DIRECTLY DIFFERENTIABLE ARCS
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By

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This thesis is a study of arcs in real projective two and three spaces from the point of view of direct (linear) differentiability. It is composed of most of the research on this subject carried on to date by Dr. Ralph Park of the University of Calgary. To ensure a greater scope of understanding, Barner arcs are the integral part of the study in the real projective plane, whereas arcs with tower are emphasized in the three dimensional case. Diagrams have been included to give a geometric background and at times much needed visual aid.
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CHAPTER I

1.1 Real Projective n-space

By a real projective n-space, \( n \geq 1 \), we mean a set \( \mathbb{P}^n \) of objects called spaces along with a 1-1 mapping of \( \mathbb{P}^n \) onto the set of all subspaces of a real \((n+1)\)-dimensional vector space. As usual one defines the inclusion relation \( L \subseteq M \) between spaces, the intersection \( L \cap M \) of spaces and the span \( L_M \) of spaces. For a collection of spaces, the intersection is denoted by \( \bigcap \, L_i \) and the span by \( \bigvee \, L_i \).

A space of \( \mathbb{P}^n \) is said to have dimension \( k \) if the vector subspace corresponding to it has dimension \( k + 1 \). Let \( \mathbb{P}^n_k \) denote the set of all \( k \)-spaces, \(-1 \leq k \leq n\). The unique \((-1)\)-space is denoted by \( \emptyset \). The elements of \( \mathbb{P}^n_0 \) and \( \mathbb{P}^n_{n-1} \) are called points and hyperplanes of \( \mathbb{P}^n \) respectively.

1.2 A topology on \( \mathbb{P}^n \)

Let \( \mathbb{R} \) be the real number system. Then \( \mathbb{R}^{n+1} \) is a real \((n+1)\)-dimensional vector space. We first define a topology on the points of \( \mathbb{P}^n \).

Denote \( \mathbb{R}^{n+1}_{\null} = \mathbb{R}^{n+1} - \{ \bar{0} \} \) where \( \bar{0} \) is the null vector of \( \mathbb{R}^{n+1} \). Since the one-dimensional subspaces of \( \mathbb{R}^{n+1} \) are the lines passing through the null vector, every non-zero vector \( \bar{x} \in \mathbb{R}^{n+1} \) determines a unique one dimensional subspace of \( \mathbb{R}^{n+1} \). We define
an equivalence relation $\mathcal{P}_n$ on $\mathcal{R}^{n+1}$ by $x \mathcal{P}_n y$ iff $x, y$ determine the same one-dimensional subspace where $x, y \in \mathcal{R}^{n+1}$.

We now identify $\mathcal{F}^n$ with $\overline{\mathcal{R}^{n+1}}/\mathcal{P}_n$ by the usual coordinization of $\mathcal{F}^n$. Then we define a topology on the points of $\mathcal{F}_n^o$, i.e. $\mathcal{F}_o^n$, to be the quotient topology of the topology of $\overline{\mathcal{R}^{n+1}}$ induced by $\mathcal{P}_n$.

We now extend the topology on $\mathcal{F}_o^n$ to the rest of $\mathcal{F}^n$. Let $(L_1)$ be a sequence of spaces in $\mathcal{F}^n$. We define $(L_1)$ to converge to a limit space $L$ of $\mathcal{F}^n$ iff each sequence $(x_i), x_i \in L$, converges to some point of $L$ and each point of $L$ is a limit of some sequence $(x_i)$ where $x_i \in L_i$. Or alternately, if $L$ and $M$ are spaces in $\mathcal{F}^n$, we define $L$ to be sufficiently close to $M$ iff each point of $L$ belongs to a sufficiently small neighbourhood of some point of $M$. Thus each $\mathcal{F}_k^n$ is compact and connected and $\mathcal{F}^n$ is compact; cf. Bourbaki.

If $L$ is a k-space, $-1 \leq k \leq n-2$, then the set $\mathcal{P}^{n-k-1}(L)$ of all spaces containing $L$ is a projective space, where the "points" in $\mathcal{P}^{n-k-1}(L)$ are all $(k+1)$-spaces of $\mathcal{F}^n$ containing $L$ and the "j-spaces" in $\mathcal{P}^{n-k-1}(L)$ are all $(k+j+1)$-spaces of $\mathcal{F}^n$ containing $L$.

It is readily seen from above that if $(L_1), (M_i)$ are two convergent sequences of spaces and $L_i \subset M_i$ for $i = 1, 2, \ldots$, then $\lim L_i \subset \lim M_i$, hence a sequence in $\mathcal{P}^{n-k-1}(L)$ converges in $\mathcal{P}^{n-k-1}(L)$ iff it converges in $\mathcal{F}^n$.

1.3 Directly differentiable arcs

Let $J$ be an ordered topological space which is isomorphic with the ordered topological space $\mathcal{R}^1$. A set $X \subset J$ is called an interval
if there exist \( p, q \in J \) with \( p < q \) such that

\[
X = (p, q), [p, q], (p, q], \text{ or } [p, q].
\]

Thus \([p, q] = \{ r \in J \mid p \leq r < q \}\). By a two sided (deleted, left, right) neighbourhood of \( p \in J \) we mean a set \( u(p) = (q, r) \) \((u'(p) = (q, p) \cup (p, r), u^-(p) = (q, p), u^+(p) = (p, r))\) where \( p \in u(p) \) and \( q < p < r \). If \( X \) is a finite subset of \( J \) we write \(|X|\) for the number of elements of \( J \).

Let a mapping \( A: J \to \mathbb{P}_n^\circ \) and a \( k \)-space \( L \) be given. For \( p \in J \) it may happen that \( \lim_{q \to p, q \neq p} A(q)L \) exists, in which case we denote it by \( A(p)/L \).

We say \( A \) is directly differentiable at \( p \) if there exist spaces

\[
A_k(p) \in \mathbb{P}_k^n, \quad -1 \leq k \leq n,
\]

such that \( A_0(p) = A(p) \) and \( A_k(p) = A(p)/A_{k-1}(p) \), \( 0 \leq k \leq n \).

Thus if \( A \) is directly differentiable at \( p \) it is also continuous at \( p \).

By an arc we mean a mapping \( A: J \to \mathbb{P}_n^\circ \) which is directly differentiable at each \( p \in J \). If \( A \) is an arc then \( A_k(p) \) is called the osculating \( k \)-space of \( A \) at \( p \), and \( p \in J \) is said to be a point of \( A \).

**1.3.1 Lemma:** Let \( A \) be an arc and \( L \) a hyperplane. For each \( p \in J \) there exists a \( u'(p) \) such that \( A(q) \not\in L \) if \( q \in u'(p) \).
Proof: Let \( p \in J \) be given. Let \( A_k(p) \) be the largest osculating space of \( p \) contained in \( L \). Assuming \( L \) meets every \( u'(p) \), there exists a sequence \( p_i \to p, \ p_i \neq p \) and \( A(p_i) \subset L \).

Then \( A_{k+1}(p) = \lim_{q \to p} A(q) A_k(p) \) is contained in \( L \), a contradiction.

**Corollary 1:** If \( n \geq 2 \), the image of an arc cannot contain a line segment.

**Corollary 2:** If \( X \subset J \) is compact, then for any hyperplane \( L, A(p) \subset L \) for only finitely many \( p \in X \).

### 1.4 Dually differentiable arcs

We have defined \( \mathbb{P}^n \) such that each \( k \)-space \( L \) corresponds to a \((k+1)\)-dimensional subspace of a real \((n+1)\) dimensional vector space (say \( V \)). If we now associate to \( L \) the corresponding \((n-k)\)-dimensional subspace of \( V^* \), the dual of \( V \), then \( \mathbb{P}^n \) is again a projective space called the dual \( \mathbb{P}^{n*} \) of \( \mathbb{P}^n \). One has \( \mathbb{P}^n_{k} = \mathbb{P}^{n-k-1*} \), \(-1 \leq k \leq n \).

Let \( A \) be an arc in \( \mathbb{P}^n \). Define a mapping \( A^*: J \to \mathbb{P}^{n*} \) by

\[
A^*(p) = A_{n-k-1}(p)
\]

for all \( p \in J \). We say \( A \) is dually differentiable if \( A^* \) is an arc in \( \mathbb{P}^{n*} \) and for all \( p \in J \)

\[
A^*_k(p) = A_{n-k-1}(p), \ -1 \leq k \leq n.
\]
If $A^*$ is an arc, then for $p \in J$, $-1 \leq k \leq n$,

$$A_{n-k}(p) = A_{k+1}(p) = A^*(p)/A^*_k(p)$$

$$= \lim_{q \rightarrow p} A^*(q)/A^*_k(p)
\quad q \neq p$$

$$= \lim_{q \rightarrow p} A_{n-1}(q) \cap A_{n-k-1}(p).$$
\quad q \neq p

Therefore, $A$ is dually differentiable iff $A_k(p) = \lim_{q \rightarrow p} A_{n-1}(q) \cap A_{k+1}(p)$
\quad $q \neq p$

for all $p \in J$, $-1 \leq k \leq n-1$.

**Lemma 4.4.1**: If $A$ is dually differentiable then $A_{n-1}$ is continuous. If $P \in F^m$ and $p \in J$, then there exists a $u'(p)$ such that $p \not\in A_{n-1}(q)$ for all $q \in u'(p)$.

**Proof**: 4.3.1.
CHAPTER II

In this and the following two chapters, all our considerations will be limited to \( \mathbb{F}^2 \) unless stated otherwise.

2.1 Projection

Let \( A \) be an arc, \( p \) a point of \( A \) and \( P \in \mathbb{F}^2_0 \). We define \( \pi(P, p) \) to be the dimension of the largest osculating space at \( p \) which does not contain \( P \). We now define

\[
\tilde{A}_k(p) = \begin{cases} 
A_k(p)P, & -1 \leq k \leq \pi(P, p) \\
A_{k+1}(p), & \pi(P, p) < k \leq 1.
\end{cases}
\]

2.1.1 Theorem: \( \tilde{A} = \{ \tilde{A}_0(p) | p \in J \} \) is an arc in \( \mathbb{F}^1(P) \) with \( \tilde{A}_k(p) \) as its osculating k-space at \( p \).

Proof: Since \( \tilde{A}_k(p) \) is a \((k+1)\)-space in \( \mathbb{F}^2 \) and \( P \in \tilde{A}_k(p), -1 \leq k \leq 1, \tilde{A}_k(p) \in \mathbb{F}^1_k(P) \). Now there exists a \( u'(p) \) such that \( A(q) \neq P \) if \( q \in u'(p) \). We may assume \( P \neq A(p) \) and \( PA(p) \) is a line. Then \( u'(p) \) exists from 1.3.1. Therefore wish to show

\[
\tilde{A}_k(p) = \tilde{A}(p)/\tilde{A}_{k-1}(p).
\]

Case 1: \(-1 \leq k \leq \pi(P, p)\). Then \( P \not\in A_k(p) \) and

\[
\tilde{A}(p)/\tilde{A}_{k-1}(p) = \lim_{q \to P} A(q) \tilde{A}_{k-1}(p) = \lim_{q \to P} A(q) A_{k-1}(p)P = A_k(p)P = \tilde{A}(p)
\]
Figure II.1
Case 2: \( \pi(P, p) < k \leq l \). Then \( P \subset A^k(p) \) and

\[
\tilde{A}(p) / A^1_{k-1}(p) = \lim_{q \to p} (A(q)P) A^1_k(p)
\]

\[
= \lim_{q \to p} A(q) A^1_k(p)
\]

\[
= A^1_{k+1}(p) = \tilde{A}^1_k(p).
\]

The arc \( \tilde{A} \) is called the projection of \( A \) from \( P \). In general, we denote \( \tilde{A} \) by \( A/P \). In Figure II,1, we have an example of projection from a point \( p \) of \( A \) where we assume \( A^1_1(p) \) does not meet \( A \) outside of \( p \).

2.2 Secants of an arc

Let \( A \) be an arc and \( L \) be a \( k \)-space. Let \( (\delta(p, L)) A^\delta_0(p, L) \) be (the dimension of) the largest osculating space at \( p \) which is contained in \( L \); thus \(-1 \leq \delta(p, L) \leq k\) and

\[
\bigvee_{p \in X} A^\delta_0(p, L)(p) \subset L
\]

for any \( X \subset J \). If the inclusion is improper we say \( L \) is a \( k \)-secant of \( X \).

In the Euclidean plane one considers secants to be lines which pass through exactly two points of a figure. But in \( P^2 \), a 1-secant may meet an arc at only one point as, for example, in Figure II. 1, taking \( L = A^1_1(p) \).
We will now consider some basic properties of secants in $\mathbb{P}^2$.

In most cases the proofs given with minimal modification will apply to results in Chapter V, Section 3.

2.2.1 Lemma: Let $P$ be a point of a k-space $L$, $\tilde{A} = A/P$. Then

$$
\tilde{\delta}(p,L) = \begin{cases} 
\delta(p,L) & \text{if } -1 \leq \delta(p,L) < \pi(P,p), \\
\delta(p,L) - 1 & \text{if } \pi(P,p) < \delta(p,L) \leq k.
\end{cases}
$$

Proof: If $\delta(p,L) < \pi(P,p)$, then $\tilde{A}_{\delta(p,L)}(p) = A_{\delta(p,L)}(p)P$.

Since $P \in L$, $\tilde{A}_{\delta(p,L)}(p) \subseteq L$ and $\delta(p,L) \leq \tilde{\delta}(p,L)$. The equality follows since $A_{\delta(p,L)+1}(p) \notin L$.

If $\pi(P,p) < \delta(p,L)$ then $\tilde{A}_{\delta(p,L)}(p) = A_{\delta(p,L)+1}(p)$ and $\tilde{\delta}(p,L) < \delta(p,L)$. But $\tilde{A}_{\delta(p,L)-1}(p) = A_{\delta(p,L)}(p)$ or $A_{\delta(p,L)-1}(p)P$.

In either case it is contained in $L$ and the result follows.

2.2.2 Lemma: A k-space $L$ with $A(p) \subset L$ is a k-secant of $X$ iff it is a (k-1)-secant of $X$ on $A/p$, where $p \in X$.

Proof: Let $X_1 = \{ q \in X \mid \pi(p,q) < \delta(q,L) \}$ and $X_2 = \{ q \in X \mid \delta(q,L) < \pi(p,q) \}$. Then $X = X_1 \cup X_2$. If $L$ is a k-secant of $X$ then

$$
L = \bigvee_{q \in X} A_{\delta(q,L)}(q)
= \bigvee_{q \in X_1} A_{\delta(q,L)}(q) \bigvee \bigvee_{q \in X_2} A_{\delta(q,L)}(q).
$$
By 2.2.1 and projection from \( p \),

\[
L = \bigvee_{q \in X} \tilde{A}_0(q, L)_1(q) \bigvee_{q \in X} \tilde{A}_0(q, L)(q)
= \bigvee_{q \in X} \tilde{A}_{0'}(q, L)(q) \bigvee_{q \in X} \tilde{A}_{0'}(q, L)(q)
= \bigvee_{q \in X} \tilde{A}_{0'}(q, L)(q).
\]

2.2.3 Theorem: The set of all \( k \)-secants of a connected set \( X \subset J \) is pointwise connected, \(-1 \leq k \leq 2\).

Proof: Since the set of 0-secants of \( X \) is the continuous image of \( X \) and there is only one 2-secant, we may restrict ourselves to the consideration of 1-secants only.

The set of 1-secants of \( X \) which contain an element \( p \in X \) is the set of 0-secants of \( X \) on \( A/p \) by 2.2.2 and hence are pointwise connected by same argument as above.

Now let \( L \) and \( M \) be 1-secants of \( X \). Take \( p, q \) in \( X \) such that \( A(p) \subset L \) and \( A(q) \subset M \). Let \( N \) be a 1-secant of \( X \) containing \( A(p) \) and \( A(q) \). Then we can construct a path from \( L \) to \( M \) by constructing a path from \( L \) to \( N \) and then from \( N \) to \( M \).

2.2.4 Let \( L \) be a \( k \)-secant of \( X \subset J \). Then

\[
k \leq \left\{ \sum_{p \in X} (s(p, L) + 1) \right\} - 1.
\]

When there is equality \( L \) is said to be an independent \( k \)-secant of \( X \).
X is said to be k-independent iff every k-secant of X is independent.

If L is an independent k-secant of X then L meets X at most k + 1 times. Otherwise if L meets X at P_0, ..., P_{k+1} distinct points then \(\sum_{i=0}^{k+1} (\delta(p_i, L) + 1) - 1 \geq (k+2) - 1 = k+1\), and

\[
\sum_{p \in X} (\delta(p, L) + 1) - 1 \geq \sum_{i=0}^{k+1} (\delta(p_i, L) + 1) - 1
\]

\[
\geq (k+2) - 1 = k+1,
\]

a contradiction.

2.25 Lemma: Let X be k-independent and L be a k-secant of X on A, \(-1 \leq k \leq 1\). Let \(A(p) \subseteq L\). Then if \(q \neq p\) in X, \(A(p) \neq A_0(q, L)(q)\).

Proof: If \(k = 0\) then \(0 = \sum_{q \in X} (\delta(q, L) + 1) - 1\) implies

\[
\sum_{q \in X} (\delta(q, L) + 1) = 1 \text{ and } L \text{ meets } X \text{ exactly once. Since } A(p) \subseteq L
\]

\[
A(p) = L \text{ and } A_0(q, L)(q) = \emptyset.
\]

If \(k = 1\) and \(A(p) \subseteq A_0(q, L)(q')\) for some \(q' \neq p\) in X then \(A(p) = A(q')\) or \(A(p) \neq A(q')\) and \(A_1(q') = L\). If \(A(p) = A(q')\) then from Figure II.2, N is a 1-secant of X but N is not independent since

\[
\sum_{q \in X} (\delta(q, N) + 1) - 1 = \delta(q', N) + \delta(p, N) + \delta(r, N) + 2 \geq 2,
\]

a contradiction.

If \(A(p) \neq A(q')\) and \(A_1(q') = L\) then
\[
\sum_{p \in X} (\delta(q,L)+1)-1 \geq \delta_1(L) + \delta(q^*,L) + 1 \geq 2;
\]

again a contradiction.

**Corollary:** If X is 1-independent, then X is 0-independent.

**2.2.6 Lemma:** If X is k-independent and p \in X then X is (k-1)-independent on A/p.

**Proof:** Let L be a (k-1)-secant of X on A/p. By 2.2.2 L is a k-secant of X on A with A(p) \subseteq L. Define \( X_1, X_2 \) as in 2.2.2 then \( X = X_1 \cup X_2 \) and

\[
k = \sum_{q \in X_1} (\delta(q,L)+1) + \sum_{q \in X_2} (\delta(q,L)+1) - 1
\]

\[
= \sum_{q \in X_1} (\delta(q,L)+2) + \sum_{q \in X_2} (\delta(q,L)+1) - 1 \quad \text{by 2.2.1.}
\]

Since \( X_1 = \{ q \in X \mid A(p) \subseteq A_\delta(q,L)(q) \} \), then \( X_1 = \{ p \} \) by 2.2.5 and

\[
k = \delta(p,L)+2 + \sum_{q \in X \setminus \{p\}} (\delta(q,L)+1) - 1
\]

and the result follows.

We have already commented upon the fact that an independent k-secant of X on A meets X at most \( k+1 \) times. The converse does not hold as can be seen in Figure II.2 for the case \( k = 1 \) and \( X = A \).

**2.3 Representations of secants**

Let L be a k-secant of X on A which meets X at \( m+1 \)
points \( p_0, \ldots, p_m \). Let \( L \) be independent. Then \( m \leq k \).

If \( m = k \) then \( \delta(p_i, L) = 0 \) for \( i = 0, \ldots, k \) and \( \{ p_0, \ldots, p_k \} \) uniquely determine \( L \). If \( m < k \) then since

\[
    k = \sum_{p \in X} (\delta(p, L) + 1) - 1
\]

there is a \( p_j \) with \( \delta(p_j, L) > 0 \). For example, if \( \delta(p_j, L) = k \) then \( L \) meets \( X \) only at \( p_j \) and \( p_j \) uniquely determines \( L \). If \( \delta(p_j, L) = k - 1 \) then there is \( p_i \neq p_j \) in \( X \) such that \( A(p_i, \ldots, p_k) \subseteq L \setminus A_{k-1}(p_j) \). Hence \( \{ p_i, p_j \} \) uniquely determine \( L \).

Thus an independent \( k \)-secant \( L \) of \( X \) on \( A \) is uniquely determined by the points at which \( L \) meets \( X \). If \( L \) is not independent then \( L \) is still determined by the points of which \( L \) meets \( X \) though no longer uniquely. As, for example, in Figure II.3, \( L \) is a \( 1 \)-secant of \( A \), not independent and \( \{ p_0, p_1 \}, \{ p_0, p_2 \}, \{ p_1, p_2 \} \) all represent; that is, determine \( L \).

Putting these observations in more precise form, we define inductively the mapping

\[
    A^k : \mathbb{P}^{k+1} \rightarrow \mathbb{P}_k^2 \quad -1 \leq k \leq 2,
\]

by requiring that

\[
    A^{k-1}(\ ) \subseteq \mathbb{P}_{k-1}^2
\]

and

\[
    A^k(p_0, \ldots, p_k) = A(p_k) / A^{k-1}(p_0, \ldots, p_{k-1}) \quad 0 \leq k \leq 2.
\]

We note that \( A^k(p, \ldots, p) = A_k(p) \).
Figure II.4
2.3.1 Theorem: \( A^k(x^{k+1}) \) is the set of all \( k \)-secants of \( X \), 
\(-1 \leq k \leq 2 \).

**Proof:** The result holds trivially for \( k = -1, 0, \text{and} 2 \). Let 
\( \{ p_0, p_1 \} \subset X \) be given. By projecting \( A \) from \( p_0 \), \( \tilde{A}(p_1) = A^1(p_0, p_1) \) 
and \( A^1(p_0, p_1) \) is a 0-secant of \( X \) on \( \tilde{A} = A/p_0 \). By 2.2.2, 
\( A^1(p_0, p_1) \) is a 1-secant of \( X \) on \( A \).

Conversely, let a 1-secant \( L \) of \( X \) on \( A \) be given. Let 
p \in X \) with \( A(p) \subset L \). By projecting \( A \) from \( p \) and 2.2.2, \( L \) is 
a 0-secant of \( X \) on \( A/p \), hence \( L = (A/p)(q) = A^1(p, q) \).

Now at this point one is tempted to say that \( X \) is \( k \)-independent 
iff \( A^k \) is 1-1. However, \( A^k \) is a mapping of tuples, hence we are now 
also concerned with the order of appearance of points in a given tuple; 
that is, permutation of the components.

The ideal situation would be to have \( A^k \) independent of 
permutations, however this is not always the case. Consider Figure II.4, 
where \( A \) is a regular arc; cf. Section 6. Let \( X = (p_0, p_3) \), then 
\[ A^1(X) = A(p_3)/A(p_0) = A(p_3)/A(p_2) = A_1(p_3) \]
but \( y = (p_3, p_0) \) is a permutation of \( X \) and 
\[ A^1(y) = A_1(p_0) \neq A_1(p_3) = A^1(X). \]

Suppose \( p \in J \) and \( x = (p_0, \ldots, p_k) \in J^{k+1}, -1 \leq k \leq 2 \). 
Define \( \gamma(p, x) = \lbrace \sum_{p_i=p} 1 \rbrace \), -1. Thus \( \gamma(p, x) \leq 6(p, A^k(x)) \).
and equality occurs iff \( A^k(x) \) is an independent \( k \)-secant of \( X \) iff the components of \( x \) uniquely determine \( A^k(x) \).

Since \( \gamma(p, x) \leq 5(p, A^k(x)) \), then \( \bigvee_{p \in X} A \gamma(p, x)(p) \subseteq A^k(x) \).

When the inclusion is improper we say \( X \) is independent.

2.3.2. Lemma: If \( A^k(x) \) is an independent \( k \)-secant of \( X \) then \( x \) is independent but not conversely.

Proof: Since \( A^k(x) \) is independent, \( \gamma(p, x) = 5(p, A^k(x)) \) and

\[
\bigvee_{p \in X} A \gamma(p, x)(p) = \bigvee_{p \in X} A 5(p, A^k(x))(p) = A^k(x).
\]

That the converse is not true; in Figure II.3, \( x = (p_0, p_1) \) is independent but \( L = A^1(x) \) is a dependent secant.

2.3.3. Lemma: If \( X \) is independent, then \( A^k(x) = A^k(y) \) if \( y \) is a permutation of \( x \in J^{k+1} \).

Proof: Since \( \gamma(p, y) = \gamma(p, x) \) for all permutations \( y \) of \( x \), then

\[
A^k(y) = \bigvee_{p \in X} A \gamma(p, y)(p) = \bigvee_{p \in X} A \gamma(p, x)(p) = A^k(x)
\]

and the result follows.

Thus by 2.3.2 and 2.3.3, if \( A^k(x) \) is an independent \( k \)-secant, then it is also independent of the permutations of \( x \). Now if \( A^k(x) \) is not \( k \)-independent then from above there is a \( p \in X \) such that \( \gamma(p, x) < 5(p, A^k(x)) \). Hence there is at least one \( z \in J^{k+1} \) with the property that \( \gamma(p, z) = 5(p, A^k(z)) \) and \( A^k(z) = A^k(x) \). Hence,
Theorem: Let $X \subset J$ and $L = A^k(x)$ be a $k$-secant of $X$ where $x \subset x^{k+1}$. Then $L$ is an independent secant of $X$ iff $L = A^k(y)$, where $y \subset y^{k+1}$, holds exactly for the permutations $y$ of $x$.

As can be readily seen in Figure II.4, $x = (p_0, p_2)$ is not independent since

$$V_{p \in A} A^V(p, x)(p) = A(p_0) A(p_2) = A(p_0) = A(p_2) \neq A^1(x).$$

In fact we have the following result.

Lemma: Let $x \subset x^{k+1}$. Then $x$ is independent for $k = 1$ [ $k = 2$ ] iff the components of $x$ are not equal [not collinear].

Proof: The result follows since the independence of $x$ is based upon the dimension of space spanned by its components.

Theorem: Let $-1 \leq k \leq 1$. Then $X \subset J$ is $k$-independent iff every $x \subset x^{k+2}$ is independent.

Proof: Let $X$ be $k$-independent and $x \subset x^{k+2}$. Then

$$L = V_{p \in X} A^V(p, x)(p) \subset A^{k+1}(x)$$

is an $h$-secant of $X$, $h \leq k + 1$. If $h = k + 1$ the result follows, so assume $h < k + 1$. Then $h \leq k \leq 1$ and $L$ is an $h$-independent secant of $X$ on $A$ by the Corollary of 2.2.5. Since $V(p, L) \geq A(p, x)$ for all $p \in X$, we have

$$h = \sum_{p \in X} (\delta(p, L) + 1) - 1$$

$$\geq \sum_{p \in X} (A(p, x) + 1) - 1.$$
But \( \gamma(p, x) = \sum_{p_i=p} 1 - 1 \), hence \( \gamma(p, x) + 1 = \sum_{p_i=p} 1 \).

Since \( x \in x^{k+2} \),

\[
\sum_{p \in X} \left( \gamma(p, x) + 1 \right) = \sum_{p \in X} \left( \sum_{p_i=p} 1 \right) = k + 2.
\]

Hence \( h \geq k+1 \); a contradiction.

Conversely, if \( L \) is a dependent \( k \)-secant, say \( L = A^k(y) \),
\( y \in x^{k+1} \), then we can always construct from \( y \) an \( x \in x^{k+2} \) which
is dependent. For example, in Figure II.3, let \( k = 1 \) and \( y = (p_0, p_1) \).
Then \( x = (y, p_2) = (p_0, p_1, p_2) \) is dependent since \( p_0, p_1, p_2 \) are
collinear.

2.4 Order

Let \( A \) be an arc and \( X \subset J, -1 \leq k \leq 2 \). Let \( k \) be fixed.
If \( S(X, L) = \{ p \in X \mid A(p) \subset L \} \) is finite for every \( k \)-space \( L \), we
say \( X \) has finite \( P^2_k \)-order.

If in addition

\[
m = \sup_{L \in \mathcal{P}^2_k} |S(X, L)|
\]

is finite, we say \( X \) has bounded \( P^2_k \)-order \( m \). By the order of \( X \),
we mean the \( P^2_1 \)-order.

If \( X \) has at least \( k + 1 \) points and is \( k \)-independent, then
it has \( P^2_k \)-order \( k + 1 \), since we can construct a \( k \)-secant through
the \( k + 1 \) points.
Any compact set $X \subset J$ is of finite order and, in particular, $A$ is locally of finite order. This follows from the corollaries if 1.3.1.

2.4.1 Theorem: If $(p,q)$ has order 2 then $[p,q)$ and $(p,q]$ are 1-independent.

Proof: First of all we note that in $P'$, if $(p,q)$ has order 1 then $[p,q)$ and $(p,q]$ are 0-independent. For if $A(p) = A(s)$ for some $s \subset (p,q)$ then $[p,s]$ will be a closed subarc of $[p,q)$ in $P'$ and by continuity of $A$ there is $r_1 \neq r_2$ in $(p,s)$ such that $A(r_1) = A(r_2)$. But then $(p,q)$ is at least of order 2 in $P^1$, a contradiction.

Our method of proof will be to show $(p,q)$ is 1-independent and then extend the independence to $[p,q)$.

Let $s \subset (p,q)$ and $L$ be a line through $s$, then $L$ meets $[p,s) \cup (s,q)$ at most once and $(p,s)$ has order 1 on $\tilde{A} = A/s$. Hence if $p < r < s < q$, then $A(r) \notin A_1(s)$ otherwise $\tilde{A}(r) = A(r) A(s) = A_1(s) = \tilde{A}(s)$, a contradiction (Figure II.5).

Let $L$ be a 1-secant of $(p,q)$, then there is $t \subset (p,q)$ such that $A(t) \subset L$. Let $r$ be the first such point in $(p,q)$. Then $(r,q)$ has order 1 on $\tilde{A} = A/r$ and for any $s \subset (r,q)$, $A(r) \notin A_1(s)$ from above. Hence $\tilde{d}(s,L) = d(s,L)$ by 2.2.1. Since $[r,q)$ is 0-independent on $A/r$,
\[ 0 = \sum_{sc(r,q)} (\delta(s,L)+1)-1 \]
\[ = (\delta(r,L)+1) + \sum_{sc(r,q)} (\delta(s,L)+1)-1 \]
\[ = (\delta(r,L)+2) + \sum_{sc(r,q)} (\delta(s,L)+1)-1. \]

Therefore \( l = \sum_{sc(r,q)} (\delta(s,L)+1)-1 \) and \( L \) is an independence secant \( sc(r,q) \) of \( [r,q] \) on \( A \). Hence \( (p,q) \) is \( l \)-independent.

We now wish to show that \( A(p) \not\subset A_1(s) \) for any \( s \in (p,q) \). Since \( (p,q) \) is \( l \)-independent on \( A \), by 2.2.6, \( (p,q) \) is \( 0 \)-independent on \( A/s \) if \( s \in (p,q) \). From the comments in the beginning of this section, \( (p,q) \) is of order \( l \) on \( A/s \) and hence \( [p,q] \) is \( 0 \)-independent on \( A/s \) and \( A(p) \not\subset A_1(s) \) for \( s \in (p,q) \).

Furthermore, \( [p,q] \) has order 2. For if \( p < p_1 < p_2 < q \) and \( p, p_1, p_2 \) are collinear then \( [p, p_2] \) is at least of order 2 on \( A/p_2 \) a contradiction. But \( [p,q] \) has order 2 on \( A \) implies \( (p,q) \) has order 1 on \( A/p \) and \( [p,q] \) is \( 0 \)-independent on \( A/p \).

Finally, let \( L \) be a \( 1 \)-secant of \( [p,q] \) with \( A(p) \subset L \). Then \( L \) is \( 0 \)-independent on \( A/p \). Since \( A(p) \not\subset A_1(s) \) for \( s \in (p,q) \), \( \delta(s,L) = \delta(s,L) \) and \( L \) is an independent secant of \( [p,q] \) on \( A \) as above. Hence \( [p,q] \) as \( 1 \)-independent on \( A \).

The symmetric argument holds for \( (p,q) \).
Corollary 1: A is an arc of order 2 iff A is 1-independent.

Corollary 2: If \((p, q)\) has order 2 on \(A\) and \(r \in [p, q]\) then \((p, q)\) has order 1 on \(A/r\).

Corollary 3: If \((p, q)\) has order 2 on \(A\) and \(s \in (p, q)\) then

\[A(p) \cap A[p, q] = A(s)\]

Let \(A\) be an arc of order 2 and \(L\) and oriented line such that \(A(p) \neq L\) for \(p \in J\). Then for each \(x \in J^2\), \(A(x) \cap L\) is a point \(x\) of \(L\). We assume there exists a point \(p_0 \neq x\) for all \(x \in J^2\). We put \((p_0, p_1) \leq (q_0, q_1)\) iff \(p_0 \leq q_0\) and \(p_1 \leq q_1\).

2.4.2 Theorem: \(\varphi\) is (strictly) monotone.

Proof: We prove the result locally and since \(J\) is connected, we can then extend the result to the whole arc.

Let \(p, p_0, q_0\) be three distinct points of the arc (Figure II.6). Since \(A\) is of order 2, \(A/p_0\) is of order 1 by 2.4.1, Cor. 2, hence the mapping \(p \rightarrow \varphi(p_0, p)\) is monotone. Similarly \(p \rightarrow \varphi(q_0, p)\) is monotone. Therefore we now need to show that the monotonicity does not depend on the choice of a given component; that is, \(p \rightarrow \varphi(p_0, p)\) and \(p \rightarrow \varphi(q_0, q)\) are monotone in the same sense.

Take \(r_1\) and \(r_2\) with \(p_0 < r_1 < r_2\) (Figure II.7). We may assume the orientation of \(L\) is such that \(\varphi(p_0, r_1) \leq \varphi(p_0, r_2)\).

Then there exists a \(u(p_0)\) such that if \(q_0 \in u(p_0)\) then
\( \psi(q_0, r_1) < \psi(q_0, r_2) \). Otherwise all of \( L \) will be the image of \( J^2 \) under \( \psi \). Thus \( p \rightarrow \psi(q_0, p) \) has the same sense for all \( q_0 \in u(p_0) \).

### 2.5 Characteristic of a point

A line \( L \) is said to support \( A \) at \( p \) if there is a line \( H_\infty \) such that \( H_\infty \not= L \), \( A(p) \not= H_\infty \) and a \( u'(p) \) such that \( A(u'(p)) \) is contained in one of the two open half spaces determined by \( L \) and \( H_\infty \). When \( L \) does not support \( A \) at \( p \) we say \( L \) cuts \( A \) at \( p \), since by 1.3.1, \( A(u'(p)) \) does not contain any segment of \( L \).

#### 2.5.1 Lemma:

Let \(-1 < k < 1\). Let \( S_k \) be the set of all lines \( L \) with \( \delta(p, L) = k \) for \( p \in J \) fixed. Then

1. \( S_k \) is connected.
2. The elements of \( S_k \) either all support or all cut \( A \) at \( p \).

**Proof:**

1. Since in a projective plane any two lines meet, every \( L \in S_{-1} \) meets \( A_1(p) \) outside \( A(p) \). Conversely, given any \( P \not= A_1(p) \) and \( Q \in A_1(p) - A(p) \), \( PQ \in S_{-1} \). Therefore \( S_{-1} \) is isomorphic to the real line \( \mathbb{R} \). Hence \( S_{-1} \) is connected.

Since \( S_1 \) is a singleton set consisting of \( A_1(p) \) it is connected.

By projection from \( p \), the elements of \( S_0 \) are projected onto \( \mathbb{P}^1 - \{ \tilde{A}(p) \} \), where \( \tilde{A} = A/p \). The result 1 now follows.
2. Let \( L_i, L \subseteq S_k \) with \( L_i \longrightarrow L \). Then there is \( u'(p) \) such that \( A(u'(p)) \notin L[L_i] \); otherwise (Figure II.8, \( k = 0 \)) there exist points \( q_i \) with \( q_i \longrightarrow p, q_i \neq p \) and integers \( j(i) \) with \( A(q_i) \subseteq L_j(i) \).

But then \( A(q_i) \subseteq L_j(i) \) by hypothesis and \( A_{k+1}(p) \subseteq L_j(i) \), a contradiction. Hence if all \( L_j \) support \([\text{cut}]\) \( A \) at \( p \) then \( L \) supports \([\text{cuts}]\) \( A \) at \( p \). Thus the set of lines of \( S_k \) which support \([\text{cut}]\) \( A \) at \( p \) is closed. But \( S_k \) is connected and result follows.

So far by the study of order, projection and secants we have been getting a global picture of arcs. But since each \( p \in J \) has its well defined osculating spaces, by the above uniformity theorem we can describe in detail a point and its neighbourhood in the arc. As it turns out there are four different types of points distinguished by 2.5.1 in the real projective plane and later we shall give examples of each.

Let \( p \in J \) and \(-1 \leq k \leq 1\). We define \( \sigma_k(p) = 0 \) or \( 1 \) according as the elements of \( S_k \) support or cut \( A \) at \( p \). Thus \( \sigma_{-1}(p) = 0 \). The characteristic \( (a_0(p), a_1(p)) \) of \( p \) is defined by taking \( a_i(p) = 1 \) or \( 2 \) and requiring that

\[
a_0(p) + a_k(p) \equiv \sigma_k(p) \pmod{2}, \quad 0 \leq k \leq 1.
\]

We also define numbers

\[
\beta_k(p) = \sum_{i=0}^{k} a_i(p), \quad 0 \leq k \leq 1
\]

and \( \beta_{-1}(p) = 0 \). Therefore \( \beta_k(p) \equiv \sigma_k(p) \pmod{2}, \quad -1 \leq k \leq 1 \).
We say \( p \) is \textbf{regular} if \( a_0(p) = 1 = \hat{a}_1(p) \). We say \( p \) is a \textbf{point of inflection} if \( a_0(p) = 1 \) and \( a_1(p) = 2 \). An arc \( A \) is regular (has at most inflections) if each point is regular (regular or an inflection). In Figures II-9 to II-12, the four different types of characteristic of a point \( p \) of an arc in \( F^2 \) are shown. In Figures II.9 [II.10] \( p \) is regular [inflection]. Whereas in Figures II.11 and II.12, the characteristic of \( p \) is \((2,1)\) and \((2,2)\) respectively. In all cases the line \( L \neq A_1(p) \).

Upon projection, say from a point \( P \), we have seen that

\[
\hat{\sigma}_k(p) = \begin{cases} 
\sigma_k(p) & \text{if } -1 \leq k \leq \pi(P,p) \\
\sigma_{k+1}(p) & \text{if } \pi(P,p) < k \leq 1,
\end{cases}
\]

where \( \hat{\sigma}_k(p) \), \(-1 \leq k \leq 1\) are the osculating spaces of \( p \) in \( \hat{A} = A/P \).

Since the \( \hat{\sigma}_k(p) \) are dependent on \( A_k(p) \) and \( \pi(P,p) \), it is fair to assume that the characteristic of \( p \) in \( \hat{A} \) will depend only on \( \pi(P,p) \) and the characteristic of \( p \) on \( A \). This is in fact the case.

**2.5.2 Theorem:** Let \( P \) be a point and \( \hat{A} = A/P \). Then

1. \[
\hat{\sigma}_k(p) = \begin{cases} 
\sigma_0(p) + \sigma_{k+1}(p) & P = A(p), -1 \leq k \leq 0 \\
\sigma_k(p) & -1 < k < \pi(P,p) \\
\sigma_{k+1}(p) & -1 < \pi(P,p) \leq k \leq 0.
\end{cases}
\]

2. \[
\hat{a}_0(p) = \begin{cases} 
a_0(p) & 0 < \pi(P,p) \\
a_0(p) + a_1(p) & 0 = \pi(P,p) \\
a_1(p) & 0 > \pi(P,p)
\end{cases}
\]
Figure II.9
Figure II.11
where the congruences are \((\text{mod } 2)\) and

\[
\beta_0(p) = \begin{cases} 
\beta_1(p) - \beta_0(p) & P = A(p) \\
\beta_0(p) & 0 < \pi(P, p).
\end{cases}
\]

Proof: Case 1. \(P = A(p)\). Let \(L\) and \(H_\infty\) be distinct lines with \(\delta(p, L) = k + 1\) and \(\delta(p, H_\infty) = 0\). Then \(\tilde{\delta}(p, L) = k\) and \(\tilde{\delta}(p, H_\infty) = 0 - 1\) in \(\tilde{A}\). Now some \(u'(p)\) is contained in the same open half-space determined by \(L\) and \(H_\infty\) iff both support or both intersect \(A\) at \(p\) iff

\[
\sigma_0(p) + \sigma_{k+1}(p) = 0 \pmod{2}.
\]

If \(-1 \leq k < \pi(P, p)\), let \(L\) and \(H_\infty\) be lines such that \(P \subset L \cap H_\infty\), \(\delta(p, L) = K\) and \(\delta(p, H_\infty) = -1\). Since \(k < \pi(P, p)\), \(\tilde{\delta}(p, L) = \delta(p, L)\) by 2.2.1. Then as above \(L\) supports \(\tilde{A}\) at \(p\) iff \(\tilde{\sigma}_{-1}(p) + \tilde{\sigma}_k(p) = 0 \pmod{2}\). Hence \(\tilde{\sigma}_k(p) = \sigma_k(p)\).

If \(-1 < \pi(P, p) \leq k \leq 0\), consider the lines \(L\) and \(H_\infty\) with \(P \subset L \cap H_\infty\), \(\delta(p, L) = k + 1\) and \(\delta(p, H_\infty) = -1\). Then \(\tilde{\delta}(p, L) = \delta(p, L) - 1 = k\) by 2.2.1 and \(\tilde{\sigma}_k(p) = \sigma_{k+1}(p)\).

Case 2. If \(0 < \pi(P, p)\), then \(\tilde{\sigma}_k(p) = \sigma_k(p)\) for \(k = -1, 0\). Hence \(\tilde{a}_0(p) = \tilde{\sigma}_0(p) = \sigma_0(p) = a_0(p) \pmod{2}\). Since \(|a_0(p) - a_0(p)| \leq 2\), the result follows. Similarly, \(\tilde{a}_0(p) = a_1(p)\) if \(\pi(P, p) < 0\).

If \(0 = \pi(P, p)\), then \(\tilde{\sigma}_0(p) = \sigma_{-1}(p)\). Since \(\sigma_{-1}(p) \equiv a_0(p) + a_1(p) \pmod{2}\), we have

\[
\tilde{a}_0(p) = a_0(p) + a_1(p) \pmod{2}.
\]
Case 3. If \( P = A(p) \) then \( \tilde{a}_0(p) = a_1(p) \). Hence
\[
\tilde{\beta}_0(p) = \tilde{a}_0(p) = a_1(p) = \beta_1(p) - \beta_0(p).
\]

2.6 Ordinary and singular points

The order of a point \( p \) is the minimum order which a neighbourhood of \( p \) can possess. A point \( p \) is ordinary if it is of order 2; otherwise \( p \) is a singularity. If there exist \( u^+(p) \), \( u^-(p) \) of order 2, we say \( p \) is elementary. An arc is ordinary [elementary] if each of its points is ordinary [elementary].

It should be noted that in \( P' \), if a point \( p \) is ordinary, then it is regular. Considering the case of the real projective plane, let \( p \in \mathcal{J} \) be ordinary. Then there is \( u(p) \) of order 2 on \( A \). Take \( q \in u(p) \), \( q \neq p \) (Figure II.13), by 2.4.1. Cor. 2, \( u(p) \) has order 1 on both \( A/p \) point \( A/q \). Writing \( \tilde{A} = A/p \) and \( \tilde{A} = A/q \), \( \tilde{a}_0(p) = a_1(p) \) by 2.5.2. Since \( A(q) \notin A_1(p) \) by 2.4.1, Cor 3, we have \( \tilde{a}_0(p) = a_0(p) \).

But \( \tilde{a}_0(p) = 1 = \tilde{a}_0(p) \), hence \( p \) is regular.

2.6.1 Lemma: An ordinary point is regular. Thus an ordinary arc is regular.

In our study of secants, it was established that \( A^k(X^{k+1}) \) is the set of all \( k \)-secants of \( X \) on \( A \), where \( X \subset \mathcal{J} \). The question that naturally arises is, when is \( A^k \) continuous? Trivially, for \( k = -1, 0 \) and 2, it is continuous into \( P^2 \). Hence we need only consider the continuity of 1-secants. But we already have a tool for the study of lines namely order.
Figure II.13
If an arc $A$ has order 2 and a line $L$ meets $A$ at distinct points $p$ and $q$. It seems intuitively true that if we hold say $p$ constant and vary $q$ continuously on $J$, $L$ will vary continuously on $A$ since $L$ can meet $A$ at most twice. This is in fact what happens.

2.6.2 Theorem: If $(p,q)$ is of order 2 then $A^k$ is continuous on $[p,q]^{k+1}$; $-1 \leq k \leq 2$.

Proof: In 2.4.1, it was shown that if $(p,q)$ has order 2 then $[p,q)$ and $(p,q]$ is 1-independent, hence also of order 2. Therefore we need only consider the case $k = 1$.

**Case 1.** $x = (p_0, p_1) \subset (p,q)^2$, $p_0 \neq p_1$. Then there exists a neighbourhood $U$ of $x$ in $[p,q)^2$ such that if $y = (q_0, q_1) \subset U$ then $q_0 \neq q_1$. Now $(p,q)$ is of order 2 hence $[p,q)$ is 1-independent and for any $r \subset (p,q)$

$$\delta(r, A^r(y)) = \bigvee (r, y) \leq 0 \quad \text{for } y \subset U.$$

Therefore

$$A^r(y) = \bigvee_{p \leq r \subset q} A^r(\delta(r, A^r(y)))(r)$$

is continuous on $U$ since $A$ and $A_{-1}$ are continuous.

**Case 2.** $x = (r,r) \subset (p,q)^2$. Consider $x_i \subset (p,q)^2$ such that $x_i \longrightarrow x$ and $A^1(x_i) \longrightarrow L$. By holding one component of the $x_i$ fixed, $A(r) \subset L$. Since $(p,q)$ is of order 2, it is ordinary and regular by 2.6.1. Hence we are considering a case as in Figure II,14. We wish to show $L = A^1(x) = A_{-1}(r)$.
We may assume \( L \) meets \([p,q]\) only at \( r \) and there exists a line \( H_{\infty} \) which does not meet \([p,q]\). Then \( p,q \) lie in the same open half-space determined by \( M = A(x_1) \) and \( H_{\infty} \) iff

a) if \( M \) meets \( A \) at two distinct points then it either supports or cuts \( A \) at both points,

or b) if \( M \) meets \( A \) exactly once then it supports \( A \) at that point;

that is

\[
\sum_{p<s<q} \sigma \delta(S,M)(S) \text{ is even.}
\]

But since \( (p,q) \) is regular and \( l \)-independent

\[
\sum_{p<s<q} x \delta(S,M)(S) \equiv \sum_{p<s<q} \left( \sum_{i=0}^{\delta(S,M)} a_i(S) \right) \pmod{2}
\]

\[
\equiv \sum_{p<s<q} \left( \sum_{i=0}^{\delta(S,M)} 1 \right) \pmod{2}
\]

\[
\equiv \sum_{p<s<q} (\delta(S,M)+1) \pmod{2}
\]

and \( l = \sum_{p<s<q} (\delta(S,M)+1)-1 \). Hence

\[
\sum_{p<s<q} (\delta(S,M)+1) \equiv 2 \pmod{2}
\]
Figure II.14
Therefore $L$ supports $A$ at $r$. If $L$ contains $A(r)$ but $L \neq A^1(r)$ then since $r$ is regular, $L$ cuts $A$ at $r$; a contradiction. Hence $L = A^1(x) = A_1(r)$.

**Case 3.** $x = (p, p) \in J^2$. Let $P$ be a point with $\pi(P, p) = 0$. Put $L = PA(r)$ for fixed $r \in (p, q)$, (Figure II,15). Since $A$ is continuous and $A(p) \neq A(r)$, there exists $u^+(p)$ such that for any $y \in (u^+(p))$, $A(y) \notin L$. Then by 2.6.2 put $\varphi(y) = A^1(y) \wedge L$ for $y \in (u^+(p))^2$. Since $\varphi(y) \neq A(r)$, $\varphi$ is monotone. Let $Q = \lim_{y \to x^+} \varphi(y)$. Since there exist $y_i \in u^+(p)^2$ such that $y_i \to x$ and $A^1(y_i) \to A^1(x)$, one has $P = Q$.

Now let $x_i$ be a sequence in $(p, q)^2$ with $x_i \to (p, p)$ and $A^1(x_i) \to M$. Since $A$ is continuous, $A(p) \subseteq M$. Now we can construct a sequence $y_i \in (p, q)^2$ such that $y_i \to x$ and $A^1(y_i) \to M$. But then since $\varphi(y_i) \subseteq A^1(y_i)$, $P = \lim \varphi(y_i) \subseteq M$ and $M = A_1(p)$.

**Corollary:** If $A$ is elementary then $A^1$ is continuous.

**Proof:** Let $p \in J$ be elementary. Then there exist $u^-(p) = (r, p)$, $u^+(p) = (p, q)$ where $r < p < q$ such that $u^+(p)$, $u^-(p)$ are of order 2. From above, $A^1$ is continuous on $(r, p]$ and $[p, q)$, hence $A^1$ is continuous on $(r, q)$.

Using 2.6.2, we can derive further properties of ordinary points.

**2.6.3 Lemma:** Let $p$ be ordinary and $\pi(P, p) = 1$ then $p$ is ordinary on $A/P$.

**Proof:** If $p$ is not ordinary on $A/P$ then for every neighbourhood $u_1(p)$, there exists a 1-secant which meets $u_1(p)$
at points say $p_{0i}, p_{1i}$. Then $p_{j1} \rightarrow p$ for $j = 0, 1$ and
$P \subseteq A^1(p_{0i}, p_{1i})$ for all $i$. But $A^1$ is continuous, by 2.6.2, and
$P \subseteq A^1(p, p) = A_1(p)$; a contradiction.

2.6.4 Lemma: Suppose $L$ is a line with $A(p) \notin L$ for all
$p \in J$. Put $\psi(p) = A_1(p) \cap L$ for $p \in J$. If $A$ is ordinary then $\psi$
is monotone.

Proof: Put $\psi(x) = A_1(x) \cap L$ if $x \in u(p)^2$ where $u(p)$
is a neighbourhood of $p$. We may take $u(p)$ to be of order 2 and by
2.6.2, $\psi(j^2) \notin L$. Hence by 2.4.2, $\psi$ and $\varphi$ are monotone.

2.6.5 Lemma: Let $A$ be of order 2. For any $P$, there exist
$p \in J$ such that $p \notin A_1(p)$.

Proof: Trivially there exists $q \in J$ such that $P \neq A(q)$.

Let $L$ be a line through $P$ with $A(q) \notin L$, (Figure II,16). Since
$A$ is continuous, there exists $u(q)$ such that $A(r) \notin L$ for $r \in U(q)$.

Put $\psi(r) = A_1(r) \cap L$. By 2.6.4, $\varphi$ is monotone. Hence there exists
some point $p$ such that $\psi(p) \neq P$; that is, $P \notin A_1(p)$. 
Figure II.16
CHAPTER III

3.1 Barner Arcs

We call an arc a Barner arc if there exists a continuous mapping

\[ B: J \rightarrow \mathbb{R}^2 \] such that

\[ \delta(p, B(x)) = \beta_{\gamma(p, x)}(p) \cdot 1 \]

for all \( p \in J \) and \( x \in J \).

In this section we shall assume \( A \) is a Barner arc unless stated otherwise. Some elementary properties of Barner arcs are immediate from the definition, namely:

\[ (3.1.1) \quad A(q) \subset B(p) \iff q = p. \]
\[ (3.1.2) \quad \delta(p, B(p)) = \beta_0(p) \cdot 1 \text{ for all } p \in J. \]
\[ (3.1.3) \quad \text{If } p \neq q \text{ then } A(p) \neq A(q). \]
\[ (3.1.4) \quad \text{If } p \neq q \text{ and } A(p) \subset A_1(q) \text{ then } \beta_0(q) = 1. \]

Proof: Since \( A(p) \subset A_1(q) \) then \( A_1(q) \neq B(q) \) and \( \delta(q, B(q)) = 0 \). Hence from (3.1.2), \( \beta_0(q) = 1 \).

In view of the above properties, Barner arcs are simple arcs; that is, arcs where \( A \) is 1-1. Moreover, through each point of the arc we can draw a line which does not meet the arc elsewhere. Since \( \beta_0(p) = a_0(p) \) is either one or two, the line \( B(p) \) may or may not be
a tangent line of \( A \) at \( p \). And \( B(p) = A_{\perp}(p) \) iff \( \beta_0(p) = a_0(p) = 2 \) iff non-tangent lines through \( p \) all support \( A \) at \( p \).

Thus, for instance, if \( A \) is regular (Figure III.1) or if \( A \) has at most infinitesimal (Figure III.2), \( B(p) \neq A_{\perp}(p) \) and \( B(p) \) cuts \( A \) at \( p \).

It is worthwhile to point out that in \( \mathbb{P}^1 \) an arc \( A \) is called a Barner arc if \( A \neq \mathbb{P}^1_0 \).

**3.1.5 Theorem:** Let \( q \in J \). If \( a_0(q) = 1 \), then \( A/q \) is a Barner arc. If \( a_0(q) = 2 \), then the restriction of \( A/q \) to either component of \( J \setminus \{ q \} \) is a Barner arc.

**Proof:** If \( a_0(q) = 1 \), then \( B(q) \neq A_{\perp}(q) \). Consider the point \( B(q) \in \hat{A} = A/q. B(q) \neq \hat{A}(q) \) and \( B(q) \neq \hat{A}(p) \) if \( p \neq q \), otherwise \( B(q) = \hat{A}(p) = A(q)A(p) \) since \( A \) is simple. But then \( A(p) \subseteq B(q) \), a contradiction. Hence \( A \) is not the whole of \( \mathbb{P}^1_0 \).

If \( a_0(q) = 2 \), then \( B(q) = A_{\perp}(q) \) and \( \hat{A}(q) = B(q) \). Trivially \( \hat{A}(q) \neq \hat{A}(p) \) for any \( p \in J \setminus \{ q \} \) and the result follows.

**3.1.6 Theorem:** Let \( p \in J \). Then \( p \) cannot have the characteristic \((2,2)\).

**Proof:** If \( p \) has characteristic \((2,2)\) then \( B(p) = A_{\perp}(p) \) supports \( A \) at \( p \). Take \( u'(p) \) and a line \( H_{\infty} \) such that \( A(p) \not\subset H_{\infty} \) and \( A(q) \not\subset A_{\perp}(p), A(q) \not\subset H_{\infty} \) for all \( q \in u'(p) \). Upon projecting from \( p \), \( a_0(p) = a_0(p) = 2 \). Thus there exist points \( p_1, p_2 \in u'(p) \) with \( p_1 < p < p_2 \) and a line \( L = A(p)A(p_1)A(p_2) \). We may assume \( A(p_1), A(p_2) \) lie on same side of \( A_{\perp}(p) \). If say \( A(p_1) \) lies between \( A(p) \) and \( A(p_2) \),
Figure III.1
then there exists $q \subset [p_1, p_2]$ such that $A(q) = A(p_1) \subset C(p_1)$ a contradiction. Similarly $A(p_2)$ does not lie between $A(p)$ and $A(p_1)$. But then both $L$ and $A_1(p)$ cannot support $A$ at $p$, (Figure III.3), since $A(p) \subset L \cap A_1(p)$.

3.1.7 Lemma: If $(p, q)$ is ordinary then $A(p) \subset A_1(q)$.

Remark: In $P^1$, if $(p, q)$ is ordinary and not equal to $P_0^1$ then $(p, q)$ is of order $1$. Otherwise there is a point of inflection on $(p, q)$ which contradicts the regularity of $(p, q)$.

Proof: If $a_0(q) = 2$, then $B(q) = A_1(q)$ and result follows.

Assume $a_0(q) = 1$ and $A(p) \subset A_1(q)$. Then $\tilde{A} = A/q$ is a Barner arc and $\tilde{A}(p) = \tilde{A}(q)$. Therefore $(p, q)$ is not ordinary on $\tilde{A}$, from above Remark. Let $p_1 \subset (p, q)$ be the first singularity on $\tilde{A}$.

Since $p_1$ is ordinary on $A$, then $A(q) \subset A_1(p_1)$ by 2.6.3. We now project $(p_1, q)$ from $p$ and in the same manner as above, we obtain $q_1 \subset (p_1, q)$ such that $A(p_1) \subset A_1(q_1)$.

Consider the set $S$ of intervals $(r, s)$ such that $(r, s) \subset (p_1, q_1)$ and $A(r) \subset A_1(s)$. Let $s_0$ be the infimum of the $s \subset (p_1, q_1)$ for which there exists an $r$ with $(r, s) \subset S$. Since $p_1$ is ordinary, there exists $u(p_1) \subset (p, q)$ such that $u(p_1)$ is of order 2 on $A$ and $A(p_1) \subset A_1(s)$ for any $s \subset u(p_1)$; hence, $p_1 < s_0$. Let $(r_i, s_i)$ be such that $(r_i, s_i) \subset S$, $r_i$ converges, to say $r_0$, and $s_i \longrightarrow s_0$.

Since $s_0$ is ordinary, $r_0 < s_0$ and $A(r_0) \subset A_1(s_0)$, as above. Repeating the argument of the preceding paragraph, we obtain: $(r, s) \subset (r_0, s_0)$ such that $A(r) \subset A_1(s)$, contradicting the definition of $s_0$. 
3.1.8 Theorem: If \((p,q)\) is ordinary then \([p,q]\) is of order 2.

Proof: If \([p,q]\) is of order 3, then there exist \(p \leq p_1 < p_2 < p_3 \leq q\) such that \(p_1, p_2, p_3\) are collinear. Since \((p,q)\) is ordinary, then \(a_0(p_2) = 1\) and \(A = A/p_2\) is Barner. Since there exists \(u(p_2)\) such that \(u(p_2)\) has order 2 on \(A\), then \(p_2\) is regular on \(A/p_2\) by 2.4.1, corollary 2. If \(r \in (p,q), r \neq p_2\) then \(A(p_2) \notin A_1(r)\) by 3.1.7, and \(r\) is ordinary on \(A/p_2\) by 2.6.3. Therefore \((p,q)\) is ordinary on \(A/p_2\). Since \(A/p_2\) is an arc in \(F^1\), \((p,q)\) is of order 1 on \(A/p_2\). But \(A(p_1) = A(p_3)\); a contradiction.

Corollary: Let \(A\) be a Barner arc. Then \(A\) is ordinary iff \(A\) is of order 2.

3.1.9 Lemma: If \(p < q < r\) and \((q,r)\) is ordinary then \(A(p) \notin A_1(q) \cap A_1(r)\).

Proof: Assume \(A(p) \subset A_1(q) \cap A_1(r)\), (Figure III.4). Since \((q,r)\) is of order 2 on \(A\), there exists \(s_0 \in (q,r)\) such that \(A(p) \notin A_1(s_0)\), by 2.6.5, and \(A_1\) is continuous on \((q,r)\), by 2.6.2. Hence there exist \(q_0, r_0\) such that \(q \leq q_0 < s_0 < r_0 \leq r\), \(A(p) \notin A_1(s)\) for \(s \in (q_0, r_0)\), and \(A(p) \subset A_1(q_0) \cap A_1(r_0)\).

Since \((q_0, r_0)\) is ordinary and \(\pi(p,s) = 1\) for \(s \in (q_0, r_0)\), \((q_0, r_0)\) as ordinary on \(A/p\) by 2.6.3.

Since \(A\) is simple, \(A(p), A(q_0),\) and \(A(r_0)\) are not colinear. Put \(L_1 = A_1(q_0), L_2 = A_1(r_0)\). Then if \(s \in (q_0, r_0)\), \(A(s) \notin L_1\) otherwise \((q_0, s_0)\) would not be ordinary on \(A/p\) from the Remark of 3.1.7. Therefore for \(s \in (q_0, r_0)\), \(A(s) \notin L_i, i = 1, 2\). Let \(L\) be a line such that
$L \cap A[q_0, r_0] = \emptyset$ and $A(p) \subseteq L$. Put

$$\varphi(s) = A_1(s) \cap L$$

for $s \in [q_0, r_0]$. Since $A_1$ is continuous so is $\varphi$; moreover, $\varphi$ is monotone on $[q_0, r_0]$ by 2.6.4. But $\varphi(q_0) = \varphi(r_0)$ and hence $\varphi$ cannot exist.

### 3.2 Regular Barner arcs

In this section, unless stated otherwise, we assume $A$ is a Barner arc with at most inflections. The following notation will be used: $H_\infty$ is a line which does not meet a given closed interval $[p, q]$. Take, for example, $H_\infty = B(r)$, $r \notin [p, q]$. Let two points $P, Q$ not in $H_\infty$ determine a line $L$. Now $L$ will meet $H_\infty$ and $P, Q$ divides $L$ into two components. Denote by $L_\infty [L_r]$ the open segment [not] containing $L \cap H_\infty$.

The major result to be proved in this section is that a regular Barner arc has order 2.

#### 3.2.1 Lemma:

Suppose $(p, q)$ has order 2 and $H_\infty$ is a line not meeting $[p, q]$. Let $L = A(p) A(q)$. Then $A_1(r) \cap L \in L_\infty$ for all $r \in (p, q)$.

**Proof:** Since $(p, q)$ has order 2, $(p, q)$ is ordinary on $A$ and $A_1(r)$ supports $A$ at $r$ for $r \in (p, q)$. Moreover, $A_1(r)$ does not meet $[p, q]$ outside of $r$ by 2.4.1, Corollary 3. Hence $A(p), A(q)$
Figure III.6
Figure III.7
lie in some open half-space determined by $H_\infty$ and $A_\perp(r)$, (Figure III.5), and the result follows.

**3.2.2 Lemma:** Suppose $p < q < r$. Let $H_\infty$ and $L = A(p) A(q)$ be given. If $(q, r)$ is of order 2 and $A(p) \subset A_\perp(r)$, then $A_\perp(s)$ meets $L_f$ for all $s \in (q, r)$.

**Proof:** Since $A(p) \subset A_\perp(r)$, $A(p) \not\subset A_\perp(s)$ for all $s \in (q, r)$, by 3.1.9. Therefore $\pi(p, s) = 1$ for all $s \in (q, r)$ and $(q, r)$ is ordinary and of order 1 on $A/p = \widetilde{A}$ by 2.6.3 and the Remark of 3.1.7. But then $\tilde{A}(q) \not\subset A(s)$ for all $s \in (q, r)$ and $\tilde{A}(s) \not\subset \tilde{A}(t)$ for all $s \neq t$ in $(q, r)$. Hence neither $A(p), A(q), A(s)$ nor $A(p), A(t), A(s)$ are collinear. Thus

$$L \cap A(q, r) = \emptyset$$

and

$$A(p) \not\subset A_\perp(s, t)$$

for $s \neq t$ in $(q, r)$. Since $A(p) \not\subset A_\perp(s)$ for $s \in (q, r)$, the restriction $s \neq t$ can be dropped. From 2.4.1, $(q, r)$ is 1-independent; thus

$$A(q) \not\subset A_\perp(s, t)$$

for $s, t$ in $(q, r)$. Since $A_\perp$ is continuous on $(q, r)$, we have $A_\perp(s, t)$ meets either $L_f$ or $L_\infty$ for all $s, t \in (q, r)$.

Since $L \cap A(q, r) = \emptyset$, we may define

$$\varphi(t) = A_\perp(r, t) \cap L$$
for all \( t \in [q, r] \), (Figure III.6). Then \( \varphi \) is monotone and continuous.

Now \( \varphi(r) = A_1(r) \cap L = A(p) \) and \( \varphi(q) = A(q) \).

Since \((q, r)\) is ordinary, it is regular and \( a_0(r) = 1 \).

Therefore \( B(r) \neq A_1(r) \) and \( B(r) \) cuts \( A \) at \( r \). Then \([p, q]\) lies in one of the open half-spaces determined by \( B(r) \) and \( H_\infty \) and \( B(r) \) meets \( L \) in \( L_\infty \). But \( \varphi(t) \neq B(r) \) for all \( t \in (q, r) \) by the definition of \( B(r) \). Since \( \varphi(t) \) moves continuously from \( A(q) \) to \( A(p) \) as \( t \) moves from \( q \) to \( r \), there is a \( t_0 \) such that

\[
\varphi(t_0) = A_1(r, t_0) \cap L \in L_\infty.
\]

Thus \( A'(s, t) \) meets \( L_\infty \) for all \( s, t \in (q, r) \).

It is interesting to note that the condition \( p < q < r \) and \( A(p) \subseteq A_1(r) \) determines the concavity (towards) of \((q, r)\) of order 2 with respect to \( P \). If we retain the original hypothesis with the exception that \( A(p) \subseteq A_1(q) \) now, then since \( A(p) \neq A_1(r) \) by \( 3.1.9 \) \((q, r)\) is concave away from \( p \) (Figure III.7).

**Lemma:** Suppose \( A \) is regular and that there exists \( p < q, q \) ordinary such that \( A(p) \subseteq A_1(q) \). Then there exist \( r, s \), with \( s \) singular, such that \( A_1(s) \) cuts \( A \) at \( r \).

**Proof:** Take \( s \) as small as possible such that \((s, q)\) is ordinary. (This \( s \) exists, otherwise \((p, q)\) is ordinary which is impossible by \( 3.1.8 \)). Therefore \( s \) is singular, \( p < s < q \), and \((s, q)\) is of order 2 on \( A \). Let \( H_\infty \) be a line not meeting \([p, q]\) and put \( P = A(s) A(q), S = A(p) A(q), Q = A(p) A(s) \), (Figure III.8).
For any \( t \in (s, q) \), \( A_\perp(t) \) meets \( P_\infty \) by 3.2.1 and \( Q_f \) by 3.2.2. Thus \( A_\perp(t) \) meets \( S_f \). Since \( A(p) \subset A_\perp(q) \), \( A(p) \not\subset A_\perp(s) \) by 3.1.9 and \( A(q) \not\subset A_\perp(c) \), since \((q, s)\) is of order 2. Therefore \( A_\perp(s) \) meets \( S_f \) since \( A_\perp \) is continuous on \((s, q)\) and \( A_\perp(t) \) meets \( S_f \)
for \( t \in (s, q) \). Since \( A_\perp(s) \) does not meet \((s, q)\), there is an \( r \in (p, s) \) such that \( A_\perp(s) \) meets \( A \) at \( r \). Since \( s \) is regular, the result follows.

It should be noted that the arc depicted in Figure III.8 is not a regular arc but an arc with at most inflections. However it is a Barner arc since through each point of the arc, a line can be drawn through that point which does not meet the arc elsewhere. It is this fact, that it is impossible to construct a regular Barner arc such that \( A(p) \subset A_\perp(q) \) for some two distinct points of the arc, which indicates that a regular Barner arc is of order 2. In fact, we shall use 3.2.3 primarily to prove that a regular Barner arc is of order 2.

3.2.4  a) In the Remark of 3.1.7, it was stated that if \( A \) is a regular arc in \( P^1 \) not equal to \( P_0^1 \) then \( A \) is of order 1. Hence, if \( A \) is a regular Barner arc in \( P^1 \) then \( A \) is of order 1.

b) Let \( A \) be a Barner arc in \( P^1 \), if \( A \) has a singularity then it is an elementary singularity.

**Proof:** If \( A \) has no elementary singularity, then it has an inflection point \( p^1_1 \). Then there is a neighbourhood \((q_1, r_1)\) of \( p^1 \) such that \( A(p) \neq A(p^1) \) for all \( p \in (q_1, p^1) \cup (p^1, r_1) \). We may assume \( A(q^1) = A(p^1) \) and \( A(p) \neq A(q^1) \) for all \( p \in (q^1, r^1) \). Since \( p^1 \) is not elementary then \((q^1, p^1)\) (and) or \((p^1, r^1)\) contains an inflection.

Let \( X_\perp \) be the interval containing an inflection and \( Y_\perp \) be the other
interval. Then \( X_1 \cap Y_1 = \emptyset \) and \( A(X_1) = A(Y_1) \).

Repeat the above argument using \( X_1 \) instead of \( J \). Thus \( p_2, q_2, r_2, X_2 \) and \( Y_2 \) are defined. Continuing indefinitely, one obtains sequences \( X_i, Y_i \) such that \( X_i \cap Y_i = \emptyset \), \( A(X_i) = A(Y_i) \), and \( \overline{X}_{i+1} \subseteq \overline{Y}_{i+1} \subseteq X_i \); \( i = 1, 2, \ldots \). It follows that \( A(X_{i+1}) = A(X_i) \subseteq A(Y_i) \), \( i = 1, 2, \ldots \). Since \( J \) is isomorphic to the set of real numbers and \( \{ X_i \} \) is a nested sequence of intervals in \( J \), then there exists a point \( P \in \bigcap_{i=1}^{\infty} X_i \).

Hence \( P = A(p) \in \bigcap_{i=1}^{\infty} A(X_i) \). But then \( P \) meets each \( Y_i \) and since the \( Y_i \) are disjoint, \( P \) meets \([q_1, r_1]\) an infinite number of times, a contradiction by 4.3.1.

**2.2.5 Lemma:** If \( A \) is not of order 2, then there exist \( p < q \) such that \( 6(p, A_1(q)) = 0 \).

**Proof:** A not of order 2 implies there exist three points \( p < p_1 < p_2 \) which are collinear. Consider \( \overline{A} = A/p \). In \( \overline{A} \), \( A(p_1) = A(p_2) \) and therefore \( (p, p_2) \) is not of order 1 on \( \overline{A} \). By 2.2.4, \( (p, p_2) \) is not regular on \( \overline{A} \).

If there exists only one inflection point \( q \) of \( \overline{A} \) on \( (p, p_2) \), then \( (p, q) \) is regular and of order 1 on \( \overline{A} \) by 2.2.4. But then \( A_1(p) \neq A(q) = A(p) A(q) \) and \( A_1(p) \neq A_1(q) \). Since \( a_0(q) = 2 \) and from 2.5.2

\[
\begin{align*}
\overline{a}_0(q) = & \begin{cases} 
  a_0(q) & 0 < \pi(p, q) \\
  a_0(q) + a_1(q) & 0 = \pi(p, q) \\
  a_1(q) & 0 > \pi(p, q)
\end{cases} 
\pmod{2}
\end{align*}
\]
0 = \pi(p, q); \text{ otherwise, } \pi(p, q) = -1 \text{ a contradiction since } A \text{ is Barner.}

or \( \pi(p, q) = 1 \) and \( \tilde{a}_0(q) = a_0(q) = 2 \), a contradiction. Since \( A(p) \subset A_1(q) \), the result follows.

If \( q_1 < q_2 \) are inflections of \( \tilde{A} \) in \( (p_1, p_2) \) then by 3.2.4, there is an ordinary point \( q_3 \) of \( \tilde{A} \) in \( (q_1, q_2) \). Take \( q_1 < q_4 < q_3 < q_5 < q_2 \) such that \( (q_4, q_5) \) is the largest regular subarc of \( (q_1, q_2) \) in \( \tilde{A} \). Then there exists a point of inflection \( q \) of \( \tilde{A} \) such that \( \tilde{A}(p) \neq \tilde{A}(q) \); otherwise \( \tilde{A}(p) = \tilde{A}(q_4) = \tilde{A}(q_5) \) and \( (q_4, q_5) \) is not regular on \( \tilde{A} \). Since \( \tilde{A}(p) \neq \tilde{A}(q) \) for some inflection \( q \) of \( \tilde{A} \), the result follows as in the preceding paragraph.

3.2.6 Lemma: If \( A \) is regular but not of order 2 then there exist \( p < q < r \), \( q \) singular and \( A(p) A(q) A(r) \) a line which cuts \( A \) at \( p \).

Proof: By 3.2.5, there exist \( s < q \) with \( \delta(s, A_1(q)) = 0 \).

Since \( a_0(s) = 1 \), \( A_1(q) \) cuts \( A \) at \( s \). By 3.2.3, we may assume that \( q \) is singular. Projecting from \( q \), \( \tilde{A}(q) = \tilde{A}(s) \) and \( \tilde{A}(q) \) cuts \( \tilde{A} \) at \( q \). Hence there exists \( r \) with \( q < r \) and \( \tilde{A}(r) \) cutting \( \tilde{A} \) at \( p < q \); that is, \( A(r) A(p) A(q) \) is a line which cuts \( A \) at \( p \).

(Figure III.9).

3.2.7 Theorem: Let \( A \) be a Barner arc. Then

a) if \( A \) is regular \( A \) is of order 2,

b) if \( A \) has at most inflections and \( A_1 \) is continuous, \( A \) has an ordinary point.

Proof: a) Assume \( A \) is not of order 2. Take \( p_1, q_1, r_1 \) with the properties of \( p, q, r \) in 3.2.6. Let \( X_1 \) be a neighbourhood of \( q_1 \)
such that \( r_1 \not\in X_1 \) and for any \( q \in X_1, A(q) A(r_1) \) meets \( A \) in a point \( p \not\in X_1 \), \( p < q_1 \). Since \( q_1 \) is a singularity, we may repeat this argument using \( X_1 \) instead of \( J \) and obtain \( p_2, q_2, r_2 \) and \( X_2 \). Continuing one obtains \( p_i, q_i, r_i \) and \( X_i \); \( i = 1, 2, \ldots \) such that \( X_{i+1} \subseteq X_i, r_i \not\in X_i, q_i < r_i \) and for \( q \in X_{i+1}, A(q) A(r_{i+1}) \) meets \( X_i \setminus X_{i+1} \) in \( p < q_{i+1}; i = 1, 2, \ldots \). We may assume \( \bigcap X_i = \{ q \} \).

Then \( q < r_i; i = 1, 2, \ldots \). Also \( A(q) A(r_{i+1}) \) meets \( X_i \) in a point \( \overset{\sim}{A} \) is regular on \( A \) it is regular on \( A = A/q \). Therefore there exists, \( u(q) = u - (q) \cup \{ q \} \cup u + (q) \) of order 1 on \( \overset{\sim}{A}; \) that is \( \overset{\sim}{A}(u(q)) \cap \overset{\sim}{A}(u(q)) = \emptyset \). Taking \( X_i \subseteq u(q) \), we obtain a contradiction.

b) Suppose \( A \) has no ordinary point. By 3.2.5, there exist \( p < q \) such that \( \delta(p, A(q)) = 0 \). Now \( A(q) \) cuts \( A \) at \( p \). Let \( X_1, Y_1 \) be disjoint neighbourhood of \( p, q \) respectively such that \( A(q) \) meets \( X_1 \) if \( q \in Y_1 \). This is possible since by hypothesis \( A \) is continuous. Repeating this process and using \( X_1 \) instead of \( J \) one obtains \( X_2, Y_2 \). Continuing indefinitely one obtains contradiction as in 3.2.4.

3.3 Existence theorems

3.3.1 Lemma: Let \( A \) be an arc such that for each \( p \in J, A_1(p) = 2 \) for at most one \( i; i = 1, 2 \). If there is a point \( P \) such that \( A = A/p \) is of order 1 and
\[ \pi(P, p) = \begin{cases} 
1 & \text{if } P \text{ regular} \\
i & \text{if } a_i(p) = 2 
\end{cases} \]

for all \( p \in J \), then \( A \) is a Baner arc.

**Proof:** Put \( B(x) = \tilde{\lambda}(x) \) for \( x \in J \). Since \( \tilde{\lambda} \) is continuous, so is \( B \). Put \( \delta = \tilde{\gamma}(p, B(x)) \) and \( \delta = \delta(p, B(x)) \). We must show

\[ \delta = \beta \gamma(p, x)(p) - 1 \]

for all \( p, x \in J \). Since \( \tilde{\lambda} \) is of order 1, \( \delta = \gamma(p, x) \).

**Case 1:** \( p \) is regular. \( \pi(P, p) = 1 \) implies \( \tilde{\delta} = \delta \) by 2.2.1 and \( \tilde{\delta} = \delta = \gamma(p, x) \). If \( \gamma(p, x) = 0 \), \( \beta \gamma(p, x)(p) - 1 = \beta_0(p) - 1 = a_0(p) - 1 = 0 \). If \( \gamma(p, x) = -1 \), \( \beta \gamma(p, x)(p) - 1 = \beta_{-1}(p) - 1 = -1 \).

Therefore the result follows.

**Case 2:** \( a_i(p) = 2 \). Then by 2.2.1,

\[ \delta = \begin{cases} 
\delta & -1 \leq \delta < \pi(P, p) = i \\
\delta - 1 & i = \pi(P, p) < \delta \leq 1.
\end{cases} \]

and

\[ \beta_k(p) = \begin{cases} 
k+1 & \text{if } -1 \leq k < i \\
k+2 & \text{if } i \leq k \leq 1.
\end{cases} \]

Therefore iff \( \delta < i \), \( \delta = \tilde{\delta} = \gamma(p, x) \) and \( \beta \gamma(p, x)(p) = \gamma(p, x) + 1 \); hence \( \delta = \beta \gamma(p, x)(p) - 1 \). If \( \delta > i \), \( \delta = \delta - 1 = \gamma(p, x) + 1 \). Thus \( \gamma(p, x) \geq i \) and \( \beta \gamma(p, x)(p) = \gamma(p, x) + 2 \). The result now follows.
3.3.2 Theorem: Barner arcs exist in the real projective plane.

Proof: Let A be an arc and let (p,q) be of order 2. By 2.6.1, (p,q) is regular on A and by 2.4.1, Corollary 2, (p,q) has order 1 on A/p. Applying 2.3.1 with P = A(p), we are finished.

In 3.2.7 b) it was proved that a Barner arc, with at most inflections on which A_1 is continuous, has an ordinary point. The remainder of this section is involved in proving a more general statement about ordinary points; namely that every arc has an ordinary point.

3.3.3 Lemma: Let W be a non-empty set of points of an arc A such that A_1 is continuous at each point of W. Then there is a subarc X which contains a point of W such that if p,q ∈ X_1 p ≠ q, p ∈ W, then A_1(p) does not cut A at q.

Proof: Suppose that for every subarc X with X ∩ W ≠ ∅, there exist p,q ∈ X, p ≠ q, p ∈ W such that A_1(p) cuts A at q. Let p_1,q_1 ∈ J be two such points. Since p_1 ∈ W and q_1 ∉ W, there are disjoint neighbourhoods X_1,Y_1 of p_1,q_1 respectively, such that if p ∈ X_1 then A_1(p) meets Y_1. Since p_1 ∈ X_1 we may repeat this construction replacing J by X_1, and so on. As in 3.2.7 b), we get a contradiction.

3.3.4 Lemma: Consider the following properties of arcs in P^1:

a) Every arc has an ordinary point.

Proof: This is trivial by 3.2.4 (b), since an arc in P^1 has at most inflections.
b) If $A$ is a Barner arc, then either $A$ is of order 1 or $A$ has an elementary inflection.

Proof: If $A$ has no singularity then $A$ is ordinary. If $A$ has a singularity then it is elementary by 3.2.4 (b).

**3.3.5 Lemma:** Let $A$ be an arc and $L$ be a line such that $A(p) \notin L$ for all $p \in J$. Let $\{ P_i | i = 1, 2, ... \}$ be a dense set of points in $L$. Then there exists a sequence $X_i; i = 1, 2, ...$ of open intervals such that $X_i$ has order 1 on $A^{(1)} = A/P_i$ and $\{ X_i \}$ is nested.

Proof: Consider $A^{(1)} = A/P_1$ an arc in $P^1$. By 3.3.4(a), there exists $p_1 \in J$ such that $p_1$ is ordinary on $A^{(1)}$. Let $X_1$ be a neighbourhood of $p_1$ such that $X_1$ has order 1 on $A^{(1)}$. Now consider $X_1$ and $P_2$ instead of $X$ and $P_1$. By 3.3.4(a), there exists $p_2 \in X_1$ such that $p_2$ is ordinary on $A^{(2)}$. Let $X_2$ be a neighbourhood of $p_2$ such that $X_2 \subset X_1$ and $X_2$ has order 1 on $A^{(2)}$. Repeating the construction for $i = 3, 4, ...$, the result follows.

**3.3.6 Lemma:** Let $A$ be an arc with at most inflections. Then there exists a point at which $A_1$ is continuous.

Proof: We may assume there exists a line $L$ such that $A(p) \notin L$ for $p \in J$. Let $\{ P_i | i = 1, 2, ... \}$ be a set of points of $L$ which is dense in $L$. By 3.3.5, there exists a nested sequence $\{ X_i \}$ of open intervals such that $X_i$ is of order 1 on $A/P_i; i = 1, 2, ...$. Take $p \in \bigcap_{i=1}^{\infty} X_i$ and put $P = A_1(p) \cap L$.

We wish to show $A_1$ is continuous at $p$, hence $\lim_{q \to p} A_1(q) \cap L = P$. 
Let \( U(P) \) be a neighbourhood of \( P \) on \( L \), say one with end points \( P_i \) \( P_j \) where \( i < j \). Take \( q < p < r \); \( q, r \in X_j \). Since \( X_j \) is of order 1 on \( A/P_j \), \( A(s) \neq A(t) \) for \( s, t \in X_j, s \neq t \). We may define a continuous path (Figure III.10) in \( P^2 \) by

\[
A_1(p) \xrightarrow{\psi_1} A(p)A(r) \xrightarrow{\psi_2} A(q)A(r) \xrightarrow{\psi_3} A_1(q)
\]

where

\[
\begin{align*}
\psi_1(s) &= A(p)A(s) & s & \in (p, r) \\
\psi_2(s) &= A(s)A(r) & s & \in (q, p) \\
\psi_3(s) &= A(q)A(s) & s & \in (q, r).
\end{align*}
\]

Since \( X_j \) is of order 1 on \( A/P_i \), \( A/P_i \), no line of this path, except possibly \( A_1(q) \) contains \( P_i \) or \( P_j \). For if say \( \psi_1(s) = A(p)A(s) \) contains \( P_i \), then \( (A/P_i)(p) = (A/P_i)(s) \) on \( A/P_i \); a contradiction. Hence \( A_1(q) \cap L \subset U(p) \) and similarly \( A_1(r) \cap L \subset U(P) \). The path is continuous since \( A \) is continuous. Since \( q \) and \( r \) are arbitrary points of \( X_j \) with \( p < q < r \), the continuity of \( A_1 \) at \( p \) follows.

In Figure III.10, the path moves along \( s_i \) as \( i \) increases. Hence \( \psi_1 \) moves from \( A_1(p) \) to \( A(p)A(r) \), \( \psi_2 \) from \( A(p)A(r) \) to \( A(q)A(r) \) and finally \( \psi_3 \) form \( A(q)A(r) \) to \( A_1(q) \).

**Corollary:** Let \( A \) be an arc with at most inflections. Let \( W \) be the set of points at which \( A_1 \) is continuous, then \( W \) is dense in \( A \).

**Proof:** If \( W \) is not dense in \( A \) there exists \( q \in J \) such that \( q \) is not the limit of any sequence of \( W \). Hence there exists \( U(q) \)
such that \( U(q) \cap W = \emptyset \). But \( U(q) \subseteq A \) has at most inflections, a contradiction by 3.3.6.

3.3.7 Lemma: Every arc contains a Barner arc with at most inflections.

Proof: We may assume there is a line \( L \) not meeting \( A \).

Let \( P_1 \neq P_2 \) be points on \( L \). By 3.3.5, there exist subarcs \( X_1, X_2 \) such that \( X_2 \subseteq X_1 \) and \( X_i \) has order 1 on \( A(i) = A/P_i, i = 1,2 \). Then \( X_2 \) has at most inflections on \( A \); for if \( p \in X_2, P_1 \) or \( P_2 \notin A_1(p) \), say \( P_1 \notin A_1(p) \). Since \( p \) is regular on \( A(1) \) and \( \pi(p,p) = 1 \), \( 1 = a_0(1)(p) = a_0(p) \) and \( p \) is at most an inflection on \( A \).

Applying 3.3.6 to \( X_2 \), there exists a point \( p \in X_2 \) such that \( A_1 \) is continuous at \( p \). Choose \( P \) and \( U(p) \subseteq X_2 \) such that \( P \notin A_1(q) \) for all \( q \in U(p) \). By 3.3.4 (a), there exists \( X \subseteq U(P) \) of order 1 on \( A/P \). Then \( X \) is a Barner arc by 3.3.1.

3.3.8 For the remainder of this section we assume \( A \) is a Barner arc with at most inflections. Moreover, the following notation will remain fixed: \( p_0 \) is a point at which \( A_1 \) is continuous, \((p_1,p_2)\) is a neighbourhood of \( p_0 \) and \( H_\infty \) a line which does not meet \([p_1,p_2]\).

\( A_1(p_0) \) meets \([p_1,p_2]\) only at \( p_0 \) and \( A(p_1) \subseteq A_1(p) \) for any \( p \in (p_0,p_2) \). Since \( A \) is a Barner arc, \( L = A(p_1) A(p_0) \) is a line. Let \( L_\infty \) be the open segment of \( L \) with end points \( A(p_1), A(p_0) \) which does [not] meet \( H_\infty \).

3.3.9 Lemma: Given the above hypothesis, \((p_0,p_2)\) has order 1 on \( A = A/p_1 \) and \( A(p_0,p_2) \cap L = \emptyset \).
Proof: Since \( \pi(p_1, p) = 1 \) for all \( p \in (p_0, p_2) \),
\( \tilde{a}_0(p) = a_0(p) = 1 \) for all \( p \in (p_0, p_2) \). \( a_0(p_1) = 1 \) and hence \( \tilde{A} \) is
Barner by 2.1.5. Therefore \( (p_0, p_2) \) is a regular Barner subarc of \( \tilde{A} \) and by 3.2.4(a), \( (p_0, p_2) \) has order 1 on \( \tilde{A} \).

If \( L \) meets \( (p_0, p_2) \) at a point \( q \), then \( \tilde{A}(p_0) = \tilde{A}(q) \) and
there exist an inflection point \( q^1 \in (p_0, q) \), a contradiction from
above.

3.3.10 Lemma: Suppose \( p_0 \) is an inflection. If there is a
\( p_3 \in (p_0, p_2) \) such that \( (p_0, p_3) \) has order 1 on \( \tilde{A} = A/p_0 \), then for
each \( p \in (p_0, p_3) \) either \( A_1(p) \) meets \( L_f \) or \( A(p_0) \subseteq A_1(p) \).

Proof: Since \( a_0(p_0) + a_1(p_0) = 3 \), \( A_1(p_0) \) cuts \( A \) at \( p_0 \)
and \( A(p_1), A(p_3) \) lie on opposite sides of \( A_1(p_0) \). Since \( A_1 \) is
continuous at \( p_0 \), we can choose \( p_4 \in (p_0, p_2) \) such that \( A(p_4), A(p_3) \)
lie on opposite sides of \( A(p_0)A(p_4) \), (Figure III,11).

Since \( (p_0, p_2) \) is of order 1 on \( A/p_0 \), \( (p_0, p_2) \) and hence
\( p_4 \) is regular on \( \tilde{A} \). Therefore \( A(p_4) \) cuts \( \tilde{A} \) at \( p_4 \) and \( A(p_0)A(p_4) \)
cuts \( A \) at \( p_4 \). Since \( A(p_0)A(p_4) \) meets \( (p_0, p_3) \) only at \( p_4 \), we
have that \( A(p_1) \) and \( A(p_0, p_4) \) lie on the same side of \( A(p_0)A(p_4) \).

Now by projecting from \( p_4 \), we find that \( (p_0, p_4) \) is not of order 1 on
\( A/p_4 \) and there exists \( p_5 \in (p_0, p_4) \) such that \( A(p_5)A(p_4) \) meets \( L_f \).

Let \( p \neq q \in (p_0, p_3) \). Since \( (p_0, p_3) \) is of order 1 on both
\( A/p \) by 2.3.2, and \( A/p_0 \) by hypothesis, neither \( A(p_0) \) nor \( A(p_1) \) lie
on \( A(p)A(q) \). Since \( A(p_4)A(p_5) \) meets \( L_f \), the continuity of \( A \)
implies that \( A(p)A(q) \) meets \( L_f \) for \( p_0 < p < q < p_3 \).
3.3.11 Theorem: Every arc \( A \) has an ordinary point.

Proof: By 3.3.7, we can consider \( A \) to be a Bärner arc with at most inflections. Let \( W \) be the set of points of \( J \) at which \( A \) is continuous. \( W \neq \emptyset \) by 3.3.6. Choose \( X \) as in 3.3.2 and the \( p_0 \in \ X \cap W \). We may take \( X = J \) and hence if \( p \in W \), \( A_1(p) \) does not cut \( A \) at any point except possibly \( p \). Note also that 3.3.8 now applies.

Case 1. \( p_0 \) is regular. Then \( A_1(p_0) \) supports \( A \) at \( p_0 \) and \( p_1p_2 \) lie on the same side of \( A_1(p_0) \). Since \( A_1 \) is continuous at \( p_0 \), choose \( p_3 \in (p_0,p_2) \) such that \( p_1p_2 \) lie on the same side of \( A_1(p) \) for all \( p \in (p_0,p_3) \). Therefore \( A_1(p) \) supports \( A \) at \( p \) for all \( p \in (p_0,p_3) \cap W \) and each \( p \in (p_0,p_3) \cap W \) is regular. We claim that \( (p_0,p_3) \) is itself regular, hence of order 2, by 3.2.7, and thus ordinary.

Let \( q \in (p_0,p_3) \), then there are points \( q_1 \in (p_0,p_3) \cap W \) such that \( q_1 \to q \) by the corollary of 3.3.6. Let \( M \) be a line of accumulation of \( A_1(q_1) \). Let \( (r_1,r_2) \subset (p_0,p_3) \) be a neighbourhood of \( q \) such that \( A_1(q) \) meets \( [r_1,r_2] \) only at \( q \). This is possible by 3.3.1. Since \( q_1 \in (p_0,p_3) \cap W \) is regular and there exists \( q_1 \in (r_1,r_2) \) such that \( r_1,r_2 \) lie on the same side of \( A_1(q_1) \) iff they lie on the same side of \( M \), \( M \) supports \( A \) at \( q \). But then since \( \alpha_0(q) = 1 \), \( M = A_1(q) \) and \( q \) is regular.

Case 2. \( p_0 \) is an inflection.

(i) If there exists \( p_3 \in (p_0,p_2) \) such that \( (p_0,p_3) \) is of order 1 on \( A/p_0 \). By 3.3.6, there exists a point \( p \in (p_0,p_3) \cap W \).
If $A_1(p)$ meets $L_f$, then $A_1(p)$ cuts $A$ at a point of $(p_1,p_2)$, a contradiction by 3.3.2, where $X = J$. Therefore by 3.3.10, $A(p_0) \subset A_1(p)$.

If $A_1(p) \neq A_1(p_0)$ then $a_0(p_0) = 1$ implies $A_1(p)$ cuts $A$ at $p_0$, a contradiction. If $A_1(p) = A_1(p_0)$ then since $p_0$ is an inflection $A_1(p)$ cuts $A$ at $p_0$ again, a contradiction. Therefore this case cannot occur.

(ii) No $u^+(p_0)$ is of order 1 on $A/p_0$. But $A/p_0$ is an arc in $R^1$ and it is Barten since $a_0(p_0) = 1$. Therefore $(p_0,p_2)$ has an elementary inflection on $A/p_0$ by 3.3.4(b). Therefore there exist points $p_3,p_4,p_5$ such that $p_0 < p_4 < p_3 < p_5 < p_2$, $p_3$ is an inflection on $A/p_0$ and $(p_4,p_3),(p_3,p_5)$ are of order 1 on $A = A/p_0$.

Since $p_3$ is at most an inflection on $A$, $a_0(p_3) = 1$ and $a_1(p_3) = 1$ or 2. Now $p_3$ is an inflection on $\hat{A}$ iff $a_0(p_3) = 2$.

But then $\hat{a}_0(p_3) \equiv a_0(p_3) + a_1(p_3) \pmod{2}$ and $\pi(p_0,p_3) = 0$; that is, $A(p_0) \subset A_1(p_3)$. Since $2 \equiv 1 + a_1(p_3) \pmod{2}$ implies $a_0(p_3) = 1$ and $p_3$ is regular on $A$. Hence $p_3$ is regular on $A/p_3$.

$A(p_0) \subset A_1(p_3)$ implies $(A/p_3)(p_0) = (A/p_3)(p_3)$ on $A/p_3$.

Therefore $p_3$ does not have a neighbourhood of order 1 on $A/p_3$ and there exists $p_6 \in (p_4,p_3) \cup (p_3,p_5)$ such that $(A/p_3)(p_6)$ is equal to some $(A/p_3)(q)$ for $q \neq p_3$. But $(A/p_3)(p_0) = (A/p_3)(p_3)$ implies that $q \in (p_1,p_0)$. Hence $A(p_6)A(p_3)$ meets $L_f$.

Suppose $p_6 \in (p_3,p_5)$. Since $(p_3,p_5)$ is of order 1 on $A/p_0$, for any $p \neq q \in (p_3,p_5)$, $A(p_0) \not\subset A(p)A(q)$. From 3.3.2, $(p_0,p_3)$ is of order 1 on $A/p_1$, hence $A(p_1) \not\subset A(p)A(q)$ also. Since $A(p_6)A(p_3)$ meets
Figure III.12
by the continuity of $A$, $A(p)A(q)$ meets $L_x$ for these $p, q$. If we remove the restriction $p \neq q$ and let $r = p, q$, then $A_1(r)$ meets $L_x \cup \{A(p_0)\}$ for all $r \in (p_3, p_5)$. Take $r_1 \neq r_2 \in (p_3, p_5) \cap W$. By the arguments of Case 2(i), we obtain $A(p_0) = A(r_1) = A(r_2)$ in $\hat{A} = A/p_0$, which contradicts $(p_3, p_5)$ of order 1 on $A$. Similarly, $p_6 \in (p_4, p_3)$ will give a contradiction, hence this case cannot happen.

Figure III.12 is not accurate construction of Case 2(ii). Since this case cannot happen, an accurate construction cannot be made. It is meant to be only a visual aid in the understanding of 3.3.11.

3.3.12 Theorem: Let $A$ be a Baner arc with at most inflections and $A_1$ continuous. Then either $A$ is of order 2 or $A$ has an elementary inflection.

Proof: Let $W$ be the set of inflections of $A$. If $W = \emptyset$, $A$ is regular and of order 2 by 3.2.7. Hence we may assume $W \neq \emptyset$. As in 3.3.11, we take $X = J$. Assume $p_0 \in W$.

Case 1. There exists $p_3 \in (p_0, p_2)$ such that $(p_0, p_3)$ has order 1 on $A/p_0$. If there exists $p \in (p_0, p_3) \cap W$, then $A_1(p)$ meets $L_x$ or $A(p_0) \subset A_1(p)$ by 3.3.10. But $A_1(p)$ meets $L_x$ implies $A_1(p)$ cuts $(p_1, p_0)$. Since $A_1$ is continuous, this is a contradiction by 3.2.2. Hence $A(p_0) \subset A_1(p)$, but since $p_0$ is a point of inflection $A_1(p)$ cuts $A$ at $p_0$, a contradiction. Therefore there is no such $p_0$.

Case 2. No $u^+(p_0)$ is of order 1 on $A/p_0$. By 3.2.7, every $u^+(p)$ contains an inflection on $A/p_0$. By 3.3.4(b), there exist $p_0 < p_4 < p_3 < p_5 < p_2$ such that $p_3$ is an inflection on $A/p_0$, every neighbourhood of $p_4, p_5$ contains an inflection on $A/p_0$, and
(p_4, p_3), (p_5, p_5) are of order 1 on A/p_0.

If p is an inflection on A/p_0 then p is regular on A and A(p_0) \subset A_1(p), by 2.2.11, Case 2. Since every neighbourhood of p_4 and p_5 contains an inflection and A_1 is continuous, A(p_0) \subset A_1(p_4) \cap A_1(p_5). Since p_3 is regular on A and A(p_0) \subset A_1(p_3), projecting from p_3 it follows that there is a p_6 \in (p_4, p_3) \cup (p_3, p_5) such that A(p_6)A(p_3) meets L_f.

Suppose p_6 \in (p_4, p_3), then as in 2.2.11 for each p \in (p_4, p_3), A_1(p) meets L_f \cup \{ A(p_0) \}. Hence (p_4, p_3) is not ordinary by 2.1.9. By 2.2.7(a), there is r_1 \in (p_4, p_3) \cap W. Since A_1(r) meets L_f \cup \{ A(p_0) \} then A_1(r_1) = A_1(p_0) as in 2.2.11. Repeating this argument for (r_1, p_3), one obtains r_2 \in (r_1, p_3) \cap W such that A_1(r_2) = A_1(p_0) = A_1(r_1). By projecting from p_0, (p_4, p_3) is not of order 1 on A/p_0, a contradiction.

Similarly p_6 \in (p_3, p_5) gives a contradiction. Therefore there exists u^+(p_0) of order 2. Symmetrically, there exists u^-(p_0) of order 2. Thus p is an elementary inflection.
CHAPTER IV

4.1 Dually Differentiable Arcs

If \( A \) is a dually differentiable arc, then by 1.4, the 'points' of \( A^* \) are the tangent lines of \( A \) and the tangent lines of \( A^* \) are the points of \( J \). Moreover \( A_1 \) is continuous on \( A \).

4.1.1 Let \( p \) be a point of the arc \( A \). Let \( P_i \in A_i(p) - A_{i+1}(p) \) for \( i = 0, 1, 2 \). Then \( \{P_0, P_1, P_2\} \) are independent and are the vertices of four open 2-simplexes, (Figure IV.1). Let \( S^- \) be that open 2-simplex which contains some \( U^-(p) \). Let \( E^-_1 \) be the open segment of \( P_0P_1 \) with end points \( P_0, P_1 \) which is an edge of \( S^- \) and \( E^-_2 \) be the open segment of \( P_0P_2 \) with end points \( P_0, P_2 \) which is not an edge of \( S^- \). Similarly we define \( E^+_i, S^+ \) for \( i = 1, 2 \).

4.1.2 Lemma: \( \sigma_k(p) = 0 \) iff \( E^-_{k+1} = E^+_k, k = 0, 1 \).

Proof: \( \sigma_0(p) = 0 \) iff \( P_0P_2 \) supports \( A \) at \( p \) iff \( S^+, S^- \) lie on the same side of \( P_0P_2 \) iff the edges of \( S^+, S^- \) on \( P_0P_1 \) are the same iff \( E^-_1 = E^+_1 \).

The result follows similarly for \( k = 1 \).

Let \( p \) be a dually differentiable point. Take \( P_i, S^+, S^-, E^+_i, E^-_i \) as in 4.1.1. By 1.4.1, there are neighbourhoods \( U^+(p), U^-(p) \) such that \( P_i \not\in A_+(q) \) for any \( q \in U^-(p) \cup U^+(p), i = 0, 1, 2 \).
Put $\mathcal{G}^-_k(p) = 0$ if $A_1(q)$ meets $E^-_k$ for all $q \in U^-(p)$ where $k = 0, 1$. Define $\mathcal{G}^+_k(p)$ similarly and put $\mathcal{G}^-_{-1}(p) = 0 = \mathcal{G}^+_{-1}(p)$.

**4.1.3. Theorem:** At any point $p$ of a dually differentiable arc

$$\mathcal{G}^-_k \equiv \mathcal{G}^-_{-k} + \mathcal{G}^+_{-k} + \mathcal{G}^-_{-1} + \mathcal{G}^+_{-1} + \mathcal{G}^+_1 \quad (\text{mod } 2)$$

for $k = 0, 1$.

**Proof:** Let $P^*_0 = P_0 P_1$, $P^*_1 = P_0 P_2$, and $P^*_2 = P_1 P_2$. Then $P^*_i$ are points in $P^2$ with $P^*_i \subset A^*_i \subset A^*_{i+1}(p)$, $0 \leq i \leq 2$. For all $q \in J$, put

$$\psi(q) = A^*(q)P^*_2$$

This is a line in $P^*$ since $P^*_2 = P_1 P_2$ is not equal to $A^*_1(q) = A^*_2(q)$ for $q \in J$. Now $\psi(p) = A^*(p)P^*_2 = A^*_1(p) \cap P_1 P_2 = P_2 \neq A^*_1(p)$. Hence

$$\mathcal{G}^-_0(p) = 0 \iff \psi(p) \text{ supports } A^* \text{ at } p \iff \text{there exists } U'(p)$$

such that $A^*(U'(p))$ lie in some open half-space determined by $\psi(p)$ and $P^*P^*_2$, since $P^*P^*_2 = P_0 P_2 \cap P_1 P_2 = P_2$ and $P^*P^*_2 \cap \overline{A}(q) = P_2 A_1(q) = A^*_{-1}(p)$.

Therefore $\mathcal{G}^-_0(p) = 0 \iff \text{there exists } U'(p)$ such that for any $q \in U'(p)$, $A_1(q)$ meets $P_1 P_2$ in the same open segment determined by $P_1, P_2$. Choose $U^+(p), U^-(p)$ such that $P_i \not\subset A_1(q)$ for $i = 0, 1, 2$ and $q \in U_i(p) = U^+(p) \cup U^-(p)$. Then $A_1(q)$ meets $P_0 P_1$ in the same open segment iff $\mathcal{G}^-_0(p) + \mathcal{G}^+_0(p) + \mathcal{G}^-_0(p) \equiv 0 \quad (\text{mod } 2)$.

If $\mathcal{G}^-_0(p) = 0$ then $E^+_1 = E^-_1$ and if $A_1(q)$ meets $E^-_1$ for
all \( q \in U'(p) \), we have \( A_1(q) \) meets \( E_1^+ \) for all \( q \in U'(p) \) \([U'(p)]\). Thus \( \sigma_0^+(p) = 0 = \sigma_0^-(p) \) and the result follows. If \( \sigma_0^+(p) = 1 \) then \( E_1^+ \neq E_1^- \) and if \( A_1(q) \) meets say \( E_1^+ \) for all \( q \in U'(p) \) then \( A_1(q) \) does not meet \( E_1^- \) for all \( q \in U'(p) \). Thus, \( \sigma_0^+(p) = 0 \) and \( \sigma_0^-(p) = 1 \) and the result follows. Similarly \( A_1(q) \) meets \( P_0 P_2 \) in the same open segment determined by \( P_0 P_2 \) for all \( q \in U'(p) \) iff \( \sigma_1(p) + \sigma_1^+(p) + \sigma_1^-(p) \equiv 0 \pmod{2} \).

Combining these results, \( A_1(q) \) meets the same open segment of \( P_1 P_2 \) for all \( q \in U'(p) \) iff \( A_1(q) \) meets the same [different] segments of \( P_0 P_1 \) and \( P_0 P_2 \) simultaneously. If this were not so then the two possible cases Figure IV.2 and Figure IV.3 indicate a contradiction. Therefore

\[
\sigma_0^* \equiv \sigma_0^- + \sigma_0^+ + \sigma_0^- + \sigma_1^+ + \sigma_1^- \pmod{2}.
\]

Similarly the result follows for \( k = 1 \).

**Corollary:** For any point \( p \) of a dually differentiable arc,

\[
a_k^* \equiv a_{1-k}^- + \sigma_{1-k}^- + \sigma_{1-k}^+ + \sigma_{1-k}^- + \sigma_{1-k}^+ \pmod{2}
\]

for \( k = 0,1 \).

**Proof:** For \( k = 0 \), \( a_0^* = a_0^* \) and \( \sigma_0 = a_0 \). Therefore

\[
a_0^* \equiv \sigma_0^- + \sigma_0^+ + \sigma_1^- + \sigma_1^+ + (\sigma_0 + \sigma_1) \pmod{2}
\]

and

\[
\sigma_0 + \sigma_1 \equiv a_0 + a_0 + a_1 = 2a_0 + a_1 \equiv a_1 \pmod{2}.
\]

For \( k = 1 \), \( \sigma_1^* \equiv \sigma_1^- + \sigma_1^+ + \sigma_1^- \pmod{2} \).
Hence

\[ a_1^* \equiv \sigma_1^- + \sigma_1^+ + \sigma_0^- a_0^* \pmod{2} \]
\[ \equiv \sigma_0^- + \sigma_0^+ + \sigma_1^- (\sigma_0^- + \sigma_0^+ + \sigma_1^- + \sigma_1^+ + a_1) \pmod{2} \]
\[ \equiv \sigma_0^- + \sigma_0^+ + a_0^- a_0^+ a_1 \pmod{2} \]
\[ \equiv a_0^- + \sigma_0^- + \sigma_0^+ \pmod{2} \]

4.2 Strong finiteness

A point \( p \) of an arc is strongly left finite if for every \((1-k)\)-space \( L, k = 0, 1 \) there exists a \( U^-(p) \) such that no \( k \)-secant of \( U^-(p) \) meets \( L \). We define strong right finiteness symmetrically. An arc is strongly finite, if each of its points is strongly finite.

4.2.1 Lemma: Let \( \Lambda = A/P, P \) a point. If \( p \) is strongly finite on \( A \) then it is strongly finite on \( \Lambda \).

Proof: Let \( L \) be a point of \( \Lambda \). Then \( L \) is a line in \( P^2 \) with \( P \in L \). \( p \) strongly finite implies there exist \( U^-(p), U^+(p) \) such that \( L \) does not meet \( A(U^-(p)), A(U^+(p)) \). Since \( P \in L \), then for \( q \in U^-(p) [U^+(p)] \), \( \Lambda(q) = A(q)P \). Hence \( \Lambda(q) \) does not meet \( L \) in \( \Lambda \) and \( p \) is strongly left [right] finite on \( \Lambda \).

Let \( p \) be a strongly left finite point of an arc. Take \( P_i, S^-, E_i^- \) an is 4.1.1 and \( U^-(p) \) such that no \( l \)-secant of \( U^-(p) \) meets
the points \( P_i; i = 0, 1, 2 \). Consider the 1-space \( P_{M(1)} \) where \( M(1) = 1 \) or 2. Let \( S^-(M(1)) \) be the open 1-simplex with vertices \( P_0, P_{M(1)} \) which meets the 1-secants of \( U^-(p) \). This makes sense since any two lines in a projective plane must meet.

**4.2.2 Theorem:** \( E_{M(1)}^- = S^-(M(1)) \) for \( M(1) = 1 \) or 2.

**Proof:** Let \( S^- \) be the open 2-simplex which contains \( U^-(p) \) and take \( q \in U^-(p) \) such that \( A(q)A(p) \) does not meet \( (q, p) \) (Figure IV.4). Then \( A(q)A(p) \) divides \( S^- \) into two open triangles and \( (q, p) \) is contained in the triangle which has \( E_1^- \) as an edge. Let \( r \in (q, p) \) such that \( A(q)A(r) \) meets \( E_1^- \) and \( E_2^- \). Thus a 1-secant of \( U^-(p) \) meets \( E_1^-, E_2^- \).

In Figure IV.5, an arc different from the one in Figure IV.4, is used in the construction to illustrate that 4.2.2 is independent of the type of construction used in its proof.

**4.2.3 Lemma:** Let \( p \) be a point of an arc \( A \). Let \( U \subset P^2_{M(1)} \) be a neighbourhood of \( A_1(p) \) and \( U^-(p) \) be given. Then there is a 1-secant \( L \) of \( U^-(p) \) such that \( L \subset U(A_1(p)) \).

**Proof:** Assume \( U \) is open. Since

\[
A_1(p) = A(p)/A(p) = \lim_{q \to p} A(q)A(p)
\]

there exists a \( q \in U^-(p) \) such that \( A(q)A(p) \subset U \). Then there exists \( U'(p) \) such that \( A(q)M \subset U \) for all \( M \subset U'(p) \). Otherwise of \( A(q)M \subset \mathring{U} \) for all \( M \subset U'(p) \), then \( \lim_{M \to p} A(q)M = A(q)A(p) \subset \mathring{U} \),

\[
M \neq P
\]
since $U$ is open, a contradiction. Since $A(q)M$ for $M \in U^1(p)$
is a 1-secant, the result follows.

4.2.4 Theorem: Let $p$ be a strongly left finite point of an arc $A$.
Then $A^1$ is continuous at $p$; that is, $A^1(p_0p_1) \rightarrow A_1(p)$ where
$p_i < p, i = 0,1$ and $(p_0, p_1) \rightarrow (p, p)$.

Proof: Since $A^1(p^2)$ is the set of all 1-secants of $A$, we
must show that given a neighbourhood $U(A_1(p))$, there exists $U^r(p)$ such that

$$A^1(U^r(p)^2) \subseteq U(A_1(p)).$$

Let $L$ be a line such that $A(p) \not\subseteq L$ for $p \in J$. Let
$P = A^1(p) \cap L$ and let $U$ be a neighbourhood of $P$ on $L$, determined
by some $U(A_1(p))$, with end points $Q_1, Q_2$; (Figure IV.6). Let $U(p)$
be a neighbourhood of $p$ such that $QM \in U(A_1(p))$ for any $Q \in U(P)$
and $M \in U(p)$.

$p$ is strongly left finite; hence, there exist $U^r_1(p), U^r_2(p)$
such that no 1-secant of $U^r_i(p)$ meets $Q_i, i = 1,2$. Let
$U^r(p) \subseteq U^r_1(p) \cap U^r_2(p)$ such that $U^r(p) \subseteq U(p)$. Then by 4.2.2, there
is a 1-secant $L$ of $U^r(p)$ such that $L$ meets $U(p)$. Since no
1-secant of $U^r(p)$ meets $Q_1$ or $Q_2$ and the set of 1-secants of
$U^r(p)$ is connected, every 1-secant of $U^r(p)$ meets $U(p)$.

4.2.5 Lemma: Let $p$ be a strongly left finite point of an arc
$A$. Let $p_i \in A_1(p) - A_{i-1}(p); i = 0,1,2$. Then
$$\lim_{q \rightarrow p} A(q) \cap P_k P_{k+1} = P_k$$
for $k = 0,1$. 
Proof: Let $U(P_k)$ be a neighbourhood of $P_k$ on $P_k^e_{k+1}; k = 0,1$. Take $U^-(p)$ such that no $h$-secant of $U^-(p)$ meets any $(1,h)$-space spanned by the $P_i$, and no $1$-secant of $U^-(p)$ meets the end points of $U(P_k)$.

**Case 1.** $k = 0$. Let $L$ be a point on $U^-(p)$, (Figure IV.7). Now $L \not\in A(p)$ hence $LA(p)$ is a line which meets $U(P_0)$ at $P_0$. Let $U(p)$ be a neighbourhood of $p$ in $A$ such that $L \not\in U(p)$ and $LM$ meets $U(P_0)$ for any $M \in U(p)$. Trivially $U^-(p) \cap U(p) \neq \emptyset$, hence there is a $1$-secant of $U^-(p)$ which meets $U(P_0)$. Since the $1$-secants of $U^-(p)$ are connected and do not meet the end points of $U(P_0)$, the result follows.

**Case 2.** $k = 1$. Follows from 4.2.3 and Figure IV.8.

### 4.2.6 Theorem

A strongly finite arc $A$ is dually differentiable and $a^*_k(p) = a_{-k}(p)$ for all $p \in J, k = 0,1$.

**Proof:** Now $A$ is dually differentiable iff $A_k(p) = \lim_{q \to p} A_1(q) \cap A_{k+1}(p), -1 \leq k \leq 1$, Let $p$ be a point of $A$ and choose $P_i; i = 0,1,2,$ and $U^-(p)$ as in 4.2.5. Therefore for any $q \in U^-(p), P_i \not\in A_1(p)$ for $i = 0,1,2$ and $A_1(q) \cap A_{k+1}(p)$ is a $k$-space where $-1 \leq k \leq 1$. Let $L$ be a $k$-space of accumulation of $A_1(q) \cap A_{k+1}(p)$, $q \to p$. Since $P_0 \ldots P_k \subset A_{k+1}(p)$, by 4.2.5

$$\lim_{q \to p} A_1(q) \cap P_k P_{k+1} = P_k \subset \lim_{q \to p} A_1(q) \cap A_{k+1}(p) = L$$

Then $P_0 \ldots P_k \subset L$ and $L = A_k(p)$. Similarly

$$\lim_{q \to p} A_1(q) \cap A_{k+1}(p) = A_k(p)$$

and hence $A$ is dually differentiable.
From 4.2.2, $A_1(q)$ meets $E_1^-$, $E_2^-$ for all $q \in U^-(p)$. Therefore $U_1^-(p) = 0 = U_2^-(p)$. Similarly $U_1^+(p) = 0 = U_2^+(p)$ and then $a_k^* = a_{1-k}$ for $k = 0, 1$. From the Corollary of 4.1.3.

4.2.7 Theorem: An elementary arc is strongly finite.

Proof: Let $p$ be a point of $A$ and $U^-(p)$ of order 2. Let $L$ be a line, then by 1.3.1 there is $U^+(p)$ such that $U^+(p) \cap U^-(p) \neq \emptyset$ and $L \cap A(U^+(p)) = \emptyset$. Therefore we need only consider points $P$ in $P^2$.

If $P = A(p)$ then since $U^-(p)$ is of order 2, no 1-secant of $U^-(p)$ meets $P$ by 2.4.1.

If $P \neq A(p)$ then $L = PA(p)$ is a line and as in preceding paragraph there exists $U_1^-(p) \subset U^-(p)$ such that $L \cap A(U_1^-(p)) = \emptyset$. Put $\varphi(x) = A_1(x) \cap L$ for $x \in (U_1^-(p))^2$, (Figure IV.9). Then $\varphi$ is monotone. Now $\lim_{r \to 0} \varphi(q,r) = A(p)$ where $q,r \in U_1^-(p)$. But $L \cap A(U_1^-(p)) = \emptyset$ implies there exists $U_2^-(p) \subset U^-(p)$ such that $\varphi(x) \neq P$ for $x \in (U_2^-(p))^2$ and the result follows.

Corollary: Let $P$ be a point. If $p$ is an elementary point of an arc $A$ then $p$ is also elementary on $\tilde{A} = A/P$.

Proof: Take $U^-(p)$ of order 2 such that no 1-secant of $U^-(p)$ meets $P$. Therefore if $\tilde{A}(q_1) = \tilde{A}(q_2)$ for $q_1 \neq q_2$ in $U^-(p)$, the 1-secant $A(q_1)A(q_2)$ contains $P$, a contradiction.

4.3 Arcs with towers

Spaces $H_k \subset E_k^2$, $-1 \leq k \leq 2$, are said to be a tower if
\( H_1 \subseteq H_0 \subseteq H_1 \subseteq H_2 \). An arc with tower is an arc for which there exist a tower satisfying

\[ A_k(p) \cap H_{l-k} = \emptyset \]

for any \( p \in J, -1 \leq k \leq 2 \).

4.3.1 Theorem: Let \( A \) be an arc with tower and \( A_1 \) continuous. Then

1. If \( A \) has at most inflections, \( A \) is a Barner arc.
2. If \( A \) is regular, \( A \) is of order 2.

Proof: Since a regular Barner arc is of order 2, we need only prove 1.

Consider \( \tilde{A} = A/H_0 \). Since \( H_0 \not\subseteq A_1(p) \) for any \( p \in J \),

\[ \tilde{a}_0(p) = a_0(p) = 1 \]. Therefore \( \tilde{A} \) is regular. Put \( \tilde{H}_0 = H_1 \). Then

\[ A(p) \cap \tilde{H}_0 = A(p)H_0 \cap H_1 = H_0 = \emptyset \] in \( \tilde{A} \) and hence \( \tilde{A} \) is not all of \( \mathbb{P}_0^{1} \). Therefore \( A \) is Barner and of order 1 by 3.2.4 (a). The result now follows from 3.3.2.

4.3.2 The Scherk-Derry duality theorem: Let \( A \) be an arc of order 2. Then \( A \) is dually differentiable and \( A^* \) has order 2.

Proof: \( A \) is of order 2, hence \( A \) is elementary and strongly finite by 4.2.7. Therefore \( A \) is dually differentiable and \( a_k^* = a_{1-k} \), \( k = 0,1 \) by 4.2.6.

\( A \) is of order 2, hence ordinary and regular by 2.6.1. But \( A \) regular and \( a_k^* = a_{1-k} \) implies \( A^* \) is regular. Since \( A_1^* = A_0, A_1^* \) is continuous. Therefore we need only show \( A^* \) is an arc with tower and result follows from 4.3.1.
Let \((q,r)\) be an interval of \(A, (q,r)\) is of order 2 and by
\[2,4,1,\] \(A(q) \not\subset A_1(p)\) for any \(p \in (q,r)\). Put \(H_0^* = A_1(q)\) and \(H_1^* = A(q)\).
Then \((q,r)\) is an arc with tower on \(A^*\). But \(q\) and \(r\) are arbitrary,
hence \(A^*\) is an arc with order 2.

**Corollary:** If \(A\) is elementary, then \(A\) is dually differentiable
and \(A^*\) is elementary.

The above theorem point out an important fact about arcs with
tower; namely, it is a local property. In other words, we can prove
many things about an arc by localizing to a point and constructing (if
possible) a neighbourhood with tower for that point. We shall see
more examples of this in Chapter VI.

Points which have neighbourhoods with tower are available.
For example, a point \(p \in J\) which is elementary and hence strongly finite.
One can easily construct a neighbourhood of a strongly finite point,
which has a tower.

Let \(L\) be a line. Put
\[
\mu(p,L) = \beta_0(p,L)(p)
\]
for all \(p \in J\). We call \(\mu(p,L)\) the multiplicity with which \(L\) meets \(A\)
at \(p\). Counting multiplicities facilitates giving precise information
about the order of point, as will be seen in \(4,3,4\).

4.3.3 **Theorem:** Let \(A\) be an arc with tower such that every regular
subarc of \(A\) is of order 2. Then
\[
\sum_{p \in J} \mu(p,L) \leq \left( \sum_{p \in J} \sum_{i=0} (a_i(p)-1) \right) + 2.
\]
Proof: If $\sum_{pcJ} \sum_{i=0}^{1} (a_i(p)-1)$ is infinite, the result holds trivially. If the theorem is true for finite number of $p$ in $J$, then

$$\sum_{p_1, \ldots, p_n} \mu(p, L) \leq \sum_{pcJ} \sum_{i=0}^{1} (a_i(p)-1) + 2$$

for all $n < \infty$. Hence if $\sum_{pcJ} \mu(p, L)$ is infinite, so is $\sum_{pcJ} \sum_{i=0}^{1} (a_i(p)-1) + 2$.

Therefore we may assume both sides are finite. But $\sum_{i=0}^{1} (a_i(p)-1) \neq 0$ for only finitely many points implies there are only finitely many non-regular points on $A$. Since by hypothesis, regular subarcs are of order 2, $A$ is elementary.

Suppose $A$ is an arc with tower in $P^1$ such that $A$ has finitely many non-regular points. Let $L$ be a point in $P^1$ and suppose $L$ meets $A$ $k$-times and $h < k$ of these are points of inflection. Then

$$\sum_{pcJ} \mu(p, L) = h + k.$$ 

But $L = A(p) k$-times implies that each of the $k-1$ intervals determined by these points contains a point of inflection. Hence $A$ has at least $h + k - 1$ points of inflection. Then

$$h + k - 1 \leq \sum_{pcJ} (a_0(p)-1)$$
and the result follows that

\[ \sum_{p \in J} \mu(p,L) \leq \sum_{p \in J} (a_0(p)-1) + 1 \]

for any \( p \in J \).

\( A \) is an arc with tower. Take \( \{H_i\} \) to be a tower with \( H_i \neq A_1(q) \) for any \( q \in J \) and \( A(q) \not\subset H_1 \) for any \( q \in J \). Let \( L \) a line be given.

If \( H_0 \subset L \) then consider \( \widetilde{A} = A/H_0 \). \( A \) is an arc with tower in \( P_1 \), as proved in 4.3.1. Since \( A \) has only finitely many non-regular points and \( \widetilde{a_0}(p) = a_0(p) \), then \( \widetilde{A} \) has only finitely many non-regular points. From above,

\[ \sum_{p \in J} \widetilde{\mu}(p,L) \leq \sum_{p \in J} (\widetilde{a_0}(p)-1)+1. \]

Since \( H_0 \not\subset A_1(p) \) for any \( p \in J \), we have \( \delta(p,L) \leq 0 \) and \( \pi(H_0,p) = 1 \) for all \( p \in J \). Then \( \widetilde{\mu} = \widetilde{\delta} = \beta_0 = \beta_0 = \mu \) by 2.2.1 and 2.5.2. Hence

\[ \sum_{p \in J} \mu(p,L) = \sum_{p \in J} (a_0(p)-1) + 1 \]

\[ = \sum_{p \in J} (a_0(p)-1) + 1 \]

\[ \leq \sum_{p \in J} \frac{1}{\sum_{i=0} (a_i(p)-1) + 2} \]
Figure IV.11
If $H_0 \not\in L_1$ put $P = L \cap H_1$ and consider $\tilde{A} = A/P$. $H_1$ does not meet $\tilde{A}$; otherwise, $H_1 = \tilde{A}(q)$ for some $q \in J$ and hence $A(q) \subset H_1$, a contradiction. Thus $\tilde{A}$ is an arc with tower and finitely many inflections. Then

$$\sum_{p \in J} \tilde{\mu}(p, L) \leq \sum_{p \in J} (\tilde{a}_0(p) - 1) + 1.$$ 

Since $A$ is elementary, $A$ is strongly finite and dually differentiable with $a_k^* = a_{1-k}$; $k = 0, 1$. Now $H_1 \not\in A = \emptyset$, therefore put

$$\psi(p) = H_1 \cap A_1(p)$$

for all $p \in J$. In $A^*$, $\psi(p)$ is the line $H_1 A^*(p)$. In Figure IV.10 and IV.11, $A$ and the corresponding $A^*$ are illustrated. Projecting from $H_1$, $\varphi = A^*/H_1 = \tilde{A}^*$ is an arc in $P^1$. Moreover $\psi$ is an arc with tower in $P^1$, since $H_0 \not\in A_1(p)$ for all $p \in J$, hence $A^*(p) \not\in H_0$ for all $p \in J$ and $H_0 \not\in \varphi$ in $P^1$. Since $A$ has finitely many non-regular points, $a_k^* = a_{1-k}$ for $k = 0, 1$, and $\tilde{a}_0^* = a_0^*$ or $a_1^*$, $\tilde{A}^*$ has finitely many inflections. Therefore

$$\sum_{p \in J} \tilde{\mu}^*(p, L') \leq \sum_{p \in J} (\tilde{a}_0^*(p) - 1) + 1$$

for any point $L'$ in $A^*$. In particular since $A^*_1(p) \cap H_1 = A(p) \cap H_1 = \emptyset$ for all $p \in J$, $\pi(H_2(p)) = 1$ and $\tilde{a}_0^*(p) = a_0^*(p) = a_1(p)$ for all $p \in J$.

Hence

$$\sum_{p \in J} \tilde{\mu}^*(p, L') \leq \sum_{p \in J} (a_1(p) - 1) + 1.$$
Let \( X = \{ p \in J | P \in A_1(p) \} \) where \( P \) is defined above.

Let \( p \in X \), then \( P \subseteq H_1 \cap L \cap A_1(p) \) and \( P = H_1 A_1(p) \) in \( A^* \).

Therefore \( L = \Lambda^*(p) \) in \( \Lambda^* \) and \( \delta^*(p,L) = 0 \). Since

\[
\hat{\mu}^*(p,L) = \hat{\beta}_0^*(p,L)(p),
\]

\[
\hat{\mu}^*(p,L) = \hat{\beta}_0^*(p) = \hat{\alpha}_0^*(p) = a_1(p).
\]

Therefore

\[
\sum_{p \in X} a_1(p) = \sum_{p \in J} \hat{\mu}^*(p,L)
\]

\[
\leq \sum_{p \in J} \hat{\mu}^*(p,L)
\]

\[
\leq \sum_{p \in J} (a_1(p)-1) + 1.
\]

Considering again \( \Lambda = A/P \) where \( P \subseteq L \cap H_1 \). If \( \delta(p,L) \leq 0 \)

for some \( p \in J \), then \( P \notin A_1(p) \) and \( \pi(P,p) = 1 \). Hence \( \hat{\mu} = \mu \) and

result follows as before. If \( \delta(p,L) = 1 \) then \( P \subseteq L = A_1(p) \) and \( p \in X \).

Since \( \delta(p,L) = \delta(p,L) - 1 \),

\[
\hat{\mu}(p,L) = \hat{\beta}_0^*(p)
\]

\[
= \hat{\alpha}_0(p) + (\mu(p,L) - a_0(p) - a_1(p))
\]

\[
= \mu(p,L) - (a_0(p) + a_1(p) + \hat{\alpha}_0(p)).
\]

Then
\[
\sum_{p \in J} \mu(p, L) = \sum_{\delta(p, L) = 0} \mu(p, L) + \sum_{\delta(p, L) = 1} \mu(p, L)
\]

\[
= \sum_{\delta(p, L) = 0} \tilde{\mu}(p, L) + \sum_{\delta(p, L) = 1} (\tilde{\mu}(p, L) + (a_0(p) + a_1(p)) - a_0(p))
\]

\[
\leq \sum_{p \in J} \tilde{\mu}(p, L) + \sum_{p \in X} ((a_0(p) + a_1(p)) - a_0(p))
\]

\[
= \sum_{p \in J} \tilde{\mu}(p, L) + \sum_{p \in X} ((a_0(p) - 1 + a_1(p)) - (a_0(p) - 1)).
\]

Now

\[
\sum_{p \in J} \tilde{\mu}(p, L) \leq \sum_{p \in J} (a_0(p) - 1) + 1
\]

\[
\leq \sum_{p \in X} (a_0(p) - 1) + \sum_{p \in X} (a_0(p) - 1) + 1
\]

and if \( p \notin X \), \( \delta(p, L) = 0 \) and \( \delta(p, p) = 1 \). Hence from above \( \tilde{a}_0(p) = a_0(p) \).

But then

\[
\sum_{p \in J} \mu(p, L) \leq \sum_{p \in J} (a_0(p) - 1) + \sum_{p \in X} a_1(p) + 1
\]

\[
\leq \sum_{p \in J} (a_0(p) - 1) + \sum_{p \in J} (a_1(p) - 1) + 2
\]

and the result follows.
4.3.4 Denk's Theorem: Let \( p \) be an elementary point of an arc.

Then \( \text{Ord}(p) = a_0(p) + a_1(p) \) where \( \text{Ord}(p) \) is defined to be the ord \( p \) counting multiplicities.

**Proof:** Take \( U^-(p), U^+(p) \) of order 2, such that \( U(p) = U^-(p) \cup \{ p \} \cup U^+(p) \) is an arc with tower. This is possible since \( A_1 \) is continuous on \( U(p) \) and \( p \) is strongly finite. Then by 4.3.3,

\[
\sum_{qcU(p)} \mu(q,L) \leq \sum_{qcU(p)} \sum_{i=0}^{1} (a_i(p)-1) + 2.
\]

Since \( q \neq p \) implies \( q \) is regular where \( q \in U^-(p) \cup U^+(p) \), \( a_i(q) - 1 = 0 \) for \( i = 0,1 \). Hence

\[
\sum_{qcU(p)} (\mu(q,L) \leq a_0(p) + a_1(p)) \text{ for all } q \in U^+(p).
\]

The equality follows by taking \( L = A_1(p) \).

**Corollary:** An elementary regular point is ordinary.

4.4 Finite arcs

A point \( p \) of an arc \( A \) is left finite, if for every \((1-k)\)-space \( L \) there exists \( U^-(p) \) such that

\[
A_k(q) \cap L = \emptyset
\]

for all \( q \in U^-(p), k = 0,1 \). Right finiteness is similarly defined. A point \( p \) is finite if it is both left and right finite. A is finite if each of its points is finite.
Trivially if \( A \) is strongly finite it is finite and in \( \mathbb{P}^1 \), strong finiteness is exactly finiteness; that is, given only two points \( p, q \) in \( \mathbb{P}^1 \), there exists \( U^-_1(\mathbb{P}) \) of \( p \) such that \( A(q) \notin A(U^-_1(\mathbb{P})) \).

4.4.1 Lemma: Let \( \tilde{A} = A/P \) where \( A \) is an arc and \( P \) a point. If \( p \) is finite on \( A \), it is finite on \( \tilde{A} \).

The proof is as in 4.2.1.

4.4.2 Theorem: Let \( p \) be left finite, and some \( U^-_1(\mathbb{P}) \) have at most inflections, then \( p \) is strongly left finite.

Proof: Let \( P \) be a point. Since \( p \) is left finite, there is \( U^-_1(\mathbb{P}) \subset U^-_1(p) \) such that \( P \notin A_1(q) \) for any \( q \subset U^-_1(p) \). Thus \( U^-_1(p) \) is regular on \( \tilde{A} = A/P \) an arc in \( \mathbb{P}^1 \). If \( \tilde{A}(U^-_1(p)) \) is equal to \( \mathbb{P}^1 \), then there is \( U^-_2(\mathbb{P}) \subset U^-_1(p) \) such that \( \tilde{A}(U^-_2(p)) \neq \mathbb{P}^1 \) and hence \( U^-_2(p) \) is Barner on \( A \) and of order 1 by 3.2.4.

Let \( M \) be a line through \( P \). Since \( P \notin A_1(q) \), \( M \notin A_1(q) \) and \( \delta(q, M) = \delta(q, M) \leq 0 \) by 2.2.1, for all \( q \subset U^-_1(p) \). If \( M \) meets \( U^-_2(p) \), then

\[
\dim_{\text{qu}U^-_2(p)} A_0(q, M)(q) \leq \dim M = 1.
\]

If \( M \) meets \( U^-_2(p) \) only once then \( M \) is not a 1-secant of \( U^-_2(p) \) since \( M \notin A(q) \) for all \( q \subset U^-_2(p) \). Assume \( M \) meets \( U^-_2(p) \) at least twice, then by 2.2.4

\[
1 \leq \sum_{\text{qu}U^-_2(p)} (\delta(q, M) + 1) - 1 = \sum_{\text{qu}U^-_2(p)} (\delta(q, M) + 1) - 1.
\]
Since $M$ is a 0-secant of $U^-(p)$ on $A$ by $\text{2.2.2}$ and $U^-(p)$ has order 1 on $A$, 

$$1 \leq \sum_{q \in U^-(p)} (\delta(q, M+1) - 1)$$

$$= 0$$

a contradiction. Hence no 1-secant of $U^-(p)$ contains $P$.

Let $L$ be a line, by $\text{1.3.1}$ there is $U^-_3(p) \subset U^-_2(p)$ such that $A(q) \notin L$ for all $q \in U^-_3(p)$. Then $p$ is strongly left-finite.

**Theorem:** Let $p$ be left finite and some $U^-(p)$ regular. Then there is a $U^-(p)$ of order 2.

**Proof:** Let $P \notin A(p)$ be a point and $U^-_1(p)$ be regular such that $P \notin A_1(q)$ for all $q \in U^-_1(p)$. Then $U^-_1(p)$ is regular on $A/P$ and there is $U^-_2(p) \subset U^-_1(p)$ of order 1 on $A/P$. Since $x(P, q) = 1$ for $q \in U^-_2(p)$, $U^-_2(p)$ is a Barner arc by $\text{3.3.1}$, and of order 2 by $\text{3.2.7}$.

**Corollary:** A regular finite arc is ordinary.

**Proof:** Since $A$ is finite, every $p \in A$ has $U^-(p), U^+(p)$ of order 2 on $A$ by $\text{4.4.3}$. Hence $p$ is regular elementary point, which is ordinary by $\text{4.3.4}$.

**Theorem:** If a finite arc $A$ has a singularity, it has an elementary singularity.

**Proof:** If $A$ has at most inflections, we may assume $A$ has an inflection point $p_1$ by the Corollary of $\text{4.4.3}$, and $A$ is strongly
finite by \(4.4.2\). Then \(A\) is dually differentiable and \(a_k^* = a_{1-k}\) for \(k = 0, 1\) by \(4.2.6\). Let \(L\) be a line with \(A(p_1) \neq L\). Since \(p\) is strongly finite we may assume \(A(p) \notin L\) for all \(p \in J\). Put

\[
\psi(p) = A_1(p) \cap L
\]

for all \(p \in J\).

As in \(4.3.3\), the mapping \(\psi\) is the projection of \(A^*\) from \(L\). Put \(\tilde{A}^* = A^*/L = \psi\). Since \(A_1^*(p) \cap L = \emptyset\), \(\tilde{a}_0^*(p) = a_0^*(p) = a_1(p)\) for all \(p \in J\). Therefore \(p_1\) is an inflection point of \(\tilde{A}^*\). Since an ordinary point is regular, \(p_1\) is a singularity of \(\tilde{A}^*\). But \(\tilde{A}^*\) is an arc in \(\mathbb{P}^1\), therefore \(\psi\) has an elementary singularity \(p_2\) by \(3.2.4\).

Let \(\tilde{U}^+(p_2)\) be of order 1 on \(\tilde{A}^*\) then \(\tilde{a}_0^*(q) = 1\) for all \(q \in \tilde{U}^+(p_2)\). Hence \(a_1(q) = 1\) for all \(q \in U^+(p_2)\) and there is \(U_1^+(p_2)\) of order 2 on \(A\) by \(4.4.3\). Similarly, there exists \(U_1^-(p_2)\) of order 2 on \(A\) and hence \(p_2\) is an elementary singularity.

Assume only that \(A\) has a singularity \(p_1\). We can again choose point \(P\) such that \(P \neq A_1(q)\) for all \(q \in J\). If \(\tilde{A} = A/P\) is ordinary then \(A\) has at most inflections and result follows. If \(\tilde{A}\) has an elementary singularity \(p_2\), take \(\tilde{U}^-(p_2), \tilde{U}^+(p_2)\) of order 1 on \(\tilde{A}\). Then both are regular on \(\tilde{A}\) and have at most inflections on \(A\). If either contains an inflection the theorem follows. Otherwise both are regular and \(p_2\) is elementary on \(A\) by \(4.4.3\). Since \(2 = a_0^*(p_2)\), \(p_2\) is an elementary singularity.
4.5 Regular simple arcs

In Chapter III, it was pointed out that a Barner arc in \( \mathbb{R}^2 \)
is simple. Hence by 3.2.7, a regular simple arc which is Barner is of
order 2. In 2.6.1, it was proved that if \( A \) is an arc of order 2 then
\( A \) is regular.

The question naturally arises if a regular arc is of order 2.
The answer is no, for in Figure IV.12, \( A \) is a regular arc which is not
of order 2. Moreover a regular simple arc which is not Barner is not
of order 2, as can be seen in Figure IV.13. The most we will be able
to show in this chapter is that a regular simple arc is ordinary.

4.5.1 Theorem: Let \((p,q)\) be an ordinary simple subarc of an arc
\( A \). Let \( \{H_i\} \) be a tower of spaces such that \( H_i \) does not meet \( A \).
If the lines \( L = A(p)H_0 \) and \( M = A(q)H_0 \) are distinct and do not cut
\((p,q)\), then \((p,q)\) is of order 2.

(It should be noted that the hypothesis does not state \( A \) is
an arc with tower \( \{H_i\} \), since no restriction is made upon \( H_0 \subseteq H_{1,i} \).)

Proof: Case 1. If \( p \) and \( q \) are ordinary. Let \( r \subset (p,q) \),
then \( r \) is elementary and strongly finite by 4.2.7. Thus there exist
\( U^+(r) \), \( U^-(r) \) such that \( H_0 \not\subset A_1(t) \) for \( t \subset U^+(r) \cup U^-(r) \). Since \( J \)
is isomorphic to \( IR \) as a topological space and \([p,q]\) is closed in
\( J \), \([p,q]\) is compact. Since \( A \) is continuous, \( A[p,q] \) is compact.
Hence there are only finitely many points \( r \subset (p,q) \) such that \( H_0 \subset A_1(r) \).
Let \( r_1 < \ldots < r_k \) be such points. Put \( p = r_0 \) and \( q = r_{k+1} \).
Put \( L_i = A(r_i)H_0 \), \( 0 \leq i \leq k+1 \), hence \( L_i = A_1(r_i) \) for \( 1 \leq i \leq k \).

Now consider \((r_{i-1}, r_i)\), \(0 \leq i \leq k+1\). Since \( H_0 \not\in A_1(r) \) for \( r \in (r_{i-1}, r_i) \), then \((r_{i-1}, r_i)\) is an arc with tower \( \{H_i\} \). Moreover \((r_{i-1}, r_i)\) is strongly finite and dually differentiable by 4.2.6.
Hence \( A_1 = A_0 \) is continuous on \((r_{i-1}, r_i)\) and \((r_{i-1}, r_i)\) is of order 2 by 4.2.1, \(0 \leq i \leq k+1\). Thus if \( k = 0, (p,q) \) is of order 2.

Suppose \( k \geq 1 \). We think of the lines distinct from \( H_1 \), through \( H_0 \), as being vertical and order so \( L < M \). Since \( L \) and \( M \) do not meet \((p,q)\), \( L \leq L_i \leq M \) for \( i = 0, \ldots, k+1 \). We may assume \((p,r_1)\) lies above the line \( A(p)A(r_1) \), (Figure IV.14).

Now \( r_1 \) is regular, hence \( A(p)A(r_1) \) and \( A(r_1)A(r_2) \) both cut \( A \) at \( r_1 \). Then \((r_1, r_2)\) lies below the line \( A(r_1)A(r_2) \); otherwise \( H_0 \not\in A_1(r_2) \), since \( r_2 \) is also regular, (Figure IV.15). But then \((r_1, r_2)\) is contained in an open half-space determined by \( L \) and \( L_1 \), hence \( L \leq L_2 < L_1 \leq M \), (Figure IV.16). Since \( A \) is simple and \( L \leq L_2 < L_1 \leq M \), then \( k \geq 2 \). But then we have \((r_2, r_3)\) lie above \( A(r_2)A(r_3) \) and \( L \leq L_2 < L_3 < L_1 \leq M \). Hence \( k \geq 3 \). Continuing, it follows that \( k \) is arbitrarily large; a contradiction.

**Case 2.** If \( p \) is not necessarily ordinary. Take \( r \in (p,q) \) such that the lines \( L \) and \( A(r)H_0 \) are distinct and do not meet \((p,r)\).

For any \( p_1 \in (p,r) \), there is a \( p_2 \in (p,p_1) \) such that \( A(p_0)H_0 \) does not meet \((p_2,r)\). But then taking \( L = A(p_2)H_0 \) and \( M = A(r_1)H_0 \), \((p_2,r)\) is of order 2 by case 1. Since \((p_1,r) \subset (p_2,r)\), \((p_1,r)\) is of order 2. But \( p_1 \) is arbitrary, hence \((p,r)\) is of order 2. Similarly \( q \) has a left neighbourhood of order 2 and the result follows.
Figure IV.14
4.5.2 Theorem: Suppose a regular simple arc has a singularity \( p_0 \).

Let \( \{ H_i \} \) be a tower of spaces such that \( H_i \cap A = \emptyset \). Then there exist points \( p_1, q_1, r_1 \) such that \( p_1 \) is singular, \( p_1 \not\in [q_1, r_1] \) and \( A(p_1)H_0 \) lies between \( A(q_1)H_0 \) and \( A(r_1)H_0 \).

Proof: There is a point \( p \) such that \( H_0 \not\subset A_1(p) \), otherwise \( A \) is a Barner arc by 3.3.1, and of order 2 by 3.2.7, a contradiction.

Case 1. \( H_0 \not\subset A_1(p) \) for some singularity \( p \). Let \( (q, r) \) be a neighbourhood of \( p \) such that \( A(q)H_0 = A(r)H_0 = L \) say. We may assume \( A_1(p) \) and \( L \) are distinct and do not meet \( (q, p) \) or \( (p, r) \). One of these intervals contains a singularity \( p_1 \); for otherwise, they are of order 2 by 4.5.1 and \( p \) is elementary. But then \( p \) is ordinary by 4.3.4, a contradiction. If \( p_1 \in (q, p) \), choose \( q_1 = p, r_1 = r \) and the result follows.

Case 2. \( H_0 \not\subset A_1(p) \) only for ordinary points. Let \( (p, q) \) be a neighbourhood of the singularity \( p_0 \), such that \( A(p)H_0 \) and \( A(q)H_0 \) are distinct and do not meet \( (p, q) \). Let \( r \in (p, q) \) such that \( H_0 \not\subset A_1(r) \) and \( A_1(r) \) does not meet \( (p, r) \), (Figure IV,16). If no such \( r \) exists, then by 4.5.1, \( (p, q) \) has order 2, a contradiction.

Take \( s \in (r, q) \) such that \( A(s) \not\subset A_1(r) \) and \( A_1(r) \) does not meet \( (r, s) \). Since \( p_0 \) is singular, we can always consider a sufficiently small \( (p, q) \) such that \( s \in (r, q) \) exists. Take \( t \in (r, s) \) such that \( H_0 \not\subset A_1(t) \) and \( A_1(t) \) does not meet \( (r, t) \) or \( (t, s) \). Take \( u \in (p, r) \) such that \( A(u) \not\subset A_1(t) \) and \( A_1(t) \) does not meet \( (u, r) \).
Then \( u < r < t < s \). By 4.5.1, \((u,s)\) contains a singularity \( p_1 \); for otherwise, \((u,s)\) is of order 2 and \( p_0 \subset (u,s) \) is ordinary. Since \( H_0 \subset A_1(r) \cap A_1(t) \), by assumption \( p_1 \neq r \) and \( p_1 \neq t \). If \( p_1 \subset (r,t) \) or \((t,s)\), we can choose \( q_1,r_1 \) appropriately and the result follows.

**4.5.3 Theorem:** Every regular simple arc \( A \) is ordinary.

**Proof:** Suppose \( A \) has a singularity \( p_0 \). Let \( \{H_i\} \) be a tower such that \( H_1 \) does not meet some neighbourhood \( U_0 \) of \( p_0 \).

Take \( p_1,q_1,r_1 \) in \( U_0 \) as in 4.5.2. Let \( U_1 \) be a neighbourhood of \( p_1 \) such that \( U_1 \subset U_0 \), \( U_1 \cap [q_1,r_1] = \emptyset \) and \( A(p)H_0 \) lies between \( A(q_1)H_0 \) and \( A(r_1)H_0 \) for all \( p \in U_1 \). Applying 4.5.2 repeatedly, one obtains \( p_1,q_1,r_1,U_1 \) such that \( p_1 \subset U_1 \), \( U_1 \subset U_{i-1} \), \( q_i \) and \( r_i \subset U_{i-1} \), \( U_i \cap [q_i,r_i] = \emptyset \) and \( A(p)H_0 \) lies between \( A(q_i)H_0 \) and \( A(r_i)H_0 \) for all \( p \in U_i \). Take \( p \subset \bigcap_{i=1}^{\infty} U_i \). Then \( A(p)H_0 \) meets each of the disjoint intervals \([q_i,r_i]\), contradicting 1.2.1.

As can be observed, the arc in Figure IV.17, is an arc with at most inflections and not regular as 4.5.2 requires.
CHAPTER V

In these last three chapters, we shall be concerned with arcs in a real projective three-space only. In many instances, results will not be proved in detail or may not be proved at all. In all these cases the proofs follow along the lines of proofs of analogous results in $\mathbb{P}^2$. The only difference in the proofs will be that of dimension. Namely, instead of considering points and lines in $\mathbb{P}^2$, one considers lines and planes in $\mathbb{P}^3$, and so on.

5.1 Projection

Let $P \subset \mathbb{P}^3_0$ and $p$ be a point of an arc $A$. Define

$$A_k(p) = \begin{cases} A_k(p) & \text{if } -1 \leq k \leq \pi(p, p) \\ A_{k+1}(p) & \text{if } \pi(p, p) < k \leq 2. \end{cases}$$

5.1.1 Theorem: $\tilde{A} = \{ A_0(p) | p \in J \}$ is an arc in $\mathbb{P}^2(P)$ with $A_k(p)$ as the osculating $k$-spaces at $p$, $-1 \leq k \leq 2$.

One should note that $\tilde{A}$ is now an arc in a real projective plane, (Figure V.1). One can generalize this definition of projection to the case of a line in $\mathbb{P}^3$. Let $L \subset \mathbb{P}^3_1$, the projection $A/L$ of $A$ from $L$ is defined by

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Figure V.1
\[(A/L)(p) = A(p)/L = \lim_{q \to p} A(q)L\]

for all \(p \in J\).

5.1.2 Theorem: \(A/L\) is an arc in \(P^1(L)\) where \(L \in P^3_1\).

**Proof:** Take a point \(P \in L\). Then \(A/P = \mathcal{A}\) is an arc in \(P^2(P)\) with \(L \in P^2_0(P)\). Thus \(A = (A/P)/L\) is an arc in \(P^1(L)\), (Figure V.2). But

\[
((A/P)/L)(p) = \lim_{q \to p} (A/p)(q)L
\]

\[= A(p)/L\]

\[= (A/L)(p).
\]

5.2 Characteristic of a point

Let \(L\) be a k-space, \(-1 \leq k \leq 3\) and let \(p \in J\). Then \(\delta(p, L)\) is the dimension of the largest osculating space of \(p\) which is contained in \(L\).

5.2.1 Lemma: Let \(P\) be a point of a k-space \(L\) and \(\mathcal{A} = A/P\). Then

\[
\tilde{\delta}(p, L) = \begin{cases} 
\delta(p, L) & \text{if } -1 \leq \delta(p, L) < \pi(P, p) \\
\delta(p, L) - 1 & \text{if } \pi(P, p) < \delta(p, L) \leq k.
\end{cases}
\]

We define a plane \(L\) to support \(A\) at \(p\) if there is a plane \(H_0\) with \(H_0 \neq L\), \(A(p) / H_0\) and a \(U'(p)\) such that \(A(U'(p))\) is
Figure V.2
contained in one of the two open half-spaces determined by \(L\) and \(H_\omega\).

When \(L\) does not support \(A\) at \(p\), we say \(L\) cuts \(A\) at \(p\).

By 1.3.1, every plane in \(P^3\) either supports or cuts a given point of an arc. Let \(S_k\) be the set of all planes \(L\) with \(\delta(p,L) = k\) for a fixed \(p \in J\).

**5.2.2 Lemma:** Let \(-1 \leq k \leq 2\). Let \(p \in J\) be fixed and the corresponding \(S_k\) be given. Then

a) \(S_k\) is connected, \(-1 \leq k \leq 2\).

b) the elements of \(S_k\) either all support or all cut \(A\) at \(p\).

**Proof:** (a) If \(k = 2\), then \(S_2 = \{A_2(p)\}\) and is connected. If \(k = 1\), then since \(A_2(p) \not\in S_1\); projecting from \(A_1(p)\), \(S_1\) is projected onto \(P^1(A_1(p)) \setminus (A/A_1(p))(p)\). Thus \(S_1\) will be isomorphic to the real line and hence connected. If \(k = 0\), let \(\tilde{A} = A/p\) and let \(\tilde{S}_0\) be defined for \(p\) in \(\tilde{A}\). Since \(\tilde{A}\) is an arc in \(P^2(p)\), it is connected by 2.5.1. But \(L \cap S_0\) iff \(L \cap \tilde{S}_0\) (Figure V.3); hence \(S_0\) is connected.

Since in \(P^3\), every plane meets a given line in at least one point, \(S_{-1}\) is exactly the set of planes of \(P^3\) which meet \(A_1(p)\) at exactly one point \(q \neq p\). For if \(L \cap S_{-1}\) then \(A_1(p) \not\subset L\). Conversely if a plane \(L\) meets \(A_1(p)\) exactly at one point \(q \neq p\), then \(\delta(p,L) = -1\). Thus the result follows for \(k = -1\).

(b) See 2.5.1.
Figure V.3
We can now characterize a given point \( p \) of an arc \( A \) by describing whether \( S_k \) cuts or supports \( A \) at \( p \); \(-1 \leq k \leq 2\).

We define for \( p \in J, \sigma_k(p) = 0 \) or 1 according to whether \( S_k \) supports or cuts \( A \) at \( p \) respectively, \(-1 \leq k \leq 2\). The characteristic \((a_0(p), a_1(p), a_2(p))\) of \( p \in J \) is defined by taking \( a_i(p) = 1 \) or 2 and requiring that

\[
\sigma_k(p) \equiv \sum_{i=0}^{k} a_i(p) \pmod{2}.
\]

Lastly we define

\[
\beta_k(p) = \sum_{i=0}^{k} a_i(p), \quad 0 \leq k \leq 2
\]

and \( \beta_{-1}(p) = 0 \).

**3.2.3 Theorem:** Let \( P \) be a point and \( \bar{A} = A/P \). Then

\[
\sigma_k(p) \equiv \begin{cases} 
\sigma_0(p) + \sigma_{k+1}(p) & \pi(P,p) = -1 \leq k \leq 1 \\
\sigma_k(p) & -1 \leq k < \pi(P,p) \\
\sigma_{k+1}(p) & -1 < \pi(P,p) < k \leq 1,
\end{cases}
\]

\[
a_k(p) \equiv \begin{cases} 
a_k(p) & 0 \leq k < \pi(P,p) \\
a_k(p) + a_{k+1}(p) & k = \pi(P,p) \\
a_{k+1}(p) & \pi(P,p) < k \leq 1,
\end{cases}
\]

where the congruences are \((\mod 2)\), and
\[ \tilde{\beta}_k(p) = \begin{cases} \beta_{k+1}(p) - \beta_0(p) & \pi(p, p) = -1 \leq k \leq 1 \\ \beta_k(p) & -1 \leq k < \pi(p, p). \end{cases} \]

In Figure V.4, \( p \) is a point on \( A \) with \( \pi(p, p) = 1 \) and characteristic \((1, 2, 1)\). The planes \( L_i \subset S_i, i = 0, 1, 2 \). Note that since \( P \notin L_1 \), \( L_1 \) is not projected into \( P^2(p) \) and the characteristic of \( p \) in \( A \) is \((1, 1)\).

Since a projection from a line was defined as a projection from a point on the line followed by a projection from the projected image of the line, one can obtain the characteristics of a projected image of a point of an arc when projecting from a line by applying the formulas in 5.2.2 twice.

Since for any \( P \neq P^1 \), \( P \) and \( P^1 \) points on \( L \) and \( p \in J \),

\[ ((A/P)/L)(p) = ((A/P^1)/L)(p), \]

the characteristics of \( p \) upon projection from \( L \) is independent of the choice of \( P \) in \( L \). In the following theorem, we shall prove this independence by direct calculation of the characteristic and at the same time derive the formula for the characteristic of a point with respect to each \( L \subset F^3 \).

5.2.4 Theorem: Let \( p \in J \) and \( L \subset F^3 \). Let \( P \) be a point on \( L \) and write \( A = A/P, A = (A/P)/L \). The characteristic of \( p \) in \( A[\tilde{A}, \tilde{A}] \) is \( (a_0(p), a_1(p), a_2(p)) [(\tilde{a}_0(p), \tilde{a}_1(p)), (\tilde{a}_0(p))] \). Then \( \tilde{a}_0(p) \) is independent of \( P \in L \).
Figure V.4
Proof: \textbf{Case 1.} \( L = A_1(p), \hat{a}_0(p) = a_2(p). \)

If \( P \neq A(p) \) then \( \pi(P,p) = 0 \) and \( \hat{a}_0(p) \equiv a_0(p) + a_2(p), \hat{a}_1(p) = a_2(p). \) Moreover, \( \hat{A}(p) = A(p)P = L \) and \( \pi(L,p) = -1. \) Hence \( \hat{a}_0(p) = \hat{a}_1(p) = a_2(p). \) If \( P = A(p) \) then \( \pi(P,p) = -1, \hat{a}_0(p) = a_1(p) \) and \( \hat{a}_1(p) = a_2(p). \) Since \( \pi(L,p) = -1, \) the result follows.

\textbf{Case 2.} \( L \neq A_1(p) \) and \( L \subset A_2(p) \)

(i) If \( L \cap A_1(p) \neq A(p) \) then \( \hat{a}_0(p) \equiv \sum_{i=0}^{2} a_i(p) \pmod{2}. \)

If \( P \neq A_1(p) \) then \( \hat{a}_0(p) = a_0(p) \) and \( \hat{a}_1(p) \equiv a_1(p) + a_2(p) \pmod{2}. \) \( \hat{A}(p) = A(p)P \neq L \) and \( \hat{A}_1(p) = A_1(p)P = A_2(P) \supset L \) implies \( \pi(L,p) = 0 \) and the result follows.

If \( P \subset A_1(p) \setminus A(p) \) then \( \hat{a}_0(p) \equiv a_0(p) + a_1(p) \pmod{2} \) and \( \hat{a}_1(p) = a_2(p). \) \( \pi(L,p) = 0 \) implies the result.

(ii) If \( L \cap A_1(p) = A(p) \) then \( \hat{a}_0(p) = a_1(p) + a_2(p) \pmod{2}. \)

If \( P \neq A_1(p) \) then \( \hat{a}_0(p) = a_0(p) \) and \( \hat{a}_1(p) \equiv a_1(p) + a_2(p) \pmod{2}. \) \( \hat{A}(p) = A(p)P = L \) and hence \( \hat{a}_0(p) = \hat{a}_1(p) \) and the result follows.

If \( P = A(p), \hat{a}_0(p) = a_1(p) \) and \( \hat{a}_1(p) = a_2(p). \) But \( P = A(p) \) implies \( \hat{A}(p) = A_1(p) \neq L. \) Hence \( \pi(L,p) = 0 \) and \( \hat{a}_0(p) = \hat{a}_0(p) + \hat{a}_1(p) \pmod{2}. \)

\textbf{Case 3.} \( L \not\in A_2(p). \)
(i) If $L \cap A_2(p) \neq A_1(p)$ then $\hat{a}_0(p) = a_0(p)$.

If $P \not\subseteq A_2(p)$, $\tilde{a}_0(p) = a_0(p)$ and $\tilde{a}_1(p) = a_1(p)$. $\hat{A}_1(p) = A_1(p)$ = PL, hence $\pi(L, p) = 1$ and $\tilde{a}_0(p) = \hat{a}_0(p) = a_0(p)$.

If $P \subseteq A_2(p)$ then $\pi(P, p) = 1$ and $\tilde{a}_0(p) = a_0(p)$, $\tilde{a}_1(p) \equiv a_1(p) + a_2(p)$ (mod 2). $\hat{A}_1(p) = A_1(p)P = A_2(p) \not\subseteq L$ and the result follows.

(ii) If $L \cap A_2(p) \subset A_1(p) \setminus A(p)$ then $\tilde{a}_0(p) = a_0(p) + a_1(p)$ (mod 2).

If $P \not\subseteq A_2(p)$, $\tilde{a}_i(p) = a_i(p)$ for $i = 0, 1$. $\tilde{A}_1(p) = A_1(p)P \supseteq L$ and $\tilde{A}(p) = A(p)P \not\subseteq L$ imply $\pi(L, p) = 0$ and $\tilde{a}_0(p) \equiv \tilde{a}_0(p) + \tilde{a}_1(p)$ (mod 2).

If $P \subseteq A_2(p)$ then $P \subseteq A_1(p)$ by assumption. Hence $\pi(P, p) = 0$ and $\tilde{a}_0(p) \equiv a_0(p) + a_1(p)$ (mod 2), $\tilde{a}_1(p) = a_1(p)$. $\hat{A}_1(p) = A_2(p) \not\subseteq L$.

(iii) If $L \cap A_2(p) = A(p)$ then $\tilde{a}_0(p) = a_1(p)$.

If $P \not\subseteq A_2(p)$ then $\tilde{a}_i(p) = a_i(p)$ for $i = 0, 1$. $\tilde{A}(p) = A(p)P = L$ hence $\pi(L, p) = -1$ and $\tilde{a}_0(p) = \tilde{a}_1(p) = a_1(p)$.

If $P \subseteq A_2(p)$ then $P = A_2(p)$ and $\tilde{a}_i(p) = a_{i+1}(p)$ for $i = 0, 1$, $\pi(L, p) = 1$ implies $\tilde{a}_0(p) = \tilde{a}_0(p) = a_1(p)$.

5.3 Secants

Let $A$ be an arc and $L$ be a k-space $-1 \leq k \leq 3$. $L$ is said to be a k-secant of $X \subset J$ if
\[ V_{p \in X} A_\delta(p, L)(p) = L \]

L is said to be an independent k-secant of X if

\[ k = \sum_{p \in X} (\delta(p, L)+1) - 1 \]

X is said to be k-independent, \(-1 \leq k \leq 3\), if every k-secant of X is independent.

We define the mapping

\[ A^k: j^{k+1} \longrightarrow \mathbb{P}^3_k \]

\(-1 \leq k \leq 3\), inductively by requiring that \(A^{-1}(\ ) = \emptyset\) and

\[ A^k(p_0, \ldots, p_k) = A(p_k)/A^{k-1}(p_0, \ldots, p_{k-1}). \]

We denote an element \((p_0, \ldots, p_k) \in j^{k+1}\) by \(x\) and define

\[ \gamma(p, x) = \sum_{p_i = p} 1 -1 \]

where \(p_i\) are the components of \(x\). We define \(x\) to be independent if

\[ V_{p \in j} A_{\gamma(p, x)}(p) = A^k(x). \]

Then using arguments similar to those found in Chapter II, the results of 2.3 and 2.4 can be extended to the case of a real projective three space.

5.3.1 Lemma: A k-space L with \(A(p) \subset L\) is a k-secant of X iff it is a \((k-1)\)-secant of X on \(A/p\), where \(p \in X\).
5.3.2 Lemma: The set of all k-secants of a connected set $X \subseteq J$ is pathwise connected, $-1 \leq k \leq 3$.

5.3.3 Lemma: Suppose $-1 \leq h \leq k \leq 2$. If $X$ is $k$-independent it is also $h$-independent.

Proof: Let $L$ be an $h$-secant of $X$. By 1.3.1, there are say $p_1, \ldots, p_r$ distinct points of $X$ such that $h + r = k$, $A(p_i) \notin LA(p_1) \cdots A(p_{i-1})$ for $i = 1, \ldots, r$ and $LA(p_1) \cdots A(p_r)$ is a $k$-secant of $X$.

If $L$ is not independent then

$$h < \sum_{p \in X} (\delta(p,L)+1)-1$$

$$= \sum_{p \in X} (\delta(p,L)+1)-1$$

Therefore

$$h+r < \sum_{p \in X} (\delta(p,L)+1) + \sum_{i=1}^{r} (\delta(p_i,A(p_i)+1)-1)$$

Since $L$ and $A(p_i)$ are contained in $LA(p_1) \cdots A(p_r)$ for $i = 1, \ldots, r$, then

$$\delta(p_i,L), \delta(p_i,A(p_i) \leq \delta(p_i,LA(p_1) \cdots A(p_r))$$

for all $p \in X$ and $i = 1, \ldots, r$. Hence

$$k = h+r < \sum_{p \in X} (\delta(p_i,LA(p_i)) \cdots A(p_r)+1)-1;$$

a contradiction.
5.3.4 Lemma: Let $X$ be $k$-independent, $-1 \leq k \leq 2$, and $L$ be a $k$-secant of $X$ on $A$ with $A(p) \subset L$. Then if $q \neq p$, $A(p) \not\subset A_6(q,L)(q)$.

5.3.5 Lemma: If $X$ is $k$-independent and $p \in X$ then $X$ is $(k-1)$-independent on $A/p$.

5.3.6 Theorem: $A^k(x^{k+1})$ is the set of all $k$-secants of $X$,

$-1 \leq k \leq 3$.

5.3.7 Theorem: Let $X \subset J$ and $x \in x^{k+1}$. Then $A^k(x)$ is an independent $k$-secant of $X$ iff $\delta(p,A^k(x)) = \gamma(p,x)$ for all $p \in X$.

iff $A^k(x) = A^k(y)$, where $y \in x^{k+1}$ holds exactly for the permutations of $X$.

5.3.8 Lemma: $x \in J^{k+1}$ for $k = 1$ [k=2, k=3] is independent iff the components of $x$ are not equal [not collinear, not coplanar].

5.3.9 Theorem: Let $-1 \leq k \leq 2$. Then $X \subset J$ is $k$-independent iff every $x \in x^{k+2}$ is independent.

5.4 Order and monotonicity

Let $A$ be an arc, $X \subset J$, and $L$ a $k$-space, $-1 \leq k \leq 3$.

If $S(x,L) = \{ p \in X | A(p) \subset L \}$ is finite for every $k$-space $L$, $k$ fixed, we say that $X$ has finite $\mathbb{P}^3_k$ order. Moreover, it

$\sup |S(x,L)|$ is bounded, then we say $X$ has bounded $\mathbb{P}^3_k$ order. Let $\mathbb{P}^3_k$.

We speak of the order of $X$ when $k = 2$.

5.4.1 Theorem: If $(p,q)$ has order 3 then $(p,q]$ and $[p,q)$ are 2-independent.
**Corollary 1:** An arc has order 3 iff it is 2-independent.

**Corollary 2:** If \((p,q)\) has order 3 on \(A\) and \(r \in [p,q]\) then \((p,q)\) has order 2 on \(A/r\).

**Corollary 3:** If \((p,q)\) has order 3 on \(A\) and \(s \in (p,q)\) then

\[ A_2(s) \cap A[p,q] = A(s). \]

Let \(A\) be an arc of order 3, \(L\) an oriented line and assume no 1-scoutant of \(A\) meets \(L\). Then for \(x \in J^3\), \(A^2(x) \cap L\) is a point \(\varphi(x)\) of \(L\), (Figure V.5). We assume there is a point \(p_\infty \in L\) such that \(p_\infty \neq \varphi(x)\) for \(x \in J^3\) and define

\[(p_0,p_1,p_2) \leq (q_0,q_1,q_2)\]

if \(p_i < q_i\) for \(i = 0,1,2\).

**5.4.2 Theorem:** \(\varphi\) is (strictly) monotone.

**Proof:** Since \(A\) has order 3, \(A\) is 2-independent by 5.4.1 and hence \(A^2\) and \(\varphi\) are symmetric for \(x \in J^3\) by 5.3.7; that is

\[ \varphi(p_0,p_1,p_2) = \varphi(p_1,p_2,p_0) = \varphi(p_2,p_0,p_1) \]

and so on where \(x = (p_0,p_1,p_2)\).

Suppose \((p_0,p_1,p_2) \leq (q_0,q_1,q_2)\). In 2.4.2, it was proved that \(\varphi\) is monotone in a real projective plane, hence for \(p \in J\)

\[ p \rightarrow \varphi(p_0,p_1,p) \]

is monotone. If \(\varphi\) is say increasing, then
\[ \psi(p_0, p_1, p_2) \leq \psi(p_0, p_1', q_2') \leq \psi(p_0, q_1', q_2') \leq \psi(q_0', q_1', q_2') \]

by symmetry of \( \psi \)

We define a point \( p \) to be ordinary if it is of order 3, where the order of \( p \) is defined to be the minimum order which a neighbourhood of \( p \) can possess. If \( p \) is not ordinary, it is said to be singular.

An elementary point has a right and left neighbourhood of order 3. A point \( p \) is regular if \( a_i(p) = 1 \) for \( i = 0, 1, 2 \) and an inflection if \( a_0(p) = a_1(p) = 1 \) and \( a_2(p) = 2 \).

5.4.3 Lemma: An ordinary point is regular.

Proof: Use 2.6.1 and 5.4.1 upon projecting from distinct points of \( U(p) \) where \( p \) is ordinary and \( U(p) \) has order 3.

5.4.4 Theorem: If \( (p, q) \) is of order 3, then \( A^k \) is continuous on \([p, q]^{k+1}, -1 \leq k \leq 2\).

Proof: The result holds trivially for \( k = -1 \) and 0. If \( k = 1 \), let \( x = (p_0, p_1) \in [p, q]^2 \) and take \( q_1, q_2 \) such that

\[ p_1 < q_1 < q_2 < q \]

for \( i = 0, 1 \). Then

\[ A^1(p_0, p_1') = A^2(q_1, p_0, p_1') \cap A^2(q_2, p_0, p_1') \]
for all \((p_0', p_1') \in [p, q]^2\). But \((A/q_1)^{\frac{1}{q}}(p_0', p_1')\) is continuous on 
\([p, q]^2\) by 2.6.2. Since

\[(A/q_1)^{\frac{1}{q}}(p_0', p_1') = A^2(q_1, p_0, p_1')\]

for \(i = 0, 1\), the result follows for \(k = 1\).

For the proof of \(k = 2\), see 2.6.2.

**Corollary:** If \(A\) is elementary, then \(A^k\) is continuous
for \(-1 \leq k \leq 2\).
CHAPTER VI

6.1 Arcs with tower

A set \( \{ H_i \} \) for \( -1 \leq i \leq 3 \) of spaces is called a tower if \( H_{-1} \subset H_0 \subset H_1 \subset H_2 \subset H_3 \). An arc with tower is an arc \( A \) for which there exists a tower \( \{ H_i \} \) such that

\[
A_k(p) \cap H_{2-k} = \emptyset
\]

for all \( p \in J, \ -1 \leq k \leq 3 \), (Figure VI.1).

In the study of arcs in a real projective plane, the most useful and important tool developed was the Barner arc. In \( \mathbb{P}^3 \), arcs with tower assume the central role in the theory resulting in a number of simplifications.

6.1.1 Lemma: Let \( A \) be a regular arc with tower in \( \mathbb{P}^n \), \( A_k \) continuous for \( -1 \leq k \leq n \leq 3 \). Let \( p \in A \), then

\( \{ H_{-1}, H_0, H_0A(p), \ldots, H_{n-1}A(p) \} \) is a tower for each of the components of \( A \) determined by \( p \).

Proof: Let \( n = 1 \). Since \( H_0 \not\subset A(p) \) for \( p \in J \) then \( H_0A(p) \) meets \( A \) only at \( p \) and the result follows.

Let, \( n = 2 \). Consider \( \tilde{A} = A/H_0 \) an arc in \( \mathbb{P}^1 \). \( H_0 \not\subset A_1(q) \), therefore \( \tilde{A}_0(q) = 1 \) for all \( q \in J \) and \( \tilde{A} \) is regular. Put \( \tilde{H}_k = H_{k+1} \), \( -1 \leq k \leq 1 \). Since

\[
\tilde{A}_k(q) \cap \tilde{H}_{-k} = A_k(q)H_0 \cap H_{1-k} = H_0,
\]

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\( \tilde{A} \) is an arc with tower \( \{ \tilde{H}_k \} \). Therefore \( \{ \tilde{H}_{-1}, \tilde{H}_0, \tilde{H}_0 A(p) \} \) is a tower for each of the components of \( \tilde{A} \) determined by \( p \) from \( n = 1 \). Since a regular arc in \( \mathbb{P}^1 \) is of order 1, if \( q \neq p \) then \( \tilde{A}(q) \neq \tilde{A}(p) \). Thus

\[
A(q)H_0 \cap A(p)H_0 = \tilde{A}(q) \cap \tilde{A}(p) = H_0
\]

and

\[
A(q) \cap H_0 A(p) = \emptyset.
\]

Hence \( \{ H_{-1}, H_0, H_0 A(p), H_1 A(p) \} \) is a tower for each component of \( A \) determined by \( p \).

Let \( n = 3 \). Again consider \( \tilde{A} = A/H_0, H_0 \neq H_2(q) \) for any \( q \in \mathcal{J} \), therefore \( \tilde{A} \) is regular. Put \( \tilde{H}_k = H_{k+1} \) for \( -1 \leq k \leq 2 \). By the same argument as above, \( \tilde{A} \) is an arc with tower \( \{ \tilde{H}_k \} \) in \( \mathbb{P}^2 \). Since

\[
\tilde{A}_1(q) = A_1(q)H_0, \quad \tilde{A}_1 \text{ is continuous and } \{ \tilde{H}_{-1}, \tilde{H}_0, \tilde{H}_0 A(p), \tilde{H}_1 A(p) \}
\]

is a tower for each component of \( \tilde{A} \) determined by \( p \) from \( n = 2 \).

Taking \( q \neq p \),

\[
A(q)H_0 \cap H_1 A(p) = \tilde{A}(q) \cap \tilde{H}_0 A(p) = H_0
\]

and hence

\[
A(q) \cap H_1 A(p) = \emptyset.
\]

By 4.3.1, \( \tilde{A} \) is of order 2 and by 2.4.1, Corollary 3,

\[
A_1(q)H_0 \cap H_0 A(p) = \tilde{A}_1(q) \cap \tilde{A}(p) = H_0
\]

Then

\[
A_1(q) \cap H_0 A(p) = \emptyset,
\]

and the result follows.
For the remainder of this section, we assume that \( A \) is a regular arc with tower and \( A_k \) continuous, \(-1 \leq k \leq 2\).

6.1.2 Lemma: For any \( p \in H_1 \setminus H_0 \), there is at most one \( p' \in J \) such that \( p \subset A_2(p') \).

Proof: Since \( H_1 \cap A_1(p) = \emptyset \) for all \( p \in J \), put
\[
\varphi(p) = H_1 \cap A_2(p).
\]

Since \( A_2 \) is continuous, so is \( \varphi \) and \( \varphi(p) \not\in H_0 \) for all \( p \in J \).

Suppose there is \( p_1 < p_2 \) in \( J \) such that \( \varphi(p_1) = \varphi(p_2) = p \). (Figure VI, 2).

Since \( A_2 \) is continuous and \( \varphi(p_1) = \varphi(p_2) \), there exists a \( q \in (p_1, p_2) \) such that \( \varphi((p_1, p_2)) \) lies in one of the closed segments of \( H_1 \) with endpoints \( H_0 \) and \( Q = \varphi(q) \). Call this segment \( S \).

Denote \( \bar{A} = A/A_1(q) \). By 5.2.4 and the regularity of \( q \), \( A_0(q) = a_2(q) = 1 \).

Since \( \bar{A} \) is an arc in \( F^1 \) taking \( H_1 = F_1 \), we have \( H_0 \not\subset \bar{A} \) and since \( A_1(q) \) cuts \( \bar{A} \) at \( q \), there is a point \( r \in J \) such that \( \bar{A}(r) \not\subset S \). But \( A(r) = A(r)A_1(q) \cap H_1 \) and since \( q \in (p_1, p_2) \) we may assume \( r \in (p_1, p_2) \).

Put \( R = A(r)A_1(q) \cap H_1 \not\subset S \) and let \( \bar{A} = A/R \).

Now \( R \not\subset S \) implies \( R \not\subset A_1(q) \) for any \( q \in (p_1, p_2) \) and hence \((p_1, p_2)\) is regular on \( \bar{A} \). Moreover \( \bar{A} \) is an arc with tower \( \{ R, H, H_2, H_3 \} \) in \( F^2 \). Since \( \bar{A}_k(p) = A_k(p)R \) for \( p \in (p_1, p_2) \), \( \bar{A}_k \) is continuous for \(-1 \leq k \leq 2\). Hence by 4.3.1, \((p_1, p_2)\) is of order 2 on \( \bar{A} \). But
\[
\bar{A}(r) = A(r)R \subset A_1(q)R = \bar{A}_1(q),
\]
a contradiction by 2.4.1.
6.1.3 Lemma: If \( p < q \) then \( A(p) \not
subseteq A(q) \).

Proof: Put \( P = A(q) \cap H \). By 6.1.2, \( P \subseteq A(r) \) iff \( r = q \).
Hence \( X = \{ r/r < q \} \) is regular on \( \hat{A} = A/P \). Now \( \{ H_1, H_2, H_3 \} \)

is a tower for \( X \) on \( \hat{A} \) and since \( \hat{A}_k = A/k \), \( \hat{A}_k \) is continuous for

\(-1 \leq k \leq 1\). Therefore \( X \) has order 2 by 4.3.1 and the result follows

from 2.4.1.

6.1.4 Theorem: If \( A \) is a regular arc with tower and \( \hat{A}_k \) is
continuous, \(-1 \leq k \leq 2\), then \( A \) is of order 3.

Proof: Assume \( A \) is not of order 3, hence there are points

\( p_0 < p_1 < p_2 < p_3 \) such that they are coplanar. Let \( X = \{ p/p > p_0 \} \).

By 6.1.3, \( A(p_0) \not
subseteq A(p) \) for any \( p \in X \), hence \( X \) is regular on
\( \hat{A} = A/p_0 \). By 6.1.1, \( \{ H_1, \ldots, H_2A(p_0) \} \) is a tower for \( X \) on \( A \),
therefore \( \{ A(p_0), H_0A(p_0), H_1A(p_0), H_2A(p_0) \} \) is a tower for \( X \) on \( \hat{A} \).
Since \( \hat{A}_k(p) = A_k(p)A(p_0) \), \( \hat{A}_k \) is continuous for \(-1 \leq k \leq 2\) and \( X \)
has order 2 on \( \hat{A} \) by 4.3.1. But \( p_1 < p_2 < p_3 \) are collinear on \( X \),
a contradiction.

6.2 Finiteness

A point \( p \) of an arc is right finite (strong right finite) if
for every \((2-k)\)-space \( L, 0 \leq k \leq 2 \), there is a \( U^+(p) \) such that no
osculating \( k \)-space (no \( k \)-secant) of \( U^+(p) \) meets \( L \). The point \( p \) is
finite (strongly finite) if it is both right and left finite (both
strong right and strong left finite), where left finiteness (strong left
finiteness) is defined similarly. An arc is finite (strongly finite)
if each of its points is finite (strongly finite).
6.2.1 Lemma: A finite (strongly finite) point \( p \) is finite (strongly finite) on any projection \( A/P \).

6.2.2 Theorem: An elementary arc \( A \) is strongly finite.

Proof: Let \( p \) be a point on the arc. If \( L \) is a plane, by 1.3.1 there exists \( U^+(p) \) such that \( L \cap A(U^+(p)) = \emptyset \). Hence we need only consider points and lines in \( P^3 \).

Let \( P \) be a point in \( P^3 \) and \( U^+(p) \) be of order 3. If \( P = A(p) \), no 2-secant of \( U^+(p) \) contains \( P \) by 5.4.1. Let \( P \neq A(p) \) and \( L = PA(p) \).

Since \( U^+(p) \) has order 3, then \( U^+(p) \) has order 2 on \( \tilde{A} = A/P \) by 5.4.1. Then \( p \) will be elementary on \( A \) and thus \( p \) is strongly finite on \( A \) by 4.2.2, and there exists \( U_1^+(p) \subset U^+(p) \) such that no 1-secant of \( U_1^+(p) \) meets \( L \). Put

\[
\varphi(x) = A^2(x) \cap L
\]

for \( x \in (U_1^+(p))^3 \), \( \varphi \) is monotone by 5.4.2. If \( q \in U_1^+(p) \) then

(Figure VI.3)

\[
\lim_{r \to p^+} \varphi(q, q, r) = A(p).
\]

Since \( P \neq A(p) \), there is a \( U_2^+(p) \subset U_1^+(p) \) such that \( \varphi(x) \neq P \) for all \( x \in (U_2^+(p))^3 \). Hence, no 2-secant of \( U_2^+(p) \) contains \( P \).

Let \( L \) be a line and choose a point \( P \in L \). By the above, choose \( U_2^+(p) \) such that no 2-secant of \( U_2^+(p) \) contains \( P \). Suppose there is a plane \( \Pi \) through \( P \) which meets \( U_2^+(p) \) at points \( P_1, P_2, P_3 \). Since \( U_2^+(p) \) is of order 3, it is 2-independent by 5.4.1 and...
$H = A(p_1)A(p_2)A(p_3)$. Then $H$ is a 2-secant of $U_2^+(p)$; a contradiction. Hence $U_2^+(p)$ has order 2 on $A/P$, it is thus elementary and strongly finite, by 4.2.7, on $A/P$. Therefore there exists $U_3^+(p) \subset U_2^+(p)$ such that no 1-secant of $U_3^+(p)$ on $A/P$ meets $L$. Since $P \in L$, the result follows.

**Corollary:** An elementary point $p$ of an arc $A$ is elementary on any projection $A/P$.

**Proof:** Let $P$ be an elementary point of $A$. Since $P$ is elementary, it is strongly finite on $A$. Therefore there exists $U^+(p)$ ($U^-(p)$) of order 3 such that no 2-secant of $U^+(p)U^-(p)$ meets $P$. Then as above, no plane through three points of $U^+(p)U^-(p)$ meets $P$ and the result follows.

**6.2.3 Lemma:** A regular elementary point $p$ is ordinary.

**Proof:** Let $\{H_4\}$ be a tower of spaces such that $A_k(p) \cap H_{2-k} = \emptyset$ for $0 \leq k \leq 2$. By 6.2.2, there exist $U^+(p)$, $U^-(p)$ of order 3 such that

$A_k(q) \cap H_{2-k} = \emptyset$

for all $q \in U^+(p) \cup U^-(p)$. Then $U(p) = U^+(p) \cup \{p\} \cup U^-(p)$ is an arc with tower which is regular by 5.4.3. $A_k$ is continuous on $U(p)$ by 5.4.4, hence $U(p)$ is of order 3 by 6.1.4.

**Corollary:** If $p$ is an ordinary point or an elementary inflection, and $\pi(P,p) = 2$ then $p$ is ordinary on $A/P$. 
It should be noted that the 2-dimensional analogue of 6.2.2 was proved by Denk's Theorem.

6.2.4 Lemma: Let \( p \in J \). Given \( U^+(p) \) and a neighbourhood \( U \subset \mathbb{P}^3 \) of \( A_k(p) \), there is a \( k \)-secant \( L \) of \( U^+(p) \) with \( L \subset U(A_k(p)) \), \( 0 \leq k \leq 2 \).

6.2.5 Theorem: Let \( p \) be a strongly right finite point of an arc \( A \). Then

\[
A^k(p_0, \ldots, p_k) \rightarrow A_k(p)
\]
as \( (p_0, \ldots, p_k) \rightarrow (p, \ldots, p) \) where \( p_i > p, i = 0, \ldots, k \).

Proof: Since \( A^k(\mathbb{P}^{k+1}) \) is the set of all \( k \)-secants of \( A \), we must show that given a neighbourhood \( U(A_k(p)) \) of \( A_k(p) \), there is a \( U^+(p) \) such that

\[
A^k((U^+(p))^{k+1}) \subset U(A_k(p)).
\]

Let \( k = 1 \). Let \( Q_1, Q_2 \) be points with \( Q_i \not\in A_1(p) \) for \( i = 1, 2 \) and \( Q_1A_1(p) \not\in Q_2A_1(p) \). Let \( \tilde{U}(Q_iA_1(p)) \) be a neighbourhood of \( Q_iA_1(p) \) in \( A/Q_i \) such that \( M_1 \cap M_2 \subset U(A_1(p)) \) if \( M_i \subset U(Q_iA_1(p)); i = 1, 2 \) (Figure VI.4).

By projection from \( Q_i \) and 4.2.4, there is \( U^+(p) \) such that every \( 1 \)-secant of \( U^+(p) \) on \( A/Q_i \) is in \( \tilde{U}(Q_iA_1(p)); i = 1, 2 \). Hence if \( M \) is a \( 1 \)-secant of \( U^+(p) \) on \( A \), then \( Q_iM \) is a \( 1 \)-secant of \( U^+(p) \) on \( A/Q_i; i = 1, 2 \). But \( M = Q_iM \cap Q_2M \) and the result follows.
Figure VI.4
Let $K = 2$. Let $L$ be a line with $A_1(p) \cap L = \emptyset$. Put $P = A_2(p) \cap L$. Let $U(P)$ be a neighbourhood of $P$ on $L$, with end points $Q_1, Q_2$. Let $U(A_1(p))$ be a neighbourhood of $A_1(p)$ such that $QM \subset U(A_2(p))$ for all $Q \in U(P)$, $M \in U(A_1(p))$, (Figure VI.5). Since $p$ is strongly finite, applying the case $k = 1$, take $U^+(p)$ such that no 2-secant of $U^+(p)$ contains $Q_1$ or $Q_2$ and every 1-secant of $U^+(p)$ is in $U(A_1(p))$. By 6.2.4, there is a 2-secant of $U^+(p)$ which meets $U(P)$. Since $U^+(p)$ is connected, every 2-secant of $U^+(p)$ meets $U(P)$ by 5.3.2.

6.2.6 Theorem: A finite arc with at most inflections is strongly finite.

6.2.7 Theorem: A regular finite arc with tower is of order 3.

Proof: By 6.2.6, a regular finite arc is strongly finite and $A_k, -1 \leq k \leq 3$ is continuous by 6.2.5. Hence it is of order 3 by 6.4.4.

Corollary: A regular finite arc is ordinary.

6.3 Behaviour of secants

In Chapter IV, we constructed a way of describing the real projective plane, depending only on a point $p$ of an arc. The method consisted of choosing independent points from the osculating spaces of $p$ and then the 2-simplices determined by these points formed a covering for the plane. We now extend this construction to $P_3$. 
Let $p$ be a point of an arc $A$. Let $P_i \subset A_i(p) - A_{i-1}(p)$, $0 \leq i \leq 3$. Then $\{P_0, P_1, P_2, P_3\}$ are independent and the vertices of 9 open 3-simplices.

If $1 \leq i \leq 3$ and $i$ is odd [even] let $E_i^+$ be the open segment of $P_{0i}$ with end points $P_0, P_i$ (Figure VI,6), which is [is not] and edge of $S^+$, where $S^+$ is that open 3-simplex which contains some $U^+(p)$. Similarly $E_i^-$ and $S^-$ are defined using some $U^-(p)$.

Let $p$ be a strongly right finite point of an arc. Choose $U^+(p)$ such that no k-secant of $U^+(p)$ meets a (2-k)-space spanned by the points $P_i$, $0 \leq i \leq 3$. Then the k-secants of $U^+(p)$ meet any (3-k)-space spanned by the $P_i$.

Consider the (3-k)-space $P_0 P_{m(1)} \ldots P_{m(3-k)}$ where $0 < m(1) < \ldots < m(3-k) \leq 3$. Let $S^+(m(1), \ldots, m(3-k))$ be the open (3-k)-simplex with vertices $P_0, P_{m(1)}, \ldots, P_{m(3-k)}$ which the k-secants of $U^+(p)$ meet. One can choose one open (3-k)-simplex since $U^+(p)$ is connected and thus the k-secants of $U^+(p)$ are connected.

6.3.1 Theorem: $E_{m(i)}^+$ is an edge of $S^+(m(1), \ldots, m(3-k))$ iff $i$ is odd; $1 \leq i \leq 3-k$, $0 \leq k \leq 2$.

Proof: For $l = 0$, $S^+(m(1), \ldots, m(s)) = S^+$ and the result follows.

Let $k = 1$. First consider Figure VI,7. Here $m(i) = i$; $i = 1,2$. Projecting from $P_1, P_0, P_2$ and $S^+$ are projected onto $P_0, P_1$ and $S^+$. Put $M = E_2^+/P_1$, then $M \neq E_1^+$ whereas $S^+(1,2)$ is projected on $S^+(1)$. Hence, $E_2^+$ is not edge of $S^+(1,2)$. Now
Figure VI.6
Figure VI.7
projecting from $P_2$ instead of $P_1$ we find that $E_1^+$ is an edge of $S^+(1,2)$.

More generally, the projections of $P_0, P_m(2)$ from $P_m(1)$ are $P_0, P_m(2)_1$ respectively. The projection of $S^+$ is $S^+_m$, hence the projection of $E_m(2)$ is the open segment of $P_0, P_m(2)_1$ different from $E_m(2)_1$. But $S^+(m(1), m(2))$ is projected onto $S^+_m(m(1), m(2))$ and hence $E_m(2)$ is not an edge of $S^+(m(1), m(2))$. By projecting from $P_m(2)$, we have that $E_m^+$ is an edge of $S^+(m(1), m(2))$.

Let $k = 2$. We now have to show that $E_{m(1)}^+ = S^+(m(1))$. Suppose $m(1) > 1$. Project $A$ from $A(q)$ into the plane $P_0P_1P_2$, since $m(1) = 2$. From $k = 1$, $E_1^+(E_2^+)$ is (is not) an edge of $S^+(1,2)$. Also $S^+ = S^+(1,2)$. Hence $E_i^+ = E_i^+$, $i = 1, 2$. Since $A_1(q)$ meets $E_1^+$ and $E_2^+$, $A_2(q)$ meets $E_1^+$ and $E_2^+$. Thus $S^+(2) = E_2^+$.

6.4 Dually differentiable arcs

Let $A$ be an arc in $P^3$. Define a mapping $A^*: J \rightarrow P^3_0$ by $A^*(p) = A_2^*(p)$ for all $p \in J$. We say $A$ is dually differentiable if $A^*$ is an arc in $P^3$ and $A_k^* = A_{2-k}$, $-1 \leq k \leq 3$.

If $A$ is dually differentiable then $A_2$ is continuous and if $p \in P^3_0$ and $p \in J$, there exists $U^1(p)$ such that $P \not\in A_2(q)$ for all $q \in U^1(p)$. Moreover, $A$ is dually differentiable iff

$$A_k^*(p) = \lim_{q \to p} A_2(q) \cap A_{k+1}(p)$$

for all $p \in J$, $-1 \leq k \leq 2$. 
Let \( p \) be a point of an arc \( A \). Take \( P_i, S^+, S^-, E^+_i, E^-_i \) as in Section 6.3. If \( A \) is dually differentiable then by the above comment, there exists \( U^+(p) \) such that \( P_i \not\subseteq A_2(q) \) for all \( q \in U^+(p), i = 0, 1, 2 \). Put \( \sigma^+_k(p) = 0 \) \( \sigma^-_k(p) = 1 \) if \( A_2(q) \) meets (does not meet) \( E^+_k \) for all \( q \in U^+(p), 0 \leq k \leq 2 \). Also put \( \sigma^-_{-1}(p) = 0 \) and define \( \sigma^-_k(p), -1 \leq k \leq 2 \), similarly.

**6.4.1 Lemma:** \( \sigma^-_k(p) = 0 \) iff \( E^+_k = E^-_{k+1}, 0 \leq k \leq 2 \).

**6.4.2 Theorem:** At any point of a dually differentiable arc,

\[
\sigma^-_k = \sigma^+_0 + \sigma^-_1 + \sigma^+_2 + \sigma^-_3 + \sigma^+_4 \pmod{2}, \quad 0 \leq k \leq 2.
\]

**Corollary:** \( a^-_k = a^-_{2-k} + \sigma^+_1 + \sigma^-_2 + \sigma^+_3 + \sigma^-_4 \pmod{2}, \quad 0 \leq k \leq 2. \)

**6.4.3 Lemma:** Let \( p \) be a strongly right finite point of an arc \( A \).

Let \( P_i \in A_i(p) - A_{i-1}(p), 0 \leq i \leq 3 \). Then

\[
\lim_{q \to p} A_2(q) \cap P_k P_{k+1} = P_k, \quad 0 \leq k \leq 2.
\]

**Proof:** Let \( U(P_k) \) be a neighbourhood of \( P_k \) on \( P_k P_{k+1} \). Take \( U^+(p) \) such that no \( h \)-secants of \( U^+(p) \) meets a \((2-h)\)-space spanned by the points \( P_i, 0 \leq h \leq 2 \) and \( 0 \leq i \leq 3 \); moreover, no 2-secant of \( U^+(p) \) contains an end point of \( U(P_k). \)

Let \( L \) be a \((1-k)\)-secant of \( U^+(p) \). Then \( L \) does not meet any \( k + 1 = 2-(1-k) \) space spanned by the \( P_i \), hence

\[
L \cap A_{k+1}(p) = \emptyset
\]
Figure VI.8
and \( L_{k}(p) \) is a plane. (See Figure VI.8 for the case \( k = 0 \).) Now \( L_{k}(p) \cap P_{k}P_{k+1} = P_{k} \) otherwise \( P_{k+1} \subseteq L_{k}(p) \) and \( L \cap A_{k+1}(p) \neq \emptyset \).

Let \( U(A_{k}(p)) \) be a neighbourhood of \( A_{k}(p) \) such that \( L \cap M = \emptyset \) and \( LM \) meets \( U(P_{k}) \) for all \( M \in U(A_{k}(p)) \). This is permissible by the continuity of \( A_{k} \); cf. 6.2.5. By 6.2.4, there is a \( k \)-secant \( M \) of \( U_{k}^{+}(p) \) such that \( M \in U(A_{k}(p)) \). Since \( LM \) meets \( U(P_{k}) \), then all \( 2 \)-secants of \( U_{k}^{+}(p) \) meets \( U(P_{k}) \) by 5.3.2 and the result follows.

**6.4.4 Theorem:** A strongly finite arc is dually differentiable and

\[
a_{k}^{*}(p) = a_{2-k}(p)
\]

for all \( p \in J, 0 \leq k \leq 2 \).

**6.4.5 The Scherk-Derry duality theorem:** The dual \( A^{*} \) of an arc of order 3 is also of order 3.

**Proof:** \( A \) is of order 3, hence elementary and strongly finite by 6.2.2. Therefore \( A \) is dually differentiable by 6.4.4. By 5.4.3 \( A \) is regular and since \( a_{k}^{*}(p) = a_{2-k}(p) \) for all \( p \in J, 0 \leq k \leq 2 \), \( A^{*} \) is also regular. By 5.4.4, \( A_{k}^{*} = A_{2-k}^{*} \) is continuous for \( 0 \leq k \leq 2 \).

Therefore by 5.4.1, for any \( p, q \in J, \{ A_{i}(p) \} \) is a tower for \( (p, q) \) on \( A \). Hence \( \{ A_{i}^{*}(p) \} \) is a tower for \( (p, q) \) on \( A^{*} \). Then \( (p, q) \) has order 3 on \( A^{*} \) by 6.1.4 and hence \( A^{*} \) has order 3.

**Corollary 1:** The dual of an elementary arc is elementary.

**Corollary 2:** Let \( A \) be of order 3. For any point \( P \) there exists \( p \in J \) such that \( P \not\in A_{2}(p) \).
Proof: If \( P \not\in A_2(p) \) for all \( p \in J \), then \( A^*(p) \subseteq P \), a plane in \( \mathbb{P}^3^* \) for all \( p \in J \); a contradiction by 1.3.1.

We have already seen in 6.2.2 that a regular elementary point is ordinary. The question then is whether there exist any elementary singular points. In 4.4.4, it was shown that if a finite arc in \( \mathbb{P}^2 \) has a singularity then it has an elementary singularity. This result can be extended to \( \mathbb{P}^3 \) using the proof of 4.4.4 with dimensional modifications. But to be able to do this, we must first consider dual projections from a line in \( \mathbb{P}^3^* \).

6.4.6 Lemma: Let \( p \) be a point of a strongly finite arc. Let \( P_i \subseteq A_i(p) - A_{i-1}(p); \ 0 \leq i \leq 3 \). The dual projection of \( A \) into the line \( P_k P_{k+1} \) has the characteristic \( (a_k(p)) \) at \( p, \ 0 \leq k \leq 2 \).

Proof: Since \( A \) is strongly finite, it is dually differentiable and \( a_k^* = a_{2-k}; \ k = 0, 1, 2 \). Keeping in mind that the line \( P_k P_{k+1} \) in \( A \) is also a line \( (P_k P_{k+1})^* \) in \( A^* \), we wish to project \( A^* \) from \( (P_k P_{k+1})^* \). In the following, we shall use 5.2.4 as a reference. Denote \( A^* = A^*/(P_0 P_1)^* \).

If \( k = 0 \), then \( (P_0 P_1)^* = (A_1(p))^* = A_1^*(p) \) and \( a_0^*(p) = a_2^*(p) = a_0(p) \).

If \( k = 1 \), \( (P_1 P_2)^* \not\subseteq A_2^*(p) \) since \( A(p) \not\subseteq P_1 P_2 \). But \( (P_1 P_2)^* \cap A_2^*(p) = P_0 P_1 P_2 = A^*(p) \), hence \( a_0^*(p) = a_1^*(p) = a_1(p) \).

If \( k = 2 \), \( A^*(p) \not\subseteq (P_2 P_3)^* \) since \( P_2 P_3 \not\subseteq A_2(p) \). Moreover, \( (P_2 P_3)^* A_1^*(p) = P_2 P_3 A_1(p) = A_3(p) = \emptyset^* \). Hence \( a_0^*(p) = a_0^*(p) = a_2(p) \).
6.4.7 Theorem: If a finite arc has a singularity it has an elementary singularity.

Let \( L \) be a plane. Put

\[
\mu(p,L) = \beta_6(p,L)(p)
\]

for all \( p \in J \). We call \( \mu(p,L) \) the multiplicity with which \( L \) meets \( A \) at \( p \).

6.4.8 Theorem: Let \( A \) be an elementary arc with tower. Then

\[
\sum_{p \in J} \mu(p,L) \leq \sum_{p \in J} \sum_{i=0}^{2} (a_i(p)-1)+3
\]

for every plane \( L \).

6.4.9 Denk's Theorem: Let \( p \) be an elementary point, the

\[
\text{Ord}(p) = \sum_{i=0}^{2} a_i(p)
\]
CHAPTER VII

7.1 Barner Arcs

An arc is Barner if there exists a continuous mapping

\[ B: J^2 \rightarrow \mathbb{R}^3 \]

such that

\[ \delta(p, B(x)) = \beta \gamma(p, x)(p) - 1 \]

for all \( p \in J, x \in J^2 \).

7.1.1 Lemma: Let \( A \) be an arc such that for each \( p \in J \), \( a_4(p) = 2 \) for at most one \( i \), \( 0 \leq i \leq 2 \). If there exists a point \( P \) such that \( A/P \) is of order 2 and

\[ \pi(P, p) = \begin{cases} 2 & \text{if } p \text{ regular} \\ i & \text{if } a_4(p) = 2 \end{cases} \]

for all \( p \in J \), then \( A \) is a Barner arc.

7.1.2 Theorem: Let \( A \) be an arc with tower and with at most inflections. Let \( A_k \) be continuous, \( 0 \leq k \leq 2 \). Then \( A \) is a Barner arc.

Proof: Let \( \{ H_i \} \) be a tower. Since \( H_0 \not\subset A_2(q) \) for any \( q \in J \), \( \tilde{A} = A/H_0 \) is a regular arc with tower \( \{ H_0, H_1, H_2, H_3 \} \). Moreover \( A_k(p) = A_k(p)H_0 \) is continuous for \( k = 1 \). Then \( A \) is of order 2 by 7.1.1, and \( A \) is Barner by 7.1.1.
It should be noted that any \((p,q)\) of order 3 where \(p,q \in J\) is Barner by 7.1.1.

For the remainder of this section we assume \(A\) is a Barner arc unless otherwise stated.

7.1.2 \(A(p) \subseteq B(p_1,p_2)\) iff \(p \in \{p_1,p_2\}\)

7.1.4 \(e(p,B(p,p)) = B_2(p) - 1\) for all \(p \in J\).

7.1.5 \(A(p) \subseteq B(p,p)\) for all \(p \in J\).

7.1.6 If \(p \neq q\) then \(A(p) \subseteq A(q)\); hence \(A\) is simple; that is, \(A\) is 1-1.

7.1.7 If \(p \neq q\) and \(A(p) \subseteq A(q)\), then \(B_2(q) = 2\).

7.1.8 **Lemma:** Let \(q \in J\). If \(a_0(q) = 1\) then \(A/q\) is a Barner arc. If \(a_0(q) = 2\), then the restriction of \(A/q\) to either component of \(J - \{q\}\) is a Barner arc.

**Proof:** Let \(y_1 \in J\) be given and put \(x = (q,y_1)\). Then put \(\hat{B}(y_1) = B(x)\) in \(\hat{A} = A/q\) and hence \(\hat{B}\) is continuous. Let \(a_0(q) = 1\).

**Case 1.** If \(p = q\) then \(\pi(q,p) = -1\) and by 5.2.2,
\[
\hat{\beta}_k(p) = \beta_{k+1}(p) - \beta_0(p).
\]
Since \(\gamma(p,y_1) = \gamma(p,x) - 1\) and \(\beta_0(p) = a_0(p) = 1\),
\[
\hat{\beta} \gamma(p,y_1)(p) = \beta \gamma(p,y_1+1)(p) - 1
\]
\[
= \beta \gamma(p,x)(p) - 1.
\]
by $5.2.1$, \( \delta(p, B(y_1)) = \delta(p, B(x)) - 1 \), hence

\[
\delta(p, B(y_1)) = (\beta \gamma(p, x)(p) - 1) - 1
\]

\[
= \beta \gamma(p, y_1)(p) - 1.
\]

**Case 2.** If \( p \neq q \) and \( A(q) \subseteq A_2(p) \). Then \( \pi(q, p) = 1 \) by $7.1.6$. But \( p \neq q \) implies

\[
\gamma(p, y_1) = \gamma(p, x) < 1.
\]

Therefore

\[
\delta(p, B(x)) = \beta \gamma(p, x) - 1 < \beta(p) - 1.
\]

and since \( A(q) \subseteq A_2(p) \), \( \beta_1(p) = 2 \) by $4.1.7$. Then

\[
\delta(p, B(x)) < 2 - 1 = 1
\]

and \( \delta(p, B(y_1)) = \delta(p, B(x)) \) by $5.2.1$. Hence

\[
\delta(p, B(y_1)) = \delta(p, B(x))
\]

\[
= \beta \gamma(p, x)(p) - 1
\]

\[
= \beta \gamma(p, y)(p) - 1.
\]

**Case 3.** If \( p \neq q \) and \( A(q) \not\subseteq A_2(p) \), then \( \pi(q, p) = 2 \) and the result follows as in **Case 2**.

Let \( a_0(q) = 1 \) and \( p = q \). Then

\[
\beta \gamma(p, y_1)(p) = \beta \gamma(p, y_1)(p) + 1(p) - \beta_0(p)
\]

\[
= \beta \gamma(p, x)(p) - 2.
\]
Therefore
\[ \hat{\delta}(p, B(y)) = \delta(p, B(x)) - 1 \]
\[ = (\beta \gamma(p, x)(p) - 1) - 1 \]
\[ = \beta \gamma(p, y_1)(p) \]
\[ \neq \beta \gamma(p, y_1)(p) - 1. \]

7.1.9 Lemma: A l-space \( L \) meets a Barner arc in at most two points.

Proof: If \( L \) meets \( A \) at points \( p_1 < p_2 < p_3 \). Then since \( A \) is simple; \( A(p_1), A(p_2) \) and \( A(p_3) \) are distinct and collinear.

Let \( x = (p_1, p_2) \), then \( A(p_1) \subseteq B(x) \) for \( i = 1, 2 \). Hence
\[ L = A(p_1) A(p_2) \subseteq B(x) \]
and finally \( A(p_3) \subseteq B(p_1, p_2) \), a contradiction.

Therefore a l-secant of a Barner arc meets the arc at most twice. If \( q \in J \), then \( p \neq q \) implies \( A(p) \notin A_1(q) \) for all \( p \in J - \{ q \} \) by 7.1.6. Hence the result:

7.1.10 Lemma: If \( A \) is Barner then \( A \) is l-independent.

7.1.11 Lemma: If \( p \in J \) then \( a_i(p) = 2 \) for at most one \( i \), \( 0 \leq i \leq 2 \).

Proof: If \( a_0(p) = 1 \) and \( a_1(p) = a_2(p) = 2 \) then \( \hat{\gamma} = A/p \)
is Barner by 7.1.8, and \( \hat{\gamma}_0(p) = a_0(p) = 2 = a_2(p) = a_1(p) \), a contradiction by 7.1.6. Therefore \( a_0(p) = 2 \). If \( a_0(p) = a_1(p) = 2 \) then by 7.1.4,
\[ \delta(p, B(p, p)) = \beta_1(p) - 1 \]
\[ = a_0(p) + a_1(p) - 1 \]
\[ = 3 \]
which is impossible since \( B(p, p) \) is a plane. Then \( a_0(p) = 2 = a_2(p) \) and \( a_1(p) = 2 \) is the only case possible.

But projecting from \( A_1(p) \), \( a_0(p) = a_2(p) = 2 \) by 5.2.4 where \( A = A/A_1(p) \). Then \( A_0(p) \) supports \( A \) at \( p \) and one obtains points \( p_1, p_2 \) such that \( A(p_1) \cap A(p_2) \). But \( a_0(p) = 2 \) implies

\[
6(p, B(p_2, p)) = B_0(p) - 1 = 1
\]

and hence \( A_1(p) \subseteq B(p_2, p) \). Therefore

\[
A(p_1) \subseteq A_1(p) \subseteq A(p_2) \subseteq B(p_2, p)
\]

a contradiction.

7.1.12 Lemma: If \( (p, q) \) is ordinary then \( A(p) \not\subseteq A_2(q) \).

Proof: If \( a_0(q) = 2 \) then \( A_2(q) = B(q, q) \) and the result follows.

Assume \( a_0(q) = 1 \) and \( A(p) \subseteq A_2(q) \). Then \( \sim A = A/q \) is Barner and \( \sim A(p) \subseteq \sim A_1(q) \). Therefore \( (p, q) \) is not ordinary on \( \sim A \) by 3.1.7. But \( (p, q) \) ordinary on \( A \) and not ordinary on \( \sim A \) implies there exists \( p_1 \in (p, q) \) such that \( A(q) \subseteq A_2(p_1) \) by the corollary of 6.2.2.

Now consider the interval \( (p_1, q) \) with \( A(q) \subseteq A_2(p_1) \).

Repeating the above argument one obtains \( q_1 \in (p_1, q) \) such that \( A(p_1) \subseteq A_2(q_1) \). Consider the set \( S \) of intervals \( (r, s) \subseteq (q, q) \) such that \( A(r) \subseteq A_2(s) \). Then by the argument of 3.1.7, one has a contradiction.

7.1.13 Theorem: If \( (p, q) \) is ordinary then \( [p, q] \) has order 3.
Proof: Suppose $P_1, P_2, P_3, P_4$ are coplanar points with

$$p \leq p_1 < p_2 < p_3 < p_4 \leq q.$$

$p_2 \in (p, q)$, hence $a_0(p_2) = 1$ by Section 5.4.3 and $A = A/p_2$ is Barner by 7.1.8. $p_2$ ordinary implies there exists $U(p_2)$ of order 3 on $A$ and hence of order 2 on $A$ by 5.4.1. Then $p_2$ is ordinary on $A$. In fact, $(p, q)$ is ordinary on $A$ since $A(p^2) \notin A_2(r)$ for $r \in (p, q)$ by the Corollary of 6.2.3. Therefore $(p, q)$ has order 2 on $A$ by 3.1.8, a contradiction since $p_2, p_3, p_4$ are collinear on $A$.

7.1.14 Theorem: If $p < q < r$ and $(q, r)$ ordinary then

$$A(p) \notin A_2(q) \cap A_2(r).$$

Proof: Since $(q, r)$ is ordinary then $(q, q)$ has order 3 by 7.1.13 and $A_k$ is continuous, $k = 1, 2$ by 5.4.4.

Assume $A(p) \subset A_2(q) \cap A_2(r)$, $p < q < r$. By 6.4.5, Corollary 2, there exists $s_0 \subset (q, r)$ such that $A(p) \notin A_2(s_0)$, (Figure VII.1).

Since $A_2$ is continuous, then there exist $q_0, r_0$ such that

$$q \leq q_0 < s_0 < r_0 \leq q_0,$$

$A(p) \notin A_2(s)$ for all $s \subset (q_0, r_0)$ and $A(p) \subset A_2(q_0) \cap A_2(r_0)$.

Now $\pi(p, s) = 2$ for $s \subset (q_0, r_0)$ implies $(q_0, r_0)$ is ordinary on $A = A/p$. Since $A$ is 1-independent, $A(p), A(q_0)$ and $A(r_0)$ are not collinear and $M = A(p) A(q_0) A(r_0)$ is a plane. Put

$L_1 = A(p) A(q_0)$ and $L_2 = A(p) A(r_0)$. If $s \subset (q_0, r_0)$ then $A_1(s) \cap L_1 = \emptyset$.

For if there exists $s^1 \subset (q_0, r_0)$ such that $A_1(s^1) \cap L_1 \neq \emptyset$ then
\[ A(q_0) = L_2 \subseteq A_1(s^1) A(p_0) \subseteq A_1(s^1) \] in \( A_1 \), a contradiction by 2.1.7 since \((q_0, s^1) \subseteq (q_0, r_0')\) is ordinary on \( A \). Therefore \( A_1(s) \cap M \) is a point for all \( s \in [q_0, r_0'] \) and does not lie on \( L_1 \) or \( L_2 \). Hence there exists a line \( L \) such that \( L \cap A_1[q_0, r_0] = \emptyset \). Put
\[
\psi(s) = A_2(s) \cap L
\]
for all \( s \in [q_0, r_0'] \). \( \psi \) is continuous since \( A_2 \) is continuous and \( \psi \) is monotone by 5.4.2. But \( \psi(q_0) = \psi(r_0) \) and hence \( \psi \) cannot exist.

### 7.2 Regular arcs

The aim of this section is to prove the following three theorems:

#### 7.2.1
If \( A \) is a regular Baner arc and \( A_1 \) is continuous then \( A \) is of order 3.

#### 7.2.2
Every arc has an ordinary point.

#### 7.2.3
If \( A \) is a Baner arc with at most inflections and \( A_2 \) is continuous, then either \( A \) is of order 3 or \( A \) has an elementary singularity.

#### 7.2.4 Lemma:
Let \( A \) be a Baner arc with at most inflections. If \( A_2 \) is continuous at \( p \), then so is \( A_1 \).

**Proof:** By 7.1.4, \( \delta(p, B(p, p)) = B_1(p) = 1 \) for all \( p \in J \).

Since \( A \) has at most inflections, \( B_1(p) = 2 \) and hence
\[
A_1(p) = A_2(p) \cap B(p, p)
\]
Therefore \( A_1 \) is continuous at \( p \).
To make our considerations easier, we shall first show that every arc contains a Barner arc with at most inflections on which $A_1$ is continuous.

**7.2.5 Lemma:** Let $A$ be an arc on which $A_1$ is continuous. Let $W$ be the set of points of $A$ at which $A_2$ is continuous. Then $W \neq \emptyset$ and $W$ is dense in $A$.

**7.2.6 Lemma:** Every arc contains a Barner arc with at most inflections on which $A_1$ is continuous.

**Proof:** We may assume there exists a plane $H_0$ not meeting $A$.

Let $P_1, P_2, P_3$ be independent points. By 3.3.11, there exist subarcs $X_1, X_2, X_3$ such that $X_3 \subseteq X_2 \subseteq X_1$ and $X_1$ has order 2 on $A/P_i$, $i = 1, 2, 3$. Then $X_3$ has at most inflections since for any $p \in X_1$ there exists $j$ with $P_j \not\in A_2(p)$. This follows from the fact the $\{P_i\}$ are independent. But $P_j \not\in A_2(p)$ implies

$$a_j^i(p) = a_i^i(p)$$

for $i = 0, 1$, where $A_j = A/P_j$. Now $X_3$ has order 2 on $A_j$ and hence is regular on $A_j$ by 2.6.1. Therefore $a_0(p) = a_1(p) = 1$ for any $p \in J$.

But $X_3$ having at most inflections implies $P_i \not\subseteq A_1(X_3)$; otherwise $\pi(P_j, p) = 0$ for some $j$ and $a_0^j(p) = a_0(p) + a_1(p) \equiv 0 \pmod{2}$. This is a contradiction since $A_j$ is regular. Then

$$A_1(p) = \bigcap_{i=1}^{3} A_1^i(p) = \bigcap_{i=1}^{3} A_1(p)P_i$$
(Figure VII.2), and thus $A_1$ is continuous on $X_3$.

By 7.2.5, there exists $p \in X_3$ such that $A_2$ is continuous at $p$. Take a point $P$ and $U(p)$ such that $p \not\in A_2(q)$ for all $q \in U(p)$. By 3.3.11, there exists $\mathcal{X} \subset U(p)$ such that $\mathcal{X}$ has order 2 on $\widetilde{A} = A/P$ and hence $X$ is Barner by 7.1.1.

For the remainder of this chapter we assume that $A$ is a Barner arc with at most inflections and that $A_1$ is continuous.

If $H_\infty$ is a plane and $P, Q$ are distinct points not on $H_\infty$, then $L_\infty(L_f)$ will denote the segment of the line $PQ$ determined by $P, Q$ which does (does not) contain $L \cap H_\infty$.

7.2.7 Lemma: Suppose $(p, q)$ is of order 3 and $H_\infty$ is a plane not meeting $[p, q]$. Put $L = A(p)A(q)$. Then $A_2(r) \cap L \subset L_f$ for all $r \in (p, q)$.

Proof: Since ordinary points are regular, for every $r \in (p, q)$, $A_2(r)$ cuts $(p, q)$ at $r$. Since $(p_1, q)$ is of order 3, $A_2(r)$ does not meet $[p, q]$ outside of $r$ by 5.4.1. Then $A(p)$ and $A(q)$ lie in different open half-spaces determined by $A_2(r)$ and $H_\infty$, and the result follows.

7.2.8 Lemma: Suppose $p < q < r$. Let $H_\infty$ be a plane not meeting $[p, r]$. Put $L = A(p)A(q)$. If $(q, r)$ is of order 3 then $A_2(s)$ meets $L_f$ for all $s \in (q, r)$.

Proof: See 3.2.2 and Figure VII.3.

7.2.9 Lemma: Suppose $A$ is a regular arc and there exist $p < q$, $q$ ordinary such that $A(p) \subset A_2(q)$. Then there exist $r$, $s$, with $s$ singular, such that $A_2(s)$ cuts $A$ at $r$. 
Proof: Take \( s \) as small as possible such that \((s, q)\) is ordinary. Then \( p < s < q \) and \( s \) is singular by \( 7.1.12 \). Hence \((s, q)\) is of order 3 by \( 7.1.13 \). Let \( H_\infty \) be a plane which does not meet \([p, q]\) and put \( P = A(s)A(q) \), \( S = A(p)A(q) \), \( Q = A(p)A(s) \), (Figure VII.4). Now for any \( t \in (s, q) \), \( A_2(t) \) meets \( P_t\) by \( 7.2.7 \) and \( Q_t\) by \( 7.2.8 \); thus \( A_2(t) \) meets \( S_\infty \), (Figure VII.5).

Since \( A(p) \subset A_2(q) \) then \( A(p) \not\subset A_2(s) \) by \( 7.1.14 \). Moreover \((s, q)\) is ordinary and hence \( A(q) \not\subset A_2(s) \) by \( 7.1.12 \). But then \( A_2(s) \) meets \( S_\infty \) and in fact cuts at \( s \), since \( s \) is regular. Since \( A_2(s) \) does not meet \((s, q)\), there exists \( r \in (p, s) \) such that \( A_2(s) \) cuts \( A \) at \( r \).

7.2.10 Lemma: If \( A \) is not of order 3 then there exist \( p, q \) with \( p < q \) such that \( \delta(p, A_2(q)) = 0 \).

Proof: \( A \) not of order 3 implies that there exist four distinct coplanar points \( p_1 < p_1 < p_2 < p_3 \). Consider \( \hat{A} = A/p \).
\( A \) has at most inflections, therefore \( a_0(p) = 1 \) and \( \hat{A} \) is Barner. Moreover \( A \) is Barner and \( A(p) \not\subset A_1(q) \), hence \( \hat{A}_1(q) = A_1(q)A(p) \) for all \( q \in (p_1, p_3) \) by \( 7.1.6 \), \( \hat{A}_1(q) = A_1(q)A(p) \) implies \( \hat{A}_1 \) is continuous on \((p, p_3)\).

Since \( \hat{a}_0(p) = a_1(p) = 1 \) and \( \hat{a}_1(p) = a_2(p) \), \( p \) is at most an inflection on \( \hat{A} \). If \( p \neq q \) then \( \pi(p, q) \geq 1 \) and \( \hat{a}_0(q) = a_0(q) = 1 \). Hence \( \hat{A} \) has at most inflections. Then \((p, p_3)\) contains at least one inflection on \( A_1 \); otherwise \((p, p_3)\) is regular and hence of order 2 on \( \hat{A} \) by \( 3.2.2 \); a contradiction since \( p_1 < p_2 < p_3 \) are collinear on \( \hat{A} \).
Figure VII.5
If there is only one inflection point $q$ of $\tilde{A}$ in $(p, p_3)$ then $(p, q)$ is regular on $\tilde{A}$ and hence of order 2 by 3.2.7. Then by 2.4.1, $\tilde{A}(p) \not\subset \tilde{A}_1(q)$ that is $A_1(p) \not\subset A_1(q)A(p)$. Now $q$ is an inflection on $\tilde{A}$, hence $\tilde{a}_0(q) = 1$, $\tilde{a}_1(q) = 2$. Then $\pi(p, q) = 1$ otherwise $\pi(p, q) = 2$ and therefore we have $\tilde{a}_1(q) = a_1(q) = 1$, a contradiction. Then $A(p) \subset A_2(q)$ and $A_1(p) \not\subset A_2(q)$ implies $\delta(p, A_2(q)) = 0$.

Suppose $q_1 < q_2$ are inflections of $\tilde{A}$ in $(p, p_3)$ By 3.3.11, there is an ordinary point $q_3$ of $\tilde{A}$ in $(q_1, q_2)$. Take $q_4, q_5$ such that

$$q_1 < q_4 < q_3 < q_5 < q_2$$

and $(q_4, q_5)$ is the largest regular subarc of $(q_1, q_2)$ in $\tilde{A}$. Then $(q_1, q_5)$ is of order 2 on $\tilde{A}$. If $\tilde{A}(p) \subset \tilde{A}_1(q)$ for all inflections $q$ of $\tilde{A}$ in $(p, p_3)$, $\tilde{A}(p) \subset \tilde{A}_1(q_4) \cap \tilde{A}_1(q_5)$, a contradiction by 3.1.9. Hence $\tilde{A}(p) \not\subset \tilde{A}(q)$ for some inflection $q$ of $A$ in $(p, p_3)$ and the result follows as above.

**7.2.11 Lemma:** If $A$ is regular but not of order 3, there exist $p, q, r$ with $p < q < r$, $q$ singular, such that $A(p) A_1(q) A(r)$ is a plane which cuts $A$ at $p$.

**Proof:** By 7.2.10, there exist $s < q$ such that $\delta(s, A_2(q)) = 0$. Since $a_2(s) = 1$, $A_2(q)$ cuts $A$ at $s$. By 7.2.9, we may assume $q$ is singular. By projection from $A_1(q)$, there is a $r$ with $q < r$ such that $A_1(q) A(r)$ cuts $A$ at a point $p < q$. 
Proof of 7.2.1: Assume $A$ is not of order 3. Take $p_1, q_1, r_1$ with properties of $p, q, r$ in 2.11. Let $X_1$ be a neighbourhood of $q_1$ such that $r_1 \not\in X_1$ and that for all $q \in X_1$, $A_1(q)$ meets $A$ at least in a point $p \not\in X_1$, $p < q_1$. Since $q_1$ is singular, we may repeat the argument using $X_1$ instead of $J$. Proceeding with the construction, we obtain a contradiction as in 3.2.7.

Corollary: Let $A$ be a regular Barner arc. If $A_2$ is continuous at $p$ the $p$ is ordinary.

Proof: Take $p_1, p_2$ with $A(p_i) \cap A_2(p)$ for $i = 1, 2$, and $A_1(p) \not\subset A_2(p)$. Since $A_2$ is continuous at $p$ take $U_1(p)$ such that $A(p_i) \not\subset A_2(q)$ for all $q \in U_1(p); i = 1, 2$. Put $A^{(i)} = A/p_i$. Then $U_1(p)$ is regular on $A^{(i)}; i = 1, 2$. By 7.2.4, $A_1$ is continuous at $p$. By 7.1.6, $A(p_i) \not\subset A_1(p)$ and thus

$$A_1^{(i)}(p) = A_1(p) A(p_i)$$

for $i = 1, 2$.

By 3.2.2, $U_1(p)$ is of order 2 on $A^{(i)}$; hence $p$ is ordinary on $A^{(i)}$ for $i = 1, 2$. In particular, there exists $U_2(p) \subset U_1(p)$ such that $A_1^{(i)}$ is continuous on $U_2(p)$ for $i = 1, 2$. Take $U(p) \subset U_2(p)$ such that

$$A_1(q) = \bigcap_{i=1}^{2} A_1^{(i)}(q)$$

for all $q \in U(p)$. Then $A_1$ is continuous on $U(p)$ and by 7.2.1, $U(p)$ is of order 3.
7.2.12 Lemma: Suppose \( A_2 \) is continuous at each point of a non-empty set \( W \). Then there is a subarc \( X \) which contains a point of \( W \) such that if \( p, q \in X, p \neq q, p \in W \) then \( A_2(p) \) does not cut \( A \) at \( q \).

The following notation will remain fixed: \( p_0 \) is a point at which \( A_2 \) is continuous, \( (p_1, p_2) \) is a neighbourhood of \( p_0 \) and \( H_\infty \) a plane which does not meet \( [p_1, p_2] \). \( A_2(p_0) \cap A[p_1, p_2] = A(p_0) \) and \( A(p_1) \not\subseteq A_2(p) \) for any \( p \in (p_0, p_2) \). \( L = A(p_0)A(p_1) \) a line since \( A \) is simple, and \( L_\infty(L_f) \) is the open segment of \( L \) with end points \( A(p_1), A(p_0) \) which meets (does not meet) \( H_\infty \).

The immediate result of the above hypothesis is that \((p_0, p_2)\) has order 2 on \( A/p_1 \) and \( A_1(p) \cap L = \emptyset \) for all \( p \in (p_1, p_2) \).

7.2.13 Lemma: Suppose \( p_0 \) is an inflection. If there is a \( p_3 \in (p_0, p_2) \) such that \((p_0, p_3)\) has order 2 on \( A/p_0 \), then for each \( p \in (p_0, p_3) \) either \( A_2(p) \) meets \( L_f \) or \( A(p_0) \subset A_2(p) \).

Proof: Since \( a_0(p) + a_1(p) + a_2(p) \equiv 0 \) (mod 2), \( p_1, p_3 \) lie on the same side of \( A_2(p_0) \). Since \( A_2(p) = \lim_{q \to p} A_1(p)A(p) \), take \( p_4 \in (p_0, p_3) \) such that \( p_1, p_3 \) lie on the same side \( A(p_0)A_1(p_4) \). Since \((p_0, p_3)\) is of order 2 on \( A/p_0 \), \( p_4 \) is regular on \( A/p_0 \). Hence by projection from \( p_0 \), \( A(p_0)A_1(p_4) \) supports \( A \) at \( p_4 \), (Figure VII.6). Now \( A(p_0)A_1(p_4) \) meets \((p_0, p_3)\) only at \( p_4 \) otherwise \((p_0, p_3)\) is not of order 2, \( p_1 \) and \((p_0, p_4)\) lie on the same side of \( A(p_0)A_1(p_4) \). Hence by projecting from \( A_1(p_4) \), there exist \( p_5 \in (p_0, p_4) \) such that \( A(p_5)A_1(p_4) \) cuts \((p_1, p_0)\); that is, \( A(p_5)A_1(p_4) \) meets \( L_f \).
Since \((p_0, p_3)\) has order 2 on \(A/p_1\) and \(A/p_0\), neither
\(A(p_1)\) nor \(A(p_0)\) lie on \(A(p) A_1(q)\) if \(p_0 < p < q < p_3\), by 2.4.1.
Thus \(A(p) A_1(q)\) meets \(L_f\) if \(p_0 < p < q < p_3\) and the result follows.

**Proof of 7.2.2:** Let \(W\) be the set of all points at which
\(A_2\) is continuous. Since we are assuming \(A\) is a Barner arc with at
most inflections on which \(A_1\) is continuous, \(W \neq \emptyset\) by 7.2.5. Choose
\(X\) according to 7.2.12, let \(p_0 \in X \cap W\). We may assume \(X = J\). Then
if \(p \in W\), \(A_2(p)\) does not cut \(A\) at any point except possibly at \(p\).

**Case 1.** \(p_0\) is regular. Then \(p_1, p_2\) lie on opposite sides
of \(A_2(p_0)\). Take \(p_3 \in (p_0, p_2)\) such that \(p_1, p_2\), lie on opposite sides
of \(A_2(p)\) for all \(p \in (p_0, p_3)\). Thus \(A_2(p)\) cuts \(A\) at \(p\) for all
\(p \in (p_0, p_3)\). Since \(A\) has at most inflections and \(A_1(p)\) does not
cut \(A\) outside of \(p\) for all \(p \in (p_0, p_3) \cap W\), then \((p_0, p_3) \cap W\) is
regular. Then \((p_0, p_3)\) is itself regular, by argument as in 3.3.11.
Hence \((p_0, p_3)\) is a regular Barner arc on which \(A_1\) is continuous, the
result now follows from 7.2.1.

**Case 2.** \(p_0\) is a point of inflection. This case leads to
a contradiction by arguments as in 3.3.11.

**Proof of 7.2.3** Let \(W\) be the set of inflections of \(A\).
We may assume \(A\) is not of order 3, hence \(W \neq \emptyset\) by 7.2.1. Let
\(X = J\) and \(p_0 \in W\).

**Case 1.** There exists \(p_3 \in (p_0, p_2)\) such that \((p_0, p_3)\) has
order 2 on \(A/p_0\). If there is a \(p \in (p_0, p_3) \cap W\), then by 7.2.13
and an argument as in 3.3.12, we obtain a contradiction. Thus \((p_0, p_3)\)
is regular and of order 3 by 7.2.1.
Case 2. No $u^+(p)$ has order 2 on $A/p_0$. By an argument as in 3.3.12, one obtains a contradiction.

