DIRECTLY DIFFERENTIABLE ARCS

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SCOPE AND CONTENTS:

This thesis is a study of arcs in real projective two and three spaces from the point of view of direct (linear) differentiability. It is composed of most of the research on this subject carried on to date by Dr. Ralph Park of the University of Calgary. To ensure a greater scope of understanding, Barner arcs are the integral part of the study in the real projective plane, whereas arcs with tower are emphasized in the three dimensional case. Diagrams have been included to give a geometric background and at times much needed visual aid.

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TABLE OF CONTENTS

CHAPTER V

CHAPTER VI

CHAPTER VII

CHAPTER I

1.1 Real Projective n-space

By a real projective n-space, $n \geq 1$, we mean a set $\underline{P}^n$ of objects called spaces along with a 1-1 mapping of $\underline{P}^n$ onto the set of all subspaces of a real (n+1)-dimensional vector space. As usual one defines the inclusion relation $L \subset M$ between spaces, the intersection $L \cap M$ of spaces and the span LM of spaces. For a collection of spaces, the intersection is denoted by $\bigcap_i L_i$ and the span by $V_i L_i$.

A space of $\underline{P}^n$ is said to have dimension k if the vector subspace corresponding to it has dimension $k + 1$. Let $\underline{P}^n_k$ denote the set of all k-spaces, $-1 \leq k \leq n$. The unique (-1)-space is denoted by $\emptyset$. The elements of $\underline{P}^n_0$ and $\underline{P}^n_{n-1}$ are called <u>points</u> and <u>hyperplanes</u> of $\underline{P}^n$ respectively.

1.2 A topology on $\underline{P}^n$

Let $\mathbb{R}$ be the real number system. Then $\mathbb{R}^{n+1}$ is a real (n+1)-dimensional vector space. We first define a topology on the points of $\underline{P}^n$.

Denote $\underline{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} - \{\bar{0}\}$ where $\bar{0}$ is the null vector of $\mathbb{R}^{n+1}$. Since the one-dimensional subspaces of $\mathbb{R}^{n+1}$ are the lines passing through the null vector, every non-zero vector $\bar{x} \in \mathbb{R}^{n+1}$ determines a unique one dimensional subspace of $\mathbb{R}^{n+1}$. We define an equivalence relation P_n on $\underline{\mathbb{R}}^{n+1}$ by $\bar{x}\, p_n\, \bar{y}$ iff $\bar{x}, \bar{y}$ determine the same one-dimensional subspace where $\bar{x}, \bar{y} \varepsilon \underline{\mathbb{R}}^{n+1}$. We now identify $\underline{P}^n$ with $\underline{\mathbb{R}}^{n+1}/p_n$ by the usual coordinatization of $\underline{P}^n$. Then we define a topology on the points of $\underline{P}^n$; i.e. $\underline{P}^n_0$, to be the quotient topology of the topology of $\underline{\mathbb{R}}^{n+1}$ induced by P_n.

We now extend the topology on $\underline{P}^n_0$ to the rest of $\underline{P}^n$. Let (L_i) be a sequence of spaces in $\underline{P}^n$. We define (L_i) to converge to a limit space L of $\underline{P}^n$ iff each sequence (x_i), $x_i \varepsilon L$, converges to some point of L and each point of L is a limit of some sequence (x_i) where $x_i \varepsilon L_i$. Or alternately, if L and M are spaces in $\underline{P}^n$, we define L to be sufficiently close to M iff each point of L belongs to a sufficiently small neighbourhood of some point of M. Thus each $\underline{P}^n_k$ is compact and connected and $\underline{P}^n$ is compact; cf. Bourbaki.

If L is a k-space, $-1 \leq k \leq n-2$, then the set $\underline{P}^{n-k-1}(L)$ of all spaces containing L is a projective space, where the "points" in $\underline{P}^{n-k-1}(L)$ are all (k+1)-spaces of $\underline{P}^n$ containing L and the "j-spaces" in $\underline{P}^{n-k-1}(L)$ are all (k+j+1)-spaces of $\underline{P}^n$ containing L.

It is readily seen from above that if (L_i), (M_i) are two convergent sequences of spaces and $L_i \subset M_i$ for $i = 1, 2, \ldots,$ then $\lim L_i \subset \lim M_i$, hence a sequence in $\underline{P}^{n-k-1}(L)$ converges in $\underline{P}^{n-k-1}(L)$ iff it converges in $\underline{P}^n$.

1.3 Directly differentiable arcs

Let J be an ordered topological space which is isomorphic with the ordered topological space $\mathbb{R}$. A set $X \subset J$ is called an <u>interval</u>

if there exist $p, q \varepsilon J$ with $p < q$ such that

$$X = (p,q),\ [p,q],\ (p,q],\ \text{or}\ [p,q].$$

Thus $[p,q) = \{ r \varepsilon J \mid p \leq r < q\}$. By a two sided (deleted, left, right) neighbourhood of $p \varepsilon J$ we mean a set $u(p) = (q,r)$ $(u'(p) = (q,p) \cup (p,r),\ u^-(p) = (q,p),\ u+(p) = (p,r))$ where $p \varepsilon u(p)$ and $q < p < r$. If X is a finite subset of J we write $|X|$ for the number of elements of J.

Let a mapping $A\colon\ J \longrightarrow P^n_o$ and a k-space L be given. For $p \varepsilon J$ it may happen that $\lim_{\substack{q \to p \\ q \neq p}} A(q)L$ exists,

in which case we denote it by $A(p)/L$.

We say A is <u>directly differentiable</u> at p if there exist spaces

$$A_k(p) \ \varepsilon\ \underline{P}^n_k\ ,\quad -1 \leq k \leq n,$$

such that $A_o(p) = A(p)$ and $A_k(p) = A(p)/A_{k-1}(p)$, $0 \leq k \leq n$. Thus if A is directly differentiable at p it is also continuous at p.

By an <u>arc</u> we mean a mapping $A\colon\ J \longrightarrow \underline{P}^n_0$ which is directly differentiable at each $p \varepsilon J$. If A is an arc then $A_k(p)$ is called the <u>osculating k-space</u> of A at p, and $p \varepsilon J$ is said to be a point of A.

<u>1.3.1 Lemma</u>: Let A be an arc and L a hyperplane. For each $p \varepsilon J$ there exists a $u'(p)$ such that $A(q) \not\varepsilon L$ if $q \varepsilon u'(p)$.

<u>Proof</u>: Let $p \in J$ be given. Let $A_k(p)$ be the largest osculating space of p contained in L. Assuming L meets every $u'(p)$, there exists a sequence $p_i \longrightarrow p$, $p_i \neq p$ and $A(p_i) \subset L$. Then $A_{k+1}(p) = \lim_{\substack{q \to p \\ q \neq p}} A(q)\, A_k(p)$ is contained in L, a contradiction.

<u>Corollary 1</u>: If $n \geq 2$, the image of an arc cannot contain a line segment.

<u>Corollary 2</u>: If $X \subset J$ is compact, then for any hyperplane L, $A(p) \subset L$ for only finitely many $p \in X$.

1.4 Dually differentiable arcs

We have defined $\underline{P}^n$ such that each k-space L corresponds to a (k+1)-dimensional subspace of a real (n+1) dimensional vector space (say V). If we now associate to L the corresponding (n-k)-dimensional subspace of V^*, the dual of V, then $\underline{P}^n$ is again a projective space called the <u>dual</u> $\underline{P}^{n^*}$ of $\underline{P}^n$. One has $\underline{P}_k^{n^*} = \underline{P}^n_{n-k-1}$, $-1 \leq k \leq n$.

Let A be an arc in P^n. Define a mapping A^*: $J \longrightarrow \underline{P}_0^{n^*}$ by

$$A^*(p) = A_{n-1}(p)$$

for all $p \in J$. We say A is <u>dually differentiable</u> if A^* is an arc in $\underline{P}^{n^*}$ and for all $p \in J$

$$A_k^*(p) = A_{n-k-1}(p), \quad -1 \leq k \leq n.$$

If A^* is an arc, then for $p \in J$, $-1 \leq k \leq n$,

$$A_{n-k}(p) = A^*_{k+1}(p) = A^*(p)/A^*_k(p)$$

$$= \lim_{\substack{q \to p \\ q \neq p}} A^*(q)\ A^*_k(p)$$

$$= \lim_{\substack{q \to p \\ q \neq p}} A_{n-1}(q) \cap A_{n-k-1}(p).$$

Therefore, A is dually differentiable iff $A_k(p) = \lim_{\substack{q \to p \\ q \neq p}} A_{n-1}(q) \cap A_{k+1}(p)$

for all $P \in J$, $-1 \leq k \leq n-1$.

1.4.1 Lemma: If A is dually differentiable then A_{n-1} is continuous. If $P \in \underline{P}^n_0$ and $p \in J$, then there exists a $u'(p)$ such that $p \not\in A_{n-1}(q)$ for all $q \in u'(p)$.

Proof: 1.3.1.

CHAPTER II

In this and the following two chapters, all our considerations will be limited to $\underline{P}^2$ unless stated otherwise.

2.1 Projection

Let A be an arc, p a point of A and $P \in \underline{P}_0^2$. We define $\pi(P, p)$ to be the dimension of the largest osculating space at p which does not contain P. We now define

$$\tilde{A}_k(p) = \begin{cases} A_k(p)P, & -1 \leq k \leq \pi(P,p) \\ & \text{if} \\ A_{k+1}(p), & \pi(P,p) < k \leq 1. \end{cases}$$

2.1.1 Theorem: $\breve{A} = \{ \breve{A}_0(p) | p \in J \}$ is an arc in $\underline{P}^1(P)$ with $\tilde{A}_k(p)$ as its osculating k-space at p.

Proof: Since $\tilde{A}_k(p)$ is a $(k+1)$-space in $\underline{P}^2$ and $P \in \tilde{A}_k(p)$, $-1 \leq k \leq 1$, $\tilde{A}_k(p) \in \underline{P}_k^1$ (P). Now there exists a $u'(p)$ such that $A(q) \neq P$ if $q \in u'(p)$. We may assume $P \neq A(p)$ and $PA(p)$ is a line. Then $u'(p)$ exists from 1. 3.1. Therefore wish to show $\breve{A}_k(p) = \breve{A}(p)/\breve{A}_{k-1}(p)$.

Case 1: $-1 \leq k \leq \pi(P,p)$. Then $P \notin A_k(p)$ and

$$\breve{A}(p)/\breve{A}_{k-1}(p) = \lim_{\substack{q \to p \\ q \neq p}} \breve{A}(q)\, \breve{A}_{k-1}(p)$$

$$= \lim_{\substack{q \to p \\ q \neq p}} A(q)\, A_{k-1}(p)P$$

$$= A_k(p)P = \breve{A}_k(p)$$

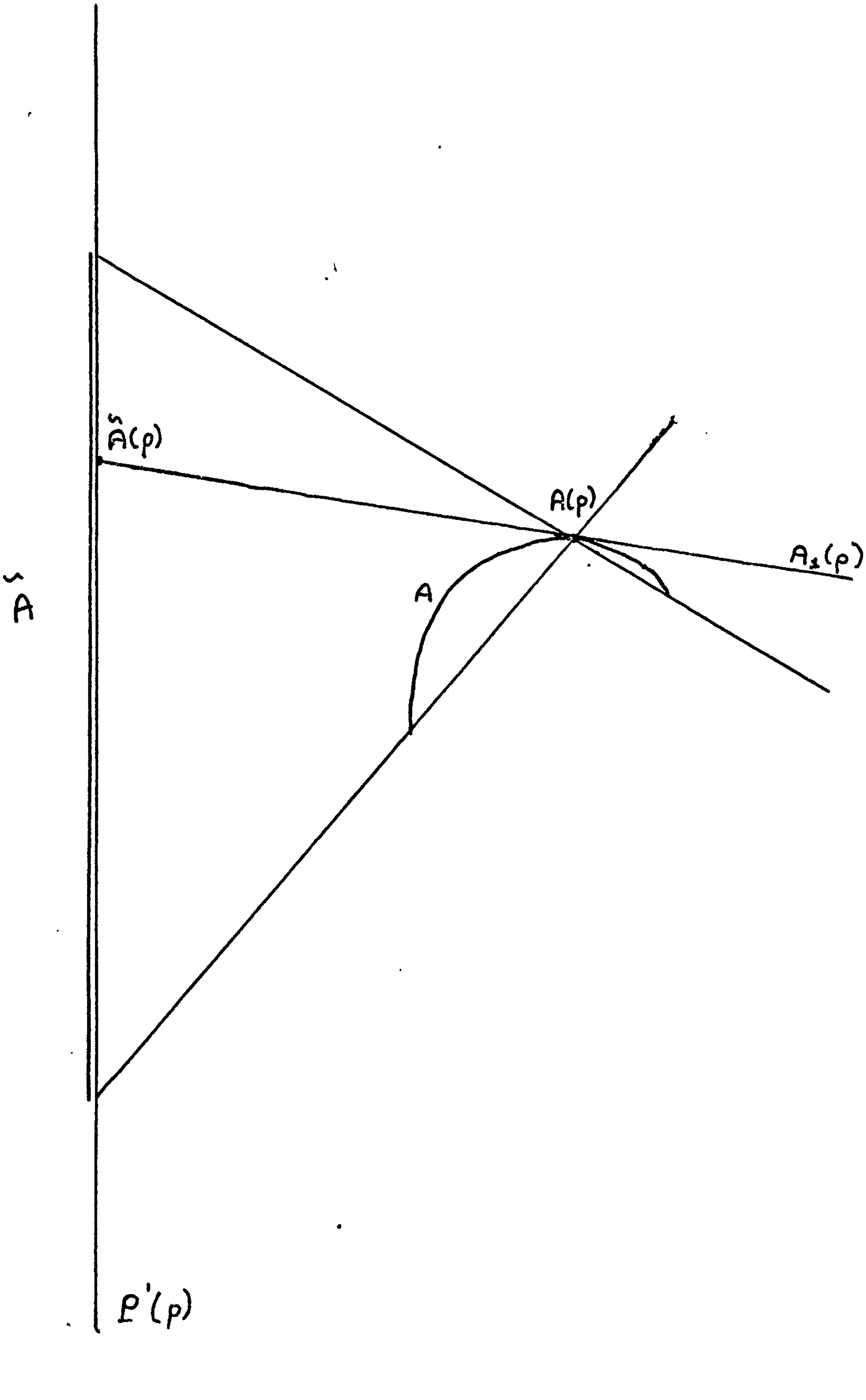


Figure II.1

Case 2: $\pi(P,p) < k \leq 1$. Then $P \in A_k(p)$ and

$$\breve{A}(p)/\breve{A}_{k-1}(p) = \lim_{\substack{q \to p \\ q \neq p}} (A(q)P)\ A_k(p)$$

$$= \lim_{\substack{q \to p \\ q \neq p}} A(q)\ A_k(p)$$

$$= A_{k+1}(p) = \breve{A}_k(p).$$

The arc $\breve{A}$ is called the projection of A from P. In general, we denote $\breve{A}$ by A/P. In Figure II.1, we have an example of projection from a point p of A where we assume $A_1(p)$ does not meet A outside of p.

2.2 Secants of an arc

Let A be an arc and L be a k-space. Let $(\delta(p,L))$ $A_{\delta(p,L)}$ be (the dimension of) the largest osculating space at p which is contained in L; thus $-1 \leq \delta(p,L) \leq k$ and

$$\bigvee_{p \in X} A_{\delta(p,L)}(p) \subset L$$

for any $X \subset J$. If the inclusion is improper we say L is a k-secant of X.

In the Euclidean plane one considers secants to be lines which pass through exactly two points of a figure. But in $\underline{P}^2$, a 1-secant may meet an arc at only one point as, for example, in Figure II. 1, taking $L = A_1(p)$.

We will now consider some basic properties of secants in $\underline{P}^2$. In most cases the proofs given with minimal modification will apply to results in Chapter V, Section 3.

2.2.1 Lemma: Let P be a point of a k-space L, $\breve{A} = A/P$. Then

$$\breve{\delta}(p,L) = \begin{cases} \delta(p,L) & \text{if } -1 \leq \delta(p,L) < \pi(P,p), \\ \delta(p,L)-1 & \text{if } \pi(P,p) < \delta(p,L) \leq k. \end{cases}$$

Proof: If $\delta(p,L) < \pi(P,p)$, then $\breve{A}_{\delta(p,L)}(p) = A_{\delta(p,L)}(p)P$. Since $P \in L$, $\breve{A}_{\delta(p,L)}(p) \subset L$ and $\delta(p,L) \leq \breve{\delta}(p,L)$. The equality follows since $A_{\delta(p,L)+1}(p) \not\subset L$.

If $\pi(P,p) < \delta(p,L)$ then $\breve{A}_{\delta(p,L)}(p) = A_{\delta(p,L)+1}(p)$ and $\breve{\delta}(p,L) < \delta(p,L)$. But $\breve{A}_{\delta(p,L)-1}(p) = A_{\delta(p,L)}(p)$ or $A_{\delta(p,L)-1}(p)P$. In either case it is contained in L and the result follows.

2.2.2 Lemma: A k-space L with $A(p) \subset L$ is a k-secant of X iff it is a (k-1)-secant of X on A/p, where $p \in X$.

Proof: Let $X_1 = \{ q \in X \mid \pi(p,q) < \delta(q,L)\}$ and $X_2 = \{ q \in X \mid \delta(q,L) < \pi(p,q)\}$. Then $X = X_1 \cup X_2$. If L is a k-secant of X then

$$L = \bigvee_{q \in X} A_{\delta(q,L)}(q)$$

$$= \bigvee_{q \in X_1} A_{\delta(q,L)}(q) \bigvee \bigvee_{q \in X_2} A_{\delta(q,L)}(q) .$$

By 2.2.1 and projection from p,

$$L = \bigvee_{q \in X_1} \tilde{A}_{\delta(q,L)-1}(q) \quad \vee \quad \bigvee_{q \in X_2} \tilde{A}_{\delta(q,L)}(q)$$

$$= \bigvee_{q \in X_1} \tilde{A}^{\sim}_{\delta(q,L)}(q) \quad \vee \quad \bigvee_{q \in X_2} \tilde{A}^{\sim}_{\delta(q,L)}(q)$$

$$= \bigvee_{q \in X} \tilde{A}^{\sim}_{\delta(q,L)}(q).$$

2.2.3 Theorem: The set of all k-secants of a connected set $X \subset J$ is pointwise connected, $-1 \leq k \leq 2$.

Proof: Since the set of 0-secants of X is the continuous image of X and there is only one 2-secant, we may restrict ourselves to the consideration of 1-secants only.

The set of 1-secants of X which contain an element $p \in X$ is the set of 0-secants of X on A/p by 2.2.2 and hence are pointwise connected by same argument as above.

Now let L and M be 1-secants of X. Take p, q in X such that $A(p) \subset L$ and $A(q) \subset M$. Let N be a 1-secant of X containing A(p) and A(q). Then we can construct a path from L to M by constructing a path from L to N and then from N to M.

2.2.4 Let L be a k-secant of $X \subset J$. Then

$$k \leq \left\{ \sum_{p \in X} (\delta(p,L) + 1) \right\} - 1.$$

When there is equality L is said to be an independent k-secant of X.

X is said to be k-<u>independent</u> iff every k-secant of X is independent.

If L is an independent k-secant of X then L meets X at most $k + 1$ times. Otherwise if L meets X at $p_0, \ldots, p_{k+1}$ distinct points then $\delta(p_i, L) \geq 0$ for $i = 0, 1, \ldots, k+1$, and

$$\sum_{p\epsilon X} (\delta(p,L)+1)-1 \geq \sum_{i=0}^{k+1} (\delta(p_i,L)+1)-1$$

$$\geq (k+2)-1 = k+1,$$

a contradiction.

<u>2.2.5 Lemma</u>: Let X be k-independent and L be a k-secant of X on A, $-1 \leq k \leq 1$. Let $A(p) \subset L$. Then if $q \neq p$ in $X, A(p) \not\subset A_{\delta(q,L)}(q)$.

<u>Proof</u>: If $k = 0$ then $0 = \sum_{q\epsilon X} (\delta(q,L)+1)-1$ implies $\sum_{q\epsilon X} (\delta(q,L)+1) = 1$ and L meets X exactly once. Since $A(p) \subset L$ then $A(p) = L$ and $A_{\delta(q,L)}(q) = \emptyset$.

If $k = 1$ and $A(p) \subset A_{\delta(q',L)}(q')$ for some $q' \neq p$ in X then $A(p) = A(q')$ or $A(p) \neq A(q')$ and $A_1(q') = L$. If $A(p) = A(q')$ then from Figure II.2, N is a 1-secant of X but N is not independent since

$$\sum_{q\epsilon X} (\delta(q,N)+1)-1 = \delta(q',N) + \delta(p,N) + \delta(r,N) + 2 \geq 2,$$

a contradiction.

If $A(p) \neq A(q')$ and $A_1(q') = L$ then

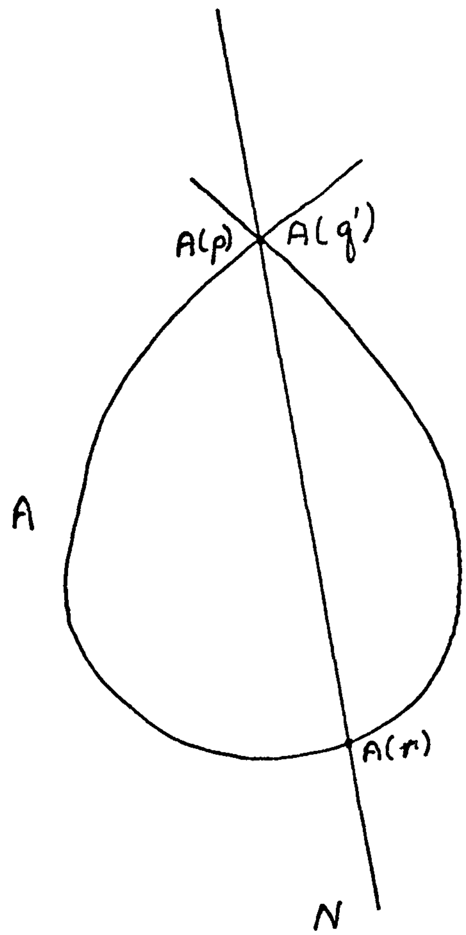


Figure II.2

$$\sum_{p\varepsilon X} (\delta(q,L)+1)-1 \geq \delta(q,L) + \delta(q',L) + 1 \geq 2;$$

again a contradiction.

<u>Corollary</u>: If X is 1-independent, then X is 0-independent.

<u>2.2.6 Lemma</u>: If X is k-independent and $p \varepsilon X$ then X is (k-1)-independent on A/p.

<u>Proof</u>: Let L be a (k-1)-secant of X on A/p. By <u>2.2.2</u> L is a k-secant of X on A with $A(p) \subset L$. Define X_1, X_2 as in <u>2.2.2</u> then $X = X_1 \cup X_2$ and

$$k = \sum_{q\varepsilon X_1} (\delta(q,L)+1) + \sum_{q\varepsilon X_2} (\delta(q,L)+1) - 1$$

$$= \sum_{q\varepsilon X_1} (\breve{\delta}(q,L)+2) + \sum_{q\varepsilon X_2} (\breve{\delta}(q,L)+1) - 1 \quad \text{by } \underline{2.2.1}$$

Since $X_1 = \left\{ q \varepsilon X \mid A(p) \subset A_{\delta(q,L)}(q)\right\}$ then $X_1 = \{ p\}$ by <u>2.2.5</u> and

$$k = \breve{\delta}(p,L)+2 + \sum_{q\varepsilon X-\{p\}} (\breve{\delta}(q,L)+1) - 1$$

and the result follows.

We have already commented upon the fact that an independent k-secant of X on A meets X at most $k+1$ times. The converse does not hold as can be seen in Figure II.2 for the case $k = 1$ and $X = A$.

<u>2.3 Representations of secants</u>

Let L be a k-secant of X on A which meets X at $m+1$

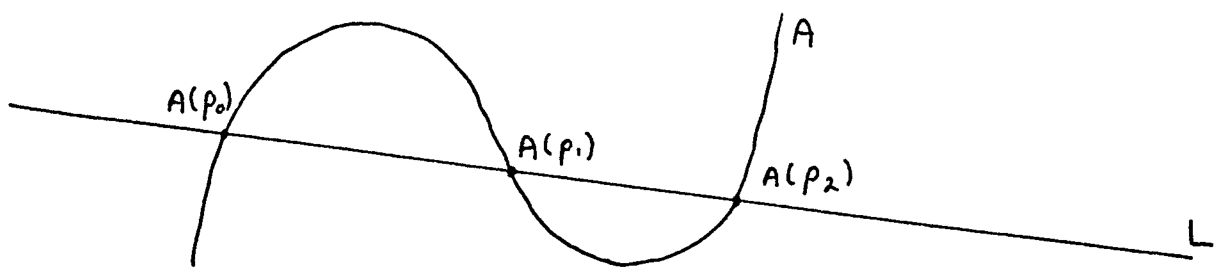


Figure II.3

points $p_0, \ldots, p_m$. Let L be independent. Then $m \leq k$.

If $m = k$ then $\delta(p_i, L) = 0$ for $i = 0, \ldots, k$ and $\{p_0, \ldots, p_k\}$ uniquely determine L. If $m < k$ then since

$$k = \sum_{p \in x} (\delta(p,L)+1) \quad -1$$

there is a p_j with $\delta(p_j, L) > 0$. For example, if $\delta(p_j, L) = k$ then L meets X only at p_j and p_j uniquely determines L. If $\delta(p_j, L) = k - 1$ then there is $p_i \neq p_j$ in X such that $A(p_i) \subset L \setminus A_{k-1}(p_j)$. Hence $\{p_i, p_j\}$ uniquely determine L.

Thus an independent k-secant L of X on A is uniquely determined by the points at which L meets X. If L is not independent then L is still determined by the points of which L meets X though no longer uniquely. As, for example, in Figure II.3, L is a 1-secant of A, not independent and $\{p_0, p_1\}$, $\{p_0, p_2\}$, $\{p_1, p_2\}$ all represent; that is, determine L.

Putting these observations in more precise form, we define inductively the mapping

$$A^k\colon J^{k+1} \longrightarrow \underline{P}^2_k \quad -1 \leq k \leq 2,$$

by requiring that

$$A^{-1}(\ ) \in \underline{P}^2_{-1}$$

and

$$A^k(p_0, \ldots, p_k) = A(p_k)/A^{k-1}(p_0, \ldots, p_{k-1}), 0 \leq k \leq 2.$$

We note that $A^k(p, \ldots, p) = A_k(p)$.

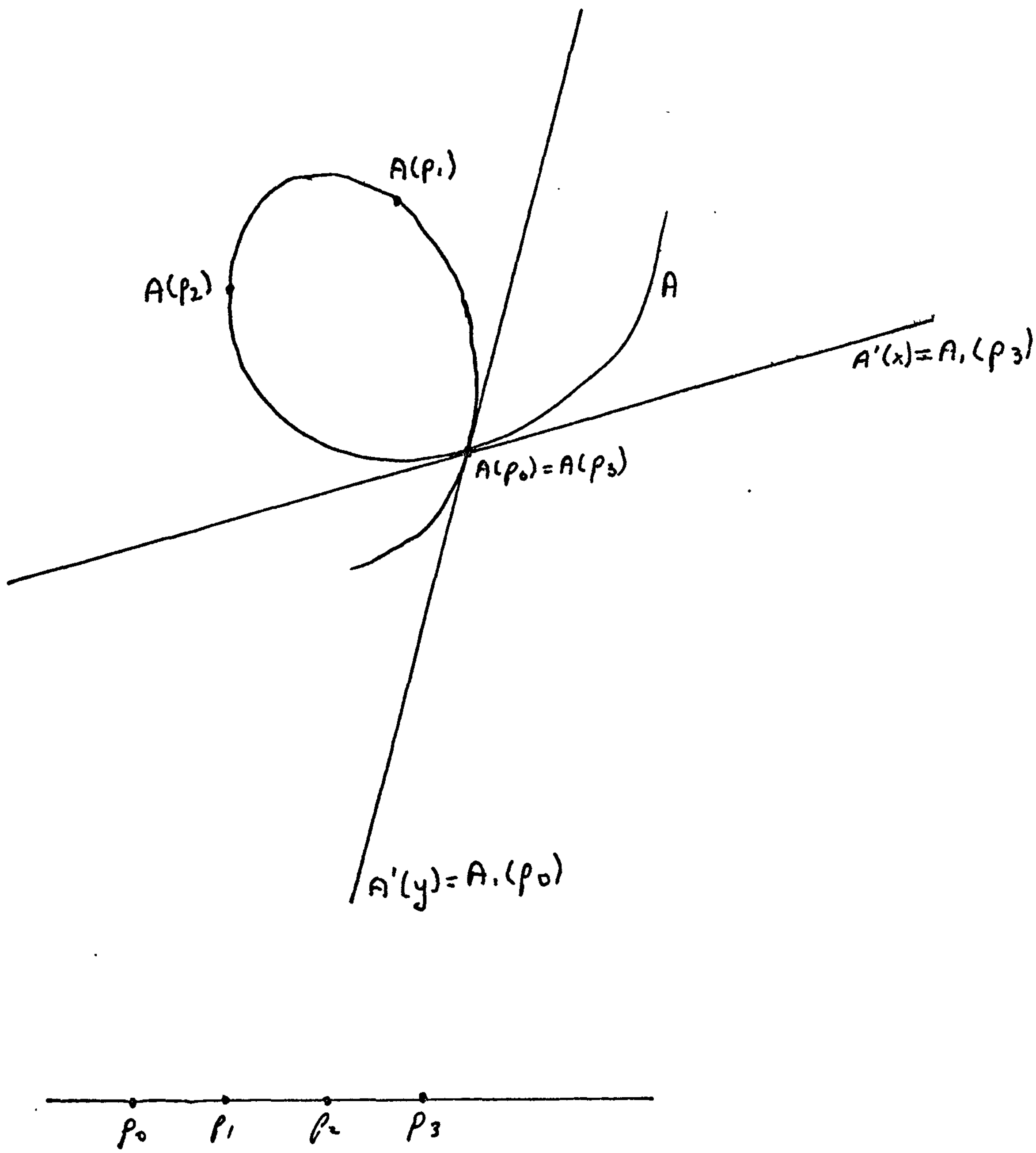


Figure II.4

<u>2.3.1 Theorem</u>: $A^k(X^{k+1})$ is the set of all k-secants of X, $-1 \leq k \leq 2$.

<u>Proof</u>: The result holds trivially for $k = -1$, 0, and 2. Let $\{p_0, p_1\} \subset X$ be given. By projecting A from p_0, $\tilde{A}(p_1) = A^1(p_0, p_1)$ and $A^1(p_0, p_1)$ is a 0-secant of X on $\hat{A} = A/p_0$. By <u>2.2.2</u>, $A'(p_0, p_1)$ is a 1-secant of X on A.

Conversely, let a 1-secant L of X on A be given. Let $p \in X$ with $A(p) \subset L$. By projecting A from p and <u>2.2.2</u>, L is a 0-secant of X on A/p, hence $L = (A/p)(q) = A'(p,q)$.

Now at this point one is tempted to say that X is k-independent iff A^k is 1-1. However, A^k is a mapping of tuples, hence we are now also concerned with the order of appearance of points in a given tuple; that is, permutation of the components.

The ideal situation would be to have A^k independent of permutations, however this is not always the case. Consider Figure II.4, where A is a regular arc; cf. Section 6. Let $X = (p_0, p_3)$, then

$$A^1(X) = A(p_3)/A(p_0) = A(p_3)/A(p_3) = A_1(p_3)$$

but $y = (p_3, p_0)$ is a permutation of X and

$$A^1(y) = A_1(p_0) \neq A_1(p_3) = A^1(X).$$

Suppose $p \in J$ and $x = (p_0, \ldots, p_k) \in J^{k+1}$, $-1 \leq k \leq 2$. Define $\gamma(p, x) = \left\{ \sum_{p_i = p} 1 \right\} -1$. Thus $\gamma(p, x) \leq \delta(p, A^k(x))$

and equality occurs iff $A^k(x)$ is an independent k-secant of X iff the components of x uniquely determine $A^k(x)$.

Since $\gamma(p, x) \leq \delta(p, A^k(x))$, then $\bigvee_{p \in X} A_{\gamma(p,x)}(p) \subset A^k(x)$. When the inclusion is improper we say X is independent.

<u>2.3.2 Lemma</u>: If $A^k(x)$ is an independent k-secant of X then x is independent but not conversely.

<u>Proof</u>: Since $A^k(x)$ is independent, $\gamma(p, x) = \delta(p, A^k(x))$ and

$$\bigvee_{p \in X} A_{\gamma(p,x)}(p) = \bigvee_{p \in X} A_{\delta(p,A^k(x))}(p) = A^k(x).$$

That the converse is not true; in Figure II.3, $x = (p_0, p_1)$ is independent but $L = A^1(x)$ is a dependent secant.

<u>2.3.3 Lemma</u>: If X is independent, then $A^k(x) = A^k(y)$ if y is a permutation of $x \in J^{k+1}$.

<u>Proof</u>: Since $\gamma(p, y) = \gamma(p, x)$ for all permutations y of x, then

$$A^k(y) \supseteq \bigvee_{p \in X} A_{\gamma(p,y)}(p) = \bigvee_{p \in X} A_{\gamma(p,x)}(p) = A^k(X)$$

and the result follows.

Thus by <u>2.3.2</u> and <u>2.3.3</u>, if $A^k(x)$ is an independent k-secant, than it is also independent of the permutations of x. Now if $A^k(x)$ is not k-independent then from above there is a $p \in X$ such that $\gamma(p,x) < \delta(p, A^k(X))$. Hence there is at least one $z \in J^{k+1}$ with the property that $\gamma(p, z) = \delta(p, A^k(z))$ and $A^k(z) = A^k(x)$. Hence, 2.3.4 Theorem: Let $X \subset J$ and $L = A^k(x)$ be a k-secant of X where $x \in X^{k+1}$. Then L is an independent secant of X iff $L = A^k(y)$, where $y \in X^{k+1}$, holds exactly for the permutations y of x.

As can be readily seen in Figure II.4, $x = (p_0, p_3)$ is not independent since

$$\bigvee_{p \in A} A_{\gamma(p,x)}(p) = A(p_0)\ A(p_3) = A(p_0) = A(p_3) \neq A^1(x).$$

In fact we have the following result.

2.3.5 Lemma: Let $x \in J^{k+1}$. Then x is independent for $k = 1$ [$k = 2$] iff the components of x are not equal [not collinear].

Proof: The result follows since the independence of x is based upon the dimension of space spanned by its components.

2.3.6 Theorem: Let $-1 \leq k \leq 1$. Then $X \subset J$ is k-independent iff every $x \in X^{k+2}$ is independent.

Proof: Let X be k-independent and $x \in X^{k+2}$. Then

$$L = \bigvee_{p \in X} A_{\gamma(p,x)}(p) \subset A^{k+1}(x) \text{ is an h-secant of } X,$$

$h \leq k + 1$. If $h = k + 1$ the result follows, so assume $h < k + 1$. Then $h \leq k \leq 1$ and L is an h-independent secant of X on A by the Corollary of 2.2.5. Since $\delta(p,L) \geq \gamma(p,x)$ for all $p \in X$, we have

$$h = \sum_{p \in x} (\delta(p,L)+1)-1$$

$$\geq \sum_{p \in x} (\gamma(p,x)+1)-1.$$

But $\gamma(p,x) = \sum_{p_i=p} 1 - 1$, hence $\gamma(p,x)+1 = \sum_{p_i=p} 1$.

Since $x \in X^{k+2}$,

$$\sum_{p\in X} (\gamma(p,x)+1) = \sum_{p\in X} \left(\sum_{p_i=p} 1 \right) = k + 2.$$

Hence $h \geq k+1$; a contradiction.

Conversely, if L is a dependent k-secant, say $L = A^k(y)$, $y \in X^{k+1}$, then we can always construct from y an $x \in X^{k+2}$ which is dependent. For example, in Figure II.3, let $k = 1$ and $y = (p_0, p_1)$. Then $x = (y, p_2) = (p_0, p_1, p_2)$ is dependent since p_0, p_1, p_2 are collinear.

2.4 Order

Let A be an arc and $X \subset J$, $-1 \leq k \leq 2$. Let k be fixed. If $S(X,L) = \{p \in X \mid A(p) \subset L\}$ is finite for every k-space L, we say X has <u>finite</u> $\underline{P}^2_k$-<u>order</u>.

If in addition

$$m = \sup_{L\in \underline{P}^2_k} |S(X,L)|$$

is finite, we say X has <u>bounded</u> $\underline{P}^2_k$-<u>order</u> m. By the order of X, we mean the $\underline{P}^2_1$-order.

If X has at least $k + 1$ points and is k-independent, then it has $\underline{P}^2_k$-order $k + 1$, since we can construct a k-secant through the $k + 1$ points.

Any compact set $X \subset J$ is of finite order and, in particular, A is locally of finite order. This follows from the corollaries if 1.3.1.

2.4.1 Theorem: If (p,q) has order 2 then $[p,q)$ and $(p,q]$ are 1-independent.

Proof: First of all we note that in $\underline{P}'$, if (p,q) has order 1 then $[p,q)$ and $(p,q]$ are 0-independent. For if $A(p) = A(s)$ for some $s \in (p,q)$ then $[p,s]$ will be a closed subarc of $[p,q)$ in $\underline{P}'$ and by continuity of A there is $r_1 \neq r_2$ in (p,s) such that $A(r_1) = A(r_2)$. But then (p,q) is at least of order 2 in $\underline{P}^1$, a contradiction.

Our method of proof will be to show (p,q) is 1-independent and then extend the independence to $[p,q)$.

Let $s \in (p,q)$ and L be a line through s, then L meets $(p,s) \cup (s,q)$ at most once and (p,s) has order 1 on $\breve{A} = A/s$. Hence if $p < r < s < q$, then $A(r) \not\subset A_1(s)$ otherwise $\breve{A}(r) = A(r)\ A(s) = A_1(s) = \breve{A}(s)$, a contradiction (Figure II.5).

Let L be a 1-secant of (p,q), then there is $t \in (p,q)$ such that $A(t) \subset L$. Let r be the first such point in (p,q). Then (r,q) has order 1 on $\breve{A} = A/r$ and for any $s \in (r,q)$, $A(r) \not\subset A_1(s)$ from above. Hence $\breve{\delta}(s,L) = \delta(s,L)$ by 2.2.1. Since $[r,q)$ is 0-independent on A/r,

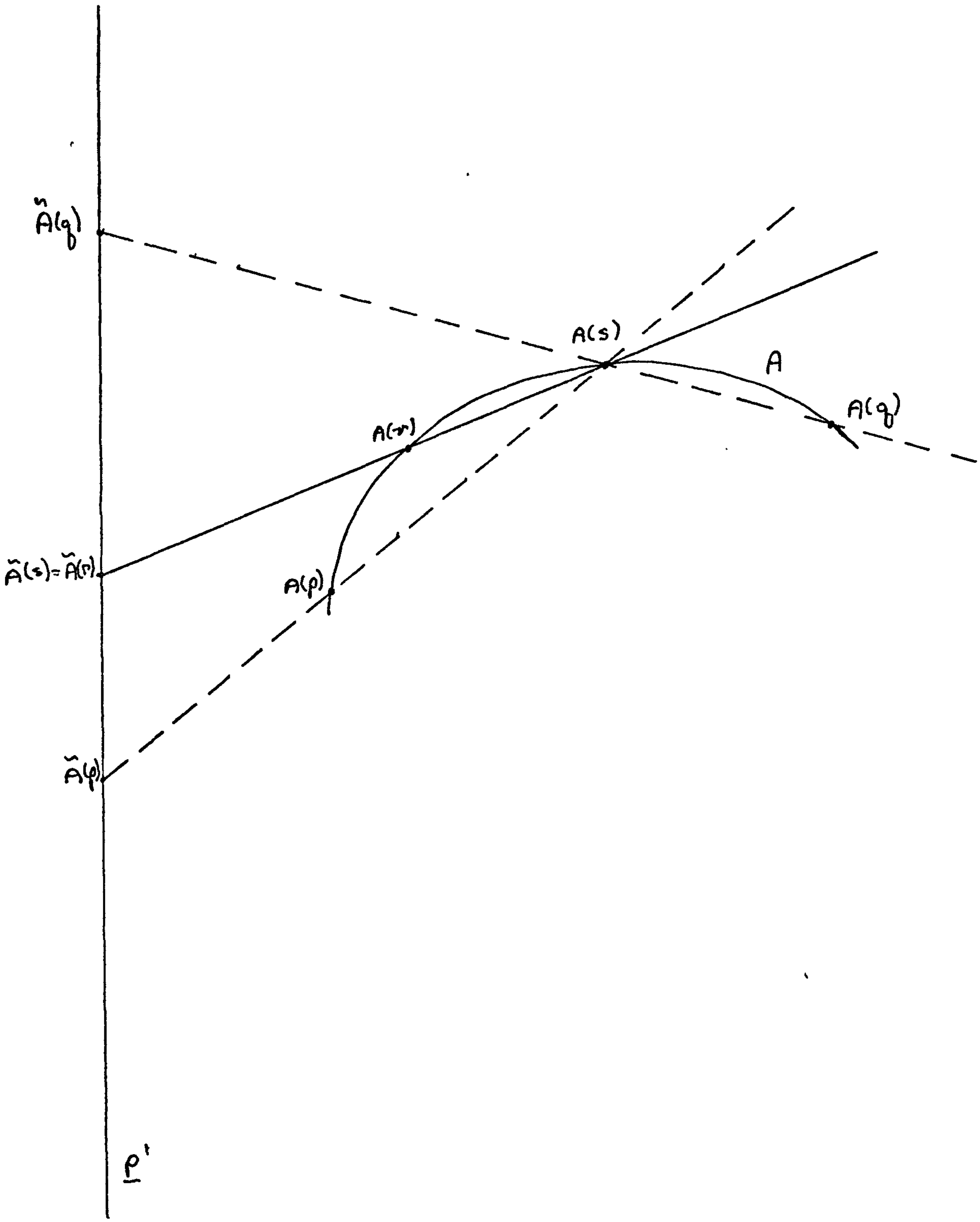


Figure II.5

$$0 = \sum_{s\varepsilon[r,q)} (\overset{\frown}{\delta}(s,L)+1)-1$$

$$= (\overset{\frown}{\delta}(r,L)+1) + \sum_{s\varepsilon(r,q)} (\overset{\frown}{\delta}(s,L)+1)-1$$

$$= (\delta(r,L)+2) + \sum_{s\varepsilon(r,q)} (\delta(s,L)+1)-1.$$

Therefore $1 = \sum_{s\varepsilon(r,q)} (\delta(s,L)+1)-1$ and L is an independence secant of $[r,q)$ on A. Hence (p,q) is 1-independent.

We now wish to show that $A(p) \not\subset A_1(s)$ for any $s \varepsilon (p,q)$. Since (p,q) is 1-independent on $\dot{A}$, by 2.2.6, (p,q) is 0-independent on A/s if $s \varepsilon (p,q)$. From the comments in the beginning of this section, (p,q) is of order 1 on A/s and hence $[p,q)$ is 0-independent on A/s and $A(p) \not\subset A_1(s)$ for $s \varepsilon (p,q)$.

Furthermore, $[p,q)$ has order 2. For if $p < p_1 < p_2 < q$ and p, p_1, p_2 are collinear then $[p, p_2)$ is at least of order 2 on A/p_2 a contradiction. But $[p,q)$ has order 2 on A implies (p,q) has order 1 on A/p and $[p,q)$ is 0-independent on A/p.

Finally, let L be a 1-secant of $[p,q)$ with $A(p) \subset L$. Then L is 0-independent on A/p. Since $A(p) \not\subset A_1(s)$ for $s \varepsilon (p,q)$, $\overset{\frown}{\delta}(s,L) = \delta(s,L)$ and L is an independent secant of $[p,q)$ on A as above. Hence $[p,q)$ as 1-independent on A.

The symmetric argument holds for $(p,q]$.

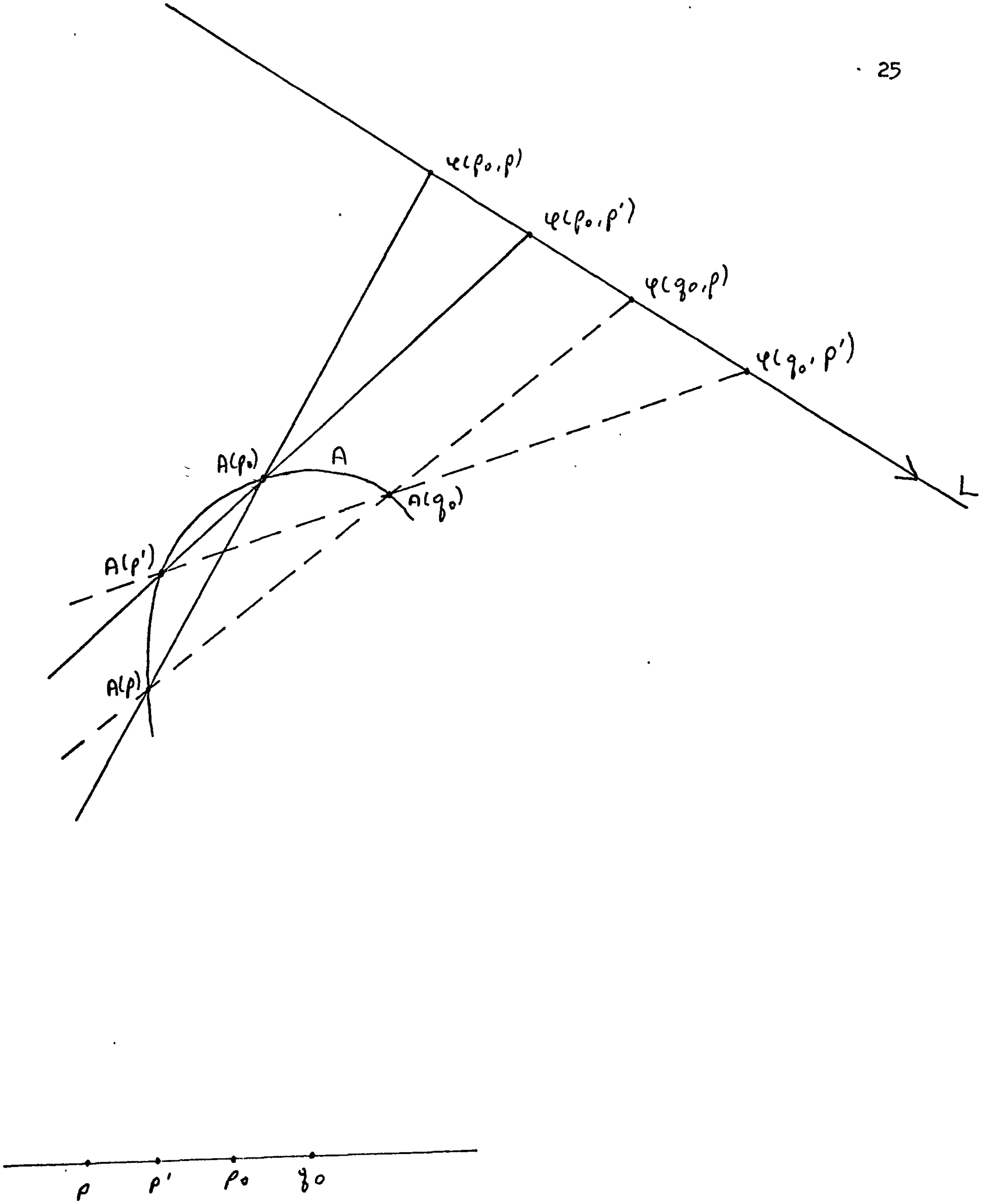


Figure II.6

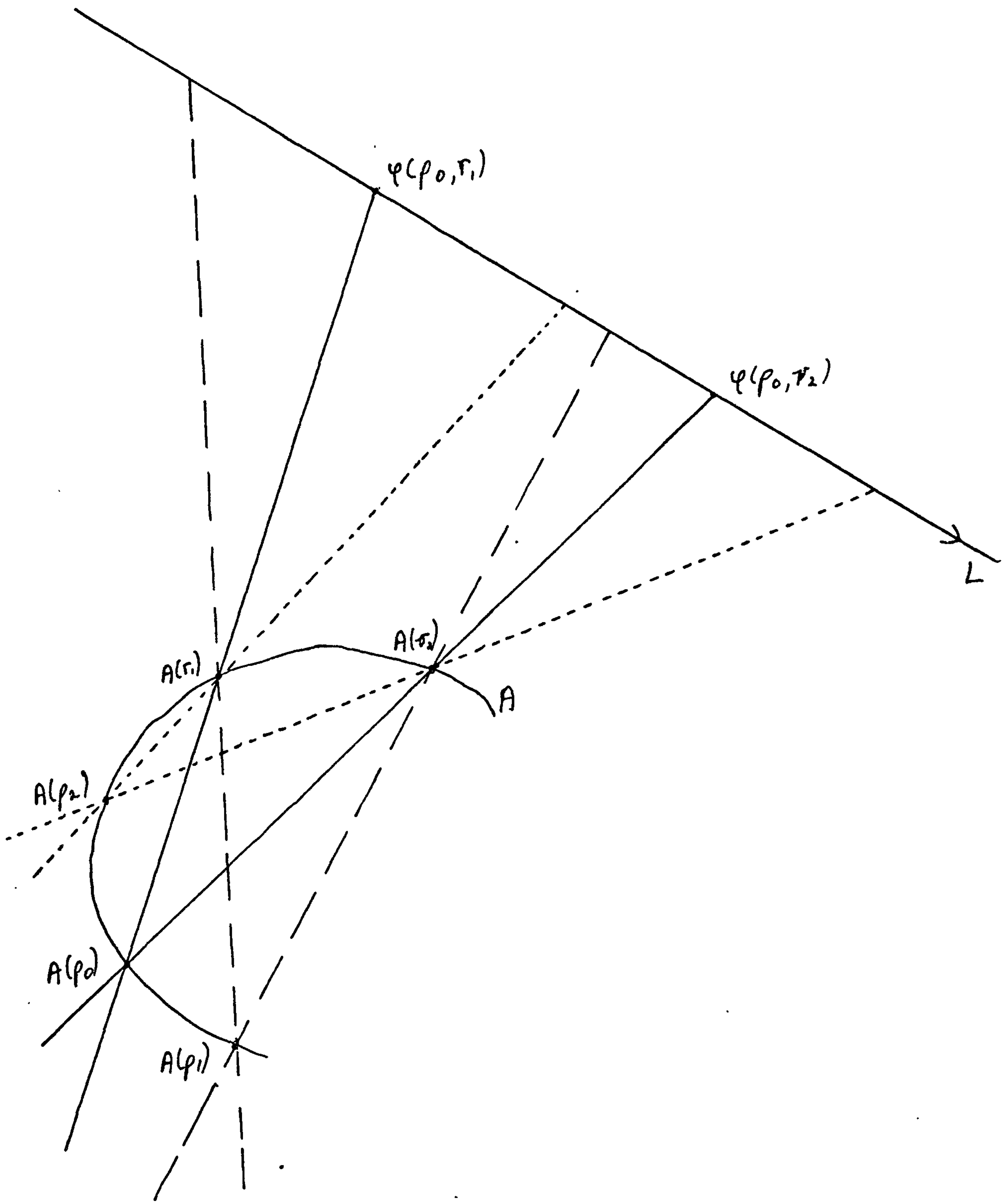


Figure II.7

<u>Corollary 1</u>: A is an arc of order 2 iff A is 1-independent.

<u>Corollary 2</u>: If (p,q) has order 2 on A and $r \in [p,q]$ then (p,q) has order 1 on A/r.

<u>Corollary 3</u>: If (p,q) has order 2 on A and $s \in (p,q)$ then

$$A_1(s) \cap A[p,q] = A(s).$$

Let A be an arc of order 2 and L and oriented line such that $A(p) \not\subset L$ for $p \in J$. Then for each $x \in J^2$, $A^1(x) \cap L$ is a point $\varphi(x)$ of L. We assume there exists a point $P_\infty \neq \varphi(x)$ for all $x \in J^2$. We put $(p_0, p_1) \leq (q_0, q_1)$ iff $p_0 \leq q_0$ and $p_1 \leq q_1$.

<u>2.4.2 Theorem</u>: φ is (strictly) monotone.

<u>Proof</u>: We prove the result locally and since J is connected, we can then extend the result to the whole arc.

Let p, p_0, q_0 be three distinct points of the arc (Figure II.6). Since A is of order 2, A/p_0 is of order 1 by <u>2.4.1</u>, Cor. 2, hence the mapping $p \longrightarrow \varphi(p_0, p)$ is monotone. Similarly $p \longrightarrow \varphi(q_0, p)$ is monotone. Therefore we now need to show that the monotonicity does not depend on the choice of a given component; that is, $p \longrightarrow \varphi(p_o, p)$ and $p \longrightarrow \varphi(q_0, q)$ are monotone in the same sense.

Take r_1 and r_2 with $p_0 < r_1 < r_2$ (Figure II.7). We may assume the orientation of L is such that $\varphi(p_0, r_1) \leq \varphi(p_0, r_2)$.

Then there exists a $u(p_0)$ such that if $q_0 \in u(p_0)$ then

$\varphi(q_0,r_1) < \varphi(q_0,r_2)$. Otherwise all of L will be the image of J^2 under φ. Thus $p \longrightarrow \varphi(q_0,p)$ has the same sense for all $q_0 \varepsilon u(p_0)$.

2.5 Characteristic of a point

A line L is said to support A at p if there is a line H_∞ such that $H_\infty \neq L$, $A(p) \not\subset H_\infty$ and a $u'(p)$ such that $A(u'(p))$ is contained in one of the two open half spaces determined by L and H_∞. When L does not support A at p we say L cuts A at p, since by 1.3.1, $A(u'(p))$ does not contain any segment of L.

2.5.1 Lemma: Let $-1 \leq k \leq 1$. Let S_k be the set of all lines L with $\delta(p,L) = K$ for $p \varepsilon J$ fixed. Then

1. S_k is connected.
2. The elements of S_k either all support or all cut A at p.

Proof: 1. Since in a projective plane any two lines meet, every $L \varepsilon S_{-1}$ meets $A_1(p)$ outside $A(p)$. Conversely, given any $P \not\subset A_1(p)$ and $Q \varepsilon A_1(p) - A(p)$, $PQ \varepsilon S_{-1}$. Therefore S_{-1} is isomorphic to the real line $\mathbb{R}$. Hence S_{-1} is connected.

Since S_1 is a singleton set consisting of $A_1(p)$ it is connected.

By projection from p, the elements of S_0 are projected onto $\underline{P}^1 - \{\tilde{A}(p)\}$, where $\tilde{A} = A/p$. The result 1 now follows.

2. Let L_i, $L \in S_k$ with $L_i \longrightarrow L$. Then there is $u'(p)$ such that $A(U'(p)) \not\subset L[L_i]$; otherwise (Figure II.8, $k = 0$) there exist points q_i with $q_i \longrightarrow p$, $q_i \neq p$ and integers $j(i)$ with $A(q_i) \subset L_{j(i)}$. But then $A(q_i)\ A_k(p) \subset L_{j(i)}$ by hypothesis and $A_{k+1}(p) \subset L$, a contradiction. Hence if all L_j support [cut] A at p then L supports [cuts] A at p. Thus the set of lines of S_k which support [cut] A at p is closed. But S_k is connected and result follows.

So far by the study of order, projection and secants we have been getting a global picture of arcs. But since each $p \in J$ has its well defined osculating spaces, by the above uniformity theorem we can describe in detail a point and its neighbourhood in the arc. As it turns out there are four different types of points distinguished by 2.5.1 in the real projective plane and later we shall give examples of each.

Let $p \in J$ and $-1 \leq k \leq 1$. We define $\sigma_k(p) = 0$ or 1 according as the elements of S_k support or cut A at p. Thus $\sigma_{-1}(p) = 0$. The characteristic $(a_0(p), a_1(p))$ of p is defined by taking $a_i(p) = 1$ or 2 and requiring that

$$a_0(p) + a_k(p) \equiv \sigma_k(p) \pmod 2, \ 0 \leq k \leq 1.$$

We also define numbers

$$\beta_k(p) = \sum_{i=0}^{k} a_i(p), \quad 0 \leq k \leq 1$$

and $\beta_{-1}(p) = 0$. Therefore $\beta_k(p) \equiv \sigma_k(p) \pmod 2$, $-1 \leq k \leq 1$.

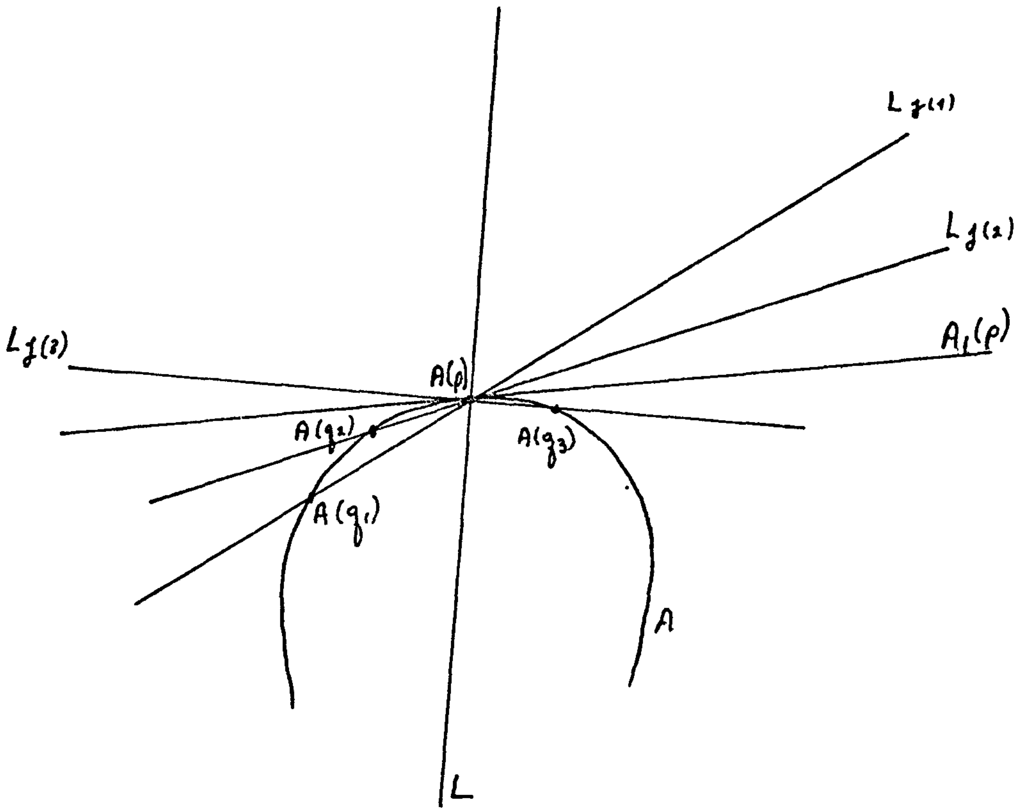


Figure II.8

We say p is <u>regular</u> if $a_0(p) = 1 = a_1(p)$. We say p is a <u>point of inflection</u> if $a_0(p) = 1$ and $a_1(p)=2$. An arc A is regular (has at most inflections) if each point is regular (regular or an inflection. In Figures II-9 to II-12, the four different types of characteristic of a point p of an arc in $\underline{P}^2$ are shown. In Figures II.9 [II.10] p is regular [inflection]. Whereas in Figures II.11 and II.12, the characteristic of p is (2,1) and (2,2) respectively. In all cases the line $L \neq A_1(p)$.

Upon projection, say from a point P, we have seen that

$$\breve{A}_k(p) = \begin{cases} A_k(p)P & \text{if } -1 \leq k \leq \pi(P,p) \\ A_{k+1}(p) & \text{if } \pi(P,p) < k \leq 1, \end{cases}$$

where $\breve{A}_k(p)$, $-1 \leq k \leq 1$ are the osculationg spaces of p in $\breve{A} = A/P$. Since the $\breve{A}_k(p)$ are dependent on $A_k(p)$ and $\pi(P,p)$, it is fair to assume that the characteristic of p in $\breve{A}$ will depend only on $\pi(P,p)$ and the characteristic of p on A. This is in fact the case.

<u>2.5.2 Theorem:</u> Let P be a point and $\breve{A} = A/P$. Then

1. $$\breve{\sigma}_k(p) \equiv \begin{cases} \sigma_0(p) + \sigma_{k+1}(p) & P = A(p),\ -1 \leq k \leq 0 \\ \sigma_k(p) & -1 \leq k < \pi(P,p) \\ \sigma_{k+1}(p) & -1 < \pi(P,p) \leq k \leq 0. \end{cases}$$

2. $$\breve{a}_0(p) \equiv \begin{cases} a_0(p) & 0 < \pi(P,p) \\ a_0(p) + a_1(p) & 0 = \pi(P,p) \\ a_1(p) & 0 > \pi(P,p) \end{cases}$$

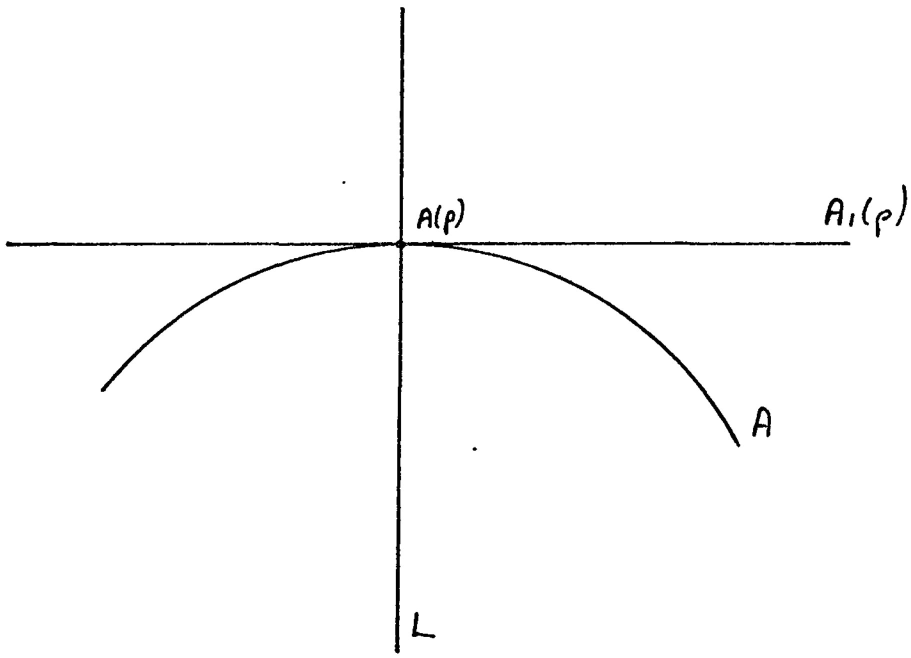


Figure II.9

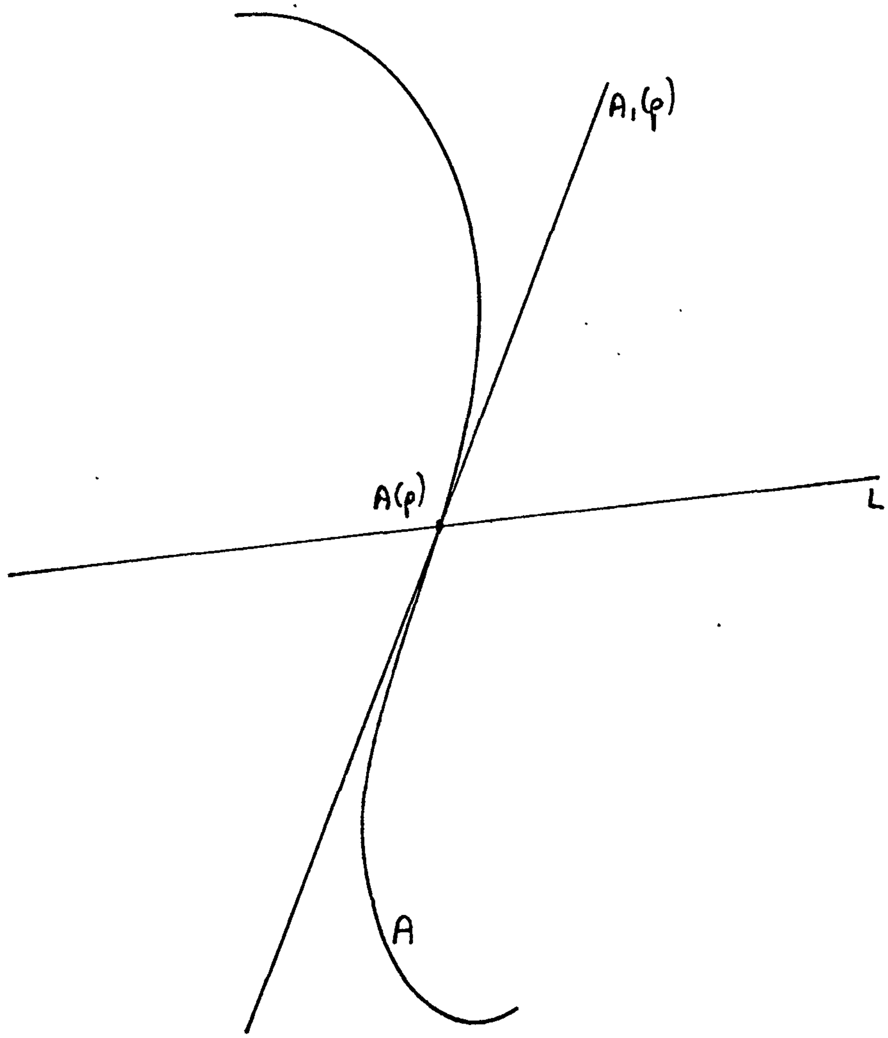


Figure II.10

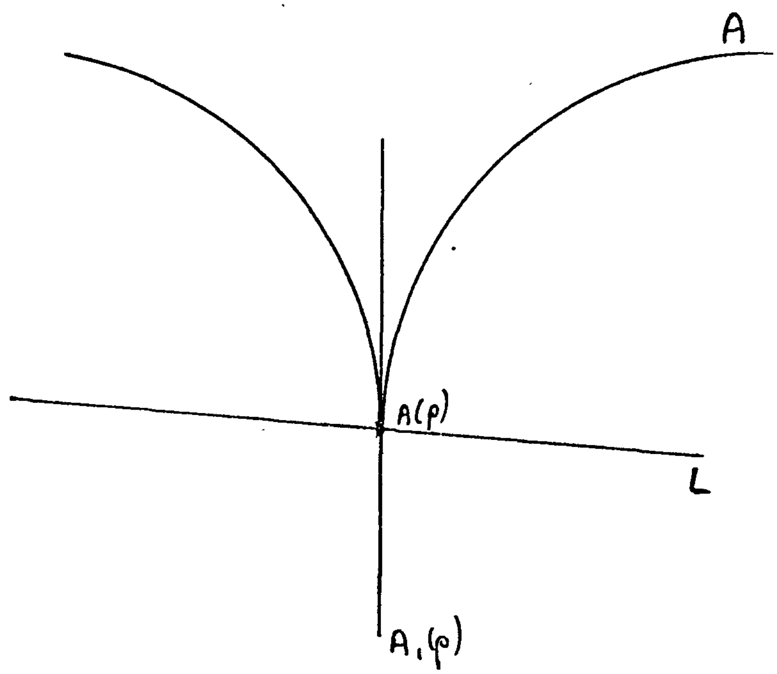


Figure II.11

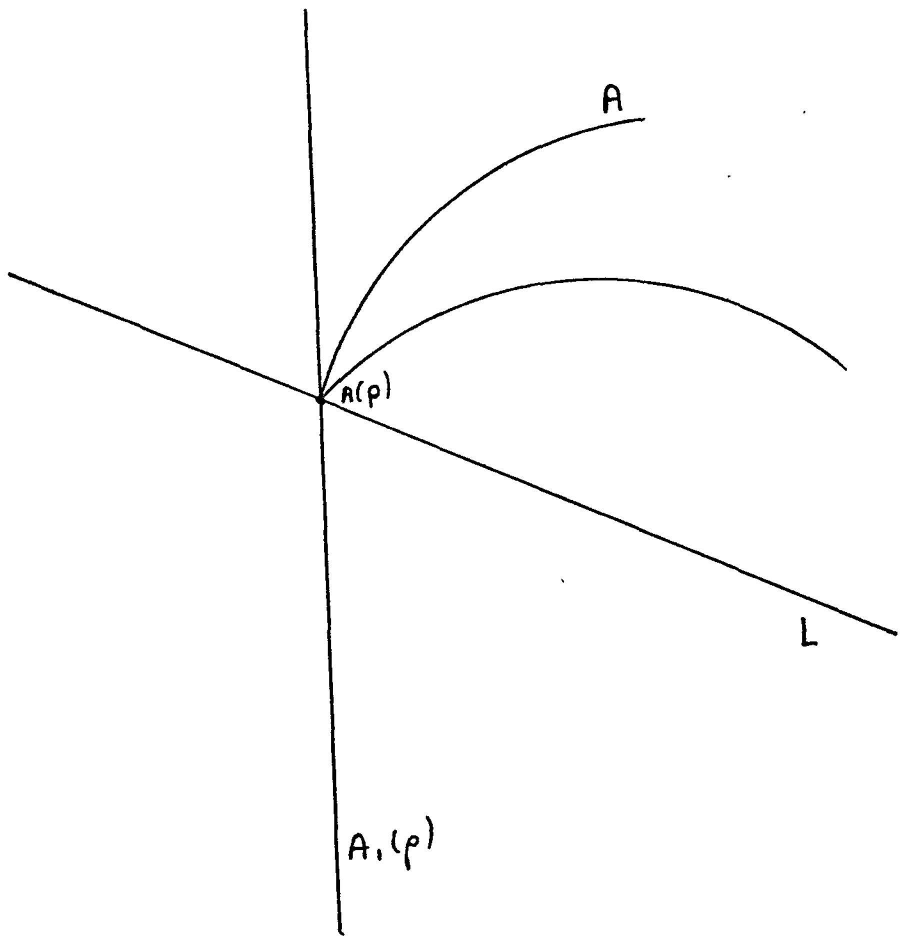


Figure II.12

where the congruences are (mod 2) and

3. $$\beta_0(p) = \begin{cases} \beta_1(p) - \beta_0(p) & P = A(p) \\ \beta_0(p) & 0 < \pi(P,p) . \end{cases}$$

<u>Proof:</u> <u>Case 1</u>. $P = A(p)$. Let L and H_∞ be distinct lines with $\delta(p,L) = k+1$ and $\delta(p,H_\infty) = 0$. Then $\breve{\delta}(p,L) = k$ and $\breve{\delta}(p,H_\infty) = 0 - 1$ in $\breve{A}$. Now some $u'(p)$ is contained in the same open half-space determined by L and H_∞ iff both support or both intersect A at p iff

$$\sigma_0(p) + \sigma_{k+1}(p) = 0 \pmod 2.$$

If $-1 \leq k < \pi(P,p)$, let L and H_∞ be lines such that $P \subset L \cap H_\infty$, $\delta(p,L) = K$ and $\delta(p, H_\infty) = -1$. Since $k < \pi(P,p)$, $\breve{\delta}(p,L) = \delta(p,L)$ by <u>2.2.1</u>. Then as above L supports $\breve{A}$ at p iff $\sigma_{-1}(p) + \sigma_k(p) = 0$ (mod 2). Hence $\breve{\sigma}_k(p) = \sigma_k(p)$.

If $-1 < \pi(P,p) \leq k \leq 0$. Consider the lines L and H_∞ with $P \subset L \cap H_\infty$, $\delta(p,L) = k + 1$ and $\delta(p,H_\infty) = -1$. Then $\breve{\delta}(p,L) = \delta(p,L)-1 = k$ by <u>2.2.1</u> and $\breve{\sigma}_k(p) = \sigma_{k+1}(p)$.

<u>Case 2</u>. If $0 < \pi(P,p)$, then $\breve{\sigma}_k(p) = \sigma_k(p)$ for $k = -1, 0$. Hence $\breve{a}_0(p) = \breve{\sigma}_0(p) = \sigma_0(p) = a_0(p)$ (mod 2). Since $|a_0(p) - a_0(p)| \leq 2$, the result follows. Similarly, $\breve{a}_0(p) = a_1(p)$ if $\pi(P,p) < 0$.

If $0 = \pi(P,p)$, then $\breve{\sigma}_0(p) = \sigma_1(p)$. Since $\sigma_1(p) \equiv a_0(p) + a_1(p)$ (mod 2), we have

$$\breve{a}_0(p) = a_0(p) + a_1(p) \pmod 2.$$

Case 3. If $P = A(p)$ then $\breve{a}_0(p) = a_1(p)$. Hence $\tilde{\beta}_0(p) = \tilde{a}_0(p) = a_1(p) = \beta_1(p) - \beta_0(p)$.

2.6 Ordinary and singular points

The order of a point p is the minimum order which a neighbourhood of p can possess. A point p is ordinary if it is of order 2; otherwise p is a singularity. If there exist $u+(p)$, $u-(p)$ of order 2, we say p is elementary. An arc is ordinary [elementary] if each of its points is ordinary [elementary].

It should be noted that in $\underline{P}'$, if a point p is ordinary, then it is regular. Considering the case of the real projective plane, let $p \varepsilon J$ be ordinary. Then there is $u(p)$ of order 2 on A. Take $q \varepsilon u(p)$, $q \neq p$ (Figure II.13), by 2.4.1. Cor. 2. $u(p)$ has order 1 on both A/p point A/q. Writing $\breve{A} = A/p$ and $\breve{\breve{A}} = A/q$. $\breve{a}_0(p) = a_1(p)$ by 2.5.2. Since $A(q) \not\subset A_1(p)$ by 2.4.1, Cor 3, we have $\breve{\breve{a}}_0(p) = a_0(p)$. But $\breve{a}_0(p) = 1 = \breve{\breve{a}}_0(p)$, hence p is regular.

2.6.1 Lemma: An ordinary point is regular. Thus an ordinary arc is regular.

In our study of secants, it was established that $A^k(X^{k+1})$ is the set of all k-secants of X on A, where $X \subset J$. The question that naturally arises is, when is A^k continuous? Trivially, for $k = -1$, 0 and 2, it is continuous into $\underline{P}^2$. Hence we need only consider the continuity of 1-secants. But we already have a tool for the study of lines namely order.

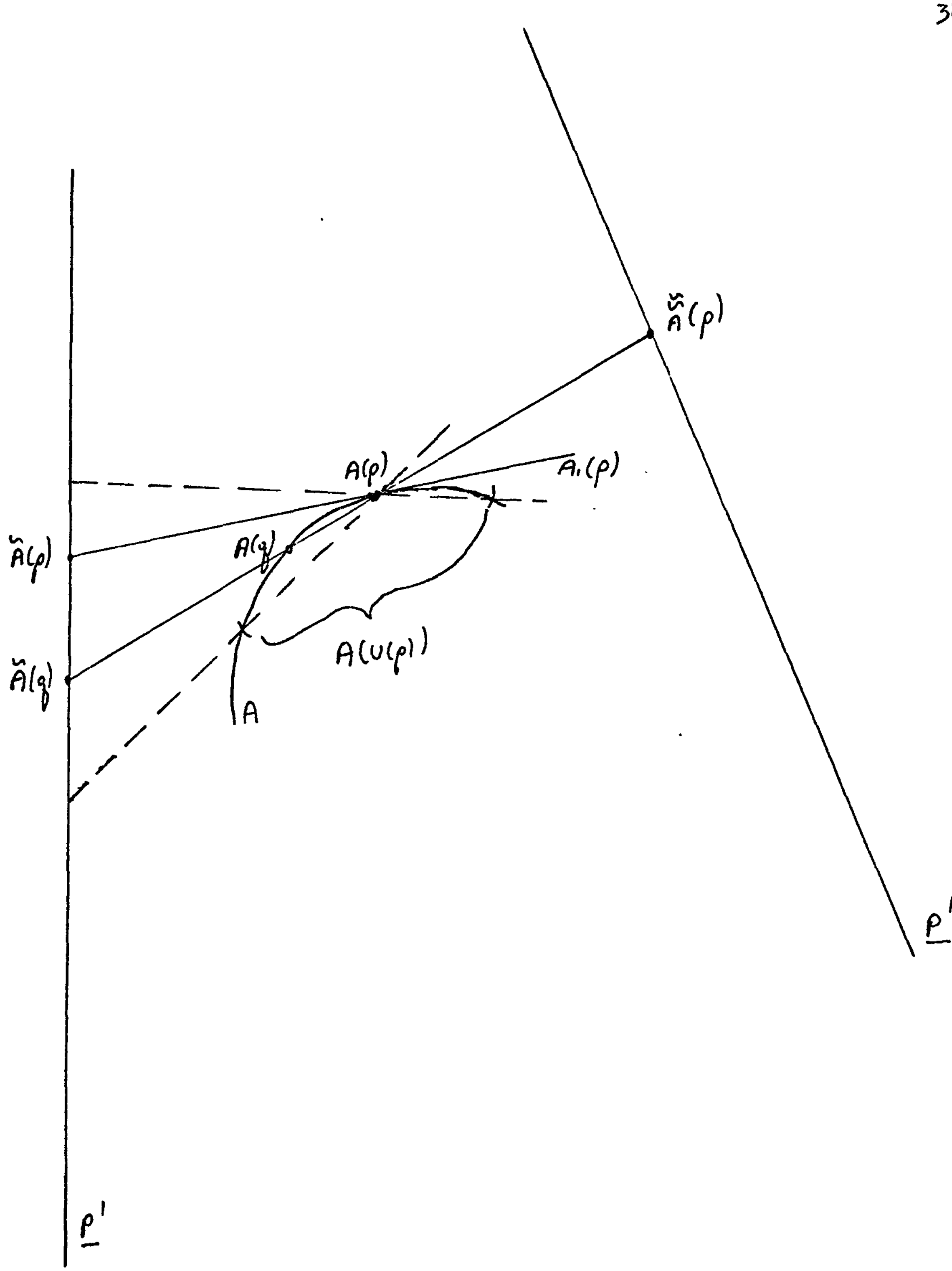


Figure II.13

If an arc A has order 2 and a line L meets A at distinct points p and q. It seems intuitively true that if we hold say p constant and vary q continuously on J, L will vary continuously on A since L can meet A at most twice. This is in fact what happens.

<u>2.6.2 Theorem</u>: If (p,q) is of order 2 then A^k is continuous on $[p,q)^{k+1}$; $-1 \leq k \leq 2$.

<u>Proof</u>: In <u>2.4.1</u>, it was shown that if (p,q) has order 2 then $[p,q)$ and $(p,q]$ is 1-independent, hence also of order 2. Therefore we need only consider the case $k = 1$.

<u>Case 1</u>. $x = (p_0, p_1) \in [p, q)^2$, $p_0 \neq p_1$. Then there exists a neighbourhood U of x in $[p,q)^2$ such that if $y = (q_0, q_1) \in U$ then $q_0 \neq q_1$. Now (p,q) is of order 2 hence $[p,q)$ is 1-independent and for any $r \in [p,q)$

$$\delta(r,A'(y)) = \Upsilon(r,y) \leq 0 \quad \text{for } y \in U.$$

Therefore

$$A'(y) = \bigvee_{p \leq r < q} A_{\delta(r,A'(y))}(r)$$

is continuous on U since A and A_{-1} are continuous.

<u>Case 2</u>. $x = (r,r) \in (p,q)^2$. Consider $x_i \in (p,q)^2$ such that $x_i \longrightarrow x$ and $A^1(x_i) \longrightarrow L$. By holding one component of the x_i fixed. $A(r) \subset L$. Since (p,q) is of order 2, it is ordinary and regular by <u>2.6.1</u>. Hence we are considering a case as in Figure II.14. We wish to show $L = A^1(x) = A_1(r)$.

We may assume L meets $[p,q]$ only at r and there exists a line H_∞ which does not meet $[p,q]$. Then p,q lie in the same open half-space determined by $M = A^{\perp}(x_i)$ and H_∞ iff

a) if M meets A at two distinct points then it either supports or cuts A at both points,

or b) if M meets A exactly once then it supports A at that point;

that is

$$\sum_{p<s<q} \sigma_{\delta(S,M)}(S) \text{ is even.}$$

But since (p,q) is regular and 1-independent

$$\sum_{p<s<q} \sigma_{\delta(S,M)}(S) \equiv \sum_{p<s<q} \left(\sum_{i=0}^{\delta(S,M)} a_i(S) \right) \pmod 2$$

$$\equiv \sum_{p<s<q} \left(\sum_{i=0}^{\delta(S,M)} 1 \right) \pmod 2$$

$$\equiv \sum_{p<s<q} (\delta(S,M)+1) \pmod 2$$

and $1 = \sum_{p<s<q} (\delta(S,M)+1)-1$. Hence

$$\sum_{p<s<q} (\delta(S,M)+1) \equiv 2 \pmod 2$$

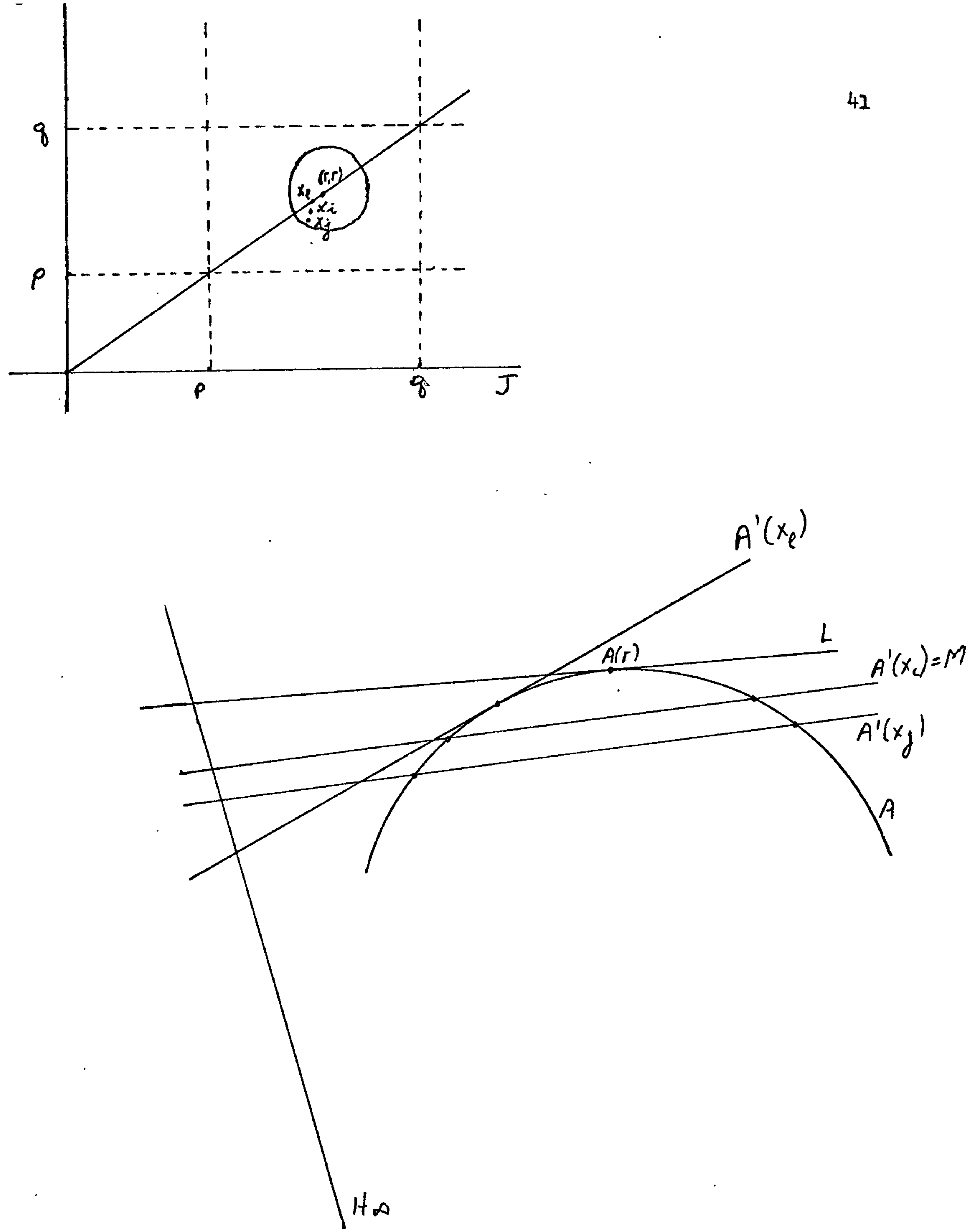


Figure II.14

Therefore L supports A at r. If L contains A(r) but $L \neq A^1(r)$ then since r is regular, L cuts A at r; a contradiction. Hence $L = A^1(x) = A_1(r)$.

Case 3. $x = (p,p) \in J^2$. Let P be a point with $\pi(P,p) = 0$. Put $L = PA(r)$ for fixed $r \in (p,q)$, (Figure II.15). Since A is continuous and $A(p) \neq A(r)$, there exists $u^+(p)$ such that for any $y \in (u^+(p))$, $A(y) \not\subset L$. Then by 2.4.2 put $\varphi(y) = A^1(y) \cap L$ for $y \in (u^+(p))^2$. Since $\varphi(y) \neq A(r)$, φ is monotone. Let $Q = \lim_{y \to x^+} \varphi(y)$. Since there exist $y_i \in u^+(p)^2$ such that $y_i \longrightarrow x$ and $A^1(y_i) \longrightarrow A^1(x)$, one has $P = Q$.

Now let x_i be a sequence in $[p,q)^2$ with $x_i \longrightarrow (p,p)$ and $A^1(x_i) \longrightarrow M$. Since A is continuous, $A(p) \subset M$. Now we can construct a sequence $y_i \in (p,q)^2$ such that $y_i \longrightarrow x$ and $A'(y_i) \longrightarrow M$. But then since $\varphi(y_i) \subset A^1(y_i)$, $P = \lim \varphi(y_i) \subset M$ and $M = A_1(p)$.

Corollary: If A is elementary then A^1 is continuous.

Proof: Let $p \in J$ be elementary. Then there exist $u^-(p) = (r,p)$, $u^+(p) = (p,q)$ where $r < p < q$ such that $u^+(p)$, $u^-(p)$ are of order 2. From above, A^1 is continuous on $(r, p]$ and $[p,q)$, hence A^1 is continuous on (r,q).

Using 2.6.2, we can derive further properties of ordinary points.

2.6.3 Lemma: Let p be ordinary and $\pi(P,p) = 1$ then p is ordinary on A/P.

Proof: If p is not ordinary on A/P then for every neighbourhood $u_i(p)$, there exists a 1-secant which meets $u_i(p)$

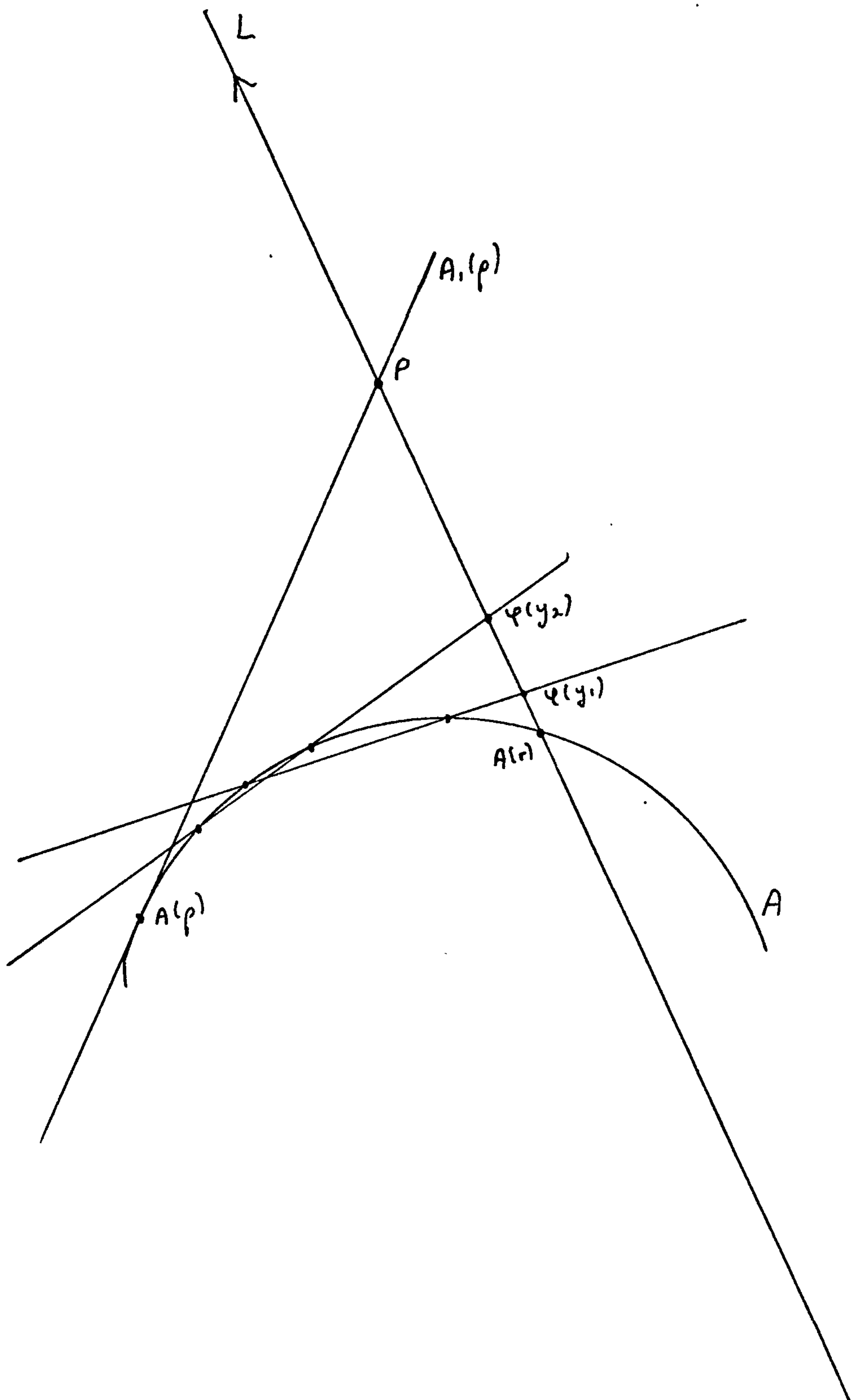


Figure II.15

at points say p_{0i}, p_{1i}. Then $p_{ji} \longrightarrow p$ for $j = 0, 1$ and $P \subset A^1(p_{0i}, p_{1i})$ for all i. But A^1 is continuous, by 2.6.2, and $P \subset A^1(p,p) = A_1(p)$; a contradiction.

2.6.4 Lemma: Suppose L is a line with $A(p) \not\subset L$ for all $p \varepsilon J$. Put $\varphi(p) = A_1(p) \cap L$ for $p \varepsilon J$. If A is ordinary then φ is monotone.

Proof: Put $\psi(x) = A^1(x) \cap L$ if $x \varepsilon u(p)^2$ where $u(p)$ is a neighbourhood of p. We may take $u(p)$ to be of order 2 and by 2.6.2, $\psi(J^2) \neq L$. Hence by 2.4.2, ψ and φ are monotone.

2.6.5 Lemma: Let A be of order 2. For any P, there exist $p \varepsilon J$ such that $P \not\subset A_1(p)$.

Proof: Trivially there exists $q \varepsilon J$ such that $P \neq A(q)$. Let L be a line through P with $A(q) \not\subset L$, (Figure II.16). Since A is continuous, there exists $u(q)$ such that $A(r) \not\subset L$ for $r \varepsilon U(q)$. Put $\varphi(r) = A_1(r) \cap L$. By 2.6.4, φ is monotone. Hence there exists some point p such that $\varphi(p) \neq P$; that is, $P \not\subset A_1(p)$.

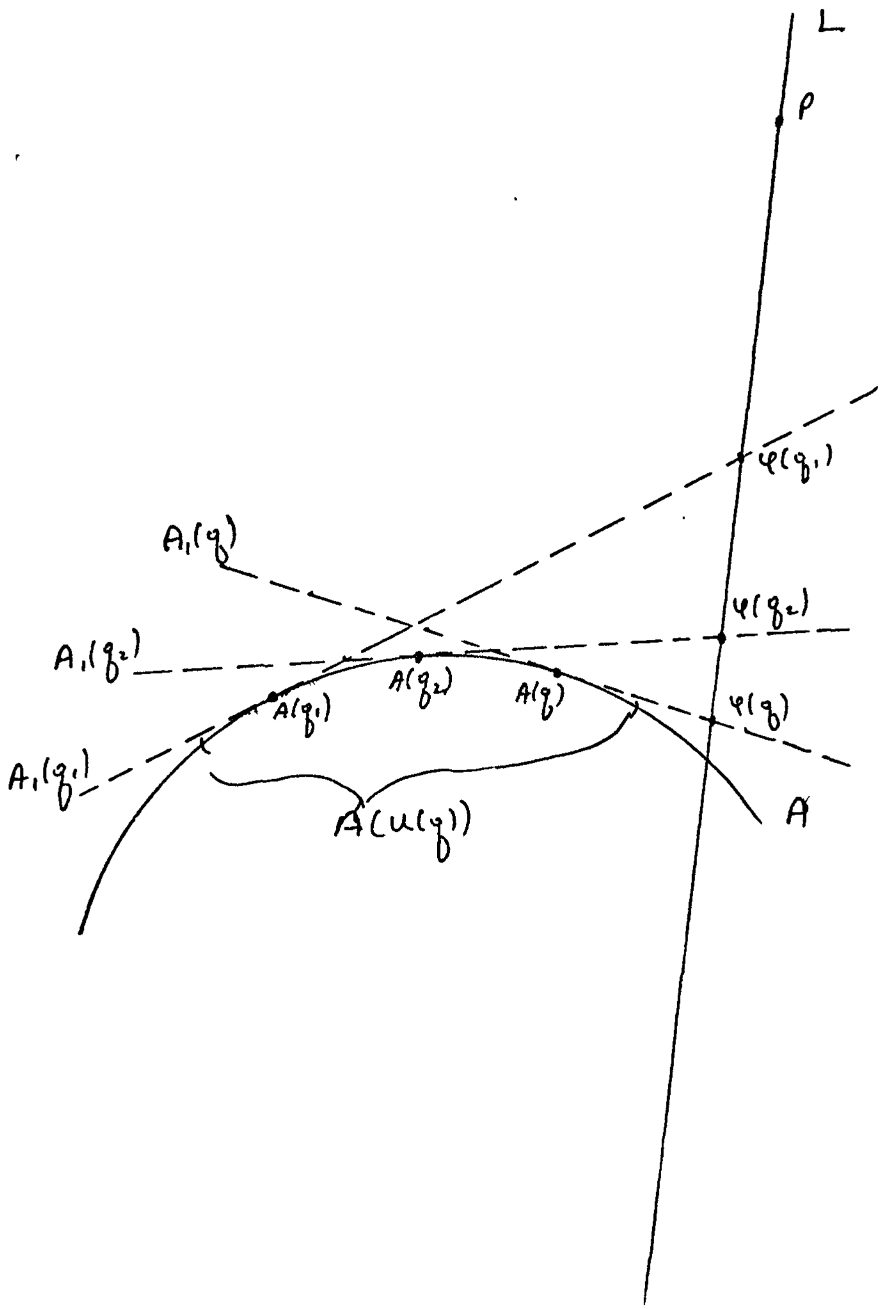


Figure II.16

CHAPTER III

3.1 Barner Arcs

We call an arc a <u>Barner arc</u> if there exists a continuous mapping $B: \; J \longrightarrow \underline{P}_1^2$ such that

$$\delta(p,B(x)) = \beta_{\gamma(p,x)}(p)-1$$

for all $p \in J$ and $x \in J$.

In this section we shall assume A is a Barner arc unless stated otherwise. Some elementary properties of Barner arcs are immediate from the definition, namely:

(3.1.1) $A(q) \subset B(p)$ iff $q = p$.

(3.1.2) $\delta(p,B(p)) = \beta_0(p)-1$ for all $p \in J$.

(3.1.3) If $p \neq q$ then $A(p) \neq A(q)$.

(3.1.4) If $p \neq q$ and $A(p) \subset A_1(q)$ then $\beta_0(q) = 1$.

<u>Proof</u>: Since $A(p) \subset A_1(q)$ then $A_1(q) \neq B(q)$ and $\delta(q,B(q)) = 0$. Hence from (3.1.2), $\beta_0(q) = 1$.

In view of the above properties, Barner arcs are <u>simple arcs</u>; that is, arcs where A is 1-1. Moreover, through each point of the arc we can draw a line which does not meet the arc elsewhere. Since $\beta_0(p) = a_0(p)$ is either one or two, the line $B(p)$ may or may not be a tangent line of A at p. And $B(p) = A_1(p)$ iff $\beta_0(p) = a_0(p) = 2$ iff non-tangent lines through p all support A at p.

Thus, for instance, if A is regular (Figure III.1) or if A has at most inflectious (Figure III.2), $B(p) \neq A_1(p)$ and $B(p)$ cuts A at p.

It is worthwhile to point out that in $\underline{P}^1$ an arc A is called a Barner arc if $A \neq \underline{P}^1_o$.

<u>3.1.5 Theorem</u>: Let $q \in J$. If $a_0(q) = 1$, then A/q is a Barner arc. If $a_0(q) = 2$, then the restriction of A/q to either component of $J \sim \{q\}$ is a Barner arc.

<u>Proof</u>: If $a_0(q) = 1$, then $B(q) \neq A_1(q)$. Consider the point $B(q)$ in $\breve{A} = A/q$. $B(q) \neq \breve{A}(q)$ and $B(q) \neq \breve{A}(p)$ if $p \neq q$, otherwise $B(q) = \breve{A}(p) = A(q)\, A(p)$ since A is simple. But then $A(p) \subset B(q)$, a contradiction. Hence A is not the whole of $\underline{P}^1_0$.

If $a_0(q) = 2$, then $B(q) = A_1(q)$ and $\breve{A}(q) = B(q)$. Trivially $\breve{A}(q) \neq \breve{A}(p)$ for any $p \in J \setminus \{q\}$ and the result follows.

<u>3.1.6 Theorem</u>: Let $p \in J$. Then p cannot have the characteristic (2,2).

<u>Proof</u>: If p has characteristic (2,2) then $B(p) = A_1(p)$ supports A at p. Take $u'(p)$ and a line H_∞ such that $A(p) \not\subset H_\infty$ and $A(q) \not\subset A_1(p)$, $A(q) \not\subset H_\infty$ for all $q \in u'(p)$. Upon projecting from p, $\breve{a}_0(p) = a_0(p) = 2$. Thus there exist points $p_1, p_2 \in u'(p)$ with $p_1 < p < p_2$ and a line $L = A(p)\, A(p_1)\, A(p_2)$. We may assume $A(p_1)$, $A(p_2)$ lie on same side of $A_1(p)$. If say $A(p_1)$ lies between $A(p)$ and $A(p_2)$,

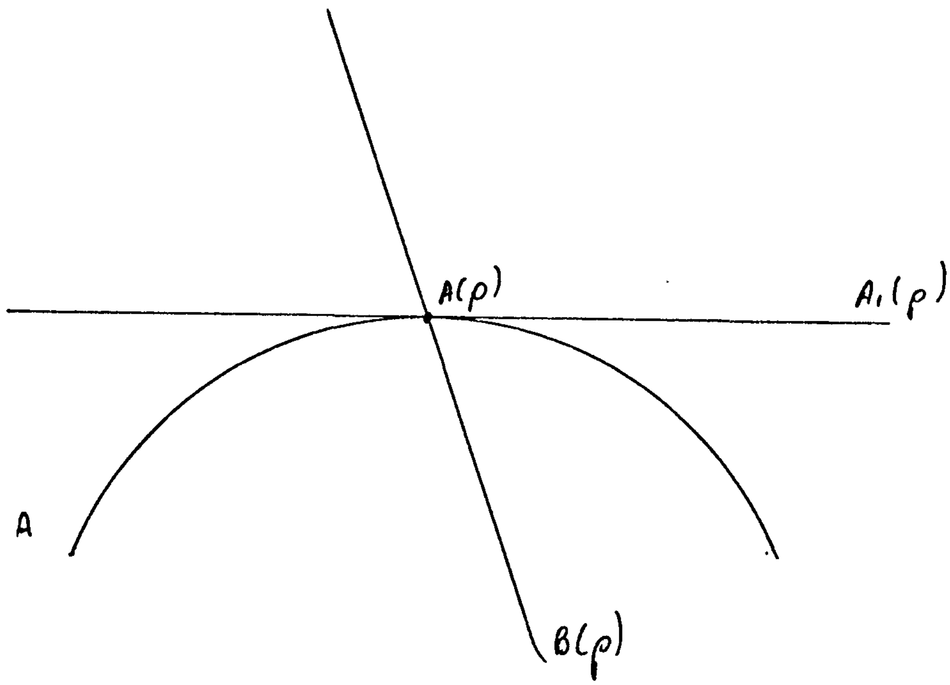


Figure III.1

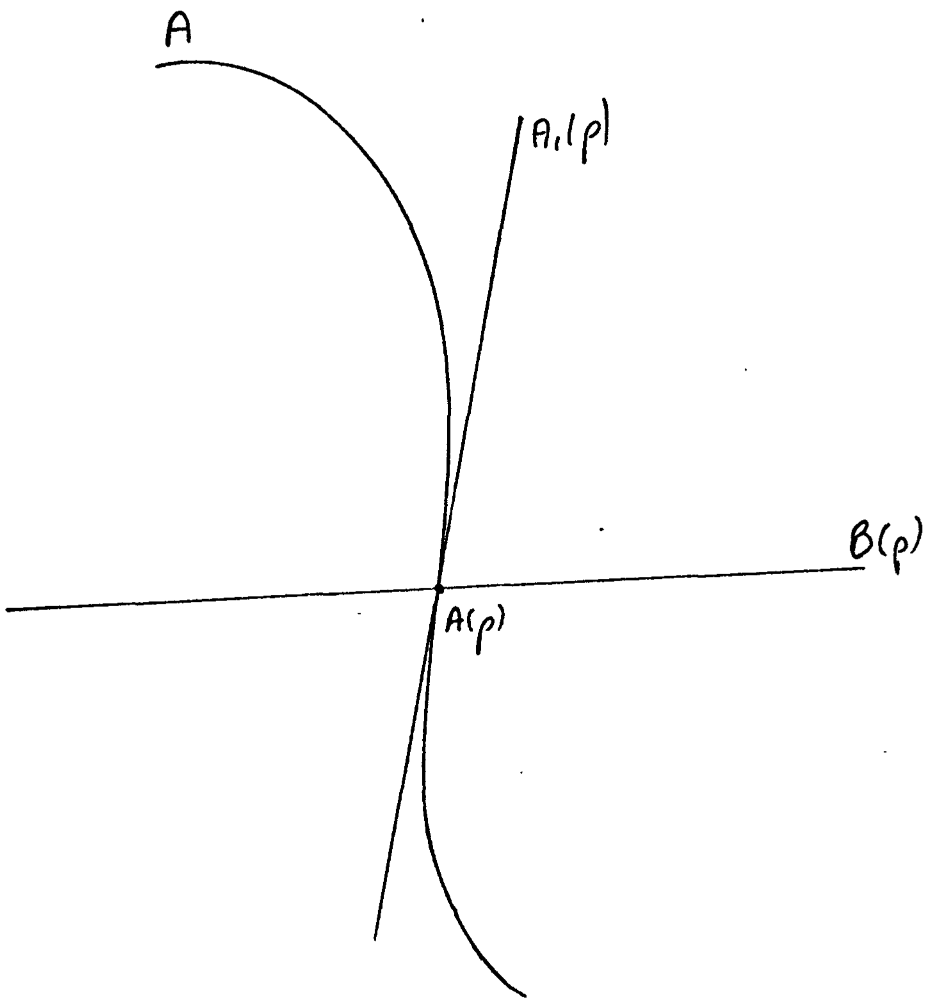


Figure III.2

then there exists $q \in [p_1, p_2]$ such that $A(q) = A(p_1) \subset B(p_1)$ a contradiction. Similarly $A(p_2)$ does not lie between $A(p)$ and $A(p_1)$. But then both L and $A_1(p)$ cannot support A at p, (Figure III.3), since $A(p) \subset L \cap A_1(p)$.

3.1.7 Lemma: If (p,q) is ordinary then $A(p) \not\subset A_1(q)$.

Remark: In $\underline{P}^1$, if (p, q) is ordinary and not equal to $\underline{P}^1_0$ then (p,q) is of order 1. Otherwise there is a point of inflection on (p,q) which contradicts the regularity of (p,q).

Proof: If $a_0(q) = 2$, then $B(q) = A_1(q)$ and result follows.

Assume $a_0(q) = 1$ and $A(p) \subset A_1(q)$. Then $\tilde{A} = A/q$ is a Barner arc and $\tilde{A}(p) = \tilde{A}(q)$. Therefore (p,q) is not ordinary on $\tilde{A}$, from above Remark. Let $p_1 \in (p,q)$ be the first singularity on $\tilde{A}$. Since p_1 is ordinary on A, then $A(q) \subset A_1(p_1)$ by 2.6.3. We now project (p_1, q) from p and in the same manner as above, we obtain $q_1 \in (p_1,q)$ such that $A(p_1) \subset A_1(q_1)$.

Consider the set S of intervals (r,s) such that $(r,s) \subset (p_1,q_1)$ and $A(r) \subset A_1(s)$. Let s_0 be the infinum of the $s \in (p_1,q_1)$ for which there exists an r with $(r,s) \in S$. Since p_1 is ordinary, there exists $u(p_1) \subset (p,q)$ such that $u(p_1)$ is of order 2 on A and $A(p_1) \not\subset A_1(s)$ for any $s \in u(p_1)$; hence, $p_1 < s_0$. Let (r_i, s_i) be such that $(r_i, s_i) \in S$, r_i converges, to say r_0, and $s_i \longrightarrow s_0$. Since s_0 is ordinary, $r_0 < s_0$ and $A(r_0) \subset A_1(s_0)$, as above. Repeating the argument of the preceeding paragraph, we obtain: $(r,s) \subset (r_0,s_0)$ such that $A(r) \subset A_1(s)$, contradicting the definition of s_0.

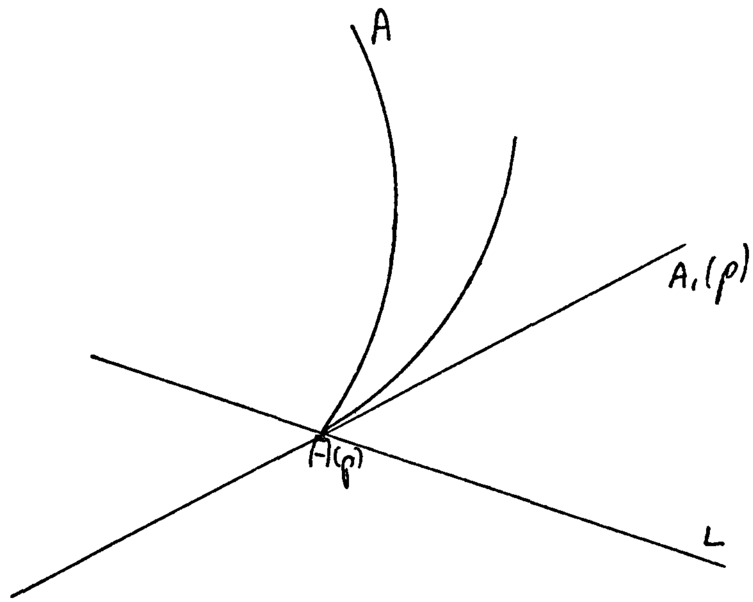


Figure III.3

<u>3.1.8 Theorem</u>: If (p,q) is ordinary then $[p,q]$ is of order 2.

<u>Proof</u>: If $[p,q]$ is of order 3, then there exist $p \leq p_1 < p_2 < p_3 \leq q$ such that p_1, p_2, p_3 are collinear. Since (p,q) is ordinary, then $a_0(p_2) = 1$ and $\breve{A} = A/p_2$ is Barner. Since there exists $u(p_2)$ such that $u(p_2)$ has order 2 on A, then p_2 is regular on A/p_2 by <u>2.4.1</u>, corollary 2. If $r \varepsilon (p,q)$, $r \neq p_2$ then $A(p_2) \not\subset A_1(r)$ by <u>3.1.7</u>, and r is ordinary on A/p_2 by <u>2.6.3</u>. Therefore (p,q) is ordinary on A/p_2. Since A/p_2 is an arc in $\underline{P}^1$, (p,q) is of order 1 on A/p_2. But $\breve{A}(p_1) = \breve{A}(p_3)$; a contradiction.

<u>Corollary</u>: Let A be a Barner arc. Then A is ordinary iff A is of order 2.

<u>3.1.9 Lemma</u>: If $p < q < r$ and (q,r) is ordinary then $A(p) \not\subset A_1(q) \cap A_1(r)$.

<u>Proof</u>: Assume $A(p) \subset A_1(q) \cap A_1(r)$, (FigureIII.4). Since (q,r) is of order 2 on A, there exists $s_0 \varepsilon (q,r)$ such that $A(p) \not\subset A_1(s_0)$, by <u>2.6.5</u>, and A_1 is continuous on (q,r), by <u>2.6.2</u>. Hence there exist q_0, r_0 such that $q \leq q_0 < s_0 < r_0 \leq r$, $A(p) \not\subset A_1(s)$ for $s \varepsilon (q_0,r_0)$, and $A(p) \subset A_1(q_0) \cap A_1(r_0)$.

Since (q_0,r_0) is ordinary and $\pi(p,s) = 1$ for $s \varepsilon (q_0,r_0)$, (q_0,r_0) as ordinary on A/p by <u>2.6.3</u>.

Since A is simple, $A(p)$, $A(q_0)$, and $A(r_0)$ are not colinear. Put $L_1 = A_1(q_0)$, $L_2 = A_1(r_0)$. Then if $s \varepsilon (q_0,r_0]$, $A(s) \not\subset L_1$ otherwise (q_0,s_0) would not be ordinary on A/p from the Remark of <u>3.1.7</u>. Therefore for $s \varepsilon (q_0,r_0)$, $A(s) \not\subset L_i$, $i = 1, 2$. Let L be a line such that

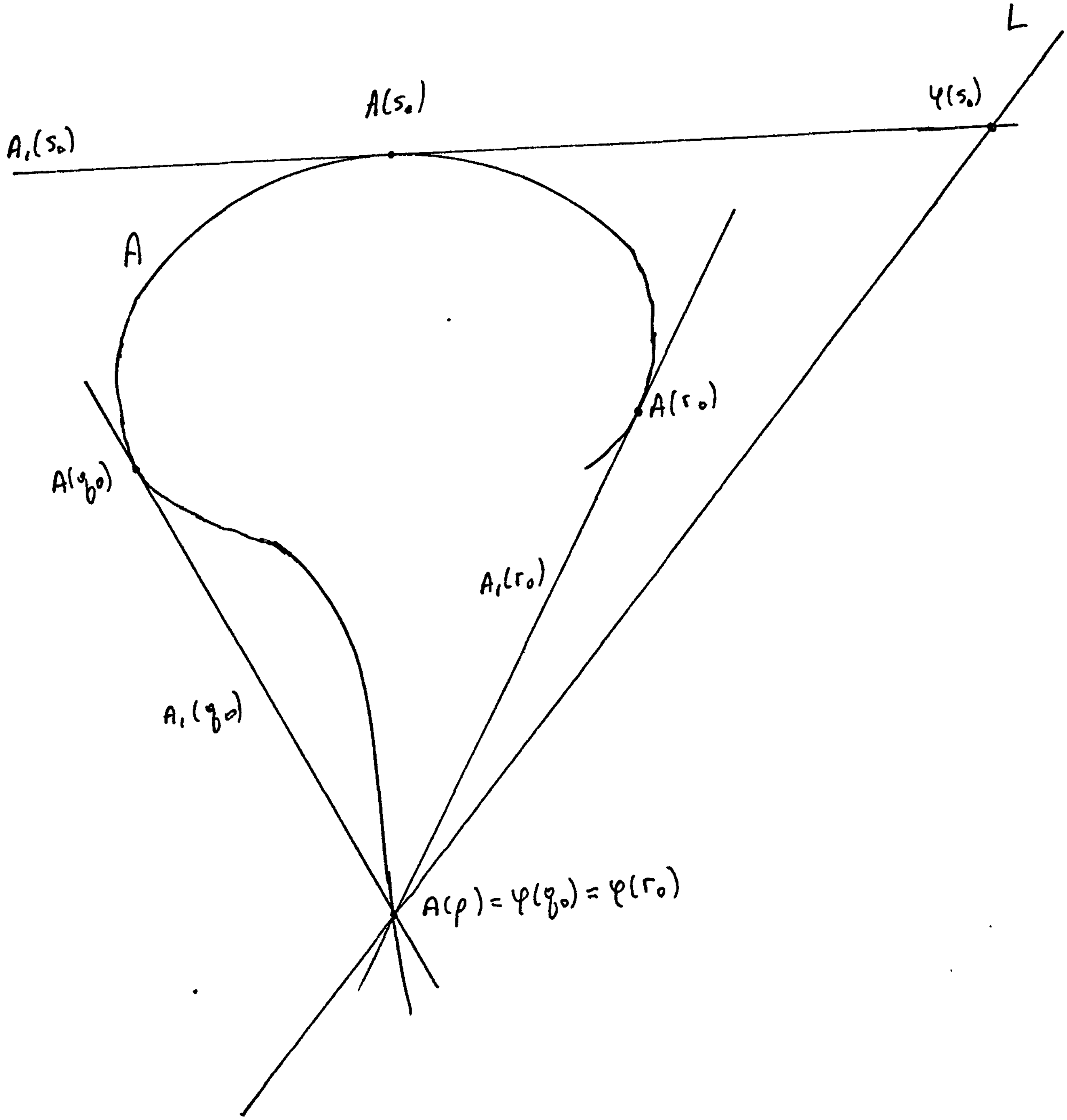


Figure III.4

$L \cap A[q_0, r_0] = \emptyset$ and $A(p) \subset L$. Put

$$\varphi(s) = A_1(s) \cap L$$

for $s \in [q_0, r_0]$. Since A_1 is continuous so is φ; moreover, φ is monotone on $[q_0, r_0]$ by 2.6.4. But $\varphi(q_0) = \varphi(r_0)$ and hence φ cannot exist.

3.2 Regular Barner arcs

In this section, unless stated otherwise, we assume A is a Barner arc with at most inflections. The following notation will be used: H_∞ is a line which does not meet a given closed interval $[p,q]$. Take, for example, $H_\infty = B(r)$, $r \notin [p,q]$. Let two points P,Q not in H_∞ determine a line L. Now L will meet H_∞ and P,Q divides L into two components. Denote by L_∞ $[L_f]$ the open segment [not] containing $L \cap H_\infty$.

The major result to be proved in this section is that a regular Barner arc has order 2.

3.2.1 Lemma: Suppose (p,q) has order 2 and H_∞ is a line not meeting $[p,q]$. Let $L = A(p)\,A(q)$. Then $A_1(r) \cap L \in L_\infty$ for all $r \in (p,q)$.

Proof: Since (p,q) has order 2, (p,q) is ordinary on A and $A_1(r)$ supports A at r for $r \in (p,q)$. Moreover, $A_1(r)$ does not meet $[p,q]$ outside of r by 2.4.1, Corollary 3. Hence $A(p)$, $A(q)$

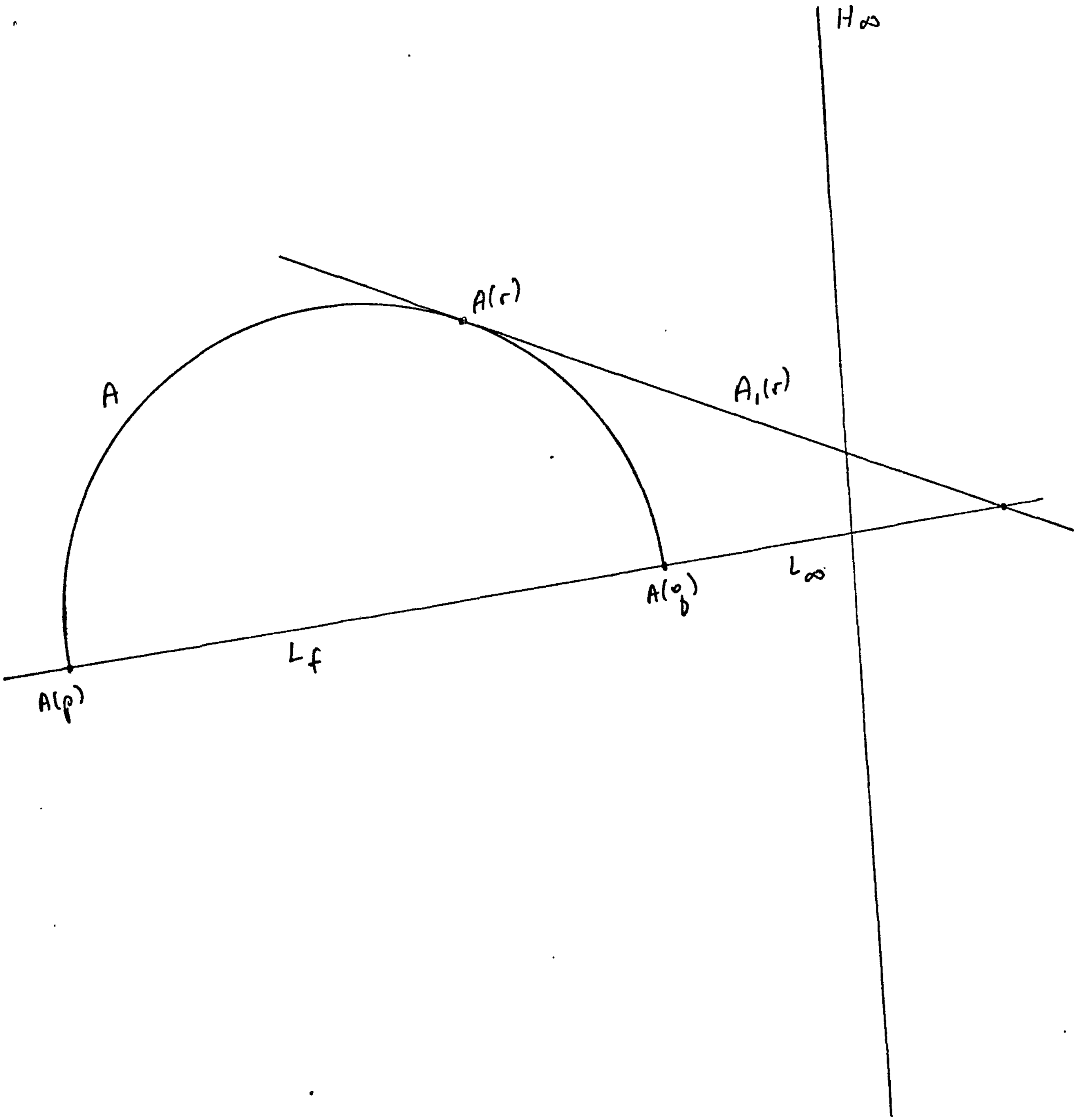


Figure III.5

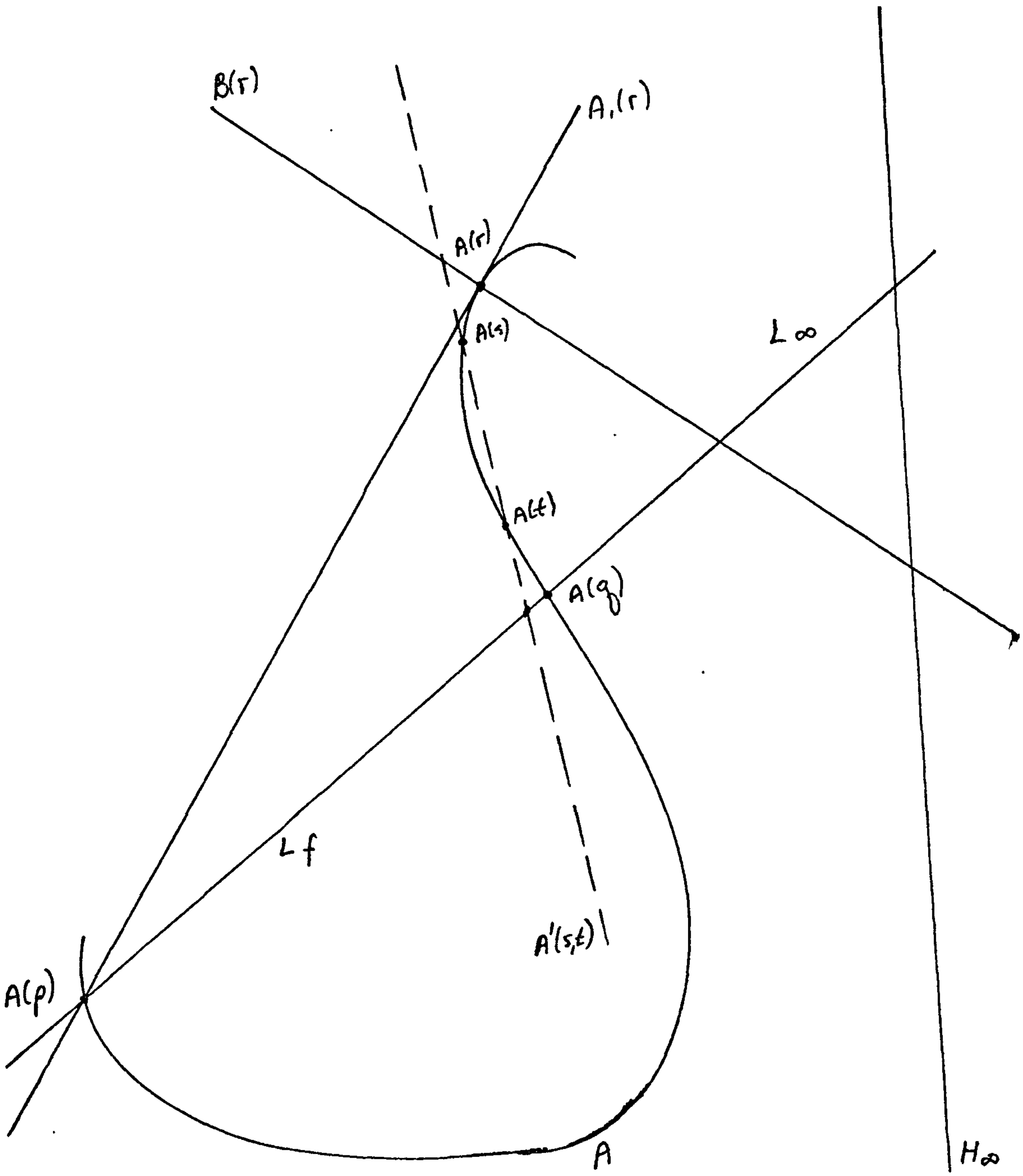


Figure III.6

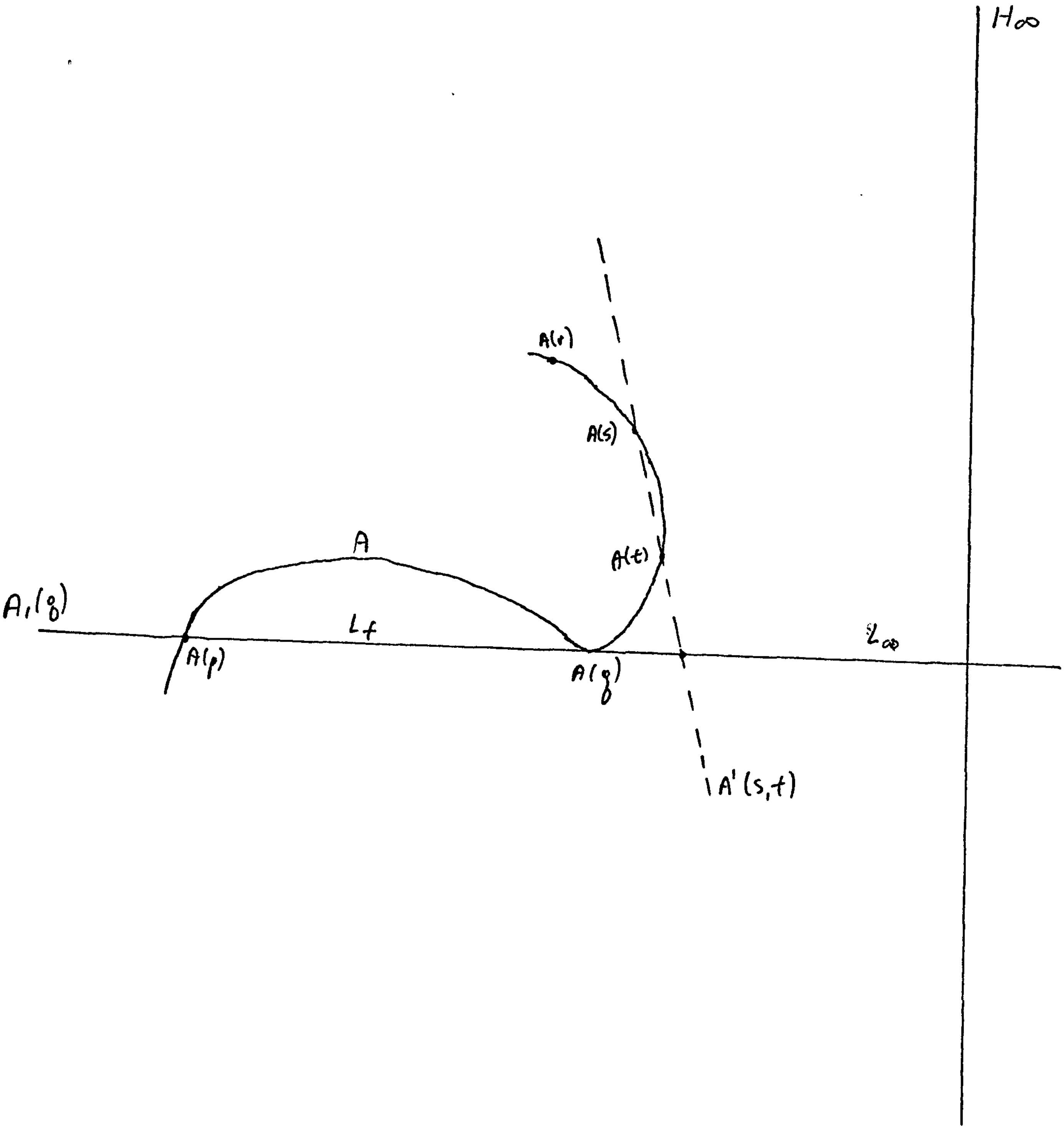


Figure III.7

lie in some open half-space determined by H_∞ and $A_1(r)$, (Figure III.5), and the result follows.

<u>3.2.2 Lemma</u>: Suppose $p < q < r$. Let H_∞ and $L = A(p)\,A(q)$ be given. If (q,r) is of order 2 and $A(p) \subset A_1(r)$, then $A_1(s)$ meets L_f for all $s \in (q,r)$.

<u>Proof</u>: Since $A(p) \subset A_1(r)$, $A(p) \not\subset A_1(s)$ for all $s \in (q,r)$, by <u>3.1.9</u>. Therefore $\pi(p,s) = 1$ for all $s \in (q,r)$ and (q,r) is ordinary and of order 1 on $A/p = \tilde{A}$ by <u>2.6.3</u> and the Remark of <u>3.1.7</u> But then $\tilde{A}(q) \neq \tilde{A}(s)$ for all $s \in (q,r)$ and $\tilde{A}(s) \neq \tilde{A}(t)$ for all $s \neq t$ in (q,r). Hence neither $A(p)$, $A(q)$, $A(s)$ nor $A(p)$, $A(t)$, $A(s)$ are collinear. Thus

$$L \cap A(q,r) = \emptyset$$

and

$$A(p) \not\subset A^1(s,t)$$

for $s \neq t$ in (q,r). Since $A(p) \not\subset A_1(s)$ for $s \in (q,r)$, the restriction $s \neq t$ can be dropped. From <u>2.4.1</u>, $[q,r)$ is 1-independent; thus

$$A(q) \not\subset A^1(s,t)$$

for s,t in (q,r). Since A^1 is continuous on (q,r), we have $A^1(s,t)$ meets either L_f or L_∞ for <u>all</u> $s,t \in (q,r)$.

Since $L \cap A(q,r) = \emptyset$, we may define

$$\varphi(t) = A^1(r,t) \cap L$$

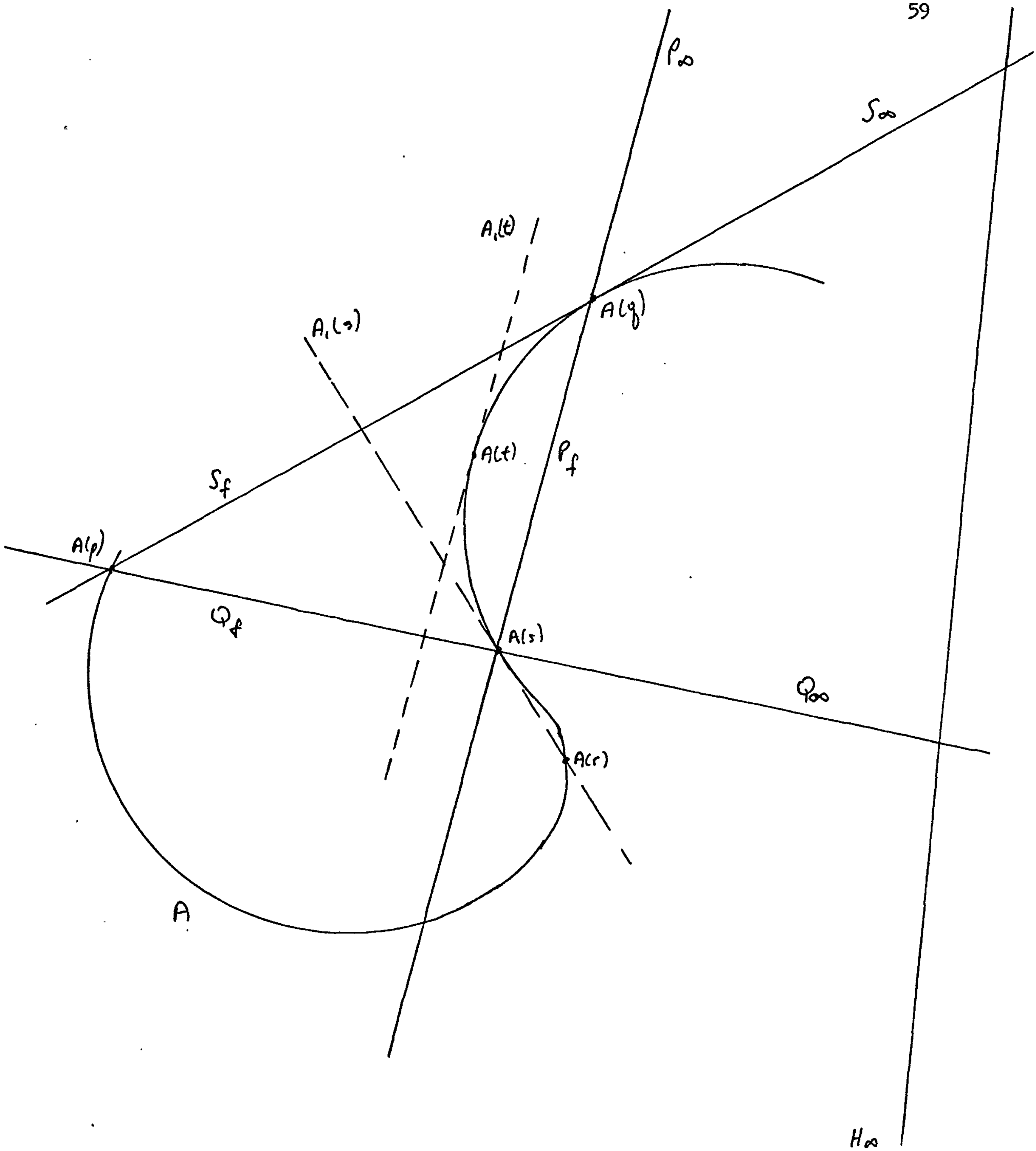


Figure III.8

for all $t \varepsilon [q,r]$, (Figure III.6). Then φ is monotone and continuous. Now $\varphi(r) = A_1(r) \cap L = A(p)$ and $\varphi(q) = A(q)$.

Since (q,r) is ordinary, it is regular and $a_0(r) = 1$. Therefore $B(r) \neq A_1(r)$ and $B(r)$ cuts A at r. Then $[p,q]$ lies in one of the open half-spaces determined by $B(r)$ and H_∞ and $B(r)$ meets L in L_∞. But $\varphi(t) \neq B(r)$ for all $t \varepsilon (q,r)$ by the definition of $B(r)$. Since $\varphi(t)$ moves continuously from $A(q)$ to $A(p)$ as t moves from q to r, there is a t_0 such that

$$\varphi(t_0) = A^1(r,t_0) \cap L \varepsilon L_f.$$

Thus $A'(s,t)$ meets L_f for all $s,t \varepsilon (q,r)$.

It is interesting to note that the condition $p < q < r$ and $A(p) \subset A_1(r)$ determines the concavity (towards) of (q,r) of order 2 with respect to P. If we retain the original hypothesis with the exception that $A(p) \subset A_1(q)$ now, then since $A(p) \not\subset A_1(r)$ by 3.1.9 (q,r) is concave away from p (Figure III.7).

3.2.3 Lemma: Suppose A is regular and that there exists $p < q$, q ordinary such that $A(p) \subset A_1(q)$. Then there exist r, s, with s singular, such that $A_1(s)$ cuts A at r.

Proof: Take s as small as possible such that (s,q) is ordinary. (This s exists, otherwise (p,q) is ordinary which is impossible by 3.1.8). Therefore s is singular, $p < s < q$, and (s,q) is of order 2 on A. Let H_∞ be a line not meeting $[p,q]$ and put $P = A(s)\ A(q)$, $S = A(p)\ A(q)$, $Q = A(p)\ A(s)$, (Figure III.8).

For any $t \varepsilon (s,q)$, $A_1(t)$ meets P_∞ by 3.2.1 and Q_f by 3.2.2. Thus $A_1(t)$ meets S_f. Since $A(p) \subset A_1(q)$, $A(p) \not\subset A_1(s)$ by 3.1.9 and $A(q) \not\subset A_1(s)$ since (q,s) is of order 2. Therefore $A_1(s)$ meets S_f since A_1 is continuous on (s,q) and $A_1(t)$ meets S_f for $t \varepsilon (s,q)$. Since $A_1(s)$ does not meet (s,q), there is an $r \varepsilon (p,s)$ such that $A_1(s)$ meets A at r. Since s is regular, the result follows.

It should be noted that the arc depicted in Figure III.8 is not a regular arc but an arc with at most inflections. However it is a Barner arc since through each point of the arc, a line can be drawn through that point which does not meet the arc elsewhere. It is this fact, that it is impossible to construct a regular Barner arc such that $A(p) \subset A_1(q)$ for some two distinct points of the arc, which indicates that a regular Barner arc is of order 2. In fact, we shall use 3.2.3 primarily to prove that a regular Barner arc is of order 2.

3.2.4 a) In the Remark of 3.1.7, it was stated that if A is a regular arc in $\underline{P}^1$ not equal to $\underline{P}^1_0$ then A is of order 1. Hence, if A is a regular Barner arc in $\underline{P}^1$ then A is of order 1.

b) Let A be a Barner arc in $\underline{P}^1$, if A has a singularity then it is an elementary singularity.

Proof: If A has no elementary singularity, then it has an inflection point p_1. Then there is a neighbourhood (q_1,r_1) of p_1 such that $A(p) \neq A(p_1)$ for all $p \varepsilon (q_1,p_1) \cup (p_1,r_1)$. We may assume $A(q_1) = A(r_1)$ and $A(p) \neq A(q_1)$ for all $p \varepsilon (q_1,r_1)$. Since p_1 is not elementary then (q_1,p_1) (and) or (p_1,r_1) contains an inflection. Let X_1 be the interval containing an inflection and Y_1 be the other interval. Then $X_1 \cap Y_1 = \emptyset$ and $A(X_1) = A(Y_1)$.

Repeat the above argument using X_1 instead of J. Thus p_2, q_2, r_2, X_2 and Y_2 are defined. Continuing indefinitely, one obtains sequences X_i, Y_i such that $X_i \cap Y_i = \emptyset$, $A(X_i) = A(Y_i)$, and $\overline{X}_{i+1}, \overline{Y}_{i+1} \subset X_i$; $i = 1, 2, \ldots$ It follows that $\overline{A(X_{i+1})} = A(\overline{X}_{i+1}) \subset A(X_i)$, $i = 1, 2, \ldots$. Since J is isomorphic to the set of real numbers and $\{X_i\}$ is a nested sequence of intervals in J, then there exists a point $P \in \bigcap_{i=1}^{\infty} X_i$.

Hence $P = A(p) \in \bigcap_{i=1}^{\infty} A(X_i)$. But then P meets each Y_i and since the Y_i are disjoint, P meets $[q_1, r_1]$ an infinite number of times, a contradiction by 1.3.1.

3.2.5 Lemma: If A is not of order 2, then there exist $p < q$ such that $\delta(p, A_1(q)) = 0$.

Proof: A not of order 2 implies there exist three points $p < p_1 < p_2$ which are collinear. Consider $\tilde{A} = A/p$. In $\tilde{A}$, $\tilde{A}(p_1) = \tilde{A}(p_2)$ and therefore (p, p_2) is not of order 1 on $\tilde{A}$. By 3.2.4, (p, p_2) is not regular on $\tilde{A}$.

If there exists only one inflection point q of $\tilde{A}$ on (p, p_2) then (p,q) is regular and of order 1 on $\tilde{A}$ by 3.2.4. But then $A_1(p) = \tilde{A}(p) \neq \tilde{A}(q) = A(p)\, A(q)$ and $A_1(p) \neq A_1(q)$. Since $\tilde{a}_0(q) = 2$ and from 2.5.2 .

$$\tilde{a}_0(q) \equiv \begin{cases} a_0(q) & 0 < \pi(p,q) \\ a_0(q) + a_1(q) & 0 = \pi(p,q) \\ a_1(q) & 0 > \pi(p,q), \end{cases} \qquad \pmod 2$$

$0 = \pi(p,q)$; otherwise, $\pi(p,q) = -1$ a contradiction since A is Barner or $\pi(p,q) = 1$ and $\tilde{a}_0(q) = a_0(q) = 2$, a contradiction. Since $A(p) \subset A_1(q)$, the result follows.

If $q_1 < q_2$ are inflections of $\hat{A}$ in $(p_1 p_2)$ then by 3.2.4, there is an ordinary point q_3 of $\hat{A}$ in (q_1,q_2). Take $q_1 \leq q_4 < q_3 < q_5 \leq q_2$ such that (q_4,q_5) is the largest regular subarc of (q_1,q_2) in $\hat{A}$. Then there exists a point of inflection q of $\hat{A}$ such that $\hat{A}(p) \neq \hat{A}(q)$; otherwise $\hat{A}(p) = \hat{A}(q_4) = \hat{A}(q_5)$ and (q_4,q_5) is not regular on $\hat{A}$. Since $\hat{A}(p) \neq \hat{A}(q)$ for some inflection q of $\hat{A}$, the result follows as in the preceding paragraph.

<u>3.2.6 Lemma</u>: If A is regular but not of order 2 then there exist $p < q < r$, q singular and $A(p)\,A(q)\,A(r)$ a line which cuts A at p.

<u>Proof</u>: By 3.2.5, there exist $s < q$ with $\delta(s, A_1(q)) = 0$. Since $a_0(s) = 1$, $A_1(q)$ cuts A at s. By 3.2.3, we may assume that q is singular. Projecting from q, $\hat{A}(q) = \hat{A}(s)$ and $\hat{A}(q)$ cuts $\hat{A}$ at q. Hence there exists r with $q < r$ and $\hat{A}(r)$ cutting $\hat{A}$ at $p < q$; that is, $A(r)\,A(p)\,A(q)$ is a line which cuts A at p. (Figure III.9).

<u>3.2.7 Theorem</u>: Let A be a Barner arc. Then

a) if A is regular A is of order 2,

b) if A has at most inflections and A_1 is continuous, A has an ordinary point.

<u>Proof</u>: a) Assume A is not of order 2. Take p_1, q_1, r_1 with the properties of p,q,r in 3.2.6. Let X_1 be a neighbourhood of q_1

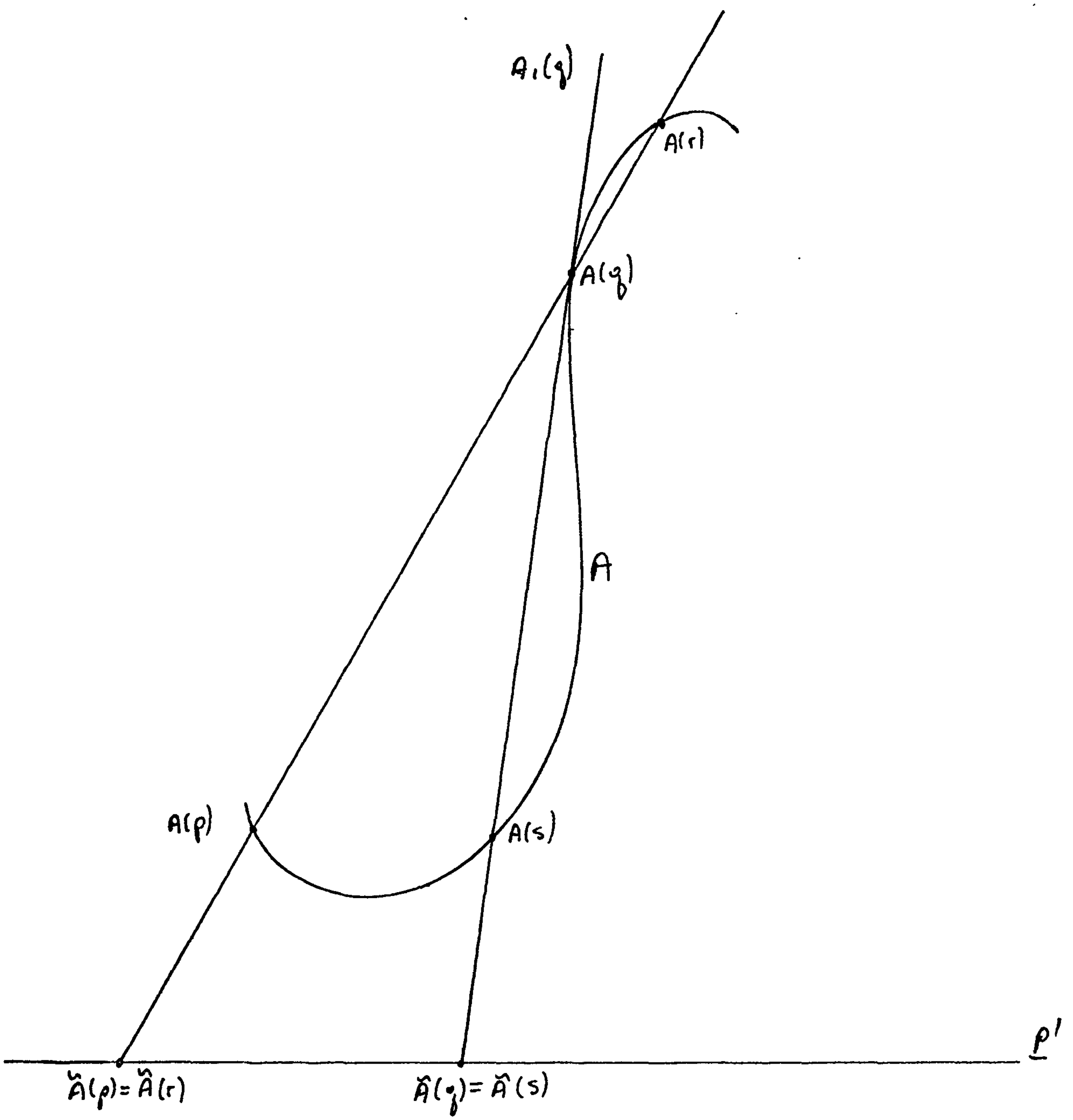


Figure III.9

such that $r_1 \not\in X_1$ and for any $q \in X_1$, $A(q)\ A(r_1)$ meets A in a point $p \not\in X_1$, $p < q_1$. Since q_1 is a singularity, we may repeat this argument using X_1 instead of J and obtain p_2, q_2, r_2 and X_2. Continuing one obtains p_i, q_i, r_i and X_i; $i = 1, 2, \ldots$ such that $\overline{X}_{i+1} \subset X_i$, $r_i \not\in X_i$, $q_i < r_i$ and for $q \in X_{i+1}$, $A(q)\ A(r_{i+1})$ meets $X_i \setminus X_{i+1}$ in $p < q_{i+1}$; $i = 1, 2, \ldots$. We may assume $\bigcap_i X_i = \{q\}$. Then $q < r_i$; $i = 1, 2, \ldots$. Also $A(q)\ A(r_{i+1})$ meets X_i in a point $p < q$, $i = 1, 2, \ldots$ Since q is regular on A it is regular on $\check{A} = A/q$. Therefore there exists, $u(q) = u-(q) \cup \{q\} \cup u+(q)$ of order 1 on $\check{A}$; that is $\check{A}(u-(q)) \cap \check{A}(u+(q)) = \emptyset$. Taking $X_i \subset u(q)$, we obtain a contradiction.

b) Suppose A has no ordinary point. By 3.2.5, there exist $p < q$ such that $\delta(p, A_1(q)) = 0$. Now $A_1(q)$ cuts A at p. Let X_1, Y_1 be disjoint neighbourhood of p, q respectively such that $A_1(q)$ meets X_1 if $q \in Y_1$. This is possible since by hypothesis A is continuous. Repeating this process and using X_1 instead of J one obtains X_2, Y_2. Continuing indefinitley one obtains contradiction as in 3.2.4.

3.3 Existence theorems

3.3.1 Lemma: Let A be an arc such that for each $p \in J$, $a_i(p) = 2$ for at most one i; $i = 1, 2$. If there is a point P such that $\check{A} = A/p$ is of order 1 and

$$\pi(P,p) = \begin{cases} 1 & \text{if} \quad P \text{ regular} \\ i & \phantom{\text{if}} \quad a_i(p) = 2 \end{cases}$$

for all $p \in J$, then A is a Barner arc.

<u>Proof</u>: Put $B(x) = \widehat{A}(x)$ for $x \in J$. Since $\widehat{A}$ is continuous, so is B. Put $\widetilde{\delta} = \widetilde{\delta}(p,B(x))$ and $\delta = \delta(p,B(x))$. We must show

$$\delta = \beta_{\gamma(p,x)}(p)-1$$

for all $p, x \in J$. Since $\breve{A}$ is of order 1, $\widehat{\delta} = \gamma(p,x)$.

<u>Case 1.</u> p is regular. $\pi(P,p) = 1$ implies $\breve{\delta} = \delta$ by <u>2.2.1</u> and $\breve{\delta} = \delta = \gamma(p,x)$. If $\gamma(p,x) = 0$, $\beta_{\gamma(p,x)}(p)-1 = \beta_0(p)-1 = a_0(p)-1 = 0$. If $\gamma(p,x) = -1$, $\beta_{\gamma(p,x)}(p)-1 = \beta_{-1}(p)-1 = -1$. There fore the result follows.

<u>Case 2.</u> $a_i(p) = 2$. Then by <u>2.2.1</u>,

$$\breve{\delta} = \begin{cases} \delta & -1 \leq \delta < \pi(P,p) = i \\ \delta-1 & i = \pi(P,p) < \delta \leq 1. \end{cases}$$

and

$$\beta_k(p) = \begin{cases} k+1 & \text{if } -1 \leq k < i \\ k+2 & \text{if } i \leq k \leq 1. \end{cases}$$

Therefore iff $\delta < i$, $\delta = \breve{\delta} = \gamma(p,x)$ and $\beta_{\gamma(p,x)}(p) = \gamma(p,x) + 1$; hence $\delta = \beta_{\gamma(p,x)}(p)-1$. If $\delta > i$, $\delta = \breve{\delta}-1 = \gamma(p,x) + 1$. Thus $\gamma(p,x) \geq i$ and $\beta_{\gamma(p,x)}(p) = \gamma(p,x) + 2$. The result now follows.

3.3.2 Theorem: Barner arcs exist in the real projective plane.

Proof: Let A be an arc and let (p,q) be of order 2. By 2.6.1, (p,q) is regular on A and by 2.4.1, Corollary 2, (p,q) has order 1 on A/p. Applying 2.3.1 with P = A(p), we are finished.

In 3.2.7 b) it was proved that a Barner arc, with at most inflections on which A_1 is continuous, has an ordinary point. The remainder of this section is involved in proving a more general statement about ordinary points; namel y that every arc has an ordinary point.

3.3.3 Lemma: Let W be a non-empty set of points of an arc A such that A_1 is continuous at each point of W. Then there is a subarc X which contains a point of W such that if $p,q \in X_1$ $p \neq q$, $p \in W$, then $A_1(p)$ does not cut A at q.

Proof: Suppose that for every subarc X with $X \cap W \neq \emptyset$, there exist $p,q \in X$, $p \neq q$, $p \in W$ such that $A_1(p)$ cuts A at q. Let $p_1,q_1 \in J$ be two such points. Since $p_1 \in W$ and $q_1 \notin W$, there are disjoint neighbourhoods X_1,Y_1 of p_1,q_1 respectively, such that if $p \in X_1$ then $A_1(p)$ meets Y_1. Since $p_1 \in X_1$ we may repeat this construction replacing J by X_1, and so on. As in 3.2.7 b), we get a contradiction.

3.3.4 Lemma: Consider the following properties of arcs in $\underline{P}^1$.

a) Every arc has an ordinary point.

Proof: This is trivial by 3.2.4 (b), since an arc in P^1 has at most inflections.

b) If A is a Barner arc, then either A is of order 1 or A has an elementary inflection.

Proof: If A ha no singularity then A is ordinary. If A has a singularity then it is elementary by 3.2.4 (b).

3.3.5 Lemma: Let A be an arc and L be a line such that $A(p) \not\subset L$ for all $p \in J$. Let $\{ P_i | i = 1,2,\ldots \}$ be a dense set of points in L. Then there exists a sequence X_i; $i = 1,2,\ldots$ of open intervals such that X_i has order 1 on $A^{(i)} = A/P_i$ and $\{ X_i \}$ is nested.

Proof: Consider $A^{(1)} = A/P_1$ an arc in $\underline{P}^1$. By 3.3.4(a), there exists $p_1 \in J$ such that p_1 is ordinary on $A^{(1)}$. Let X_1 be a neighbourhood of p_1 such that X_1 has order 1 on $A^{(1)}$. Now consider X_1 and P_2 instead of X and P_1. By 3.3.4(a), there exists $p_2 \in X_1$ such that p_2 is ordinary on $A^{(2)}$. Let X_2 be a neighbourhood of p_2 such that $\overline{X}_2 \subset X_1$ and X_2 has order 1 on $A^{(2)}$. Repeating the construction for $i = 3,4,\ldots,$ the result follows.

3.3.6 Lemma: Let A be an arc with at most inflections. Then there exists a point at which A_1 is continuous.

Proof: We may assume there exists a line L such that $A(p) \not\subset L$ for $p \in J$. Let $\{ P_i | i = 1,2,\ldots \}$ be a set of points of L which is dense in L. By 3.3.5, there exists a nested sequence $\{ X_i \}$ of open intervals such that X_i is of order 1 on A/P_i; $i = 1,2,\ldots$. Take $p \in \bigcap_{i=1}^{\infty} X_i$ and put $P = A_1(p) \cap L$.

We wish to show A_1 is continuous at p, hence $\lim_{q \to p} A_1(q) \cap L = P$.

Let $U(P)$ be a neighbourhood of P on L, say one with end points P_i, P_j where $i < j$. Take $q < p < r$; $q, r \varepsilon X_j$. Since X_j is of order 1 on A/P_j, $A(s) \neq A(t)$ for $s, t \varepsilon X_j$, $s \neq t$. We may define a continuous path (Figure III.10) in $\underline{P}_1^2$ by

$$A_1(p) \xrightarrow{\varphi_1} A(p)A(r) \xrightarrow{\varphi_2} A(q)A(r) \xrightarrow{\varphi_3} A_1(q)$$

where

$$\begin{aligned} \varphi_1(s) &= A(p)A(s) & & s \varepsilon (p,r) \\ \varphi_2(s) &= A(s)A(r) & \text{if} \quad & s \varepsilon (q,p) \\ \varphi_3(s) &= A(q)A(s) & & s \varepsilon (q,r). \end{aligned}$$

Since X_j is of order 1 on A/P_i, A/P_j, no line of this path, except possibly $A_1(q)$ contains P_i or P_j. For if say $\varphi_1(s) = A(p)A(s)$ contains P_i, then $(A/P_i)(p) = (A/P_i)(s)$ on A/P_i; a contradiction. Hence $A_1(q) \cap L \varepsilon \overline{U(p)}$ and similarly $A_1(r) \cap L \varepsilon \overline{U(P)}$. The path is continuous since A is continuous. Since q and r are arbitrary points of X_j with $p < q < r$, the continuity of A_1 at p follows.

In Figure III.10, the path moves along s_i as i increases. Hence φ_1 moves from $A_1(p)$ to $A(p)A(r)$, φ_2 from $A(p)A(r)$ to $A(q)A(r)$ and finally φ_3 form $A(q)A(r)$ to $A_1(q)$.

<u>Corollary</u>: Let A be an arc with at most inflections. Let W be the set of points at which A_1 is continuous, then W is dense in A.

<u>Proof</u>: If W is not dense in A there exists $q \varepsilon J$ such that q is not the limit of any sequence of W. Hence there exists $U(q)$

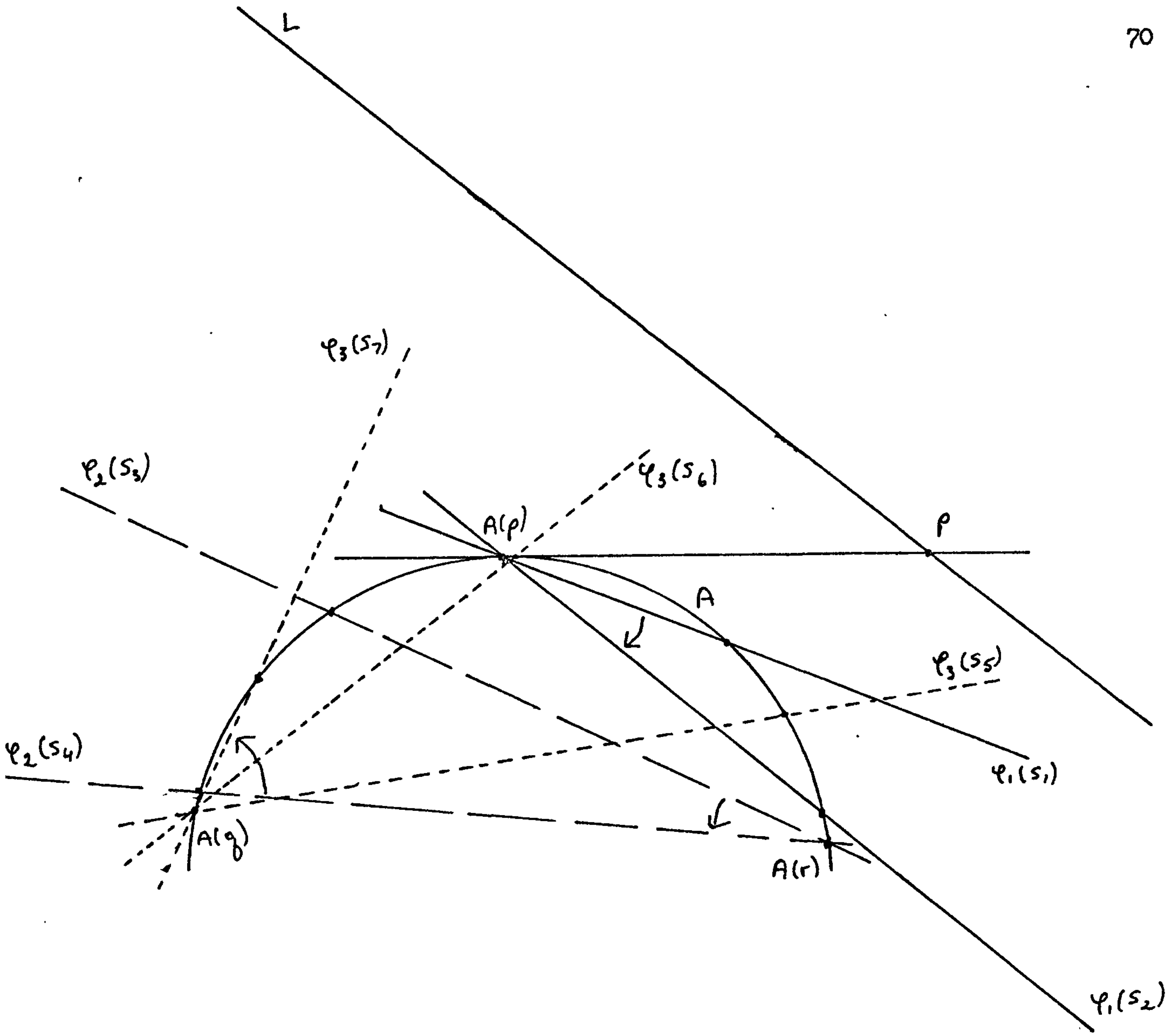


Figure III.10

such that $U(q) \cap W = \emptyset$. But $U(q) \subset A$ has at most inflections, a contradiction by 3.3.6.

3.3.7 Lemma: Every arc contains a Barner arc with at most inflections.

Proof: We may assume there is a line L not meeting A. Let $P_1 \neq P_2$ be points on L. By 3.3.5, there exist subarcs X_1, X_2 such that $X_2 \subset X_1$ and X_i has order 1 on $A^{(i)} = A/P_i$, $i = 1,2$. Then X_2 has at most inflections on A; for if, $p \in X_2$, P_1 or $P_2 \notin A_1(p)$, say $P_1 \notin A_1(p)$. Since p is regular on $A^{(1)}$ and $\pi(P,p) = 1$, $1 = a_0^{(1)}(p) = a_0(p)$ and p is at most an inflection on A.

Applying 3.3.6 to X_2, there exists a point $p \in X_2$ such that A_1 is continuous at p. Choose P and $U(p) \subset X_2$ such that $P \notin A_1(q)$ for all $q \in U(p)$. By 3.3.4 (a), there exists $X \subset U(P)$ of order 1 on A/P. Then X is a Barner arc by 3.3.1.

3.3.8 For the remainder of this section we assume A is a Barner arc with at most inflections. Moreover, the following notation will remain fixed: p_0 is a point at which A_1 is continuous. (p_1,p_2) is a neighbourhood of p_0 and H_∞ a line which does not meet $[p_1,p_2]$. $A_1(p_0)$ meets $[p_1,p_2]$ only at p_0 and $A(p_1) \subset A_1(p\ )$ for any $p \in (p_0,p_2)$. Since A is a Barner arc, $L = A(p_1)\ A(p_0)$ is a line. Let $L_\infty[L_f]$ be the open segment of L with end points $A(p_1), A(p_0)$ which does [not] meet H_∞.

3.3.9 Lemma: Given the above hypothesis, (p_0,p_2) has order 1 on $\tilde{A} = A/p_1$ and $A(p_0,p_2) \cap L = \emptyset$.

<u>Proof</u>: Since $\pi(p_1,p) = 1$ for all $p \varepsilon (p_0,p_2)$, $\breve{a}_0(p) = a_0(p) = 1$ for all $p \varepsilon (p_0,p_2)$. $a_0(p_1) = 1$ and hence $\breve{A}$ is Barner by <u>3.1.5</u>. Therefore (p_0,p_2) is a regular Barner subarc of $\breve{A}$ and by <u>3.2.4</u>(a), (p_0,p_2) has order 1 on $\breve{A}$.

If L meets (p_0,p_2) at a point q, then $\breve{A}(p_0) = \breve{A}(q)$ and there exist an inflection point $q^1 \varepsilon (p_0,q)$, a contradiction from above.

<u>3.3.10 Lemma</u>: Suppose p_0 is an inflection. If there is a $p_3 \varepsilon (p_0,p_2)$ such that (p_0,p_3) has order 1 on $\breve{A} = A/p_0$, then for each $p \varepsilon (p_0,p_3)$ either $A_1(p)$ meets L_f or $A(p_0) \subset A_1(p)$.

<u>Proof</u>: Since $a_0(p_0) + a_1(p_0) = 3$, $A_1(p_0)$ cuts A at p_0 and $A(p_1)$, $A(p_3)$ lie on opposite sides of $A_1(p_0)$. Since A_1 is continuous at p_0, we can choose $p_4 \varepsilon (p_0,p_2)$ such that $A(p_1)$,$A(p_3)$ lie on opposite sides of $A(p_0)A(p_4)$, (Figure III.11).

Since (p_0,p_2) is of order 1 on A/p_0, (p_0,p_2) and hence p_4 is regular on $\breve{A}$. Therefore $\breve{A}(p_4)$ cuts $\breve{A}$ at p_4 and $A(p_0)A(p_4)$ cuts A at p_4. Since $A(p_0)A(p_4)$ meets (p_0,p_3) only at p_4, we have that $A(p_1)$ and $A(p_0,p_4)$ lie on the same side of $A(p_0)A(p_4)$. Now by projecting from p_4, we find that (p_0,p_4) is not of order 1 on A/p_4 and there exists $p_5 \varepsilon (p_0,p_4)$ such that $A(p_5)A(p_4)$ meets L_f.

Let $p \neq q \varepsilon (p_0,p_3)$. Since (p_0,p_3) is of order 1 on both A/p by <u>3.3.9</u>, and A/p_0 by hypothesis, neither $A(p_0)$ nor $A(p_1)$ lie on $A(p)A(q)$. Since $A(p_4)A(p_5)$ meets L_f, the continuity of A implies that $A(p)A(q)$ meets L_f for $p_0 < p < q < p_3$.

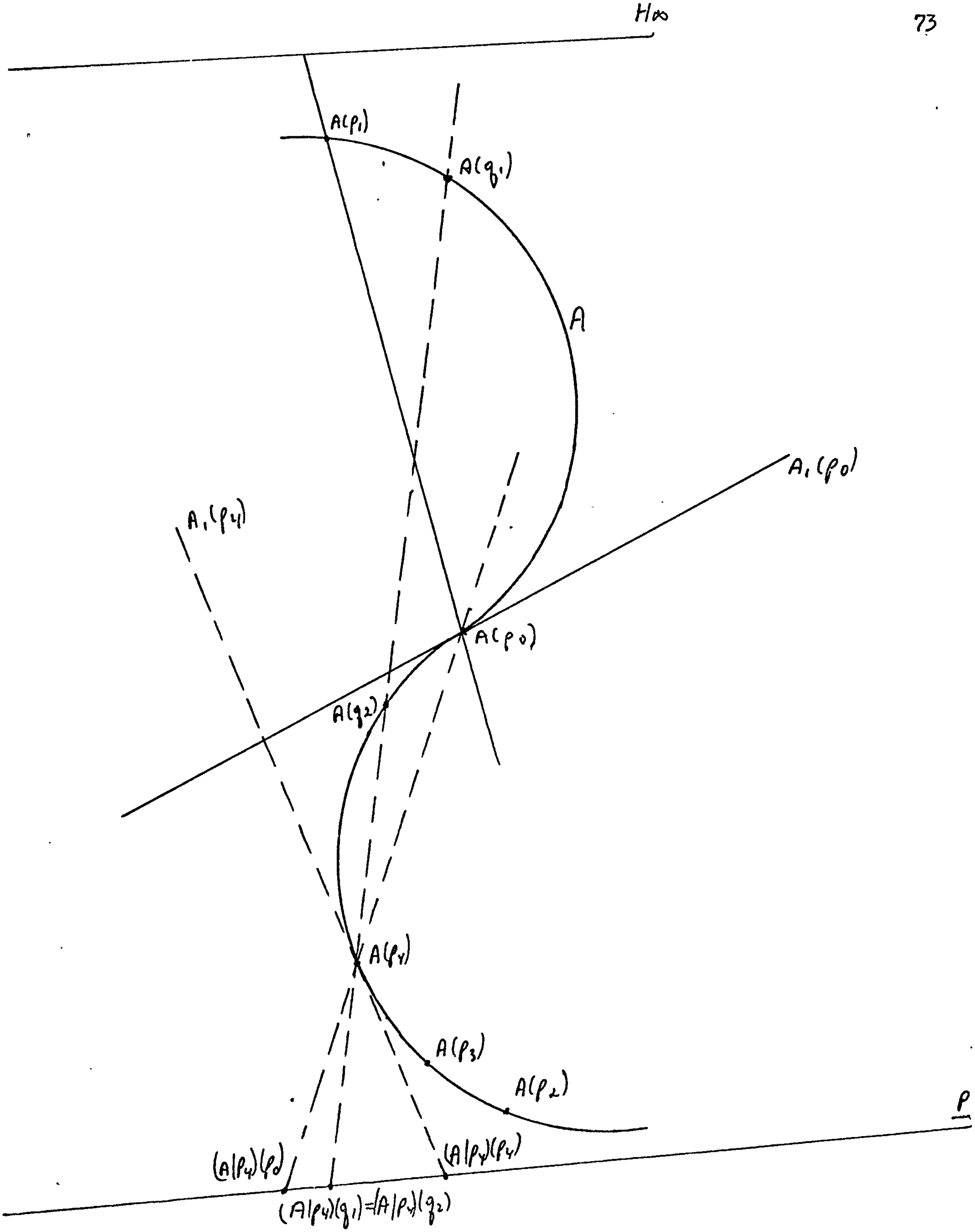


Figure III.11

3.3.11 Theorem: Every arc A has an ordinary point.

Proof: By 3.3.7, we can consider A to be a Barner arc with at most inflections. Let W be the set of points of J at which A_1 is continuous. $W \neq \emptyset$ by 3.3.6. Choose X as in 3.3.3 and the $p_0 \in X \cap W$. We may take $X = J$ and hence if $p \in W$, $A_1(p)$ does not cut A at any point except possibly p. Note also that 3.3.8 now applies.

Case 1. p_0 is regular. Then $A_1(p_0)$ supports A at p_0 and $p_1 p_2$ lie on the same side of $A_1(p_0)$. Since A_1 is continuous at p_0, choose $p_3 \in (p_0, p_2)$ such that $p_1 p_2$ lie on the same side of $A_1(p)$ for all $p \in (p_0, p_3)$. Therefore $A_1(p)$ supports A at p for all $p \in (p_0, p_3) \cap W$ and each $p \in (p_0, p_3) \cap W$ is regular. We claim that (p_0, p_3) is itself regular, hence of order 2, by 3.2.7, and thus ordinary.

Let $q \in (p_0, p_3)$, then there are points $q_i \in (p_0, p_3) \cap W$ such that $q_i \longrightarrow q$ by the corollary of 3.3.6. Let M be a line of accumulation of $A_1(q_i)$. Let $(r_1, r_2) \subset (p_0, p_3)$ be a neighbourhood of q such that $A_1(q)$ meets $[r_1, r_2]$ only at q. This is possible by 1.3.1. Since $q_i' \in (p_0, p_3) \cap W$ is regular and there exists $q_i \in (r_1, r_2)$ such that r_1, r_2 lie on the same side of $A_1(q_i')$ iff they lie on the same side of M, M supports A at q. But then since $a_0(q) = 1$, $M = A_1(q)$ and q is regular.

Case 2. p_0 is an inflection.

(i) If there exists $p_3 \in (p_0, p_2)$ such that (p_0, p_3) is of order 1 on A/p_0. By 3.3.6, there exists a point $p \in (p_0, p_3) \cap W$.

If $A_1(p)$ meets L_f then $A_1(p)$ cuts A at a point of (p_1,p_2), a contradiction by 3.3.3, where $X = J$. Therefore by 3.3.10, $A(p_0) \subset A_1(p)$. If $A_1(p) \neq A_1(p_0)$ then $a_0(p_0) = 1$ implies $A_1(p)$ cuts A at p_0, a contradiction. If $A_1(p) = A_1(p_0)$ then since p_0 is an inflection $A_1(p)$ cuts A at p_0 again, a contradiction. Therefore this case cannot occur.

(ii) No $u^+(p_0)$ is of order 1 on A/p_0. But A/p_0 is an arc in P^1 and it is Barner since $a_0(p_0) = 1$. Therefore (p_0,p_2) has an elementary inflection on A/p_0 by 3.3.4(b). Therefore there exist points p_3,p_4,p_5 such that $p_0 < p_4 < p_3 < p_5 < p_2$, p_3 is an inflection on A/p_0 and $(p_4,p_3),(p_3,p_5)$ are of order 1 on $A = A/p_0$.

Since p_3 is at most an inflection on A, $a_0(p_3) = 1$ and $a_1(p_3) = 1$ or 2. Now p_3 is an inflection on $\hat{A}$ iff $\hat{a}_0(p_3) = 2$. But then $\hat{a}_0(p_3) \equiv a_0(p_3) + a_1(p_3) \pmod 2$ and $\pi(p_0,p_3) = 0$; that is, $A(p_0) \subset A_1(p_3)$. Since $2 \equiv 1 + a_1(p_3) \pmod 2$ implies $a_0(p_3) = 1$ and p_3 is regular on A. Hence p_3 is regular on A/p_3.

$A(p_0) \subset A_1(p_3)$ implies $(A/p_3)(p_0) = (A/p_3)(p_3)$ on A/p_3. Therefore p_3 does not have a neighbourhood of order 1 on A/p_3 and there exists $p_6 \in (p_4,p_3) \cup (p_3,p_5)$ such that $(A/p_3)(p_6)$ is equal to some $(A/p_3)(q)$ for $q \neq p_3$. But $(A/p_3)(p_0) = (A/p_3)(p_3)$ implies that $q \in (p_1,p_0)$. Hence $A(p_6)A(p_3)$ meets L_f.

Suppose $p_6 \in (p_3,p_5)$. Since (p_3,p_5) is of order 1 on A/p_0, for any $p \neq q \in (p_3,p_5)$, $A(p_0) \not\subset A(p)A(q)$. From 3.3.9, (p_0,p_3) is of order 1 on A/p_1, hence $A(p_1) \not\subset A(p)A(q)$ also. Since $A(p_6)A(p_3)$ meets

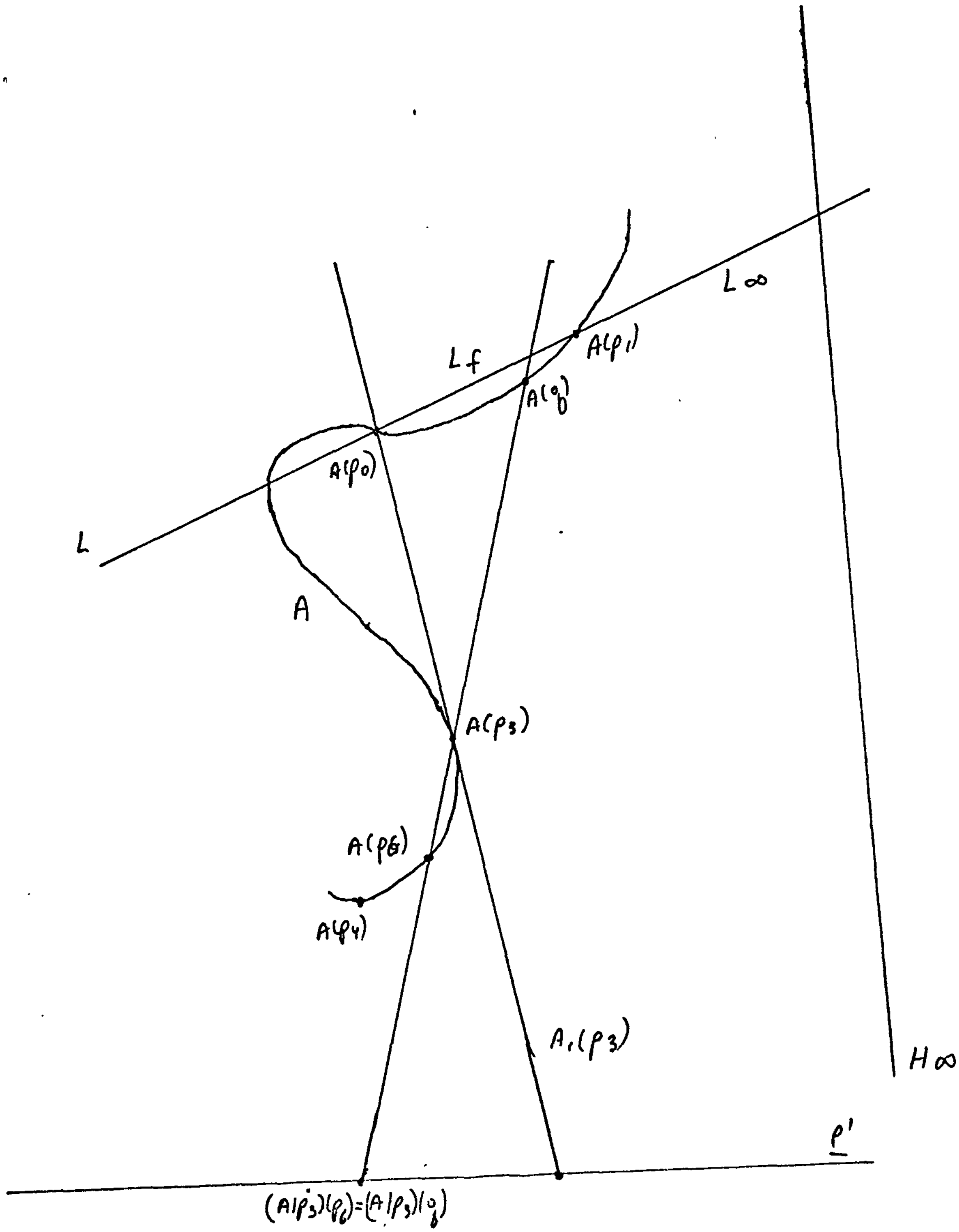


Figure III.12

L_f, by the continuity of A, $A(p)A(q)$ meets L_f for these p,q. If we remove the restriction $p \neq q$ and let $r = p,q$, then $A_1(r)$ meets $L_f \cup \{A(p_0)\}$ for all $r \in (p_3,p_5)$. Take $r_1 \neq r_2 \in (p_3,p_5) \cap W$. By the arguments of Case 2(i), we obtain $A(p_0) = A(r_1) = A(r_2)$ in $\breve{A} = A/p_0$, which contradicts (p_3,p_5) of order 1 on A. Similarily, $p_6 \in (p_4,p_3)$ will give a contradiction, hence this case cannot happen.

Figure III.12 is not accurate construction of Case 2(ii). Since this case cannot happen, an accurate construction cannot be made. It is meant to be only a visual aid in the understanding of 3.3.11.

3.3.12 Theorem: Let A be a Barner arc with at most inflections and A_1 continuous. Then either A is of order 2 or A has an elementary inflection.

Proof: Let W be the set of inflections of A. If $W = \emptyset$, A is regular and of order 2 by 3.2.7. Hence we may assume $W \neq \emptyset$. As in 3.3.11, we take $X = J$. Assume $p_0 \in W$.

Case 1. There exists $p_3 \in (p_0,p_2)$ such that (p_0,p_3) has order 1 on A/p_0. If there exists $p \in (p_0,p_3) \cap W$, then $A_1(p)$ meets L_f or $A(p_0) \subset A_1(p)$ by 3.3.10. But $A_1(p)$ meets L_f implies $A_1(p)$ cuts (p_1,p_0). Since A_1 is continuous, this is a contradiction by 3.3.3. Hence $A(p_0) \subset A_1(p)$, but since p_0 is a point of inflection $A_1(p)$ cuts A at p_0, a contradiction. Therefore there is no such p_0.

Case 2. No $u^+(p_0)$ is of order 1 on A/p_0. By 3.2.7, every $u^+(p)$ contains an inflection on A/p_0. By 3.3.4(b), there exist $p_0 < p_4 < p_3 < p_5 < p_2$ such that p_3 is an inflection on A/p_0, every neighbourhood of p_4,p_5 contains an inflection on A/p_0, and

(p_4,p_3), (p_3,p_5) are of order 1 on A/p_0.

If p is an inflection on A/p_0 then p is regular on A and $A(p_0) \subset A_1(p)$, by 3.3.11, Case 2. Since every neighbourhood of p_4 and p_5 contains an inflection and A_1 is continuous, $A(p_0) \subset A_1(p_4) \cap A_1(p_5)$. Since p_3 is regular on A and $A(p_0) \subset A_1(p_3)$, projecting from p_3 it follows that there is a $p_6 \in (p_4,p_3) \cup (p_3,p_5)$ such that $A(p_6)A(p_3)$ meets L_f.

Suppose $p_6 \in (p_4,p_3)$, then as in 3.3.11 for each $p \in (p_4,p_3)$, $A_1(p)$ meets $L_f \cup \{A(p_0)\}$. Hence (p_4,p_3) is not ordinary by 3.1.9. By 3.2.7(a), there is $r_1 \in (p_4,p_3) \cap W$. Since $A_1(r)$ meets $L_f \cup \{A(p_0)\}$ then $A_1(r_1) = A_1(p_0)$ as in 3.3.11. Repeating this argument for (r_1,p_3), one obtains $r_2 \in (r_1,p_3) \cap W$ such that $A_1(r_2) = A_1(p_0) = A_1(r_1)$. By projecting from p_0, (p_4,p_3) is not of order 1 on A/p_0, a contradiction.

Similarly $p_6 \in (p_3,p_5)$ gives a contradiction. Therefore there exists $u^+(p_0)$ of order 2. Symmetrically, there exists $u^-(p_0)$ of order 2. Thus p is an elementary inflection.

CHAPTER IV

4.1 Dually Differentiable Arcs

If A is a dually differentiable arc, then by 1.4, the 'points' of A^* are the tangent lines of A and the tangent lines of A^* are the points of J. Moreover A_1 is continuous on A.

4.1.1 Let p be a point of the arc A. Let $P_i \in A_i(p) - A_{i-1}(p)$ for $i = 0, 1, 2$. Then $\{P_0, P_1, P_2\}$ are independent and are the vertices of four open 2-simpleces, (Figure IV.1). Let S^- be that open 2-simplex which contains some $U^-(p)$. Let E_1^- be the open segment of P_0P_1 with end points P_0, P_1 which is an edge of S^- and E_2^- be the open segment of P_0P_2 with end points P_0, P_2 which is not an edge of S^-. Similarly we define E_i^+, S^+ for $i = 1,2$.

4.1.2 Lemma: $\sigma_k(p) = 0$ iff $E_{k+1}^- = E_{k+1}^+$, $k = 0, 1$.

Proof: $\sigma_0(p) = 0$ iff P_0P_2 supports A at p iff S^+, S^- lie on the same side of P_0P_2 iff the edges of S^+, S^- on P_0P_1 are the same iff $E_1^- = E_1^+$.

The result follows similarly for $k = 1$.

Let p be a dually differentiable point. Take P_i, S^+, S^-, E_i^+, E_i^- as in 4.1.1. By 1.4.1, there are neighbourhoods $U^+(p)$, $U^-(p)$ such that $P_i \not\in A_i(q)$ for any $q \in U^-(p) \cup U^+(p)$, $i = 0, 1,2$.

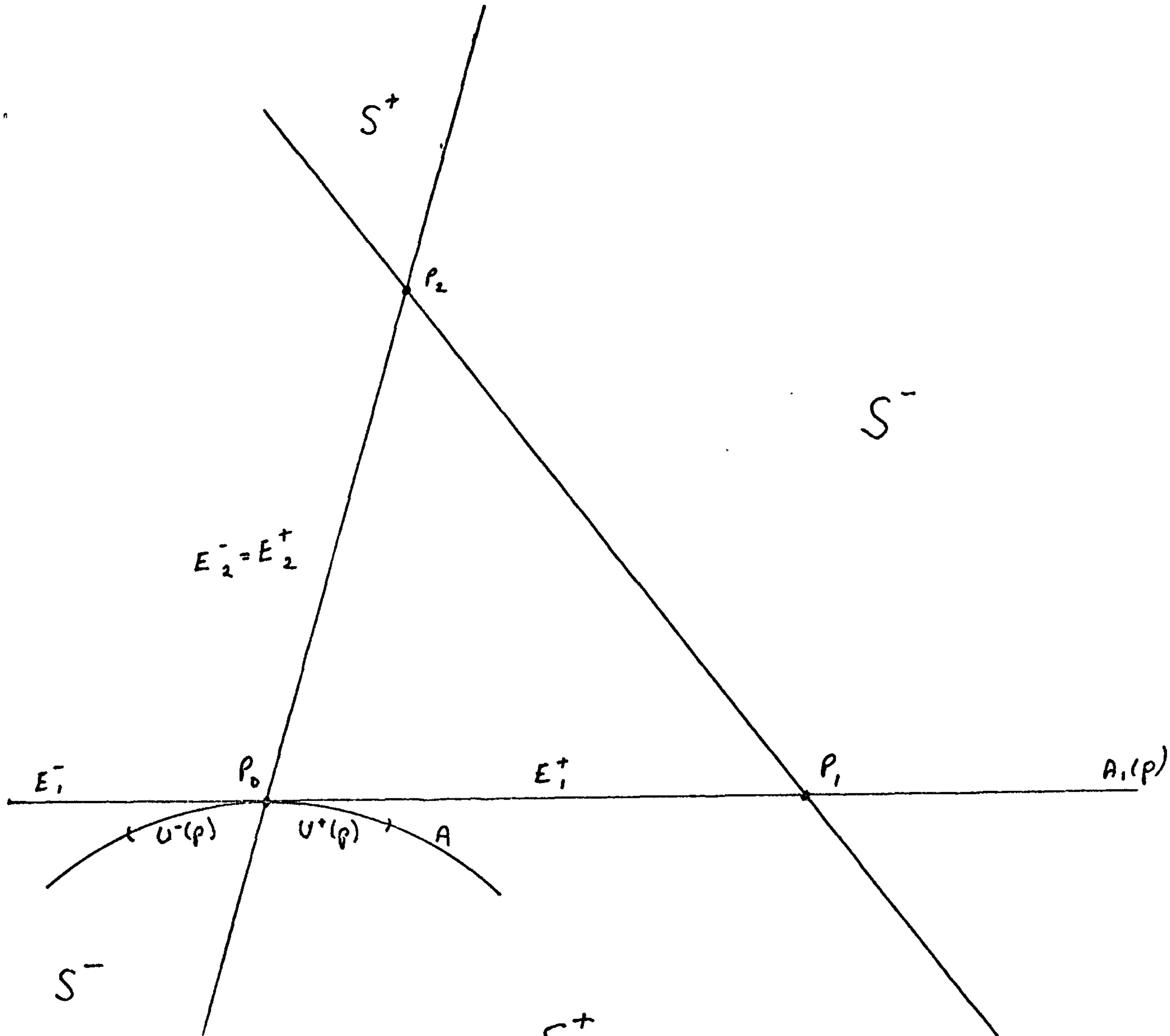


Figure IV.1

Put $\sigma^-_k(p) = 0$ [$\sigma^-_k(p) = 1$] if $A_1(q)$ meets [does not meet] E^-_{k+1} for all $q \in U^-(p)$ where $k = 0,1$. Define $\sigma^+_k(p)$ similarly and put $\sigma^+_{-1}(p) = 0 = \sigma^-_{-1}(p)$.

4.1.3. Theorem: At any point p of a dually differentiable arc

$$\sigma^*_k \equiv \sigma^-_{-k} + \sigma_{-k} + \sigma^+_{-k} + \sigma^-_1 + \sigma^+_1 + \sigma_1 \pmod 2$$

for $k = 0,1$.

Proof: Let $P^*_0 = P_0P_1$, $P^*_1 = P_0P_2$, and $P^*_2 = P_1P_2$. Then P^*_i are points in $\underline{P}^{2*}$ with $P^*_i \subset A^*_i \subset A^*_{i-1}(p)$, $0 \leq i \leq 2$. For all $q \in J$, put

$$\varphi(q) = A^*(q)P^*_2$$

This is a line in $\underline{P}^*_2$ since $P^*_2 = P_1P_2$ is not equal to $A_1(q) = A^*(q)$ for $q \in J$. Now $\varphi(p) = A^*(p)P^*_2 = A_1(p) \cap P_1P_2 = P_1 \neq A^*_1(p)$. Hence $\sigma^*_0(p) = 0$ iff $\varphi(p)$ supports A^* at p iff there exists $U'(p)$ such that $A^*(U'(p))$ lie in some open half-space determined by $\varphi(p)$ and $P^*_1P^*_2$, since $P^*_1P^*_2 = P_0P_2 \cap P_1P_2 = P_2$ and $P^*_1P^*_2 \cap A^*(q) = P_2A_1(q) = A^*_{-1}(p)$.

Therefore $\sigma^*_0(p) = 0$ iff there exists $U'(p)$ such that for any $q \in U'(p)$, $A_1(q)$ meets P_1P_2 in the same open segment determined by P_1,P_2. Choose $U^+(p)$, $U^-(p)$ such that $P_i \notin A_1(q)$ for $i = 0,1,2$ and $q \in U^1(p) = U^+(p) \cup U^-(p)$. Then $A_1(q)$ meets P_0P_1 in the same open segment iff $\sigma_0(p) + \sigma^+_0(p) + \sigma^-_0(p) \equiv 0 \pmod 2$.

If $\sigma_0(p) = 0$ then $E^+_1 = E^-_1$ and if $A_1(q)$ meets E^-_1 for all $q \in U'(p)$, we have $A_1(q)$ meets $E_1^-[E_1^+]$ for all $q \in U^-(p)$ $[U^+(p)]$. Thus $\sigma_0^+(p) = 0 = \sigma^+(p)$ and the result follows. If $\sigma_0(p) = 1$ then $E_1^+ \neq E_1^-$ and if $A_1(q)$ meets say E_1^+ for all $q \in U'(p)$ then $A_1(q)$ does not meet E_1^- for all $q \in U'(p)$. Thus, $\sigma_0^+(p) = 0$ and $\sigma_0^-(p) = 1$ and the result follows. Similarly $A_1(q)$ meets P_0P_2 in the same open segment determined by P_0P_2 for all $q \in U^1(p)$ iff $\sigma_1(p) + \sigma_1^+(p) + \sigma_1^-(p) \equiv 0 \pmod 2$.

Combining these results, $A_1(q)$ meets the same open segment of P_1P_2 for all $q \in U'(p)$ iff $A_1(q)$ meets the same [different] segments of P_0P_1 and P_0P_2 simultaneously. If this were not so then the two possible cases Figure IV.2 and Figure IV.3 indicate a contradiction. Therefore

$$\sigma_0^* \equiv \sigma_0^- + \sigma_0^+ + \sigma_0 + \sigma_1^- + \sigma_1^+ + \sigma_1 \pmod 2.$$

Similarly the result follows for $k = 1$.

Corollary: For any point p of a dually differentiable arc,

$$a_k^* \equiv a_{1-k} + \sigma_{1-k}^- + \sigma_{1-k}^+ + \sigma_{-k}^- + \sigma_{-k}^+ \pmod 2$$

for $k = 0,1$.

Proof: For $k = 0$, $\sigma_0^* = a_0^*$ and $\sigma_0 = a_0$. Therefore

$$a_0^* \equiv \sigma_0^- + \sigma_0^+ + \sigma_1^- + \sigma_1^+ + (\sigma_0 + \sigma_1) \pmod 2$$

and

$$\sigma_0 + \sigma_1 \equiv a_0 + a_0 + a_1 \equiv 2a_0 + a_1 \equiv a_1 \pmod 2.$$

For $k = 1$, $\sigma_1^* \equiv \sigma_1^- + \sigma_1^+ + \sigma_1 \pmod 2$.

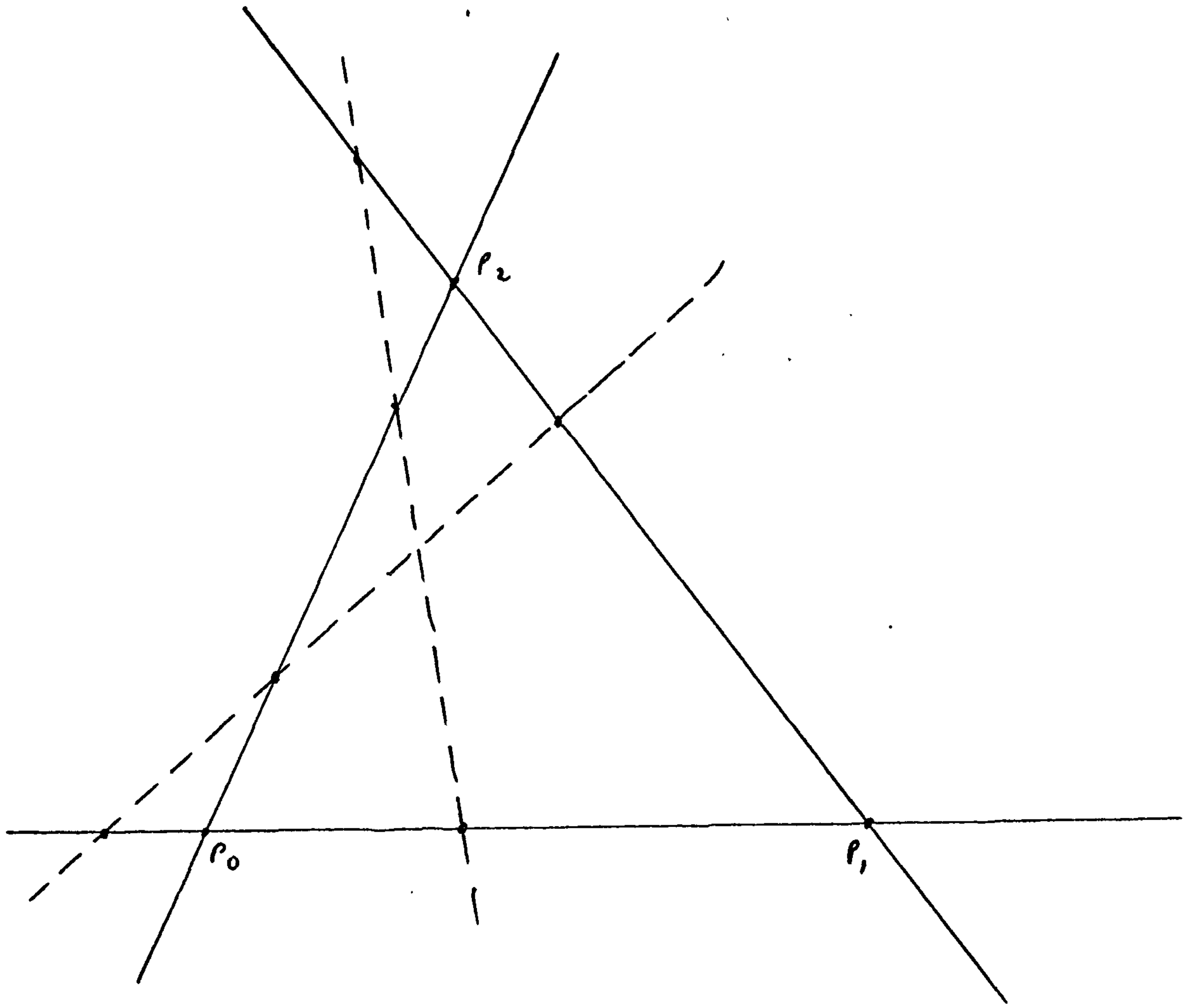


Figure IV.2

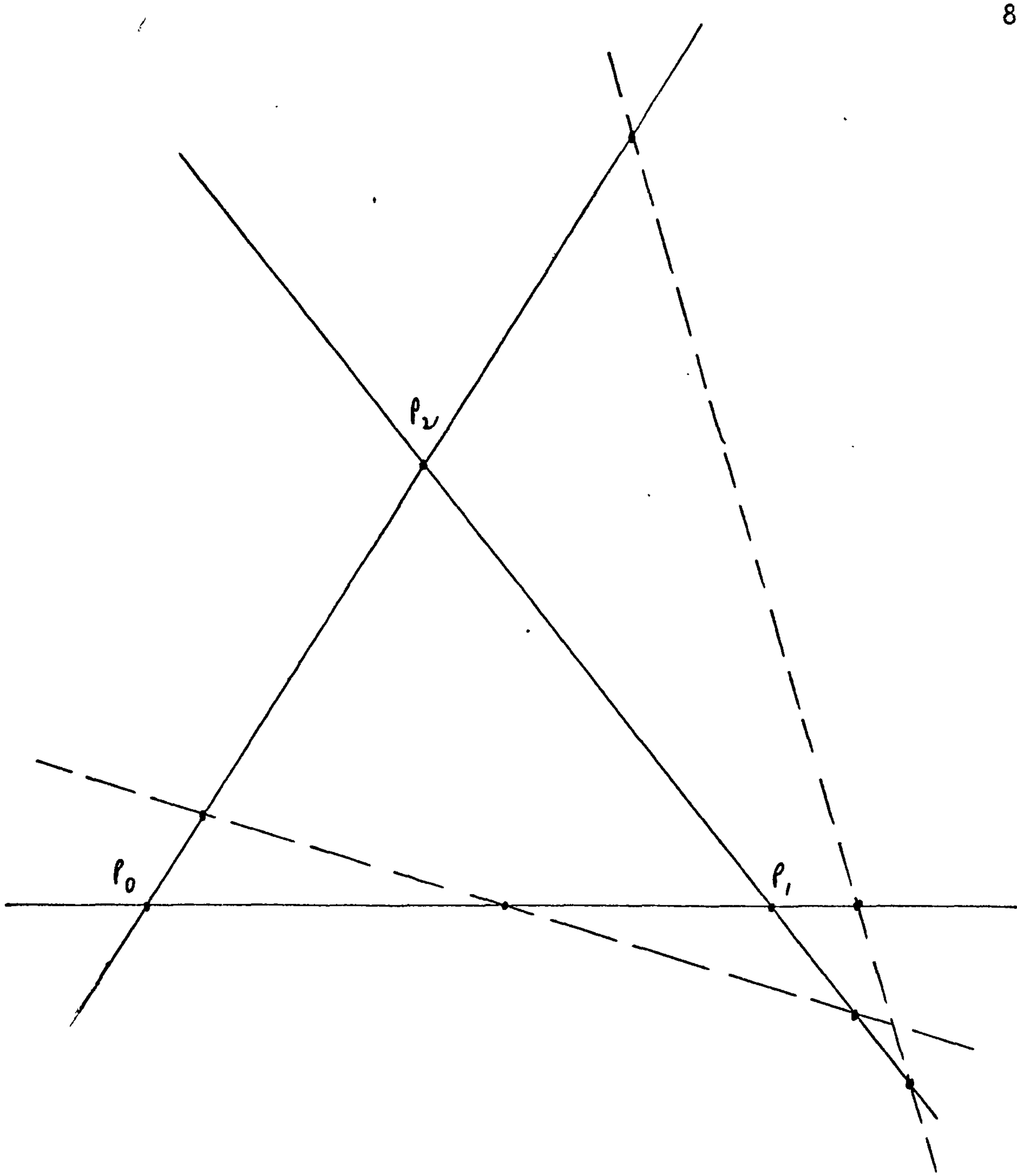


Figure IV.3

Hence

$$a_1^* \equiv \sigma_1^- + \sigma_1^+ + \sigma_1 - a_0^* \quad (\text{mod } 2)$$

$$\equiv \sigma_1^- + \sigma_1^+ + \sigma_1 - (\sigma_0^- + \sigma_0^+ + \sigma_1^- + \sigma_1^+ + a_1) \quad (\text{mod } 2)$$

$$\equiv \sigma_0^- + \sigma_0^+ + \sigma_1 + a_1 \quad (\text{mod } 2)$$

$$\equiv \sigma_0^- + \sigma_0^+ + a_0 + a_1 + a_1 \quad (\text{mod } 2)$$

$$\equiv a_0 + \sigma_0^- + \sigma_0^+ \quad (\text{mod } 2)$$

4.2 Strong finiteness

A point p of an arc is strongly left finite if for every (1-k)-space L, k = 0, 1; there exists a $U^-(p)$ such that no k-secant of $U^-(p)$ meets L. We define strong right finiteness symmetrically. A points is stongly finite if it is both strongly left and right finite. An arc is strongly finite, if each of its points is strongly finite.

4.2.1 Lemma: Let $\breve{A} = A/P$, P a point. If p is strongly finite on A the it is strongly finite on $\breve{A}$.

Proof: Let L be a point of $\breve{A}$. Then L is a line in P^2 with $P \varepsilon L$. p strongly finite implies there exist $U^-(p)$, $U^+(p)$ such that L does not meet $A(U^-(p))$, $A(U^+(p))$. Since $P \varepsilon L$, then for $q \varepsilon U^-(p)$ [$U^+(p)$], $\breve{A}(q) = A(q)P$. Hence $\breve{A}(q)$ does not meet L in $\breve{A}$ and p is strongly left [right] finite on $\breve{A}$.

Let p be a strongly left finite point of an arc. Take P_i, S^-, E_i^- an is 4.1.1 and $U^-(p)$ such that no 1-secant of $U^-(p)$ meets the points P_i; $i = 0,1,2$. Consider the 1-space $P_0P_{M(1)}$ where $M(1) = 1$ or 2. Let $S^-(M(1))$ be the open 1-simplex with vertices $P_0, P_{M(1)}$ which meets the 1-secants of $U^-(p)$. This makes sense since any two lines in a projective plane must meet.

<u>4.2.2 Theorem</u>: $E^-_{M(1)} = S^-(M(1))$ for $M(1) = 1$ or 2.

<u>Proof</u>: Let S^- be the open 2-simplex which contains $U^-(p)$ and take $q \in U^-(p)$ such that $A(q)A(p)$ does not meet (q,p) (Figure IV.4). Then $A(q)A(p)$ divides S^- into two open triangles and (q,p) is contained in the triangle which has E^-_1 as an edge. Let $r \in (q,p)$ such that $A(q)A(r)$ meets E^-_1 and E^-_2. Thus a 1-secant of $U^-(p)$ meets E^-_1, E^-_2.

In Figure IV.5, an arc different from the one in Figure IV.4, is used in the construction to illustrate that <u>4.2.2</u> is independent of the type of construction used in its proof.

<u>4.2.3 Lemma</u>: Let p be a point of an arc A. Let $U \subset \underline{P}^2_1$ be a neighbourhood of $A_1(p)$ and $U^-(p)$ be given. Then there is a 1-secant L of $U^-(p)$ such that $L \in U(A_1(p))$.

<u>Proof</u>: Assume U is open. Since

$$A_1(p) = A(p)/A(p) = \lim_{\substack{q \to p \\ q \neq p}} A(q)A(p)$$

there exists a $q \in U^-(p)$ such that $A(q)A(p) \in U$. Then there exists $U'(p)$ such that $A(q)M \in U$ for all $M \in U'(p)$. Otherwise of $A(q)M \in \complement U$ for all $M \in U'(p)$, then $\lim_{\substack{M \to P \\ M \neq P}} A(q)M = A(q)A(p) \in \complement U$,

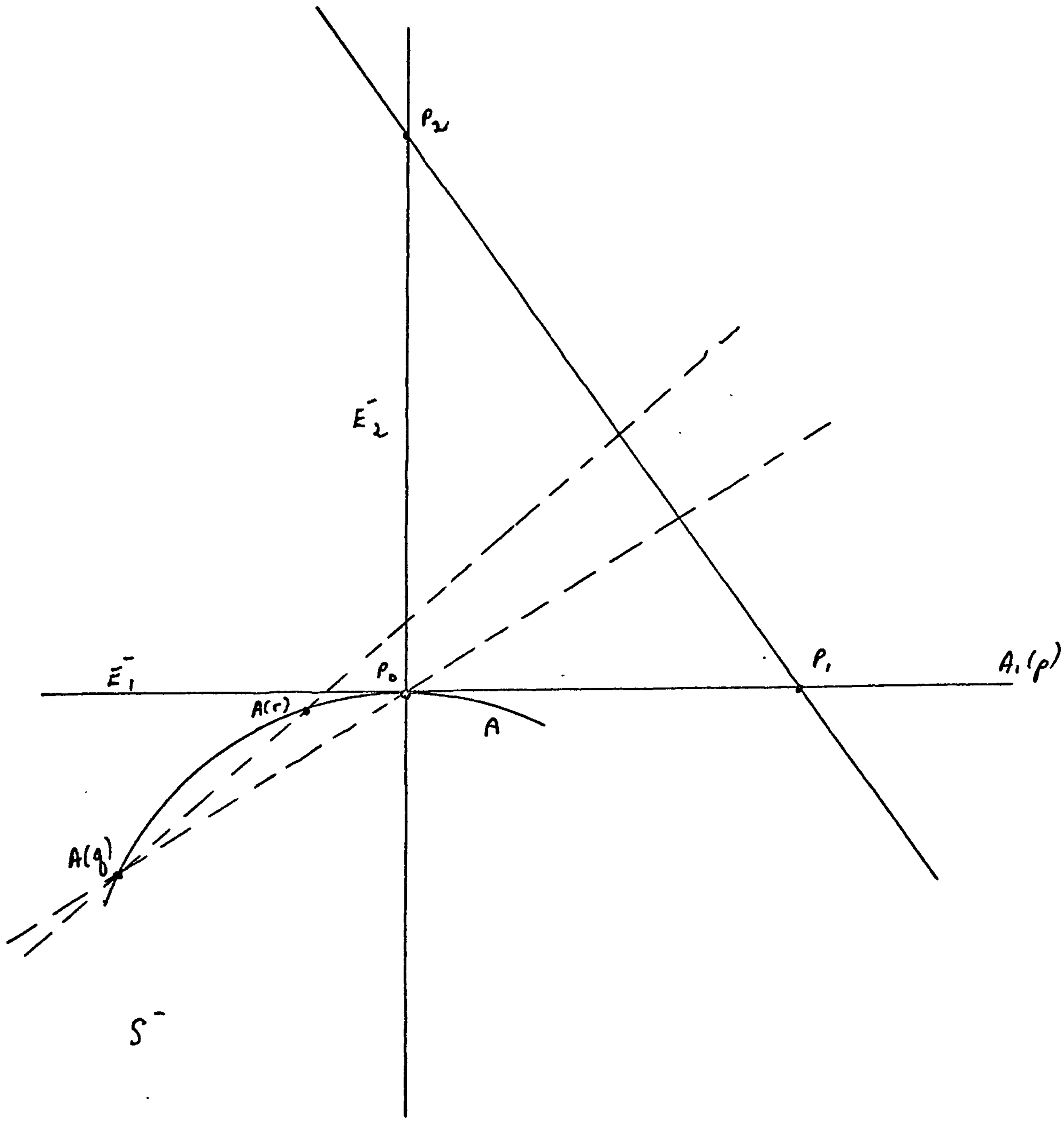


Figure IV.4

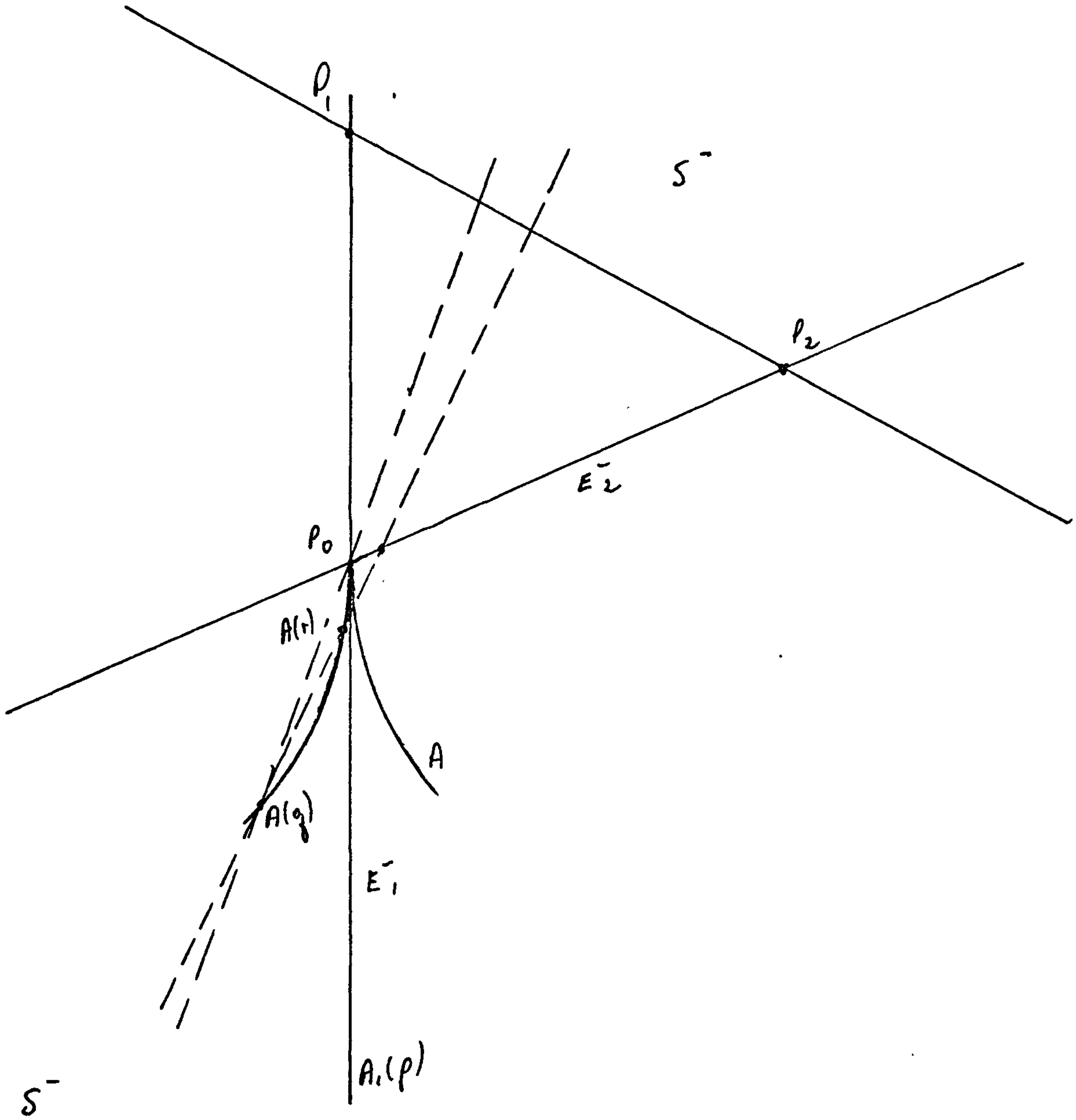


Figure IV.5

since U is open, a contradiction. Since $A(q)M$ for $M \in U'(p)$ is a 1-secant, the result follows.

<u>4.2.4 Theorem</u>: Let p be a strongly left finite point of an arc A. Then A^1 is continuous at p; that is, $A^1(p_0,p_1) \longrightarrow A_1(p)$ where $p_i < p$, $i = 0,1$ and $(p_0,p_1) \longrightarrow (p,p)$.

<u>Proof</u>: Since $A^1(J^2)$ is the set of all 1-secants of A, we must show that given a neighbourhood $U(A_1(p))$, there exists $U^-(p)$ such that

$$A^1(U^-(p)^2) \subset U(A_1(p)).$$

Let L be a line such that $A(p) \not\subset L$ for $p \in J$. Let $P = A_1(p) \cap L$ and let U be a neighbourhood of P on L, determined by some $U(A_1(p))$, with end points Q_1, Q_2; (Figure IV.6). Let $U(p)$ be a neighbourhood of p such that $QM \in U(A_1(p))$ for any $Q \in U(P)$ and $M \in U(p)$.

p is strongly left finite; hence, there exist $U_1^-(p)$, $U_2^-(p)$ such that no 1-secant of $U_i^-(p)$ meets Q_i; $i = 1,2$. Let $U^-(p) \subset U_1^-(p) \cap U_2^-(p)$ such that $U^-(p) \subset U(p)$. Then by <u>4.2.3</u>, there is a 1-secant L of $U^-(p)$ such that L meets $U(p)$. Since no 1-secant of $U^-(p)$ meets Q_1 or Q_2 and the set of 1-secants of $U^-(p)$ is connected, every 1-secant of $U^-(p)$ meets $U(p)$.

<u>4.2.5 Lemma</u>: Let p be a strongly left finite point of an arc A. Let $P_i \in A_i(p) - A_{i-1}(p)$; $i = 0,1,2$. Then $\lim_{q \to p^-} A_1(q) \cap P_kP_{k+1} = P_k$ for $k = 0,1$.

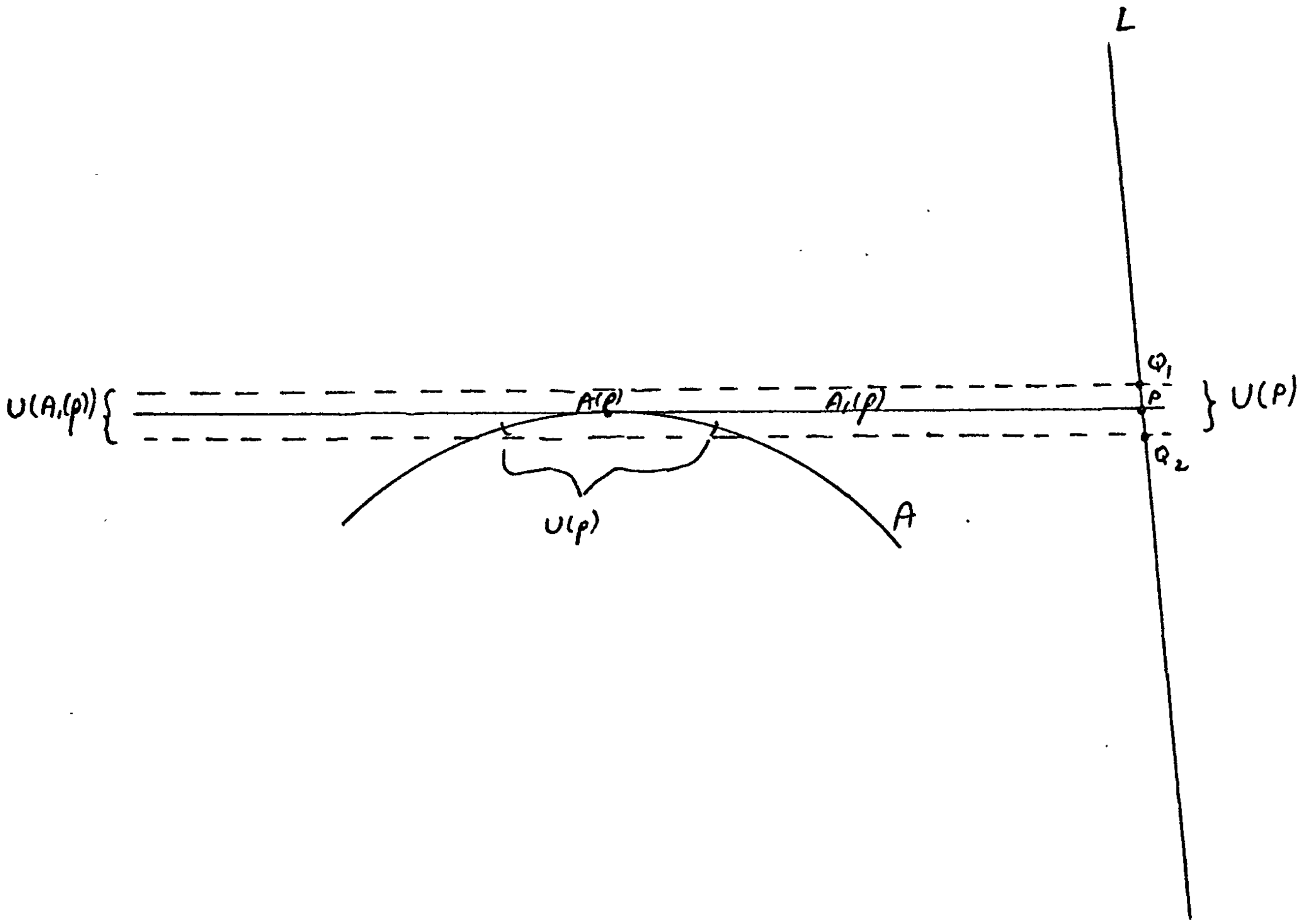


Figure IV.6

<u>Proof:</u> Let $U(P_k)$ be a neighbourhood of P_k on P_kP_{k+1}; $k = 0,1$. Take $U^-(p)$ such that no h-secant of $U^-(p)$ meets any (1-h)-space spannned by the P_i, and no 1-secant of $U^-(p)$ meets the end points of $U(P_k)$.

<u>Case 1.</u> $k = 0$. Let L be a point on $U^-(p)$, (Figure IV.7). Now $L \neq A(p)$ hence $LA(p)$ is a line which meets $U(P_0)$ at P_0. Let $U(p)$ be a neighbourhood of p in A such that $L \notin U(p)$ and LM meets $U(P_0)$ for any $M \in U(p)$. Trivially $U^-(p) \cap U(p) \neq \emptyset$, hence there is a 1-secant of $U^-(p)$ which meets $U(P_0)$. Since the 1-secants of $U^-(p)$ are connected and do not meet the end points of $U(P_0)$, the result follows

<u>Case 2.</u> $k = 1$. Follows from <u>4.2.3</u> and Figure IV.8.

<u>4.2.6 Theorem:</u> A strongly finite arc A is dually differentiable and $a_k^*(p) = a_{1-k}(p)$ for all $p \in J$, $k = 0,1$.

<u>Proof:</u> Now A is dually differentiable iff $A_k(p) = \lim_{\substack{q \to p \\ q \neq p}} A_1(q) \cap A_{k+1}(p)$, $-1 \leq k \leq 1$,. Let p be a point of A and choose P_i; $i = 0,1,2$, and $U^-(p)$ as in <u>4.2.5</u>. Therefore for any $q \in U^-(p)$, $P_i \notin A_1(p)$ for $i = 0,1,2$ and $A_1(q) \cap A_{k+1}(p)$ is a k-space where $-1 \leq k \leq 1$. Let L be a k-space of accumulation of $A_1(q) \cap A_{k+1}(p)$, $q \longrightarrow p^-$. Since $P_0 \ldots P_{k+1} \subset A_{k+1}(p)$, by <u>4.2.5</u>

$$\lim_{q \to p^-} A_1(q) \cap P_kP_{k+1} = P_k \subset \lim_{q \to p^-} A_1(q) \cap A_{k+1}(p) = L$$

Then $P_0 \ldots P_k \subset L$ and $L = A_k(p)$. Similarly

$$\lim_{q \to p^+} A_1(q) \cap A_{k+1}(p) = A_k(p)$$

and hence A is dually differentiable.

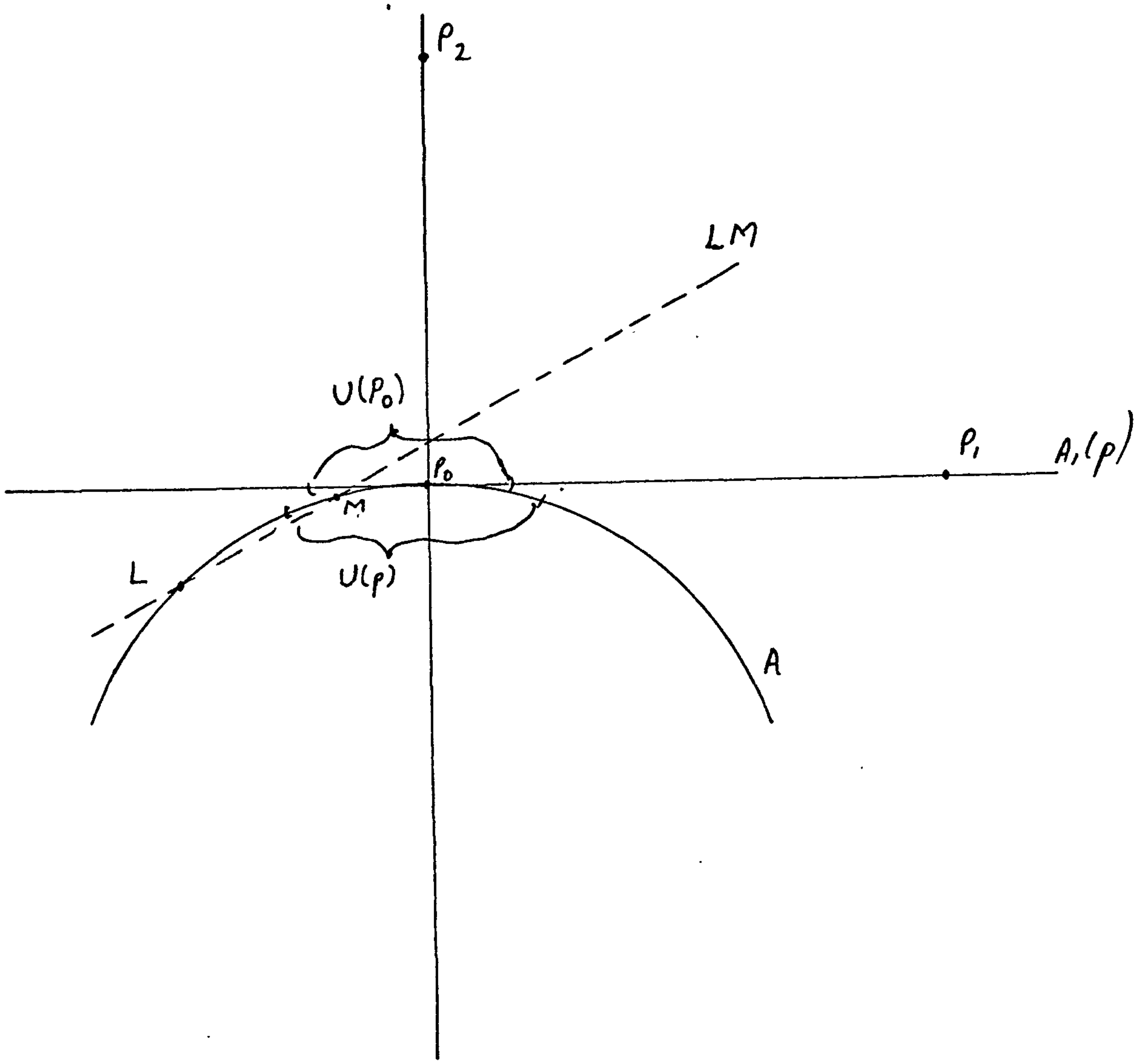


Figure IV.7

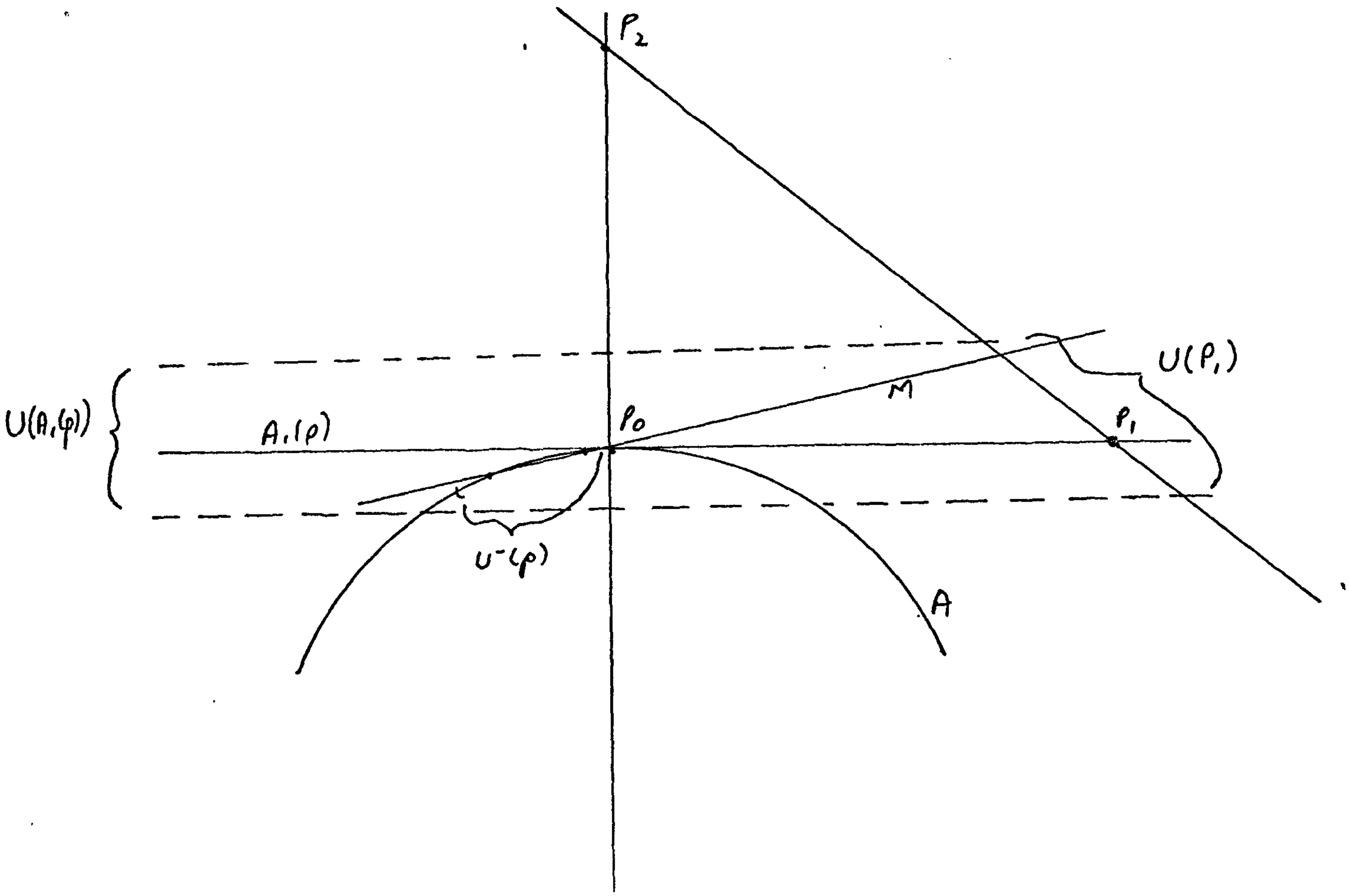


Figure IV.8

From 4.2.2, $A_1(q)$ meets E_1^-, E_2^- for all $q \in U^-(p)$. Therefore $\sigma_1^-(p) = 0 = \sigma_2^-(p)$. Similarly $\sigma_1^+(p) = 0 = \sigma_2^+(p)$ and then $a_k^* = a_{1-k}$ for $k = 0,1$ from the Corollary of 4.1.3.

4.2.7 Theorem: An elementary arc is strongly finite.

Proof: Let p be a point of A and $U^-(p)$ of order 2. Let L be a line, then by 1.3.1 there is $U'(p)$ such that $U'(p) \cap U^-(p) \neq \emptyset$ and $L \cap A(U'(p)) = \emptyset$. Therefore we need only consider points P in $\underline{P}^2$.

If $P = A(p)$ then since $U^-(p)$ is of order 2, no 1-secant of $U^-(p)$ meets P by 2.4.1.

If $P \neq A(p)$ then $L = PA(p)$ is a line and as in preceeding paragraph there exists $U_1^-(p) \subset U^-(p)$ such that $L \cap A(U_1^-(p)) = \emptyset$. Put $\varphi(x) = A^1(x) \cap L$ for $x \in (U_1^-(p))^2$, (Figure IV.9). Then φ is monotone. Now $\lim_{r \to p^-} \varphi(q,r) = A(p)$ where $q,r \in U_1^-(p)$. But $L \cap A(U_1^-(p)) = \emptyset$ implies there exists $U_2^-(p) \subset U_1^-(p)$ such that $\varphi(x) \neq P$ for $x \in (U_2^-(p))^2$ and the result follows.

Corollary: Let P be a point. If p is an elementary point of an arc A then p is also elementary on $\breve{A} = A/P$.

Proof: Take $U^-(p)$ of order 2 such that no 1-secant of $U^-(p)$ meets P. Therefore if $\breve{A}(q_1) = \breve{A}(q_2)$ for $q_1 \neq q_2$ in $U^-(p)$, the 1-secant $A(q_1)A(q_2)$ contains P, a contradiction.

4.3 Arcs with towers

Spaces $H_k \in \underline{P}_k^2$, $-1 \leq k \leq 2$, are said to be a tower if

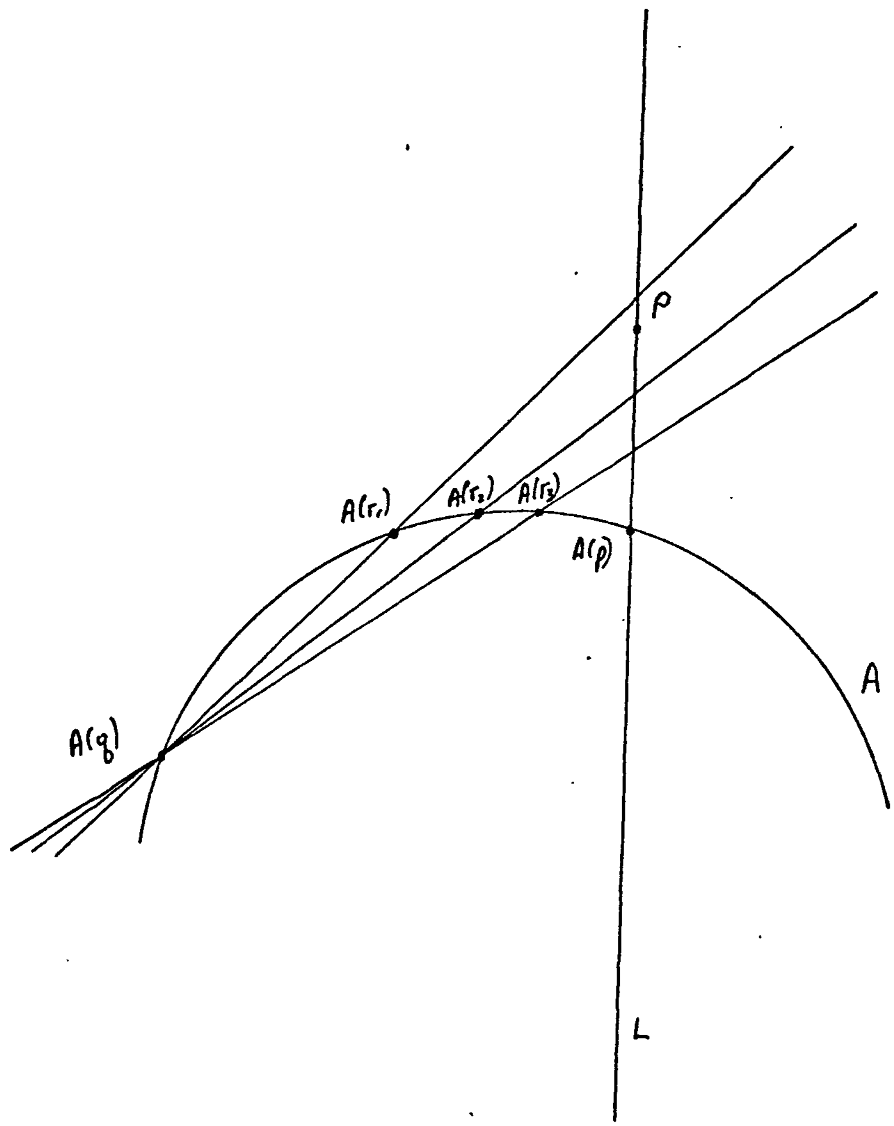


Figure IV.9

$H_{-1} \subset H_0 \subset H_1 \subset H_2$. An arc with tower is an arc for which there exist a tower satisfying

$$A_k(p) \cap H_{1-k} = \emptyset$$

for any $p \in J$, $-1 \leq k \leq 2$.

4.3.1 Theorem: Let A be an arc with tower and A_1 continuous. Then 1. If A has at most inflections, A is a Barner arc.

2. If A is regular, A is of order 2.

Proof: Since a regular Barner arc is of order 2, we need only prove 1.

Consider $\breve{A} = A/H_0$. Since $H_0 \not\subset A_1(p)$ for any $p \in J$, $\breve{a}_0(p) = a_0(p) = 1$. Therefore $\breve{A}$ is regular. Put $\breve{H}_0 = H_1$. Then $\breve{A}(p) \cap \breve{H}_0 = A(p)H_0 \cap H_1 = H_0 = \emptyset$ in $\breve{A}$ and hence $\breve{A}$ is not all of $\underline{P}_0^1$. Therefore A is Barner and of order 1 by 3.2.4 (a). The result now follows from 3.3.7.

4.3.2 The Scherk-Derry duality theorem: Let A be an arc of order 2. Then A is dually differentiable and A^* has order 2.

Proof: A is of order 2, hence A is elementary and strongly finite by 4.2.7. Therefore A is dually differentiable and $a_k^* = a_{1-k}$, $k = 0,1$ by 4.2.6.

A is of order 2, hence ordinary and regular by 2.6.1. But A regular and $a_k^* = a_{1-k}$ implies A^* is regular. Since $A_1^* = A_0$, A_1^* is continuous. Therefore we need only show A^* is an arc with tower and result follows from 4.3.1.

Let (q,r) be an interval of A. (q,r) is of order 2 and by 2.4.1, $A(q) \not\subset A_1(p)$ for any $p \in (q,r)$. Put $H_0^* = A_1(q)$ and $H_1^* = A(q)$. Then (q,r) is an arc with tower on A^*. But q and r are abitrary, hence A^* is an arc with order 2.

Corallary: If A is elementary, then A is dually differentiable and A^* is elementary.

The above theorem point out an important fact about arcs with tower; namely, it is a local property. In other words, we can prove many things about an arc by localizing to a point and constructing (if possible) a neighbourhood with tower for that point. We shall see more examples of this in Chapter VI.

Points which have neighbourhoods with tower are available. For example, a point $p \in J$ which is elementary and hence strongly finite. One can easily construct a neighbourhood of a strongly finite point, which has a tower.

Let L be a line. Put

$$\mu(p,L) = \beta_{\delta(p,L)}(p)$$

for all $p \in J$. We call $\mu(p,L)$ the multiplicity with which L meets A at p. Counting multiplicities facilitates giving precise information about the order of point, as will be seen in 4.3.4.

4.3.3 Theorem: Let A be an arc with tower such that every regular subarc of A is of order 2. Then

$$\sum_{p \in J} \mu(p,L) \leq \left(\sum_{p \in J} \sum_{i=0} (a_i(p)-1) \right) +2.$$

<u>Proof</u>: If $\sum_{p\varepsilon J} \sum_{i=0}^{1} (a_i(p)-1)$ is infinite, the result holds trivially. If the theorem is true for finite number of p in J, then

$$\sum_{p_1,\ldots,p_n} \mu(p,L) \leq \sum_{p\varepsilon J} \sum_{i=0}^{1} (\alpha_i(p)-1) + 2$$

for all $n < \infty$. Hence if $\sum_{p\varepsilon J} \mu(p,L)$ is infinite, so is $\sum_{p\varepsilon J} \sum_{i=0}^{1} (\alpha_i(p)-1) + 2$.

Therefore we may assume both sides are finite. But $\sum_{i=0}^{1} (a_i(p)-1) \neq 0$ for only finitely many points implies there are only finitely many non-regular points on A. Since by hypothesis, regular subarcs are of order 2; A is elementary.

Suppose A is an arc with tower in $\underline{P}^1$ such that A has finitely many non-regular points. Let L be a point in $\underline{P}^1$ and suppose L meets A k-times and h < k of these are points of inflection. Then

$$\sum_{p\varepsilon J} \mu(p,L) = h + k.$$

But $L = A(p)$ k-times implies that each of the k-1 intervals determined by these points contains a point of inflection. Hence A has at least $h + k - 1$ points of inflection. Then

$$h + k - 1 \leq \sum_{p\varepsilon J} (a_0(p)-1)$$

and the result follows that

$$\sum_{p\varepsilon J} \mu(p,L) \leq \sum_{p\varepsilon J} (a_0(p)-1) + 1$$

for any $p \varepsilon J$.

A is an arc with tower. Take $\{H_i\}$ to be a tower with $H_0 \not\subset A_1(q)$ for any $q \varepsilon J$ and $A(q) \not\subset H_1$ for any $q \varepsilon J$. Let L a line be given.

If $H_0 \subset L$ then consider $\breve{A} = A/H_0$. $\breve{A}$ is an arc with tower in $\underline{P}^1$, as proved in 4.3.1. Since A has only finitely many non-regular points and $\breve{a}_0(p) = a_0(p)$, then $\breve{A}$ has only finitely many non-regular points. From above,

$$\sum_{p\varepsilon J} \breve{\mu}(p,L) \leq \sum_{p\varepsilon J} (\breve{a}_0(p)-1)+1.$$

Since $H_0 \not\subset A_1(p)$ for any $p \varepsilon J$, we have $\delta(p,L) \leq 0$ and $\pi(H_0,p) = 1$ for all $p \varepsilon J$. Then $\breve{\mu} = \breve{\beta}_{\breve{\delta}} = \breve{\beta}_{\delta} = \beta_{\delta} = \mu$ by 2.2.1 and 2.5.2. Hence

$$\sum_{p\varepsilon J} \mu(p,L) = \sum_{p\varepsilon J} (\breve{a}_0(p)-1) + 1$$

$$= \sum_{p\varepsilon J} (a_0(p)-1) + 1$$

$$\leq \sum_{p\varepsilon J} \sum_{i=0}^{1} (a_i(p)-1) + 2$$

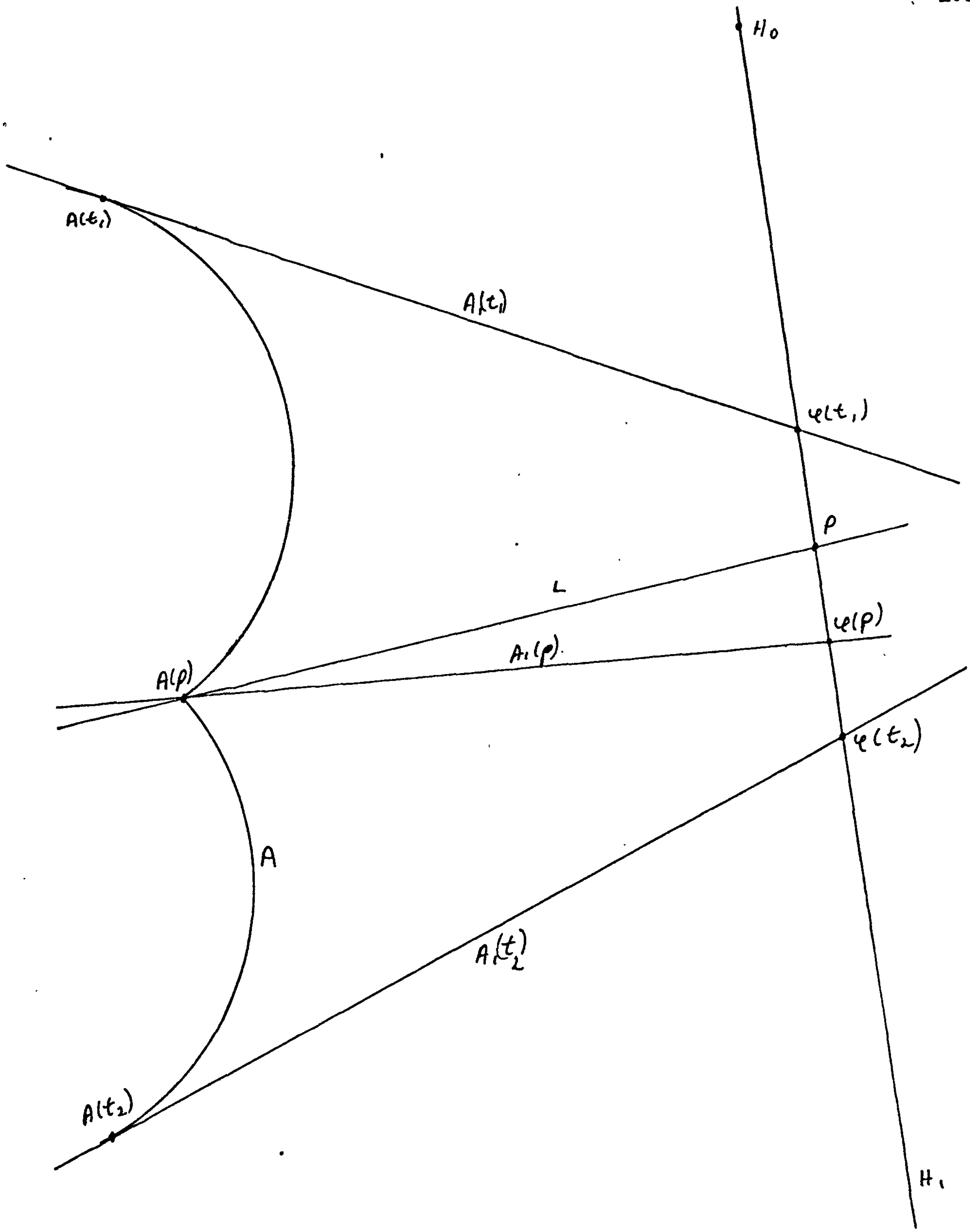


Figure IV.10

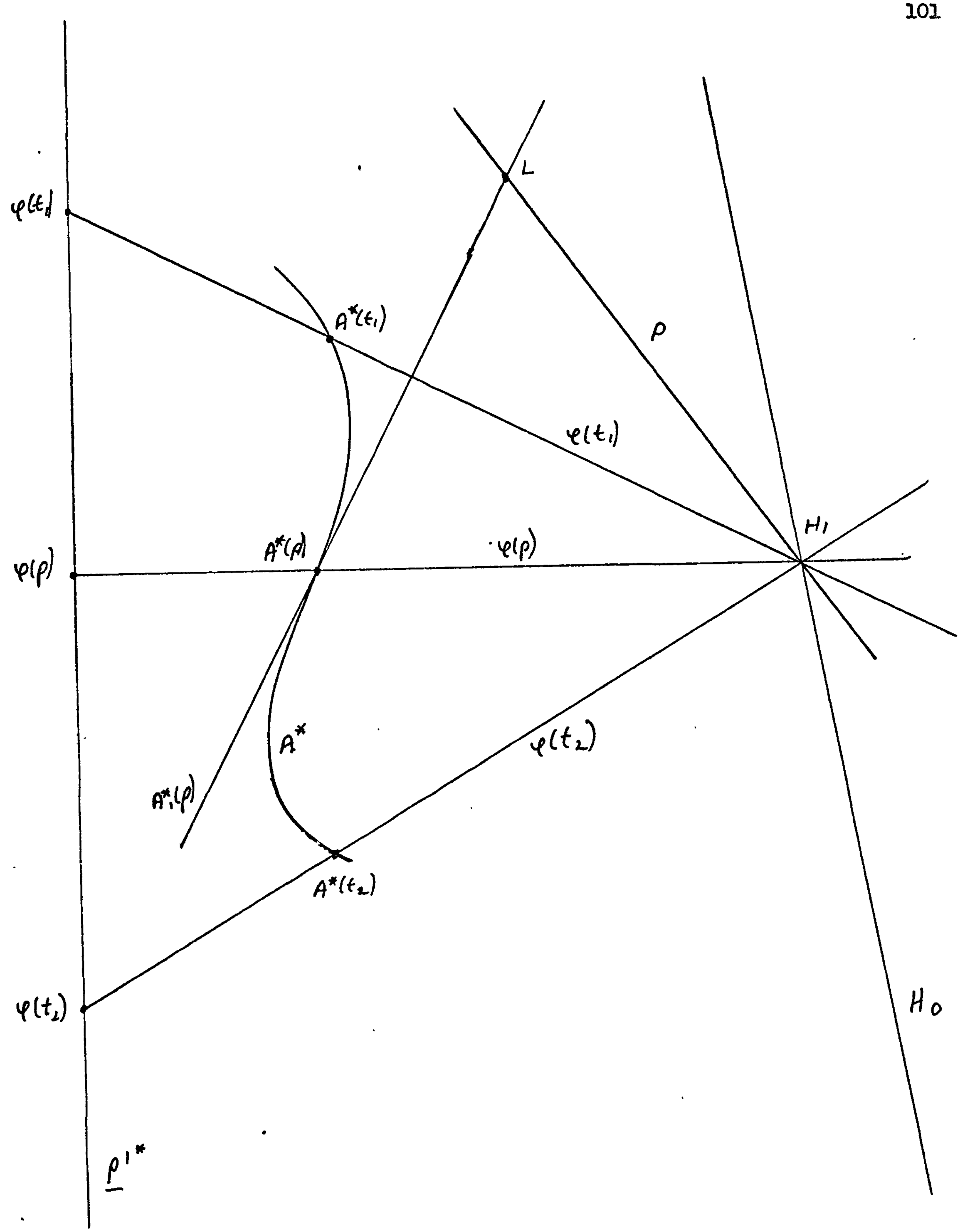


Figure IV.11

If $H_0 \not\subset L_1$ put $P = L \cap H_1$ and consider $\breve{A} = A/P$. H_1 does not meet $\breve{A}$; otherwise, $H_1 = \breve{A}(q)$ for some $q \in J$ and hence $A(q) \subset H_1$, a contradiction. Thus $\breve{A}$ is an arc with tower and finitely many inflections. Then

$$\sum_{p \in J} \breve{\mu}(p,L) \leq \sum_{p \in J} (\breve{a}_0(p)-1) + 1.$$

Since A is elementary, A is strongly finite and dually differentiable with $a_k^* = a_{1-k}$; $k = 0, 1$. Now $H_1 \cap A = \emptyset$, therefore put

$$\varphi(p) = H_1 \cap A_1(p)$$

for all $p \in J$. In A^*, $\varphi(p)$ is the line $H_1A^*(p)$. In Figure IV.10 and IV.11, A and the corresponding A^* are illustrated. Projecting from H_1, $\varphi = A^*/H_1 = \breve{A}^*$ is an arc in $\underline{P}^{1*}$. Moreover φ is an arc with tower in $\underline{P}^{1*}$, since $H_0 \not\subset A_1(p)$ for all $p \in J$, hence $A^*(p) \not\subset H_0$ for all $p \in J$ and $H_0 \not\subset \varphi$ in $\underline{P}^{1*}$. Since A has finitely many non-regular points, $a_k^* = a_{1-k}$ for $k = 0,1$, and $\breve{a}_0^* = a_0^*$ or a_1^*, $\breve{A}^*$ has finitely many inflections. Therefore

$$\sum_{p \in J} \breve{\mu}^*(p,L') \leq \sum_{p \in J} (\breve{a}_0^*(p)-1) + 1$$

for any point L' in A^*. In particular since $A_1^*(p) \cap H_1 = A(p) \cap H_1 = \emptyset$ for all $p \in J$, $\pi(H_1,p) = 1$ and $\breve{a}_0^*(p) = a_0^*(p) = a_1(p)$ for all $p \in J$. Hence

$$\sum_{p \in J} \breve{\mu}^*(p,L') \leq \sum_{p \in J} (a_1(p)-1) + 1.$$

Let $X = \left\{ p \varepsilon J \middle| P \varepsilon A_1(p) \right\}$ where P is defined above. Let $p \varepsilon X$, then $P \subset H_1 \cap L \cap A_1(p)$ and $P = H_1 L A^*(p)$ in A^*. Therefore $L = \breve{A}^*(p)$ in $\breve{A}^*$ and $\breve{\delta}^*(p,L) = 0$. Since

$$\breve{\mu}^*(p,L) = \breve{\beta}^*_{\breve{\delta}^*(p,L)}(p),$$

$$\breve{\mu}^*(p,L) = \breve{\beta}^*_0(p) = \breve{a}^*_0(p) = a_1(p). \text{ Therefore}$$

$$\sum_{p\varepsilon X} a_1(p) = \sum_{p\varepsilon J} \breve{\mu}^*(p,L)$$

$$\leq \sum_{p\varepsilon J} \breve{\mu}^*(p,L)$$

$$\leq \sum_{p\varepsilon J} (a_1(p)-1) + 1 .$$

Considering again $\breve{A} = A/P$ where $P \varepsilon L \cap H_1$, If $\delta(p,L) \leq 0$ for some $p \varepsilon J$, then $P \not\subset A_1(p)$ and $\pi(P,p) = 1$. Hence $\breve{\mu} = \mu$ and result follows as before. <u>If $\delta(p,L) = 1$ then $P \varepsilon L = A_1(p)$ and $p \varepsilon X$</u>. Since $\breve{\delta}(p,L) = \delta(p,L) - 1$,

$$\breve{\mu}(p,L) = \breve{\beta}_0(p)$$

$$= \breve{a}_0(p) + (\mu(p,L) - a_0(p) - a_1(p))$$

$$= \mu(p,L) - (a_0(p) + a_1(p) + \breve{a}_0(p)).$$

Then

$$\sum_{p\varepsilon J} \mu(p,L) = \sum_{\delta(p,L)\leq 0} \mu(p,L) + \sum_{\delta(p,L)=1} \mu(p,L)$$

$$= \sum_{\delta(p,L)=0} \breve{\mu}(p,L) + \sum_{\delta(p,L)=1} (\breve{\mu}(p,L)+(a_0(p)+a_1(p))-\breve{a}_0(p))$$

$$\leq \sum_{p\varepsilon J} \breve{\mu}(p,L) + \sum_{p\varepsilon X} ((a_0(p)+a_1(p))-\breve{a}_0(p))$$

$$= \sum_{p\varepsilon J} \breve{\mu}(p,L) + \sum_{p\varepsilon X} ((a_0(p)-1+a_1(p))-(\breve{a}_0(p)-1)).$$

Now

$$\sum_{p\varepsilon J} \breve{\mu}(p,L) \leq \sum_{p\varepsilon J} (\breve{a}_0(p)-1)+1$$

$$\leq \sum_{p\varepsilon X} (\breve{a}_0(p)-1) + \sum_{p\not\varepsilon X} (\breve{a}_0(p)-1) + 1$$

and if $p \not\varepsilon X$, $\delta(p,L) = 0$ and $\delta(P,p) = 1$. Hence from above $\breve{a}_0(p) = a_0(p)$. But then

$$\sum_{p\varepsilon J} \mu(p,L) \leq \sum_{p\varepsilon J} (a_0(p)-1) + \sum_{p\varepsilon X} a_1(p) + 1$$

$$\leq \sum_{p\varepsilon J} (a_0(p)-1) + \sum_{p\varepsilon J} (a_1(p)-1) + 2$$

and the result follows.

4.3.4 Denk's Theorem: Let p be an elementary point of an arc. Then $\mathrm{Ord}(p) = a_0(p) + a_1(p)$ where $\mathrm{Ord}(p)$ is defined to be the ord p counting multiplicities.

Proof: Take $U^-(p)$, $U^+(p)$ of order 2, such that $U(p) = U^-(p) \cup \{p\} \cup U^+(p)$ is an arc with tower. This is possible since A_1 is continuous on $U(p)$ and p is strongly finite. Then by 4.3.3,

$$\sum_{p \in U(p)} \mu(q,L) \leq \sum_{q \in U(p)} \sum_{i=0}^{1} (a_i(p)-1) + 2.$$

Since $q \neq p$ implies q is regular where $q \in U^-(p) \cup U^+(p)$, $a_i(q) - 1 = 0$ for $i = 0,1$. Hence

$$\sum_{q \in U(p)} (\mu(q,L) \leq a_0(p) + a_1(p).$$

The equality follows by taking $L = A_1(p)$.

Corollary: An elementary regular point is ordinary.

4.4 Finite arcs

A point p of an arc A is left finite, if for every (1-k)-space L there exists $U^-(p)$ such that

$$A_k(q) \cap L = \emptyset$$

for all $q \in U^-(p)$, $k = 0,1$. Right finiteness is similarly defined. A point p is finite if it is both left and right finite. A is finite if each of its points is finite.

Trivially if A is strongly finite it is finite and in $\underline{P}^1$, strong finiteness is exactly finiteness; that is, given only two points p, q in $\underline{P}^1$, there exists $U^-(p)$ of p such that $A(q) \not\subset A(U^-(p))$.

4.4.1 Lemma: Let $\breve{A} = A/P$ where A is an arc and P a point. If p is finite on A, it is finite on $\breve{A}$.

The proof is as in 4.2.1.

4.4.2 Theorem: Let p be left finite, and some $U^-(p)$ have at most inflections, then p is strongly left finite.

Proof: Let P be a point. Since p is left finite, there is $U_1^-(p) \subset U^-(p)$ such that $P \not\subset A_1(q)$ for any $q \in U_1^-(p)$. Thus $U_1^-(p)$ is regular on $\breve{A} = A/P$ an arc in $\underline{P}^1$. If $\breve{A}(U_1^-(p))$ is equal to P_0^1, then there is $U_2^-(p) \subset U_1^-(p)$ such that $\breve{A}(U_2^-(p)) \neq P_0^1$ and hence $U_2^-(p)$ is Barner on $\breve{A}$ and of order 1 by 3.2.4.

Let M be a line through P. Since $P \not\subset A_1(q)$, $M \not\supset A_1(q)$ and $\breve{\delta}(q,M) = \delta(q,M) \leq 0$ by 2.2.1, for all $q \in U^-(p)$. If M meets $U_2^-(p)$, then

$$\dim \bigvee_{q \in U_2^-(p)} A_{\delta(q,M)}(q) \leq \dim M = 1.$$

If M meets $U_2^-(p)$ only once then M is not a 1-secant of $U_2^-(p)$ since $M \not\supset A(q)$ for all $q \in U_2^-(p)$. Assume M meets $U_2^-(p)$ at least twice, then by 2.2.4

$$1 \leq \sum_{q \in U_2^-(p)} (\delta(q,M)+1) - 1$$

$$= \sum_{q \in U_2^-(p)} (\breve{\delta}(q,M)+1) - 1.$$

Since M is a 0-secant of $U_2^-(p)$ on $\breve{A}$ by 2.2.2 and $U_2^-(p)$ has order 1 on $\breve{A}$,

$$1 \leq \sum_{q \in U_2^-(p)} (\breve{\delta}(q, M+1) - 1$$

$$= 0$$

a contradiction. Hence no 1-secant of $U_2^-(p)$ contains P.

Let L be a line, by 1.3.1 there is $U_3^-(p) \subset U_2^-(p)$ such that $A(q) \not\subset L$ for all $q \in U_3^-(p)$. Then p is strongly left-finite.

4.4.3 Theorem: Let p be left finite and some $U^-(p)$ regular. Then there is a $U^-(p)$ of order 2.

Proof: Let $P \neq A(p)$ be a point and $U_1^-(p)$ be regular such that $P \not\subset A_1(q)$ for all $q \in U_1^-(p)$. Then $U_1^-(p)$ is regular on A/P and there is $U_2^-(p) \subset U_1^-(p)$ of order 1 on A/P. Since $\kappa(P,q) = 1$ for $q \in U_2^-(p)$, $U_2^-(p)$ is a Barner arc by 3.3.1, and of order 2 by 3.2.7.

Corollary: A regular finite arc is ordinary.

Proof: Since A is finite, every $p \in A$ has $U^-(p)$, $U^+(p)$ of order 2 on A by 4.4.3. Hence p is regular elementary point, which is ordinary by 4.3.4.

4.4.4 Theorem: If a finite arc A has a singularity, it has an elementary singularity.

Proof: If A has at most inflections, we may assume A has an inflection point p_1 by the Corollary of 4.4.3, and A is strongly finite by 4.4.2. Then A is dually differentiable and $a_k^* = a_{1-k}$ for $k = 0,1$ by 4.2.6. Let L be a line with $A(p_1) \not\subset L$. Since p is strongly finite we may assume $A(p) \not\subset L$ for all $p \in J$. Put

$$\varphi(p) = A_1(p) \cap L$$

for all $p \in J$.

As in 4.3.3, the mapping φ is the projection of A^* from L. Put $\breve{A}^* = A^*/L = \varphi$. Since $A_1^*(p) \cap L = \emptyset$, $\breve{a}_0^*(p) = a_0^*(p) = a_1(p)$ for all $p \in J$. Therefore p_1 is an inflection point of $\breve{A}^*$. Since an ordinary point is regular, p_1 is a singularity of $\breve{A}^*$. But $\breve{A}^*$ is an arc in $\underline{P}^{1*}$, therefore φ has an elementary singularity p_2 by 3.2.4. Let $U^+(p_2)$ be of order 1 on $\breve{A}^*$ then $\breve{a}_0^*(q) = 1$ for all $q \in U^+(p_2)$. Hence $a_1(q) = 1$ for all $q \in U^+(p_2)$ and there is $U_1^+(p_2)$ is of order 2 on A by 4.4.3. Similarly, there exists $U_1^-(p_2)$ of order 2 on A and hence p_2 is an elementary singularity.

Assume only that A has a singularity p_1. We can again choose point P such that $P \not\in A_1(q)$ for all $q \in J$. If $\breve{A} = A/P$ is ordinary then A has at most inflections and result follows. If $\breve{A}$ has an elementary singularity p_2, take $U^-(p_2)$, $U^+(p_2)$ of order 1 on $\breve{A}$. Then both are regular on $\breve{A}$ and have at most inflections on A. If either contains an inflection the theorem follows. Otherwise both are regular and p_2 is elementary on A by 4.4.3. Since $2 = \breve{a}_0(p_2) = a_0(p_2)$, p_2 is an elementary singularity.

4.5 Regular simple arcs

In Chapter III, it was pointed out that a Barner arc in $\underline{P}^2$ is simple. Hence by 3.2.7, a regular simple arc which is Barner is of order 2. In 2.6.1, it was proved that if A is an arc of order 2 then A is regular.

The question naturally arises if a regular arc is of order 2. The answer is no, for in Figure IV.12, A is a regular arc which is not of order 2. Moreover a regular simple arc which is not Barner is not of order 2, as can be seen in Figure IV.13. The most we will be able to show in this chapter is that a regular simple arc is ordinary.

4.5.1 Theorem: Let (p,q) be an ordinary simple subarc of an arc A. Let $\{H_i\}$ be a tower of spaces such that H_1 does not meet A. If the lines $L = A(p)H_0$ and $M = A(q)H_0$ are distinct and do not cut (p,q), then (p,q) is of order 2.

(It should be noted that the hypothesis does not state A is an arc with tower $\{H_i\}$, since no restriction is made upon $H_0 \varepsilon H_1$.)

Proof: Case 1. If p and q are ordinary. Let $r \varepsilon (p,q)$, then r is elementary and strongly finite by 4.2.7. Thus there exist $U^+(r)$, $U^-(r)$ such that $H_0 \not\varepsilon A_1(t)$ for $t \varepsilon U^+(r) \cup U^-(r)$. Since J is isomorphic to $\mathbb{R}$ as a topological space and $[p,q]$ is closed in J, $[p,q]$ is compact. Since A is continuous, $A[p,q]$ is compact. Hence there are only finitely many points $r \varepsilon (p,q)$ such that $H_0 \varepsilon A_1(r)$. Let $r_1 < \ldots < r_k$ be such points. Put $p = r_0$ and $q = r_{k+1}$.

Put $L_i = A(r_i)H_0$, $0 \leq i \leq k+1$, hence $L_i = A_1(r_i)$ for $1 \leq i \leq k$.

Now consider (r_{i-1},r_i), $0 \leq i \leq k+1$. Since $H_0 \notin A_1(r)$ for $r \in (r_{i-1},r_i)$, then (r_{i-1},r_i) is an arc with tower $\{H_i\}$. Moreover (r_{i-1},r_i) is strongly finite and dually differentiable by <u>4.2.6</u>. Hence $A_1 = A_0^*$ is continuous on (r_{i-1},r_i) and (r_{i-1},r_i) is of order 2 by <u>4.3.1</u>; $0 \leq i \leq k+1$. Thus if $k = 0$, (p,q) is of order 2.

Suppose $k \geq 1$. We think of the lines distinct from H_1, through H_0, as being vertical and order so $L < M$. Since L and M do not meet (p,q), $L \leq L_i \leq M$ for $i = 0,\dots, k+1$. We may assume (p,r_1) lies above the line $A(p)A(r_1)$, (Figure IV.14).

Now r_1 is regular, hence $A(p)A(r_1)$ and $A(r_1)A(r_2)$ both cut A at r_1. Then (r_1,r_2) lies below the line $A(r_1)A(r_2)$; otherwise $H_0 \notin A_1(r_2)$, since r_2 is also regular, (Figure IV.15). But then (r_1,r_2) is contained in an open half-space determined by L and L_1, hence $L \leq L_2 < L_1 \leq M$, (Figure IV.16). Since A is simple and $L \leq L_2 < L_1 \leq M$, then $k \geq 2$. But then we have (r_2,r_3) lie above $A(r_2)A(r_3)$ and $L \leq L_2 < L_3 < L_1 \leq M$. Hence $k \geq 3$. Continuing, it follows that k is arbitrarily large; a contradiction.

<u>Case 2</u>. If p is not necessarily ordinary. Take $r \in (p,q)$ such that the lines L and $A(r)H_0$ are distinct and do not meet (p,r). For any $p_1 \in (p,r)$, there is a $p_2 \in (p,p_1)$ such that $A(p_0)H_0$ does not meet (p_2,r). But then taking $L = A(p_2)H_0$ and $M = A(r_1)H_0$, (p_2,r) is of order 2 by case 1. Since $(p_1,r) \subset (p_2,r)$, (p_1,r) is of of order 2. But p_1 is arbitrary, hence (p,r) is of order 2. Similarly q has a left neighbourhood of order 2 and the result follows.

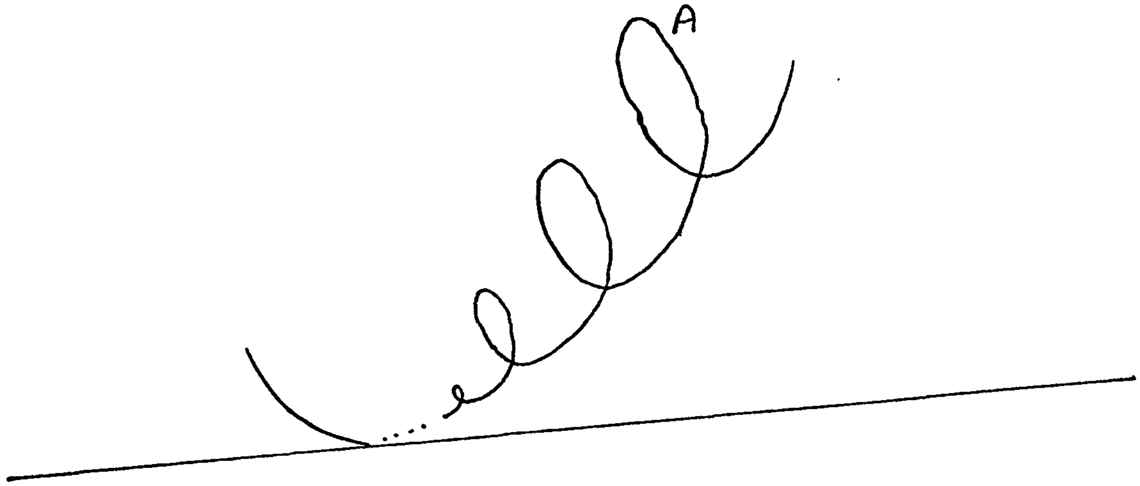


Figure IV.12

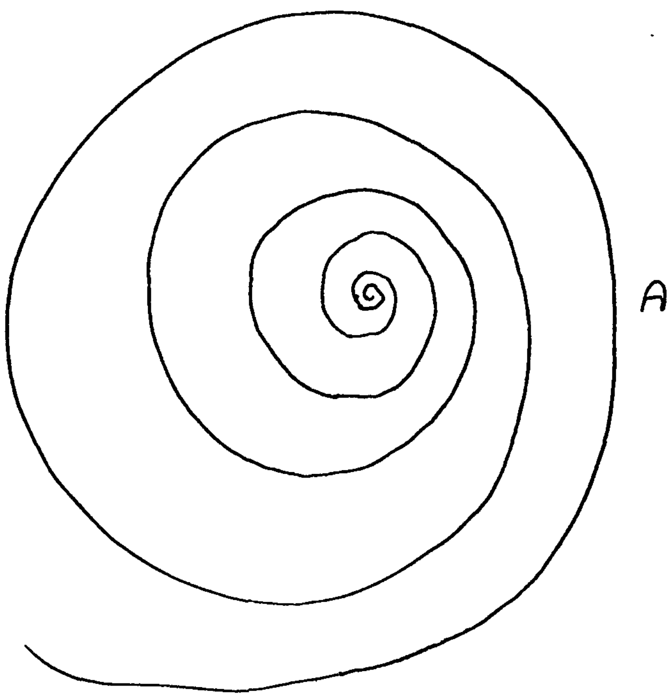


Figure IV.13

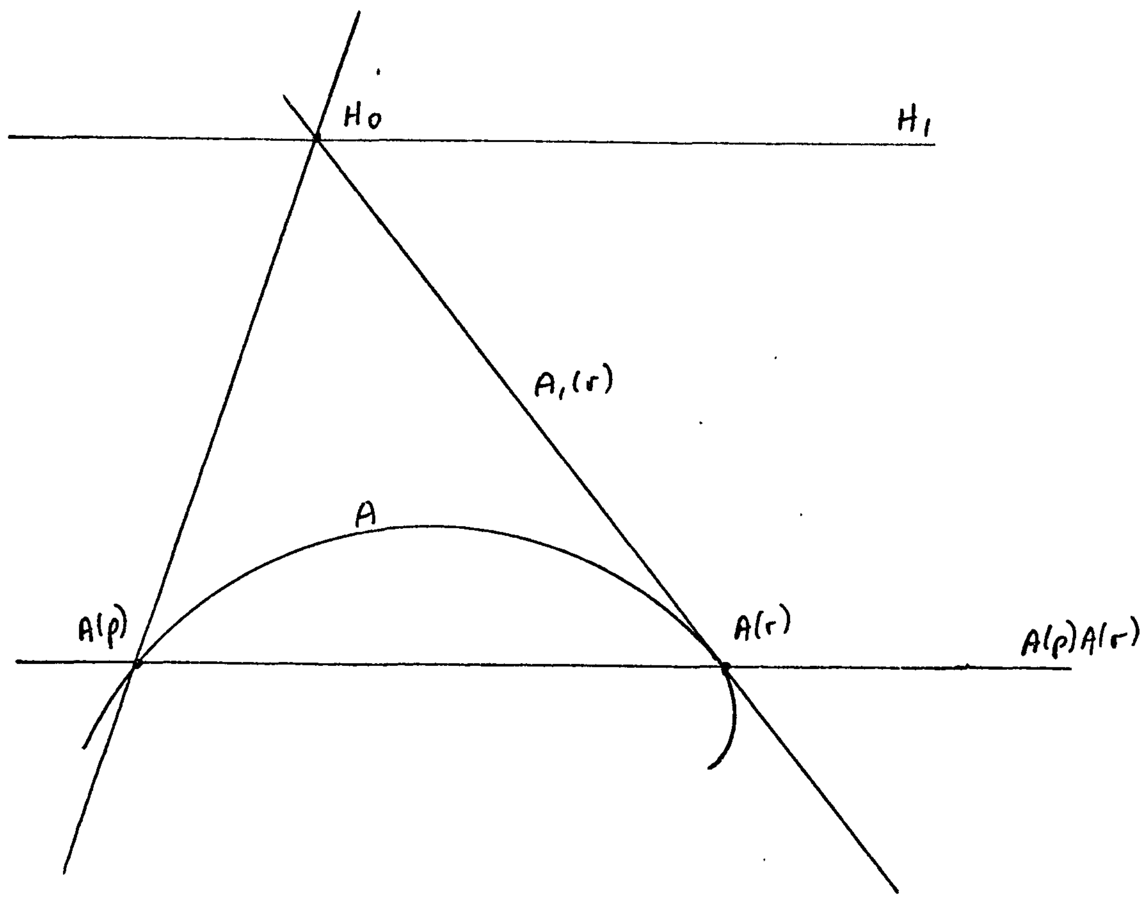


Figure IV.14

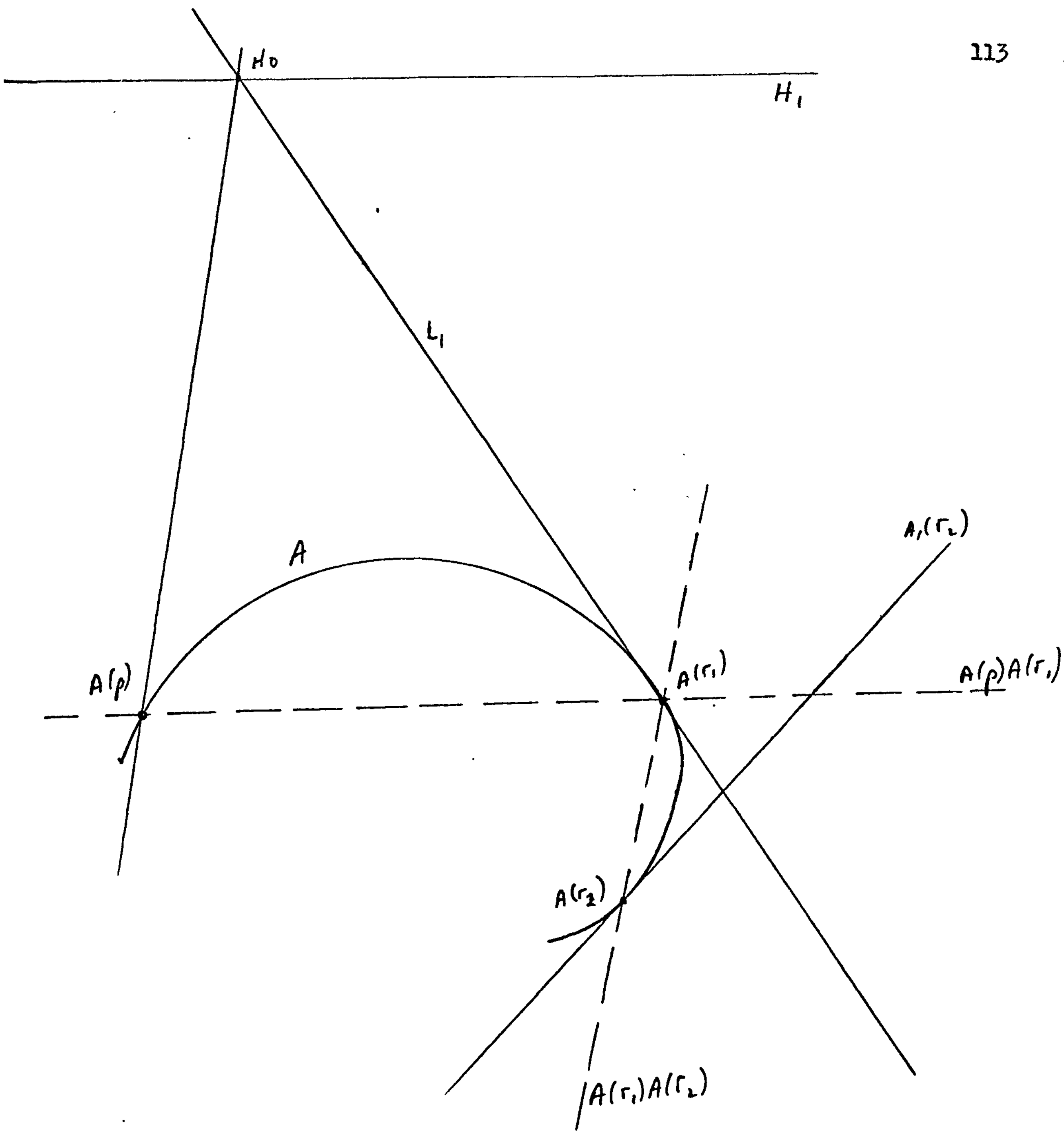


Figure IV.15

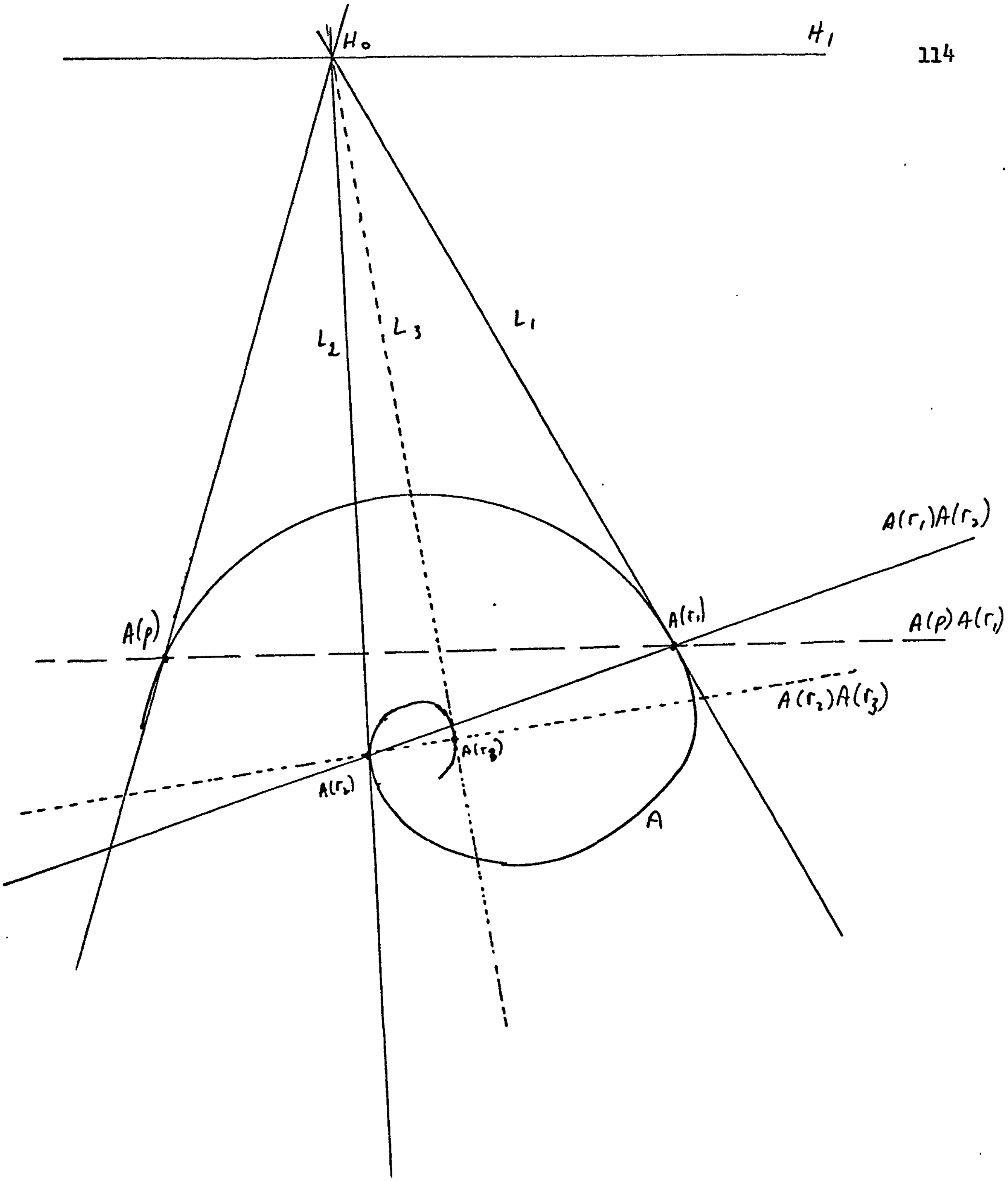


Figure IV.16

4.5.2 Theorem: Suppose a regular simple arc has a singularity p_0. Let $\{H_i\}$ be a tower of spaces such that $H_1 \cap A = \emptyset$. Then there exist points p_1, q_1, r_1 such that p_1 is singular, $p_1 \notin [q_1, r_1]$ and $A(p_1)H_0$ lies between $A(q_1)H_0$ and $A(r_1)H_0$.

Proof: There is a point p such that $H_0 \in A_1(p)$, otherwise A is a Barner arc by 3.3.1, and of order 2 by 3.2.7, a contradiction.

Case 1. $H_0 \in A_1(p)$ for some singularity p. Let (q,r) be a neighbourhood of p such that $A(q)H_0 = A(r)H_0 = L$ say. We may assume $A_1(p)$ and L are distinct and do not meet (q,p) or (p,r). One of these intervals contains a singularity p_1; for otherwise, they are of order 2 by 4.5.1 and p is elementary. But then p is ordinary by 4.3.4, a contradiction. If $p_1 \in (q,p)$, choose $q_1 = p$, $r_1 = r$ and the result follows.

Case 2. $H_0 \in A_1(p)$ only for ordinary points. Let (p,q) be a neighbourhood of the singularity p_0, such that $A(p)H_0$ and $A(q)H_0$ are distinct and do not meet (p,q). Let $r \in (p,q)$ such that $H_0 \in A_1(r)$ and $A_1(r)$ does not meet (p,r), (Figure IV.16). If no such r exists, then by 4.5.1, (p,q) has order 2, a contradiction.

Take $s \in (r,q)$ such that $A(s) \in A_1(r)$ and $A_1(r)$ does not meet (r,s). Since p_0 is singular, we can always consider a sufficiently small (p,q) such that $s \in (r,q)$ exists. Take $t \in (r,s)$ such that $H_0 \in A_1(t)$ and $A_1(t)$ does not meet (r,t) or (t,s). Take $u \in (p,r)$ such that $A(u) \subset A_1(t)$ and $A_1(t)$ does not meet (u,r).

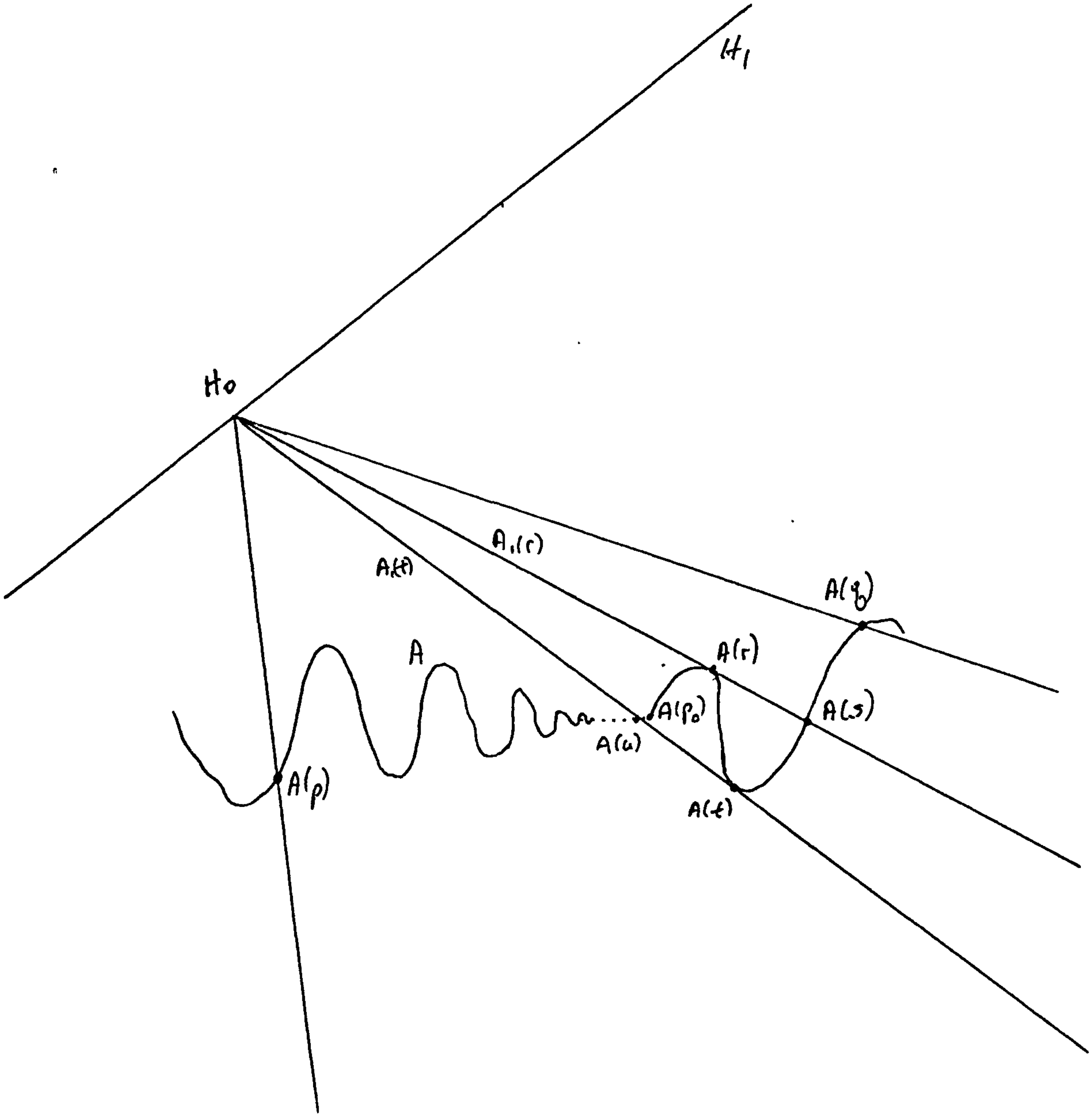


Figure IV.17

Then $u < r < t < s$. By 4.5.1, (u,s) contains a singularity p_1; for otherwise, (u,s) is of order 2 and $p_0 \in (u,s)$ is ordinary. Since $H_0 \in A_1(r) \cap A_1(t)$, by assumption $p_1 \neq r$ and $p_1 \neq t$. If $p_1 \in (r,t)$ or (t,s), we can choose q_1, r_1 appropriately and the result follows.

4.5.3 Theorem: Every regular simple arc A is ordinary.

Proof: Suppose A has a singularity p_0. Let $\{H_i\}$ be a tower such that H_1 does not meet some neighbourhood U_0 of p_0. Take p_1, q_1, r_1 in U_0 as in 4.5.2. Let U_1 be a neighbourhood of p_1 such that $\overline{U}_1 \subset U_0$, $U_1 \cap [q_1, r_1] = \emptyset$ and $A(p)H_0$ lies between $A(q_1)H_0$ and $A(r_1)H_0$ for all $p \in U_1$. Applying 4.5.2 repeatedly, one obtains p_i, q_i, r_i, U_i such that $p_i \in U_i$, $\overline{U}_i \subset U_{i-1}$, q_i and $r_i \in U_{i-1}$, $U_i \cap [q_i, r_i] = \emptyset$ and $A(p_0)H_0$, lies between $A(q_i)H_0$ and $A(r_i)H_0$ for all $p \in U_i$. Take $p \in \bigcap_{i=1}^{\infty} U_i$. Then $A(p)H_0$ meets each of the disjoint intervals $[q_i, r_i]$, contradicting 1.3.1.

As can be observed, the arc in Figure IV.17, is an arc with at most inflections and not regular as 4.5.2 requires.

CHAPTER V

In these last three chapters, we shall be concerned with arcs in a real projective three-space only. In many instances, results will not be proved in detail or may not be proved at all. In all these cases the proofs follow along the lines of proofs of analogous results in $\underline{P}^2$. The only difference in the proofs will be that of dimension. Namely, instead of considering points and lines in $\underline{P}^2$, one considers lines and planes in $\underline{P}^3$, and so on.

5.1 Projection

Let $P \varepsilon \underline{P}_0^3$ and p be a point of an arc A. Define

$$\breve{A}_k(p) = \begin{cases} A_k(p)P & \text{if } -1 \leq k \leq \pi(P,p) \\ A_{k+1}(p) & \text{if } \pi(P,p) < k \leq 2. \end{cases}$$

5.1.1 Theorem: $\breve{A} = \{ \breve{A}_0(p) | p \varepsilon J \}$ is an arc in $\underline{P}^2(P)$ with $\breve{A}_k(p)$ as the osculating k-spaces at p, $-1 \leq k \leq 2$.

One should note that $\breve{A}$ is now an arc in a real projective plane, (Figure V.1). One can generalize this definition of projection to the case of a line in $\underline{P}^3$. Let $L \varepsilon \underline{P}_1^3$, the projection A/L of A from L is defined by

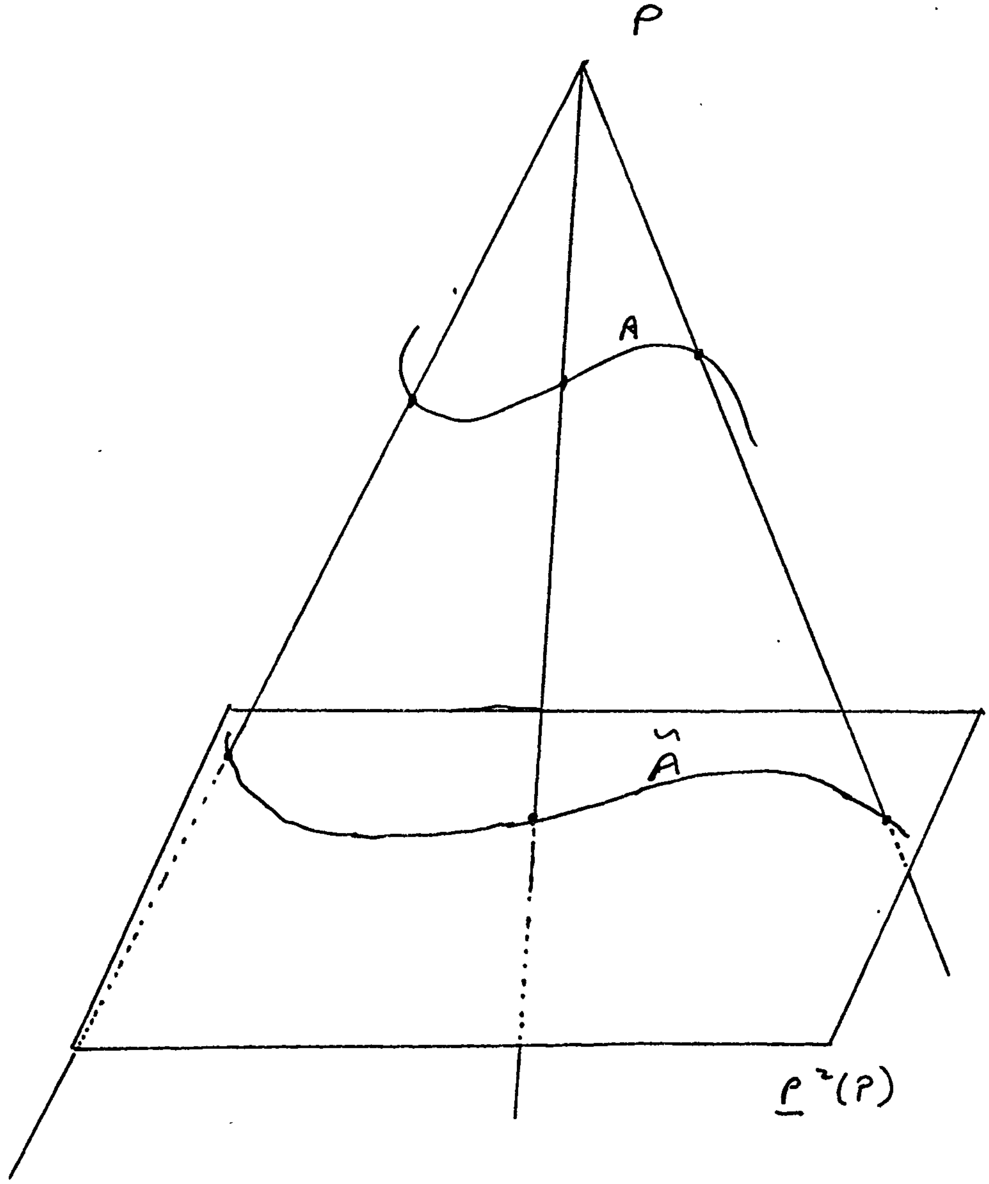


Figure V.1

$$(A/L)(p) = A(p)/L = \lim_{\substack{q \to p \\ q \neq p}} A(q)L$$

for all $p \in J$.

<u>5.1.2 Theorem</u>: A/L is an arc in $\underline{P}^1(L)$ where $L \in \underline{P}^3_{-1}$.

<u>Proof</u>: Take a point $P \in L$. Then $A/P = \breve{A}$ is an arc in $\underline{P}^2(P)$ with $L \in \underline{P}^2_0(P)$. Thus $\breve{\breve{A}} = (A/P)/L$ is an arc in $\underline{P}^1(L)$, (Figure V.2). But

$$((A/P)/L)(p) = \lim_{\substack{q \to p \\ q \neq p}} (A/p)(q)L$$

$$= A(p)/L$$

$$= (A/L)(p).$$

<u>5.2 Characteristic of a point</u>

Let L be a k-space, $-1 \leq k \leq 3$ and let $p \in J$. Then $\delta(p,L)$ is the dimension of the largest osculating space of p which is contained in L.

<u>5.2.1 Lemma</u>: Let P be a point of a k-space L and $\breve{A} = A/P$. Then

$$\breve{\delta}(p,L) = \begin{cases} \delta(p,L) & \text{if} \quad -1 \leq \delta(p,L) < \pi(P,p) \\ \delta(p,L)-1 & \text{if} \quad \pi(P,p) < \delta(p,L) \leq k. \end{cases}$$

We define a plane L to support A at p if there is a plane H_∞ with $H_\infty \neq L$, $A(p) \not\in H_\infty$ and a $U'(p)$ such that $A(U'(p))$ is

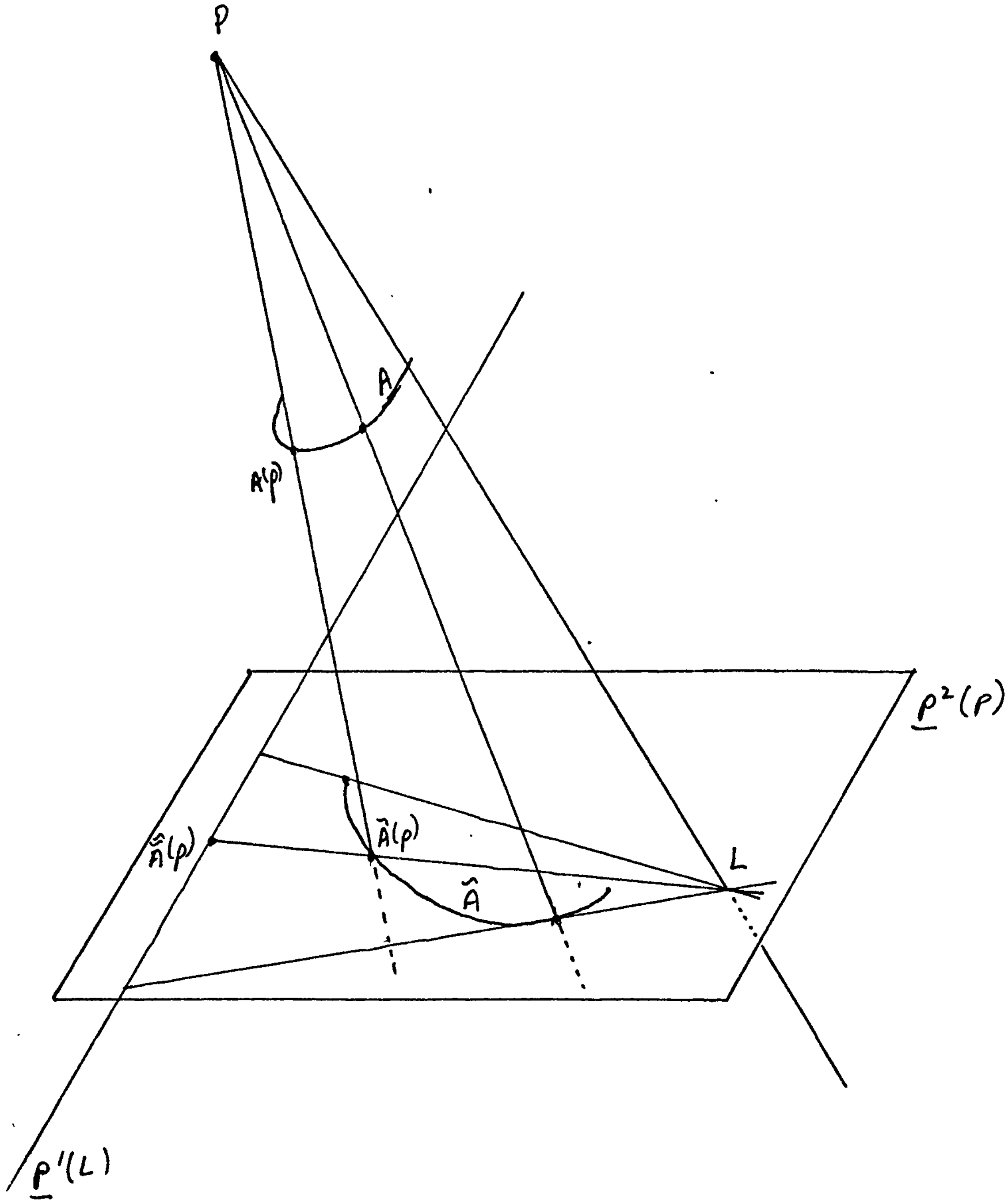


Figure V.2

contained in one of the two open half-spaces determined by L and H_∞. When L does not support A at p, we say L cuts A at p.

By 1.3.1, every plane in $\underline{P}^3$ either supports or cuts a given point of an arc. Let S_k be the set of all planes L with $\delta(p,L) = k$ for a fixed $p \in J$.

5.2.2 Lemma: Let $-1 \leq k \leq 2$. Let $p \in J$ be fixed and the corresponding S_k be given. Then

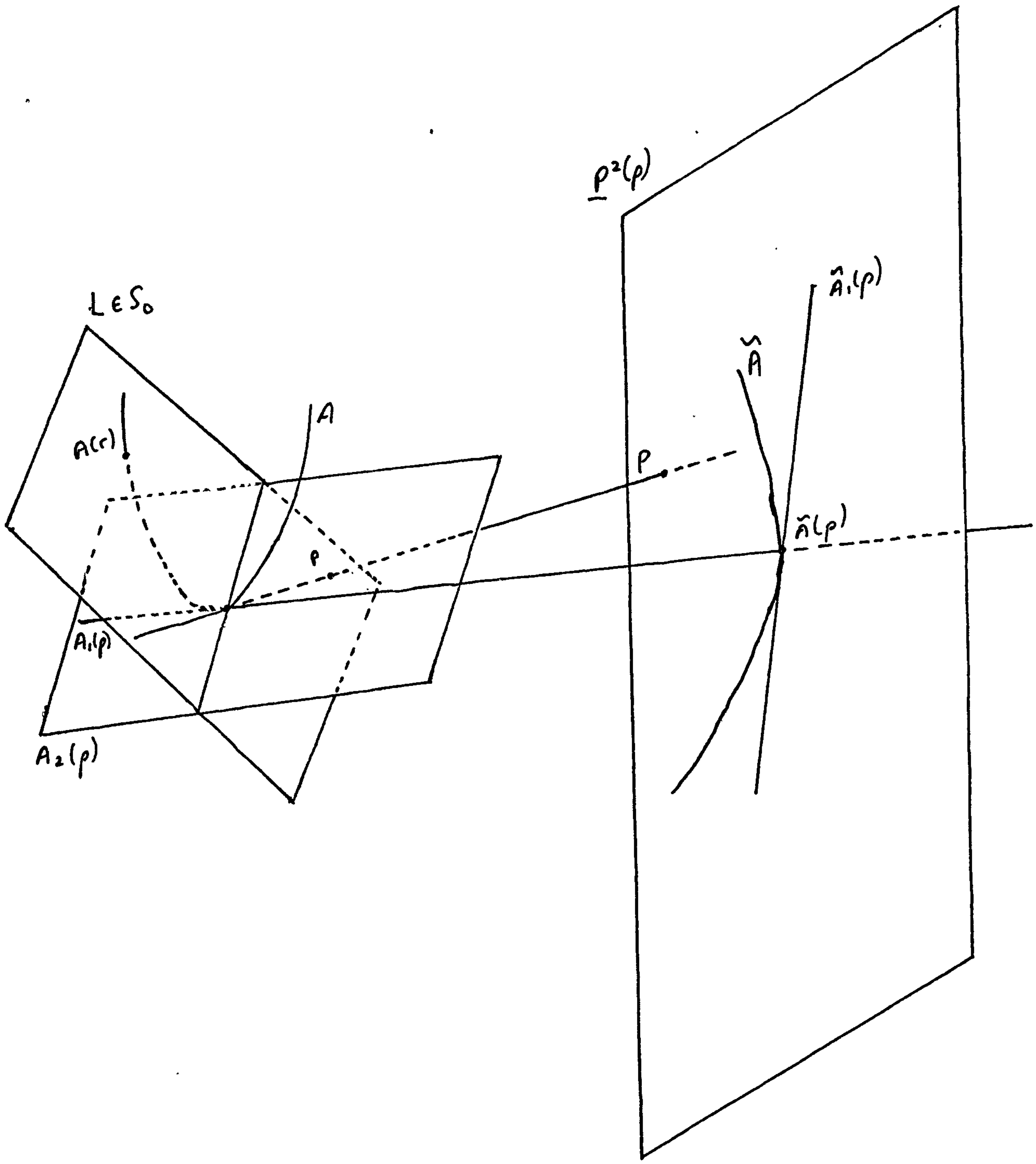
a) S_k is connected, $-1 \leq k \leq 2$.

b) the elements of S_k either all support or all cut A at p.

Proof: (a) If $k = 2$, then $S_2 = \{ A_2(p) \}$ and is connected. If $k = 1$, then since $A_2(p) \notin S_1$; projecting from $A_1(p)$, S_1 is projected onto $\underline{P}^1(A_1(p)) \setminus (A/A_1(p))(p)$. Thus S_1 will be isomorphic to the real line and hence connected. If $k = 0$, let $\hat{A} = A/p$ and let $\hat{S}_{-1}$ be defined for p in $\hat{A}$. Since $\hat{A}$ is an arc in $\underline{P}^2(p)$, it is connected by 2.5.1. But $L \in S_0$ iff $L \in \hat{S}_{-1}$, (Figure V.3); hence S_0 is connected.

Since in $\underline{P}^3$, every plane meets a given line in at least one point, S_{-1} is exactly the set of planes of $\underline{P}^3$ which meet $A_1(p)$ at exactly one point $q \neq p$. For if $L \in S_{-1}$ then $A_1(p) \not\subset L$. Conversely if a plane L meets $A_1(p)$ exactly at one point $q \neq p$, then $\delta(p,L) = -1$. Thus the result follows for $k = -1$.

(b) See 2.5.1.



$\underline{P}^2(\rho)$
$L \in S_0$
$A(r)$
A
P
p
$A_1(\rho)$
$A_2(\rho)$
$\tilde{A}_1(\rho)$
$\tilde{A}$
$\tilde{A}(\rho)$

Figure V.3

We can now characterize a given point p of an arc A by describing whether S_k cuts or supports A at p; $-1 \leq k \leq 2$.

We define for $p \varepsilon J$, $\sigma_k(p) = 0$ or 1 according to whether S_k supports or cuts A at p respectively, $-1 \leq k \leq 2$. The characteristic $(a_0(p), a_1(p), a_2(p))$ of $p \varepsilon J$ is defined by taking $a_i(p) = 1$ or 2 and requiring that

$$\sigma_k(p) \equiv \sum_{i=0}^{k} a_i(p) \pmod 2.$$

Lastly we define

$$\beta_k(p) = \sum_{i=0}^{k} a_i(p), \quad 0 \leq k \leq 2$$

and $\beta_{-1}(p) = 0$.

<u>3.2.3 Theorem</u>: Let P be a point and $\breve{A} = A/P$. Then

$$\breve{\sigma}_k(p) \equiv \begin{cases} \sigma_0(p) + \sigma_{k+1}(p) & \pi(P,p) = -1 \leq k \leq 1 \\ \sigma_k(p) & -1 \leq k < \pi(P,p) \\ \sigma_{k+1}(p) & -1 < \pi(P,p) < k \leq 1, \end{cases}$$

$$\breve{a}_k(p) \equiv \begin{cases} a_k(p) & 0 \leq k < \pi(P,p) \\ a_k(p) + a_{k+1}(p) & k = \pi(P,p) \\ a_{k+1}(p) & \pi(P,p) < k \leq 1, \end{cases}$$

where the congruences are (mod 2), and

$$\breve{\beta}_k(p) = \begin{cases} \beta_{k+1}(p) - \beta_0(p) & \pi(P,p) = -1 \leq k \leq 1 \\ \beta_k(p) & -1 \leq k < \pi(P,p). \end{cases}$$

In Figure V.4, p is a point on A with $\pi(P,p) = 1$ and characteristic $(1,2,1)$. The planes $L_i \in S_i$, $i = 0,1,2$. Note that since $P \notin L_1$, L_1 is not projected into $\underline{P}^2(P)$ and the characteristic of p in $\breve{A}$ is $(1,1)$.

Since a projection from a line was defined as a projection from a point on theline followed by a projection from the projected image of the line, one can obtain the characteristics of a projected image of a point of an arc when projecting from a line by applying the formulas in <u>5.2.3</u> twice.

Since for any $P \neq P^1$, P and P^1 points on L and $p \in J$

$$((A/P)/L)(p) = ((A/P^1)/L)(p),$$

the characteristics of p upon projection from L is independent of the choice of P in L. In the following theorem, we shall prove this independence by direct calculation of the characteristic and at the same time derive the formula for the characteristic of a point with respect to each $L \in \underline{P}^3_1$.

<u>5.2.4 Theorem</u>: Let $p \in J$ and $L \in \underline{P}^3_1$. Let P be a point on L and write $\breve{A} = A/P$, $\breve{\breve{A}} = (A/P)/L$. The characteristic of p in $A[\breve{A},\breve{\breve{A}}]$ is $(a_0(p), a_1(p), a_2(p))$ $[(\breve{a}_0(p), \breve{a}_1(p)), (\breve{\breve{a}}_0(p))]$. Then $\breve{\breve{a}}_0(p)$ is independent of $P \in L$.

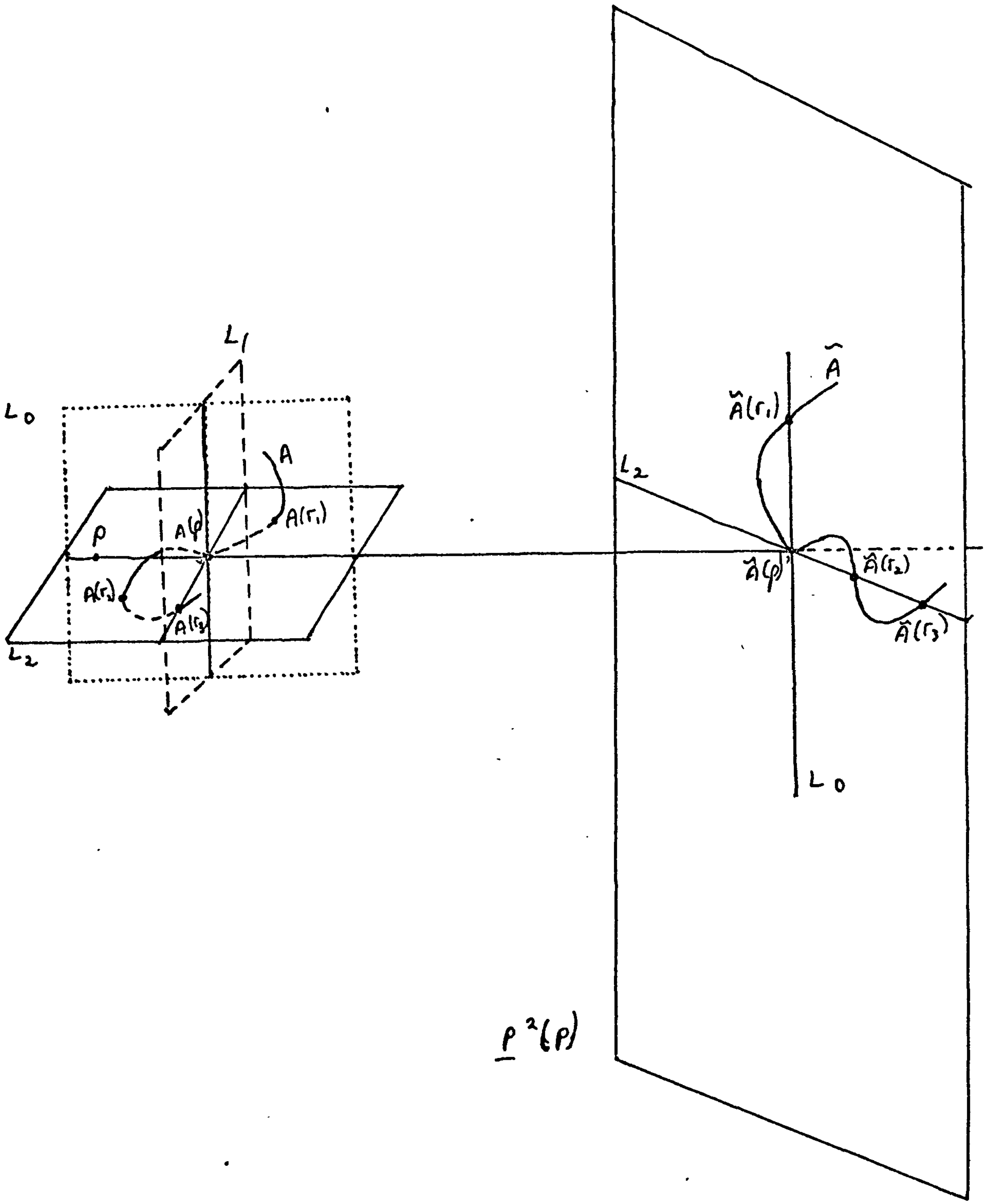

L_1
L_0
A
$A(r_1)$
p
$A(p)$
$A(r_2)$
$A(r_3)$
L_2
$\tilde{A}$
$\tilde{A}(r_1)$
L_2
$\tilde{A}(p)$
$\tilde{A}(r_2)$
$\tilde{A}(r_3)$
L_0
$\underline{P}^2(p)$

Figure V.4

<u>Proof:</u> <u>Case 1</u>. $L = A_1(p)$, $\check{\hat{\check{a}}}_0(p) = a_2(p)$.

If $P \neq A(p)$ then $\pi(P,p) = 0$ and $\check{a}_0(p) \equiv a_0(p) + a_1(p)$, $\hat{a}_1(p) = a_2(p)$. Moreover, $\check{A}(p) = A(p)P = L$ and $\pi(L,p) = -1$. Hence $\check{\hat{a}}_0(p) = \check{a}_1(p) = a_2(p)$. If $P = A(p)$ then $\pi(P,p) = -1$, $\check{a}_0(p) = a_1(p)$ and $\check{a}_1(p) = a_2(p)$. Since $\pi(L,p) = -1$, the result follows.

<u>Case 2</u>. $L \neq A_1(p)$ and $L \subset A_2(p)$

(i) If $L \cap A_1(p) \neq A(p)$ then $\check{\hat{a}}_0(p) \equiv \sum_{i=0}^{2} a_i(p) \pmod 2$.

If $P \not\subset A_1(p)$ then $\check{a}_0(p) = a_0(p)$ and $\check{a}_1(p) \equiv a_1(p) + a_2(p) \pmod 2$. $\check{A}(p) = A(p)P \neq L$ and $\check{A}_1(p) = A_1(p)P = A_2(P) \supset L$ implies $\pi(L,p) = 0$ and the result follows.

If $P \in A_1(p) \setminus A(p)$ then $\check{a}_0(p) \equiv a_0(p) + a_1(p) \pmod 2$ and $\hat{a}_1(p) = a_2(p)$. $\pi(L,p) = 0$ implies the result.

(ii) If $L \cap A_1(p) = A(p)$ then $\check{\hat{a}}_0(p) = a_1(p) + a_2(p) \pmod 2$.

If $P \not\subset A_1(p)$ then $\check{a}_0(p) = a_0(p)$ and $\check{a}_1(p) \equiv a_1(p) + a_2(p) \pmod 2$. $\check{A}(p) = A(p)P = L$ and hence $\check{\hat{a}}_0(p) = \check{a}_1(p)$ and the result follows.

If $P = A(p)$, $\check{a}_0(p) = a_1(p)$ and $\check{a}_1(p) = a_2(p)$. But $P = A(p)$ implies $\check{A}(p) = A_1(p) \neq L$. Hence $\pi(L,p) = 0$ and $\check{\hat{a}}_0(p) = \check{a}_0(p) + \check{a}_1(p) \pmod 2$.

<u>Case 3</u>. $L \not\subset A_2(p)$.

(i) If $L \cap A_2(p) \not\in A_1(p)$ then $\tilde{\tilde{a}}_0(p) = a_0(p)$.

If $P \not\in A_2(p)$, $\tilde{a}_0(p) = a_0(p)$ and $\tilde{a}_1(p) = a_1(p)$. $\tilde{A}_1(p) = A_1(p)) = PL$, hence $\pi(L,p) = 1$ and $\tilde{\tilde{a}}_0(p) = \tilde{a}_0(p) = a_0(p)$.

If $P \varepsilon A_2(p)$ then $\pi(P,p) = 1$ and $\tilde{a}_0(p) = a_0(p)$, $\tilde{a}_1(p) \equiv a_1(p) + a_2(p) \pmod 2$. $\tilde{A}_1(p) = A_1(p)P = A_2(p) \not\supset L$ and the result follows.

(ii) If $L \cap A_2(p) \varepsilon A_1(p) \setminus A(p)$ then $\tilde{\tilde{a}}_0(p) = a_0(p) + a_1(p)$ (mod 2).

If $P \not\in A_2(p)$, $\tilde{a}_i(p) = a_i(p)$ for $i = 0, 1$. $\tilde{A}_1(p) = A_1(p)P \supset L$ and $\tilde{A}(p) = A(p)P \not\supset L$ imply $\pi(L,p) = 0$ and $\tilde{\tilde{a}}_0(p) \equiv \tilde{a}_0(p) + \tilde{a}_1(p)$ (mod 2).

If $P \varepsilon A_2(p)$ then $P \varepsilon A_1(p)$ by assumption. Hence $\pi(P,p) = 0$ and $\tilde{a}_0(p) \equiv a_0(p) + a_1(p) \pmod 2$, $\tilde{a}_1(p) = a_1(p)$. $\tilde{A}_1(p) = A_2(p) \not\supset L$, hence $\pi(L,p) = 1$ and $\tilde{\tilde{a}}_0(p) = \tilde{a}_0(p)$.

(iii) If $L \cap A_2(p) = A(p)$ then $\tilde{\tilde{a}}_0(p) = a_1(p)$.

If $P \not\in A_2(p)$ then $\tilde{a}_i(p) = a_i(p)$ for $i = 0,1$. $\tilde{A}(p) = A(p)P = L$ hence $\pi(L,p) = -1$ and $\tilde{\tilde{a}}_0(p) = \tilde{a}_1(p) = a_1(p)$.

If $P \varepsilon A_2(p)$ then $P = A_2(p)$ and $\tilde{a}_i(p) = a_{i+1}(p)$ for $i = 0,1$. $\pi(L,p) = 1$ implies $\tilde{\tilde{a}}_0(p) = \tilde{a}_0(p) = a_1(p)$.

5.3 Secants

Let A be an arc and L be a k-space $-1 \le k \le 3$. L is said to be a k-*secant* of $X \subset J$ if

$$\bigvee_{p\varepsilon X} A_{\delta(p,L)}(p) = L$$

L is said to be an <u>independent</u> k-<u>secant</u> <u>of</u> X if

$$k = \sum_{p\varepsilon X} (\delta(p,L)+1) - 1$$

X is said to be k-<u>independent</u>, $-1 \leq k \leq 3$, if every k-secant of X is independent.

We define the mapping

$$A^k:\ J^{k+1} \longrightarrow \underline{P}^3_k,$$

$-1 \leq k \leq 3$, inductively by requiring that $A^{-1}(\ ) = \emptyset$ and

$$A^k(p_0, \ldots, p_k) = A(p_k)/A^{k-1}(p_0, \ldots, p_{k-1}).$$

We denote an element $(p_0, \ldots, p_k) \varepsilon J^{k+1}$ by x and define

$$\gamma(p,x) = \sum_{p_i=p} 1 \quad -1$$

where p_i are the components of x. We define x to be <u>independent</u> if

$$\bigvee_{p\varepsilon J} A_{\gamma(p,x)}(p) = A^k(x).$$

Then using arguments similar to those found in Chapter II, the results of 2.3 and 2.4 can be extended to the case of a real projective three space.

<u>5.3.1 Lemma</u>: A k-space L with $A(p) \subset L$ is a k-secant of X iff it is a (k-1)-secant of X on A/p, where $p \varepsilon X$.

5.3.2 Lemma: The set of all k-secants of a connected set $X \subset J$ is pathwise connected, $-1 \leq k \leq 3$.

5.3.3 Lemma: Suppose $-1 \leq h \leq k \leq 2$. If X is k-independent it is also h-independent.

Proof: Let L be an h-secant of X. By 1.3.1, there are say $p_1, \ldots, p_r$ distinct points of X such that $h + r = k$, $A(p_i) \not\subset LA(p_1)\ldots A(p_{i-1})$ for $i = 1, \ldots, r$ and $LA(p_1)\ldots A(p_r)$ is a k-secant of X.

If L is not independent then

$$h < \sum_{p \in X} (\delta(p,L)+1)-1$$

$$= \sum_{\substack{p \in X \\ p \neq p_i}} (\delta(p,L)+1)-1$$

Therefore

$$h+r < \sum_{\substack{p \in X \\ p \neq p_i}} (\delta(p,L)+1) + \sum_{i=1}^{r} (\delta(p_i,A(p_i)+1)-1$$

Since L and $A(p_i)$ are contained in $LA(p_1)\ldots A(p_r)$ for $i = 1,\ldots,r$, then

$$\delta(p,L),\ \delta(p_i,A(p_i)) \leq \delta(p,LA(p_1)\ldots A(p_r))$$

for all $p \in X$ and $i = 1,\ldots, r$. Hence

$$k = h+r < \sum_{p \in X} (\delta(p,LA(p_1)\ldots A(p_r)+1)-1;$$

a contradiction.

5.3.4 Lemma: Let X be k-independent, $-1 \leq k \leq 2$, and L be a k-secant of X on A with $A(p) \subset L$. Then if $q \neq p$, $A(p) \not\subset A_{\delta(q,L)}(q)$.

5.3.5 Lemma: If X is k-independent and $p \in X$ then X is (k-1)-independent on A/p.

5.3.6 Theorem: $A^k(X^{k+1})$ is the set of all k-secants of X, $-1 \leq k \leq 3$.

5.3.7 Theorem: Let $X \subset J$ and $x \in X^{k+1}$. Then $A^k(x)$ is an independent k-secant of X iff $\delta(p, A^k(x)) = \gamma(p,x)$ for all $p \in X$. iff $A^k(x) = A^k(y)$, where $y \in X^{k+1}$ holds exactly for the permutations of X.

5.3.8 Lemma: $x \in J^{k+1}$ for $k = 1$ [$k=2$, $k=3$] is independent iff the components of x are not equal [not collinear, not coplanar].

5.3.9 Theorem: Let $-1 \leq k \leq 2$. Then $X \subset J$ is k-independent iff every $x \in X^{k+2}$ is independent.

5.4 Order and monotonicity

Let A be an arc, $X \subset J$, and L a k-space, $-1 \leq k \leq 3$. If $S(X,L) = \{ p \in X \mid A(p) \subset L \}$ is finite for every k-space L, k fixed; we say that X has finite $\underline{P}_k^3$ order. Moreover, it $\sup_{L \in \underline{P}_k^3} |S(X,L)|$ is bounded, then we say X has bounded $\underline{P}_k^3$ order.

We speak of the order of X when $k = 2$.

5.4.1 Theorem: If (p,q) has order 3 then $(p,q]$ and $[p,q)$ are 2-independent.

Corollary 1: An arc has order 3 iff it is 2-independent.

Corollary 2: If (p,q) has order 3 on A and $r \in [p,q]$ then (p,q) has order 2 on A/r.

Corollary 3: If (p,q) has order 3 on A and $s \in (p,q)$ then

$$A_2(s) \cap A[p,q] = A(s).$$

Let A be an arc of order 3, L an oriented line and assume no 1-secant of A meets L. Then for $x \in J^3$, $A^2(x) \cap L$ is a point $\varphi(x)$ of L, (Figure V.5). We assume there is a point $P_\infty \in L$ such that $P_\infty \neq \varphi(x)$ for $x \in J^3$ and define

$$(p_0,p_1,p_2) \leq (q_0,q_1,q_2)$$

if $p_i < q_i$ for $i = 0,1,2$.

5.4.2 Theorem: φ is (strictly) monotone.

Proof: Since A has order 3, A is 2-independent by 5.4.1 and hence A^2 and φ are symmetric for $x \in J^3$ by 5.3.7; that is

$$\varphi(p_0,p_1,p_2) = \varphi(p_1,p_2,p_0) = \varphi(p_2,p_0,p_1)$$

and so on where $x = (p_0,p_1,p_2)$.

Suppose $(p_0,p_1,p_2) \leq (q_0,q_1,q_2)$. In 2.4.2, it was proved that φ is monotone in a real projective plane, hence for $p \in J$

$$p \longrightarrow \varphi(p_0,p_1,p)$$

is monotone. If φ is say increasing, then

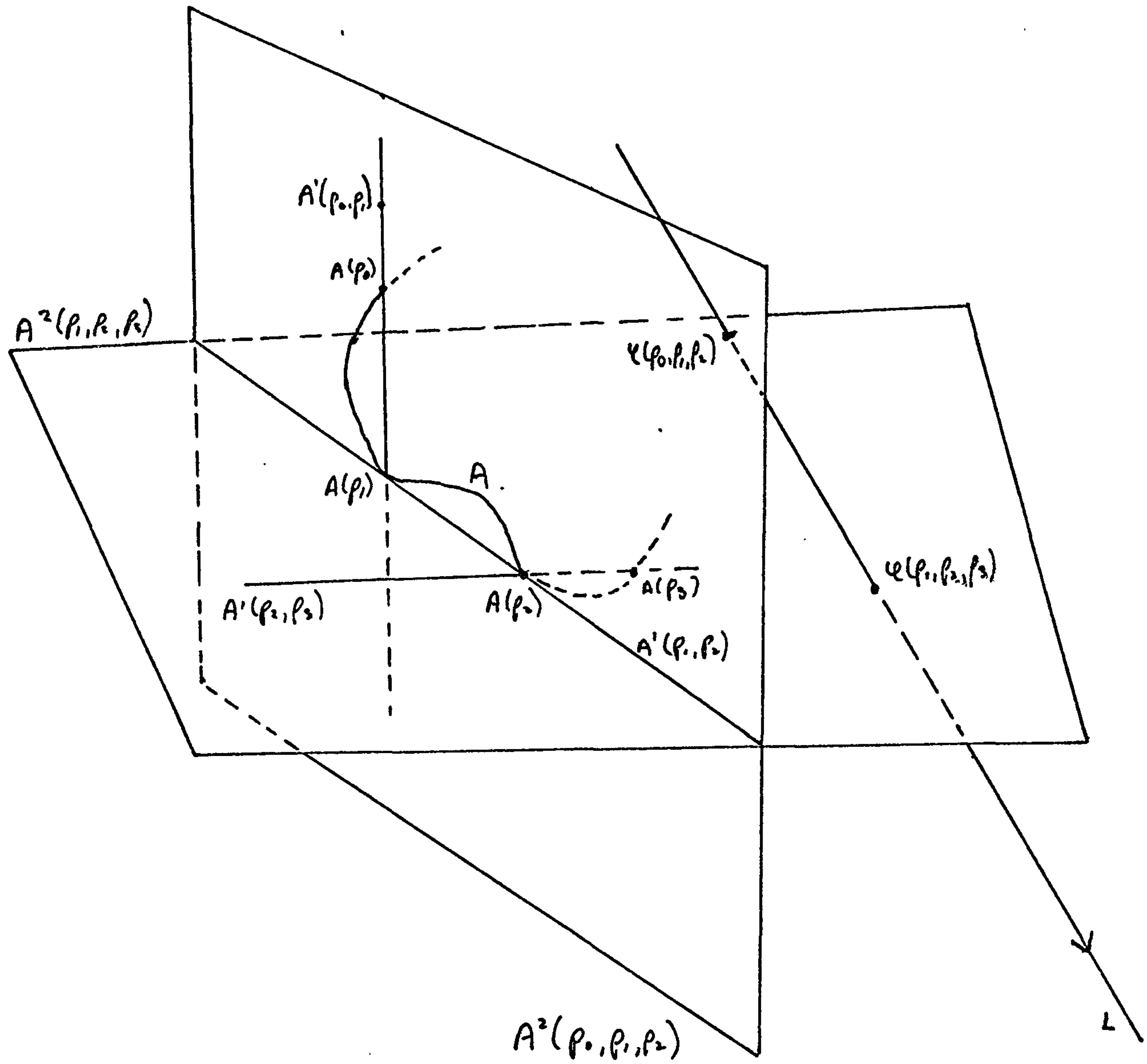

A'(ρ₀,ρ₁)
A(ρ₀)
A²(ρ₁,ρ₂,ρ₃)
φ(ρ₀,ρ₁,ρ₂)
A(ρ₁)
A
A(ρ₃)
φ(ρ₁,ρ₂,ρ₃)
A'(ρ₂,ρ₃)
A(ρ₂)
A'(ρ₁,ρ₂)
A²(ρ₀,ρ₁,ρ₂)
L

Figure V.5

$$\varphi(p_0,p_1,p_2) \leq \varphi(p_0,p_1,q_2)$$
$$\leq \varphi(p_0,q_1,q_2)$$
$$\leq \varphi(q_0,q_1,q_2)$$

by symmetry of φ

We define a point p to be ordinary if it is of order 3, where the order of p is defined to be the minimum order which a neighbourhood of p can possess. If p is not ordinary, it is said to be singular. An elementary point has a right and left neighbourhood of order 3. A point p is regular if $a_i(p) = 1$ for $i = 0,1,2$ and an inflection if $a_0(p) = a_1(p) = 1$ and $a_2(p) = 2$.

<u>5.4.3 Lemma</u>: An ordinary point is regular.

<u>Proof</u>: Use <u>2.6.1</u> and <u>5.4.1</u> upon projecting from distinct points of $U(p)$ where p is ordinary and $U(p)$ has order 3.

<u>5.4.4 Theorem</u>: If (p,q) is of order 3, then A^k is continuous on $[p,q)^{k+1}$, $-1 \leq k \leq 2$.

<u>Proof</u>: The result holds trivially for $k = -1$ and 0. If $k = 1$, let $x = (p_0,p_1) \in [p,q)^2$ and take q_1,q_2 such that

$$p_i < q_1 < q_2 < q$$

for $i = 0,1$. Then

$$A^1(p_0',p_1') = A^2(q_1,p_0',p_1') \cap A^2(q_2,p_0',p_1')$$

for all $(p_0', p_1') \in [p,q)^2$. But $(A/q_i)^1(p_0',p_1')$ is continuous on $[p,q)^2$ by 2.6.2. Since

$$(A/q_i)^1(p_0',p_1') = A^2(q_i,p_0',p_1')$$

for $i = 0,1$, the result follows for $k = 1$.

For the proof of $k = 2$, see 2.6.2.

Corollary: If A is elementary, then A^k is continuous for $-1 \leq k \leq 2$.

CHAPTER VI

6.1 Arcs with tower

A set $\{ H_i | -1 \leq i \leq 3 \}$ of spaces is called a tower if $H_{-1} \subset H_0 \subset H_1 \subset H_2 \subset H_3$. An arc with tower is an arc A for which there exists a tower $\{ H_i \}$ such that

$$A_k(p) \cap H_{2-k} = \emptyset$$

for all $p \in J$, $-1 \leq k \leq 3$, (Figure VI.1).

In the study of arcs in a real projective plane, the most useful and important tool developed was the Barner arc. In $\underline{P}^3$, arcs with tower assume the central role in the theory resulting in a number of simplifications.

6.1.1 Lemma: Let A be a regular arc with tower in P^n, A_k continuous for $-1 \leq k \leq n \leq 3$. Let $p \in A$, then $\{ H_{-1}, H_0, H_0A(p), \ldots, H_{n-1}A(p) \}$ is a tower for each of the components of A determined by p.

Proof: Let $n = 1$. Since $H_0 \not\in A(p)$ for $p \in J$ then $H_0A(p)$ meets A only at p and the result follows.

Let, $n = 2$. Consider $\tilde{A} = A/H_0$ an arc in $\underline{P}^1$. $H_0 \not\in A_1(q)$, therefore $\tilde{a}_0(q) = 1$ for all $q \in J$ and $\tilde{A}$ is regular. Put $\tilde{H}_k = H_{k+1}$, $-1 \leq k \leq 1$. Since

$$\tilde{A}_k(q) \cap \tilde{H}_{-k} = A_k(q)H_0 \cap H_{1-k} = H_0,$$

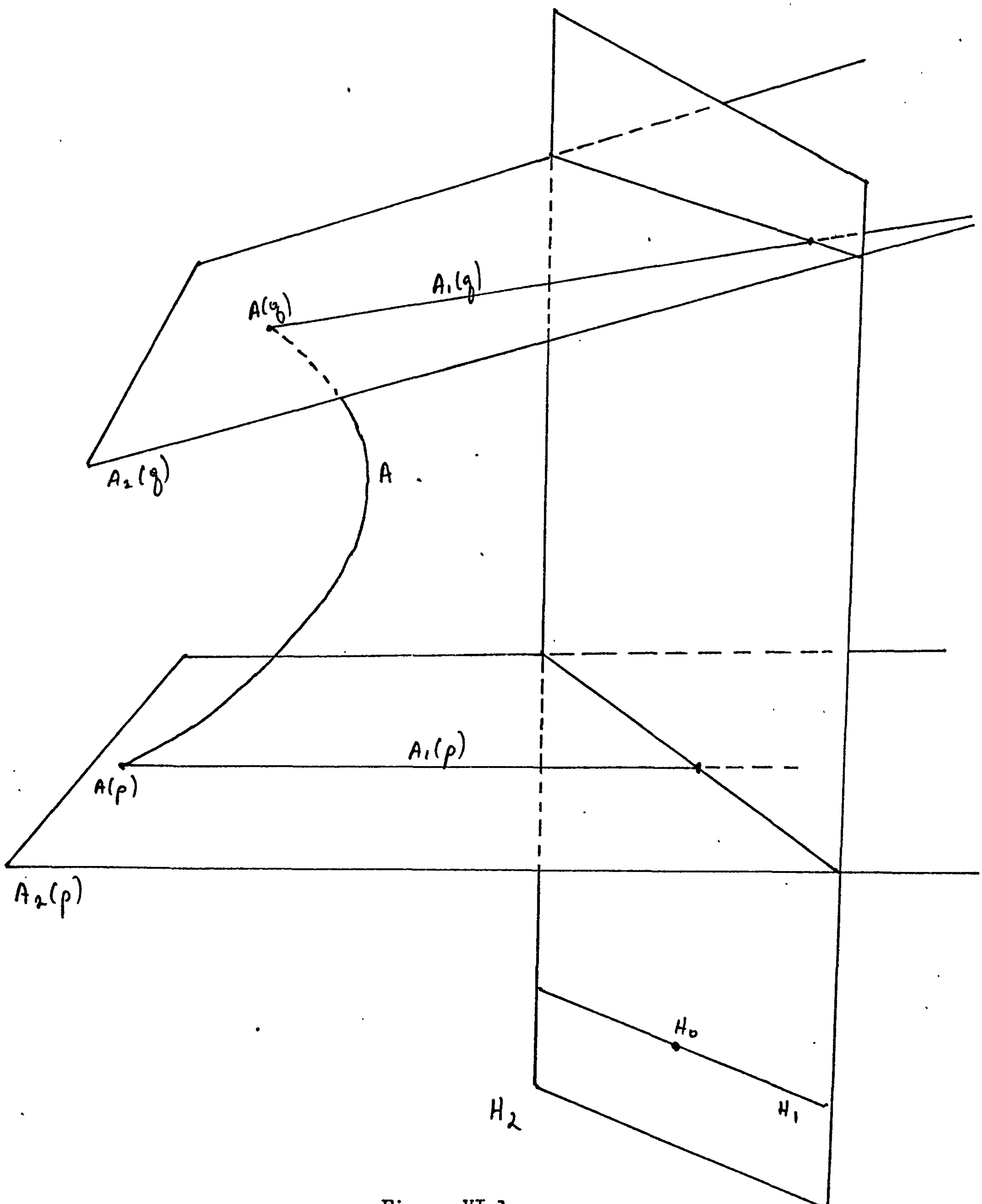


Figure VI.1

$\tilde{A}$ is an arc with tower $\{\hat{H}_k\}$. Therefore $\{\tilde{H}_{-1}, \tilde{H}_0, \tilde{H}_0 A(p)\}$ is a tower for each of the components of $\tilde{A}$ determined by p from n = 1. Since a regular arc in $\underline{P}^1$ is of order 1, if $q \neq p$ then $\tilde{A}(q) \neq \tilde{A}(p)$. Thus

$$A(q)H_0 \cap A(p)H_0 = \hat{A}(q) \cap \hat{A}(p) = H_0$$

and

$$A(q) \cap H_0 A(p) = \emptyset.$$

Hence $\{H_{-1}, H_0, H_0 A(p), H_1 A(p)\}$ is a tower for each component of A determined by p.

Let n = 3. Again consider $\tilde{A} = A/H_0$, $H_0 \not\subset A_2(q)$ for any $q \in J$, therefore $\tilde{A}$ is regular. Put $\tilde{H}_k = H_{k+1}$ for $-1 \leq k \leq 2$. By the same argument as above, $\hat{A}$ is an arc with tower $\{\hat{H}_k\}$ in $\underline{P}^2$. Since $\tilde{A}_1(q) = A_1(q)H_0$, $\tilde{A}_1$ is continuous and $\{\tilde{H}_{-1}, \tilde{H}_0, \tilde{H}_0 A(p), \tilde{H}_1 A(p)\}$ is a tower for each component of $\tilde{A}$ determined by p from n = 2. Taking $q \neq p$,

$$A(q)H_0 \cap H_1 A(p) = \hat{A}(q) \cap \hat{H}_0 A(p) = H_0$$

and hence

$$A(q) \cap H_1 A(p) = \emptyset.$$

By <u>4. 3.1</u>, $\hat{A}$ is of order 2 and by <u>2.4.1</u>, Corollary 3,

$$A_1(q)H_0 \cap H_0 A(p) = \hat{A}_1(q) \cap \hat{A}(p) = H_0.$$

Then

$$A_1(q) \cap H_0 A(p) = \emptyset,$$

and the result follows.

For the remainder of this section, we assume that A is a regular arc with tower and A_k continuous, $-1 \leq k \leq 2$.

6.1.2 Lemma: For any $P \in H_1 \setminus H_0$, there is at most one $p \in J$ such that $P \subset A_2(p)$.

Proof: Since $H_1 \cap A_1(p) = \emptyset$ for all $p \in J$, put

$$\varphi(p) = H_1 \cap A_2(p).$$

Since A_2 is continuous, so is φ and $\varphi(p) \neq H_0$ for all $p \in J$. Suppose there is $p_1 < p_2$ in J such that $\varphi(p_1) = \varphi(p_2) = p$, (Figure VI.2).

Since A_2 is continuous and $\varphi(p_1) = \varphi(p_2)$, there exists a $q \in (p_1,p_2)$ such that $\varphi((p_1,p_2))$ lies in one of the closed segments of H_1 with end points H_0 and $Q = \varphi(q)$. Call this segment S. Denote $\hat{A} = A/A_1(q)$. By 5.2.4 and the regularity of q, $\hat{a}_0(q) = a_2(q) = 1$. Since $\hat{A}$ is an arc in $\underline{P}^1$ taking $H_1 = \underline{P}^1$, we have $H_0 \not\subset \hat{A}$ and since $\hat{A}(q)$ cuts $\hat{A}$ at q, there is a point $r \in J$ such that $\hat{A}(r) \not\subset S$. But $\hat{A}(r) = A(r)A_1(q) \cap H_1$ and since $q \in (p_1,p_2)$ we may assume $r \in (p_1,p_2)$. Put $R = A(r)A_1(q) \cap H_1 \not\subset S$ and let $\breve{A} = A/R$.

Now $R \not\subset S$ implies $R \not\subset A_2(q)$ for any $q \in (p_1,p_2)$ and hence (p_1,p_2) is regular on $\breve{A}$. Moreover $\breve{A}$ is an arc with tower $\{R, H, H_2, H_3\}$ in $\underline{P}^2$. Since $\breve{A}_k(p) = A_k(p)R$ for $p \in (p_1,p_2)$, $\breve{A}_k$ is continuous for $-1 \leq k \leq 2$. Hence by 4.3.1, (p_1,p_2) is of order 2 on $\breve{A}$. But

$$\breve{A}(r) = A(r)R \subset A_1(q)R = \breve{A}_1(q),$$

a contradiction by 2.4.1.

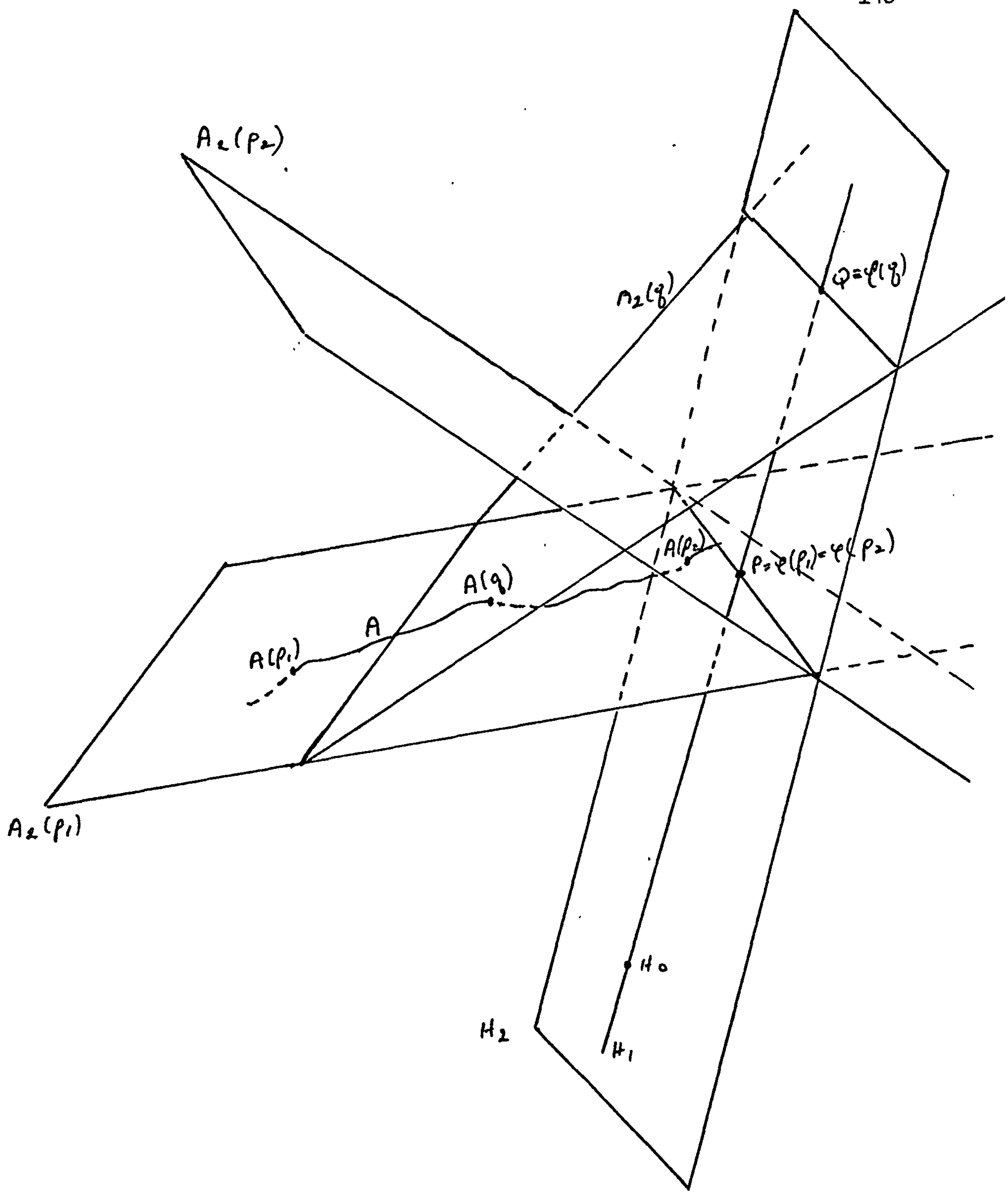


Figure VI.2

6.1.3 Lemma: If $p < q$ then $A(p) \not\subset A_2(q)$.

Proof: Put $P = A_2(q) \cap H_1$. By 6.1.2, $P \varepsilon A_2(r)$ iff $r = q$. Hence $X = \{ r/r < q \}$ is regular on $\tilde{A} = A/P$. Now $\{P_1, H_1, H_2, H_3\}$ is a tower for X on $\tilde{A}$ and since $\tilde{A}_k = A_k P$, $\tilde{A}_k$ is continuous for $-1 \leq k \leq 1$. Therefore X has order 2 by 4.3.1 and the result follows from 2.4.1.

6.1.4 Theorem: If A is a regular arc with tower and A_k is continuous, $-1 \leq k \leq 2$, then A is of order 3.

Proof: Assume A is not of order 3, hence there are points $p_0 < p_1 < p_2 < p_3$ such that they are coplanar. Let $X = \{ p/p > p_0 \}$. By 6.1.3, $A(p_0) \not\subset A_2(p)$ for any $p \varepsilon X$, hence X is regular on $\tilde{A} = A/p_0$. By 6.1.1, $\{ H_{-1}, \ldots, H_2 A(p_0) \}$ is a tower for X on A, therefore $\{A(p_0), H_0 A(p_0), H_1 A(p_0), H_2 A(p_0) \}$ is a tower for X on $\tilde{A}$. Since $\tilde{A}_k(p) = A_k(p)\, A(p_0)$, $\tilde{A}_k$ is continuous for $-1 \leq k \leq 2$ and X has order 2 on $\tilde{A}$ by 4.3.1. But $p_1 < p_2 < p_3$ are collinear on X, a contradiction.

6.2 Finiteness

A point p of an arc is right finite (strong right finite) if for every $(2-k)$-space L, $0 \leq k \leq 2$, there is a $U^+(p)$ such that no osculating k-space (no k-secant) of $U^+(p)$ meets L. The point p is finite [strongly finite] if it is both right and left finite [both strong right and strong left finite], where left finiteness (strong left finiteness) is defined similarly. An arc is finite (strongly finite) if each of its points is finite (strongly finite).

6.2.1 Lemma: A finite (strongly finite) point p is finite (strongly finite) on any projection A/P.

6.2.2 Theorem: An elementary arc A is strongly finite.

Proof: Let p be a point on the arc. If L is a plane, by 1.3.1 there exists $U^+(p)$ such that $L \cap A(U^+(p)) = \emptyset$. Hence we need only consider points and lines in $\underline{P}^3$.

Let P be a point in $\underline{P}^3$ and $U^+(p)$ be of order 3. If $P = A(p)$, no 2-secant of $U^+(p)$ contains P by 5.4.1. Let $P \neq A(p)$ and $L = PA(p)$.

Since $U^+(p)$ has order 3, then $U^+(p)$ has order 2 on $\hat{A} = A/P$ by 5.4.1. Then p will be elementary on A and thus p is strongly finite on A by 4.2.7, and there exists $U_1^+(p) \subset U^+(p)$ such that no 1-secant of $U_1^+(p)$ meets L. Put

$$\varphi(x) = A^2(x) \cap L$$

for $x \in (U_1^+(p))^3$, φ is monotone by 5.4.2. If $q \in U_1^+(p)$ then (Figure VI.3)

$$\lim_{r \to p^+} \varphi(q,q,r) = A(p).$$

Since $P \neq A(p)$, there is a $U_2^+(p) \subset U_1^+(p)$ such that $\varphi(x) \neq P$ for all $x \in (U_2^+(p))^3$. Hence, no 2-secant of $U_2^+(p)$ contains P.

Let L be a line and choose a point $P \in L$. By the above, choose $U_2^+(p)$ such that no 2-secant of $U_2^+(p)$ contains P. Suppose there is a plane H through P which meets $U_2^+(p)$ at points p_1, p_2, p_3. Since $U_2^+(p)$ is of order 3, it is 2-independent by 5.4.1 and

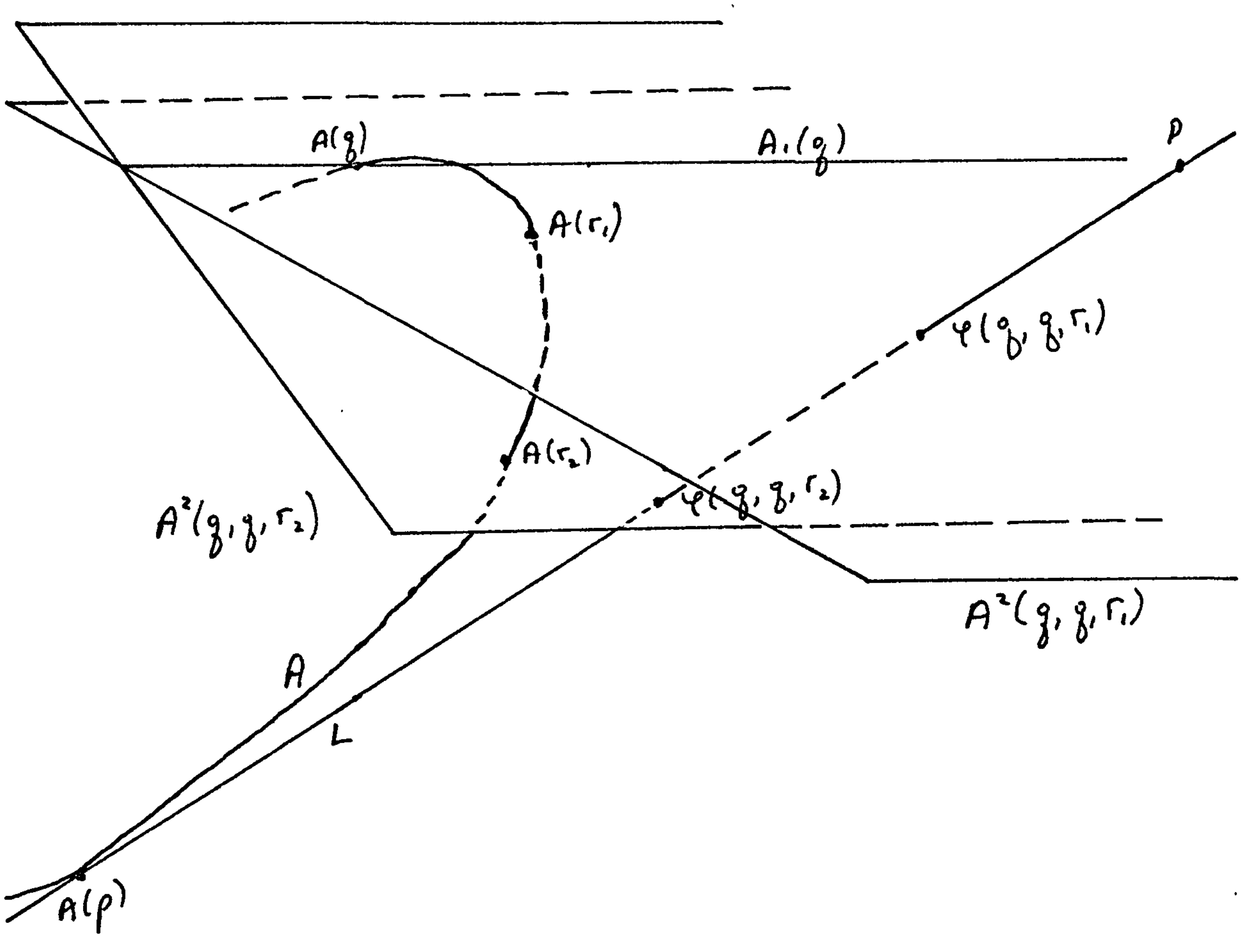


Figure VI.3

$H = A(p_1)A(p_2)A(p_3)$. Then H is a 2-secant of $U_2^+(p)$; a contradiction. Hence $U_2^+(p)$ has order 2 on A/P, it is thus elementary and strongly finite, by 4.2.7, on A/P. Therefore there exists $U_3^+(p) \subset U_2^+(p)$ such that no 1-secant of $U_3^+(p)$ on A/P meets L. Since $P \varepsilon L$, the result follows.

Corollary: An elementary point p of an arc A is elementary on any projection A/P.

Proof: Let P be an elementary point of A. Since P is elementary, it is strongly finite on A. Therefore there exists $U^+(p)$ $(U^-(p))$ of order 3 such that no 2-secant of $U^+(p)(U^-(p))$ meets P. Then as above, no plane through three points of $U^+(p)(U^-(p))$ meets P and the result follows.

6.2.3 Lemma: A regular elementary point p is ordinary.

Proof: Let $\{H_i\}$ be a tower of spaces such that $A_k(p) \cap H_{2-k} = \emptyset$ for $0 \leq k \leq 2$. By 6.2.2, there exist $U^+(p)$, $U^-(p)$ of order 3 such that

$$A_k(q) \cap H_{2-k} = \emptyset$$

for all $q \varepsilon U^+(p) \cup U^-(p)$. Then $U(p) = U^+(p) \cup \{p\} \cup U^-(p)$ is an arc with tower which is regular by 5.4.3 . A_k is continuous on $U(p)$ by 5.4.4, hence $U(p)$ is of order 3 by 6.1.4.

Corollary: If p is an ordinary point or an elementary inflection, and $\pi(P,p) = 2$ then p is ordinary on A/P.

It should be noted that the 2 dimensional analogue of 6.2.2 was proved by Denk's Theorem.

6.2.4 Lemma: Let $p \in J$. Given $U^+(p)$ and a neighbourhood $U \subset \underline{P}^3$ of $A_k(p)$, there is a k-secant L of $U^+(p)$ with $L \in U(A_k(p))$, $0 \leq k \leq 2$.

6.2.5 Theorem: Let p be a strongly right finite point of an arc A. Then

$$A^k(p_0, \ldots, p_k) \longrightarrow A_k(p)$$

as $(p_0, \ldots, p_k) \longrightarrow (p, \ldots, p)$ where $p_i > p$, $i = 0, \ldots, k$.

Proof: Since $A^k(J^{k+1})$ is the set of all k-secants of A, we must show that given a neighbourhood $U(A_k(p))$ of $A_k(p)$, there is a $U^+(p)$ such that

$$A^k((U^+(p))^{k+1}) \subset U(A_k(p)).$$

Let $k = 1$. Let Q_1, Q_2 be points with $Q_i \not\in A_i(p)$ for $i = 1,2$ and $Q_1A_1(p) \neq Q_2A_1(p)$. Let $\hat{U}(Q_iQ_1(p))$ be a neighbourhood of $Q_iA_1(p)$ in A/Q_i such that $M_1 \cap M_2 \in U(A_1(p))$ if $M_i \in U(Q_iA_1(p))$; $i = 1,2$ (Figure VI.4).

By projection form Q_i and 4.2.4, there is $U^+(p)$ such that every 1-secant of $U^+(p)$ on A/Q_i is in $\hat{U}(Q_iA_1(p))$; $i = 1,2$. Hence if M is a 1-secant of $U^+(p)$ on A, then Q_iM is a 1-secant of $U^+(p)$ on A/Q_i; $i = 1,2$. But $M = Q_1M \cap Q_2M$ and the result follows.

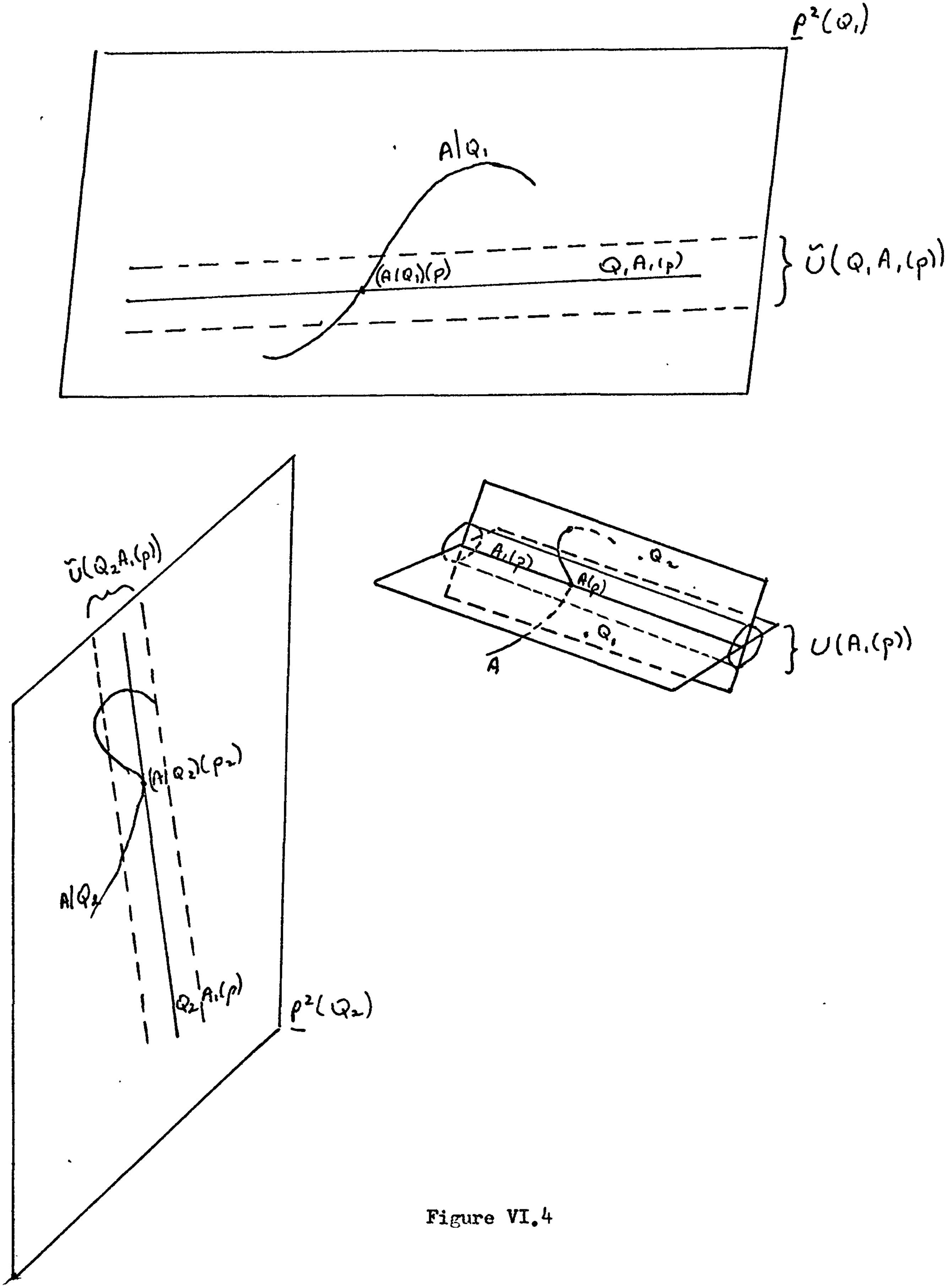

$\underline{P}^2(Q_1)$
$A|Q_1$
$\check{U}(Q_1A_1(p))$
$(A|Q_1)(p)$
$Q_1A_1(p)$
$A_1(p)$
Q_2
$A(p)$
Q_1
A
$U(A_1(p))$
$\check{U}(Q_2A_1(p))$
$(A|Q_2)(p_2)$
$A|Q_2$
$Q_2A_1(p)$
$\underline{P}^2(Q_2)$

Figure VI.4

Let $K = 2$. Let L be a line with $A_1(p) \cap L = \emptyset$. Put $P = A_2(p) \cap L$. Let $U(P)$ be a neighbourhood of P on L, with end points Q_1, Q_2. Let $U(A_1(p))$ be a neighbourhood of $A_1(p)$ such that $QM \in U(A_2(p))$ for all $Q \in U(P)$, $M \in U(A_1(p))$, (Figure VI.5). Since p is strongly finite, applying the case $k = 1$, take $U^+(p)$ such that no 2-secant of $U^+(p)$ contains Q_1 or Q_2 and every 1-secant of $U^+(p)$ is in $U(A_1(p))$. By 6.2.4, there is a 2-secant of $U^+(p)$ which meets $U(P)$. Since $U^+(p)$ is connected, every 2-secant of $U^+(p)$ meets $U(P)$ by 5.3.2.

6.2.6 Theorem: A finite arc with at most inflections is strongly finite.

6.2.7 Theorem: A regular finite arc with tower is of order 3.

Proof: By 6.2.6, a regular finite arc is strongly finite and A_k, $-1 \leq k \leq 3$ is continuous by 6.2.5. Hence it is of order 3 by 6.1.4.

Corollary: A regular finite arc is ordinary.

6.3 Behaviour of secants

In Chapter IV, we constructed a way of describing the real projective plane, depending only on a point p of an arc. The method consisted of choosing independent points from the osculating spaces of p and then the 2-simplices determined by these points formed a covering for the plane. We now extend this construction to $\underline{P}^3$.

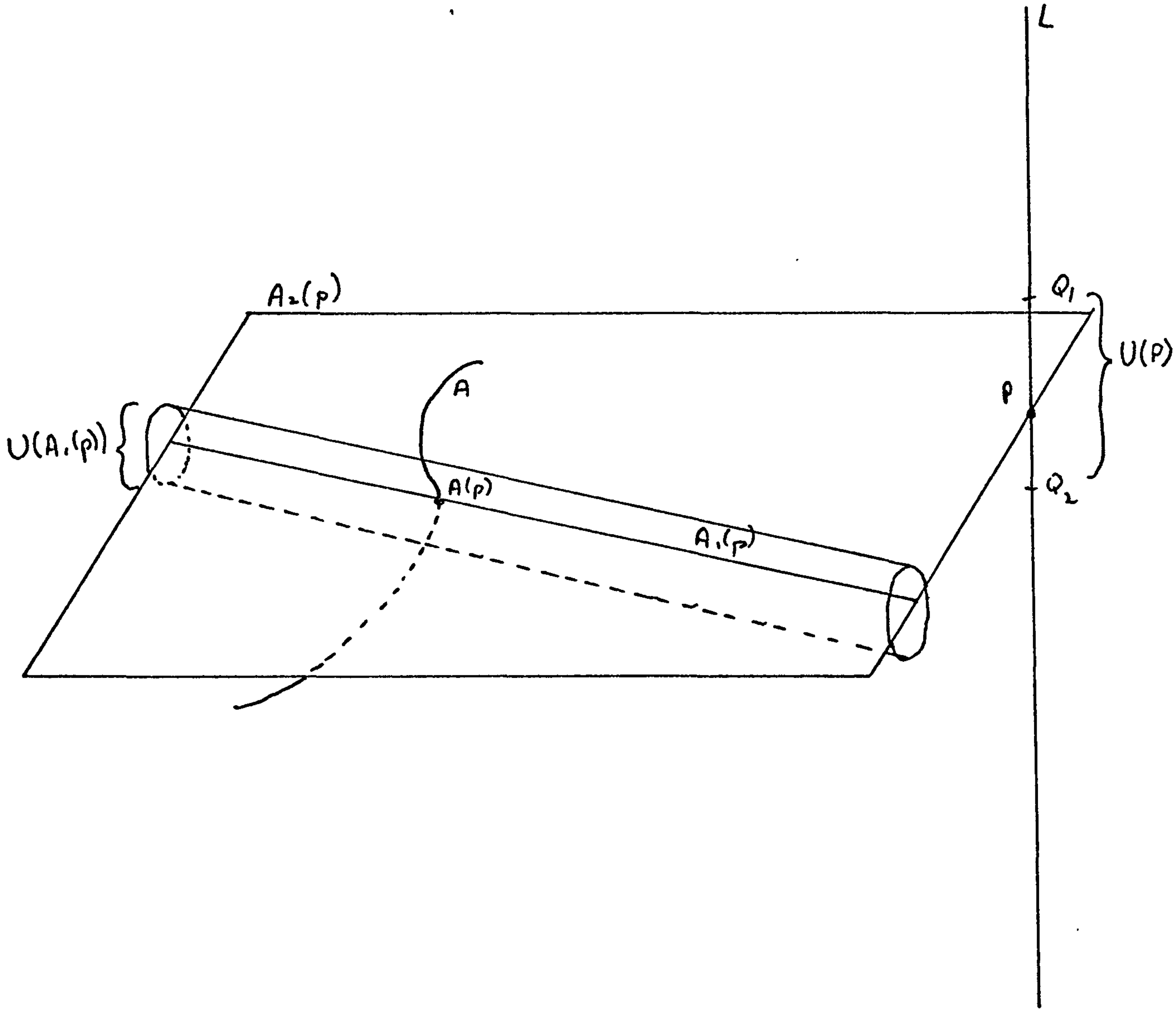


Figure VI.5

Let p be a point of an arc A. Let $P_i \in A_i(p) - A_{i-1}(p)$, $0 \leq i \leq 3$. Then $\{P_0, P_1, P_2, P_3\}$ are independent and the vertices of 9 open 3-simplices.

If $1 \leq i \leq 3$ and i is odd [even] let E_i^+ be the open segment of P_0P_i with end points P_0, P_i, (Figure VI.6), which is [is not] and edge of S^+, where S^+ is that open 3-simplex which contains some $U^+(p)$. Similarly E_i^- and S^- are defined using some $U^-(p)$.

Let p be a strongly right finite point of an arc. Choose $U^+(p)$ such that no k-secant of $U^+(p)$ meets a (2-k)-space spanned by the points P_i, $0 \leq i \leq 3$. Then the k-secants of $U^+(p)$ meet any (3-k)-space spanned by the P_i.

Consider the (3-k)-space $P_0P_{m(1)} \cdots P_{m(3-k)}$ where $0 < m(1) < \ldots < m(3-k) \leq 3$. Let $S^+(m(1), \ldots, m(3-k))$ be the open (3-k)-simple with vertices $P_0, P_{m(1)}, \ldots, P_{m(3-k)}$ which the k-secants of $U^+(p)$ meet. One can choose one open (3-k)-simplex since $U^+(p)$ is connected and thus the k-secants of $U^+(p)$ are connected.

<u>6.3.1 Theorem</u>: $E^+_{m(i)}$ is an edge of $S^+(m(1), \ldots, m(3-k))$ iff i is odd; $1 \leq i \leq 3-k$, $0 \leq k \leq 2$.

<u>Proof</u>: For $1 = 0$, $S^+(m(1), \ldots, m(s) = S^+$ and the result follows.

Let $k = 1$. First consider Figure VI.7. Here $m(i) = i$; $i = 1,2$. Projecting from P_1; P_0, P_2 and S^+ are projected onto $\hat{P}_0, \hat{P}_1$ and $\check{S}^+$. Put $M = E_2^+/P_1$, then $M \neq \hat{E}_1^+$ whereas $S^+(1,2)$ is projected on $\check{S}^+(1)$. Hence, E_2^+ is not edge of $S^+(1,2)$. Now

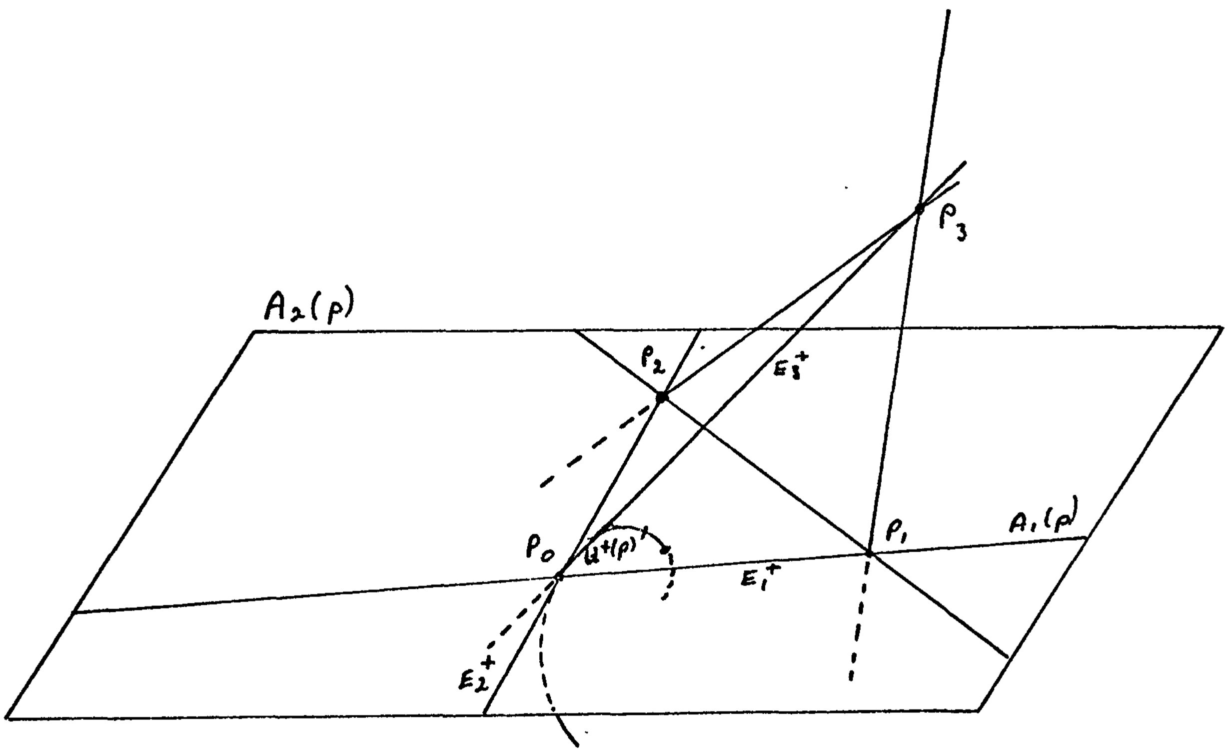


Figure VI.6

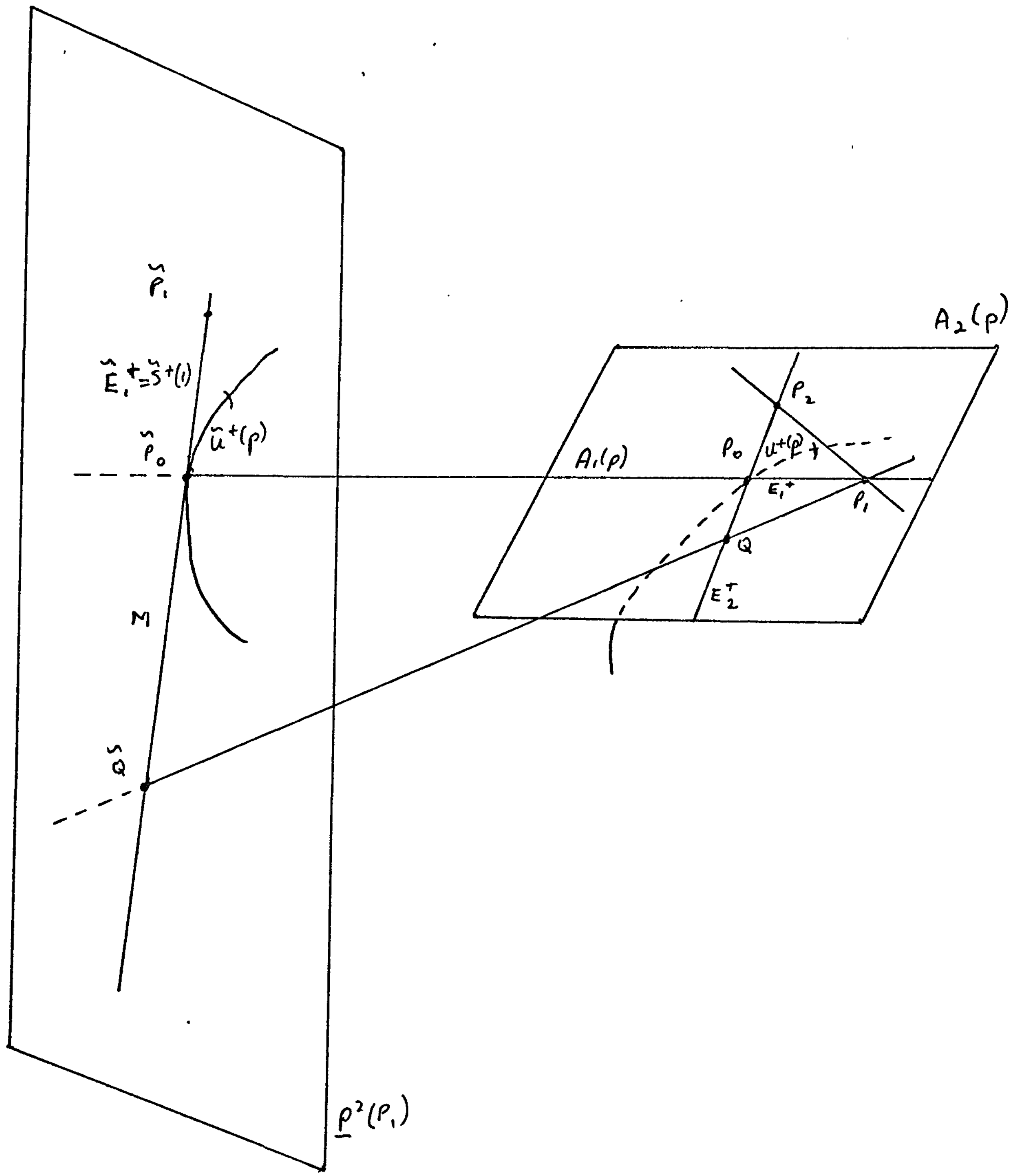

$\tilde{P}_1$
$\tilde{E}_1^+ = \tilde{S}^+(1)$
$\tilde{P}_0$
$\tilde{U}^+(\rho)$
M
$\tilde{Q}$
$\underline{P}^2(P_1)$
$A_2(\rho)$
$A_1(\rho)$
P_2
P_0
$U^+(\rho)$
E_1^+
P_1
Q
E_2^+

Figure VI.7

projecting from P_2 instead of P_1 we find that E_1^+ is an edge of $S^+(1,2)$.

More generally, the projections of $P_0, P_{m(2)}$ from $P_{m(1)}$ are $\breve{P}_0, \breve{P}_{m(2)-1}$ respectively. The projection of S^+ is $\hat{S}^+$, hence the projection of $E^+_{m(2)}$ is the open segment of $\breve{P}_0\breve{P}_{m(2)-1}$ different from $\hat{E}^+_{m(2)-1}$. But $S^+(m(1),m(2))$ is projected onto $\breve{S}^+(m(1),m(2))$ and hence $E^+_{m(2)}$ is not an edge of $S^*(m(1),m(2))$. By projecting from $P_{m(2)}$, we have that $E^+_{m(1)}$ is an edge of $S^+(m(1),m(2))$.

Let $k = 2$. We now have to show that $E^+_{m(1)} = S^+(m(1))$. Suppose $m(1) > 1$. Project A from $A(q)$ into the plane $P_0P_1P_2$, since $m(1) = 2$. From $k = 1$, $E_1^+(E_2^+)$ is (is not) an edge of $S^+(1,2)$. Also $\breve{S}^+ = S^+(1,2)$. Hence $\breve{E}_i^+ = E_i^+$, $i = 1,2$. Since $\breve{A}_1(q)$ meets $\breve{E}_1^+$ and $\breve{E}_2^+$, $A_2(q)$ meets E_1^+ and E_2^+. Thus $S^+(2) = E_2^+$.

6.4 Dually differentiable arcs

Let A be an arc in $\underline{P}^3$. Define a mapping $A^*: J \longrightarrow \underline{P}_0^{3*}$ by $A^*(p) = A_2(p)$ for all $p \in J$. We say A is dually differentiable if A^* is an arc in $\underline{P}^3$ and $A_k^* = A_{2-k}$, $-1 \leq k \leq 3$.

If A is dually differentiable then A_2 is continuous and if $P \in \underline{P}_0^3$ and $p \in J$, there exists $U^1(p)$ such that $P \notin A_2(q)$ for all $q \in U^1(p)$. Moreover, A is dually differentiable iff

$$A_k(p) = \lim_{q \to p} A_2(q) \cap A_{k+1}(p)$$

for all $p \in J$, $-1 \leq k \leq 2$.

Let p be a point of an arc A. Take $P_i, S^+, S^-, E_i^+, E_i^-$ as in Section <u>6.3</u>. If A is dually differentiable then by the above comment, there exists $U^1(p)$ such that $P_i \notin A_2(q)$ for all $q \in U^1(p)$, $i = 0,1,2$. Put $\sigma_k^+(p) = 0$ [$\sigma_k^+(p) = 1$] if $A_2(q)$ meets (does not meet] E_{k+1}^+ for all $q \in U^+(p)$, $0 \leq k \leq 2$. Also put $\sigma_{-1}^+(p) = 0$ and define $\sigma_k^-(p)$, $-1 \leq k \leq 2$, similarly.

<u>6.4.1 Lemma</u>: $\sigma_k(p) = 0$ iff $E_{k+1}^+ = E_{k+1}^-$, $0 \leq k \leq 2$.

<u>6.4.2 Theorem</u>: At any point of a dually differentiable arc,

$$\sigma_k^* = \sigma_{1-k}^+ + \sigma_{1-k}^- + \sigma_{1-k} + \sigma_2^+ + \sigma_2^- + \sigma_2 \pmod 2,$$

$0 \leq k \leq 2$.

<u>Corollary</u>: $$a_k^* = a_{2-k} + \sigma_{1-k}^+ + \sigma_{1-k}^- + \sigma_{2-k}^+ + \sigma_{2-k}^- \pmod 2,$$

$0 \leq k \leq 2$.

<u>6.4.3 Lemma</u>: Let p be a strongly right finite point of an arc A. Let $P_i \in A_i(p) - A_{i-1}(p)$, $0 \leq i \leq 3$. Then

$$\lim_{q \to p^+} A_2(q) \cap P_k P_{k+1} = P_k, \quad 0 \leq k \leq 2.$$

<u>Proof</u>: Let $U(P_k)$ be a neighbourhood of P_k on $P_k P_{k+1}$. Take $U^+(p)$ such that no h-secants of $U^+(p)$ meets a (2-h)-space spanned by the points P_i, $0 \leq h \leq 2$ and $0 \leq i \leq 3$; moreover, no 2-secant of $U^+(p)$ contains an end point of $U(P_k)$.

Let L be a (1-k)-secant of $U^+(p)$. Then L does not meet any $k + 1 = 2-(1-k)$ space spanned by the P_i, hence

$$L \cap A_{k+1}(p) = \emptyset$$

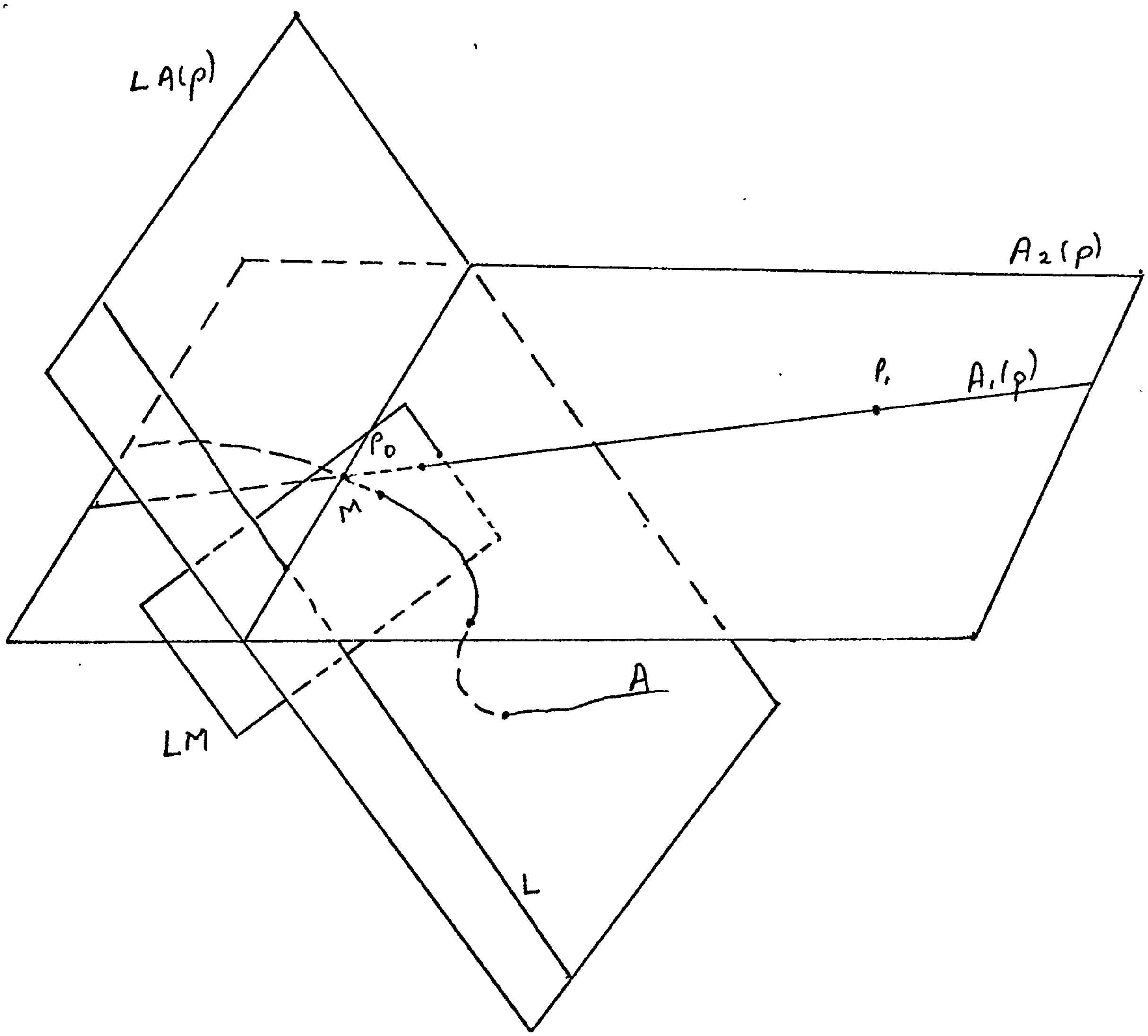


Figure VI.8

and $LA_k(p)$ is a plane. (See Fi~ure VI.8 for the case $k = 0$.) Now $LA_k(p) \cap P_k P_{k+1} = P_k$ otherwise $P_{k+1} \subset LA_k(p)$ and $L \cap A_{k+1}(p) \neq \emptyset$. Let $U(A_k(p))$ be a neighbourhood of $A_k(p)$ such that $L \cap M = \emptyset$ and LM meets $U(P_k)$ for all $M \varepsilon U(A_k(p))$. This is permissible by the continuity of A_k; cf. 6.2.5. By 6.2.4, there is a k-secant M of $U^+(p)$ such that $M \varepsilon U(A_k(p))$. Since LM meets $U(P_k)$, then all 2-secants of $U^+(p)$ meets $U(P_k)$ by 5.3.2 and the result follows.

6.4.4 Theorem: A strongly finite arc is dually differentiable and

$$a_k^*(p) = a_{2-k}(p)$$

for all $p \varepsilon J$, $0 \leq k \leq 2$.

6.4.5 The Scherk-Derry duality theorem: The dual A^* of an arc of order 3 is also of order 3.

Proof: A is of order 3, hence elementary and strongly finite by 6.2.2. Therefore A is dually differentiable by 6.4.4. By 5.4.3 A is regular and since $a_k^*(p) = a_{2-k}(p)$ for all $p \varepsilon J$, $0 \leq k \leq 2$, A^* is also regular. By 5.4.4, $A_k^* = A_{2-k}$ is continuous for $0 \leq k \leq 2$. Therefore by 5.4.1, for any $p,q \varepsilon J$, $\{ A_i(p) \}$ is a tower for (p,q) on A. Hence $\{ A_i^*(p) \}$ is a tower for (p,q) on A^*. Then (p,q) has order 3 on A^* by 6.1.4 and hence A^* has order 3.

Corollary 1: The dual of an elementary arc is elementary.

Corollary 2: Let A be of order 3. For any point P there exists $p \varepsilon J$ such that $P \notin A_2(p)$.

Proof: If $P \varepsilon A_2(p)$ for all $p \varepsilon J$, then $A^*(p) \subset P$, a plane in $\underline{P}^{3*}$ for all $p \varepsilon J$; a contradiction by 1.3.1.

We have already seen in 6.2.3 that a regular elementary point is ordinary. The question then is whether there eixst any elementary singular points. In 4.4.4, it was shown that if a finite arc in $\underline{P}^2$ has a singularity then it has an elementary singularity. This result can be extended to $\underline{P}^3$ using the proof of 4.4.4 with dimensional modifications. But to be able to do this, we must first consider dual projections from a line in $\underline{P}^{3*}$.

6.4.6 Lemma: Let p be a point of a strongly finite arc. Let $P_i \varepsilon A_i(p) - A_{i-1}(p)$; $0 \leq i \leq 3$. The dual projection of A into the line $P_k P_{k+1}$ has the characteristic $(a_k(p))$ at p, $0 \leq k \leq 2$.

Proof: Since A is strongly finite, it is dually differentiable and $a_k^* = a_{2-k}$; $k = 0,1,2$. Keeping in mind that the line $P_k P_{k+1}$ in A is also a line $(P_k P_{k+1})^*$ in A^*, we wish to project A^* from $(P_k P_{k+1})^*$. In the following, we shall use 5.2.4 as a reference. Denote $\hat{A}^* = A^*/(P_0P_1)^*$.

If $k = 0$, then $(P_0P_1)^* = (A_1(p))^* = A_1^*(p)$ and $\hat{a}_0^*(p) = a_2^*(p) = a_0(p)$.

If $k = 1$, $(P_1P_2)^* \not\subset A_2^*(p)$ since $A(p) \not\subset P_1P_2$. But $(P_1P_2)^* \cap A_2^*(p) = P_0P_1P_2 = A^*(p)$, hence $\hat{a}_0^*(p) = a_1^*(p) = a_1(p)$.

If $k = 2$, $A^*(p) \not\subset (p_2p_3)^*$ since $P_2P_3 \not\subset A_2(p)$. Moreover, $(P_2P_3)^* \quad A_1^*(p) = p_2p_3 \; A_1(p) = A_3(p) = \emptyset^*$. Hence $\hat{a}_0^*(p) = a_0^*(p) = a_2(p)$.

<u>6.4.7 Theorem</u>: If a finite arc has a singularity it has an elementary singularity.

Let L be a plane. Put

$$\mu(p,L) = \beta_{\delta(p,L)}(p)$$

for all $p \in J$. We call $\mu(p,L)$ the <u>multiplicity</u> with which L meets A at p.

<u>6.4.8 Theorem</u>: Let A be an elementary arc with tower. Then

$$\sum_{p \in J} \mu(p,L) \leq \sum_{p \in J} \sum_{i=0}^{2} (a_i(p)-1)+3$$

for every plane L.

<u>6.4.9 Denk's Theorem</u>: Let p be an elementary point, the

$$\mathrm{Ord}(p) = \sum_{i=0}^{2} a_i(p).$$

CHAPTER VII

7.1 Barner Arcs

An arc is Barner if there exists a continuous mapping

$$B: J^2 \longrightarrow \underline{P}_2^3$$

such that

$$\delta(p, B(x)) = \beta_{\gamma(p,x)}(p) - 1$$

for all $p \in J$, $x \in J^2$.

7.1.1 Lemma: Let A be an arc such that for each $p \in J$, $a_i(p) = 2$ for at most one i; $0 \leq i \leq 2$. If there exists a point P such that A/P is of order 2 and

$$\pi(P,p) = \begin{cases} 2 & \text{if } p \text{ regular} \\ i & \text{if } a_i(p) = 2 \end{cases}$$

for all $p \in J$, then A is a Barner arc.

7.1.2 Theorem: Let A be an arc with tower and with at most inflections. Let A_k be continuous, $0 \leq k \leq 2$. Then A is a Barner arc.

Proof: Let $\{H_i\}$ be a tower. Since $H_0 \not\subset A_2(q)$ for any $q \in J$. $\breve{A} = A/H_0$ is a regular arc with tower $\{H_0, H_1, H_2, H_3\}$. Moreover $\breve{A}_k(p) = A_k(p)H_0$ is continuous for $k = 1$. Then $\breve{A}$ is of order 2 by 4.3.1, and A is Barner by 7.1.1

It should be noted that any (p,q) of order 3 where $p,q \in J$ is Barner by 7.1.1.

For the remainder of this section we assume A is a Barner arc unless otherwise stated.

7.1.3 $A(p) \in B(p_1,p_2)$ iff $p \in \{p_1,p_2\}$

7.1.4 $\delta(p,B(p,p)) = B_1(p)-1$ for all $p \in J$.

7.1.5 $A_1(p) \subset B(p,p)$ for all $p \in J$.

7.1.6 If $p \neq q$ then $A(p) \not\subset A_1(q)$; hence A is simple; that is, A is 1-1.

7.1.7 If $p \neq q$ and $A(p) \subset A_2(q)$, then $B_1(q) = 2$.

7.1.8 Lemma: Let $q \in J$. If $a_0(q) = 1$ then A/q is a Barner arc. If $a_0(q) = 2$, then the restriction of A/q to either component of $J - \{q\}$ is a Barner arc.

Proof: Let $y_1 \in J$ be given and put $x = (q,y_1)$. Then put $\hat{B}(y_1) = B(x)$ in $\hat{A} = A/q$ and hence $\hat{B}$ is continuous. Let $a_0(q) = 1$.

Case 1. If $p = q$ then $\pi(q,p) = -1$ and by 5.2.3, $\hat{\beta}_k(p) = \beta_{k+1}(p) - \beta_0(p)$. Since $\gamma(p,y_1) = \gamma(p,x) -1$ and $\beta_0(p) = a_0(p) = 1$,

$$\hat{\beta}_{\gamma(p,y_1)}(p) = \beta_{\gamma(p,y_1)+1}(p) -1$$

$$= \beta_{\gamma(p,x)}(p) -1.$$

by 5.2.1, $\hat{\delta}(p, \breve{B}(y_1)) = \delta(p,B(x)) - 1$, hence

$$\breve{\delta}(\breve{p},B(y_1)) = (\beta_{\gamma(p,x)}(p)-1) - 1$$
$$= \breve{\beta}_{\gamma(p,y_1)}(p) - 1.$$

Case 2. If $p \neq q$ and $A(q) \subset A_2(p)$. Then $\pi(q,p) = 1$ by 7.1.6. But $p \neq q$ implies

$$\gamma(p,y_1) = \gamma(p,x) < 1.$$

Therefore

$$\delta(p,B(x) = \beta_{\gamma(p,x)} - 1 < \beta_1(p) - 1.$$

and since $A(q) \subset A_2(p)$, $\beta_1(p) = 2$ by 4.1.7. Then

$$\delta(p,B(x)) < 2 - 1 = 1$$

and $\breve{\delta}(p,\hat{B}(y_1)) = \delta(p,B(x))$ by 5.2.1. Hence

$$\breve{\delta}(p,\breve{B}(y_1)) = \delta(p,B(x))$$
$$= \beta_{\gamma(p,x)}(p) - 1$$
$$= \breve{\beta}_{\gamma(p,y)}(p) - 1.$$

Case 3. If $p \neq q$ and $A(q) \not\subset A_2(p)$, then $\pi(q,p) = 2$ and the result follows as in Case 2.

Let $a_0(q) = 1$ and $p = q$. Then

$$\breve{\beta}_{\gamma(p,y_1)}(p) = \beta_{\gamma(p,y_1)+1}(p) - \beta_0(p)$$
$$= \beta_{\gamma(p,x)}(p) - 2.$$

Therefore

$$\breve{\delta}(p,\breve{B}(y)) = \delta(p,B(x))-1$$
$$= (\beta_{\gamma(p,x)}(p)-1)-1$$
$$= \breve{\beta}_{\gamma(p,y_1)}(p)$$
$$\neq \breve{\beta}_{\gamma(p,y_1)}(p)-1.$$

<u>7.1.9 Lemma</u>: A 1-space L meets a Barner arc in at most two points.

<u>Proof</u>: If L meets A at points $p_1 < p_2 < p_3$. Then since A is simple; $A(p_1)$, $A(p_2)$ and $A(p_3)$ are distinct and collinear. Let $x = (p_1,p_2)$, then $A(p_i) \subset B(x)$ for $i = 1,2$. Hence $L = A(p_1)\ A(p_2) \subset B(x)$ and finally $A(p_3) \subset B(p_1,p_2)$, a contradiction.

Therefore a 1-secant of a Barner arc meets the arc at most twice. If $q \in J$, then $p \neq q$ implies $A(p) \not\subset A_1(q)$ for all $p \in J - \{q\}$ by <u>7.1.6</u>. Hence the result:

<u>7.1.10 Lemma</u>: If A is Barner then A is 1-independent.

<u>7.1.11 Lemma</u>: If $p \in J$ then $a_i(p) = 2$ for at most one i, $0 \leq i \leq 2$.

<u>Proof</u>: If $a_0(p) = 1$ and $a_1(p) = a_2(p) = 2$ then $\breve{A} = A/p$ is Barner by <u>7.1.8</u>, and $\hat{a}_0(p) = a_1(p) = 2 = a_2(p) = \breve{a}_1(p)$, a contradiction by <u>3.1.6</u>. Therefore $a_0(p) = 2$. If $a_0(p) = a_1(p) = 2$ then by <u>7.1.4</u>,

$$\delta(p,B(p,p)) = \beta_1(p)-1$$
$$= a_0(p) + a_1(p)-1$$
$$= 3$$

which is impossible since $B(p,p)$ is a plane. Then $a_0(p) = 2 = a_2(p)$ and $a_1(p) = 2$ is the only case possible.

But projecting from $A_1(p)$, $\tilde{\tilde{a}}_0(p) = a_2(p) = 2$ by 5.2.4 where $\tilde{\tilde{A}} = A/A_1(p))$. Then $\tilde{\tilde{A}}_0(p)$ supports $\tilde{\tilde{A}}$ at p and one obtains points p_1, p_2 such that $A(p_1) \quad A_1(p)\ A(p_2)$. But $a_0(p) = 2$ implies

$$\delta(p, B(p_2,p)) = B_0(p)-1 = 1$$

and hence $A_1(p) \subset B(p_2,p)$. Therefore

$$A(p_1) \subset A_1(p)\ A(p_2) \subset B(p_2,p)$$

a contradiction.

7.1.12 Lemma: If (p,q) is ordinary then $A(p) \not\subset A_2(q)$.

Proof: If $a_0(q) = 2$ then $A_2(q) = B(q,q)$ and the result follows.

Assume $a_0(q) = 1$ and $A(p) \subset A_2(q)$. Then $\tilde{A} = A/q$ is Barner and $\tilde{A}(p) \subset \tilde{A}_1(q)$. Therefore (p,q) is not ordinary on $\tilde{A}$ by 3.1.7. But (p,q) ordinary on A and not ordinary on $\tilde{A}$ implies there exists $p_1 \in (p,q)$ such that $A(q) \subset A_2(p_1)$ by the corollary of 6.2.2.

Now consider the interval (p_1,q) with $A(q) \subset A_2(p_1)$. Repeating the above argument one obtains $q_1 \in (p_1,q)$ such that $A(p_1) \subset A_2(q_1)$. Consider the set S of intervals $(r,s) \subset (q,q)$ such that $A(r) \subset A_2(s)$. Then by the argument of 3.1.7, one has a contradiction.

7.1.13 Theorem: If (p,q) is ordinary then $[p,q]$ has order 3.

<u>Proof</u>: Suppose p_1, p_2, p_3, p_4 are coplanar points with

$$p \leq p_1 < p_2 < p_3 < p_4 \leq q.$$

$p_2 \in (p,q)$, hence $a_0(p_2) = 1$ by <u>5.4.3</u> and $\hat{A} = A/p_2$ is Barner by <u>7.1.8</u>. p_2 ordinary implies there exists $U(p_2)$ of order 3 on A and hence of order 2 on $\hat{A}$ by <u>5.4.1</u>. Then p_2 is ordinary on $\hat{A}$. In fact, (p,q) is ordinary on $\hat{A}$ since $A(p_2) \not\subset A_2(r)$ for $r \in (p,q)$ by the Corollary of <u>6.2.3</u>. Therefore (p,q) has order 2 on $\hat{A}$ by <u>3.1.8</u>, a contradiction since p_2, p_3, p_4 are collinear on $\hat{A}$.

<u>7.1.14 Theorem</u>: If $p < q < r$ and (q,r) ordinary then

$$A(p) \not\subset A_2(q) \cap A_2(r).$$

<u>Proof</u>: Since (q,r) is ordinary then (q,q) has order 3 by <u>7.1.13</u> and A_k is continuous, $k = 1,2$ by <u>5.4.4</u>.

Assume $A(p) \subset A_2(q) \cap A_2(r)$, $p < q < r$. By <u>6.4.5</u>, Corollary 2, there exists $s_0 \in (q,r)$ such that $A(p) \not\subset A_2(s_0)$, (Figure VII.1). Since A_2 is continuous, then there exist q_0, r_0 such that $q \leq q_0 < s_0 < r_0 \leq q_0$, $A(p) \not\subset A_2(s)$ for all $s \in (q_0, r_0)$ and $A(p) \subset A_2(q_0) \cap A_2(r_0)$.

Now $\pi(p,s) = 2$ for $s \in (q_0, r_0)$ implies (q_0, r_0) is ordinary on $\hat{A} = A/p$. Since A is 1-independent, $A(p)$, $A(q_0)$ and $A(r_0)$ are not collinear and $M = A(p)\ A(q_0)\ A(r_0)$ is a plane. Put $L_1 = A(p)\ A(q_0)$ and $L_2 = A(p)\ A(r_0)$. If $S \in (q_0, r_0]$ then $A_1(S) \cap L_1 = \emptyset$. For if there exists $s^1 \in (q_0, r_0]$ such that $A_1(S^1) \cap L_1 \neq \emptyset$ then

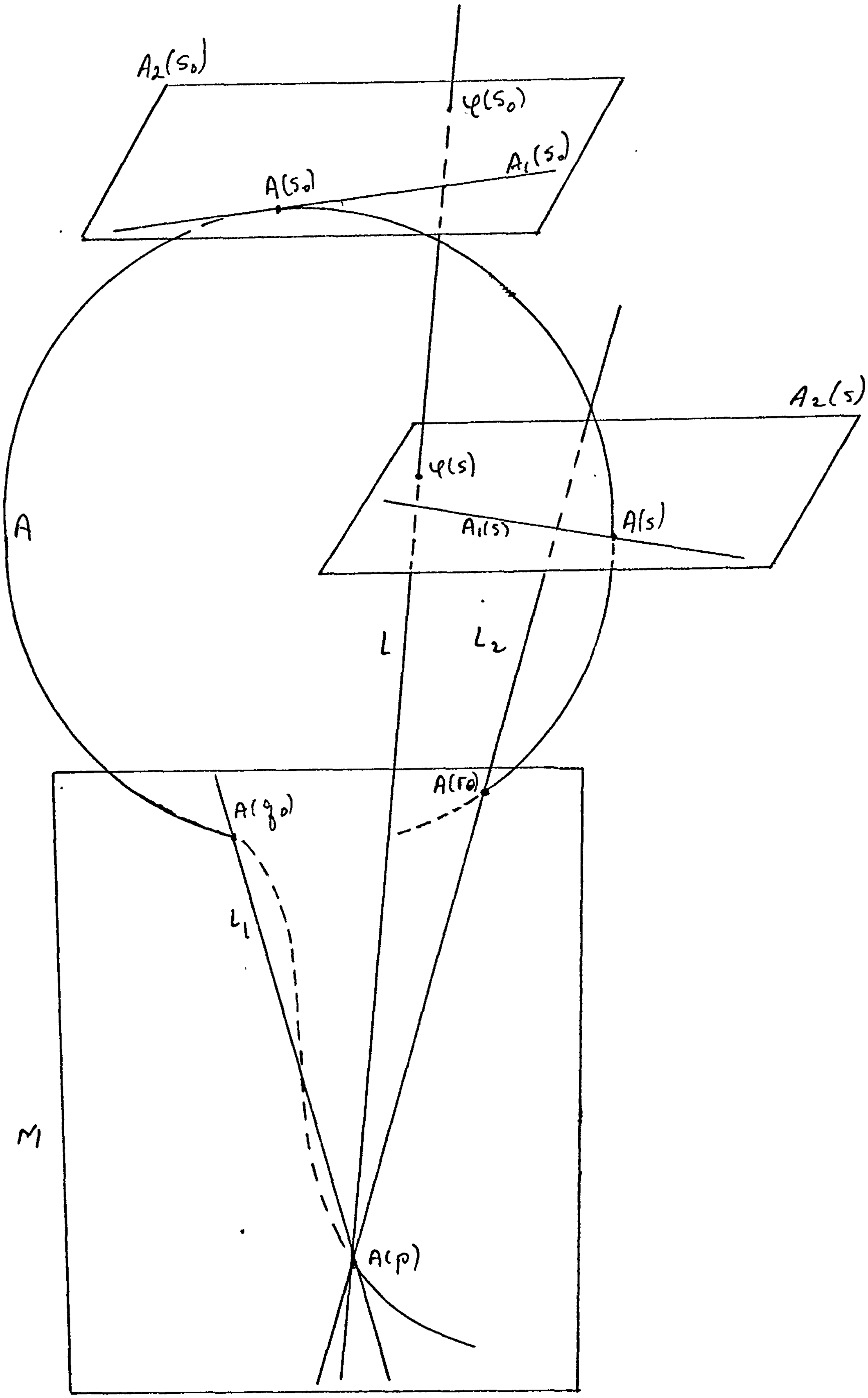


Figure VII.1

$\breve{A}(q_0) = L_1 \subset A_1(S^1)\ A(p_0) \subseteq \breve{A}_1(S^1)$ in $\breve{A}$, a contradiction by 3.1.7 since $(q_0, S^1) \subset (q_0, r_0)$ is ordinary on $\breve{A}$. Therefore $A_1(S) \cap M$ is a point for all $S \in [q_0, r_0]$ and does not lie on L_1 or L_2. Hence there exists a line L such that $L \cap A_1\,[q_0, r_0] = \emptyset$. Put

$$\varphi(S) = A_2(S) \cap L$$

for all $s \in [q_0, r_0]$. φ is continuous since A_2 is continuous and φ is monotone by 5.4.2. But $\varphi(q_0) = \varphi(r_0)$ and hence φ cannot exist.

7.2 Regular arcs

The aim of this section is to prove the following three theorems:

7.2.1 If A is a regular Barner arc and A_1 is continuous then A is of order 3.

7.2.2 Every arc has an ordinary point.

7.2.3 If A is a Barner arc with at most inflections and A_2 is continuous, then either A is of order 3 or A has an elementary singularity.

7.2.4 Lemma: Let A be a Barner arc with at most inflections. If A_2 is continuous at p, then so is A_1.

Proof: By 7.1.4, $\delta(p, B(p,p)) = B_1(p) - 1$ for all $p \in J$. Since A has at most inflections, $\beta_1(p) = 2$ and hence

$$A_1(p) = A_2(p) \cap B(p,p)$$

Therefore A_1 is continuous at p.

To make our considerations easier, we shall first show that every arc contains a Barner arc with at most inflections on which A_1 is continuous.

7.2.5 Lemma: Let A be an arc on which A_1 is continuous. Let W be the set of points of A at which A_2 is continuous. Then $W \neq \emptyset$ and W is dense in A.

7.2.6 Lemma: Every arc contains a Barner arc with at most inflections on which A_1 is continuous.

Proof: We may assume there exists a plane H_∞ not meeting A. Let P_1, P_2, P_3 be independent points. By 3.3.11, there exist subarcs X_1, X_2, X_3 such that $X_3 \subset X_2 \subset X_1$ and X_i has order 2 on A/P_i, $i = 1,2,3$. Then X_3 has at most inflections since for any $p \in X_1$ there exists j with $p_j \not\in A_2(p)$. This follows from the fact the $\{P_i\}$ are independent. But $P_j \not\in A_2(p)$ implies

$$\tilde{a}_i^j(p) = a_i(p)$$

for $i = 0,1$, where $\tilde{A}^{(j)} = A/P_j$. Now X_3 has order 2 on $\tilde{A}^j$ and hence is regular on $\tilde{A}^j$ by 2.6.1. Therefore $a_0(p) = a_1(p) = 1$ for any $p \in J$.

But X_3 having at most inflections implies $P_i \not\subset A_1(X_3)$; $i = 1,2,3$; otherwise $\pi(P_j, p) = 0$ for some j and $\tilde{a}_0^{(j)}(p) \equiv a_0(p) + a_1(p) \equiv 0 \pmod 2$. This is a contradiction since $\tilde{A}^{(j)}$ is regular. Then

$$A_1(p) = \bigcap_{i=1}^{3} \tilde{A}_1^{(i)}(p) = \bigcap_{i=1}^{3} A_1(p)P_i$$

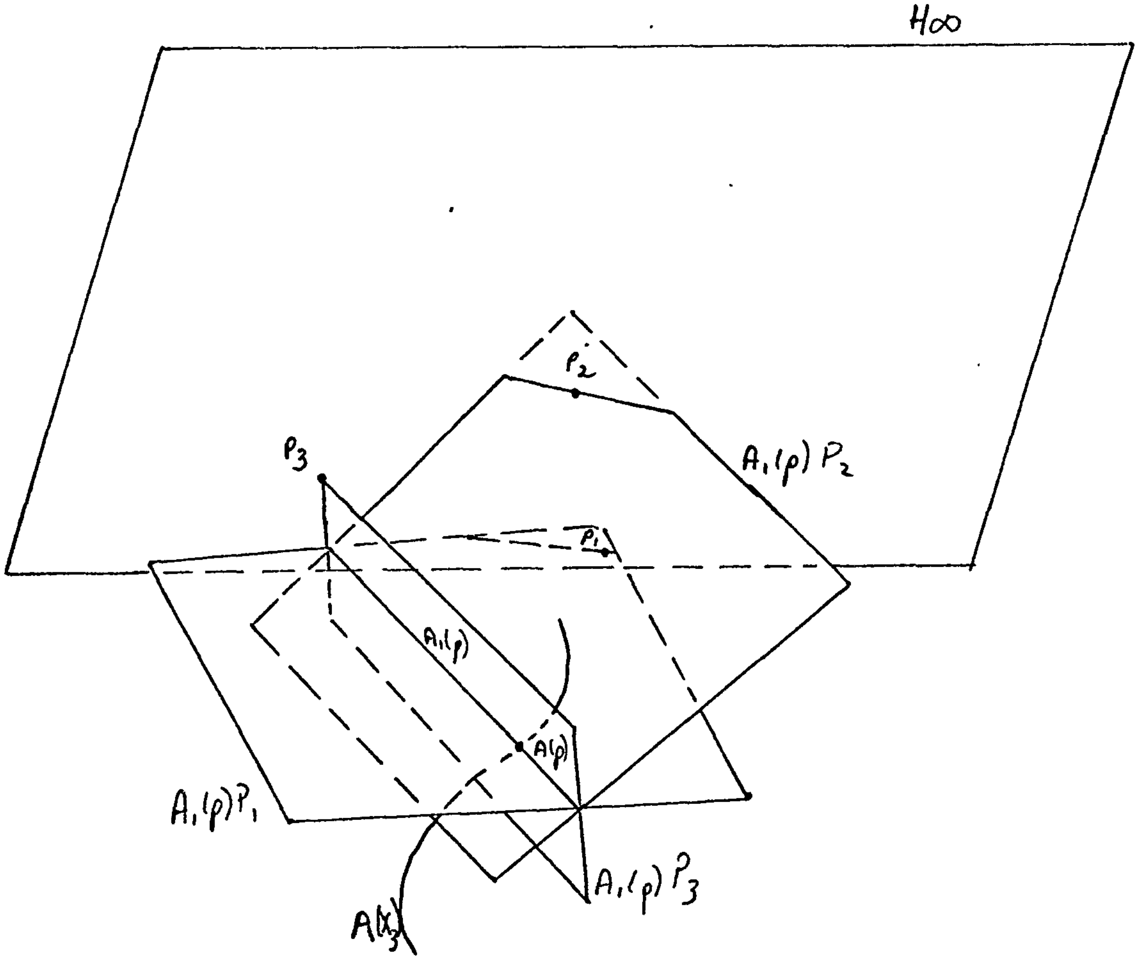


Figure VII.2

(Figure VII.2), and thus A_1 is continuous on X_3.

By 7.2.5, there exists $p \in X_3$ such that A_2 is continuous at p. Take a point P and U(p) such that $P \not\subset A_2(q)$ for all $q \in U(p)$. By 3.3.11, there exists $X \subset U(p)$ such that X has order 2 on $\tilde{A} = A/P$ and hence X is Barner by 7.1.1.

For the remainder of this chapter we assume that A is a Barner arc with at most inflections and that A_1 is continuous.

If H_∞ is a plane and P,Q are distince points not on H_∞, then $L_\infty(L_f)$ will denote the segment of the line PQ determined by P,Q which does (does not) contain $L \cap H_\infty$.

7.2.7 Lemma: Suppose (p,q) is of order 3 and H_∞ is a plane not meeting [p,q]. Put $L = A(p)A(q)$. Then $A_2(r) \cap L \in L_f$ for all $r \in (p,q)$.

Proof: Since ordinary points are regular, for every $r \in (p,q)$, $A_2(r)$ cuts (p,q) at r. Since (p_1,q) is of order 3, $A_2(r)$ does not meet [p,q] outside of r by 5.4.1. Then A(p) and A(q) lie in different open half-spaces determined by $A_2(r)$ and H_∞, and the result follows.

7.2.8 Lemma: Suppose $p < q < r$. Let H_∞ be a plane not meeting [p,r]. Put $L = A(p)A(q)$. If (q,r) is of order 3 then $A_2(s)$ meets L_f for all $s \in (q,r)$.

Proof: See 3.2.2 and Figure VII.3.

7.2.9 Lemma: Suppose A is a regular arc and there exist $p < q$, q ordinary such that $A(p) \subset A_2(q)$. Then there exist r, s, with s singular, such that $A_2(s)$ cuts A at r.

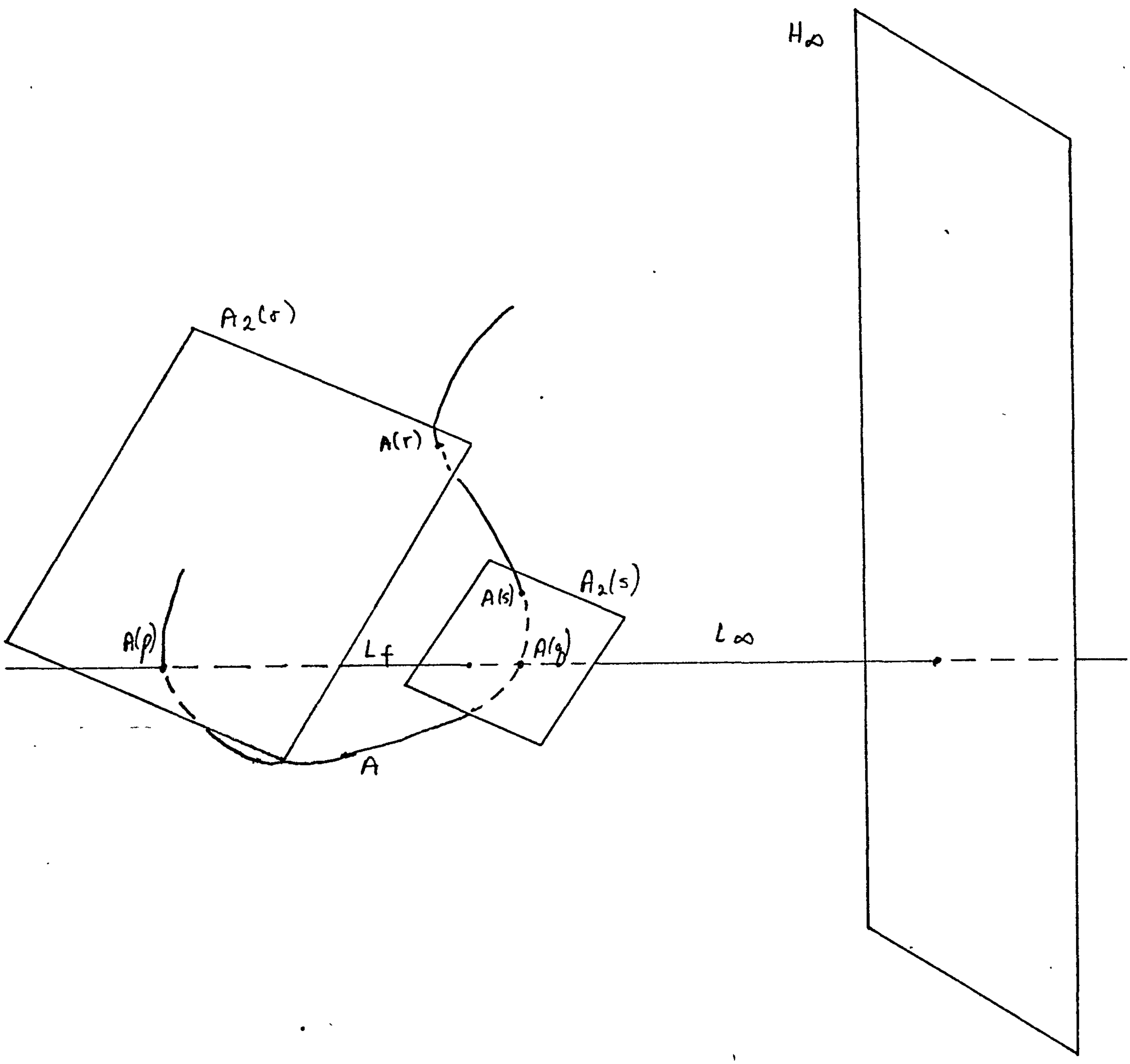


Figure VII.3

Proof: Take s as small as possible such that (s,q) is ordinary. Then $p < s < q$ and s is singular by 7.1.12. Hence (s,q) is of order 3 by 7.1.13. Let H_∞ be a plane which does not meet [p,q] and put $P = A(s)A(q)$, $S = A(p)A(q)$, $Q = A(p)A(s)$, (Figure VII.4). Now for any $t \varepsilon (s,q)$, $A_2(t)$ meets P_t by 7.2.7 and Q_f by 7.2.8; thus $A_2(t)$ meets S_∞, (Figure VII.5).

Since $A(p) \subset A_2(q)$ then $A(p) \not\subset A_2(s)$ by 7.1.14. Moreover (s,q) is ordinary and hence $A(q) \not\subset A_2(s)$ by 7.1.12. But then $A_2(s)$ meets S_∞ and in fact cuts at s, since s is regular. Since $A_2(s)$ does not meet (s_1,q), there exists $r \varepsilon (p,s)$ such that $A_2(s)$ cuts A at r.

7.2.10 Lemma: If A is not of order 3 then there exist p,q with $p < q$ such that $\delta(p,A_2(q)) = 0$.

Proof: A not of order 3 implies that there exist four distinct coplanar points $p_1 < p_1 < p_2 < p_3$. Consider $\hat{A} = A/p$. A has at most inflections, therefore $a_0(p) = 1$ and $\hat{A}$ is Barner. Moreover A is Barner and $A(p) \not\subset A_1(q)$, hence $\hat{A}_1(q) = A_1(q)A(p)$ for all $q \varepsilon (p_1p_3)$ by 7.1.6, $\hat{A}_1(q) = A_1(q)A(p)$ implies $\hat{A}_1$ is continuous on (p,p_3).

Since $\hat{a}_0(p) = a_1(p) = 1$ and $\hat{a}_1(p) = a_2(p)$, p is at most an inflection on $\hat{A}$. If $p \neq q$ then $\pi(p,q) \geq 1$ and $\hat{a}_0(q) = a_0(q) = 1$. Hence $\hat{A}$ has at most inflections. Then (p,p_3) contains at least one inflection on A; otherwise (p,p_3) is regular and hence of order 2 on $\hat{A}$ by 3.2.7; a contradiction since $p_1 < p_2 < p_3$ are collinear on $\hat{A}$.

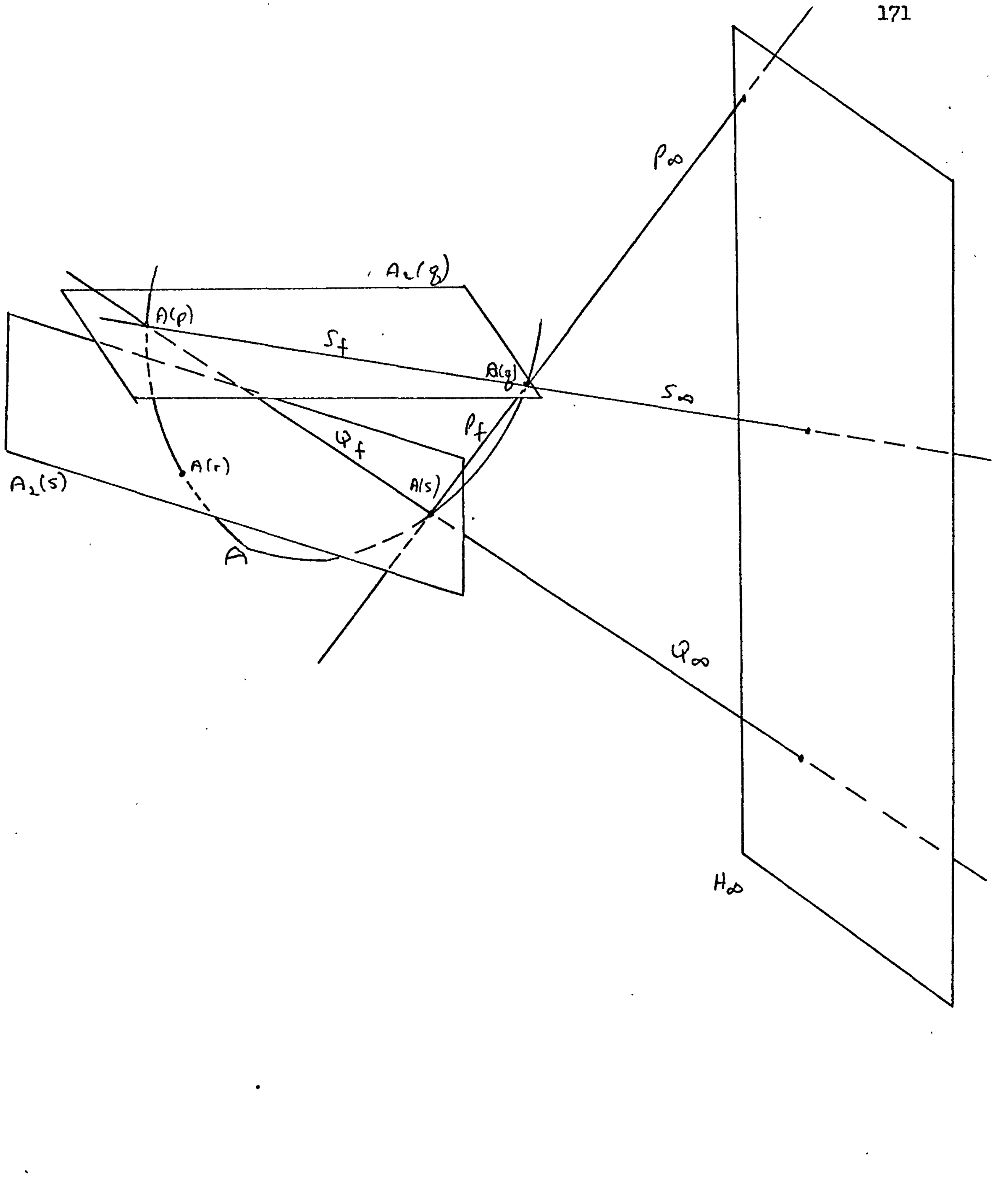


Figure VII.4

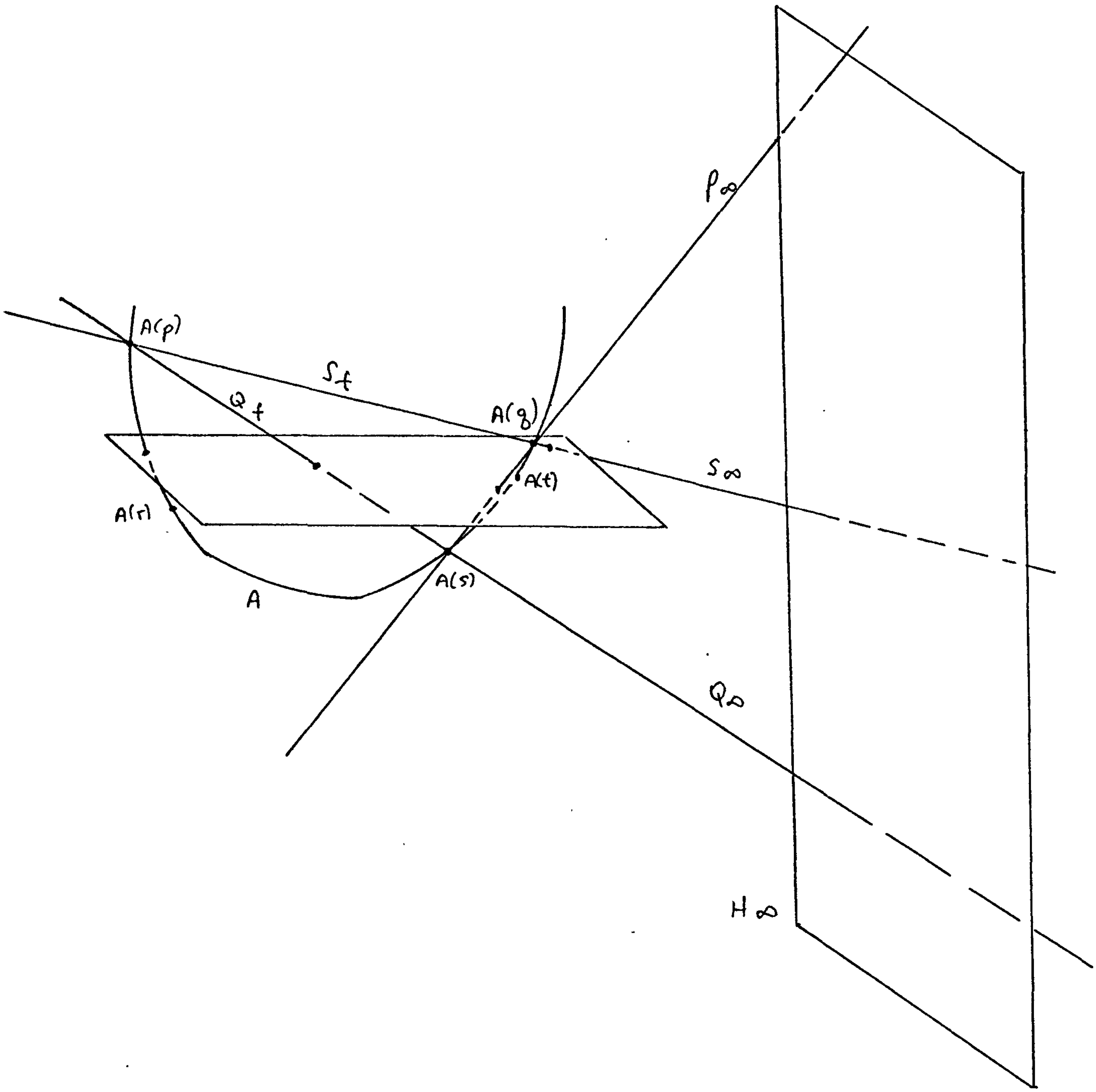


Figure VII.5

If there is only one inflection point q of $\breve{A}$ in (p,p_3) then (p,q) is regular on $\breve{A}$ and hence of order 2 by 3.2.7. Then by 2.4.1, $\breve{A}(p) \not\subset \breve{A}_1(q)$ that is $A_1(p) \not\subset A_1(q)A(p)$. Now q is an inflection on $\breve{A}$, hence $\breve{a}_0(q) = 1$, $\breve{a}_1(q) = 2$. Then $\pi(p,q) = 1$ otherwise $\pi(p,q) = 2$ and therefore we have $\breve{a}_1(q) = a_1(q) = 1$, a contradiction. Then $A(p) \subset A_2(q)$ and $A_1(p) \not\subset A_2(q)$ implies $\delta(p,A_2(q)) = 0$.

Suppose $q_1 < q_2$ are inflections of $\breve{A}$ in (p,p_3) By 3.3.11, there is an ordinary point q_3 of $\breve{A}$ in (q_1,q_2). Take q_4,q_5 such that

$$q_1 \leq q_4 < q_3 < q_5 \leq q_2$$

and (q_4,q_5) is the largest regular subarc of (q_1,q_2) in $\breve{A}$. Then (q_1,q_5) is of order 2 on $\breve{A}$. If $\breve{A}(p) \in \breve{A}_1(q)$ for all inflections q of $\breve{A}$ in (p,p_3), $\breve{A}(p) \in \breve{A}_1(q_4) \cap \breve{A}_1(q_5)$, a contradiction by 3.1.9. Hence $\breve{A}(p) \notin \breve{A}_1(q)$ for some inflection q of A in (p,p_3) and the result follows as above.

7.2.11 Lemma: If A is regular but not of order 3, there exist p,q,r with $p < q < r$, q singular, such that $A(p)\ A_1(q)\ A(r)$ is a plane which cuts A at p.

Proof: By 7.2.10, there exist $s < q$ such that $\delta(s,A_2(q)) = 0$. Since $a_2(s) = 1$, $A_2(q)$ cuts A at s. By 7.2.9, we may assume q is singular. By projection from $A_1(q)$, there is a r with $q < r$ such that $A_1(q)\ A(r)$ cuts A at a point $p < q$.

Proof of 7.2.1: Assume A is not of order 3. Take p_1, q_1, r_1 with properties of p, q, r in 2.11. Let X_1 be a neighbourhood of q_1 such that $r_1 \notin X_1$ and that for all $q \in X_1$, $A_1(q)\ A(r_1)$ meets A at least in a point $p \notin X_1$, $p < q_1$. Since q_1 is singular, we may repeat the argument using X_1 instead of J. Proceeding with the construction, we obtain a contradiction as in 3.2.7.

Corollary: Let A be a regular Barner arc. If A_2 is continuous at p the p is ordinary.

Proof: Take p_1, p_2 with $A(p_i) \not\subset A_2(p)$ for $i = 1,2$, and $A_1(p)\ A(p_1) \neq A_1(p)\ A(p_2)$. Since A_2 is continuous at p take $U_1(p)$ such that $A(p_i) \not\subset A_2(q)$ for all $q \in U_1(p)$; $i = 1,2$. Put $A^{(i)} = A/p_i$. Then $U_1(p)$ is regular on $A^{(i)}$; $i = 1,2$. By 7.2.4, A_1 is continuous at p. By 7.1.6, $A(p_i) \not\subset A_1(p)$ and thus

$$A_1^{(i)}(p) = A_1(p)\ A(p_i)$$

for $i = 1,2$.

By 3.2.7, $U_1(p)$ is of order 2 on $A^{(i)}$; hence p is ordinary on $A^{(i)}$ for $i = 1,2$. In particular, there exists $U_2(p) \subset U_1(p)$ such that $A_1^{(i)}$ is continuous on $U_2(p)$ for $i = 1,2$. Take $U(p) \subset U_2(p)$ such that

$$A_1(q) = \bigcap_{i=1}^{2} A_1^{(i)}(q)$$

for all $q \in U(p)$. Then A_1 is continuous on $U(p)$ and by 7.2.1, $U(p)$ is of order 3.

7.2.12 Lemma: Suppose A_2 is continuous at each point of a non-empty set W. Then there is a subarc X which contains a point of W such that if $p,q \in X$, $p \neq q$, $p \in W$ then $A_2(p)$ does not cut A at q.

The following notation will remain fixed: p_0 is a point at which A_2 is continuous, (p_1,p_2) is a neighbourhood of p_0 and H_∞ a plane which does not meet $[p_1,p_2]$. $A_2(p_0) \cap A[p_1,p_2] = A(p_0)$ and $A(p_1) \not\subset A_2(p)$ for any $p \in (p_0,p_2)$. $L = A(p_0)A(p_1)$ a line since A is simple, and $L_\infty(L_f)$ is the open segment of L with end points $A(p_1)$, $A(p_0)$ which meets (does not meet) H_∞.

The immediate result of the above hypothesis is that (p_0,p_2) has order 2 on A/p_1 and $A_1(p) \cap L = \emptyset$ for all $p \in (p_1,p_2)$.

7.2.13 Lemma: Suppose p_0 is an inflection. If there is a $p_3 \in (p_0,p_2)$ such that (p_0,p_3) has order 2 on A/p_0, then for each $p \in (p_0,p_3)$ either $A_2(p)$ meets L_f or $A(p_0) \subset A_2(p)$.

Proof: Since $a_0(p) + a_1(p) + a_2(p) \equiv 0 \pmod 2$, p_1, p_3 lie on the same side of $A_2(p_0)$. Since $A_2(p) = \lim_{\substack{q \to p^+ \\ q \neq p}} A_1(p)A(p)$, take $p_4 \in (p_0,p_3)$ such that p_1,p_3 lie on the same side $A(p_0)A_1(p_4)$. Since (p_0,p_3) is of order 2 on A/p_0, p_4 is regular on A/p_0. Hence by projection from p_0, $A(p_0)A_1(p_4)$ supports A at p_4, (Figure VII.6). Now $A(p_0)A_1(p_4)$ meets (p_0,p_3) only at p_4 otherwise (p_0,p_3) is not of order 2, p_1 and (p_0, p_4) lie on the same side of $A(p_0)A_1(p_4)$. Hence by projecting from $A_1(p_4)$, there exist $p_5 \in (p_0,p_4)$ such that $A(p_5)A_1(p_4)$ cuts (p_1,p_0); that is, $A(p_5)A_1(p_4)$ meets L_f.

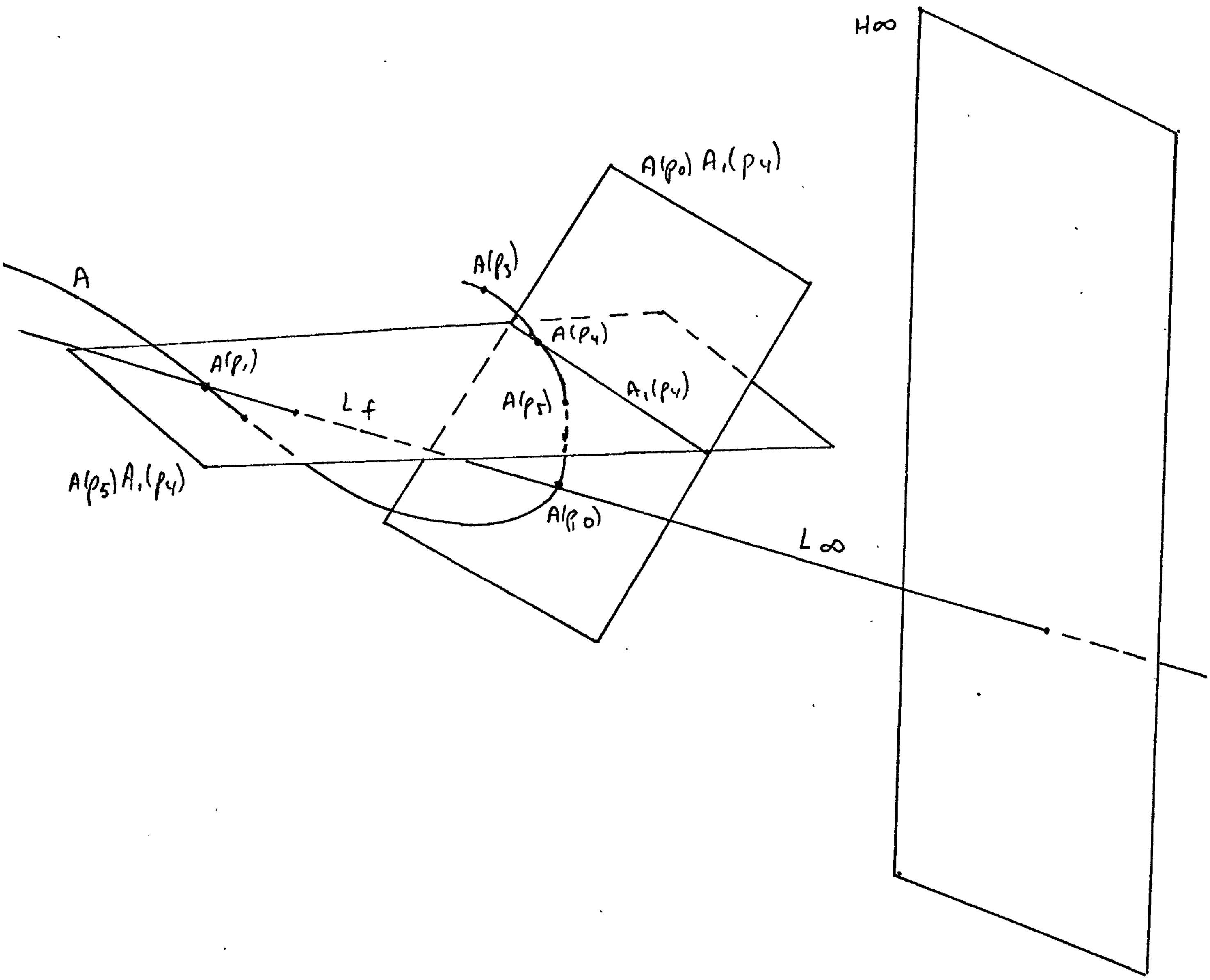

H_∞
$A(\rho_0)\ A_1(\rho_4)$
$A(\rho_3)$
A
$A(\rho_4)$
$A(\rho_1)$
L_f
$A(\rho_5)$
$A_1(\rho_4)$
$A(\rho_5) A_1(\rho_4)$
L_∞

Figure VII.6

Since (p_0, p_3) has order 2 on A/p_1 and A/p_0, neither $A(p_1)$ nor $A(p_0)$ lie on $A(p)\, A_1(q)$ if $p_0 < p < q < p_3$, by 2.4.1. Thus $A(p)A_1(q)$ meets L_f if $p_0 < p < q < p_3$ and the result follows.

Proof of 7.2.2: Let W be the set of all points at which A_2 is continuous. Since we are assuming A is a Barner arc with at most inflections on which A_1 is continuous, $W \neq \emptyset$ by 7.2.5. Choose X according to 7.2.12, let $p_0 \in X \cap W$. We may assume $X = J$. Then if $p \in W$, $A_2(p)$ does not cut A at any point except possibly at p.

Case 1. p_0 is regular. Then p_1, p_2 lie on opposite sides of $A_2(p_0)$. Take $p_3 \in (p_0, p_2)$ such that p_1, p_2, lie on opposite sides of $A_2(p)$ for all $p \in (p_0, p_3)$. Thus $A_2(p)$ cuts A at p for all $p \in (p_0, p_3)$. Since A has at most inflections and $A_1(p)$ does not cut A outside of p for all $p \in (p_0, p_3) \cap W$, then $(p_0, p_3) \cap W$ is regular. Then (p_0, p_3) is itself regular, by argument as in 3.3.11. Hence (p_0, p_3) is a regular Barner arc on which A_1 is continuous, the result now follows from 7.2.1.

Case 2. p_0 is a point of inflection. This case leads to a contradiction by arguments as in 3.3.11.

Proof of 7.2.3 Let W be the set of inflections of A. We may assume A is not of order 3, hence $W \neq \emptyset$ by 7.2.1. Let $X = J$ and $p_0 \in W$.

Case 1. There exists $p_3 \in (p_0, p_2)$ such that (p_0, p_3) has order 2 on A/p_0. If there is a $p \in (p_0, p_3) \cap W$, then by 7.2.13 and an argument as in 3.3.12, we obtain a contradiction. Thus (p_0, p_3) is regular and of order 3 by 7.2.1.

Case 2. No $U^{+}(p)$ has order 2 on A/p_0. By an argument as in 3.3.12, one obtains a contradiction.

BIBLIOGRAPHY

1. N. Bourbaki, General Topology Part 2, Addison-Wesley, London (1966), 44-53.

2. R. Park, On Barner Arcs and Curves, Unpublished doctoral dissertation, University of Toronto, 1967.

3. R. Park, Topics in Direct Differential Geometry, Private Communication, University of Calgary, 1971.

4. P. Scherk, Über Differenzierbare Kurven Und Bögen, Casopis pest. mat. a fys. (1937), 165-191.