deal more has been accomplished than simply proposing a new type of controller for this particular plant, or for other reactors of its kind. Hence, only a fleeting comparison between the results achieved by the proposed adaptive controller and the performance of the existing control system has been made.

As a result of this thesis, a method now exists for the adaptive control of high-order nonlinear systems with time-varying parameters, provided a suitable general purpose computer is available to implement the control system. The nuclear power reactor is an important example of such a system, hence its extensive use in demonstrating the capabilities of the proposed method.

At the heart of the new technique is the use of certain numerical optimization algorithms, known as search routines, to find a low-order linear model of the plant. The parameters of the model are chosen to minimize the deviation between the observed system response and that of the model, in any desired sense. For such a model, the optimal feedback parameters can be computed readily for integral quadratic cost functions or if the desired response can be described by a rational transfer function having no finite zeros. For the more general least pth and minimax objectives, the search

routines are again used to find the optimal controller coefficients of the model. While the same control system will result in suboptimal performance of the plant, provided the model gives a sufficiently accurate representation of the system response, the difference between optimal model cost and suboptimal system cost will be negligible for practical purposes. It should also be realized, that an adaptive system never quite reaches the optimum, hence the approach of successively improved suboptimal controllers is quite valid.

In deriving the adaptive controller, two important contributions have been made, in the areas of modelling and suboptimal control of high-order linear systems. Previously proposed methods, for reducing the order of the system describing differential equations, have been restricted to either qualitative criteria for this purpose, or only a least squares type objective could be minimized. Using the modelling technique presented in this thesis, no restriction is placed on the error criterion.

While the possibility of representing a high-order system by a substantially lower-order one is useful in its own right; it has been demonstrated in this thesis that the optimal controller derived for the low-order model can be used for the suboptimal control of the high-order system. Furthermore, for systems having no finite zeros, a new approach has been proposed for obtaining the optimal feedback controller for least pth and minimax cost functions. For systems having finite zeros, suboptimal performance may be realized by the above-mentioned modelling technique.

A great advantage of the proposed technique of designing a suboptimal controller is, that the whole process can be realized in the form of a digital computer programme. All the designer has to specify is the system equations or, alternatively, a set of input-output data that characterizes the system, as well as the desired closed-loop performance. The latter may be an integral or a sum of the p th power of samples of the input and the output as well as their derivatives; or the desired time-domain response may be specified in terms of the step response of a rational transfer function having no finite zeros.

Because of the wide range of topics covered in this thesis, from the modelling and suboptimal control of high-order linear systems to the adaptive control of a nuclear reactor, many of these areas need a considerable amount of additional research effort. As an extension of the modelling work described in this thesis, the application of various new search routines to this problem has already been undertaken^[44]. Some of the optimal models derived for the seventh-order system have been used in the investigation of suboptimal control of high-order systems^[45]. Both of these areas are potentially open for additional research: models of order greater than two may be needed in certain applications, or more than one model of first- and second-order should be used to approximate a given system, in a piecewise manner. The corresponding suboptimal controller would then have coefficients which change at the preassigned instants. An extension of the proposed method of optimal control for least p th and minimax cost functions, to cover the case of responses with finite zeros,

would also be an important contribution.

Instead of using a rational transfer function to approximate the optimal response of the system, the use of other mathematical functions should be considered. Polynomials are an obvious candidate, and the application of orthogonal functions could also lead to useful results. Legendre polynomials have already been tested in such an application^[22] and the use of Fourier series for signal representation is well-known. For the use of special purpose digital controllers, the unique properties of Walsh functions^{[47],[48]} could be used to great advantage, in modelling the system as well as representing the desired performance.

Turning our attention now to the adaptive aspect of our problem, the most important improvements in the proposed method should accrue from the application of more efficient numerical optimization methods to both updating the model parameters and to reoptimize the controller coefficients. The use of gradient search routines and new concepts in optimizing minimax objectives^{[49],[50]} should lead to improvements in the accuracy of locating the minimum. There is also a great need for search techniques which can produce significant reductions in the cost function in only a few steps, and requiring word lengths and fast access memory storage appropriate to process computers.

The proposed solution of the reactor control problem also suggests many areas of useful research. The consideration of noise is perhaps the foremost of these. The effect of noisy observations on identifying the model parameters and on the resultant optimal controller should be investigated. Regulation of the power level,

despite disturbances within the plant and in the measurements, is another important practical problem.

With the increasing size of nuclear power reactors, spatial effects become increasingly significant. The approach presented here, based on a point kinetic model will need to be modified, to take into account the interaction of the various control zones of the reactor. Maintaining a uniform flux density and preventing the build-up of xenon spatial oscillations is also an important task of the control system. The optimal start-up and shut-down control of the reactor open up further areas for the use of modern control theory.

Perhaps the single most notable contribution of the work presented in this thesis is the application of some well established theoretical results to a practical problem. While it is not claimed that all the questions which arise in the adaptive control of nuclear reactors have been fully answered, the approach presented here and the results which have been obtained, should be a considerable contribution towards achieving that goal.

REFERENCES

- [1] S.C. Gupta and L. Hasdorf: "Fundamentals of Automatic Control", Wiley, 1970.
- [2] H. Chestnut and R.W. Mayer: "Servomechanisms and Regulating System Design", Wiley, 1959.
- [3] K. Ogata: "State Space Analysis of Control Systems", Prentice-Hall, 1967.
- [4] D.G. Schultz and J.L. Melsa: "State Functions and Linear Control Systems", McGraw-Hill, 1967.
- [5] G.T. Bereznai: "Adaptive Control of Nuclear Reactors Using a Digital Computer", M.Eng. Thesis, McMaster University, 1969.
- [6] L.E. Weaver: "Reactor Dynamics and Control", Americal Elsevier, 1968.
- [7] W.C. Lipinski: "Optimal Digital Computer Control of Nuclear Reactors", Argonne National Laboratory Report ANL-7530, 1969.
- [8] G.T. Bereznai and N.K. Sinha: "Adaptive Control and Identification of Nuclear Reactors", Preprints of the Second Prague IFAC Symposium on Identification and Process Parameter Estimation, June 1970.
- [9] C.D. Gregg King: "Nuclear Power Systems", Macmillan, 1964.
- [10] R. Bellman: "Dynamic Programming", Princeton University Press, 1957.
- [11] L.S. Pontryagin, V.G. Battyanskii, R.V. Gamkrelidze and E.F. Mishcenko: "The Mathematical Theory of Optimal Processes", Wiley, 1962.

- [12] A.P. Sage: "Optimum Systems Control", Prentice-Hall, 1968.
- [13] K. Monta: "Time Optimal Digital Computer Control of Nuclear Reactors", Journal of Nuclear Science and Technology, vol. 3, no. 6, pp. 227-236, June 1966.
- [14] M. Ash: "Optimal Shutdown Control of Nuclear Reactors", Academic Press, 1966.
- [15] B.V. Dasarathy: "Optimal Control in Nonlinear Systems", IEEE Trans., vol. AC-15, no. 6, pp. 690-692, Dec. 1970.
- [16] G.N.T. Lock and M. Enns: "Optimal Control Trajectories with Minimax Objective Functions by Linear Programming", IEEE Trans., vol. AC-12, no. 6, pp. 749-752, Dec. 1967.
- [17] S.J. Asseo: "Optimal Control of a Servo Derived from Nonquadratic Performance Criteria", IEEE Trans., vol. AC-14, no. 4, pp. 404-407, Aug. 1969.
- [18] D. Tabak: "Applications of Mathematical Programming Techniques in Optimal Control: A Survey", IEEE Trans., vol. AC-15, no. 6, pp. 688-690, Dec. 1970.
- [19] A.R.M. Noton, P. Dyer and C.A. Markland: "Numerical Computation of Optimal Control", IEEE Trans., vol. AC-12, no. 1, pp. 59-66, Feb. 1967.
- [20] J.R. Lamarsh: "Introduction to Nuclear Reactor Theory", Addison-Wesley, 1966.
- [21] R.L. Murray, C.R. Bingham and C.F. Martin: "Reactor Kinetics Analysis by an Inverse Method", Nucl. Sci. and Eng., vol. 18, pp. 481-490, 1964.

- [22] W.M. Stacey: "Application of Variational Synthesis to the Optimal Control of a Point Reactor Model", Nucl. Sci. and Eng., vol. 33, pp. 257-260, 1968.
- [23] M. Enns: "Optimal Control of a Nuclear Reactor Power Plant", Lecture Notes in Operation Research and Mathematical Economics, vol. 14: Computing Methods in Optimization Problems, pp. 77-91, Springer-Verlag, Heidelberg, 1969.
- [24] L.E. Weaver and D.G. Schultz: "Reactor Control via State Variable Feedback", as above, pp. 180-191.
- [25] W.C. Lipinski and A.G. Vacroux: "Optimal Digital Computer Control of Nuclear Reactors", IEEE Trans., vol. NS-17, no. 1, pp. 510-516, Feb. 1970.
- [26] E.J. Davison: "A Method for Simplifying Linear Dynamic Systems", IEEE Trans., vol. AC-11, pp. 93-101, Jan. 1966.
- [27] N.K. Sinha and J.C. Wismath: "Reduction of High-Order Linear Dynamic Systems", Trans. of the Eng. Inst. of Canada, vol. 14, no. C-5, June 1971.
- [28] D.A. Pierre: "Optimization Theory with Applications", Wiley, 1969.
- [29] J.W. Bandler: "Optimization Methods for Computer-Aided Design", IEEE Trans., vol. MTT-17, no. 8, pp. 533-552, Aug. 1969.
- [30] R. Hooke and T.A. Jeeves: "'Direct Search' Solution of Numerical and Statistical Problems", J. ACM, vol. 8, pp. 212-229, April 1961.
- [31] R. Fletcher: "A New Approach to Variable Metric Algorithms", Computer Journal, vol. 13, pp. 317-322, Aug. 1970.

- [32] R. Fletcher and M.J.D. Powell: "A Rapidly Convergent Descent Method for Minimization", Computer Journal, vol. 6, pp. 163-168, June 1963.
- [33] B. Ramaswami and K. Ramar: "Transformation to Phase-variable Canonical Form", IEEE Trans., vol. AC-13, pp. 746-747, Dec. 1968.
- [34] J.W. Bandler and P.A. Macdonald: "Optimization of Microwave Networks by Razor Search", IEEE Trans., vol. MTT-17, pp. 552-562, Aug. 1969.
- [35] E.J. Davison: "A New Method for Simplifying Large Linear Dynamic Systems", IEEE Trans., vol. AC-13, pp. 214-215, April 1968.
- [36] D. Mitra: "The Reduction of Complexity of Linear Time-invariant Dynamic Systems", Proc. 4th IFAC Congress, Warsaw, Poland, pp. 19-33, June 1969.
- [37] C.F. Chen and L.S. Shieh: "A Novel Approach to Linear Model Simplification", Preprints of the 1968 JACC, Ann Arbor, Mich., pp. 454-460, June 1968.
- [38] J.H. Anderson: "Geometrical Approach to Reduction of Dynamic Systems", Proc IEE, vol. 114, pp. 1014-1018, July 1967.
- [39] N.K. Sinha and W. Pille: "A New Method for Reduction of Dynamic Systems", Int. J. of Control, vol. 14, pp. 111-118, July 1971.
- [40] R.C. Dorf: "Modern Control Systems", Addison-Wesley, 1967, pp. 167-168.
- [41] A.V. Balakrishnan and V. Peterka: "Identification in Automatic Control Systems", Automatica, vol. 5, pp. 817-829, 1969.

- [42] K.J. Astrom and P. Eykoff: "System Identification", Survey Paper, 2nd Prague IFAC Symposium, June 1970.
- [43] N.K. Sinha and W. Pille: "On-line Parameter Estimation using Matrix Pseudoinverse", Proc. IEE, vol. 118, no. 8, pp. 1041-1046, Aug. 1971.
- [44] J.W. Bandler, N.D. Markettos and N.K. Sinha: "Optimum Approximation of High-order Systems by Low-order Models Using Recent Gradient Methods", Proceedings of the 9th Annual Allerton Conference on Circuits and Systems, Monticello, Illinois, Oct. 1971.
- [45] N.K. Sinha and H. deBruin: "Application of Low-order Models for Near-optimal Control of High-order Systems", to be presented at the 5th Hawaii International Conference on System Sciences, January 1971.
- [46] F.T. Man and H.W. Smith: "Design of Linear Regulators Optimal for Time-Multiplied Performance Indices", IEEE Trans., vol. AC-14, no. 5, pp. 527-529, Oct. 1969.
- [47] H.F. Harmuth: "Transmission of Information by Orthogonal Functions", Springer-Verlag, 1969.
- [48] K.H. Siemens and R. Kitai: "Digital Walsh-Fourier Analysis of Periodic Waveforms", IEEE Trans., vol. IM-18, no. 4, pp. 316-321, Dec. 1969.
- [49] J.W. Bandler and T.V. Srinivasan: "A New Gradient Algorithm for Minimax Optimization of Networks and Systems", Proceedings of 14th Midwest Symposium on Circuit Theory, Denver, Colorado, May 1971, pp. 16.5.1-16.5.11.

- [50] J.W. Bandler and C. Charalambous: "Practical Least pth Approximation with Extremely Large Values of p", Proceedings of the 5th Asilomar Conference on Circuits and Systems, Pacific Grove, California, Nov. 1971.

BIBLIOGRAPHY

MODERN CONTROL THEORY

- C.H. Schley and I. Lee: "Optimal Control Computation by the Newton-Raphson Method and the Riccati Transformation", IEEE Trans., vol. AC-12, no. 2, pp. 139-144, April 1967.
- L.S. Lasdon, S.K. Mitter and A.D. Warren: "The Conjugate Gradient Method for Optimal Control Problems", IEEE Trans., vol. AC-12, no. 2, pp. 132-138, April 1967.
- A.V. Balakrishnan: "On a New Computing Technique in Optimal Control", SIAM Journal of Control, vol. 6, no. 2, pp. 149-162, 1968.
- T.L. Johnson and M. Athans: "On the Design of Optimal Constrained Dynamic Compensators for Linear Constant Systems", IEEE Trans., vol. AC-15, no. 6, pp. 658-660, Dec. 1970.
- W.E. Bosarge and O.G. Johnson: "Direct Method Approximation to the State Regulator Control Problem Using a Ritz-Trefftz Suboptimal Control", IEEE Trans., vol. AC-15, no. 6, pp. 627-631, Dec. 1970.
- R.L. Kosut: "Suboptimal Control of Linear Time-Invariant Systems Subject to Control Structure Constraints", IEEE Trans., vol. AC-15, no. 5, pp. 557-563, Oct. 1970.
- M.R. Chidambara and R.B. Schainker: "Lower Order Generalized Aggregated Model and Suboptimal Control", IEEE Trans., vol. AC-16, no. 2, pp. 175-180, April 1971.
- J.H. Burghart: "A Technique for Suboptimal Feedback Control of Nonlinear Systems", IEEE Trans., AC-14, no. 5, pp. 530-532, Oct. 1969.

- W.L. Garrard, N.H. McClamroch and L.G. Clark: "An Approach to Sub-optimal Feedback Control of Nonlinear Systems", Int. J. of Control, vol. 5, no. 5, pp. 425-435, 1967.
- B. Freidland: "A Technique of Quasi-Optimum Control", J. of Basic Engineering, pp. 437-443, June 1966.
- T.C. Hsiao: "Low Order Design for High Order Adaptive Control Systems", Proc. of the 1970 IEEE System Science and Cybernetics Conf., Pittsburgh, Pa., pp. 154-158, Oct. 1970.
- M.A. Schurer and K. Steiglitz: "Adaptive Step Size Random Search", IEEE Trans., vol. AC-13, no. 3, pp. 270-276, June 1968.
- R.S. Ratner and D.G. Luenberger: "Performance-Adaptive Renewal Policies for Linear Systems", IEEE Trans., vol. AC-14, no. 4, pp. 344-351, Aug. 1969.
- A.E. Pearson and K.-S.V.R. Vanguri: "A Synthesis Procedure for Parameter Adaptive Control Systems", IEEE Trans., vol. AC-16, no. 5, pp. 440-449, Oct. 1971.

NUCLEAR REACTOR CONTROL

- T.W. Kerlin: "Stability Extrema in Nuclear Power Systems with Design Uncertainties", Nucl. Sci. and Eng., vol. 27, pp. 120-130, 1967.
- T.A. Porshing: "Numerical Solution of the Reactor Kinetics Equations by Approximate Exponentials", Nucl. Sci. and Eng., vol. 25, pp. 183-188, 1966.
- F. Bevilacqua and E.A. Yuile: "Application of Digital Computer Systems in Nuclear Power Plants", IEEE Trans., vol. NS-16, no. 1, pp. 196-206, Feb. 1969.

J.A. Luopa and R.D. Anderton: "Direct Digital Control of a Nuclear Power Reactor", IEEE Trans., vol. NS-17, no. 1, pp. 510-516, Feb. 1970.

A.P. Sage and G.W. Masters: "Identification and Modeling of Nuclear Reactors", IEEE Trans., vol. NS-14, no. 1, pp. 279-285, Feb. 1967.

W. Ciechanowicz: "On Transient Digital Control of Large Nuclear Power Reactors", Nucl. Sci. and Eng., vol. 31, pp. 465-473, 1968.

J.C. Venerus: "Estimation of the Dynamic Reactivity Using Digital Kalman Filtering", Nucl. Sci. and Eng., vol. 40, pp. 199-205, 1970.

R.C. Kern, W.C. Coppersmith and Z.R. Rosztoczy: "A Three-Dimensional Method for Designing Studies of Xenon-Induced Spatial Power Oscillations", Nucl. Appl. and Tech., vol. 9, no. 1, pp. 70-82, July 1970.

T.J. Marciniak: "Time-Optimal Digital Control of Zero-Power Nuclear Reactors", Nucl. Appl. and Tech., vol. 8, no. 5, pp. 401-416, May 1970.

L.R. Peterson and L.E. Weaver: "A Graphical Design of an Optimum Control System to Minimize Boiling Reactor Noise", Nucl. Sci. and Eng., vol. 21, pp. 40-48, 1965.

J.R. Sheff and R.W. Albrecht: "The Space Dependence of Reactor Noise", Nucl. Sci. and Eng., vol. 24, pp. 246-259, 1966.

T.W. Kerlin: "Stability Extrema in Nuclear Power Systems with Design Uncertainties", Nucl. Sci. and Eng., vol. 27, pp. 120-130, 1967.

W.M. Stacey: "Optimal Control of Xenon-Power Spatial Transients",

Nucl. Sci. and Eng., vol. 33, pp. 162-168, 1968.

C.L. Partain, et. al.: "On-line Digital Data Handling and Optimal

Control Investigations at the Missouri University Research

Reactor", IEEE Trans., vol. NS-18, no. 1, pp. 395-402, Feb. 1971.

P.F. Schweizer: "Design of Specifically Optimal Filters for Nuclear

Reactor Systems", IEEE Trans., vol. NS-18, no. 1, pp. 403-

407, Feb. 1971.

C.G. Love: "A Nonlinear Dynamic Optimization Technique for Controlling

Xenon Induced Oscillations in Large Nuclear Reactors", IEEE

Trans., vol. NS-18, no. 1, pp. 408-411, Feb. 1971.

PUBLICATIONS

Papers by the author, related to this thesis and accepted for publication.

1. G.T. Bereznai, "A Modern Approach to Nuclear Reactor Control".
Presented at the ANS Niagara-Finger Lakes Section Student Conference, May 24-25, 1970, Toronto. (Awarded Second prize.)
2. G.T. Bereznai and N.K. Sinha, "Adaptive Control and Identification of Nuclear Reactors". Proceedings of the 2nd Prague IFAC Symposium on Identification and Process Parameter Estimation, Czechoslovakia, June 1970, pp. 6.7.1-6.7.7.
3. G.T. Bereznai and N.K. Sinha, "A New Minimax Objective for Automated Design". Electronics Letters, vol. 6, no. 26, December 1970, pp. 847-849.
4. G.T. Bereznai, "Suboptimum Control of Power Transients in Nuclear Reactors". Presented at the ANS Niagara-Finger Lakes Section Student Conference, March 26-27, 1971, State University of New York at Buffalo. (Awarded second prize.)
5. G.T. Bereznai and N.K. Sinha, "Computer-aided Design of Suboptimal Feedback Controllers using Optimal Low-order Models". Electronics Letters, vol. 7, no. 15, July 1971, pp. 440-442.
6. G.T. Bereznai and N.K. Sinha, "Adaptive Control of Nuclear Reactors using a Digital Computer". IEEE Transactions on Nuclear Science, vol. NS-18, August 1971, pp. 387-397.

7. G.T. Bereznaï and N.K. Sinha, "System Identification Using Search Routines". Proceedings of the Fifth Asilomar Conference on Circuits and Systems, November 1971.
8. N.K. Sinha and G.T. Bereznaï, "Optimum Approximation of High-order Systems by Low-order Models". International Journal of Control, vol. 14, no. 5, November 1971, pp. 951-960.
9. G.T. Bereznaï and N.K. Sinha, "Adaptive Control of Large-Scale Nonlinear Systems using Optimal Low-order Models". To be presented at the 1971 IEEE Decision and Control Conference, Miami, Florida, December 1971.
10. G.T. Bereznaï and N.K. Sinha, "An Inverse Approach to Computer-Aided Feedback System Design". To be presented at the 5th Hawaii International Conference on System Sciences, Honolulu, January 1972.
11. G.T. Bereznaï and N.K. Sinha, "Application of Optimum Low-order Models to the Adaptive Control of Nuclear Reactors". To be presented at the 5th IFAC World Congress, Paris, June 1972.

APPENDIX

Numerical values of the Douglas Point reactor model parameters, which have been defined in Chapter 3.

$$G = 945$$

$$\tau_m = 0.159 \text{ second}$$

$$f(v) = 0.02 \frac{\text{mk}}{\text{sec-volt}} \quad |v| \leq 15$$

$$= \frac{0.3}{v} \frac{\text{mk}}{\text{sec-volt}} \quad |v| > 15$$

$$T_c = 4.54 \text{ mk at 100\% FP}$$

$$\tau_T = 12.5 \text{ seconds}$$

$$\ell = 7.216 \cdot 10^{-4} \text{ second}$$

$$\beta = 4.867 \cdot 10^{-3}$$

Delayed group	β_i (%)	λ_i (sec ⁻¹)
2	0.05667	1.61
3	0.16067	0.457
4	0.14200	0.154
5	0.11067	0.0315
6	0.01667	0.0125